Finite-Difference Techniques for Financial Derivatives

Computational Finance

Scope of lecture

- The Black-Scholes PDE
- Concept of Finite-Difference Methods (FDM)
  - **Taylor Expansion**
  - **Several Basic Schemes**
  - **Equivalence Lax Theorem**
- Stability analysis (Von Neumann, Fourier decomposition)
- Analysing the order of convergence

The PDE is linear-hyperbolic

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

+

INITIAL & BOUNDARY CONDITIONS

V: Asset Value

S: Asset Underlying Value

r: Riskless (or arbitrage free) Interest rate

σ ***: Volatility (Standard deviation) of price (movement)***

Risk Terminology for Black-Scholes

Time-rate term or Theta (Θ)

Convective term or Delta (Δ)

Diffusion term or Gamma (Γ)

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

Rewritten Black Scholes in Risk Parameters:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV$$

Why using Finite Difference

■ When the number of dimensions is small, Finite-Difference may be faster than Monte Carlo (even when variance reduction techniques are used in MC).

■ In Finite-Difference, effort and accuracy always scale in the same way, in Monte Carlo scaling may jump (e.g. $(1/n)^{1/2}$ to $(1/n)^{3/2}$).

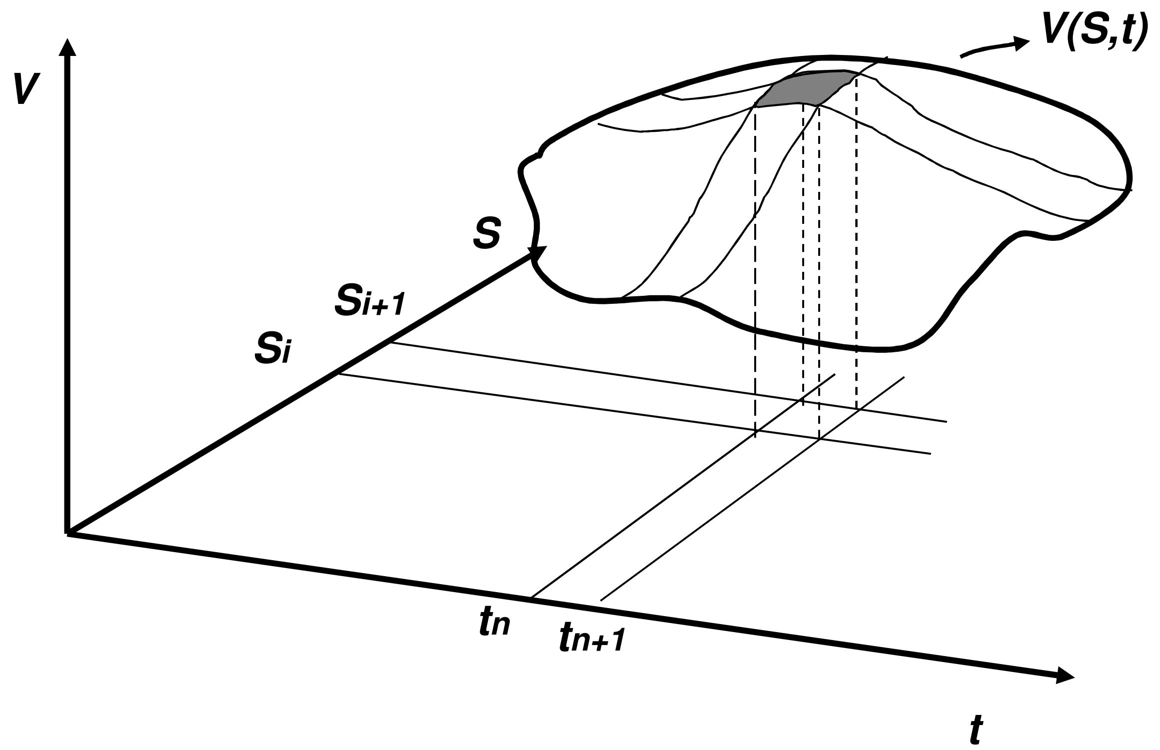
■ It can handle early exercise, discrete sampling, complex boundaries and barriers.

■ Finite-Difference are ideally suited for simultaneous solutions of multiple instruments

Disadvantage: only feasible for low dimensional problems < 4

We are looking for a surface in 3 dimensions

- **Divide the interval [0,T] in N equal sized subintervals (equidistancy)**
- **Divide the interval [0,Smax] in N equal sized subintervals (equidistancy)**



Transformation of Black-Scholes

For convenience the Black - Scholes is transformed to a constant coefficient PDE by introducing

$X = \ln S$ **and** $\tau = T - t$ **Where T is the exercise time**

The Black - scholes then turn into

$$\frac{\partial V}{\partial \tau} = \left(r - \frac{1}{2}\sigma^2 \right)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X^2} - rV$$

EQUATION IS SOLVED NUMERICALLY BY DISCRETE VARIANT

Examples:

1D expansion of the Asset Value in the Underlying Value

$$V(X+\Delta X,\tau)=V(X,\tau)+\Delta X\,\frac{\partial V(X,\tau)}{\partial X}+\frac{1}{2!}\Delta X^2\,\frac{\partial^2 V(X,\tau)}{\partial X^2}+\cdots$$

2D expansion of the Asset Value in the Underlying Value and Time

$$V(X+\Delta X,\tau+\Delta\tau)=V(X,\tau)+\Delta X\,\frac{\partial V(X,\tau)}{\partial X}+\Delta\tau\,\frac{\partial V(X,\tau)}{\partial \tau}+$$

$$\frac{1}{2!}\Delta X^2\,\frac{\partial^2 V(X,\tau)}{\partial X^2}+\frac{1}{2!}\Delta\tau^2\,\frac{\partial^2 V(X,\tau)}{\partial \tau^2}+\Delta\tau\Delta X\,\frac{\partial^2 V(X,\tau)}{\partial \tau\Delta X}+\cdots$$

Some approximation of temporal and spatial derivatives

$$\frac{\partial V}{\partial \tau}\left(i\Delta X, n\Delta \tau\right) \approx \frac{V_i^{n+1} - V_i^n}{\Delta \tau}$$

Forward difference

$$\frac{\partial V}{\partial x}\left(i\Delta X, n\Delta \tau\right) \approx \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta X}$$

Centered difference

$$\frac{\partial^2 V}{\partial X^2} \approx \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta X^2}$$

Centered difference

The Euler-Forward scheme

Substituting the former gives the FTCS (Forward Time Centered Scheme

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \left(r - \frac{1}{2}\sigma^2\right)\frac{1}{2\Delta X}\left(V_{i+1}^n - V_{i-1}^n\right) + \frac{1}{2}\sigma^2 \frac{1}{\Delta X^2}\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n\right) - rV_i^n$$

The scheme is 1st-order in time and 2nd-order in space.

The Euler-Backward scheme

The Euler-Forward or FTCS scheme can be made implicit in time, also known as the BTCS (Backward Time Centered Scheme)

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \left(r - \frac{1}{2}\sigma^2\right)\frac{1}{2\Delta X}\left(V_{i+1}^{n+1} - V_{i-1}^{n+1}\right) + \frac{1}{2}\sigma^2\frac{1}{\Delta X^2}\left(V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}\right) - rV_i^{n+1}$$

The scheme is still 1st-order in time and 2nd-order in space. And advantage is that it is more stable, due to increase of numerical diffusion.

Crank-Nicolson; Mixed Explicit/Implicit Method

Another way of obtaining 2nd-order accuracy in time is by constructing an averaged scheme from FTCS and BTCS

$$\frac{V_i^{n+1}-V_i^{n}}{\Delta\tau}=\left(r-\frac{1}{2}\sigma^2\right)\frac{1}{2\Delta X}\left(V_{i+1}^{n+1}-V_{i-1}^{n+1}+V_{i+1}^{n}-V_{i-1}^{n}\right)+$$

$$\frac{1}{2}\sigma^2\frac{1}{\Delta X^2}\left(V_{i+1}^{n+1}-2V_i^{n+1}+V_{i-1}^{n+1}+V_{i+1}^{n}-2V_i^{n}+V_{i-1}^{n}\right)-\frac{r}{2}\left(V_i^{n+1}+V_i^{n}\right)$$

Like the BTCS, the Crank-Nicolson scheme must be solved by solving matrix equations.

Generalization of Methods; Theta scheme

All former discretization schemes are family of the following general scheme:

$$\frac{V_i^{n+1}-V_i^n}{\Delta\tau}=\left(r-\frac{1}{2}\sigma^2\right)\left(\theta\frac{V_{i+1}^{n+1}-V_{i-1}^{n+1}}{2\Delta X}+(1-\theta)\frac{V_{i+1}^{n}-V_{i-1}^{n}}{2\Delta X}\right)+$$

$$\frac{1}{2}\sigma^2\left(\theta\frac{V_{i+1}^{n+1}-2V_i^{n+1}+V_{i-1}^{n+1}}{\Delta X^2}+(1-\theta)\frac{V_{i+1}^{n}-2V_i^{n}+V_{i-1}^{n}}{\Delta X^2}\right)-r\left(\theta V_i^{n+1}+(1-\theta)V_i^n\right)$$

$\theta = 1/2$ **Crank-Nicolson Scheme**

$\theta = 0$ **FTCS**

$\theta = 1$ **BTCS**

Basic Definitions; Equivalence Theorem of Lax

Consistency expresses that discretized equations should tend to the differential equations to which they are related when $\Delta\tau$ and ΔX tend to zero.

CONSISTENCY

CONDITION ON STRUCTURE OF NUMERICAL FORMULATION

Discretized Equation $\longleftrightarrow$ **Differential Equation**

Basic Definitions; Equivalence Theorem of Lax

Stability requires that difference scheme should not allow error to grow indefinitely, that is, to be amplified, as we progress from one time step $\Delta\tau$ to another.

STABILITY

↓

CONDITION ON SOLUTION OF NUMERICAL SCHEME

Numerical Solution ⟷ **Exact Solution of Discretized Equation**

Convergence requires that the numerical solution V_i^n aproach the exact solution $V\left(i\Delta X, n\Delta\tau\right)$ of the differential equation at any point and time, when $\Delta\tau$ and ΔX tend to zero.

CONVERGENCE

→ CONDITION ON SOLUTION OF NUMERICAL SCHEME

Numerical Solution ⟷ Exact Solution of Differetial Equation

Von Neumann method of stability analysis

$$V_i^n = V(i\Delta X, n\Delta\tau) + \varepsilon_i^n$$

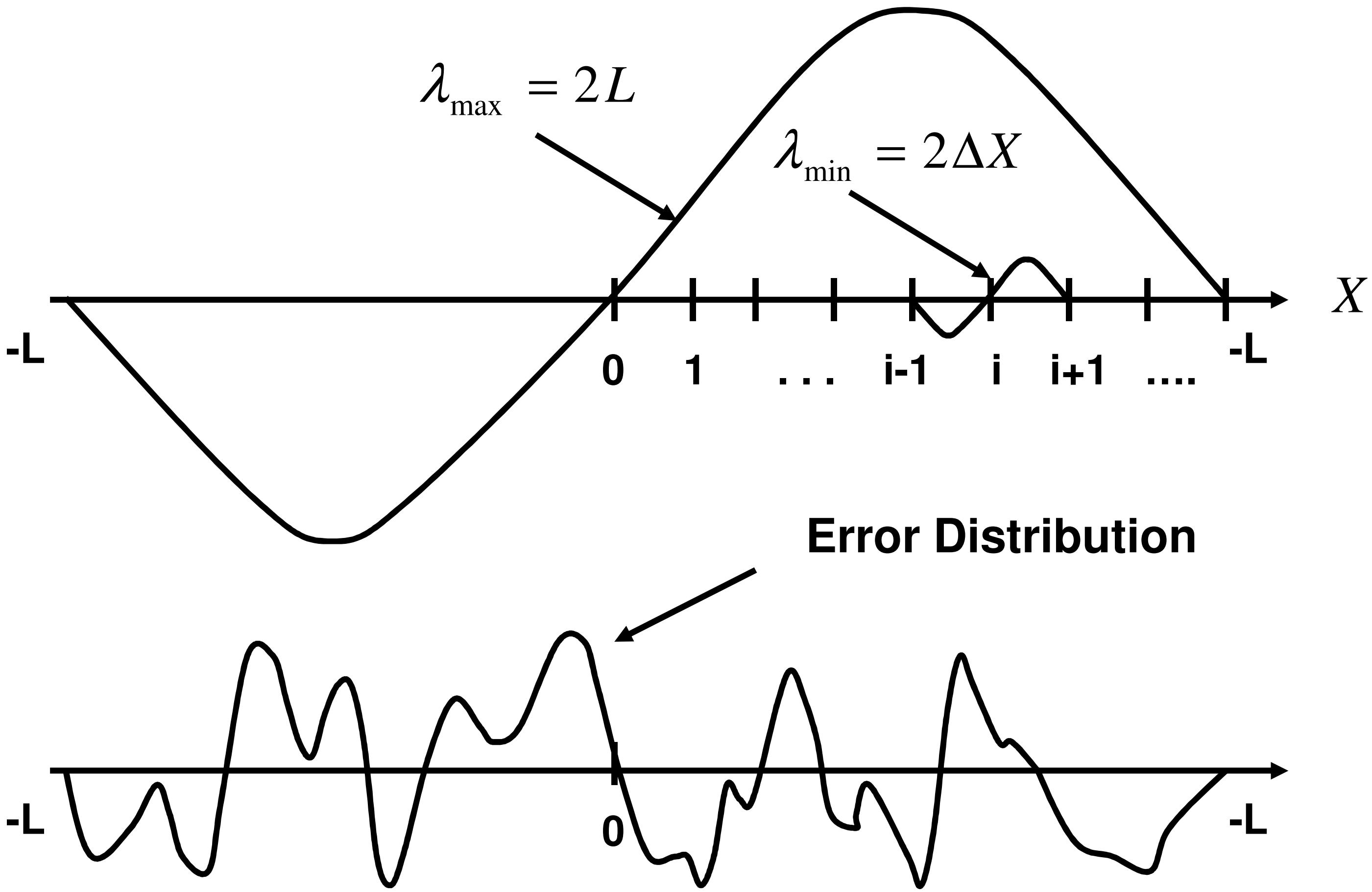
V_i^n : Computed Solution from FD Scheme

$V(i\Delta X, n\Delta\tau)$: Exact Solution

ε_i^n : Error at time level n mesh point i

The idea is to find a condition that bounds the Error ε_i^n as one advances in time.

Fourier representation of the error on interval (-L,L)



Stability Analysis

Now introduce:

$$\varepsilon_i^n = A^n(k)\exp(JKi\Delta X)$$

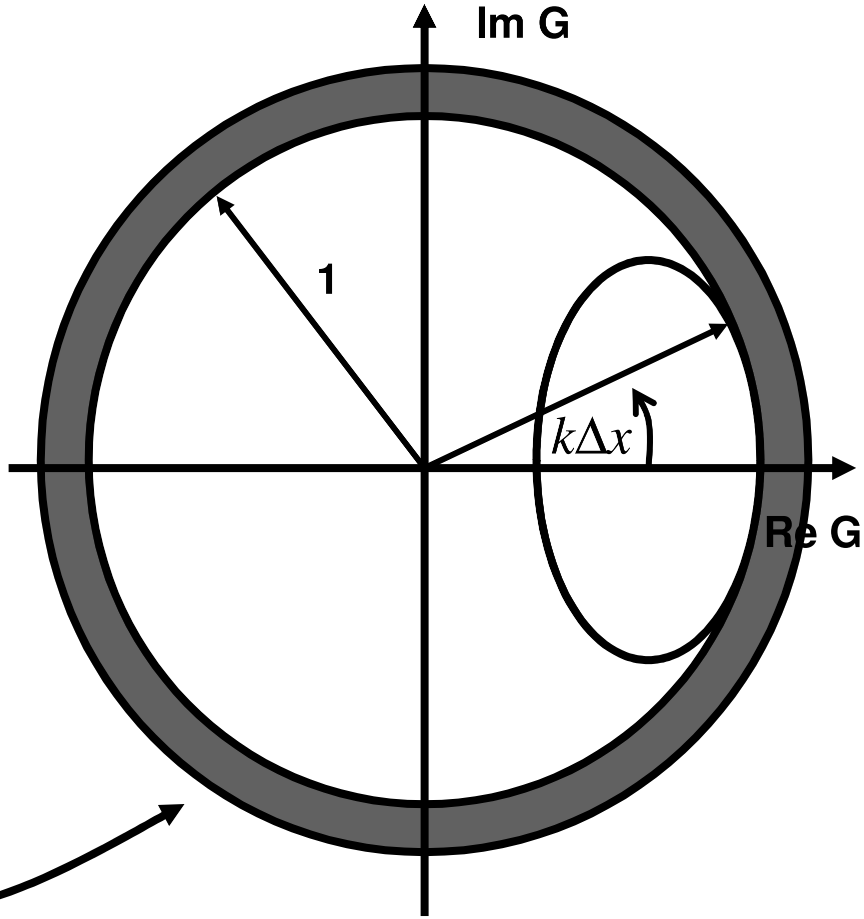
$$J = \sqrt{-1}$$

Where k is a wavenumber

For stability it is required that:

$$|G| = \left|\frac{A_i^{n+1}}{A_i^n}\right| \leq 1$$

For all $ki\Delta x$



Region of instability

Typical Polar plot of a stability diagram

Applying the Fourier representation of the error in the Crank-Nicolson scheme one can derive the following amplification factor.

$$G(\beta)=\frac{1-\sigma^2\dfrac{\Delta\tau}{\Delta X^2}\sin^2\dfrac{\beta}{2}+\left(r-\dfrac{\sigma^2}{2}\right)\dfrac{\Delta\tau}{2Xx}J\sin\beta-\dfrac{r}{2}\Delta\tau}{1+\sigma^2\dfrac{\Delta\tau}{\Delta X^2}\sin^2\dfrac{\beta}{2}-\left(r-\dfrac{\sigma^2}{2}\right)\dfrac{\Delta\tau}{2\Delta X}J\sin\beta+\dfrac{r}{2}\Delta\tau}$$

Wherein:

$\beta = k\Delta X$ **Provided that** $\longrightarrow$ $0 \le \beta \le \pi$

The amplification factor, being a complex function is rewritten as follows:

$$G(\beta)=\frac{(1-a)+Jb}{(1+a)-Jb}=\frac{(1-a)+Jb}{(1+a)-Jb}*\frac{(1+a)+Jb}{(1+a)+Jb}=$$

$$=\frac{(1-a)^2-b^2}{(1+a)^2+b^2}+J\,\frac{2b}{(1+a)^2+b^2}$$

Wherein:

$$a=\sigma^2\frac{\Delta\tau}{\Delta X^2}\sin^2\frac{\beta}{2}+\frac{r}{2}\Delta\tau \quad \textbf{and} \quad b=\left(r-\frac{\sigma^2}{2}\right)\frac{\Delta\tau}{2\Delta X}J\sin\beta$$

Then it is easy to see from the former that the stability condition requires that

$$|G(\beta)|^2 = \frac{\left(1-\sigma^2 \frac{\Delta\tau}{\Delta X^2}\sin^2\frac{\beta}{2} - \frac{r}{2}\Delta\tau\right)^2 + \left(r-\frac{\sigma^2}{2}\right)^2 \frac{\Delta\tau^2}{4\Delta X^2}\sin^2\beta}{\left(1+\sigma^2 \frac{\Delta\tau}{\Delta X^2}\sin^2\frac{\beta}{2} + \frac{r}{2}\Delta\tau\right)^2 + \left(r-\frac{\sigma^2}{2}\right)^2 \frac{\Delta\tau^2}{4\Delta X^2}\sin^2\beta} \leq 1$$

You are encouraged to do this yourself at home!

Errors in the Finite-Difference method arise from several sources and can be classified in:

- Modelling errors
- Discretization errors
- Errors for solving matrix equations
- Rounding errors

At this point we are just interest in the Discretization errors stemming from truncating the Taylor expansions

Analysing order of convergence

As an example we take the Crank-Nicolson scheme applied to the Diffusion Equation

$$\frac{\Phi}{\partial t} = \kappa \frac{\partial^2 \Phi}{\partial X^2}$$

Where κ is the diffusion coefficient

The discretized Diffusion Equation becomes:

$$\frac{\Phi_i^{n+1} - \Phi_i^n}{\Delta \tau} = \frac{1}{4}\kappa \left(\frac{\Phi_{i+1}^{n+1} - 2\Phi_i^{n+1} + \Phi_{i-1}^{n+1}}{\Delta X^2} + \frac{\Phi_{i+1}^n - 2\Phi_i^n + \Phi_{i-1}^n}{\Delta X^2} \right)$$

Analysing order of convergence

For this scheme it is convenient to have a symmetric expansion

$$\Phi_i^{n+1/2} = \Phi\left(i\Delta X, \left(n+1/2\right)\Delta\tau\right)$$

$$\Phi_i^{n+1} = \Phi\left(i\Delta X, \left(n+1/2\right)\Delta\tau + 1/2\Delta\tau\right) = \Phi_i^{n+1/2} + \frac{1}{2}\Delta\tau \left.\frac{\partial\Phi}{\partial\tau}\right|_i^{n+1/2} + \frac{1}{8}\Delta\tau^2 \left.\frac{\partial^2\Phi}{\partial\tau^2}\right|_i^{n+1/2} + \frac{1}{48}\Delta\tau^3 \left.\frac{\partial^3\Phi}{\partial\tau^3}\right|_i^{n+1/2} + \cdots$$

$$\Phi_i^{n} = \Phi\left(i\Delta X, \left(n+1/2\right)\Delta\tau - 1/2\Delta\tau\right) = \Phi_i^{n+1/2} - \frac{1}{2}\Delta\tau \left.\frac{\partial\Phi}{\partial\tau}\right|_i^{n+1/2} + \frac{1}{8}\Delta\tau^2 \left.\frac{\partial^2\Phi}{\partial\tau^2}\right|_i^{n+1/2} - \frac{1}{48}\Delta\tau^3 \left.\frac{\partial^3\Phi}{\partial\tau^3}\right|_i^{n+1/2} + \cdots$$

Perform similar technique for other terms in the discretized Diffusion Equation: The expansion becomes more tedious for the variables at location i+1 and i-1.

Analysing order of convergence

As an example:

$$\Phi_{i+1}^{n+1} = \Phi\big((i+1)\Delta X, (n+1/2)\Delta\tau + 1/2\Delta\tau\big) = \Phi_i^{n+1/2} + \left(\Delta X \frac{\partial \bullet}{\partial X} + \frac{1}{2}\Delta\tau \frac{\partial \bullet}{\partial \tau}\right) \cdot \Phi \Big|_i^{n+1/2} +$$

$$\frac{1}{2!}\left(\Delta X \frac{\partial \bullet}{\partial X} + \frac{1}{2}\Delta\tau \frac{\partial \bullet}{\partial \tau}\right)^2 \cdot \Phi \Big|_i^{n+1/2} + \frac{1}{3!}\left(\Delta X \frac{\partial \bullet}{\partial X} + \frac{1}{2}\Delta\tau \frac{\partial \bullet}{\partial \tau}\right)^3 \cdot \Phi \Big|_i^{n+1/2} + \cdots$$

$$= \Phi_i^{n+1/2} + \Delta X \frac{\partial \Phi}{\partial X}\Big|_i^{n+1/2} + \frac{1}{2}\Delta\tau \frac{\partial \Phi}{\partial \tau}\Big|_i^{n+1/2} + \frac{1}{2}\Delta X^2 \frac{\partial^2 \Phi}{\partial X^2}\Big|_i^{n+1/2} + \frac{1}{2}\Delta X \Delta\tau \frac{\partial^2 \Phi}{\partial X \partial \tau}\Big|_i^{n+1/2} + \frac{1}{8}\Delta\tau^2 \frac{\partial^2 \Phi}{\partial \tau^2}\Big|_i^{n+1/2} +$$

$$+ \frac{1}{6}\Delta X^3 \frac{\partial^3 \Phi}{\partial X^3}\Big|_i^{n+1/2} + \frac{1}{4}\Delta X^2 \Delta\tau \frac{\partial^3 \Phi}{\partial X^2 \partial \tau}\Big|_i^{n+1/2} + \frac{1}{8}\Delta X \Delta\tau^2 \frac{\partial^3 \Phi}{\partial X \partial \tau^2}\Big|_i^{n+1/2} + \frac{1}{48}\Delta\tau^2 \frac{\partial^3 \Phi}{\partial \tau^3}\Big|_i^{n+1/2} + \cdots$$

Doing this for all the terms, one can show that

$$\frac{\Phi}{\partial t}\Big|_i^{n+1/2} = \kappa \frac{\partial^2 \Phi}{\partial X^2}\Big|_i^{n+1/2} + O\left(\Delta t^2\right) + O\left(\Delta X^2\right)$$

Analysing order of convergence

This the Crank-Nicolson scheme is 2nd-order accurate in time and space.

The same can be shown for the Black-Scholes equation, by taking into account convective term and source term.