Statistics 630 - Assignment 4
(partial solutions)

1. Exer. 2.7.4.

   (a) The total double integral of $f(x,y)$ must be 1. So

   $$1 = \int_0^1 \int_0^1 (2x^2y + Cy^5)\,\mathrm{d}y\mathrm{d}x = \int_0^1 (x^2 + \frac{C}{6})\,\mathrm{d}x = \frac{2+C}{6},$$

   implying $C = 4$. [Note: you can integrate in either order.] The marginal pdf for X was found implicitly above (the inner integral): $f_X(x) = x^2 + \frac{2}{3}$, for $0 < x < 1$. The marginal pdf for Y is found likewise: $f_Y(y) = \frac{2}{3}y + 4y^5$, for $0 < y < 1$.

   $$\mathsf{P}(X \le 0.8, Y \le 0.6) = \int_0^{0.8} \int_0^{0.6} (2x^2y + 4y^5)\,\mathrm{d}y\mathrm{d}x = 0.086323.$$

   (d) $C = \frac{40^6}{36}$. $f_Y(y) = \frac{6}{4}(\frac{x}{4})^5$, $0 < x < 4$, and $f_X(x) = \frac{6}{10}(\frac{y}{10})^5$, $0 < y < 10$.

   $$\mathsf{P}(X \le 0.8, Y \le 0.6) = \Big(\frac{0.8}{4}\Big)^6 \Big(\frac{0.6}{10}\Big)^6.$$

2. Exer. 2.7.8.

   (a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \frac{x^2+2}{9}$, if $-2 < x < 1$; otherwise $f_X(x) = 0$. [Check that this is nonnegative and integrates to 1.]

   (b) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \frac{y+1}{12}$, if $0 < y < 4$; otherwise $f_Y(y) = 0$. [Check that this is nonnegative and integrates to 1.]

   (c) $\mathsf{P}(Y < 1) = \int_0^1 \frac{y+1}{12}\,\mathrm{d}y = \frac{1}{8}$.

   (d) $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,\mathrm{d}s\mathrm{d}t$

   $$= \begin{cases} 0 & \text{if } x < -2 \text{ or } y < 0, \\ \frac{1}{36}\Big(\frac{x^3+8}{3}y + (x+2)\frac{y^2}{2}\Big) & \text{if } -2 \le x < 1 \text{ and } 0 \le y < 4, \\ \frac{x^3+6x+20}{27} & \text{if } -2 \le x < 1 \text{ and } y \ge 4, \\ \frac{y^2+2y}{24} & \text{if } x \ge 1 \text{ and } 0 \le y < 4, \\ 1 & \text{if } x \ge 1 \text{ and } y \ge 4. \end{cases}$$

   [Check that this is continuous (at $x = -2, 1$ and at $y = 0, 4$) and that the mixed partial derivative at all (x,y) is the pdf.]

   Exer. 2.7.9. For parts (a) and (b), take care with the constraint on the variables. For example,

   $$f_Y(y) = \int_0^y \frac{(x^2+y)}{4}\,dx = \frac{y^3+3y^2}{12} \quad \text{for } 0 < y < 2.$$

   For part (c) you can either use the marginal above or you can integrate the joint pdf (in the other order), whichever seems easier (or do both to check):

   $$\mathsf{P}(Y < 1) = \int_0^1 \int_x^1 \frac{(x^2+y)}{4}\,dydx = \cdots = \frac{5}{48}.$$

(continued next page)

Exer. 2.7.16. (a) $C = 2$. (Take note that the pdf is positive only if $y > x$.)

(b) $f_X(x) = \int_x^\infty 2\mathrm{e}^{-x-y}\,\mathrm{d}y = 2\mathrm{e}^{-2x}$ for $x > 0$. So $X \sim$ exponential(2).
$f_Y(y) = \int_0^y 2\mathrm{e}^{-x-y}\,\mathrm{d}x = 2\mathrm{e}^{-y}(1 - \mathrm{e}^{-y})$ for $y > 0$.

3. Exer. 2.7.10. (a) $X \sim \text{normal}(3, 4)$. (b) $Y \sim \text{normal}(5, 16)$. (c) No, because $\rho = .5 \neq 0$ (see discussion after Example 2.7.9).

4. It is actually easiest to do part (b) along the way to doing part (a). Following the book's hint to let $x_1 = (1 - x_2)u$ in the integral over x_1 from 0 to $1 - x_2$,

$$\begin{aligned} f_{X_2}(x_2) &= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^{1-x_2} x_1^{\alpha_1-1} x_2^{\alpha_2-1} (1 - x_1 - x_2)^{\alpha_3-1}\,dx_1 \\ &= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3) x_2^{\alpha_2-1}(1-x_2)^{\alpha_1+\alpha_3-1}}{\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_3)} \int_0^1 \frac{\Gamma(\alpha_1+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_3)} u^{\alpha_1-1}(1-u)^{\alpha_3-1}\,du \\ &= \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_3)} x_2^{\alpha_2-1}(1-x_2)^{\alpha_1+\alpha_3-1}, \end{aligned}$$

because the integral is the total of a beta(α_1, α_3) pdf. We then see that X_2 has a beta$(\alpha_2, \alpha_1 + \alpha_2)$ pdf, which of course also integrates to 1, proving part (a).

5. Exer. 2.8.2. It may help to make a little table of the joint pmf. More formally, we can write

$$p_{X,Y}(x, y) = \frac{1}{16}\Big(1 + 3I_{\{(-2,5)\}}(x, y) + 7I_{\{(9,3)\}}(x, y)\Big) I_{\{-2,9,13\}}(x) I_{\{3,5\}}(y),$$

which does not factor into separate functions for x and y. So X and Y are not independent. We also have

$$p_X(x) = \frac{1}{16}\Big(5I_{\{-2\}}(x) + 9I_{\{9\}}(x) + 2I_{\{13\}}(x)\Big), \quad p_Y(y) = \frac{1}{8}\Big(5I_{\{3\}}(y) + 3I_{\{5\}}(y)\Big).$$

Exer. 2.8.5. For example, $\mathsf{P}(Y = 4|X = 9) = \frac{1/9}{1/9+2/9+3/9} = \frac{1}{6}$.

Exer. 2.8.14. (Recall Exer. 2.7.8 for the marginal pdfs.)

(a) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x^2+y}{4(x^2+2)}$ for $0 < y < 4$. [Check: integrate over y to get 1.]

(b) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{x^2+y}{3(y+1)}$ for $-2 < x < 1$. [Check: integrate over x to get 1.]

(c) No. $f_{Y|X}(y|x)$ is not independent of x (and thus not equal to $f_Y(y)$). Likewise, $f_{X|Y}(x|y)$ is not independent of y. [You can also note that the joint pdf does not factor.]

6. $F_Y(y) = y^\alpha$ for $0 < y < 1$. Thus, $F_{Y_{(1)}}(y) = 1 - (1 - F_Y(y))^n = 1 - (1 - y^\alpha)^n$ and $F_{Y_{(n)}}(y) = F_Y(y)^n = y^{n\alpha}$, again for $0 < y < 1$.

7. $F_Y(y) = 1 - e^{-y^\alpha}$ and $f_Y(y) = \alpha y^{\alpha-1} e^{-y^\alpha}$ for $y > 0$. So $F_{Y_{(1)}}(y) = 1 - e^{-ny^\alpha}$ and $f_{Y_{(1)}}(y) = n\alpha y^{\alpha-1} e^{-ny^\alpha}$ for $y > 0$. This is a Weibull$(\alpha, n^{-1/\alpha})$ distribution (using the formulation that has the scale parameter for the second parameter).