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STAT 630
Assignment 6
10/18/21

View lectures 16-20:

1.) (A) Chp 3 Exercise 3.3.25. You can do this directly from the joint pmf, but here is a simpler alternative approach. First observe that $X_i \sim \text{Binomial}(n, \theta_i)$ (recall Example 2.8.5 in the text) & $X_i + X_j \sim \text{Binomial}(n, \theta_i + \theta_j)$ for $i \neq j$ (why? - think about combining categories). Then use these facts and properties of variance and covariance to get two expressions for $var(X_i + X_j)$ which you can use to solve for the desired covariance.

3.3.25: S that $(X_1, X_2, X_3) \sim \text{Multinomial}(n, \theta_1, \theta_2, \theta_3)$. Prove that
$var(X_i) = n\theta_i(1-\theta_i)$; $cov(X_i, X_j) = -n\theta_i\theta_j$ when $i \neq j$
(Hint: Recall problem 3.1.23)

① Note from problem 3.1.23 that:

IF $(X_1, X_2, X_3) \sim \text{Multinomial}(n, \theta_1, \theta_2, \theta_3)$

then $X_i \sim \text{Binomial}(n, \theta_i)$

Thus, we know $\boxed{var(X_i) = n\theta_i(1-\theta_i)}$

② Note by Theorem 3.3.4 that
For any R.Vs. X & Y:
$var(X+Y) = var(X) + var(Y) + 2cov(X,Y)$

- Let $Y = X_i + X_j$; then $Y \sim \text{Binomial}(n, \theta_i + \theta_j)$

$$var(Y) = n(\theta_i+\theta_j)(1-(\theta_i+\theta_j)) \underset{\text{by Theorem 3.3.4}}{=} n\theta_i(1-\theta_i) + n\theta_j(1-\theta_j) + 2cov(X_i, X_j)$$

$$(n\theta_i + n\theta_j)(1-\theta_i-\theta_j) = n\theta_i - n\theta_i^2 + n\theta_j - n\theta_j^2 + 2cov(X_i, X_j)$$

$$\cancel{n\theta_i} - \cancel{n\theta_i^2} - n\theta_i\theta_j + \cancel{n\theta_j} - n\theta_i\theta_j - \cancel{n\theta_j^2} = \cancel{n\theta_i} - \cancel{n\theta_i^2} + \cancel{n\theta_j} - \cancel{n\theta_j^2} + 2cov(X_i, X_j)$$

$$-2n\theta_i\theta_j = 2cov(X_i, X_j)$$

$$\boxed{-n\theta_i\theta_j = cov(X_i, X_j) \quad \text{QED}}$$

(b) Use the above to find the $corr(X_i, X_j)$. How does the correlation change w/ n?

Recall Def 3.3.4: The correlation of two R.V.s X & Y is given by

$$corr(X,Y) = \frac{cov(X,Y)}{SD(X)SD(Y)} = \frac{cov(X,Y)}{\sqrt{var(X)\,var(Y)}}$$

$$= \frac{-n\theta_i\theta_j}{\sqrt{n^2\theta_i\theta_j(1-\theta_i)(1-\theta_j)}}$$

$$\boxed{corr(X_i, X_j) = \frac{-\theta_i\theta_j}{\sqrt{\theta_i\theta_j(1-\theta_i)(1-\theta_j)}}}$$

$var(X_i)\,var(X_j) =$
$n\theta_i(1-\theta_i)\,n\theta_j(1-\theta_j)$
$n^2\theta_i\theta_j(1-\theta_i)(1-\theta_j)$
$n^2\theta_i\theta_j(1-\theta_j-\theta_i+\theta_i\theta_j)$
$n^2(\theta_i\theta_j - \theta_i\theta_j^2 - \theta_i^2\theta_j + \theta_i^2\theta_j^2)$

The correlation does not change w/ n

2) (a) Show that the variance for the beta(a,b) distribution is $\frac{ab}{(a+b)^2(a+b+1)}$.

[Recall: $var(X) = E[X^2] - E[X]^2$]

$$E[X] = \frac{1}{B(\alpha,\beta)} \int_0^1 x(x^{\alpha-1})(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{\alpha}{\alpha+\beta}$$

$$E[X^2] = \frac{1}{B(\alpha,\beta)} \int x^2(x^{\alpha-1})(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\alpha(\alpha+1)\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)(\alpha+\beta+1)\Gamma(\alpha+\beta)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$var(X) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

(b) $f(X_1, X_2) \sim$ Dirichlet (a_1, a_2, a_3) (recall exercise 2.7.17). It can be shown (and you may assume) that $X_1+X_2 \sim beta(a_1+a_2, a_3)$. Use an argument similar to part (a) of problem 1 to find $cov(X_1,X_2)$

- $Var(X_1+X_2) = var(X_1) + var(X_2) + 2cov(X_1,X_2)$ [use this to find cov]

[NOTE: Recall from problem 2.7.17 part b that for $(X_1,X_2) \sim$ Dirichlet (a_1,a_2,a_3)
$X_1 \sim beta(a_1, a_2+a_3)$, $X_2 \sim (a_2, a_1+a_3)$]

$$var(X_1+X_2) = \frac{(a_1+a_2)a_3}{(a_1+a_2+a_3)^2(a_1+a_2+a_3+1)} = \frac{a_1(a_2+a_3)}{(a_1+a_2+a_3)^2(a_1+a_2+a_3+1)} + \frac{a_2(a_1+a_3)}{(a_1+a_2+a_3)^2(a_1+a_2+a_3+1)} + 2cov(X_1,X_2)$$

$$\frac{a_1a_3 + a_2a_3 - a_1a_2 - \cancel{a_1a_3} - a_2a_1 - a_2a_3}{(a_1+a_2+a_3)^2(a_1+a_2+a_3+1)} = 2cov(X_1,X_2)$$

$$\frac{-2a_1a_2}{(a_1+a_2+a_3)^2(a_1+a_2+a_3+1)} \cdot 2cov(X_1,X_2)$$

$$\boxed{cov(X_1,X_2) = \frac{-a_1a_2}{(a_1+a_2+a_3)^2(a_1+a_2+a_3+1)}}$$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
$\Gamma(\alpha+2) = \alpha(\alpha+1)\Gamma(\alpha)$
$\Gamma(\alpha+n) = (\alpha+n-1)!\,\Gamma(\alpha)$

3) Chp 3 Exercises 3.4.5, 3.4.12, 3.4.16:

3.4.5: Let $Y = 3X+4$ compute $M_Y(s)$ interms of M_X

(see chp 3 notes slide 54 properties of mgf ⊙)

⊙ Let a & b be constants and M_X be the mfg of X. Then the mgf of $Y = aX+b$ is

$$M_Y(s) = E[e^{sY}] = e^{bs} M_X(as)$$

$$\Rightarrow \boxed{M_Y(s) = E[e^{sY}] = e^{4s} M_X(3s)}$$

3.4.12: Let $X \sim \text{Geom}(\theta)$

(a) Compute $M_X(s)$

$$M_X(s) = E[e^{sX}] = \sum_{x=0} e^{sx}\,\theta(1-\theta)^x = \theta \sum e^{sx}(1-\theta)^x$$

• Let $a = \theta$, $r = e^s(1-\theta)$

$$\boxed{M_X(s) = \frac{\theta}{1-(1-\theta)e^s}}$$

(b) Use M_X to compute $E[X]$.

$E[X] = M_X'(0)$

$$M_X'(s) = \frac{d}{ds} M_X(s) = \frac{d}{ds}\left[\frac{\theta}{1-(1-\theta)e^s}\right] = \theta \frac{d}{ds}\left[(1-(1-\theta)e^s)^{-1}\right]$$

$$= \theta \frac{-1}{(1-(1-\theta)e^s)^2} \cdot \frac{d}{ds}\left[1-(1-\theta)e^s\right] = \frac{-\theta}{(1-(1-\theta)e^s)^2} \cdot -e^s(1-\theta)$$

$$M_X'(s) = \frac{\theta e^s(1-\theta)}{(1-(1-\theta)e^s)^2} \Rightarrow M_X'(0) = \frac{\theta(1-\theta)}{(1-(1-\theta))^2} = \boxed{\frac{1-\theta}{\theta} = M_X'(s) = E[X]}$$

(c) Use M_X to compute $Var(X)$

$E[X^2] = M_X''(0)$

$$M_X''(s) = \frac{d}{ds}\left[\theta(1-\theta)e^s(1-(1-\theta)e^s)^{-2}\right]$$
$$= \theta(1-\theta)\frac{d}{ds}\left[e^s(1-(1-\theta)e^s)^{-2}\right]$$
$$= \theta(1-\theta)\left[e^s(1-(1-\theta)e^s)^{-2} + e^s \frac{d}{ds}\left[(1-(1-\theta)e^s)^{-2}\right]\right]$$
$$= \theta(1-\theta)e^s(1-(1-\theta)e^s)^{-2} + 2\theta e^s(1-\theta)(1-(1-\theta)e^s)^{-3}(1-\theta)e^s$$

$$M_X''(0) = \theta(1-\theta)(1-(1-\theta))^{-2} + 2\theta(1-\theta)(1-(1-\theta))^{-3}(1-\theta)$$
$$= \frac{\theta(1-\theta)}{\theta^2} + \frac{2\theta(1-\theta)^2}{\theta^3} = \frac{\theta(1-\theta)+2(1-\theta)^2}{\theta^2} = \frac{\theta-\theta^2+2-4\theta+2\theta^2}{\theta^2}$$

↳

3.) 3.4.12 (contd):

(c) contd

$$E[X^2] = M''_X(0) = \frac{\theta - \theta^2 + 2 - 4\theta + 2\theta^2}{\theta^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{\theta - \theta^2 + 2 - 4\theta + 2\theta^2}{\theta^2} + \frac{-1 + 2\theta - \theta^2}{\theta^2}$$

$$= \frac{2 - 3\theta + \theta^2 - 1 + 2\theta - \theta^2}{\theta^2} = \boxed{\frac{1-\theta}{\theta^2} = Var(X)}$$

Exercise 3.4.16: Let Y be distributed according to the Laplace distribution (see problem 2.4.22)

$f_Y(y) = e^{-|y|}/2$; $(-\infty < y < \infty)$

(a) Compute $M_Y(s)$:

$$M_Y(s) = \int_{-\infty}^{0} \tfrac{1}{2} e^{sy} e^{y}\, dy = \lim_{a \to -\infty} \int_a^0 \tfrac{1}{2} e^{y(s+1)}\, dy = \lim_{a\to-\infty} \left[\frac{1}{2(s+1)} e^{y(s+1)} \right]_{y=a}^{y=0}$$

$$= \lim_{a\to-\infty} \frac{1}{2(s+1)}\left[1 - e^{a(s+1)}\right] = \frac{1}{2(s+1)} \quad \text{for } -\infty < y < 0$$

$$M_Y(s) = \int_0^{\infty} \tfrac{1}{2} e^{sy} e^{-y}\, dy = \lim_{b\to\infty} \int_0^b \tfrac{1}{2} e^{y(s-1)}\, dy = \lim_{a\to\infty} \left[\frac{1}{2(s-1)} e^{-y(1-s)}\right]_{y=0}^{y=b}$$

$$M_Y(s) = \frac{1}{2-2s} \quad \text{for } 0 < y < \infty, \ \text{for } s < 1$$

$$M_Y(s) = \frac{1}{2s+2} + \frac{1}{2-2s} = \frac{2-2s+2s+2}{(2s+2)(2-2s)} = \frac{4}{4(1-s^2)} = \frac{1}{1-s^2} \quad \text{for } s < 1$$

$$\boxed{M_Y(s) = \frac{1}{1-s^2} \quad \text{for } |s| < 1}$$

(b) Use M_Y to compute the mean of Y.

$E[Y] = M'_Y(0)$

- $M'_Y(s) = \frac{d}{ds}\left[(1-s^2)^{-1}\right] = -(1-s^2)^{-2}(-2s) = \frac{2s}{(1-s^2)^2}$

$\boxed{M'_Y(0) = 0 = E[Y]}$

(c) Use M_Y to compute the variance of Y.

$Var(Y) = E[Y^2] - E[Y]^2$

- $E[Y^2] = M''_Y(0)$
- $M''_Y(s) = \frac{d}{ds}\left[2s(1-s^2)^{-2}\right] = 2(1-s^2)^{-2} + (2s)(-2(1-s^2)^{-3})(-2s)$

$= 2(1-s^2)^{-2} + 8s^2(1-s^2)^{-3}$

$Var(Y) = E[Y^2] - E[Y]^2 = 2 - 0 = \boxed{2 = Var(Y)}$

4.) Chp 3 Exercise 3.4.20, 3.4.23; (why is it necessary that $t < \lambda$?)

Exercise 3.4.20: Prove that the moment generating function of the Gamma(α, λ) distribution is given by $\lambda^\alpha / (\lambda - t)^\alpha$ when $t < \lambda$

$$f_X(x) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} \quad \text{for } x > 0 \quad \text{where } \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

$$M_X(s) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} x^{\alpha-1} e^{-\lambda x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-x(\lambda-t)} \cdot x^{\alpha-1} \cdot dx$$

let $u = x(\lambda - t)$, $\Rightarrow$ $x = \frac{u}{\lambda - t}$; $dx = \frac{1}{\lambda - t} du$

$$M_X(s) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{1}{\lambda-t} u\right)^{\alpha-1} e^{-u} \frac{1}{\lambda-t} du = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{(\lambda-t)^{\alpha-1}} \cdot \frac{1}{(\lambda-t)} \underbrace{\int_0^\infty u^{\alpha-1} e^{-u} du}_{= \Gamma(\alpha)}$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{(\lambda-t)^\alpha} \Gamma(\alpha) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha}$$

$$\boxed{M_X(s) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha}}$$

Exercise 3.4.23: $\S$ that $X_i \sim$ Gamma(α_i, λ) and $X_1, \dots, X_n$ are independent. Using moment generating functions, determine the distribution of $Y = \sum_{i=1}^n X_i$

① IF $X_i \sim$ Gamma(α_i, λ) then we know (from Exercise 3.4.20) that

$$M_{X_i}(s) = \frac{\lambda^{\alpha_i}}{(\lambda-t)^{\alpha_i}}$$

② We know from chp 3 notes (slide 54) Properties of MGFs ③ that if we

- let $X_1, \dots, X_n$ be independent r.v.s w/ mgfs $M_1, \dots, M_n$. Then the mgf M of $X_1 + \dots + X_n$ is $M(s) = \prod_{i=1}^n M_i(s)$

$$M(s) = \prod_{i=1}^n \frac{\lambda^{\alpha_i}}{(\lambda-t)^{\alpha_i}} = \left(\frac{\lambda}{\lambda-t}\right)^{\sum_{i=1}^n \alpha_i} \Rightarrow \boxed{Y \sim \text{Gamma}\left(\sum_{i=1}^n \alpha_i, \lambda\right)}$$

5.) Chp 3 Exercises 3.5.4, 3.5.11 (note textbook sol is wrong), 3.5.16

Exercise 3.5.4: Let $p_{X,Y}$ be the following:

$$p_{X,Y}(x,y) = \begin{cases} 1/11 & ; x=-4, y=2 \\ 2/11 & ; x=-4, y=3 \\ 4/11 & ; x=-4, y=7 \\ 1/11 & ; x=6, y=2 \\ 1/11 & ; x=6, y=3 \\ 1/11 & ; x=6, y=7 \\ 1/11 & ; x=6, y=13 \\ 0 & ; \text{o.w} \end{cases}$$

(a.) compute $E[X|Y=2] = (\tfrac{1}{2})(-4) + (\tfrac{1}{2})(6) = \boxed{E[X|y=2] = 1}$

(b) compute $E[X|Y=3] = (\tfrac{2}{3})(-4) + (\tfrac{1}{3})(6) = \boxed{E[X|y=3] = -2/3}$

(c) compute $E[X|y=7] = (\tfrac{4}{5})(-4) + (\tfrac{1}{5})(6) = \boxed{E[X|y=7] = -2}$

(d) compute $E[X|Y=13] = (1)(6) = \boxed{E[X|Y=13] = 6}$

(e) compute $E[X|Y]$

$$\boxed{E[X|y] = \begin{cases} 1, & y=2 \\ -2/3, & y=3 \\ 7, & y=-2 \\ 6, & y=13 \end{cases}}$$

5.) (Contd)

Exercise 3.5.11: $\S$ X & Y are jointly absolutely continuous, w/ joint density function

$$f_{XY}(x,y) = \begin{cases} \left(\frac{6}{19}\right)(x^2+y^3) & 0<x<2 \wedge 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(a) Compute $E[X]$:

$$E[X] = \int_0^2\int_0^1 x\left(\frac{6}{19}\right)(x^2+y^3)\,dy\,dx = \left(\frac{6}{19}\right)\int_0^2\int_0^1 x^3+xy^3\,dy\,dx$$

$$= \left(\frac{6}{19}\right)\int_0^2\left[yx^3+\frac{x}{4}y^4\right]_{y=0}^{y=1}dx = \left(\frac{6}{19}\right)\left[\frac{x^4}{4}+\frac{x^2}{8}\right]_{x=0}^{x=2}$$

$$= \left(\frac{6}{19}\right)\left(\frac{36}{8}\right) = \frac{27}{19}$$

$$\boxed{E[X] = 27/19}$$

(b) Compute $E[Y]$:

$$E[Y] = \int_0^1\int_0^2 y\left(\frac{6}{19}\right)(x^2+y^3)\,dx\,dy = \left(\frac{6}{19}\right)\int_0^1\int_0^2 yx^2+y^4\,dx\,dy$$

$$= \left(\frac{6}{19}\right)\int_0^1\left[y\frac{x^3}{3}+xy^4\right]_{x=0}^{x=2}dy = \left(\frac{6}{19}\right)\int_0^1 \frac{8}{3}y+2y^4\,dy = \left(\frac{6}{19}\right)\left[\frac{4}{3}y^2+\frac{2}{5}y^5\right]_{y=0}^{y=1}$$

$$E[Y] = \left(\frac{26}{15}\right)\left(\frac{6}{19}\right) = \frac{156}{285} = \frac{52}{95}$$

$$\boxed{E[Y] = \frac{52}{95}}$$

(c) Compute $E[X|Y]$:

> **Definition 3.5.4:** Let X & Y be jointly absolutely continuous random variables w/ joint density $f_{XY}(x,y)$. Then the conditional expectation of X, given $Y=y$ is:
>
> $$E[X|Y=y] = \int_{x\in\mathbb{R}^1} x\, f_{X|Y}\, f(x|y)\,dx = \int_{x\in\mathbb{R}^1} x\,\frac{f_{(x,y)}X,Y}{f_Y(y)}\,dy$$

- $$f_Y(y) = \int_0^2\left(\frac{6}{19}\right)(x^2+y^3)\,dx = \left(\frac{6}{19}\right)\left[\frac{x^3}{3}+xy^3\right]_{x=0}^{x=2} = \frac{6}{19}\left[\frac{8}{3}+2y^3\right]$$

$$f_{X|Y}(x|y) = \frac{f_{(x,y)}(x,y)}{f_Y(y)} = \frac{\left(\frac{6}{19}\right)(x^2+y^3)}{\left(\frac{6}{19}\right)\left(\frac{8}{3}+2y^3\right)} = \frac{3(x^2+y^3)}{8+6y^3}$$

- $$E[X|Y=y] = \int_0^2 \frac{3x^3+3xy^3}{8+6y^3}\,dx = \frac{1}{8+6y^3}\int_0^2 3x^3+3xy^3\,dx = \frac{1}{8+6y^3}\left[\frac{3}{4}x^4+\frac{3}{2}x^2y^3\right]_{x=0}^{x=2}$$

$$= \frac{1}{8+6y^3}\left[12+6y^3\right] = \frac{12+6y^3}{8+6y^3} = \boxed{\frac{6+3y^3}{4+3y^3} = E[X|Y=y]}$$

5.) (Cont'd)

Exercise 3.5.16: $(X|Y=y) \sim$ Gamma(α, y) and that the marginal distribution of Y is given by $\frac{1}{Y} \sim \exp(\lambda)$. Determine $E(X)$

$$f_{X|Y}(x|y=y) = \frac{1}{\Gamma(\alpha)} y^{\alpha} x^{\alpha-1} e^{-yx}$$

[• We know: $E[X|Y=y] = \frac{\alpha}{y}$ (from previous work, we know if $X \sim$ Gamma$(\alpha, \lambda) \Rightarrow E[X] = \frac{\alpha}{\lambda}$)]

• Theorem 3.5.2 (Theorem of total expectation)

If X & Y are R.V.s

then $E[E[X|Y]] = E[X]$

• Example 3.5.7 (in book) states

$$E[E[X|Y]] = \int_{-\infty}^{\infty} E[X|Y=y] f_Y(y)\,dy$$

* Don't need this *

• $E[X] = E[E[X|Y]] = E[\alpha/y] = \alpha E[1/y] = \boxed{\alpha/\lambda = E[X]}$

3.5.11 (cont'd)

(d) Compute $E[Y|X]$:

$$f_X(x) = \int_0^1 \frac{6}{19}(x^2+y^3)\,dy = \frac{6}{19}\left[yx^2 + y^4/4\right]_{y=0}^{y=1} = \frac{6}{19}\left[x^2 + \frac{1}{4}\right]$$

$$f_{Y|X}(y|x) = \frac{(6/19)(x^2+y^3)}{(6/19)(x^2+1/4)} = \boxed{\frac{x^2+y^3}{x^2+1/4}}$$

$$E[Y|X] = \int_0^1 y\left(\frac{x^2+y^3}{x^2+1/4}\right)dy = \frac{1}{x^2+1/4}\int_0^1 yx^2+y^4\,dy = \frac{1}{x^2+1/4}\left[\frac{y^2x^2}{2} + \frac{y^5}{5}\right]_{y=0}^{y=1} = \frac{x^2/2+1/5}{x^2+1/4}$$

$$\boxed{E[Y|X] = \frac{10x^2+4}{20x^2+5}}$$

e) Verify directly that $E[E[X|Y]] = E[X]$

• $E[E[X|Y]] = \int_0^1 \left(\frac{6+3y^3}{4+3y^3}\right)\left(\frac{6}{19}\left(2y^3+\frac{8}{3}\right)\right)dy = \left(\frac{6}{19}\right)\int_0^1 \left(\frac{6+3y^3}{4+3y^3}\right)\left(2y^3+8/3\right)dy$

$= (6/19)\int_0^1 \frac{(6+3y^3)(2y^3+8/3)}{4+3y^3}\,dy = \left(\frac{6}{19}\right)\int_0^1 \frac{2(3y^3+4)(y^3+2)}{4+3y^3}\,dy$

$= \left(\frac{6}{19}\right)\int_0^1 2y^3+4\,dy = \left(\frac{6}{19}\right)\left[y^4/2 + 4y\right]_{y=0}^{y=1} = \left(\frac{6}{19}\right)\left(\frac{1}{2}\right)$

$\boxed{E[E[X|Y]] = \frac{27}{19} = E[X]}$

(f) Verify directly that $E[E[Y|X]] = E[Y]$

• $E[E[Y|X]] = \int_0^2 \left(\frac{10x^2+4}{20x^2+5}\right)\left(\frac{6}{19}\right)(x^2+1/4)\,dx = \left(\frac{6}{190}\right)\int_0^2 5x^2+2\,dx$

$= \frac{6}{190}\left[\frac{5}{3}x^3 + 2x\right]_{x=0}^{x=2} = \frac{6}{190}\left[\frac{40}{3} + \frac{12}{3}\right] = \left(\frac{6}{190}\right)\left(\frac{52}{3}\right)$

$\boxed{E[E[Y|X]] = \frac{52}{95} = E[Y]}$

6.) Let $T \sim \exp(\lambda)$ and $(U|T=y) \sim \text{Unif}[0,T]$. Find the unconditional mean and variance of U.

[* NOTE: $E[U] = E[E[U|T]]$ *]

- $E[U|T] = T/2 \Rightarrow E[E[U|T]] = E[T/2] = \frac{1}{2}E[T]$
- $E[T] = \int_0^\infty \lambda t e^{-\lambda t}\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt.$

[• recall: integration by parts: $\int fg' = fg - \int f'g$]

· let $f = t,\ g' = e^{-\lambda t} \Rightarrow f' = 1,\ g = -\frac{1}{\lambda}e^{-\lambda t}$

$\Rightarrow E[T] = \lambda \int_0^\infty t e^{-\lambda t}\,dt = \lambda \left[-\frac{1}{\lambda} t e^{-\lambda t} \Big|_{t=0}^{t=\infty} - \int_0^\infty -\frac{1}{\lambda} e^{-\lambda t}\,dt \right]$

$= -te^{-\lambda t}\Big|_{t=0}^{t=\infty} - \int_0^\infty -e^{-\lambda t}\,dt = 0 - \left[\frac{1}{\lambda}e^{-\lambda t}\right]_{t=0}^{t=\infty} = \frac{1}{\lambda} = E[T]$

- $E[U] = E[E[U|T]] = E\left[\frac{1}{2\lambda}\right] = \boxed{1/2\lambda = E[U]}$

[NOTE: By theorem 3.5.6: For Random Variables X & Y
$Var(X) = Var(E[X|Y]) + E[Var(X|Y)]$]

- $Var(U|T) = \frac{t^2}{12}$ [if $X \sim \text{unif}(a,b) \Rightarrow Var(X) = \frac{(b-a)^2}{12}$]
- $E[Var(U|T)] = E[t^2/12] = \frac{1}{12}E[T^2]$

$E[T^2] = \int_0^\infty \lambda t^2 e^{-\lambda t}\,dt$ (let $f = t^2,\ g' = e^{-\lambda t}$)

$= -e^{-\lambda t}t^2\Big|_{t=0}^{t=\infty} - \lambda\int_0^\infty -\frac{2}{\lambda}e^{-\lambda t}t\,dt = -e^{-\lambda t}t^2\Big|_{t=0}^{t=\infty} + 2\int_0^\infty te^{-\lambda t}\,dt$

· letting $u = t,\ v' = e^{-\lambda t}$

$= -e^{-\lambda t}t^2\Big|_{t=0}^{t=\infty} + 2\left[-\frac{1}{\lambda}te^{-\lambda t}\Big|_{t=0}^{t=\infty} - \int_0^\infty -1/\lambda\, e^{-\lambda t}\,dt\right] = \frac{2}{\lambda^2} = E[T^2]$

$E[Var(U|T)] = \frac{1}{12}\left(\frac{2}{\lambda^2}\right) = \frac{1}{6\lambda^2}$

- $Var(E[U|T]) = Var\left(\frac{T}{2}\right) = \frac{1}{4}Var(T) \cdot \frac{1}{4\lambda^2}$ [if $X \sim \text{Exp}(\lambda) \Rightarrow Var(X) = \frac{1}{\lambda^2}$]
- $Var(U) = Var(E[U|T]) + E[Var(U|T)]$

$= \frac{1}{4\lambda^2} + \frac{2}{12\lambda^2} = \boxed{\frac{5}{12\lambda^2} = Var(U)}$

7.) ~~Chp 3 Exercise 3.6.10~~: add (c) compare the band in part (b) to the exact probability.

3.6.10: § W has density function $f(w) = 3w^2$ for $0 < w < 1$, o.w. $f(w) = 0$.

(a) compute $E[W]$.

$$E[W] = \int_0^1 w\, 3w^2\, dw = 3\int_0^1 w^3\, dw = \frac{3}{4}\left[w^4\right]_{w=0}^{w=1} = \boxed{3/4 = E[W]}$$

(b) what bound does chebychevs inequality give for $P(|W - E[W]| \geq 1/4)$?

$$P(|W - 3/4| \geq 1/4) \leq \frac{\sigma_w^2}{(1/4)^2}$$

- $E[W^2] = \int_0^1 3w^4\, dw = \int_0^1 \frac{3}{5}w^5\, dw = \frac{3}{5}$
- $Var(W) = \sigma_w^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{48}{80} - \frac{45}{80} = \frac{3}{80}$

$$P(|W - 3/4| \geq 1/4) \leq \frac{(3/80)}{(1/16)} = \frac{48}{80}$$

(c) compare the band in part (b) to the exact probability.

$$P(|W - 3/4| \geq 1/4) = P(0 \leq w \leq 1/4) = P(w \leq 1/4)$$

- $P(w \leq 1/4) = \int_0^1 3w^2\, dw = \left[w^3\right]_{w=0}^{w=1/4} = 1/64$

$$\boxed{P(|W - 3/4| \geq 1/4) = 1/64 \leq \frac{48}{80}}$$

8.) Chp 4 Exercises 4.2.10 & 4.2.11

Exercise 4.2.10: Let Z_n be the sum of the squares of the numbers showing when we roll n fair dice. Find (w/proof) a number m s.t. $\frac{1}{n} Z_n \xrightarrow{P} m$

(Hint use the weak law of large numbers)

- By the weak law of large numbers;

$$\left(\frac{1}{n}\right) Z_n \to m$$

$$\bar{X} = \frac{(1^2+2^2+ \cdots +n^2)}{n}$$

- As $n \to \infty$ $\bar{X} \to \mu$ (the population mean)

8.) (cont'd)

Exercise 4.2.11 Consider flipping n fair nickels and n fair dimes. Let X_n equal 4 times the number of nickels showing heads, plus 5 times the number of dimes showing heads. Find (w/ proof) a number r s.t. $\frac{1}{n} X_n \xrightarrow{P} r$

E[# of nickels showing heads] = $n/2$
E[# of dimes showing heads] = $n/2$ } If n is even

$\Rightarrow X_n = 4\left(\frac{n}{2}\right) + 5\left(\frac{n}{2}\right) = 2n + \frac{5n}{2} = \frac{4n}{2} + \frac{5n}{2} = \boxed{\frac{9n}{2}}$

as $n \to \infty$ $\quad \frac{X_n}{n} = \frac{9n}{2}/n = \boxed{\frac{9}{2} = r}$

IF n is odd:

$X_n = 4 \times \left(\frac{n-1}{2}\right) + 5 \times \left(\frac{n+1}{2}\right)$ or $X_n = 4 \times \left(\frac{n+1}{2}\right) + 5 \times \left(\frac{n-1}{2}\right)$

$= 2(n-1) + \frac{5n+5}{2}$ or $2n+2 + \frac{5n-5}{2}$

$= \frac{9n+1}{2}$ or $\frac{9n-1}{2}$

Now as $n \to \infty$ $\quad \frac{X_n}{n} \to \frac{9}{2} + \frac{1}{2n} = \frac{9}{2} \Rightarrow \boxed{r = 9/2}$

or $\quad \frac{X_n}{n} \to \frac{9}{2} - \frac{1}{2n} = 9/2$