Jeroen Hofman

2513225
Homework 3
SPF 2012

1 Exercise 1

We have an 'all-or-nothing' contract with delivery date T with strike K, where the payoff is 0 if the underlying stock exceeds L during the lifetime of the contract, otherwise it is zero.

If we want to compute the price at $t \in [0, T]$, under the restriction that $S(s) < ln(L)$ for all $s \leq t$, then we can use Proposition 18.4 by noting that we want to compute the probability that the process $X = lnS$ reaches at least a level of $ln(L)$ during the interval $[t, T]$. This is equivalent to 1 minus the probability of having a running maximum of $ln(L)$. This probability is given by:

$$(1 - F_{M(T)}(ln(L))) - (1 - F_{M(t)}(ln(L))) = F_{M(t)}(ln(L)) - F_{M(T)}(ln(L))$$

where $F_{M(t)}(ln(L))$ is given by Proposition 18.4. Since the payoff will be paid out at time T and we want to determine the price at t, we have to discount back from T to obtain the price $V(t)$ of the 'all-or-nothing' option at time t:

$$V(t) = \exp\left(r(t - T)\right)(F_{M(t)}(ln(L)) - F_{M(T)}(ln(L)))$$

where r is the riskfree interest rate as given by the market.

2 Exercise 2

Let $T > 0$. We have the following problem:

$$\begin{aligned} \frac{\partial F(t,x)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F(t,x)}{\partial x^2} &= 0,\ x \in \mathbb{R},\ t \in [0, T] \\ F(T, x) &= x^3, x \in \mathbb{R} \end{aligned}$$

We follow example 5.7 from the book. From Proposition 5.5 we have that:

$$F(t, x) = E_{t,x}[X_T^3]$$

with

$$\begin{aligned} dX_s &= \sigma dW_s \\ X_t &= x \end{aligned}$$

giving

$$X_T = x + \sigma[W_T - W_t]$$

Now we compute $E_{t,x}[X_T^3]$ by inserting the solution above:

$$\begin{aligned}
E[X_T^3] &= E[x^3] + E[\sigma^3 W_{T-t}^3] + E[3x\sigma^2 W_{T-t}^2] + E[3x^2 W_{T-t}] \\
&= x^3 + \sigma^3 E[W_{T-t}^3] + 3x\sigma^2 E[W_{T-t}] + 3x^2 E[W_{T-t}] \\
&= x^3 + \sigma^3 E[W_{T-t}^3] + 3x\sigma^2 (T-t) + 0
\end{aligned}$$

We now compute $E[W_{T-t}^3]$ using Ito's lemma with $Z(t, X(t)) = x^3$:

$$\begin{aligned}
dZ(t) &= 3x^2 dW(t) + 0.5 * 6x dW(t) \\
&= 3W^2(t) dW(t) + 3W(t) dt
\end{aligned}$$

Now integrate and take the expectation:

$$\begin{aligned}
E[Z(t)] &= 3\int_t^T E[W^2(s)] dW(s) + 3\int_t^T E[W(s)] ds \\
&= 3\int_t^T s dW(s) + 0 \\
&= 0
\end{aligned}$$

So we obtain:

$$F(t,x) = E_{t,x}[X_T^3] = x^3 + 3x\sigma^2(T-t)$$

We can check the correctness by computing the PDE:

$$\begin{aligned}
\frac{\partial F(t,x)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F(t,x)}{\partial x^2} &= 0 \\
-3x\sigma^2 + 3x\sigma^2 &= 0
\end{aligned}$$

3 Exercise 3

We are given the process:

$$\begin{aligned}
&dX(t) = X(t)dt + dW(t), \; t \geq 0 \\
&X(0) = 0
\end{aligned}$$

The process $Y(t) = e^t \int_0^t e^s dW(s)$ satisfies this PDE, which can be seen by applying the product rule of Ito's lemma:

$$
\begin{aligned}
dY(t) = d\left(e^t \int_0^t e^s dW(s)\right) &= \int_0^t e^s dW(s) de^t + e^t d\left(\int_0^t e^s dW(s)\right) + de^t d\left(\int_0^t e^s dW(s)\right) \\
&= e^t \int_0^t e^s dW(s) dt + e^t e^{-t} dW(t) + e^t dt e^{-t} dW(t) \\
&= Y(t)dt + dW(t)
\end{aligned}
$$

since $dtdW(t) = 0$.

Also we have $Y(0) = 0$. If we use Lemma 4.15 we can now immediately conclude that $E[Y(t)] = 0$ and $Var[Y(t)] = e^{2t} \int_0^t e^{-2s} ds = -0.5e^{2t}(e^{-2t} - 1) = 0.5(e^{2t} - 1)$.