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Homework 2
SPF 2012

1 Exercise 1

We are given a process X with a stochastic differential:

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t) \tag{1}$$

where σ and μ are adapted processes. Furthermore we have $Z(t) = X^{-1}(T)$. By taking $\mu(t) = \alpha X(t)$ and $\sigma(t) = \sigma X(t)$ we obtain $dX(t) = \mu(t)dt + \sigma(t)dW(t)$. Furthermore we take $f(t, X(t)) = X^{-1}(t)$. Then according to Theorem 4.10 and Lemma 4.11 from Björk f (and hence Z) follows a stochastic differential equation given by:

$$df(t, X(t)) = \left[\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right] dt + \sigma\frac{\partial f}{\partial x}dW(t) \tag{2}$$

By plugging dX in the equation we obtain:

$$\begin{aligned} dZ &= \left[\frac{\partial Z}{\partial t} + \mu\frac{\partial Z}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 Z}{\partial x^2}\right] dt + \sigma\frac{\partial Z}{\partial x}dW(t) && (3)\\ &= \left[0 + -X^{-2}\alpha X + \frac{1}{2}2X^{-3}\sigma^2 X^2\right] dt + (-X^2)\sigma X dW && (4)\\ &= \left(\sigma^2 - \frac{\alpha}{2}\right) X^{-1}dt - \sigma X^{-1}dW && (5)\\ &= \left(\sigma^2 - \frac{\alpha}{2}\right) Zdt - \sigma ZdW && (6) \end{aligned}$$

2 Exercise 2

Given $m(t) = E[X(t)]$, with $dX(t) = \alpha X(t)dt + \sigma(t)dW(t)$. If we integrate and take the expected value then we obtain:

$$\begin{aligned} E[X(t)] &= E\left[X_0 + \alpha\int_0^t X(s)ds + \int_0^t \sigma(s)dW(s)\right] && (7)\\ &= X_0 + \alpha E\left[\int_0^t X(s)ds\right] + E\left[\int_0^t \sigma(s)dW(s)\right] && (8) \end{aligned}$$

According to Proposition 4.4 the second integral is zero if:

- $\sigma(s)$ is an adapted process

- $\int_0^t E[\sigma^2]ds < \infty$

If we assume this to hold, then we are only left with the first integral. For the first integral we can take the expectation inside:

$$E[X(t)] = X_0 + \alpha \int_0^t E[X(s)]ds \tag{9}$$

If we take the time derivative of this and use that $E[X(t)] = m(t)$, we obtain:

$$\dot{m}(t) = Z_0 m(t) \tag{10}$$

This can be solved to give:

$$m(t) = Z_0 e^{\alpha t} \tag{11}$$

3 Exercise 3

We are given a stochastic process:

$$Z(t) = \frac{W^2(t)}{t},\ t \geq 1 \tag{12}$$

According to Lemma 4.9 this process is a martingale iff its Ito differential has no t-dependency. Similarly to Exercise 1 we are given a stochastic differential equation $dX(t) = \mu(t)dt + \sigma(t)dW(t)$ where $\mu = 0$ and $\sigma = 1$. Furthermore we have $f(t, X(t)) = Z(X(t)) = \frac{X^2(t)}{dt}$. We use Theorem 4.10 and Proposition 4.11 just as in Exercise 1 to get:

$$\begin{aligned}
dZ &= \left[\frac{\partial Z}{\partial t} + \mu\frac{\partial Z}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 Z}{\partial x^2}\right]dt + \sigma\frac{\partial Z}{\partial x}dW(t) & (13)\\
&= \left[\frac{-W^2(t)}{t^2} + 0 + \frac{1}{2}\frac{2}{t}\right]dt + \frac{2W(t)}{t}dW(t) & (14)\\
&= \left[\frac{-W^2(t)}{t^2} + \frac{1}{t}\right]dt + \frac{2W(t)}{t}dW(t) & (15)
\end{aligned}$$

Now it seems that this system has a systematic drift and hence is not a martingale in the strict sense. Note however, that the expectation of the dt term is 0 since $E[W^2(s)] = t$, which makes this system behave 'almost' like a martingale.