Stat 641 Fall 2021

Solutions for Assignment 4

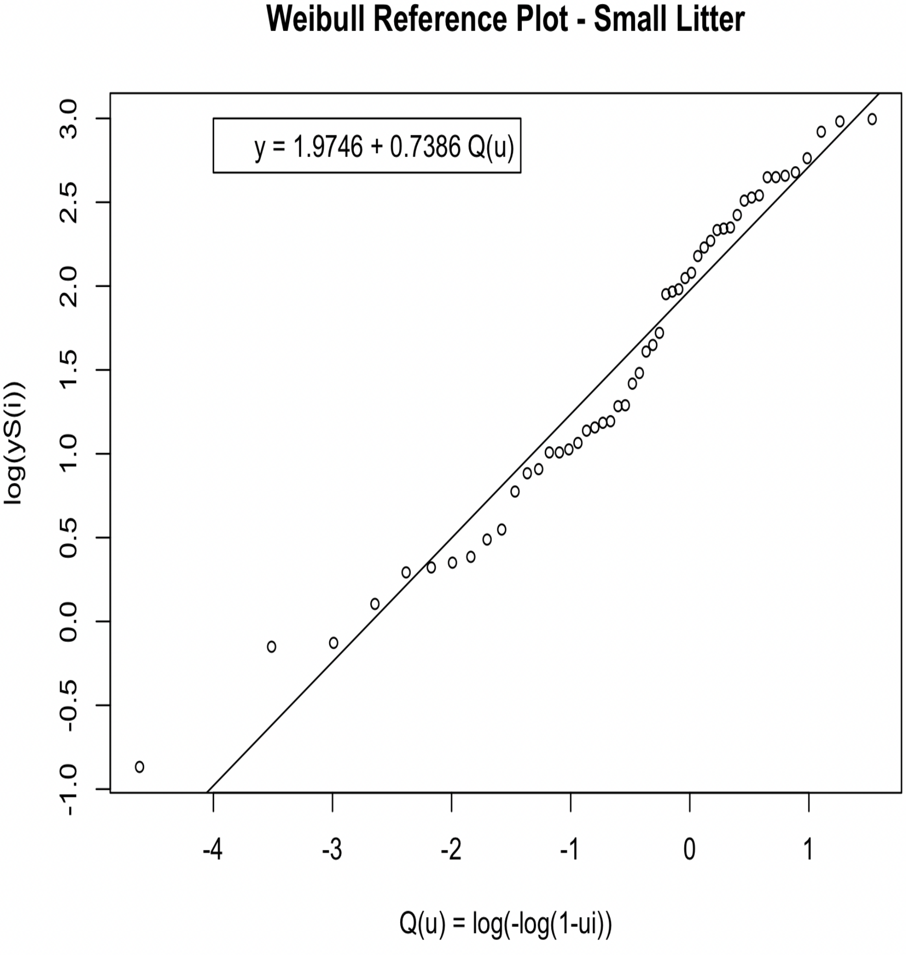
P1. (50 points)

1. See R code at the end of this document. For the Small Litter Size we obtain:

   a. A 10% trimmed mean would involve averaging the middle
   $K(.10) = 51 - [(51)(.1)] - [(51)(.1) + 1] + 1 = 51 - [5.1] - [6.1] + 1 = 41$ values in the data set yielding:
   $\hat{\mu}_{(.1)} = \frac{1}{41}\sum_{i=6}^{46} Y_{(i)} = \frac{1}{41}(257.89) = 6.29 = mean(yS, trim = .1)$ whereas the untrimmed mean is
   $\hat{\mu} = \frac{1}{51}\sum_{i=1}^{51} Y_{(i)} = \frac{1}{51}(351.18) = 6.89$

   The untrimmed mean is somewhat larger than the 10% trimmed mean which would indicate that there a few large outliers in the data. In fact, examining the sorted data, we have three relatively large data values in the data set: 18.55, 19.73, 20.00

   b. A Weibull reference distribution plot is displayed:



The plot indicates a reasonably good fit of the Small Litter data to a Weibull distribution. The graphical estimates are

$$\hat{\gamma} = 1/.7386 = 1.3539 \qquad \hat{\alpha} = e^{1.9746} = 7.2037$$

The MLE of the Weibull parameters from R are

$$\hat{\gamma} = \text{Weibull Shape} = 1.265583 \qquad \hat{\alpha} = \text{Weibull Scale} = 7.4268$$

A fairly good match to the graphical estimates.

c. Using the MLE estimates: $P[Y_L > 15] = 1 - F(15) \approx e^{-(15/7.4268)^{1.265583}} = .0877$

Using the Graphical estimates: $P[Y_L > 15] = 1 - F(15) \approx e^{-(30/7.2037)^{1.3539}} = .0672$ about 23% less than the MLE

The distribution-free estimate would be $P[Y > 15] = 1 - \hat{F}(15) = 1 - 47/51 = .0784$ about 11% less than the MLE

d. The distribution-free estimates are

$\hat{\mu} = \text{sample mean} = \bar{Y}_S = 6.8859$ $\quad$ $\hat{\sigma} = \text{sample stand. dev.} = S_{Y_S} = 5.46$

Using the formulas on page 6 in HO 6, we have for the Weibull distribution, with MLE's from R:

$$\widehat{\mu} = \widehat{\alpha}\Gamma\left(1 + \frac{1}{\hat{\gamma}}\right) = (7.4268)\Gamma\left(1 + \frac{1}{1.265583}\right) = 6.8981$$

$$\widehat{\sigma} = \widehat{\alpha}\sqrt{\Gamma\left(1 + \frac{2}{\hat{\gamma}}\right) - \Gamma^2\left(1 + \frac{1}{\hat{\gamma}}\right)} = 7.4268\sqrt{\Gamma\left(1 + \frac{2}{1.265583}\right) - \Gamma^2\left(1 + \frac{1}{1.265583}\right)} = 5.4882$$

The MLE estimates of μ and σ are very close to the distribution-free estimates thus lending evidence that the Weibull model is the correct model for this data.

e. The distribution-free estimates are

$\hat{\tilde{\mu}} = \hat{Q}(.5) = \text{sample median} = Y_{(26)} = 5.00$ $\quad$ Using R-function, quantile(yS,.5,type=5) = 5.00

$\widehat{IQR} = \text{sample IQR} = \hat{Q}(.75) - \hat{Q}(.25) = Y_{(.75n+.5)} - Y_{(.25n+.5)} = Y_{(38.75)} - Y_{(13.25)} \Rightarrow$

$\hat{Q}(.75) - \hat{Q}(.25) = (.25 * Y_{(38)} + .75 * Y_{(39)}) - (.75 * Y_{(13)} + .25 * Y_{(14)}) = 10.4625 - 2.545 = 7.9175$

Using R-function, $quantile(yL, .75, type = 5) - quantile(yL, .25, type = 5) = 10.4625 - 2.545 = 7.9175$

Using the formula for the quantile function from a Weibull distribution:

$Q(u) = \alpha(-log(1-u))^{1/\gamma}$ along with MLE from R for α and γ we have

$\widehat{\tilde{\mu}} = \hat{Q}(.5) = \hat{\alpha}(-log(1-.5))^{1/\hat{\gamma}} = 7.42681(-log(1-.5))^{1/1.265583} = 5.56$

$\widehat{IQR} = \hat{Q}(.75) - \hat{Q}(.25) = \hat{\alpha}(-log(1-.75))^{1/\hat{\gamma}} - \hat{\alpha}(-log(1-.25))^{1/\hat{\gamma}} = 6.8387$

Equivalently, using the R quantile function for the Weibull distribution, we have

$\widehat{\tilde{\mu}} = qweibull(.5, 1.265583, 7.42681) = 5.56$

$\widehat{IQR} = qweibull(.75, 1.265583, 7.42681) - qweibull(.25, 1.265583, 7.42681) = 6.8387$

The MLE estimate of the median based on the Weibull model is close to the distribution-free estimate (5.56 to 5.00) but there is substantial difference between the two estimates of the IQR (6.8387 to 7.9175). This may be due to the IQR reflecting only the fit of the data in the middle of the distribution.

2. For Large Litter Data: $\hat{\mu}_L = 10.39$ $\hat{\sigma}_L = 9.15$

   For Small Litter Data: $\hat{\mu}_S = 6.89$ $\hat{\sigma}_S = 5.46$

3. For Large Litter Data: $\hat{\tilde{\mu}}_L = 7.93$ $M\hat{A}D_L = 7.95$

   For Small Litter Data: $\hat{\tilde{\mu}}_S = 5.00$ $M\hat{A}D_S = 5.23$

4. For the Small Litter Data, the pdf appeared to be just slightly right skewed so the mean should be only slightly larger than the median (6.89 vs 5.00) and the standard deviation somewhat larger than MAD (5.46 vs 5.23). The larger than expected difference in the Mean and Median was very surprising considering that S and MAD were so close in value.

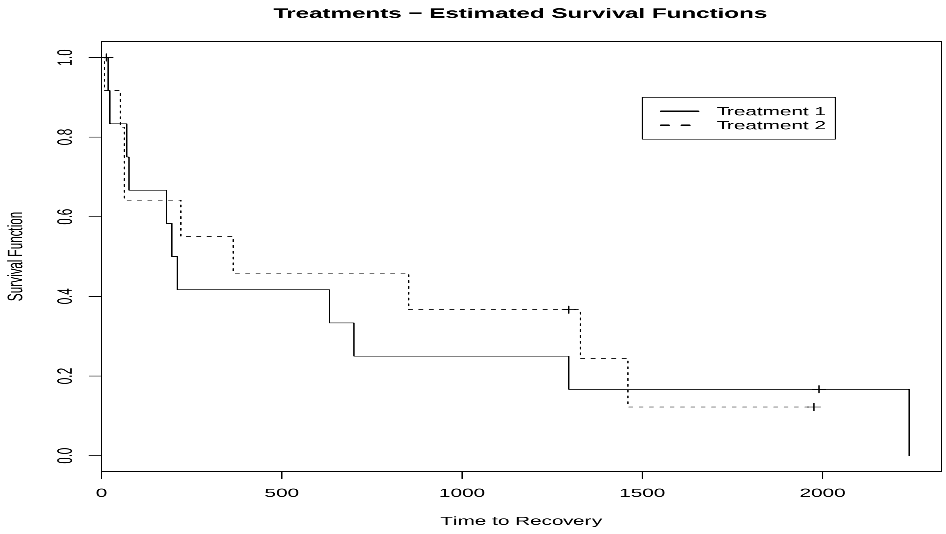
   For the Large Litter Data, the pdf appeared to be just more right skewed so the mean should be larger than the median (10.39 vs 7.93) and the standard deviation somewhat larger than MAD (9.15 vs 7.95). I was somewhat surprised that there was not a larger difference between S and MAD considering the 4 or 5 rather large values in the Large Litter data set.

Based on the right skewness of the estimated pdf for the Large Litter data and the goal of the study was to compare the Small to the Large Litter relative brain weights, I would select (Median, MAD) to represent the location and scale in the two data sets.

5. From the given data, it would appear that larger relative brain weights are associated with Larger Litter sizes. It would be much more informative to have the actual litter sizes associated with each relative brain weight as opposed to having the groupings into just small and large litters.

P2. (30 points) Using the times to recovery (or censoring) for the 25 patients we obtain:

1. The estimate survival functions for the two Treatments are given in the following plot:



2. From the R output we have

```
                G=1
time n.risk n.event survival std.err lower 95% CI upper 95% CI
  18     12       1    0.917  0.0798       0.7729        1.000
  23     11       1    0.833  0.1076       0.6470        1.000
  70     10       1    0.750  0.1250       0.5410        1.000
  76      9       1    0.667  0.1361       0.4468        0.995
 180      8       1    0.583  0.1423       0.3616        0.941
 195      7       1    0.500  0.1443       0.2840        0.880
 210      6       1    0.417  0.1423       0.2133        0.814
 632      5       1    0.333  0.1361       0.1498        0.742
 700      4       1    0.250  0.1250       0.0938        0.666
1296      3       1    0.167  0.1076       0.0470        0.591
2240      1       1    0.000     NaN           NA           NA
                G=2
time n.risk n.event survival std.err lower 95% CI upper 95% CI
   8     12       1    0.917  0.0798       0.7729        1.000
  52     10       1    0.825  0.1128       0.6311        1.000
  63      9       2    0.642  0.1441       0.4132        0.996
 220      7       1    0.550  0.1499       0.3224        0.938
 365      6       1    0.458  0.1503       0.2410        0.872
```

```
  852       5       1     0.367  0.1456       0.1684        0.798
 1328       3       1     0.244  0.1392       0.0801        0.746
 1460       2       1     0.122  0.1110       0.0206        0.724
> print(results, print.rmean=TRUE)
Call: survfit(formula = Surv(T, ST) ~ G)
    records n.max n.start events *rmean *se(rmean) median 0.95LCL 0.95UCL
G=1      13    13      13     11    657        229    202      76      NA
G=2      12    12      12      9    731        216    365      63      NA
```

Note that for G=1, the table reports the median as 202 but $\hat{S}(195) = .5$ from the output of the K-M estimator of the survival function. According to our definition of the quantile function, $\hat{Q}(u) = inf\{t : \hat{S}(t) \leq 1 - u\}$, the median would be 195.

The estimated mean and median are smaller for Treatment 1 (G=1) than for Treatment 2 (G=2).

3. Based on the median time to recovery, Treatment 1 would be the more effective treatment. The mean times to recovery are much larger than the median times due to a few very large values in both treatment groups. But, Treatment 1 still has a smaller mean the Treatment 2. However, as we will discuss in future handouts, when the standard errors of the estimators are taken into account, there may not be significant evidence of a difference in the two treatments.

P4. (20 points)

1. **A or B -** Because the true stress for the censored specimens are greater than or equal to $t_C = 500$ psi

2. **D -** Because the amount of water retained at 90 days would be less than the amount of water retained at death.

3. **A or B -** Because the study is terminated at a fixed time, 30 days

4. **A or C -** Because the study was terminated after a pre-selected number of fires

5. **A -** Because brake failure mileage for the censored automobiles are greater than the miles traveled at the end of the study.

V. (10 Bonus points)

- Bonus 1. (5 points)

$$\lim_{\alpha \to .5} \mu_{(\alpha)} = \frac{\lim_{\alpha \to .5} \int_{Q(\alpha)}^{Q(1-\alpha)} y f(y) dy}{\lim_{\alpha \to .5} (1-2\alpha)} = \frac{0}{0}$$

Apply $l^{'}$Hopital's Rule:

$$\lim_{\alpha \to .5} \mu_{(\alpha)} = \frac{\lim_{\alpha \to .5} \frac{d}{d\alpha} \int_{Q(\alpha)}^{Q(1-\alpha)} y f(y) dy}{\lim_{\alpha \to .5} \frac{d}{d\alpha}(1-2\alpha)}$$

$$= \frac{\lim_{\alpha \to .5} [Q(1-\alpha) f(Q(1-\alpha))(-1) Q^{'}(1-\alpha) - Q(\alpha) f(Q(\alpha)) Q^{'}(\alpha)]}{-2}$$

$$= \frac{-2Q(.5) f(Q(.5)) Q^{'}(.5)}{-2} = Q(.5)$$

Therefore, $Q^{'}(u) = \frac{1}{f(Q(u))}$

- Bonus 2. (5 points)

i. The likelihood function is given by

$$L(\beta, \theta; t_1, t_2, \ldots, t_n) = \prod_{i=1}^{n} g(t_i;\ \beta, \theta) = \prod_{i=1}^{n} \beta e^{-\beta(t_i - \theta)} I(t_i \geq \theta)$$

ii. As a function of θ, the likelihood increases as θ increase, until $\theta \geq min(t_1, t_2, \ldots, t_n)$ after which the likelihood becomes 0.

- Therefore, the MLE for θ is $\hat{\theta} = min(t_1, t_2, \ldots, t_n)$

  The log-likelihood function is given by

$$l(\beta, \theta; t_1, t_2, \ldots, t_n) = log(L(\beta, \theta; t_1, t_2, \ldots, t_n)) = nlog(\beta) - \beta \left(\sum_{i=1}^{n} t_i - n\theta \right) \quad \text{for all} \quad t_i \geq \theta$$

  The log-likelihood evaluated at $\hat{\theta}$ is

$$l = l(\beta, \hat{\theta}; t_1, t_2, \ldots, t_n) = nlog(\beta) - \beta \left(\sum_{i=1}^{n} t_i - n\hat{\theta} \right) \quad \text{for all} \quad t_i \geq \hat{\theta}$$

$$\frac{dl}{d\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} (t_i - \hat{\theta}) \quad \text{and} \quad \frac{d^2 l}{d\beta^2} = \frac{-n}{\beta^2} < 0$$

  Setting $\frac{dl}{d\beta}$ equal to 0 and solving for β yields

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} (t_i - \hat{\theta})} \quad \text{which is a maximum because 2nd derivative was negative}$$

iii. $P[T > 40] = e^{-\hat{\beta}(40 - \hat{\theta})} = e^{-.1637(40-30)} = 0.1946$

```
####
#### P1
####

##
## (1)
##

library(MASS)
yS = c(0.42,    0.86,    0.88,    1.11,    1.34,    1.38 ,   1.42,    1.47,    1.63,
   1.73,    2.17,    2.42,    2.48,    2.74,    2.74,    2.79,    2.90,    3.12,
   3.18,    3.27,    3.30,    3.61 ,   3.63,    4.13 ,   4.40,    5.00,    5.20,
   5.59,    7.04,    7.15,    7.25,    7.75,    8.00,    8.84,    9.30 ,   9.68,
   10.32,   10.41,   10.48,   11.29,   12.30,   12.53,   12.69,   14.14,   14.15,
   14.27 ,  14.56,   15.84,   18.55,   19.73,   20.00)
yL =
 c( 0.94 ,  1.26 ,  1.44  , 1.49 ,  1.63 ,  1.80 ,  2.00 ,  2.00 ,  2.56,
    2.58 ,  3.24 ,  3.39  , 3.53 ,  3.77 ,  4.36 ,  4.41 ,  4.60 ,  4.67,
    5.39 ,  6.25 ,  7.02 ,  7.89 ,  7.97 ,  8.00 ,  8.28 ,  8.83 ,  8.91,
    8.96 ,  9.92 , 11.36 , 12.15,  14.40 , 16.00 , 18.61 , 18.75 , 19.05,
   21.00 , 21.41 , 23.27 , 24.71,  25.00 , 28.75 , 30.23 , 35.45 )

nS <- length(yS)
nL <- length(yL)

## (a)
yS  = sort(yS)
ySt = yS[c(-(1:5),-(nS - 0:4))]
meanS = mean(yS)
trim.meanS = mean(ySt)
trimmedmean = mean(yS,trim=.10)

## (b)
i <- 1:nS
ui <- (i - 0.5) / nS
QW <- log(-log(1 - ui))
plot(QW, log(yS), main="Weibull Reference Plot - Small Litter",cex=.75,lab=c(7,11,7),
       xlab="Q(u) = log(-log(1-ui))",
       ylab="log(yS(i))")
abline(lm(log(yS) ~ QW))
legend(-4,3.0,"y = 1.9746 + 0.7386 Q(u)")

mle_weib <- fitdistr(yS,"weibull")
gamma_hat <- 1.2655827
alpha_hat <- 7.4268097

## (c)
exp(-(15 / alpha_hat) ^ gamma_hat)

## (d)
mu_hat <- alpha_hat * gamma(1 + 1 / gamma_hat)
sigma_hat <- sqrt(alpha_hat ^ 2 * (gamma(1 + 2 / gamma_hat) -
  (gamma(1 + 1 / gamma_hat)) ^ 2))

mean(yS)
sd(yS)

## (e)
med_hat <- alpha_hat * (-log(1 - 0.5)) ^ (1 / gamma_hat)
Q1_hat <- alpha_hat * (-log(1 - 0.25)) ^ (1 / gamma_hat)
Q3_hat <- alpha_hat * (-log(1 - 0.75)) ^ (1 / gamma_hat)
IQR_hat <- Q3_hat - Q1_hat

0.5 * nS + 0.5
0.25 * nS + 0.5
0.75 * nS + 0.5
med_df <- yS[26]
Q1_df <- 0.75 * yS[13] + 0.25 * yS[14]
Q3_df <- 0.25 * yS[38] + 0.75 * yS[39]
median(yS)
quantile(yS, c(0.25, 0.75), type = 5)
```

```
IQR_df <- Q3_df - Q1_df

##
## (2)
##

xbar_S <- mean(yS)
s_S <- sd(yS)
xbar_L <- mean(yL)
s_L <- sd(yL)

##
## (3)
##

med_L <- mean(yL)
iqr_L <- quantile(yL, 0.75) - quantile(yL, 0.25)
mad_L <- mad(yL)
mad_S <- mad(yS)

####
#### P2
####

library(survival)

T = c( 180, 632, 2240, 195, 76, 70, 13, 1990, 18, 700, 210, 1296, 23, 8, 852,  52, 220, 63,   8, 1976,1296,1460,63,1328,365)
ST = c(  1,   1,    1,   1,  1,  1,  0,    0,  1,   1,   1,    1,  1, 0,   1,   1,   1,  1,   1,    0,    0,   1, 1,   1,  1)
G =  c( rep(1,13),rep(2,12))

out = cbind(T,ST,G)
Surv(T, ST)

results <- survfit(Surv(T, ST) ~ G)
summary(results)
print(results, print.rmean=TRUE,rmean="individual",mark.time=True)

par(lab=c(15,20,4))
plot(results,ylab="Survival Function",xlab="Time to Recovery",mark.time=TRUE,
main="Treatments - Estimated Survival Functions",lty=1:2 )
legend(1500,.9,c("Treatment 1","Treatment 2"),lty=1:2,lwd=2)
```