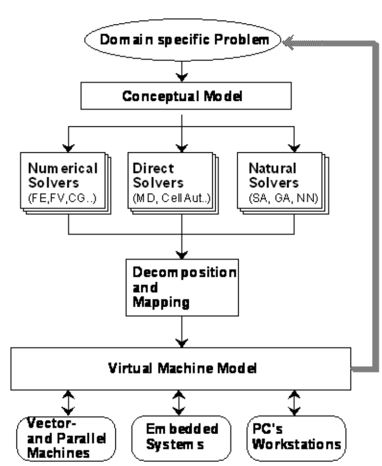
Part 3

Modeling and simulation of waves

the Vibrating String

Simulation and Modeling

Abstraction	->	Conceptual Model
Transformation Model	->	Computer Specific
Implementation	->	Computer
Simulation	->	Problem

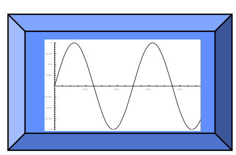


A Guitar String

- We have a string, of length L, suspended at both ends, and held under a tension force T.
- The string has a mass per length unit μ (kg/meter).
- At rest, the string is straight between its suspension points.
- What happens if we pluck the string?

Standing waves

- We observe standing waves



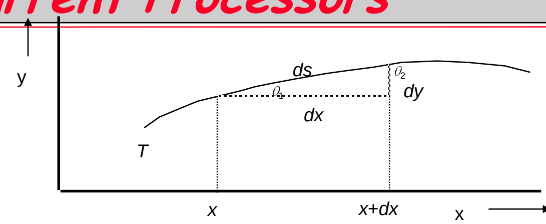
- Denote $y(x,t)$ the amplitude of the wave, as a function of position x along the string, and as function of time.
- For a standing wave, $y(x,t) = y(x,t+T)$, with a period T.

A model for waves in a guitar string

- We make the following assumptions:
  - 1. A long string (i.e. $L >> d$, with d the diameter) is considered as a truly 1 dimensional string.
  - 2. The tension force is constant along the length of the string.
  - 3. For (2) to hold, the amplitude in the string must be assumed to be very small.
- We will analyze the force balance in a small piece of the string, and apply Newton's law (F = ma).
- In this way we derive a PDE (partial differential equation) for $y(x,t)$.

A string Concert for Concurrent Processors

Derivation of the wave equation



$$T\sin(\theta_2) - T\sin(\theta_1) \rightarrow T\left(\left.\frac{\partial y}{\partial x}\right|_{x+dx} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = F$$

with T the tension in the string

For small angles

$$F = ma = \mu ds\left(\frac{\partial^2 y}{\partial t^2}\right)$$

Divide by μdx, for small string pull: $ds \sim dx$

$$\frac{T}{\mu}\frac{\left(\left.\frac{\partial y}{\partial x}\right|_{x+dx} - \left.\frac{\partial y}{\partial x}\right|_{x}\right)}{dx} = \frac{\partial^2 y}{\partial t^2}$$

A string concert ...

for dx->0 and substituting $T/\mu = C^2$

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$$

This is a linear homogeneous hyperbolic partial differential equation of second order.

II Analytical Solution :

For a string with a length L, a solution of the wave equation is

$$y(x,t) = B \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{m\pi Ct}{L}\right), m = 0,1,2,\ldots$$

A general solution is now expressed as

$$y(x,t) = \sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{m\pi Ct}{L}\right)$$

The coefficients B_m are determined by the initial condition $y(x,0) = f(x)$.

$$B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

A String Concert ...

Each term in the series can be interpreted as a normal or eigen mode of vibration. The frequency of the mode is determined by the cosine term, the sine term is the amplitude of the mode.

The frequency of mode m is given by

$$\nu_m = \frac{m}{2L} \sqrt{\frac{T}{\mu}}$$

Since all frequencies are multiples of n1, this can be interpreted as a musical tone (piano, guitar).

A String Concert ...

III Numerical Solution (Finite Difference) :

Rewrite the Wave Equation to : $\frac{1}{C^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$

Define y_j as the solution at nodal point j:

$$\frac{\partial^2 y_j}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{y_j(t+\Delta t) - y_j(t)}{\Delta t}\right)$$

$$\frac{\partial^2 y_j}{\partial t^2} = \frac{1}{\Delta t^2}\left(y_j(t-\Delta t) - 2y_j(t) + y_j(t+\Delta t)\right)$$

An analog expression for x can be found. This results in:

$$\frac{1}{C^2\Delta t^2}\left(y_j(t-\Delta t) - 2y_j(t) + y_j(t+\Delta t)\right) - \frac{1}{\Delta x^2}\left(y_{j-1}(t) - 2y_j(t) + y_{j+1}(t)\right) = 0$$

The resulting finite difference scheme is

$$y_j(t+\Delta t) = 2y_j(t) - y_j(t-\Delta t) + \tau^2\left(y_{j-1}(t) - 2y_j(t) + y_{j+1}(t)\right)$$

where $\tau = \frac{C\Delta t}{\Delta x}$

Explicit Finite Difference Scheme

$$y_j^{n+1} = 2y_j^n - y_j^{n-1} + \tau^2\left(y_{j-1}^n - 2y_j^n + y_{j+1}^n\right)$$

- Explicit computation of amplitude on new time *(n+1)* as a function of amplitudes on previous times (*n*, *n*-1).
- Give as initial condition $y_j^{-1} = y_j^0 = f(x_j)$
- For stability $\tau \leq 1$ (see next slides).

Requirements for finite difference scheme

- Consistency
  - requirement for any algebraic approximation to a partial differential equation to reproduce the partial differential equation in the limit of an infinitesimal time step and grid spacing.
- Computed amplitudes must resemble real solutions.
  - Accuracy
    - Local errors in computation must be small
  - Stability
    - Small errors should not be amplified

Consistency

- Below follows the proof that our FD scheme is consistent with the wave equation

$$\lim_{\Delta x\to 0}\frac{\left(y_{j-1}^n-2y_j^n+y_{j+1}^n\right)}{\Delta x^2}\underbrace{=}_{\text{apply Taylor series}}\lim_{\Delta x\to 0}\frac{1}{\Delta x^2}($$

$$y_j^n-\Delta x\,\partial y/\partial x+\frac{1}{2}\Delta x^2\,\partial^2 y/\partial x^2-\frac{1}{3!}\Delta x^3\,\partial^3 y/\partial x^3+\frac{1}{4!}\Delta x^4\,\partial^4 y/\partial x^4+o(\Delta x^5)$$

$$-2y_j^n$$

$$y_j^n-\Delta x\,\partial y/\partial x+\frac{1}{2}\Delta x^2\,\partial^2 y/\partial x^2+\frac{1}{3!}\Delta x^3\,\partial^3 y/\partial x^3+\frac{1}{4!}\Delta x^4\,\partial^4 y/\partial x^4+o(\Delta x^5))=$$

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x^2}(\Delta x^2\,\partial^2 y/\partial x^2+\frac{1}{4!}\Delta x^4\,\partial^4 y/\partial x^4+o(\Delta x^5))=$$

$$\lim_{\Delta x\to 0}(\partial^2 y/\partial x^2+\frac{1}{4!}\Delta x^2\,\partial^4 y/\partial x^4+o(\Delta x^3))=\partial^2 y/\partial x^2$$

- Likewise for the time derivative. Subsitution into the FD scheme then results in the original wave equation (do this yourself !!)

Accuracy

- Accuracy is concerned with local errors. Local errors arise from two sources.
  - Roundoff errors
    - See lectures on number representation: $|\delta| \leq \varepsilon$; $\varepsilon \equiv \frac{1}{2}\beta^{1-p}$
  - Truncation errors
    - caused by representing continuous variables by discrete sets of values.
- Generally, roundoff errors are much smaller than truncation errors, and provided the scheme is stable (see below) they can usually be ignored.
- Truncation errors are usually described in terms of the difference between the differential and algebraic equations.
  - A measure of the smallness of truncation errors is given by the order of the difference scheme.
  - In our case $O(\Delta x^4)$ in spatial discretization and $O(\Delta t^4)$ for the temporal discretization.

Stability

- Stability is concerned with the propagation of errors.
- A numerical method is stable if a small error at any stage does not lead to a larger cumulative error.
- In our scheme we can prove that it is stable if $\tau \leq 1$.
- This the famous Courant stability condition.
- Sequel of this lecture is concerned with stability of numerical schemes.

Complex numbers

- But first, a refreshment in complex numbers.

The imaginary number $i : i \equiv \sqrt{-1}$; so $i^2 = -1$
Note that with a a positive real number, we can
write $\sqrt{-a} = i\sqrt{a}$.
A complex number z is $z = a + ib$, where a and b
are real numbers. We call a the real part of z, denoted as
$a = \mathrm{Re}(z)$ and b the imaginary part, denoted as $b = \mathrm{Im}(z)$.
The absolute value of a complex numbers is $|z| = \sqrt{a^2 + b^2}$.
Finally, we will need the following : $e^{ix} = \cos(x) + i\sin(x)$.
So, this allows us to express cos, sin as $\mathrm{Re}(e^{ix})$ or $\mathrm{Im}(e^{ix})$.

Von Neumann Stability Analysis

- Decompose the solution in modes
  - i.e. a collection of waves, a bit like our exact solution to the 1D wave problem.
- The amplitudes of all modes should never grow indefinitely.
- Write the modes as

$$y_j^n = \xi^n e^{ik_x \Delta x j}$$

- k_x is the wavenumber.
- Stable scheme $\Leftrightarrow$ $|\xi| \leq 1$.
- Procedure is to substitute the mode solution in the FD scheme, compute ξ, and then demand that $|\xi| \leq 1$.

First, a simpler example

- Consider: $\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$
- Forward Euler difference: $\left.\frac{\partial y}{\partial t}\right|_{j,n} = \frac{y_j^{n+1} - y_j^n}{\Delta t} + O(\Delta t)$
- Central difference : $\left.\frac{\partial y}{\partial x}\right|_{j,n} = \frac{y_{j+1}^n - y_{j-1}^n}{2\Delta x} + O(\Delta x^2)$
- FTCS (forward time centered space scheme)

$$y_j^{n+1} = y_j^n - \frac{v\Delta t}{2\Delta x}\left(y_{j+1}^n - y_{j-1}^n\right)$$

  - Nice and simple, but…
  - … not stable (see next slide)

Stability of FTCS

- Substitute the mode expression

$$\xi^{n+1}e^{ik_x\Delta xj} = \xi^n e^{ik_x\Delta xj} - \frac{v\Delta t}{2\Delta x}\left(\xi^n e^{ik_x\Delta x(j+1)} - \xi^n e^{ik_x\Delta x(j-1)}\right)$$

- Divide by $\xi^n e^{ik_x\Delta xj}$

$$\xi = 1 - \frac{v\Delta t}{2\Delta x}\left(e^{ik_x\Delta x} - e^{-ik_x\Delta x}\right)$$

$$= 1 - \frac{v\Delta t}{2\Delta x}\left(\cos(k_x\Delta x) + i\sin(k_x\Delta x) - \cos(k_x\Delta x) + i\sin(k_x\Delta x)\right)$$

$$= 1 - i\frac{v\Delta t}{\Delta x}\sin(k_x\Delta x)$$

$$|\xi|^2 = 1 + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2(k_x\Delta x)$$

- Amplification factor ξ always > 1 (for k_x > 0)

Lax Method

- FTCS can be made stable in a simple way (due to Lax). Replace in the time derivative

$$y_j^n \rightarrow \frac{1}{2}\left(y_{j+1}^n + y_{j-1}^n\right)$$

- Scheme now becomes

$$y_j^{n+1} = \frac{1}{2}\left(y_{j+1}^n + y_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(y_{j+1}^n - y_{j-1}^n\right)$$

- Using the same procedure we find (do this yourself !!)

$$\xi = \cos(k_x \Delta x) - i\frac{v\Delta t}{\Delta x}\sin(k_x \Delta x)$$

Stability Lax method

- So, we find for the norm of the amplification factor

$$|\xi|^2 = \cos^2(k_x \Delta x) + \left(\frac{v\Delta t}{\Delta x}\right)^2 \sin^2(k_x \Delta x)$$

$$= \cos^2(k_x \Delta x) + \sin^2(k_x \Delta x) + (\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1)\sin^2(k_x \Delta x)$$

$$= 1 + (\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1)\sin^2(k_x \Delta x)$$

- Next, we demand $|\xi| \le 1$ or $-1 \le |\xi|^2 - 1 \le 0$

$$|\xi|^2 - 1 = (\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1)\sin^2(k_x \Delta x) \le (\left(\frac{v\Delta t}{\Delta x}\right)^2 - 1)$$

$$\Rightarrow \frac{v\Delta t}{\Delta x} \le 1$$

- This is the Courant stability condition

Back to the FD scheme for the wave equation

- So, we had

$$y_j^{n+1} = 2y_j^n - y_j^{n-1} + \tau^2 \left(y_{j-1}^n - 2y_j^n + y_{j+1}^n \right)$$

- Try to compute the amplification factor yourself, and compare with the result that is derived below.
- Substitution of the modes gives

$$\xi^{n+1} e^{ik_x \Delta x j} = 2\xi^n e^{ik_x \Delta x j} - \xi^{n-1} e^{ik_x \Delta x j} + \tau^2 \left(\xi^n e^{ik_x \Delta x (j+1)} - 2\xi^n e^{ik_x \Delta x j} + \xi^n e^{ik_x \Delta x (j-1)} \right)$$

- Divide by $\xi^n e^{ik_x \Delta x j}$

$$\xi = 2 - \xi^{-1} + \tau^2 \left(e^{ik_x \Delta x} - 2 + e^{-ik_x \Delta x} \right)$$

Stability FD scheme for wave

- Work out the exponentials

$$e^{ik_x \Delta x} - 2 + e^{-ik_x \Delta x} = \cos(k_x \Delta x) + i\sin(k_x \Delta x) - 2 + \cos(k_x \Delta x) - i\sin(k_x \Delta x)$$
$$= 2(\cos(k_x \Delta x) - 1) = -4\sin^2(1/2\, k_x \Delta x)$$
$$\text{where we used } \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$$

- So, we find

$$\xi = 2 - \xi^{-1} - 4\tau^2 \sin^2(1/2\, k_x \Delta x)$$
$$\text{with } \alpha = \tau \sin(1/2\, k_x \Delta x) \text{ we find}$$
$$\xi^2 - (2 - 4\alpha^2)\xi + 1 = 0$$

- We can solve this equation using the standard a,b,c formula (next slide)

Stability wave FD continued

$$\xi = \frac{(2-4\alpha^2) \pm \sqrt{(2-4\alpha^2)^2 - 4}}{2}$$

$$= \frac{(2-4\alpha^2) \pm \sqrt{(4-16\alpha^2+16\alpha^4) - 4}}{2}$$

$$= 1 - 2\alpha^2 \pm 2\alpha\sqrt{\alpha^2 - 1}$$

- Next, consider two cases
- Case I: $\alpha \le 1$, ξ is now complex (because $\alpha^2 - 1$ under the square root is negative)

$$\xi = 1 - 2\alpha^2 \pm 2\alpha\sqrt{\alpha^2 - 1} = 1 - 2\alpha^2 \pm 2i\alpha\sqrt{1-\alpha^2}$$

$$|\xi|^2 = (1-2\alpha^2)^2 + 4\alpha^2(1-\alpha^2) = 1$$

- So, in this case the amplification factor is always 1, so scheme is always stable.

Stability wave FD continued

- Case I continued:
- So, for $\alpha \le 1$ the FD scheme is always stable.
- In other words:

$$\alpha = \tau \sin(1/2\, k_x \Delta x) \le \tau \le 1, \text{ or}$$

$$\frac{c\Delta t}{\Delta x} \le 1$$

- Case II: $\alpha \ge 1$, ξ is now real

$$\xi = 1 - 2\alpha^2 \pm 2\alpha\sqrt{\alpha^2 - 1}$$

- The – solution is always < -1, so $|\xi| > 1$, and in this case the scheme is always unstable.

Consequence of stability

- Stability analysis showed that

$$\frac{c\Delta t}{\Delta x} \leq 1$$

- Consequence:
  - A choice for Δx limits the maximum time step to $\Delta t \leq \Delta x/c$
  - Now, the grid spacing is determined by the wavelength λ of wave that must be represented accurately. A rule of thumb is that one needs 10 discrete points to represent a wavelength λ. With $\lambda = 2L/m$, with m the mode of the wave to be represented, we have $\Delta x = L/5m$ and therefore $\Delta t \leq L/5mc$.
  - Usually one takes the maximum timestep, i.e. put $\tau = 1$.

A special feature for τ = 1

- Our finite difference scheme has a very interesting feature, for $\tau = 1$ it has no truncation errors.
  - Try this yourself in the simulation, compare with exact solution and you will observe this unexpected, and indeed, quite special behavior (this is certainly not true for other discretization schemes).
  - Let us try to understand why this is the case.

$$y_{j+1}^{n} = y_j^n + \Delta x \frac{\partial y}{\partial x} + \frac{1}{2}\Delta x^2 \frac{\partial^2 y}{\partial x^2} + \frac{1}{3!}\Delta x^3 \frac{\partial^3 y}{\partial x^3} + \frac{1}{4!}\Delta x^4 \frac{\partial^4 y}{\partial x^4} + \dots$$

$$y_{j-1}^{n} = y_j^n - \Delta x \frac{\partial y}{\partial x} + \frac{1}{2}\Delta x^2 \frac{\partial^2 y}{\partial x^2} - \frac{1}{3!}\Delta x^3 \frac{\partial^3 y}{\partial x^3} + \frac{1}{4!}\Delta x^4 \frac{\partial^4 y}{\partial x^4} + \dots$$

addition results in

$$\frac{\partial^2 y}{\partial x^2} = \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{\Delta x^2} - \frac{2}{4!}\Delta x^2 \frac{\partial^4 y}{\partial x^4} - \frac{2}{6!}\Delta x^4 \frac{\partial^6 y}{\partial x^6} + \dots$$

τ = 1 continued

- Likewise for the time derivative

$$\frac{\partial^2 y}{\partial t^2} = \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2} - \frac{2}{4!}\Delta t^2 \frac{\partial^4 y}{\partial t^4} - \frac{2}{6!}\Delta t^4 \frac{\partial^6 y}{\partial t^6} + \ldots$$

- Substituting these expressions in the wave equation gives

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2} - c^2 \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{\Delta x^2}$$
$$-2\left\{\frac{\Delta t^2}{4!}\frac{\partial^4 y}{\partial t^4} - c^2 \frac{\Delta x^2}{4!}\frac{\partial^4 y}{\partial x^4} + \frac{\Delta t^4}{6!}\frac{\partial^6 y}{\partial t^6} - c^2 \frac{\Delta x^4}{4!}\frac{\partial^6 y}{\partial x^6} + \ldots\right\}$$

- So, if the term in curly brackets is zero, the finite difference equation is exactly equal to the wave equation. This happens for τ = 1.

τ = 1 continued

- Now, for exact solutions we have

$$\frac{\partial^4 y}{\partial t^4} = \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\partial^2}{\partial t^2}\left(c^2 \frac{\partial^2 y}{\partial x^2}\right) = c^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 y}{\partial t^2}\right) = c^2 \frac{\partial^2}{\partial x^2}\left(c^2 \frac{\partial^2 y}{\partial x^2}\right) = c^2 \frac{\partial^4 y}{\partial x^4}$$

- So that

$$\frac{\Delta t^2}{4!}\frac{\partial^4 y}{\partial t^4} - c^2 \frac{\Delta x^2}{4!}\frac{\partial^4 y}{\partial x^4} = \frac{c^2}{4!}\left(c^2 \Delta t^2 - \Delta x^2\right)\frac{\partial^4 y}{\partial x^4}$$

- So, for τ = 1 this leading error term is equal to zero !
- The same is true for the higher order error terms (try to prove this yourself).

- This concludes our proof that for τ = 1 we have an 'exact' FD scheme.