Convergence of the Unscented Kalman Filter for Satellite Re-Entry under Bounded Kalman Gain

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The UKF ideally gives us predictions of the satellite's landing site, but it's not guaranteed to converge [1]. Hence we must find conditions under which the UKF converges so that we can tune our model accordingly.

1 General Bounds on the Kalman Gain

We start with a dynamical system and measurement function of the form:

$$x_{n+1} = f(x_n) + \delta_n \tag{1}$$
$$y_n = h(x_n) + \varepsilon_n, \tag{2}$$

where $f : \mathbb{R}^M \to \mathbb{R}^M$ is a smooth function and $\delta_i, \varepsilon_i > 0 \;\; \forall i \in \{1, 2, \dots N\}, \;\; n \in \{1, 2, \dots N\}$.

The update step for the prediction $\hat{x}_{n+1}$ is:

$$K_n = P_{x_n y_n}(P_{y_n y_n})^{-1} \tag{3}$$
$$\hat{x}_{n+1} = \hat{x}_{n+1|n} + K_{n+1}(y_{n+1} - \hat{y}_{n+1|n}), \tag{4}$$

where $P_{x_n y_n}$ is the prior covariance matrix of x_n & y_n and K_n is the Kalman gain at the n^{th} instant. To show the UKF converges, we define the error as: $e_n = x_n - \hat{x}_n$. Looking at the long-term evolution of e_n we can find conditions under which the UKF converges. Linearising (1) about $\hat{x}_n$[1] gives us:

$$f(x_n) \approx f(\hat{x}_n) + J_n^f(x_n - \hat{x}_n), \tag{5}$$

Now the prediction can be approximated using the prior state estimate $\hat{x}_{n+1|n} = f(\hat{x}_n)$ and (5) as:

$$x_{n+1} = f(\hat{x}_n) + J_n^f(x_n - \hat{x}_n) + \delta_n = \hat{x}_{n+1|n} + J_n^f(x_n - \hat{x}_n) + \delta_n$$
$$\Rightarrow e_{n+1|n} = x_{n+1} - \hat{x}_{n+1|n} \approx J_n^f(x_n - \hat{x}_n) + \delta_n,$$

Next we linearise (2) about our prior state estimate $\hat{x}_{n+1|n}$:

$$h(x_{n+1}) \approx h(\hat{x}_{n+1|n}) + J_n^h(x_{n+1|n} - \hat{x}_{n+1|n}) = h(\hat{x}_{n+1|n}) + J_n^h e_{n+1|n}.$$

Now we can redefine our measurement as:

$$\begin{aligned} y_{n+1} &= h(x_{n+1}) + \varepsilon_{n+1} \approx h(\hat{x}_{n+1|n}) + J_n^h e_{n+1|n} + \varepsilon_{n+1} \\ &= \hat{y}_{n+1|n} + J_n^h e_{n+1|n} + \varepsilon_{n+1} \\ \Rightarrow y_{n+1} - \hat{y}_{n+1|n} &\approx J_n^h e_{n+1|n} + \varepsilon_{n+1} \end{aligned} \tag{6}$$

[1] $J_n^s = \frac{\partial s}{\partial x}|_{x=\hat{x}}$

Substituting (6) into (4) gives:

$$\begin{aligned}
\hat{x}_{n+1} &= \hat{x}_{n+1|n} + K_{n+1}\left(J^h_{n+1} e_{n+1|n} + \varepsilon_{n+1}\right)\\
\Rightarrow e_{n+1} &= x_{n+1} - \hat{x}_{n+1}\\
&= \hat{x}_{n+1} - \hat{x}_{n+1|n} - K_{n+1}(y_{n+1} - y_{n+1|n})\\
&= J^f_n e_n + \delta_n - K_{n+1}(J^h_n e_{n+1|n} + \varepsilon_{n+1})\\
&= J^f_n e_n + \delta_n - K_{n+1}(J^h_n \left(J^f_n e_n + \delta_n\right) + \varepsilon_{n+1})\\
&= (I - K_{n+1} J^h_n) J^f_n e_n + (I - K_{n+1} J^h_n)\delta_n - K_{n+1}\varepsilon_{n+1}. \qquad (7)
\end{aligned}$$

We now define an error transition matrix $\Phi_{n+1} := (I - K_{i+1} J^h_i) J^f_i$ so that:

$$e_{n+1} = \Phi_{n+1} e_0 + \text{noise} \qquad (8)$$

By the the boundedness of K[2], the triangle inequality and Minkowski's inequality we can bound $\|\Phi_{n+1}\|$[3][4]:

$$\begin{aligned}
\left\|J^f_n\right\| &\leq \left(\|J_{21}\|^2 + \|J_{22}\|^2 + 1\right)^{\frac{1}{2}} \qquad (9)\\
\left\|I - K_n J^h_n\right\| &\leq \|I\| + \left(\|K_n\| \cdot \left\|J^h_n\right\|\right) = 1 + \|K_n\| \cdot \left\|J^h_n\right\| \leq 1 + \kappa \left\|J^h_n\right\| \qquad (10)\\
\|\Phi_{n+1}\| &\leq \left\|I - K_{n+1} J^h_n\right\| \cdot \left\|J^f_n\right\|\\
&\leq (1+\kappa) \cdot \left(\|J_{21}\|^2 + \|J_{22}\|^2 + 1\right)^{\frac{1}{2}} \qquad (11)
\end{aligned}$$

Our system evolves over time, so a simple eigenvalue analysis will only tell us the stability of the n^{th} state. To see the divergence or convergence of two nearby trajectories we must look at the largest Lyapunov exponent $\lambda_{\max}$ of $\Phi_n + 1$.

$$\begin{aligned}
\lambda_{max} &= \lim_{n\to\infty} \frac{1}{n} \log \left\| \prod_{i=0}^{n-1} \Phi_i \right\| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|\Phi_i\|\\
&\leq \lim_{n\to\infty} \sup \log \left[\left\|I - K_{n+1} J^h_n\right\| \cdot \left\|J^f_n\right\|\right]\\
&\leq \lim_{n\to\infty} \sup \log \left[(1 + \kappa \cdot \left\|J^h_n\right\|) \cdot \left\|J^f_n\right\|\right] \qquad (12)
\end{aligned}$$

In order for $\lambda_{max} < 0$, we must have $(1 + \kappa \cdot \left\|J^h_n\right\|) \cdot \left\|J^f_n\right\| < 1$:

$$\begin{aligned}
\left\|I - K_{n+1} \cdot J^h_n\right\| &< \frac{1}{\left\|J^f_n\right\|}\\
\Rightarrow K_{n+1} &\in \left(\frac{1}{\|J^h_n\|} - \frac{1}{\|J^h_n\| \cdot \left\|J^f_n\right\|}, \frac{1}{\|J^h_n\|} + \frac{1}{\|J^h_n\| \cdot \left\|J^f_n\right\|}\right). \qquad (13)
\end{aligned}$$

Now we have bounds on the Kalman gain for a general system, we can find the specific convergence criterion for an example model.

2 Bounds on the Kalman Gain for a De-Orbiting Satellite

We start by defining our system of ODEs:

[2] $\exists\ \kappa > 0$ s.t. K_n is bounded $\|K_n\| \leq \kappa$

[3] $\|\cdot\|$ denotes the operator norm.

[4] Assuming measurement noise δ_n and model noise ε_n are both also bounded

$$\frac{\partial^2 \vec{x}}{\partial t^2} = -\frac{GM_E}{r^2} + \frac{1}{M_{\text{sat}}} f_{\text{drag}}(\vec{x}, \vec{v}) + \frac{1}{M_{\text{sat}}} \vec{F}_{\text{other}}, \tag{14}$$

and our measurement function:

$$h(\vec{x}, \vec{v}) = (x - x_{radar}, y - y_{radar}, z - z_{radar}, v_x, v_y, v_z) + \varepsilon, \tag{15}$$

where $\varepsilon \sim \mathcal{N}(\vec{0}, \Sigma)$. In this case we will simplify the model by assuming that only gravitational and drag forces are acting on the satellite, as they dominate all other force terms such as tidal or electromagnetic forces.

The Jacobian of our model, (14), J_n^f is:

$$\begin{gathered} \begin{pmatrix} 0 & I_{3\times 3} \\ J_{21} & J_{22} \end{pmatrix}, \\ J_{21} = -GM_E \left(\frac{1}{r^3} I_{3\times 3} + \frac{1}{r^5} \vec{x}\vec{x}^\top \right) + \frac{C_D A \tilde{\rho}_0}{2M_{sat} H} e^{-\frac{r-R_E}{H}} \frac{\vec{x}}{r} \parallel \vec{v} \parallel \vec{v}^\top \\ J_{22} = \frac{C_D A \tilde{\rho}_0}{2M_{sat}} \left(\frac{\vec{v}\vec{v}^\top}{\parallel \vec{v} \parallel} + \parallel \vec{v} \parallel I_{3\times 3} \right). \end{gathered}$$

Since the measurement function[5] is a translation, J_n^h is simply the identity matrix $I_{6\times 6}$.

Our atmospheric density has a maximum at $r = R_E \ \Rightarrow \rho(x) \leq \rho_0$, allowing us to bound $\|J_{21}\|$ & $\|J_{22}\|$:

$$\begin{aligned} \|J_{21}\| &\leq \frac{2GM_E}{r^3} + \frac{\rho_0 C_D A}{2M_{sat} H} \|\vec{v}\|^2 \\ \|J_{22}\| &\leq \left\| \frac{\rho_0 C_D A}{2M_{sat} H} \left(\frac{\|\vec{v}\|^2}{\|\vec{v}\|} + \|\vec{v}\| \right) \right\| = \frac{\rho_0 C_D A}{M_{sat} H} \cdot \|\vec{v}\|. \end{aligned} \tag{16}$$

For our de-orbiting system to satisfy $\lambda_{max} < 0$, we must have $(\|I - K_{n+1}\|) \cdot \left(\|J_{21}\|^2 + \|J_{22}\|^2 + 1 \right)^{\frac{1}{2}} < 1$. Combining (12), (16) and (17) gives our condition:

$$\begin{aligned} \|I - K_{n+1}\| &< \frac{1}{\sqrt{\|J_{21}\|^2 + \|J_{22}\|^2 + 1}} \\ &= \frac{1}{\sqrt{\left(\frac{2GM_E}{r^3} + \frac{\rho_0 C_D A}{2M_{sat} H} \|\vec{v}\|^2 \right)^2 + \left(\frac{\rho_0 C_D A}{M_{sat} H} \cdot \|\vec{v}\| \right)^2 + 1}}, \end{aligned} \tag{17}$$

$$\Rightarrow K_{n+1} \in \left(1 - \frac{1}{\sqrt{\|J_{21}\|^2 + \|J_{22}\|^2 + 1}}, 1 + \frac{1}{\sqrt{\|J_{21}\|^2 + \|J_{22}\|^2 + 1}} \right). \tag{18}$$

Physically this means that if our variance is too high or too low, then our Kalman gain (3) will be out of the convergence range (18):

1. If K_{n+1} is too low, the UKF becomes overconfident and won't converge to the correct solution (in this case the landing location of the satellite).
2. If K_{n+1} is too high then the estimates $\hat{x}_{n+1}$ become too noisy themselves and cause the filter to diverge.

[5] The measurement function in (2) takes radar measurements in spherical coordinates and transforms them back to the inertial rest frame in Cartesian coordinates.

References

[1] Jiuchao Feng and Chi K. Tse. An unscented-transform-based filtering algorithm for noisy nonlinear systems. In *Proceedings of the IEEE International Symposium on Circuits and Systems (ISCAS)*, pages 1349–1352. IEEE, 2007.