Statistics 630 – Final Exam
Partial Solutions

These solutions are relatively complete but (except for the multiple choice questions) they may not include all the steps or details I would expect. As always, please rework your solutions to the best of your ability.

1. (d). $\frac{1}{n}\sum_{i=1}^n X_i^2$ estimates θ^2. [$\overline{X}$ estimates $\mathsf{E}(X_i)$, which might not even depend on θ.]

2. (b). Increasing (just) α makes it easier to reject H_0. So rejection of H_0 is more likely, no matter what the parameter value is.

3. (d). The normal approximation does not apply: think about how likely are small integer values such as 0, 1, 2.

4. (c). The MLE for $\sigma = 1/\lambda$ is $\overline{V}$, so this is what to use in the standard error estimate.

5. (e).

6. (a). $L(\beta) = \beta^n \prod_{i=1}^n T_i^{-2} \exp\Big(-\beta\sum_{i=1}^n T_i^{-1}\Big)$, so $\sum_{i=1}^n T_i^{-1}$ is sufficient by the factorization theorem.

7. (c).

8. (e). By iterated expectation $\mathsf{E}(X) = \mathsf{E}(\mathsf{E}(X|Y)) = \mathsf{E}(Y+Y^2) = \mathsf{E}\Big(\frac{1}{n}\sum_{i=1}^n (Y_i + Y_i^2)\Big)$.

9. (a) $\mathsf{E}\Big(\frac{2W_i^2}{\beta^2}\Big) = 3$ so $\mathsf{E}(\widehat{\beta}^2) = \frac{\beta^2}{3}\mathsf{E}\Big(\frac{1}{n}\sum_{i=1}^n \frac{2W_i^2}{\beta^2}\Big) = \beta^2$. Since $0 < \mathsf{Var}(\widehat{\beta}) = \mathsf{E}(\widehat{\beta}^2) - \mathsf{E}(\widehat{\beta})^2$, it must be that $\mathsf{E}(\widehat{\beta}) < \beta$. Thus, $\widehat{\beta}^2$ is unbiased but $\widehat{\beta}$ is biased.

   (b) The score function is $S(\beta) = \frac{2}{\beta^3}\sum_{i=1}^n W_i^2 - \frac{3n}{\beta}$. Note that $\frac{2}{\beta^2}\sum_{i=1}^n W_i^2 \sim$ chi-square$(3n)$, with mean $3n$. Either compute

$$I_n(\beta) = \mathsf{Var}(S(\beta)) = \frac{1}{\beta^2}\mathsf{Var}\Big(\frac{2}{\beta^2}\sum_{i=1}^n W_i^2\Big) = \frac{6n}{\beta^2}$$

   or

$$I_n(\beta) = -\mathsf{E}\Big(\tfrac{dS(\beta)}{d\beta}\Big) = \mathsf{E}\Big(\frac{6}{\beta^4}\sum_{i=1}^n W_i^2 - \frac{3n}{\beta^2}\Big) = \frac{6n}{\beta^2}.$$

   (c) For some function $h(\cdot)$, the log-likelihood is $\ell(\beta) = h(W_1,\ldots,W_n) - 3n\log\beta - \frac{3n\widehat{\beta}^2}{2\beta^2}$. H_0 is rejected for $G^2 > \chi^2_{1-\alpha}(1)$, where

$$G^2 = 2(\ell(\widehat{\beta}) - \ell(\beta_0)) = 3n(\widehat{\beta}^2/\beta_0^2 - \log(\widehat{\beta}^2/\beta_0^2) - 1).$$

   [Or use the gamma distribution of $\widehat{\beta}^2/\beta_0^2$ under H_0 to get c_α such that $\mathsf{P}(G^2 > c_\alpha) = \alpha$.]

10. First, $Y = \sum_{i=1}^n X_i \sim$ Poisson$(n\lambda)$ and we can just work with the distribution for Y. So, letting y be the observed value of Y and ignoring what does not depend on λ, the posterior pdf is

$$f(\lambda|y) \propto (n\lambda)^y e^{-n\lambda}\, e^{-\lambda},$$

    which (as a function of λ) is proportional to the gamma$(y+1, n+1)$ pdf. The posterior mean is therefore $\frac{y+1}{n+1}$.

11. Use gamma integrals to compute

$$\begin{aligned}
\mathsf{E}(S+T) &= \frac{1}{2}\int_0^\infty \int_0^\infty (s^2+2st+t^2)e^{-s-t}\,dsdt \\
&= \frac{1}{2}\int_0^\infty s^2e^{-s}\,ds \int_0^\infty e^{-t}\,dt + \int_0^\infty se^{-s}\,ds \int_0^\infty te^{-t}\,dt + \frac{1}{2}\int_0^\infty e^{-s}\,ds \int_0^\infty t^2e^{-t}\,dt \\
&= \frac{(2!)(0!)+2(1!)(1!)+(0!)(2!)}{2} = 3.
\end{aligned}$$