Finite-Difference Techniques for Financial Derivatives

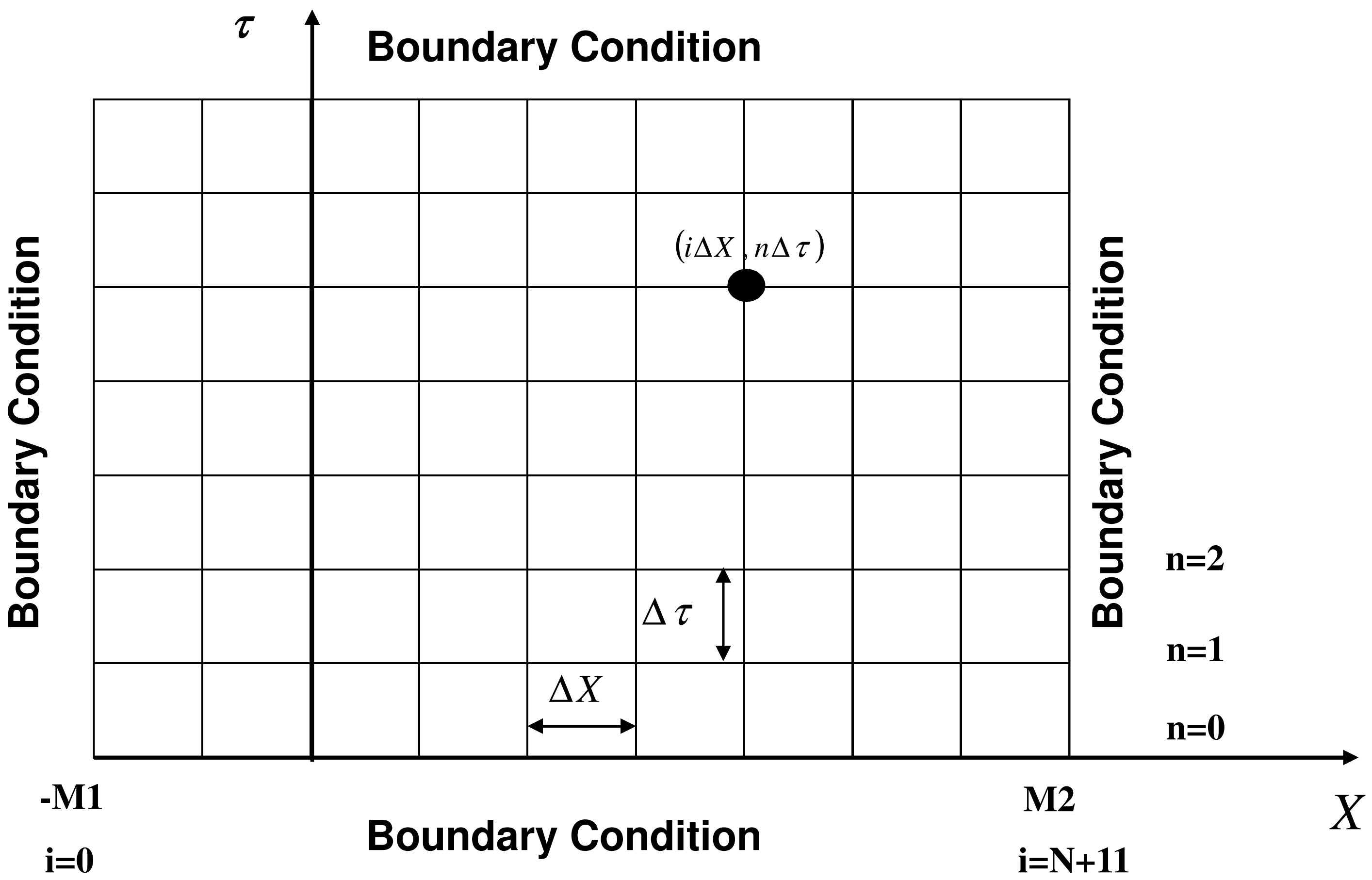
Computational Finance

Scope of lecture

- Solving the discretized equations: direct and iterative methods
- Higher Dimensional Problems
- Some examples (European option, Interest rate option)
- A step forward; Finite Volume Method

Solving the equations by CN-Scheme

Solve the discretized Black-Scholes equation on the following 2D mesh



Solving the equations by CN-Scheme

The system of equations obtained from the discretized PDE can generically be written as:

$$a_1 V_{i+1}^{n+1} + a_0 V_i^{n+1} + a_{-1} V_{i-1}^{n+1} = b_1 V_{i+1}^{n} + b_0 V_i^{n} + b_{-1} V_{i-1}^{n} \qquad \forall i = 1,2,\cdots,N \qquad n = 0,1,\cdots$$

The above representation is suitable to write in a matrix form. The resulting matrix for the 1D Black-Scholes equation is a so-called tri-diagonal matrix

$$\begin{pmatrix} a_0 & a_{-1} & 0 & \cdots & \cdots & 0 \\ a_1 & a_0 & a_{-1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & a_{-1} & a_0 \end{pmatrix} \begin{pmatrix} V_1^{n+1} \\ V_2^{n+1} \\ \vdots \\ \vdots \\ V_N^{n+1} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_N \end{pmatrix}$$

Solving the equations by CN-Scheme

The matrix coefficient are

$$c_1 = b_1 V_2^n + b_0 V_1^n + b_{-1} V_0^n - a_{-1} V_0^{n+1}$$

$$c_N = b_1 V_{N+1}^n + b_0 V_N^n + b_{-1} V_{N-1}^n - a_{-1} V_{N+1}^{n+1}$$

$$c_i = b_1 V_{i+1}^n + b_0 V_i^n + b_{-1} V_{i-1}^n, \qquad \forall i = 2, \cdots, N-1$$

Solving the matrix equation can be done in several ways. A common method is the so-called Thomas Algorithm, which is a direct method.

- **Construct a Upper triangular matrix**
- **Construct a Lower triangular Null matrix**
- **Back substitution or Gauss Elimination**

Example of European Option

The equation that describes a European Put Option is the standard Black-Scholes equation

$$\frac{\partial V}{\partial \tau} = \left(r - \frac{1}{2}\sigma^2 \right)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X^2} - rV$$

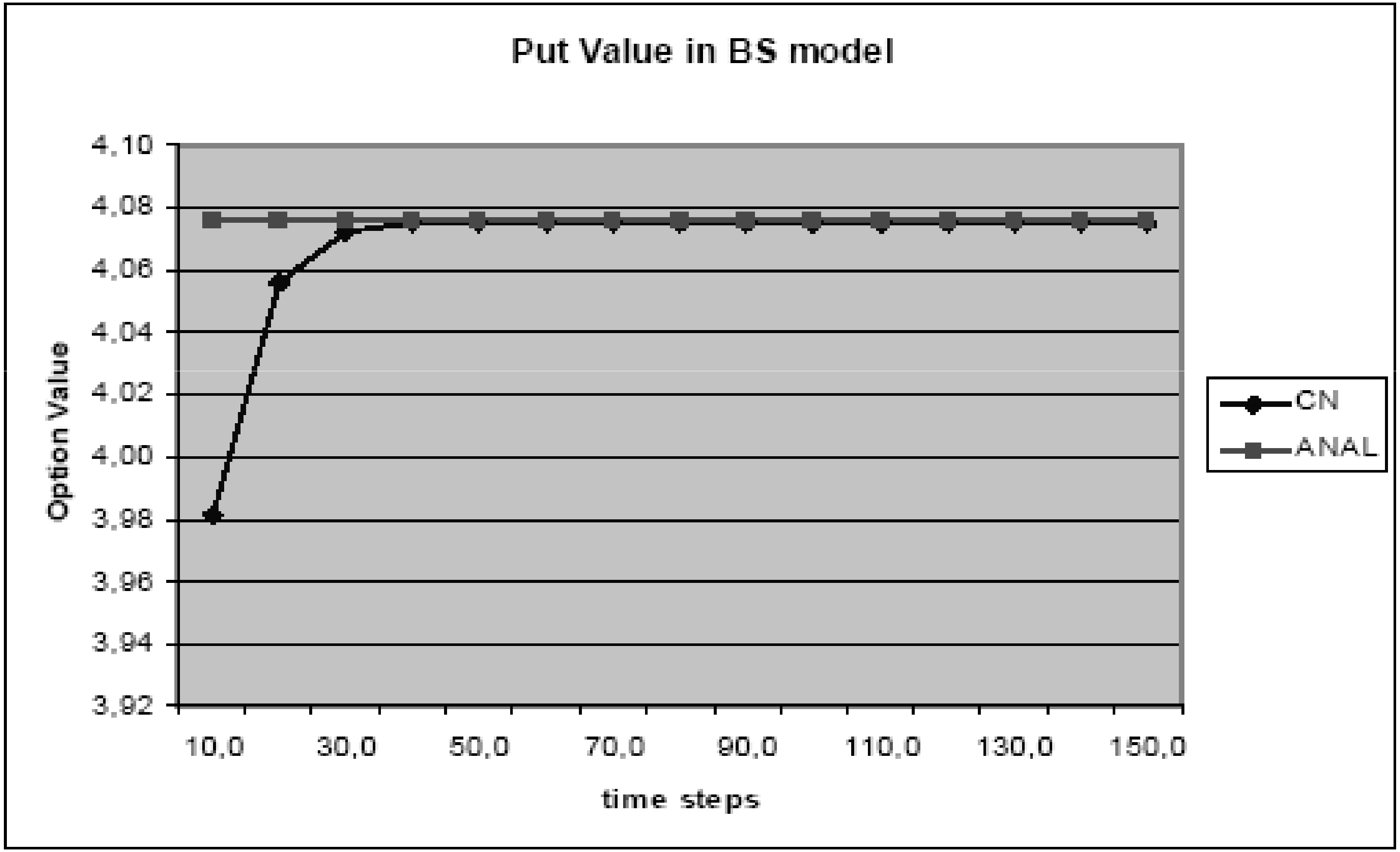
The corresponding boundary condition at expiration time is

$$\max\left(K - S_T, 0\right)$$

Market parameters:

S = 50, K = 50, r = 0.1, σ = 0.4, T = 5/12, Analytical value= 4.076

Example of European Option



Example of Interest Rate Option

Another example is pricing a zero-coupon bond under a stochastic interest rate model

As model for the interest rate, the Hull & White model is taken, i.e.

$$dr = \left(\phi(t) - ar\right)dt + \sigma dW_r$$

Combined with the bond model and using Ito's lemma yield a PDE for the price $P(r,t)$ of a zero-coupon bond

$$\frac{\partial P}{\partial t} + \left(\phi(t) - ar\right)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} = rP$$

Example of Interest Rate Option

For this model problem a closed form solution exist, so numerical comparison is again possible:

$$\phi(t) = \frac{\partial f}{\partial t} + af + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

In this formulation we have no time-dependant coefficients at the grid-nodes.

As boundary condition for a put option on a zero-coupon bond (a caplet) we have the following:

$$\max\left(K - P_T, 0\right)$$

Example of Interest Rate Option

As initial condition we take

$$r\left(t\right)=0.08-0.05e^{-0.18t}$$

Option maturity : 5.5 years

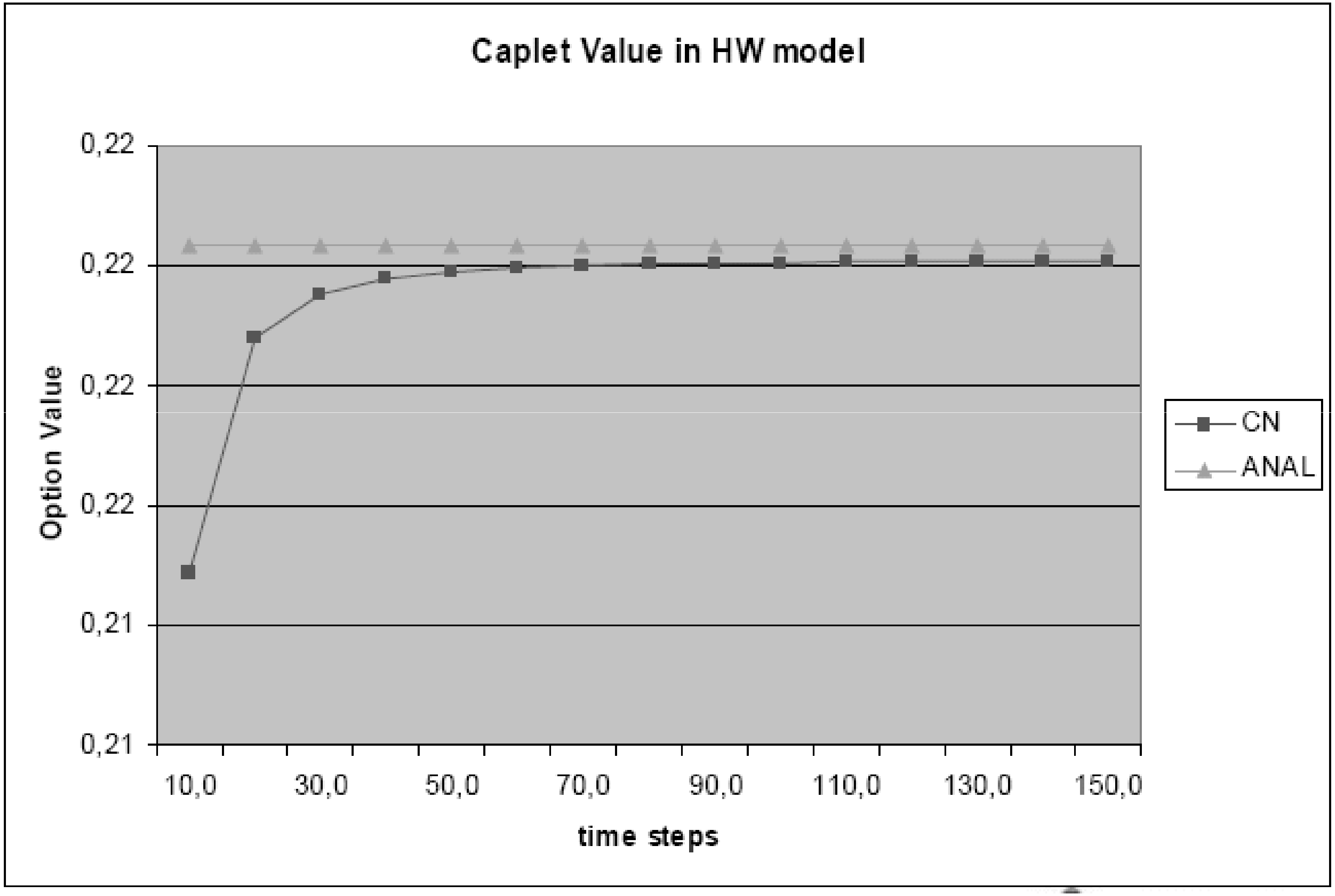
Bond maturity : 6 years

K (ATM Forward Bond) = 0.9605

A = 0.15, Notional = 100, nr. spatial steps = 200

σ **= 0.01 , Analytical value = 0.2160793**

Example of Interest Rate Option



Higher dimensional problems

It should be noted that for higher dimensional systems, the matrix equation becomes more complex. Iterative Techniques like Succesive Over Relaxation (SOR), Conjugate Gradient (CG) or even Multi-Grid (MG) Methods are applied.

Some examples in pricing derivatives contract that may lead to the above, occur especially in Fixed Income area. Also in areas where market parameters are modelled more advanced (interest rate, volatility.

Some examples of the models will follow:

Bond Option

The following PDE describes a bond option described with a stochastic (one-factor) spot rate model

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0, \qquad t < T$$

$$\frac{\partial B}{\partial t} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2} + (u - \lambda w)\frac{\partial B}{\partial r} - rB = 0, \qquad t < T$$

Boundary condition link above equations with eachother

$$B(r, T_B) = F \qquad\qquad V(r, T_V) = \max(\eta(B(r, T_V) - X), 0)$$

Multi-factor interest rate model

Bonds may be priced with a stochastic interest rate model, that can be decomposed into a short-rate r and a long-rate l .

$$\mathrm{dr} = \beta_{\mathrm{r}}(\mathrm{r}, \mathrm{l}, \mathrm{t})\mathrm{dt} + \eta_{\mathrm{r}}(\mathrm{r}, \mathrm{l}, \mathrm{t})\mathrm{dW}_{\mathrm{r}}$$
$$\mathrm{dl} = \beta_{\mathrm{l}}(\mathrm{r}, \mathrm{l}, \mathrm{t})\mathrm{dt} + \eta_{\mathrm{l}}(\mathrm{r}, \mathrm{l}, \mathrm{t})\mathrm{dW}_{\mathrm{l}}$$

Applying Ito's Lemma

$$\frac{\partial B}{\partial t} + \frac{1}{2}\eta_r^2 \frac{\partial^2 B}{\partial r^2} + \rho \eta_r \eta_r \frac{\partial^2 B}{\partial r \partial l} + \frac{1}{2}\eta_l^2 \frac{\partial^2 B}{\partial l^2} +$$
$$(\beta_r - \lambda_r \eta_r)\frac{\partial B}{\partial r} + (\beta_l - \lambda_l \eta_l)\frac{\partial B}{\partial l} = rB$$

In this model mot only the underlying value S is modelled as a GBM but also the volatility ν

$$\mathrm{dS} = \mathrm{S(r - D)dt} + \mathrm{S}\sqrt{\nu}\,\mathrm{dW_1}$$

$$\mathrm{d}\nu = \mathrm{a}(\nu)\,\mathrm{dt} + \mathrm{b}(\nu)\,\mathrm{dW_2}$$

Applying Ito's Lemma

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\nu\frac{\partial^2 V}{\partial S^2} + \rho S\sqrt{\nu}b(\nu)\frac{\partial^2 V}{\partial S\partial\nu} + \frac{1}{2}b(\nu)^2\frac{\partial^2 V}{\partial\nu^2} +$$

$$S(r - D)\frac{\partial V}{\partial S} + a(\nu)\frac{\partial V}{\partial\nu} = rV$$

The resulting matrix becomes a penta-diagonal system with a main diagonal D, two upper diagonal U and two lower diagonal L

$$\begin{pmatrix} D_1 & U_{11} & U_{12} & \cdots & \cdots & 0 \\ L_{11} & D_2 & U_{21} & U_{22} & \cdots & 0 \\ L_{12} & L_{21} & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & U_{n-1,n-1} \\ 0 & \cdots & \cdots & \ddots & L_{n-1,n-1} & D_n \end{pmatrix} \begin{pmatrix} V_1^{n+1} \\ V_2^{n+1} \\ \vdots \\ \vdots \\ V_N^{n+1} \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ \vdots \\ C_N \end{pmatrix}$$

Iterative methods are suitable and efficient to tackle above systems

A step Forward; Finite Volume Method

■ The Model equations can be treated as "conservation laws".

■ The technique discretizes the integral formulation of the "conservation laws" directly in the "physical space", i.e. time, underlying value, market parameters (volatility, interest rate etc.).

■ The method gives considerable flexibility compared to Finite Difference Method: arbitrary mesh (non-equidistance, free shape), varying rules and accuracy.

■ By direct integration basic quantities are conserved. Thus arbitrage free model, will stay numerically arbitrage free.

However, the method cannot be directly applied. The model equation has to be “rewritten” in integral form. An example take a European option with a stochastic volatility model:

$$\frac{\partial V}{\partial t}+\frac{1}{2}S^2\nu\frac{\partial^2 V}{\partial S^2}+\rho S\sqrt{\nu}b(\nu)\frac{\partial^2 V}{\partial S\partial\nu}+$$
$$\frac{1}{2}b(\nu)^2\frac{\partial^2 V}{\partial \nu^2}+S(r-D)\frac{\partial V}{\partial S}+a(\nu)\frac{\partial V}{\partial \nu}=rV$$

The above equation has to be written in the form

$$\frac{\partial V}{\partial t}+\nabla\cdot\left(\bar{A}\nabla V+\vec{b}V\right)=cV$$

A step Forward; Finite Volume Method

The following quantities need to be determined

$$\overline{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad c$$

Then write the equation in integral form

$$\iiint_{\Omega} \frac{\partial V}{\partial t} d\Omega + \oiint_{\partial\Omega} \left(\overline{A} \nabla V + \vec{b} V \right) dS = \iiint_{\Omega} c V \, d\Omega$$

Becomes quite tedious task when dimension become larger than 4, but then again other techniques, like Monte Carlo are in favor