Expected Mean Square Rules when $n_{ijk} \equiv r$

The following rules for determining the expected mean squares apply to equally replicated designs under the set the last term equal to 0 restrictions for all fixed effects terms in the model. The rules will be illustrated for an experiment with

factors F_1 and F_2 having a and b fixed levels, respectively, and factor F_3 having c random levels.

1. Write out the linear model for the experiment:

   MODEL: $y_{ijkl} = \mu + \tau_i + \gamma_j + c_k + (\tau\gamma)_{ij} + (\tau c)_{ik} + (\gamma c)_{jk} + (\tau\gamma c)_{ijk} + e_{l(ijk)}$

   Note that the replication source of variation is nested within the ijk treatment combination.

2. Construct a two-way table with

   a. The first column containing an entry for each source of variation in the model, excluding μ, include the number of levels of each factor, and whether the factor is Fixed "F" or Random "R"

   b. A column for each random variance component, σ or fixed variance component Q

   c. Above each column write "R" if the component is a variance or "F" if the component is a fixed levels treatment difference

Factor F_1-Fixed, Factor F_2-Fixed, Factor F_3-Random

		F	F	R	F	R	R	R	R
SV	Levels	Q_{F_1}	Q_{F_2}	$\sigma^2_{F_3}$	$Q_{F_1*F_2}$	$\sigma^2_{F_1*F_3}$	$\sigma^2_{F_2*F_3}$	$\sigma^2_{F_1*F_2*F_3}$	$\sigma^2_{e(F_1,F_2,F_3)}$
F_1-F	a								
F_2-F	b								
F_3-R	c								
$F_1 * F_2$-F	ab								
$F_1 * F_3$-R	ac								
$F_2 * F_3$-R	bc								
$F_1 * F_2 * F_3$-R	abc								
Error(F_1, F_2, F_3)-R	$abcr$								

3. For each row, place the following values under each variance component:

   a. In the column for a **fixed** variance component place a 0 in all rows **except** for the row where the source of variation exactly matches the subscripts of the **fixed** variance component. For this row divide the number consisting of r times the number of levels of all factors, *rabc* in this example, by the number of levels of the factors in the subscript of the variance component, then place the resulting number in the row of the SV that exactly matched the subscripts of the **fixed** variance component.

      E.g, For $F_1 * F_2$ where both F_1 and F_2 have fixed levels, divide $abcr$ by ab yielding the value cr. Then place cr in row for $F_1 * F_2$ under $Q_{F_1*F_2}$

      E.g., For SV F_1, a 0 would be placed in the row for F_1 under $Q_{F_1*F_2}$ but bcr would be placed in the row for F_1 under Q_{F_1}

   b. If the source of variation (SV) is not a part of the subscript of a variance component, then place a 0 in the row for the SV and column of that variance component.

      E.g., For SV F_1, a 0 would be placed under $\sigma_{F_2*F_3}$ in the row for F_1

   c. If the source of variation is a part of or the complete subscript of a **random** variance component then divide the number consisting of r times the number of levels of all factors, *rabc* in this example, by the number of levels of the factors in the subscript of the variance component, then place the resulting number in the row for that SV under the random variance component.

      E.g., For SV F_1 and random variance component $\sigma^2_{F_1*F_3}$, divide $abcr$ by ac yielding the value br. Then place place br under $\sigma^2_{F_1*F_3}$ in the row for F_1

   d. Place a 1 in each row of the column under $\sigma^2_e = \sigma^2_{e(F_1,F_2,F_3)}$ because all sources of variation are contained in the subscript and $abcr$ divided by $abcr$ is 1.

4. After completing all entries in the table, the expected mean square for each source of variation is obtained by multiplying the entries in the row corresponding to each source of variation by the corresponding variance component.

Factor F_1-Fixed, Factor F_2-Fixed, Factor F_3-Random

		F	F	R	F	R	R	R	R
SV	Levels	Q_{F_1}	Q_{F_2}	$\sigma^2_{F_3}$	$Q_{F_1*F_2}$	$\sigma^2_{F_1*F_3}$	$\sigma^2_{F_2*F_3}$	$\sigma^2_{F_1*F_2*F_3}$	σ^2_e
F_1	a	bcr	0	0	0	br	0	r	1
F_2	b	0	acr	0	0	0	ar	r	1
F_3	c	0	0	abr	0	br	ar	r	1
$F_1 * F_2$	ab	0	0	0	cr	0	0	r	1
$F_1 * F_3$	ac	0	0	0	0	br	0	r	1
$F_2 * F_3$	bc	0	0	0	0	0	ar	r	1
$F_1 * F_2 * F_3$	abc	0	0	0	0	0	0	r	1
Error	$abcr$	0	0	0	0	0	0	0	1

SV	Expected Mean Square
F_1	$bcrQ_{F_1} + br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
F_2	$acrQ_{F_2} + ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
F_3	$abr\sigma^2_{F_3} + br\sigma^2_{F_1*F_3} + ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_2$	$crQ_{F_1*F_2} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_3$	$br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_2 * F_3$	$ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_2 * F_3$	$r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
Error	σ^2_e

Factor F_1-Fixed, Factor F_2-Random, Factor F_3-Random

SV	Levels	F Q_{F_1}	R $\sigma^2_{F_2}$	R $\sigma^2_{F_3}$	R $\sigma^2_{F_1*F_2}$	R $\sigma^2_{F_1*F_3}$	R $\sigma^2_{F_2*F_3}$	R $\sigma^2_{F_1*F_2*F_3}$	R σ^2_e
F_1	a	bcr	0	0	cr	br	0	r	1
F_2	b	0	acr	0	cr	0	ar	r	1
F_3	c	0	0	abr	0	br	ar	r	1
$F_1 * F_2$	ab	0	0	0	cr	0	0	r	1
$F_1 * F_3$	ac	0	0	0	0	br	0	r	1
$F_2 * F_3$	bc	0	0	0	0	0	ar	r	1
$F_1 * F_2 * F_3$	abc	0	0	0	0	0	0	r	1
Error	$abcr$	0	0	0	0	0	0	0	1

SV	Expected Mean Square
F_1	$bcrQ_{F_1} + cr\sigma^2_{F_1*F_2} + br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
F_2	$acr\sigma^2_{F_2} + cr\sigma^2_{F_1*F_2} + ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
F_3	$abr\sigma^2_{F_3} + br\sigma^2_{F_1*F_3} + ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_2$	$cr\sigma^2_{F_1*F_2} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_3$	$br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_2 * F_3$	$ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_2 * F_3$	$r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
Error	σ^2_e