Financial Mathematics for Insurance

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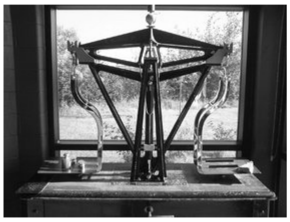
Static Replication

Recent News

Pensioenakkoord bevat fundamenteel misverstand tussen waardering en rendement

Tjitsger Hulshoff - 16 aug 2011 - Pensioen - 841 keer bekeken



Een pensioenuitkering is te zien als een soort obligatie. De waarde daarvan wordt bepaald door de actuele rente (*yield*) op deze obligatie. Dat het risicoprofiel van de beleggingen anders is – en in de toekomst mogelijk een hoger rendement dan deze rente genereert is voor de waardering van de verplichtingen niet relevant. Dit fundamentele verschil tussen waardering en rendement lijkt door de onderhandelaars van het pensioenakkoord niet juist begrepen, stelt Tjitsger Hulshoff. In dit akkoord is als waarderingsrente het verwachte rendement gekozen, wellicht om de zaken rooskleuriger voor te stellen dan ze zijn.

Recent News

Actuarieel Genootschap

De heer dr. P.H. Omtzigt
Rapporteur Pensioenen Europa
namens de Regering en de Tweede Kamer
Postbus 20018
2500 EA 's Gravenhage

Utrecht, 23 augustus 2012

Betreft: De Ultimate Forward Rate Methodiek / Notitie van het AG

Een gunstige eigenschap van de methodiek is dat het de volatiliteit van de waarde van de verplichtingen reduceert en daarmee op zich bijdraagt aan de stabiliteit van pensioenfondsen. Invoering van de UFR methode zou op dit moment een daling van de verplichtingen en daarmee tot een stijging van de dekkingsgraden tot gevolg kunnen hebben.

De UFR-methode heeft ogenschijnlijk duidelijke voordelen maar de methode kent ook een aantal minder gunstige eigenschappen, zo zouden fondsen zich rijk kunnen rekenen en kan de invoering van de UFR ongewenste generatie-effecten kan hebben. Het is zaak deze effecten vooraf duidelijk inzichtelijk te maken. Deze effecten zouden ertoe kunnen leiden dat de voordelen, die aan de methodiek verbonden zijn, teniet worden gedaan.

Hoogachtend,

Drs. R.K. Sagoenie
Voorzitter AG

Goals of the course

At the end of the course you should

- Understand the theoretical background of dynamic models needed for market-consistent valuation
- Be able to apply the basic techniques to insurance products and retirement provisions
- Know which crucial assumptions underlie the theory, and be able to see when these are problematic

Role of the actuary
Kinds of actuaries

Bühlmann made a distinction between three kinds of actuaries:

- First kind: all processes are deterministic (as in tradtional life insurance actuarial practice, with fixed valuation interest rate and fixed mortality table)
- Second kind: the insurance processes are stochastic (as in current P&C actuarial practice, with fixed interest rate and stochastic processes for claims)
- Third kind: both insurance processes and financial components are stochastic (we want to move to that direction in this course)

Role of the actuary
Towards active risk management

Passive risk management was built on conservative assumptions and waiting for better times when it went wrong (Equitable case in UK)

Active risk management is required:

- Translate changing environment directly in numbers
- Respond to a changing environment (low interest rates: extend the duration; low capital: take less risk,…)
- Even if life insurance is long term business, we can act in the short term to protect the long term
- If the actuary wants to remain central in risk management he must change to the “third kind”

Basic idea fair value
Road map for calculations

1 If you have reliable market prices use these

Reliable: liquid markets

2 If you have no market prices, but are able to replicate, then the price of the replicating portfolio is the price of the instrument

Replication: cash flow matching

3 If neither 1 or 2 applies, make a model to price the instrument as consistent as possible. Use of a Market Value Margin (MVM) is necessary

MVM: prudence where it is inevitable, but make calculation explicit

Integrating Insurance with Mathematical Finance

Mathematical Finance is a quantitative approach to investment and financial risk management problems, using mathematical modelling and analysis techniques.

– Why mathematical models possible for financial products
  (i.e. why not simply question of supply and demand) ?

  Internal consistency constraints (economic equilibrium theory)

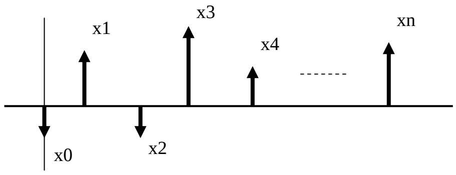
– Why useful to apply mathematical analysis to investment problem
  (i.e. why not simply pick suitable financial products directly) ?

  Optimization, due to non-trivial balance between risk and return.

Interest: The Time Value of Money

Cashflows assumed to be fixed and known in advance, at fixed equidistant points in time.

Notation: $(x_{0,} x_{1,}, \ldots, x_n)$ for cashflows at time 0,1,…,n



$$PV(x_0, x_1, \ldots, x_n) = \sum_{k=0}^{n} \frac{x_k}{(1+r)^k}$$

with r the fixed rate of interest per period.

Concept of Ideal Bank with fixed rate ("rekenrente")

Bank which is willing to swap cashflows schemes if they have the same present value:

- No Transaction Costs
- Constant rate of interest r in the future
- Same rate r for borrowing and lending
- No default risk or change in credit quality

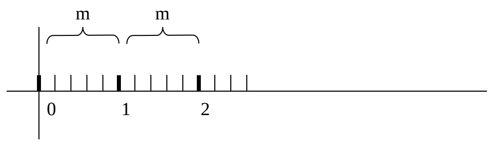
Clearly this is an idealization; we remove unrealistic aspects one by one in the modelling process.

Compounding

m	payments a year
r/m	interest paid per period

then yearly interest r' calculated by
or other way around

$$1 + r = \left(1 + \frac{r^{(m)}}{m}\right)^m$$



If $m \to \infty$ then <u>continuous compounding</u> at time t (in years)

$$\left(1+\frac{r}{m}\right)^{mt} \xrightarrow{m\to\infty} e^{rt}$$

Effective Rates

Compounding makes effective rate slightly higher, due to reinvestment:

Example:

5% annually	$(1.05)^1 = 1.05000$
5% semi-annually	$(1.025)^2 = 1.05063$
5% quarterly	$(1.0125)^4 = 1.05094$
5% monthly	$(1+\frac{0.05}{12})^{12} = 1.05116$
5% continuously	$e^{0.05} = 1.05127$

$$r^{(m)} = 0.05$$

Example : Compounding

LAS VEGAS Een vaste klant van het Palace Station Hotel and Casino in Las Vegas heeft zondag een record-jackpot van 27,6 miljoen dollar (51,9 miljoen gulden) in de wacht gesleept. De vrouw, die uit de omgeving van de gokstad komt, had zo'n 300 dollar verspeeld voordat de grote prijs er uitrolde. Ze is van plan de opbrengst te delen met haar drie kinderen en haar kleinkind. De winnares zal 25 jaar lang jaarlijks 1,1 miljoen dollar (ruim 2 miljoen gulden) uitbetaald krijgen.

Volkskrant, 1999

Discounting not taken into consideration

Example : Compounding

LAS VEGAS Een vaste klant van het Palace Station Hotel and Casino in Las Vegas heeft zondag een record-jackpot van 27,6 miljoen dollar (51,9 miljoen gulden) in de wacht gesleept. De vrouw, die uit de omgeving van de gokstad komt, had zo'n 300 dollar verspeeld voordat de grote prijs er uitrolde. Ze is van plan de opbrengst te delen met haar drie kinderen en haar kleinkind. De winnares zal 25 jaar lang jaarlijks 1,1 miljoen dollar (ruim 2 miljoen gulden) uitbetaald krijgen.

$$1.1 \sum_{k=0}^{24} \frac{1}{(1.07)^k} = 13.7$$

Effects of discounting can be dramatic, and misuse can lead to serious errors.

Ideal Bank is willing to transform cashflow scheme into another cashflow scheme as long as Present Values are the same.

Example:

(1,0,0)	equivalent with	(0, 1+r, 0)
(3,2,0)	equivalent with	(0, 0, $3(1+r)^2 + 2(1+r)$)

<u>Project Evaluation</u>

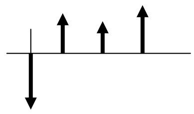
Comparing projects $(x_{0,} x_{1,} ..., x_n)$ and $(y_{0,} y_{1,} ..., y_n)$

take project with highest PV since

$(x_{0,} x_{1,} ..., x_n)$	equivalent with	$(PV_{x,} 0,...,0)$
$(y_{0,} y_{1,} ..., y_n)$	equivalent with	$(PV_{y,} 0,...,0)$

Typically $x_0 < 0$ (invest now)

$x_1 ... x_n \geq 0$ (return later)



Internal Rate of Return

Different Performance Evaluation criterion: <u>Internal rate of Return</u>

IRR = Rate of interest that would make PV of cashflows zero

So IRR of cashflow stream $(x_{0,} x_{1,} ..., x_n)$ is equal to r* if

$$\boxed{0 \quad = \quad x_0 + \frac{x_1}{(1+r^*)} + \frac{x_2}{(1+r^*)^2} + ... + \frac{x_n}{(1+r^*)^n}}$$

If $r^* \geq r$ then sensible investment compared to ideal bank account

$r^* < r$ then it is better to put money in ideal bank account

NPV vs. IRR

NPV

- Advantage: Easy to calculate
  Linear
- Disadvantage: What r to use in the real world ?

IRR

- Advantage: No ideal bank rate r needs to be specified
- Disadvantage: Hard (or even impossible !) to calculate
  Nonlinear

Applying the two methods to the same investment problem may give conflicting recommendations, but they usually agree.

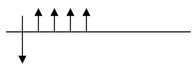
Example: Real Estate Project

Short Term Project: Invest \$ 500.000 now and receive \$ 150.000 in the next 4 years, at the end of each year.

Long Term Project: Invest \$ 500.000 now and receive \$ 82.000 in the next 8 years, at the end of each year.

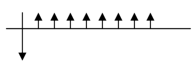
Interest: 4%

$$NPV_1 = -500.000 + \sum_{k=1}^{4} 150.000(1.04)^{-k} = 44.484,-$$



$$NPV_1 = -500.000 + \sum_{k=1}^{8} 82.000(1.04)^{-k} = 52.085,-$$

$$0 = -500.000 + \sum_{k=1}^{4} 150.000(1+r_1^*)^{-k} \quad \Rightarrow \quad r_1^* = 7.7\%$$



$$0 = -500.000 + \sum_{k=1}^{8} 82.000(1+r_2^*)^{-k} \quad \Rightarrow \quad r_2^* = 6.4\%$$

Issues

- Reinvestment ?
  (what do I do with cashflows I receive during the project)
- Missed Opportunities ?
  (how long is the invested money tight up)
- One-term project or repeated investment possible ?
  (discounting of future rewards after this project)

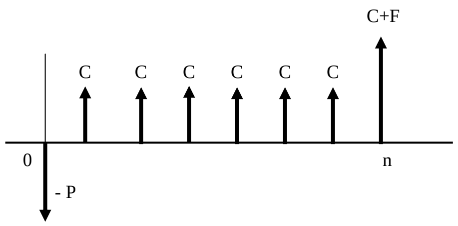
Typical for Life Insurance : uncertainties in time horizon for investment !

If there would indeed be a known, eternally fixed interest rate from an infinitely creditworthy institution without operational risk which charges the same for borrowing and lending, then there would not be a problem since reinvestment risk would not be an issue !

But there isn't such an institution...

Bond Markets

Investing in Treasury Bonds = series of promised cashflows



Period Half a year in US (semi-annual payments), year in NL

F	Face Value or Principal (Example: $ 10.000,-)
n	Number of periods until maturity
C	Coupon payment (usually specified as a percentage of F)
P	Price of the bond (usually specified as a percentage of F)

Bond Markets

Main investment vehicle for large institutions such as life insurance companies; highly liquid.

Issues
- What is a fair price for a bond, given its specifications (i.e. given its cashflows ?
- What investment risks do we run when holding bonds ?
- How can we reduce these risks ?

Methods
- Assessing bonds, not comparing prices but rates of return (yield)
- Quantify how bond prices react to market changes
- Use these to reduce risk, by designing a robust portfolio of bonds

Valuation of Cashflows

To carry out this program we need a proper valuation method.

Step 1 Value of a Perpetual Annuity

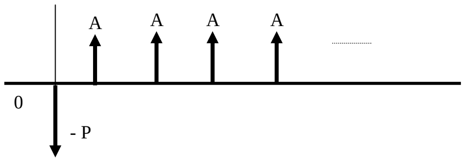
Step 2 Value of an Ordinary Annuity

Step 3 Value of a Zero Coupon Bond

Step 4 Value of a Bond

Assume r is the fixed interest rate per period
coupons paid once a year in NL so then period is 1 year
coupons paid every six months in US, so there period is ½ year

Step 1 : Valuation of a Perpetual Annuity



$$P = \frac{A}{1+r} + \frac{A}{(1+r)^2} + \frac{A}{(1+r)^3} + \ldots.$$

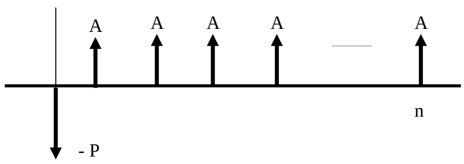
$$= \frac{A}{1+r} + \frac{1}{1+r}\left(\frac{A}{1+r} + \frac{A}{(1+r)^2} + \ldots\right)$$

$$= \frac{A}{1+r} + \frac{1}{1+r}P$$

$$\boxed{P = \frac{A}{r}} \qquad a_{\overline{\infty}|r}$$

$$\Rightarrow \quad P\left(1 - \frac{1}{1+r}\right) = \frac{A}{1+r}$$

Step 2 : Valuation of an Annuity

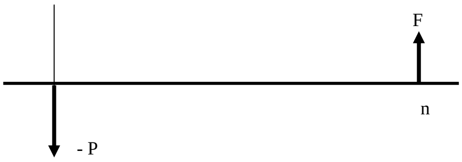


$$P = \frac{A}{r} - \frac{A}{r}\frac{1}{(1+r)^n}$$

$$= \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

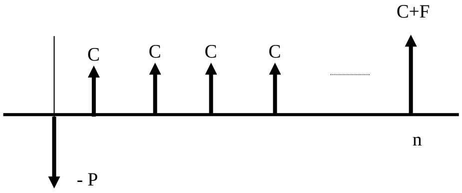
$$\boxed{A = \frac{rP}{1-(1+r)^{-n}}} \qquad a_{\overline{n}|r}$$

Step 3 : Valuation of a Zero Coupon Bond



$$P = \frac{F}{(1+r)^n}$$

Step 4 : Valuation of a Bond

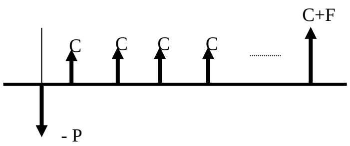


$$P = \frac{C}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{F}{(1+r)^n}$$

Yield

But what is the appropriate rate of interest per period ?

Price P of bond = Present value of cashflows (coupons + face value)



so this investment scheme has a PV of zero for a certain interest rate λ which is the IRR for the bond

Yield λ of bond = Internal Rate of Return for bond investment

Comparison of different bonds should be based on IRR (i.e. λ) instead of NPV (i.e. P), since ‘similar’ bonds should have ‘similar’ yields.

Bond Pricing Formula

$$P = \frac{F}{\left(1+\frac{\lambda}{m}\right)^n} + \frac{C}{\lambda}\left(1-\frac{1}{\left(1+\frac{\lambda}{m}\right)^n}\right)$$

n	Number of periods until maturity	contract
F	Face value	contract
C	Coupon payment per year	contract
m	Number of periods in a year	contract
λ	Yield (yearly IRR)	market
P	Price	market

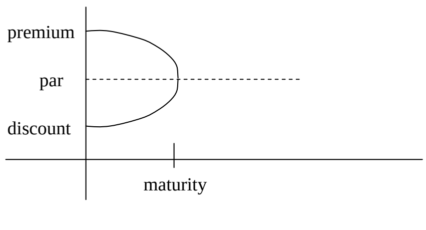
P known for fixed contract ➔ λ can be calculated
λ known for fixed contract ➔ P can be calculated

$C = \lambda F \Rightarrow P = F$	par bond	quote = 100
$C > \lambda F \Rightarrow P > F$	premium bond	quote > 100
$C < \lambda F \Rightarrow P < F$	discount bond	quote < 100

Bond price changes due to passage of time

In the bond pricing formula the value of n (number of payment periods remaining) gets smaller.

Assume that yield λ stays constant over the time period considered:



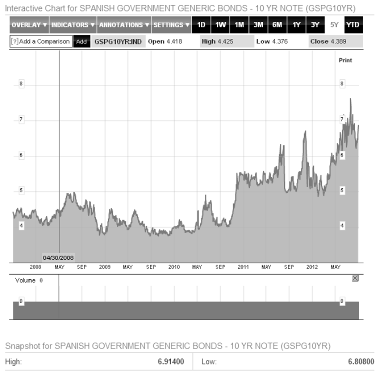
- Size of discount or premium decreases as time progresses.
- This happens at an increasing speed.

Bond price changes due to Change in credit quality

Problem of default: Non-treasury bond issuer may not pay because of financial difficulties (Russia, 1999)

Moody's / Standard & Poor's credit ratings

Lower rating implies higher default risk implies higher required yield than on Treasuries the yield spread)



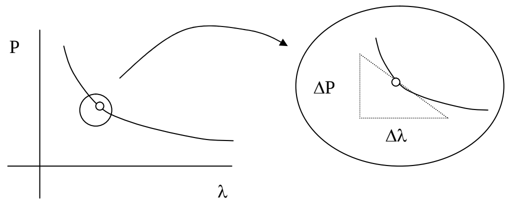
Bond Price Volatility Risk

- If investment goal is safeguarding known, fixed cashflows then Treasuries pose no risk, corporate bonds only default risk
- If cashflow obligations in future may change, then part of bond portfolio will have to be bought/sold later, or reinvested:
  - Interest Rate Risk (bond value decreases due to yield change)
  - Reinvestment Risk (coupons reinvested against worse yields)
  - Inflation Risk (cashflows need to be higher to compensate)
  - Liquidity Risk (bid-ask spread if hard to sell/buy bond)

First two risks due to yield changes.

Needed: quantification of influence of yield changes on price.

Price Sensitivity to Yield



$$\Delta P \approx \frac{\partial P}{\partial \lambda}\Delta\lambda$$

Tangent (slope)

$$P = \frac{C}{\left(1+\frac{\lambda}{m}\right)} + \frac{C}{\left(1+\frac{\lambda}{m}\right)^2} + \ldots. + \frac{C+F}{\left(1+\frac{\lambda}{m}\right)^n}$$

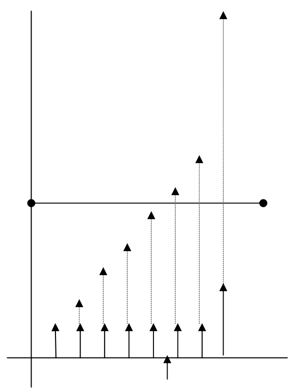
$$\frac{\partial P}{\partial \lambda} = \frac{\left(-\frac{1}{m}\right)C}{\left(1+\frac{\lambda}{m}\right)^2} + \frac{\left(-\frac{2}{m}\right)C}{\left(1+\frac{\lambda}{m}\right)^3} + \ldots. + \frac{\left(-\frac{n}{m}\right)(C+F)}{\left(1+\frac{\lambda}{m}\right)^{n+1}}$$

$$\frac{\partial P/\partial\lambda}{P} = -\frac{1}{\left(1+\frac{\lambda}{m}\right)}\underbrace{\left(\frac{\left(\frac{1}{m}\right)C}{\left(1+\frac{\lambda}{m}\right)^1} + \frac{\left(\frac{2}{m}\right)C}{\left(1+\frac{\lambda}{m}\right)^2} + \ldots. + \frac{\left(\frac{n}{m}\right)(C+F)}{\left(1+\frac{\lambda}{m}\right)^n}\right)\frac{1}{P}}_{Macaulay\quad Duration}$$

Calculating Price Changes due to Market Movements

$$\boxed{\frac{\Delta P}{P} \approx -D_M\,\Delta\lambda}$$

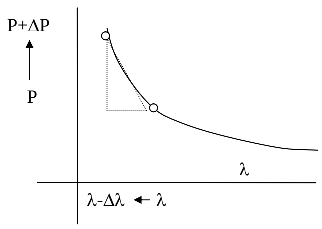
$$D_M = \frac{\sum_{k=0}^{n} t_k PV(t_k) \Big/ \sum_{k=0}^{n} PV(t_k)}{\left(1+\frac{\lambda}{m}\right)}$$



Modified Duration

‘Cash-weighted time to maturity’

Visual Interpretation of Duration



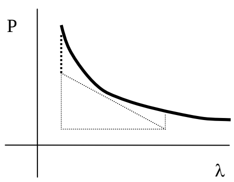
Knowing modified duration, we can predict change in price from change in yield.

Example: Price 90, Yield 7 %, Mod. Dur 5.0 yrs.
If yield changes from 7 % to 6.5 % then

$$\begin{aligned}\Delta P &= -D_m \cdot P \cdot \Delta\lambda \\ &= -5 \cdot 90 \cdot (-0.005) \\ &= 2.25\end{aligned}$$

so new price will be approximately 92.25

Visual Interpretation



If yield goes up/down, approximation based on duration underestimates new price due to curvature of price-yield graph.

Extra correction: Convexity

$$C = \frac{1}{P}\frac{\partial^2 P}{\partial \lambda^2}$$

$$\boxed{\frac{\Delta P}{P} = -D_m \Delta\lambda + \tfrac{1}{2} C (\Delta\lambda)^2}$$

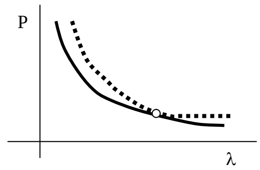
Convexity

Convexity is the <u>second derivative</u> of the price of a bond with respect to its yield,

So it is the <u>derivative of the duration</u> with respect to its yield

Visual interpretation: curvature of price-yield curve gives <u>extra correction term</u> which shows deviation from the first order (straight line) approximation

Value of convexity (pure positive influence !)



Calculating Convexity

Take second derivative, then use definition $C = \frac{1}{P}\frac{\partial^2 P}{\partial \lambda^2}$

$$
\begin{aligned}
\frac{\partial^2 P}{\partial \lambda^2} &= \frac{\partial}{\partial \lambda}\left(\frac{\partial P}{\partial \lambda}\right) \\
&= \frac{\partial}{\partial \lambda}\left(\frac{\left(-\frac{1}{m}\right)C}{\left(1+\frac{\lambda}{m}\right)^2} + \frac{\left(-\frac{2}{m}\right)C}{\left(1+\frac{\lambda}{m}\right)^3} + \ldots. + \frac{\left(-\frac{n}{m}\right)(C+F)}{\left(1+\frac{\lambda}{m}\right)^{n+1}}\right) \\
&= \frac{\left(-\frac{1}{m}\right)\left(-\frac{2}{m}\right)C}{\left(1+\frac{\lambda}{m}\right)^3} + \frac{\left(-\frac{2}{m}\right)\left(-\frac{3}{m}\right)C}{\left(1+\frac{\lambda}{m}\right)^4} + \ldots. + \frac{\left(-\frac{n}{m}\right)\left(-\frac{n+1}{m}\right)(C+F)}{\left(1+\frac{\lambda}{m}\right)^{n+2}} \\
&= \left(\frac{1}{m^2}\right)\left(\sum_{i=1}^{n}\frac{i(i+1)C}{\left(1+\frac{\lambda}{m}\right)^{i+2}} + \frac{n(n+1)F}{\left(1+\frac{\lambda}{m}\right)^{n+2}}\right)
\end{aligned}
$$

Note: first factor 1/m² makes sure convexity is measured in years².

Example: Convexity Adjustment

6% coupon bond maturing in 25 yrs, selling to yield 9%

Investment Calculator:	Modified Duration (yrs)	10.62
	Convexity (yrs)	182.92

Required yield changes from 9% to 11% (200 bp) then

Percentage price change due to duration	$-(10.62)(0.02)$	= -21.24 %
Percentage price change due to convexity	$\frac{1}{2}(182.92)(0.02)^2$	= +3.66%
Total approximation for price change		= -17.58 %

Required yield changes from 9% to 7% (-200 bp) then

Percentage price change due to duration	$-(10.62)(-0.02)$	= +21.24 %
Percentage price change due to convexity	$\frac{1}{2}(182.92)(0.02)^2$	= +3.66%
Total approximation for price change		= +24.90 %

Approximations are quite good: real values are -18.03 % and +25.46%

Duration Matching

Given the liabilities, create an asset (bond) portfolio which has

- Same PV as the liabilities
  (i.e. enough value now in portfolio to pay liabilities)
- Same (Modified) Duration as liabilities
  (i.e. if yields change we will <u>still</u> have enough)

Portfolio:
invest in a_1 bonds with price P_1 and duration D_1
invest in a_2 bonds with price P_2 and duration D_2
…
invest in a_n bonds with price P_n and duration D_n

PV portfolio $a_1P_1 + a_2P_2 + \dots + a_nP_n$

Duration portfolio $$\frac{a_1P_1 \cdot D_1 + a_2P_2 \cdot D_2 + \dots + a_nP_n \cdot D_n}{PV_{portfolio}}$$

Duration Matching

PV portfolio $$PV_{Liabilities} = a_1 \cdot P_1 + a_2 \cdot P_2 + ... + a_n \cdot P_n$$

Duration portfolio $$D_{Liabilities} = a_1 \cdot \frac{P_1 D_1}{PV} + a_2 \cdot \frac{P_2 D_2}{PV} + ... + a_n \cdot \frac{P_n D_n}{PV}$$

Linear Problem: (Easily) Solvable.

Result A portfolio which is protected against small changes in yield (i.e. against moderate interest rate risk).

Earlier Assumption: One discount rate for all maturities

In practice

$\boxed{\text{Yields depend on time to maturity}}$

since people want higher returns on money that is tied up for a longer period (missed opportunities !)

Important for immunization, where one yield was assumed, and therefore one sensitivity to yield changes, for bonds with different maturities !

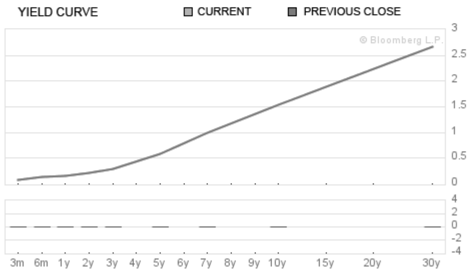
Yield as a function of time to maturity: yield curve.

Government Bonds

U.S. UK Germany Japan Hong Kong Australia Brazil

U.S. Treasuries

	COUPON	MATURITY	PRICE/YIELD	PRICE/YIELD CHANGE	TIME
3-Month	0.000	11/29/2012	0.07 / 0.07	0.000 / 0.000	04:33
6-Month	0.000	02/28/2013	0.13 / 0.13	0.000 / 0.000	04:33
12-Month	0.000	08/22/2013	0.16 / 0.16	0.000 / 0.000	04:33
2-Year	0.250	08/31/2014	100-01+ / 0.22	0-00 / 0.000	04:33
3-Year	0.250	08/15/2015	99-28+ / 0.29	0-00 / 0.000	04:33
5-Year	0.625	08/31/2017	100-05½ / 0.59	0-00 / 0.000	04:33
7-Year	1.000	08/31/2019	100-00+ / 1.00	0-00 / 0.000	04:33
10-Year	1.625	08/15/2022	100-22+ / 1.55	0-00 / 0.000	04:33
30-Year	2.750	08/15/2042	101-19 / 2.67	0-00 / 0.000	04:33



Shortcomings of the Yield Curve

Yield curve based on US Treasuries

- No default risk
- Highly Liquid (especially, on-the-run issues)
- Quick incorporation of news (efficient market)

Benchmark for bond pricing <u>but</u> bonds with same maturity and credit quality still have different yields, due to differences in <u>coupons</u>

Therefore:

- Treat bond as package of zero-coupon bonds
- Determine yield-curve for zero-coupon bonds
- This gives each time its own distinct discount factor

Zero Rates (or "Spot Rates")

Zero rate for a certain time = Yield on zero-coupon bond of that maturity
= Rate which gives appropriate discount factor for that time

Example

Zero-coupon bond matures in 10 yrs, and trades at 54.48 quote. Then 10-yr zero rate z_{10} is determined by

$$100\left(1+\frac{z_{10}}{2}\right)^{-20} = 54.48 \quad \Rightarrow \quad z_{10} = 6.17\%$$

and discount factor for 10-yr cash flows will be $d_{10} = \left(1+\frac{z_{10}}{2}\right)^{-20} = 0.5448$

Present value of cash flow stream $(x_0, x_1, \ldots, x_n)$ is $\boxed{PV = x_0 + x_1 d_1 + \ldots + x_n d_n}$

Calculation of zero Rates

Bootstrapping

Market Data

6 months Note		Price	97.44
1 year Note		Price	94.88
9 months Bond	Coupon 5%	Price	99.38
2 year Bond	Coupon 6 %	Price	100.89

$$97.44 = \frac{100}{\left(1+\frac{z_{0.5}}{2}\right)^1} \quad \Rightarrow \quad z_{0.5} = 5.25\%$$

$$94.88 = \frac{100}{\left(1+\frac{z_1}{2}\right)^2} \quad \Rightarrow \quad z_1 = 5.32\%$$

$$99.38 = \frac{2.5}{\left(1+\frac{z_{0.5}}{2}\right)^1} + \frac{2.5}{\left(1+\frac{z_1}{2}\right)^2} + \frac{102.5}{\left(1+\frac{z_{1.5}}{2}\right)^3} \quad \Rightarrow \quad z_{1.5} = 5.44\%$$

$$100.89 = \frac{3}{\left(1+\frac{z_{0.5}}{2}\right)^1} + \frac{3}{\left(1+\frac{z_1}{2}\right)^2} + \frac{3}{\left(1+\frac{z_{1.5}}{2}\right)^3} + \frac{103}{\left(1+\frac{z_2}{2}\right)^4} \quad \Rightarrow \quad z_2 = 5.53\%$$

Calculation of Bond Price from Zero Rates

Price of a 7% coupon Treasury Bond maturing in 2 yrs should be, according to these zero rates:

$$P = \frac{7/2}{\left(1+\frac{0.00525}{2}\right)^1} + \frac{7/2}{\left(1+\frac{0.0532}{2}\right)^2} + \frac{7/2}{\left(1+\frac{0.0544}{2}\right)^3} + \frac{100+7/2}{\left(1+\frac{0.0553}{2}\right)^4} = 102.76$$

Method is slightly more accurate than when only yield curve would have been used, and the yield for this 2-yr bond would be assumed to equal the 2-yr bond yield in the market.

Bond's yield is 'average' of the zero rates (i.e. the zero yields) over its lifetime.
Can calculate 'at par coupon' C_n: coupon which makes nominal value = market value i.e.

$$1 = C_n \cdot (d_1 + d_2 + \ldots + d_n) + 1 \cdot d_n \quad \Leftrightarrow \quad C_n = \frac{1 - d_n}{d_1 + d_2 + \ldots + d_n}$$

also called 'par swap rates'. This gives <u>other method</u> to define term structure !
Bonds with par coupons are called 'par bonds'.

Replication of a new Bond

We know (from some source) the market value of the next four bonds.

We want to value the stream of cashflows in the last column, using the known market values in the first row

	Bond 1	Bond 2	Bond 3	Bond 4	
Value	995,22	962,95	1.360,83	778,35	????????
CF 1	1.040,00	30,00	500,00	232,00	2.242,00
CF 2		1.030,00	500,00	224,00	194,00
CF 3			500,00	216,00	1.216,00
CF 4				208,00	208,00

Market-Consistent Valuation of a new Bond

Look for a combination of the known bonds that replicates the unknown bond

Start with latest CF: you need 1 time bond #4
Third payment: you already have 216,00, so you need 2 times bond #3
Second payment: you already have 1.224,00 so you need -1 times bond #2
First payment: you already have 1.202,00 so you need 1 times bond #1

Market value: 1*995,22-1*962,95+2*1.360,83+778,35 = 3.532,38

Unique price! No “opinion”! Principle of ‘no arbitrage’ (notice assumptions)

cashflows #5 =
1*cashflows #4 + 2*cashflows #3 -1*cashflows $2 + 1*cashflows #1

then price #5 =
1*price #4 + 2*price #3 -1*price $2 + 1*price #1

Market-Consistent Valuation of Annuity Policy

Market Value should reflect time value of money

1. Take market prices of traded securities
2. Use risk-free, liquid instruments to derive term structure of interest rates
3. Use term structure to price liabilities (with fixed cash flows)

Example of valuation of life insurance annuity

Age 66, Maturity 4 years
Annual payments (4 payments left) of 1000,- EUR
Survival probabilities 1Y: 0.99 2Y: 0.98 3Y: 0.97 4Y: 0.95

4 Bonds in the market

Coupon Bond, Mat. 1Y, 4%,	MV = 995,22 EUR
Coupon Bond, Mat. 2Y, 3%,	MV = 962,95 EUR
Annuity, Mat. 3Y, €500,	MV = 1360,83 EUR
Equal Redemption, 4Y, €800, 4%,	MV = 778,35 EUR

Market-Consistent Valuation of Annuity Policy

Market

Value	995,22	962,95	1.360,83	778,35
CF 1	1.040,00	30,00	500,00	232,00
CF 2		1.030,00	500,00	224,00
CF 3			500,00	216,00
CF 4				208,00

$$\begin{aligned} 995.22 &= 1040\, d_1 \\ 962.95 &= 30\, d_1 + 1030\, d_2 \\ 1360.83 &= 500\, d_1 + 500\, d_2 + 500\, d_3 \\ 778.35 &= 232\, d_1 + 224\, d_2 + 216\, d_3 + 208\, d_4 \end{aligned}$$

$$\begin{bmatrix} 995.22 \\ 962.95 \\ 1360.83 \\ 778.35 \end{bmatrix} = \begin{bmatrix} 1040 & 0 & 0 & 0 \\ 30 & 1030 & 0 & 0 \\ 500 & 500 & 500 & 0 \\ 232 & 224 & 216 & 208 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1040 & 0 & 0 & 0 \\ 30 & 1030 & 0 & 0 \\ 500 & 500 & 500 & 0 \\ 232 & 224 & 216 & 208 \end{bmatrix}^{-1} \begin{bmatrix} 995.22 \\ 962.95 \\ 1360.83 \\ 778.35 \end{bmatrix} = \begin{bmatrix} 0.95694 \\ 0.90703 \\ 0.85769 \\ 0.80723 \end{bmatrix}$$

Market-Consistent Valuation of Annuity Policy

Market

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1040 & 0 & 0 & 0 \\ 30 & 1030 & 0 & 0 \\ 500 & 500 & 500 & 0 \\ 232 & 224 & 216 & 208 \end{bmatrix}^{-1} \begin{bmatrix} 995.22 \\ 962.95 \\ 1360.83 \\ 778.35 \end{bmatrix} = \begin{bmatrix} 0.95694 \\ 0.90703 \\ 0.85769 \\ 0.80723 \end{bmatrix}$$

Age 66, Maturity 4 years
Annual payments (4 payments left) of 1000,- EUR
Survival probabilities 1Y: 0.99 2Y: 0.98 3Y: 0.97 4Y: 0.95

Use survival probabilities to find expected cash flow

Discount expected cash flow using discount factors

Annuity = 1000 *(0.99 * 0.95694+
0.98 * 0.90703+
0.97 * 0.85769+
0.95 * 0.80723)
= 3435,09 EUR

Static Replication

Notice general principle here:

- We can replicate a cashflow pattern in terms of existing assets in the market
- Price of cashflows is price of replicating portfolio
- Discount rates are useful tool to calculate price directly but to determine them we need up-to-date market information
- Asset pricing here consists of pricing new asset in terms of assets which have already been priced