Statistics 630 – Exam I
Friday, 22 February 2013

Printed Name: ______________________ **Email:** ______________________

INSTRUCTIONS FOR THE STUDENT:

1. You have exactly 75 minutes to complete the exam.
2. There are 4 pages including this cover sheet and the formula sheets.
3. There are 5 questions. Each question is worth 10 points.
4. Please write out the answers to the exam questions on *blank* sheets of paper. Start each question on a *separate* sheet of paper (and return them in order).
5. Answer all questions fully, explaining your steps. You may refer to theorems by name/description rather than by its number in the book.
6. *Do not use a calculator.* You may leave answers in forms that can easily be put into a calculator such as $\frac{12}{19}$, $\binom{40}{5}$, e^{-3}, $\Phi(1.5)$, etc.
7. Do not discuss or provide any information to anyone concerning any of the questions on this exam or your solutions until I post my solutions.
8. You may use the *attached formula sheets.* No other resources are allowed. Do not use the textbook, the class notes or the formula sheets that were posted online.

I attest that I spent no more than 75 minutes to complete the exam. I used only the materials allowed above. I did not receive assistance from or provide assistance to anyone either before or while taking this exam.

Student's Signature ______________________________

INSTRUCTIONS FOR THE PROCTOR:

1. Download the exam from the student's account on WebAssign and print it. The student should not view the exam content until it is time to start.
2. The student may take 75 minutes to complete the exam. Record the time at which the student starts the exam: ______________________
3. Record the time at which the student ends the exam: ______________________
4. Immediately after the student completes the exam, please scan *this page* and the student's *solutions* to a *single, unsecured* PDF file and oversee the student uploading it to WebAssign. Be sure the solutions are legible and in order. (You have 105 minutes from the time the exam was downloaded, allowing 30 minutes for printing, scanning and uploading. Please contact Penny Jackson or Kim Ritchie for any necessary delay.)
5. Collect all pages of this exam and the student's solution. Do not allow the student to take any page. You may return them to the student *one week* after the exam.

I attest that the student and I have followed the INSTRUCTIONS FOR THE STUDENT and PROCTOR listed above and that the exam was downloaded, printed, scanned into a PDF file and uploaded to WebAssign in my presence.

Proctor's Signature ______________________________

1. A random variable X has probability density function

$$f_X(x) = c(4 - x^2)I_{[1,2]}(x)$$

for some constant c.

Find c and the cumulative distribution function for X, for each $x \in (-\infty, \infty)$.

2. A small town has 4200 children and 500 of these have not been vaccinated for measles. Also, 2100 children are vaccinated for both polio and measles and 400 are not vaccinated for either polio or measles. Suppose one child is sampled at random.

   (a) Is the event $M =$ "being vaccinated for measles" independent of $P =$ "being vaccinated for polio"?

   (b) If a randomly selected child is vaccinated for measles, what is the chance it is also not vaccinated for polio?

3. Scores are recorded for two questions in a survey. Suppose S is the smaller score and T is the larger. $S = T$ is possible. It happens that the joint probability mass function for (S, T) is

$$p(s,t) = \begin{cases} \frac{1}{21} & s \leq t,\ s \in \{1, \ldots, 6\},\ t \in \{1, \ldots, 6\}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal pmf for T and the conditional pmf for S, given $T = t$.

4. A very busy professor juggles three classes. He spends 20% of his time grading for class U, 30% for class M and 50% for class P. If he's grading for class U, there is a 50% chance he is pulling his hair out, while it's only a 25% chance or a 10% chance for classes M or P, respectively.

   A student wanders in at random and finds the professor pulling out his hair. Which class was he most likely to have been grading for? Which is least likely?

5. A random variable Y has cumulative distribution function $F_Y(y) = \frac{y}{1+y}$ for $y \geq 0$ (and $= 0$ otherwise).

   Define a new random variable $X = F_Y(Y) = \frac{Y}{1+Y}$. Show that the cdf for X is

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ x & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x \geq 1, \end{cases}$$

and find the pdf for X.

The course lecturer, Dr. Wehrly, has reviewed and approved this exam.

Formulas for Exam I

permutations $P_{n,k} = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$.

combinations $C_{n,k} = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

complement and union $\mathsf{P}(A^c) = 1 - \mathsf{P}(A)$; $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$.

conditional probability $\mathsf{P}(A \,|\, B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)}$.

independent events $\mathsf{P}(A \cap B) = \mathsf{P}(A)\mathsf{P}(B)$.

total probability $\mathsf{P}(A) = \sum_{k=1}^{n} \mathsf{P}(A \cap B_k) = \sum_{k=1}^{n} \mathsf{P}(A \,|\, B_k)\mathsf{P}(B_k)$ if $B_1, \ldots, B_n$ are disjoint, $\bigcup_{k=1}^{n} B_k = S$.

Bayes' rule $\mathsf{P}(B_j \,|\, A) = \dfrac{\mathsf{P}(A \,|\, B_j)\mathsf{P}(B_j)}{\sum_{k=1}^{n} \mathsf{P}(A \,|\, B_k)P(B_k)}$ if $B_1, \ldots, B_n$ are disjoint and $\bigcup_{k=1}^{n} B_k = S$.

cdf of random variable $F_X(x) = \mathsf{P}(X \le x) = \sum_{y \le x} p_X(y)$ if X is discrete;
$F_X(x) = \mathsf{P}(X \le x) = \int_{-\infty}^{x} f_X(y)\,dy$ if X is continuous.
$\mathsf{P}(a < X \le b) = F_X(b) - F_X(a)$.

pmf of random variable $p_X(x) = \mathsf{P}(X = x)$ if X is discrete.

pdf of random variable $f_X(x) = \frac{d}{dx} F_X(x)$ if X is continuous.

quantile function $Q_X(p)$ such that $F(Q_X(p)) = p$ if X is continuous. Otherwise, $Q_X(p)$ satisfies $F_X(x) \le p \le F(Q_X(p))$ for all $x < Q_X(p)$.

distribution of a function of X $F_Y(y) = \mathsf{P}(h(X) \le y)$ for $Y = h(X)$.

If X is a discrete rv or $h(x)$ takes only countably many values then Y has pmf $p_Y(y) = \mathsf{P}(h(X) = y)$.

If X is a continuous rv and $h(x)$ is a continuous function then Y has pdf $f_Y(y) = \frac{d}{dy}\mathsf{P}(h(X) \le y)$.

binomial theorem $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$.

geometric sum $\sum_{k=n}^{\infty} a^k = \frac{a^n}{1-a}$ if $-1 < a < 1$.

exponential expansion $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$.

integral of a power function $\int_u^v x^a\,dx = \frac{v^{a+1} - u^{a+1}}{a+1}$ if $a \ne -1$, and $\int_u^v x^{-1}\,dx = \log_e(v/u)$.

integral of an exponential function $\int_u^v e^{ax}\,dx = \frac{1}{a}\left(e^{av} - e^{au}\right)$.

gamma integral $\int_0^{\infty} x^{a-1} e^{-x}\,dx = \Gamma(a) = (a-1)!$ for $a > 0$.

integral of exponential of a quadratic $\int_{-\infty}^{\infty} e^{a+bx-cx^2}\,dx = \sqrt{\frac{\pi}{c}}\, e^{b^2/(4c)+a}$ for $c > 0$.

discrete uniform(N) pmf $p(x) = \frac{1}{N}$ for $x = 1, 2, \ldots, N$.

hypergeometric(N, M, n) pmf $p(x) = \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$ for $x = 0, 1, \ldots, n$, $M \leq N$.

binomial(n, θ) pmf $p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$ for $x = 0, 1, \ldots, n$, $0 < \theta < 1$.

geometric(θ) pmf $p(x) = \theta(1-\theta)^x$ for $x = 0, 1, 2, \ldots$, $0 < \theta < 1$.

negative binomial(r, θ) pmf $p(x) = \binom{r-1+x}{r-1}\theta^r(1-\theta)^x$ for $x = 0, 1, 2, \ldots$, $0 < \theta < 1$.

Poisson(λ) pmf $p(x) = \frac{\lambda^x}{x!}\,e^{-\lambda}$ for $x = 0, 1, 2, \ldots$, $\lambda > 0$.

uniform(a, b) pdf $f(x) = \frac{1}{b-a}$ for $a < x < b$.

exponential(λ) pdf $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, $\lambda > 0$.

gamma(α, λ) pdf $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\,x^{\alpha-1}e^{-\lambda x}$ for $x > 0$, $\lambda > 0$, $\alpha > 0$.

normal(μ, σ^2) pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-(x-\mu)^2/(2\sigma^2)}$ for $-\infty < x < \infty$, $\sigma^2 > 0$.

Weibull(α, β) pdf $f(x) = \frac{\alpha}{\beta}\,(x/\beta)^{\alpha-1}e^{-(x/\beta)^\alpha}$ for $x > 0$, $\alpha > 0$, $\beta > 0$.

beta(a, b) pdf $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,x^{a-1}(1-x)^{b-1}$ for $0 < x < 1$, $a > 0$, $b > 0$.

joint cdf $F_{X,Y}(x, y) = \mathsf{P}(\{X \leq x\} \cap \{Y \leq y\})$.

joint pmf $p_{X,Y}(x, y) = \mathsf{P}(\{X = x\} \cap \{Y = y\})$, $F_{X,Y}(x, y) = \sum_{u \leq x}\sum_{v \leq y} p_{X,Y}(u, v)$.

joint pdf $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y}F_{X,Y}(x, y)$, $F_{X,Y}(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u, v)\,dvdu$.

marginal pmf/pdf $p_X(x) = \sum_y p_{X,Y}(x, y)$, $p_Y(y) = \sum_x p_{X,Y}(x, y)$;
$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dy$, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx$.

conditional pmf/pdf $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$; $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.

independent random variables $p(x, y) = p_X(x)p_Y(y)$ for all (x, y) if (X, Y) is discrete;
$f(x, y) = f_X(x)f_Y(y)$ for all (x, y) if (X, Y) is continuous.

discrete convolution $p_{X+Y}(z) = \sum_x p_X(x)p_Y(z - x)$ for independent X, Y.

continuous convolution $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z - x)\,dx$ for independent X, Y.

cdf of minimum $F_{\min(X_1,\ldots,X_n)}(u) = 1 - (1 - F_{X_1}(u)) \times \cdots \times (1 - F_{X_n}(u))$ for independent $X_1, \ldots, X_n$.

cdf of maximum $F_{\max(X_1,\ldots,X_n)}(u) = F_{X_1}(u) \times \cdots \times F_{X_n}(u)$ for independent $X_1, \ldots, X_n$.