Statistics 630 - Assignment 3
(partial solutions)

1. Exer. 2.3.18. Let X_1 be the number of calls in the first two minutes and let X_2 be the number of calls in the next two minutes.
   (a) $\mathsf{P}(X_1 = 5) = \frac{4^5 e^{-4}}{5!}$.
   (b) By independence, $\mathsf{P}(\{X_1 = 5\} \cap \{X_2 = 5\}) = \mathsf{P}(X_1 = 5)\mathsf{P}(X_2 = 5) = \left(\frac{4^5 e^{-4}}{5!}\right)^2$. [We will simply write $\mathsf{P}(X_1 = 5, X_2 = 5)$ for the left-hand expression above, in the future.]
   (c) e^{-20}.

2. Exer. 2.4.4. (a) $2xI_{[0,1]}(x)$. (b) $(n+1)x^n I_{[0,1]}(x)$. (c) $\frac{3}{4\sqrt{2}} x^{1/2} I_{[0,2]}(x)$.

3. Exer. 2.4.19. Use a change of variables $y = x^\alpha$ with $\frac{dy}{dx} = \alpha x^{\alpha-1}$. Thus,

$$\int_0^\infty \alpha x^{\alpha-1} e^{-x^\alpha} \, dx = \int_0^\infty e^{-y} \, dy = 1.$$

6. (d) $y_{.40} = -8.5067$, $y_{.77} = -6.5223$. [Please do not overly round answers!]

7. Exer. 2.5.8. (a) 0.9473. (b) 0. (c) $\frac{3}{4}$. This is an example that has *both* discrete and continuous components. [The expression given in the textbook is not a cdf because it is *decreasing* on $(\frac{1}{2}, 1)$.]

9. Exer. 2.5.19. $\Phi(-x) = 1 - \Phi(x)$. This is valid for all real x.

10. Exer. 2.5.21. (a) Use the same change of variables as in Exer. 2.4.19 (or the chain rule) to get $F(x) = (1 - e^{-x^\alpha}) I_{[0,\infty)}(x)$.
    (b) Solve $F(x_p) = p$ to get $x_p = (-\log(1-p))^{1/\alpha}$.

11. Exer. 2.5.24. (a) $F(x) = \frac{1}{2}e^x$ for $x < 0$ and $F(x) = 1 - \frac{1}{2}e^{-x}$ for $x \geq 0$.
    (b) $x_p = \log(2p)$ for $p < \frac{1}{2}$ and $x_p = -\log(2(1-p))$ for $p \geq \frac{1}{2}$.

12. Exer. 2.6.4. Two methods can be used.
    (i) $F_Y(y) = \mathsf{P}(X \leq y/c) = 1 - e^{-\lambda y/c}$ which is the exponential(λ/c) cdf.
    (ii) Using the result on slide 58 (Chapter 2 - Univariate), $f_Y(y) = \frac{1}{c} f_X(y/c) = \frac{\lambda}{c} e^{-\lambda y/c}$ which is the exponential(λ/c) pdf.

    On the use of Theorem 2.6.2 in the book. A simple mnemonic is

$$f_Y(y) \, dy = f_X(x) \, dx,$$

    which leads naturally to $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$, where we express $x = h^{-1}(y)$ in terms of y and take absolute value of the differential when $h(x)$ is decreasing. *Do not forget the assumptions:* X has absolutely continuous distribution (with a pdf), and h is 1-1 and differentiable over the range of X.

    Exer. 2.6.9. Use the note above. The answers in the book are *incomplete*: they do not indicate the ranges of the new random variables, $[0, 4]$ and $[0, \sqrt{2}]$, respectively. [Please see the book's errata for the solution to (b).]

Exer. 2.6.18. $X = Y^{1/\beta}$. So $\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = \frac{1}{\beta} y^{1/\beta - 1}$ and hence

$$f_Y(y) = f_X(x)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = \alpha(y^{1/\beta})^{1/\alpha - 1}\mathrm{e}^{-(Y^{1/\beta})^{\alpha}} \times \frac{1}{\beta} y^{1/\beta - 1} = \frac{\alpha}{\beta} y^{\alpha/\beta - 1}\mathrm{e}^{-y^{\alpha/\beta}}, \quad \text{for } y > 0.$$

We note that this is the Weibull($\frac{\alpha}{\beta}$) pdf. Therefore, $Y \sim$ Weibull($\frac{\alpha}{\beta}$).

[In the case $\beta < 0$ and we take the absolute value of the expression above, we would have the pdf for a Frechét distribution.]