Financial Mathematics for Insurance

Michel Vellekoop

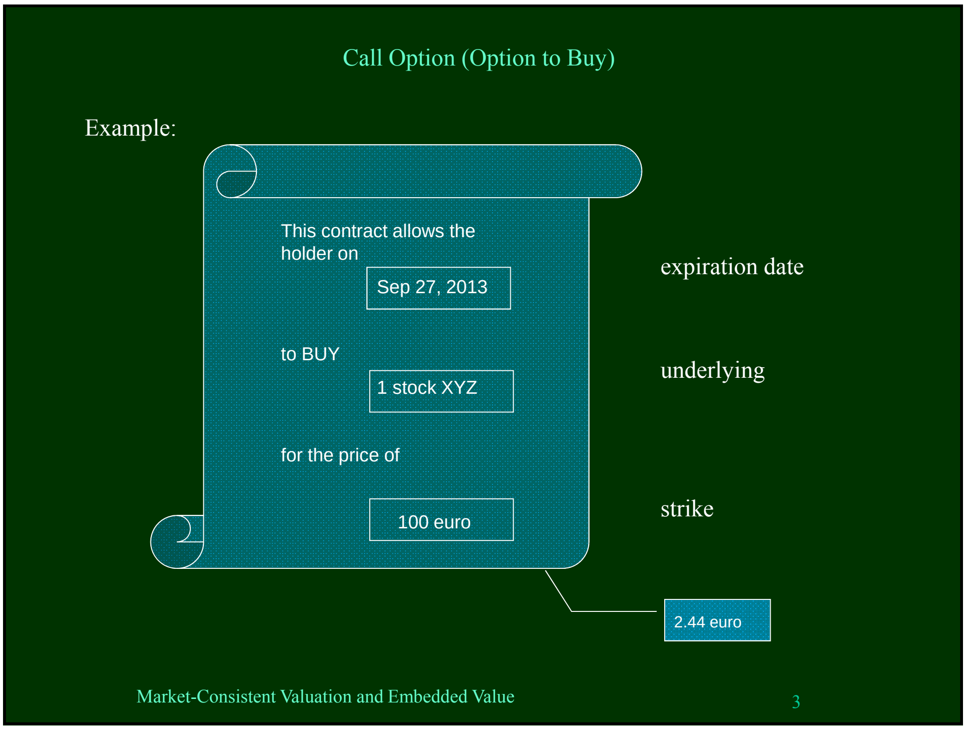
III
Contingent Claims on Equity

Pricing Models for Derivatives

Vanilla Call Option on XYZ stock with maturity Sep 27, 2013 and strike price 100 euro.

Gives the holder the right (not the obligation) to buy XYZ stock on Sep 27, 2013 for a price of 100 euro.

What is a fair price for such an option contract ?



Call Option (Option to Buy)

Example:

This contract allows the holder on

Sep 27, 2013

to BUY

1 stock XYZ

for the price of

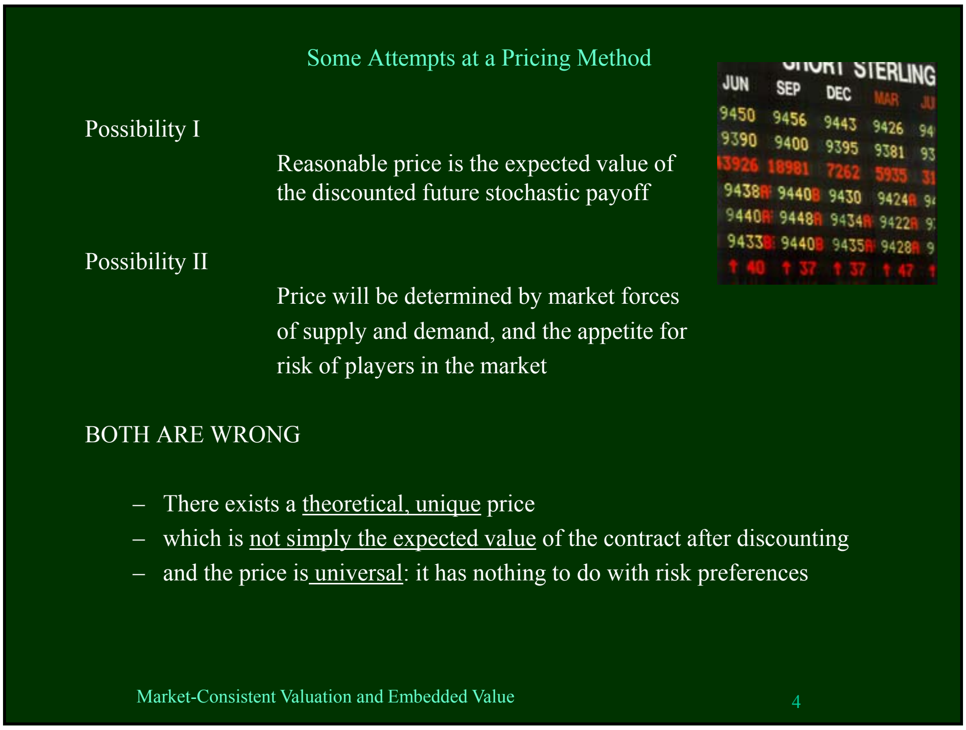
100 euro

expiration date

underlying

strike

2.44 euro



Some Attempts at a Pricing Method

Possibility I

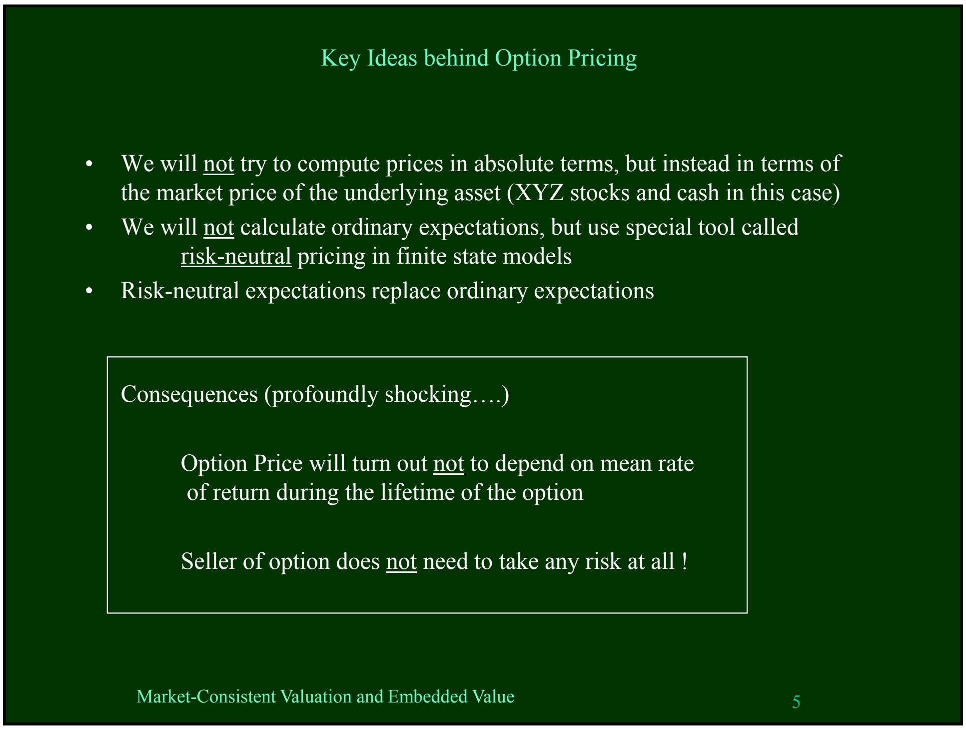
Reasonable price is the expected value of the discounted future stochastic payoff

Possibility II

Price will be determined by market forces of supply and demand, and the appetite for risk of players in the market

BOTH ARE WRONG

- There exists a theoretical, unique price
- which is not simply the expected value of the contract after discounting
- and the price is universal: it has nothing to do with risk preferences



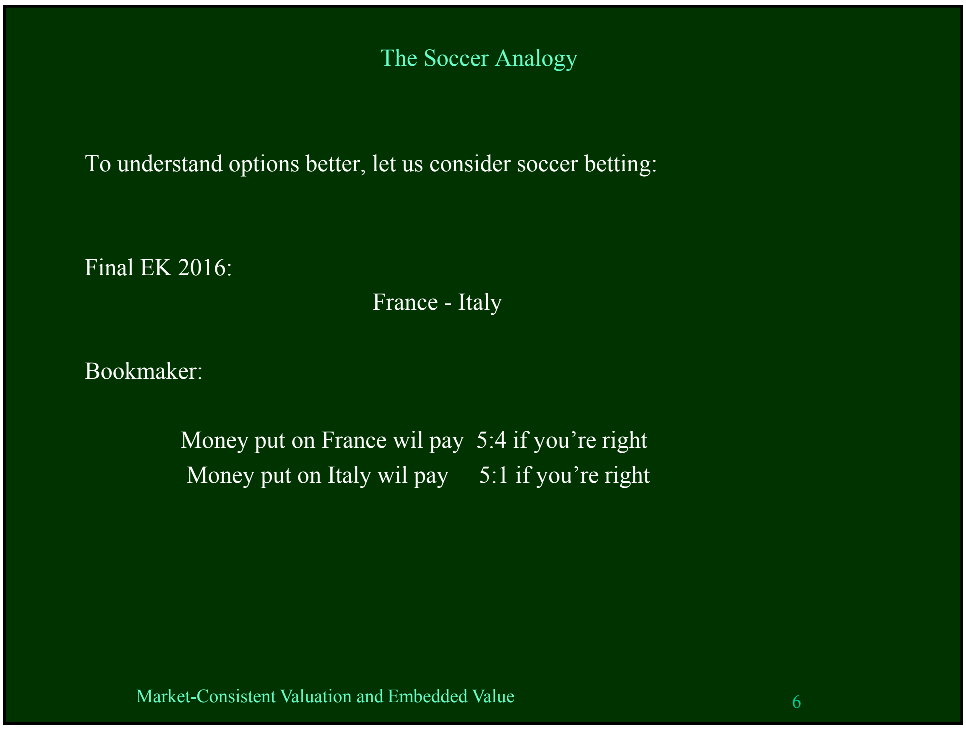
Key Ideas behind Option Pricing

- We will not try to compute prices in absolute terms, but instead in terms of the market price of the underlying asset (XYZ stocks and cash in this case)
- We will not calculate ordinary expectations, but use special tool called risk-neutral pricing in finite state models
- Risk-neutral expectations replace ordinary expectations

Consequences (profoundly shocking….)

Option Price will turn out not to depend on mean rate of return during the lifetime of the option

Seller of option does not need to take any risk at all !



The Soccer Analogy

To understand options better, let us consider soccer betting:

Final EK 2016:

France - Italy

Bookmaker:

Money put on France wil pay 5:4 if you're right
Money put on Italy wil pay 5:1 if you're right

Risk in the Betting Contracts

Suppose

900 people bet on France for 1,-
100 people bet on Italy for 1,-

Profit if France wins

$$1000 - \tfrac{5}{4}(900) = -125$$

Profit if Italy wins

$$1000 - \tfrac{5}{1}(100) = +500$$

So considerable risk in the value of the betting contracts, due to the uncertainty in the (stochastic) payoff after the game

Risk in the Betting Contracts

Suppose

500 people bet on France for 1,-
500 people bet on Italy for 1,-

Profit if France wins

$$1000 - \tfrac{5}{4}(500) = +375$$

Profit if Italy wins

$$1000 - \tfrac{5}{1}(500) = -1500$$

So even more considerable risk in the value of the betting contracts

How to value bookmaker’s contracts ?

Even if you know how many people will take what bet,
profit/loss still depends on outcome of the game, a stochastic payoff

How to value the bookmaker’s accounts ?

- Taking expectations ?
  NO, and who knows probabilities anyway ?
- Defining utility functions ?
  POSSIBLE, but different people will then assign different prices.
- After all, still problem of risk.

Elimination of Risk

Use Market information

Suppose we know
750 people will bet on France for 1,-
250 people will bet on Italy for 1,-

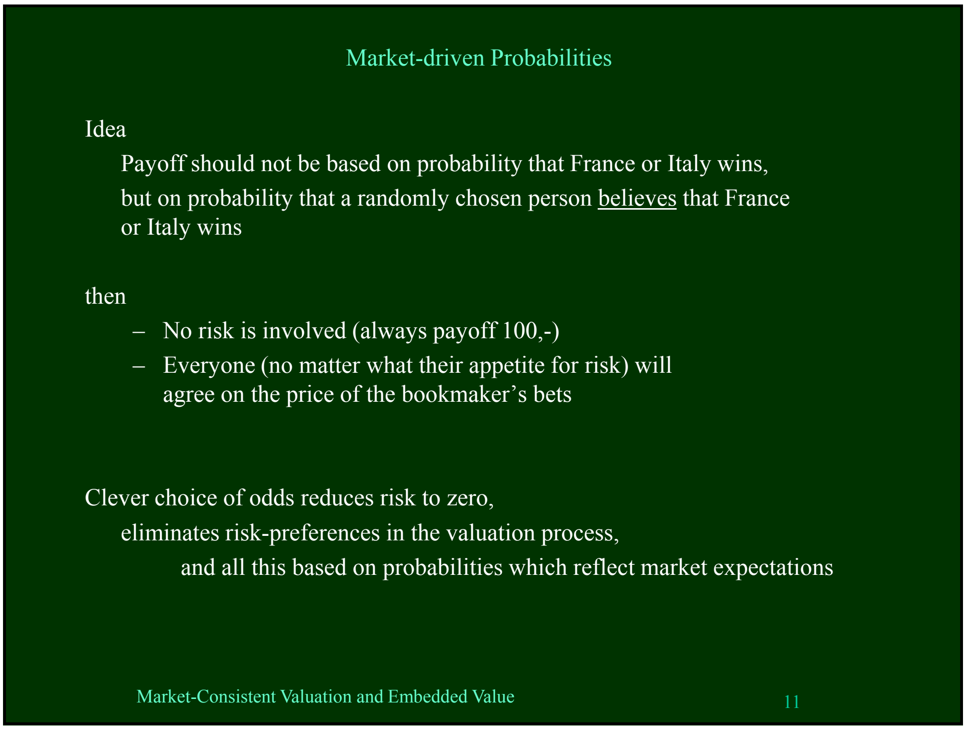
Then take as payoff

France wins	6 : 5
Italy wins	18 : 5

Then no risk is involved !

France wins: $1000-\frac{6}{5}(750)=+100$

Italy wins: $1000-\frac{18}{5}(250)=+100$



Market-driven Probabilities

Idea

Payoff should not be based on probability that France or Italy wins, but on probability that a randomly chosen person <u>believes</u> that France or Italy wins

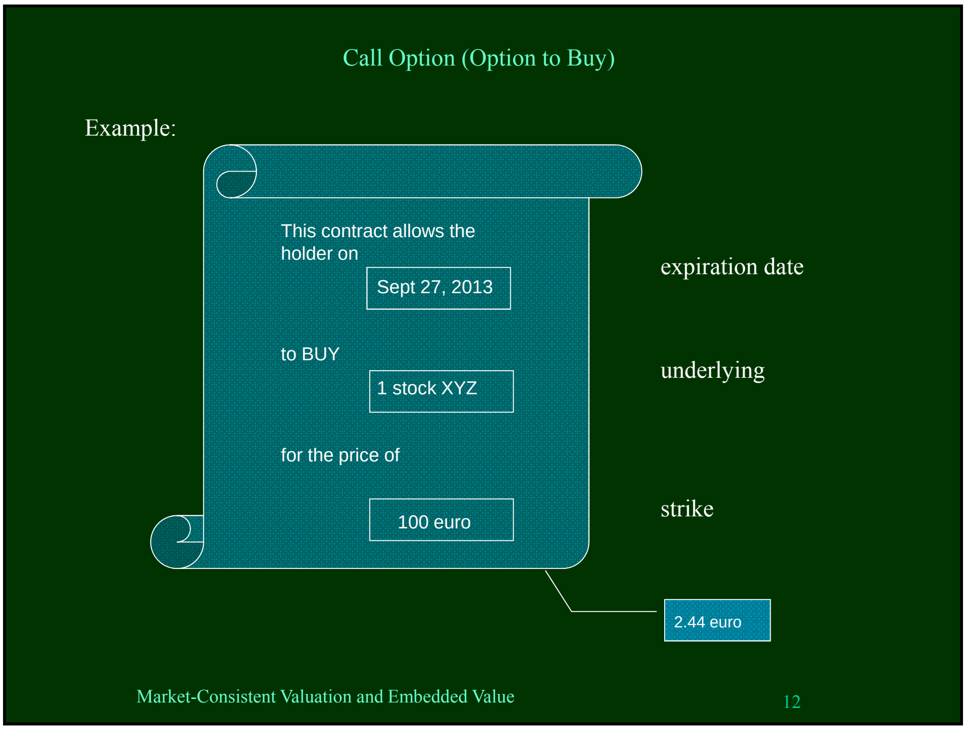
then

- No risk is involved (always payoff 100,-)
- Everyone (no matter what their appetite for risk) will agree on the price of the bookmaker's bets

Clever choice of odds reduces risk to zero,

eliminates risk-preferences in the valuation process,

and all this based on probabilities which reflect market expectations



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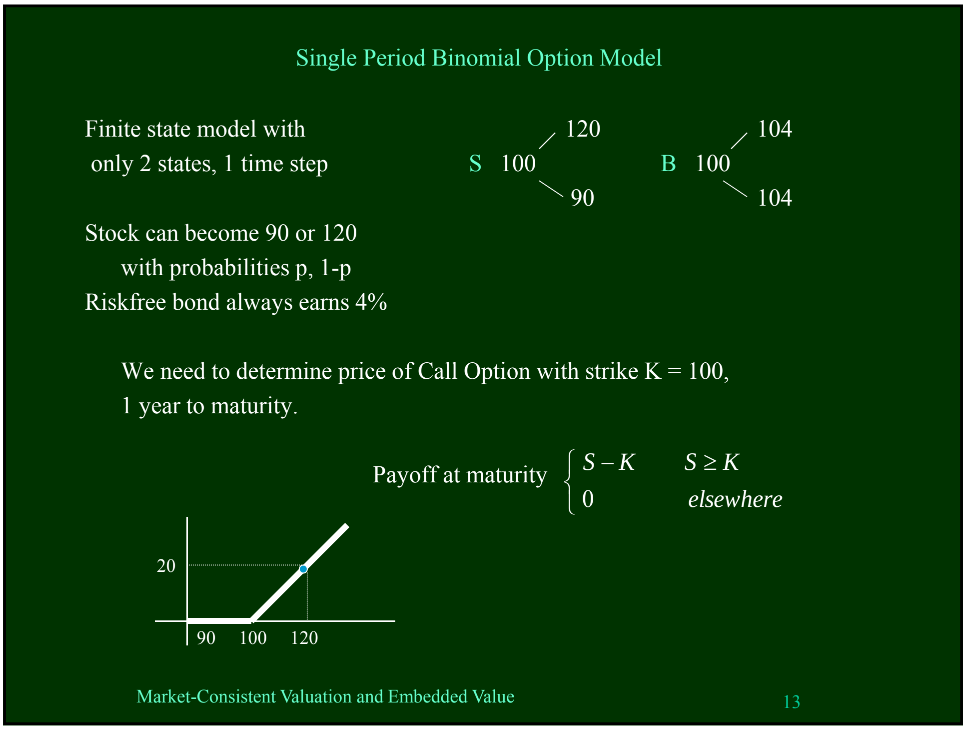


Single Period Binomial Option Model

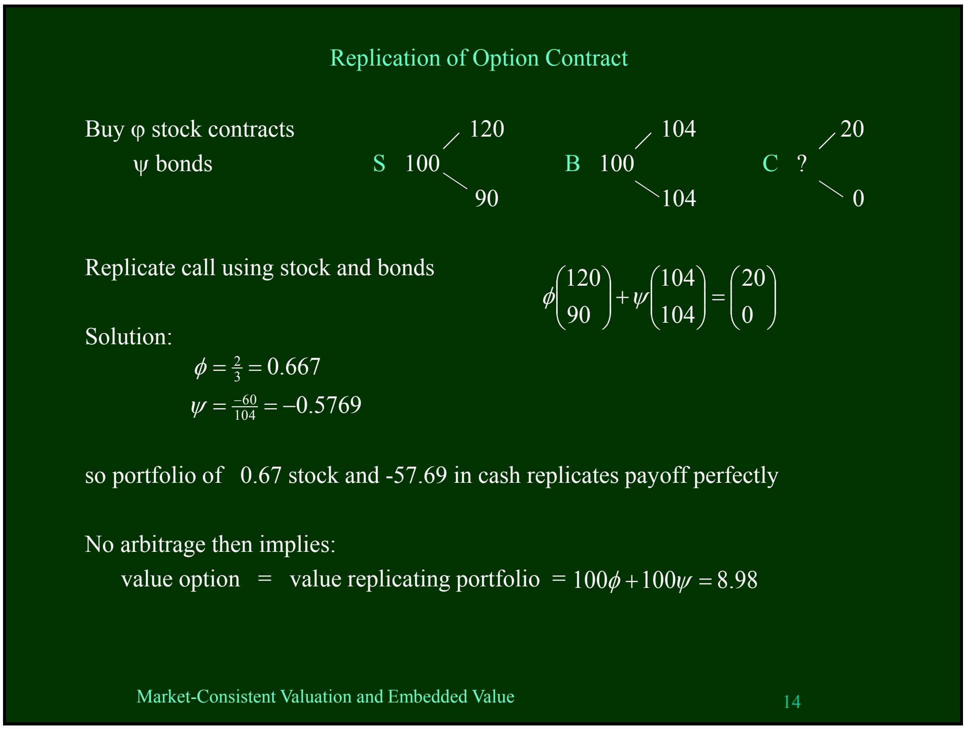
Finite state model with
only 2 states, 1 time step

$$S\ 100 \begin{array}{l} \nearrow 120 \\ \searrow 90 \end{array} \qquad B\ 100 \begin{array}{l} \nearrow 104 \\ \searrow 104 \end{array}$$

Stock can become 90 or 120
with probabilities p, 1-p
Riskfree bond always earns 4%

We need to determine price of Call Option with strike K = 100,
1 year to maturity.

Payoff at maturity $\begin{cases} S-K & S \geq K \\ 0 & elsewhere \end{cases}$



Replication of Option Contract

Buy φ stock contracts
ψ bonds

$$S\ 100 \begin{array}{l} \nearrow 120 \\ \searrow 90 \end{array} \qquad B\ 100 \begin{array}{l} \nearrow 104 \\ \searrow 104 \end{array} \qquad C\ ? \begin{array}{l} \nearrow 20 \\ \searrow 0 \end{array}$$

Replicate call using stock and bonds

$$\phi \begin{pmatrix} 120 \\ 90 \end{pmatrix} + \psi \begin{pmatrix} 104 \\ 104 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \end{pmatrix}$$

Solution:

$$\phi = \tfrac{2}{3} = 0.667$$

$$\psi = \tfrac{-60}{104} = -0.5769$$

so portfolio of 0.67 stock and -57.69 in cash replicates payoff perfectly

No arbitrage then implies:
value option = value replicating portfolio $= 100\phi + 100\psi = 8.98$

Disappearance of objective probabilities

Note:

> The probability of going up/down (p and 1-p)
> never even entered our calculations !

Investor who buys ‘naked’ option runs huge risk :

Invest $ 1000 in stock (10 contracts)		
If stock goes up	Value $ 1200	Profit 20 %
If stock goes down	Value $ 900	Loss -10 %
Invest $ 1000 in options (111 contracts)		
If stock goes up	Value $ 2220	Profit 122 %
If stock goes down	Value $ 0	Loss -100 %

However: bank who sells option <u>does not run any risk at all</u> !

Hedged Options Bookkeeping

Bank that sells option does not run any risk, since it never writes ‘naked’ option:

Bank sells 100 options	+ 898,-
Bank buys 67 stock	- 6700,-
Bank Account	- 5802,-

If price goes up : 100 people want to buy stock for 100,- when it costs 120,-

Bank receives 100 times 100,-	+ 10.000,-
Bank pays back loan (5802 plus 4%)	- 6.034,-
Bank buys 33 extra stocks (33 times 120,-)	- 3.960,-
	6,-

If price goes down: nobody wants to exercise option since stock costs 90,-

Bank sells its stock (67 times 90,-)	+ 6.030,-
Bank pays back loan (5802 plus 4%)	- 6.034,-
	- 4,-

Riskfree Option trading

Conclusion:

Holding portfolio of only options extremely risky,
but selling option, while
buying replicating portfolio
is entirely riskfree !

The amount of stock one has to hold per contract is the hedge ratio Δ .

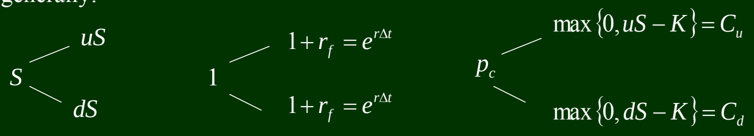
In this case we saw that $\Delta = \frac{2}{3}$ so

for 100 options, we should buy $100\Delta = 67$ stock contracts, and put the rest of the option selling profit in a bank account

for 200 options, we should buy $200\Delta = 133$ stock contracts, and put the rest of the option selling profit in a bank account

General Model for One-Step Replication

More generally:



S, uS, dS

1, $1+r_f = e^{r\Delta t}$, $1+r_f = e^{r\Delta t}$

p_c, $\max\{0, uS-K\} = C_u$, $\max\{0, dS-K\} = C_d$

General Model for One-Step Replication

More generally:

$$S \quad \begin{matrix} \nearrow uS \\ \searrow dS \end{matrix} \qquad\qquad 1 \quad \begin{matrix} \nearrow 1+r_f = e^{r\Delta t} \\ \searrow 1+r_f = e^{r\Delta t} \end{matrix} \qquad\qquad p_c \quad \begin{matrix} \nearrow \max\{0, uS-K\} = C_u \\ \searrow \max\{0, dS-K\} = C_d \end{matrix}$$

$$\begin{aligned} \phi(uS) + \psi(e^{r\Delta t}) &= C_u \\ \phi(dS) + \psi(e^{r\Delta t}) &= C_d \\ \hline \phi S(u-d) &= C_u - C_d \end{aligned} \qquad \Rightarrow \qquad \phi = \frac{C_u - C_d}{S(u-d)} \quad \left(= \frac{\Delta C}{\Delta S}\right)$$

$$\begin{aligned} \psi &= e^{-r\Delta t}(C_u - \phi\, uS) \\ &= e^{-r\Delta t}\left(\frac{uC_d - dC_u}{u-d}\right) \end{aligned}$$

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Price today : $p_c = \phi\, S + \psi 1$

$$\begin{aligned} \psi &= e^{-r\Delta t}(C_u - \phi\, uS) \\ &= e^{-r\Delta t}\left(\frac{uC_d - dC_u}{u-d}\right) \end{aligned}$$

Risk-Neutral Pricing

So

$$p_c = \phi S + \psi 1 = \frac{C_u - C_d}{u-d} + \frac{(uC_d - dC_u)e^{-r\Delta t}}{u-d}$$

$$= e^{-r\Delta t}\left[\frac{e^{r\Delta t}-d}{u-d}C_u + \frac{u-e^{r\Delta t}}{u-d}C_d\right]$$

$$= e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]$$

with $q = \frac{e^{r\Delta t}-d}{u-d} \in]\,0,1\,[$

Risk-Neutral Pricing

So

$$p_c = \phi S + \psi 1 = \frac{C_u - C_d}{u-d} + \frac{(uC_d - dC_u)e^{-r\Delta t}}{u-d}$$

$$= e^{-r\Delta t}\left[\frac{e^{r\Delta t}-d}{u-d}C_u + \frac{u-e^{r\Delta t}}{u-d}C_d\right]$$

$$= e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]$$

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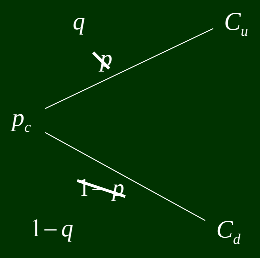
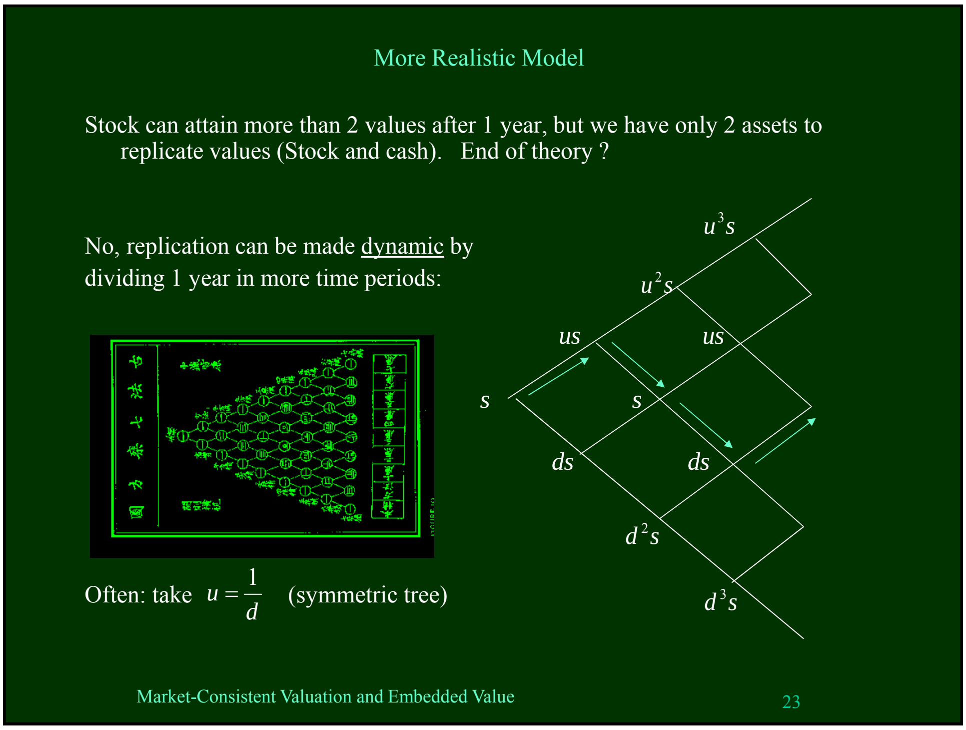


Once more:

price = discounted version of expected value under new probabilities q instead of p

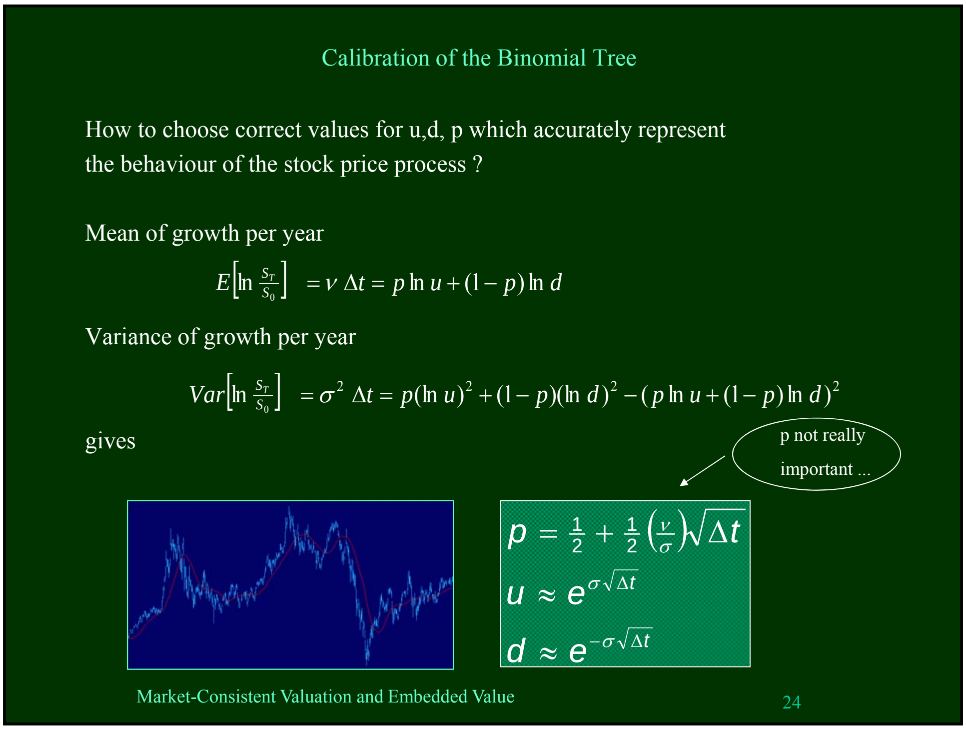


More Realistic Model

Stock can attain more than 2 values after 1 year, but we have only 2 assets to replicate values (Stock and cash). End of theory ?

No, replication can be made <u>dynamic</u> by dividing 1 year in more time periods:

Often: take $u = \dfrac{1}{d}$ (symmetric tree)



Calibration of the Binomial Tree

How to choose correct values for u,d, p which accurately represent the behaviour of the stock price process ?

Mean of growth per year

$$E\left[\ln \tfrac{S_T}{S_0}\right] = \nu\,\Delta t = p \ln u + (1-p)\ln d$$

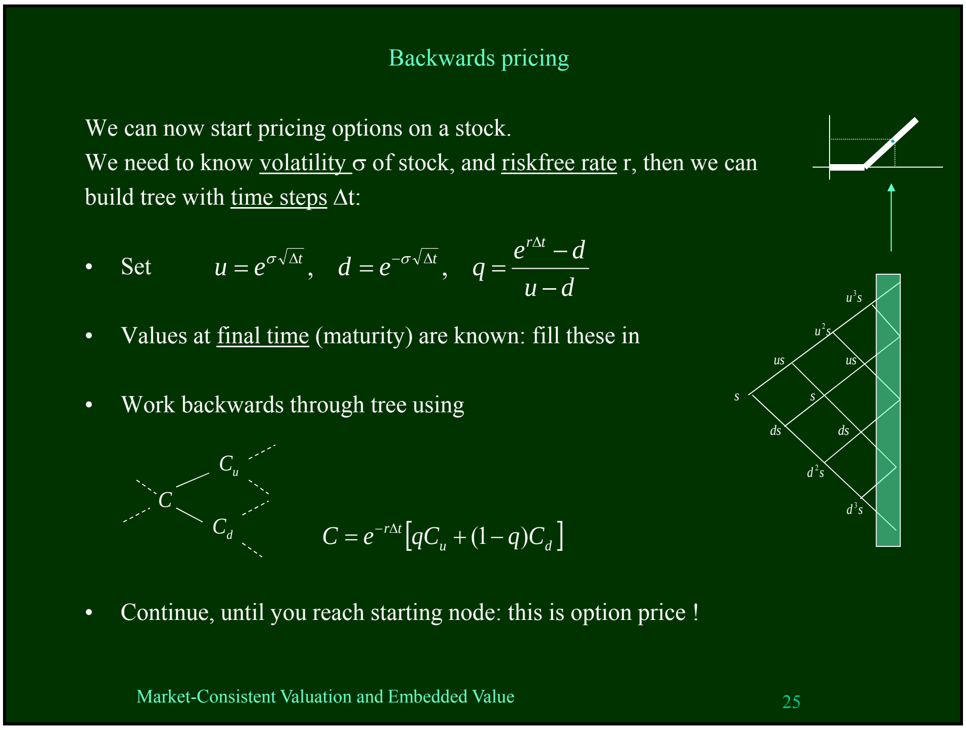
Variance of growth per year

$$Var\left[\ln \tfrac{S_T}{S_0}\right] = \sigma^2\,\Delta t = p(\ln u)^2 + (1-p)(\ln d)^2 - (p\ln u + (1-p)\ln d)^2$$

gives

p not really important ...

$$p = \tfrac{1}{2} + \tfrac{1}{2}\left(\tfrac{\nu}{\sigma}\right)\sqrt{\Delta t}$$
$$u \approx e^{\sigma\sqrt{\Delta t}}$$
$$d \approx e^{-\sigma\sqrt{\Delta t}}$$



Backwards pricing

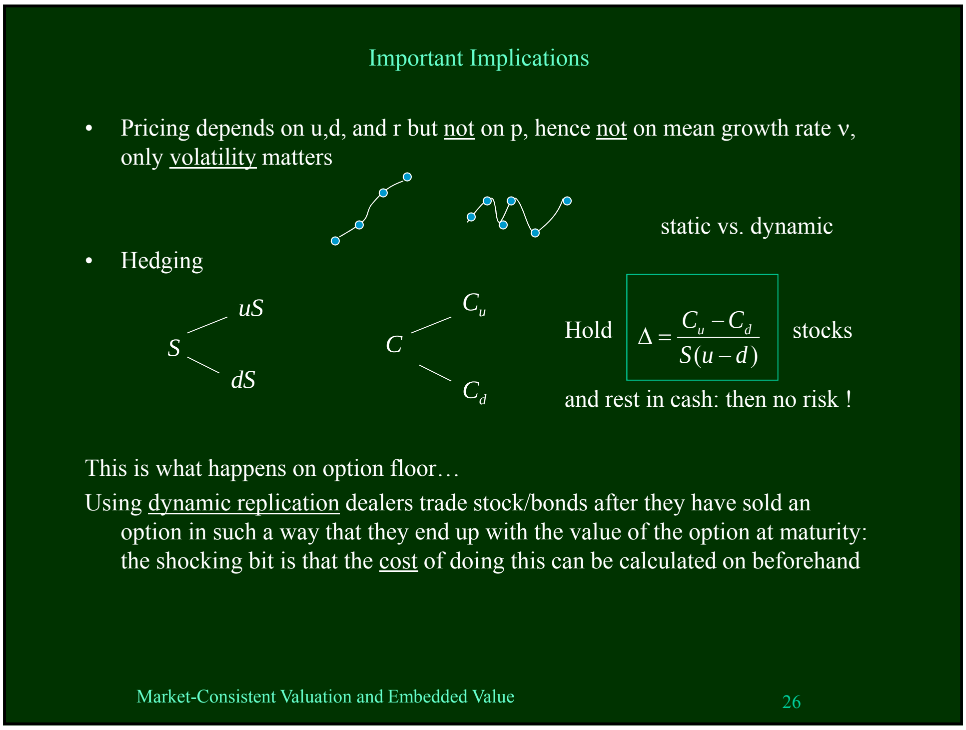
We can now start pricing options on a stock.

We need to know volatility σ of stock, and riskfree rate r, then we can build tree with time steps Δt:

- Set $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, $q = \dfrac{e^{r\Delta t} - d}{u - d}$
- Values at final time (maturity) are known: fill these in
- Work backwards through tree using

$$C = e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]$$

- Continue, until you reach starting node: this is option price !



Important Implications

- Pricing depends on u,d, and r but not on p, hence not on mean growth rate ν, only volatility matters
- Hedging

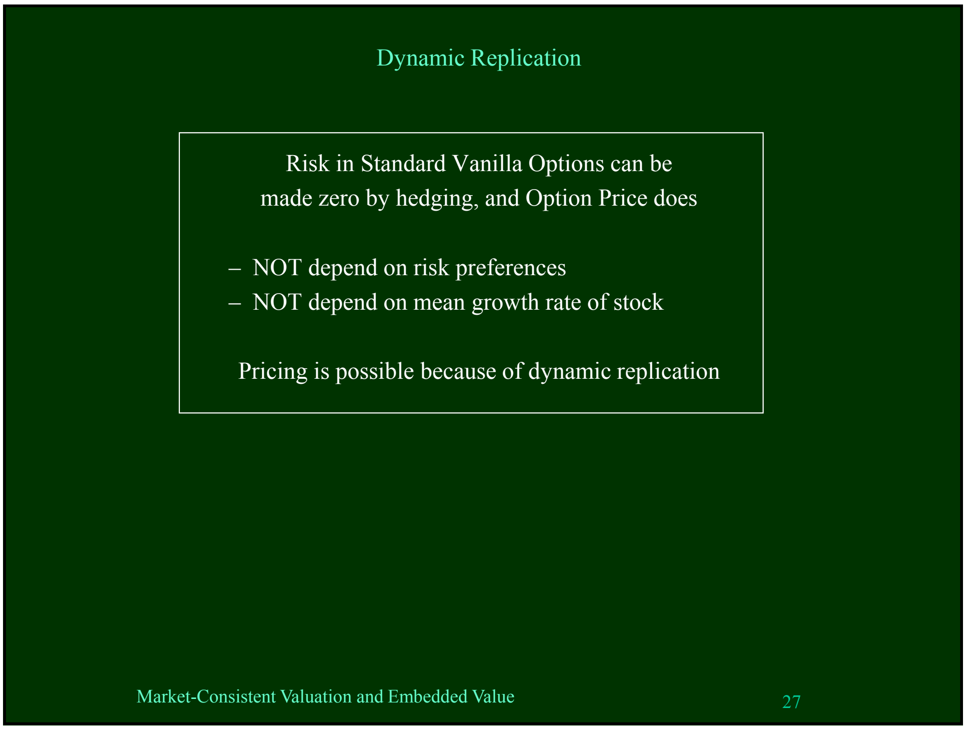
static vs. dynamic

Hold $\boxed{\Delta = \dfrac{C_u - C_d}{S(u-d)}}$ stocks

and rest in cash: then no risk !

This is what happens on option floor…

Using dynamic replication dealers trade stock/bonds after they have sold an option in such a way that they end up with the value of the option at maturity: the shocking bit is that the cost of doing this can be calculated on beforehand

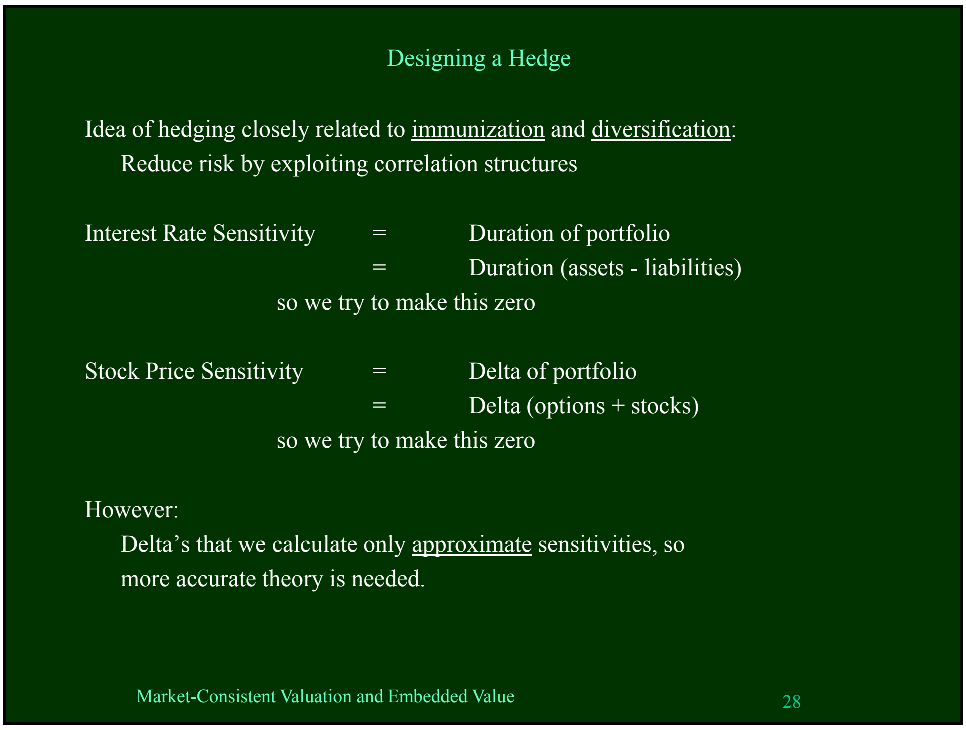


Dynamic Replication

Risk in Standard Vanilla Options can be made zero by hedging, and Option Price does

- NOT depend on risk preferences
- NOT depend on mean growth rate of stock

Pricing is possible because of dynamic replication



Designing a Hedge

Idea of hedging closely related to immunization and diversification:
Reduce risk by exploiting correlation structures

Interest Rate Sensitivity = Duration of portfolio
= Duration (assets - liabilities)
so we try to make this zero

Stock Price Sensitivity = Delta of portfolio
= Delta (options + stocks)
so we try to make this zero

However:
Delta's that we calculate only approximate sensitivities, so more accurate theory is needed.

Pricing an American Put option

American put is right (not obligation) to sell underlying stock against predetermined strike price at <u>any time up to</u> maturity

So at any time t we face choice

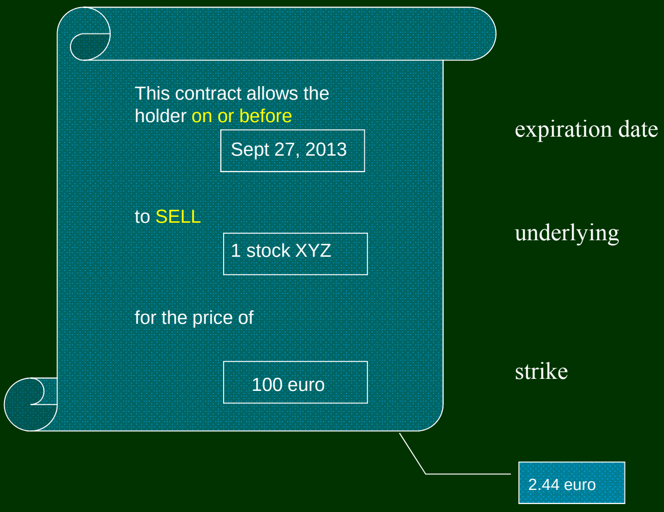
- Exercise option: we receive the value $K - S_t$
- Hold on to option, and see if exercising it later can be more advantageous

Solution:

- Fill in option values at end of tree (maturity)
- Work backwards as before, but if value of exercise $K - S_t$ exceeds calculated backward value, use the right to exercise !

American Put Option

Example:



This contract allows the holder on or before

Sept 27, 2013

to SELL

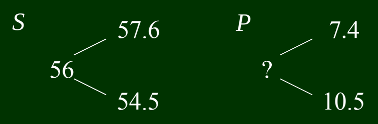
1 stock XYZ

for the price of

100 euro

Example: American Put pricing

$\Delta t = \frac{1}{12}$
$K = 65$
$q = 0.518$
$r = 4\%$



No exercise ("European") $e^{-r\Delta t}\left[7.4q + 10.5(1-q)\right] = 8.86$
Exercise $K - S_t = 65.0 - 56.0 = 9.00$

So value at node is 9.00 and at this node exercising is optimal !

In Excel: change $e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]$
into $\max\left\{K - S_t,\ e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]\right\}$

and color all cells where $p = K - S$ red to see if you should exercise or not.

Equity derivative:
Unit-Linked Insurance Product

Unit-Linked Guarantee:

We invest in a risky asset (let's say a stock or stock index S) for T years, but we get the guarantee that our rate of return over the investment period is at least g. So we are sure that a certain minimal rate of return for our investment will either be earned by the index or otherwise be compensated by the insurance company.

So if return over period [0,T] equals

$$S_T \ / \ S_0$$

then we will earn the return

$$\max(S_T \ / \ S_0, \ (1+g)^T)$$

Equity derivatives (tree)
Unit-Linked with maturity guarantee

Example

Maturity 2 year & 4 time steps:	T = 2 $\Delta t = 0.5$
Volatility 15%	$\sigma = 0.15$
Guaranteed rate 4%, half year compounded	g = 0.04

5% interest rate discrete so

$$e^{-r\Delta t} = (1 + 0.05\Delta t)^{-1} = 0.9756$$

Stock dynamics

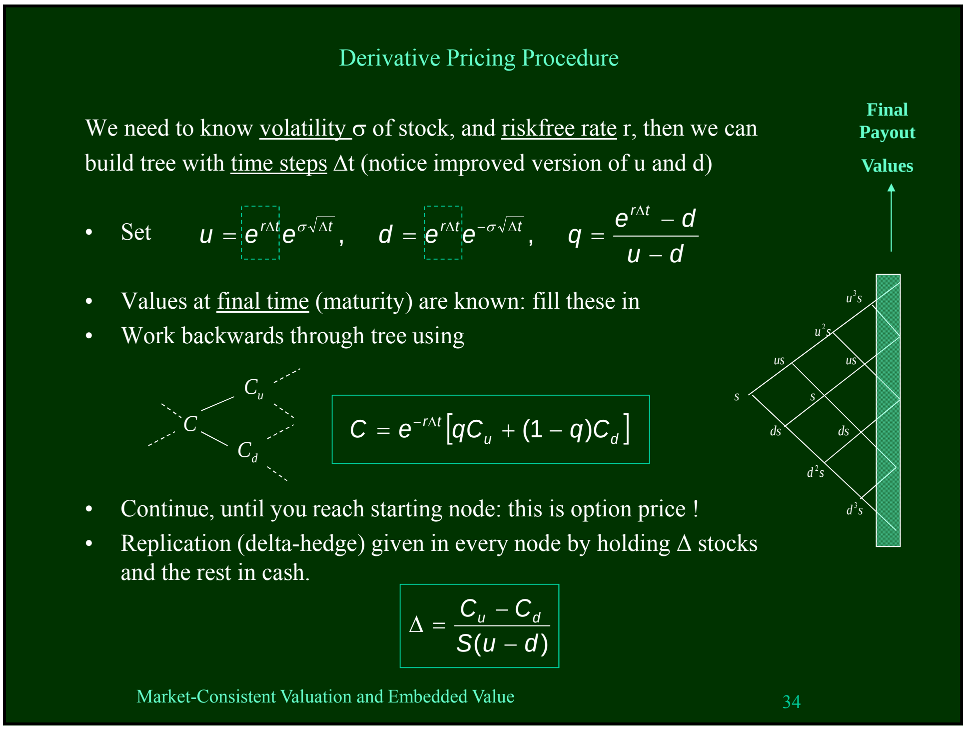
$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.15\sqrt{0.5}} = 1.1397$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1 / u = 0.9218$$

and

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{1/0.9756 - 0.9218}{1.1397 - 0.9218} = 0.4735$$

Guaranteed amount at maturity = 100*(1.02^4) = 108.24



Derivative Pricing Procedure

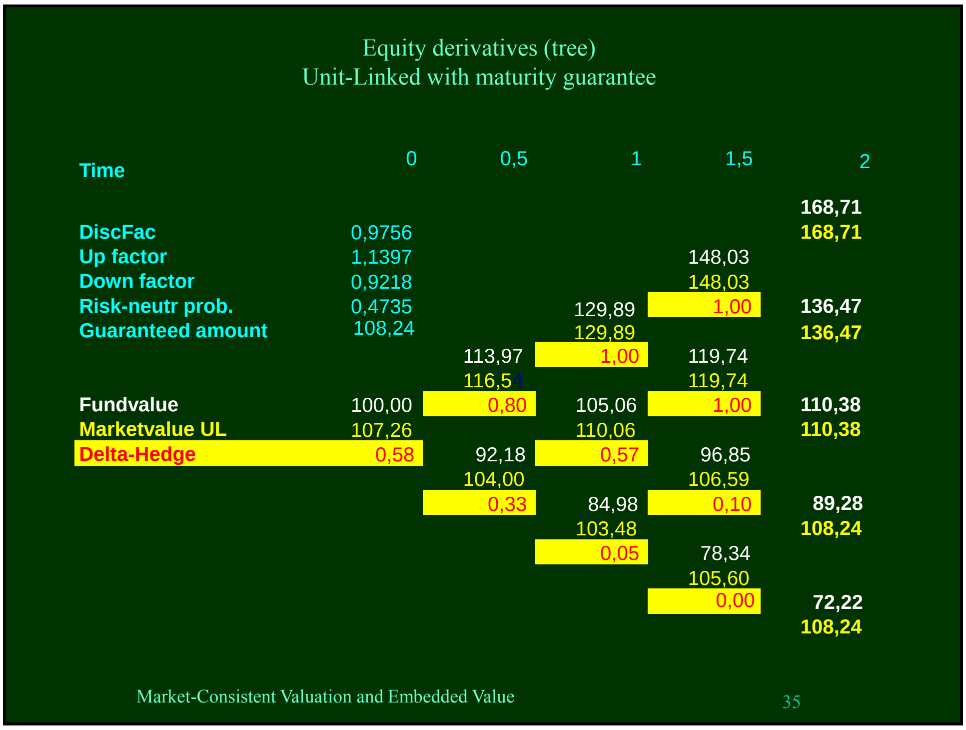
We need to know volatility σ of stock, and riskfree rate r, then we can build tree with time steps Δt (notice improved version of u and d)

- Set $u = \boxed{e^{r\Delta t}} e^{\sigma\sqrt{\Delta t}}, \quad d = \boxed{e^{r\Delta t}} e^{-\sigma\sqrt{\Delta t}}, \quad q = \frac{e^{r\Delta t} - d}{u - d}$
- Values at final time (maturity) are known: fill these in
- Work backwards through tree using

$$\boxed{C = e^{-r\Delta t}\left[qC_u + (1 - q)C_d\right]}$$

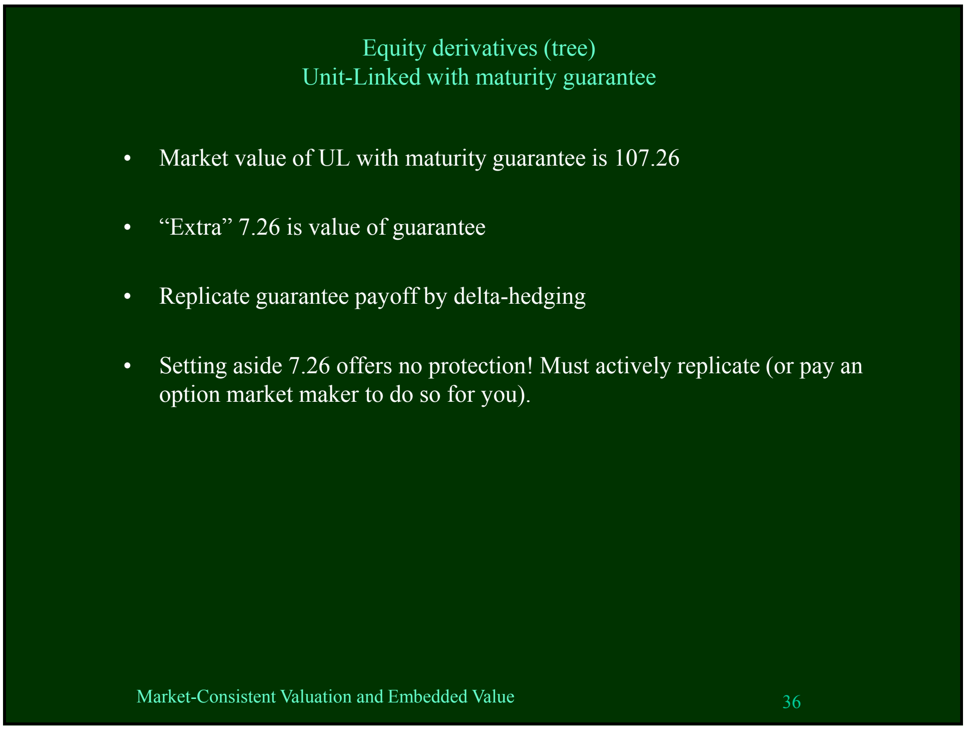
- Continue, until you reach starting node: this is option price !
- Replication (delta-hedge) given in every node by holding Δ stocks and the rest in cash.

$$\boxed{\Delta = \frac{C_u - C_d}{S(u - d)}}$$



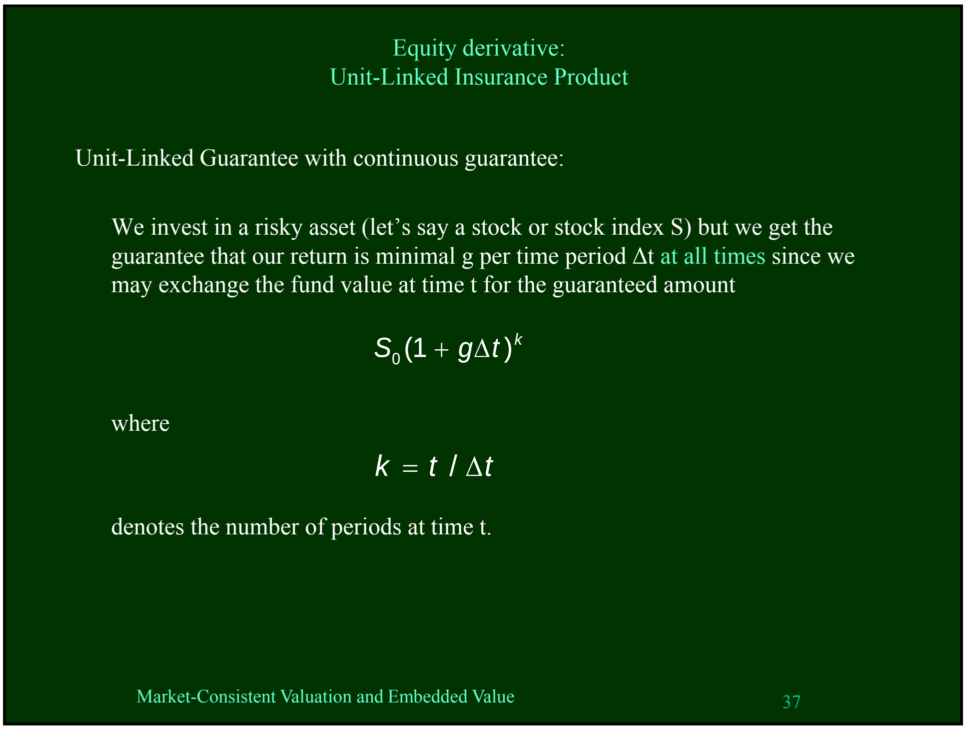
Equity derivatives (tree)
Unit-Linked with maturity guarantee

Time	0	0,5	1	1,5	2
					168,71
DiscFac	0,9756				**168,71**
Up factor	1,1397			148,03	
Down factor	0,9218			148,03	
Risk-neutr prob.	0,4735		129,89	1,00	**136,47**
Guaranteed amount	108,24		129,89		**136,47**
		113,97	1,00	119,74	
		116,54		119,74	
Fundvalue	100,00	0,80	105,06	1,00	**110,38**
Marketvalue UL	107,26		110,06		**110,38**
Delta-Hedge	0,58	92,18	0,57	96,85	
		104,00		106,59	
		0,33	84,98	0,10	**89,28**
			103,48		**108,24**
			0,05	78,34	
				105,60	
				0,00	**72,22**
					108,24



Equity derivatives (tree)
Unit-Linked with maturity guarantee

- Market value of UL with maturity guarantee is 107.26
- “Extra” 7.26 is value of guarantee
- Replicate guarantee payoff by delta-hedging
- Setting aside 7.26 offers no protection! Must actively replicate (or pay an option market maker to do so for you).



Equity derivative:
Unit-Linked Insurance Product

Unit-Linked Guarantee with continuous guarantee:

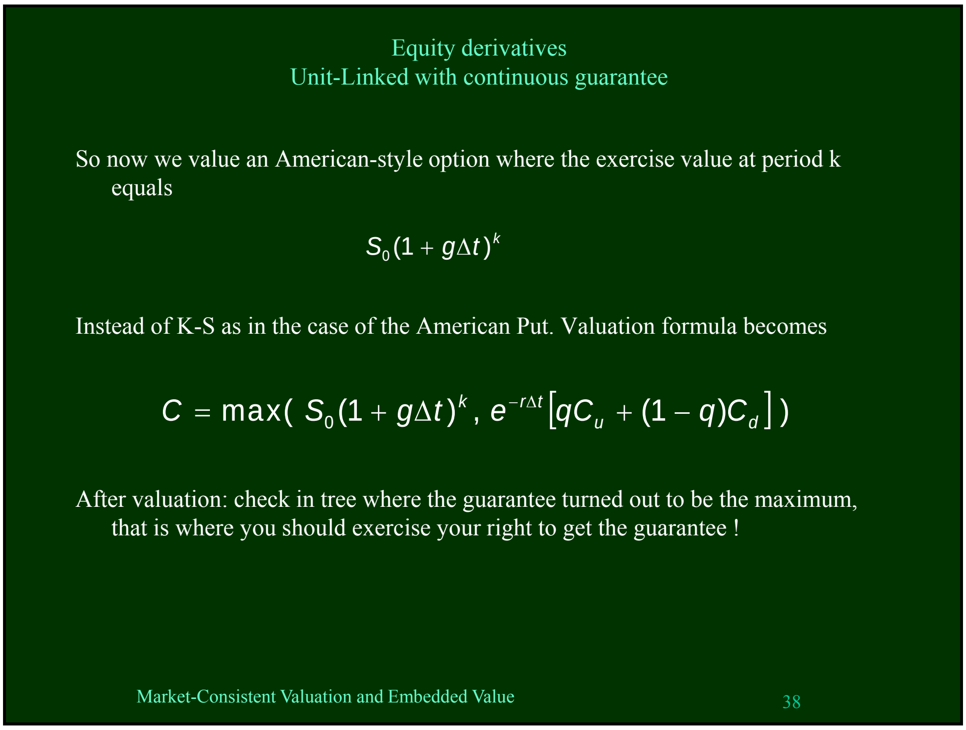
We invest in a risky asset (let’s say a stock or stock index S) but we get the guarantee that our return is minimal g per time period Δt at all times since we may exchange the fund value at time t for the guaranteed amount

$$S_0(1+g\Delta t)^k$$

where

$$k = t / \Delta t$$

denotes the number of periods at time t.



Equity derivatives
Unit-Linked with continuous guarantee

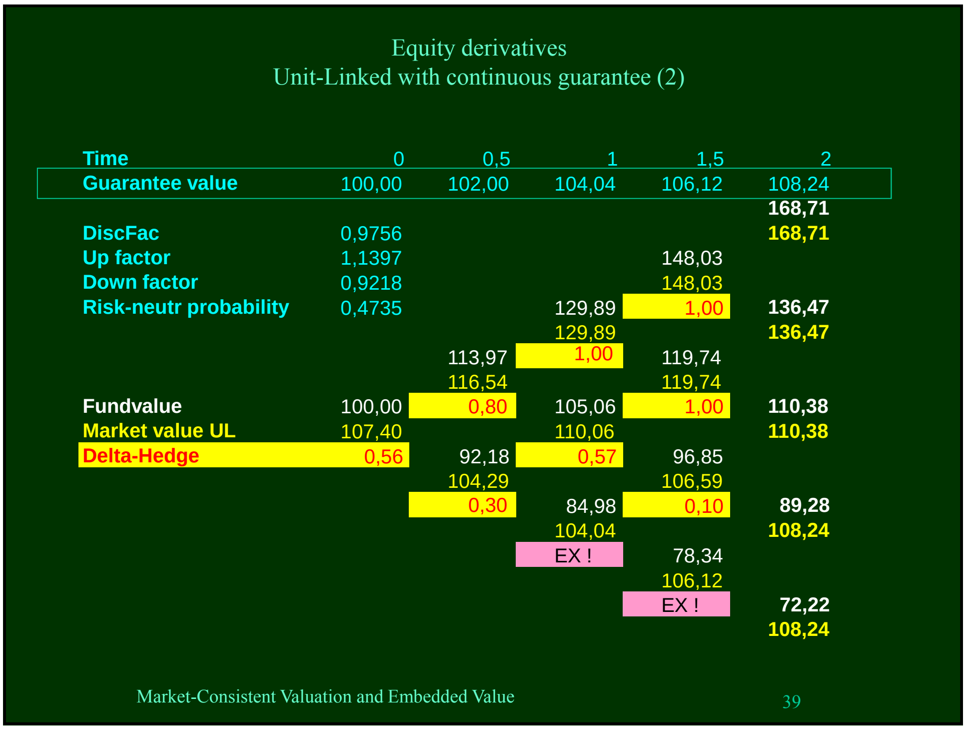
So now we value an American-style option where the exercise value at period k equals

$$S_0(1+g\Delta t)^k$$

Instead of K-S as in the case of the American Put. Valuation formula becomes

$$C = \max(\ S_0(1+g\Delta t)^k\,,\ e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]\ )$$

After valuation: check in tree where the guarantee turned out to be the maximum, that is where you should exercise your right to get the guarantee !



Equity derivatives
Unit-Linked with continuous guarantee (2)

Time	0	0,5	1	1,5	2
Guarantee value	100,00	102,00	104,04	106,12	108,24
					168,71
DiscFac	0,9756				**168,71**
Up factor	1,1397			148,03	
Down factor	0,9218			148,03	
Risk-neutr probability	0,4735		129,89	1,00	**136,47**
			129,89		**136,47**
		113,97	1,00	119,74	
		116,54		119,74	
Fundvalue	100,00	0,80	105,06	1,00	**110,38**
Market value UL	107,40		110,06		**110,38**
Delta-Hedge	0,56	92,18	0,57	96,85	
		104,29		106,59	
		0,30	84,98	0,10	**89,28**
			104,04		**108,24**
			EX !	78,34	
				106,12	
				EX !	**72,22**
					108,24

Equity derivatives
Unit-Linked with continuous guarantee (3)

- Market value UL with continuous guarantee is 107.40
- Higher value than maturity guarantee 107.26
- Extra value of "switching" in UL!
- Again, replicate guarantee payoff by delta-hedging

If we take smaller and smaller timesteps Δt , we can rehedge our portfolio more and more often so run less and less risk.

What happens when Δt goes to zero (i.e. we take more and more timesteps, so n goes to infinity ?)

Riskneutral Pricing Principle

Notice that not only did we not need the probability of stocks going up or down (what we called p earlier) but in our pricing mechanism the riskneutral probabilities are such that stocks earn the riskfree rate:

Since

We find

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad q = \frac{e^{r\Delta t} - d}{u - d}$$

$$E[S_{t+1}/S_t] = qu + (1-q)d \qquad = \boxed{e^{r\Delta t}}$$

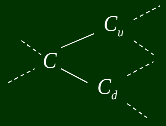
$$Var[S_{t+1}/S_t] = qu^2 + (1-q)d^2 - (qu + (1-q)d)^2 = \sigma^2\Delta t + O(\Delta t^2)$$

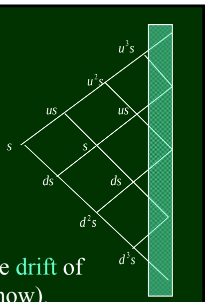
So riskneutral dynamics on tree

- Keeps standard deviation per unit of time (i.e. volatility) the same
- but changes the mean rate of return to the riskfree rate

Principle an be used for Pricing by (Riskneutral) Monte Carlo simulation !

Riskneutral pricing and Deflators





$$C = e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]$$

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad q = \frac{e^{r\Delta t} - d}{u - d}$$

So we take expectations under different probabilities q which make the drift of the stock (which we did not need to know) equal to r (which we know).

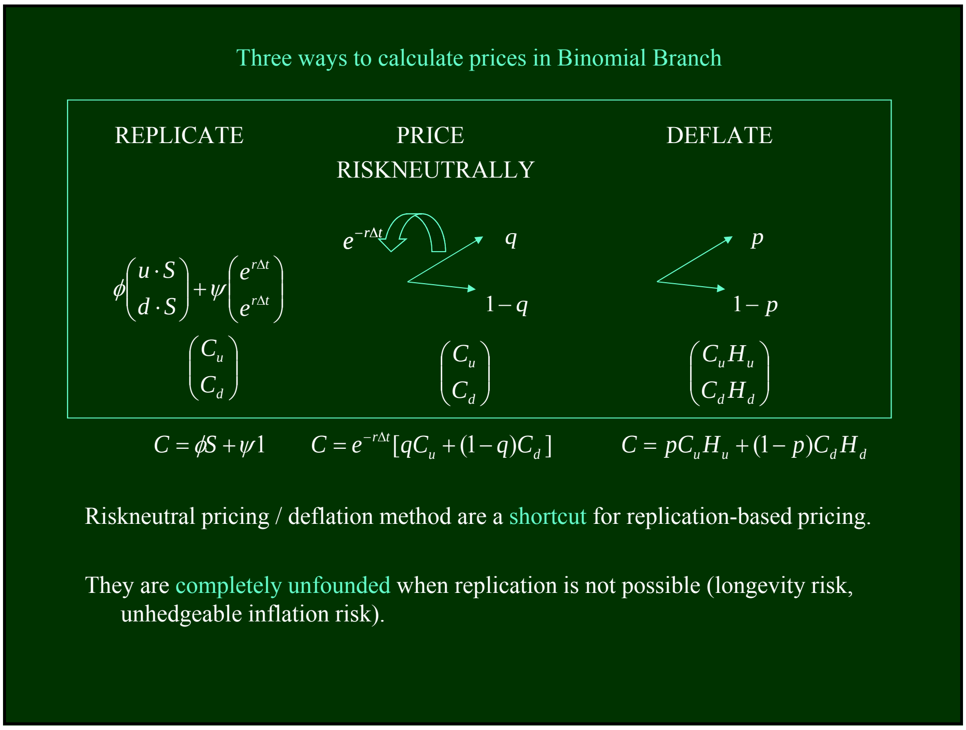
Alternative way of looking at it: we take ordinary expectations but distort ('deflate') the payoff values:

$$C = e^{-r\Delta t}[qC_u \qquad + (1-q)C_d \qquad]$$
$$= \qquad p[C_u H_u] + (1-p)[C_d H_d]$$

With deflator process H (here μ is the mean rate of return of the asset)

Sharpe Ratio of Stock

$$\boxed{\begin{aligned} H_u &= \frac{q}{p}e^{-r\Delta t} \approx e^{-\lambda\sqrt{\Delta t} + \eta\Delta t} & \lambda &= \frac{\mu - r}{\sigma} \\ H_d &= \frac{1-q}{1-p}e^{-r\Delta t} \approx e^{+\lambda\sqrt{\Delta t} + \eta\Delta t} & \eta &= -\tfrac{1}{2}(\mu + r) + \frac{\mu^2 - r^2}{2\sigma^2} \end{aligned}}$$



Three ways to calculate prices in Binomial Branch

REPLICATE

PRICE RISKNEUTRALLY

DEFLATE

$$\phi\begin{pmatrix}u\cdot S\\ d\cdot S\end{pmatrix}+\psi\begin{pmatrix}e^{r\Delta t}\\ e^{r\Delta t}\end{pmatrix}$$

$$\begin{pmatrix}C_u\\ C_d\end{pmatrix}$$

$e^{-r\Delta t}$

q

$1-q$

$$\begin{pmatrix}C_u\\ C_d\end{pmatrix}$$

p

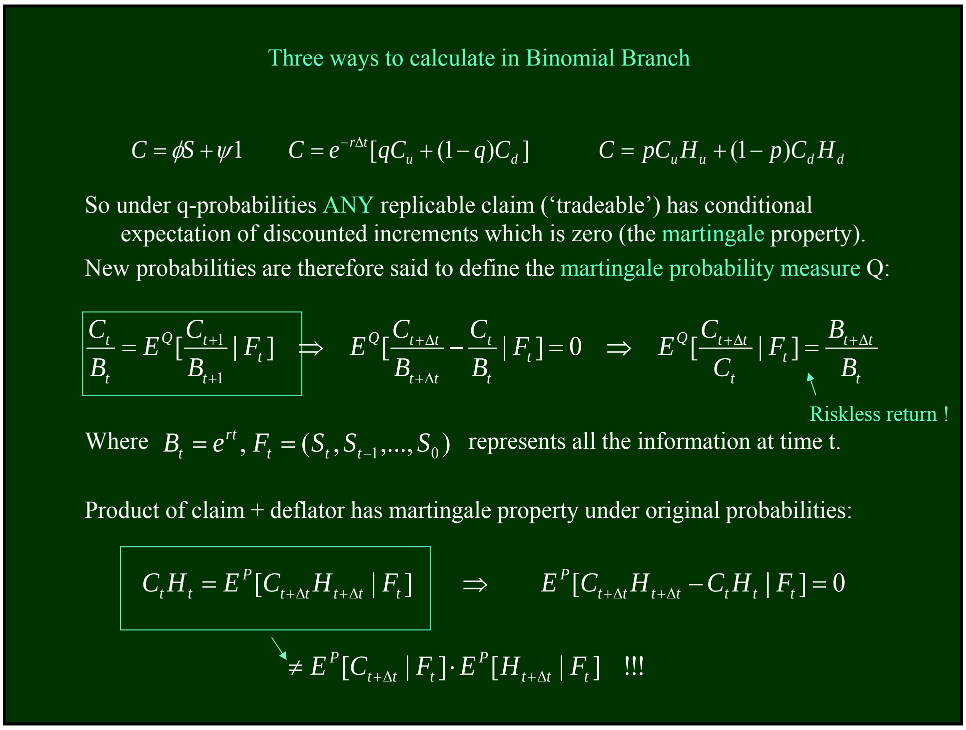
$1-p$

$$\begin{pmatrix}C_uH_u\\ C_dH_d\end{pmatrix}$$

$$C=\phi S+\psi 1 \qquad C=e^{-r\Delta t}[qC_u+(1-q)C_d] \qquad C=pC_uH_u+(1-p)C_dH_d$$

Riskneutral pricing / deflation method are a shortcut for replication-based pricing.

They are completely unfounded when replication is not possible (longevity risk, unhedgeable inflation risk).



Three ways to calculate in Binomial Branch

$$C=\phi S+\psi 1 \qquad C=e^{-r\Delta t}[qC_u+(1-q)C_d] \qquad C=pC_uH_u+(1-p)C_dH_d$$

So under q-probabilities ANY replicable claim (‘tradeable’) has conditional expectation of discounted increments which is zero (the martingale property).

New probabilities are therefore said to define the martingale probability measure Q:

$$\boxed{\frac{C_t}{B_t}=E^Q[\frac{C_{t+1}}{B_{t+1}}\,|\,F_t]} \Rightarrow E^Q[\frac{C_{t+\Delta t}}{B_{t+\Delta t}}-\frac{C_t}{B_t}\,|\,F_t]=0 \Rightarrow E^Q[\frac{C_{t+\Delta t}}{C_t}\,|\,F_t]=\frac{B_{t+\Delta t}}{B_t}$$

Riskless return !

Where $B_t=e^{rt}, F_t=(S_t,S_{t-1},...,S_0)$ represents all the information at time t.

Product of claim + deflator has martingale property under original probabilities:

$$\boxed{C_tH_t=E^P[C_{t+\Delta t}H_{t+\Delta t}\,|\,F_t]} \Rightarrow E^P[C_{t+\Delta t}H_{t+\Delta t}-C_tH_t\,|\,F_t]=0$$

$$\neq E^P[C_{t+\Delta t}\,|\,F_t]\cdot E^P[H_{t+\Delta t}\,|\,F_t] \quad !!!$$

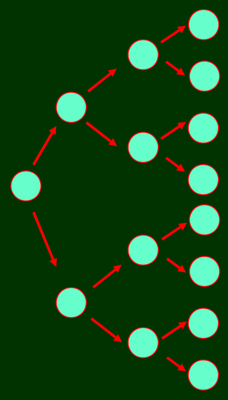
Market Completeness

We can only price new assets in terms of assets which have already been priced.

If there is no market for certain risks (longevity risk, Dutch inflation risk) choice of q, H etc. is arbitrary.
Question of existence vs. uniqueness of q (or H).

On positive side: we can price any claim with payoffs in all nodes.

Complete market: i.e. every claim with payoffs which depend on history of stock price path can be replicated.



Increasingly More Accurate Replication

Notice that

$$S_{k+1} / S_k = R_k = \begin{cases} u & \text{prob } q \\ d & \text{prob } 1 - q \end{cases}$$

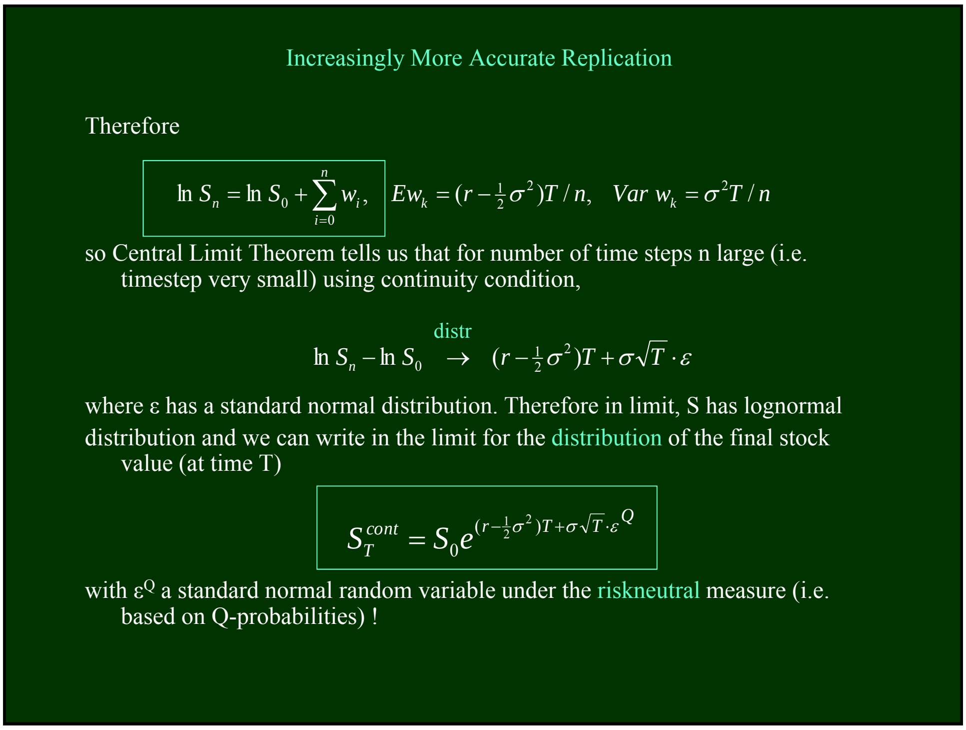
so

$$\ln S_{k+1} = \ln S_k + w_k , \quad w_k = \ln R_k = \begin{cases} r\Delta t + \sigma\sqrt{\Delta t} & \text{prob } q \\ r\Delta t - \sigma\sqrt{\Delta t} & \text{prob } 1 - q \end{cases}$$

and the w(k) are independent samples from the same distribution with

$$\begin{aligned} Ew_k &= r\Delta t + (2q - 1)\sigma\sqrt{\Delta t} \\ &= r\Delta t + (2\frac{e^{r\Delta t}(1 - e^{-\sigma\sqrt{\Delta t}})}{e^{r\Delta t}(e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}})} - 1)\sigma\sqrt{\Delta t} \\ &\approx r\Delta t + (2\frac{-(-\sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t)}{\sigma\sqrt{\Delta t} - (-\sigma\sqrt{\Delta t})} - 1)\sigma\sqrt{\Delta t} \\ &\approx (r - \frac{1}{2}\sigma^2)\Delta t \end{aligned}$$

$$\begin{aligned} \text{Var } w_k &= \text{Var}(w_k - r\Delta t) \\ &= \sigma^2\Delta t - [(2q - 1)\sigma\sqrt{\Delta t}]^2 \\ &\approx \sigma^2\Delta t \end{aligned}$$



Increasingly More Accurate Replication

Therefore

$$\boxed{\ln S_n = \ln S_0 + \sum_{i=0}^{n} w_i,}\quad Ew_k = (r-\tfrac{1}{2}\sigma^2)T/n,\quad Var\ w_k = \sigma^2 T/n$$

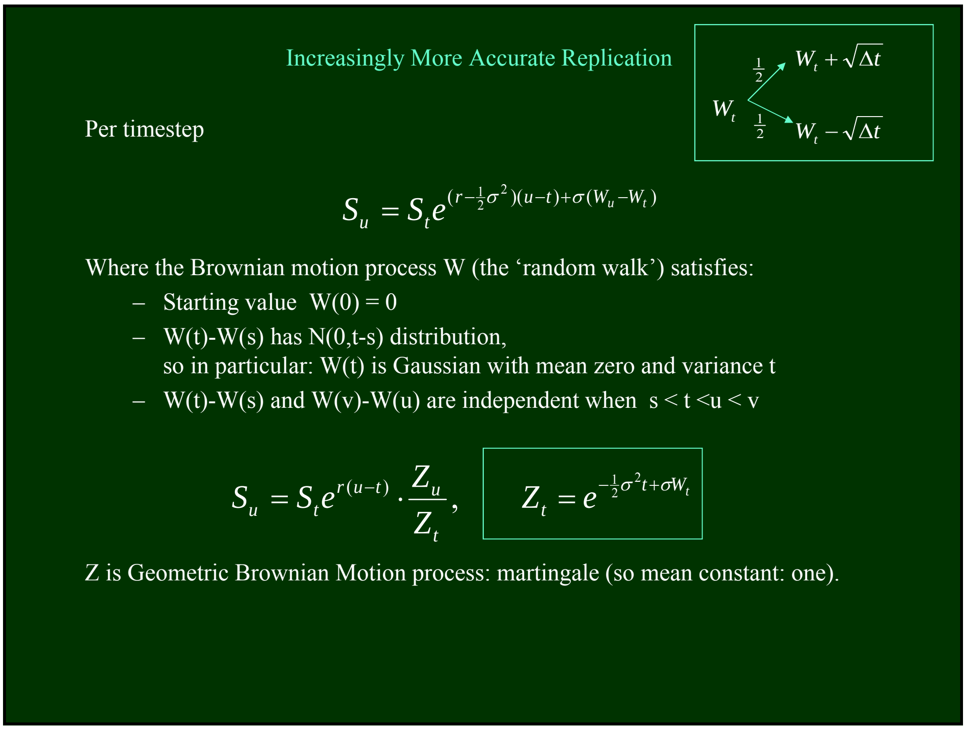
so Central Limit Theorem tells us that for number of time steps n large (i.e. timestep very small) using continuity condition,

$$\ln S_n - \ln S_0 \xrightarrow{\text{distr}} (r-\tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\cdot\varepsilon$$

where ε has a standard normal distribution. Therefore in limit, S has lognormal distribution and we can write in the limit for the distribution of the final stock value (at time T)

$$\boxed{S_T^{cont} = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\cdot\varepsilon^Q}}$$

with ε^Q a standard normal random variable under the riskneutral measure (i.e. based on Q-probabilities) !



Increasingly More Accurate Replication

$$\begin{tikzcd} & W_t+\sqrt{\Delta t} \\ W_t \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \\ & W_t-\sqrt{\Delta t} \end{tikzcd}$$

Per timestep

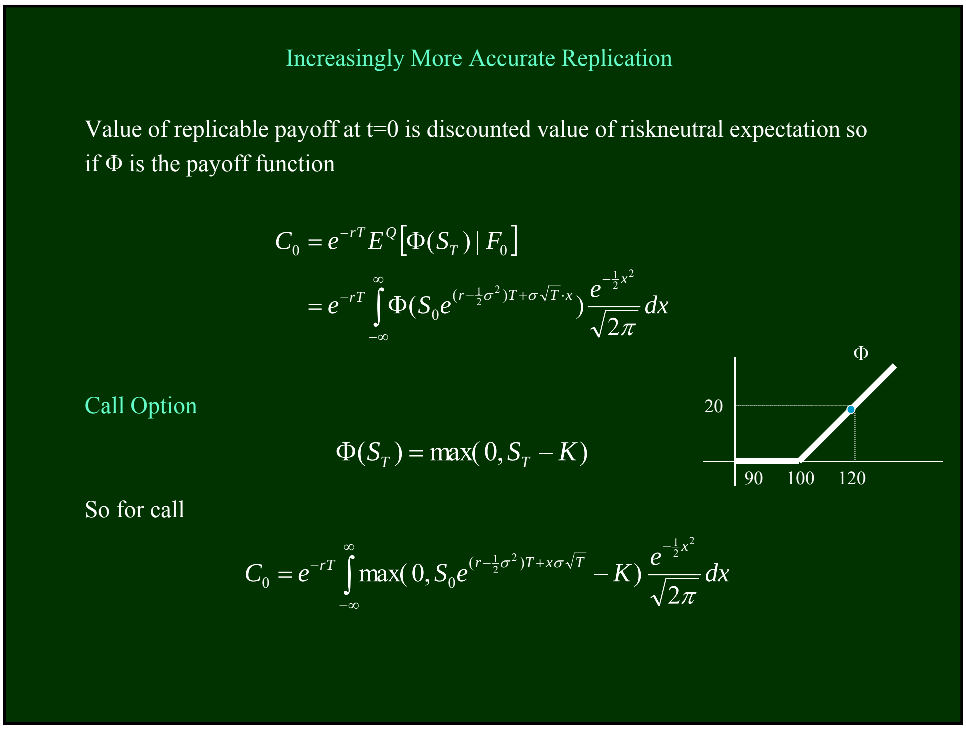
$$S_u = S_t e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W_u-W_t)}$$

Where the Brownian motion process W (the ‘random walk’) satisfies:

- Starting value W(0) = 0
- W(t)-W(s) has N(0,t-s) distribution,
  so in particular: W(t) is Gaussian with mean zero and variance t
- W(t)-W(s) and W(v)-W(u) are independent when s < t <u < v

$$S_u = S_t e^{r(u-t)}\cdot\frac{Z_u}{Z_t},\qquad \boxed{Z_t = e^{-\frac{1}{2}\sigma^2 t+\sigma W_t}}$$

Z is Geometric Brownian Motion process: martingale (so mean constant: one).



Increasingly More Accurate Replication

Value of replicable payoff at t=0 is discounted value of riskneutral expectation so if Φ is the payoff function

$$C_0 = e^{-rT}E^Q\left[\Phi(S_T)\,|\,F_0\right]$$

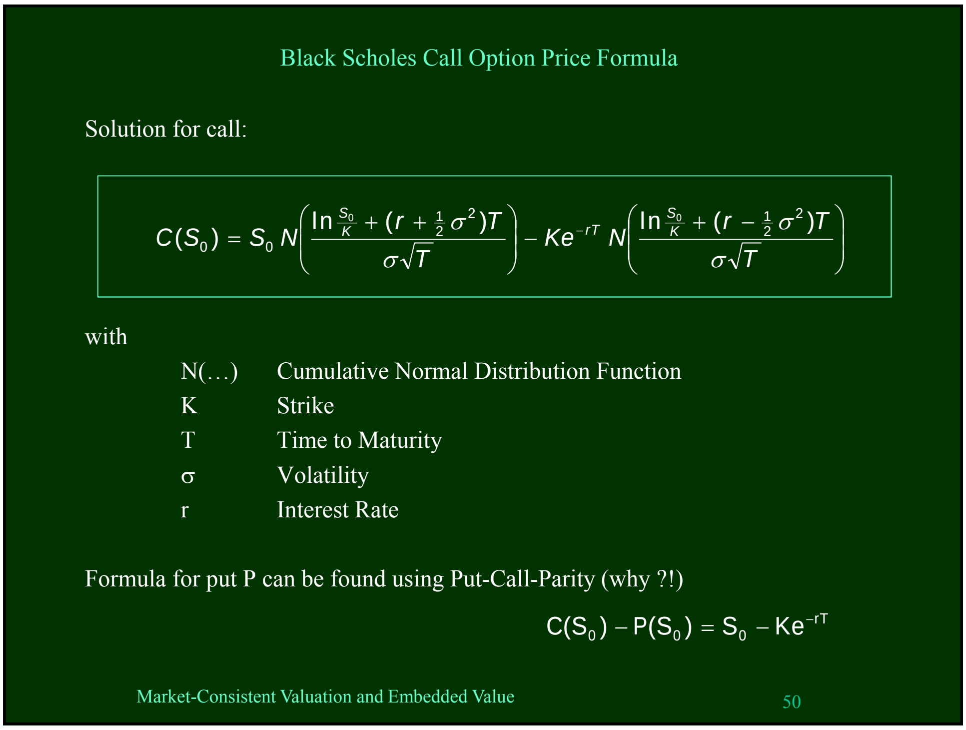
$$= e^{-rT}\int_{-\infty}^{\infty}\Phi(S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\cdot x})\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx$$

Call Option

$$\Phi(S_T) = \max(0, S_T - K)$$

So for call

$$C_0 = e^{-rT}\int_{-\infty}^{\infty}\max(0, S_0e^{(r-\frac{1}{2}\sigma^2)T+x\sigma\sqrt{T}} - K)\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx$$



Black Scholes Call Option Price Formula

Solution for call:

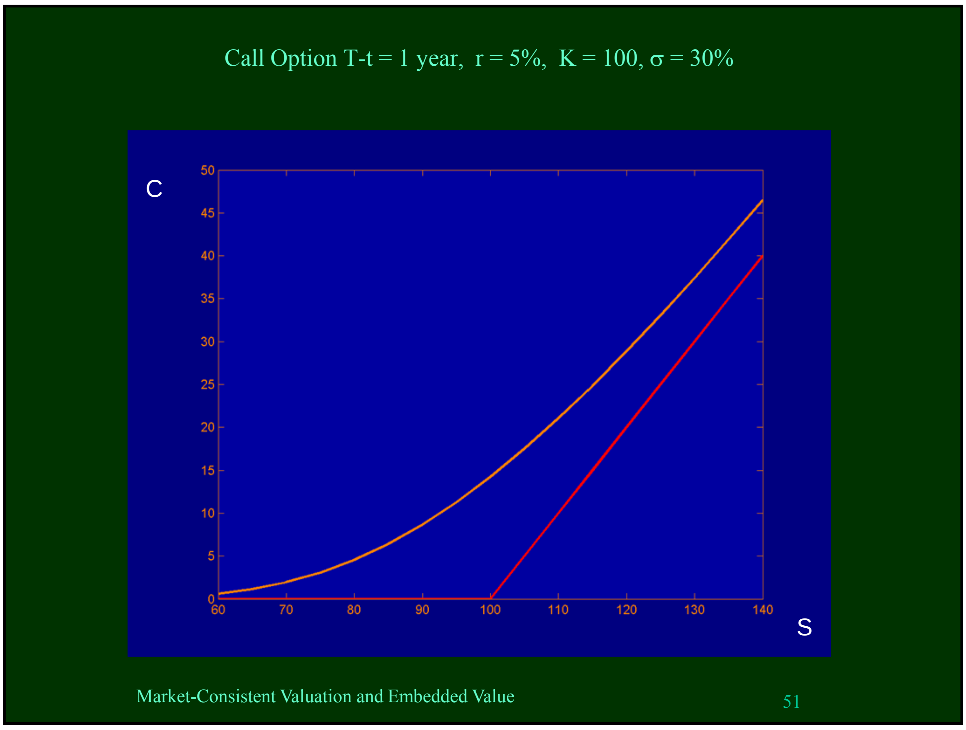
$$C(S_0) = S_0\,N\left(\frac{\ln\frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\,N\left(\frac{\ln\frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

with

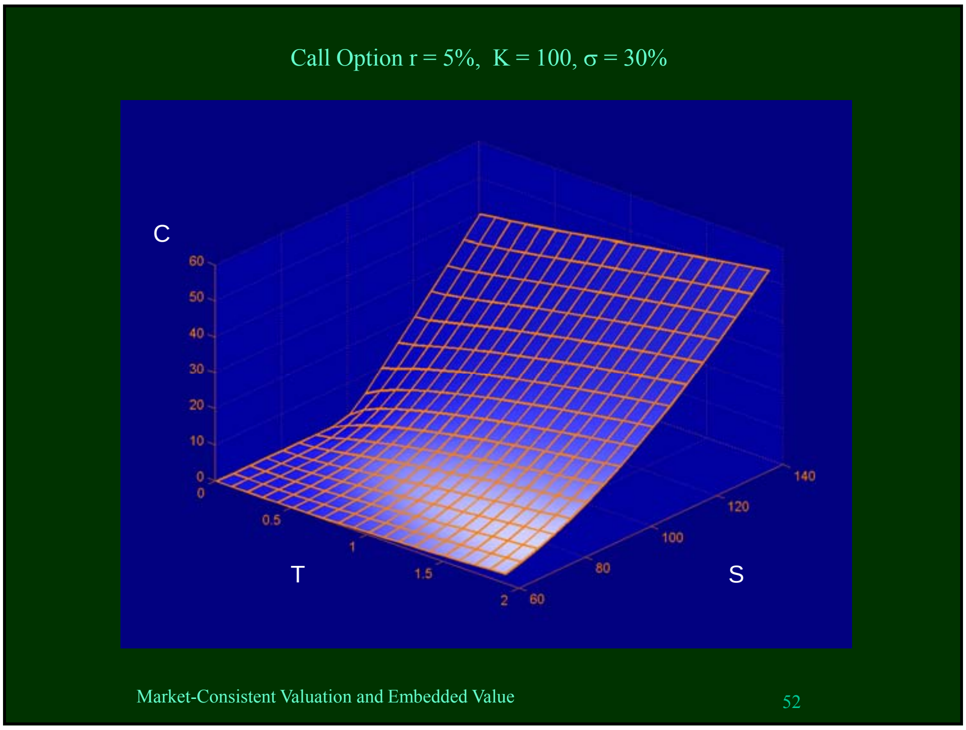
N(…)	Cumulative Normal Distribution Function
K	Strike
T	Time to Maturity
σ	Volatility
r	Interest Rate

Formula for put P can be found using Put-Call-Parity (why ?!)

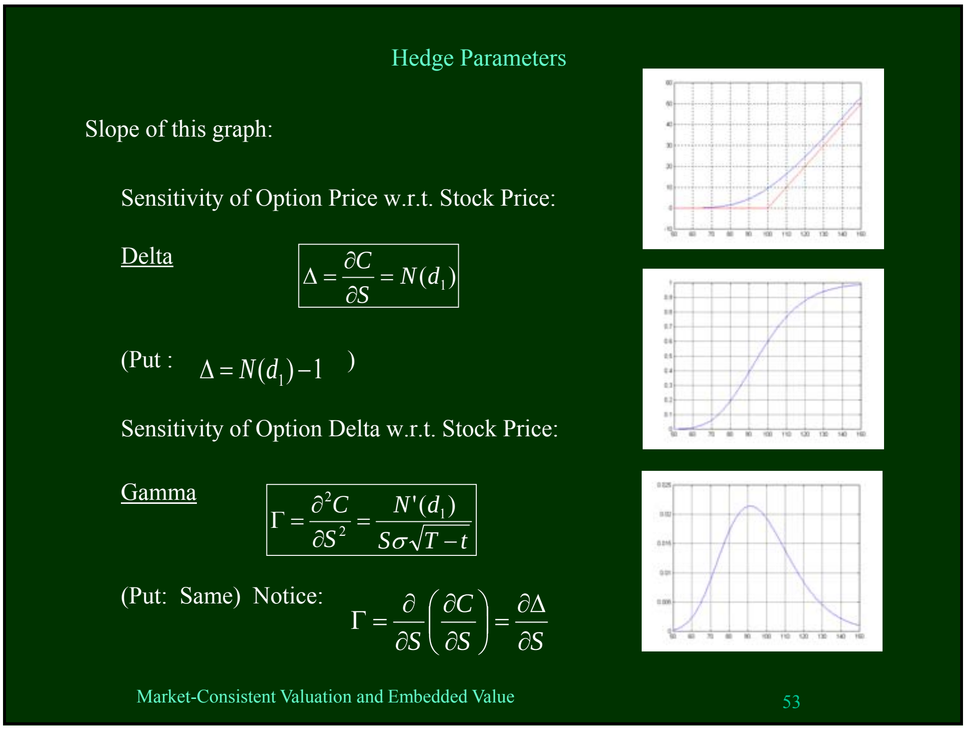
$$C(S_0) - P(S_0) = S_0 - Ke^{-rT}$$



Call Option T-t = 1 year, r = 5%, K = 100, σ = 30%



Call Option r = 5%, K = 100, σ = 30%



Hedge Parameters

Slope of this graph:

Sensitivity of Option Price w.r.t. Stock Price:

<u>Delta</u>

$$\boxed{\Delta = \frac{\partial C}{\partial S} = N(d_1)}$$

(Put : $\Delta = N(d_1) - 1$)

Sensitivity of Option Delta w.r.t. Stock Price:

<u>Gamma</u>

$$\boxed{\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}}$$

(Put: Same) Notice:

$$\Gamma = \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial \Delta}{\partial S}$$

Equity derivatives in practice: Volatility varies over time

Last	16.91
High on 10/24/08	89.53
Average	31.56
Low on 05/19/08	15.82