Statistics 630 – Exam II
Friday, 5 April 2013

Printed Name: ________________________ **Email:** ________________________

INSTRUCTIONS FOR THE STUDENT:

1. You have exactly 75 minutes to complete the exam.
2. There are 5 pages including this cover sheet and the formula sheets.
3. There are 6 questions. Each question is worth 10 points.
4. Please write out the answers to the exam questions on *blank* sheets of paper. Start each question on a *separate* sheet of paper (and return them in order).
5. Answer all questions fully, explaining your steps. You may refer to theorems by name/description rather than by its number in the book.
6. *Do not use a calculator.* You may leave answers in forms that can easily be put into a calculator such as $\frac{12}{19}$, $\binom{40}{5}$, e^{-3}, $\Phi(1.5)$, etc.
7. Do not discuss or provide any information to anyone concerning any of the questions on this exam or your solutions until I post my solutions.
8. You may use the *attached formula sheets.* No other resources are allowed. Do not use the textbook, the class notes or the formula sheets that were posted online.

I attest that I spent no more than 75 minutes to complete the exam. I used only the materials allowed above. I did not receive assistance from or provide assistance to anyone either before or while taking this exam.

Student's Signature ____________________________________

INSTRUCTIONS FOR THE PROCTOR:

1. Download the exam from the student's account on WebAssign and print it. The student should not view the exam content until it is time to start.
2. The student may take 75 minutes to complete the exam. Record the time at which the student starts the exam: ________________________
3. Record the time at which the student ends the exam: ________________________
4. Immediately after the student completes the exam, please scan *this page* and the student's *solutions* to a *single, unsecured* PDF file and oversee the student uploading it to WebAssign. Be sure the solutions are legible and in order. (You have 105 minutes from the time the exam was downloaded, allowing 30 minutes for printing, scanning and uploading. Please contact Penny Jackson or Kim Ritchie for any necessary delay.)
5. Collect all pages of this exam and the student's solution. Do not allow the student to take any page. You may return them to the student *one week* after the exam.

I attest that the student and I have followed the INSTRUCTIONS FOR THE STUDENT and PROCTOR listed above and that the exam was downloaded, printed, scanned into a PDF file and uploaded to WebAssign in my presence.

Proctor's Signature ____________________________________

For several of these problems, referring to the formula sheets may be helpful.

1. $T_1, \ldots, T_n$ are iid random variables with mean $\gamma^{1/2}$ and variance 2γ. Identify a moment estimator for γ based on the equation $\mathsf{E}(T_i) = \gamma^{1/2}$ and determine its bias. Hint: if $S(\underline{T})$ is a statistic then $\mathsf{E}(S(\underline{T})^2) = \mathsf{Var}(S(\underline{T})) + (\mathsf{E}(S(\underline{T})))^2$.

2. Find the maximum likelihood estimator (MLE) of θ for a random sample of size n from the pdf
$$f(x|\theta) = \sqrt{\frac{\theta}{\pi}}\, e^{-\theta(x+x^2+1/4)}, \quad x \in \mathbb{R}, \theta > 0.$$

3. The conditional distribution of Y, given X, is gamma$(5, 1/X)$, and the distribution of X is gamma(2,2). Find $\mathsf{E}(Y)$ and $\mathsf{Var}(Y)$.

4. Suppose X and Y are independent random variables with moment generating functions $M_X(s) = \mathsf{E}(e^{sX}) = e^{2s^2}$ and $M_Y(s) = \mathsf{E}(e^{sY}) = e^{s^2-2s}$, respectively. Find the moment generating function for $V = Y - 2X$.

5. The maximum likelihood estimator of λ for a random sample of $n = 7$ gamma$(3, \lambda)$ random variables is $\hat{\lambda} = \frac{3}{\bar{X}}$, where $\bar{X}$ is the sample mean. Assume you also know $\mathsf{E}(\hat{\lambda}) = \frac{21}{20}\lambda$ and $\mathsf{Var}(\hat{\lambda}) = \frac{441}{7600}\lambda^2$. (Those first two sentences are given – *do not prove them!*) Consider an alternative estimator $c\hat{\lambda}$. Write down an expression for its mean squared error, $\mathsf{MSE}(c\hat{\lambda})$. (No need to simplify the algebra.)

6. Suppose $X_1, \ldots, X_5$ are independent normal(0,3) random variables. Identify the distributions of $R = \frac{X_1^2+\cdots+X_5^2}{3}$, $S = \frac{4X_1^2}{X_2^2+\cdots+X_5^2}$ and $T = \frac{2X_1}{\sqrt{X_2^2+\cdots+X_5^2}}$.

The course lecturer, Dr. Wehrly, has reviewed and approved this exam.

Formulas for Exam II

Bayes' rule $\mathsf{P}(B_j \,|\, A) = \frac{\mathsf{P}(A\,|\,B_j)\mathsf{P}(B_j)}{\sum_{k=1}^n \mathsf{P}(A\,|\,B_k)\mathsf{P}(B_k)}$ if $B_1,\dots,B_n$ are disjoint and $\bigcup_{k=1}^n B_k = S$.

quantile function $Q_X(p)$ satisfies $F_X(x) \le p \le F(Q_X(p))$ for all $x < Q_X(p)$. $F(Q_X(p)) = p$ if X is a continuous rv.

distribution of a function of X $F_Y(y) = \mathsf{P}(h(X) \le y)$ for $Y = h(X)$.

If X is a discrete rv or $h(x)$ takes only countably many values then Y has pmf $p_Y(y) = \mathsf{P}(h(X) = y)$.

If X is a continuous rv and $h(x)$ is a continuous function then Y has pdf $f_Y(y) = \frac{d}{dy}\mathsf{P}(h(X) \le y)$.

binomial theorem $\sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = (a+b)^n$.

geometric sum $\sum_{k=n}^\infty a^k = \frac{a^n}{1-a}$ if $-1 < a < 1$.

exponential expansion $\sum_{k=0}^\infty \frac{a^k}{k!} = e^a$.

integral of a power function $\int_u^v x^a\,dx = \frac{v^{a+1}-u^{a+1}}{a+1}$ if $a \ne -1$, and $\int_u^v x^{-1}\,dx = \log_e(v/u)$.

integral of an exponential function $\int_u^v e^{ax}\,dx = \frac{1}{a}\left(e^{av} - e^{au}\right)$.

gamma integral $\int_0^\infty x^a e^{-x}\,dx = \Gamma(a+1) = a!$ for $a > -1$.

integral of exponential of a quadratic $\int_{-\infty}^\infty e^{a+bx-cx^2}\,dx = \sqrt{\frac{\pi}{c}}\, e^{b^2/(4c)+a}$ for $c > 0$.

Bernoulli pmf $p(x) = (1-\theta)^{1-x}\theta^x I_{\{0,1\}}(x)$ for $0 < \theta < 1$, same as binomial$(1,\theta)$.

beta(a,b) pdf $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1} I_{(0,1)}(x)$ for $a > 0$, $b > 0$; $\mathsf{E}(X) = \frac{a}{a+b}$ $\mathsf{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$.

binomial(n,θ) pmf $p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x} I_{\{0,1,\dots,n\}}(x)$ for $0 < \theta < 1$. $\mathsf{E}(X) = n\theta$, $\mathsf{Var}(X) = n\theta(1-\theta)$, $m(s) = (1-\theta+\theta e^s)^n$.

chi-square(n) same as gamma$(\frac{n}{2},\frac{1}{2})$, the distribution of $Z_1^2 + \cdots + Z_n^2$ for iid standard normal $Z_1,\dots,Z_n$. $\mathsf{E}(X) = n$, $\mathsf{Var}(X) = 2n$.

In particular, if $X_1,\dots,X_n \overset{\text{iid}}{\sim}$ normal(μ,σ^2) then $\frac{(n-1)S^2}{\sigma^2} \sim$ chi-square$(n-1)$.

discrete uniform(N) pmf $p(x) = \frac{1}{N} I_{\{1,2,\dots,N\}}(x)$. $\mathsf{E}(X) = \frac{N+1}{2}$, $\mathsf{Var}(X) = \frac{N^2-1}{12}$.

exponential(λ) pdf $f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$ for $\lambda > 0$, same as gamma$(1,\lambda)$. $\mathsf{E}(X) = \frac{1}{\lambda}$, $\mathsf{Var}(X) = \frac{1}{\lambda^2}$.

F(m,n) the distribution of $\frac{X/m}{Y/n}$ where $X \sim$ chi-square(m), $Y \sim$ chi-square(n), independent. $\mathsf{E}(X) = \frac{n}{n-2}$ if $n > 2$.

gamma(α,λ) pdf $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x)$ for $\lambda > 0$, $\alpha > 0$; $\mathsf{E}(X) = \frac{\alpha}{\lambda}$, $\mathsf{Var}(X) = \frac{\alpha}{\lambda^2}$, $m(s) = \left(\frac{\lambda}{\lambda-s}\right)^\alpha$ if $s < \lambda$.

geometric(θ) pmf $p(x) = \theta(1-\theta)^x I_{\{0,1,2,\ldots\}}(x)$ for $0 < \theta < 1$, same as negative binomial$(1, \theta)$. $\mathsf{E}(X) = \frac{1-\theta}{\theta}$, $\mathsf{Var}(X) = \frac{1-\theta}{\theta^2}$.

hypergeometric(N, M, n) pmf $p(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} I_{\{0,1,\ldots,n\}}(x)$ for $M < N$. $\mathsf{E}(X) = np$ where $p = \frac{M}{N}$, $\mathsf{Var}(X) = \frac{N-n}{N-1} np(1-p)$.

negative binomial(r, θ) pmf $p(x) = \binom{r+x-1}{r-1}\theta^r(1-\theta)^x I_{\{0,1,2,\ldots\}}(x)$ for $0 < \theta < 1$. $\mathsf{E}(X) = \frac{r(1-\theta)}{\theta}$, $\mathsf{Var}(X) = \frac{r(1-\theta)}{\theta^2}$, $m(s) = \left(\frac{\theta}{1-(1-\theta)e^s}\right)^r$ if $s < -\log(1-\theta)$.

normal(μ, σ^2) pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} I_{(-\infty,\infty)}(x)$ for $\sigma^2 > 0$; $\mathsf{E}(X) = \mu$, $\mathsf{Var}(X) = \sigma^2$, $m(s) = e^{\mu s + \sigma^2 s^2/2}$.

Poisson(λ) pmf $p(x) = \frac{\lambda^x}{x!} e^{-\lambda} I_{\{0,1,2,\ldots\}}(x)$ for $\lambda > 0$. $\mathsf{E}(X) = \lambda$, $\mathsf{Var}(X) = \lambda$, $m(s) = e^{\lambda(e^s-1)}$.

t(n) the distribution of $\frac{Z}{\sqrt{Y/n}}$ where $Z \sim$ normal$(0,1)$, $Y \sim$ chi-square(n), independent. $\mathsf{E}(X) = 0$, $\mathsf{Var}(X) = \frac{n}{n-2}$ if $n > 2$.

In particular, if $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ normal(μ, σ^2) then $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim$ t$(n-1)$.

uniform(a, b) pdf $f(x) = \frac{1}{b-a} I_{(a,b)}(x)$ for $a < b$. $\mathsf{E}(X) = \frac{a+b}{2}$, $\mathsf{Var}(X) = \frac{(b-a)^2}{12}$.

Weibull(α, β) pdf $f(x) = \frac{\alpha}{\beta} (x/\beta)^{\alpha-1} e^{-(x/\beta)^\alpha} I_{(0,\infty)}(x)$ for $\alpha > 0$, $\beta > 0$. $\mathsf{E}(X^k) = \beta^k \Gamma(1 + \frac{k}{\alpha})$.

joint pmf/cdf $p(x, y) = \mathsf{P}(\{X = x\} \cap \{Y = y\})$, $F(x, y) = \sum_{u \le x} \sum_{v \le y} p(u, v)$.

joint pdf/cdf $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$, $F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, dv du$.

marginal pmf/pdf $p_X(x) = \sum_y p(x, y)$; $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$.

conditional pmf/pdf $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$; $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.

independent random variables $p(x, y) = p_X(x)p_Y(y)$ if (X, Y) is discrete; $f(x, y) = f_X(x)f_Y(y)$ if (X, Y) is continuous.

discrete convolution $p_{X+Y}(z) = \sum_x p_X(x)p_Y(z-x)$ for independent X, Y.

continuous convolution $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\, dx$ for independent X, Y.

cdf of minimum $F_{\min(X_1,\ldots,X_n)}(u) = 1 - \prod_{i=1}^{n}(1 - F_{X_i}(u))$ for independent $X_1, \ldots, X_n$.

cdf of maximum $F_{\max(X_1,\ldots,X_n)}(u) = \prod_{i=1}^{n} F_{X_i}(u)$ for independent $X_1, \ldots, X_n$.

expectation for a discrete rv $\mathsf{E}(h(X)) = \sum_x h(x)p_X(x)$.

expectation for a continuous rv $\mathsf{E}(h(X)) = \int_{-\infty}^{\infty} h(x)f_X(x)\, dx$.

mean and variance $\mu_X = \mathsf{E}(X)$; $\sigma_X^2 = \mathsf{Var}(X) = \mathsf{E}((X-\mu_X)^2) = \mathsf{E}(X^2) - \mu_X^2$.

standard deviation $\sigma_X = \sqrt{\mathsf{Var}(X)}$.

covariance and correlation $\mathsf{Cov}(X,Y) = \mathsf{E}((X-\mu_X)(Y-\mu_Y)) = \mathsf{E}(XY) - \mu_X\mu_Y$; $\mathsf{Corr}(X,Y) = \frac{\mathsf{Cov}(X,Y)}{\sigma_X\sigma_Y}$.

For independent X and Y, $\mathsf{Cov}(X,Y) = \mathsf{Corr}(X,Y) = 0$.

expectation of a sum $\mathsf{E}(a_1X_1 + \cdots + a_nX_n) = a_1\mathsf{E}(X_1) + \cdots + a_n\mathsf{E}(X_n)$.

expectation of a product If $X_1, \ldots, X_n$ are independent, $\mathsf{E}\left(\prod_{i=1}^n h_i(X_i)\right) = \prod_{i=1}^n \mathsf{E}(h_i(X_i))$.

variance of a sum $\mathsf{Var}(aX+bY) = a^2\mathsf{Var}(X) + 2ab\mathsf{Cov}(X,Y) + b^2\mathsf{Var}(Y)$.

variance of a sum of independent rvs $\mathsf{Var}(a_1X_1 + \cdots + a_nX_n) = a_1^2\mathsf{Var}(X_1) + \cdots + a_n^2\mathsf{Var}(X_n)$.

moments k-th moment is $\mu_k = \mathsf{E}(X^k)$, $k = 1, 2, \ldots$

moment generating function $m_X(s) = \mathsf{E}(e^{sX})$; $\mathsf{E}(X^k) = \frac{d^k}{ds^k} m_X(s)\Big|_{s=0}$.

mgf of a sum If X and Y are independent, $m_{aX+bY}(s) = \mathsf{E}(e^{(aX+bY)s}) = m_X(as)m_Y(bs)$.

conditional expectation $\mathsf{E}(h(Y)|X=x) = \sum_y h(y)p_{Y|X}(y|x)$ or $\mathsf{E}(h(Y)|X=x) = \int_{-\infty}^{\infty} h(y)f_{Y|X}(y|x)\,dy$.

iterated expectation $\mathsf{E}(h(Y)) = \mathsf{E}(\mathsf{E}(h(Y)|X))$.

conditional variance $\mathsf{Var}(Y|X) = \mathsf{E}(Y^2|X) - (\mathsf{E}(Y|X))^2$.

variance partition formula $\mathsf{Var}(Y) = \mathsf{E}(\mathsf{Var}(Y|X)) + \mathsf{Var}(\mathsf{E}(Y|X))$.

Markov's inequality $\mathsf{P}(|X| \geq x) \leq \frac{\mathsf{E}(|X|)}{x}$ for $x > 0$.

Chebyshev's inequality $\mathsf{P}(|X-\mu_X| \geq x) \leq \frac{\mathsf{Var}(X)}{x^2}$ for $x > 0$.

sample mean, variance, k-th moment $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$; $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$; $m_k = \frac{1}{n}\sum_{i=1}^n X_i^k$.

unbiased sample variance $S^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

law of large numbers (averages) For iid $X_1, X_2, \ldots$ with mean μ, $\mathsf{P}(|\bar{X}_n - \mu| > \epsilon) \to 0$ as $n \to \infty$, each $\epsilon > 0$.

central limit theorem For iid $X_1, X_2, \ldots$ with mean μ and variance σ^2, $\mathsf{P}\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\right) = \mathsf{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \leq z\right) \to \Phi(z)$, as $n \to \infty$, where $\Phi(z)$ is the normal(0,1) cdf.

method of moments for iid sample match the k-th population moment $\mathsf{E}_\theta(X^k)$ with the k-th sample moment m_k, and solve for the desired parameter estimates.

maximum likelihood for iid sample maximize the likelihood function $L(\theta|X_1, \ldots, X_n) = \prod_{i=1}^n f_\theta(X_i)$ or the log-likelihood $\ell(\theta|X_1, \ldots, X_n) = \log L(\theta|X_1, \ldots, X_n) = \sum_{i=1}^n \log f_\theta(X_i)$.

If $\log L(\theta)$ is differentiable and concave at θ, the MLE is a solution to $S(\theta) = \frac{d}{d\theta}\log L(\theta) = 0$. (For a multidimensional parameter θ this is a system of equations.)

bias and standard error $\mathsf{Bias}_\theta(\hat{\theta}) = \mathsf{E}_\theta(\hat{\theta}) - \theta$; $\mathsf{SE}_\theta(\hat{\theta}) = \sqrt{\mathsf{Var}_\theta(\hat{\theta})}$.

mean squared error $\mathsf{MSE}_\theta(\hat{\theta}) = \mathsf{E}_\theta((\hat{\theta} - \theta)^2) = \mathsf{Var}_\theta(\hat{\theta}) + (\mathsf{Bias}_\theta(\hat{\theta}))^2$.