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ASYMPTOTICS OF THE PRICE OSCILLATIONS OF A EUROPEAN CALL OPTION IN A TREE MODEL

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It is well known that the price of a European vanilla option computed in a binomial tree model converges toward the Black-Scholes price when the time step tends to zero. Moreover, it has been observed that this convergence is of order $1/n$ in usual models and that it is oscillatory. In this paper, we compute this oscillatory behavior using asymptotics of Laplace integrals, giving explicitly the first terms of the asymptotics. This allows us to show that there is no asymptotic expansion in the usual sense, but that the rate of convergence is indeed of order $1/n$ in the case of usual binomial models since the second term (in $1/\sqrt{n}$) vanishes. The next term is of type $C_2(n)/n$, with $C_2(n)$ some explicit bounded function of n that has no limit when n tends to infinity.

KEY WORDS: European options, oscillations, binomial trees, asymptotic expansions, Laplace integral.

1. INTRODUCTION

There are mainly three kinds of methods to compute the price of financial derivatives: tree methods, numerical methods for solving partial differential equations, and Monte Carlo methods. The tree methods are the simplest and are frequently used, mainly because they are easy to understand and can be used safely. Actually, their simplicity can precisely be considered as being a good reason to use them.

Moreover, if one considers the Black-Scholes model, which is also very popular, it is well known that the price (which we shall denote by BS) computed with this continuous time model is close to the price obtained with the tree model when the number n of time steps is large, as BS is the limit of the tree model price. If one is, as we are, partial to tree methods, continuous-time model prices can be understood as useful approximations of discrete-time model prices.

And yet only few results exist on the convergence, such as its speed (how many steps are needed to obtain some given precision?), its nature, monotonic or oscillatory (does

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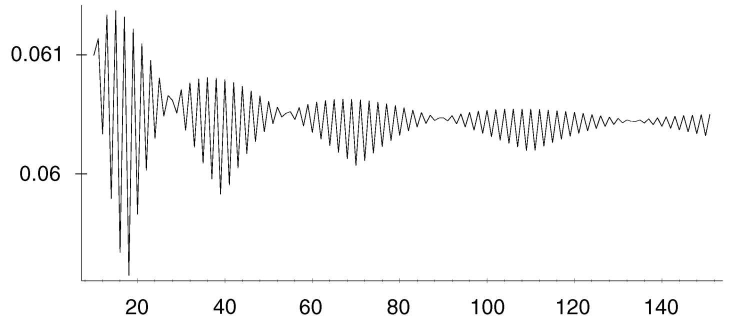

FIGURE 1.1. Price $C(n)$ of a European call option as a function of the number n of time steps (the horizontal line is the Black-Scholes price, toward which $C(n)$ tends as n tends to infinity). One can observe scalloped lines with cusp points on the two enveloping lines and intervals where the distance to the limit decreases as n increases followed by intervals where this distance increases again (same model as in Figure 4.3).

the obtained result underestimate or overestimate the limit?), or concerning the effect of the position of the nodes of the tree with respect to the barrier (in case it is a barrier option) or to the exercise price (is it advisable to choose the tree in such a way that some nodes coincide with these values?).

One may understand why the answers to these questions are not easy by looking at Figure 1.1, which shows the price $C(n)$ of a European call option plotted as a function of the number n of time steps. One observes rather irregular oscillations, and yet the values of $C(n)$ for even values of n and those for odd values of n seem to line up along two curves that envelop the oscillations. Furthermore these curves exhibit amazing cusp points (scallops).

One way to answer these questions would be to give the equations of these two curves, or at least to compute an approximation of them. This is precisely the aim of this paper. To that purpose, we introduce a new method for computing the asymptotics of the price of an option as a function of the number n of time steps. It consists of replacing the binomial sums that this price exhibits with Euler integrals, and giving estimates of these integrals using an extended Laplace method. This method, quite natural for asymptotics of Euler integrals, leads to explicit results but involves heavy straightforward computations with Gaussian integrals. Thanks to a technical result (Theorem 3.4), these computations can be passed to `Maple` (the worksheet is available on our webpage: http://www.math.unice.fr/~diener).

The existence of oscillations in the price $C(n)$ is known by users of tree methods and has been pointed out by various authors (Boyle 1986; Hilliard and Schwartz 1996, who usually have tried to benefit from the oscillations to improve their computations of the price, yet complete explanations have not been provided. Leisen and Reimer (1996) have given a nice explanation of the scallops.

In order to prove the $\frac{1}{n}$ convergence, the Markovian method introduced by Talay and Tubaro (1990), and described in Kloeden and Platen (1992) has been followed. With this technique Talay and Tubaro established order $\frac{1}{n}$ convergence of $\mathbb{E}(f(S_T^n))$, where S_t^n is a discretization of a solution of a stochastic differential equation such as $dS_t = \sigma S_t dW_t$, for any smooth function f. However, as Leisen and Reimer (1996) pointed out, this cannot be used as such here for a call option payoff function that is only piecewise linear; but in their published version Leisen and Reimer did not provide the complete proof on how to overcome this difficulty.[1] A convergence of order $\frac{1}{n}$ when f is smooth is also underlined by Heston and Zhou (2000). The Talay and Tubaro method takes advantage of the fact that, if one considers the random walk as a discretization (according to an Euler scheme) of a continuous-time stochastic process, an expectation like $\mathbb{E}(f(S_T))$ can be considered as the value at one point of the solution of some partial differential equation. This yields an estimate of the difference with the limiting value of the scheme as the sum of approximation errors along one solution of this equation, using in a Taylor expansion the control that one has on successive derivatives of the solutions. Generalized by Bally and Talay (1996a, 1996b), this method actually gives an asymptotic expansion in powers of $\frac{1}{n}$, even in the case when f is merely measurable, but only in the case when the Euler scheme uses the increments of a Brownian motion. This is not the case for a binomial walk for which the increments are Bernoulli random variables. It is also with this idea that Gobet (1999), for the case of barrier options and under these same general hypothesis for f, obtained a nontrivial order 1 asymptotic expansion, in powers of $\frac{1}{\sqrt{n}}$. Lamberton (1999) has also obtained an estimate of order $\frac{1}{n}$ in case f is merely a Lipschitz continuous function, with bounded second-order derivative (in the sense of distributions), but under the assumption that the discrete underlying asset is of type $\sum_{s=1}^{n} X_s/\sqrt{n}$, for a family of i.i.d. random variables X_s such that $\mathbb{E}(X) = \mathbb{E}(X^3) = 0$, $\mathbb{E}(X^2) = 1$, and $\mathbb{E}(X^4) < \infty$, conditions that are also not satisfied here, as the choice of the martingale probability $p(n)$ has to be different from $\frac{1}{2}$ (it is a function of n).

We will show, among other things, that in the case of the usual binomial models the difference with the limit BS, for a European call option, has the form $\frac{C_2(n)}{n} + o(\frac{1}{n})$, where $C_2(n)$ is a function that remains bounded as n tends to infinity. In the case of the Cox-Rubinstein (1985) model, Walsh and Walsh (2002) obtained a similar result using Skorohod embedding. An explicit computation of $C_2(n)$ provides an excellent approximation of the equation of the two enveloping curves that one perceives in Figure 1.1. The asymptotics of the price obtained shows that the rate of convergence of $C(n)$ is indeed of order $\frac{1}{n}$ (and not of order $\frac{1}{\sqrt{n}}$ as was suggested by Heston and Zhou 2000), but we will see that the function $C_2(n)$ has no limit as n tends to infinity; in Section 4 we give an explicit expression in two particular cases (Corollaries 4.1 and 4.2). This explains why attempts to show the existence of an asymptotic expansion of type BS $+ \frac{C_2}{n} + o(\frac{1}{n})$ remained fruitless, for such an expansion cannot exist if $C_2(n)$ has no limit, as is the case here. In fact, we shall see that the price $C(n)$ is equal to the difference of two terms, the rate of convergence of each, taken separately, is of order $\frac{1}{\sqrt{n}}$, and the rate of convergence of $C(n)$ is of order $\frac{1}{n}$ only because of the cancellation, in the difference, of the two terms of order $\frac{1}{\sqrt{n}}$, that precisely balance each other. Additionally, this cancellation phenomenon does not occur for higher order terms, and the expansion, beyond the term $\frac{C_2(n)}{n}$ exhibits a term $\frac{C_3(n)}{n\sqrt{n}}$ that generally does not vanish. Hence, the expansion is not an expansion in integer powers of $\frac{1}{n}$.

2. A GENERAL BINOMIAL MODEL

In this section we first recall how to price a European vanilla option in a general Cox-Ross-Rubinstein binomial model, and we introduce some notations that will be used in the sequel. Then we state the main result of this paper: this price admits, with respect to

[1] The authors provide on demand a complementary paper. This 21-page paper is very technical; moreover, some proofs seem to contain gaps.

the number n of time steps, an asymptotic expansion of a somewhat unusual type, which we shall call an *asymptotic expansion with bounded coefficients.*

2.1. Model for the Underlying Asset

Using the approach of Cox, Ross, and Rubinstein (1979), that was inspired by a suggestion of the economist W. Sharpe, one adopts the following *finite random walk* as the dynamics for the price (S_t) of the underlying asset:

- a finite set of time instants $t \in \{0, \delta t, \ldots, n\delta t\} =: [0..T]_{\delta t}$, with $T = n\delta t$ (and, thus, $\delta t = T/n$)
- an initial value S_0 (at $t = 0$)
- a dynamic characterized by the existence, for each time step, of exactly two possibilities for the next step, the present price S_t of the asset being multiplied by a factor $U_{t+\delta t} := \frac{S_{t+\delta t}}{S_t}$ either equal to u (for *up*) or to d (for *down*), with the condition

  $$d < e^{r\delta t} < u, \tag{2.1}$$

  where $r \geq 0$ is the riskless interest rate.
- the factors $U_t = \frac{S_t}{S_{t-\delta t}}$ are i.i.d. Bernoulli random variables. One puts $p := P(U_t = u)$ and, consequently, $1 - p = P(U_t = d)$.

The natural choice for p is defined below; with this choice the price process (S_t) is, therefore, for each $t = \nu\delta t$, a binomial random variable, assuming $\nu + 1$ values:

$$S_\nu^j := S_0 u^j d^{\nu - j}, \qquad j = 0, \ldots, \nu,$$

with probability

$$P(S_{\nu\delta t} = S_\nu^j) = \binom{\nu}{j} p^j (1-p)^{\nu - j}.$$

2.2. The "Exact Formula" for the Price of a European Option

Let (C_t) be the price, at time t, of a European option with exercise date $T = n\delta t$ and payoff function $\varphi(S_T)$. In the (discrete) Cox-Ross-Rubinstein model, this price is equal, as for the (continuous) Black-Scholes model, to the value of a self-financing portfolio of final value $C_T = \varphi(S_T)$. Under the hypothesis of absence of arbitrage, simple reasoning allows computation of the value of such a *hedging portfolio* by backward induction from its final value. The price is then just the (conditional) expectation of the present value of the payoff $e^{-r(T-t)}\varphi(S_T)$, provided the probability p is chosen such that

$$pu + (1-p)d = e^{rT/n}. \tag{2.2}$$

With this value of p, the process $\tilde{S}_t := e^{-rt} S_t$ becomes a martingale. This probability p is usually called the martingale probability or risk-neutral probability.

As one can observe, this probability p depends on n, as does the term $e^{rT/n}$:

$$p = p(n) = \frac{e^{rT/n} - d}{u - d}. \tag{2.3}$$

As is shown below, u and d will also be chosen depending on n, as they will be expressed as a function of $\delta t = T/n =: \delta t(n)$. Consequently, $S_\nu^j := S_0 u^j d^{\nu-j} = S_\nu^j(n)$ will also depend on n. But we shall no longer write this dependance on n, and shall adopt the notations

$$\delta t, \quad u, \quad d, \quad S_\nu^j, \quad p, \quad \text{and, later,} \quad k \quad \text{and} \quad q$$

for $\delta t(n)$, $u(n)$, $d(n)$, $S_\nu^j(n)$, $p(n)$, $k(n)$, and $q(n)$.

Now, denote by $C(n)$ the price, at time *0*, of a European option with payoff $\varphi(S_T)$, when $T = n\delta t$. One has

$$C(n) = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \varphi(S_0 u^j d^{n-j}). \tag{2.4}$$

In particular, for a call option, with $\varphi(S_T) = (S_T - K)^+ (= S_T - K \text{ if } S_T \geq 0, \text{else } 0)$, one can write this value as the difference of two terms

$$C(n) = S_0 \sum_{j=k}^{n} \binom{n}{j} \left(pue^{-r\frac{T}{n}}\right)^j \left((1-p)de^{-r\frac{T}{n}}\right)^{n-j} - Ke^{-rT} \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j},$$

the sums beginning at k, where $k = k(n)$ is the smallest integer j such that $S_0 u^j d^{n-j} > K$. Let

$$q = q(n) = pue^{-r\frac{T}{n}}. \tag{2.5}$$

From the martingale relation (2.2), one deduces that $(1-p)de^{-r\frac{T}{n}} = 1 - q$, and thus the price $C(n)$ can finally be written as

$$C(n) = S_0 \Phi(n, k, q) - Ke^{-rT} \Phi(n, k, p), \tag{2.6}$$

where Φ denotes the *incomplete binomial sum*

$$\Phi(n, k, p) := \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}. \tag{2.7}$$

The formula (2.6) for the option price $C(n)$, called *exact pricing formula*by Cox and Rubinstein, is very similar to the famous Black-Scholes formula. As for this formula, one recognizes in (2.6) the two parts of the hedging portfolio, one in the underlying asset and one in cash. There is however an important difference between the two formulas. Here $C(n)$ depends on the integer parameter n, whereas the second is independent of n. It is well known that, when n tends to infinity, the "limit" of the Cox-Ross-Rubinstein model is the Black-Scholes model (see, e.g., Musiela and Rutkowski 1997 or Van den Berg 2000). Therefore, one has

$$\lim_{n \to \infty} C(n) = \text{BS},$$

where BS denotes the price of a call option for the continuous model; that is,

$$\text{BS} := S_0 \mathcal{N}(d_1) - Ke^{-rT} \mathcal{N}(d_2)$$

with $d_1 = (\ln \frac{S_0}{K} + (r + \frac{\sigma^2}{2})T)/\sigma\sqrt{T}$, $d_2 = (\ln \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T)/\sigma\sqrt{T}$, and $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy$. Of course we will find this result again in our asymptotic computations. But to understand the oscillations that can be observed (see Figure 1.1), one has to study more carefully the difference $C(n) - \text{BS}$ when n goes to infinity.

2.3. The Main Result

Cox and Rubinstein introduced the discrete model where $u = e^{+\frac{\sigma}{\sqrt{n}}}$ and $u = 1/d = e^{-\frac{\sigma}{\sqrt{n}}}$, which is natural in view of the continuous lognormal model, and the condition $d < e^{r\delta t} = e^{r\frac{T}{n}} < u$ is also satisfied for large enough n. For such a model, the first idea for studying the price $C(n)$ when n tends to infinity is to look for an asymptotic *expansion in powers* of $1/\sqrt{n}$ (or $1/n$?) of this function of n. As a matter of fact, we shall see that *there exists no expansion in the usual meaning*, even in such a simple case; nevertheless we shall also see that it is possible to compute explicitly an asymptotic approximation of this price in a somewhat more general meaning, provided u and d have some asymptotic properties.

Therefore we make the following assumptions on the random walk (S_t):

(2.8) $\qquad u$ and d have a convergent expansion in powers of $\dfrac{1}{\sqrt{n}}$ of type

$$(2.9)\qquad \begin{aligned} u(n) = u &:= 1 + \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n} + O\left(\frac{1}{n\sqrt{n}}\right), \\ d(n) = d &:= 1 - \frac{\sigma}{\sqrt{n}} + \frac{\nu}{n} + O\left(\frac{1}{n\sqrt{n}}\right), \qquad \sigma \neq 0. \end{aligned}$$

These assumptions include the Cox-Rubinstein model where $\mu = \nu = \sigma^2/2$ (see Corollary 4.1). As we shall see, the effect of the coefficients μ and ν, which are known to disappear in the limit, affects in fact the oscillations of the price (see Figure 4.4 in Section 4).

If (2.8) and (2.9) hold for u and d, the quantities p and q in the "exact formula" (2.6) for $C(n)$ also have an asymptotic expansion in powers of $1/\sqrt{n}$, so it seems reasonable to expect an expansion in powers of $1/\sqrt{n}$ for $C(n)$. Actually the difficulty comes from k; by definition $k = k(n)$ is the smallest integer j such that $S_0 u^j d^{n-j} > K$. Now introduce the quantity

$$(2.10)\qquad a(n) := \frac{\ln(K/S_0) - n \ln d(n)}{\ln u(n) - \ln d(n)},$$

which is the solution of $S_0 u^a d^{n-a} = K$. We can rewrite $k(n)$ as

$$(2.11)\qquad k(n) = [a(n)] + 1 = a(n) + 1 - \{a(n)\},$$

where [.] denotes the integer part and {.} the fractional part. Under our assumptions on u and d, $a(n)$ has indeed an asymptotic expansion, but this is no longer the case for $\{a(n)\}$, which has no limit when n tends to infinity, nor has the function $k(n)$. Nevertheless the fractional part $\{a(n)\}$ remains of course bounded between 0 and 1. So, according to (2.11), the presence in formula (2.6) of such a term as $\{a(n)\}$, which has no expansion but remains bounded, leads us to introduce an extended asymptotic calculus.

DEFINITION 2.1. Let $(f_i)_{i\geq 0}$ be a sequence of bounded functions of $\varepsilon > 0$; we shall say that a function $f(\varepsilon)$ has an asymptotic expansion in powers of ε with coefficients $(f_i)_{i\geq 0}$ if, for any $m \geq 0$,

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-m}\left(f(\varepsilon) - \sum_{i=0}^{m} f_i(\varepsilon)\varepsilon^i\right) = 0.$$

The term $f_i(\varepsilon)\varepsilon^i$ is called the *term of order i* of the expansion.

Let us stress that there is no uniqueness for the expansion with bounded coefficients of a given function. Observe that the values $f(\varepsilon)$ and $f_i(\varepsilon)$ may be numbers, but also, more generally, may be elements of a normed vector space, such as the space $\mathcal{L}^1$ of integrable functions on $\mathbb{R}$, as will be the case in the proof of the technical Theorem 3.4.

THEOREM 2.1. *Assume u and d satisfy assumptions (2.8) and (2.9). Then the price $C(n)$ of a European call option for the Cox-Ross-Rubinstein model has an* asymptotic expansion with bounded coefficients *of type*

$$C(n) = C_0 + \frac{C_1}{\sqrt{n}} + \frac{C_2(n)}{n} + \frac{C_3(n)}{n\sqrt{n}} + \cdots$$

with $C_0 = \mathrm{BS}$,

$$C_1 = \frac{\mu - \nu}{2\sqrt{2\pi}} T S_0 e^{-\frac{d_1^2}{2}}, \quad \text{with } d_1 = \left(\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T\right) \Big/ \sigma\sqrt{T}, \tag{2.12}$$

and, for $i = 2, 3, \ldots$, $C_i(n)$ are bounded functions that can be computed explicitly from the corresponding expansions of u and d by the extended Laplace method described below. This shows in particular that C_1 does not depend on n and is zero if and only if $\mu = \nu$.

As mentioned in the Introduction, the enveloping curves are described in first approximation by $C_2(n)$. Its value in the general case is rather lengthy (a sum of 40 terms or so) and we shall give it in two particular cases; see Corollaries 4.1 and 4.2. The next section is devoted to the proof of this theorem, up to the proof of the technical Theorem 3.4 which is postponed to Appendix A.

3. PROOF OF THE MAIN RESULT: AN EXTENDED LAPLACE METHOD

The proof of Theorem 2.1 to which this section is devoted consists in giving an integral representation of the incomplete binomial sums that show up in the classical form of the price of the vanilla options in discrete models. These integrals turn out to be Laplace integrals; therefore it is possible to use the Laplace method to compute their asymptotics, which are known to give particularly accurate approximations even for not so large values of the parameter (here n). Recall that the price $C(n)$ is given by the *exact formula* (2.6):

$$C(n) = S_0\Phi(n, k(n), q(n)) - Ke^{-rT}\Phi(n, k(n), p(n)),$$

where Φ is the incomplete binomial sum $\Phi(n, k, p) := \sum_{j=k}^{n} \binom{n}{j} p^j(1-p)^{n-j}$.

3.1. An Integral Version of the CRR Formula

The first idea needed to compute this expansion is contained in the following lemma, which turns out to be a *magic formula*.[2] This formula will allow us to rewrite the exact formula (2.6) in an *integral form*.

LEMMA 3.1. *For all $n \in \mathbb{N}$, and all k, $0 < k \leq n$, one has the following identity:*

$$\sum_{j=k}^{n} \binom{n}{j} p^j(1-p)^{n-j} = k\binom{n}{k}\int_0^p y^{k-1}(1-y)^{n-k}\,dy.$$

[2] We thank Adri Olde Daalhuis, who drew our attention on the virtues of the incomplete Beta function (Abramowitz and Stegung (1965, pp. 263, 944).

Proof. This identity follows immediately from an elementary $(n-k)$-fold integration by parts of its right member. □

Using this formula, the price (2.6) can thus be written in the following integral form, to which we will be able to apply an adapted version of the *Laplace method*. This new version of the Cox-Ross-Rubinstein exact formula is a key point in the proof of Theorem 2.1.

PROPOSITION 3.2 (INTEGRAL VERSION OF THE COX-ROSS-RUBINSTEIN FORMULA).

$$C(n) = k(n)\binom{n}{k(n)} \times \left(S_0 \int_0^{q(n)} y^{k(n)-1}(1-y)^{n-k(n)}\, dy - Ke^{-rT} \int_0^{p(n)} y^{k(n)-1}(1-y)^{n-k(n)}\, dy \right). \tag{3.1}$$

3.2. A Frozen Parameter

The second important point[3] for the asymptotic computations that we tackle here—and this will be of the utmost importance when leaving the task to `Maple`—is to deal with a *frozen* parameter κ, as we will explain now. First, introduce the notation $\bar{\kappa}(n) := \{a(n)\}$ for the fractional part of the quantity $a(n)$ defined by (2.10). As u and d have an expansion in powers of $\frac{1}{\sqrt{n}}$ of type (2.9), an elementary computation shows that the quantities p, q, a, defined by (2.3), (2.5), (2.10), and the integer k have an asymptotic expansion in powers of $\frac{1}{\sqrt{n}}$ (with bounded coefficients, in the case of k). The next statement follows from an elementary computation. The proof is left to the reader.

PROPOSITION 3.3. *Assume that u and d satisfy (2.9), then*

$$\begin{aligned}
p(n) &= \frac{1}{2} + \frac{p_1}{\sqrt{n}} + \frac{p_2}{n} + O\left(\frac{1}{n\sqrt{n}}\right), \\
q(n) &= \frac{1}{2} + \frac{q_1}{\sqrt{n}} + \frac{q_2}{n} + O\left(\frac{1}{n\sqrt{n}}\right), \\
a(n) &= \frac{1}{2}n + a_{-1}\sqrt{n} + a_0 + O\left(\frac{1}{\sqrt{n}}\right), \\
k(n) &= \frac{1}{2}n + a_{-1}\sqrt{n} + a_0 + 1 - \bar{\kappa}(n) + O\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= \frac{\sqrt{T}}{4\sigma}(2r - (\mu + \nu)) \\
q_1 &= \frac{\sqrt{T}}{4\sigma}(2r - (\mu + \nu) + 2\sigma^2) \\
a_{-1} &= \frac{\sqrt{T}}{4\sigma}\left(\frac{2}{T}\ln(K/S_0) - (\mu + \nu) + \sigma^2\right) \\
a_0 &= -\frac{T}{8\sigma^2}\left(\frac{2}{T}\ln(K/S_0) - (\mu + \nu) - \sigma^2\right)(\mu - \nu).
\end{aligned}$$

[3] It was Laurent Prigneaux, student of the Ecole des Mines de Paris, and trainee at the project Omega of Inria Sophia-Antipolis during the spring of 1999, who first introduced a notation into his computations that "froze" what we denote here by $\bar{\kappa}(n)$.

Now, as it is impossible to expand $\bar{\kappa}(n)$, the idea of dealing with a frozen parameter consists of performing all computations of the asymptotic expansion of $C(n)$ always considering $\bar{\kappa}(n)$ as fixed and equal to κ, and to "remember" only at the end that $\kappa = \bar{\kappa}(n)$ is a bounded function of n without limit. This turns the asymptotic expansion we obtain into an asymptotic expansion with bounded coefficients (bounded but varying with n as they depend on κ).

Let us introduce some notations. Define[4]

$$k(n, \kappa) := \frac{n}{2} + a_{-1}\sqrt{n} + \bar{\alpha}\left(\frac{1}{\sqrt{n}}\right) + 1 - \kappa, \tag{3.2}$$

with $\bar{\alpha}(\varepsilon) = \sum_{i=0}^{\infty} a_i \varepsilon^i$ analytic at $\varepsilon = 0$; κ is a parameter, and $k(n) = k(n, \bar{\kappa}(n))$. Let

$$c(n, \kappa) := \frac{2^{1-n}}{\sqrt{n}} k(n, \kappa) \binom{n}{k(n, \kappa)} \tag{3.3}$$

$$\theta(n, y, \kappa) := 2^{n-1}\sqrt{n}\, y^{k(n,\kappa)-1}(1-y)^{n-k(n,\kappa)} \tag{3.4}$$

$$I^p(n, \kappa) := \int_0^{p(n)} \theta(n, y, \kappa)\, dy, \tag{3.5}$$

and similarly for $I^q(n, \kappa)$. Formula (3.1) can be written

$$C(n) = S_0 c(n, \bar{\kappa}(n)) I^q(n, \bar{\kappa}(n)) - Ke^{-rT} c(n, \bar{\kappa}(n)) I^p(n, \bar{\kappa}(n)). \tag{3.6}$$

In order to get the asymptotic expansion with bounded coefficients that we want, we first compute an ordinary asymptotic expansion of the quantity

$$C(n, \kappa) := S_0 c(n, \kappa) I^q(n, \kappa) - Ke^{-rT} c(n, \kappa) I^p(n, \kappa) \tag{3.7}$$

uniformly valid for all κ, $\kappa \in \mathcal{K}$, $\mathcal{K} = [0, 1]$ compact (see Wasow 1965), *uniformly asymptotic series*. When, finally, we substitute $\bar{\kappa}$ for κ, the uniformly asymptotic series will become the announced expansion with (not constant but) bounded coefficients, as these coefficients will be continuous functions of the parameter κ (actually polynomial functions).

3.3. Applying the Extended Laplace Method

To compute an expansion of the integral

$$\int_0^{p(n)} y^{k(n)-1}(1-y)^{n-k(n)}\, dy,$$

notice first that, as the first term of the asymptotic expansion of $k(n)$ is $\frac{n}{2}$, it is natural to rewrite the integrand in the following way:

$$y^{k(n)-1}(1-y)^{n-k(n)} = (y(1-y))^{\frac{n}{2}} \left(\frac{y}{1-y}\right)^{k(n)-\frac{n}{2}-1} \frac{1}{1-y},$$

[4] We use here, for simplicity, the same character for different functions, when they do not have the same number of variables. For example, we saw that the choice of $u(n)$ or $d(n)$ leads to a definition of $\bar{\kappa}(n)$. Now, for any function f of *two* variables n and κ, we define $f(n)$ (with only *one* variable) by $f(n) := f(n, \bar{\kappa}(n))$.

Let $h(y) := \frac{1}{2}\ln(y(1-y))$ and $g(y) := (\frac{y}{1-y})^{k(n)-\frac{n}{2}-1}\frac{1}{1-y}$; the integral is thus a *Laplace integral:*

$$\int_0^p e^{nh(y)}g(y)\,dy. \tag{3.8}$$

Let us first recall the principle of the usual Laplace method. One performs the change of variable $Y = (y-y_0)\sqrt{n}$, which is a blowup centered at the maximum y_0 of the function $h(y)$. With the new variable Y, the integral becomes, with $y_0 = \frac{1}{2} = p_0$,

$$\int_0^p e^{nh(y)}g(y)\,dy = \int_{-\frac{\sqrt{n}}{2}}^{(p-\frac{1}{2})\sqrt{n}} e^{nh(\frac{1}{2}+\frac{Y}{\sqrt{n}})}g\left(\frac{1}{2}+\frac{Y}{\sqrt{n}}\right)\frac{dY}{\sqrt{n}}$$

and, thus, as $h'(\frac{1}{2}) = 0$,

$$\int_0^p e^{nh(y)}g(y)\,dy = \frac{e^{nh(\frac{1}{2})}}{\sqrt{n}}\int_{-\frac{\sqrt{n}}{2}}^{(p-\frac{1}{2})\sqrt{n}} e^{-\frac{Y^2}{2}(-h''(\frac{1}{2})+\dots)}g\left(\frac{1}{2}+\frac{Y}{\sqrt{n}}\right)dY.$$

A Taylor expansion of h and g leads to an asymptotic expansion in powers of $\frac{1}{\sqrt{n}}$ of the integrand of this new integral; after integrating term by term, one gets an expansion with Gaussian integrals as coefficients. It suffices then to check that term-by-term integration leads indeed to an asymptotic expansion for the integral.

In the case under consideration here, the change of variable $Y = (y-\frac{1}{2})\sqrt{n}$ leads (3.5) to the integral

$$I^p(n,\kappa) = \int_{-\frac{\sqrt{n}}{2}}^{P(n)} \Theta(n, Y, \kappa)\,dY, \tag{3.9}$$

where

$$P(n) := \left(p(n) - \frac{1}{2}\right)\sqrt{n} \tag{3.10}$$

and

$$\Theta(n, Y, \kappa) := \left(1 - \frac{(2Y)^2}{n}\right)^{\frac{n}{2}}\left(\frac{1+\frac{2Y}{\sqrt{n}}}{1-\frac{2Y}{\sqrt{n}}}\right)^{k(n,\kappa)-\frac{n}{2}-1}\frac{1}{1-\frac{2Y}{\sqrt{n}}}. \tag{3.11}$$

Now, using $\exp\circ\ln = \mathrm{Id}$, one gets

$$\Theta(n, Y, \kappa) = \frac{1}{1-2Y/\sqrt{n}}\exp\Bigg[\frac{n}{2}\ln(1-(2Y)^2/n) + (k(n,\kappa)-n/2-1)(\ln(1+2Y/\sqrt{n}) - \ln(1-2Y/\sqrt{n})\Bigg].$$

As, by (3.2), $k(n,\kappa) - n/2 - 1 = a_{-1}\sqrt{n} + \bar{\alpha}(1/\sqrt{n}) - \kappa$, this makes it possible, for any integer $i_0 \geq 0$, to compute explicitly a formal expansion $\Theta_{i_0}(n, Y, \kappa)$ of $\Theta(n, Y, \kappa)$ in powers of $1/\sqrt{n}$ up to order i_0. So, for any fixed Y and κ, Θ_{i_0} is such that

$$\Theta(n, Y, \kappa) = \Theta_{i_0}(n, Y, \kappa) + O\left(\left(\frac{1}{\sqrt{n}}\right)^{i_0+1}\right).$$

The task of determining explicitly Θ_0, Θ_1, and so forth, can be left to `Maple`. In this way one gets

$$\Theta_0(n, Y, \kappa) = \exp\big(-2Y^2 + 4a_{-1}Y\big) =: F(Y), \tag{3.12}$$

$$\Theta_1(n, Y, \kappa) = \Theta_0(n, Y, \kappa) + 2(2a_0 + 1 - \kappa) Y \frac{F(Y)}{\sqrt{n}}, \tag{3.13}$$

and

$$\Theta_2(n, Y, \kappa) = \Theta_1(n, Y, \kappa) + 4\left(-Y^4 + \frac{4}{3}a_{-1}Y^3 + 2\left(a_0^2 + a_0(1-2\kappa) + \kappa(\kappa-1) + \frac{1}{2}\right)\right.$$
$$\left. \times\ Y^2 + a_1 Y\right)\frac{F(Y)}{n}.$$

To compute any expansion of $I^p(n, \kappa)$ (or $I^q(n, \kappa)$) as expressed in (3.9) we may use Θ_{i_0}; this is the purpose of the following technical theorem.

THEOREM 3.4 (TECHNICAL). *Assume $u(n)$ and $d(n)$ satisfy the hypotheses (2.8) and (2.9) and $p = p(n)$ given by (2.3), as in Theorem 2.1, and let $\mathcal{K} := [0, 1]$. Then the integral $I^p(n, \kappa)$ has an expansion in powers of $1/\sqrt{n}$, uniformly valid for $\kappa \in \mathcal{K}$. More precisely, if $P_{i_0}(n)$ and $\Theta_{i_0}(n, Y, \kappa)$ denote the truncated at order i_0 asymptotic expansion in powers of $1/\sqrt{n}$ of $P(n)$ and $\Theta(n, Y, \kappa)$, then, uniformly for $\kappa \in \mathcal{K}$,*

$$I^p(n, \kappa) = \int_{-\infty}^{P_{i_0}(n)} \Theta_{i_0}(n, Y, \kappa)\, dY + O(1/(\sqrt{n})^{i_0+1}). \tag{3.14}$$

Moreover one has $P_{i_0}(n) := p_1 + \frac{p_2}{n^{1/2}} + \cdots + \frac{p_{i_0+1}}{n^{i_0/2}}$, where $p_0, p_1, p_2, \ldots$ denote the coefficients in the expansion of $p(n)$ and

$$\Theta_{i_0}(n, Y, \kappa) = e^{-2Y^2 + 4a_{-1}Y} \sum_{i=0}^{i_0} \Psi_i(Y, \kappa)\left(\frac{1}{\sqrt{n}}\right)^i,$$

where the $\Psi_i(Y, \kappa)$ are polynomial in Y and κ.

The expansion of the integral in (3.14) can be performed term by term.

We shall show this result in the Appendix.

3.4. Proof of Theorem 2.1

Proof. In order to prove that $C(n)$ has an asymptotic expansion with bounded coefficients, recall that by (3.6) we have

$$C(n) = S_0 c(n, \bar{\kappa}(n)) I^q(n, \bar{\kappa}(n)) - Ke^{-rT} c(n, \bar{\kappa}(n)) I^p(n, \bar{\kappa}(n)),$$

with $\bar{\kappa}(n) = \{a(n)\}$, and $c(n, \kappa)$ defined in (3.3) by

$$c(n, \kappa) = \frac{2^{1-n}}{\sqrt{n}} k(n, \kappa) \binom{n}{k(n, \kappa)}.$$

By (3.14) in the Technical Theorem 3.4, the expansion of $I^p(n, \kappa)$ up to order i_0 is given by

$$I^p(n, \kappa) = \int_{-\infty}^{P_{i_0}(n)} \Theta_{i_0}(n, Y, \kappa)\, dY + O\left(\tfrac{1}{(\sqrt{n})^{i_0+1}}\right),$$

and similarly for $I^q(n, \kappa)$. Thus, to end the proof, it suffices to compose the expansion (3.2) of $k(n, \kappa) = n/2 + a_{-1}\sqrt{n} + \bar{\alpha}(1/\sqrt{n}) + 1 - \kappa$ with the expansion of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ given by Stirling's formula, and observe that $C(n)$ is a sum of two products of functions admitting expansions uniformly valid for $\kappa \in [0, 1]$, and finally replace κ by $\bar{\kappa}(n)$.

For the effective values of C_0, C_1, and $C_2(n)$, we have used `Maple`, which, up to some elementary rearrangings, provided the announced results. (The reader may find our `Maple` worksheet on our webpage.) The crucial point in guiding `Maple` in its work was to suggest it use term-by-term integration, as this is allowed by the Technical Theorem 3.4.

4. MORE RESULTS AND COMMENTS

4.1. Application to Most Popular Models

Our main theorem (Theorem 2.1) states the existence of an expansion

$$C(n) = C_0 + \frac{C_1}{\sqrt{n}} + \frac{C_2(n)}{n} + \frac{C_3(n)}{n\sqrt{n}} + \cdots$$

that can be computed explicitly (e.g., using `Maple`) whatever the choice of u and d, provided that they have an asymptotic expansion in powers of $1/\sqrt{n}$ of type (2.9). This assumption is very general and is satisfied by most of the models that have been studied in the literature—for example, the Cox Rubinstein (1985) model

$$u = e^{\frac{\sigma\sqrt{T}}{\sqrt{n}}} \quad \text{and} \quad d = e^{\frac{-\sigma\sqrt{T}}{\sqrt{n}}},$$

the Tian (1993) model

$$u = \frac{1}{2}Rv(v + 1 + \sqrt{v^2 + 2v - 3}) \quad \text{and} \quad d = \frac{1}{2}Rv(v + 1 - \sqrt{v^2 + 2v - 3}),$$

where $R = e^{\frac{rT}{n}}$ and $v = e^{\frac{\sigma^2 T}{n}}$, or the Van den Berg (2000) model

$$u = 1 + \sigma\sqrt{\frac{T}{n}} + \mu\frac{T}{n} \quad \text{and} \quad d = 1 - \sigma\sqrt{\frac{T}{n}} + \mu\frac{T}{n}.$$

But notice that Theorem 2.1 cannot be applied to the Jarrow-Rudd (1983) model, even if the existence of an expansion of the same type probably still exists in that case. Indeed, even if the Jarrow-Rudd model does satisfy our assumptions on u and d as

$$u = e^{(r-\frac{\sigma^2}{2})\frac{T}{n} + \frac{\sigma\sqrt{T}}{\sqrt{n}}} \quad \text{and} \quad d = e^{(r-\frac{\sigma^2}{2})\frac{T}{n} - \frac{-\sigma\sqrt{T}}{\sqrt{n}}},$$

a problem arises from the choice of $p(n) = \frac{1}{2}$ that does not satisfy the martingale relation (2.2), because p should be

$$p(n) = \frac{1}{2} + \frac{1}{24}\sigma^3 T^{3/2}\frac{1}{n\sqrt{n}} + O\left(\frac{1}{n^2\sqrt{n}}\right)$$

in that case. Therefore, even if $1/2$ coincides with the martingale value of $p(n)$ up to order 2 (in $\frac{1}{n}$), this does not suffice to allow us to apply the theorem here.

The main consequence of the theorem is the fact that on the one hand an asymptotic expansion in the usual sense (with constant coefficients) exists up to order 1 (because C_1 is independent of n—and is zero if $\mu = \nu$), but on the other hand, even for the simplest models, such an expansion does not exist for any order greater than 1 because $C_2(n)$, $C_3(n)$, ..., do depend on n and do not converge to any limit.

4.2. The Zero-Order Term

It is well known that the standard version of the Central Limit (CL) theorem (the one that can be found in usual books on probability) is not general enough to handle convergence of the CRR price $C(n)$ to the Black-Scholes price when n tends to infinity. The reason for this is that, although $C(n)$ can be expressed as a function of the sum of $\{X_{\delta t}, \ldots, X_{n\delta t}\}$, where $X_t := \ln U_t$ are independent Bernoulli random variables taking the two values $\ln u(n)$ and $\ln d(n)$ with probability $p(n)$ and $1 - p(n)$ respectively, they are not identically distributed as they depend on n. Nevertheless the Lindeberg-Feller extension of the CL theorem is sufficient here for, according to this extension, such a triangular family of random variables converges in distribution to a normally distributed random variable in $\mathcal{N}(m, \sigma)$ provided that this family is bounded by a nonrandom sequence vanishing as n tends to infinity and that both the expectation and variance converge, to m and σ^2 respectively. For $C(n)$, under the hypothesis of the theorem, these two moments can be written as

$$p(n)\ln u(n) + (1 - p(n))\ln d(n) = \left(r - \frac{\sigma^2}{2}\right)T - \frac{\sigma}{2}T^{3/2}(\mu - \nu)\frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$

$$p(n)(1 - p(n))(\ln u(n) - \ln d(n))^2 = \sigma^2 T + \sigma T^{3/2}(\mu - \nu)\frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right),$$

and thus this implies the convergence in distribution to the Black-Scholes price. In particular, the zero-order term of the expansion of $C(n)$ we have computed could have been obtained without computations by using, as usual, the Lindeberg-Feller extension of the CL theorem and it depends only on the choice of σ in u and d satisfying (2.9).

4.3. First-Order Term

The fact that the zero-order term does not depend on the exact value of u and d is a consequence of the facts that only the two first coefficients of the expansion of u and d, namely 1 and $\pm\sigma$, are relevant for C_0 and that they coincide in all models satisfying assumption (2.9). But when we look at higher order terms, for instance C_1 or C_2, then higher order terms of u and d become relevant, namely the second-order coefficients or apparent drift term μ and ν, and they are not necessarily the same for two different models. This explains why, even when working with no-arbitrage models, the (apparent) drift terms are relevant in the CRR setting. Figure 4.1 shows that the CRR price and the Tian price are not equal even if they both converge to the BS price.

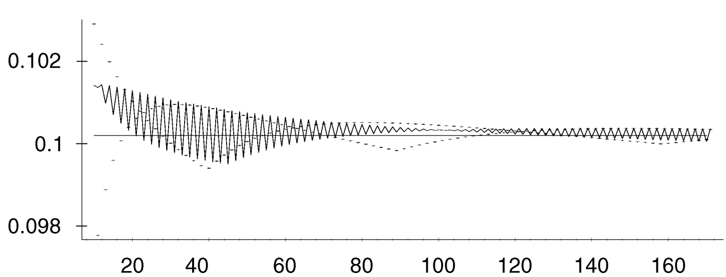

FIGURE 4.1. Price of a call option as a function of n for two different models: the Cox-Rubinstein model and the Tian model. In order to distinguish the two prices, the CRR price of Figure 1.1 is represented here just with points. For each n these two prices are different, even if both tend to the BS price (represented as a horizontal line) each exhibiting, when n varies, its own scalloped pattern.

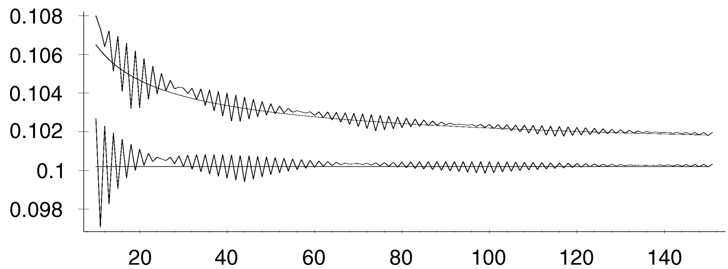

FIGURE 4.2. Prices of a call option as a function of n for two models of type $u(n) = 1 + \sigma/\sqrt{n} + \mu/n$ and $d(n) = 1 - \sigma/\sqrt{n} + \nu/n$. The upper oscillating line corresponds to the case $\nu \neq \mu$; the lower one corresponds to the case $\nu = \mu$. The nonoscillating lines correspond in each case to the first-order asymptotic (i.e., up to terms in $1/\sqrt{n}$). One observes that this asymptotic reduces to the limit BS (horizontal line) in the case $\mu = \nu$, and if $\mu \neq \nu$ the term in $1/\sqrt{n}$ does not oscillate. In other words, in any case the amplitude of the oscillations is of order $1/n$. Observe the slow convergence; note that all lines tend to the same value BS, even if $1/\sqrt{n}$ convergence tends to hide this fact in the figure.

Let us just add some comments on the explicit expression (2.12) of C_1 we have obtained.

- The expression of C_1, and more precisely its factor $(\mu - \nu)$, shows that the convergence of the binomial price to the Black-Scholes price is actually of order $\frac{1}{n}$ for all models, provided that they satisfy $\mu = \nu$, as is the case for all usual ones. But for models with $\mu \neq \nu$ this will no longer be true as can be seen in Figure 4.2, for in this case, the rate of convergence is $\frac{1}{\sqrt{n}}$, hence much slower than $\frac{1}{n}$. Nevertheless, considering models with $\mu \neq \nu$ can be viewed from a practical point of view like an artificial extension except perhaps in a more general setting like models involving proportional transaction costs of similar magnitude as the apparent trend $\mu\delta t$ or as the interest $r\delta t$ on one time period.
- The difference $C(n) - \text{BS}$ is of order $\frac{1}{n}$ provided that $\mu = \nu$ is compatible with the result obtained by Leisen and Reimer (1996). Indeed, they have computed an upper bound for this difference of the form

$$C(n) - \text{BS} \leq C^{ste}\left(n\left(m_n^2 + m_n^3 + p_n\right) + \frac{1}{n}\right)$$

  with $m_n^2 = pu^2 + (1-p)d^2 - e^{2(r-\frac{\sigma^2}{2})\frac{T}{n}}$, $m_n^3 = pu^3 + (1-p)d^3 - e^{3(r-\frac{\sigma^2}{2})\frac{T}{n}}$, and $p_n = p\ln u(u-1)^3 + (1-p)\ln d(d-1)^3$. In our general model, it easy to show that

$$n\left(m_n^2 + m_n^3 + p_n\right) + \frac{1}{n} = 4\sigma T^{3/2}(\mu - \nu)\frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right).$$

  The given upper bound for $C(n) - \text{BS}$ is thus of order $\frac{1}{n}$ if and only if $\mu = \nu$ (as soon as the model belongs to our general model).
- In a more recent paper, Leisen and Reimer (2000) discussed the conclusions obtained by Heston and Zhou (2000) where these latter authors proved that the rate of convergence of $C(n)$ to BS is $\frac{1}{\sqrt{n}}$ and produced some arguments allowing them to consider that this can probably not be improved. One knows now that this is not true (except if $\mu \neq \nu$). But if one considers separately the expansions of the two terms of $C(n)$ (recall that $C(n) = S_0\Phi(n,k,q) - Ke^{-rT}\Phi(n,k,p)$) then, as Leisen

and Reimer pointed out, the order of any of them is $\frac{1}{\sqrt{n}}$ and thus the claim of Heston and Zhou becomes true for each term separately. More precisely, the same computations we have performed show that, under the same hypothesis as for Theorem 2.1, the asymptotic expansions with bounded coefficients of each term

$$S_0\Phi(n,k,q) = S_0\mathcal{N}(d_1) + \frac{\Phi_1(n)}{\sqrt{n}} + \cdots$$

$$Ke^{-rT}\Phi(n,k,p) = Ke^{-rT}\mathcal{N}(d_2) + \frac{\Phi_1(n)}{\sqrt{n}} + \cdots$$

have the same coefficient for the term in $\frac{1}{\sqrt{n}}$. In the difference, the two first-order terms cancel out and this produces the $\frac{1}{n}$ convergence toward the limit. Notice that the coefficient $\Phi_1(n)$ depends on n because its expression contains $\kappa(n)$. Thus, thanks to this cancellation the price $C(n)$ admits an asymptotic expansion (with constant coefficients) up to order 1; $C(n)$ is in fact the difference of two terms that do not admit such expansions separately.

- In the particular case of the Cox-Rubinstein model, one can observe that the order 1 term can be obtained using the result of Gobet (1999); indeed a call option can be viewed as a down-and-in call with barrier L equal to the present price S_0 of the underlying asset. In this case, the "overshoot" (i.e., the difference between the barrier L and the closest node line equal to or below L) vanishes, and this overshoot appears as a factor of the $\frac{1}{\sqrt{n}}$-order term in the expansion given by Gobet. This is consistent with our result for C_1, which vanishes also, as $\mu = \nu$ in the case of the Cox-Rubinstein model.

4.4. Second-Order Term and Higher Order Terms

The explicit formulas given by the two following corollaries allow us to test the quality of the approximation of the price by the asymptotic expansion with bounded coefficient given by Theorem 2.1. For any value n, we will compare the quantity $C(n)$, its limit when n tends to infinity (or zero-order approximation), and its second-order approximation given by the truncated expansion.

COROLLARY 4.1 (COX-RUBINSTEIN MODEL). *If the underlying asset is such that* $u = e^{\frac{\sigma}{\sqrt{n}}}$ *and* $d = e^{\frac{-\sigma}{\sqrt{n}}}$, *then* $\mu = \nu = \frac{1}{2}\sigma^2$ *and the price at* $t = 0$ *of a call option with value* $(S_0u^jd^{n-j} - K)^+$ *at time* $T = n\delta t$ *satisfies the formula*

$$C(n,\kappa) = \mathrm{BS} - S_0e^{-\frac{d_1^2}{2}}\sqrt{\frac{2}{\pi}}\{\sigma\sqrt{T}\kappa(\kappa-1) + D_1\}\frac{1}{n} + O\left(\frac{1}{n\sqrt{n}}\right),$$

where BS is the Black-Scholes price, $d_1 := \frac{1}{\sigma\sqrt{T}}(\ln\frac{S_0}{K} + (r + \frac{\sigma^2}{2})T)$,

$$D_1 := \frac{1}{96\sigma\sqrt{T}}\left(4\left(\ln\frac{S_0}{K}\right)^2 - 8rT\ln\frac{S_0}{K} + 3T(4\sigma^2 - 12rT^2 - \sigma^4T)\right),$$

and $\kappa = \kappa(n)$ *is the fractional part of* $a(n) := \frac{\ln(K/S_0) - n\ln d(n)}{\ln u(n) - \ln d(n)}$.

Figure 4.3 shows, at (a), the second-order approximation of $C(n)$ as a function of n, plotted separately for odd and even values of n, and, at (b), the plottings of the price $C(n)$, together with its second-order approximation. One observes that it is difficult to see the difference between $C(n)$ and its second-order approximation, even for very small values of n. For the same values of the parameters as for Figure 4.3, one has the following data:

n	$C(n)$	Zero-order approx. (BS)	Second-order approx.
50	0.06060695478	0.06040088125	0.06060099755
150	0.06031978406	0.06040088125	0.06031902046

The formula of Corollary 4.1 allows us to understand the oscillations in the price of a call option and the cusps that can be observed for specific values of n. Indeed, recall that the function $\bar{\kappa}(n)$ is the fractional part of $a(n) = \frac{\ln(K/S_0)-n\ln d}{\ln u-\ln d}$, which has an asymptotic expansion of type $a(n) = \frac{1}{2}n + a_{-1}\sqrt{n} + O(\frac{1}{\sqrt{n}})$ in the model under consideration here. The first term gives no contribution to κ for even n (as its fractional part is zero), and, on the contrary, brings a contribution $\frac{1}{2}$ to κ for odd n. This explains the oscillations of order $\frac{1}{n}$ between the even and the odd values of n in the price $C(n)$. Moreover, for values of n with the same parity, when $a(n)$ is far from an integer value, its fractional part κ changes continuously, but κ will have a discontinuity each time $a(n)$ crosses an integer. It is easy to check on the definition of $a(n)$ that this happens at the cusp points on the picture,

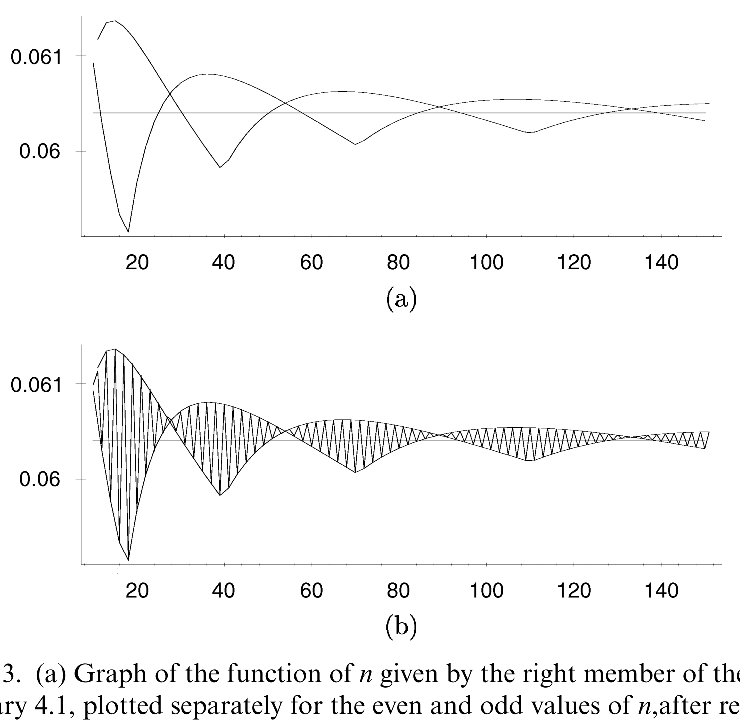

FIGURE 4.3. (a) Graph of the function of n given by the right member of the formula of Corollary 4.1, plotted separately for the even and odd values of n,after replacing κ by its value as a function of n (κ is the fractional part of $a(n)$), thus neglecting the tail-term $O(\frac{1}{n\sqrt{n}})$. The chosen model is the Cox-Rubinstein one (i.e., $u = \exp(\sigma/\sqrt{n})$ and $d = 1/u$); the values of the parameter are $T = 1$, $S_0 = 1$, $K = 1.1$, $r = 0.05$, and $\sigma = 0.2$. (b) Simultaneous plotting of the value of $C(n)$ given by the Cox and Rubinstein exact formula (formulas 2.6 and 2.7) and its second-order approximation already plotted at (a). The difference between exact and second-order approximation is difficult to see. The horizontal line is for the Black-Scholes price BS, limit of $C(n)$ when n tends to infinity (or zero-order approximation).

which give it its scalloped aspect. Observe that, in the case of the at-the-money call—that is, when $K = S_0$—one has $a(n) = \frac{n}{2}$ (as $u = 1/d$ in the model under consideration here) and thus the price $C(n)$ oscillates, as in the general case, but the asymptotic expansion with bounded coefficient is much easier to compute (see Diener and Diener 1999) as κ is simply equal to 0 for even n and equal to $\frac{1}{2}$ for odd n. Thus, there is no scallop in this case.

We finally provide the explicit second-order term for the Van den Berg model.

COROLLARY 4.2. *If the underlying asset is such that* $u = 1 + \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ *and* $d = 1 - \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$, *then the price at* $t = 0$ *of a European call with value* $(S_0 u^j d^{n-j} - K)^+$ *at* $T = n\delta t$ *satisfies*

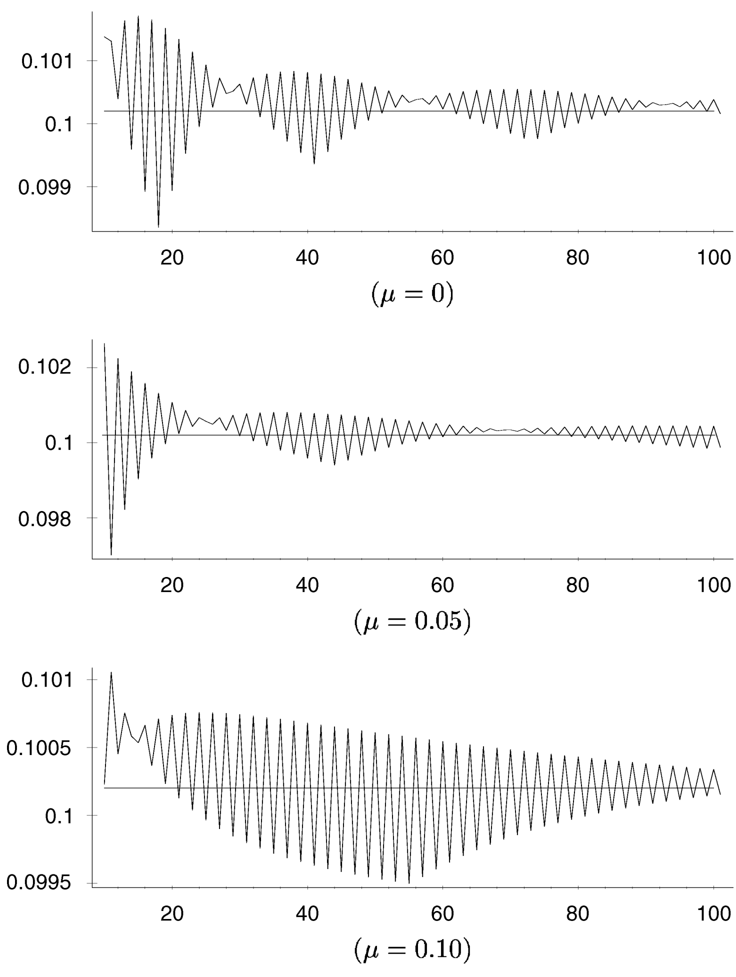

FIGURE 4.4. Oscillations of the option price (formulas (2.6) and (2.7)) in a Cox-Ross-Rubinstein model, where $u = 1 + \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ and $d = 1 - \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ for three different values of the (apparent) drift μ. The values of the other parameters are $T = 1$, $S_0 = 1$, $K = 1.1$, $r = 0.05$, and $\sigma = 0.3$.

the formula

$$C(n,\kappa) = \mathrm{BS} + S_0 e^{-\frac{d_1^2}{2}} \sqrt{\frac{2}{\pi}} \{\sigma\sqrt{T}\kappa(\kappa-1) + D_2\} \frac{1}{n} + O\left(\frac{1}{n\sqrt{n}}\right),$$

where BS *stands for the Black-Scholes price*, $d_1 := \frac{1}{\sigma\sqrt{T}}(\ln \frac{S_0}{K} + (r + \frac{\sigma^2}{2})T)$,

$$D_2 := \frac{1}{96\sigma\sqrt{T}} \left(4\left(\ln \frac{S_0}{K}\right)^2 + 8T(2\mu - r - \sigma^2)\ln \frac{S_0}{K} \right. \\ \left. + 32r(2\mu - \sigma^2)T^2 - 36r^2T^2 + 24\mu(\sigma^2 - \mu)T^2 + 3\sigma^2 T(4 - 3\sigma^2 T) \right),$$

and $\kappa = \bar{\kappa}(n)$ *is the fractional part of* $a(n) := \frac{\ln(K/S_0) - n\ln d(n)}{\ln u(n) - \ln d(n)}$.

Figure 4.4 shows the oscillations of the CRR price $C(n)$ for three different values of the parameter μ. In this model, as for the Cox-Rubinstein model discussed above, one can check that the expansion obtained with the asymptotic formula (Corollary 4.2), neglecting the tail $O(\frac{1}{n\sqrt{n}})$, gives excellent approximations of the price $C(n)$. The figure shows the influence of the (apparent) drift μ on the CRR prices, influence that disappears in the BS limit.

5. CONCLUSION

In this paper we have shown that it is possible to compute explicitly the first terms of an *asymptotic expansion with bounded coefficients* for the price of a European call option in a binomial tree model. The second term of this expansion (or the first term after the limit BS) that we obtained confirms the (already-known) fact that the rate of convergence toward the limit is $1/n$ for most usual models. One observes that, even if this second term is $\mathcal{O}(1/n)$, the expansion is in fact in powers of $1/\sqrt{n}$, the first term after the limit being usually zero. The precise expression makes also explicit how this gap depends on the choice of u and d (so as of the value of μ which does not appear in the limit), and thus on the chosen tree model.

Moreover, the obtained expansion shows clearly why the research devoted to conventional expansion of the CRR price could not succeed: Such an expansion cannot exist.

We have good reason to hope that it will be possible to adapt the method we introduced here to compute expansions of the price of options other than vanilla options and make explicit to which extent continuous-time models give good approximations of prices for tree models. We have already been able to reach this target for usual barrier options for CRR models (Diener and Diener 2001) and have begun work on some trinomial trees models that are commonly used to price such options.

A. APPENDIX: PROOF OF THE TECHNICAL THEOREM 3.4

We want to compute the asymptotics of $I^p(n,\kappa) = \int_0^{p(n)} \theta(n,y,\kappa)\,dy$, with

$$\theta(n,y,\kappa) = 2^{n-1}\sqrt{n}\,y^{k(n,\kappa)-1}(1-y)^{n-k(n,\kappa)},$$

as they are defined in (3.5) and (3.4). The first step consists of showing that one can, without changing the asymptotic expansion, reduce the domain of integration $[0, p(n)]$

to a carefully chosen interval $[y_-(n), p(n)]$, which, among other things, tends to $\{1/2\}$. This step is necessary for the proof, but does not need to be explicitly computed; we will show in a later step that, after replacing the integrand by its truncated expansion, we can change once again the domain of integration to a third one (typically $(-\infty, p]$) on which the computations may be performed in a naive way.

We shall make use of the notation

$$\left(\frac{1}{\sqrt{n}}\right)^{\infty}$$

to denote a function of n and κ with identically zero expansion in powers of $\frac{1}{\sqrt{n}}$ in the space $\mathcal{L}^{\infty}_{\mathcal{K}}$ of all continuous functions on $\mathcal{K} = [0, 1]$, normed by $\|f\|^{\infty}_{\mathcal{K}} := \mathrm{Sup}_{\kappa\in\mathcal{K}}|f(\kappa)|$.

A1. Localizing the Integral

LEMMA A.1. *There exists a sequence of real numbers* $y_-(n) \in [0, p(n)] \cap [0, \frac{1}{2})$, *with* $\lim_{n\to+\infty} y_-(n) = \frac{1}{2}$ *and* $e^{-Y_-^2(n)} = (\frac{1}{\sqrt{n}})^{\infty}$, *where* $Y_-(n) := \sqrt{n}(y_-(n) - \frac{1}{2})$, *such that*

$$\hat{I}(n, \kappa) := \int_0^{y_-(n)} \theta(n, y, \kappa)\, dy = \left(\frac{1}{\sqrt{n}}\right)^{\infty} \quad \text{in } \mathcal{L}^{\infty}_{\mathcal{K}}. \tag{A.1}$$

In particular, $\lim_{n\to+\infty} Y_-(n) = -\infty$.

Proof. We will show that we can choose $y_-(n)$ such that $\theta(n, y, \kappa) \le \sqrt{n}^{-\sqrt{n}}$ for any $y \in [0, y_-(n)]$ and any $\kappa \in \mathcal{K}$. To that purpose, consider, for any integer $N > 0$,

$$\mathcal{Y}_N := \left\{\zeta \in \left[0, \tfrac{1}{2}\right] \mid \forall n \ge N \quad \forall y \in [0, \zeta] \quad \forall \kappa \in \mathcal{K} \quad \theta(n, y, \kappa) \le \sqrt{n}^{-\sqrt{n}}\right\}.$$

We build in this way an increasing sequence of sets $(\mathcal{Y}_N)_{N\ge1}$, compact because θ is continuous, and nonempty as $0 \in \mathcal{Y}_N$ for all N. Let

$$y_-(N) := \mathrm{Sup}\, \mathcal{Y}_N.$$

The sequence $y_-(N)$ is increasing by construction, and $y_-(N) \in \mathcal{Y}_N$ by compactness of $\mathcal{Y}_N$, so $\theta(n, y, \kappa) \le \sqrt{n}^{-\sqrt{n}}$ for all $y \in [0, y_-(N)]$, and, therefore,

$$0 \le \hat{I}(n, \kappa) := \int_0^{y_-(n)} \theta(n, y, \kappa)\, dy \le \int_0^{y_-(n)} \sqrt{n}^{-\sqrt{n}}\, dy \le \frac{1}{2}\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = \left(\frac{1}{\sqrt{n}}\right)^{\infty}.$$

To be sure that $e^{-Y_-^2(n)} = (\frac{1}{\sqrt{n}})^{\infty}$ for $Y_-(n) := \sqrt{n}(y_-(n) - \frac{1}{2})$, we have to make sure that $y_-(n)(\le \frac{1}{2})$ does not become too close to $\frac{1}{2}$, which is true after having possibly replaced $y_-(n)$ by $\hat{y}_-(n) := \mathrm{Min}\{y_-(n), \frac{1}{2} - (\frac{1}{n})^{\frac{1}{3}}\}$. So the proof of the lemma is reduced to the proof of the following scholium.

SCHOLIUM A.2. *The sequence* $y_-(N) := \mathrm{Sup}\mathcal{Y}_N$ *tends to* $\frac{1}{2}$.

Proof. This can easily be shown using nonstandard analysis (see, e.g., Diener and Diener 1995 for a tutorial on nonstandard analysis): as $(y_-(N))_{N\ge1}$ is a standard sequence, it suffices to show that $y_-(N) \simeq \frac{1}{2}$ for any infinitely large N, which, in turn, by permanence, reduces to show that for any infinitely large n and any $y \in [0, \frac{1}{2}]$ such that $y \not\simeq \frac{1}{2}$ one has indeed $\theta(n, y, \kappa) \le \frac{1}{\sqrt{n}^{\sqrt{n}}}$.

To check this, let z be such that $y = \frac{1}{2} - z$; thus $0 \leq z \leq \frac{1}{2}$, and $y \simeq \frac{1}{2}$ if and only if $z \simeq 0$. We have

$$\begin{aligned}\theta(n, y, \kappa) &= 2^{n-1}\sqrt{n}y^{k(n,\kappa)-1}(1-y)^{n-k(n,\kappa)}\\ &= 2^{n-1}\sqrt{n}\left(\frac{1}{2}-z\right)^{\frac{n}{2}+l\sqrt{n}}\left(\frac{1}{2}+z\right)^{\frac{n}{2}-l\sqrt{n}}\frac{2}{1+2z} = \sqrt{n}e^{-n\psi(z)},\end{aligned}$$

where, by (3.2), $l := (k(n,\kappa) - 1 - \frac{n}{2})/\sqrt{n} = a_{-1} + \frac{1}{\sqrt{n}}(\bar{\alpha}(\frac{1}{\sqrt{n}}) - \kappa)$ is limited (i.e., not infinitely large), and

$$\psi(z) := \ln\frac{1}{\sqrt{1-4z^2}} - \frac{l}{\sqrt{n}}\ln\frac{1-2z}{1+2z} - \frac{1}{n}\ln\frac{1}{1+2z}.$$

It is elementary to see that ψ is a function such that, for any considered z such that $z \not\simeq 0$, one has $\psi(z) > 0$ and $\psi(z) \not\simeq 0$. So, with $\omega := \sqrt{n}$, which is infinitely large, we have $\sqrt{n}e^{-n\psi(z)}/\sqrt{n}^{-\sqrt{n}} = \exp\left(-\omega^2\left(\psi(z) - \frac{\omega+1}{\omega}\frac{1}{\omega}\ln\omega\right)\right) \simeq 0 \leq 1$, and thus $\theta(n, y, \kappa) =: \sqrt{n}e^{-n\psi(z)} \leq \sqrt{n}^{-\sqrt{n}}$.

A2. End of the Proof of the Technical Theorem

Let us choose $y_-(n)$ and $Y_-(n)$ as in Lemma A.1. Let

$$\tilde{I}^p(n,\kappa) := \int_{y_-(n)}^{p(n)} \theta(n, y, \kappa)\, dy. \tag{A.2}$$

Lemma A.1 says precisely that $\tilde{I}^p(n,\kappa)$ has the same $\frac{1}{\sqrt{n}}$-expansion as $I^p(n,\kappa)$. On the interval $[y_-(n), p(n)]$ on which the integral $\tilde{I}^p(n,\kappa)$ is taken, the Laplace method may be applied after proper adapting; define Y by

$$y = \frac{1}{2} + \varepsilon Y, \quad \text{with } \varepsilon = \varepsilon(n) := \frac{1}{\sqrt{n}},$$

which, in particular, changes the "infinitesimal" interval $[y_-(n), p(n)]$ into the "infinitely large" one $[Y_-(n), P(n)]$, with

$$P = P(n) := \frac{1}{\varepsilon}\left(p - \frac{1}{2}\right) = p_1 + p_2\varepsilon + \cdots + p_{i_0+1}\varepsilon^{i_0} + O(\varepsilon^{i_0+1}) = P_{i_0} + O(\varepsilon^{i_0+1}),$$

where

$$P_{i_0} := p_1 + p_2\varepsilon + \cdots + p_{i_0+1}\varepsilon^{i_0}. \tag{A.3}$$

In this way, we obtain that

$$\tilde{I}^p(n,\kappa) \quad = \int_{Y_-(n)}^{P} \Theta(n, Y, \kappa)\, dY$$

with

$$\begin{aligned}\Theta(n, Y, \kappa) &:= e^{\frac{1}{\varepsilon^2}\ln(1-4\varepsilon^2Y^2)}e^{\frac{a_{-1}}{\varepsilon}\ln\frac{1+2\varepsilon Y}{1-2\varepsilon Y}}e^{(\bar{\alpha}(\varepsilon)-\kappa)\ln\frac{1+2\varepsilon Y}{1-2\varepsilon Y}}e^{-\ln(1+2\varepsilon)}\\ &=: H(Y)D(n, Y, \kappa),\end{aligned} \tag{A.4}$$

with

$$H(Y) := e^{-2Y^2+4a_{-1}Y},$$

and

$$\begin{aligned} D(n, Y, \kappa) &:= \Theta(n, Y, \kappa)/H(Y) = e^{\frac{1}{\varepsilon^2}(4\varepsilon^2 Y^2+\ln(1-4\varepsilon^2 Y^2))} \\ &\quad \times e^{\frac{a_{-1}}{\varepsilon}(-4\varepsilon Y+\ln\frac{1+2\varepsilon Y}{1-2\varepsilon Y})} e^{(\bar{\alpha}(\varepsilon)-\kappa)\ln\frac{1+2\varepsilon Y}{1-2\varepsilon Y}} e^{-\ln(1+2\varepsilon)} \\ &= e^{-\beta(\varepsilon,\varepsilon Y, Y,\kappa)}, \end{aligned}$$

with

$$\beta(\varepsilon, \xi, Y, \kappa) = Y^2\beta_1(\xi) + Y\beta_2(\xi) + \kappa\beta_3(\xi) + \beta_4(\varepsilon, \xi),$$

where, for $i = 1.4$, the β_is are analytic functions, and $\beta_1(0) = \beta_2(0) = \beta_3(0) = 0 = \beta_4(0, \xi)$. Now we can see that

$$\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} e^{-\beta(\varepsilon,\varepsilon Y, Y,\kappa)} = e^{-\beta(\varepsilon,\varepsilon Y, Y,\kappa)}\psi_i(\varepsilon, \varepsilon Y, Y, \kappa),$$

where the ψ_i are polynomials in the two last variables (Y, κ), with coefficients that are analytic in the first two ones $(\varepsilon, \xi = \varepsilon Y)$. Now, by Taylor's formula, for each integer $i_0 > 0$, each $Y \in [Y_-(n), P(n)]$, and each $\kappa \in \mathcal{K}$, there exists a $\tau = \tau(i_0, Y, \kappa) \in (0, 1)$ such that

$$\text{(A.5)} \qquad \begin{aligned} D(n, Y, \kappa) = e^{-\beta(\varepsilon,\varepsilon Y, Y,\kappa)} &= \sum_{i=0}^{i_0} \psi_i(0, 0, Y, \kappa)\varepsilon^i \\ &\quad + \varepsilon^{i_0+1} e^{-\beta(\tau\varepsilon,\tau\varepsilon Y, Y,\kappa)}\psi_{i_0+1}(\tau\varepsilon, \tau\varepsilon Y, Y, \kappa). \end{aligned}$$

Moreover, as $\lim_{n\to+\infty} \mathrm{Sup}_{(Y,\kappa)\in\mathbb{R}\times\mathcal{K}} |\tau\varepsilon Y \mathbb{I}_{[Y_-(n),P(n)]}(Y)| = 0$, we have

$$\lim_{n\to+\infty} \psi_{i_0+1}(\tau\varepsilon, \tau\varepsilon Y, Y, \kappa) = \psi_{i_0+1}(0, 0, Y, \kappa)$$

with uniform convergence of the coefficients of this polynomial on $[Y_-(n), P(n)]$, and

$$\lim_{n\to+\infty} \mathrm{Sup}_{(Y,\kappa)\in\mathbb{R}\times\mathcal{K}} \left|\beta(\tau\varepsilon, \tau\varepsilon Y, Y, \kappa)\mathbb{I}_{[Y_-(n),P(n)]}(Y)\right| = 0.$$

Thus,

$$\text{(A.6)} \qquad e^{-\beta(\tau\varepsilon,\tau\varepsilon Y, Y,\kappa)}\psi_{i_0}(\tau\varepsilon, \tau\varepsilon Y, Y, \kappa)\mathbb{I}_{[Y_-(n),P(n)]}(Y) = O(1) \text{ in } \mathcal{L}^{1,\infty},$$

where $\mathcal{L}^{1,\infty}$ is the space of continuous functions of (Y, κ) with norm

$$\|f\|_{1,\infty} = \mathop{\mathrm{Sup}}_{\kappa\in\mathcal{K}} \int |f(Y, \kappa)|\, dM(Y), \quad \text{with} \quad dM(Y) := H(Y)dY = e^{-2Y^2+4a_{-1}Y}\, dY.$$

Finally, let

$$\Psi_i(Y, \kappa) := \psi_i(0, 0, Y, \kappa).$$

Denoting by $O_{\mathcal{K}}(1)$ any quantity with bounded norm in $\mathcal{L}^{\infty}_{\mathcal{K}}$, uniformly for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\text{(A.7)}\qquad \tilde{I}^p(n,\kappa) &= \int D(n,\, Y,\kappa)\mathbb{I}_{[Y_-(n),\,P(n)]}\, dM(Y)\\
&= \sum_{i=0}^{i_0} \varepsilon^i \int_{Y_-(n)}^{P(n)} \Psi_i(Y,\kappa)\, dM(Y)\\
&\quad + \varepsilon^{i_0+1} \int e^{-\beta(\tau\varepsilon,\tau\varepsilon Y,Y,\kappa)}\psi_{i_0+1}(\tau\varepsilon,\,\tau\varepsilon Y,\, Y,\kappa)\mathbb{I}_{[Y_-(n),\,P(n)]} dM(Y)\\
&= \sum_{i=0}^{i_0} \varepsilon^i \int_{-\infty}^{P(n)} \Psi_i(Y,\kappa) dM(Y) + \varepsilon^{i_0+1} O_{\mathcal{K}}(1),
\end{aligned}
$$

because

$$\int_{-\infty}^{Y_-(n)} \Psi_i(Y,\kappa)\, dM(Y) = \left(\frac{1}{\sqrt{n}}\right)^{\infty} \quad \text{in } \mathcal{L}^{\infty}_{\mathcal{K}},$$

as $\Psi_i(Y,\kappa)$ is a polynomial in Y with continuous (polynomial) functions in κ as coefficients, and as $e^{-Y_-^2} = (\frac{1}{\sqrt{n}})^{\infty}$. Finally, putting pieces together using (A.1) and (A.5), we get

(A.8)

$$
\begin{aligned}
I^p(n,\kappa) &= \hat{I}^p(n,\kappa) + \tilde{I}^p(n,\kappa)\\
&= \left(\frac{1}{\sqrt{n}}\right)^{\infty} + \sum_{i=0}^{i_0} \varepsilon^i \int_{-\infty}^{P(n)} \Psi_i(Y,\kappa) e^{-2Y^2+4a_{-1}Y}\, dY + \varepsilon^{i_0+1} O_{\mathcal{K}}(1)\\
&= \sum_{i=0}^{i_0} \varepsilon^i \int_{-\infty}^{P_{i_0}} \Psi_i(Y,\kappa) e^{-2Y^2+4a_{-1}Y}\, dY + \sum_{i=0}^{i_0} \varepsilon^i \int_{P_{i_0}}^{P(n)} \Psi_i(Y,\kappa) e^{-2Y^2+4a_{-1}Y}\, dY\\
&\quad + \varepsilon^{i_0+1} O_{\mathcal{K}}(1), \quad \text{with } P_{i_0} \text{ as in } (A.3)\\
&= \int_{-\infty}^{P_{i_0}} \sum_{i=0}^{i_0} \Psi_i(Y,\kappa)\varepsilon^i e^{-2Y^2+4a_{-1}Y}\, dY + \varepsilon^{i_0+1} O_{\mathcal{K}}(1)\\
&\quad + \varepsilon^{i_0+1} O_{\mathcal{K}}(1), \quad \text{as} \quad P(n) - P_{i_0} = \varepsilon^{i_0+1} O(1)\\
&= \int_{-\infty}^{P_{i_0}} \sum_{i=0}^{i_0} \Psi_i(Y,\kappa)\varepsilon^i e^{-2Y^2+4a_{-1}Y}\, dY + \varepsilon^{i_0+1} O_{\mathcal{K}}(1)\\
&= \int_{-\infty}^{P_{i_0}} \Theta_{i_0}(n,\, Y,\kappa)\, dY + \varepsilon^{i_0+1} O_{\mathcal{K}}(1), \quad \text{by (A.4), (A.5), and the definition}\\
&\quad \text{of } \Theta_{i_0}(n,\, Y,\kappa).
\end{aligned}
$$

This shows (3.14), and (A.8) shows that the expansion of the integral in (3.14) can indeed be performed term by term, as $\varepsilon = 1/\sqrt{n}$ and

$$\int_{-\infty}^{P_{i_0}} \sum_{i=0}^{i_0} \Psi_i(Y,\kappa)\varepsilon^i e^{-2Y^2+4a_{-1}Y}\, dY = \sum_{i=0}^{i_0} \left(\frac{1}{\sqrt{n}}\right)^i \int_{-\infty}^{P_{i_0}} \Psi_i(Y,\kappa) e^{-2Y^2+4a_{-1}Y}\, dY.$$

This ends the proof of the technical theorem.

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