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STAT 630
Assignment 4
6/1/21

1.) Chp 2 Exercises 2.7.3, 2.7.4 (a, d - it suffices to express C as a fraction)

2.7.3: B: $p_{x,y}(x,y) = \begin{cases} 1/5 & x=2, y=3 \\ 1/5 & x=3, y=2 \\ 1/5 & x=-3, y=-2 \\ 1/5 & x=-2, y=-3 \\ 1/5 & x=17, y=19 \\ 0 & \text{o.w.} \end{cases}$

(a) $p_X(x) = \begin{cases} 1/5 & \text{for } x \in (2,3,-3,-2,17) \\ 0 & \text{o.w.} \end{cases}$

(b) $p_Y(y) = \begin{cases} 1/5 & \text{for } y \in (3,2,-2,-3,19) \\ 0 & \text{o.w.} \end{cases}$

(c) $\boxed{P(Y>X) = 3/5}$

(d) $\boxed{P(Y=X) = 0}$

(e) $\boxed{P(XY<0) = 0}$

2.7.4: For each of the following joint density functions $f_{X,Y}$, find the value of C and compute $f_X(x)$, $f_Y(y)$; $P(X \le 0.8, Y \le 0.6)$

(a) $f_{X,Y}(x,y) = \begin{cases} 2x^2y + cy^5 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$

- $F_{X,Y}(x,y) = \int_0^1 \int_0^1 2x^2y + cy^5 \, dx\, dy = \int_0^1 \left[\frac{2}{3}x^3y + cxy^5\right]_{x=0}^{x=1} dy$

$= \int_0^1 \frac{2}{3}y + cy^5 \, dy = \frac{1}{3}y^2 + \frac{c}{6}y^6 \Big|_{y=0}^{y=1} = \frac{2}{6} + \frac{c}{6} = 1$

$\Rightarrow \boxed{c=4}$

- $f_X(x) = \int_0^1 2x^2y + 4y^5 \, dy = x^2y^2 + \frac{4}{6}y^6 \Big|_0^1 = x^2 + 4/6$

$\boxed{f_X(x) = \begin{cases} x^2 + 2/3 & ; x \in [0,1] \\ 0 & ; \text{o.w.} \end{cases}}$

- $f_Y(y) = \int_0^1 2x^2y + 4y^3 \, dx = \frac{2}{3}x^3y + 4xy^3 \Big|_{x=0}^{x=1} = \frac{2}{3}y + 4y^3$

$\boxed{f_Y(y) = \begin{cases} \frac{2}{3}y + 4y^3 & ; y \in [0,1] \\ 0 & ; \text{o.w.} \end{cases}}$

2.7.4 (cont'd)

$$P(X \le 0.8, Y \le 0.6) = \int_0^{0.6}\int_0^{0.8} 2x^2y + 4y^5\,dx\,dy = \int_0^{0.6} \frac{2}{3}x^3y + 4xy^5 \Big|_{x=0}^{x=0.8} dy$$

$$= \int_0^{0.6} \frac{2}{3}(0.8)^3 y + 4(0.8)y^5\,dy = \frac{1}{3}(0.8)^3y^2 + \frac{4}{6}(0.8)y^6 \Big|_{y=0}^{y=0.6}$$

$$= \frac{2}{3}(0.8)^3(0.6)^2 + \frac{2}{3}(0.8)(0.6)^6 = \boxed{0.0863}$$

(d) $f_{X,Y}(x,y) = \begin{cases} Cx^5y^5 & ;\ 0 \le x \le 4;\ 0 \le y \le 10 \\ 0 & ;\ \text{o.w.} \end{cases}$

- $$F_{X,Y}(x,y) = \int_0^{10}\int_0^{4} Cx^5y^5\,dx\,dy = \int_0^{10} \frac{C}{6}x^6y^5 \Big|_{x=0}^{x=4} dy$$

$$= \int_0^{10} \frac{4096C}{6}y^5\,dy = \frac{4096C}{36}y^6\Big|_0^{10} = \frac{4096(1{,}000{,}000)C}{36} = 1$$

$$\Rightarrow \boxed{C = 36/(4096 * 1{,}000{,}000)}$$

- $$f_X(x) = \int_0^{10} Cx^5y^5\,dy = \frac{C}{6}x^5y^6\Big|_0^{10} \Rightarrow f_X(x) = \begin{cases} \frac{1000000C}{6}x^5 & ;\ 0 \le x \le 4 \\ 0 & ;\ \text{o.w.} \end{cases}$$

$$\Rightarrow \boxed{f_X(x) = \begin{cases} \frac{6}{4096}x^5 & ;\ 0 \le x \le 4 \\ 0 & ;\ \text{o.w.} \end{cases}}$$

- $$f_Y(y) = \int_0^4 Cx^5y^5\,dx = \frac{C}{6}x^6y^5\Big|_{x=0}^{x=4} = \left(\frac{6}{1000000}\right)y^5$$

$$\boxed{f_Y(y) = \begin{cases} \left(\frac{6}{1000000}\right)y^5 & ;\ 0 \le y \le 10 \\ 0 & ;\ \text{o.w.} \end{cases}}$$

- $$P(X \le 0.8, Y \le 0.6) = \int_0^{0.8}\int_0^{0.6} Cx^5y^5\,dy\,dx = \int_0^{0.8} \frac{C}{6}x^5y^6\Big|_0^{0.6} dx$$

$$= \int_0^{0.8} \frac{C(0.6)^6}{6}x^5\,dx = \boxed{\frac{C(0.6)^6(0.8)^6}{36}}$$

2.) 2.7.8: Let X & Y have joint density $f_{X,Y}(x,y) = (x^2+y)/36$ for $x \in (-2,1)$ & $y \in (0,4)$; o.w. $f_{XY}(x,y) = 0$.

(a) The marginal density $f_X(x)$ $\forall x \in \mathbb{R}^1$

$$f_X(x) = \frac{1}{36}\int_0^4 x^2+y\,dy = \frac{1}{36}\left[x^2y + \frac{1}{2}y^2\right]_{y=0}^{y=4} = \frac{1}{36}\left[4x^2+8\right] = \frac{1}{9}x^2 + \frac{2}{9}$$

$$\boxed{f_X(x) = \begin{cases} \frac{x^2+2}{9} & ; x \in (-2,1) \\ 0 & \text{o.w.} \end{cases}}$$

(b) The marginal density $f_Y(y)$ $\forall y \in \mathbb{R}$.

$$f_Y(y) = \frac{1}{36}\int_{-2}^{1} x^2+y\,dx = \frac{1}{36}\left[\frac{x^3}{3} + yx\right]_{x=-2}^{x=1} = \frac{1}{36}\left[\left(\frac{1}{3}+y\right) - \left(-\frac{8}{3} - 2y\right)\right]$$

$$= \frac{1}{36}\left[3+3y\right] = \frac{1+y}{12}$$

$$\boxed{f_Y(y) = \begin{cases} (1+y)/12 & ; y \in (0,4) \\ 0 & ; \text{o.w} \end{cases}}$$

(c) $P(Y<1) = \frac{1}{12}\int_0^1 1+y\,dy = \frac{1}{12}\left[y + \frac{1}{2}y^2\right]_{y=0}^{y=1} = \frac{1}{8}$

$\boxed{P(Y<1) = 1/8}$

(d) $F_{XY}(x,y) = \frac{1}{36}\int_0^y\int_{-2}^x x^2+y\,dx\,dy = \frac{1}{36}\int_0^y \frac{x^3}{3} + xy\Big|_{x=-2}^{x=x} = \frac{1}{36}\int_0^y\left[\left(\frac{x^3}{3}+xy\right) - \left(-\frac{8}{3} - 2y\right)\right]dy$

$$= \frac{1}{36}\int_0^y \frac{x^3}{3} + xy + 2y + 8/3\,dy = \frac{1}{36}\left[\frac{x^3y}{3} + \frac{xy^2}{2} + y^2 + \frac{8y}{3}\right]_{y=0}^{y=y}$$

$$\boxed{F_{XY}(x,y) = \begin{cases} \frac{1}{36}\left[\frac{x^3y}{3} + \frac{xy^2}{2} + y^2 + \frac{8y}{3}\right] & ; x \in (-2,1) \wedge y \in (0,4) \\ 0 & ; \text{o.w.} \end{cases}}$$

2.7.9: Let X & Y have joint density $f_{XY}(x,y) = \frac{1}{4}(x^2+y)$ for $0<x<y<2$. o.w. $f_{XY}(x,y)=0$.

(a) $f_X(x) = \frac{1}{4}\int_x^2 x^2+y\,dy = \frac{1}{4}\left[x^2y+\frac{1}{2}y^2\right]_{y=x}^{y=2} = \frac{1}{4}\left[(2x^2+4)-(x^3+\frac{1}{2}x^2)\right]$

$= \frac{1}{4}\left[\frac{3}{2}x^2 - x^3 + 4\right] = \frac{3}{8}x^2 - x^3 + 1$

$$\boxed{f_X(x) = \begin{cases} 3/8x^2 - x^3 + 1 & ; \ x \in (0,2) \\ 0 & ; \ \text{o.w.} \end{cases}}$$

(b) $f_Y(y) = \frac{1}{4}\int_0^y x^2+y\,dx = \frac{1}{4}\left[\frac{1}{3}x^3+xy\right]_{x=0}^{x=y} = \frac{1}{4}\left[\frac{1}{3}y^3+y^2\right]$

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{12}y^3 + \frac{1}{4}y^2 & ; \ y \in (0,2) \\ 0 & ; \ \text{o.w.} \end{cases}}$$

(c) $P(Y<1) = \int_0^1 \frac{1}{12}y^3 + \frac{1}{4}y^2\,dy = \frac{1}{48}y^4 + \frac{1}{12}y^3\Big|_0^1 = \frac{5}{48}$

$$\boxed{P(Y<1) = 5/48}$$

2.) (Cont'd) 2.7.16: β that the joint density $f_{X,Y}$ is given by

$$f_{X,Y}(x,y) = Ce^{-(x+y)} \quad \text{for} \quad 0<x<y<\infty \text{ ; } 0 \text{ o.w.}$$

(a) Determine C s.t. $f_{X,Y}$ is a density:

$$1 = \int_0^\infty \int_0^y Ce^{-(x+y)}\,dx\,dy = -C\int_x^\infty e^{-(x+y)}\Big|_{x=0}^{x=y} dy$$

$$= -C\int_0^\infty e^{-2y} - e^{-y}\,dy = -C\left[-\tfrac{1}{2}e^{-2y} + e^{-y}\right]_{y=0}^{y=\infty}$$

$$= C\left[\frac{e^{-2y}}{2} - e^{-y}\right]_{y=0}^{y=\infty} = C\left[\left(\lim_{y\to\infty}\left(\frac{1}{2e^{2y}} - \frac{1}{e^{y}}\right)\right) - \left(\frac{e^{-2\cdot 0}}{2} - e^{-0}\right)\right]$$

$$1 = C\left(\tfrac{1}{2}\right) \Rightarrow \boxed{C=2}$$

(b) Compute the marginal densities of X & Y.

- $$f_X(x) = \int_x^\infty Ce^{-(x+y)}\,dy = \lim_{a\to\infty}\left(\int_x^a Ce^{-(x+y)}\,dy\right)$$

[*~~For practice~~: Let $u = -(x+y)$; $\frac{du}{dy} = -1 \Rightarrow dy = -du$

bounds $\Rightarrow$ $y=x \to u=-2x$, $y=a \to u=-x-a$

$$f_X(x) = \int_{-2x}^{-x-a} -Ce^{u}\,du = -Ce^{u}\Big|_{u=-2x}^{u=-x-a} = -C\left(e^{-x-a} - e^{-2x}\right)$$]

$$\lim_{a\to\infty}\left(-C\left(e^{-x-a} - e^{-2x}\right)\right) = \lim_{a\to\infty} \cancel{-Ce^{-x-a}}^{\,0} + \lim_{a\to\infty} Ce^{-2x}$$

$$\boxed{f_X(x) = \begin{cases} 2e^{-2x} & ; x\in(0,\infty) \\ 0 & ; \text{o.w.}\end{cases}}$$

- $$f_Y(y) = \int_0^y Ce^{-(x+y)}\,dx = -C\left[e^{-(x+y)}\right]_{x=0}^{x=y} = -C\left[e^{-2y} - e^{-y}\right]$$

$$\boxed{f_Y(y) = \begin{cases} 2(e^{-y} - e^{-2y}) & ; y\in(0,\infty) \\ 0 & \text{o.w.}\end{cases}}$$

3) Chp 2 Exercise 2.7.10. You may use the result (w/o proof) in exercise 2.7.13

2.7.10: Let X & Y have Bivariate-Normal (3, 5, 2, 4, 1/2) distribution.

(a) Specify the marginal distribution of X.

Using the result from 2.7.13, we know that
IF $X, Y \sim BNorm(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$
then: $X \sim N(\mu_1, \sigma_1)$ w/
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

Thus, $X \sim (3, 2)$ w/
$$\boxed{f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-(x-3)^2/8} \;;\; x \in (-\infty, \infty)}$$

(b) Specify the marginal distribution of Y.

* Using the result from 2.7.13
$Y \sim (5, 4)$ w/
$$\boxed{f_Y(y) = \frac{1}{4\sqrt{2\pi}} e^{-(y-5)^2/32} \;;\; y \in (-\infty, \infty)}$$

(c) Are X & Y independent? Why or why not?

No, looking at Example 2.7.9, we see that
for $X, Y \sim BNorm(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$
X & Y are independent iff $\rho = 0$. Here
$\rho = 0.5 \neq 0 \Rightarrow$ X & Y are NOT independent.

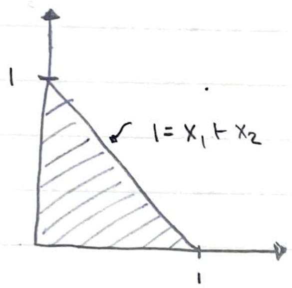
4.) Chp 2 Exercise 2.7.17; [This is a continuous analogue to the multinomial distribution]

(Dirichlet$(\alpha_1,\alpha_2,\alpha_3)$ distribution. Let (X_1, X_2) have the joint density

$$f_{X_1,X_2}(x_1,x_2) = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1} x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}$$

for $x_1 \geq 0$, $x_2 \geq 0$; $0 \leq x_1 + x_2 \leq 1$. A Dirichlet distribution is often applicable when X_1, X_2 and $1-X_1-X_2$ correspond to random proportion.

(a) Prove f_{X_1,X_2} is a density. (Hint: Sketch the region where f_{X_1,X_2} is nonnegative, integrate out x_1 first by making the transformation $u = x_1/(1-x_2)$ in this integral and use 2.4.10 from problem 2.4.24



• NOTE: $0 \leq x_1 + x_2 \leq 1 \Rightarrow 0 \leq x_1 \leq 1 - x_2$

$0 \leq x_2 \leq 1$

• To prove $f_{X_1,X_2}(x_1,x_2)$ is a density WTS:

$$\int_0^1 \int_0^{1-x_2} f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 = 1.$$

$$\int_0^1\int_0^{1-x_2} f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 = \int_0^1\int_0^{1-x_2} \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x_1^{\alpha_1-1} x_2^{\alpha_2-1}(1-x_1-x_2)^{\alpha_3-1}\,dx_1\,dx_2$$

• Let $u = \frac{x_1}{1-x_2} \Rightarrow x_1 = u(1-x_2)$; $du = \frac{dx_1}{1-x_2} \Rightarrow dx_1 = (1-x_2)\,du$

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^1\int_0^1 [u(1-x_2)]^{\alpha_1-1} x_2^{\alpha_2-1}(1-u(1-x_2)-x_2)^{\alpha_3-1}(1-x_2)\,du\,dx_2$$

call this C

$$= \left\{\frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\right\} \int_0^1\int_0^1 (1-x_2)^{\alpha_1-1} u^{\alpha_1-1} x_2^{\alpha_2-1} (\underbrace{1-u-x_2+ux_2}_{(1-u)(1-x_2)})^{\alpha_3-1}(1-x_2)\,du\,dx_2$$

$$= C\int_0^1\int_0^1 (1-x_2)^{\alpha_1-1} u^{\alpha_1-1} x_2^{\alpha_2-1}(1-u)^{\alpha_3-1}(1-x_2)^{\alpha_3-1}(1-x_2)\,du\,dx_2$$

$$= C\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_3-1}\,du \int_0^1 x_2^{\alpha_2-1}(1-x_2)^{\alpha_1+\alpha_2-1}\,dx_2$$ (can split up integral by like terms b/c both integrals go to constants)

[• Note: $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}$ (from pg 37 in Chp 2 notes)

in the first integral, we have $a = \alpha_1$; $b = \alpha_3$;

in the second integral, we have $a = \alpha_2$; $b = \alpha_1+\alpha_2-1$]

$$= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \text{Beta}(\alpha_1,\alpha_3)\,\text{Beta}(\alpha_2,\alpha_1+\alpha_3)$$ $\rightarrow X_2 \sim \text{Beta}(\alpha_2, \alpha_1+\alpha_3)$

$$= \left[\frac{\cancel{\Gamma(\alpha_1+\alpha_2+\alpha_3)}}{\cancel{\Gamma(\alpha_1)}\cancel{\Gamma(\alpha_2)}\cancel{\Gamma(\alpha_3)}}\right]\left[\frac{\cancel{\Gamma(\alpha_1)}\cancel{\Gamma(\alpha_3)}}{\cancel{\Gamma(\alpha_1+\alpha_3)}}\right]\left[\frac{\cancel{\Gamma(\alpha_2)}\cancel{\Gamma(\alpha_1+\alpha_3)}}{\cancel{\Gamma(\alpha_2)}\cancel{\Gamma(\alpha_1+\alpha_2+\alpha_3)}}\right]$$

$= 1.$ $\therefore$ $f_{X_1,X_2}(x_1,x_2)$ is a density. QED.

4) (contd)

(b) We showed in part(a) that $X_2 \sim \text{Beta}(\alpha_2, \alpha_1+\alpha_3)$

If we change our u-sub we did in pt 1 to $u = \frac{x_2}{1-x_1}$. Then by symmetry,

we get

$$\int_0^1 \int_0^{1-x_1} f_{X_1,X_2}(x_1,x_2)\, dx_2\, dx_1 = \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \underbrace{\text{Beta}(\alpha_1, \alpha_2+\alpha_3)} \text{Beta}(\alpha_2, \alpha_3)$$

$$\Rightarrow \boxed{X_1 \sim \text{Beta}(\alpha_1, \alpha_2+\alpha_3)}$$

5.) (Cont'd)

2.8.3: $\not\exists$ X & Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} \tfrac{12}{49}\left(2+x+xy+4y^2\right) & x\in[0,1] \wedge y\in[0,1] \\ 0 & \text{o.w.} \end{cases}$$

(a.) Compute $f_X(x)$ $\forall x\in\mathbb{R}$.

- $f_X(x) = \frac{12}{49}\int_0^1 2+x+xy+4y^2\,dy = \frac{12}{49}\left[2y+xy+\frac{xy^2}{2}+\frac{4y^3}{3}\right]_{y=0}^{y=1}$

$= \frac{12}{49}\left[2+x+\frac{x}{2}+\frac{4}{3}\right] = \frac{24}{49}+\frac{16}{49}+\frac{3}{2}x = \frac{40}{49}+\frac{3}{2}x$

- $f_X(x) = \begin{cases} 40/49+\frac{3}{2}x & ; x\in[0,1] \\ 0 & ; \text{o.w.} \end{cases}$

(b) Compute $f_Y(x)$ $\forall y\in\mathbb{R}$

- $f_Y(y) = 12/49\int_0^1 2+x+xy+4y^2\,dx = \frac{12}{49}\left[2x+\frac{x^2}{2}+\frac{x^2y}{2}+4xy^2\right]_{x=0}^{x=1}$

$= \frac{12}{49}\left[2+\frac{1}{2}+\frac{1}{2}y+4y^2\right] = \frac{12}{49}\left[\frac{5}{2}+\frac{1}{2}y+4y^2\right]$

$= \frac{30}{49}+\frac{6}{49}y+\frac{48}{49}y^2$

$$\boxed{f_Y(y) = \begin{cases} \frac{30}{49}+\frac{6}{49}y+\frac{48}{49}y^2 & ; x\in[0,1] \\ 0 & ; \text{o.w.} \end{cases}}$$

(c) Determine whether or not X & Y are independent.

X & Y are independent if $f_X(x) * f_X(y) = f_{XY}(x,y)$

5.) (Cont'd)

2.8.7 (a,d): For each of the following joint density functions $f_{X,Y}$ compute the conditional density $f_{Y|X}(y|x)$ & determine whether or not X & Y are independent.

NOTE: $P_{Y|X}(y|x) = P(Y=y \mid X=x)$
$= P[(X=x)\cap(Y=y)] \,/\, P(X=x)$
$= P_{X,Y}(x,y) \,/\, P_X(x)$

(a) $f_{XY}(x,y) = \begin{cases} 2x^2y + Cy^5 & ; \ x\in[0,1],\ y\in[0,1] \\ 0 & ; \ \text{o.w.} \end{cases}$

- $1 = \int_0^1\int_0^1 2x^2y + Cy^5 \, dy\, dx = \int_0^1 x^2y^2 + \frac{c}{6}y^6 \Big|_{y=0}^{y=1} dx$

$= \int_0^1 x^2 + \frac{c}{6}\, dx = \left[\frac{x^3}{3} + \frac{c}{6}x\right]_{x=0}^{x=1} = \frac{2}{6} + \frac{1}{6}c = 1$

$\Rightarrow 4/6 = \frac{1}{6}c \Rightarrow \boxed{c=4}$

- $f_X(x) = \int_0^1 2x^2y + 4y^5\, dy = x^2y^2 + \frac{4}{6}y^6\Big]_{y=0}^{y=1} = x^2 + 4/6$

$\boxed{f_X(x) = \begin{cases} x^2 + 4/6 & ; \ x\in[0,1] \\ 0 & ; \ \text{o.w.} \end{cases}}$

- $f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \boxed{\frac{2x^2y + 4y^5}{x^2 + 4/6}}$

- $f_Y(y) = \int_0^1 2x^2y + 4y^5\, dx = \frac{2}{3}x^3y + 4xy^5\Big|_0^1 = \underline{\frac{2}{3}y + 4y^5}$

- $\boxed{X \text{ \& } Y \text{ are not independent as } f_X(x) * f_Y(y) \neq f_{XY}(x,y)}$

5.) Cont'd: 2.8.7

(d) $f_{XY}(x,y) = \begin{cases} Cx^5y^5 & ; \quad x \in [0,4] \wedge y \in [0,10] \\ 0 & ; \quad \text{o.w.} \end{cases}$

- $1 = \int_0^4 \int_0^{10} Cx^5y^5 \, dy \, dx = \frac{c}{6} \int_0^4 \left[x^5 y^6 \right]_{y=0}^{y=10} dx$

$= c\left(\frac{1000000}{6}\right) \int_0^4 x^5 \, dx = \frac{1000000\,c}{36} \left[x^6 \right]_{x=0}^{x=4} = \frac{1000000\,(4096)}{36} c = 1$

$\boxed{C = 36/4{,}096{,}000{,}000}$

- $f_X(x) = \int_0^{10} Cx^5y^5 \, dy = \frac{c}{6} \left[x^5 y^6 \right]_{y=0}^{y=10} = \boxed{\frac{10^6 c}{6} x^5}$

$$\boxed{f_X(x) = \begin{cases} \frac{6}{4600} x^5 & ; \quad x \in [0,4] \\ 0 & ; \quad \text{o.w.} \end{cases}}$$

- $f_Y(y) = C \int_0^4 x^5 y^5 \, dx = \begin{cases} \frac{6}{1000000} y^5 & ; \quad y \in [0,10] \\ 0 & ; \quad \text{o.w.} \end{cases}$

- $f_{Y|X}(y|x) = \frac{\frac{\cancel{36}\ 6}{(4096)(100000)} \cancel{x^5} y^5}{\left(\frac{6}{4600}\right) \cancel{x^5}} = \left(\frac{6}{1000000}\right) y^5$

$$\boxed{f_{Y|X}(y|x) = \begin{cases} \left(\frac{6}{1000000}\right) y^5 & ; \quad y \in [0,10] \\ 0 & ; \quad \text{o.w.} \end{cases}}$$

5.) (cont'd)

2.8.14: Let X & Y have joint density $f_{XY}(x,y) = \frac{(x^2+y)}{36}$ $\forall$ $x \in (-2,1)$, $y \in (0,4)$
$f_{XY}(x,y) = 0$.

(a). Compute the conditional density $f_{Y|X}(y|x)$ $\forall$ $x,y \in \mathbb{R}$

- $f_X(x) = \frac{1}{36}\int_0^4 x^2+y \, dy = \frac{1}{36}\left[x^2y + \frac{1}{2}y^2\right]_{y=0}^{y=4} = \frac{1}{36}\left[4x^2+8\right]$

$= \frac{1}{9}x^2 + \frac{2}{9}$

$$\boxed{f_X(x) = \begin{cases} 1/9\, x^2 + 2/9 & ; \quad x \in (-2,1) \\ 0 & ; \quad \text{o.w.} \end{cases}}$$

- $f_Y(y) = \frac{1}{36}\int_{-2}^{1} x^2+y \, dx = \left[x^3/3 + xy\right]_{x=-2}^{x=1}$

$= \frac{1}{36}\left[\left(\frac{1}{3}+y\right) - \left(-\frac{8}{3} - 2y\right)\right] = \frac{1}{36}\left[3+3y\right]$

$$\boxed{f_Y(y) = \begin{cases} 1/12\,(y+1). & ; \quad y \in (0,4) \\ 0 & ; \quad \text{o.w.} \end{cases}}$$

$$f_{Y|X} = \frac{\left(\frac{1}{36}\right)(x^2+y)}{\left(\frac{1}{9}\right)(x^2+2)} = \frac{(1/4)(x^2+y)}{(x^2+2)}$$

$$\boxed{f_{Y|X}(y|x) = \begin{cases} \frac{x^2+y}{4x^2+8} & ; \quad y \in (0,4) \\ 0 & ; \quad \text{o.w.} \end{cases}}$$

(b) compute the conditional density $f_{X|Y}$ $\forall$ $x,y \in \mathbb{R}$

$$f_{X|Y}(x,y) = \frac{(1/36)(x^2+y)}{(1/12)(y+1)} = \frac{x^2+y}{3y+3}$$

$$\boxed{f_{X|Y}(x|y) = \begin{cases} \frac{x^2+y}{3y+3} & ; \quad x \in (-2,1) \\ 0. & ; \quad \text{o.w.} \end{cases}}$$

(c) Are X & Y independent? Why or why not?

No, X & Y aren't independent. We can see this is true by the fact that $f_X(x) \neq f_{X|Y}(x|y)$

5.) (Contd)

2.8.15: Let X & Y have joint density $f_{X,Y}(x,y) = \frac{1}{4}(x^2+y)$ $\forall$ $0<x<y<2$

o.w. $f_{X,Y}(x,y) = 0$

$$f_X(x) = \frac{1}{4}\int_x^2 (x^2+y)\,dy = \frac{1}{4}\left[x^2y + \frac{1}{2}y^2\right]_{y=x}^{y=2} = \frac{1}{4}\left[(2x^2+2)-(x^3+\frac{1}{2}x^2)\right]$$

$$\boxed{f_X(x) = \begin{cases} \frac{1}{4}\left[\frac{3}{2}x^2 - x^3 + 2\right] & ; \quad x \in (0,2) \\ 0 & ; \quad \text{o.w} \end{cases}}$$

$$f_Y(y) = \frac{1}{4}\int_0^y (x^2+y)\,dx = \frac{1}{4}\left[\frac{1}{3}x^3 + xy\right]_{x=0}^{x=y} = \frac{1}{12}y^3 + y^2$$

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{12}y^3 + y^2 & ; \quad y \in (0,2) \\ 0 & ; \quad \text{o.w.} \end{cases}}$$

(a) The conditional density $f_{X|Y}(x|y)$ $\forall x,y \in \mathbb{R}$ w/ $f_Y(y)>0$

$$f_{X|Y}(x|y) = \frac{(1/4)(x^2+y)}{(1/12)(y^3+y^2)} = \frac{3x^2+3y}{y^3+y^2}$$

$$\boxed{f_{X|Y}(x|y) = \begin{cases} \frac{3x^2+3y}{y^3+y^2} & ; \quad 0<x<y \\ 0 & ; \quad \text{o.w.} \end{cases}}$$

(b) The conditional density $f_{Y|X}(y|x)$ $\forall x,y \in \mathbb{R}$ w/ $f_X(x)>0$

$$f_{Y|X}(y|x) = \frac{\cancel{(1/4)}(x^2+y)}{\cancel{(1/4)}\left[3/2\,x^2 - x^3 + 2\right]} = \frac{x^2+y}{\frac{3}{2}x^2 - x^3 + 2}$$

$$\boxed{f_{Y|X}(y|x) = \begin{cases} \frac{x^2+y}{3/2\,x^2 - x^3 + 2} & ; \quad x<y<2 \\ 0 & ; \quad \text{o.w} \end{cases}}$$

(c) No, they're not independent b/c $f_Y(y) \neq f_{Y|X}(y|x)$

$F_X(\alpha, 1) = x^{\alpha}$

slide 97 & 98

6.) β: $Y_1, \dots Y_n$ is a random sample from the beta $(\alpha, 1)$ distribution.
Find the cdfs & pdfs for $Y_{(1)} = \min(Y_1, \dots Y_n)$ & $Y_{(n)} = \max(Y_1, \dots Y_n)$
Are either of these beta RVs?

$$B(\alpha, 1) = \int_0^1 x^{\alpha-1}(1-x)^{1-1}\,dx = \int_0^1 x^{\alpha-1}\,dx = \frac{1}{\alpha}x^{\alpha}\Big|_0^1 = \frac{1}{\alpha}$$

$$f(x) = \alpha x^{\alpha-1}(1-x)^{b-1} I_{(0,1)}(x)$$
$$= \alpha x^{\alpha-1}(1-x)^0 \; ; \; x \in (0,1)$$
$$= \alpha x^{\alpha-1}$$

from pg 37 in chp 2 notes

$$\boxed{F_X(x) = \int_0^1 \alpha x^{\alpha-1} = x^{\alpha}}$$

- Find the cdf of the minimum $V = Y_{(1)}$ (see pg 97 in chp 2 notes)

$$F_V(v) = P(V \le v) = 1 - P(V > v) = 1 - P(Y_1 > v, \dots, Y_n > v)$$
$$= 1 - (1 - F_Y(v))^n$$

note: $F_Y(v) = v^{\alpha}$

$$\boxed{F_{(V)}(v) = 1 - (1 - v^{\alpha})^n}$$

$\boxed{\bullet \text{ Let } v^{\alpha} = x \; ; \text{ then } F_X(x) = 1 - (1-x)^n = I_x(1, n)}$ (see wiki pg for properties)

- Find pdf of $F_V(v) = F_{(Y_1)}(v)$

$$f_V(v) = F'_{(V)}(v) = \frac{d}{dv}\left[1 - (1 - v^{\alpha})^n\right] = -n(1 - v^{\alpha})^{n-1} \cdot -\alpha v^{\alpha-1}$$

$$\boxed{f_V(v) = \alpha n v^{\alpha-1}(1 - v^{\alpha})^{n-1}}$$

- Find the cdf of the maximum $U = Y_{(n)}$ (see pg 98 in chp 2 notes)

$$F_U(u) = P(U \le u) = P(Y_1 \le u, Y_2 \le u, \dots Y_n \le u)$$
$$= F_Y(u)^n$$

note: $F_Y(u) = u^{\alpha}$ (see wiki pg)

$\boxed{F_U(u) = u^{\alpha n}}$ note: $\boxed{\text{Let } x = u^{\alpha} \text{ then } F_X(x) = x^n = I_x(n, 1)}$

- Find the pdf of $F_U(u) = F_{(Y_n)}(u)$

$$f_U(u) = F'_U(u) = \frac{d}{du}\left[u^{\alpha n}\right] = \alpha n u^{\alpha n - 1}$$

$$\boxed{f_U(u) = \alpha n u^{\alpha n - 1}}$$

7) $\oint$ $Y_1, \dots Y_n$ is a random sample from the Weibull $(\alpha, 1)$ distribution (recall exercises 2.4.19 & 2.5.21). Find the cdf & pdf for $Y_{(1)} = \min(Y_{(1)}, \dots Y_{(n)})$. Show that this is another Weibull distribution.

[NOTE: (from Exercise 2.4.19) (Weibull(α) distribution). Consider, for $\alpha > 0$ fixed, the function given by $f(x) = \alpha x^{\alpha-1} e^{-x^\alpha}$ for $x \in (0, \infty)$ and 0 oth

- $f_X(x) = \alpha x^{\alpha-1} e^{-x^\alpha} \Rightarrow F_X(x) = \int_0^x \alpha x^{\alpha-1} e^{-x^\alpha} = -e^{-x^\alpha} \Big|_{x=0}^{x=x}$

$$= -e^{-x^\alpha} - (-e^{0^\alpha})$$

$$\underline{F_X(x) = 1 - e^{-x^\alpha}}$$

- Find cdf for $V = Y_{(1)}$ (see pg 97 in Chap 2 notes)

$$F_V(v) = P(V \le v) = 1 - P(V > v) = 1 - P(Y_{(1)} > v, \dots Y_{(n)} > v)$$

$$= 1 - (1 - F_Y(v))^n$$

$$= 1 - (1 - (1 - e^{-v^\alpha}))^n$$

$$= 1 - (e^{-v^\alpha})^n$$

$$\boxed{F_V(v) = 1 - e^{-v^{\alpha n}}}$$

note: $F_V(v)$ is the cdf for a R.V. $V \sim$ Weibull $(\lambda = 1, k = \alpha n)$

- Find pdf $f_V(v)$:

$$f_V(v) = F_V'(v) = \frac{d}{dv}\left[1 - e^{-v^{\alpha n}}\right] = \boxed{\alpha n e^{-v^{\alpha n}} v^{\alpha n - 1}}$$

5.) (cont'd).

2.8.5: $ X & Y have joint probability function:

$$P_{XY}(x,y) = \begin{cases} 1/9 & ; x=-4, y=-2 \\ 2/9 & x=5, y=-2 \\ 3/9 & x=9, y=-2 \\ 2/9 & x=9, y=0 \\ 1/9 & x=9, y=4 \\ 0 & \text{o.w.} \end{cases}$$

(a) Compute $P(Y=4|X=9)$

$$P(Y=4|X=9) = \frac{P(Y=4)\wedge P(X=9)}{P(X=9)} = \frac{(1/9)}{(2/3)} = \boxed{\frac{1}{6}}$$

(b) compute $P(Y=-2|X=9) = \frac{P(Y=-2)\wedge P(X=9)}{P(X=9)} = \frac{(3/9)}{(2/3)} = \boxed{\frac{1}{2}}$

(c) compute $P(Y=0|X=-4) = \frac{P(Y=0)\wedge P(X=-4)}{P(X=-4)} = \frac{0}{(1/9)} = \boxed{0}$

(d) compute $P(Y=-2|X=5) = \frac{P(Y=-2)\wedge P(X=5)}{P(X=5)} = \frac{(2/9)}{(2/9)} = \boxed{1}$

(e) compute $P(X=5|Y=-2) = \frac{P(X=5)\wedge P(Y=-2)}{P(Y=-2)} = \frac{(2/9)}{(6/9)} = \boxed{\frac{1}{3}}$

5.) Chp 2 Exercises 2.8.2, 2.8.3, 2.8.5, 2.8.7 (a, d), 2.8.14, 2.8.15

2.8.2: $\ni$ X & Y have joint probability function

$$p_{X,Y}(x,y) = \begin{cases} 1/16 & x=-2,\ y=3 \\ 4/16 & x=-2,\ y=5 \\ 8/16 & x=9,\ y=3 \\ 1/16 & x=9,\ y=5 \\ 1/16 & x=13,\ y=3 \\ 1/16 & x=13,\ y=5 \\ 0 & \text{o.w.} \end{cases}$$

a.) Compute $p_X(x)$ $\forall x \in \mathbb{R}^1$

$$p_X(x) = \begin{cases} 5/16 & ;\ x=-2 \\ 9/16 & ;\ x=9 \\ 2/16 & ;\ x=13 \\ 0 & ;\ \text{o.w.} \end{cases}$$

$\frac{10}{16} \cdot \frac{5}{16} = \frac{50}{256} = \frac{25}{128} \neq \frac{1}{16}$

b.) Compute $p_Y(y)$ $\forall y \in \mathbb{R}^1$

$$p_Y(y) = \begin{cases} 10/16 & ;\ y=3 \\ 6/16 & ;\ y=5 \\ 0 & ;\ \text{o.w.} \end{cases}$$

c.) Determine whether or not X & Y are independent.

$P(Y=3 \wedge X=-2) = 1/16 \neq \frac{50}{256} = P(X=-2) \cdot P(Y=3)$

$\Rightarrow$ $\boxed{\text{X \& Y are } \underline{\text{not}} \text{ independent}}$