Sparse Grid Method in the LIBOR Market Model. Option Valuation and the 'Greeks'

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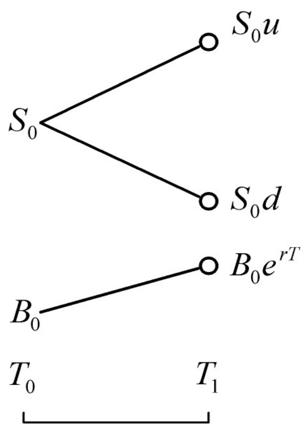
Outline of the Presentation

- Overview of Asset Pricing
    - what is an option?
    - what is option pricing via no-arbitrage?
- LIBOR Market Economy and its PDE
- Sparse Grid and its Combination Technique
- Numerical Results
- Conclusions and Suggestions for Future Work

One-Step Binary Tree.



B : zero-coupon cash bond

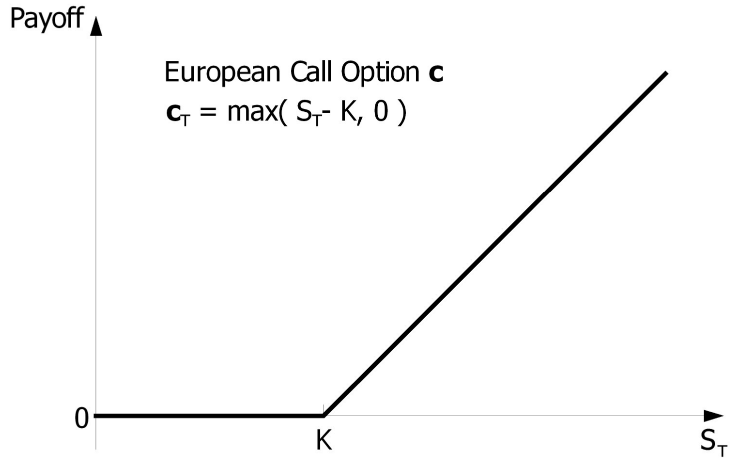
S : a stock

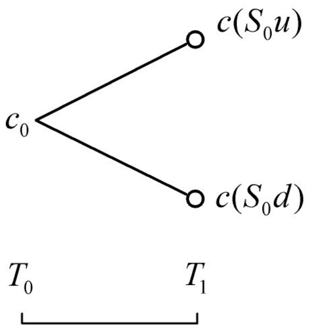
u : upward movement in stock

d : downward movement in stock

r : risk-free rate

$$u > e^{rT} > d$$





$c(S_0u)$: option payoff in case of u

$c(S_0d)$: option payoff in case of d

c_0 : option value at T_0

How can we determine the option price at time zero?

No-arbitrage pricing

Sloppy definition

Arbitrage is a portfolio of underlying assets that gives a chance to positive payouts without initial investment and guarantees no risk of loosing money.

If we can find a portfolio $\boldsymbol{\theta} = \begin{pmatrix} \Delta \\ \psi \end{pmatrix}$ such that

$$c(S_0 u) = \Delta S_0 u + \psi B_0 e^{rT}$$
$$c(S_0 d) = \Delta S_0 d + \psi B_0 e^{rT}$$

Then, the fair price of c_0 must equal :

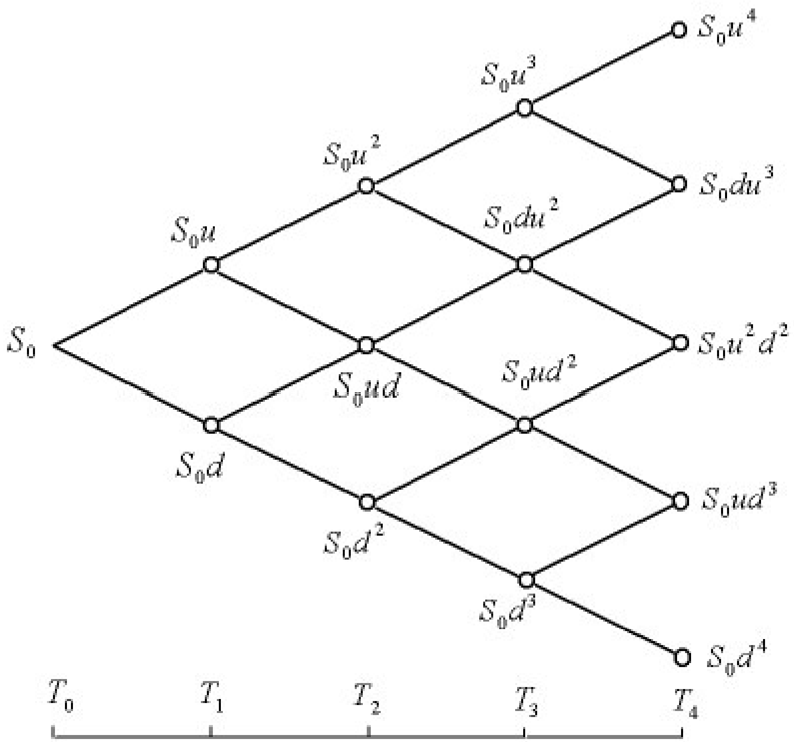
$$c_0 = \begin{pmatrix} S_0 \\ B_0 \end{pmatrix} \cdot \boldsymbol{\theta}^t$$

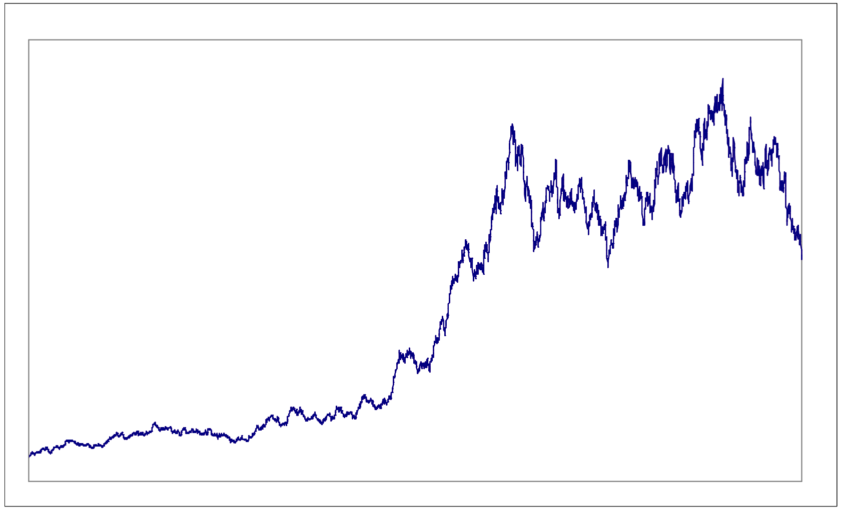
or the cost of setting up $\boldsymbol{\theta}$ at time zero

Key result

Equilibrium price of a derivative instrument is determined by the cost of the setup of its self-financing, replicating portfolio.

From Recombinant Tree to Continuous Time Models





$$dS = \mu S dt + \sigma S dW$$

Black-Scholes Model. Hedge Portfolio.

The idea governing the model is to eliminate risk by taking reverse positions in a derivative and its underlying asset.

$$\Pi = -c + \Delta S$$

Since this position is risk-free, by no-arbitrage principle it must earn a risk-free rate:

$$\mathrm{d}\Pi = r\Pi\,\mathrm{d}t$$

The setup of a hedge portfolio needs cash input. We can generate this cash by borrowing:

$$\Pi_0 = -\psi B_0$$

Now we can obtain a process for a option by differentiating both sides:

$$\mathrm{d}\Pi = \mathrm{d}(-c + \Delta S) = \mathrm{d}(-\psi B)$$

and rearranging:

$$\mathrm{d}c = \Delta\,\mathrm{d}S + r(c - \Delta S)dt$$

Black-Scholes Partial Differential Equation

On the other hand, the process for *c* is a function of time and stock price process. Thus, by Ito's Lemma:

$$\mathrm{d}c = \frac{\partial c}{\partial t}\mathrm{d}t + \frac{\partial c}{\partial S}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 \mathrm{d}t$$

From the previous slide we know a different expression for the same process and two must be the same:

$$\frac{\partial c}{\partial S}\mathrm{d}S + \left[\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right]\mathrm{d}t = \Delta \mathrm{d}S + \left[r(c - \Delta S)\right]\mathrm{d}t$$

Now stock process terms cancel out and, by rearranging, we get:

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial S}rS + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 = rc$$

which is a one-dimensional B-S PDE

'Greeks'

The derivation of BS PDE leads to the result:

$$\Delta = \frac{\partial c}{\partial S}$$

The first greek letter – delta – shows how sensitive the value of the option is to the changes in the underlying asset.

Other sensitivities include:

$$\Gamma = \frac{\partial^2 c}{\partial S^2}$$

Gamma – measures non-linearity or curvature in the option price

$$\theta = \frac{\partial c}{\partial t}$$

Theta – measures time decay in the option price

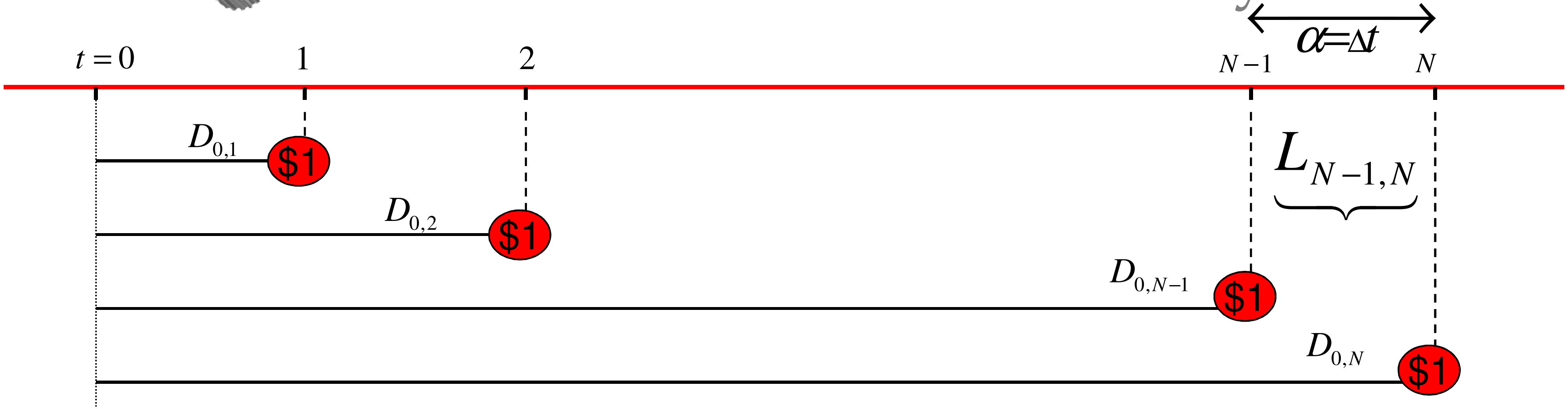
$$\mathrm{V} = \frac{\partial c}{\partial \sigma}$$

Vega – measures sensitivity to changes in volatility of the underlying.
(in BS assumed constant, in practice not so!)

Thus, BS PDE from the previous slide:

$$\theta + \Delta r S + \tfrac{1}{2}\Gamma \sigma^2 S^2 - rc = 0$$

LIBOR Market Model Economy



$$L_{N-1,N}(0) = \frac{1}{\alpha_{N-1,N}}\left(\frac{D_{0,N-1} - D_{0,N}}{D_{0,N}}\right)$$

$$D_{0,N-1} = \left(1 + \alpha_{N-1,N} L_{N-1,N}(0)\right) D_{0,N}$$

LMM assumes :

$$dL_{N-1,N}(t) = \sigma_{N-1,N} L_{N-1,N}(t)\, dW^N$$

LIBOR Market Model PDE

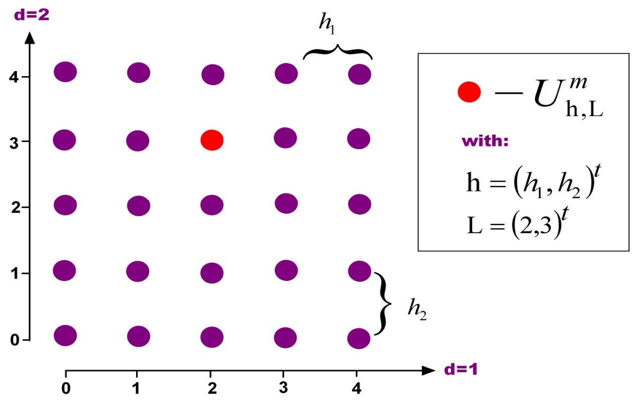
$$\frac{\partial \tilde{U}}{\partial t} + \sum_{i=1}^{N-1} \mu(t) L_i \frac{\partial \tilde{U}}{\partial L_i} + \tfrac{1}{2} \sum_{i,j=1}^{N-1} \frac{\partial^2 \tilde{U}}{\partial L_i \partial L_j} \rho_{i,j} \sigma_i \sigma_j L_i L_j = 0$$

with terminal boundary conditions given by the payoff function of the product:

$$\tilde{U}(T_N, \mathrm{L}) = p(L(T_N))$$

Solving LMM PDE – Finite Difference Method

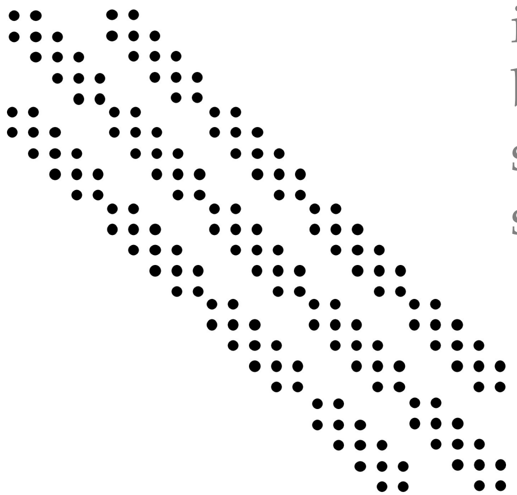
1. Discretize first and second derivative terms on a mesh of points.



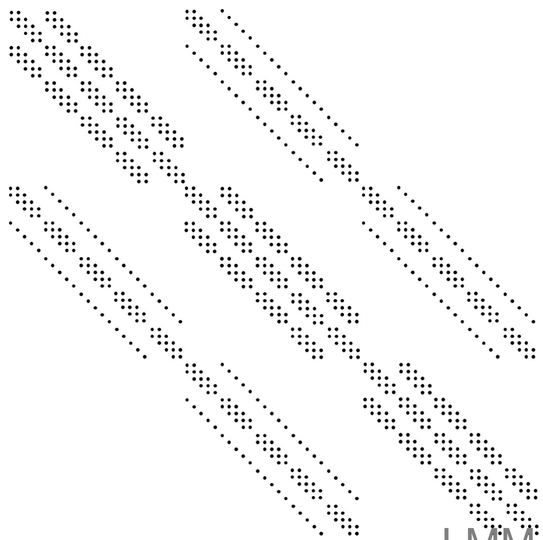
2. Crank-Nicolson discretisation scheme

3. Regrouping terms to arrive at the system of discrete equations :

$$\mathrm{Ax} = \mathrm{b}$$



4. Solve the system iteratively accounting for boundary conditions and stopping times if there are such.



Solving LMM PDE – Finite Difference Method

Curse of dimensionality :
the number of degrees of freedom grows Exponentially!!

2D	3D	5D
65536	16,7x10^6	10,9x10^11

256 points in each dim.

Possible solution – pricing LMM derivatives by **Monte Carlo** simulation

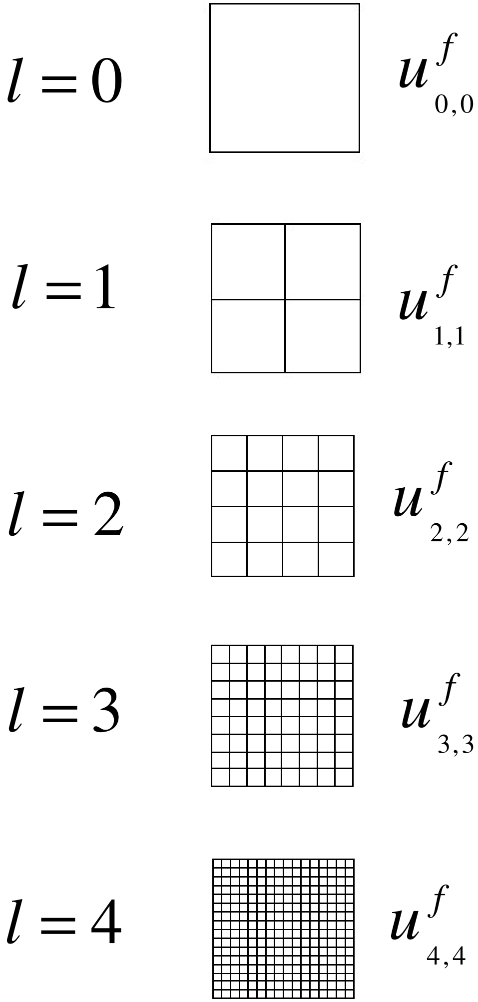
Drawbacks of MC:
1. Time-consuming
2. Point-wise method – each simulation gives a value for a specific set of spot rates
3. Hard to price options with an early-exercise feature.
4. It is hard to find 'Greeks'.

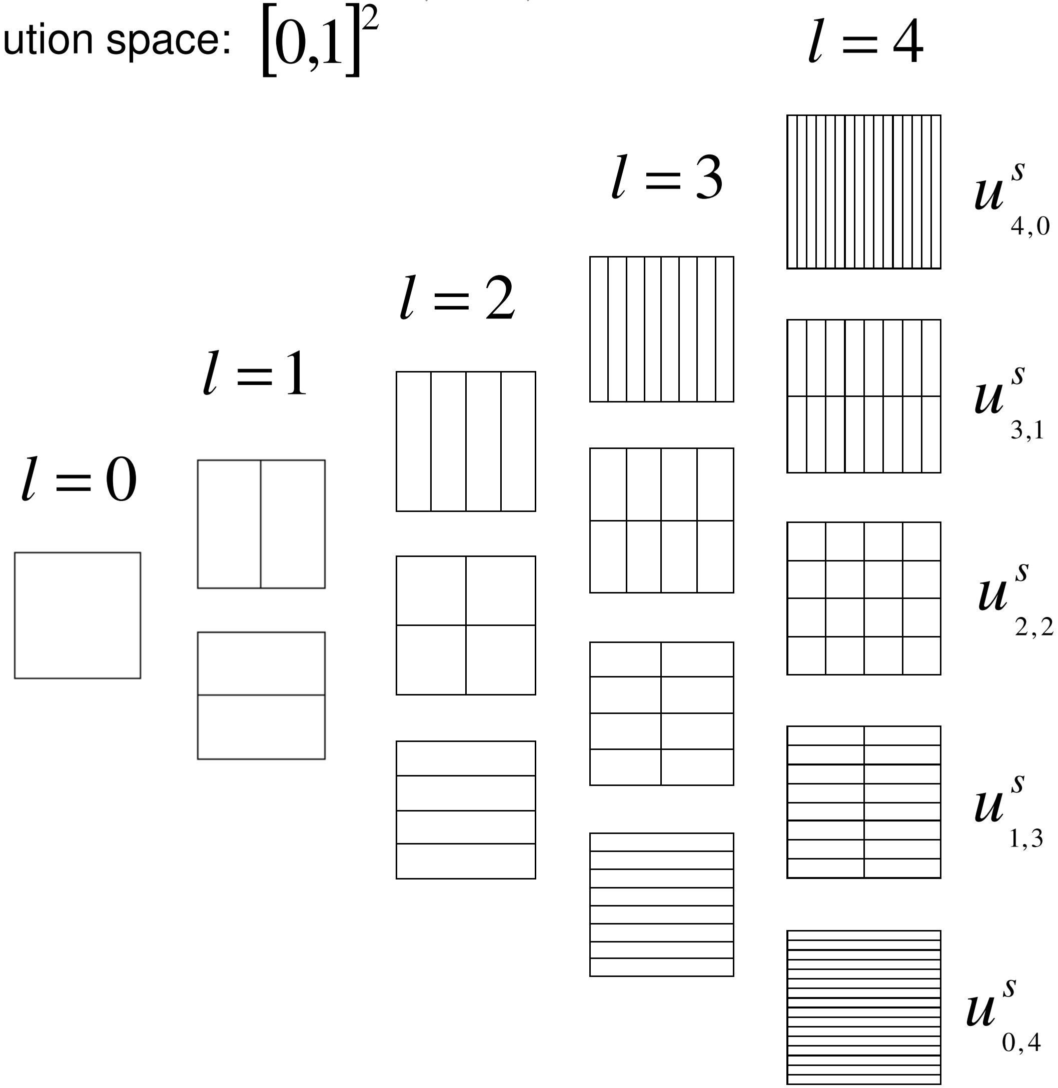
How to revive FD with all its advantages?
Advanced discretization techniques such as..

Sparse Grid Method (2D)

Solution space: $[0,1]^2$

$n = 2^l + 1$ i.e. n^d points total

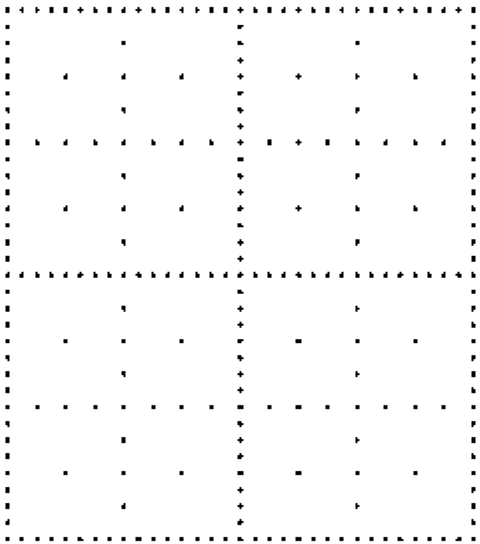


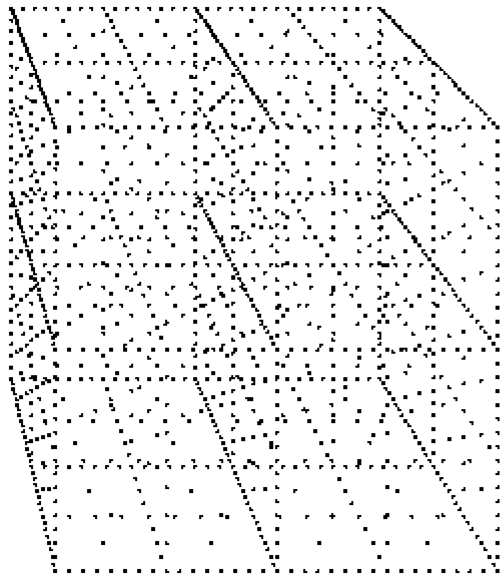


$\begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix}$ grids with n points each

Sparse Grid Method

Sparse grid function representation in 2D and 3D:





	Full Grid	Sparse Grid
Degrees of freedom	2^{2l}	$O(2^l l^{d-1})$
Accuracy	$O(2^{-2l})$	$O(2^{-2l}\log(2^l))$

Sparse Grid Combination Technique

In this master project we have used a combination technique approach introduced in [Griebel et al.,1992]:

2D formula:

$$\hat{u}^{c}_{n,n} = \sum_{i+j=n+1} u_{i,j} - \sum_{i+j=n} u_{i,j}$$

3D formula:

$$\hat{u}^{c}_{n,n} = \sum_{i+j=n+2} u_{i,j} - 2\sum_{i+j=n+1} u_{i,j} + \sum_{i+j=n} u_{i,j}$$

Numerical Results. Valuation and Greeks

Threefold objective for practical work:

1. Independent verification of the findings regarding option valuation for 2D and 3D cases of J.Blackham and exploration of a larger parameter space.
2. Investigation of the behaviour of sparse grid method in case of the discontinuous payoff.

3 A more rigorous study of the quality of the sparse 'Greeks'.

Test cases under consideration:

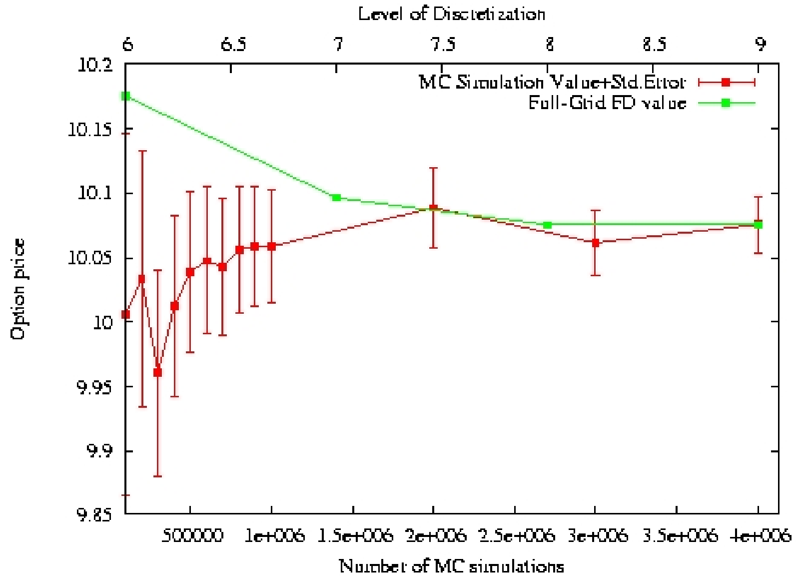
- 2D Chooser Option
- 3D Bermudan Swaption
- 2D Digital Option

Pricing frameworks:

- Monte Carlo
- Finite Difference Solver

2D CHOOSER OPTION

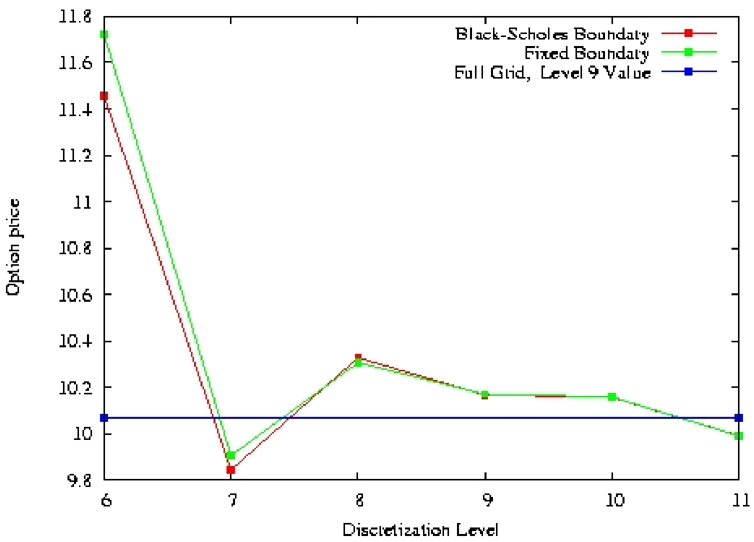
Monte Carlo vs. Full Grid FD



Payoff:

$$V(T_3) = \alpha_2(\max(L_1, L_2) - K)_+$$

Full Grid vs. Sparse Grid convergence



Level	Full time (in sec.)	Sparse time (in sec.)	Full grid points	Sparse grid points
5	1.5	0.36	1086	257
6	6.6	0.88	4225	577
7	30.1	2.16	16641	1281
8	176.4	5.18	66049	2817
9	651.4	12.89	2631169	6145
10		31.64	1050625	13313
11		74.86	4198401	28673

Surface Plots and the ‘Greeks’.

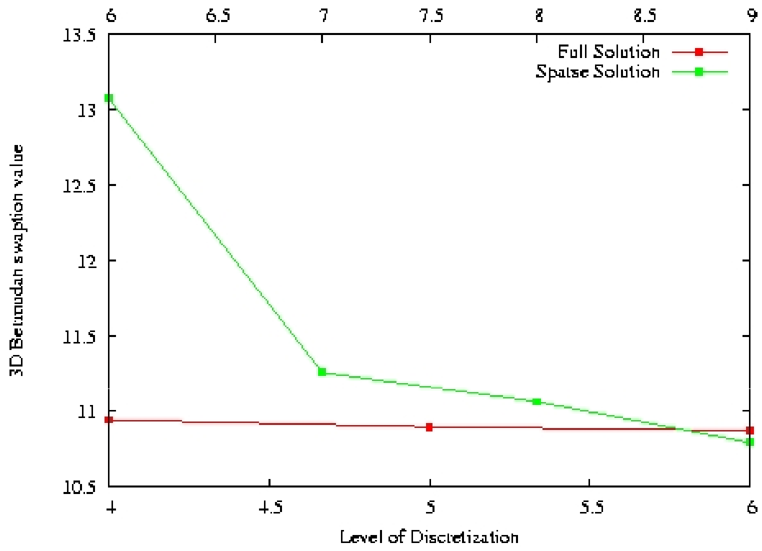
1. Full Grid solution surface
2. Sparse Grid solution surface
3. Error Surface Plot
4. Delta Surface
5. Vega Surface

Observations:

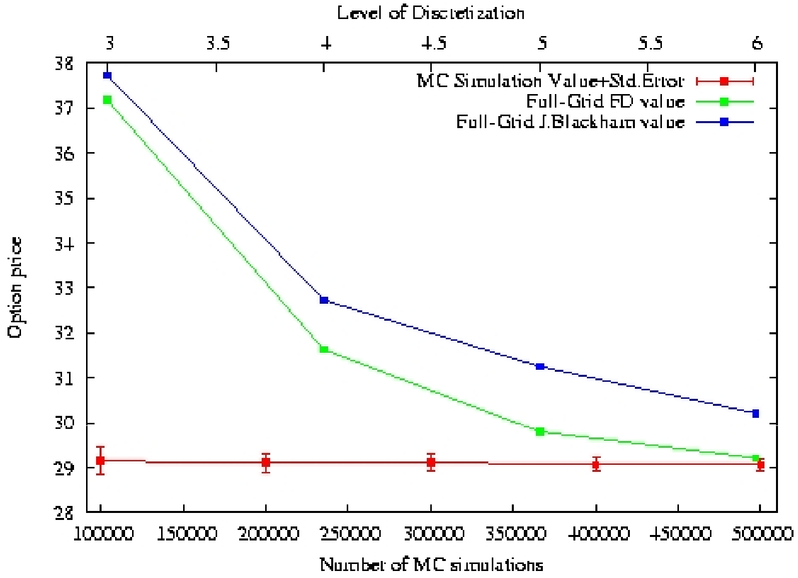
1. Surface plots for full and sparse solutions show close resemblance.
2. Error Surface Plot shows distribution of the error and
3. ‘Wiggles’ are much more distinct when we solve for delta and vega.

3D BERMUDAN SWAPTION.

Full vs. Sparse solution



MC vs. FD and Blackham's

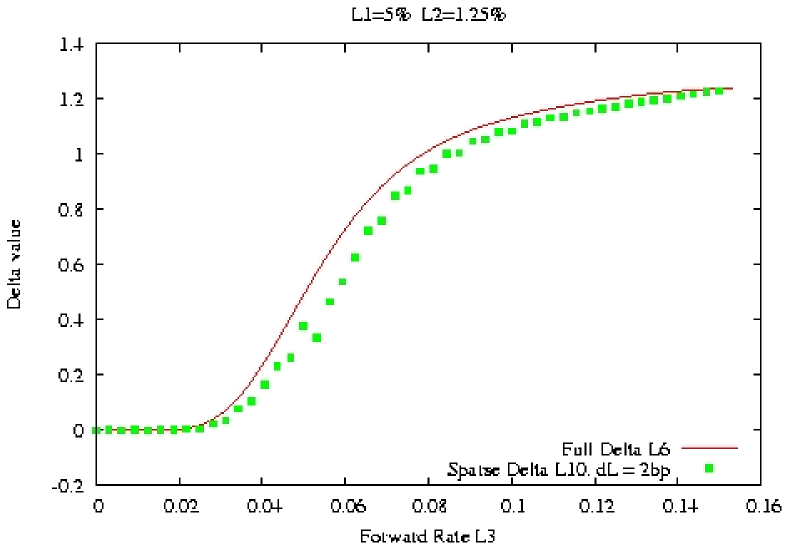


Level	Full time (in sec.)	Sparse time (in sec.)	Full grid points	Sparse grid points
4	11.94		4913	593
5	121.11	1.51	35937	1505
6	1107.16	5.27	274625	3713
7		17.48	2146689	8961
8		54.39	16974593	21249
9		158.39	135005697	49665

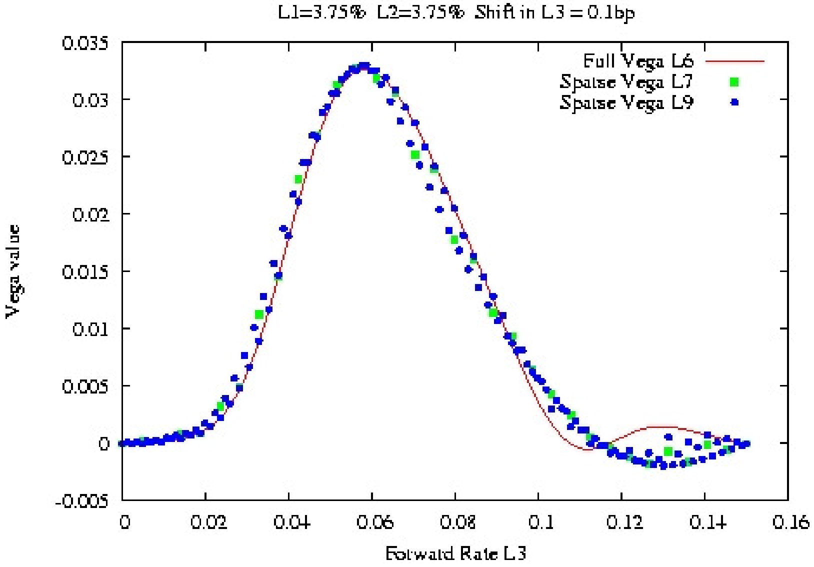
SURFACE PLOTS and GREEKS.

1. Full Grid solution surface
2. Sparse Grid solution surface
3. Delta Surface
4. Vega Surface

Full Delta vs. Sparse Delta



Full Vega vs. Sparse Vega

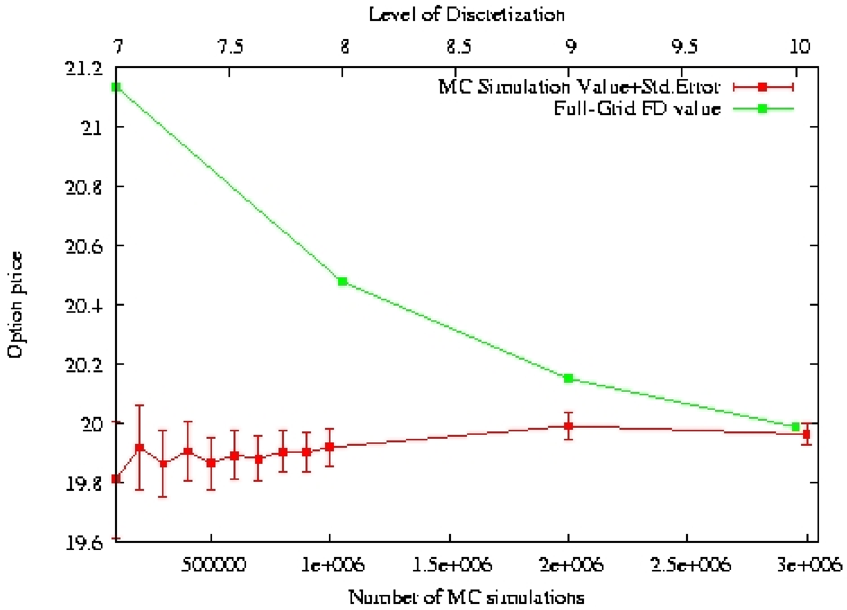


2D DIGITAL OPTION

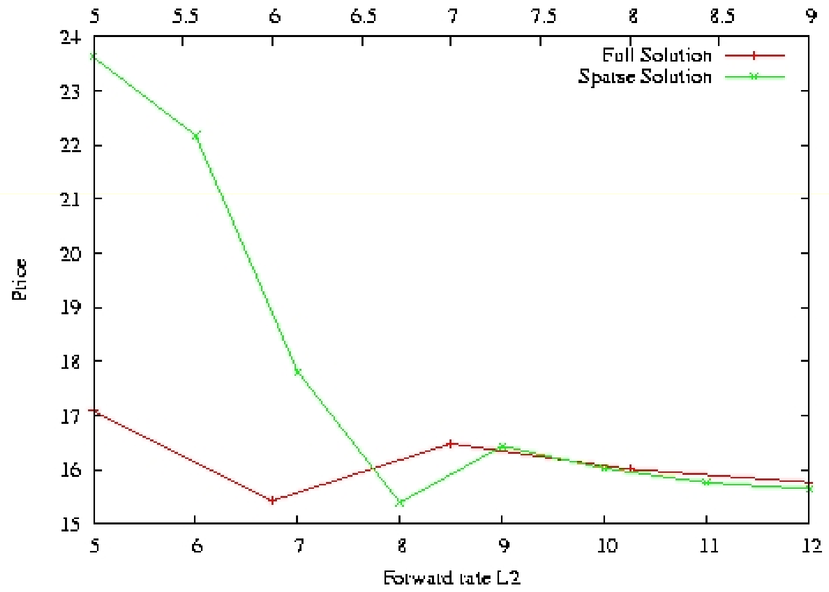
Payoff:

$$V(T)=\begin{cases} 0.02 & \text{if } L_1 > K \ \textbf{OR} \ L_2 > K \\ 0 & \text{otherwise} \end{cases}$$

Full FD vs. MC



Full vs. Sparse

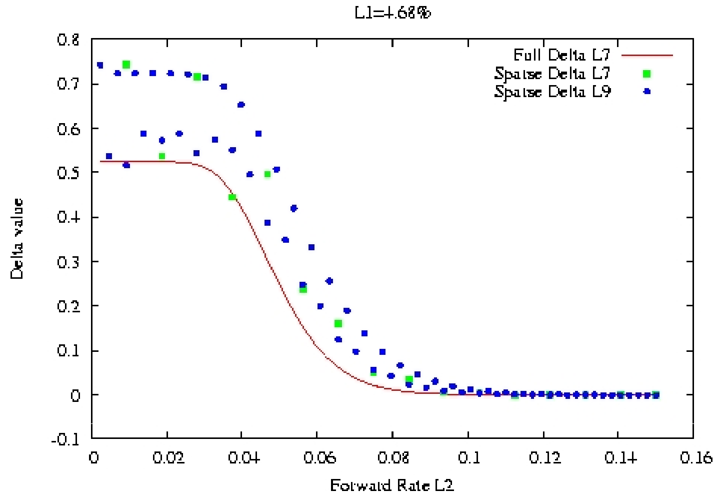


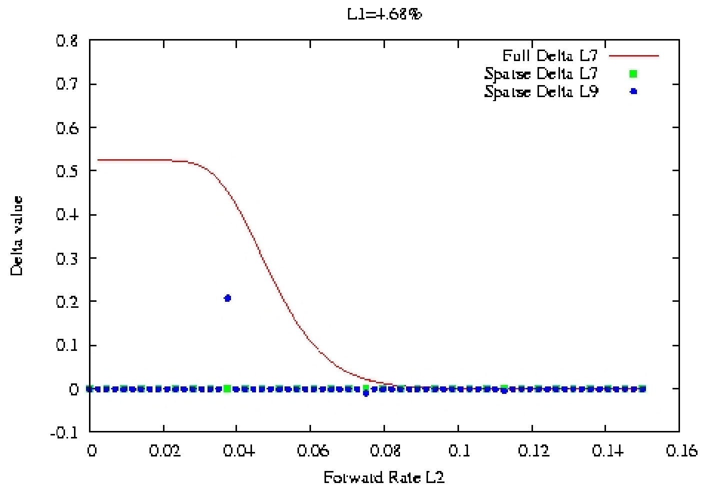
Full Solution Surface

Sparse Solution Surface

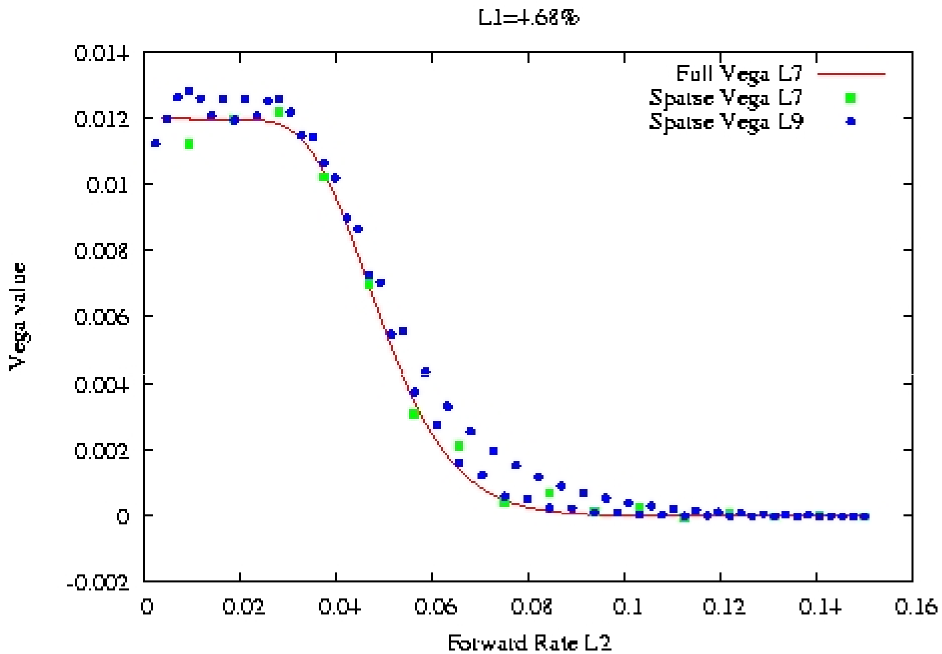
2D DIGITAL OPTION Cont'D. GREEKS.

1. Digital Delta Surface





2. Digital Vega Surface



Quantification error shows in sparse grid delta.

CONCLUSION and SUGGESTIONS FOR FUTURE WORK

CONCLUSIONS:

- Valuation results of a sparse grid method are of acceptable quality
- The "Greeks" of the options are disturbed by the 'wiggles' effect of the sparse grid combination technique.
- A large quantification error is present in the sparse grid solution for the digital delta sensitivity.

We would continue by studying:

1. Effect of the Iterative Solver choice (such as MultiGrid)
2. Effect of the Higher order of FD discretization (4^{th} order FD)
3. Different Combination techniques
4. Adaptive Sparse Grids
5. Battling Quantification Error in discontinuous payoff – coordinate transformation procedure.
6. More careful mathematical error analysis.
7. Parallel implementation of the combination technique.
8. Etc..

Questions?