Financial Mathematics for Insurance

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II

Dynamic Replication

Present Value and Internal Rate of Return (Yield)

IRR = Rate of interest that would make PV of cashflows zero

So IRR of cashflow stream $(x_{0,} x_{1,} ..., x_n)$ is equal to r* if

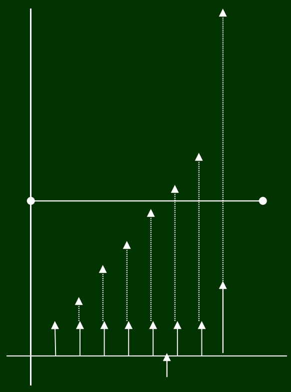
$$0 \quad = \quad x_0 + \frac{x_1}{(1+r^*)} + \frac{x_2}{(1+r^*)^2} + ... + \frac{x_n}{(1+r^*)^n}$$

If $r^* \geq r$ then sensible investment compared to ideal bank account
$r^* < r$ then it is better to put money in ideal bank account

Calculating Price Changes due to Market Movements

$$\boxed{\frac{\Delta P}{P} \approx -D_M \, \Delta\lambda}$$

$$D_M = \frac{\sum_{k=0}^{n} t_k PV(t_k) \Big/ \sum_{k=0}^{n} PV(t_k)}{\left(1+\frac{\lambda}{m}\right)}$$



Modified Duration
'Cash-weighted time to maturity'

Duration Matching

PV portfolio
Duration portfolio

$$PV_{Liabilities} = a_1 \cdot P_1 + a_2 \cdot P_2 + \ldots + a_n \cdot P_n$$

$$D_{Liabilities} = a_1 \cdot \tfrac{P_1 D_1}{PV} + a_2 \cdot \tfrac{P_2 D_2}{PV} + \ldots + a_n \cdot \tfrac{P_n D_n}{PV}$$

Linear Problem: (Easily) Solvable.

Result A portfolio which is protected against small changes in yield (i.e. against moderate interest rate risk).

Calculation of Bond Price from Zero Rates

Spot rate (‘zero rate’)

$$PV = \sum_{k=0}^{n} x_k \left(1+\tfrac{z_k}{m}\right)^{-k}$$

Bond’s yield is ‘average’ of the zero rates
(i.e. the zero yields) over its lifetime.

Can calculate ‘at par coupon’ C_n: coupon which makes nominal value = market value i.e.

$$1 = C_n \cdot (d_1 + d_2 + \ldots + d_n) + 1 \cdot d_n \quad \Leftrightarrow \quad C_n = \frac{1 - d_n}{d_1 + d_2 + \ldots + d_n}$$

also called ‘par swap rates’. This gives other method to define (same) term structure.
Bonds with par coupons are called ‘par bonds’.

Static Replication

General principle we used:

- We can replicate a cashflow pattern in terms of existing assets in the market
- Price of cashflows is price of replicating portfolio
- Discount rates are useful tool to calculate price directly but to determine them we need up-to-date market information
- Asset pricing here consists of pricing new asset in terms of assets which have already been priced

We will now

- Use this to derive prices of FRA’s, swaps
- Extend all this to dynamic form of replication, by considering equity

Forward Rates

- Zero rates allow pricing of any fixed-income instrument
- How zero rates change will determine how prices will change
- Therefore natural question: what can be said about future zero rates based on today's zero rates ?

Future zero rates may seem unpredictable, but market's consensus on future zero rates can be (partially) recovered from today's rates.

Tools: Economic Equilibrium Theory
Absence of Arbitrage Principle

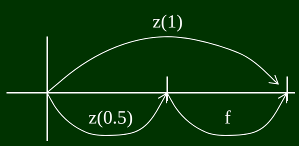
Example : Rollover Investment

Suppose investor wants to invest his money for 1 year.
Assume 6 month's zero rate is 5.25%, and 1-year zero rate is 5.50%.
He can choose from 2 alternatives:

- Invest the money against 1-year zero rate
- Invest the money against 6-months zero rate, and when it matures in 6 months, reinvest for another 6 months

To analyze the second alternative, we need to know the future 6 months zero rate, 6 months from now, which we will call f.

Arbitrage Theory



Look at the total return on one dollar for both alternatives

– 1 year investment $\left(1+\frac{z_1}{2}\right)^2$

– 2×6 months investment $\left(1+\frac{z_{0.5}}{2}\right)\left(1+\frac{f}{2}\right)$

Arbitrage Theory: these two must be equal !

Otherwise riskfree profit would be possible by either

– selling 1 year instrument, buying 6 months instruments
– buying 1 year instrument, selling 6 months instruments

Calculating Forward Rates

So

$$\left(1+\frac{z_{0.5}}{2}\right)\left(1+\frac{f}{2}\right)=\left(1+\frac{z_1}{2}\right)^2$$

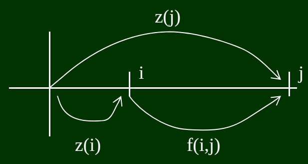
which implies that

$$f=2\left(\frac{\left(1+\frac{1}{2}z_1\right)^2}{\left(1+\frac{1}{2}z_{0.5}\right)}-1\right)=2\left(\frac{\left(1+\frac{1}{2}(0.0550)\right)^2}{\left(1+\frac{1}{2}(0.0525)\right)}-1\right)=0.0575$$

Conclusion: 6 month rate 5.25 %
1 year rate 5.50 %
then forward rate between 6 months and 1 year in the future must (theoretically !) equal 5.75 %

Forward Rates: the general case

In general:



Forward rate f(i,j) between period i and j must satisfy

$$\left(1+\frac{z_i}{m}\right)^i\left(1+\frac{f_{i,j}}{m}\right)^{(j-i)} = \left(1+\frac{z_j}{m}\right)^j$$

- In practice forward rates are not perfect predictors for future rates
- However: good indicators of market consensus, against which investor must set his own expectations

We often call f(i,i+1) the <u>short</u> rate (not to be confused with <u>zero</u> rate !)

$$r_i = f_{i,i+1}$$

Review of Interest Rate Modelling (at time zero)

zero rate	z(k)	yearly rate between now (time zero) and time k
forward rate	f(i,j)	yearly rate between <u>future</u> times i and j
short rate	r(i)	yearly rate between time i and next time (i+1)
discount rate	d(0,k)	today's price of k-maturity zero-coupon bond

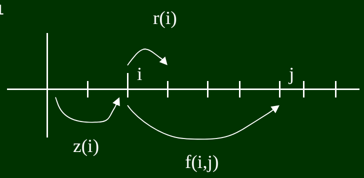
so

$$d(0,k) = \frac{1}{(1+z_k)^k} \qquad r_i = f_{i,i+1}$$

$$(1+f_{i,j})^{j-i} = (1+r_i)(1+r_{i+1})\ldots(1+r_{j-1})$$

$$(1+z_k)^k = (1+r_0)(1+r_1)\ldots(1+r_{k-1})$$

$$d(0,i) = (1+r_i)d(0,i+1)$$



Review of Interest Rate Modelling

zero rate	z(t,k)	yearly rate between now (time t) and time k
forward rate	f(t,i,j)	yearly rate between <u>future</u> times i and j as seen at t
short rate	r(t,i)	yearly rate between time i and time (i+1) as seen at t
discount rate	d(t,k)	price of k-maturity zero-coupon bond at time t

so:

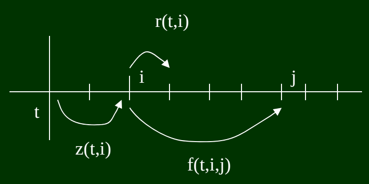
$$d(t,k) = \frac{1}{(1+z(t,k))^{(k-t)}}$$

$$r(t,i) = f(t,i,i+1)$$

$$(1+f(t,i,j))^{j-i} = (1+r(t,i))\cdot(1+r(t,i+1))\ldots\cdot(1+r(t,j-1))$$

$$(1+z(t,k))^{k-t} = (1+r(t,t))\cdot(1+r(t,t+1))\ldots\cdot(1+r(t,k-1))$$

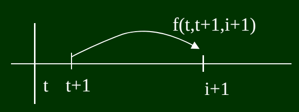
$$d(t,i) = (1+r(t,i))\cdot d(t,i+1)$$



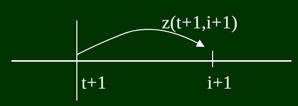
Expectations Hypothesis

Forward rates follow from zero rates, and define <u>expected future zero rates</u>.

$$z(t+1,i+1) = f(t,t+1,i+1)$$



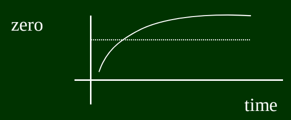
So future term structure can in principle
be calculated from current term structure !



Pure Expectations:	No other systematic factors then expected future short rates determine forward rates
Biased Expectations:	Other factors do have some influence

Bond Price Volatility Revisited

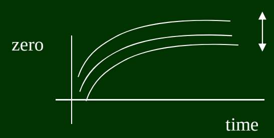
zerorates z_k define maturity-dependent discount factors d_k
No longer one single yield λ used for all discounting



$$PV = \sum_{k=0}^{n} x_k \left(1 + \tfrac{\lambda}{m}\right)^{-k}$$

$$PV = \sum_{k=0}^{n} x_k \left(1 + \tfrac{z_k}{m}\right)^{-k}$$

"Sensitivity to yield changes" now translated to
"Sensitivity to parallel changes in zero rate curve"



$$\lambda \quad \rightarrow \quad \lambda + \Delta\lambda$$
$$z_k \rightarrow \quad z_k + \lambda$$

Quasi-Modified Duration

$$P(\lambda) \quad = \quad x_1\left(1 + \tfrac{z_1+\lambda}{m}\right)^{-1} + x_2\left(1 + \tfrac{z_2+\lambda}{m}\right)^{-2} + \ldots + x_n\left(1 + \tfrac{z_n+\lambda}{m}\right)^{-n}$$

$$\frac{\partial P}{\partial \lambda} \quad = \quad \left(-\tfrac{1}{m}\right) x_1\left(1 + \tfrac{z_1+\lambda}{m}\right)^{-2} + \left(-\tfrac{2}{m}\right) x_2\left(1 + \tfrac{z_2+\lambda}{m}\right)^{-3} + \ldots + \left(-\tfrac{n}{m}\right) x_n\left(1 + \tfrac{z_n+\lambda}{m}\right)^{-(n+1)}$$

so <u>quasi-modified duration</u> D_Q defined as

No simplification !

$$D_Q \quad = \quad -\frac{\partial P(0)}{\partial \lambda} \cdot \frac{1}{P(0)} \quad = \quad \frac{\sum_{k=0}^{n} \frac{k}{m} x_k \left(1 + \frac{z_k}{m}\right)^{-(k+1)}}{\sum_{k=0}^{n} x_k \left(1 + \frac{z_k}{m}\right)^{-k}}$$

leading to

$$\boxed{\frac{\Delta P}{P} = -D_Q \cdot \Delta\lambda}$$

for <u>parallel shifts</u> !

Forward Rate Agreement (FRA)

Example

Company intends to buy another company in 6 months time, and has invested money to do so: will obtain $ 1 million in 6 months
Then news arrives that deal will be postponed by 6 months

Interest Rate Risk: there will be $ 1 million idly on bank account between six months and one year from now
- if rates fall, company loses total return
- if rates rise, company earns total return

Company does not like this risk, so it would like to be sure about its return over the period, without downside risk or upside potential.

Constructing a forward

What forward rate can we construct ?

	Rate
6 months	5.52%
1 year	5.62%

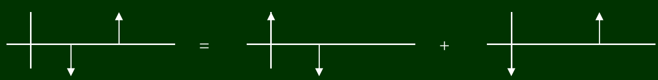
Construction:

Invest money to meet 1 year obligation (m=2 now)
by borrowing against 6 months cash you receive

Borrow today against 6 month cash $$\frac{1.000.000}{1+\frac{0.0552}{2}} = 973.141$$

Invest this today for 12 months $$973.141\,(1+\tfrac{0.0562}{2})^2 = 1.028.600$$

so (yearly !) forward rate is $$2\,\frac{1.028.600 - 1.000.000}{1.000.000} = 5.72\%$$



Forward Rate Agreement

Instead of this (rather cumbersome) construction, you can enter an FRA (Forward Rate Agreement), a derivative which

- Specifies a forward rate for a given time period (the deal rate), and a notional amount
- Later, at the beginning of that time period, the actual interest rate for that period is observed (the benchmark rate)
- The contract pays you the difference (benchmark-deal) times the notional, times the period under consideration (and if this difference is negative, YOU pay to the issuer !)

Interest Rate Derivative, which eliminates risk of future rates.

- Borrowers buy FRA (if rates rise, the FRA pays the difference)
- Investors sell FRA (if rates drop, they earn on the FRA contract)

Smaller credit risk than earlier construction, since only differences are paid.

Swap

Swap agreement to exchange cashflows in the future,
usually based on future values of market variables

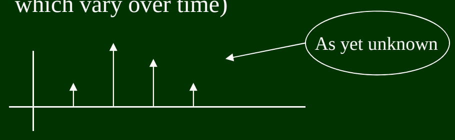
Futures, FRA's : exchange of cashflows on <u>one</u> single future date

Most popular is interest rate swap between two parties A and B

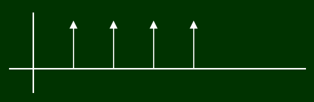
- Company A faces future interest rate payments on a loan which are floating (i.e. they depend on, possibly changing, market conditions)
- It would like to have <u>fixed</u> interest rates
- It therefore ask B (a Bank) to pay the floating rates for them
- in return, A pays B a certain fixed rate which is determined by B
- no cash changes hands initially (in the usual case)

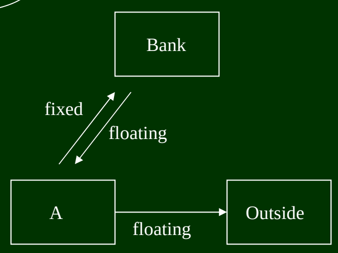
Vanilla Interest Rate Swap

You (A) face interest payments which are floating (i.e. dependent on market rates which vary over time)



you would like known rates, fixed in advance





Swap Party B(ank) swaps payments with party A

- B pays A floating rates (after period in which they occur !)
- A pays B fixed rate r (‘swap rate’, determined by B)

Swap

Standard convention for EURO:

Floating: 6-month EURIBOR

Fixed: annual payments of some fixed rate

Fixed rate determines name (point of view), no seller/buyer

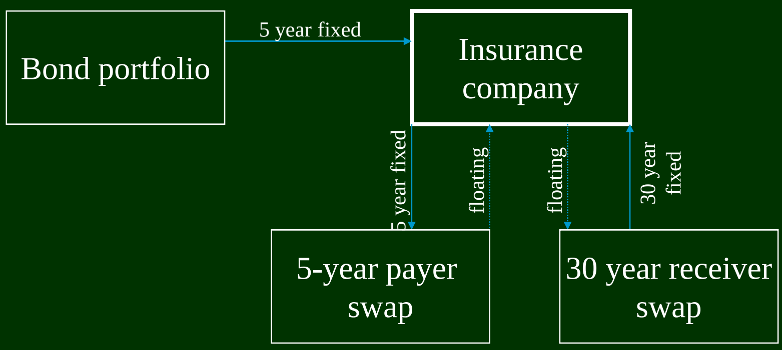
Payer swap = pay fixed rate and receive floating

Receiver swap = receive fixed rate and pay floating

Swap

Combination of both creates possibility to invest in long maturity coupon bonds

Exchange 5-year coupon bond into 30-year coupon bond : change duration !



Cash flows for Plain Vanilla Swap

Example

Four-year swap initiated on January 1, 2002 where A pays B 5% semi-annually on notional principal of 100 million euro and receives LIBOR (London Interbank Offer Rate) rate on same principal

SWAP

Date	LIBOR Rate	Floating Received	Fixed Paid	Net Received
January 1, 2002	4.44%			
July 1, 2002	4.56%	2,220,000	-2,500,000	-280,000
January 1, 2003	5.22%	2,280,000	-2,500,000	-220,000
July 1, 2003	5.45%	2,610,000	-2,500,000	110,000
January 1, 2004	5.55%	2,725,000	-2,500,000	225,000
July 1, 2004	5.92%	2,775,000	-2,500,000	275,000
January 1, 2005	6.14%	2,960,000	-2,500,000	460,000

Perspective of A

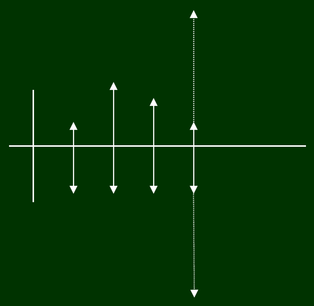
Note
- No uncertainty about first exchange of payments
- Principal is never exchanged
- Payments of floating rates occur after the period to which they apply

Swaps as Bond Positions

Cashflows not mutually exchanged usually : only differences

No payment of principal, so less credit risk (just as in FRA)

If we include a (virtual) mutual exchange of principal at the end, we find an interesting interpretation:



In swap, party A is

Long a floating-rate bond, and
Short a fixed-rate (coupon) bond

(explains why we pay floating rate after the relevant time period)

Interest rate swaps : Market value

Calculate market value of fixed and floating payments
Assume fixed swap rate y at times k=1,…, N

Market value fixed payments:

$$V_{fix,t} = y \sum_{k=1}^{N} D_{k,t}$$

Market value floating payments:

Replication by principal in deposit account:

$$V_{flo,t} = D_{0,t} - D_{N,t}$$

Total market value swap contract:

Payer swap: $V_{flo,t} - V_{fix,t}$
Receiver swap: $V_{fix,t} - V_{flo,t}$

Interest rate swaps : Par Swap Rate

Market convention:

quote fixed rate which sets $V_{swap}=0 \Leftrightarrow V_{flo,t} = V_{fix,t}$

this rate is called the Par Swap Rate

$$\left.\begin{aligned} V_{flo,t} &= D_{0,t} - D_{N,t} \\ V_{fix,t} &= y_{N,t} \sum_{k=1}^{N} D_{k,t} \end{aligned}\right\} \Rightarrow y_{N,t} = \frac{D_{0,t} - D_{N,t}}{\sum_{k=1}^{N} D_{k,t}}$$

$y_{N,t}$ is Par Swap Rate for time to maturity N

Formulas also hold for forward swap contracts (where t<0, i.e. first cashflow is not directly 1 period from now but later).

Interest rate swaps
Par Swap Rate (2)

Use Par Swap Rate to determine market value of swap contract: fixed rate K

Current Par Swap Rate = $y_{N,t}$

Only difference in fixed payments, hence:

$$\begin{aligned} V_{payer,t} &= V_{payer,t} - V_{payer,t}^{PAR} = V_{float,t} - V_{fix,t} - (V_{float,t}^{PAR} - V_{fix,t}^{PAR}) \\ &= V_{fix,t}^{PAR} - V_{fix,t} \\ &= (y_{N,t} - K)\left(\sum_{k=1}^{N} D_{k,t}\right) \\ V_{receiver,t} &= V_{receiver,t} - V_{receiver,t}^{PAR} = V_{fix,t} - V_{float,t} - (V_{fix,t}^{PAR} - V_{float,t}^{PAR}) \\ &= V_{fix,t} - V_{fix,t}^{PAR} \\ &= (K - y_{N,t})\left(\sum_{k=1}^{N} D_{k,t}\right) = -V_{payer,t} \end{aligned}$$

Using swaps to find Zero Curve

How to calculate zero-rates from swap rates?

1) Determine discount factors from swap rates

$$y_{N,t} = \frac{D_{0,t} - D_{N,t}}{\sum_{k=1}^{N} D_{k,t}}$$

2) Determine zero rates from discount factors

$$D_{k,t} = (1 + z_{k,t})^{-k} \Rightarrow z_{k,t} = D_{k,t}^{-\frac{1}{k}} - 1$$

Start with shortest maturity (with $D_{0,t}=1$ if not forward starting)
and then use induction using earlier calculated discount rates.

Interest rate swaps
Zero curve (2)

Shortest maturity

$$y_{1,t} = \frac{1 - D_{1,t}}{D_{1,t}} \quad \Rightarrow \quad D_{1,t} = \frac{1}{1 + y_{1,t}}$$

Then using earlier rates $D_{1,t} \ldots D_{N-1,t}$ we find

$$y_{N,t} = \frac{1 - D_{N,t}}{\left(\sum_{k=1}^{N-1} D_{k,t}\right) + D_{N,t}} \quad \Rightarrow \quad D_{N,t} = \frac{1 - \left(\sum_{k=1}^{N-1} D_{k,t}\right) y_{N,t}}{\left(1 + y_{N,t}\right)}$$

Market Quotes for Par Swap Rates

Broker quotes, for illustration (not professional): swap-rates.com

Year	05/09/12	31/07/12
1 Year	0.500	0.000
2 Year	0.510	2.740
3 Year	0.620	2.870
5 Year	0.980	0.620
7 Year	1.360	0.710
10 Year	1.780	1.240
12 Year	1.980	1.410
15 Year	2.170	1.550
20 Year	2.260	1.680
25 Year	2.290	1.790
30 Year	2.310	1.970

Bond Forward Contract

Contract to buy (obligatory !) a certain bond (cash flows c(k) at times k) at a specific future time M (before its maturity) for a specific price, the <u>forward price</u> F

Forward price chosen in such a way that value of contract is zero today

What is fair value for forward price F ?

Holder must buy contract X from seller at time M for price of F

0,-

Having bond now =

Getting coupons until time M

\+ Getting bond at time M

$$P = \sum_{k=0}^{M-1} c(k) D_{k,0} + F\ D_{M,0}$$

so

$$\boxed{F = \frac{P - \sum_{k=0}^{M-1} c(k) D_{k,0}}{D_{M,0}}}$$

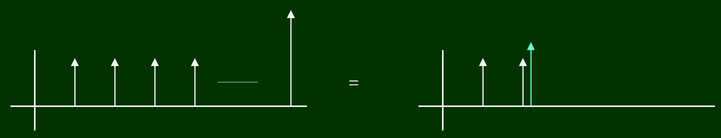
Forward Contract Pricing

Example

Forward on 10-yr bond (8% coupon, selling for 92 now, coupon date was yesterday) for delivery in 1 year, just after coupon.

Current yields for 6 months and 1 year bills are 7% and 8%

Then forward price F satisfies



$$92 = \frac{4}{1.035} + \frac{4+F}{(1.04)^2} \quad \Rightarrow \quad F = 91.32$$

Bond Future Contract

Future contract specifies price F(0) for bond at future time (like forward)

Following time period: new future price F(1)

Contract changes to future contract with price F(1)

You receive/pay the difference F(1)-F(0) in cash

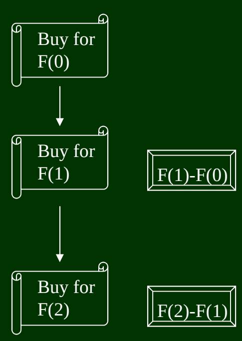
Following time period: new future price F(2)

Contract changes to future contract with price F(2)

You receive/pay the difference F(2)-F(1)

and so on …

All this is taken care of by clearinghouse.



Valuation Structures

Forwards, futures, FRA's and swaps in interest rate market

- take away risk by fixing future interest rates or future prices of interest rate instruments
- and this is usually done in such a way that no cash exchange is needed when contract is made (zero initial costs)
- costless reduction of risk is possible since entering the contract protects against adverse market movements by giving up possible profits due to favourable market movements

Valuation of instruments ('fair price') is possible using today's term structure, since all cash flows in future and corresponding discount factors are known today.