Stat 642 - Solutions for Assignment 9

Problem I. (30 points) : For a 2^{n-p} fractional factorial design we have the following results:

c1. The fraction of the full design is 2^{-p}.

c2. There are p generators.

c3. The number of Generalized Interactions (Implicit Generators) equals the number of newly created contrasts by given p generators. So, the number of generalized interactions is $\sum_{k=2}^{p}\binom{p}{k} = 2^p - p - 1$ for $p \geq 2$ and 0 for $p = 1$.

c4. The number of effects in each of the $2^{n-p} - 1$ alias sets is given by $1 + p + (2^p - p - 1) = 2^p$. Therefore, each effect has $2^p - 1$ aliases, that is, effect is confounded with $2^p - 1$ effects.

c5. The number of experimental units required to conduct the experiment is 2^{n-p} .

Design	c1.	c2.	c3.	c4.	c5.
2^{6-2}	$\frac{1}{4}$	2	1	3	16
2^{7-3}	$\frac{1}{8}$	3	4	7	16
2^{7-4}	$\frac{1}{16}$	4	11	15	8

Problem II. (16 points) Construct a 2^{7-3} fractional factorial using ABCF, CDEF, and ADFG as the design generators. We will randomly select +1 for ABCF, +1 for CDEF, and -1 for ADFG.

The implicit generators are ABDE, BCDG, ACEG, BEFG (see part b.). Therefore use E=ABD, F=ABC, and G=-BCD in the following R code to obtain the specified design:

```
install.packages("FrF2")
library(FrF2)
design = FrF2(nruns=16,nfactors=7,generators=c("ABD","ABC","-BCD"))
design
design$y = c(1:nrow(design))
alias_sets = aliases(lm(y~(.)^5,data=design))
alias_sets
```

1. The 16 treatments are given in the following table:

TRT	A	B	C	D	E	F	G	ABCF	CDEF	ADFG	ABDE	ACEG	BCDG	BEFG
1	1	1	1	1	1	1	-1	1	1	-1	1	-1	-1	-1
2	1	1	1	-1	-1	1	1	1	1	-1	1	-1	-1	-1
3	1	1	-1	1	1	-1	1	1	1	-1	1	-1	-1	-1
4	1	1	-1	-1	-1	-1	-1	1	1	-1	1	-1	-1	-1
5	1	-1	1	1	-1	-1	1	1	1	-1	1	-1	-1	-1
6	1	-1	1	-1	1	-1	-1	1	1	-1	1	-1	-1	-1
7	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	-1	-1
8	1	-1	-1	-1	1	1	1	1	1	-1	1	-1	-1	-1
9	-1	1	1	1	-1	-1	-1	1	1	-1	1	-1	-1	-1
10	-1	1	1	-1	1	-1	1	1	1	-1	1	-1	-1	-1
11	-1	1	-1	1	-1	1	1	1	1	-1	1	-1	-1	-1
12	-1	1	-1	-1	1	1	-1	1	1	-1	1	-1	-1	-1
13	-1	-1	1	1	1	1	1	1	1	-1	1	-1	-1	-1
14	-1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	-1
15	-1	-1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	-1
16	-1	-1	-1	-1	-1	-1	1	1	1	-1	1	-1	-1	-1

From the above matrix, we can see that the effects given by the generators and the implicit generators are not estimable.

2. $(ABCF) \times (CDEF) = ABDE$; $(ABCF) \times (ADFG) = BCDG$;
$(ADFG) \times (CDEF) = ACEG$; $(ABCF) \times CDEF) \times (ADFG) = BEFG$;
and so the implicit generators (generalized interactions) are ABDE, BCDG, ACEG, BEFG.

3. Resolution IV since the length of the shortest generator (including the implicit generators) is 4. Also, no main effect is confounded with any other main effect or two-factor interaction, but two-factor interactions are confounded with one another. See the alias sets in part d.

4. From the defining contrasts and implicit generators,

$I = ABCF = CDEF = ADFG = ABDE = BCDG = ACEG = BEFG$, the $2^{n-p} - 1 = 15$ alias sets are

Set								
0	I	$ABCF$	$CDEF$	$ADFG$	$ABDE$	$BCDG$	$ACEG$	$BEFG$
1	A	BCF	$ACDEF$	DFG	BDE	$ABCDG$	CEG	$ABEFG$
2	B	ACF	$BCDEF$	$ABFG$	ADE	CDG	$ABCEG$	EFG
3	C	ABF	DEF	$ACFG$	$ABCDE$	BDG	AEG	$CBEFG$
4	D	$ABCDF$	CEF	AFG	ABE	BCG	$ACDEG$	$BDEFG$
5	E	$ABCEF$	CDF	$ADEFG$	ABD	$BCDEG$	ACG	BFG
6	F	ABC	CDE	ADG	$ABDEF$	$BCDFG$	$ACEFG$	BEG
7	G	$ABCFG$	$CDEFG$	ADF	$ABDEG$	BCD	ACE	BEF
8	AB	CF	$ABCDEF$	$BDFG$	DE	$ACDG$	$BCEG$	$AEFG$
9	AC	BF	$ADEF$	$CDFG$	$CBDE$	$ABDG$	EG	$ABCEFG$
10	AD	$BCDF$	$ACEF$	FG	BE	$ABCG$	$CDEG$	$ABDEFG$
11	AE	$BCEF$	$ACDF$	$DEFG$	BD	$ABCDEG$	CG	$ABFG$
12	AF	BC	$ACDE$	DG	$ABDE$	$ABCDFG$	$CEFG$	$ABEG$
13	AG	$ABCFG$	$ACEFG$	DF	$BDEG$	$ABCD$	CE	$ABEF$
14	BG	$ACFG$	$BCDEFG$	$ABDF$	$ADEG$	CD	$ABCE$	EF
15	ABG	CFG	$ABCDEFG$	BDF	DEG	ACD	BCE	AEF

Problem III. (8 points) Consider a 2^{5-2}_{III} fractional factorial design with defining relationship $I_1 = ABC$ and $I_2 = BCDE$.

1. Alias sets obtained from the generating effects and the implicit generator are

Alias Set	I	=	ABC	=	BCDE	=	ADE
1	A	=	BC	=	ABCDE	=	DE
2	B	=	AC	=	CDE	=	ABDE
3	C	=	AB	=	BDE	=	ACDE
4	D	=	ABCD	=	BCE	=	AE
5	E	=	ABCE	=	BCD	=	AD
6	BD	=	ACD	=	CE	=	ABE
7	BE	=	ACE	=	CD	=	ABD.

The alias sets and treatments can be obtained using R-code by letting E' = A and A' = E. The generators are now BCE' and A'BCD and R-code can now be used with the command

design = FrF2(nruns=8,nfactors=5,generators=c(”ABC”,”BC”))

because D=A'BC and E'=BC. After obtaining the Treatments and Alias sets in the R output, reverse the symbols, A to E and E to A to obtain the treatments corresponding to the generators ABC and BCDE.

But factors A, B, and C do not interact with one another and factors C, D, and E do not interact with one another, that is,

the following two-way interactions are considered to be negligible: AB, AC, BC, CD, CE, DE and the 3-way interactions ABC and CDE are negligible.

By assumption all 3-way, 4-way and 5-way interactions are negligible.

Therefore, the main effects of A, B, and C and the 2-way interactions BD and BE can be estimated because all the effects that they are confounded with are negligible.

That is, the main effects of A, B and C and the 2-way interactions BD and BE can be estimated without being confounded with any other effects. However, the main effects of D and E are confounded with 2-way interactions which are not negligible, AE and AD, respectfully.

2. To improve the Design given in part 1., use the generators $I_1 = ABC$ and $I_2 = CDE$. The implicit generator is then $I_3 = ABDE$ and the resolution is still III. However, in this design, main effects are no longer confounded with any non-negligible two-factor interactions or other main effects. However, AD is confounded with BE and AE is confounded with BD, so not all of the two factor interactions can be estimated unless more of the two-factor interactions can be determined to be negligible prior to running the experiment.

Problem IV. . (12 points) In a Resolution VIII, 2^{n-p} design:

1. Main Effects (s=1) are not confounded with effects having fewer than 8-1=7 elements, that is, with any main effects, two-way interactions, three-way interactions, four-way interactions, five-way interactions, and six-way interactions.

2. Two-way interactions (s=2) are not confounded with effects having fewer than 8-2=6 elements, that is, with main effects, two-way interactions, three-way interactions, four-way interactions, and five-way interactions.

3. Three-way interactions (s=3) are not confounded with effects having fewer than 8-3=5 elements, that is, with main effects, two-way interactions, three-way interactions, and four-way interactions.

4. Four-way interactions (s=4) are not confounded with effects having fewer than 8-4=4 elements, that is, with main effects, two-way interactions, and three-way interactions elements.

Problem V. (10 points) Using the SAS program fractional,main.sas or R-code from Handout 10, a 1/8 fraction of a 2^8 (a 2^{8-3} Design) factorial experiment with resolution IV is obtained using the +1 levels of the three generators: $ABFG$, $ACFH$ and $ABCDEF$.

TRT	A	B	C	D	E	F	G	H	$ABFG$	$ACFH$	$ABCDEF$
(I)	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1
efgh	-1	-1	-1	-1	1	1	1	1	1	1	1
dfgh	-1	-1	-1	1	-1	1	1	1	1	1	1
de	-1	-1	-1	1	1	-1	-1	-1	1	1	1
cfg	-1	-1	1	-1	-1	1	1	-1	1	1	1
ceh	-1	-1	1	-1	1	-1	-1	1	1	1	1
cdh	-1	-1	1	1	-1	-1	-1	1	1	1	1
cdefg	-1	-1	1	1	1	1	1	-1	1	1	1
bfh	-1	1	-1	-1	-1	1	-1	1	1	1	1
beg	-1	1	-1	-1	1	-1	1	-1	1	1	1
bdg	-1	1	-1	1	-1	-1	1	-1	1	1	1
bdeh	-1	1	-1	1	1	1	-1	1	1	1	1
bch	-1	1	1	-1	-1	-1	1	1	1	1	1
bcef	-1	1	1	-1	1	1	-1	-1	1	1	1
bcdf	-1	1	1	1	-1	1	-1	-1	1	1	1
bcdegh	-1	1	1	1	1	-1	1	1	1	1	1
af	1	-1	-1	-1	-1	1	-1	-1	1	1	1
aegh	1	-1	-1	-1	1	-1	1	1	1	1	1
adgh	1	-1	-1	1	-1	-1	1	1	1	1	1
adef	1	-1	-1	1	1	1	-1	-1	1	1	1
acg	1	-1	1	-1	-1	-1	1	-1	1	1	1
acefh	1	-1	1	-1	1	1	-1	1	1	1	1
acdfh	1	-1	1	1	-1	1	-1	1	1	1	1
acdeg	1	-1	1	1	1	-1	1	-1	1	1	1
abh	1	1	-1	-1	-1	-1	-1	1	1	1	1
abefg	1	1	-1	-1	1	1	1	-1	1	1	1
abdfg	1	1	-1	1	-1	1	1	-1	1	1	1
abdeh	1	1	-1	1	1	-1	-1	1	1	1	1
abcfgh	1	1	1	-1	-1	1	1	1	1	1	1
abce	1	1	1	-1	1	-1	-1	-1	1	1	1
abcd	1	1	1	1	-1	-1	-1	-1	1	1	1
abcdefgh	1	1	1	1	1	1	1	1	1	1	1

Problem VI. (24 points) Instant Soup Problem:

1. The design has resolution = V because we have a 2^{5-1} factorial design ($16 = 2^4 = 2^{5-1}$) with generator $I = ABCDE$, which is of length 5. We know this is the generator because the 16 selected treatments all have $ABCDE = +1$.

2. E=ABCD, that is, the value for E in any given row is obtained by multiplying the values of A, B, C, and D in that row.

3. The alias structure is identical to the alias structure in Table 4 on page 15 in Handout 10.

4. Because main effects are confounded with four-factor interactions and two-factor interactions are confounded with three-factor interactions, it is necessary to assume that three-factor or higher interactions are negligible in order to be able to estimate main effects and two-factor interactions free of any other effects.

5. The estimated main effects, two-way interactions, and the estimated standard errors for the estimated Main effects and Two-way interactions are computed as follows , where r is the number of runs in the experiment. :

   Main Effects: $\bar{Y}_+ - \bar{Y}_-$ = (Sum of Data associated with + for Factor)/(r/2) - (Sum of Data associated with - for Factor)/(r/2)

   Main Effect of Factor A: $(10.39)/8 - (9.23)/8 = 0.145$

   SE of Main Effects: $\widehat{SE}(\bar{Y}_+ - \bar{Y}_-) = \sqrt{\widehat{Var}(\bar{Y}_+ - \bar{Y}_-)} = \sqrt{\frac{2\hat{\sigma}_e^2}{r/2}} = \sqrt{\frac{4\hat{\sigma}_e^2}{16}} = \hat{\sigma}_e/2$

   Two-way Interaction: $[\bar{Y}_{++} - \bar{Y}_{+-})] - [\bar{Y}_{-+} - \bar{Y}_{--}]$ = (Sum of Data associated with + in product of two Factors)/(r/4) - (Sum of Data associated with - in product of two Factors)/(r/4)

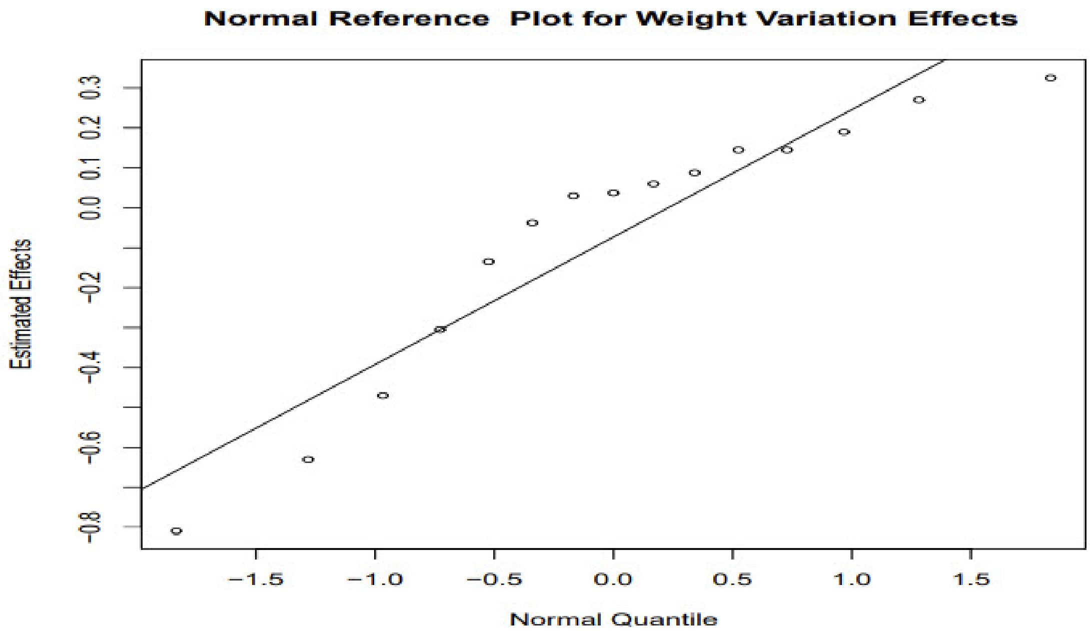
   Two-way Interaction of AB: $(9.87 - 9.75)/(16/4) = .03$

   SE of Two-way Interaction: $\widehat{SE}([\bar{Y}_{++} - \bar{Y}_{+-})] - [\bar{Y}_{-+} - \bar{Y}_{--}]) = \sqrt{\widehat{Var}([\bar{Y}_{++} - \bar{Y}_{+-})] - [\bar{Y}_{-+} - \bar{Y}_{--}])} = \sqrt{\frac{4\hat{\sigma}_e^2}{r/4}} = \sqrt{\frac{16\hat{\sigma}_e^2}{16}} = \hat{\sigma}_e$

   There are 5 df for main effects and 10 df for two-way interactions. This leaves 16-1-5-10 = 0 df for error. Therefore, $\hat{\sigma}_e = \sqrt{MSE}$ cannot be computed. Thus, the standard errors for both the estimated main effects and estimated two-way interactions cannot be estimated. The estimates of main effects and two-way interactions are given below:

A(.145)	B(.0875)	C(.0375)	D(-.0375)	E(-.47)
AB(.03)	AC(.19)	AD(.06)	AE(-.305)	BC(-.135)
BD(.325)	BE(-.81)	CD(.145)	CE(.27)	DE(-.63)

- From normal probability plot, it would appear that E, BE, BD, and DE are effects which may impact the weight variation. Thus, a more detailed experiment could be run with just the factors B, D, and E.



6. From the 16 runs shown in the given design, a 1/4 fraction of a 2^5 factorial experiment with defining equations $I = -ABC = -ABCDE$ can be obtained by taking those treatments having $ABC = -1$

Run	A	B	C	D	E	ABC	y
12	-1	-1	-1	-1	-1	-1	1.13
1	-1	-1	-1	1	1	-1	0.78
16	-1	1	1	-1	-1	-1	1.18
9	-1	1	1	1	1	-1	0.76
4	1	-1	1	-1	-1	-1	1.28
2	1	-1	1	1	1	-1	1.10
3	1	1	-1	-1	-1	-1	1.70
10	1	1	-1	1	1	-1	0.62

- However, note that the treatments listed above constitute a 2^{5-2} design (quarter fraction of the complete 2^5 design) with defining generators $I = ABCDE = ABC$ and implied generator $I = DE$. Thus the above design has resolution 2 (RES=II) and the main effects of factors D and E are confounded. The researcher would have an improved design by using the generators $I = ABD = CDE$ with implied generator $I = ABCE$. The resolution would then be 3 (RES=III).