Stochastic Finance

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■ Content

REFERENCE

■ Chapter I A Brief Introduction to Finance

■ I.1 Finance, the Finance System and Financial Assets

Finance is defined as the practice of manipulating and managing money. It is concerned with the process, institutions, markets, and instruments involved in the transfer of money among and between individuals, businesses, and governments. Financial decisions are with two distinctive features: spreading over time, and with uncertainty.

To implement their financial decisions, people make use of *the financial system*, which is defined as the set of markets and other institutions used for financial contracting and the exchange of assets and risks. The financial system includes the markets for stocks, bonds, and other financial instruments; financial intermediaries (such as banks and insurance companies); financial-service firms (such as financial advisory firms); and the regulatory bodies that govern all of these institutions.

The basic types of *financial assets* are bond, stock, etc and derivatives - forward, futures and option, etc. They are categorized into underlying assets and derivative assets.

■ *Underlying Assets*

The underlying assets can be stocks, bonds, currency, commodities, and other financial assets, or combinations of these. The traditional stock and bond markets raise necessary capital for corporations and governments, and the foreign exchange market facilitates international trade and investment.

▪ *Stocks*

Stocks represent the claim of the owners of a firm. Stocks are issued by corporations and can be traded in the stock market. *Common stock* usually entitles the shareholder to vote in the election of directors and other matters. *Preferred stock* generally does not confer voting rights but it has a prior claim on assets and earnings: dividends must be paid on preferred stock before any can be paid on common stock.

▪ *Bonds*

Bonds are issued by anyone who borrows money - firms, governments, etc. They are fixed-income instruments because they promise to pay fixed sums of cash in the future. Bondholders have an *IOU* (I owe you) from the issuer, but no corporate ownership privileges, as stockholders do.

■ *Derivative Assets*

Derivatives are financial instruments that derive their value from the prices of one or more other assets such as stocks, bonds, foreign currencies, or commodities. Derivatives serve as tools for managing risks associated with these underlying assets. The most common types of derivatives are options and futures.

▪ *Forwards and Futures*

A forward is a financial contract in which two parties agree to buy and sell a certain amount of the underlying commodity or financial asset at a prespecified price at a specified time in the future. The specified time is called the *time-to-maturity* of the forward contract and the price specified in the contract is called the forward price.

A futures contract is a standardized forward contract traded in an exchange. To avoid the shortcomings of the forwards that each party cannot change his/her mind to reverse the position specified in the contract, in a futures contract, both the time to maturity and the amount of the underlying asset to be delivered in each contract are standardized so that the futures contract can be traded in a market place. Thus, if one party wants to change position, he/she can buy or sell in the futures market.

▪ *Options*

An option is a financial contract, which provides the holder with the right to buy or sell a certain amount of the underlying asset at a prespecified price at or before a specified time in the future. Similar to the forward and futures contracts, the time specified in an option contract is called the time-to-maturity of the option. The price specified in the contract is called the *exercise price* or the *strike price* of the option.

Unlike a forward or futures contract, an option contract gives its holder the *right*, but not the *obligation* to buy or sell the underlying asset. The two most common types of options are *calls* and *puts*. A call option is an option to purchase the underlying asset, while a put option is an option to sell.

Everyone would like to hold options because they provide a positive likelihood for the holder to make a profit. But he/she has to pay the price. The cost for this likelihood is the money paid by the buyer of option to the seller to compensate the latter's possible losses. This money is called the *premium* of the option, or simply the *option price*.

The following example may help you to understand some of the concepts of derivatives:

You decide to buy a new car and you go to the dealer's showroom. You select a certain type of car, and the dealer tells you that if you place the order today and place a deposit, then you can take delivery of the car in 3 months time. If in 3 months time the price of the model has decreased or increased, it doesn't matter. When the agreement between you and the dealer is reached, you have entered into a **forward** contract: you have the **right** and **also the obligation** to buy the car in 3 months.

Instead, suppose the car you selected is on offer at $ 30,000 but you must buy it today. You don't have that amount of cash today and it will take a week to organize a loan. You could offer the dealer a deposit, for example $200, if he will just keep the car for a week and hold the price. During the week, you might discover a second dealer offering an identical model for a lower price, then you don't take up your option with the first dealer. At the end of the week the $ 200 is the dealer's whether you buy the car, or not. In this case, you have entered an **option** contract, a **call** option here. It means that you have the **right** to buy the car in a week, but **not the obligation**. The **expiration time** is one week from now, the **strike price** is $30,000.

In this example, for both the forwards and option contracts described, delivery of the car is for a future date and the prices of the deposit and option are based on the **underlying asset** - the car.

■ I.2 Important Financial Concepts

I.2.1. Time Value of Money

Time value of money refers to the fact that money in hand today is worth more than the expectation of the same amount to be received in the future. Money has a time value because of the opportunity to earn interest or the cost of paying interest on borrowed capital.

We begin our study of the time value of money with the concept of *compounding* - the process of going from today's value, or *present value* (PV), to *future value* (FV). Future value is the amount of money an investment will grow to at some date in the future by earning interest at some compound rate. If r is the interest rate and n is the number of years, the future value of a present value is given by:

$$FV = PV(1 + r)^n$$

Calculating the present values of given future amounts is called *discounting*, the reverse of compounding. Because the interest is paid only at a finite number of discrete time intervals, we refer to it as discrete compounding. Consider an amount A invested for n years. If the rate is compounded once per annum, the terminal value of the investments is

$$A(1 + r)^n$$

If it is compounded m times per annum, the terminal value of the investment is

$$A(1 + \frac{r}{m})^{mn}$$

The limit as m tends to infinity is known as continuous compounding and it can be shown that

$$lim_{m \to \infty} A(1 + \frac{r}{m})^{mn} = Ae^{rn}$$

So, compounding a sum of money at a continuously compounded rate r for n years involves multiplying it by e^{rn}. Discounting involves multiplying by e^{-rn}. For finite m the continous compounding rate r_c and the discrete compouding rate r are different.

From

$$A(1 + \frac{r}{m})^{mn} = Ae^{r_c n},$$

it follows that

$$r_c = m \ln(1 + \frac{r}{m}).$$

The time-value-of -money concept provides a tool for making investment decisions. The most common decision rule is the *net- present- value* (*NPV*) rule, which is a way of comparing the value of money now with the value of money in the future. The *NPV* is the difference between the present value of all future cash inflows minus the present value of all investments. The rule says that accept any project if its *NPV* is positive; reject a project if its *NPV* is negative.

I.2.2. Risk and Return

In the most basic sense, risk can be defined as the chance of financial loss or, more formally, the *variability* of returns associated with a given asset. Most financial decision-makers are risk-averse because they require higher expected returns as compensation for taking greater risk.

Risk can be assessed from a behavioral point of view using sensitivity analysis and probability distributions. Sensitivity analysis uses the range of estimated return to obtain a sense of the variability among outcomes. The greater the range for a given asset, the more variability, or risk, it is said to have. Although the use of sensitivity analysis is rather crude, it does provide decision-makers with a feel for the behavior of returns. This behavioral insight can be used to assess roughly the risk involved.

In addition to the range, the risk of an asset can be measured quantitatively using statistics.

Probability distributions of outcome provide a more quantitative, yet behavioral, insight into an asset's risk. Here we consider two statistics: the standard deviation σ and the coefficient of variation *CV*, which measures the dispersion around the expected value of a return $\overline{k}$.

$$\overline{k} = \sum_{i=1}^{n} k_i \times p_i$$

where k_i = *return for the i-th outcome*

p_i = *probability of occurrence of the i-th outcome*

n = *number of outcomes considered*

$$\sigma = \sqrt{\sum_{i=1}^{n} (k_i - \overline{k})^2 \times p_i}$$

$$CV = \sigma_k / \overline{k}$$

The two statistics can be used to measure the risk (i.e., variability) of asset returns. In general, the higher the standard deviation and the coefficient of variation, the greater the risk.

The return on a portfolio is calculated as a weighted average of the returns on the individual assets from which it is formed. Letting w_j equals the proportion of the portfolio's total dollar value represented by asset *j,* and k_j equals the return on asset *j*, the portfolio return, k_p is

$$k_p = (w_1 \times k_1) + (w_2 \times k_2) + \cdots + (w_n \times k_n) = \sum_{j=1}^{n} w_j \times k_j$$

I.2.3. Valuation

Asset valuation is the process of estimating how much an asset is worth, and is at the heart of much of financial decision-making.

The book value of an asset or a liability as reported in a firm's financial statements often differs from its current market value. In making most financial decisions, it is a good idea to start by assuming that for assets that are bought and sold in competitive markets, *price* is a pretty accurate reflection of fundamental value. This assumption is generally warranted precisely because there are many well-informed professionals looking for mispriced assets who profit by eliminating discrepancies between the market prices and the fundamental values of assets. This process is called **arbitrage**, the purchase and immediate sale of equivalent assets in order to earn a sure profit from a difference in their prices.

Valuation process links risk and return to determine the worth of an asset. The key inputs to the valuation process include *cash flows* (returns), *timing*, and the *required return* (risk compensation).

Because risk describes the chance that an expected outcome will not be realized, the level of risk associated with a given cash flow can significantly affect its value. In general, the greater the risk of a cash flow, the lower its value. Turning to the opposite point of view, to compensate the risk, the holders of the assets require greater return rate, or discount rate in terms of present value calculation.

Here we show the *basic valuation model*. Simply stated, the value of any asset is the present value of all future cash flows it is expected to provide over the relevant time period. The value of an asset is therefore determined by discounting the expected cash flows back to their present value, using the required return as the appropriate discount rate. Utilizing the present value techniques, we can express the value of any asset at time zero, V_0, as

$$V_0 = \frac{CF_1}{(1+k)^1} + \frac{CF_2}{(1+k)^2} + \cdots + \frac{CF_n}{(1+k)^n}$$

where V_0 = *value of the asset at time zero*

CF_t = *cash flow expected at the end of year t*

k = *appropriate required return (discount rate)*

n = *relevant time period*

The principle of this basic valuation model can be applied into the valuation of bond and stock:

- *Bond Valuation*

Bonds are long-term debt instruments used by business and government to raise large sums of money, typically from a diverse group of lenders. The value of a bond is the present value of the contractual payments its issuer is obligated to make from the current time until it matures. The appropriate discount rate would be the required return, k_b, which depends on prevailing interest rates and risk. The basic equation for the value, B_0, of a bond is given by

$$B_0 = I \times \left[\sum_{t=1}^{n} \frac{1}{(1+k_b(t))^t}\right] + M \times \left[\frac{1}{(1+k_b(n))^n}\right]$$

where $B_0 =$ *value of the bond at time* zero

$I =$ *coupon paid by the issuer*

$n =$ *number of years to maturity*

$M =$ *nominal value of the bond*

$k_b(t) =$ *discount rate*

The value of a bond in the marketplace is rarely equal to its nominal value. A variety of forces in the economy as well as the passage of time tend to affect value, which are really in no way controlled by bond issuer or investors.

▪ *Common Stock Valuation*

Like bonds, the value of a share of common stock is equal to the present value of all future benefits it is expected to provide. Simply stated, the value of a share of common stock is equal to the present value of all future dividends it is expected to provide over an infinite time horizon:

$$S_0 = \frac{D_1}{(1+k_s)^1} + \frac{D_2}{(1+k_s)^2} + \cdots + \frac{D_\infty}{(1+k_s)^\infty}$$

where $S_0 =$ *value of common stock at time zero*

$D_t =$ *per share dividend expected at the end of year t*

$k_s =$ *required return on a common stock*

From the formula, we can see that any action of increasing the level of expected return without changing risk should increase share value, and vice versa. Similarly, any action that increases risk (required return) will reduce share value, and vice versa.

■ I.3 Derivatives and Risk Management

I.3.1 The Functions of Derivatives

The most obvious function of derivatives is to facilitate the reallocation of exposure to risk among market participants. Basically there are two strategies to implement the risk reallocation: **hedging** and **speculating**.

Hedging can be generally understood as a financial activity to either reduce or eliminate the risk of an underlying asset by making the appropriate *offsetting* derivative transaction. Such activities may involve certain costs, i.e. the cost for buying the futures or options.

Let us take the above car-purchasing example. If during the week you discover a second dealer offering an identical model for $ 29,000 then you don't take up your option with the first dealer. The total cost of buying the car is now $29,000 + $200 = 29,200: cheaper than the first price you were offered, i.e $300,000. If you cannot find the car at a cheaper price and buy the car from the first dealer, then the car will cost a total of $300,200. If you decide not to buy at all you will lose $200 to the car dealer. But anyway, in this example, both in the forward contract case and the option one, you are *hedging* against a price rise in the car - it eliminates the risk of buying a car more expensive than $300,000.

Speculating is somewhat the opposite activity to hedging. Whereas the aim of hedging is to reduce or eliminate risk by sacrificing some capital, the aim of speculating is to obtain higher returns by taking higher risks.

Again take the above example. Suppose that the car you have bought a call option for is very much in demand and there is a sudden price rise to $33,000. One colleague of yours also wants the same car and hears that you have an option to buy the car for $30,000 in a week's time. You can sell the option to your colleague for $400. This means that the car dealer still gets his sale, your friend gets the car he wants and you make $200 on selling your option. In this case you have *speculated* on your contract and made a 200%.

In the following we present the pricing principles and methods for futures and options. Fortunately, the prices of forwards and futures contracts on the same underlying assets with the same time to expiration are generally very close to each other, although they are normally different because of taxes, transaction costs, and other factors. Thus, here we can simply regard futures as the same as forward contracts for convenience of understanding and analysis.

I.3.2. Forward Pricing

Different from some of their underlying assets, such like copper, etc, futures can be produced and stored at very low cost, so we can ignore those costs completely in deriving parity relations between spot (current) and future prices.

Consider a forward contract (or simply forward) on a share of a stock that pays no dividend. The contract is the promise to deliver a share at some specified delivery date at a specified delivery price. Let us denote this forward price by F, and the spot price at the beginning of the stock is S. The relation between F and S is

$$S = F/(1+r)^T$$

where r is the risk-free interest rate, such as the interest rate offered by a bank if you deposit the amount of money in it, T is the maturity of the forward contract in number of years. This pricing formula guarantees that there are no *arbitrage* opportunities in this market. An arbitrage opportunity is the situation when a zero net initial investment will result in a profit in the future with probability one. If e.g. $S\ (1+r)^T < F$, an arbitrageur could make a sure profit by borrowing an amount of money (S), buying a stock in the market and taking a *short* position (agreeing to sell the stock in the future) in the forward contract. At the maturity of the contract, the stock will be sold for F, the loan compounded with interest (equal to $S\ (1+r)^T$) will be paid, and thus a net profit of $F - S(1+r)^T$. In financial modeling the major assumption is that markets are arbitrage free. This is a very reasonable assumption because possible arbitrage opportunities are generally not expected to exist for a long time.

Question:

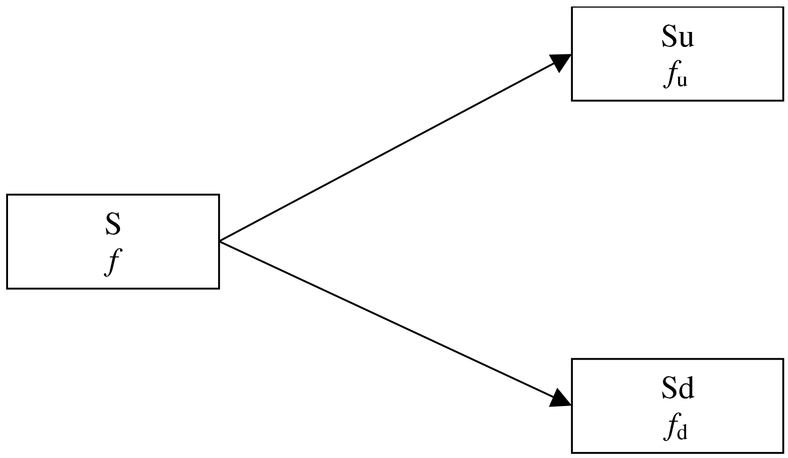
What is an appropriate arbitrage strategy if $S\ (1+r)^T > F$?

I.3.3. Option Pricing

The valuation or pricing of options is one of the basic activities in derivative finance. The basic principle lies in that, the present value of an option is the *expected* value of the *payoff, i.e. the value*, of the option at the expiration date *discounted* at the risk-free interest rate. In this section we will explain what is actually meant by this statement in more detail.

■ Binomial Tree Model

Suppose we want to price a **European call option,** the payoff of the option is the difference between the underlying asset price at maturity and strike price of the underlying asset. We consider a very simple world with two time-steps, the initial time ($t=0$), and the time of maturity ($t=T$). The price of the stock can go either up (Su) or down (Sd). If the price goes up, the value of the option at time $t=T$ is f_u, and if the price goes down, its value is f_d. This system is also known as a binomial tree of level one.



The main idea now is to construct a *risk-less* portfolio consisting of a long position in a number of shares of the stock (say Δ) and a short position in (*i.e.*, writing) a call option contract, such that the value of the portfolio at time $t=T$ is known with certainty (a risk-less portfolio). The value of Δ that guarantees that the final value of the portfolio is the same for the up and down situation is

$$\Delta Su - f_u = \Delta Sd - f_d ,$$

and thus

$$\Delta = \frac{f_u - f_d}{S(u-d)}$$

Construction of such a risk-free portfolio is called the *replicating portfolio strategy*. Given the up and down movement of the stock price, the value of the portfolio at the time of maturity can be determined with no uncertainty. The value at of the portfolio at $t=0$ is then given by (continuous discounting)

$$\Delta S - f = e^{-rT} (\Delta Su - f_u)$$

and thus by substituting the value for Δ, the following expression can be derived for the price of the option

$$f = e^{-rT} (pf_u + (1-p) f_d) \text{ with } p = \frac{e^{rT} - d}{u-d}$$

It should be noted that p can be interpreted as a probability. The price of the option is then given by the expectation of the *payoff, i.e the value,* of the option at the expiration date, discounted by the risk-free interest rate (due to the time value of money). Taking the expectation operator based on the probabilities p is also known as taking the expectation under the risk-neutral measure. Moreover, the probability that the stock price will go up or down in the real world is irrelevant for the price of the option. In the absence of arbitrage opportunities prices obtained in a risk-neutral world are the real world prices.

In summary, for an **European call option,** the pricing model is

$$C = e^{-r(T-t)} E[max(S_T - K, 0)]$$

where S_T is the spot price of the underlying stock at expiration, K the strike price, T the expiration time, t the present time, r the risk-free interest rate and C, the present price of the option. This pricing model will enable us to determine the price of a wide variety of option by means of analytical and numerical methods as will be demonstrated in the next sections. The payoff (or value) of the option at expiration, max (S_T - K, 0), can be understood in this way: if the stock price at expiration is at or below the strike price, then the payoff of the option is zero, i.e the call option will expire worthless, because nobody wants to buy a stock at a price higher than market price; if it is greater than the strike price, then the payoff is simply the difference between the spot price and the strike price. The difference between the underlying asset price at maturity and strike price is called the intrinsic value of the call option.

In the market several other types of options are traded, e.g **Asian option**, **American option**, **lookback option** etc.

They are distinctive in different aspects: how to calculate payoff, whether an option can only be exercised on the expiration data, etc.

Here we present the celebrated Black-Scholes formula for calculation of the option price on a stock. The derivation process of it is shown in Appendix.

■ Black-Scholes Option Pricing Formula

The most popular model in options pricing is the *Black-Scholes model* in which the underlying asset price is assumed to be lognormally distributed. As in most other theoretical models, Black and Scholes made many assumptions. We list a few important ones here:

(i). The underlying asset price is lognormally distributed;

(ii). The underlying asset pays no dividend;

(iii). There are no transaction costs in buying or selling the underlying asset or option;

(iv). The short-term interest rate is known and is constant through time.

With these assumptions, Black and Scholes obtained the following formula for the price of a European call option:

$$C = SN(d_1) - Ke^{-rT} N(d)$$

where

$$d = \frac{ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = d + \sigma\sqrt{T}$$

C is the price of a call option, K is the strike price of the underlying asset, r is the annual interest rate, σ is the volatility of the annual return of the underlying asset, T is the time-to-maturity in number or fraction of year(s), $N(x)$ is the value of the cumulative function of the standard normal distribution at x.

■ Put-Call Parity

There is a important relationship between a call and its corresponding put option prices with the same strike price and time-to-maturity, which can be used to calculate the put option price:

$$P = C - S + Ke^{-rT}.$$

The derivation of this relationship can be found in the appendix.

■ Hedging Parameters

The Black-Scholes formula is clearly a function of five factors: S, K, T, r and σ. S and r can be observed from the market; K and T are specified in the option contract. σ, however, is neither specified in the option contract nor directly observable from the market. We have to estimate this volatility value using historical data of the underlying asset in order to use the Black -Scholes formula.

There are a few popular terms characterizing the sensitivity of options prices to the changes of these factors. These sensitivities are often named by Greek letters. They play an important role in both trading activities and risk management in financial institutions.

Delta ($\delta = \partial C / \partial S$) measures how fast its price changes with the price of its underlying asset. Its discrete counterpart has been discussed in the Binomial Tree subsection. Vega ($v = \partial C / \partial \sigma$) measures how fast the option's price changes with the volatility of its underlying asset. Theta ($\theta = \partial C / \partial T$)) measures the sensitivity of its price with respect to the time to maturity. Rho ($\rho = \partial C / \partial r$) measures the sensitivity of the option's value with respect to the fluctuation of the interest rate. Gamma ($\gamma = \partial \delta / \partial S$) measures how fast the option's delta changes with the price of its underlying asset.

I.3.4 Risk Management

Risk management deals with risk reduction, basing on the principle of benefit-cost tradeoff and information available. There are four basic techniques available for reducing risk:

- Risk avoidance: A consciousness not to be exposed to a particular risk.
- Loss prevention and control: Actions taken to reduce the likelihood or the severity of losses.
- Risk retention: Absorbing the risk and covering losses out of one's own resources.
- Risk transfer: Transferring the risk to others who are willing to accept it in order to earn possible profit.

One of the greatest roles of the financial system is transferring risk. Three most commonly used methods for transferring risk are **hedging, insuring**, and **diversifying**.

One is said to hedge a risk when the action taken to reduce one's exposure to a loss also causes one to give up of the possibility of a gain. For example, farmers who sell their future crops before the harvest at a fixed price to eliminate the risk of a low price at harvest time, also give up the possibility of profiting from high prices at harvest time.

Insuring means paying a premium to avoid losses. By buying insurance, you substitute a sure loss for the possibility of a larger loss if you do not insure.

Diversification: diversifying means holding many risky assets instead of concentrating all of your investment in only one. Its meaning is captured by the familiar saying: "Don't put all your eggs in one basket." The diversification principle is that by diversifying across risky assets, people can sometimes achieve a reduction in their overall risk exposure with no reduction in their expected return.

■ Chapter II Computational Finance in General

■ II.1 Computational Finance

Computational finance refers generally to the application of computational techniques to finance. It is a rather broad field, including many diverse areas of applied mathematics, computer science and economics. Nowadays, computational finance has become an integral part of modeling, analysis, and decision-making in financial industry.

Many models used in finance end up in formulation of highly mathematical problems. Solving these equations exactly in closed form is impossible as the experience in other fields suggests. Therefore, we have to look for efficient numerical algorithms in solving complex problems such as option pricing, risk analysis, portfolio management, etc.

Especially, in the finance market, financial models used in online information processing are generally large and dynamic, normally ending up in large number of equations, say, 1000 to 100,000 equations. Solving these models is, due to their size, computationally intensive. For this type of problems, high performance computing is needed to generate solutions quickly in order to respond adequately on rapid changes in the financial market.

Recently, enabled by powerful computational tools, computational finance is able to proceed in novel directions: new stochastic methods, neural networks, chaos theory, genetic algorithms, artificial life, simulated annealing and control of dynamic systems etc. These new computational techniques have greatly enhanced our understanding of complex economic and financial behaviors.

■ II.2 Some Active Fields

II.2.1 Stochastic Simulation

The financial market is random by nature, e.g the stock prices, interest rates of bonds, foreign exchange rates, etc present typical stochastic behaviors. Understanding this financial probability process can help us to develop correct valuation models for estimating expected returns and volatilities and their effects on asset and derivative prices, which are essential in financial decision making. The financial stochastic dynamics is mathematically modeled with stochastic calculus and other stochastic methods.

Many complex financial problems exist for which analytical formulae are not possible. Stochastic simulation (Monte Carlo simulation) provides flexible methods for solving these types of problem. Monte Carlo simulation is advantageous at dealing easily with multiple random factors, implementing some more realistic asset pricing processes, such as that with jumps in asset prices, and that for exotic path-dependent options. The main drawback of Monte Carlo method is its computational inefficiency. However, new research and recent innovations in both technologic and algorithmic aspects have dramatically reduced computational time of Monte Carlo simulation by increasing speed and efficiency, and have been enhancing its attractiveness for modern financial computation.

II.2.2 Complexity Theory

The most important characteristics of chaos theory are clearly contradictory to the fundamental notions of the Newtonian world view of science. For instance, it demonstrates that many types of phenomena (fractal phenomena) are not susceptible to definitive measurement because they lack a characteristic scale, and is hence contradictory to the Newtonian world view which portrays the universe as a clockwork mechanism susceptible to precise measurement, prediction, and control.

The new vision of chaos theory or complexity theory is undermining the methodological perspective advocated by capital markets researchers most notably with respect to the assumptions of linearity and predictability. It is also challenging the validity of the most fundamental or traditional theories and models: EMH (efficient market hypothesis), MPT (modern portfolio theory), the CAPM (the capital asset pricing model), which underlie most of the research about the relationship between market prices and accounting information. The new vision is also breeding new theories of stock price behavior that are quite inconsistent with the traditional capital markets models and it has led to the emergence of a fractal market hypothesis.

II.2.3. Agent-Based Simulation

Agent-based computational finance models economic markets with large numbers of interacting agents, relying heavily on computational tools to push beyond the restrictions of analytic methods. This new research methodology stresses interactions and learning dynamics in groups of traders who learn about the relations between prices and market information.

In this framework, traders are made up from a very diverse set of types and behaviors. To make the situation more complex, the population of agent types, or the individual behaviors themselves, are allowed to change over time in response to past performance. The finance market is thought of as an adaptive non-linear network whose primary building blocks are adaptive agents. Also, the research achievements on *increasing return* and *path dependency* contribute to this evolutionary perspective of finance and economy.

II.2.4. Neural Networks

Neural networks are computing models often used for pattern classification and pattern recognition problems. They can learn from examples or experiences and particularly noted for their flexible function mapping ability, and achieves a high degree of prediction accuracy. Therefore, financial application areas that require pattern matching, classification, and prediction, such as bankruptcy prediction, loan evaluation, credit scoring, and bond rating, are fruitful candidate areas for neural network technology.

Especially, financial forecasting is always and will remain difficult because such data are greatly influenced by economical, political, international and even natural shocks. Neural networks are data-driven self-adaptive methods in that there are few *a priori* assumptions about the model form for a problem under study. Unlike linear regression analysis, which is limited to the linear function mappings, neural networks are able to discover complex nonlinear relationships in the data. These unique features make them valuable for solving many practical problems such as option price, stocks price, foreign exchange rate forecasting, etc in terms of accuracy, adaptability, robustness, effectiveness, and efficiency.

■ Chapter III Stochastic Modeling and Simulation in Finance

■ III.1 Stochastic Modeling

III.1.1 The Random Nature of the Financial Market

The values of the major indices and the graphs of them are quoted frequently on newspapers, television news bulletins. To many people these 'mountain ranges' showing the variation of the value of an asset or index with time is an excellent example of the 'random walk'.

We cannot predict tomorrow's values of asset prices. The past history of the asset value is there as a financial time series for us to examine as much as we want: but we cannot use it to forecast the next move that the asset will make. This does not mean that it tells us nothing. We know from a statistical analysis of it what are the likely jumps in asset price, what are their mean and variance and, generally, what is the likely distribution of future asset prices.

According to the e*fficient market hypothesis,* which states that market prices reflect the knowledge and expectations of all investors, asset prices must move randomly. This basically tells two things:

- The past history is fully reflected in the present price, which does not hold any further information;
- Markets respond immediately to any new information about an asset price.

Thus the modeling of asset prices is really about modeling the arrival of new information which affects the price. With the two assumptions above, changes in the asset price are a M*arkov process*.

Now suppose that at time t the asset price is S. Let us consider a small subsequent time interval dt, during which S changes to $S + dS$. We can model the corresponding *return* on the asset, dS / S , into two parts. One is a predictable, deterministic return akin to the return on money invested in a risk-free bank. It gives a contribution

$$\mu dt$$

to the return dS / S, where μ is a measure of the average rate of growth of the asset price, also known as the drift. Normally, μ is taken to be a constant.

The second contribution to dS / S models the random change in the asset price in response to external effects, such as unexpected news. It is represented by a random sample drawn from a normal distribution with mean zero and adds a term

$$\sigma dz$$

to dS / S . Here σ is a number called the *volatility*, which measures the standard deviation of the returns. The quantity dz is the sample from a normal distribution, which is discussed further below.

Putting these contributions together, we obtain the stochastic differential equation

$$\frac{dS}{S} = \sigma dz + \mu dt \qquad \text{(III.1)}$$

which is the mathematical representation of our simple recipe for generating asset prices.

The only symbol in the above equation whose role is not yet entirely clear is *dz*. If we were to cross out the term involving *dz*, by taking $\sigma = 0$, we would be left with the ordinary differential equation

$$\frac{dS}{S} = \mu dt \qquad \text{(III.2)}$$

or

$$\frac{dS}{dt} = \mu S \qquad \text{(III.3)}$$

When μ is constant this can be solved exactly to give exponential growth in the value of the asset, i.e.

$$S = S_0 e^{\mu(t - t_0)} \qquad \text{(III.4)}$$

where S_0 is the value of the asset at $t = t_0$. Thus if $\sigma = 0$ the asset price is totally deterministic and we can predict the future price of the asset with certainty.

The term *dz*, which contains the randomness that is certainly a feature of asset prices, is known as a *Wiener process*. It has the following properties:

- *dz* is a random variable, drawn from a normal distribution;
- the mean of *dz* is zero;
- the variance of *dz* is *dt*.

One way of writing this is

$$dz = \phi \sqrt{dt} \qquad \text{(III.5)}$$

where ϕ is a random variable drawn from a *standardized normal distribution*. The standardized distribution has zero mean, unit variance and a probability density function given by

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$$

for $-\infty < \phi < \infty$. If we define the expectation operator ξ by

$$\xi[F(.)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\phi) e^{-\frac{1}{2}\phi^2} d\phi \qquad \text{(III.6)}$$

for any function *F*, then

$$\xi[\phi] = 0 \qquad \text{(III.7)}$$

and

$$\xi[\phi^2] = 1 \qquad \text{(III.8)}$$

Equation (III.1) is a particular example of a random walk. It does not refer to the past history of the asset price; the next asset price ($S + dS$) depends solely on today's price and hence it is a Markov process.

III.1.2 Modeling with Stochastic Calculus

In modeling, analyzing and predicting activities in finance, greater and greater emphasis has been placed upon stochastic methods: Ito's lemma, stochastic differential equations, stochastic stability and stochastic control. Such methods are expected to capture the various complexities, measurement errors and uncertainties that are associated with financial reality. Among them, Ito's lemma is the basic stochastic calculus rule for computing stochastic differentials of composite random functions.

■ Ito's Lemma

Let $u(X,t)$ be a continuous, non-random function with continuous partial derivatives and $X(t)$ a stochastic process defined by

$$dX(t) = a(X, t)dt + b(X, t)dz(t) \qquad \text{(III.9)}$$

where $dz(t)$ is the standard Wiener process. Then the stochastic process $Y(t) = u(X(t),t)$ has the following form of stochastic differential

$$dY(t) = \left(\frac{\partial u}{\partial t} + a(X, t)\frac{\partial u}{\partial X} + \frac{1}{2} b(X, t)^2 \frac{\partial^2 u}{\partial X^2}\right)dt + b(X,t)\frac{\partial u}{\partial X} dz(t) \qquad \text{(III.10)}$$

The derivation of Ito's Lemma is given in Appendix. Here we simply use it to model stochastic processes. In the following we show two examples of stochastic modeling:

(1) Geometric Brownian Motion *model for asset price dynamics*

The asset model we discussed above is

$$\frac{dS}{S} = \sigma dz + \mu dt \qquad \text{(III.11)}$$

With the help of the corresponding expression of Ito's lemma

$$df = \frac{\partial f}{\partial S}\sigma S\, dz + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt \qquad \text{(III.12)}$$

we can quickly derive a probability density function for the asset price, S(t). Setting

$$f = ln(S) \qquad \text{(III.13)}$$

we have

$$\frac{\partial f}{\partial S} = \frac{1}{S} \qquad \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} \qquad \frac{\partial f}{\partial t} = 0$$

Substituting them into the expression of Ito's Lemma III.12 results in

$$d\,ln(S) = \sigma\,dz + (\mu - \frac{1}{2}\,\sigma^2)dt \qquad \text{(III.14)}$$

i.e

$$ln(S_t)\text{- } ln(S_0)= \ \sigma\,dz + (\mu - \frac{1}{2}\,\sigma^2)dt \qquad \text{(III.15)}$$

Finally we get

$$S_t = S_0\,Exp(\ \sigma\,dz + (\mu - \frac{1}{2}\,\sigma^2)\ dt\ ) \qquad \text{(III.16)}$$

It shows that the asset price follows a *lognormal distribution*.

(2) Derivative securities pricing model

Again we start at the asset model

$$\frac{dS}{S} = \sigma dz + \mu dt \qquad \text{(III.17)}$$

Suppose that f is the price of a derivative contingent on S, which must be a function of S and t. Ito's lemma says

$$df{=}\frac{\partial f}{\partial S}\,\sigma S\,dz + \left(\mu S\,\frac{\partial f}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt \qquad \text{(III.18)}$$

An important step in derivatives valuation is the elimination of randomness *(replication portfolio strategy)*: the two random walks in S (equation III.17) and f (equation III.18) are both driven by the single random variable dz. We can exploit this fact to construct a third variable Π whose variation $d\Pi$ is wholly deterministic during the small time period dt.

Let Δ be a constant number during the timestep dt and let

$$\Pi = -f\ + \Delta S \qquad \text{(III.19)}$$

We can write

$$d\Pi = -df\ + \Delta dS$$

$$= -\sigma S\frac{\partial f}{\partial S}\,dz - \left(\mu S\,\frac{\partial f}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}\right)dt$$
$$+\Delta(\ \sigma S dz + \mu S dt\ )$$

$$= -\ \sigma S\ (\ \frac{\partial f}{\partial S}\ - \Delta)dz$$
$$-(\ \mu S(\frac{\partial f}{\partial S}\ - \Delta)\text{-}\frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 f}{\partial S^2}\ - \frac{\partial f}{\partial t}\ )dt \qquad \text{(III.20)}$$

Now, by choosing $\Delta = \partial f / \partial S$, we can make the coefficient of dz vanish. So, It follows that by choosing a portfolio of the stock and the derivative, the Wiener process can be eliminated

The appropriate portfolio is

$$-1: \textit{derivative}$$

$$+\frac{\partial f}{\partial S}: \textit{shares of stock}$$

The holder of this portfolio is *short* (agreeing to sell in the future) one derivative and *long* (agreeing to buy in the future) an amount $\partial f / \partial S$ of shares. Defining Π as the value of the portfolio, we have

$$\Pi = -f + \frac{\partial f}{\partial S} S \qquad \text{(III.21)}$$

and

$$d\Pi = -df + \frac{\partial f}{\partial S} dS \qquad \text{(III.22)}$$

Substituting equations (III.17) and (III.18) into equation (III.22) yields

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2}\right) dt \qquad \text{(III.23)}$$

Because this equation does not involve dz, the portfolio must be riskless during time Δt. The assumptions listed in the preceding section imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by shorting the risk-free securities and using the proceeds to buy the portfolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free securities. It follows that

$$d\Pi = r\Pi dt \qquad \text{(III.24)}$$

where r is the risk-free interest rate. Substituting from equation (III.21) and (III.23), this becomes

$$\left(-\frac{\partial f}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2}\right)dt = r\left(-f + S \frac{\partial f}{\partial S}\right)dt \qquad \text{(III.25)}$$

So that

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - rf = 0 \qquad \text{(III.26)}$$

This parabolic partial differential equation is called the *Black-Scholes-Merton differential equation*. Combining it with particular boundary conditions of European call, we can derive Black-Scholes formula as shown in Chapter I. The complete derivation is in Appendix A.2.

■ III.2 Stochastic Simulation

III.2.1 The Principle of Monte Carlo Simulation for Finance

The Monte Carlo method simulates the random movement of the asset prices and provides a probabilistic solution to the option pricing problems. Since most derivative pricing problems can be formulated as the valuation of the discounted expectation of the terminal payoff function, the Monte Carlo simulation becomes naturally an effective numerical tool for pricing derivative securities whose analytic closed form solutions do not exist, e.g. Asian option pricing.

Consider a derivative dependent on a single underlying asset S that provides a payoff at time T. Assuming that interest rates are constant, we can value the derivative by following the steps below:

1. Sample a random path for S in a risk-neutral world (the risk preferences of the investors do not affect the price): $S_t = S_0 \mathrm{Exp}(\sigma dz + (\mu - 1/2\, \sigma^2)\, dt)$.

2. Calculate the payoff, for an European call option, it is $\max(S_T - X, 0)$.

3. Discount the expected payoff at the risk-free rate to get an estimate of the value of the derivative $c = e^{-r(T-t)} \max(S_T - X, 0)$.

4. Repeat steps 1 and 2 to get many sample values of the payoff.

5. Calculate the mean of the values to get the price of the derivative: $\hat{c} = 1 / M \sum_{i=1}^{M} c_i$.

The major advantage of the Monte Carlo approach is its ease to accommodate complicated terminal payoff function in a derivative pricing model. For example, the terminal payoff may depend on the average of the asset price over certain time interval (Asian options) or the extremum value of the asset price over some period of time (Lookback options). It is quite straightforward to obtain the average or extremum value in the simulated price path in individual simulation runs, and this represents a instinctive advantage. The main drawback of the Monte Carlo simulation is the demand for a large number of simulation trials in order to achieve a high level of accuracy. However, viewing from another perspective, practitioners dealing with a newly invented option may obtain an estimate of its price using the Monte Carlo approach through brute force simulation, rather than risking themselves in the construction of an analytic pricing model for the new option. Furthermore, when the dimensionality of the problem in high (e.g. in the case of interest rate derivatives) the Monte Carlo approach is far more efficient compared to other methods *(curse of dimensionality).*

III.2.2 Application Examples Implemented in Mathematica

III.2.2.1 Simulation of Stock Price Behavior

Here we simulate stock price behavior formulated by (Refer to III.1.1)

$$\frac{dS}{S} = \sigma\, dz + \mu\, dt$$

or

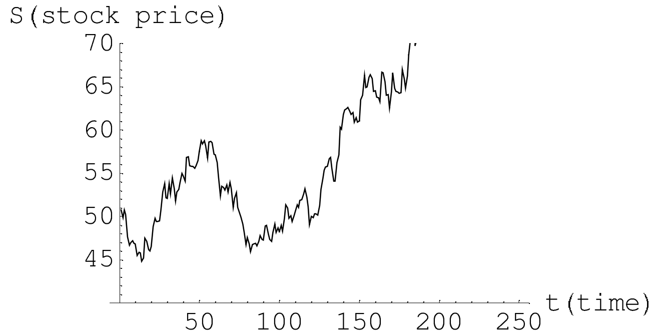
$$S_{t+1} = S_t + \sigma S dz + \mu S dt, \quad t = 0, 1, \ldots n$$

In[450]:=
```
Needs["Statistics`NormalDistribution`"]
```

In[451]:=
```
t = 250; S = 49; r = 0.1; dt = 0.01; σ = 0.2;
DATA =
  Table[S = S + S * (r * dt + Random[NormalDistribution[]] * σ * Sqrt[dt]), {t}];
```

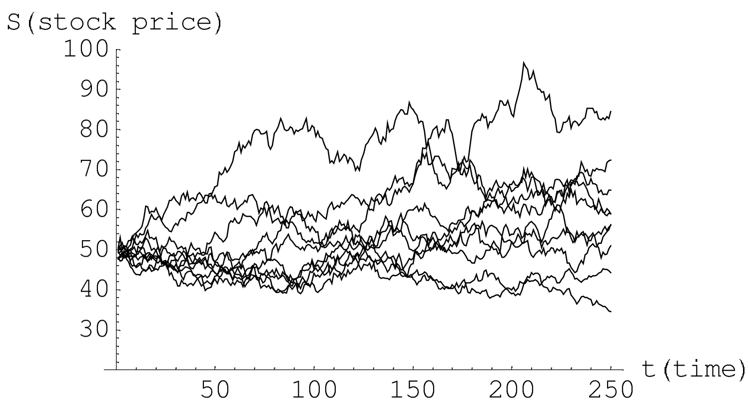
```
In[453]:= Take[DATA, 100];

In[454]:= ListPlot[DATA, PlotJoined → True, PlotRange → {40, 70},
           AxesLabel → {"t(time)", "S(stock price)"}]
```




```
Out[454]= - Graphics -
```

The above code is for only one path of the random stock movement. Repeating the procedure for many times and putting the results together we get the following multiple-path graph. The principle behind this is important for Monte Carlo simulation for finance. As that we stated in III.2.1 and that we will see late, Monte Carlo simulation needs a number of times of random procedure followed by somehow diverse results, of which the mean is what we seek after.



■ Appendix

■ A.1 The Derivation of Ito's Lemma

A completely rigorous derivation or proof of Ito's lemma is beyond the scope of this notebook. We now present a non-rigorous definition of Ito's lemma using the Taylor series formula. For a smooth function $G(x, t)$, the normal Taylor series expansion goes as

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} \Delta t^2 + \cdots \qquad (A.1)$$

For a non-stochastic process, when $\Delta x \to 0$, $\Delta t \to 0$, the above equation becomes

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt \qquad (A.2)$$

We now extend equation (A.2) to cover functions of variables following Ito processes. Suppose that a variable, x, follows the Ito process

$$dx = a(x, t)\, dt + b(x, t)\, dz \qquad (A.3)$$

the discretised form of which is

$$\Delta x = a(x, t)\, \Delta t + b(x, t)\, \epsilon \sqrt{\Delta t} \qquad (A.4)$$

where ϵ is a standard normal variable. From this equation,

$$\Delta x^2 = a^2 \Delta t^2 + 2\, ab\epsilon \Delta t^{\frac{3}{2}} + b^2 \epsilon^2 \Delta t = b^2 \epsilon^2 \Delta t + \mathcal{O}\left(\Delta t^{\frac{3}{2}}\right) \qquad (A.5)$$

This shows that the term involving Δx^2 in equation (A.1) has a component that is of order Δt and cannot be ignored.

The variance of a standardized normal distribution is 1. This means that

$$E(\epsilon^2) - [E(\epsilon)]^2 = 1$$

Because $E(\epsilon) = 0$, it follows that $E(\epsilon^2) = 1$. The expected value of $\epsilon^2 \Delta t$ is of order Δt^2 and that as a result of this, we can treat $\epsilon^2 \Delta t$ as nonstochastic and equal to its expected value of Δt as Δt tends to zero. It follows from equation (A.5) that Δx^2 becomes nonstochastic and equals to $b^2 \Delta t$ as Δt tends to zero. Taking limits as Δx and Δt tend to zero in equation (A.1), and using this last result, we obtain

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt \qquad (A.6)$$

This is Ito's lemma. Substituting for dx from equation (A.3), equation (A.6) becomes

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2\right) dt + \frac{\partial G}{\partial x} b dz \qquad (A.7)$$

■ A.2 The Derivation of Black-Scholes Formula

There are several assumptions involved in the derivation of the Black-Scholes formula:

(1) trading takes place continuously in time. (2) the riskless interest rate r is known and constant over time. (3) the asset pays no dividend. (4) there are no transaction costs in buying or selling the asset or the option, and no taxes. (5) the assets are perfectly divisible. (6) short selling (sale of a security not own by the seller) with full use of proceeds (funds given to a borrower after all costs and fees are deducted) is possible. (7) there are no riskless arbitrage (profiting from differences in price when the same asset is traded on two or more markets) opportunities.

One common way of pricing a derivative is to form a self-financing, replicating hedging strategy for it. Self-financing means that the portfolio produced by the strategy must not itself take up any money apart from a possible initial investment. As we will soon see, this initial investment will be the price of the security that the strategy is replicating. The term replicating means that the strategy must replicate the payoff of the security we are trying to price. Further, the portfolio produced by the strategy should always produce the same result regardless of price changes in the underlying security. In other words, the value of the portfolio generated by the strategy should be deterministic and cannot have a stochastic component (except for the stochastic components of the underlying securities of the derivative). This explains the term 'hedging' used for the strategy.

The value of a derivative is its expected future value discounted at the risk-free interest rate. This is exactly the same result that we would obtain if we assumed that the world was risk-neutral. In such a world, investors would require no compensation for risk. This means that the expected return on all securities would be the risk-free interest rate. This is a very useful principle as it states that we can assume that the world is risk-neutral when calculating option prices. The result would still be correct in the real world even if (as is most probably the case) it is not risk-neutral.

We derive the put-call parity relation using this principle. To do so, use the fact that the sum of the payoff of a long call and a short put option with the same strike price and maturity (and, of course, on the same underlying securities) is given by S - K. Hence, the value of the call and put option is given by

$$e^{-r(T-t)}\,E[S - K] = e^{-r(T-t)}(E[S] - K) = S - Ke^{-r(T-t)} \qquad \text{(A.8)}$$

the last equality coming from risk-neutral valuation. This gives us

$$C - P = S - Ke^{-r(T-t)} \ or \ C + Ke^{-r(T-t)} = P + S \qquad \text{(A.9)}$$

We need one more piece of information before we can derive the Black-Scholes formula. This is Ito's lemma: if a variable x follows a stochastic process of the form

$$dx = a(x,t)dt + b(x,t)dz \qquad \text{(A.10)}$$

then any smooth function $G(x, t)$ follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz \qquad \text{(A.11)}$$

We are now in a position to present a derivation of the Black-Scholes formula. Review the Black-Scholes equation derived in Chapter III:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} = rf \qquad \text{(A.12)}$$

We have shown in Chapter III that if $\frac{dS}{S} = \sigma dz + \mu dt$, then

$$ln(S_t) - ln(S_0) = \sigma dz + (\mu - \frac{1}{2}\sigma^2)dt \qquad (A.13)$$

i.e, S follows a lognormal distribution:

$$lnS \sim N\left[ln\, S_0 + \left(\phi - \frac{\sigma^2}{2}\right)t,\ \sigma\sqrt{t}\right] \qquad (A.14)$$

The principle of risk-neutral valuation implies that the present value of the option is the expected final value E [max (S-K, 0)] of the option discounted at the risk-free interest rate. So, we have

$$c = e^{-r(T-t)}\, E\,[max(S-K,\ 0)] = e^{-r(T-t)} \int_K^{\infty} (S-K)\, g(S)\, dS \qquad (A.15)$$

where $g(S)$, the probability density function is given by (A.14) which can be explicitly written as

$$g(S) = \frac{1}{\sigma S\sqrt{2\pi t}} \exp\left(-\frac{\left(\ln\left(\frac{S}{S_0}\right)\right) - \left(r - \frac{\sigma^2}{2}\right)t^2}{2\sigma^2 t}\right) \qquad (A.16)$$

where μ has been replaced by r in accordance with the principle of risk-neutral valuation. We can easily verify that this solution satisfies the principle of risk-neutral valuation by evaluating $E\,(S) = \int_K^{\infty} Sg(S)\, dS = S_0\, e^{rt}$.

The value of the integral (A.15) can be found with a bit of algebraic manipulation and the Black-Scholes formula for standard European call option is

$$c = SN((d_1) - Ke^{-r(T-t)}\, N(d_2) \qquad (A.17)$$

$$where \qquad d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},\quad d_2 = \frac{ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

and $N(x)$ is the cumulative standard normal distribution.

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