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STAT 630
HW #3
9/24/21

- Material covered is primarily from Lectures 6-9 (9/10/21 - 9/17/21) and chp 2 of the textbook.

1.) Chp 2: Exercise 2.3.18.

The number of arrivals at a queue for some specific unit of time t can be modeled by a Poisson(λt) distribution and is such that the number of arrivals in nonoverlapping periods are independent. λt is the average number of arrivals during a time period of length t and so λ is the rate of arrivals per unit of time.

β telephone calls arrive at a help line at the rate of 2 per minute (i.e. $\lambda = 2$). A poisson process provides a good model.

(a) what is the probability that 5 calls arrive in the next 2-minutes

- $\lambda t = 2(2) = 4.$

$$\boxed{P(X=5) = (e^{-4}4^5)/5! = 0.1562934519}$$

(b) What is the probability 5 calls arrive in the next two minutes & 5 more calls arrive in the following 2 minutes.

$$P(X_t = 5 \wedge X_{t+1} = 5) = P(X_t = 5)P(X_{t+1}=5 \mid X_t = 5)$$

$$= \underbrace{P(X_t=5)P(X_{t+1}=5)}$$

[B/c # of calls in nonoverlapping periods are ind]

$$= ((e^{-4}4^5)/5!)((e^{-4}4^5)/5!)$$

$$\boxed{P(X_t=5, X_{t+1}=5) = 0.0244276431}$$

(c) What is the prob no calls arrive during a 10-minute period.

- Let Y be the # of calls in a 10 minute period.

$\lambda t = 2(10) = 20.$

$$\boxed{P(Y=0) = (e^{-20}20^0)/20! = 0.000000002061154}$$

NOTE: B/c of independence: IF X = # of calls in a 1 minute period.

$$P(Y=0) = (P(X=0))^{10}$$

2) Chp 2: Exercise 2.4.4 (a, b, c). For each part, once you have a value for c, write an expression for $f(x)$ that is valid $\forall x \in \mathbb{R}^1$, using an indicator function as needed. Also - find the cdf for each.

• Establish for which constants c the following are densities.

Recall: Def 2.4.2: Let $f: \mathbb{R}^1 \to \mathbb{R}^1$ be a function. Then f is a density function if:

① $f(x) \geq 0 \ \forall x \in \mathbb{R}^1$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(a) $f(x) = cx$ on $(0,1)$ and 0 otherwise.

• $1 = \int_0^1 cx\,dx = \frac{c}{2}x^2\Big|_0^1 = \frac{1}{2}c = 1 \Rightarrow \boxed{c=2}$

$$f(x) = \begin{cases} 2x & ; \ 0 < x < 1 \\ 0 & ; \ \text{o.w.} \end{cases}$$

$$F(x) = \begin{cases} x^2 & ; \ 0 < x < 1 \\ 0 & ; \ \text{o.w.} \end{cases}$$

(b) $f(x) = cx^n$ on $(0,1)$; 0 o.w. for $n \in \mathbb{N} \cup \{0\}$

• $1 = \int_0^1 cx^n\,dx = \frac{c}{n+1} x^{n+1}\Big|_{x=0}^{x=1} = \frac{c}{n+1} = 1 \Rightarrow \boxed{c = n+1}$

$$f(x) = \begin{cases} (n+1)x^n & ; \ x \in (0,1) \wedge n \in \mathbb{N} \cup \{0\} \\ 0 & ; \ \text{o.w.} \end{cases}$$

$$F(x) = \begin{cases} x^{n+1} & ; \ x \in (0,1) \wedge n \in \mathbb{N} \cup \{0\} \\ 0 & ; \ \text{o.w.} \end{cases}$$

2) (Continued)

(c) $f(x) = cx^{1/2}$ on $(0,2)$ & 0 o.w.

• $1 = \int_0^2 cx^{1/2}\,dx = \frac{2c}{3} x^{3/2} \Big|_0^2 = (4\sqrt{2}/3)c = 1 \Rightarrow \boxed{c = \frac{3\sqrt{2}}{8}}$

$$\boxed{\begin{array}{l} f(x) = \begin{cases} \frac{3\sqrt{2}}{8} x^{1/2} & ; x \in (0,2) \\ 0 & ; \text{o.w.} \end{cases} \\ \\ F(x) = \begin{cases} \frac{\sqrt{2}}{4} x^{3/2} & ; x \in (0,2) \\ 0 & ; \text{o.w.} \end{cases} \end{array}}$$

3.) Chp 2: Exercise 2.4.19. Use an indicator function to give an expression for $f(x)$ that is valid $\forall x \in \mathbb{R}$.

• (Weibull (α) distribution) Consider for $\alpha > 0$ fixed, the function given by $f(x) = \alpha x^{\alpha-1} e^{-x^\alpha}$ for $0 < x < \infty$ and 0 o.w. Prove $f(x)$ is a density function.

Recall: DEF 2.4.2: Let $f: \mathbb{R}^1 \to \mathbb{R}^1$ be a function. Then f is a density function if:

① $f(x) \geq 0 \ \forall x \in \mathbb{R}^1$

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Proof: (WTS: $f(x) = \alpha x^{\alpha-1} e^{-x^\alpha}$ for $x \in (0,\infty)$ and 0 o.w. w/ $\alpha > 0$ is a pdf.)

① WTS: $f(x) \geq 0 \ \forall x \in \mathbb{R}^1$

• By def $\alpha > 0$ & $x > 0$, Thus $\alpha x^{\alpha-1} > 0 \ \forall$ valid x, α.

• $\frac{d}{dx}[e^{-x^\alpha}] = -\alpha x^{\alpha-1} e^{-x^\alpha}$ which is negative $\forall$ valid α, x.

Thus e^{-x^α} is a strictly decreasing function and will hit its minimum value as $x \to \infty$. $\lim_{x\to\infty} e^{-x^\alpha} = 0$. Thus $e^{-x^\alpha} \geq 0 \ \forall x, \alpha$.

• B/c the product of positive numbers is always positive

$\alpha x^{\alpha-1} e^{-x^\alpha} \geq 0$ for $x \in (0,\infty) \land \alpha > 0$.

Thus condition 1 is satisfied.

② WTS: $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

• B/c $f(x)$ is defined piecewise we'll split it up into two components.

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^\infty \alpha x^{\alpha-1} e^{-x^\alpha}\,dx = 0 - e^{-x^\alpha}\Big|_0^\infty$

$= \lim_{x\to\infty} -e^{-x^\alpha} - (-1) = 1 \Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$

Thus $f(x)$ satisfies both condition ① & ② and is a valid pdf. QED.

4) Chp 2 Exercise 2.4.22: Hint - split the integral for the cases $x \leq 0$ & $x > 0$.

- (Laplace Distribution) Consider the function given by $f(x) = e^{-|x|}/2$ for $-\infty < x < \infty$ and 0 o.w. Prove that f is a density function.

Proof: WTS $f(x) = e^{-|x|}/2$ for $x \in (-\infty, \infty)$ & 0 o.w. is a valid pdf.

① WTS $f(x) = e^{-|x|}/2 \geq 0 \quad \forall x \in (-\infty, \infty)$

- Graphing $e^{-|x|}/2$, we can see that it is a strictly decreasing function over the interval $(0, \infty)$ and strictly increasing over the interval $(-\infty, 0)$.
- Thus, $e^{-|x|}/2$ will take its min values as $x \to \infty$ and $x \to -\infty$ on the intervals $(0,\infty)$ & $(-\infty, 0)$ respectively.

$\lim_{x \to \infty} e^{-|x|}/2 = 0$; $\lim_{x \to -\infty} e^{-|x|}/2 = 0$. $\Rightarrow \underline{f(x) \geq 0 \text{ always}}$

② WTS $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{-\infty}^{\infty} e^{-|x|}/2\, dx = \int_{-\infty}^{0} e^{x}/2\, dx + \int_{0}^{\infty} e^{-x}/2\, dx = \tfrac{1}{2} e^{x}\Big|_{-\infty}^{0} - \tfrac{1}{2} e^{-x}\Big|_{0}^{\infty}$$

$$= 1/2\big((1-0) - (0-1)\big) = 1 \Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-|x|}/2 = 1}$$

Conditions ① & ② are satisfied, thus $f(x) = e^{-|x|}/2$ is a valid pdf QED.

5) Chp 2, Exercise 2.5.3 (a, c, d, f, g). Give reasons if you say "no".

- For each of the following functions, determine F is a valid CDF. (i.e. if F satisfies properties (a)-(d) of Theorem 2.5.2.

Recall

Theorem 2.5.2: Let F_X be the CDF of a random variable X. Then:

(a) $0 \leq F_X(x) \leq 1 \quad \forall x$

(b) $F_X(x) \leq F_X(y)$ whenever $x \leq y$ (i.e. $F_X(x)$ is increasing)

(c) $\lim_{x \to \infty} F_X(x) = 1$; (d) $\lim_{x \to -\infty} F_X(x) = 0$.

(a) $F(x) = x$; $\forall x \in \mathbb{R}$. $\boxed{\text{No; Let } x = 1.1. \text{ Then } F(x) = 1.1 > 1 \Rightarrow F(x) \text{ violates (a)}}$

(c) $F(x) = \begin{cases} 0 & ; x < 0 \\ x^2 & ; 0 \leq x \leq 1 \\ 1 & ; x > 1 \end{cases}$ $\boxed{\text{yes}}$

(d) $F(x) = \begin{cases} 0 & ; x < 0 \\ x^2 & ; 0 \leq x \leq 3 \\ 1 & ; x > 3 \end{cases}$

NO, let $x = 2 \Rightarrow F(x) = 4 > 1 \Rightarrow F(x)$ violates (a)

Let $x = 2$, $y = 4 \Rightarrow y > x$ but $F(y) = 1 < 4 = F(x) \Rightarrow F(x)$ violates (b)

(f) $F(x) = \begin{cases} 0 & ; x < 1 \\ x^2/9 & ; 1 \leq x \leq 3 \\ 1 & ; x > 3 \end{cases}$ $\boxed{\text{yes}}$

(g) $F(x) = \begin{cases} 0 & ; x < -1 \\ x^2/9 & ; -1 \leq x \leq 3 \\ 1 & ; x > 3 \end{cases}$

NO, let $x = -1$ & $y = 0$. Then $y > x$ but $F(y) = 0 < 1/9 = F(x) \Rightarrow F(x)$ violates (b)

6.) Chp2 Exercise 2.5.5: Use the pnorm function in R. Add.

(d) Find the 40^{th} & 77^{th} percentiles. Use the Qnorm function.

Let $y \sim (\mu = -8, \sigma^2 = 4)$ compute each of the following

(a) $P(y \le -5) = \boxed{0.9331928}$

(b) $P(-2 \le y \le 7) = \boxed{0.001349898}$

(c) $P(y \ge 3) = \boxed{0.00000001898956}$

(d) qnorm $(0.40) = \boxed{-8.506694}$; qnorm$(0.77) = \boxed{-6.522306}$

7.) Chp 2 Exercise 2.5.8. Note: it should say $F_y(y) = 1 - (1-y)^3$ for $y \in [1/2, 1]$.

Why is the cdf shown in the book not a valid cdF?

$$p: F_y(y) = \begin{cases} y^3 & ; y \in [0, 1/2) \\ 1-(1-y^3); & y \in [1/2, 1] \text{ (profs. version)} \\ 1 - y^3 & ; y \in [1/2, 1] \text{ (books version)} \end{cases}$$

(a) $P(1/3 < y < 3/4) = P(y \le 3/4) - P(y \le 1/3) = \boxed{0.947337963}$

(b) $P(y = 1/3) = \boxed{0}$

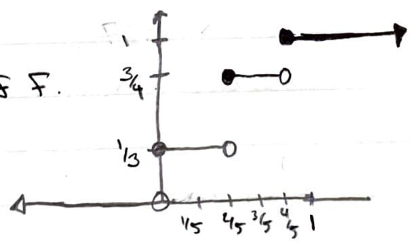
(c) $P(y = 1/2) = \boxed{0}$

(d) Why is the cdf given in the book not a valid cdf.

Let $x = 0.5$, $y = 1$; $x \le y$ but $F(x) > F(y) \Rightarrow$ F_x violates (b) from theorem 2.5.2

8.) Chp 2 Exercise 2.5.13: Let $F(x) = 0$ for $x < 0$ w/ $F(x) = \begin{cases} 0 & x < 0 \\ 1/3 & x \in [0, 2/5) \\ 3/4 & x \in [2/5, 4/5) \\ 1 & x > 4/5 \end{cases}$

(a) Sketch a graph of F.



c.) If X has CDF equal to F, compute:

- $P(X > 4/5) = 0$
- $P(-1 < X < 1/2) = 3/4$
- $P(X = 2/5) = 5/12$
- $P(X = 4/5) = 1/4$

(b) Prove F is a valid CDF

(a) $0 \le F_x(x) \le 1$: True by definition

(b) $F_x(x) \le F_y(y)$, for $x \le y$. True by def

(c) $\lim_{x \to \infty} F(x) = 1$. ; If $x \to \infty$ then $x > 4/5$ at some point $\Rightarrow F(x) = 1$

(d) $\lim_{x \to -\infty} F(x) = 0$; If $x \to -\infty$ then $x < 0$ at some point $\Rightarrow F(x) = 0$

9.) Chp 2 Exercise 2.5.19: Let Φ be as in Definition 2.5.2. Derive a formula for $\Phi(-x)$ in terms of $\Phi(x)$. Hint let $s=-t$ in (2.5.2)).

[DEF (2.5.2): Φ stands for the cdf of a standard normal distribution; defined by:
$$\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt \quad \text{for } x \in \mathbb{R}$$
]

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)\,dt = \int_{-\infty}^{\infty} \phi(t)\,dt - \int_{x}^{\infty} \phi(t)\,dt$$

NOTE: ① $\int_{-\infty}^{\infty} \phi(t)\,dt = 1$.

② $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-(t^2)/2} = \phi(-t)$

③ $\Phi(-x) = \int_{-\infty}^{-x} \phi(t)\,dt$ change bounds on integral: if $s=-t$; $f(t) = \int_a^b g$, $f(s) = \int_{-a}^{-b} g$

④ $\int_{-\infty}^{-x} \phi(t)\,dt = \int_{x}^{\infty} \phi(t)\,dt$ (by symmetry of normal dist)

$$\Phi(x) = \underbrace{\int_{-\infty}^{\infty} \phi(t)\,dt}_{①} - \underbrace{\int_{x}^{\infty} \phi(t)\,dt}_{③\ \&\ ④}$$

$$\Phi(x) = 1 - \Phi(-x) \Rightarrow \boxed{\Phi(-x) = 1 - \Phi(x)}$$

10.) ~~Chap 2 Exercise 2.5.21~~: Add (b) find the quantile function.

(a) Determine the distribution function for the Weibull (α) distribution from Problem 2.4.19.

$$f(x) = \begin{cases} \alpha x^{\alpha-1} e^{-x^{\alpha}} & ; \quad 0 < x < \infty \\ 0 & ; \quad \text{o.w.} \end{cases}$$

$$F(x) = \int \alpha x^{\alpha-1} e^{-x^{\alpha}} \, dx = -e^{-x^{\alpha}} + C$$

[NOTE: $\lim_{x \to 0^+} F(x) = 0$, gives us an initial condition to make this a valid cdf. Need this to be the case so that F is an increasing function.]

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} -e^{-x^{\alpha}} + C = 0 \iff -1 + C = 0 \Rightarrow \boxed{C = 1}$$

$$\boxed{F(x) = \begin{cases} 1 - e^{-x^{\alpha}} & ; \quad 0 < x < \infty \\ 0 & \quad \text{o.w.} \end{cases}}$$

(b) Determine the quantile function.

$Q(x) = F^{-1}(x)$:

$$x = 1 - e^{-Q(x)^{\alpha}} \iff e^{-Q(x)^{\alpha}} = 1 - x \iff -Q(x)^{\alpha} = \ln(1-x)$$

$$\boxed{Q(x) = -\ln(1-x)^{1/\alpha}}$$

11.) Chap 2 Exercise 2.5.24: Add (B) Find the quantile function.

(a) Determine the distribution function for the Laplace distribution from problem 2.4.22.

$$f(x) = \begin{cases} e^{-|x|}/2 & ; \quad -\infty < x < \infty \\ 0 & ; \quad \text{o.w.} \end{cases}$$

$$F(x) = \int_{-\infty}^{\infty} e^{-|x|}/2 \, dx = \int_{-\infty}^{0} \frac{e^{x}}{2} \, dx + \int_{0}^{\infty} e^{-x}/2 \, dx = e^{x}/2 \Big|_{-\infty}^{0} - e^{-x}/2 \Big|_{0}^{\infty}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

[NOTE: $f(x)$ is symmetric about the line $x = 0$. Thus, the cdf at $x = 0$ should be equal to 0.5. For negative values of x, this works out, however for the nonnegative values of x it does not. This can be fixed by making C in the indefinite integral $\int_0 e^{-x}/2 \, dx = -e^{-x}/2 + C$, equal to 1.]

$$F(x) = \begin{cases} 1 - \frac{1}{2} e^{-x} & ; \quad 0 < x < \infty \\ \frac{1}{2} e^{x} & ; \quad -\infty < x \leq 0 \end{cases}$$

11.) (continued)

(b) Find the quantile function.

$$Q(x) = F^{-1}(x)$$

for $x \in (0,\infty)$: $x = 1 - \frac{1}{2}e^{-Q(x)}$

$\frac{1}{2}e^{-Q(x)} = 1 - x$

$e^{-Q(x)} = 2 - 2x$

$-Q(x) = \ln(2-2x)$

$Q(x) = -\ln(2-2x)$

for $x \in (-\infty, 0)$: $x = \frac{1}{2}e^{Q(x)} \Rightarrow 2x = e^{Q(x)}$

$Q(x) = \ln(2x)$

$$\boxed{Q(x) = \begin{cases} -\ln(2-2x) & ; x \in (0,\infty) \\ \ln(2x) & ; x \in (-\infty, 0] \end{cases}}$$

12.) Chp 2 Exercises 2.6.1, 2.6.4, 2.6.9, 2.6.18: Assume $\beta > 0$ for 2.6.18.

NOTE: Theorem 2.6.2: Let X be an arbitrary continuous R.V. w/ density function f_X. Let $Y = h(X)$ where $h: \mathbb{R}^1 \to \mathbb{R}^1$ is a function that is differentiable and strictly increasing. Then Y is also absolutly continuous and its density function f_Y is given by:

$$f_Y(y) = f_X(h^{-1}(y)) / |h'(h^{-1}(y))|$$

where h' is the derivative of h and where $h^{-1}(y)$ is the unique number x s.t. $h(x) = y$.

2.6.2: Let $X \sim \text{Unif}[L,R]$. Let $Y = cX + d$ where $c > 0$. Prove that $Y \sim \text{Unif}[cL+d, cR+d]$.

- $h(x) = cX + d$; $f_X(x) = \begin{cases} \frac{1}{R-L} & ; L \le x \le R \\ 0 & ; \text{o.w.} \end{cases}$

- note: $h'(x) = c > 0 \Rightarrow h(x)$ is strictly increasing $\Rightarrow$ Theorem 2.6.2 applies.
- $h'(y) = c$, $h^{-1}(y) = \left(\frac{y-d}{c}\right)$

- $f_Y(y) = f_X(h^{-1}(y)) / |h'(h^{-1}(y))| = \left(\frac{1}{R-L}\right)/c = \frac{1}{cR-cL}$

$f_Y(y) = \begin{cases} \frac{1}{cR-cL} & \text{for } L \le \left(\frac{y-d}{c}\right) \le R \Rightarrow cL+d \le y \le cR+d \\ 0 & \text{o.w.} \end{cases}$

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{cR-cL} & ; cL+d \le y \le cR+d \\ 0 & ; \text{o.w.} \end{cases} \Rightarrow Y \sim \text{Unif}[cL+d, cR+d]}$$

12.) (Cont'd)

2.6.4: Let $X \sim Exp(\lambda)$. Let $Y = cX$ w/ $c>0$. Prove that $Y \sim Exp(\lambda/c)$.

- $h'(y) = c > 0 \Rightarrow h(y)$ strictly increasing $\Rightarrow$ Theorem 2.6.2 applies.
- $h'(y) = c$, $h^{-1}(y) = y/c$

- $f_y(y) = f_X(h^{-1}(y)) / |h'(h^{-1}(y))|$

$$= \lambda e^{-y(y/c)} / c = \left(\frac{\lambda}{c}\right) e^{-(\lambda/c)y}$$

$$\boxed{f_y(y) = \left(\frac{\lambda}{c}\right) e^{-(\lambda/c)y}}$$

2.6.9. Let X have a density function $f_X(x) = \begin{cases} x^3/4 & ; 0<x<2 \\ 0 & ; \text{o.w.} \end{cases}$

a.) Let $y = x^2$. Compute the density function $f_y(y)$ for Y.

- $h(x) = 2x > 0 \quad \forall x \in (0,2) \Rightarrow h$ is strictly increasing over domain $\Rightarrow$ Theorem 2.6.2 applies.
- $h'(y) = 2y$, $h^{-1}(y) = \sqrt{y}$

$$f_y(y) = (\sqrt{y}^3/4) / 2(\sqrt{y}) = \frac{y}{8} \text{ for } 0 < \sqrt{y} < 2$$

$$\boxed{f_y(y) = \begin{cases} y/8 & ; 0<y<4 \\ 0 & ; \text{o.w.} \end{cases}}$$

(b) Let $z = \sqrt{x}$. Compute the density function $f_Z(z)$ for Z.

- $h'(x) = \frac{1}{2\sqrt{x}} > 0 \quad \forall x \in (0,2) \Rightarrow h$ is strictly increasing $\Rightarrow$ theorem 2.6.2 applies.
- $h^{-1}(z) = z^2$, $h'(z) = \frac{1}{2\sqrt{x}}$

- $f_z(z) = \left(\frac{(z^2)^3}{4}\right) \Big/ \left(\frac{1}{2z}\right) = \boxed{y^7/2}$

- $f_z(z) = \begin{cases} y^7/2 & ; 0 < y^2 < 2 \\ 0 & ; \text{o.w.} \end{cases}$

- $\boxed{f_z(z) = \begin{cases} y^7/2 & , 0<y<\sqrt{2} \\ 0 & ; \text{o.w.} \end{cases}}$

12.) (Contd)

2.6.18: β that $X \sim$ Weibull (α) (see problem 2.4.(a)). Determine the distribution of $Y = X^{\beta}$. Assume $\beta > 0$.

- $f(x) = \alpha x^{\alpha-1} e^{-x^{\alpha}}$
- $Y = X^{\beta}$ w/ $\beta > 0$.
- $h'(y) = \beta x^{\beta-1} > 0 \Rightarrow h$ is strictly increasing $\Rightarrow$ Theorem 2.6.2 applies.
- $h^{-1}(y) = x^{1/\beta}$

$$f_Y(y) = f_X(h^{-1}(y)) / |h'(h^{-1}(y))|$$

$$= \frac{\alpha (y^{1/\beta})^{(\alpha-1)} e^{-(y^{1/\beta})^{\alpha}}}{|\beta(y^{1/\beta})^{\beta-1}|}$$ * B/c $\beta > 0$, can get rid of abs value in denom.

$$= \frac{\alpha y^{1/\beta(\alpha-1)} e^{-y^{\alpha/\beta}}}{\beta y^{1/\beta(\beta-1)}}$$

$$\boxed{f_Y(y) = \left(\frac{\alpha}{\beta}\right) y^{\frac{1}{\beta}(\alpha-1) - \frac{1}{\beta}(\beta-1)} e^{-y^{\alpha/\beta}}}$$