# Financial Mathematics for Insurance

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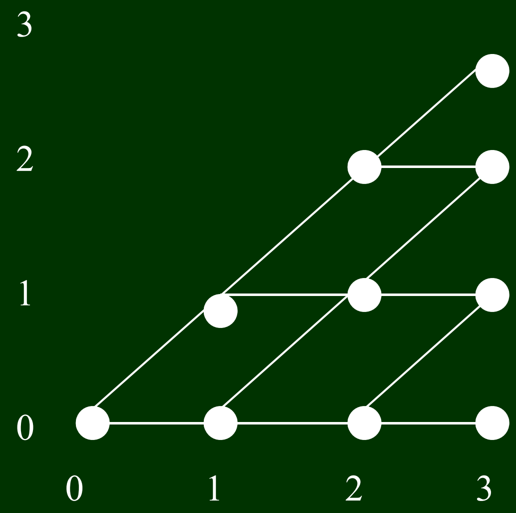
## V

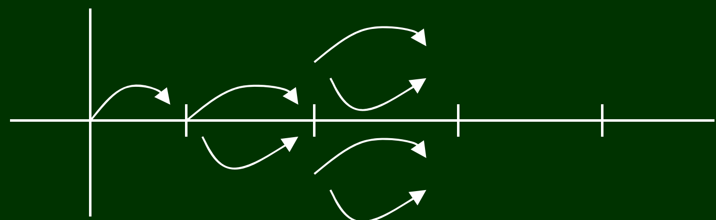
## Non-financial Risk

# Pricing Interest Rate Derivatives

- We need to introduce stochasticity in interest rates
- but this cannot be done in arbitrary manner: must be arbitrage-free




Idea Binomial Tree for short rates



- Risk-neutral probabilities are set to ½ (so not calculated using replication)
- short rates r(t,i) on nodes must be chosen according to some model (how much volatility i.e. stochasticity).

## Stochastic Dynamics for Swap Rates

Can we (as for equity) take a limit and arrive at a lognormal distribution?
Possible for short rate, but this does NOT make swaprates lognormal.

Often in practice: simplest continuous time model for whatever rate:

Black's Model (1976)

Fundamental Assumptions: (today = time t)

- European Call Option on underlying V (Bond price, future, swap rate)
- Underlying has lognormal distribution at maturity T, with a standard deviation of ln V equal to $\sigma\sqrt{T - t}$
- Discount rate for option's matuirty are taken non-stochastic (even though interest rates are …)

# Black Formula

Call Option price according to Black model

$$C = d_T(t)\left\lfloor F\ N(\tilde{d}_1) - K\ N(\tilde{d}_2)\right\rfloor$$

$$\tilde{d}_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \qquad \tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T-t}$$

| | |
|---|---|
| T | Time to maturity of option |
| F | Forward price for V of a contract with maturity T |
| K | Strike price of option |
| d | Riskfree discount rates |
| σ | volatility (see previous slide) |

## Swaptions' Market value

Swaption is European option on par swap rate $y_{N,To}$ with strike price $K$.

Payer swaption = call option on $y_{N,To}$

Receiver swaption = put option on $y_{N,To}$

Black formula for payer swaption with principal L and volatility σ:

$$V_{payswptn,t} = L\left(\sum_{k=1}^{N} D_{T_k,t}\right)\left(y_{T_0,T_N,t}N(d_1) - KN(d_2)\right)$$

$$V_{recswptn,t} = L\left(\sum_{k=1}^{N} D_{T_k,t}\right)\left(KN(-d_2) - y_{T_0,T_N,t}N(-d_1)\right)$$

with

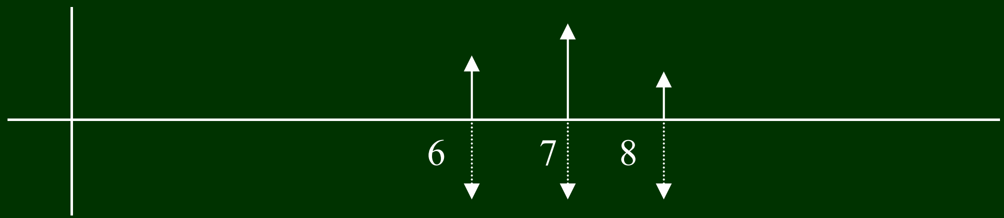
$$y_{T_0,T_N,t} = \frac{D_{T_0,t} - D_{T_N,t}}{\sum_{k=1}^{N} D_{T_k,t}}, \qquad d_{1,2} = \frac{\ln\left(y_{T_0,T_N,t}/K\right) \pm \frac{1}{2}\sigma^2 (T_0 - t)}{\sigma\sqrt{T_0 - t}}$$

# Swaption Pricing using Black's model

Example

Valuation of a swaption which gives holder the right to pay
6.2 % in 3-year swap starting in 5 years on \$100 nominal.
Current Term structure is flat at 6.0 %.
Assume that the volatility for the swap rate is 20% annually.

Payments




so equivalent to series of cashflows worth 100 max(y-K,0)
at times 6,7,8 where y is swap rate at maturity (5 yrs), K is 6.2 %.

## Calculation of Swaption Price

Total Value will be

$$\left(100\sum_{i=6}^{8} D_{i,0}\right)\max\{y-K,0\}$$

which is now priced at

$$\left(100\sum_{i=6}^{8} D_{i,0}\right)\left[F\ N(d_1)-K\ N(d_2)\right]$$

so with

$$D_{i,0}=(1.06)^{-i},\quad \boxed{F=6\%,}\quad K=6.2\%,\quad \sigma=20\%,\quad T_0-t=5$$

we find price of the swaption : 1.96

## Swaptions in Insurance products

Having (some) pricing method for swaptions is important,

since they are often embedded in insurance products which guarantee an investment return (GIC) and include profit - sharing…

Swaption payoff

$$\max\{y - K, 0\}$$

is actually value of a profit-sharing feature !

## Profit sharing
## Contract agreement

"The Profit sharing is based on the interest rates. Every premium and technical interest rates on previous premiums are invested in bonds at the than prevailing market rates. Profit sharing is the coupon income above 3% and is distributed annually to the policyholder."

Interpretation:

Couponrate = par swap rate

Above 3% = option!

Principal = 3% investments in bonds

Stylized example of interest rate driven profit sharing

Distinguish:

- Fixed cash flows
- Interest Rate-dependent cash flows
  - Market value of interest rate-dependent cash flows by option formulas (Black's model)

# Profit sharing
## Cash flows (3%)

| | CashFlows | Obligatie |
|---|---|---|
| T=0 | € 10.000,00 | € 10.000,00 |
| T=1 | € 10.000,00 | € 10.300,00 |
| T=2 | € 10.000,00 | € 10.609,00 |
| T=3 | € 10.000,00 | € 10.927,27 |
| T=4 | € 10.000,00 | € 11.255,09 |
| T=5 | -€ 54.684,10 | |

Actuarial equivalence at 3% interest. We observe real discount factors in market:

| | CashFlows | DiscFac |
|---|---|---|
| T=0 | € 10.000,00 | 1,00000 |
| T=1 | € 10.000,00 | 0,97761 |
| T=2 | € 10.000,00 | 0,94782 |
| T=3 | € 10.000,00 | 0,91204 |
| T=4 | € 10.000,00 | 0,87346 |
| T=5 | -€ 54.684,10 | 0,83454 |
| | | |
| | Marktwaarde | € 1.472,96 |

# Profit sharing

Profit sharing at $T$=0:

$$\text{Payoff at time } 0 = \max\{y_{0,5,0} - 3\% \text{ , } 0\}\left(\sum_{k=1}^{5} D_{k,0}\right)$$

Par swap rente: $y_{0,5,0}$ = 3,64%

Sum DF = 4,5455

Value: -10.000 * 0,64% * 4,5455 = -290,91

Profit sharing at $T$=1:

Payer Swaption with strike of 3%

Calculate market value with Black-Scholes formula (*Need implied volatility !)*

$$\text{Payoff at time } 1 = \max\left\{ y_{1,5,1} - 3\%\ ,\ 0 \right\} \left( \sum_{k=2}^{5} D_{k,1} \right)$$

Plug into option formula:

$y_{1,5,0} = 4{,}01\%$; $K = 3\%$; $\sigma = 25\%$

sum DF = 3,5679

$N(d_1)$=0,9007 and $N(d_2)$=0,8498

Market value $t$=0:

-10.300,00 * 3,5679 * (0,0401 * 0,9007 - 0,03 * 0,8498)= -390,40

## Profit sharing : Value entire contract

| Hoofdsom | Optie | Vol | SomDF | FwdSwap | Swaption |
|---|---|---|---|---|---|
| € 10.000,00 | 0x5 | 0% | 4,5455 | 3,64% | -€ 290,91 |
| € 10.300,00 | 1x4 | 25% | 3,5679 | 4,01% | -€ 390,40 |
| € 10.609,00 | 2x3 | 20% | 2,6200 | 4,32% | -€ 380,89 |
| € 10.927,27 | 3x2 | 15% | 1,7080 | 4,54% | -€ 291,09 |
| € 11.255,09 | 4x1 | 10% | 0,8345 | 4,66% | -€ 156,52 |

| | |
|---|---|
| **Cash Flows** | € 1.472,96 |
| **Winstdeling** | -€ 1.509,80 |
| **Marktwaarde Totaal** | -€ 36,84 |

# Direct Contracts on Rates

More convenient: derivative contracts directly on rates.

Suppose you're short a floating note with three-monthly LIBOR interest payment. Interest Rate Cap provides insurance against rate rising above certain level.

Cap with total life T on principal L with 'reset dates' $t_1, t_2, \ldots t_{n+1} = T$ and cap rate K pays

$$L(t_{k+1} - t_k)\max(r_k - K, 0)$$

at time $t_{k+1}$ (k=1..n) when the observed LIBOR rate for the period $[t_k, t_{k+1}]$ equals $r_k$ (so payments have natural time lag !)

A cap is therefore a series of caplets: call options on the LIBOR rate.
Similar: put option on LIBOR rate is called a floorlet.

## Continuous Time

Can we, as in Black-Scholes case, get easier expressions for simple products by taking limits and switch to continuous time analogon ?

Yes, but techniques become more technical. Approaches:

- Black's model
  Assume simple continuous-time distribution (lognormal) for relevant interest rate in the future (a short rate, a swap rate) at one time in the future .

- Hull-White model
  Assume mean-reversion model for short rates in future with Gaussian increments.

- LIBOR market and LIBOR swap market model
  Model for discrete rates in continuous time.

Usually simuation-based

- Heath-Jarrow-Morton model
  Model for forward rates in continuous time.

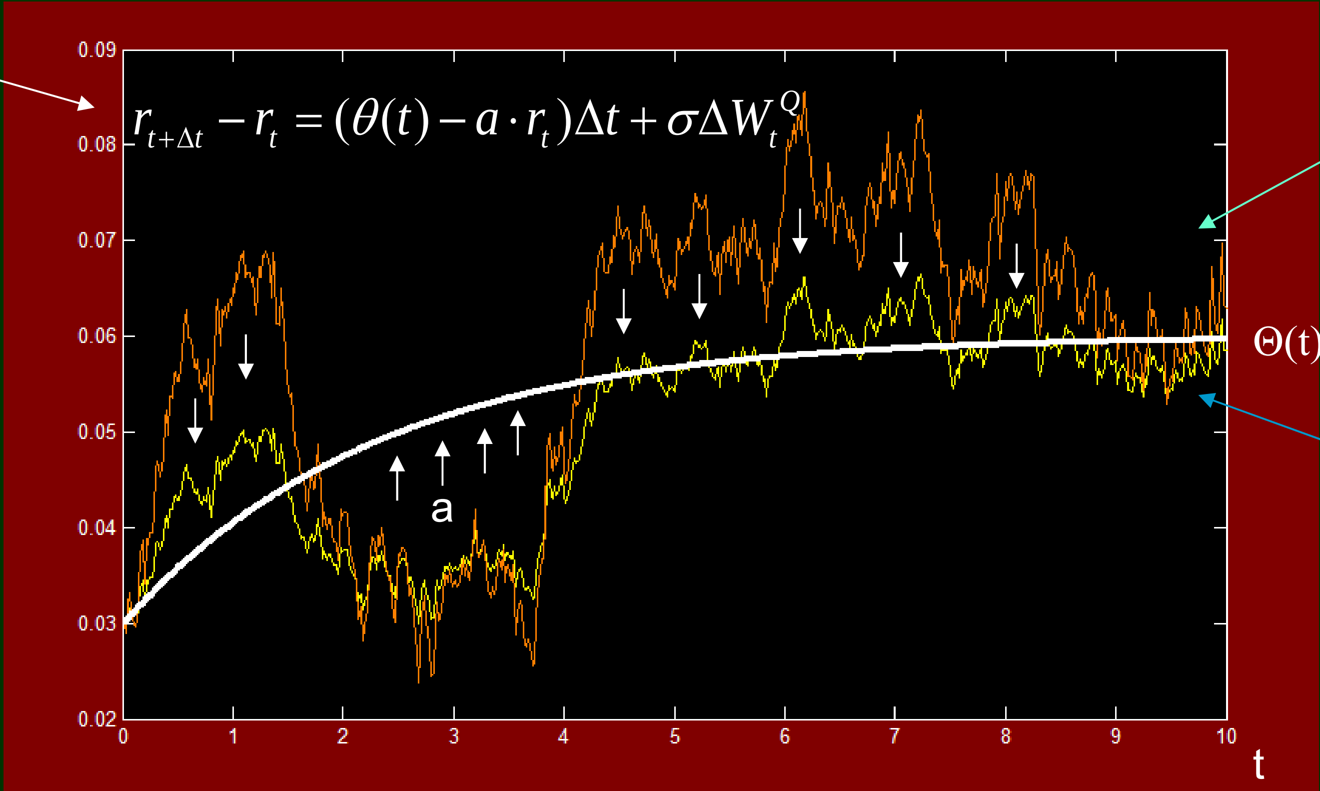
Most general: includes short rate models

## Hull-White model

Essential idea:

- short rate with mean-reversion to time-varying deterministic curve
- this ‘average’ curve can be used to fit the term structure
- volatility parameter σ and mean reversion speed a two free parameters to model uncertainty (random shocks + tendency to return to average)

Riskneutral !




High σ

Θ(t)/a

Low σ

t

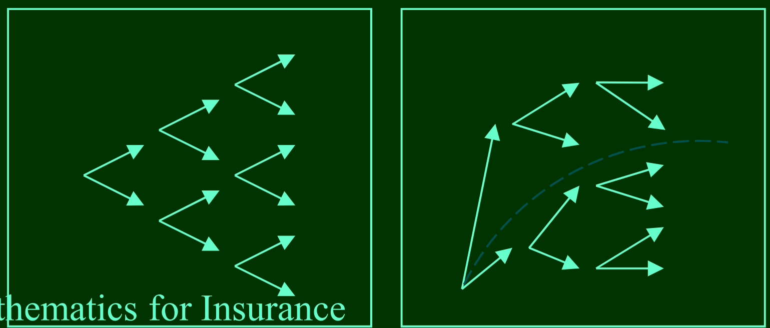
## Why popular ?

Advantages

- Later short rates have Gaussian distribution: explicit formulas
- Term structure can be fitted perfectly, due to time-varying parameter $\Theta(t)$
- Two parameters to model stochasticity have clear interpretation
- Closed-form formulas for bond prices, calls and puts on zero-coupon and coupon bonds (and therefore caps, floors), European swaptions
- Therefore: possible to calibrate on those important instruments

Disadvantages

- Gaussian short rate implies negative rate with nonzero probability
- Limit capability of matching swaption volatilities (only 2 parameters)
- More complicated to construct tree for short rate



## Hull-White model

Earlier tree models that we have seen do not exihibit mean reversion.

That makes continuous-time limit analysis simple (increments are iid so Central Limit applies). Can we create short rate model with mean reversion which still fits term structure?

Deterministic framework we had for ZC bond at time t with maturity T

$$p(t,T) = e^{-r(T-t)}$$

Natural generalisation would be

$$p(t,T) = e^{A(t,T) - r_t B(t,T)}$$

Under Gaussian dynamics for interest rate, so distribution increment (given info at t):

$$r_{t+\Delta t} - r_t \cong N(\mu(r_t, t)\Delta t, \sigma^2(r_t, t)\Delta t)$$

# Once more: One-Step Replication

Start in discrete time with 2 tradeables: bonds with maturity in 1 and 2 time steps:

$$
p(t,t+\Delta t)\begin{array}{l} \nearrow\; p(t+\Delta t,t+\Delta t)=1 \\ \searrow\; p(t+\Delta t,t+\Delta t)=1 \end{array}
\qquad\qquad
p(t,t+2\Delta t)\begin{array}{l} \nearrow\; p(t+\Delta t,t+2\Delta t) \\ \searrow\; p(t+\Delta t,t+2\Delta t) \end{array}
$$

Can be simplified to multiple of

$$
1\begin{array}{l} \overset{\frac{1}{2}}{\nearrow}\; e^{r_t\Delta t} \\ \underset{\frac{1}{2}}{\searrow}\; e^{r_t\Delta t} \end{array}
\qquad\qquad
1\begin{array}{l} \overset{\frac{1}{2}}{\nearrow}\; p^u \\ \underset{\frac{1}{2}}{\searrow}\; p^d \end{array}
$$

$$
\begin{aligned}
p^u &= \exp\left(\Delta A - r_t\Delta B - B(t+\Delta t,t+2\Delta t)[\mu(r_t,t)\Delta t+\sigma(r_t,t)\sqrt{\Delta t}\,]\right)\\
p^d &= \exp\left(\Delta A - r_t\Delta B - B(t+\Delta t,t+2\Delta t)[\mu(r_t,t)\Delta t-\sigma(r_t,t)\sqrt{\Delta t}\,]\right)
\end{aligned}
$$

We then apply pricing by replication as before. (Note: tree not recombining !)

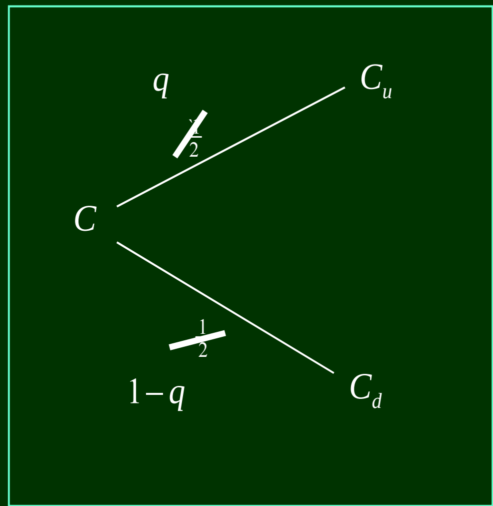
# Martingale Probabilities and Change of Drift

$$1 \begin{cases} \xrightarrow{\frac{1}{2}} \; e^{r_t \Delta t} \\ \xrightarrow{\frac{1}{2}} \; e^{r_t \Delta t} \end{cases} \qquad\qquad 1 \begin{cases} \xrightarrow{\frac{1}{2}} \; p^{u} \\ \xrightarrow{\frac{1}{2}} \; p^{d} \end{cases}$$

Pricing rule now gives

$$C = e^{-r_t \Delta t}(qC_u + (1-q)C_d)$$

$$q = \frac{e^{r_t \Delta t} - p^d}{p^u - p^d}$$




As before, every tradeable (and in particular all ZC bonds) now become martingales after discounting under the new probabilities, so

$$E^Q\left[\frac{p(t+1,T)}{p(t,T)} \,\Big|\, r_t\right] = e^{r_t \Delta t}, \quad \text{for all T}$$

## Martingale Probabilities and Change of Drift

Let us now make explicit choice for drift and volatility to build in mean reversion:

$$\boxed{\begin{aligned} r_{t+\Delta t} - r_t &= (\theta(t) - a \cdot r_t)\Delta t + \sigma \Delta W_t^Q \\ p(t,T) &= \exp[A(t,T) - r_t B(t,T)] \end{aligned}}$$

We exploit martingale property

$$Ee^{aZ} = e^{\frac{1}{2}a^2}$$

$$\begin{aligned} 1 &= e^{-r_t \Delta t} E^Q[\tfrac{p(t+\Delta t,T)}{p(t,T)} \mid r_t] \\ &= E^Q[\exp\big(-r_t\Delta t + \Delta A - (r_{t+\Delta t} - r_t)B(t+\Delta t,T) - r_t \Delta B\big) \mid r_t] \\ &= \exp\big(-r_t\Delta t + \Delta A - (\theta(t) - ar_t)B(t+\Delta t,T)\Delta t + \tfrac{1}{2}B^2(t+\Delta t,T)\sigma^2\Delta t - r_t\Delta B\big) \\ &= \exp\big(-r_t\underbrace{\big[\Delta t - aB(t+\Delta t,T)\Delta t + \Delta B\big]}_{} + \underbrace{\big[\Delta A - \theta(t)B(t+\Delta t,T)\Delta t + \tfrac{1}{2}B^2(t+\Delta t,T)\sigma^2\Delta t\big]}_{}\big) \end{aligned}$$

to find

$$\begin{aligned} A(t+\Delta t,T) - A(t,T) &= \theta(t)B(t+\Delta t,t)\Delta t - \tfrac{1}{2}B^2(t+\Delta t,t)\sigma^2\Delta t, \quad && A(T,T) = 0 \\ B(t+\Delta t,T) - B(t,T) &= -\Delta t + aB(t+\Delta t,t)\Delta t, && B(T,T) = 0 \end{aligned}$$

This allows us to fit the term structure. Indeed, we can solve for B explicitly

$$B(t,T) = (1-(1-a\Delta t)^{(T-t)/\Delta t})/a$$

and A numerically. Then current rates can be used to fit θ(t) since

$$p(0,T) = \exp[A(0,T) - r_0 B(0,T)]$$

known

known

known once a has been chosen

known in terms of θ(t)
once σ has been chosen

$$\boxed{\begin{aligned}\theta(T) &= aF_T(0) + \tfrac{F_{T+\Delta t}(0)-F_T(0)}{\Delta t} + \tfrac{\sigma^2}{2a}\left(1-(1-a\Delta t)^{1+2T/\Delta t}\right),\\ F_T(0) &= -\tfrac{1}{\Delta t}\ln\tfrac{p(0,T+\Delta t)}{p(0,T)}\end{aligned}}$$

So: mean reversion speed and volatility need to be chosen, then mean reversion level allows us to fit term structure by (again) shifting of short rates.

Affine model

$$p(t,T) = e^{A(t,T) - B(t,T) r_t}$$

determ. determ. stoch.

So

- All bond prices deterministic function of the same stochastic factor (the short rate), and thus perfectly correlated
- Short rate has normal distribution so zero coupon bond prices have lognormal disribution
- Last property implies that we should be able to get explicit formulas for call and put options on bonds

## Non hedgeable risks

We know want to look explicitly at risk that cannot be replicated perfectly.
For the fair value valuation we can use a three stage approach

1. Use market values when available
2. Model to market when you can find similar instruments with reliable market prices
3. Use a model for the risk type when there are no reliable links to financial markets

First two mainly applicable for pricing *financial* risks.

Examples of risks without reliable market prices as quoted on open markets

- Mortality risks
- P&C risks like motor, fire, catastrophe, nuclear plants, terrorism
- Operational risks
- Financial risks in non complete markets (inflation)

# Non-parallel stochastic shocks

Simple but non-parallel stochastic dynamics in term structure:

One factor short rate models (example: Hull-White)

$$r(t+\Delta t) - r(t) = a(\Theta(t) - r(t))\Delta t + \sigma_r \sqrt{\Delta t} \cdot \varepsilon(t)$$

Simple but non-parallel stochastic dynamics in force of mortality μ:

Single principal component model (example: Lee-Carter)

$$\ln \mu_x(t+1) - \ln \mu_x(t) = \beta_x(\theta + \sigma_\kappa \cdot \delta(t))$$

Mean reversion in r, not in μ (long term consequences !)
We will avoid smoothing, or the making of structural assumptions.

## Recall from yesterday : Fitting Lee-Carter model

- Assume given: exposures E and deaths D for ages x and years t

- Assume Lee-Carter mortality dynamics (general component)

$$\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t$$

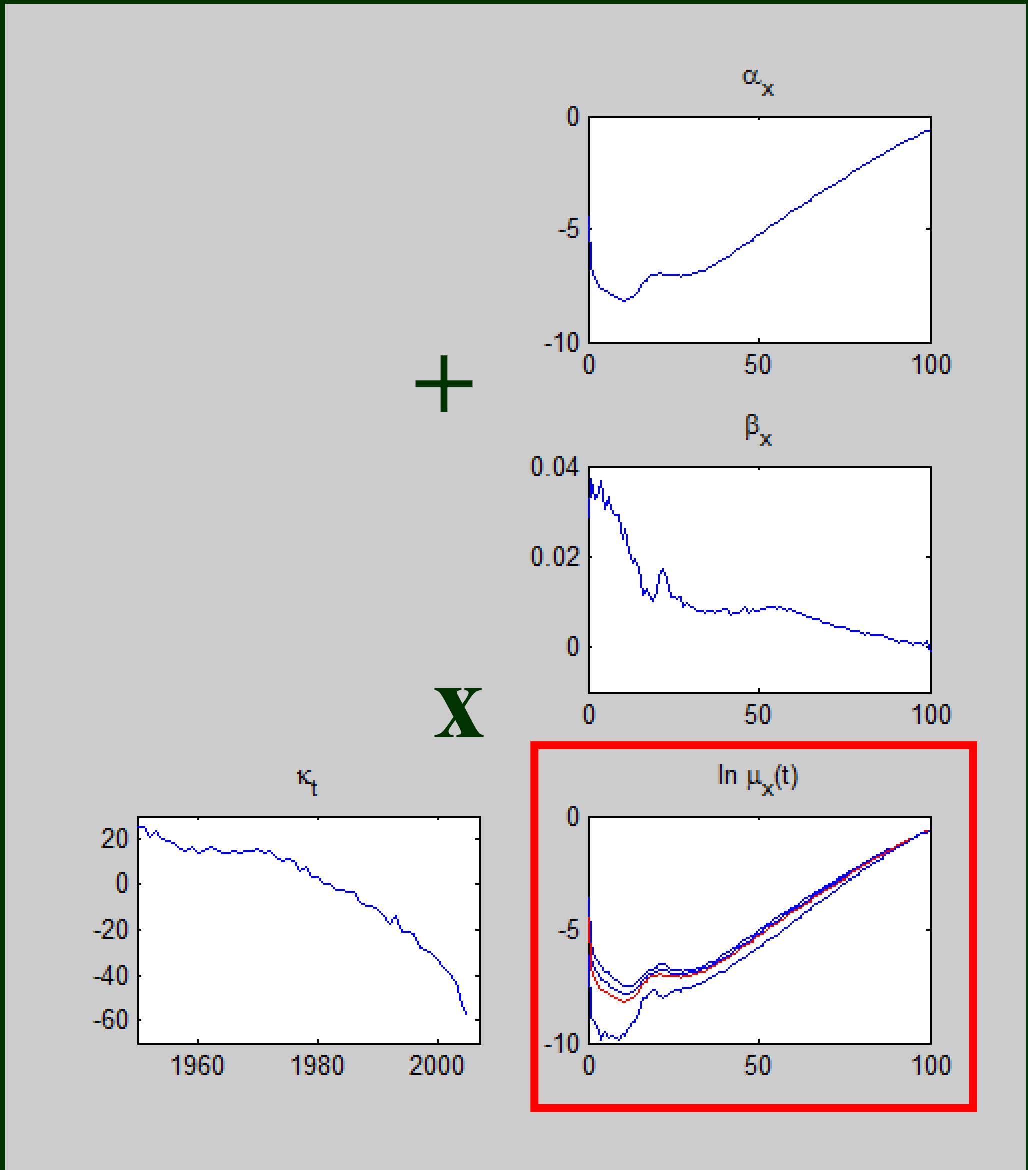
- Assume Poisson distribution of deaths, conditioned on mortality rates

$$P(D_{x,t} = d \mid \mu_x(t)) = \frac{(\mu_x(t) E_{x,t})^d}{d!} e^{-\mu_x(t) E_{x,t}}$$

so we maximize log likelihood which equals some irrelevant constants plus

$$\sum_t \sum_x D_{x,t}(\alpha_x + \beta_x \kappa_t) - E_{x,t} e^{(\alpha_x + \beta_x \kappa_t)}$$

HMDB Dutch Males
1948 - 2005




$$\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t$$

# Lee-Carter model

- Second stap is identification of time series for kappa.

- Plot suggest random walk with constant drift

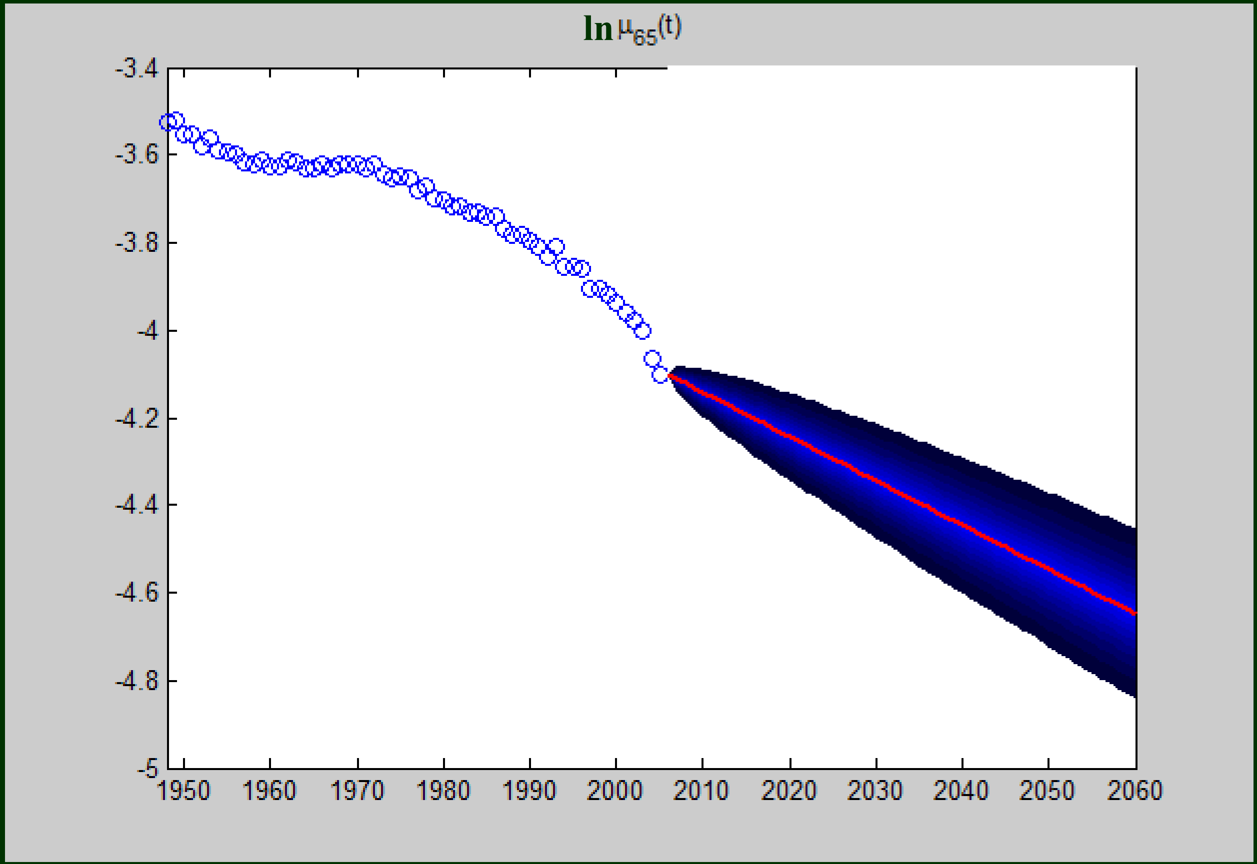
$$\kappa_{t+1} - \kappa_t = \theta + \sigma_\kappa \varepsilon_t$$

  with iid standard Gaussian series ε.

- This gives square-root growth of confidence interval since variance of κ grows linearly in time.

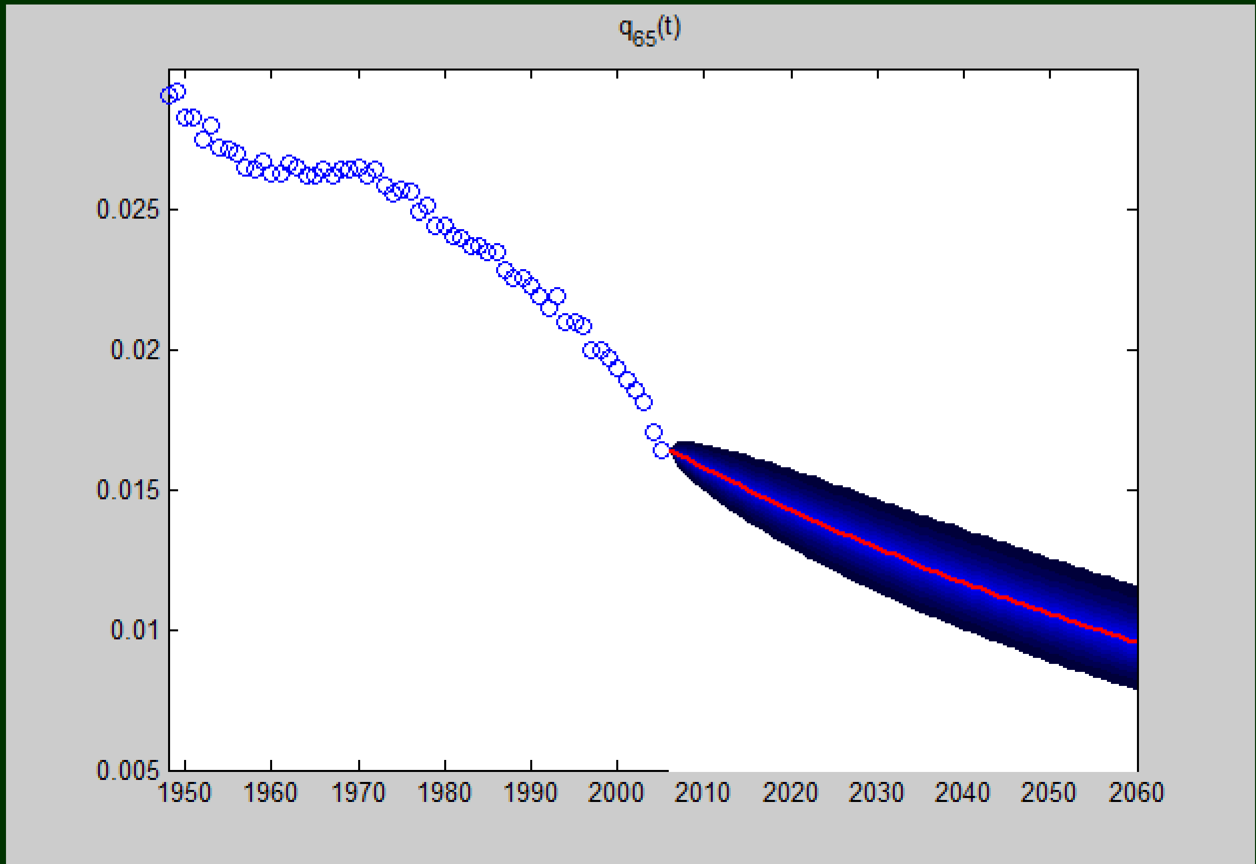
- Easy to create fan charts for (log of) force of mortality, but more important for applications is uncertainty in survival curve.

HMDB Dutch Males
1948 - 2005



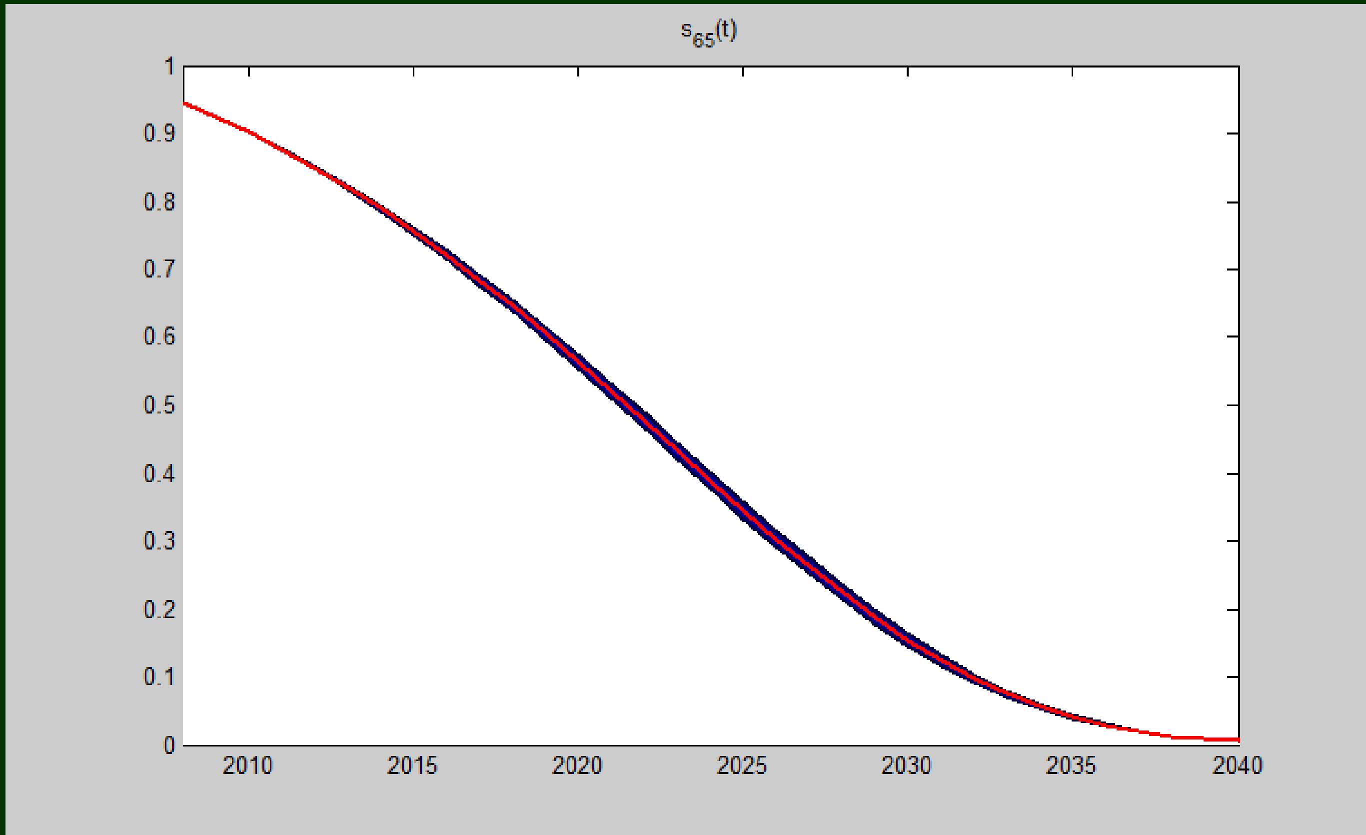
HMDB Dutch Males
1948 - 2005

$$q_x(t) = 1 - e^{-e^{\ln \mu_x(t)}}$$



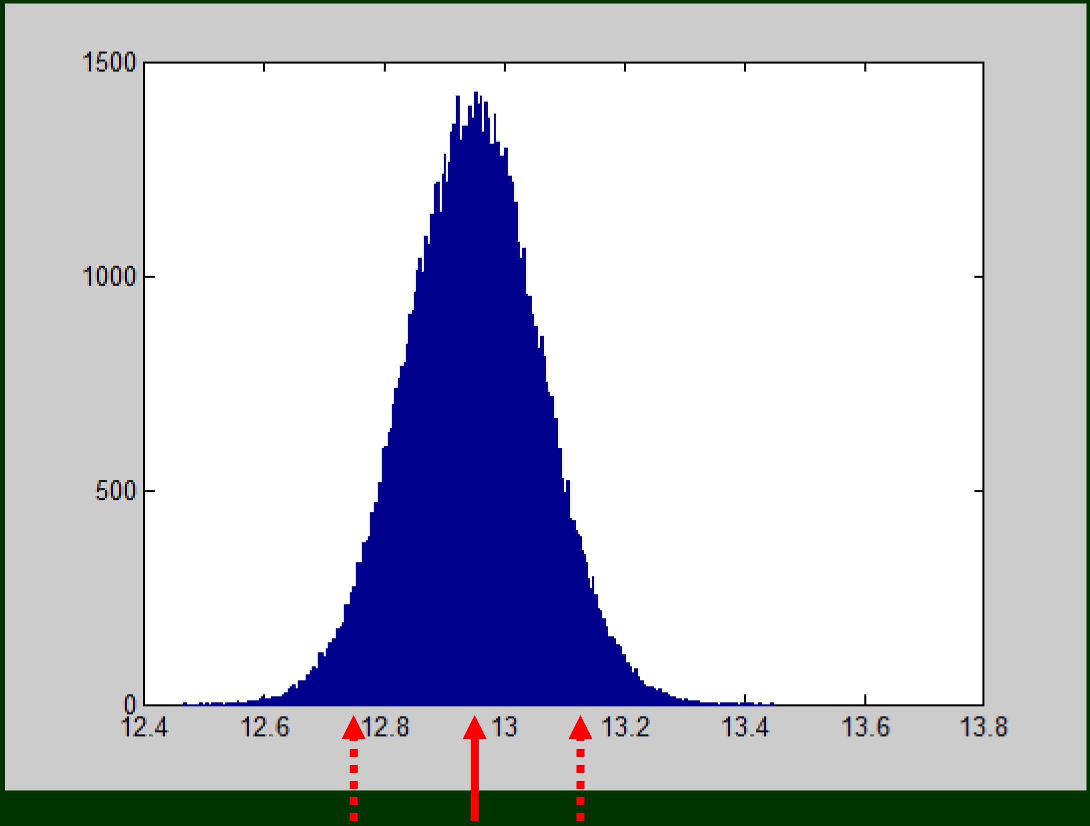
$$s_x(t) = e^{-\sum_{i=0}^{t-1} e^{\ln \mu_{x+i}(i)}}$$

HMDB Dutch Males
1948 - 2005



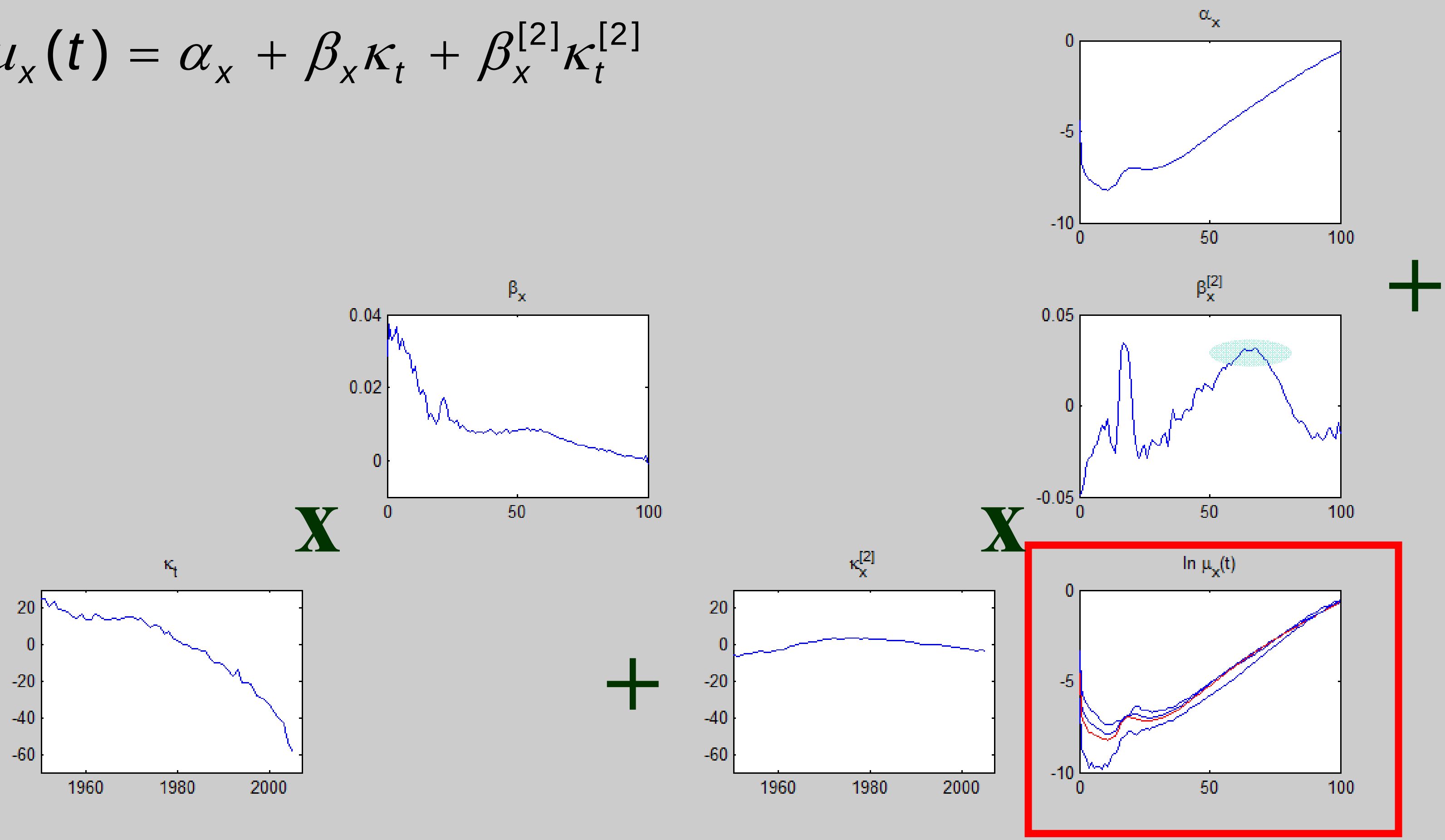
HMDB Dutch Males
1948 - 2005

$$A_x = \sum_{t=0}^{\infty} e^{-rt+\sum_{i=0}^{t-1} e^{\ln \mu_{x+i}(i)}}$$



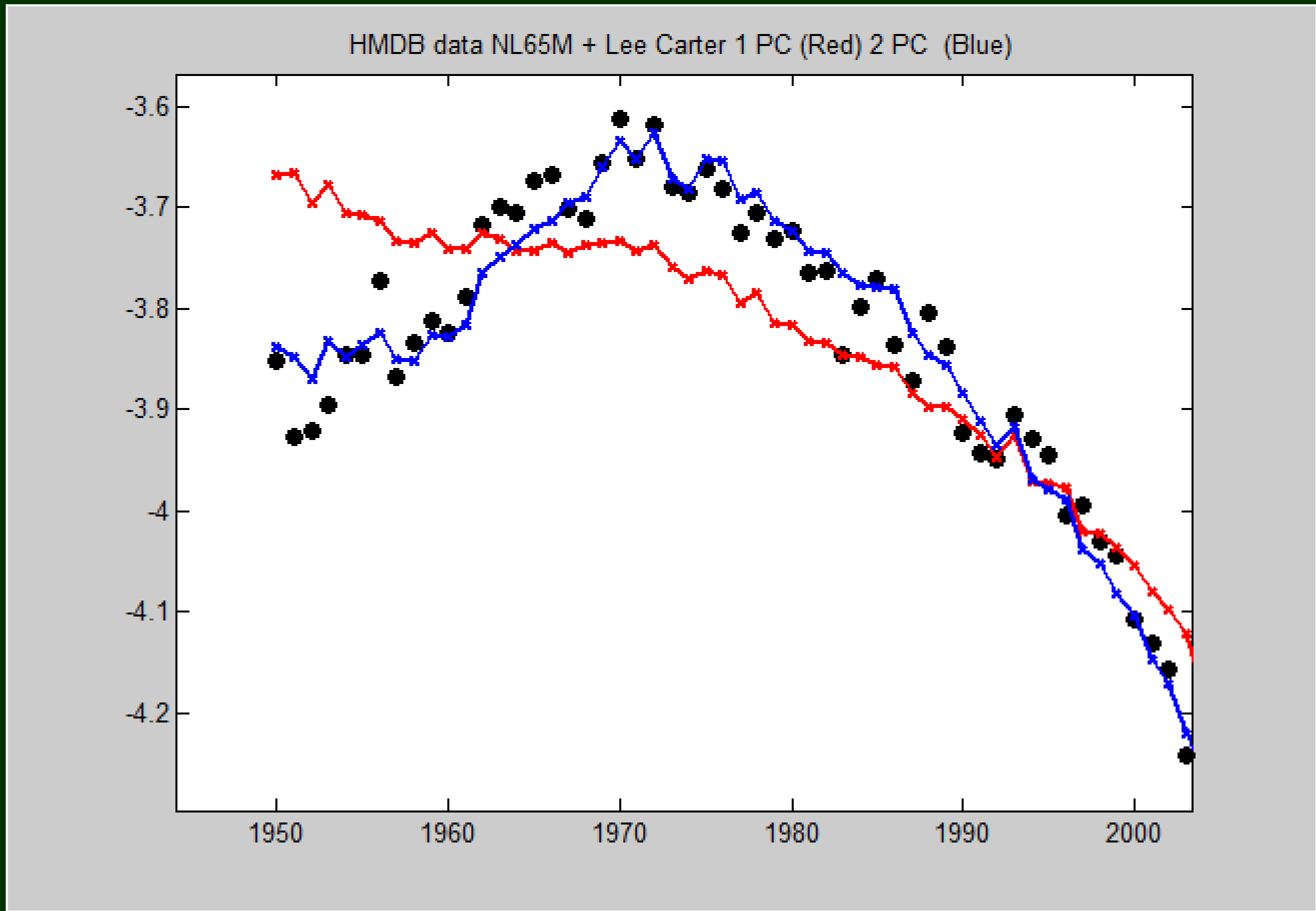
HMDB Dutch Males
1948 - 2005

$$\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t + \beta_x^{[2]} \kappa_t^{[2]}$$



# Second Principal Component

Small effect in time series may give substantial improvement in fit for particular age:



# Price of Longevity Risk

- Most insurers / pension funds are short longevity.
- Longevity risk cannot be diversified away if it is consistent across ages. It is therefore a risk which should be compensated in price.
- Since no (liquid) market for longevity products exist, it is impossible to use riskneutral pricing methodology.
- Since risk cannot be neutralized, universal fair price cannot be obtained so it must be based on risk appetite:
  - Use explicit model for risk aversion (eg. utility)
  - Use actuarial premium principle (eg. variance)
  - Use asset performance criterion (eg. Sharpe ratio)

## Indifference Pricing Principle

We need a pricing principle for longevity risk, which cannot be based on a standardized liquid market (since it does not yet exist).

We illustrate the issues in a stylized example where we use the principle of *indifference pricing*: we assume an economic agent has a utility function U which measures his/her risk preferences.

Indifference pricing means that

- Expected utility of agent selling an annuity (and receiving premium), and
- expected utility of agent without selling the annuity

should be the same (i.e. price is such that agent is *indifferent*, in terms of expected utility of discounted wealth, towards selling or not selling the annuity).

# Utility functions

Stochastic amount of money: would you prefer

- receiving 10,-
- receiving 1.000.000,- with prob. 1:100.000 and receiving nothing otherwise? (lottery)

How to rank random amounts of money (i.e. Preferences)?

<u>Simple observation:</u>

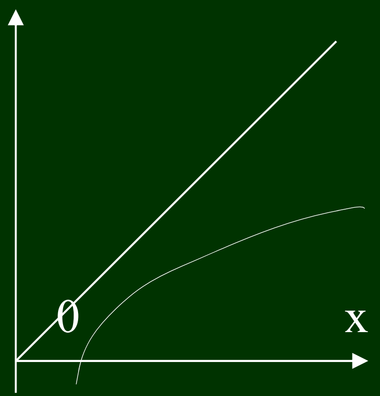
- Expected values of these two alternatives are the same
- Second has risk, first not so here <u>no</u> compensation for risk

## How to rank random amounts of money ?

Naive solution (ranking based on <u>expected values</u>) would not
Incorporate risk, see Lottery example

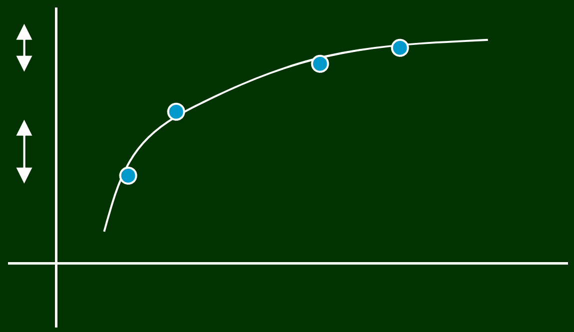
Utility function U measure “happiness” with wealth, and we
rank stochastic payoffs X based on E[U(X)]

U(x)    U(x) = x    U(x) = x    Risk-neutral
( ranking based on mean only )

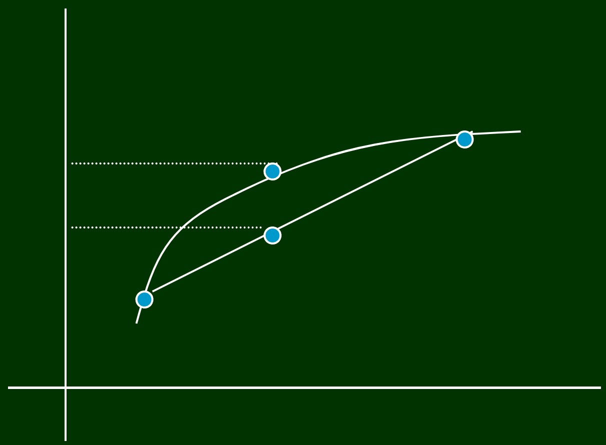



U(x) = ln(x)    U(x) = ln(x)    Risk-averse

# Concavity & Risk Aversion



Concave Utility: less happy with extra euro if hou have many already



Concave utility:

Alternative 1: - receive x with prob. p
- receive y with prob. 1 - p

<u>or</u>

Alternative 2: - receive p x + (1- p) y for sure

Expected Utility Alternative 1 $\leq$ Expected Utility Alternative 2

$$p\,U(x) + (1 - p)\,U(y) \leq 1 \cdot U(p\,x + (1 - p)\,y)$$

# Risk Aversion

Risk aversion of utility function U is defined as $$a(x) = -U''(x)/U'(x)$$

Based on idea of indifference pricing. Suppose somebody with wealth x is offered a small stochastic payoff Y with zero mean i.e. EY=0. What price will he/she ask to compensate for this payoff Y ? The price p should be such that his/her expected utility does not become worse so

$$EU(x) \leq EU(x+Y+p)$$

but since Y and p are small we then approximate this as

$$\begin{aligned} EU(x) \leq EU(x+Y+p) &\approx E[U(x)+U'(x)(Y+p)+\tfrac{1}{2}U''(x)(Y+p)^2] \\ &= EU(x)+U'(x)(EY+p)+\tfrac{1}{2}U''(x)E(Y+p)^2] \\ &= EU(x)+U'(x)p+\tfrac{1}{2}U''(x)EY^2] \end{aligned}$$

so

$$\begin{aligned} p &\geq -\tfrac{1}{2}U''(x)/U'(x)\cdot Var(Y) \\ &= \tfrac{1}{2}a(x)\cdot Var(Y) \end{aligned}$$

## Pricing Longevity Risk : stylized annuity

Consider annuities

- which pay value of 1 at the end of each year if the policyholder is still alive at that time,
- with first payment exactly one year from now.

Define

| | |
|---|---|
| M | the number of policy holders (indicated by i=1...M), |
| x(i) | age of policyholder i, $\tau$(i) his/her time of death, and |
| d(0,k) | discount rates between year 0 (today) and later years k=1,2,3,..,$\omega$=122. |

Then A(i), the discounted value of payments to policyholder i, equals

$$A(i) = \sum_{k=1}^{\omega} d(0,k)\, 1_{\tilde{T}_i > k}$$

## Pricing Longevity Risk : An annuity

Value of all annuities together (total liability annuity provider) is

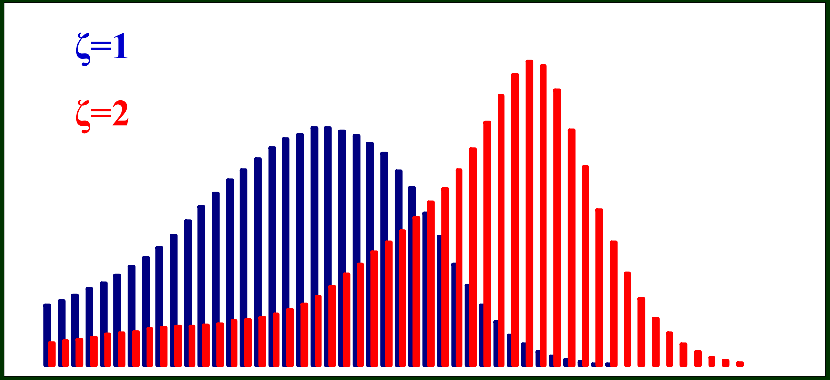
$$A = \sum_{i=1}^{M} A(i) = \sum_{i=1}^{M} \sum_{k=1}^{\omega} d(0,k)\, 1_{\tilde{T}_i > k}$$

Assume that the survival probabilities themselves are stochastic

$$P(\tilde{T}_i > k) = {}_k p_{x(i)}(\zeta)$$

where ς is a stochastic variable from a finite number of scenarios: ς has some probability distribution on {1,2,3,...,Z}

$$P(\zeta = m) = \hat{p}_m$$



## Indifference Pricing Principle for Annuity

If we denote the premium paid by policyholder i by P(i) and define

$$P = \sum_{i=1}^{M} P(i)$$

to be the total premium, equating expected utility with and without the annuity contract thus leads to this equation for annuity seller:

$$E[U(w + P - A)] = E[U(w)]$$

| | | |
|---|---|---|
| with | U | utility function of annuity seller |
| | w | initial wealth of annuity seller |
| | A | total present value of all annuity payoffs |
| | P | total present value of all annuity premia |

# Indifference Price under Exponential Utility

We choose exponential utility function

$$U(x) = -e^{-\gamma x} / \gamma$$

with risk aversion parameter γ. We then find

$$\tfrac{-1}{\gamma} E[e^{-\gamma(w+P-A)}] = \tfrac{-1}{\gamma} E[e^{-\gamma w}]$$

and hence

$$e^{-\gamma w} e^{-\gamma P} E[e^{\gamma A}] = e^{-\gamma w}$$

or

$$\boxed{P = \frac{1}{\gamma} \ln E[e^{\gamma A}]}$$

## Approximate Indifference Price

For $\gamma$ small we can use cumulant approximation

$$
\begin{aligned}
P &= \frac{1}{\gamma}\ln E[e^{\gamma A}] \\
&\approx \frac{1}{\gamma}\left(\gamma E[A] + \tfrac{1}{2}\gamma^2 Var[A]\right) \\
&= E[A] + \tfrac{1}{2}\gamma Var[A]
\end{aligned}
$$

or

$$
\sum_{i=1}^{M} P(i) = E[\sum_{i=1}^{M} A(i)] + \tfrac{1}{2}\gamma Var[\sum_{i=1}^{M} A(i)]
$$

To simplify expression

$$\sum_{i=1}^{M} P(i) = E[\sum_{i=1}^{M} A(i)] + \tfrac{1}{2}\gamma Var[\sum_{i=1}^{M} A(i)]$$

we write

$$\sum_{i=1}^{M} P(i) = E\left[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d(0,k) 1_{\tau(i)\geq k}\right] + \tfrac{1}{2}\gamma Var\left[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d(0,k) 1_{\tau(i)\geq k}\right]$$

$$= E\left[\sum_{k=1}^{\omega} d(0,k) \sum_{i=1}^{M} {}_{k}p_{x(i)}(\varsigma)\right] + \boxed{\tfrac{1}{2}\gamma Var\left[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d(0,k) 1_{\tau(i)\geq k}\right]}$$

## Sources of uncertainty

First term represents expected value of payments, based on (avaraged) deterministic actuarial approach. More interesting is the second term (risk premium):

$$\tfrac{1}{2}\gamma Var\left[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d(0,k)1_{\tau(i)\geq k}\right]$$

We have randomness due to

- Uncertainty about trend in mortality (scenario ς unknown)
- Uncertainty about number of deaths *given* scenario ς

Tool to distinguish these is conditional variance decomposition rule:

$$Var(X) = E\left[Var(X \mid Y)\right] + Var(E[X \mid Y])$$

# Variance decomposition rule

$$\boxed{Var(X) = E\left[Var(X \mid Y)\right] + Var(E[X \mid Y])}$$

Proof:

$$E[Var(X \mid Y)] = E[E(X^2 \mid Y) - E(X \mid Y)^2]$$

$$Var[E(X \mid Y)] = E[E(X \mid Y)^2] - \left[E(E(X \mid Y))\right]^2$$

and summing these gives

$$E[E(X^2 \mid Y) - \left[EX\right]^2] = EX^2 - \left[EX\right]^2 = Var(X)$$

## Decomposition of annuity risk premium

We decompose

$$\tfrac{1}{2}\gamma\, Var\left[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d(0,k) 1_{\tau(i)\geq k}\right] =$$

$$\tfrac{1}{2}\gamma\, E\left( Var\left[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d(0,k) 1_{\tau(i)\geq k} \middle| \varsigma \right]\right) + \tfrac{1}{2}\gamma\, Var\left( E\left[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d(0,k) 1_{\tau(i)\geq k} \middle| \varsigma \right]\right)$$

and assume that given the mortality scenario, policyholders times of death are independent, so

$$\tfrac{1}{2}\gamma\, Var\left[\sum_{i=1}^{M}\sum_{k=1}^{\omega} d(0,k) 1_{\tau(i)\geq k}\right] =$$

$$\tfrac{1}{2}\gamma \sum_{i=1}^{M} E\left( Var\left[\sum_{k=1}^{\omega} d(0,k) 1_{\tau(i)\geq k} \middle| \varsigma \right]\right) + \tfrac{1}{2}\gamma\, Var\left( \sum_{k=1}^{\omega} d(0,k) \sum_{i=1}^{M} {}_k p_{x(i)}(\varsigma) \right)$$

## Decomposition of annuity risk premium

Notice different elements in annuity risk premium:

$$\underbrace{\tfrac{1}{2}\gamma}_{\text{Risk aversion}}\left[\sum_{i=1}^{M} \underbrace{E\left(Var\left[\sum_{k=1}^{\omega} d(0,k)1_{\tau(i)\geq k}\middle|\varsigma\right]\right)} + \underbrace{Var\left(\sum_{k=1}^{\omega} d(0,k)\sum_{i=1}^{M} {}_k p_{x(i)}(\varsigma)\right)}\right]$$

Risk aversion

Average over all scenarios of variance in payoff to policyholder i,

(Charged to policyholder i)

EXTRA variance over all policyholders due to joint dependence on mortality trend

(Charged how?)

## Decomposition of annuity risk premium

Assume we charge per age cohort i.e assume x(i)=x (all same age)

$$\tfrac{1}{2}\gamma\left[\sum_{i=1}^{M} E\left(Var\left[\sum_{k=1}^{\omega} d(0,k)1_{\tau(i)\geq k}\middle|\varsigma\right]\right) + Var\left(\sum_{k=1}^{\omega} d(0,k)\sum_{i=1}^{M} {}_k p_{x(i)}(\varsigma)\right)\right]$$

$$= \tfrac{1}{2}\gamma\left[\sum_{i=1}^{M} E\left(Var\left[\sum_{k=1}^{\omega} d(0,k)1_{\tau(i)\geq k}\middle|\varsigma\right]\right) + M^2\, Var\left(\sum_{k=1}^{\omega} d(0,k)\,{}_k p_x(\varsigma)\right)\right]$$

$$= \sum_{i=1}^{M} \tfrac{1}{2}\gamma\left[\underbrace{E\left(Var\left[\sum_{k=1}^{\omega} d(0,k)1_{\tau(i)\geq k}\middle|\varsigma\right]\right)}_{\text{Average over all scenarios of variance in payoff}} + \underbrace{M\, Var\left(\sum_{k=1}^{\omega} d(0,k)\,{}_k p_x(\varsigma)\right)}_{\text{EXTRA variance term which scales with number of policyholders M (and is zero if there is no uncertainty in trend)}}\right]$$

Charged per Policyholder:

Average over all scenarios of variance in payoff

EXTRA variance term which scales with number of policyholders M (and is zero if there is no uncertainty in trend)

## Decomposition of annuity risk premium

Technicalities less important than financial-economic elements:

- Need to define trade off risk/return when pricing in incomplete markets
- When calculating *future* reserves for payments interest rates may change so then explicit mixture between stochastic mortality (unhedgeable risk) and stochastic interest rates (risk which can be hedged)
- Uncertainty in trend creates extra risk which cannot be diversified away and must be (intra- or intergenerationally) shared.

Interesting mathematically but also very relevant for practical issues w.r.t. intergenerational fairness of pension funds and solvency requirements for insurers !