Statistics 630 - Assignment 7
(partial solutions)

1. Exer. 4.2.12. In my simulation I got 0.17738 with $n = 20$ and 0.27658 with $n = 50$. Since the true mean is $\mu = 0.20$, this suggests that $\bar{X}_n$ is more likely to be close to μ when the sample size is larger, which is in line with the law of large numbers. [The exact chances (see, for example, Exer. 4.4.12 below) are 0.17634 and 0.27608, respectively.]

2. Exer. 4.4.4. $\mathsf{P}(W_n \le w) = \frac{w(1+w/2n)}{1+1/2n} \to w$, as $n \to \infty$, for $0 \le w \le 1$. The limit is the uniform(0,1) cdf. Note: this is convergence *only* of the distributions, not of the random variables.

3. Exer. 4.4.12. First, $\mu = \sigma = 2$. Second, the central limit theorem says $\mathsf{P}(\bar{X}_n \le 2.5) \approx \Phi\Big(\frac{2.5-2}{2/\sqrt{n}}\Big)$. Thus we get (a) $\Phi(1.00) = 0.84134$ if $n = 16$, (b) $\Phi(1.50) = 0.93319$ if $n = 36$, (c) $\Phi(2.50) = 0.99379$ if $n = 100$.
   Since each $X_i \sim$ exponential(1/2) it can be shown that $\bar{X}_n \sim$ gamma$(n, n/2)$. Using this distribution to compute $\mathsf{P}(\bar{X}_n \le 2.5)$ exactly, we get (d) 0.84349 if $n = 16$, (e) 0.92578 if $n = 36$, (f) 0.99062 if $n = 100$.

4. Exer. 4.4.16. My simulation (with $N = 10^5$) got $\mathsf{P}(\bar{X}_{30} \le -5) \approx 0.50133$. Since $\mu = -5$ here, the CLT would give approximate probability 0.500. [In fact, 0.500 is the *actual* probability due to the symmetry of the uniform distribution.]

5. Exer. 4.6.2.

   (a) Z has mean $\mu_Z = 4\mu_X - \mu_Y/3 = \frac{43}{3}$ and variance $\sigma_Z^2 = 16\sigma_X^2 + \sigma_Y^2/9 = \frac{722}{9}$. Since (X, Y) is bivariate normal (due to the independence), Z has normal distribution.

   (b) $\mathsf{Cov}(X, Z) = 4\,\mathsf{Cov}(X, X) - \mathsf{Cov}(X, Y)/3 = 4(5) - 0/3$. [This does not require knowing that X and Y have normal distributions.]

   Exer. 4.6.7. $C = \sqrt{n}$. Note that n is the degrees of freedom parameter for the chi-square random variable inside the square root sign.

6. Exer. 4.6.10.
   (a) chi-square(1), (b) chi-square(2), (c) t(3), (d) F(1,3), (e) F(70,30).

7. $f(x_1, \ldots, x_n) = \lambda^n \times \exp\Big(-\lambda \sum_{i=1}^n x_i\Big)$.

8. $p(t_1, \ldots, t_n) = (1-\theta)^n \prod_{i=1}^n \binom{4}{t_i} \times \Big(\frac{\theta}{1-\theta}\Big)^{\sum_{i=1}^n t_i}$.

9. $f(x_1, \ldots, x_n) = (2\beta)^{-n} \times \exp\Big(-\frac{1}{\beta} \sum_{i=1}^n |x_i - \mu|\Big)$.

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10. Here are the histogram and box-plot from my simulations with $n = 20$ and $n = 50$, respectively.