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STAT 630
Homework #2
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The material covered by this assignment is primarily in lectures 4-7 and covers Chapters 1-2 of the textbook.

1.) (a) Prove Bonferroni's inequality: $P(A \cap B) \geq P(A) + P(B) - 1$

Ask if I must prove for any event A $P(A) \leq 1$.

Proof: Let A & B be two events in the sample space S. Then we know from the inclusion-exclusion principle for two events (Theorem 1.3.3 in Probability & Statistics the science of uncertainty) that

① $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Rearranging ①, we get:

② $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

We know, for any event E, $P(E) \leq 1$, thus

③ $P(A \cup B) \leq 1$

From ② & ③ we get

④ $P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$

$P(A \cap B) \geq P(A) + P(B) - 1$. QED

(b) In statistics, we often talk about the event that our statistical procedure will lead to a correct (true) conclusion. If A & B are such events (for two different procedures, but in the same experiment) and each has probability 0.95. According to Bonferroni's inequality, what can we say about the chance that both procedures lead to correct conclusions.

- From Bonferroni's inequality we get

$P(A \cap B) \geq 0.95 + 0.95 - 1$

$\boxed{P(A \cap B) \geq 0.90}$

c) Extrapolate to three events. Specifically, if 3 statistical procedures each have probability $1 - \alpha/3$ of resulting in a correct conclusion. Show that the probability that all 3 are correct is at least $1 - \alpha$.

If: $P(A) = P(B) = P(C) = 1 - \alpha/3$

Then $P(A \cap B \cap C) = P((A \cap B) \cap C) \geq P(A \cap B) + P(C) - 1$ (by Bonferroni's inequality)

$\geq P(A) + P(B) + P(C) - 2$ (by Bonferroni's inequality)

$\geq 3(1 - \frac{\alpha}{3}) - 2$

$\geq 3 - \alpha - 2$

$\boxed{P(A \cap B \cap C) \geq 1 - \alpha}$

2.) Chapter exercises 1.5.9, 1.5.14, 1.5.18(a,b,c)

1.5.9: S we roll 2-fair sided dice, one red, one blue.

- Let A be the event that the two dice show the same value
- Let B " " sum to 12
- Let C " " red die shows 4
- Let D " " blue die shows 4.

a) Are A & B independent? $\boxed{\text{NO}}$

$P(A) = 6(1/6)^2 = 1/6 \neq 1 = P(A|B)$

b.) Are A & C independent? $\boxed{\text{Yes}}$

$P(A) = 1/6 = P(A|C)$

c.) Are A & D independent? $\boxed{\text{Yes}}$

$P(A) = 1/6 = P(A|D)$

d.) Are C & D independent? $\boxed{\text{Yes}}$

$P(C) = 1/6 = P(C|D).$

e.) Are A, C & D independent? $\boxed{\text{NO}}$

$P(A \cap C \cap D) = P(A)\,P(C|A)\,P(D|A,C)$

$= (1/6)(1/6)(1) = \frac{1}{6^2} \neq \frac{1}{6^3} = P(A)P(C)P(D)$

1.5.14: Prove $P(A \cap B) = P(A)P(B) \iff P(A^c \cap B) = P(A^c)\,P(B)$

Proof:

$\Rightarrow$ WTS: $P(A \cap B) = P(A)P(B) \Rightarrow P(A^c \cap B) = P(A^c)\,P(B)$

S: $P(A \cap B) = P(A)P(B)$

$\cdot\ P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\,P(B)$

$= (1 - P(A))P(B) = P(A^c)\,P(B)$

$P(A^c \cap B) = P(A^c)P(B)$. QED.

$\Leftarrow$ WTS: $P(A^c \cap B) = P(A^c)\,P(B) \Rightarrow P(A \cap B) = P(A)P(B)$.

S: $P(A^c \cap B) = P(A^c)\,P(B)$.

$P(A \cap B) = P(A|B)P(B) = (1 - P(A^c|B))\,P(B)$

$= (1 - P(A^c))P(B) = P(A)\,P(B)$

$P(A \cap B) = P(A)\,P(B)$ QED.

2.) (Cont'd)

1.5.18: (Monty Hall Problem): β there are three doors, labeled A, B & C. A new car is behind one of the three doors but you don't know which. You select one of the doors, door A. The host opens one of the doors B or C as follows: If the car is behind B, then they open C; if the car is behind C, then they open B and if the car is behind A, then they open B or C w/ probability 1/2. The host then gives you the option of either sticking w/ your original door choice (A) or switching to the remaining unopened door. β for definiteness that the host opens door B.

a) If you stick w/ your original choice (i.e A), conditional on having opened door B, what is your probability of winning? (Hint: first condition on the true location of the car. Then use theorem 1.5.2)

Let i = door letter, A = player's choice B = Car's location, C = host selection.

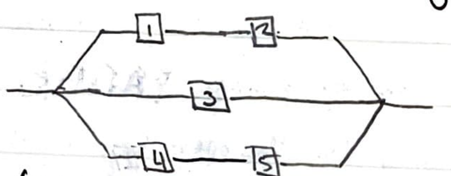
$$P(B_a \mid A_a \cap C_b) = \frac{P(B_a \cap A_a \cap C_b)}{P(A_a \cap C_b)} = \frac{P(C_b \mid B_a \cap A_a)P(B_a \cap A_a)}{P(C_b \mid A_a)P(A_a)} = \boxed{1/3}$$

b) If you switch to the remaining door, conditional on the host having opened door B, what is your prob of winning?

$$P(B_c \mid A_a \cap C_b) = \frac{P(B_c \cap A_a \cap C_b)}{P(A_a \cap C_b)} = \frac{P(C_b \mid B_c \cap A_a)P(B_c \cap A_a)}{P(C_b \mid A_a)P(A_a)} = \boxed{2/3}$$

c) No, I've seen this problem before. We could check the result using 2 dice. Let {1,2} represent door A, {3,4} represent door B and {5,6} represent door C. The "host" then rolls one of the dice and the number the die lands on represents the door which the "car" is placed behind. Then the other die is rolled and whichever number the die lands on represents the door which the player selects, Then the player rolls the die again, if high {4,5,6} the player switches doors and if low {1,2,3} the player stays.

3.) The system shown below has 5 components which act independently. Each component fails w/ probability P. Find the probability the system fails



- Let i be the event component i fails.
- $F = \{1 \cup 2\} \cap \{3\} \cap \{4 \cup 5\}$
- $P(F) = P(\{1 \cup 2\} \cap \{3\} \cap \{4 \cup 5\})$

$$= (P(1) + P(2) - P(1 \cap 2))\, P(3)\, (P(4) + P(5) - P(4 \cap 5))$$
$$= (P(1) + P(2) - P(1)P(2))\, P(3)\, (P(4) + P(5) - P(4)P(5))$$
$$= P(2P - P^2)(2P - P^2)$$
$$= P(4P^2 - 4P^3 + P^4) = \boxed{4P^3 - 4P^4 + P^5}$$

4.) If a parent has genotype Aa, he transmits either A or a to an offspring, each w/ probability 1/2. The gene he transmits to one offspring is <u>independent</u> of the gene he transmits to all other offspring. Consider a parent w/ three children (labled 1,2,3) & the following events: B = {1 and 2 have same gene}, C = {2 & 3 have same gene}, D = {1 & 3 have the same gene}. Show that all these events are pairwise independent, but not <u>mutually independent</u>.

- Let A_i be the event that a given child gets an A.

$$P(A_1) = P(A_2) = P(A_3) = 0.25$$

- Let $B = \{A_1 \cap A_2\}$

$$P(B) = (A_1 | A_2)\, P(A_2) = P(A_1)P(A_2) = 0.0625$$

- Let $C = \{A_i \cap A_3\}$

$$P(C) = (A_1 | A_3)\, P(A_3) = P(A_1)\, P(A_3) = 0.0625$$

- Let $D = \{A_2 \cap A_3\}$

$$P(D) = (A_2 | A_3)\, P(A_3) = P(A_2)\, P(A_3) = 0.0625$$

- $P(B \cap C \cap D) = P(B)\, P(C|B)\, P(D|B,C)$

$$(0.0625)(0.0625)\,1 \neq P(B)P(C)P(D) \Rightarrow \boxed{\text{NOT Independent}}$$

5.) Chp 2 Exercise 2.1.5:

(a) Let A & B be events & let $X = I_A \cdot I_B$. Is X an indicator variable? If yes, then of what event?

Yes, X is an indicator variable for the event $\{A \cap B\}$.

(b) Show that $I_{A \cup B} = \max(I_A, I_B)$.

$$I_{A \cup B} = \begin{cases} 1 & s \in A \cup B \\ 0 & s \notin A \cup B \end{cases},$$

If $s \in A \cap s \notin B$, $I_{A\cup B} = 1 = \max(I_A, I_B)$

If $s \notin A \cap s \in B$, $I_{A\cup B} = 1 = \max(I_A, I_B)$

If $s \notin A \cap s \notin B$, $I_{A\cup B} = 0 = \max(I_A, I_B)$

If $s \in A \cap s \in B$, $I_{A\cup B} = 1 = \max(I_A, I_B)$

6) Chapter 2 Exercise 2.1.8. In fact, compute $W(s)$ & $Z(s)$ $\forall s \in S$

Let $S = \{1,2,3,4,5\}$, $X = I_{\{1,2,3\}}$, $Y = I_{\{2,3\}}$, $Z = I_{\{3,4,5\}}$.

$W = X - Y + Z$

(a)-(c): compute $\boxed{W(1) = 1 \; ; \; Z(1) = 0}$

compute $\boxed{W(2) = 0 \; ; \; Z(2) = 0}$

compute $\boxed{W(3) = 2 \; ; \; Z(3) = 1}$

compute $\boxed{W(4) = 1 \; ; \; Z(4) = 1}$

compute $\boxed{W(5) = 1 \; ; \; Z(5) = 1}$

(d) Determine whether or not $W \geq Z$?

$\boxed{\text{Yes; } W \geq Z}$

7) Chp 2 exercise 2.2.4: β we roll one fair six-sided die, and let Z be the number showing. Let $W = Z^3+4$, and let $V=\sqrt{Z}$

(a) Compute $P(W=w)$ $\forall w \in \mathbb{R}$.

$$P(W=w) = \begin{cases} 1/6 & \text{for } w \in [5, 12, 31, 68, 129, 220] \\ 0 & \text{o.w.} \end{cases}$$

(b.) compute $P(V=v)$ $\forall v \in \mathbb{R}$

$$P(V=v) = \begin{cases} 1/6 & \text{for } v \in [1, 1.414214, 1.732051, 2, 2.236068, 2.449490] \\ 0 & \text{o.w} \end{cases}$$

(c.) Compute $P(ZW=x)$ $\forall x \in \mathbb{R}$

$$P(ZW=x) = \begin{cases} 1/6 & \text{for } x \in [5, 24, 93, 272, 645, 1320] \\ 0 & \text{o.w} \end{cases}$$

(d) compute $P(VW=y)$ $\forall y \in \mathbb{R}$

$$P(VW=y) = \begin{cases} 1/6 & \text{for } y \in [5, 16.97056, 53.69358, 136, 288.45277, 538.88794] \\ 0 & \text{o.w.} \end{cases}$$

(e.) compute $P(V+W=r)$ $\forall r \in \mathbb{R}$

$$P(V+W=r) = \begin{cases} 1/6 & \text{for } y \in [6, 13.41421, 32.73204, 70, 131.23607, 222.44949] \\ 0 & \text{o.w.} \end{cases}$$

8.) Chp 2: exercise 2.3.9, 2.3.10, 2.3.13, 2.3.14:

2.3.9: Let $Z \sim$ Neg-Binomial$(3, 0.25)$. Compute $P(Z \le 2)$

- $P(Z=2) = \binom{3-1+2}{3-1}(1/4)^3(3/4)^2 = \boxed{\binom{4}{2}(0.25)^3(0.75)^2}$
- $P(Z=1) = \binom{3-1+1}{3-1}(1/4)^3(3/4)^1 = \boxed{\binom{3}{2}(0.25)^3(0.75)^1}$
- $P(Z=0) = \binom{3-1+0}{3-1}(1/4)^3(3/4)^0 = \boxed{\binom{2}{2}(0.25)^3}$

$P(Z \le 2) = P(Z=0) + P(Z=1) + P(Z=2) = \boxed{0.103515625}$

8.) (Cont'd):

2.3.10: Let $X \sim \text{Geom}(0.2)$, compute $P(X^2 \leq 15)$.

$$P(X^2 \leq 15) = P(-\sqrt{15} \leq X \leq \sqrt{15})$$

NOTE: Since X can only take non-negative integer values, we can rewrite this as:

$$P(X^2 \leq 15) = P(X \leq 3)$$

- $P(X \leq 3) = (\frac{1}{5})(\frac{4}{5})^3 + (\frac{1}{5})(\frac{4}{5})^2 + (\frac{1}{5})(\frac{4}{5}) + 0.2$

$\boxed{P(X \leq 3) = 0.5904}$

2.3.13: Let $X \sim \text{HGeom}(20, 7, 8)$. What is the probability $X=3$? What is the probability $X=8$?

- $\boxed{P(X=3) = \frac{\binom{7}{3}\binom{13}{5}}{\binom{20}{5}} = 0.3575851}$
- $\boxed{P(X=8) = 0}$

2.3.14: Suppose that a symmetrical die is rolled 20 independent times, & each time we record whether or not the event $\{2,3,5,6\}$ has occurred.

(a) What is the distribution of the number of times this event occurs in 20 rolls?

Binomial$(20, 2/3)$.

(b) $P(X=5) = \binom{20}{5}(2/3)^5(1/3)^{15} = 0.0001422881$

9.) Suppose that a basketball player sinks a basket from a certain position on the court w/ probability 0.35.

(a) What is the probability that the player sinks three baskets in 10 indp. throws.

$\boxed{P(X=3) = \binom{10}{3}(0.35)^3(0.65)^7 = 0.2522196}$

(b) $\boxed{P(X=10) = (0.35)(0.65)^9}$

(c) $P(X=8) = \binom{2-1+8}{2-1}(0.35)^2(0.65)^8$

$\boxed{P(X=8) = \binom{9}{1}(0.35)^2(0.65)^8}$

10.) First, note that as N gets larger, the Binomial distribution becomes a better approximation of the hypergeometric distribution. The largest difference btwn the binomial and all the hypergeom distributions is where the # of successes is equal to 6. For $N = 50$, the absolute difference btwn the two distributions at $X=6$ is. 0.02923595. For $N=100$ the absolute difference btwn the two distributions at $X=6$ is 0.01349013. For $N=1000$ the absolute difference btwn the two distributions at $X=6$ is 0.001262926.