Statistics 630 – Exam II
Wednesday, 28 October 2020

Printed Name: ________________________ **Email:** ________________________

INSTRUCTIONS FOR THE STUDENT:

1. You have 50 minutes to complete the exam (after taking a moment to read these instructions). Please indicate your start time: ____________________ ,

   and your end time: ____________________
2. There are 7 pages including this cover sheet and the formula sheets.
3. Questions 1–4 are multiple choice and worth 5 points each.
4. Questions 5–7 require solutions to be worked out and are 10 points each. Please write out your answers in *the spaces provided*, explaining your steps. You may refer to theorems by name/description rather than by its number in the book.
5. If you *cannot* print out the exam, please write your answers on blank sheet of paper – in order.
6. You may use the *attached formula sheets*. No other resources are allowed. Do not use the textbook, the class notes, homework or formula sheets that were posted online.
7. *Do not use a calculator.* You may leave answers in forms that can easily be put into a calculator such as $\frac{12}{19}$, $\binom{40}{5}$, e^{-3}, $\Phi(1.5)$, etc.
8. Do not discuss or provide any information to anyone concerning any of the questions on this exam until your solutions are returned or I post my solutions.

I attest that I spent no more than 50 minutes to complete the exam. I used only the materials allowed above. I did not receive assistance from or provide assistance to anyone either before or while taking this exam.

Student's Signature ____________________________________

Questions 1–4 are multiple choice: circle the single correct answer. No partial credit!

1. (5 points) X, Y and Z are independent random variables with normal(1,3) distribution. Then $T = X - 2Y + 2Z$ has

   (a) normal(1,3) distribution.

   (b) normal(5,15) distribution.

   (c) normal(1,9) distribution.

   (d) normal(5,27) distribution.

   (e) normal(1,27) distribution.

2. (5 points) X has moment generating function $m(s) = \frac{1}{3}(\mathrm{e}^{-s+s^2} + 2\mathrm{e}^{s+2s^2})$. The mean of X is

   (a) $\int_0^\infty s\,\frac{1}{3}(\mathrm{e}^{-s+s^2} + 2\mathrm{e}^{s+2s^2})\,\mathrm{d}s$.

   (b) 0.

   (c) $\frac{1}{3}((2s-1)\mathrm{e}^{-s+s^2} + (6s+2)\mathrm{e}^{s+2s^2})$.

   (d) $\frac{1}{3}$.

   (e) 1.

3. (5 points) The negative binomial$(3, p)$ distribution has mean $\frac{3(1-p)}{p}$ and variance $\frac{3(1-p)}{p^2}$. Let $T_1, \ldots, T_n$ be a simple random sample from this distribution. A method of moments estimator for p is

   (a) $\frac{1}{1+\bar{T}/3}$.

   (b) $\frac{3(1-\bar{T})}{\bar{T}}$.

   (c) $\frac{\hat{\sigma}^2}{\bar{T}}$.

   (d) $\frac{3(1-p)}{\bar{T}}$.

   (e) $1 - \bar{T}/3$.

4. (5 points) Assume as in the previous problem. An asymptotic (approximate) distribution for $\bar{T}$ is

   (a) normal(0,1).

   (b) $\text{normal}\Big(\frac{3(1-p)}{p}, \frac{3(1-p)}{p^2}\Big)$.

   (c) $\text{normal}\Big(\frac{3(1-p)}{p}, \frac{3(1-p)}{np^2}\Big)$.

   (d) $\text{normal}\Big(\frac{3(1-p)}{p}, \frac{\sqrt{3(1-p)}}{\sqrt{n}p}\Big)$.

   (e) $\text{normal}\Big(0, \frac{\sqrt{3(1-p)}}{\sqrt{n}p}\Big)$.

Provide solutions to Questions 5–7, to the point of a calculable expression.

5. (10 points) Assume $Y_1, Y_2, \ldots, Y_n$ is an iid sample from the Poisson(λ) distribution. Write down the log-likelihood for the sample and use it to get the MLE for λ. Show your work.

6. (10 points) (R, S) has joint pdf $f(r, s) = \frac{36}{13}(rs + r^2 s^2)$, $0 < r < 1$, $0 < s < 1$. Find the conditional pdf for R, given $S = 1$, and use it to get $\mathsf{E}(R \mid S = 1)$.

7. (10 points) Suppose $X_1, \ldots, X_n$ is a simple random sample from gamma$(\alpha, 1)$. Since $\mathsf{E}(X_i) = \mathsf{Var}(X_i) = \alpha$, a method of moments estimator for the standard deviation $\sigma = \alpha^{1/2}$ is $\tilde{\sigma} = (\bar{X})^{1/2}$. Use the fact that $\bar{X} \sim$ gamma$(n\alpha, n)$ to give expressions for $\mathsf{Bias}(\tilde{\sigma})$ and $\mathsf{MSE}(\tilde{\sigma})$. (Show appropriate integrals at least, simplify if you can.)

Formulas for Exam II

Bayes' rule $\mathsf{P}(B_j \mid A) = \frac{\mathsf{P}(A|B_j)\mathsf{P}(B_j)}{\sum_{k=1}^n \mathsf{P}(A|B_k)\mathsf{P}(B_k)}$ if $B_1, \ldots, B_n$ are disjoint and $\bigcup_{k=1}^n B_k = S$.

quantile function $Q_X(p)$ satisfies $F_X(x) \leq p \leq F(Q_X(p))$ for all $x < Q_X(p)$. $F(Q_X(p)) = p$ if X is a continuous rv.

distribution of a function of X $F_Y(y) = \mathsf{P}(h(X) \leq y)$ for $Y = h(X)$.

If X is a discrete rv or $h(x)$ takes only countably many values then Y has pmf $p_Y(y) = \mathsf{P}(h(X) = y)$.

If X is a continuous rv and $h(x)$ is a continuous function then Y has pdf $f_Y(y) = \frac{\mathrm{d}x}{\mathrm{d}y}\,\mathsf{P}(h(X) \leq y)$.

binomial theorem $\sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = (a+b)^n$.

geometric sum $\sum_{k=n}^\infty a^k = \frac{a^n}{1-a}$ if $-1 < a < 1$.

exponential expansion $\sum_{k=0}^\infty \frac{a^k}{k!} = \mathrm{e}^a$.

integral of a power function $\int_u^v x^a \,\mathrm{d}x = \frac{v^{a+1} - u^{a+1}}{a+1}$ if $a \neq -1$, and $\int_u^v x^{-1}\, dx = \log_e(v/u)$.

integral of an exponential function $\int_u^v \mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}\left(\mathrm{e}^{av} - \mathrm{e}^{au}\right)$.

gamma integral $\int_0^\infty x^a \mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(a+1) = a!$ for $a > -1$.

integral of exponential of a quadratic $\int_{-\infty}^\infty \mathrm{e}^{a+bx-cx^2}\, dx = \sqrt{\frac{\pi}{c}}\, \mathrm{e}^{b^2/(4c)+a}$ for $c > 0$.

Bernoulli pmf $p(x) = (1-\theta)^{1-x}\theta^x I_{\{0,1\}}(x)$ for $0 < \theta < 1$, same as binomial$(1, \theta)$.

beta(a, b) pdf $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1} I_{(0,1)}(x)$ for $a > 0$, $b > 0$; $\mathsf{E}(X) = \frac{a}{a+b}$ $\mathsf{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$.

binomial(n, θ) pmf $p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x} I_{\{0,1,\ldots,n\}}(x)$ for $0 < \theta < 1$. $\mathsf{E}(X) = n\theta$, $\mathsf{Var}(X) = n\theta(1-\theta)$, $m(s) = (1 - \theta + \theta\mathrm{e}^s)^n$.

chi-square(n) same as gamma$(\frac{n}{2}, \frac{1}{2})$, the distribution of $X = Z_1^2 + \cdots + Z_n^2$ for iid standard normal $Z_1, \ldots, Z_n$. $\mathsf{E}(X) = n$, $\mathsf{Var}(X) = 2n$.

In particular, if $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{normal}(\mu, \sigma^2)$ then $\frac{(n-1)S^2}{\sigma^2} \sim \text{chi-square}(n-1)$.

discrete uniform(N) pmf $p(x) = \frac{1}{N}\, I_{\{1,2,\ldots,N\}}(x)$. $\mathsf{E}(X) = \frac{N+1}{2}$, $\mathsf{Var}(X) = \frac{N^2-1}{12}$.

exponential(λ) pdf $f(x) = \lambda\mathrm{e}^{-\lambda x} I_{(0,\infty)}(x)$ for $\lambda > 0$, same as gamma$(1, \lambda)$. $\mathsf{E}(X) = \frac{1}{\lambda}$, $\mathsf{Var}(X) = \frac{1}{\lambda^2}$.

F(m, n) the distribution of $W = \frac{X/m}{Y/n}$ where $X \sim \text{chi-square}(m)$, $Y \sim \text{chi-square}(n)$, independent. $\mathsf{E}(W) = \frac{n}{n-2}$ if $n > 2$.

gamma(α, λ) pdf $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\mathrm{e}^{-\lambda x} I_{(0,\infty)}(x)$ for $\lambda > 0$, $\alpha > 0$; $\mathsf{E}(X) = \frac{\alpha}{\lambda}$, $\mathsf{Var}(X) = \frac{\alpha}{\lambda^2}$, $m(s) = \left(\frac{\lambda}{\lambda - s}\right)^\alpha$ if $s < \lambda$.

geometric(θ) pmf $p(x) = \theta(1-\theta)^x I_{\{0,1,2,\ldots\}}(x)$ for $0 < \theta < 1$, same as negative binomial$(1,\theta)$. $\mathsf{E}(X) = \frac{1-\theta}{\theta}$, $\mathsf{Var}(X) = \frac{1-\theta}{\theta^2}$.

hypergeometric(N, M, n) pmf $p(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} I_{\{0,1,\ldots,n\}}(x)$ for $M < N$. $\mathsf{E}(X) = np$ where $p = \frac{M}{N}$, $\mathsf{Var}(X) = \frac{N-n}{N-1}\, np(1-p)$.

negative binomial(r, θ) pmf $p(x) = \binom{r+x-1}{r-1}\theta^r(1-\theta)^x I_{\{0,1,2,\ldots\}}(x)$ for $0 < \theta < 1$. $\mathsf{E}(X) = \frac{r(1-\theta)}{\theta}$, $\mathsf{Var}(X) = \frac{r(1-\theta)}{\theta^2}$, $m(s) = \left(\frac{\theta}{1-(1-\theta)\mathrm{e}^s}\right)^r$ if $s < -\log(1-\theta)$.

normal(μ, σ^2) pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)} I_{(-\infty,\infty)}(x)$ for $\sigma^2 > 0$; $\mathsf{E}(X) = \mu$, $\mathsf{Var}(X) = \sigma^2$, $m(s) = \mathrm{e}^{\mu s + \sigma^2 s^2/2}$.

Poisson(λ) pmf $p(x) = \frac{\lambda^x}{x!}\,\mathrm{e}^{-\lambda} I_{\{0,1,2,\ldots\}}(x)$ for $\lambda > 0$. $\mathsf{E}(X) = \lambda$, $\mathsf{Var}(X) = \lambda$, $m(s) = \mathrm{e}^{\lambda(\mathrm{e}^s - 1)}$.

t(n) the distribution of $T = \frac{Z}{\sqrt{Y/n}}$ where $Z \sim$ normal$(0,1)$, $Y \sim$ chi-square(n), independent. $\mathsf{E}(T) = 0$, $\mathsf{Var}(T) = \frac{n}{n-2}$ if $n > 2$. In particular, if $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ normal(μ, σ^2) then $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim$ t$(n-1)$.

uniform(a, b) pdf $f(x) = \frac{1}{b-a}\, I_{(a,b)}(x)$ for $a < b$. $\mathsf{E}(X) = \frac{a+b}{2}$, $\mathsf{Var}(X) = \frac{(b-a)^2}{12}$.

Weibull(α, β) pdf $f(x) = \frac{\alpha}{\beta}\,(x/\beta)^{\alpha-1}\mathrm{e}^{-(x/\beta)^\alpha} I_{(0,\infty)}(x)$ for $\alpha > 0$, $\beta > 0$. $\mathsf{E}(X^k) = \beta^k\,\Gamma(1 + \frac{k}{\alpha})$.

joint pmf/cdf $p(x,y) = \mathsf{P}(\{X = x\} \cap \{Y = y\})$, $F(x,y) = \sum_{u \le x}\sum_{v \le y} p(u,v)$.

joint pdf/cdf $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$, $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,\mathrm{d}v\mathrm{d}u$.

marginal pmf/pdf $p_X(x) = \sum_y p(x,y)$; $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y$.

conditional pmf/pdf $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$; $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.

independent random variables $p(x,y) = p_X(x)p_Y(y)$ if (X,Y) is discrete; $f(x,y) = f_X(x)f_Y(y)$ if (X,Y) is continuous.

discrete convolution $p_{X+Y}(z) = \sum_x p_X(x)p_Y(z-x)$ for independent X, Y.

continuous convolution $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,\mathrm{d}x$ for independent X, Y.

cdf of minimum $F_{\min(X_1,\ldots,X_n)}(u) = 1 - \prod_{i=1}^{n}(1 - F_{X_i}(u))$ for independent $X_1, \ldots, X_n$.

cdf of maximum $F_{\max(X_1,\ldots,X_n)}(u) = \prod_{i=1}^{n} F_{X_i}(u)$ for independent $X_1, \ldots, X_n$.

expectation for a discrete rv $\mathsf{E}(h(X)) = \sum_x h(x)p_X(x)$.

expectation for a continuous rv $\mathsf{E}(h(X)) = \int_{-\infty}^{\infty} h(x)f_X(x)\,\mathrm{d}x$.

mean and variance $\mu_X = \mathsf{E}(X)$; $\sigma_X^2 = \mathsf{Var}(X) = \mathsf{E}((X-\mu_X)^2) = \mathsf{E}(X^2) - \mu_X^2$.

standard deviation $\sigma_X = \sqrt{\mathsf{Var}(X)}$.

covariance and correlation $\mathsf{Cov}(X,Y) = \mathsf{E}((X-\mu_X)(Y-\mu_Y)) = \mathsf{E}(XY) - \mu_X\mu_Y$; $\mathsf{Corr}(X,Y) = \frac{\mathsf{Cov}(X,Y)}{\sigma_X\sigma_Y}$. For independent X and Y, $\mathsf{Cov}(X,Y) = \mathsf{Corr}(X,Y) = 0$.

expectation of a sum $\mathsf{E}(a_1X_1 + \cdots + a_nX_n) = a_1\mathsf{E}(X_1) + \cdots + a_n\mathsf{E}(X_n)$.

expectation of a product If $X_1, \ldots, X_n$ are independent, $\mathsf{E}\Big(\prod_{i=1}^n h_i(X_i)\Big) = \prod_{i=1}^n \mathsf{E}(h_i(X_i))$.

variance of a sum $\mathsf{Var}(aX + bY) = a^2\,\mathsf{Var}(X) + 2ab\,\mathsf{Cov}(X,Y) + b^2\,\mathsf{Var}(Y)$.

variance of a sum of independent rvs $\mathsf{Var}(a_1X_1+\cdots+a_nX_n) = a_1^2\,\mathsf{Var}(X_1)+\cdots+a_n^2\,\mathsf{Var}(X_n)$.

moments k-th moment is $\mu_k = \mathsf{E}(X^k)$, $k = 1, 2, \ldots$

moment generating function $m_X(s) = \mathsf{E}(\mathrm{e}^{sX})$; $\mathsf{E}(X^k) = \frac{\mathrm{d}x^k}{\mathrm{d}s^k}\, m_X(s)\Big|_{s=0}$.

mgf of a sum If X and Y are independent, $m_{aX+bY}(s) = \mathsf{E}(\mathrm{e}^{(aX+bY)s}) = m_X(as)m_Y(bs)$.

conditional expectation $\mathsf{E}(h(Y)|X=x) = \sum_y h(y)p_{Y|X}(y|x)$ or
$\mathsf{E}(h(Y)|X=x) = \int_{-\infty}^{\infty} h(y)f_{Y|X}(y|x)\,dy$.

iterated expectation $\mathsf{E}(h(Y)) = \mathsf{E}(\mathsf{E}(h(Y)|X))$.

conditional variance $\mathsf{Var}(Y|X) = \mathsf{E}(Y^2|X) - (\mathsf{E}(Y|X))^2$.

variance partition formula $\mathsf{Var}(Y) = \mathsf{E}(\mathsf{Var}(Y|X)) + \mathsf{Var}(\mathsf{E}(Y|X))$.

Markov's inequality $\mathsf{P}(|X| \geq x) \leq \frac{\mathsf{E}(|X|)}{x}$ for $x > 0$.

Chebyshev's inequality $\mathsf{P}(|X - \mu_X| \geq x) \leq \frac{\mathsf{Var}(X)}{x^2}$ for $x > 0$.

sample mean, variance, k-th moment $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$; $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$;
$m_k = \frac{1}{n}\sum_{i=1}^n X_i^k$.

unbiased sample variance $S^2 = \frac{n}{n-1}\,\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

law of large numbers For iid $X_1, X_2, \ldots$ with mean μ, $\bar{X}_n \to \mu$ as $n \to \infty$.

central limit theorem For iid $X_1, X_2, \ldots$ with mean μ and variance σ^2,
$\mathsf{P}\Big(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\Big) = \mathsf{P}\Big(\frac{X_1+\cdots+X_n - n\mu}{\sqrt{n}\sigma} \leq z\Big) \to \Phi(z)$ (normal(0,1) cdf), as $n \to \infty$.

method of moments for iid sample match the k-th population moment $\mathsf{E}_\theta(X^k)$ with the k-th sample moment m_k, and solve for the desired parameter estimates.

maximum likelihood for iid sample maximize the likelihood function $L(\theta|X_1,\ldots,X_n) = \prod_{i=1}^n f_\theta(X_i)$ or the log-likelihood $\ell(\theta|X_1,\ldots,X_n) = \log L(\theta|X_1,\ldots,X_n) = \sum_{i=1}^n \log f_\theta(X_i)$.

If $\log L(\theta)$ is differentiable and concave at θ, the MLE is a solution to $S(\theta) = \frac{d}{d\theta}\log L(\theta) = 0$. (For a multidimensional parameter θ this is a system of equations.)

bias and standard error $\mathsf{Bias}_\theta(\hat{\theta}) = \mathsf{E}_\theta(\hat{\theta}) - \theta$; $\mathsf{SE}_\theta(\hat{\theta}) = \sqrt{\mathsf{Var}_\theta(\hat{\theta})}$.

mean squared error $\mathsf{MSE}_\theta(\hat{\theta}) = \mathsf{E}_\theta((\hat{\theta} - \theta)^2) = \mathsf{Var}_\theta(\hat{\theta}) + (\mathsf{Bias}_\theta(\hat{\theta}))^2$.

consistency An estimator $\tilde{\theta}_n$ is consistent for θ if $\tilde{\theta}_n \to \theta$ as $n \to \infty$. $\mathsf{MSE}_\theta(\tilde{\theta}_n) \to 0$ implies consistency. If $\tilde{\theta}$ is consistent for θ and $g(x)$ is continuous then $g(\tilde{\theta})$ is consistent for $g(\theta)$.