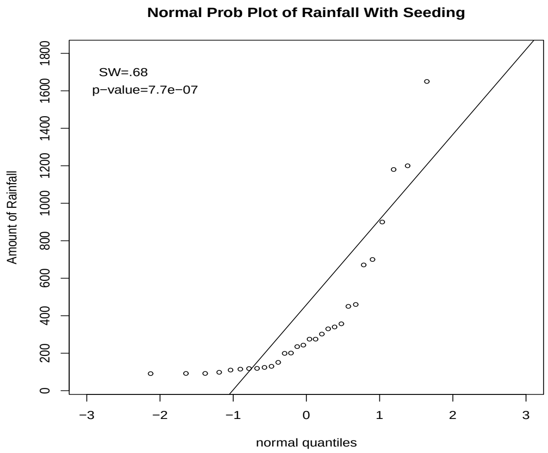
Stat 641 Fall 2021
Solutions for Homework 6

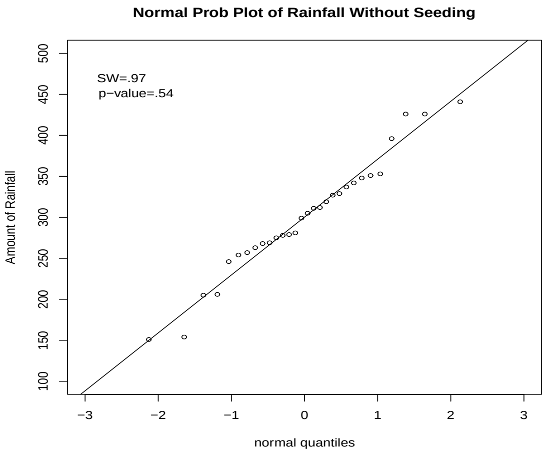
Problem P1. (16 points)

Need to determine if the two data sets come from a normal distribution and if not, can we transform data to normality:

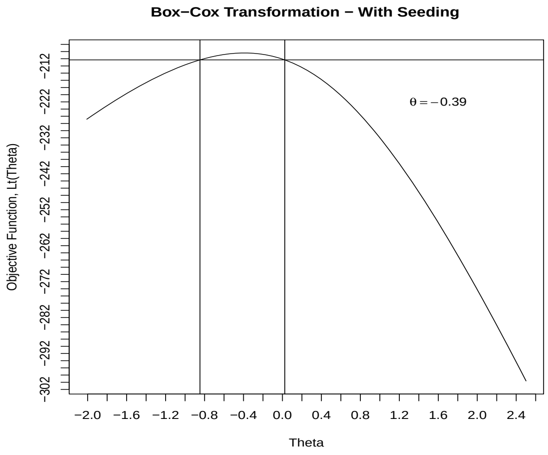
i. Clouds With Seeding: non-normal from plot, SW=.679 and p-value=0.0000008 from Shapiro-Wilk.

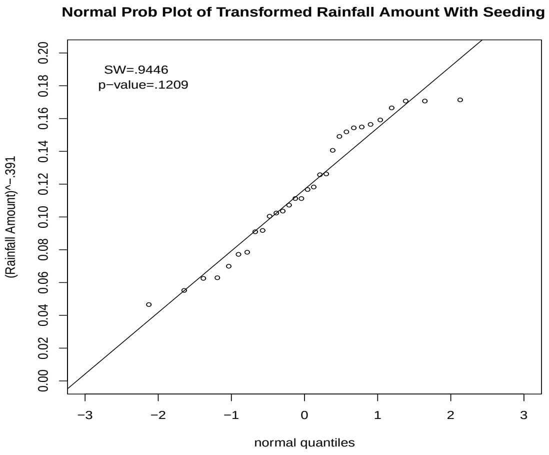
ii. Clouds Without Seeding: normal from plot, SW=.970 and p-value=0.539 from Shapiro-Wilk.





iii. Box-Cox transformation for With Seeding Data and Normal Probability Plot - Transformed With Seeding Data:





1. For a population with a normal distribution, 95/95 lower tolerance bound and 95/95 upper tolerance bound are respectively,

$$L_{0.95,0.95} = \bar{X} - K_{0.95,0.95}S, \qquad L_{0.95,0.95} = \bar{X} + K_{0.95,0.95}S,$$

where for n=30, $K_{0.95,0.95} = 2.220$ and $S = \sqrt{\frac{1}{n-1}\sum_i (X_i - \bar{X})^2}$.

i. For the With Seeding data (Y_1), using the Box-Cox transformation we have that the Shapiro-Wilk's test of $X = g(Y_1) = Y_1^{-.391}$ has a p-value = .121 along with the normal reference plot on the previous page indicates that the transformed rainfall has approximately a normal distribution.

This is a decreasing function, therefore, a 95/95 lower tolerance bound for the distribution of Y_1 is the inverse transformation of the 95/95 upper tolerance bound for the distribution of X:

$$\bar{X} + K_{0.95,0.95}S = 0.1168 + (2.220)(0.0382) = 0.202.$$

Therefore, 95/95 lower tolerance interval on the rainfall from clouds With Seeding, Y_1, is

$$(g^{-1}(0.2016) = (0.2016)^{-1/0.39}, \infty) = (60.72,\ \infty).$$

- Note the sample size for our data is too small, n=30, to use a distribution free technique when $\gamma = .95$, P=.95 and n=30 from Table.

ii. Without Seeding data (Y_2) is normally distributed. Thus, 95/95 lower tolerance bound is

$$\bar{Y_2} - K_{0.95,0.95}S = 300.2667 - (2.220)(71.3031) = 141.97.$$

Therefore, 95/95 lower tolerance interval on the rainfall from clouds Without Seeding is $(141.97,\ \infty)$.

2. i. With Seeding data is not normally distributed and transformations are not in general appropriate for generating C.I. for a population mean, the studentized bootstrap C.I. will be implemented.

the 95% CI for the mean (μ_1) rainfall from clouds With Seeding is (thL, thU) = (293.21, 817.37).

If you assumed that $n = 30$ was large enough to apply asymptotic results you will obtain the following results:

$$\bar{Y_1} \pm t_{29,0.025}\frac{S_{y_1}}{\sqrt{n}} = 458.6 \pm (2.045)\frac{552.2462}{\sqrt{30}} = (253.41,\ 665.79).$$

I would not be very confident in the validity of this CI due to the data being highly skewed with a small sample size n = 30. Note that the asymptotic CI and the studentized bootstrap CI are very different: (293.21, 817.37) versus (253.41, 665.79).

ii. Under the normality of Without Seeding data, the 95% CI for the mean (μ_2) rainfall Without Seeding is

$$\bar{Y_2} \pm t_{29,0.025}\frac{S_{y_2}}{\sqrt{n}} = 300.2667 \pm (2.045)\frac{71.3031}{\sqrt{30}} = (273.64,\ 326.89).$$

If you apply the studentized bootstrap method to the Without Seeding data, the 95% CI for the mean (μ_2) rainfall Without Seeding is (274.19, 326.90) versus the parametric CI, (273.64, 326.89), a close agreement between the two C.I.s.

3. For the median we construct a CI for the transformed data and then apply the inverse transformation to this CI to obtain the CI for the median in the original scale. This method works for any quantile because if $X = g(Y)$ then $Q_X(u) = g(Q_Y(u))$ thus if $(L_X,\ U_X)$ is a $100(1-\alpha)\%$ C.I. for $Q_X(u)$ then $(g^{-1}(L_X),\ g^{-1}(U_X)$ is a C.I. for $Q_X(u)$ provide that g is an increasing function. If g is a decreasing function then reverse the endpoints to obtain the CI on $Q_Y(u)$.

i.a From P1- part 1. of this assignment, we have shown that the With Seeding data (Y_1), $X = g(Y_1) = Y_1^{-.391}$ has approximately a normal distribution. In the normal distribution, the mean and median are the same parameter so we can use the formula for a CI on the mean of a normally distributed population for the transformed data:

$$\bar{X_1} \pm t_{29,0.025}\frac{S_{x_1}}{\sqrt{n}} = .1168 \pm (2.045)\frac{.0382}{\sqrt{30}} = (.1025375,\ 0.1310625).$$

Thus, we have that $((0.1310625)^{-1/.391},\ (.1025375)^{-1/.391}) = (180.75,\ 338.62)$ is a 95% C.I. on the median rainfall With Seeding.

i.b Alternatively, a distribution-free C.I. on the median is given by: Using our R-code or Table VII.3, we have $r = 10$. Thus a distribution-free 95% CI on the median rainfall With Seeding is

$$(Y_{(r)}, Y_{(n-r+1)}) = (Y_{(10)}, Y_{(21)}) = (130,\ 357).$$

- Note the two C.I.'s are somewhat different: (180.75, 338.62) vs (130, 357) but fairly close considering we have only n=30 data values.

ii. For the Without Seeding rainfall, the data is from normal distribution and so the mean is equal to the median. Thus, the answer should be the same as in part 2, (273.64, 326.89).

The distribution-free 95% CI on the median rainfall With Seeding is $(Y_{(r)}, Y_{(n-r+1)}) = (Y_{(10)}, Y_{(21)}) = (269,\ 329)$

Note the C.I. for the median, (269, 329), is in close agreement with the C.I. for the mean, (273.64, 326.89) for the Without Seeding data. We would expect this to be true because the Without Seeding data has a normal distribution and hence the mean and median are the same parameter.

However, this is not true for the With Seeding data, (293.21, 817.37) for the mean and (130, 357) for the median. This is due to the heavy skewness in the With Seeding data.

4. It would appear that Seeding produces a few very large rainfalls (1180, 1200, 1650, 2550) compared to Unseeded (all values less than 500) however the median rainfalls are about the same for Seeding and Unseeded. Furthermore, with the exception of a few very large rainfalls, the distribution of the Seeded clouds tended to produce smaller rainfalls than the Unseeded clouds.

Problem P2. (10 points) Failure Stress of impregnated carbon fibers:

1. First analysis without specifying the distribution of the stress to failure values using the Kaplan-Meier Estimator.

   i. Using the provided R code, the estimate of the average stress to failure for the carbon fibers from the R code is 2.8311 with a standard error of 0.0249. We could obtain a 95% CI for μ using the expression

   $$\hat{\mu} \pm z_{\alpha/2}\widehat{SE}(\hat{\mu}) = 2.8311 \pm 1.96(.0249) = (2.78, 2.88)$$

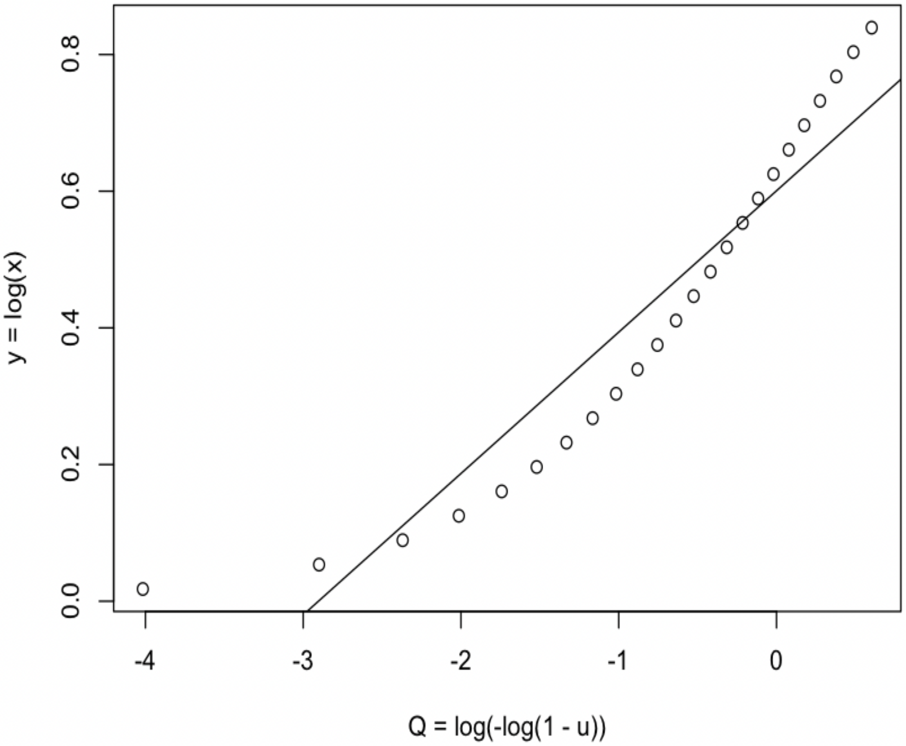
   The problem with the CI is that it is an asymptotic results and with $n = 28$ an asymptotic result may be questionable.

   ii. The provided R code yields (thL, thU) = (2.784, 2.880) as a 95% for the mean which is nearly identical to the asymptotic result. This is very surprising in that the bootstrap just averaged the censored values as if their actual value was 3.

2. The Weibull Reference plot will be modified to take into account the fact that 4 of the 28 data values are Type I censored.

   For the probability plot, make the transformation $Y_i = ln(W_i)$ and then plot $\big(Q(u_i), Y_{(i)}\big)$ for the 24 uncensored values using $u_i = \frac{i-.5}{28}$ $i = 1, \ldots, 24$ and the quantile function for the standard extreme value distribution: $Q(u_i) = log(-log(1-u_i))$. Note that we only plotted the 24 uncensored values but used n=28 in computing u_i.

   From the plot, it would appear that the Weibull distribution provides a fair fit to the data.



Also, using the modified Anderson-Darling statistic on just the uncensored values yields a value of AD = 0.1508 with a p-value between 0.474 and 0.637. This, together with the Reference Distribution plot suggests that there is a reasonable fit of the Weibull model to the data set.

i. **Exact C.I.** To obtain a confidence interval explicitly for the mean from a Weibull distribution is very involved due to the complex relationship between the mean μ and the parameters α and γ:

   $\mu = \alpha\Gamma\left(1+\frac{1}{\gamma}\right)$ and variance $\sigma^2 = \alpha^2\Gamma\left(1+\frac{2}{\gamma}\right) - \left(\alpha\Gamma\left(1+\frac{1}{\gamma}\right)\right)^2$.

   The parametric bootstrap 95% C.I. for the mean would be (2.729, 2.884). This C.I. has endpoints nearly identical to the Kaplan-Meier C.I.: (2.782, 2.880).

Problem P3. (9 points)

1. The expected count for the Group $0-10$ is less than 1 which invalidates the Chi-square approximation. Need to combine groups $0-10$ and $11-15$ to obtain $E_1 = .12 + 2.43 = 2.55$; $O_1 = 2+1 = 3$; $\frac{(O_1-E_1)^2}{E_1} = \frac{(3-2.55)^2}{2.55} = .08 \quad \Rightarrow \quad \chi^2 = 35.39 - 30.25 - .85 + .08 = 4.37$ with $df = 6-1 = 5$. Thus, $p-value = Pr[\chi^2_5 \geq 4.37] \approx pchisq(4.37, 5) = 0.497 \quad \Rightarrow$ Poisson model provides an excellent fit to the data.

2. $\hat{\lambda} = \bar{Y} = 27.7$

   $Pr\left[\frac{\sqrt{n}(\bar{Y}-\lambda)}{\sqrt{\lambda}} \leq Z_{\alpha/2}\right] \approx 1-\alpha$. Thus, we have

   $\sqrt{n}|\bar{Y}-\lambda| \leq \sqrt{\lambda}Z_{\alpha/2} \quad \Rightarrow \lambda = \frac{2\bar{Y}+\frac{1}{n}Z^2_{\alpha/2} \pm \sqrt{(2\bar{Y}+\frac{1}{n}Z^2_{\alpha/2})^2 - 4\bar{Y}^2}}{2} \quad \Rightarrow$

   An approximate $100(1-\alpha)$% C.I. for λ is

   $\bar{Y} + \frac{1}{2n}Z^2_{\alpha/2} \pm Z_{\alpha/2}\sqrt{\frac{1}{n}\bar{Y} + \frac{1}{4n^2}Z^2_{\alpha/2}} = 27.7 + \frac{1}{2*200}1.96^2 \pm 1.96\sqrt{\frac{1}{200}27.7 + \frac{1}{4*200^2}1.96^2} = (26.98, 28.44)$

A less accurate approximate $100(1-\alpha)\%$ C.I. for λ is the basic Wald C.I. with $\hat{\lambda} = \bar{Y}$:

$\bar{Y} \pm Z_{\alpha/2}\sqrt{\bar{Y}/n} = 27.7 \pm 1.96\sqrt{27.7/200} = (26.97,\ 28.43)$

Problem P4. (15 points)

1. Let Y be the lifetime of epoxy strands and p be the probability that an epoxy strand will survive for 300 hours. Therefore, the parameter to be estimated is $p = P(Y \geq 300)$.

   Because $\min\{n\widehat{p}, n(1-\widehat{p})\} \geq 5$ and $n > 40$, we can use the Agresti-Coull CI for p:

   $\widehat{p} = 22/100 = 0.22$ and so $\tilde{X} = 22 + (2.58^2)/2 = 25.3282$, and $\tilde{n} = 100 + 2.58^2 = 106.6564$.

   Thus, $\tilde{p} = \tilde{X}/\tilde{n} = 0.2375$ and the 99% Agresti-Coull CI on p is

$$0.2375 \pm (2.58)\sqrt{\frac{(0.2375)(0.7625)}{106.6564}} = (0.1312, 0.3438).$$

- Alternatively, from the Kaplan-Meier Estimator:

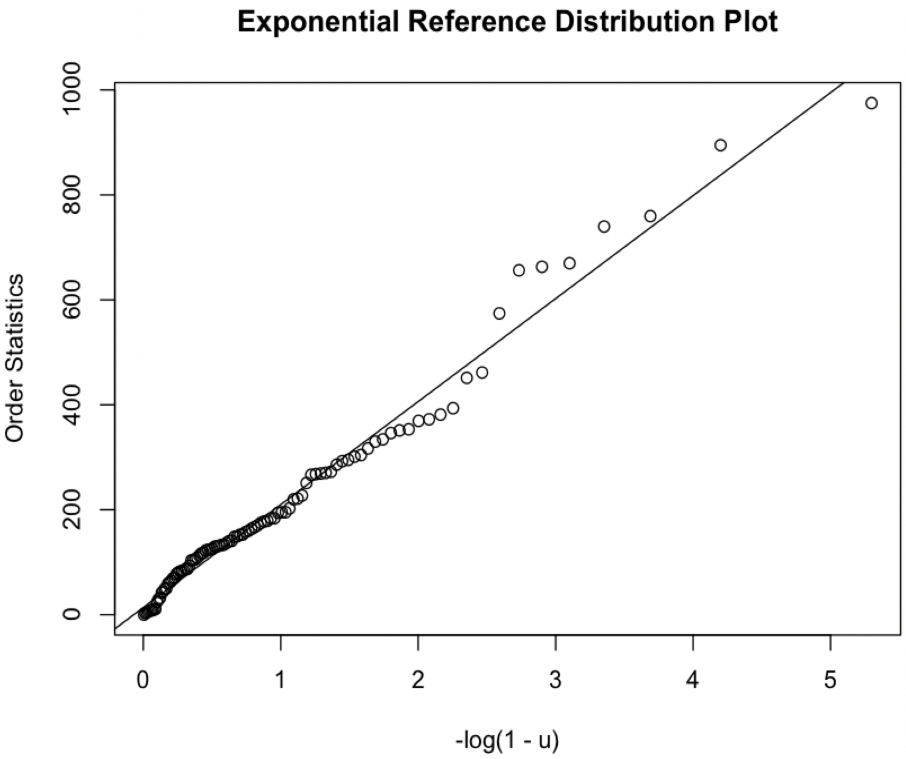
```
L = c(data)
n =  length(L)
wc = c(rep(1,100))
cordsurv = survfit(Surv(L,wc)~1,conf.type="log-log")
summary(cordsurv)
print(cordsurv,print.rmean=TRUE)
Call: survfit(formula = Surv(L, wc) ~ 1, conf.type = "log-log")

   time n.risk n.event survival std.err lower 95% CI upper 95% CI
  295.10     23       1     0.22 0.04142     0.144852       0.3053
  301.10     22       1     0.21 0.04073     0.136546       0.2943
```

Using linear interpolation, $\hat{p} = .212$ with 95% C.I. (.138, .296)

2. From the following quantile plot, the exponential model appears to fit

   AD=2.115 which yields $.005 < p - value < .010$. This indicates a poor fit of an exponential model with $\hat{\beta} = \bar{T} = 209.1838$.



i. Using the results for an exponential distribution, a 95/99 lower tolerance bound is

$$L_{0.95,0.99} == -\hat{\beta}\left[\frac{2n}{\chi^2_{1-.99}}\right] log(.95) = -(209.1838)\left[\frac{200}{156.432}\right](\log 0.95) = 13.718.$$

Therefore, a 95/99 lower tolerance interval on the average lifetime of the epoxy strands is (13.718, ∞).

ii. A distribution-free 95/99 lower tolerance interval for n=100 has m=1 from Table on page 51 in Handout 11.

Thus, the distribution-free lower tolerance interval would be $(Y_{(1)},\ \infty) = (.18,\ \infty)$.

With n=100, the distribution-free interval is not very informative.

3. Using the methodology for an exponential distribution, a 95% PI for Y_{101} is given by

$$(\bar{Y}F_{0.025,2,200},\ \bar{Y}F_{0.975,2,200}) = ((209.1838)(0.0253),\ (209.1838)(3.758)) = (5.3,\ 786.1).$$

Problem P5. (20 points) Strength of Braided Cord:

1. The parameter of interest is $p = P[S < 50]$. From the data, $\hat{p} = \frac{51}{56} = .91$, $n \cdot min(\hat{p}, 1-\hat{p}) = 5.1 > 5$, and $n = 56 > 40 \quad \Rightarrow$ Use Agresti-Coull C.I.

$\tilde{X} = 51 + (1.96^2)/2 = 25.3282$, and $\tilde{n} = 56 + 1.96^2 = 59.8416$.

Thus, $\tilde{p} = \tilde{X}/\tilde{n} = 0.8843$ and the 95% Agresti-Coull CI on p is

$$0.8843 \pm (1.96)\sqrt{\frac{(0.8843)(1-.8843)}{59.8416}} = (0.803,\ 0.965).$$

Alternatively, the Clopper-Pearson C.I. would be obtained from y=51, n=56,

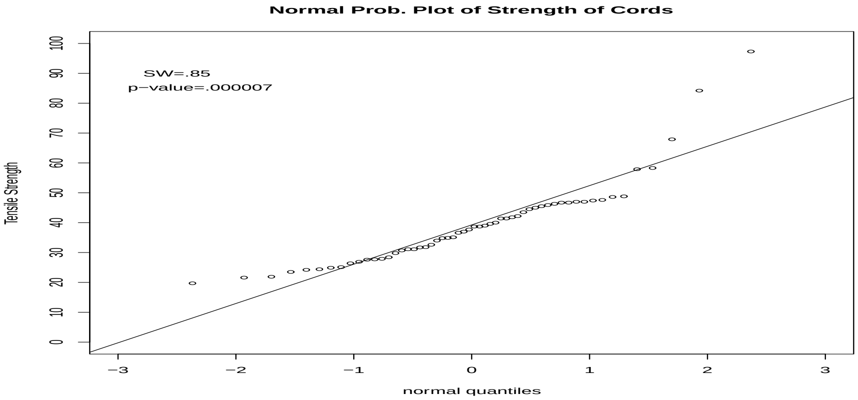
$$P_L = \frac{1}{1+\left(\frac{6}{51}\right)F_{12,102,.025}} = \frac{1}{1+\left(\frac{6}{51}\right)(2.074684)} = .804$$

$$P_U = \frac{\left(\frac{52}{5}\right)F_{104,10,.025}}{1+\left(\frac{52}{5}\right)F_{104,10,.025}} = \frac{\left(\frac{52}{5}\right)(3.149015)}{1+\left(\frac{52}{5}\right)(3.149015)} = .970$$

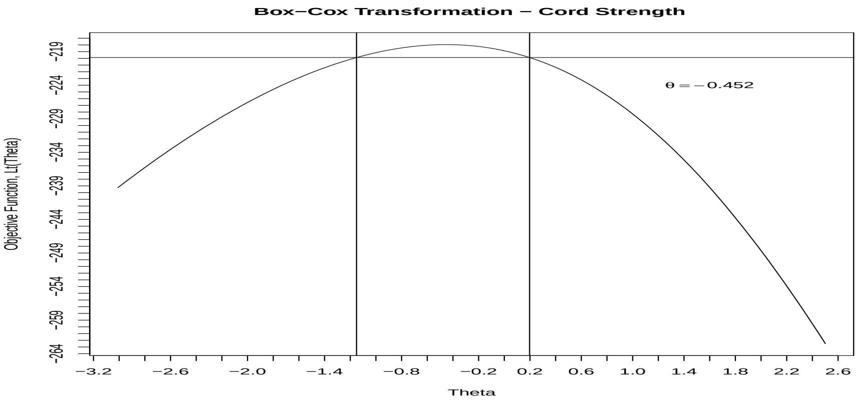
Clopper-Pearson C.I is (.804, .970)

- The two C.I.'s are in close agreement: (0.803, 0.965) vs (.804, .970) which agree to the second decimal place.
- However, the Wald C.I. is $0.911 \pm 1.96\sqrt{(.911)(1-.991)/56} = (.836,\ .985)$ which is considerable different from the other two C.I.s.

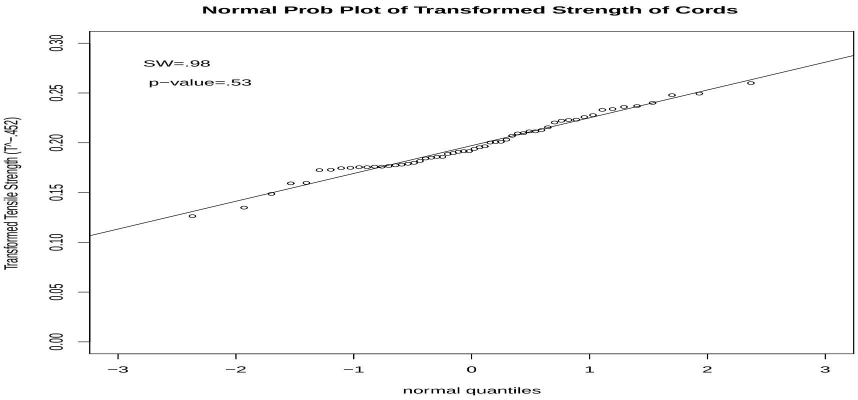
2. The normal probability plot and a value for the Shapiro-Wilk's test of W = .853 which yields a p-value of 6.915e-06 indicate that a normal distribution would not be a good model for data.



As was done in P1-3, a Box-Cox transformation will be obtained and a CI will be placed on the median of the transformed which will be inverted to obtain a CI for the median for the cord strength.



From the Box-Cox methodology, the transformation $X = Y^{-.452}$ yields a p-value of .52 from the Shapiro-Wilk test which indicates an excellent fit of a normal distribution to the data. The following normal reference distribution plot confirms the fit:



In the normal distribution, the mean and median are the same parameter so we can use the formula for a CI on the mean of a normally distributed population for the transformed data:

$$\bar{X} \pm t_{55,0.025} \frac{S_x}{\sqrt{n}} = .1971 \pm (1.673) \frac{.02813}{\sqrt{56}} = (0.1908111,\ 0.2033889).$$

Thus, we have that $((0.2033889)^{-1/.452},\ (0.1908111)^{-1/.452}) = (33.90,\ 39.05)$ is a 95% C.I. on the median of the cord strengths.

- Alternatively, a nonparametric C.I. will be constructed.

Using Table VII.3 in Handout 11 or the R code on page 31 of Handout 11, we obtain k=21, therefore a 95% C.I. on Median is

$[Y_{(21)},\ Y_{(56-21+1)}] = [Y_{(21)},\ Y_{(36)}] = (32.6,\ 41.8)$ with a coverage of 95.6%.

The two CI's are in somewhat agreement: (33.90, 39.05) vs (32.6, 41.8)

3. Using the transformation: $X = Y^{-.452}$, the Tolerance Interval will be obtained using the normal based procedures:

   A (.90, .95) T.I. for the distribution of X is given by

   $$\bar{X} \pm K_{0.90,0.95} S_X = 0.1971307 - (1.98)(0.02813393) = (0.1414255,\ 0.2528359)$$

   Inverting the endpoints of the T.I. for the X distribution yields the following T.I. for the strength distribution:

   $$((0.2528359)^{-1/.452},\ (0.1414255)^{-1/.452}) = (20.95,\ 75.75)$$

- Alternatively, a Nonparametric T.I. is obtained as follows. From the Table in Handout 11 for Nonparametric Tolerance Interval, we obtain for n=56 that m=2 which implies that

  the $(P, \gamma) = (.9, .95)$ tolerance interval is $[Y_{(1)}, Y_{(56)}] = [19.7,\ 97.3]$

- Note that the distribution-free T.I. is considerable wider than the parametric T.I.:

  [19.7, 97.3] versus (20.95, 75.75). With n=56, the distribution-free T.I. is not very informative in that it uses the minimum and maximum values in the data set to provide the specified coverage.

Problem P6. (3 points each) Multiple Choice:

1. **- D** The transformation to normality yields an excellent fit therefore just invert the endpoints to obtain the requested Tolerance Interval.

2. **- D** The bootstrap is often used when there is not an appropriate parametric procedure.

3. **- C** $n = \hat{\sigma}^2\ Z^2_{\alpha/2}/\Delta^2 = \left((30)(2.576)/10\right)^2 = 59.7$. Therefore, take n=60.

4. **- B**

5. **- B** See page 60 in Handout 11

6. **- C** If the data are highly correlated, then S^2/n underestimates $Var(\bar{X})$. This results in a C.I. for μ which is too narrow and hence has a smaller coverage probability than the nominal level of confidence.

7. **- C or D** See page 53 in Handout 11.

8. **- D or E** If the population distribution is highly right skewed, the sampling distribution of $(n-1)S^2/\sigma^2$ does not have a chi-squared distribution. In fact, the distribution will be more right skewed than the chi-squared distribution. This will result in a C.I. having lower coverage probability than the stated level of confidence. The asymptotic results would not hold for n=10 observations. There are no distribution-free methods for construction a C.I. for σ. Transformations are generally not advisable for constructing C.I.'s for μ or σ. The bootstrap C.I. for σ would not be very useful based on only n=10 observations from a population with a highly skewed distribution. Probably the best option is to tell the researcher that the sample size needs to be increased.

```
####
#### (1)
####

x_y <- c(151, 450, 124, 235, 357, 110, 302, 671, 118, 115, 275, 275, 2550, 243, 201, 199,
  130, 119, 92, 91, 92, 98, 1650, 1200, 1180, 900, 700, 460, 340, 330)
x_n <- c(246, 268, 275, 348, 305, 311, 206, 279, 426, 269, 257, 299, 337, 329, 319, 312,
  327, 342, 351, 205, 151, 426, 154, 353, 396, 441, 254, 263, 278, 281)
n_y <- length(x_y)
n_n <- length(x_n)

##
## Are data normally distributed?
##

## Normal quantile plot and Shapiro-Wilks test for x_y
x_y <- sort(x_y)
u <- (1:n_y - 0.5) / n_y
z <- qnorm(u)

plot(z, x_y, main = "Normal Prob Plot of Rainfall With Seeding",
  xlab = "Normal Quantiles", ylab = "Amount of Rainfall", ylim = c(50, 1800),
  xlim = c(-3, 3))
abline(lm(x_y ~ z))

shapiro.test(x_y)
text(-2.5, 1700, "SW = 0.68")
text(-2.2, 1600, "p-value = 7.7e-07")

## Normal quantile plot and Shapiro-Wilks test for x_n
x_n <- sort(x_n)
u <- (1:n_n - 0.5) / n_n
z <- qnorm(u)

plot(z, x_n, main = "Normal Prob Plot of Rainfall Without Seeding",
  xlab = "Normal Quantiles", ylab = "Amount of Rainfall", ylim = c(100, 500),
  xlim = c(-3, 3))
abline(lm(x_n ~ z))

shapiro.test(x_n)
text(-2.5, 470, "SW = 0.97")
text(-2.3, 450, "p-value = 0.54")

##
## Box-Cox transformation for With Seeding data
##

l <- 0
theta_seq <- seq(-3, 3, by = 0.001)

y_y <- log(x_y)
s_0 <- sum(y_y)
v_0 <- var(y_y)
for(i in 1:length(theta_seq)) {
  if(abs(theta_seq[i]) < 1e-10) {
    l[i] <- -s_0 - (n_y / 2) * (log(2 * pi * v_0) + 1)
    theta_seq[i] <- 0
  } else {
    x_theta <- (x_y ^ theta_seq[i] - 1) / theta_seq[i]
    v_1 <- var(x_theta)
    l[i] <- (theta_seq[i] - 1) * s_0 - (n_y / 2) * (log(2 * pi * v_1) + 1)
  }
}

i_max <- which.max(l)
theta_seq[i_max]

plot(theta_seq, l, xlab = "theta", ylab = "L(theta)", type = "l")
abline(v = theta_seq[i_max])

## 95% CI for theta
which_ci <- (1:length(theta_seq))[l[i_max] - l <= 0.5 * qchisq(0.95, 1)]
theta_ci <- theta_seq[c(min(which_ci), max(which_ci))]
abline(v = theta_ci[1])
abline(v = theta_ci[2])

## Normal quantile plot of transformed With Additive data
tx_y <- x_y ^ -0.391
tx_y <- sort(tx_y)
u <- (1:n_y - 0.5) / n_y
z <- qnorm(u)

plot(z, tx_y, main = "Normal Prob Plot of Transformed Rainfall With Seeding",
  xlab = "Normal Quantiles", ylab = "(Rainfall Amount)^-0.391", ylim = c(0, 0.2),
  xlim = c(-3, 3))
abline(lm(tx_y ~ z))

shapiro.test(tx_y)
text(-2.3, 0.19, "SW = 0.9446")
text(-2.2, 0.18, "p-value = 0.1209")

##
## (1)
##

K <- 2.220

## For seeded data, create upper bound for transformed data, then back-transform to
## original scale to get lower bound.
U_y <- mean(tx_y) + K * sd(tx_y)
L_y <- U_y ^ (-1 / 0.391)

## Unseeded data (already Normally distributed).
L_n <- mean(x_n) - K * sd(x_n)

##
## (2)
##
```

```
## Studentized bootstrap for With Additive data
th_est_D <- mean(x_y)
sd_est_D <- sd(x_y)

a <- (1 - 0.95) / 2
B <- 9999

th_est_B <- sd_est_B <- Z_B <- numeric(B)
for(b in 1:B) {
  x_B <- sample(x_y, replace = TRUE)
  th_est_B[b] <- mean(x_B)
  sd_est_B[b] <- sd(x_B)
  Z_B[b] <- sqrt(n_y) * (th_est_B[b] - th_est_D) / sd_est_B[b]
}

Z_B <- sort(Z_B)
L_Z <- Z_B[(B + 1) * a]
U_Z <- Z_B[(B + 1) * (1 - a)]
th_L <- th_est_D - U_Z * sd_est_D / sqrt(n_y)
th_U <- th_est_D - L_Z * sd_est_D / sqrt(n_y)

th_L
th_U

## Without Additive data already Normally distributed, so just use t-based interval
mean(x_n) + c(-1, 1) * qt(0.975, n_n - 1) * sd(x_n) / sqrt(n_n)

## For illustration purposes, here is the studentized bootstrap CI
th_est_D <- mean(x_n)
sd_est_D <- sd(x_n)

a <- (1 - 0.95) / 2
B <- 9999

th_est_B <- sd_est_B <- Z_B <- numeric(B)
for(b in 1:B) {
  x_B <- sample(x_n, replace = TRUE)
  th_est_B[b] <- mean(x_B)
  sd_est_B[b] <- sd(x_B)
  Z_B[b] <- sqrt(n_n) * (th_est_B[b] - th_est_D) / sd_est_B[b]
}

Z_B <- sort(Z_B)
L_Z <- Z_B[(B + 1) * a]
U_Z <- Z_B[(B + 1) * (1 - a)]
th_L <- th_est_D - U_Z * sd_est_D / sqrt(n_n)
th_U <- th_est_D - L_Z * sd_est_D / sqrt(n_n)

th_L
th_U

##
## (3)
```

```
##

## Apply t-based interval to transformed With Additive data to get an interval on its
## median, then back-transform.
med_interval_ty <- mean(tx_y) + c(-1, 1) * qt(0.975, n_y - 1) * sd(tx_y) / sqrt(n_y)
med_interval_y <- c(med_interval_ty[2] ˆ (-1 / 0.391), med_interval_ty[1] ˆ (-1 / 0.391))

## Again, t-based interval for Without Additive data
mean(x_n) + c(-1, 1) * qt(0.975, n_n - 1) * sd(x_n) / sqrt(n_n)

## Distribution-free CIs
L <- 0.95
P <- 0.5

f_df_CI <- function(n, L, P) {
  s <- ceiling(n * P) - 1
  r <- floor(n * P) + 1
  cov <- 0

  while(s < n - 1 && r > 1 && cov < L) {
    s <- s + 1
    cov <- pbinom(s - 1, n, P) - pbinom(r - 1, n, P)
    if(cov >= L)
      break;
    r <- r - 1
    cov <- pbinom(s - 1, n, P) - pbinom(r - 1, n, P)
  }

  return(list("r" = r, "s" = s, "cov" = cov))
}

f_df_CI(n_y, L, P)
f_df_CI(n_n, L, P)

####
#### (2)
####

library(MASS)
library(survival)

x <- c(2.526, 2.546, 2.628, 2.669, 2.869, 2.710, 2.731, 2.751, 2.771, 2.772, 2.782,
  2.789, 2.793, 2.834, 2.844, 2.854, 2.875, 2.876, 2.895, 2.916, 2.919, 2.957, 2.977,
  2.988, 3, 3, 3, 3)
n <- length(x)
delta <- c(rep(1, n - 4), rep(0, 4))

##
## (1)
##

## Asymptotic CI based on survfit
surv_fit <- survfit(Surv(x, delta) ˜ 1, conf.type = "log-log")
summary(surv_fit)
```

```
print(surv_fit, print.rmean = TRUE)

2.8311 + c(-1, 1) * 1.96 * 0.0249

## Studentized bootstrap
theta_D <- mean(x)
SD_D <- sd(x)
alpha <- 0.025
B <- 9999

Z_B <- theta_B <- SD_B <- numeric(B)
for(b in 1:B) {
  x_B <- sample(x, replace = TRUE)
  theta_B[b] <- mean(x_B)
  SD_B[b] <- sd(x_B)
  Z_B[b] <- sqrt(n) * (theta_B[b] - theta_D) / SD_B[b]
}
Z_B <- sort(Z_B)
Z_L <- Z_B[(B + 1) * alpha]
Z_U <- Z_B[(B + 1) * (1 - alpha)]
CI_L <- theta_D - SD_D * Z_U / sqrt(n)
CI_U <- theta_D - SD_D * Z_L / sqrt(n)

##
## (2)
##

## Weibull reference distribution plot, only using the uncensored values
y <- log(x)[1:24]
y <- sort(y)
u <- (1:24 - 0.5) / n
Q_u <- log(-log(1 - u))
plot(Q_u, u, xlab = "Q = log(-log(1 - u))", ylab = "y = log(x)",
  main = "Weibull Reference Plot - Stress to Failure")
abline(lm(u ~ Q_u))

## Anderson-Darling GOF test, only using the uncensored values
weib_fit <- fitdistr(x[1:24], "weibull")
U <- pweibull(sort(x[1:24]), shape = weib_fit$est[1], scale = weib_fit$est[2])
AD <- -24 - (1 / 24) * sum((2 * (1:24) - 1) * log(U)) -
  (1 / 24) * sum((2 * 24 + 1 - 2 * (1:24)) * log(1 - U))
AD_adj <- AD * (1 + 0.2 / sqrt(24))

## Parametric bootstrap. We will use the censored values here and survreg to estimate the
## Weibull parameters.
surv_reg <- survreg(Surv(x, delta) ~ 1, dist = "weibull")
summary(surv_reg)
names(surv_reg)
lambda_D <- exp(surv_reg$coef)
k_D <- 1 / surv_reg$scale

mean_D <- lambda_D * gamma(1 + 1 / k_D)
var_D <- lambda_D ^ 2 * (gamma(1 + 2 / k_D) - (gamma(1 + 1 / k_D)) ^ 2)
```

```
B <- 9999
k_B <- lambda_B <- mean_B <- var_B <- Z_B <- numeric(B)
for(b in 1:B) {
  x_B <- rweibull(n, shape = k_D, scale = lambda_D)

  ## Create censoring
  x_B[x_B >= 3] <- 3
  delta_B <- rep(1, n)
  delta_B[x_B == 3] <- 0

  surv_reg_B <- survreg(Surv(x_B, delta_B) ~ 1, dist = "weibull")
  lambda_B[b] <- exp(surv_reg_B$coef)
  k_B[b] <- 1 / surv_reg_B$scale

  mean_B[b] <- lambda_B[b] * gamma(1 + 1 / k_B[b])
  var_B[b] <- lambda_B[b] ^ 2 * (gamma(1 + 2 / k_B[b]) - (gamma(1 + 1 / k_B[b])) ^ 2)
  Z_B[b] <- sqrt(n) * (mean_B[b] - mean_D) / sqrt(var_B[b])
}
Z_B <- sort(Z_B)
Z_L <- Z_B[250]
Z_U <- Z_B[9750]
mean_L <- mean_D - sqrt(var_D) * Z_U / sqrt(n)
mean_U <- mean_D - sqrt(var_D) * Z_L / sqrt(n)

####
#### (3)
####

n <- 200
lambda_hat <- 27.7

CI <- lambda_hat + (1 / (2 * n)) * 1.96 ^ 2 + c(-1, 1) * 1.96 *
  sqrt((1 / n) * lambda_hat + (1 / (4 * n ^ 2)) * 1.96 ^ 2)

####
#### (4)
####

x <- c(.18, 3.1, 4.2, 6.0, 7.5, 8.2, 8.5, 10.3, 10.6, 24.2, 29.6, 31.7, 41.9, 44.1, 49.5,
  50.1, 59.7, 61.7, 64.4, 69.7, 70.0, 77.8, 80.5, 82.3, 83.5, 84.2, 87.1, 87.3, 93.2,
  103.4, 104.6, 105.5, 108.8, 112.6, 116.8, 118.0, 122.3, 123.5, 124.4, 125.4, 129.5,
  130.4, 131.6, 132.8, 133.8, 137.0, 140.2, 140.9, 148.5, 149.2, 152.2, 152.9, 157.7,
  160.0, 163.6, 166.9, 170.5, 174.9, 177.7, 179.2, 183.6, 183.8, 194.3, 195.1, 195.3,
  202.6, 220.0, 221.3, 227.2, 251.0, 266.5, 267.9, 269.2, 270.4, 272.5, 285.9, 292.6,
  295.1, 301.1, 304.3, 316.8, 329.8, 334.1, 346.2, 351.2, 353.3, 369.3, 372.3, 381.3,
  393.5, 451.3, 461.5, 574.2, 656.3, 663.0, 669.8, 739.7, 759.6, 894.7, 974.9)
n <- length(x)

X <- sum(x >= 300)
p_hat <- X / n

##
## (1)
##
```

```
## Agresti-Coull CI
X_tilde <- X + 0.5 * qnorm(0.995) ˆ 2
n_tilde <- n + qnorm(0.995) ˆ 2
p_tilde <- X_tilde / n_tilde

CI <- p_tilde + c(-1, 1) * qnorm(0.995) * sqrt(p_tilde * (1 - p_tilde) / n_tilde)

## Kaplan-Meier CI. Linear interpolation of CIs for survival times 295.1 and 301.1.
delta <- rep(1, n)
surv_fit <- survfit(Surv(x, delta) ˜ 1, conf.type = "log-log")
summary(surv_fit)

plot(c(0.1449, 0.1365), c(0.3053, 0.2943))
lines(c(0.1449, 0.1365), c(0.3053, 0.2943))

w <- (300 - 295.1) / (301.1 - 295.1)
LO <- 0.1449 * (1 - w) + 0.1365 * w
HI <- 0.3053 * (1 - w) + 0.2943 * w
points(LO, HI, pch = 20)

##
## (2)
##

## Exponential distribution reference plot
u <- (1:n - 0.5) / n
Q <- -log(1 - u)
x_sort <- sort(x)

plot(Q, x_sort, xlab = "-log(1 - u)", ylab = "Order Statistics",
  main = "Exponential Reference Distribution Plot")
abline(lm(x_sort ˜ Q))

## Anderson-Darling GOF statistic
beta_hat <- mean(x)
U <- pexp(x_sort, 1 / beta_hat)

AD <- -n - (1 / n) * sum((2 * (1:n)) * log(U)) -
  (1 / n) * sum((2 * n + 1 - 2 * (1:n)) * log(1 - U))
AD_adj <- AD * (1 + 0.6 / n)

## Lower tolerance bound assuming exponential distribution
K <- 2 * n / qchisq(0.01, 2 * n)
W <- -beta_hat * K * log(0.95)

##
## (3)
##

LO <- beta_hat * qf(0.025, 2, 2 * n)
HI <- beta_hat * qf(0.975, 2, 2 * n)

####
```

```
#### (5)
####

x <- c(19.7, 21.6, 21.9, 23.5, 24.2, 24.4, 24.9, 25.1, 26.4, 26.9, 27.6, 27.7, 27.9,
  28.4, 29.8, 30.7, 31.1, 31.1, 31.7, 31.8, 32.6, 34.0, 34.8, 34.9, 35.1, 36.6, 37.0,
  37.7, 38.7, 38.7, 39.0, 39.6, 40.0, 41.4, 41.4, 41.8, 42.2, 43.5, 44.5, 45.0, 45.5,
  45.9, 46.3, 46.7, 46.7, 47.0, 47.0, 47.4, 47.6, 48.6, 48.8, 57.9, 58.3, 67.9, 84.2,
  97.3)
n <- length(x)

##
## (1)
##

X <- sum(x < 50)
p_hat <- X / n

## Agresti-Coull CI
X_tilde <- X + 0.5 * qnorm(0.975) ^ 2
n_tilde <- n + qnorm(0.975) ^ 2
p_tilde <- X_tilde / n_tilde

CI <- p_tilde + c(-1, 1) * qnorm(0.975) * sqrt(p_tilde * (1 - p_tilde) / n_tilde)

##
## (2)
##

## If the data are Normally distributed, can just compute a CI for the mean, since it is
## the same parameter as the median in that case.
x_sort <- sort(x)
u <- (1:n - 0.5) / n
Q <- qnorm(u)

plot(Q, x_sort, xlab = "Normal Quantiles", ylab = "Sample Quantiles",
  main = "Normal Reference Distribution Plot\nRaw Data")
abline(lm(x_sort ~ Q))

shapiro.test(x)

## Do Box-Cox transformation
l <- 0
theta_seq <- seq(-3, 3, by = 0.001)

y <- log(x)
s_0 <- sum(y)
v_0 <- var(y)
for(i in 1:length(theta_seq)) {
  if(abs(theta_seq[i]) < 1e-10) {
    l[i] <- -s_0 - (n / 2) * (log(2 * pi * v_0) + 1)
    theta_seq[i] <- 0
  } else {
    x_theta <- (x ^ theta_seq[i] - 1) / theta_seq[i]
    v_1 <- var(x_theta)
    l[i] <- (theta_seq[i] - 1) * s_0 - (n / 2) * (log(2 * pi * v_1) + 1)
  }
}

i_max <- which.max(l)
theta_seq[i_max]

plot(theta_seq, l, xlab = "theta", ylab = "L(theta)", type = "l")
abline(v = theta_seq[i_max])

## 95% CI for theta
which_ci <- (1:length(theta_seq))[l[i_max] - l <= 0.5 * qchisq(0.95, 1)]
theta_ci <- theta_seq[c(min(which_ci), max(which_ci))]
abline(v = theta_ci[1])
abline(v = theta_ci[2])

## Transformed data
y <- x ^ (-0.452)
shapiro.test(y)

plot(Q, sort(y), xlab = "Normal Quantiles", ylab = "Sample Quantiles",
  main = "Normal Reference Distribution Plot\nTransformed Data")
abline(lm(sort(y) ~ Q))

## Normal-based CI on mean / median
CI_y <- mean(y) + c(-1, 1) * qt(0.975, n - 1) * sd(y) / sqrt(n)
CI_x <- c(CI_y[2] ^ (-1 / 0.452), CI_y[1] ^ (-1 / 0.452))

## Nonparametric CI
L <- 0.95
P <- 0.50
s <- ceiling(n * P) - 1
r <- floor(n * P) + 1
cov <- 0

while(s < n - 1 && r > 1 && cov < L) {
  s <- s + 1
  cov <- pbinom(s - 1, n, P) - pbinom(r - 1, n, P)
  if(cov >= L)
    break;
  r <- r - 1
  cov <- pbinom(s - 1, n, P) - pbinom(r - 1, n, P)
}

r
s
x_sort[c(r, s)]
cov

##
## (3)
##

## Tolerance interval on Normal transformed data
```

```
mu_hat <- mean(y)
sd_hat <- sd(y)

## Linear interpolation of constant K
w <- (56 - 50) / (60 - 50)
K <- 1.999 * (1 - w) + 1.960 * w

CI_y <- mu_hat + c(-1, 1) * K * sd_hat

## Back-transform to original units
CI_x <- c(CI_y[2] ^ (-1 / 0.452), CI_y[1] ^ (-1 / 0.452))
```