

~~START~~ Monday 2/28/22 (week 7, lecture 18)  
@ 15 min mark.  
**HANDOUT # 6**

## Experiments Investigating Variance Components

1. Random Factor Levels
2. Examples
3. Random Effects Model
4. Point Estimators of Variance Components
5. How to Deal with Negative Estimates of Variances
6. C.I. for Variances, Equal Sample Sizes
7. Test Statistics, Equal Sample Sizes
8. Power and Sample Size Determination for Random Effects Models
9. Example: Manufacture of Fish Nets
10. Subsampling to Estimate Within EU Variability
11. Supplemental Reading: Design & ANOVA Book - Chapter 17

## Random Effects

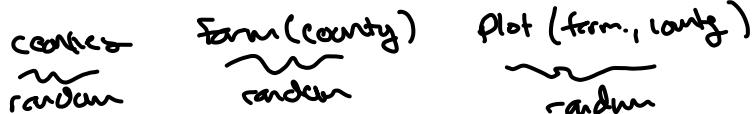
In the previous models we examined, the only random term was  $e_{ij}$  with the remaining terms fixed, but unknown population parameters:

1. the treatment means,  $\mu_i$ , in the Cell Means Model:  $y_{ij} = \mu_i + e_{ij}$
2. the mean and treatment effects,  $\mu$  and  $\tau_i$ , in the Effects Model:  $y_{ij} = \mu + \tau_i + e_{ij}$
3.  $\sigma_e$  in both Cell Means and Effects Model:  $e_{ij} \sim N(0, \sigma_e^2)$

In this handout we will examine various sources of variation that arise from the manner in which the experiment was conducted. The random terms in the model, that represent the various sources of variation in the experiment, will alter the manner in which we analyze the data and the types of inferences that are made from the data to the population.

**EXAMPLE 1:** A survey is to be designed to estimate the variability in yield per acre of cotton across farms in Texas. Twenty counties are randomly selected from the 254 counties in Texas. Next, several farms are randomly selected within each county. At each farm, several plots of 1-acre each are randomly selected to determine the yield. The individual measurements of yield on each plot would depend on the variability across counties and the variability across farms within a county. This type of experimental design provides inferences which are considerably different from the inferences that would be made to compare cotton production at a few designated counties within Texas.

*EV: plots = M0*



**EXAMPLE 2:** A company produces fiber optic cable for the communication industry. This industry needs cable which has a very consistent tension across long lengths of cable. The company is planning on purchasing new machinery used to bind the cable. There are three major manufacturers of the machines. The fiber optic company wants to determine if there is major differences in the quality of the product from the three manufacturers. A study is designed by randomly selecting 5 binding machines from each of the three manufacturers and then produces 10 cables of length 1000 yards from each of the 15 machines. The quality of the signal transmitted through the cable is then measured. Research Question: Is there a difference in quality due to

1. Manufacturer Differences *f<sub>red</sub>*

*EV: cable.*

2. Variability in Machines From the Same Manufacturer

*Machines (Manufacturer) - Random*

3. Variability in Cable Produced From the Same Machine

*cables (Manufacturer, Machine) - Random*

How do the inferences differ in the above study from a study in which a single machine is obtained from each manufacturer?

**EXAMPLE 3:** There are hundreds of laboratories that are federally qualified to produce assessments of the level of *e. coli* in meat. A consumer research group wants to determine if there is a difference in the accuracy of the determinations across the many laboratories or are the laboratories all essentially the same in the quality of the determinations. A study is designed by

1. Randomly selecting 10 laboratories from the list of all laboratories. *Lab - random*
2. Randomly assigning 20 meat samples with a known level of *e. coli* to each of the selected laboratories. *meat (lab) - random* *EV: meat = mu*.
3. Measure the difference between the known value of *e. coli* and the determination made by the laboratory.

How would the inferences differ in this study from the possible inferences running the study at a few preselected laboratories?

**Definition** A factor has **random effects** if the levels of the factor to be used in the experiment or study are randomly selected from a population of potential levels or the levels have varying experimental conditions.

Blocking factors typically have **random levels**, days - many factors affecting the responses occur from day to day. Agricultural fields - factors such as small differences in fertility, slope, drainage, etc. could impact the responses.

The researcher is interested in all levels of the factor but because of time or economical constraints it is not possible to examine all possible levels. Thus, the responses are observed on a possibly small subset of all possible factor levels (treatments) but inferences are to be about all the levels (treatments).

In Example 1, the researchers are interested in all counties in Texas but will base their inferences on a sample of counties, farms, and plots within these farms.

In Example 2, the company would like to have information about all machines at each of the three Manufacturers but will make inferences based on a limited sample of such machines. If they find large differences in the selected machines, then they may decide to expand the sample to include a larger proportion of the population of machines at each of the three manufactures.

In Example 3, the researchers are interested in the variability across the population of Laboratories. If there is very little difference across the selected Laboratories, then the consumer group may be willing to conclude that government certification has resulted in a homogeneous set of laboratories with very little difference in their measurement error. Whereas, if large differences are found, they may expand their study to determine why the laboratories are producing such a range of differences in the accuracy of their determination of the level of *e. coli* and to determine if these types of differences persist across the population of all such laboratories.

## ~~A~~ Model for Random Effects:

A random effects experiment (study) has the following elements:

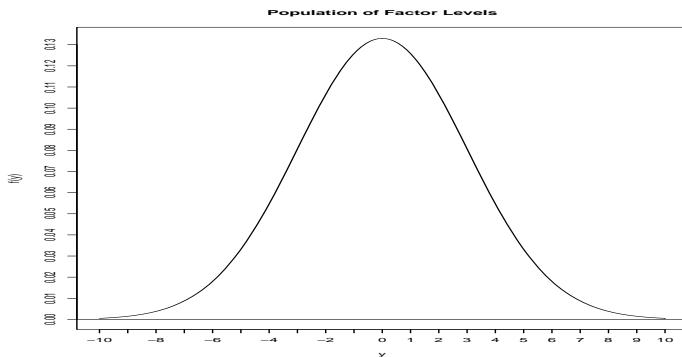
1. Randomly select  $t$  treatments (factor levels) from a population of all treatments of interest.
2. Randomly assign  $n_i$  experimental units (EU's) to the  $i$ th treatment or randomly select  $r_i$  EU's from the  $i$ th treatment population.
3. Observe or record  $y_{ij}$ , the response from the  $j$ th EU receiving the  $i$ th treatment or the response from the  $j$ th EU randomly selected from the  $i$ th treatment population.



**Random Effects Model:**  $y_{ij} = \mu + A_i + e_{ij} \quad i = 1, 2, \dots, t; \quad j = 1, 2, \dots, n_i,$

*A<sub>i</sub> ~ N(0, σ<sub>A</sub><sup>2</sup>)*

where  $A_i$  is the the  $i$ th random treatment effect with  $A_i$  distributed  $N(0, \sigma_A^2)$ , and  $\mu$  is the mean response over all treatments in the treatment population. The  $e_{ij}$ 's are iid  $N(0, \sigma_e^2)$  with  $A_i$ 's and  $e_{ij}$ 's independently distributed. Thus,  $A_i$  and  $e_{ij}$  are random variables, but  $\mu$ ,  $\sigma_A$  and  $\sigma_e$  are fixed but unknown population parameters which will be estimated using the observed data.



The goal of the experiment is determine if there are treatment differences in the population of treatments, not just in the treatments observed in the study:

$$H_0 : \text{No Treatment Differences} \quad \text{versus} \quad H_1 : \text{Treatment Differences Exist}$$

In terms of the population parameters, this is equivalent to testing:

$$H_0 : \sigma_A = 0 \quad \text{Versus} \quad H_1 : \sigma_A > 0$$

If  $\sigma_A = 0$ , then  $A_i = 0$  for all treatments and not just the  $t$  treatments selected for the given experiment. Also, if  $\sigma_A = 0$ , then  $y_{ij} = \mu + e_{ij}$ . Hence, there is a single value in the population of treatment means,  $\mu$ . Hence, all treatments have the same mean response,  $\mu$ .

Note that the mean of the  $i$ th treatment is  $\mu_i = \mu + A_i$ . However, there is very little interest in comparing the treatment means associated with the  $t$  treatments selected for the experiment:  $\mu_i$  versus  $\mu_k$ , for  $i, k = 1, \dots, t$ . The  $t$  treatment means are just the means of a small randomly selected group of treatments, whereas the researcher is interested in the difference in the mean responses across all treatments in the population of treatments.

For example, the consumer group is **not** interested in the difference between the accuracy of measuring *e. coli* in just the 10 Laboratories randomly selected for the experiment. They are interested in testing if there is a difference in **all labs** under federal certification.

Next, we will examine how we can interpret the Random Effects Model as a Fixed Effects Model:

The Random Effects Model:

$$y_{ij} = \mu + A_i + e_{ij} \quad i = 1, \dots, t; \quad j = 1, \dots, n_i \quad \text{with } A_i \text{ and } e_{ij} \text{ independent}$$

can be interpreted conditionally as a

Cell Means Model:

$$y_{ij} = M_i + e_{ij} \quad i = 1, \dots, t; \quad j = 1, \dots, n_i \quad \text{with } M_i \text{ and } e_{ij} \text{ independent}$$

Let  $M_1, M_2, \dots, M_t$  be the population mean responses associated with the  $t$  randomly selected treatments.

The  $M_i$  are iid  $N(\mu, \sigma_\mu^2)$  random variables where  $\mu$  is the average of all treatments means in the population of treatments and  $\sigma_\mu$  is the standard deviation in the population of means.

Thus, conditional on the value of  $M_i$ , we have the following

1.  $E[y_{ij}|M_i] = E[M_i + e_{ij}|M_i] = E[M_i|M_i] + E[e_{ij}|M_i] = M_i + 0 = M_i$  and
2.  $Var(y_{ij}|M_i) = Var(M_i + e_{ij}|M_i) = Var(M_i|M_i) + Var(e_{ij}|M_i) = 0 + Var(e_{ij}) = \sigma_e^2 \Rightarrow$
3. Summarizing,  $y_{ij}$ 's **conditional on**  $M_i$  are distributed as  $N(M_i, \sigma_e^2)$  random variable.

That is, conditional on the values of the treatment means  $M_1, M_2, \dots, M_t$ ,

the  $y_{ij}$ 's are independently distributed as  $N(M_i, \sigma_e^2)$  r.v.'s.

4. The unconditional distribution of the  $y_{ij}$ 's is given by

- $E[y_{ij}] = E_{M_i}[E[y_{ij}|M_i]] = E_{M_i}[M_i] = \mu$
- $Var(y_{ij}) = Var_{M_i}(E[y_{ij}|M_i]) + E_{M_i}[Var(y_{ij}|M_i)] = Var_{M_i}(M_i) + E_{M_i}[\sigma_e^2] = \sigma_\mu^2 + \sigma_e^2$

**4** In regards to the types of inferences that are generally desired, it is more intuitive to use the **Random Effects Model**:

$$y_{ij} = \mu + A_i + e_{ij}$$

1.  $A_i$ 's are distributed iid  $N(0, \sigma_A^2)$  independently of  $e_{ij}$ 's which are distributed  $N(0, \sigma_e^2)$
2.  $E[y_{ij}] = \mu$  but conditional on the selected treatments we have  $E[y_{ij}|A_i] = \mu + A_i$
3.  $\sigma_y^2 = Var(y_{ij}) = Var(A_i + e_{ij}) = \sigma_A^2 + \sigma_e^2$
4.  $y_{ij}$ 's are distributed  $N(\mu, \sigma_y^2)$
5. For  $j \neq h$ ,

$$\begin{aligned} Cov(y_{ij}, y_{ih}) &= Cov(\mu + A_i + e_{ij}, \mu + A_i + e_{ih}) \\ &= Cov(A_i, A_i) + Cov(A_i, e_{ih}) + Cov(e_{ij}, A_i) + Cov(e_{ij}, e_{ih}) \\ &= Var(A_i) + 0 + 0 + 0 = \sigma_A^2 > 0 \end{aligned}$$

Thus, the  $n_i$  observations from a given treatment are positively correlated.

6. For  $j \neq h$  and  $i \neq k$ ,

$$\begin{aligned} Cov(y_{ij}, y_{kh}) &= Cov(\mu + A_i + e_{ij}, \mu + A_k + e_{kh}) \\ &= Cov(A_i, A_k) + Cov(A_i, e_{kh}) + Cov(e_{ij}, A_k) + Cov(e_{ij}, e_{kh}) \\ &= 0 + 0 + 0 + 0 = 0 \end{aligned}$$

Thus, observations on different treatments are independent (uncorrelated jointly normally distributed r.v.'s are independent).

7. The IntraClass Correlation Coefficient,  $\rho_{IC}$ , for  $j \neq h$  is

$$\rho_{IC} = Corr(y_{ij}, y_{ih}) = \frac{Cov(y_{ij}, y_{ih})}{\sqrt{Var(y_{ij})Var(y_{ih})}} = \frac{\sigma_A^2}{\sqrt{\sigma_A^2 + \sigma_e^2}\sqrt{\sigma_A^2 + \sigma_e^2}} = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_e^2} = \frac{\tau}{1+\tau} \text{ where } \tau = \frac{\sigma_A^2}{\sigma_e^2}$$

This is the correlation between two observations receiving the same treatment (or two observations from the same treatment population).

## How do the Random Effects and Fixed Effects Experiments/Models Differ?

1. In Fixed Effects Experiments, there are only  $t$  treatments of interest to the researcher.
  - In the Random Effect Experiments, there is a population (large number) of treatments, from which the researcher randomly selects  $t$  treatments for use in the experiment. The researcher is interested in making inferences concerning the population of treatments, not just the  $t$  treatments used in the experiment.
2. In Fixed Effects Model, the Treatment Effects,  $\tau_i$  are fixed population parameters with  $\mu_i = E[y_{ij}] = \mu + \tau_i$ . Thus,  $H_o : \text{No Treatment Difference}$  is equivalent to  $H_o : \mu_1 = \dots = \mu_t$  which is equivalent to  $H_o : \tau_1 = \dots = \tau_t = 0$
3. In the Fixed Effects Model, the  $n = n_1 + \dots + n_t$  observations  $y_{ij}$  are independent random variables.
  - In the Random Effects Model, there is independence between observations from different treatments but the  $n_i$  observations from the same treatment are correlated, with value  $\rho_{IC}$ .
4. To simulate observations from the Cell Means and Fixed Effects Models:
  - (a) For the Cell Means model:  $y_{ij} = \mu_i + e_{ij}$ :
  - (b) Specify values for  $\mu_1, \dots, \mu_t$  and  $\sigma_e$ ,
  - (c) Generate  $n$  independent observations from a  $N(0, 1)$  distribution, and designate them as  $e_{ij}^*$  for  $i = 1, \dots, t$  and  $j = 1, \dots, n_i$ .
  - (d) Finally, let  $y_{ij} = \mu_i + \sigma_e e_{ij}^* = \mu_i + e_{ij}$
  - (e) Alternatively, generate  $n$  independent observations from a  $N(0, \sigma_e^2)$  distribution, designate them as  $e_{ij}$  for  $i = 1, \dots, t$  and  $j = 1, \dots, n_i$ , and let  $y_{ij} = \mu_i + e_{ij}$ .
  - (f) For the Fixed Effects Model,  $y_{ij} = \mu + \tau_i + e_{ij}$  specify values for  $\mu$  and  $\tau_1, \dots, \tau_t$  and  $\sigma_e$ , generate  $n$  independent observations from a  $N(0, \sigma_e^2)$  distribution, designate them as  $e_{ij}$  for  $i = 1, \dots, t$  and  $j = 1, \dots, n_i$ , then obtain  $y_{ij} = \mu + \tau_i + e_{ij}$

5. To simulate observations from the Random Effects Model:

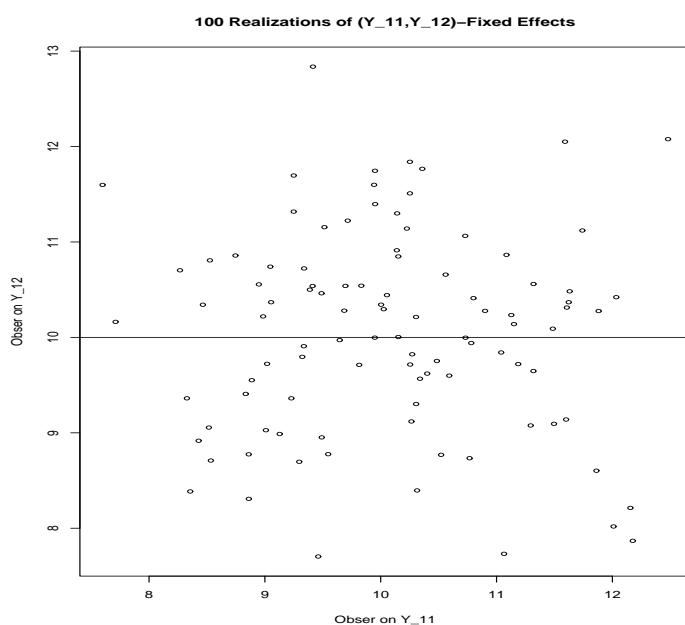
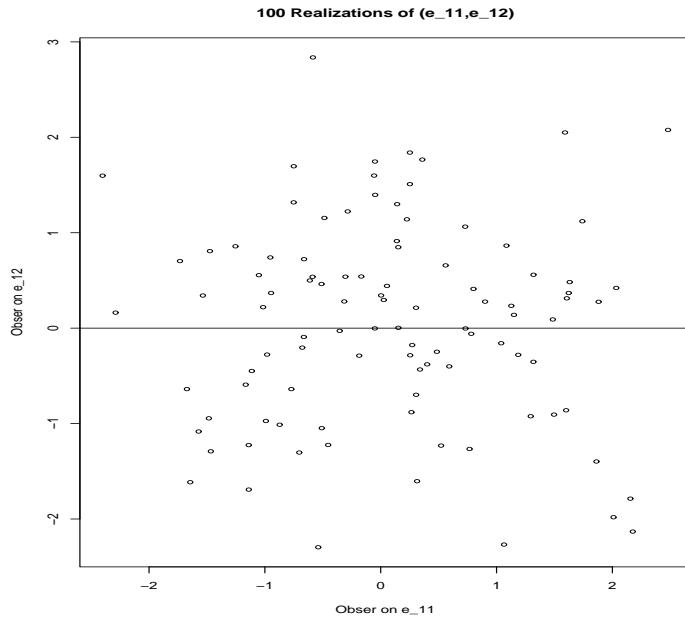
- (a) Specify values for  $\mu$ ,  $\sigma_A$  and  $\sigma_e$
- (b) Generate  $t$  independent observations from a  $N(0, \sigma_A^2)$  distribution:  $A_1, \dots, A_t$
- (c) Generate  $n = n_1 + n_2 + \dots + n_t$  independent observations from a  $N(0, \sigma_e^2)$  distribution:  
 $e_{11}, e_{12}, \dots, e_{1n_1}, \dots, e_{t1}, \dots, e_{tn_t}$ ,  
where  $A_i$ 's and  $e_{ij}$ 's are independently generated
- (d) Let  $y_{ij} = \mu + A_i + e_{ij}$

Note,  $A_i$  is common to  $y_{i1}, y_{i2}, \dots, y_{in_i}$ .

Thus,  $y_{i1}, y_{i2}, \dots, y_{in_i}$  are correlated.

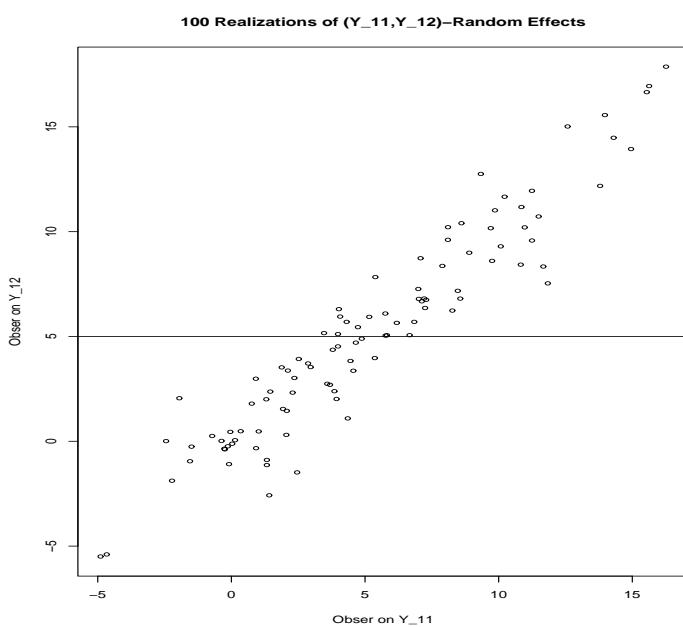
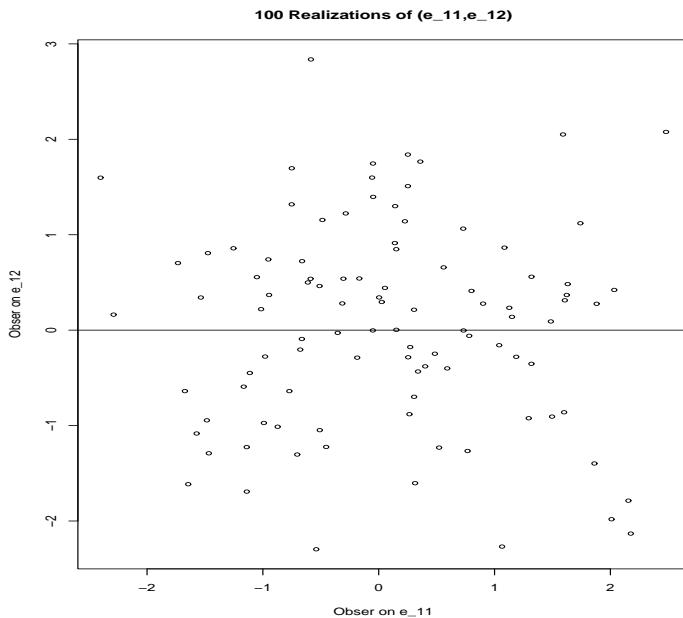
The following two plots display 100 realizations of the pair  $(e_{11}, e_{12})$  distributed  $N(0, 1)$  and the corresponding observations  $(y_{11}, y_{12})$  in a CRD having fixed effects with an overall treatment mean of 5 and fixed treatment effect of 5: (observations from same treatment)

$$y_{11} = \mu + \tau_1 + e_{11} = 5 + 5 + e_{11} \quad \text{and} \quad y_{12} = \mu + \tau_1 + e_{12} = 5 + 5 + e_{12}$$



The following two plots display 100 realizations of the pair  $(e_{11}, e_{12})$  distributed  $N(0, 1)$  and the corresponding observations  $(y_{11}, y_{12})$  in a CRD having random effects with a treatment mean of 5 and random effect  $A_1$  distributed  $N(0, 5^2)$ .

$$y_{11} = \mu + A_1 + e_{11} = 5 + A_1 + e_{11} \quad \text{and} \quad y_{12} = \mu + A_1 + e_{12} = 5 + A_1 + e_{12}$$



10

STOP Monday 2/28/22 (Week 7), leave 18)

# START Friday 3/4/22 (Week 7, Lecture 1a)

## Estimation of Variance Components

In the random effects model:  $y_{ij} = \mu + A_i + e_{ij}$  there are two components to the overall variance of the  $y_{ij}$ 's

Treatment Variance:  $Var(A_i) = \sigma_A^2$  and Error Variance:  $Var(e_{ij}) = \sigma_e^2$

Overall Variance:  $\sigma_y^2 = Var(y_{ij}) = Var(A_i + e_{ij}) = \sigma_A^2 + \sigma_e^2$

Note: A naive estimator of  $\sigma_y^2$  would be  $\hat{\sigma}_y^2 = \frac{1}{n-1} \sum_{i=1}^t \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2$  ~~- SOT~~

However, this estimator would not take into account that we have multiple treatments and thus multiple population distributions for the  $y_{ij}$ 's.

### Point Estimation - Using Method of Moments (MOM)

The partition of  $SS_{TOT}$  into two components:  $SS_{TRT}$  and  $SSE$  will use the same formulas as were used in the Fixed Effects Model:

$$MSE = \frac{1}{n-t} \sum_{i=1}^t \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2 \quad \text{and} \quad MS_{TRT} = \frac{1}{t-1} \sum_{i=1}^t n_i (\bar{y}_{i.} - \bar{y}_{..})^2$$

Using the principles of MOM, the expected values of  $MS_{TRT}$  and  $MSE$  will be equated to their realizations and the resulting equations will yield the MOM estimates of the variance components:  $\sigma_A^2$  and  $\sigma_e^2$ .

$$1. E[MSE] = \sigma_e^2 \Rightarrow MSE = \hat{\sigma}_e^2$$

Derivation of the above result:

$$\begin{aligned} y_{ij} &= \mu + A_i + e_{ij} \Rightarrow \bar{y}_{i.} = \mu + A_i + \bar{e}_{i.} \Rightarrow \\ y_{ij} - \bar{y}_{i.} &= e_{ij} - \bar{e}_{i.} = \underbrace{(1 - \frac{1}{n_i}) e_{ij}}_{\text{independent}} - \frac{1}{n_i} \sum_{k \neq j} e_{ik} \Rightarrow \text{co of these two terms is } 0. \\ E[(y_{ij} - \bar{y}_{i.})^2] &= Var(e_{ij} - \bar{e}_{i.}) = Var \left( \left(1 - \frac{1}{n_i}\right) e_{ij} - \frac{1}{n_i} \sum_{k \neq j} e_{ik} \right) \\ &= \left(1 - \frac{1}{n_i}\right)^2 Var(e_{ij}) + \frac{1}{n_i^2} Var \left( \sum_{k \neq j} e_{ik} \right) \\ &= \left(1 - \frac{1}{n_i}\right)^2 \sigma_e^2 + \frac{1}{n_i^2} (n_i - 1) \sigma_e^2 = \left(1 - \frac{1}{n_i}\right) \sigma_e^2 \end{aligned}$$

$$\begin{aligned}
E[MSE] &= E \left[ \frac{1}{n-t} \sum_{1=i}^t \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2 \right] \\
&= \frac{1}{n-t} \sum_{1=i}^t \sum_{j=1}^{n_i} E[(y_{ij} - \bar{y}_{i.})^2] \\
&= \frac{1}{n-t} \sigma_e^2 \sum_{i=1}^t \sum_{j=1}^{n_i} (1 - \frac{1}{n_i}) \\
&= \frac{1}{n-t} \sigma_e^2 (n-t) = \sigma_e^2
\end{aligned}$$

$$2. E[MS_{TRT}] = r_o \sigma_A^2 + \sigma_e^2 \Rightarrow MS_{TRT} = r_o \hat{\sigma}_A^2 + \hat{\sigma}_e^2 \Rightarrow$$

$$\hat{\sigma}_A^2 = \frac{1}{r_o} (MS_{TRT} - MSE)$$

where  $r_o = \frac{1}{t-1} \left( n - \sum_{i=1}^t \frac{n_i^2}{n} \right)$

If  $n_1 = n_2 = \dots = n_t = r$ , then  $r_o = r$

Derivation of the above result: From the model  $y_{ij} = \mu + A_i + e_{ij} \Rightarrow$

$$\bar{y}_{i.} = \mu + A_i + \bar{e}_{i.} \quad \text{and} \quad \bar{y}_{..} = \mu + \bar{A}_{..} + \bar{e}_{..}, \quad \text{with } \bar{A}_{..} = \frac{1}{n} \sum_{i=1}^t n_i A_i \Rightarrow$$

$$\bar{y}_{i.} - \bar{y}_{..} = (\underbrace{A_i - \bar{A}_{..}}_{\text{independent from } e_{ij}}) + (\underbrace{\bar{e}_{i.} - \bar{e}_{..}}_{\text{independent from } e_{ij}}) \quad \text{and}$$

$$(A_i - \bar{A}_{..}) = \left( (1 - \frac{n_i}{n}) A_i - \frac{1}{n} \sum_{k \neq i}^t n_k A_k \right)$$

$$(\bar{e}_{i.} - \bar{e}_{..}) = \left( (\frac{1}{n_i} - \frac{1}{n}) \sum_{j=1}^{n_i} e_{ij} - \frac{1}{n} \sum_{k \neq i}^t \sum_{j=1}^{n_k} e_{kj} \right) \Rightarrow$$

$$\begin{aligned}
E[(A_i - \bar{A}_.)^2] = Var(A_i - \bar{A}_.) &= Var\left(\left(1 - \frac{n_i}{n}\right) A_i - \frac{1}{n} \sum_{k \neq i}^t n_k A_k\right) \\
&= \left(1 - \frac{n_i}{n}\right)^2 Var(A_i) + \frac{1}{n^2} \sum_{k \neq i}^t n_k^2 Var(A_k) \\
&= \left[\left(1 - \frac{n_i}{n}\right)^2 + \frac{1}{n^2} \sum_{k \neq i}^t n_k^2\right] \sigma_A^2 \\
&= \left(1 - \frac{2n_i}{n} + \frac{1}{n^2} \sum_{k=1}^t n_k^2\right) \sigma_A^2
\end{aligned}$$

$$\begin{aligned}
E[(\bar{e}_{i.} - \bar{e}_{..})^2] = Var(e_{i.} - \bar{e}_{..}) &= Var\left(\left(\frac{1}{n_i} - \frac{1}{n}\right) \sum_{j=1}^{n_i} e_{ij} - \frac{1}{n} \sum_{k \neq i}^t \sum_{j=1}^{n_k} e_{kj}\right) \\
&= \left(\frac{1}{n_i} - \frac{1}{n}\right)^2 \sum_{j=1}^{n_i} Var(e_{ij}) + \frac{1}{n^2} \sum_{k \neq i}^t \sum_{j=1}^{n_k} Var(e_{kj}) \\
&= \left(\frac{1}{n_i} - \frac{1}{n}\right)^2 n_i \sigma_e^2 + \frac{1}{n^2} \sum_{k \neq i}^t n_k \sigma_e^2 \\
&= \left[\left(\frac{1}{n_i} - \frac{1}{n}\right)^2 n_i + \frac{1}{n^2} \sum_{k \neq i}^t n_k\right] \sigma_e^2 \\
&= \left(\frac{1}{n_i} - \frac{1}{n}\right) \sigma_e^2
\end{aligned}$$

$$\begin{aligned}
E[MS_{TRT}] &= \frac{1}{t-1} \sum n_i E[(\bar{y}_{i.} - \bar{y}_{..})^2] \\
&= \frac{1}{t-1} \sum_{i=1}^t n_i (E[(A_i - \bar{A}_.)^2] + E[(\bar{e}_{i.} - \bar{e}_{..})^2] + 2E[(A_i - \bar{A}_.)(\bar{e}_{i.} - \bar{e}_{..})]) \\
&= \frac{1}{t-1} \sum_{i=1}^t n_i \left(\left(1 - \frac{2n_i}{n}\right) + \frac{1}{n^2} \sum_{k=1}^t n_k^2\right) \sigma_A^2 + \frac{1}{t-1} \sum_{i=1}^t n_i \left(\frac{1}{n_i} - \frac{1}{n}\right) \sigma_e^2 + 0 \\
&= \frac{1}{t-1} \left(n - \frac{1}{n} \sum_{i=1}^t n_i^2\right) \sigma_A^2 + \frac{(t-1)\sigma_e^2}{t-1} = r_o \sigma_A^2 + \sigma_e^2
\end{aligned}$$

One of the problems with the MOM estimator of  $\sigma_A^2$  is that it could be negative. The following are suggestions from a widely used design book for dealing with this situation:

**Courses of Action for Dealing with Negative Variance Estimates:  $\hat{\sigma}_A^2 < 0$**

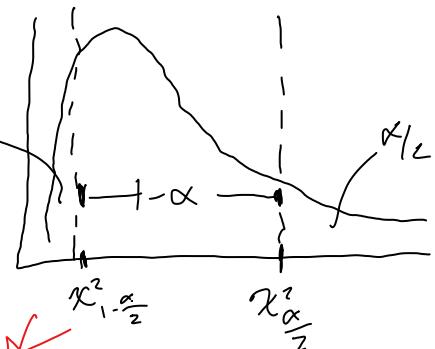
1. Accept the estimate as evidence of a true value of zero and use zero as the estimate, recognizing that the estimator is no longer unbiased.
- Bad Choice. A negative estimate is an indication of problems in the data or the procedure.
2. Retain the negative estimate, recognizing that subsequent calculations using the results may not make much sense
- Ridiculous suggestion: A negative estimate of a parameter which is non-negative.
3. Interpret the negative estimate as an indication of an incorrect model and/or outliers in the data set
4. Utilize a method different from the MOM for estimation of variance components
  - Use MLE or REML methods (See Handout # 9)
5. Collect more data and analyze the new data separately or in conjunction with the existing data and hope that the increased information will yield positive estimates.
  - Not a good suggestion. What if you continue to obtain negative estimates. When does the process end?????

## Confidence Intervals for Variance Components: $\sigma_A^2$ , and $\sigma_e^2$ :

In the random effects model:  $y_{ij} = \mu + A_i + e_{ij}$  we have

$A_i$ 's iid  $N(0, \sigma_A^2)$  r.v.s and  $e_{ij}$ 's iid  $N(0, \sigma_e^2)$  r.v.s with point estimators:

$$\hat{\sigma}_A^2 = \frac{1}{r_o} (MS_{TRT} - MSE) \quad \text{and} \quad \hat{\sigma}_e^2 = MSE$$



1.  $100(1 - \alpha)\%$  Confidence Interval for  $\sigma_e^2$ :

$$\left[ \frac{SSE}{\chi_{\alpha/2, n-t}^2}, \frac{SSE}{\chi_{1-\alpha/2, n-t}^2} \right]$$

where  $\chi_{\alpha/2, n-t}^2$  and  $\chi_{1-\alpha/2, n-t}^2$  are the upper and lower  $\alpha/2$  percentiles, respectively, from the Chi-square distribution with  $df = df_E = n - t$

This follows from the Pivot:  $\frac{1}{\sigma_e^2} SSE$

which has a Chi-square distribution with  $df = df_E = n - t$

*In Completely Randomized Design.*  
will talk about df for split plot  
& other designs later.

Derivation:

We have shown that  $\frac{1}{\sigma_e^2} SSE$  has a Chi-square distribution with  $df = df_E = n - t$ . Therefore,

$$1 - \alpha = P \left[ \chi_{1-\alpha/2}^2 \leq \frac{1}{\sigma_e^2} SSE \leq \chi_{\alpha/2, n-t}^2 \right] \Rightarrow$$

$$1 - \alpha = P \left[ \frac{1}{\chi_{1-\alpha/2}^2} \geq \frac{1}{SSE} \sigma_e^2 \geq \frac{1}{\chi_{\alpha/2, n-t}^2} \right] \Rightarrow$$

$$1 - \alpha = P \left[ \frac{SSE}{\chi_{1-\alpha/2}^2} \geq \sigma_e^2 \geq \frac{SSE}{\chi_{\alpha/2, n-t}^2} \right] \Rightarrow$$

2. Conservative  $100(1 - \alpha)\%$  Confidence Interval for  $\sigma_A^2$ :

$$\left[ \frac{\frac{1}{r} SS_{TRT} \left( 1 - \frac{F_U}{F_o} \right)}{C_U}, \frac{\frac{1}{r} SS_{TRT} \left( 1 - \frac{F_L}{F_o} \right)}{C_L} \right],$$

- Interval requires equal sample sizes,  $n_1 = \dots = n_t = r$ ,
- Interval is conservative (true coverage probability at least  $100(1 - \alpha)\%$ )
- $F_U = F_{\alpha/4, t-1, n-t} = qf(1 - \alpha/4, t - 1, n - t)$ ,  $F_L = F_{1-\alpha/4, t-1, n-t} = \frac{1}{F_{\alpha/4, n-t, t-1}}$  are upper and lower  $\alpha/4$  percentiles, respectively, from the F-distribution with  $df = t - 1, n - t$ ,
- $F_o = \frac{MS_{TRT}}{MSE}$  from the AOV, ~~\*~~
- $C_U = \chi_{\alpha/4, t-1}^2$ , and  $C_L = \chi_{1-\alpha/4, t-1}^2$  are the upper and lower  $\alpha/4$  percentiles, respectively, from the Chi-square distribution with  $df = df_{TRT} = t - 1$

### Derviation:

(a)  $\frac{1}{\sigma_e^2 + r\sigma_A^2} SS_{TRT}$  has a Chi-square distribution with  $df = t - 1$

$$1 - \frac{\alpha}{2} = P \left[ C_L \leq \frac{SS_{TRT}}{\sigma_e^2 + r\sigma_A^2} \leq C_U \right] \Rightarrow$$

$$1 - \frac{\alpha}{2} = P \left[ \frac{1}{C_L} \geq \frac{\sigma_e^2 + r\sigma_A^2}{SS_{TRT}} \geq \frac{1}{C_U} \right] \Rightarrow$$

$$1 - \frac{\alpha}{2} = P \left[ \frac{SS_{TRT}}{C_L} \geq \sigma_e^2 + r\sigma_A^2 \geq \frac{SS_{TRT}}{C_U} \right] \Rightarrow$$

$\left[ \frac{SS_{TRT}}{C_U}, \frac{SS_{TRT}}{C_L} \right]$  is a  $100(1 - \alpha/2)\%$  confidence interval for  $\sigma_e^2 + r\sigma_A^2$

(b)

$$F_o \frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} = \frac{MS_{TRT}}{MSE} \frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} = \frac{\left( \frac{SS_{TRT}}{\sigma_e^2 + r\sigma_A^2} \right) / (t - 1)}{\left( \frac{SSE}{\sigma_e^2} \right) / (n - t)}$$

has an F-distribution with  $df = t - 1, n - t$

$$P \left[ F_L \leq F_o \frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} \leq F_U \right] = 1 - (\alpha/4 + \alpha/4) \Rightarrow$$

$$1 - \frac{\alpha}{2} = P \left[ \frac{1}{F_L} \geq \frac{1}{F_o} \frac{\sigma_e^2 + r\sigma_A^2}{\sigma_e^2} \geq \frac{1}{F_U} \right] \Rightarrow$$

$$1 - \frac{\alpha}{2} = P \left[ \frac{F_o}{F_L} \geq \frac{\sigma_e^2 + r\sigma_A^2}{\sigma_e^2} \geq \frac{F_o}{F_U} \right] \Rightarrow$$

$\left[ \frac{F_o}{F_U}, \frac{F_o}{F_L} \right]$  is a  $100(1 - \alpha/2)$  confidence interval for  $\frac{\sigma_e^2 + r\sigma_A^2}{\sigma_e^2}$

(c) Define the event  $A = \left\{ \frac{SS_{TRT}}{C_U} \leq \sigma_e^2 + r\sigma_A^2 \leq \frac{SS_{TRT}}{C_L} \right\}$ , then

$$P[A] = 1 - \alpha/2$$

(d) Define the event  $B = \left\{ \frac{F_o}{F_U} \leq \frac{\sigma_e^2 + r\sigma_A^2}{\sigma_e^2} \leq \frac{F_o}{F_L} \right\}$ , then

$$P[B] = 1 - \alpha/2$$

$$B = \left\{ \frac{F_U}{F_o} \geq \frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} \geq \frac{F_L}{F_o} \right\}$$

$$B = \left\{ 1 - \frac{F_U}{F_o} \leq \frac{r\sigma_A^2}{\sigma_e^2 + r\sigma_A^2} \leq 1 - \frac{F_L}{F_o} \right\}$$

$$B = \left\{ \frac{1}{r}(\sigma_e^2 + r\sigma_A^2) \left( 1 - \frac{F_U}{F_o} \right) \leq \sigma_A^2 \leq \frac{1}{r}(\sigma_e^2 + r\sigma_A^2) \left( 1 - \frac{F_L}{F_o} \right) \right\}$$

(e)  $A \cap B \subseteq \left\{ \frac{1}{r} \frac{SS_{TRT}}{C_U} \left( 1 - \frac{F_U}{F_o} \right) \leq \sigma_A^2 \leq \frac{1}{r} \frac{SS_{TRT}}{C_L} \left( 1 - \frac{F_L}{F_o} \right) \right\} = C.I.$

(f)  $P[C.I.] \geq P[A \cap B] = 1 - P[A^c \cup B^c] \geq 1 - (P[A^c] + P[B^c]) = 1 - (\frac{\alpha}{2} + \frac{\alpha}{2}) = 1 - \alpha$

3. In *PROC MIXED*, SAS provides an approximate (large sample) C.I. when the  $n_i$ 's are unequal.

## Test of $H_0 : \sigma_A = 0$ versus $H_1 : \sigma_A > 0$

Using our results from Quadratic Forms and the F-Distribution, we have the following:

If  $W_1$  and  $W_2$  are independent random variables such that  $W_1/E[W_1]$  and  $W_2/E[W_2]$  have chi-square distributions with  $df = \nu_1$  and  $\nu_2$ , respectively, then

$$F = \frac{\left(\frac{W_1}{E[W_1]}\right)/\nu_1}{\left(\frac{W_2}{E[W_2]}\right)/\nu_2}$$

has an F-distribution with  $df = \nu_1, \nu_2$ .

Using our results on Quadratic Forms, if  $n_1 = n_2 = \dots = n_t = r$ , then

1.  $SS_{TRT}/(\sigma_e^2 + r\sigma_A^2)$  has a chi-square distribution with  $df = t - 1$
2.  $SSE/\sigma_e^2$  has a chi-square distribution with  $df = n - t$
3.  $SS_{TRT}$  and  $SSE$  are independent
4. Therefore,

$$F = \frac{\frac{SS_{TRT}}{(t-1)(\sigma_e^2 + r\sigma_A^2)}}{\frac{SSE}{(n-t)\sigma_e^2}} = \frac{MS_{TRT}}{MSE} \frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} = \frac{MS_{TRT}}{MSE} \frac{1}{1 + r\tau} \quad \text{with } \tau = \frac{\sigma_A^2}{\sigma_e^2}$$

has an F-distribution with  $df = t - 1, n - t$ .

5. Under  $H_0 : \sigma_A^2 = 0 \Rightarrow \tau = 0 \Rightarrow \frac{1}{1+r\tau} = 1$  which implies that

$\frac{MS_{TRT}}{MSE}$  has a central F-distribution with  $df = t - 1, n - t$ .

6. Under  $H_1 : \sigma_A^2 > 0$ ,  $\frac{1}{1+r\tau} \neq 1$  which implies that

$\frac{MS_{TRT}}{MSE} \frac{1}{1+r\tau}$  has a central F-distribution with  $df = t - 1, n - t$  and hence we need to know the value of  $\tau$  in order to compute power.

7. Decision Rule: Reject  $H_0$  at level  $\alpha$  if  $F_o = \frac{MS_{TRT}}{MSE} \geq F_{\alpha, t-1, n-t}$

8.  $p-value = P[F \geq F_o] = 1 - G(F_o) = 1 - pf(F_o, t - 1, n - t)$ ,

where  $G(\cdot)$  is the cdf of a central F-distribution with  $df = t - 1, n - 1$  and  $F_o$  is the computed value of  $\frac{MS_{TRT}}{MSE}$ .

We are able to compute the p-value because this computation is done under the condition that  $H_0$  is true, that is,  $\tau = 0$ .

9. With  $\tau = \sigma_A^2/\sigma_e^2$ , the test of  $H_0 : \sigma_A = 0$  versus  $H_1 : \sigma_A > 0$  has power function given by

$$\begin{aligned}
\text{Power at specified } \tau = \gamma(\tau) &= P[\text{Reject } H_0 \text{ for specified } \tau] \\
&= P\left[\frac{MS_{TRT}}{MSE} \geq F_{\alpha,t-1,n-t}\right] \\
&= P\left[\frac{MS_{TRT}}{MSE} \frac{1}{1+r\tau} \geq \frac{1}{1+r\tau} F_{\alpha,t-1,n-t}\right] \\
&= 1 - G\left(\frac{1}{1+r\tau} F_{\alpha,t-1,n-t}\right) \\
&= 1 - pf\left(\frac{1}{1+r\tau} F_{\alpha,t-1,n-t}, t-1, n-t\right) \text{ using R} \\
&= 1 - PROBF\left(\frac{1}{1+r\tau} F_{\alpha,t-1,n-t}\right) \text{ using SAS}
\end{aligned}$$

where  $G(\cdot)$  is the cdf of a central F-distribution with  $df = t-1, n-t$

10. For specified values of  $\alpha$  and  $\tau$ , the central F-distribution can be used to calculate power and sample sizes.

**EXAMPLE: Suppose**  $t = 5$ ,  $r = 4$ ,  $\alpha = .05$ , **and from previous studies**  $\sigma_e \approx 2.18$

Determine the power of an  $\alpha = .05$  F-test of  $H_0 : \sigma_A = 0$  versus  $H_1 : \sigma_A > 0$

to determine that  $\sigma_A \geq 1.5$ :

Need to calculate  $\tau = \frac{\sigma_A^2}{\sigma_e^2}$ .

From the above information, we have

$$\tau = \frac{\sigma_A^2}{\sigma_e^2} = \frac{(1.5)^2}{(2.18)^2} = 0.4734 \text{ and } F_{.05,5-1,20-5} = F_{.05,4,15} = qf(1 - .05, 4, 15) = 3.056$$

$$\begin{aligned}
\gamma(\sigma_A = 1.5) = \gamma(\tau = .4734) &= 1 - G_{t-1,n-t}\left(\frac{1}{1+(4)(.4734)} 3.056\right) \\
&= 1 - G_{4,15}(1.056) = 1 - .588 = .412
\end{aligned}$$

$$G_{4,15}(1.056) = pf(1.056, 4, 15) \text{ using R or}$$

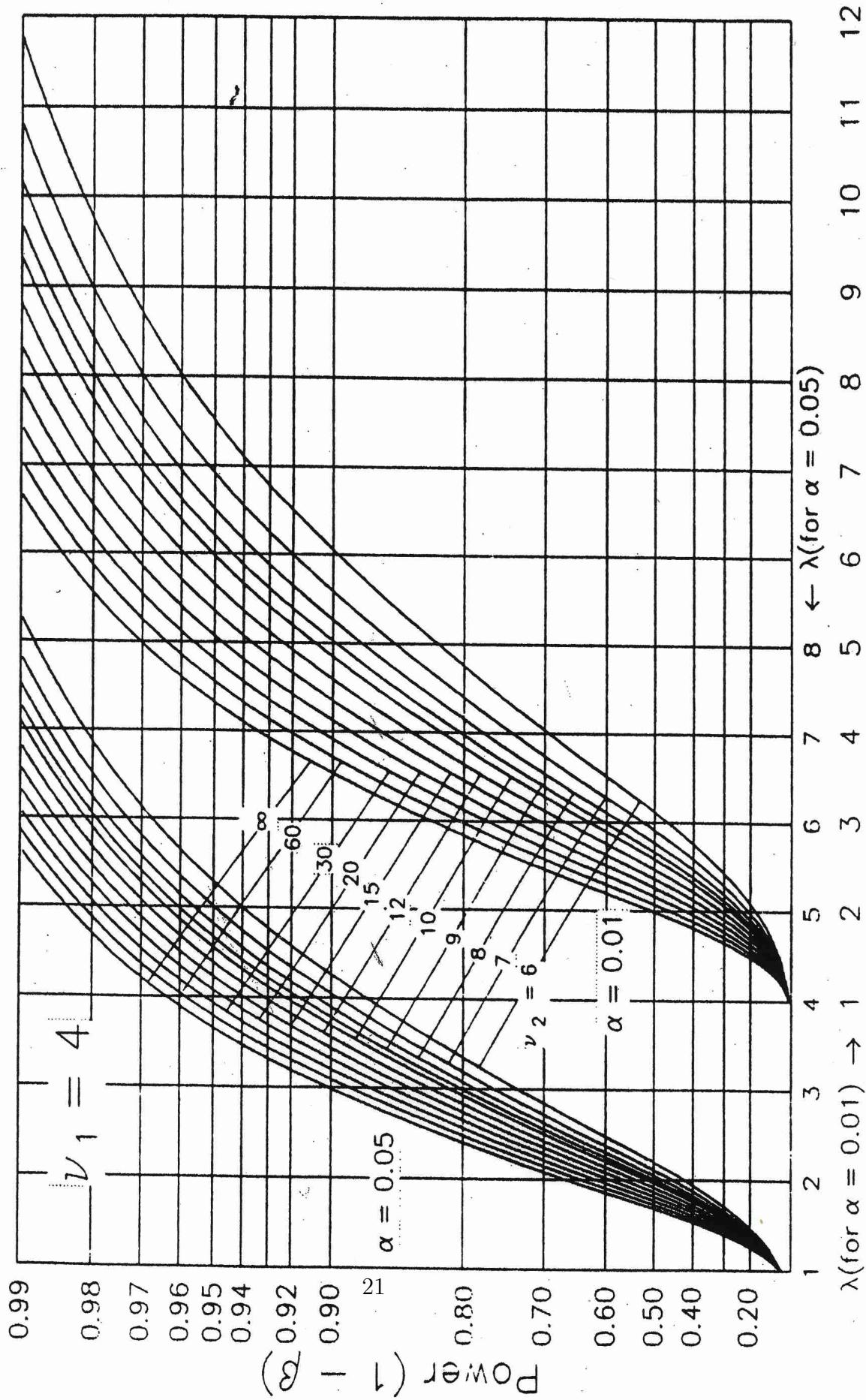
$$G_{4,15}(1.056) = PROBF(1.056, 4, 15) \text{ using SAS}$$

The graph given on the next page is Table X from the textbook.

We can use this graph to find the power:

1. Compute  $\nu_1 = t - 1 = 5 - 1 = 4$  and  $\nu_2 = n - t = (5)(4) - 5 = 15$
2. Compute  $\lambda = \sqrt{1 + \frac{r\sigma_A^2}{\sigma_e^2}} = \sqrt{1 + r\tau} = \sqrt{1 + (4)(.4734)} = 1.70$
3. Use the set of graphs with  $\nu_1 = 4$ , locate graph for  $\alpha = .05$  and  $\nu_2 = 15$
4. Intersect a vertical line at  $\lambda = 1.70$  on the  $\alpha = .05$  horizontal scale with the  $\nu_2 = 15$  line and read the power=0.412 on the vertical scale.

**Table X**  $F$  test power curves for random effects model analysis of variance



## Determine the Minimum Number of Reps, $r$ , to Achieve a Specified Power for Fixed Number of Treatments, $t$

For a fixed number of treatments  $t$ , determine the minimum  $r$  such that  $\gamma(\tau) \geq \gamma_o$  whenever  $\tau \geq \tau_o$ .

That is, determine the minimum number of replications such that a level  $\alpha$  test of  $H_0 : \sigma_A = 0$  vs  $H_1 : \sigma_A > 0$  will have power of at least  $\gamma_o$  to detect that  $\sigma_A \geq \sigma_e \sqrt{\tau_o}$ .

That is, determine the minimum number of replications such that a level  $\alpha$  test will have probability of Type II error of at most  $\beta_o = 1 - \gamma_o$  whenever  $\sigma_A \geq \sigma_e \sqrt{\tau_o}$ .

1. Specify a value of  $t$
2. Specify a value of  $\alpha$
3. Specify a value for  $\gamma_o$
4. Specify a value for  $\tau_o$  or
5. Specify a value for  $\sigma_A$  and an estimate of  $\sigma_e$

Using an iterative procedure, determine the minimum value of  $r$  such that  $\gamma(\tau) \geq \gamma_o$  whenever  $\tau \geq \tau_o$ .

The SAS or R program that was used in the fixed effects case can be easily modified for the random effects.

Just use the central F-cdf in place of the noncentral F-cdf.

repsize,randomeffects,fixedt.SAS

## How to select the value of $\tau_o$

In a Quality Control Study, the process engineer is interested in finding Factors A such that  $\sigma_A$  is large relative to  $\sigma_y$ .

That is,  $\sigma_A$  is the variability that is Detectable (Special Cause Variability) or variability due to an *Identifiable* problem in the process.

The other component in  $\sigma_y^2 = \sigma_A^2 + \sigma_e^2$  is  $\sigma_e$ , which is referred to as *Common Cause* variability or the variability that is just an inherent component in the design of the process.

Now if  $\frac{\sigma_A}{\sigma_y} > C$ , then the engineer wants to have a strong chance of detecting that Factor A is an important component in  $\sigma_y$ .

Thus, we have

$$\begin{aligned}\frac{\sigma_A}{\sigma_y} > C &\Rightarrow \sigma_A^2 > C^2 \sigma_y^2 = C^2 (\sigma_A^2 + \sigma_e^2) \Rightarrow \\ \sigma_A^2 (1 - C^2) &> C^2 \sigma_e^2 \Rightarrow \\ \tau = \frac{\sigma_A^2}{\sigma_e^2} &> \frac{C^2}{1 - C^2} = \tau_o.\end{aligned}$$

Thus, we can take  $\tau_o = \frac{C^2}{1 - C^2}$

### SAS Code for Sample Size Calculations

```
options ps=55 ls=70 nocenter nodate;
SAS Program to compute sample size for experiments with random effects with
Fixed number of treatments,t, we want to determine number of reps, r for specified power;
*repsize,randomeffects,fixedt.SAS;
data;
input r @@;
sigma_e=2.7566;
sigma_a=2.5;
t=5;
u1=t-1; u2=t*(r-1);
gamma=.85;
alpha=.05;
tau=(sigma_a/sigma_e)**2;
lambda=sqrt(1+(r)*(tau));
Fcr=finv(1-alpha,u1,u2);
C=(1/lambda)**2)*Fcr;
power=1-probf(C,u1,u2);
cards;
5 6 7 8 9
run;
proc print; var r u1 u2 C Fcr lambda power;
run;
```

**EXAMPLE:** Suppose  $t = 5$ ,  $\alpha = .05$ ,  $\gamma_o = .85$ , and  $\tau_o = .8225$

That is, for a Random Effects Experiment with  $t = 5$  treatments, determine the minimum number of replications per treatment such that an  $\alpha = .05$  test with have at least an 85% chance to reject  $H_0$  and declare  $\sigma_A > 0$  whenever  $\sigma_A^2 > .8225\sigma_e^2$ .

Using the graphs in Table X and  $G_{t-1,t(r-1)}(\cdot)$  cdf for central F-distribution,  $\text{df} = t - 1, t(r - 1)$ , necessary specifications are as follows:

$$t = 5, \quad \alpha = .05, \quad \gamma_o = .85 \quad \checkmark$$

$$\nu_1 = t - 1 = 5 - 1 = 4, \quad \nu_2 = t(r - 1) = 5(r - 1), \quad \lambda_o^2 = 1 + r\tau_o = 1 + .8225r$$

$$\gamma(\tau_o) = 1 - G_{t-1,t(r-1)}\left(\frac{1}{1+r\tau_o}F_{\alpha,t-1,t(r-1)}\right) = 1 - G_{t-1,t(r-1)}\left(\frac{1}{\lambda_o^2}F_{\alpha,t-1,t(r-1)}\right)$$

Iteratively, determine the smallest value of  $r$  such that  $\gamma(.8225) \geq .85$

From Table X or SAS Output, we obtain the following values:

r	$\nu_1 = 4$	$\nu_2 = 5(r - 1)$	$F_{.05,4,5(r-1)}$	$\lambda = \sqrt{1 + .8225r}$	$\gamma(.8225)$
7	4	30	2.6896	2.60	$1 - G_{4,30}(.3980) = .8085$
8	4	35	2.6415	2.75	$1 - G_{4,35}(.3485) = .8433$
9	4	40	2.6060	2.90	$1 - G_{4,40}(.3101) = .8695$

Thus, the minimum value of  $r$  is 9. Thus, an  $\alpha = .05$  test of the difference in  $t = 5$  randomly selected treatments would require 9 reps per treatment to achieve a power of at least 0.85 whenever  $\sigma_A^2 > .8225\sigma_e^2$ .

The following R function, RepSizeRandEffectFixedt-Findr.R in eCampus, can also be used to determine value of r:

```

repfixt <- function(alpha, gamma, t, sigma_a, sigma_e)
{
  r      <- 1
  power <- 0
  nu1   <- t-1
  while(power < gamma) {
    r      <- r+1
    nu2   <- t*(r-1)
    tau    <- (sigma_a/sigma_e)^2
    lambda <- sqrt(1+r*tau)
    Fcv   <- qf(1-alpha, nu1, nu2)
    C      <- (1/lambda^2)*Fcv
    power <- 1-pf(C, nu1, nu2)
  }
  print(cbind(r, nu1, nu2, lambda, power)) }

repfixt(.05,.85,5,2.5,2.7566)

      r  nu1  nu2      lambda      power
[1,] 9    4    40    2.898697  0.8694947

```



## Determine the Minimum Number of Treatments, $t$ , to Achieve Specified Power for Fixed Number of Reps, $r$

For a fixed number of reps  $r$ , determine the minimum number of treatments  $t$  to randomly select such that  $\gamma(\tau) \geq \gamma_o$  whenever  $\tau \geq \tau_o$ .

1. Specify a value of  $r$
2. Specify a value of  $\alpha$
3. Specify a value for  $\gamma_o$
4. Specify a value for  $\tau_o$  or
5. Specify a value for  $\sigma_A$  and an estimate of  $\sigma_e$

Using an iterative procedure, determine the minimum value of  $t$  such that  $\gamma(\tau) \geq \gamma_o$  whenever  $\tau \geq \tau_o$ .

### SAS Code for Sample Size Calculations

```
SAS Program) to compute sample size for experiments with random effects  
*Fix number of reps, r, then determine number of treatments, t for Specified Power;  
*trtsize,randomeffects,fixedr.SAS;  
data;  
input t @@;  
r=5;  
u1=t-1;  
u2=t*(r-1);  
sigma_e=2.7566;  
sigma_a=2.5;  
gamma=.85;  
alpha=.05;  
tau=(sigma_a/sigma_e)**2;  
lambda=sqrt(1+(r)*(tau));  
Fcr=finv(1-alpha,u1,u2);  
C=(1/lambda**2)*Fcr;  
power=1-probf(C,u1,u2);  
cards;  
3 4 5 6 7 8 9  
run;  
proc print; var t u1 u2 Fcr lambda power;  
run;
```

**EXAMPLE:** Suppose  $r = 5$ ,  $\alpha = .05$ ,  $\gamma_o = .85$ , and  $\tau_o = .8225$

That is, for a Random Effects Experiment with  $r = 5$  replications, determine the minimum number of treatments such that an  $\alpha = .05$  test with have at least an 85% chance to reject  $H_0$  and declare  $\sigma_A > 0$  whenever  $\sigma_A^2 > .8225\sigma_e^2$ .

Using Table X, the following specifications are needed:

1.  $r = 5$
2.  $\alpha = .05$  and  $\gamma_o = .85$
3.  $\nu_1 = t - 1$  and  $\nu_2 = n - t = rt - t = t(r - 1) = t(5 - 1) = 4t$
4.  $\lambda^2 = 1 + r\tau = 1 + (.8225)(5) = 5.1125 \Rightarrow \lambda = 2.26$
5.  $\gamma(\tau_o) = 1 - G_{t-1,t(r-1)}\left(\frac{1}{1+r\tau_o}F_{\alpha,t-1,t(r-1)}\right) = 1 - G_{t-1,t(r-1)}\left(\frac{1}{\lambda^2}F_{\alpha,t-1,t(r-1)}\right)$
6. Iteratively, determine the smallest value of  $t$  such that  $\gamma(.8225) \geq .85$

From Table X or SAS Output, we obtain the following values:

t	$\nu_1 = t - 1$	$\nu_2 = 4t$	$F_{.05,t-1,4t}$	$\lambda = 2.26$	$\gamma(.8225)$
7	6	28	2.4453	2.26	$1 - G_{6,28}(.4783) = .8187$
8	7	32	2.3127	2.26	$1 - G_{7,32}(.4524) = .8611$

Thus, the minimum value of  $t$  is 8. Thus, an  $\alpha = .05$  test would require randomly selecting  $t = 8$  treatments along with  $r = 5$  reps per treatment to achieve a power of at least 0.85 whenever  $\sigma_A^2 > .8225\sigma_e^2$ .

The following R function, RepSizeRandEffectFixedr-Findt.R in eCampus, can also be used to determine value of  $t$ :

```
repfixr <- function(alpha, gamma, r, tau)
{
  t      <- 2
  power <- 0
  while(power < gamma) {
    t      <- t+1
    nu1   <- t-1
    nu2   <- t*(r-1)
    lambda <- sqrt(1+r*tau)
    Fcr   <- qf(1-alpha, nu1, nu2)
    C     <- (1/lambda^2)*Fcr
    power <- 1-pf(C, nu1, nu2)
  }
  print(cbind(r, t, nu1, nu2, Fcr, lambda, power)) }
repfixr(.05,.85,5,.8225)

  r  t  nu1  nu2      Fcr    lambda      power
[1,] 5  8    7  32  2.312741  2.261084  0.8611072
```

STOP Friday 3/4/22 (Week 7, lecture 1a)

START Monday 3/7/22 (Week 8, lecture 20)

## EXAMPLE: Fishing Nets

A manufacturer of commercial fishing nets produces the net material on a large number of machines. The company would like the machines to be as homogeneous as possible in order to produce netting that has a very uniform strength. The company is concerned that there is considerable amount of material that must be discarded, reworked, or downgraded to a lower quality product. This results in loss of revenue to the company. The process engineer, after examining the maintenance records of the machines, suspects that there may be a larger variation in material strength between machines relative to the usual variation in material produced from the same machine. The two sources of variation are measured by  $\sigma_M^2$ , the detectable variation between machines, referred to as the "Special Cause" variability, the variability due to Machine Differences. The second source of variability is measured by  $\sigma_e^2$ , the background noise or variability in the process, referred to as the "Common Cause" variability. The engineer decides to run a small pilot study by randomly selecting four machines at random and then selecting five samples of material from each of the selected machines for strength determinations. The 20 samples of material were then analyzed in random order. The results are given here.

Machine	Strength Measurements: $y_{ij}$					Mean: $\bar{y}_i$
	1	2	3	4	5	
1	128	127	129	126	128	127.6
2	121	120	123	122	125	122.2
3	126	125	127	125	124	125.4
4	125	126	129	128	127	127.0

$$y_{ij} = \mu + M_i + \epsilon_{ij}$$

$i = 1, \dots, 4$   
 $j = 1, \dots, 5$

Assumptions:  $\epsilon_{ij} \sim N(0, \sigma_e^2)$   
 $M_i \sim N(0, \sigma_M^2)$

The above data was analyzed using the following SAS code (fishnet-random.sas):

```
* fishnet-random.sas;
ods html; ods graphics on;
option ls=80 ps=55 nocenter nodate;
title 'One-way ANOVA with Random Factor';

data fishnet;
array Y Y1-Y5;
input MACH $ Y1-Y5; do over Y; S=Y; output; end;
drop Y1-Y5;
label MACH = 'MACHINE' S = 'STRENGTH OF MATERIAL';
cards;
M1 128 127 129 126 128
M2 121 120 123 122 125
M3 126 125 127 125 124
M4 125 126 129 128 127
RUN;
```

when machines were fixed (random)  
what was our assumption?

$$\sum_{i=1}^4 M_i = 0 \quad (\sum_{i=1}^4 \epsilon_{ij} = 0)$$

$$\Leftrightarrow M_{\bar{y}} = 0$$

$$H_0: \sigma_M^2 = 0$$

$$H_a: \sigma_M^2 > 0$$

```

title 'ANALYSIS USING PROC MIXED-Type 3 \& REML';
PROC MIXED METHOD=TYPE3 CL; → MOM Asking for CI for parameters
CLASS MACH; → CL In class statement, we indicate all our categorical variables.
MODEL S = / RESIDUALS COVB ;
RANDOM MACH/CL ALPHA=.05;
RUN; → ML
PROC MIXED METHOD= REML CL; response = fixed effects
CLASS MACH;
MODEL S = / RESIDUALS COVB ; Want residuals and covariance test
RANDOM MACH/CL ALPHA=.05; In random statement, we include all our random factors.
RUN;
ods graphics off; ods html close;
Output from SAS:

```

• We will talk about why we prefer REML vs Type 3 (MOM)

(MLR)

## ANALYSIS USING PROC MIXED-TYPE3

### The Mixed Procedure

Model Information	
Data Set	WORK.FISHNET
Dependent Variable	S
Covariance Structure	Variance Components
Estimation Method	Type 3 <i>MOM</i>
Residual Variance Method	Factor
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class Level Information		
Class	Levels	Values
MACH	4	M1 M2 M3 M4

Dimensions	
Covariance Parameters	2
Columns in X	1
Columns in Z	4
Subjects	1
Max Obs per Subject	20

Number of Observations	
Number of Observations Read	20
Number of Observations Used	20
Number of Observations Not Used	0

Type 3 Analysis of Variance								
Source	DF	Sum of Squares	Mean Square	Expected Mean Square	Error Term	Error DF	F Value	Pr > F
MACH	3	87.750000	29.250000	Var(Residual) - Var(MACH)	MS(Residual)	16	13.30	0.0001
Residual	16	35.200000	2.200000	Var(Residual)	.	.	.	.

## ANALYSIS USING PROC MIXED-TYPE3

### The Mixed Procedure

Covariance Parameter Estimates				
Cov Parm	Estimate	Alpha	Lower	Upper
MACH	5.4100	0.05	-3.9567	14.7767
Residual	2.2000	0.05	1.2203	5.0958

Fit Statistics	
-2 Res Log Likelihood	79.7
AIC (Smaller is Better)	83.7
AICC (Smaller is Better)	84.4
BIC (Smaller is Better)	82.4

Covariance Matrix for Fixed Effects		
Row	Effect	Col1
1	Intercept	1.4625

Solution for Random Effects									
Effect	MACHINE	Estimate	Std Err Pred	DF	t Value	Pr >  t	Alpha	Lower	Upper
MACH	M1	1.8958	1.2875	16	1.47	0.1603	0.05	-0.8336	4.6252
MACH	M2	-3.0980	1.2875	16	-2.41	0.0286	0.05	-5.8274	-0.3686
MACH	M3	-0.1387	1.2875	16	-0.11	0.9155	0.05	-2.8681	2.5907
MACH	M4	1.3409	1.2875	16	1.04	0.3131	0.05	-1.3885	4.0703

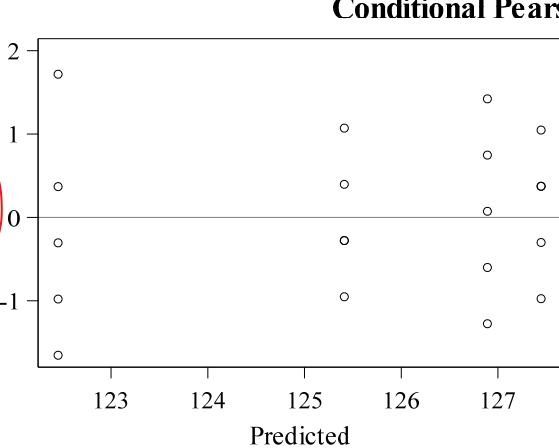
## ANALYSIS USING PROC MIXED-TYPE3

### The Mixed Procedure

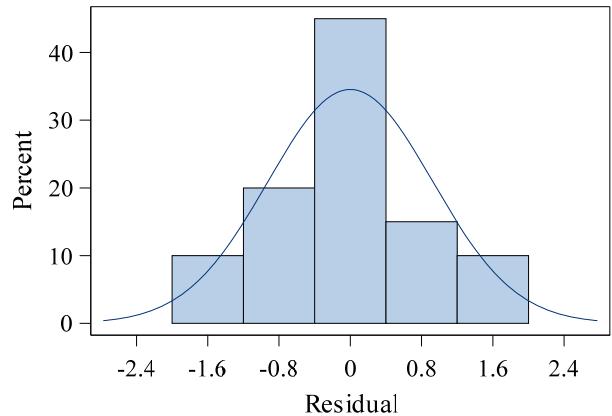
Checking assumptions of residuals

$$\textcircled{1} \quad \epsilon \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$$

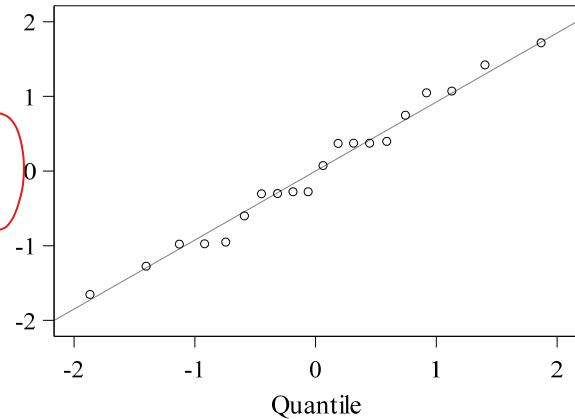
*Heterogeneity not violated*



*Symetric*



*Normality not violated*



Residual Statistics	
Observations	20
Minimum	-1.653
Mean	67E-16
Maximum	1.7179
Std Dev	0.9241

Fit Statistics	
Objective	79.658
AIC	83.658
AICC	84.408
BIC	82.431

## ANALYSIS USING PROC MIXED-TYPE3

### The Mixed Procedure

Model Information	
Data Set	WORK.FISHNET
Dependent Variable	S
Covariance Structure	Variance Components
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class Level Information		
Class	Levels	Values
MACH	4	M1 M2 M3 M4

Covariance Parameter Estimates				
Cov Parm	Estimate	Alpha	Lower	Upper
MACH	5.4100	0.05	1.6283	107.22
Residual	2.2000	0.05	1.2203	5.0958

$\sigma^2_M$   
shaded  
or Residual

\* Reason Both the numbers  
are the same & b/c I  
have equal # of replicates.  
More on this in H.O.Q.

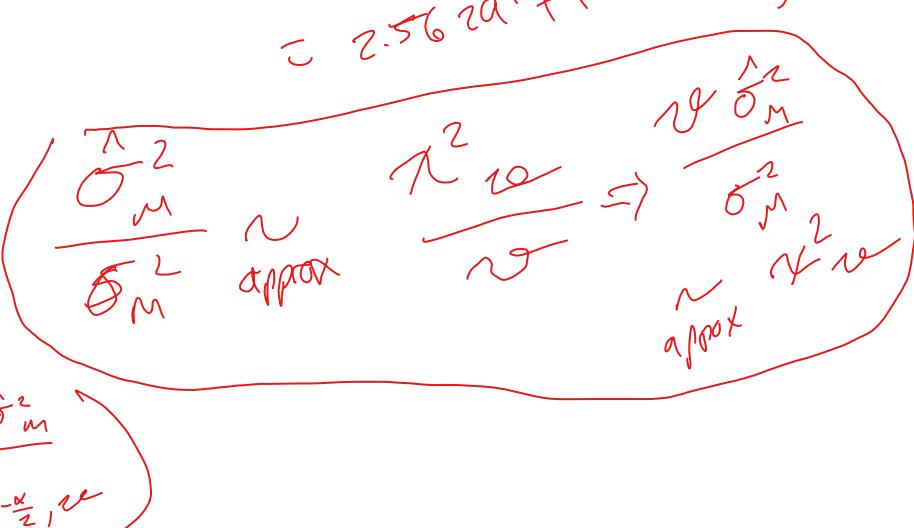
CI doesn't contain 0.

different b/c df is different.

$\Sigma \in \text{df} = \text{scatterwise calc for } \sigma^2_M$

$$\Sigma = \frac{(MS_{TYPE} - MSE)^2}{n-t}$$

$$\frac{MS_{TYPE}^2}{t-1} + \frac{MSE^2}{n-t} = \frac{(29.25 - 2.2)^2}{(29-25-2.2)} = \frac{24.76}{\frac{25.75^2 + 12}{16}} = 2.562071$$



$$P\left(\chi^2_{\frac{\alpha}{2}, n} \leq \frac{v \hat{\sigma}_M^2}{\hat{\sigma}_M^2} \leq \chi^2_{1-\frac{\alpha}{2}, n}\right) = 1 - \alpha$$

$$\Rightarrow CI: \left( \frac{v \hat{\sigma}_M^2}{\chi^2_{\alpha/2, n}}, \frac{v \hat{\sigma}_M^2}{\chi^2_{1-\alpha/2, n}} \right)$$

The following R code - fishnet-random.R - yields the same results:

```
install.packages("lme4")

library(lme4)

mach = as.factor(c(rep("M1",5),rep("M2",5),rep("M3",5),rep("M4",5)))
strngt =
c(128, 127, 129, 126, 128,
  121, 120, 123, 122, 125,
  126, 125, 127, 125, 124,
  125, 126, 129, 128, 127)

data <- data.frame(strngt,mach)

#treat Machine as a fixed effect to obtain Sum of Squares
fixmach = lm(strngt ~ mach)

anova(fixmach)
```

Output from R:  
Analysis of Variance Table

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
mach	3	87.75	29.25	13.296	0.0001301 ***
Residuals	16	35.20	2.20		

Same as ANOVA table on pg 29

#treat Machine as a random effect to obtain estimates of variances

```
ranmach = lmer(strngt ~ 1+(1|mach),data)

summary(ranmach)
```

Output from R:

Random effects:  

Groups	Name	Variance	Std.Dev.
mach	(Intercept)	5.41	2.326
Residual		2.20	1.483

  
Number of obs: 20, groups: mach, 4

↑ not given CIs for  $\hat{\sigma}_m^2$ ;  $\hat{\sigma}_e^2$  in  
we are in SAS.  
→ Equivalent to covariance param estimates b66 in SAS

Fixed effects:

	Estimate	Std. Error	t value
(Intercept)	125.550	1.209	103.8

$$y_{ij} = \mu + \mu_i + \epsilon_{ij}$$

$\downarrow$   
 $\sigma_M^2$

$\downarrow$   
 $\sigma_e^2$

From the above, there is significant ( $p - value < .0001$ ) evidence that  $\sigma_M > 0$ . Furthermore,  $\hat{\sigma}_M$  is considerably larger than  $\hat{\sigma}_e$ . An estimate of the variation in the strength measurements can be computed as

$$\hat{\sigma}_y^2 = \hat{\sigma}_M^2 + \hat{\sigma}_e^2 = 5.41 + 2.2 = 7.61.$$

This implies that the proportion of the total variation in the strength measurements attributed to the differences in the machines and the proportion attributed to the within machine variation are

Proportion Due to Machine Differences:  $\frac{\hat{\sigma}_M^2}{\hat{\sigma}_y^2} = \frac{5.41}{7.61} = 71.1\%$

Proportion Due to Within Machine Differences:  $\frac{\hat{\sigma}_e^2}{\hat{\sigma}_y^2} = \frac{2.2}{7.61} = 28.9\%$

Thus, the major source of variation is due to differences in the machines. This may be caused by faulty setup of the machine, ineffective supervision of the operators of the machine, infrequent or faulty maintenance, or poorly trained operators. Based on this pilot study the process engineer would like to design a more extensive study of the hundreds of machines used to produce the fish net material. A sample size calculation is needed to determine how many machines to sample and how many replications per machine. The engineer has decided on the number of replications needed per machine but wants assistance in determining how many machines to select.

## C.I.'s and Tests of Hypotheses when $n_i$ 's are unequal

When the sample sizes are unequal, the F-statistic  $\frac{MS_{TRT}}{MSE}$  does not have a multiple times a F-distribution, as in the equal sample size case. The random variables,  $MS_{TRT}$  and  $MSE$  are still independent and  $\frac{1}{\sigma_e^2} MSE$  still has a chi-square distribution, but  $MS_{TRT}$  is no longer a constant times a chi-square distributed random variable. However, it is still possible to use the statistic  $F = \frac{MS_{TRT}}{MSE}$  to test of  $H_0 : \sigma_A = 0$  versus  $H_1 : \sigma_A > 0$  because under  $H_0$  Wald(1940) has shown that  $\frac{MS_{TRT}}{MSE}$  has a Central F-Distribution with  $df = t - 1, n - t$ . Wald defines the following statistic:

$$w_i = \frac{\sigma_e^2}{Var(\bar{y}_{i.})} = \frac{\sigma_e^2}{\sigma_A^2 + \frac{\sigma_e^2}{n_i}} = \frac{n_i}{1 + n_i \tau} = \frac{1}{\frac{1}{n_i} + \tau}$$

$$h(\tau) = \sum_{i=1}^t w_i \left( \bar{y}_{i.} - \frac{\sum_{i=1}^t w_i \bar{y}_{i.}}{\sum_{i=1}^t w_i} \right)^2$$

Wald then shows that

$$F^* = \frac{h(\tau)/(t-1)}{MSE}$$

has an F-distribution with  $df = t - 1, n - t$ .

However, under  $H_0 : \sigma_A = 0 \Rightarrow \tau = \sigma_A^2 / \sigma_e^2 = 0$ ,

$$\tau = 0 \Rightarrow w_i = \frac{n_i}{1 + n_i \tau} = n_i \Rightarrow h(\tau) = SS_{TRT} \Rightarrow F^* = \frac{MS_{TRT}}{MSE}$$

Thus, under  $H_0$ ,  $F^* = F$ . Thus, we can use the following decision rule:

Reject  $H_0 : \sigma_A = 0$  if  $F_o \geq F_{\alpha, t-1, n-t}$  or

Reject  $H_0 : \sigma_A = 0$  if  $p-value = 1 - G(F_o) \leq \alpha$  where  $G(\cdot)$  is the cdf of a central F-distribution with  $df = t - 1, n - t$  and  $F_o$  is the observed value of  $\frac{MS_{TRT}}{MSE}$

However, the power function of the test is intractable, because the distribution of  $\frac{MS_{TRT}}{MSE}$  under  $H_1 : \sigma_A > 0$  is unknown. Simulations could be used to obtain approximations to the power function for given specifications of the sample sizes and parameters.

**Confidence Intervals for  $\sigma_e^2$  and  $\sigma_A^2$ :**

1.  $100(1 - \alpha)\%$  Confidence interval for  $\sigma_e^2$ :

$$\left[ \frac{SSE}{\chi_{\alpha/2, n-t}^2}, \frac{SSE}{\chi_{1-\alpha/2, n-t}^2} \right]$$

The above interval is an exact  $100(1 - \alpha)\%$  confidence interval even for unequal sample sizes, that is,  $n_i$ s not all the same value.

2.  $100(1 - \alpha)\%$  Approximate Confidence interval for  $\sigma_A^2$ : \*

$$\left[ \frac{kL(MS_{TRT}^*)}{f_1(1 + kL)}, \frac{kU(MS_{TRT}^*)}{f_3(1 + kU)} \right]$$

$$f_1 = \chi_{\alpha/2, t-1}^2, \quad f_2 = F_{\alpha/2, t-1, n-t}, \quad f_3 = \chi_{1-\alpha/2, t-1}^2, \quad f_4 = F_{1-\alpha/2, t-1, n-t}$$

$$m = \min(n_1, \dots, n_t), \quad M = \max(n_1, \dots, n_t), \quad k = \left( \sum_{i=1}^t \frac{1}{n_i} \right)^{-1}$$

$$MS_{TRT}^* = \frac{1}{t-1} \sum_{i=1}^t \left( \bar{y}_{i \cdot} - \frac{1}{t} \sum_{i=1}^t \bar{y}_{i \cdot} \right)^2, \quad L = \frac{MS_{TRT}^*}{f_2 MSE} - \frac{1}{m}, \quad U = \frac{MS_{TRT}^*}{f_4 MSE} - \frac{1}{M}$$

For details, see Burdick & Eickman(1986), *Journal of Statistical Computation and Simulation*, Vol. 26, pp. 205-219 and Burdick, Birch & Grabbill(1986), *Journal of Statistical Computation and Simulation*, Vol. 26, pp. 259-272.

## Subsampling in a Fixed Effects Experiment

In some experiments the researcher wants to obtain additional information concerning the variability of the response within the individual EU's. The EU may be subdivided and/or random samples of information are obtained on each EU.

### EXAMPLE: Turf Grass Investigation

A national golf association is interested in determining if the four leading root growth stimulators generate different mean root growth. A test area on a golf course is selected and twenty-four homogeneous plots of grass of equal size were identified. The four stimulators were then applied to six plots of grass each. After two months of treatment with the stimulators, three cores of a uniform size were randomly selected from each of the 24 plots. The cores were then analyzed in a random order for root weight in grams. The cores were selected to determine the amount of variability of the root growth over the plots. A large number of cores were misplaced and the experiment became unbalanced in that there are not six plots per stimulator and there are not three cores per plot. The reason that some of the cores are missing was not related to the treatments.

Stimulator	Plots						Mean $\bar{y}_{i..}$
	1	2	3	4	5	6	
S1 (means)	3.3,3.4,3.5 (3.4)	3.1,3.5,3.0 (3.2)	3.2,3.1,3.4 (3.233)	3.3,2.9,3.0 (3.067)	3.3,3.3,3.1 (3.233)	.	3.2267
S2 (means)	3.8,3.7,4.0 (3.833)	3.5,3.8,3.9 (3.733)	3.6,3.4,3.8 (3.6)	3.4,3.7 (3.55)	3.6,3.7,3.6 (3.633)	3.5,3.9 (3.7)	3.6779
S3 (means)	3.8,3.9,4.0 (3.9)	3.6,3.7,3.8 (3.7)	3.3,3.4 (3.35)	3.6,3.7 (3.65)	3.5,3.9 (3.7)	3.4,3.7 (3.55)	3.6516
S4 (means)	4.3,4.3,4.4 (4.333)	4.1,3.9,3.8 (3.933)	4.2,4.1,3.9 (4.067)	3.7,3.9,4.0 (3.867)	.	.	4.05

There are four treatments, the root growth stimulators, a fixed number of levels.

The EU's are the 21 (24-3) plots of land, a random effect.

The 57 (72-15) cores are subsamples, a random factor.

 If the four root growth stimulators were randomly selected from a long list of root growth stimulators, then this would be a random effect also.

Thus, the sources of variation in the root weights are the four Stimulators (fixed effect), the Plots nested within Stimulators (random effect), and the Cores nested within Plots within Stimulators (random effect).

## Models for Subsampling Experiments

### Case 1: Fixed Treatment Effect:

Suppose that there are  $t$  treatments which are the only treatments of interest to the researcher, there are  $n_i$  E.U.'s randomly assigned to treatment  $i$ , there are  $m_{ij}$  subsamples on the  $j$ th E.U. receiving treatment  $i$ , and  $y_{ijk}$  is the response on the  $k$ th subsample from the  $j$ th E.U. receiving treatment  $i$ .

$$y_{ijk} = \mu + \tau_i + e_{ij} + d_{ijk}, \quad \text{for } i = 1, \dots, t; \quad j = 1, \dots, n_i \quad k = 1, \dots, m_{ij}$$

1.  $\mu$  is the overall treatment mean;
2.  $\tau_i$  is the effect of  $i$ th treatment;
3.  $e_{ij}$  is random effect of  $(i, j)$  E.U.;
4.  $d_{ijk}$  is random effect of  $(i, j, k)$  subsample;
5. Set  $\tau_t = 0 \Rightarrow \mu = \mu_t; \quad \tau_i = \mu_i - \mu_t; \quad i = 1, \dots, t-1,$
6.  $e_{ij}$ 's are iid  $N(0, \sigma_e^2)$
7.  $d_{ijk}$  are iid  $N(0, \sigma_d^2)$
8.  $e_{ij}$ 's and  $d_{ijk}$ 's are independent
9.  $\mu_i = E[y_{ijk}] = \mu + \tau_i$
10.  $\sigma_y^2 = Var[y_{ijk}] = \sigma_e^2 + \sigma_d^2$
11. The multiple responses on the same EU are correlated:

$$\begin{aligned} Cov(y_{ijk}, y_{i'j'k'}) &= E[(y_{ijk} - \mu - \tau_i)(y_{i'j'k'} - \mu - \tau_{i'})] \\ &= E[e_{ij}e_{i'j'}] + E[d_{ijk}d_{i'j'k'}] + E[e_{ij}d_{i'j'k'}] + E[e_{i'j'}d_{ijk}] \end{aligned}$$

*↙ ↘  $e_{ij}, e_{i'j'} \{ d_{ijk}, d_{i'j'k'} \text{ are ind.}$*

Thus, we have  $Cov(y_{ijk}, y_{i'j'k'}) = E[e_{ij}e_{i'j'}] + E[d_{ijk}d_{i'j'k'}] + 0 \Rightarrow$

$$\begin{aligned} E[e_{ij}e_{i'j'}] &= E[e_{ij}]E[e_{i'j'}] \\ &= 0 * 0 \\ &= 0 \end{aligned}$$

$$Cov(y_{ijk}, y_{i'j'k'}) = \begin{cases} 0 & \text{if } i \neq i' \text{ or } j \neq j' \text{ (different EU's)} \\ \sigma_e^2 & \text{if } i = i', j = j', k \neq k' \text{ (same EU's, different Obs)} \end{cases}$$

**Goal:** Test for differences in the treatments

Cell Means Model: Test  $H_0 : \mu_1 = \mu_2 = \dots = \mu_t$  versus  $H_1 : \text{not all } \mu_i \text{ are equal}$

Effects Model: Test  $H_0 : \tau_1 = \tau_2 = \dots = \tau_t = 0$  versus  $H_1 : \text{at least one } \tau_i \neq 0$

Both Models: Test  $H_0 : \theta_\tau = 0$  versus  $H_1 : \theta_\tau \neq 0$ , where

$$\theta_\tau = \frac{1}{t-1} \sum_{i=1}^t m_{i\cdot} (\mu_i - \bar{\mu})^2 \text{ for Cell Means Model}$$

$$\theta_\tau = \frac{1}{t-1} \sum_{i=1}^t m_{i\cdot} (\tau_i)^2 \text{ for Effects Model}$$

where  $m_{i\cdot} = \sum_{j=1}^{n_i} m_{ij}$ , the number of subsamples for the  $i$ th treatment.

## Case 2: Random Treatment Effects:

Treatments are randomly selected from a population of treatments.

Model:  $y_{ijk} = \mu + A_i + e_{ij} + d_{ijk}$ , for  $i = 1, \dots, t; j = 1, \dots, n_i; k = 1, \dots, m_{ij}$

1.  $\mu$  is the overall treatment mean;
2.  $A_i$  is the effect of  $i$ th treatment;
3.  $A_i$ 's are distributed iid  $N(0, \sigma_A^2)$ ;
4.  $e_{ij}$  is random effect of  $(i, j)$  E.U.;
5.  $e_{ij}$ 's are distributed iid  $N(0, \sigma_e^2)$ ,
6.  $d_{ijk}$  is random effect of  $(i, j, k)$  subsample;
7.  $d_{ijk}$  are distributed iid  $N(0, \sigma_d^2)$
8.  $A_i$ 's,  $e_{ij}$ 's and  $d_{ijk}$ 's are mutually independent
9.  $\mu_i = E[y_{ijk}] = \mu$
10.  $\sigma_y^2 = Var[y_{ijk}] = Var[\mu + A_i + e_{ij} + d_{ijk}] = \sigma_A^2 + \sigma_e^2 + \sigma_d^2$
11.  $Cov(y_{ijk}, y_{i'j'k'}) = E[(y_{ijk} - \mu)(y_{i'j'k'} - \mu)] = E[A_i A_{i'}] + E[e_{ij} e_{i'j'}] + E[d_{ijk} d_{i'j'k'}] + 0 \Rightarrow$

$$Cov(y_{ijk}, y_{i'j'k'}) = \begin{cases} 0 & \text{if } i \neq i' \text{ (different TRT's)} \\ \sigma_A^2 & \text{if } i = i', j \neq j' \text{ (same TRT's, different EU's)} \\ \sigma_A^2 + \sigma_e^2 & \text{if } i = i', j = j', k \neq k' \text{ (same TRT's, same EU's, diff. Obs)} \end{cases}$$

**Goal:** Obtain estimates of the variance components and evaluate the relative sizes of

$$\sigma_A^2, \quad \sigma_e^2, \quad \sigma_d^2$$

## Sum of Squares for Both Fixed and Random Effects Models

Define the following Sum of Squares with the following notation:  $t$  = number of treatments,  
 $n_i$  = number of EU's assigned to Trt  $i$ ,  $n_{..} = \sum_{i=1}^t n_i$  = total number of EU's  
 $m_{ij}$  = number of subsamples taken on the  $j$ th EU assigned to Trt  $i$ ,  
 $m_{i..} = \sum_{j=1}^{n_i} m_{ij}$  = number of data values obtained from Trt  $i$

1. Corrected Sum of Squares Total:  $SS_{TOT}$
- Pooled Variation of  $y_{ijk}$ 's ignoring treatments, reps, and subsamples

$$SS_{TOT} = \sum_{i=1}^t \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} (y_{ijk} - \bar{y}_{...})^2$$

with  $df = \sum_{ij} m_{ij} - 1 = n - 1$

2. Sum of Squares Subsample Within Rep: Sub(Rep,Trt)

- Pooled Variation Within Subsamples for Same EU -

Variation Within EU or Due to Measurement Error

$$SS_{S(R,T)} = \sum_{i=1}^t \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} (y_{ijk} - \bar{y}_{ij.})^2$$

with  $df = \sum_{i=1}^t \sum_{j=1}^{n_i} (m_{ij} - 1) = n - n_{..}$   $n_{..} = \text{total } n \text{ not } \# \text{ of EU's}$

3. Sum of Squares Rep within Treatment: R(T)

- Pooled Variation Between Replications Within Same Treatment

$$SS_{R(T)} = \sum_{i=1}^t \sum_{j=1}^{n_i} m_{ij} (\bar{y}_{ij.} - \bar{y}_{i..})^2$$

with  $df = \sum_{i=1}^t (n_i - 1) = n_{..} - t$  = Number of EU's - Number of Treatments

4. Sum of Squares Treatment: TRT

- Variation Between the  $t$  Treatments

$$SS_{TRT} = \sum_{i=1}^t m_{i..} (\bar{y}_{i..} - \bar{y}_{...})^2$$

with  $df = t - 1$

To determine the proper test statistic, it is necessary to derive the Expected Mean Squares:  
**Expected Mean Squares:**

$$1. \quad E[MS_{TRT}] = \sigma_d^2 + C_1\sigma_e^2 + C_2\theta_\tau \text{ (Fixed Treatment Effects)}$$

$$E[MS_{TRT}] = \sigma_d^2 + C_1\sigma_e^2 + C_2\sigma_A^2 \text{ (Random Treatment Effects)}$$

$$2. \quad E[MS_{R(T)}] = \sigma_d^2 + C_3\sigma_e^2 \text{ (Random and Fixed Treatment Effects)}$$

$$3. \quad E[MS_{S(R,T)}] = \sigma_d^2 \text{ (Random and Fixed Treatment Effects)}$$

The constants  $C_i$  are defined as:

$$C_1 = \frac{1}{t-1} (A - \frac{B}{n}), \quad C_2 = \frac{1}{t-1} (n - \frac{D}{n}), \quad C_3 = \frac{1}{n-t} (n - A)$$

$$\text{where } A = \sum_{i=1}^t \sum_{j=1}^{n_i} \frac{m_{ij}^2}{m_{i.}}, \quad B = \sum_{i=1}^t \sum_{j=1}^{n_i} m_{ij}^2, \quad D = \sum_{i=1}^t m_{i.}^2.$$

Under  $H_0 : \theta_\tau = 0$ ,

$$E[MS_{TRT}] = \sigma_d^2 + C_1\sigma_e^2$$

$$E[MS_{R(T)}] = \sigma_d^2 + C_3\sigma_e^2$$

$$E[MS_{S(R,T)}] = \sigma_d^2$$

*STOP Monday 3/7/22 (Week 8, lecture 20)*

~~STRT WEDNESDAY 3/9/22 (Week 8, Lecture 21)~~

## Case 1: Equal Number of Subsamples

When  $m_{ij} = m$  for all  $(i, j)$ , equal number of subsamples for all reps,

$$A = \sum_{i=1}^t \sum_{j=1}^{n_i} \frac{m^2}{mn_i} = tm$$

$$B = \sum_{i=1}^t \sum_{j=1}^{n_i} m^2 = \sum_{i=1}^t n_i m^2 = mn \text{ because } n = \sum_{i=1}^t \sum_{j=1}^{n_i} m = m \sum_{i=1}^t n_i$$

$$D = \sum_{i=1}^t m_i^2 = \sum_{i=1}^t (mn_i)^2 = m^2 \sum_{i=1}^t n_i^2$$

$$C_1 = \frac{1}{t-1} (A - \frac{B}{n}) = \frac{1}{t-1} (tm - \frac{mn}{n}) = m$$

$$C_2 = \frac{1}{t-1} (n - \frac{D}{n}) = \frac{1}{t-1} \left( trm - \frac{t(rm)^2}{n} \right) = rm \text{ (If also } n_i = r)$$

$$C_3 = \frac{1}{n-t} (n - A) = \frac{1}{n-t} (mn - tm) = m$$

and hence  $C_1 = C_3$  which implies that under  $H_o$ ,  $E[MS_{TRT}] = E[MS_{R(T)}]$ .

Thus,  $F = \frac{MS_{TRT}}{MS_{R(T)}}$  is a test statistic for testing

either  $H_o : \theta_\tau = 0$  vs  $H_1 : \theta_\tau > 0$  or testing  $H_o : \sigma_A = 0$  vs  $H_1 : \sigma_A > 0$

In either case,  $F = \frac{MS_{TRT}}{MS_{R(T)}}$  has an F-distribution with  $df = t-1, n. - t$  under  $H_o$ .

Thus, reject  $H_o : \theta_\tau = 0$  if  $F_o \geq F_{\alpha, t-1, n.-t}$

and reject  $H_o : \sigma_A = 0$  if  $F_o \geq F_{\alpha, t-1, n.-t}$

Also, we can compute  $p-value = 1 - G(F_o)$ , where  $G(\cdot)$  is the cdf of an F-distribution with  $df = t-1, n. - t$

and  $F_o$  is the computed value of  $\frac{MS_{TRT}}{MS_{R(T)}}$

Thus, reject  $H_o$  is p-value  $\leq \alpha$ .

Under  $H_1 : \theta_\tau > 0$ ,  $F = \frac{MS_{TRT}}{MS_{R(T)}}$  has a non-central F-distribution with  $df = t-1, t(n.-1)$  and non-centrality parameter

$$\Delta = \frac{mr \sum_{i=1}^t (\mu_i - \bar{\mu})^2}{\sigma_d^2 + r\sigma_e^2} = \frac{mr \sum_{i=1}^t (\mu_i - \bar{\mu})^2}{\sigma_{S(R,T)}^2 + r\sigma_{R(T)}^2}$$

This result will be used to determine the power function and in computing sample size when we have subsampling. The power function is given by

$$\gamma(\Delta) = 1 - G(F_{\alpha, t-1, t(n.-1)}) = 1 - pf(F_\alpha, t-1, t(n.-1), \Delta)$$

where  $G$  is the cdf of a non-central F-distribution with  $df = t-1, t(n.-1)$  and non-centrality parameter

$$\Delta = \frac{mr \sum_{i=1}^t (\mu_i - \bar{\mu})^2}{\sigma_d^2 + r\sigma_e^2}, \text{ when we restrict } m_{ij} = m, n_i = r \Rightarrow n = trm$$

## Case 2: Unequal Number of Subsamples

When  $n_{ij}$ 's are not all equal, then  $C_1 \neq C_3$  and  $\frac{MS_{TRT}}{MS_{R(T)}}$  does not have an F-distribution.

An approximate test statistic can be constructed by considering a linear combination of  $MS_{R(T)}$  and  $MS_{S(R)}$ :

$M = a_1 MS_{R(T)} + a_2 MS_{S(R)}$  such that under  $H_o$  we have  $E[M] = E[MS_{TRT}]$ .

That is, select  $a_1$  and  $a_2$  such that

$$E(M) = a_1 E[MS_{R(T)}] + a_2 E[MS_{S(R)}] = E[MS_{TRT}] \Rightarrow$$

$$a_1(\sigma_d^2 + C_3\sigma_e^2) + a_2\sigma_d^2 = \sigma_d^2 + C_1\sigma_e^2 \Rightarrow$$

$$a_1 + a_2 = 1 \text{ and } a_1 C_3 = C_1 \Rightarrow a_1 = \frac{C_1}{C_3} \quad a_2 = 1 - \frac{C_1}{C_3}$$

Thus, a potential test statistic is  $F = \frac{MS_{TRT}}{M}$ .

We need to determine the distribution of  $F$  under  $H_o$ :

### Satterthwaite Approximation:

Let  $M_1, \dots, M_k$  be independent Mean Squares from the AOV having  $df = \nu_1, \dots, \nu_k$ .

$$\text{Let } M = \sum_{i=1}^k a_i M_i \text{ and } \nu = \frac{M^2}{\sum_{i=1}^k \frac{(a_i M_i)^2}{\nu_i}}$$

Then  $\frac{\nu M}{E[M]}$  has an approximate chi-square distribution with  $df = \nu$

Applying the Satterthwaite Approximation to  $F = \frac{MS_{TRT}}{M}$  yields the result that  $F$  has under  $H_o$  an approximate F-distribution with  $df = t - 1, \nu$ , where

$$\nu = \frac{M^2}{\frac{\left(\frac{C_1}{C_3}\right)^2 (MS_{R(T)})^2}{(n-t)} + \frac{\left(1 - \frac{C_1}{C_3}\right)^2 (MS_{S(R)})^2}{(n-n.)}}$$

The above result is justified as follows

1.  $\frac{SS_{TRT}}{E[MS_{TRT}]}$  has a chi-square distribution with  $df = t - 1$
2.  $\frac{\nu M}{E[M]}$  has approximately a chi-square distribution with  $df = \nu$
3.  $MS_{TRT}$  is independent of  $MS_{R(T)}$  and  $MS_{S(R)}$  and hence of  $M$
4. Under  $H_o$ ,  $E[M] = E[MS_{TRT}]$
5. Under  $H_o$ ,  $F = \frac{MS_{TRT}}{M} = \frac{SS_{TRT}/(t-1)E[MS_{TRT}]}{\nu M/\nu E[M]} \stackrel{\mathcal{D}}{\rightarrow} \frac{\chi_{t-1}^2/(t-1)}{\chi_{\nu}^2/\nu}$

has approximately an F-distribution with  $df = t - 1, \nu$

Thus, reject  $H_o : \theta_\tau = 0$  or reject  $H_o : \sigma_A = 0$  if  $F \geq F_{\alpha, t-1, \nu}$  or

with  $p-value \approx 1 - G(F) = 1 - pf(F, t - 1, \nu)$ , where  $G(\cdot)$  is the cdf of an F-distribution with  $df = t - 1, \nu$

## Estimation of $\mu_i$ in Fixed Effects Model

When the treatments have fixed levels, then estimates of  $\mu_i$ 's are given as follows:

$n_i$  E.U.'s observed on treatment  $i$ ,  $m_{ij}$  subsamples on the  $j$ th E.U. receiving treatment  $i$ ,

The LSE of  $\mu_i$  is  $\hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \bar{y}_{ij} \neq \bar{y}_{i..}$  the mean of all measurements on Treatment  $i$

\* If there are equal numbers of subsamples, i.e.,  $m_{ij} = m$  then

$$\bar{y}_{i..} = \frac{1}{m_{i..}} \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} y_{ijk} = \frac{1}{mn_i} \sum_{j=1}^{n_i} \sum_{k=1}^m y_{ijk} = \frac{1}{n_i} \sum_{j=1}^{n_i} \frac{1}{m} \sum_{k=1}^m y_{ijk} = \frac{1}{n_i} \sum_{j=1}^{n_i} \bar{y}_{ij} = \hat{\mu}_i$$

This results from the fact that the subsamples are not valid replications of the treatment means. In fact, the AOV  $F$ -test for testing differences in the treatment means is equivalent to running an AOV on the means over the subsample means:  $\bar{y}_{ij}$ .

\* To obtain a C.I. for  $\mu_i$  we need to compute the variance of  $\hat{\mu}_i$

$$\begin{aligned} Var(\hat{\mu}_i) &= Var\left(\frac{1}{n_i} \sum_{j=1}^{n_i} \bar{y}_{ij}\right) \\ &= Var\left(\frac{1}{n_i} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} \sum_{k=1}^{m_{ij}} (\mu + \tau_i + e_{ij} + d_{ijk})\right) \\ &= Var\left(\bar{e}_{i..} + \frac{1}{n_i} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} \sum_{k=1}^{m_{ij}} d_{ijk}\right) \\ &= Var(\bar{e}_{i..}) + Var\left(\frac{1}{n_i} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} \sum_{k=1}^{m_{ij}} d_{ijk}\right) \\ &= \frac{\sigma_e^2}{n_i} + \frac{1}{n_i^2} \sum_{j=1}^{n_i} \frac{1}{m_{ij}^2} \sum_{k=1}^{m_{ij}} \sigma_d^2 = \frac{\sigma_e^2}{n_i} + \frac{\sigma_d^2}{n_i^2} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} \end{aligned}$$

\* When we have equal number of subsamples and equal number of reps, that is,  $m_{ij} = m$  and  $n_i = r$ , we have

$$Var(\hat{\mu}_i) = \frac{\sigma_e^2}{n_i} + \frac{\sigma_d^2}{n_i^2} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} = \frac{\sigma_e^2}{r} + \frac{\sigma_d^2}{r^2} \sum_{j=1}^r \frac{1}{m} = \frac{\sigma_d^2 + m\sigma_e^2}{rm} = \frac{EMSS_{R(T)}}{rm}$$

Therefore, in the case when  $m_{ij} = m$  and  $n_i = r$ , our estimator of the standard error of  $\hat{\mu}_i$  would be

$$\widehat{SE}(\hat{\mu}_i) = \sqrt{\frac{MS_{R(T)}}{rm}}$$

When we have an unequal number of subsamples and/or an unequal number of reps, that is,  $m_{ij} \neq m$  and/or  $n_i \neq r$ , then to obtain an estimate of  $Var(\hat{\mu}_i)$ , it is necessary to first obtain estimates of  $\sigma_e^2$  and  $\sigma_d^2$ :

Expected MS of the subsamples nested within the replicates.

Using an AOV-MOM procedure, we have

$$E[MS_{S(R)}] = \sigma_d^2 \Rightarrow \hat{\sigma}_d^2 = MS_{S(R)}$$

$$E[MS_{R(T)}] = \sigma_d^2 + C_3\sigma_e^2 \Rightarrow \hat{\sigma}_e^2 = \frac{MS_{R(T)} - MS_{S(R)}}{C_3}$$

Expected MS of the replicates nested within the treatments.

An approximate  $100(1 - \alpha)\%$  confidence interval for  $\mu_i$  is given by  $\hat{\mu}_i \pm t_{\alpha/2, \nu} \widehat{SE}(\hat{\mu}_i)$  where

$$\widehat{SE}(\hat{\mu}_i) = \sqrt{\frac{\hat{\sigma}_e^2}{n_i} + \frac{\hat{\sigma}_d^2}{n_i^2} \sum_{j=1}^{n_i} \frac{1}{m_{ij}}}$$

and  $\nu$  is obtained from the Satterthwaite :

$$\nu = \frac{\left( MS_{R(T)} / (C_3 * n_i) + MS_{S(R)} * (C_3 * \sum_{j=1}^{n_i} \frac{1}{m_{ij}} - n_i) / (C_3 * n_i^2) \right)^2}{\frac{(MS_{R(T)} / (C_3 * n_i))^2}{n_i - t} + \frac{(MS_{S(R)} * (C_3 * \sum_{j=1}^{n_i} \frac{1}{m_{ij}} - n_i) / (C_3 * n_i^2))^2}{n - n_i}}$$

Multiple comparisons of the  $t$  treatment means can be obtained using an approximate Tukey procedure after noting that  $Var(\bar{y}_{k..} - \bar{y}_{h..}) = \sigma_e^2 \left( \frac{1}{n_k} + \frac{1}{n_h} \right) + \sigma_d^2 \left( \frac{1}{n_k^2} \sum_{j=1}^{n_k} \frac{1}{m_{kj}} + \frac{1}{n_h^2} \sum_{j=1}^{n_h} \frac{1}{m_{hj}} \right)$

### ✓ Approximate Tukey-Kramer HSD Procedure:

Use Proc Mixed with the command: LSMEANS TRT/ADJUST=TUKEY

1. State there is significant evidence at level  $\alpha_o$  that the treatment means  $\mu_k$  and  $\mu_h$  are different if  $|\hat{\mu}_k - \hat{\mu}_h| \geq HSD(t, \nu)$ , where  $\nu$  is calculated in a similar fashion as was done above for  $\widehat{SE}(\hat{\mu}_i)$  using the Satterthwaite Approximation
2. Simultaneous  $100(1 - \alpha_o)\%$  confidence intervals for the absolute difference in all pairs of treatment means  $\mu_k - \mu_h$  are given by

$$|\bar{y}_{k..} - \bar{y}_{h..}| \pm HSD(t, \nu) \text{ where } HSD(t, \nu) \approx q(\alpha_o, t, \nu) \sqrt{\frac{1}{2} [\widehat{SE}(\bar{y}_{k..} - \bar{y}_{h..})]^2}$$

$$\begin{aligned} HSD(t, \nu) &= q(\alpha_o, t, \nu) \sqrt{\frac{1}{2} \left( \frac{\hat{\sigma}_e^2}{n_k} + \frac{\hat{\sigma}_d^2}{n_k^2} \sum_{j=1}^{n_k} \frac{1}{m_{kj}} + \frac{\hat{\sigma}_e^2}{n_h} + \frac{\hat{\sigma}_d^2}{n_h^2} \sum_{j=1}^{n_h} \frac{1}{m_{hj}} \right)} \\ &= q(\alpha_o, t, \nu) \sqrt{m \hat{\sigma}_e^2 + \hat{\sigma}_d^2} \sqrt{\frac{1}{2} \left( \frac{1}{mn_k} + \frac{1}{mn_h} \right)} \quad \text{when } m_{kj} = m_{hj} = m \\ &= q(\alpha_o, t, \nu) \sqrt{m \hat{\sigma}_e^2 + \hat{\sigma}_d^2} \sqrt{\frac{1}{mn}} \quad \text{when } m_{kj} = m_{hj} = m, \quad n_k = n_h = n \end{aligned}$$

In general,  $\nu$  is the  $df$  from the Satterthwaite Approximation and  $\nu = n - t$  when  $m_{ij} = m$   
 $q(\alpha_o, t, \nu)$  is given in Table VII in **Tables for Exams-Homework1** or using R:  $q(\alpha_o, t, \nu) = qtukey(1 - \alpha_o, t, \nu)$

These results will be illustrated through our turfgrass example on page 33. The following SAS code will be applied to this example.

```
* turfgrass_subsample.sas;
option ls=72 ps=55 nocenter nodate;
title 'One-way ANOVA with Subsampling Design';

DATA TURF; ARRAY X X1-X3;
INPUT STIM $ PLOT X1-X3 @@;
DO OVER X;Y=X;OUTPUT; END;DROP X1-X3;
LABEL STIM ='STIMULATOR' Y= 'ROOT WEIGHT';
cards;
S1 1 3.3 3.4 3.5 S1 2 3.1 3.5 3.0
S1 3 3.2 3.1 3.4 S1 4 3.3 2.9 3.0
S1 5 3.3 3.3 3.1 S1 6 . . .
S2 1 3.8 3.7 4.0 S2 2 3.5 3.8 3.9
S2 3 3.6 3.4 3.8 S2 4 3.4 3.7 .
S2 5 3.6 3.7 3.6 S2 6 3.5 3.9 .
S3 1 3.8 3.9 4.0 S3 2 3.6 3.7 3.8
S3 3 3.3 3.4 . S3 4 3.6 . 3.7
S3 5 3.5 . 3.9 S3 6 3.4 . 3.7
S4 1 4.3 4.3 4.4 S4 2 4.1 3.9 3.8
S4 3 4.2 4.1 3.9 S4 4 3.7 3.9 4.0
S4 5 . . . S4 6 . . .
RUN;

TITLE 'ANALYSIS USING PROC MIXED-Type1';
PROC MIXED CL METHOD=TYPE1;
CLASS STIM PLOT;
MODEL Y = STIM/ DDFM=SAT RESIDUAL;
RANDOM PLOT(STIM)/CL ALPHA=.05;
LSMEANS STIM/c1 ADJUST=TUKEY;
RUN;
```

## ANALYSIS USING PROC MIXED-TYPE1

### The Mixed Procedure

Model Information	
Data Set	WORK.TURF
Dependent Variable	Y
Covariance Structure	Variance Components
Estimation Method	Type 1
Residual Variance Method	Factor
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Satterthwaite

Class Level Information		
Class	Levels	Values
STIM	4	S1 S2 S3 S4
PLOT	6	1 2 3 4 5 6

Number of Observations	
Number of Observations Read	72
Number of Observations Used	57
Number of Observations Not Used	15

Type 1 Analysis of Variance								
Source	DF	Sum of Squares	Mean Square	Expected Mean Square	Error Term	Error DF	F Value	Pr > F
STIM	3	4.613973	1.537991	Var(Residual) + 2.7964 Var(PLOT(STIM)) + Q(STIM)	1.0375 MS(PLOT(STIM)) - 0.0375 MS(Residual)	16.477	23.50	<.0001
PLOT(STIM)	17	1.089185	0.064070	Var(Residual) + 2.6954 Var(PLOT(STIM))	MS(Residual)	36	2.34	0.0159
Residual	36	0.986667	0.027407	Var(Residual)	.	.	.	.

Covariance Parameter Estimates				
Cov Parm	Estimate	Alpha	Lower	Upper
PLOT(STIM)	0.01360	0.05	-0.00301	0.03021
Residual	0.02741	0.05	0.01812	0.04624

$F = \frac{MS_{STIM}}{MS_{Residual}}$   
 Satterwhite

48  
 MOM estimates.  
 ⇒ why CI still  
 (only 0 error term)  
 test says ≠ 0.

## ANALYSIS USING PROC MIXED-TYPEI

### The Mixed Procedure

Fit Statistics	
-2 Res Log Likelihood	-15.4
AIC (Smaller is Better)	-11.4
AICC (Smaller is Better)	-11.1
BIC (Smaller is Better)	-9.0

Type 3 Tests of Fixed Effects				
Effect	Num DF	Den DF	F Value	Pr > F
STIM	3	16.8	22.49	<.0001

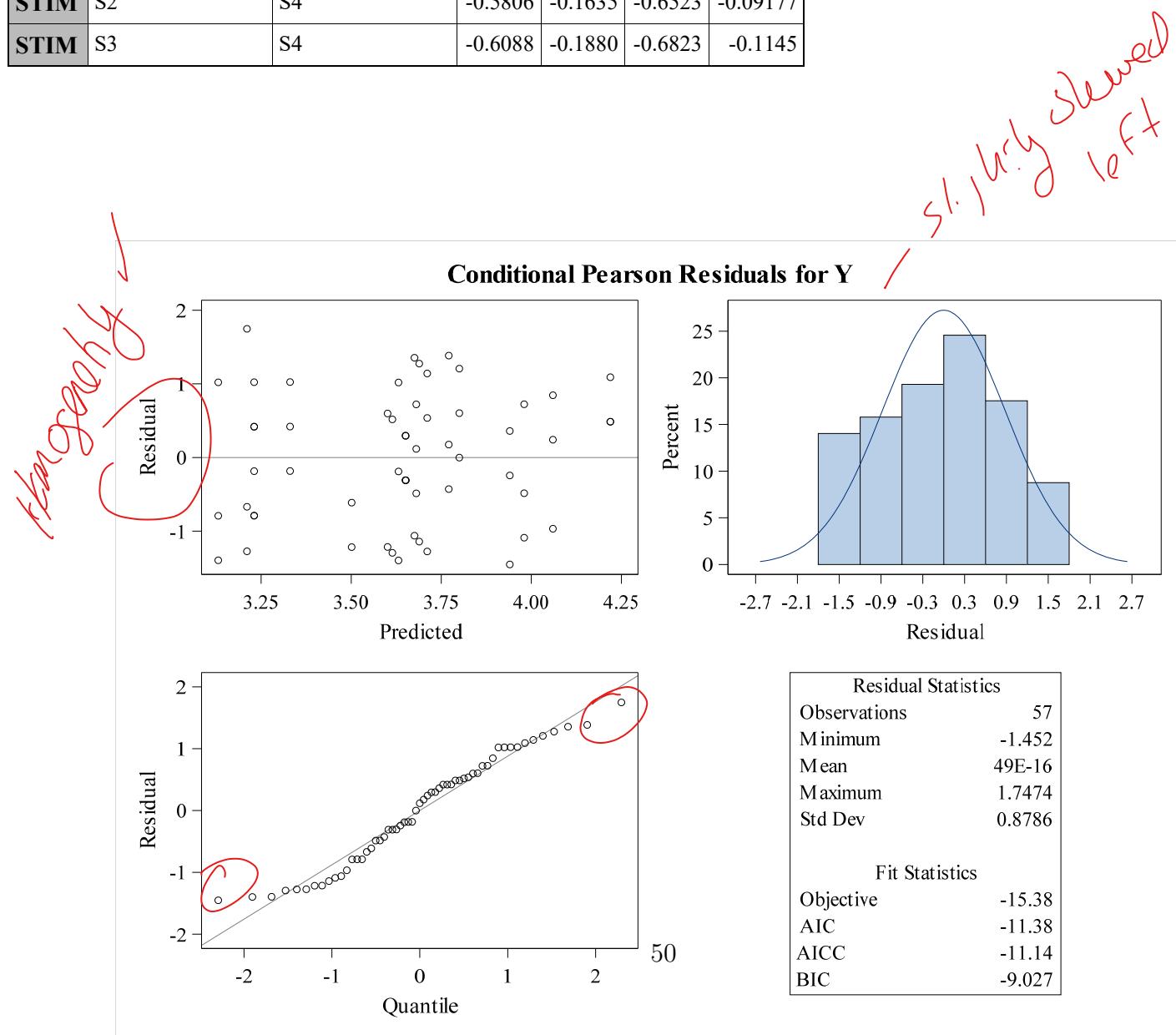
Least Squares Means									
Effect	STIMULATOR	Estimate	Standard Error	DF	t Value	Pr >  t	Alpha	Lower	Upper
STIM	S1	3.2267	0.06744	15.6	47.85	<.0001	0.05	3.0834	3.3699
STIM	S2	3.6780	0.06335	17.2	58.06	<.0001	0.05	3.5444	3.8115
STIM	S3	3.6516	0.06531	19.4	55.91	<.0001	0.05	3.5151	3.7881
STIM	S4	4.0500	0.07540	5.6	53.72	<.0001	0.05	3.8898	4.2102

Differences of Least Squares Means										
Effect	STIMULATOR	STIMULATOR	Estimate	Standard Error	DF	t Value	Pr >  t	Adjustment	Adj P	Alpha
STIM	S1	S2	-0.4513	0.09253	16.3	-4.88	0.0002	Tukey-Kramer	0.0008	0.05
STIM	S1	S3	-0.4249	0.09388	17.3	-4.53	0.0003	Tukey-Kramer	0.0016	0.05
STIM	S1	S4	-0.8233	0.1012	15.6	-8.14	<.0001	Tukey-Kramer	<.0001	0.05
STIM	S2	S3	0.02635	0.09099	18.3	0.29	0.7754	Tukey-Kramer	0.9912	0.05
STIM	S2	S4	-0.3720	0.09848	16.2	-3.78	0.0016	Tukey-Kramer	0.0075	0.05
STIM	S3	S4	-0.3984	0.09975	17.1	-3.99	0.0009	Tukey-Kramer	0.0048	0.05

## ANALYSIS USING PROC MIXED-TYPE1

### The Mixed Procedure

Differences of Least Squares Means						
Effect	STIMULATOR	STIMULATOR	Lower	Upper	Adj Lower	Adj Upper
<b>STIM</b>	S1	S2	-0.6471	-0.2555	-0.7146	-0.1879
<b>STIM</b>	S1	S3	-0.6227	-0.2271	-0.6921	-0.1578
<b>STIM</b>	S1	S4	-1.0382	-0.6084	-1.1112	-0.5354
<b>STIM</b>	S2	S3	-0.1646	0.2173	-0.2326	0.2853
<b>STIM</b>	S2	S4	-0.5806	-0.1635	-0.6523	-0.09177
<b>STIM</b>	S3	S4	-0.6088	-0.1880	-0.6823	-0.1145



Statement from SAS Documentation:

PROC GLM uses only the information pertaining to expected mean squares when you specify the TEST option in the RANDOM statement and, even then, only in the extra F tests produced by the RANDOM statement.

Other features in the GLM procedure -

including the results of the LSMEANS and ESTIMATE statements -

assume that all effects are fixed, so that all tests and

Pestimability checks for these statements are based on a fixed effects model, even when you use a RANDOM statement.

**Therefore, you should use the MIXED procedure to compute tests involving these features that take the random effects into account;**

see the section "PROC GLM versus PROC MIXED for Random Effects Analysis" and Chapter 46, "The MIXED Procedure," for more information.

#### PROC GLM versus PROC MIXED for Random-Effects Analysis

Other SAS procedures that can be used to analyze models with random effects include the MIXED and VARCOMP procedures. Note that, for these procedures, the random-effects specification is an integral part of the model, affecting how both random and fixed effects are fit; for PROC GLM, the random effects are treated in a post hoc fashion after the complete fixed-effect model is fit. This distinction affects other features in the GLM procedure, such as the results of the LSMEANS and ESTIMATE statements.

These features assume that all effects are fixed, so that all tests and estimability checks for these statements are based on a fixed-effects model, even when you use a RANDOM statement. Standard errors for estimates and LS-means based on the fixed-effects model might be significantly smaller than those based on a true random-effects model; in fact, some functions that are estimable under a true random-effects model might not even be estimable under the fixed-effects model. Therefore, you should use the MIXED procedure to compute tests involving these features that take the random effects into account;

## Class Level Information

Class	Levels	Values
-------	--------	--------

STIM	4	S1 S2 S3 S4
PLOT	6	1 2 3 4 5 6

## Number of Observations

Number of Observations Read	72
Number of Observations Used	57
Number of Observations Not Used	15

## Covariance Parameter Estimates

Cov Parm	Estimate	Alpha	Lower	Upper
PLOT(STIM)	0.01362	0.05	0.005376	0.07839
Residual	0.02648	0.05	0.018710	0.04036

Use  
as opposed  
to MM  
above.

## Type 3 Tests of Fixed Effects

Effect	Num	Den	F Value	Pr > F
	DF	DF		
STIM	3	17	22.50	<.0001

Least Squares Means

Effect	STIMULATOR	Standard				
		Estimate	Error	DF	Lower	Upper
STIM	S1	3.2267	0.06743	17	3.1380	3.3153
STIM	S2	3.6779	0.06334	17	3.5892	3.7608
STIM	S3	3.6516	0.06529	17	3.5654	3.7489
STIM	S4	4.0500	0.07539	17	3.9509	4.1491

Differences of Least Squares Means

Effect	STIMULATOR	STIMULATOR	Standard					
			Estimate	Error	DF	t Value	Pr >  t	
STIM	S1	S2	-0.4513	0.09251	17	-7.67	<.0001	
STIM	S1	S3	-0.4249	0.09386	17	-7.12	<.0003	
STIM	S1	S4	-0.8233	0.10110	17	-13.06	<.0001	
STIM	S2	S3	0.02637	0.09097	17	0.30	0.7754	
STIM	S2	S4	-0.3721	0.09846	17	-6.03	0.0015	
STIM	S3	S4	-0.3984	0.09973	17	-6.14	0.0009	
Effect	STIMULATOR	STIMULATOR	Adjustment		Adj P	Alpha	Lower	Upper
			Tukey-Kramer		<.0007	0.05	-0.6465	-0.2561
			Tukey-Kramer		<.0015	0.05	-0.6229	-0.2269
			Tukey-Kramer		<.0001	0.05	-1.0367	-0.6099
			Tukey-Kramer		0.9912	0.05	-0.1656	0.2183
			Tukey-Kramer		0.0074	0.05	-0.5798	-0.1643
			Tukey-Kramer		0.0047	0.05	-0.6088	-0.1880
			Adj	Adj				
Effect	STIMULATOR	STIMULATOR	Lower	Upper				
			-0.7142	-0.1883				
			-0.6917	-0.1581				
			-1.1108	-0.5358				
			-0.2322	0.2849				
			-0.6519	-0.09217				
			-0.6819	-0.1149				

From the above we can conclude that the only pair of Stimulators which are not significantly different are S2 and S3. Thus we have the following groupings:

Simulator	S1	S3	S2	S4
Groupings	a	b	b	c

R Code to analyze the Turf Grass Example { turfgrass\_sub.R

```
library(lsmeans)
library(ggplot2)
library(lme4)

stim = as.factor(c(rep("S1",15),rep("S2",16),rep("S3",14),rep("S4",12)))
plotS1 = as.factor(c(rep("1",3),rep("2",3),rep("3",3),rep("4",3),rep("5",3)))
plotS2 = as.factor(c(rep("1",3),rep("2",3),rep("3",3),rep("4",2),rep("5",3),rep("6",2)))
plotS3 = as.factor(c(rep("1",3),rep("2",3),rep("3",2),rep("4",2),rep("5",2),rep("6",2)))
plotS4 = as.factor(c(rep("1",3),rep("2",3),rep("3",3),rep("4",3)))
plot = as.factor(c(plotS1,plotS2,plotS3,plotS4))
rootwt =
c(3.3,3.4,3.5,3.1,3.5,3.0,3.2,3.1,3.4,3.3,2.9,3.0,3.3,3.3,3.1,
  3.8,3.7,4.0,3.5,3.8,3.9,3.6,3.4,3.8,3.4,3.7,3.6,3.7,3.6,3.5,3.9,
  3.8,3.9,4.0,3.6,3.7,3.8,3.3,3.4,3.6,3.7,3.5,3.9,3.4,3.7,
  4.3,4.3,4.4,4.1,3.9,3.8,4.2,4.1,3.9,3.7,3.9,4.0)

ranmod = lmer(rootwt ~ 1 + stim + (1|plot:stim))

summary(ranmod)
aovranmod = anova(ranmod)
aovranmod
lsmeans(ranmod, pairwise ~ stim, adjust=c("tukey"))

#treat plot as fixed effects to obtain SS's

fixmod = lm(rootwt ~ stim + plot:stim)
aovfixmod = aov(fixmod)
summary(aovfixmod)
```

The output from R is given below:

```
Linear mixed model fit by REML [‘lmerMod’]
Random effects:
 Groups Name Variance Std.Dev.
 plot:stim (Intercept) 0.01362 0.1167
 Residual 0.02732 0.1653
Number of obs: 57, groups: plot:stim, 21
```

```
Analysis of Variance Table
 Df Sum Sq Mean Sq F value
 stim 3 1.8442 0.61474 22.499
```

↑  
dikari

than what  
we get in SAS

```
$lsmeans
stim lsmean      SE   df lower.CL upper.CL
S1     3.23 0.0674 15.4     3.08    3.37
S2     3.68 0.0634 17.1     3.54    3.81
S3     3.65 0.0654 19.2     3.51    3.79
S4     4.05 0.0754 15.4     3.89    4.21
```

Degrees-of-freedom method: kenward-roger

Confidence level used: 0.95

\$contrasts

	contrast	estimate	SE	df	t.ratio	p.value
S1 - S2	-0.4513	0.0926	16.2	-4.876	0.0008	
S1 - S3	-0.4249	0.0939	17.1	-4.524	0.0015	
S1 - S4	-0.8233	0.1011	15.4	-8.140	<.0001	
S2 - S3	0.0264	0.0911	18.1	0.290	0.9913	
S2 - S4	-0.3721	0.0985	16.1	-3.777	0.0080	
S3 - S4	-0.3984	0.0998	16.9	-3.993	0.0048	

Degrees-of-freedom method: kenward-roger

P value adjustment: tukey method for comparing a family of 4 estimates

```
#treat plot as fixed to obtain AOV sum of squares:
#This output has incorrect F-statistics.
#Correct F-tests is given above in AOV Table (22.499)
```

```
fixmod = lm(rootwt ~ stim + plot:stim)
```

```
summary(fixmod)
anova(fixmod)
lsmeans(fixmod, "stim")
```

Analysis of Variance Table

```
Response: rootwt
          Df Sum Sq Mean Sq F value    Pr(>F)
stim       3 4.6140 1.53799 56.1159 1.192e-13 ***
stim:plot 17 1.0892 0.06407  2.3377   0.0159 *
Residuals 36 0.9867 0.02741
```

A *fixed*  
plan what veget  
( $\times$  2).

Finished Nedusky 3/22 (Week 8, Lab 2)