Statistics 630 – Final Exam
Friday, 3 May 2013

Printed Name: __________________________ **Email:** __________________________

INSTRUCTIONS FOR THE STUDENT:

1. You have exactly 2 hours to complete the exam.
2. There are 7 pages including this cover sheet and the formula sheets.
3. There are 7 questions. Each question is worth 10 points.
4. Please write out the answers to the exam questions on *blank* sheets of paper. Start each question on a *separate* sheet of paper.
5. Answer all questions fully, explaining your steps. You may refer to theorems by name/description rather than by its number in the book.
6. *Do not use a calculator.* You may leave answers in forms that can easily be put into a calculator such as $\frac{12}{19}$, $\binom{40}{5}$, e^{-3}, $\Phi(1.5)$, etc.
7. Do not discuss or provide any information to anyone concerning any of the questions on this exam or your solutions until I post my solutions.
8. You may use the *attached formula sheets.* No other resources are allowed. Do not use the textbook, the class notes or the formula sheets that were posted online.

I attest that I spent no more than 2 hours to complete the exam. I used only the materials described above. I did not receive assistance from anyone either before or while taking this exam.

Student's Signature _______________________________________

INSTRUCTIONS FOR THE PROCTOR:

1. Download the exam from the student's account on WebAssign and print it. The student should not view the exam content until it is time to start.
2. The student may take 2 hours to complete the exam. Record the time at which the student starts the exam: __________________________
3. Record the time at which the student ends the exam: __________________________
4. Immediately after the student completes the exam, please scan *this page* and the student's *solutions* to a *single, unsecured* PDF file and oversee the student uploading it to WebAssign. Be sure the solutions are legible and in order. (You have 150 minutes from the time the exam was downloaded, allowing 30 minutes for printing, scanning and uploading. Please contact Penny Jackson or Kim Ritchie for any necessary delay.)
5. Collect all pages of this exam at its conclusion. Do not allow the student to take any page. You may return them to the student *one week* after the exam.

I attest that the student and I have followed the INSTRUCTIONS FOR THE STUDENT and PROCTOR listed above and that the exam was downloaded, printed, scanned into a .pdf file and uploaded to WebAssign in my presence.

Proctor's Signature _______________________________________

Be sure to answer all parts of each question.

1. Suppose we have a random sample $W_1, \ldots, W_n$ from a distribution with pdf
$$f(w|\theta) = \frac{1}{3}\,\theta w^{\theta-1} + \frac{2}{3}\,(2\theta+1)w^{2\theta} \quad \text{for } 0 < w < 1,\ \theta > 0.$$
   (a) Find $\mathsf{E}(W)$.
   (b) Use it to identify a method of moments estimator for θ.

2. Let $X_1, \ldots, X_n$ be iid random variables with probability density
$$f(x|\lambda) = \frac{\lambda^2(x+1)}{\lambda+1}\,e^{-\lambda x} \quad \text{for } x > 0,\ \lambda > 0.$$
   (a) Determine the Fisher information $I_n(\lambda) = nI_1(\lambda)$.
   (b) Provide the rejection criterion for a size α score test of $H_0 : \lambda = 1$ versus $H_a : \lambda \neq 1$.

3. Consider the following cdf.
$$F(x) = \begin{cases} \dfrac{x}{4}, & 0 \le x < 1, \\[2ex] \dfrac{5x-2}{12}, & 1 \le x < 2, \\[2ex] \dfrac{x}{3}, & 2 \le x < 3. \end{cases}$$
   (a) Is F discrete, absolutely continuous or neither?
   (b) If discrete, what is the pmf? If absolutely continuous, what is the pdf?

4. Suppose $T_1, \ldots, T_n$ are iid gamma$(\alpha, 3)$ random variables and $\tilde{\alpha} = 3\bar{X}$ is an estimator for α.
   (a) Find the asymptotic distribution of $\sqrt{\tilde{\alpha}}$ (for example, by the delta method).
   (b) Use it to construct a γ level Wald confidence interval for $\sqrt{\alpha}$.

5. $X_1, \ldots, X_n$ is a random sample from the normal$(0, \frac{1}{2\theta})$ distribution (that is, with pdf $f(x) = \sqrt{\frac{\theta}{\pi}}\,e^{-\theta x^2}$ for $x \in \mathbb{R}$). Suppose also that θ has a gamma$(\alpha, 1)$ prior distribution. Find the posterior distribution and mean for θ.

6. (U, V) has bivariate distribution with joint density
$$f(u, v) = 2e^{-u-v} I_{(0,u)}(v) I_{(v,\infty)}(u).$$
   Note: $U > V$.
   (a) Find the marginal distribution of V.
   (b) Find the conditional distribution of U, given $V = v$.

7. Suppose $Y_1, \ldots, Y_n$ are iid Poisson(λ). Recall that $S = \sum_{i=1}^n Y_i$ has Poisson$(n\lambda)$ distribution. Show that $e^{t_n S}$, with $t_n = \log((n-1)/n)$, is an unbiased estimator of $e^{-\lambda} = \mathsf{P}(Y_i = 0)$. Hint: what is the mgf for Poisson$(n\lambda)$?

Formulas for Final Exam

Bayes' rule $\mathsf{P}(B_j \,|\, A) = \frac{\mathsf{P}(A \,|\, B_j)\mathsf{P}(B_j)}{\sum_{k=1}^n \mathsf{P}(A \,|\, B_k)\mathsf{P}(B_k)}$ if $B_1, \ldots, B_n$ are disjoint and $\bigcup_{k=1}^n B_k = \Omega$.

quantile function $Q_X(p)$ satisfies $F_X(x) \le p \le F(Q_X(p))$ for $x < Q_X(p)$. If X is a continuous rv then $F(Q_X(p)) = p$.

distribution of a function of X $F_Y(y) = \mathsf{P}(h(X) \le y)$ for $Y = h(X)$.

If X is a discrete rv or $h(x)$ takes only countably many values then Y has pmf $p_Y(y) = \mathsf{P}(h(X) = y)$.

If X is a continuous rv and $h(x)$ is a continuous function then Y has pdf $f_Y(y) = \frac{d}{dy}\mathsf{P}(h(X) \le y)$.

binomial theorem $\sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = (a+b)^n$.

geometric sum $\sum_{k=n}^\infty a^k = \frac{a^n}{1-a}$ if $-1 < a < 1$.

exponential expansion $\sum_{k=0}^\infty \frac{a^k}{k!} = e^a$.

gamma integral $\int_0^\infty x^a e^{-bx}\,dx = \Gamma(a+1)b^{-a-1} = a!\,b^{-a-1}$ for $a > -1$, $b > 0$.

Bernoulli pmf $p(x) = (1-\theta)^{1-x}\theta^x I_{\{0,1\}}(x)$ for $0 < \theta < 1$, same as binomial$(1,\theta)$.

beta(a,b) **pdf** $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} I_{(0,1)}(x)$ for $a > 0$, $b > 0$; $\mathsf{E}(X) = \frac{a}{a+b}$ $\mathsf{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$.

binomial(n,θ) **pmf** $p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x} I_{\{0,1,\ldots,n\}}(x)$ for $0 < \theta < 1$. $\mathsf{E}(X) = n\theta$, $\mathsf{Var}(X) = n\theta(1-\theta)$, $m(s) = (1-\theta+\theta e^s)^n$.

chi-square(n) $f(x) = \frac{x^{(n-2)/2}e^{-x/2}}{2^{n/2}\Gamma(n/2)} I_{(0,\infty)}(x)$ for $n > 0$, same as gamma$(\frac{n}{2}, \frac{1}{2})$. $\mathsf{E}(X) = n$, $\mathsf{Var}(X) = 2n$. The distribution of $Z_1^2 + \cdots + Z_n^2$ for iid standard normal $Z_1, \ldots, Z_n$.

In particular, if $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ normal(μ, σ^2) then $\frac{(n-1)S^2}{\sigma^2} \sim$ chi-square$(n-1)$.

discrete uniform(N) **pmf** $p(x) = \frac{1}{N} I_{\{1,2,\ldots,N\}}(x)$. $\mathsf{E}(X) = \frac{N+1}{2}$, $\mathsf{Var}(X) = \frac{N^2-1}{12}$.

exponential(λ) **pdf** $f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$ for $\lambda > 0$, same as gamma$(1,\lambda)$. $\mathsf{E}(X) = \frac{1}{\lambda}$, $\mathsf{Var}(X) = \frac{1}{\lambda^2}$.

F(m,n) the distribution of $\frac{X/m}{Y/n}$ where $X \sim$ chi-square(m), $Y \sim$ chi-square(n), independent. $\mathsf{E}(X) = \frac{n}{n-2}$ if $n > 2$.

gamma(α,λ) **pdf** $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x} I_{(0,\infty)}(x)$ for $\lambda > 0$, $\alpha > 0$; $\mathsf{E}(X) = \frac{\alpha}{\lambda}$, $\mathsf{Var}(X) = \frac{\alpha}{\lambda^2}$, $m(s) = \left(\frac{\lambda}{\lambda - s}\right)^\alpha$ if $s < \lambda$.

geometric(θ) **pmf** $p(x) = \theta(1-\theta)^x I_{\{0,1,2,\ldots\}}(x)$ for $0 < \theta < 1$, same as negative binomial$(1,\theta)$. $\mathsf{E}(X) = \frac{1-\theta}{\theta}$, $\mathsf{Var}(X) = \frac{1-\theta}{\theta^2}$.

hypergeometric(N, M, n) pmf $p(x) = \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} I_{\{0,1,\ldots,n\}}(x)$ for $M < N$. $\mathsf{E}(X) = np$ where $p = \frac{M}{N}$, $\mathsf{Var}(X) = \frac{N-n}{N-1}\, np(1-p)$.

negative binomial(r, θ) pmf $p(x) = \binom{x-1}{r-1}\theta^r(1-\theta)^x I_{\{0,1,2,\ldots\}}(x)$ for $0 < \theta < 1$. $\mathsf{E}(X) = \frac{r(1-\theta)}{\theta}$, $\mathsf{Var}(X) = \frac{r(1-\theta)}{\theta^2}$, $m(s) = \left(\frac{\theta}{1-(1-\theta)e^s}\right)^r$ if $s < -\log(1-\theta)$.

normal(μ, σ^2) pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/(2\sigma^2)} I_{(-\infty,\infty)}(x)$ for $\sigma^2 > 0$; $\mathsf{E}(X) = \mu$, $\mathsf{Var}(X) = \sigma^2$, $m(s) = e^{\mu s + \sigma^2 s^2/2}$.

Poisson(λ) pmf $p(x) = \frac{\lambda^x}{x!}\, e^{-\lambda} I_{\{0,1,2,\ldots\}}(x)$ for $\lambda > 0$. $\mathsf{E}(X) = \lambda$, $\mathsf{Var}(X) = \lambda$, $m(s) = e^{\lambda(e^s-1)}$.

t(n) the distribution of $\dfrac{Z}{\sqrt{Y/n}}$ where $Z \sim$ normal$(0,1)$, $Y \sim$ chi-square(n), independent. $\mathsf{E}(X) = 0$, $\mathsf{Var}(X) = \frac{n}{n-2}$ if $n > 2$.

In particular, if $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ normal(μ, σ^2) then $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim \mathrm{t}(n-1)$.

uniform(a, b) pdf $f(x) = \frac{1}{b-a}\, I_{(a,b)}(x)$ for $a < b$. $\mathsf{E}(X) = \frac{a+b}{2}$, $\mathsf{Var}(X) = \frac{(b-a)^2}{12}$.

Weibull(α, β) pdf $f(x) = \frac{\alpha}{\beta}\,(x/\beta)^{\alpha-1} e^{-(x/\beta)^\alpha} I_{(0,\infty)}(x)$ for $\alpha > 0$, $\beta > 0$. $\mathsf{E}(X^k) = \beta^k\,\Gamma(1+\frac{k}{\alpha})$.

marginal pmf/pdf $p_X(x) = \sum_y p(x,y)$; $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

conditional pmf/pdf $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$; $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.

independent random variables $p(x,y) = p_X(x)p_Y(y)$ if (X,Y) is discrete; $f(x,y) = f_X(x)f_Y(y)$ if (X,Y) is continuous.

discrete convolution $p_{X+Y}(z) = \sum_x p_X(x)p_Y(z-x)$ for independent X, Y.

continuous convolution $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,dx$ for independent X, Y.

covariance and correlation $\mathsf{Cov}(X,Y) = \mathsf{E}((X-\mu_X)(Y-\mu_Y)) = \mathsf{E}(XY) - \mu_X\mu_Y$; $\mathsf{Corr}(X,Y) = \frac{\mathsf{Cov}(X,Y)}{\sigma_X\sigma_Y}$. For independent X and Y, $\mathsf{Cov}(X,Y) = \mathsf{Corr}(X,Y) = 0$.

expectation of a sum $\mathsf{E}(a_1X_1 + \cdots + a_nX_n) = a_1\mathsf{E}(X_1) + \cdots + a_n\mathsf{E}(X_n)$.

expectation of a product If $X_1, \ldots, X_n$ are independent, $\mathsf{E}\left(\prod_{i=1}^n h_i(X_i)\right) = \prod_{i=1}^n \mathsf{E}(h_i(X_i))$.

variance of a sum $\mathsf{Var}(aX+bY) = a^2\mathsf{Var}(X) + 2ab\mathsf{Cov}(X,Y) + b^2\mathsf{Var}(Y)$.

variance of a sum of independent rvs $\mathsf{Var}(a_1X_1 + \cdots + a_nX_n) = a_1^2\mathsf{Var}(X_1) + \cdots + a_n^2\mathsf{Var}(X_n)$.

moments k-th moment is $\mu_k = \mathsf{E}(X^k)$, $k = 1, 2, \ldots$

moment generating function $m_X(s) = \mathsf{E}(e^{sX})$; $\mathsf{E}(X^k) = \frac{d^k}{ds^k}\, m_X(s)\Big|_{s=0}$.

mgf of a sum If X and Y are independent, $m_{aX+bY}(s) = \mathsf{E}(e^{(aX+bY)s}) = m_X(as)m_Y(bs)$.

conditional expectation $\mathsf{E}(h(Y)|X=x) = \sum_y h(y)p_{Y|X}(y|x)$ or $\mathsf{E}(h(Y)|X=x) = \int_{-\infty}^{\infty} h(y)f_{Y|X}(y|x)\,dy$.

iterated expectation $\mathsf{E}(h(Y)) = \mathsf{E}(\mathsf{E}(h(Y)|X))$, $\mathsf{E}(g(X)h(Y)) = \mathsf{E}(g(X)\mathsf{E}(h(Y)|X))$.

conditional variance $\mathsf{Var}(Y|X) = \mathsf{E}(Y^2|X) - (\mathsf{E}(Y|X))^2$.

variance partition formula $\mathsf{Var}(Y) = \mathsf{E}(\mathsf{Var}(Y|X)) + \mathsf{Var}(\mathsf{E}(Y|X))$.

Markov's inequality $\mathsf{P}(|X| \geq x) \leq \frac{\mathsf{E}(|X|)}{x}$ for $x > 0$.

Chebyshev's inequality $\mathsf{P}(|X - \mu_X| \geq x) \leq \frac{\mathsf{Var}(X)}{x^2}$ for $x > 0$.

sample mean, variance, k-th moment $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$; $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$; $m_k = \frac{1}{n}\sum_{i=1}^n X_i^k$. The unbiased sample variance is $S^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

law of large numbers (averages) For iid $X_1, X_2, \ldots$ with mean μ, $\mathsf{P}(|\bar{X}_n - \mu| > \epsilon) \to 0$ as $n \to \infty$, each $\epsilon > 0$.

central limit theorem For iid $X_1, X_2, \ldots$ with mean μ and variance σ^2, $\mathsf{P}\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\right) = \mathsf{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \leq z\right) \to \Phi(z)$, as $n \to \infty$, where $\Phi(z)$ is the normal(0,1) cdf.

bias and standard error $\mathsf{Bias}(\hat{\theta}) = \mathsf{E}(\hat{\theta}) - \theta$; $\mathsf{SE}(\hat{\theta}) = \sqrt{\mathsf{Var}(\hat{\theta})}$.

mean squared error $\mathsf{MSE}(\hat{\theta}) = \mathsf{E}((\hat{\theta} - \theta)^2) = \mathsf{Var}(\hat{\theta}) + (\mathsf{Bias}(\hat{\theta}))^2$.

consistency $\hat{\theta}$ is consistent if $\mathsf{MSE}(\hat{\theta}) \to 0$ as $n \to \infty$.

method of moments for iid sample match the k-th population moment $\mathsf{E}(X^k)$ with the k-th sample moment m_k, and solve for the desired parameter estimates.

likelihood function $L(\theta|X_1, \ldots, X_n) = \prod_{i=1}^n f_\theta(X_i)$ for iid sample $\underline{X} = (X_1, \ldots, X_n)$.

maximum likelihood for iid sample maximize the likelihood function $L(\theta|X_1, \ldots, X_n) = \prod_{i=1}^n f_\theta(X_i)$ or the log-likelihood $\ell(\theta|X_1, \ldots, X_n) = \log L(\theta|X_1, \ldots, X_n) = \sum_{i=1}^n \log f_\theta(X_i)$.

If $\log L(\theta)$ is differentiable and concave at θ, the MLE is a solution to $S(\theta) = \frac{d}{d\theta}\log L(\theta) = 0$. (For a multidimensional parameter θ this is a system of equations.)

score function $S(\theta|X_1, \ldots, X_n) = \frac{d}{d\theta}\,\ell(\theta)$.

Fisher information $I_n(\theta) = \mathsf{Var}(\frac{d}{d\theta}\,\ell(\theta)) = -\mathsf{E}(\frac{d^2}{d\theta^2}\,\ell(\theta))$, if ℓ has two derivatives.

For an iid sample, $I_n(\theta) = nI_1(\theta)$ and $I_1(\theta) = \mathsf{Var}(\frac{d}{d\theta}\log f_\theta(X_1)) = -\mathsf{E}(\frac{d^2}{d\theta^2}\log f_\theta(X_1))$.

sufficient statistic $T = T(\underline{X})$ is sufficient if $L(\theta|\underline{X}) = h(\underline{X})g(T(\underline{X}), \theta)$ for some functions $h(\underline{x})$ and $g(t, \theta)$.

exponential family The pdf/pmf has the form $f_X(x|\theta) = d(\theta)h(x)e^{c(\theta)t(x)}$ for all x, θ. In this case, with an iid random sample, $T(\underline{\mathrm{X}}) = \sum_{i=1}^n t(X_i)$ is a sufficient statistic and $I_n(\theta) = n(c'(\theta))^2 \mathsf{Var}(t(X_1))$.

asymptotics for MLE Assuming Fisher information exists and $\hat{\theta}$ is the MLE, $\hat{\theta} \to \theta$ in probability and $\sqrt{I_n(\theta)}(\hat{\theta} - \theta) \to \text{normal}(0, 1)$ in distribution as $n \to \infty$.

asymptotic normality $\hat{\theta}$ is asymptotic normal(θ, V_n) if $\frac{\hat{\theta}-\theta}{\sqrt{V_n}} \to \text{normal}(0, 1)$ in distribution as $n \to \infty$. V_n may depend on θ or other parameters. If $\hat{V}_n$ is an estimator such that $\hat{V}_n/V_n \to 1$ then $\frac{\hat{\theta}-\theta}{\sqrt{\hat{V}_n}} \to \text{normal}(0, 1)$ in distribution.

delta method If $g(\theta)$ is continuously differentiable and estimator $\hat{\theta}$ is asymptotic normal(θ, V_n), then $g(\hat{\theta})$ is asymptotic normal$(g(\theta), (g'(\theta))^2 V_n)$.

level γ confidence interval $(L(\underline{\mathrm{X}}), U(\underline{\mathrm{X}}))$ such that $\mathsf{P}_\theta(L(\underline{\mathrm{X}}) \le \theta \le U(\underline{\mathrm{X}})) = \gamma$.

confidence interval from pivot If $h(\underline{\mathrm{X}}, \theta)$ has a distribution that does not depend on θ, a level γ confidence interval is defined by $\{\theta : h(\underline{\mathrm{X}}, \theta) \in A\}$ where $\mathsf{P}_\theta(h(\underline{\mathrm{X}}, \theta) \in A) = \gamma$.

Wald confidence interval If $\hat{\theta}$ is asymptotic normal(θ, V_n) and $\hat{V}_n$ is an estimator for V_n, an approximate level γ confidence interval for θ has endpoints $\hat{\theta} \pm z_{(1+\gamma)/2}\sqrt{\hat{V}_n}$, where $z_{(1+\gamma)/2}$ is the $(1+\gamma)/2$ quantile of the normal(0,1) dist.

score confidence interval For MLE $\hat{\theta}$, an approximate level γ confidence interval defined by $\{\theta : -z_{(1+\gamma)/2} \le (\frac{d}{d\theta}\ell(\theta))/\sqrt{I_n(\theta)} \le z_{(1+\gamma)/2}\}$, where $z_{(1+\gamma)/2}$ is the $(1+\gamma)/2$ quantile of the normal(0,1) dist.

A related method is the interval given by $\{\theta : -z_{(1+\gamma)/2} \le \sqrt{I_n(\theta)}(\hat{\theta} - \theta) \le z_{(1+\gamma)/2}\}$.

Type I and II errors, level and power A Type I error is rejecting H_0 when it is true. The level of a test is $\alpha = \max_{\theta \in H_0} \mathsf{P}_\theta(H_0 \text{ is rejected})$ computed with values of θ such that H_0 true.

A Type II error is not rejecting H_0 when H_a is true. The power of a test is $\beta = \beta(\theta) = \mathsf{P}_\theta(H_0 \text{ is rejected})$ computed with parameter value θ (satisfying H_a).

***P*-value** The smallest level α for which H_0 will still be rejected – it is a statistic (function of the data).

Neyman-Pearson likelihood ratio test For simple hypotheses $H_0 : \theta = \theta_0$ vs. $H_a : \theta = \theta_1$, reject H_0 if $LR = \frac{L(\theta_1)}{L(\theta_0)} \ge c_\alpha$ where $\mathsf{P}(LR \ge c_\alpha) = \alpha$ when H_0 is true.
If, for each c, $LR \ge c \iff T \ge k$ (or $R \ge c \iff T \le k$) for some statistic T and some value k then it suffices to find k_α such that $\mathsf{P}(T \ge k_\alpha) = \alpha$ (resp., $\mathsf{P}(T \le k_\alpha) = \alpha$) when H_0 is true.

generalized likelihood ratio test For hypotheses H_0 and H_a about parameter θ and MLE $\hat{\theta}$, reject H_0 if $LR = \frac{L(\hat{\theta})}{\max_{\theta \in H_0} L(\theta)} \ge c_\alpha$ where $\max_{\theta \in H_0} \mathsf{P}(LR \ge c_\alpha) = \alpha$.

If $H_0 : \theta = \theta_0$ and $H_a : \theta \ne \theta_0$ then $LR = \frac{L(\hat{\theta})}{L(\theta_0)}$.

uniformly most powerful test A test is UMP if it has maximum possible power for every parameter value θ that satisfies H_a.
In particular, if the test is the same as the Neyman-Pearson test for each θ satisfying H_a then it is UMP.

Wald test If $\hat{\theta}$ is asymptotic normal(θ, V_n) and $\hat{V}_n$ is an estimator for V_n, reject $H_0 : \theta = \theta_0$ when $\frac{|\hat{\theta}-\theta_0|}{\sqrt{\hat{V}_n}} \geq z_{1-\alpha/2}$, where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ quantile of the normal(0,1) dist.
Equivalently, reject H_0 if $\frac{(\hat{\theta}-\theta_0)^2}{\hat{V}_n} \geq \chi^2_{1,1-\alpha}$.
Important case: $\hat{\theta}$ is the MLE and $\hat{V}_n = 1/I_n(\hat{\theta})$.

(asymptotic) score test For MLE $\hat{\theta}$, reject $H_0 : \theta = \theta_0$ when $\frac{|\frac{d}{d\theta}\,\ell(\theta_0)|}{\sqrt{I_n(\theta_0)}} \geq z_{1-\alpha/2}$, where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ quantile of the normal(0,1) dist. Equivalently, reject H_0 if $\frac{(\frac{d}{d\theta}\,\ell(\theta_0))^2}{I_n(\theta_0)} \geq \chi^2_{1,1-\alpha}$.
A related test is to reject $H_0 : \theta = \theta_0$ when $\sqrt{I_n(\theta_0)}\,|\hat{\theta}-\theta_0| \geq z_{1-\alpha/2}$, and $\hat{\theta}$ is the MLE.

asymptotic likelihood ratio test Using the generalized LR statistic, reject $H_0 : \theta = \theta_0$ when $2\log(LR) \geq z^2_{1-\alpha/2} = \chi^2_{1,1-\alpha}$.

test equivalent to interval Define a test from an interval (or an interval from a test) by: reject $H_0 : \theta = \theta_0$ at level $\alpha \iff \theta_0$ is not in the $1-\alpha$ confidence interval.

prior and posterior distributions If the prior density (or pmf) for θ is $f_\Theta(\theta)$ then the posterior density (or pmf) is $f_\Theta(\theta|\underline{X}) = c(\underline{X}) f_{\underline{X}}(\underline{X}|\theta) f_\Theta(\theta)$, with $c(\underline{X})$ chosen so that $f_\Theta(\theta|\underline{X})$ is a proper pdf (pmf) in θ.

Bayes estimator Either the mean or the mode of the posterior distribution.

Bayes γ credible interval An interval $(L(\underline{X}), U(\underline{X}))$ such that, under the posterior distribution, $\mathsf{P}(L(\underline{X}) \leq \theta \leq U(\underline{X})\,|\underline{X}) = \gamma$.
The interval is HPD (highest posterior density) if it equals the set $\{\theta : f_\Theta(\theta|\underline{X}) \geq c\}$ for some constant c.