Market-Consistent Valuation and Embedded Value

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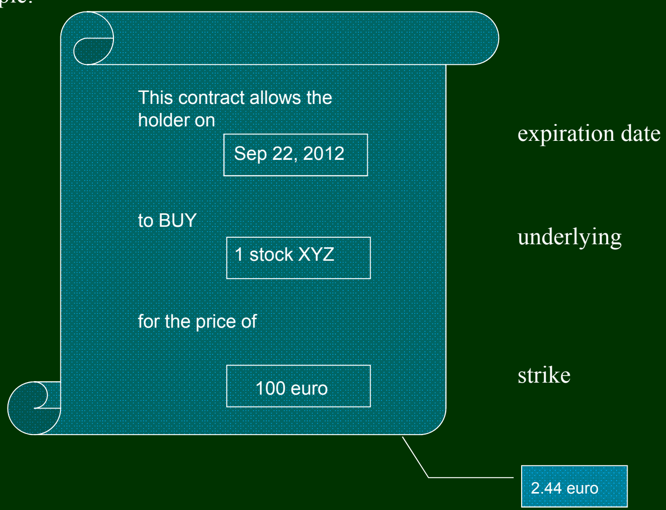
IV
Embedded Interest Rate Options

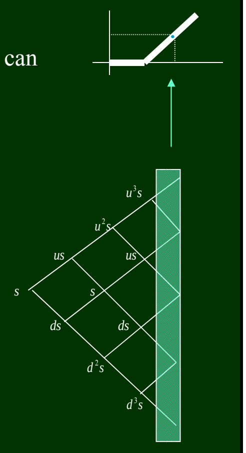
Call Option (Option to Buy)

Example:

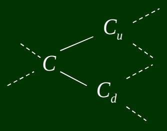


Backwards pricing

We need to know volatility σ of stock, and riskfree rate r, then we can build tree with time steps Δt:



- Set $\quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad q = \dfrac{e^{r\Delta t} - d}{u - d}$
- Values at final time (maturity) are known: fill these in
- Work backwards through tree using



$$C = e^{-r\Delta t}\left[qC_u + (1-q)C_d\right]$$

- Continue, until you reach starting node: this is option price !

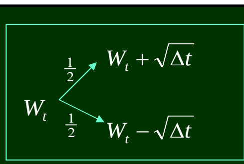
Three ways to calculate prices in Binomial Branch

REPLICATE	PRICE RISKNEUTRALLY	DEFLATE
$\phi\begin{pmatrix} u\cdot S \\ d\cdot S\end{pmatrix} + \psi\begin{pmatrix} e^{r\Delta t} \\ e^{r\Delta t}\end{pmatrix}$	$e^{-r\Delta t}$ q, $1-q$	p, $1-p$
$\begin{pmatrix} C_u \\ C_d\end{pmatrix}$	$\begin{pmatrix} C_u \\ C_d\end{pmatrix}$	$\begin{pmatrix} C_u H_u \\ C_d H_d\end{pmatrix}$
$C = \phi S + \psi 1$	$C = e^{-r\Delta t}[qC_u + (1-q)C_d]$	$C = pC_uH_u + (1-p)C_dH_d$

Riskneutral pricing / deflation method are a shortcut for replication-based pricing.

They are completely unfounded when replication is not possible (longevity risk, unhedgeable inflation risk).

Increasingly More Accurate Replication



Per timestep

$$S_u = S_t e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(W_u - W_t)}$$

Where the Brownian motion process W (the 'random walk') satisfies:

- Starting value W(0) = 0
- W(t)-W(s) has N(0,t-s) distribution,
  so in particular: W(t) is Gaussian with mean zero and variance t
- W(t)-W(s) and W(v)-W(u) are independent when s < t <u < v

$$S_u = S_t e^{r(u-t)} \cdot \frac{Z_u}{Z_t}, \qquad \boxed{Z_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}}$$

Z is Geometric Brownian Motion process: martingale (so mean constant: one).

Equity derivative: Unit-Linked Insurance Product

Revisit our earlier Unit-Linked Guarantee: assuming start price of stock and fund is 100 then we will receive

$$\max(S_T, S_0(1+g)^T) = \max(S_T, 108.24)$$

which we can rewrite as

$$\max(S_T, 108.24) = S_T + \max(0, 108.24 - S_T)$$

i.e. stock + Put with strike 108.24 or as

$$\max(S_T, 108.24) = 108.24 + \max(0, S_T - 108.24)$$

i.e. 108.24 cash at time 2 + Call with strike 108.24

Equity derivative:
Unit-Linked Insurance Product

Stock + Put with strike K=108.24 : has value at time zero

$$S_0 + P(S_0) = 100 + 7.43 = 107.43$$

108.24 cash at time 2 + Call witjh strike K=108.24 : has value at time zero

$$Ke^{-rT} + C(S_0) = 98.07 + 9.37 = 107.43$$

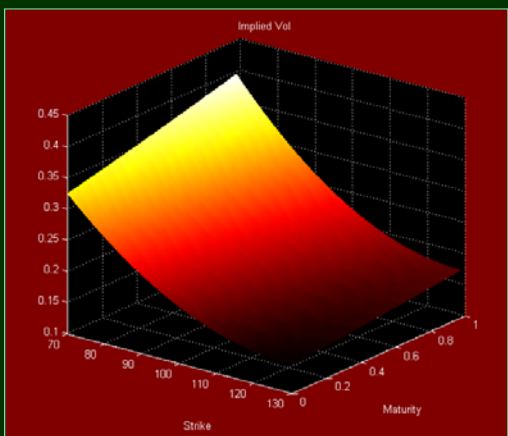
Binomial Tree gave 107.26 as good approximation, but this is of course a lot quicker.....!

IMPORTANT : Can only use Black-Scholes formula for European Options !

Equity derivatives in practice: Volatility varies over strikes !

Lognormal model for stock prices was result of repeated multiplication in a model in which there is one fixed volatility (percentage standard deviation per unit of time).

Option prices on the same stock but with different strikes are not consistent with the assumption of one fixed volatility per stock. Instead the Black-Scholes implied volatiliy is a function of the strike.



Theoretcial Problem:

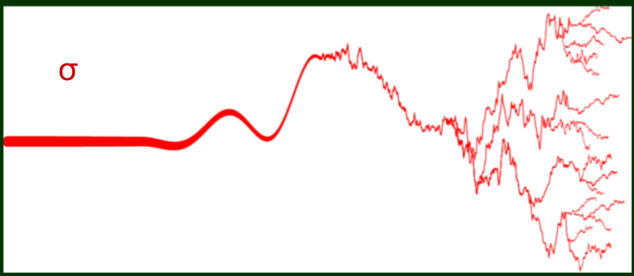
If volatility is a property of the stock process, how can it vary with the strike of options ?

Volatilty Skew or Smile

Solution:

We have to drop assumption of lognormal distribution, which corresponds to same standard deviation per time unit at all times and for all stock values.

Volatility is not constant but varies over time and may vary stochastically (due to dependence on stock values itself, or due to other stochastic factors)



More complicated contracts

Closed-form formulas usually not possible or needed. Writing prices as expectations (with q-probabilities or H-deflators) opens possibility of Monte Carlo simulation based pricing.

Simulate from t=0 to t=T=NΔt (with H_0=1)

$$\Delta W_t = \sqrt{\Delta t} \cdot \varepsilon_t$$
$$\varepsilon_t \text{ iid } N(0,1)$$

$$S_{t+\Delta t} = S_t(1 + \mu\Delta t + \sigma\Delta W_t)$$
$$B_{t+\Delta t} = B_t(1 + r\Delta t)$$
$$H_{t+\Delta t} = H_t(1 - r\Delta t - \lambda\Delta W_t)$$

Repeat M times and call simulated end values at t=T

$$(S_T^i, B_T^i, H_T^i), \quad i = 1 \ldots M$$

then price of claim Φ equals approximately

$$\boxed{C = E^P[\Phi(S_T)H_T] \approx \frac{1}{M}\sum_{i=1}^{M}\Phi(S_T^i)H_T^i}$$

Alternative

One can also directly start using riskneutral framework, then μ and H not needed.

Simulate from t=0 to t=T=NΔt

$$
\begin{aligned}
S_{t+\Delta t} &= S_t(1+\overset{r}{\cancel{\mu}}\Delta t + \sigma \Delta W_t^{Q}) \\
B_{t+\Delta t} &= B_t(1+r\Delta t) \\
&\cancel{H_{t+\Delta t} = H_t(1-r\Delta t - \lambda \Delta W_t)}
\end{aligned}
$$

Repeat M times and call simulated end values at t=T

$$(S_T^i, B_T^i, \cancel{H_T^i}), \quad i = 1 \ldots M$$

then price of claim Φ equals approximately

$$\boxed{C = E^Q[\Phi(S_T)\frac{B_0}{B_T}] \approx \frac{1}{M}\sum_{i=1}^{M}\Phi(S_T^i)\frac{B_0}{B_T^i}}$$

Alternative

For this simple model we do not need to simulate step-by-step but can calculate

$$\boxed{C \approx \frac{1}{M}\sum_{i=1}^{M}\Phi(S_T^i)H_T^i}$$

Immediately using

$$
\begin{aligned}
S_T &= S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} \\
B_T &= B_0 e^{rT} \\
H_T &= H_0 e^{(-r - \frac{1}{2}\lambda^2)T - \lambda W_T}
\end{aligned}
$$

But simulation scheme and payoffs can be very general, for example

$$
\begin{aligned}
S_{t+\Delta t} &= S_t(1+\mu(t,S_t)\Delta t + \sigma(t,S_t)\Delta W_t) \\
B_{t+\Delta t} &= B_t(1+r(t)\Delta t) \\
H_{t+\Delta t} &= H_t(1-r(t)\Delta t - \lambda(t,S_t)\Delta W_t)
\end{aligned}
\qquad \Phi = \max(0, \frac{1}{n}\sum_{i=1}^{n}\frac{S_T}{S_{t_i}} - K)
$$

Complete and Incomplete markets

We see that riskneutral pricing is a lot easier since the expected rate of return never enters our calculations.

However, these calculations are only meaningful if we know that contracts can be replicated.

If not, one should use original probabilities and the hedge will not be perfect so there is residual risk.

How that residual risk is priced cannot be inferred from market prices so other approach is needed. (Example: Cost-of-Capital, or Utility Indifference).

Review of Interest Rate Modelling

zero rate	z(k)	yearly rate between now (time zero) and time k
forward rate	f(i,j)	yearly rate between <u>future</u> times i and j
short rate	r(i)	yearly rate between time i and next time (i+1)
discount rate	d(0,k)	today's price of k-maturity zero-coupon bond

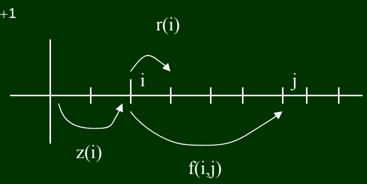
so

$$d(0,k) = \frac{1}{(1+z_k)^k} \qquad r_i = f_{i,i+1}$$

$$(1+f_{i,j})^{j-i} = (1+r_i)(1+r_{i+1})\dots(1+r_{j-1})$$

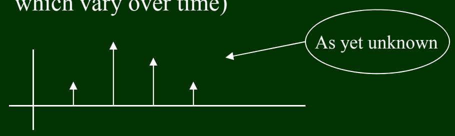
$$(1+z_k)^k = (1+r_0)(1+r_1)\dots(1+r_{k-1})$$

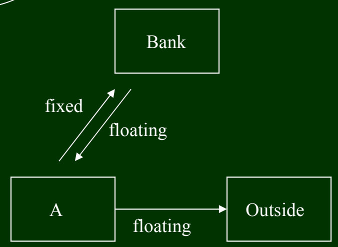
$$d(0,i) = (1+r_i)d(0,i+1)$$



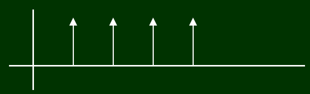
Vanilla Interest Rate Swap

You (A) face interest payments which are floating (i.e. dependent on market rates which vary over time)





you would like known rates, fixed in advance

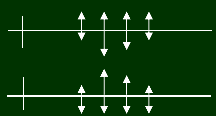


Swap Party B(ank) swaps payments with party A

- B pays A floating rates (after period in which they occur !)
- A pays B fixed rate r ('swap rate', determined by B)

Interest rate swaps : FORWARD Par Swap Rate

If we start swap at time m<n (i.e. payments at m+1, m+2, …, n):



$$P_{rec} = P_{fix} - P_{float} = s\sum_{k=m+1}^{n} d_k - d_m + d_n$$

$$P_{pay} = -P_{fix} + P_{float} = P_{rec}$$

Fair forward swap rate ('the' swap rate) $s_{m,n}$ between m and n is the value such that

$$P_{rec} = P_{pay} = 0$$

and it follows that this swap rate for n periods, s(m,n), is given by

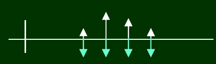
$$s_{m,n} = \frac{d_m - d_n}{d_{m+1} + d_{m+2} + \ldots + d_n}$$

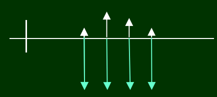
Interest rate swaps: value after initial time

Can use Par Swap Rate to determine market value of swap contract: fixed rate s
Current Par Swap Rate = s(m,n)

Only difference in fixed payments, hence:

$$
\begin{aligned}
P_{pay} &= P_{pay} - P_{pay}^{PAR} \\
&= P_{flo} - P_{fix} - (P_{flo}^{PAR} - P_{fix}^{PAR}) \\
&= P_{fix}^{PAR} - P_{fix} \\
&= \left(s_{m,n} - s\right)\left(\sum_{k=m+1}^{n} d_k\right)
\end{aligned}
$$





and of course

$$P_{rec} = -P_{pay}$$

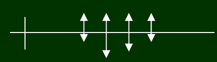
Swap Dynamics

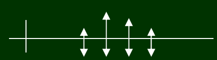
Valuation after initial time makes us realize that swaprate itself changes over time since discount rates change over time.

Therefore better to explicitly address this in our notation for forward fair swaprate

$$s_{m,n}(t) = \frac{d_m(t) - d_n(t)}{d_{m+1}(t) + d_{m+2}(t) + \ldots + d_n(t)}$$

and value of swaps at different rates





$$
\begin{aligned}
&P_{pay}(t) = \left(s_{m,n}(t) - s\right)\left(\sum_{k=m+1}^{n} d_k(t)\right) \\
&P_{rec}(t) = -P_{pay}(t)
\end{aligned}
$$

Valuation Structures

Forwards, futures, FRA's and swaps in interest rate market

- take away risk by fixing future interest rates or future prices of interest rate instruments
- and this is usually done in such a way that no cash exchange is needed when contract is made (zero initial costs)
- costless reduction of risk is possible since entering the contract protects against adverse market movements by giving up possible profits due to favourable market movements

Valuation of instruments ('fair price') is possible using today's term structure, since all cash flows in future and corresponding discount factors are known today.

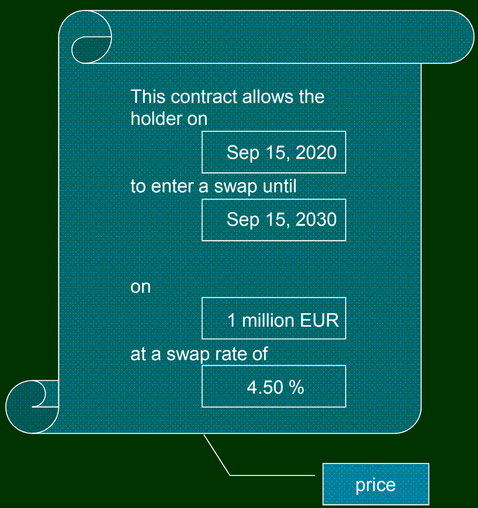
Swaption

This is no longer true for an option to swap later (swap option = swaption).

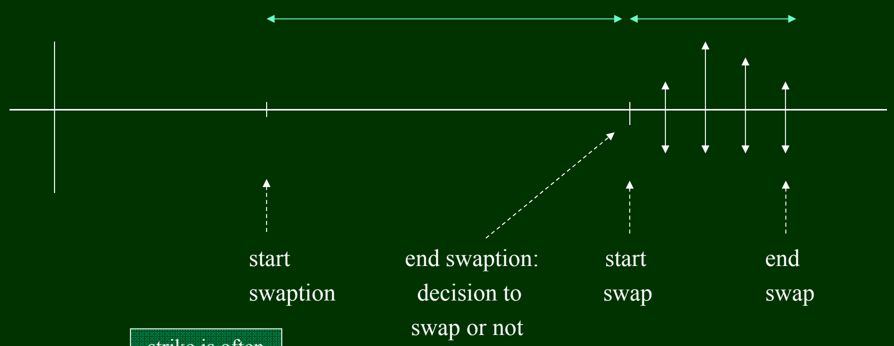
Payer/Receiver Swaption: right (not the obligation) to pay/receive fixed rate K on swap with start date m and maturity n.

Time line: $t < m < n$.

Buy today (time t) the right (not obligation) to enter a forward starting swap $[m\ ,n]$ with fixed rate K



Lifetime of swaption and underlying swap



strike is often fair rate at the time

Use swap if swap cashflows have positive value:

Payer exercises if strike is lower than swaprate at time

Receiver exercises if strike is higher than swaprate at time

Swaptions

- Exercise payer swaption at m if fixed rate you need to pay (K) is lower than the swaprate at that time s(m,n) so when s(m,n)>K

  Market value at swaption maturity date m (per unit notional):

$$P_{payswptn,m} = \max\{s_{m,n} - K, 0\}\left(\sum_{k=m+1}^{n} d_k(m)\right)$$

- Exercise receiver swaption if fixed rate that you will receive (K) is higher than the swaprate at that time s(m,n) so when s(m,n)<K

  Market value at swaption maturity date m (per unit notional):

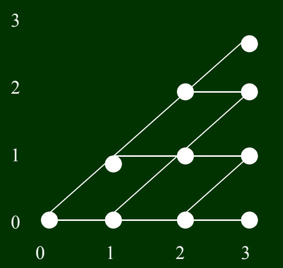
$$P_{recswptn,m} = \max\{K - s_{m,n}, 0\}\left(\sum_{k=m+1}^{n} d_k(m)\right)$$

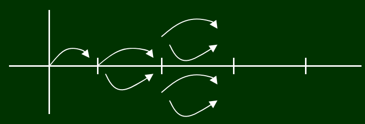
This gives value at maturity, but how can we determine price before that time?

Pricing Interest Rate Derivatives

- We need to introduce stochasticity in interest rates
- but this cannot be done in arbitrary manner: must be arbitrage-free

Idea Binomial Tree for short rates

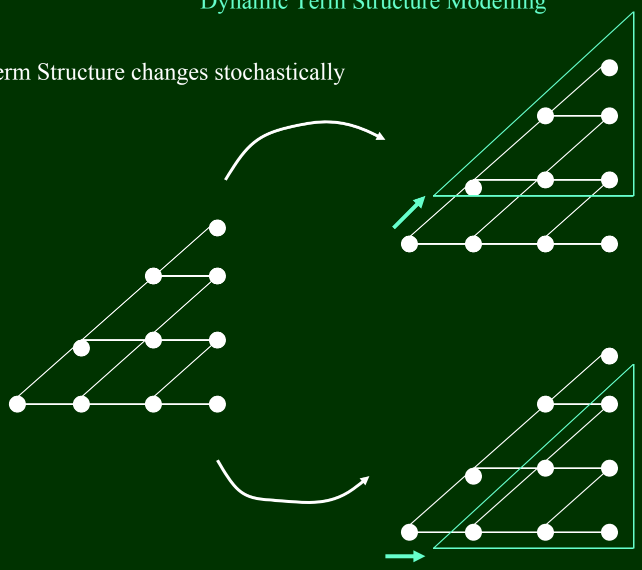




- Risk-neutral probabilities are set to ½ (so not calculated using replication)
- short rates r(t,i) on nodes must be chosen according to some model (how much volatility i.e. stochasticity).

Dynamic Term Structure Modelling

Term Structure changes stochastically



Issues in Short Rate Lattice Modelling

Main Questions

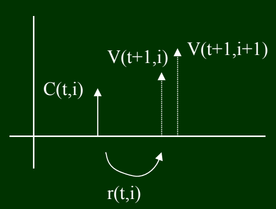
- How can we price an interest rate derivative on the tree ?
- Is such a pricing method guaranteed arbitrage-free ?
- How can we find the current Term Structure on this tree ?
- How can we fit the current Term Structure in the design of the tree ?

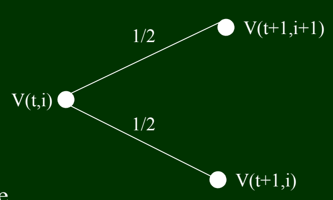
Pricing on the Short Rate Lattice

Call value of the security at node (t,i) : V(t,i)
Suppose security pays C(t,i) cashflow at node (t,i)

Risk-Neutral Valuation

$$\boxed{V_{t,i} = \frac{\frac{1}{2}\left(V_{t+1,i} + V_{t+1,i+1}\right)}{1 + r_{t,i}} + C_{t,i}}$$





As before:
Find end values, then
work backwards through tree.

Example: Short Rate Tree

Excel Calculation: SmallShortTree.xls

Other issues now resolved automatically:

Arbitrage-free ?

Yes: future payoffs positive (i.e. i get something in future) then price today strictly positive (i.e. do not get it for free !)

Can we find Term Structure ?

Yes: since we can price zero-coupon bonds with all maturities k, and then we have

$$P_k = (1+s_k)^{-k} \quad \Rightarrow \quad s_k = (P_k)^{-1/k} - 1$$

Can we fit Term Structure ?

Yes: using Solver, but then we need simpler way to calculate all zero-coupon bond prices

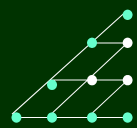
Elementary Prices

Revival of old trick:

Elementary Price Po(t,i) price of contract which pays 1 at node (t,i) and zero elsewhere

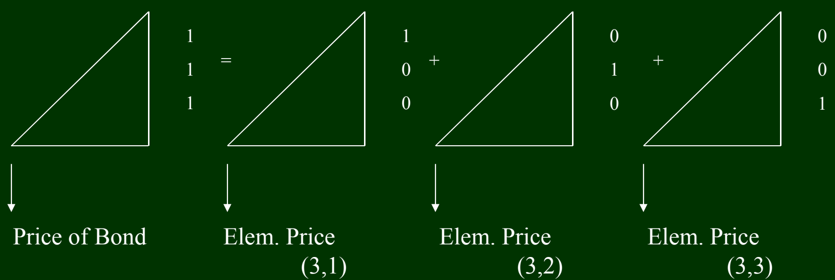
Po(k+1,s) will have value $\frac{\frac{1}{2}}{1+r_{k,s-1}}$ and $\frac{\frac{1}{2}}{1+r_{k,s}}$ at predecessor values

$$\boxed{P_0(k+1,s) = \frac{\frac{1}{2}P_0(k,s-1)}{1+r_{k,s-1}} + \frac{\frac{1}{2}P_0(k,s)}{1+r_{k,s}}}$$



$$P_0(0,0) = 1, \quad P_0(k+1,0) = \frac{\frac{1}{2}P_0(k,0)}{1+r_{k,0}}, \quad P_0(k+1,k+1) = \frac{\frac{1}{2}P_0(k,k)}{1+r_{k,k}}$$

Zero Coupon Bond Prices from Elementary Prices



so finding Term Structure

- First compute elementary prices
- Summing elementary prices in columns gives zero-coupon bond prices
- Spotrates then easy to find using $s_k = (P_k)^{-1/k} - 1$

Example: Elementary Prices

Excel Calculation: ElementaryPrices.xls

Matching the Term Structure

- Choose model to assign short rates r(k,s) where model may depend on unknown parameters

  Examples:

  Ho-Lee Model

  $$r_{k,s} = a_k + b_k s$$

  $$\begin{array}{ccc} & & a_2 + 2b_2 \\ & a_1 + b_1 & a_2 + b_2 \\ a_0 & a_1 & a_2 \end{array}$$

  Black-Derman-Toy Model

  $$r_{k,s} = a_k e^{b_k s}$$

  $$\begin{array}{ccc} & & a_2\left(e^{b_2}\right)^2 \\ & a_1 e^{b_1} & a_2 e^{b_2} \\ a_0 & a_1 & a_2 \end{array}$$

- Then calculate short rates (Term Structure) using elementary prices
- Use Solver to find correct values for a(k) and b(k) to match short rates to the values that you want.

Ho-Lee Model

Excel Calculation: HoLeeTree.xls

So we can now fit a Term Structure and we know how to price bonds.

Can we now price Interest rate Derivatives ?

Pricing

Pricing Bond

- Fill in payoff tree with coupons & final values at all columns
- Make second tree, fill in final payoffs and use backward formula (or directly apply elementary prices)

Pricing European option on Bonds

- Fill in payoff tree at column which represents maturity of option
  (for example: max{0,Bondprice-Strike} or something similar)
- Use backward formula (or directly apply elementary prices)

Excel Calculation: BondOption.xls

Pricing

Pricing Bond Forward

- Fill in payoff tree at maturity of forward M = value bond at maturity M
- Use backward formula (or directly apply elementary prices) to find value of contract at time zero
- You will not pay at time zero (forward contract costs zero !) but at time M, so discount forward to time M (multiply by $(1+s_M)^M$)

Pricing Bond Future

- Fill in final values of bond at maturity of option
- Working backwards: future price at node F then expected value in coming period is $\frac{1}{2}(F_{up}-F)+\frac{1}{2}(F_{down}-F)$ discounted, and this should be zero so we must have $F=\frac{1}{2}(F_{up}+F_{down})$
- So work backwards through tree, taking averages without discounting

Future and Forward Price Calculations

Future and Forward price can be shown to be equal when interest rates are non-stochastic, and Term Structure dynamics followed expectations hypothesis.

This is no longer true when rates are stochastic.
Future involves cashflows before maturity : interest rate levels influence a future contract, but not a forward contract !

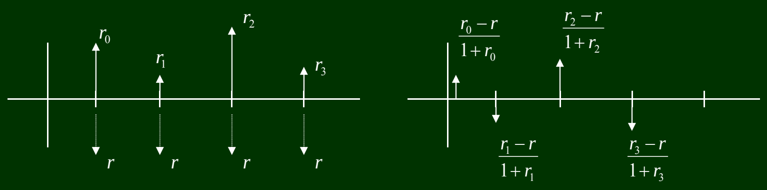
Excel Calculation: BondForward.xls

Excel Calculation: BondFuture.xls

Vanilla Swap Pricing

Pricing a Vanilla Swap

- Swap pays short rate after period where short rate applied, so need transformation to one period earlier



- Put these payoff values in tree at every node
- Create second tree for valuation, using backwards formula (or elementary prices).

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Swaption Pricing

Pricing a Swaption

- Take the swaptree we just derived, with the value of the swap at all nodes
- Create a second tree with payoff of swaption at maturity, and then work backwards (or use elementary prices)

Excel Calculation: SmallSwapTree.xls

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Stochastic Dynamics for Swap Rates

Can we (as for equity) take a limit and arrive at a lognormal distribution?
Possible for short rate, but this does NOT make swaprates lognormal.

Often in practice: simplest continuous time model for whatever rate:

Black's Model (1976)

Fundamental Assumptions: (today = time t)

- European Call Option on underlying V (Bond price, future, swap rate)
- Underlying has lognormal distribution at maturity T, with a standard deviation of ln V equal to $\sigma\sqrt{T-t}$
- Discount rate for option's matuirty are taken non-stochastic (even though interest rates are …)

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Black Formula

Call Option price according to Black model

$$C = d_T(t)\left[F\ N(\tilde{d}_1) - K\ N(\tilde{d}_2)\right]$$

$$\tilde{d}_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \qquad \tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T-t}$$

T	Time to maturity of option
F	Forward price for V of a contract with maturity T
K	Strike price of option
d	Riskfree discount rates
σ	volatility (see previous slide)

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