Statistics 630 - Assignment 8
(partial solutions)

1. Exer. 6.2.4. $p(x) = \frac{\theta^x e^{-\theta}}{x!}$.

   (a) The log-likelihood is $\ell(\theta) = -n\theta + \log(\theta)\sum_{i=1}^n X_i - \sum_{i=1}^n \log(x_i!)$, and the score function is $S(\theta) = -n + \frac{1}{\theta}\sum_{i=1}^n X_i$, which is decreasing in θ and has root $\widehat{\theta} = \overline{X}$.

   (b) The only term in the log-likelihood above that has both θ and the sample involves the sample solely through the value of $\sum_{i=1}^n X_i$. [This is the factorization theorem expressed in terms of the log-likelihood.]

   (c) Since $\mathsf{E}(X) = \mathsf{Var}(X) = \theta$ we have $\mathsf{E}(\widehat{\theta}) = \theta$ (so that $\mathsf{Bias}(\widehat{\theta}) = 0$), and $\mathsf{MSE}(\widehat{\theta}) = \mathsf{Var}(\widehat{\theta}) = \frac{\theta}{n}$.

   (d) The MLE for θ^2 is $\widehat{\theta}^2 = \overline{X}^2$. Since $\mathsf{E}(\overline{X}^2) = (\mathsf{E}(\overline{X}))^2 + \mathsf{Var}(\overline{X}) = \theta^2 + \frac{1}{n}\theta$, we have $\mathsf{Bias}(\widehat{\theta}^2) = \frac{1}{n}\theta$.

   (e) $e^{-\widehat{\theta}}$.

2. Exer. 6.2.7. Note that $f(x) = \alpha x^{\alpha-1}$ for $0 < x < 1$.

   (a) The score equation is $S(\alpha) = \frac{n}{\alpha} + \sum_{i=1}^n \log(X_i) = 0$, yielding MLE $\widehat{\alpha} = -\frac{n}{\sum_{i=1}^n \log(X_i)}$. The score function is decreasing, so $\widehat{\alpha}$ does indeed maximize the log-likelihood.

   (b) $\sum_{i=1}^n \log(X_i)$.

   (c) $\mathsf{Var}(X) = \frac{\alpha}{(\alpha+1)^2(\alpha+2)}$ has MLE $\frac{\widehat{\alpha}}{(\widehat{\alpha}+1)^2(\widehat{\alpha}+2)}$.

   (d) Since $\mathsf{E}(X) = \frac{\alpha}{\alpha+1}$, a method of moments estimator for α solves $\frac{\widetilde{\alpha}}{\widetilde{\alpha}+1} = \overline{X}$, namely, $\widetilde{\alpha} = \frac{\overline{X}}{1-\overline{X}}$.

3. (a) $L(\beta) = \frac{3^n}{\beta^{3n}}\prod_{i=1}^n W_i^2 \times I_{[\max(W_1,\ldots,W_n),\infty)}(\beta)$, which is decreasing in β starting from $\max(W_1,\ldots,W_n)$. Thus the MLE is $\widehat{\beta} = \max(W_1,\ldots,W_n)$.

   (b) $\mathsf{E}(W) = \frac{3\beta}{4}$, so a MOME for β is $\widetilde{\beta} = \frac{4\overline{W}}{3}$. [Note: if $\widetilde{\beta} < \widehat{\beta}$ then $\widetilde{\beta}$ is in contradiction to the data. This can certainly happen with some positive probability.]

4. Exer. 6.2.12. Since the value of $\mu = \mu_0$ is assumed to be known, this is a constant and not a parameter. We work with the likelihood as a function of σ (rather than σ^2).

   (a) $\ell(\sigma) = \frac{n\log(2\pi)}{2} - n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu_0)^2$. So $S(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^n (X_i - \mu_0)^2 = 0$ leads to MLE $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \mu_0)^2$. [Check: $\left.\frac{dS(\sigma)}{d\sigma}\right|_{\sigma=\widehat{\sigma}} < 0$.]

   (b,c) The MLE here is unbiased since $\mathsf{E}((X-\mu_0)^2) = \sigma^2$. Observe that $\frac{n\widehat{\sigma}^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{X_i - \mu_0}{\sigma}\right)^2$ has chi-square(n) distribution with variance $2n$. Hence, $\mathsf{Var}(\widehat{\sigma}^2) = \mathsf{MSE}(\widehat{\sigma}^2) = \frac{2\sigma^4}{n}$. On the other hand, recall from the notes that $\mathsf{Var}(S^2) = \mathsf{MSE}(S^2) = \frac{2\sigma^4}{n-1}$. So the MLE here is marginally better (i.e., has smaller MSE) than the conventional unbiased estimator S^2.

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5. Exer. 6.2.8. The log-likelihood is $\ell(\beta) = n\log(\beta) + (\beta - 1)\sum_{i=1}^n \log(X_i) - \sum_{i=1}^n X_i^\beta$. To find the MLE, solve the score equation $S(\beta) = \frac{n}{\beta} + \sum_{i=1}^n \log(X_i) - \sum_{i=1}^n (\log(X_i) X_i^\beta) = 0$.

6. Exer. 6.2.19. Note: technically the *sample* is n values of categories 1, 2 or 3. Here, we have reduced that to the category counts (X_1, X_2, X_3), which can be seen to be a sufficient statistic.

   (a) The distribution is multinomial$(\theta^2, 2\theta(1-\theta), (1-\theta)^2)$.

   (b,c) $\ell(\theta) = (2X_1 + X_2)\log(\theta) + (2X_3 + X_2)\log(1-\theta) + \log(n!) - \log(X_1!X_2!X_3!)$. The score equation is $S(\theta) = \frac{2X_1+X_2}{\theta} - \frac{2X_3+X_2}{1-\theta} = 0$, which leads to the MLE $\widehat{\theta} = \frac{2X_1+X_2}{2(X_1+X_2+X_3)} = \frac{2X_1+X_2}{2n}$. Note that $S(\theta)$ is decreasing.

7. $T = \sum_{i=1}^n X_i \sim \text{gamma}(n, \lambda)$ from which we can find

$$\mathsf{E}(1/T) = \int_0^\infty \frac{1}{t}\,\frac{\lambda^n t^{n-1} \mathrm{e}^{-\lambda t}}{(n-1)!}\,\mathrm{d}t = \cdots = \frac{\lambda}{n-1}.$$

Similarly, $\mathsf{E}(1/T^2) = \frac{\lambda^2}{(n-1)(n-2)}$. (Obviously, we need $n > 2$ for this to be finite.) Therefore,

$$\begin{aligned} \mathsf{MSE}(L_a) &= (\mathsf{Bias}(L_a))^2 + \mathsf{Var}(L_a) = \Big(\frac{a\lambda}{n-1} - \lambda\Big)^2 + \frac{a^2\lambda^2}{(n-1)(n-2)} - \Big(\frac{a\lambda}{n-1}\Big)^2 \\ &= \lambda^2\Big(\frac{1}{(n-1)(n-2)}\,a^2 - \frac{2}{n-1}\,a + 1\Big), \end{aligned}$$

which is minimized with $a = n-2$, giving value $\frac{1}{n-1}$. [Which is sensible as then $L_a = \frac{a}{T}$ is approximately the MLE for very large n.]