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Homework 1
9/10/21
STAT 630

1.) Chp Exercises 1.2.2 & 1.2.9

1.2.2: $\because$ $S = \{1,2,3,4,5,6,7,8\}$ w/ $P(\{s\}) = 1/8 \ \forall s \in S$

a.) what is $P(\{1,2\})$

let $A = \{1,2\}$

$\boxed{P(A) = 1/4}$

b.) what is $P(\{1,2,3\})$

let $B = \{1,2,3\}$

$\boxed{P(B) = 3/8}$

c.) list all events A s.t. $P(A) = 1/2$

let $S = \{1,2,3,4,5,6,7,8\}$

Then $A = \{\{i,j,k,l\} \mid (i<j<k<l) \wedge i,j,k,l \in S\}$

$\boxed{\text{There are } \binom{8}{4} = 70 \text{ such sets}}$

1.2.9: $\because$: $S = \{1,2,3,4\}$, $P(\{1\}) = 1/12$, $P(\{1,2\}) = 1/6$

& $P(\{1,2,3\}) = 1/3$. Compute $P(\{1\}), P(\{2\}), P(\{3\})$ & $P(\{4\})$.

Let $A = P(\{1\})$, $B = P(\{2\})$, $C = P(\{3\})$, $D = P(\{4\})$

① $P(A) = 1/12$

② $P(A) + P(B) = 2/12 \overset{①}{\Rightarrow} P(B) = 1/12$

③ $P(A) + P(B) + P(C) = 4/12 \overset{①,②}{\Rightarrow} P(C) = 1/6$

$P(A) + P(B) + P(C) + P(D) = 1 \overset{①,②,③}{\Rightarrow} P(D) = 2/3$

$\boxed{P(\{1\}) = 1/12,\ P(\{2\}) = 1/12,\ P(\{3\}) = 1/6,\ P(\{4\}) = 2/3}$

2) Two 6-sided dice are thrown sequentially & the values they show are recorded.

(a) List the sample space: Let S be the sample space.

$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

b.) List the outcomes that make up the following events:

- $A = \{(i,j) \mid (i+j \geq 9) \land (i,j \in [1,2,3,4,5,6])\}$

  $\boxed{A = \{(3,6), (4,5), (4,6), (5,4), (5,5), (5,6), (6,3), (6,4), (6,5), (6,6)\}}$

- $B = \{(i,j) \mid (i > j) \land (i,j \in [1,2,3,4,5,6])\}$

  $\boxed{B = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}}$

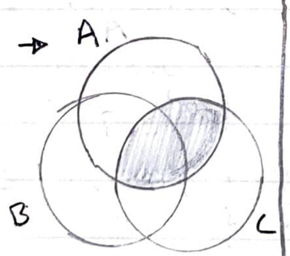
- $C = \{(i,j) \mid j = 4 \land (i,j \in [1,2,3,4,5,6]\}$

  $\boxed{C = \{(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)\}}$

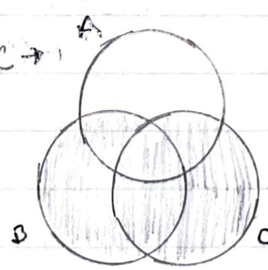
c.) List the elements of the following events:

- $A \cap C = \{(5,4), (6,4)\}$
- $B \cup C = \{(2,1), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}$
- $A \cap (B \cup C) = \{(5,4), (6,3), (6,4), (6,5)\}$

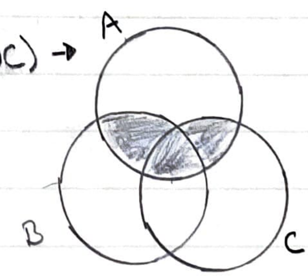
d.) $A \cap C \rightarrow$



$B \cup C \rightarrow$



$A \cap (B \cup C) \rightarrow$



2) (continued)

e) Assume the outcomes are equally likely and find the probabilities of the events in part (c)

- $P(A\cap C) = \frac{|A\cap C|}{|S|} = \frac{2}{36} = \underline{1/18 = P(A\cap C)}$
- $P(B\cup C) = \frac{|B\cup C|}{|S|} = \frac{18}{36} = \underline{1/2 = P(B\cup C)}$
- $P(A\cap(B\cup C)) = \frac{|A\cap(B\cup C)|}{|S|} = 4/36 = \underline{1/9 = P(A\cap(B\cup C))}$

f.) Can $P(A\cap C)$ be found by multiplying the probabilities of A & C?

$\boxed{\text{No b/c } A \,\&\, C \text{ are not independent events. i.e. } P(C) = 1/6 \neq 1/5 = P(C|A)}$

g.) Imagine this experiment being repeated many times. What would be the long-term proportion of all the experiments for which the sum of the two dice is 7?

- Let $A = \{(i,j) \mid (i+j=7) \wedge (i,j \in [1,2,3,4,5,6])\}$

$\boxed{P(A) = 1/6}$

3.) Chp 1. exercises 1.3.2, 1.3.4, 1.3.8, 1.3.10 (a)

<u>1.3.2</u>: β Al watches 6 o'clock news 2/3 of the time, 11 o'clock news 1/2 of the time and watches both 1/3 of the time.

For a randomly selected day, what is the probability that Al watches only the 6 o'clock news? For a randomly selected day, what is the probability that Al watches neither news?

- Let A be the event Al watches the 6 o'clock news

  B be the event Al watches the 11 o'clock news.

- Given: $P(A) = 2/3$, $P(B) = 1/2$, $P(A\cap B) = 1/3$
- <u>WTF</u>: ① $P(A\cap B^c)$, ② $P(A^c\cap B^c) = P((A\cup B)^c) = 1 - P(A\cup B)$

- $P(A\cup B) = P(A) + P(B) - P(A\cap B) = 2/3 + 1/2 - 1/3 = 5/6.$

  $P(A\cup B) = 5/6 \Rightarrow \boxed{P(A^c\cap B^c) = 1/6}$

$A = (A\cap B)\cup(A\cap B^c) \Rightarrow$

$P(A) = P(A\cap B) + P(A\cap B^c)$

$2/3 = 1/3 + P(A\cap B^c) \Rightarrow \boxed{P(A\cap B^c) = 1/3}$

(Contd)

1.3.4: β your right knee is sore 15% of the time & your left knee is sore 10% of the time. What is the largest possible percentage of time that at least one of your knees is sore? What is the smallest possible percentage at least one of your knees is sore.

- Let A be the event that one of your knees are sore
  R be the event that your right knee is sore
  L " " left " "

- Then $\boxed{0.15 \leq A \leq 0.25.}$

Justification: • Assume only one knee can be sore at a time.
i.e. $P(R \cap L) = 0$, then
$P(L \cup R) = P(L) + P(R) - P(L \cap R) = 0.10 + 0.15 - 0 = 0.25$
• Now assume if your left knee is sore, then so is the right.
i.e. $P(R|L) = 1$.
Then $P(L \cap R) = P(R|L) * P(L) = 1 * 0.1 = 0.1$
$P(L \cup R) = P(L) + P(R) - P(L \cap R) = 0.10 + 0.15 - 0.10 = 0.15$

1.3.8: β: 55% of students are female of which 4/5 (44% of total pop WL) have long hair and 45% are male of which 1/3 (15% of total pop ML) have long hair. What is the probability that a student chosen at random will either be female or have long hair (or both)?

- Let F be the event a student is female &
- Let L be the event a student has long hair.
- Given: $P(F) = 0.55$, $P(L) = 0.59$, $P(F \cap L) = 0.44$
- WTF: $P(F \cup L)$

$P(F \cup L) = P(F) + P(L) - P(F \cap L)$
$= 0.55 + 0.59 - 0.44$
$\boxed{P(F \cup L) = 0.70}$

3) (Cont'd)

1.3.10: Generalize the principle of inclusion-exclusion as follows:

(a) Suppose there are three events A, B, C. Prove that

$$P(A\cup B\cup C) = P(A)+P(B)+P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

Proof:

$P(A\cup B\cup C) = P((A\cup B)\cup C)$. By the inclusion-exclusion principle for two events (Theorem 1.3.3 in Probability & Statistics) we know that $P((A\cup B)\cup C) = P(A\cup B) + P(C) - P((A\cup B)\cap C)$. Applying Theorem 1.3.3 to $P(A\cup B)$ and the distributive law for sets to $P((A\cup B)\cap C)$ we can rewrite the above expression as:

$$P(A) + P(B) - P(A\cap B) - P((A\cap C)\cup(B\cap C)).$$

Again, applying theorem 1.3.3 to $P((A\cap C)\cup(B\cap C))$ we can rewrite the above expression as:

$$P(A)+P(B) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P((A\cap C)\cap(B\cap C))$$

Finally, applying the distributive law for sets to $P((A\cap C)\cap(B\cap C))$ we get

$$P(A\cup B\cup C) = P(A)+P(B) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C).$$ QED

4) Chp 1.4 Exercises 1.4.1, 1.4.6, 1.4.11, 1.4.12

1.4.1: Suppose we roll eight fair six-sided dice.

(a) What's the probability that all eight dice show a 6?

- Let $A = \{6,6,6,6,6,6,6,6\}$

$$\boxed{P(A) = (1/6)^8}$$

(b) What's the probability that all eight dice show the same number?

- Let A be the event that all eight dice show the same number.

$$\boxed{P(A) = 6(1/6)^8 = (1/6)^7}$$

(c) What's the probability that the sum of the eight dice is equal to 9?

- Let $A = \{(x_1,\dots,x_8) : (x_1+x_2+\dots+x_8 = 9) \wedge (x_1,\dots,x_8 \in [1,\dots,6])\}$

$$\boxed{P(A) = \binom{8}{1}/6^8}$$

4) (cont'd)

1.4.6a: $ we pick two cards at random from an ordinary 52 card deck. What is the probability that the sum of the values of the two cards is at least 4?

- Let $A = \{\{i,j\} : (i+j \geq 4) \wedge (i,j \in [1,2,\ldots,10])\}$
- Find $P(A^c)$, where $A^c = \{\{i,j\} : (i+j \leq 3) \wedge (i,j \in [1,2,\ldots,10])\}$
  - If Ace of first draw: 4 (options for 1st card) • 7 (options for second card) = 28 (total options)
  - If 2 on first draw: 4 (options for 1st card) • 4 (options for second card) = 16 (total options)
- $P(A^c) = \dfrac{(4 \cdot 7)+(4 \cdot 4)}{52 \cdot 51} \approx 0.01659$
- $P(A) = 1 - P(A^c) = 0.9834$

1.4.11 Consider two urns, labeled urn #1 and urn #2. $ urn #1 has 5 red & 7 blue balls. $ Urn #2 has 6 red and 12 blue balls. $ we pick 3 balls at random from each urn. What is the probability that all 6 balls chosen are of the same color?

$$\boxed{\frac{5 \cdot 4 \cdot 3 \cdot 6 \cdot 5 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 18 \cdot 17 \cdot 16} + \frac{7 \cdot 6 \cdot 5 \cdot 12 \cdot 11 \cdot 10}{12 \cdot 11 \cdot 10 \cdot 18 \cdot 17 \cdot 16} \approx 0.044}$$

alt method:

$$\boxed{\frac{\binom{5}{3}\binom{6}{3} + \binom{7}{3}\binom{12}{3}}{\binom{12}{3}\binom{18}{3}} \approx 0.044}$$

1.4.13. $ we roll a fair six-sided die and flip three fair coins. What is the probability that the total # of heads is equal to the number showing on the die. Let E be the event described above.

- Let A be the event that we roll a 1 and get 1 head in the three coin flips.
- Let B be the event that we roll a 2 and get 2 heads in the three coin flips.
- Let C be the event that we roll a 3 and get 3 heads in the three coin flips.
- Let D be the event that we roll a # ≥ 4 and get ≥ 4 heads in the three coin flips
- B/c A, B, C & D are disjoint events, $P(E) = P(A) + P(B) + P(C) + P(D)$
  - $P(A) = 1/6 \cdot \binom{3}{1}(0.5)^3 = 0.0625$, $P(B) = 1/6\binom{3}{2}(0.5)^3 = 0.0625$
  - $P(C) = 1/6\binom{3}{3}(0.5)^3 = 0.0208\overline{3}$, $P(D) = 0$.

$$\boxed{P(E) = 2(0.0625) + 0.02083 = 0.14583}$$

5.) Long ago we posted grades identified by the last 4 digits of a students social security number. Assume each of the 10000 configurations of 0000 to 9999 are equally likely.

(a) (Done in R)* Estimate the probability that at least two students in a class of 100 share the same 4 digits.

$\boxed{0.38611}$

(b) Compute the actual probability and compare it to your estimate.

- First compute the probability no two have the same number

$$\frac{P_{k,10000}}{10000^k} = \frac{(10000!/(10000-k)!)}{10000^k}$$

- The probability at least two (in a class of 100) have the same birthday:

$$P(100) = 1 - \frac{(10000!/(10000-100)!)}{10000^{100}}$$

$\boxed{P(100) = 0.391434}$

(c) What is the smallest enrollment for which the probability at least two students have the same 4 digits is at least 0.50?

(Done in R*)

- $\boxed{119 \text{ people}}$

6.) Chp 1 exercises 1.5.1, 1.5.8, 1.5.10

1.5.1: $ that we roll four six-sided dice.

ask about showing work -> ? (a) What is the probability that the first dice shows 2, conditional on the event that exactly 3 dice show 2?

- Let A be the event the first dice shows 2
- Let B " " exactly 3 dice show 2

$\boxed{P(A|B) = 18/24 = 0.75}$

6) (contd)

1.5.1 (contd):

(6) What is the conditional probability that the first dice shows 2, conditional on the event that at least three dice show 2?

- Let A be the event that the first dice shows 2
- Let B " " at least three dice show 2.

NOTE: WTF: $P(A|B) = \frac{P(A\cap B)}{P(B)}$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)}$$

- $P(A) = 1/6 \Leftrightarrow P(A^c) = 5/6$
- $P(B|A) = \binom{3}{2}(1/6)^2(5/6) + \binom{3}{3}(1/6)^3(5/6)^0$
- $P(B|A^c) = \binom{3}{3}(1/6)^3$

$$P(A|B) = \frac{\left(\binom{3}{2}(1/6)^2(5/6) + (1/6)^3\right)(1/6)}{\left(\binom{3}{2}(1/6)^2(5/6) + (1/6)^3\right)(1/6) + (1/6)^3(5/6)} \approx 0.7619$$

$\boxed{P(A|B) = 16/21}$

1.5.8: Suppose the probability of snow is 20% and the probability of a traffic accident is 10%. Suppose further that the conditional probability of an accident, given that it snows is 40%. What is the conditional prob that it snows, given there is an accident?

- Let A be the event that it snows
- Let B be the event there is an accident

GIVEN: $P(A) = 0.20$ $P(A|B) = ?$

$P(B) = 0.10$ $P(B|A) = 0.40$

- $P(B|A) = \frac{P(A\cap B)}{P(A)} \Leftrightarrow 0.4 = \frac{P(A\cap B)}{0.2} \Leftrightarrow P(A\cap B) = 0.08$
- $P(A|B) = \frac{P(A\cap B)}{P(B)} \Leftrightarrow P(A|B) = \frac{0.08}{0.01} \Leftrightarrow \boxed{P(A|B) = 0.80}$

6.) (contd)

1.5.10: Consider two urns, labeled urn #1 and urn #2. $ as in Exercise 1.4.11 that urn #1 has 5 red and 7 blue balls, urn #2 has 6 red and 12 blue balls and that we pick 3 balls uniformly at random from each of the two urns. Conditional on the fact that all 6 balls chosen are the same color, what is the conditional probability that the color is red?

- Let A be the event that all the balls selected are red.
- Let B be the event that all the balls selected are the same color

- WTF: $P(A|B)$

NOTE: $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)\,P(B|A)}{P(B)}$

- $P(A) = \left[\binom{5}{3}\binom{6}{3}\right] / \left[\binom{12}{3}\binom{18}{3}\right] = 0.001114082$
- $P(B) = \left[\binom{5}{3}\binom{6}{3} + \binom{7}{3}\binom{12}{3}\right] / \left[\binom{12}{3}\binom{18}{3}\right] = 0.0440062389$
- $P(B|A) = 1$

$\boxed{P(A|B) \approx 0.02532}$

7.) Chp 1 Exercise 1.5.7:

$ a baseball pitcher throws fastballs 80% of the time and curveballs 20% of the time. $ a batter hits a HR on 8% of all fastballs and on 5% of all curveballs. What is the probability that the batter will hit a HR on the next pitch.

- Let A be the event that the batter hits a home run on the next pitch
- Let B " " pitcher throws a curve "
- Let C " " fastball "

WTF. $P(A)$. NOTE: Use LOTP.

$$P(A) = P(A|B)P(B) + P(A|C)P(C)$$
$$= 0.05(0.20) + 0.08(0.80)$$

$\boxed{P(A) = 0.074}$

7.) (Cont'd)

b) ∮ we are given that the batter hits a HR. What is the conditional probability that he was thrown a curve?

- Let A be the event the batter was thrown a curve
- Let B be the event the batter hit a home run.

WTF: $P(A|B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{(0.05)(0.20)}{0.074} = 0.\overline{135}$$

$$\boxed{P(A|B) = 0.\overline{135}}$$

c.) ∮ The batter doesn't hit a HR. What is the probability he was thrown a curve?

- Let A be the event the batter was thrown a curveball.
- Let B be the event the batter did not hit a home run.

WTF: $P(A|B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{(0.95)(0.20)}{0.926} \approx 0.20518$$

$$\boxed{P(A|B) \approx 0.20518}$$