Statistics 630 – Exam I
Partial Solutions

These solutions are relatively complete but (except for the multiple choice questions) they may not include all the steps or details I would expect. **Please rework your solutions** to the best of your ability, paying close attention to notation, until you are satisfied you can do them on your own. You can post questions on the discussion board or see me during my office hours if you still have difficulty or doubts. [Comments in square brackets are not part of the solution.]

1. (d). Since $\mu = 1$ and $\sigma^2 = 2$, each $\frac{(X_i-1)^2}{2}$ is a chi-square(1) random variable, and they are independent.

2. (e). $\mathsf{Cov}(R,S) = (.5)\sqrt{\mathsf{Var}(R)\,\mathsf{Var}(S)}$. Use the general result about the variance of a sum of random variables (in the formulas attached with the exam).

3. (d). $\mathsf{Bias}(\widetilde{T}_n) = -\frac{1}{n}\mathsf{E}(T_n) = -\frac{\theta}{n}$ and $\mathsf{Var}(\widetilde{T}_n) = (1-\frac{1}{n})^2\,\mathsf{Var}(T_n) = (1-\frac{1}{n})^2\frac{\theta}{n+1}$.

4. (b). $\frac{\overline{N}-16}{\sqrt{16/100}}$ is approximately standard normal. [I blew this for the on-campus class as the problem indicated the wrong interval. It should have stated "between 15.6 and 17".]

5. First, let $y = \frac{w}{\beta}$. Then

$$\mathsf{E}(W) = \int_0^\infty w\,\frac{4}{\sqrt{\pi}}\beta^{-3}w^2 e^{-(w/\beta)^2}\,dw = \frac{4\beta}{\sqrt{\pi}}\int_0^\infty y^3 e^{-y^2}\,dy,$$

which answers the first part. Then, letting $x = y^2$,

$$C = \frac{4}{\sqrt{\pi}}\int_0^\infty y^3 e^{-y^2}\,dy = \frac{2}{\sqrt{\pi}}\int_0^\infty x e^{-x}\,dx = \frac{2}{\sqrt{\pi}},$$

where the last integral is $\Gamma(2) = 1! = 1$.

6. The log-likelihood is

$$\ell(\beta) = \sum_{i=1}^n \log(4W_i^2/\sqrt{\pi}) - 3n\log\beta - \sum_{i=1}^n (W_i/\beta)^2.$$

So the score function is

$$S(\beta) = \frac{d\ell(\beta)}{d\beta} = -\frac{3n}{\beta} + \frac{2}{\beta^3}\sum_{i=1}^n W_i^2.$$

Setting $S(\widehat{\beta}) = 0$ gives $\widehat{\beta} = \left(\frac{2}{3n}\sum_{i=1}^n W_i^2\right)^{1/2}$.

7. First, for $0 \le x \le 1$,

$$f_X(x) = \int_0^1 (2xy - x - y + 1.5)\,dy = 1.$$

[So X is uniform(0,1) and, by symmetry of the variables, Y is also uniform(0,1).] Then

$$\mathsf{E}(Y|X=x) = \int_0^1 y\frac{f_{X,Y}(x,y)}{f_X(x)}\,dy = \int_0^1 y(2xy - x - y + 1.5)\,dy = \frac{x}{6} + \frac{5}{12}.$$

[As a check: $\mathsf{E}(Y) = \mathsf{E}(\mathsf{E}(Y|X)) = \mathsf{E}(\frac{X}{6} + \frac{5}{12}) = \frac{1}{2}$.]