Statistics 630 - Assignment 10

(due Monday, 29 November 2021)

Note: many of the problems refer to exercises in Section 7.1, but the (added) questions also involve material from Section 7.2.1–2.

1. Exercises 7.1.1. Note: samples of size 2, namely $\{(1,1),(1,2),(2,1),(2,2)\}$. (The book's solution is not correct.) You may construct an appropriate table. For each sample, identify which value of θ has the highest posterior probability (the posterior mode).

   Add the following.

   (b) Repeat for samples of size 3 (there are eight of them). For each sample, identify which value of θ has the highest posterior probability.

2. Exercise 7.1.2. (The problem refers to *Example* 7.1.1.) Add the following.

   (b) Suppose $\alpha = \beta = 2$ and $n = 50$, $\bar{x} = 0.56$. What is the posterior distribution for θ, and what are the posterior mean and variance?

   (c) Show that the mode of a beta(a, b) pdf is $\frac{a-1}{a+b-2}$ and use that to determine the posterior mode for part (b).

   (d) Plot, in the same graph, the prior and posterior pdf's for the situation in part (b).

   (e) For the situation in part (b), find the 90% and 95% credible intervals. You may use the equal-tails method (using the qbeta function in R to get the quantiles). [But if you are ambitious you can also try to find the HPD intervals. This would require dbeta and trial-and-error of some sort.]

3. Exercise 7.1.3. Add the following.

   (b) Repeat, except with $\sigma_0^2 = 4$.

   (c) Repeat, except with $\sigma_0^2 = 4$ and $\tau_0^2 = 2$.

   (d) Show and compute the 99% credible interval for μ for case (c).

4. Exercise 7.1.4. Add the following.

   (b) Find the posterior mean and variance.

   (c) Suppose the prior distribution has $\alpha = 2$, $\beta = 3$, and the sample data give $n = 20$, $\sum_{i=1}^{20} x_i = 27$. Determine the posterior distribution and its mean and variance.

   (d) Find the 95% (equal-tails) credible interval given the data in part (c). (Use the qgamma function in R.)

5. Exercise 7.1.9.

6. Exercise 7.2.10. Note: β_0 is the second parameter for the gamma prior. Do not confuse that with the parameter λ of the exponential distribution. Add the following.

   (b) Consider two cases: (1) the mean of the gamma prior is 4.0 and the standard deviation is 1.0, and (2) the mean of the gamma prior is 4.0 and the standard deviation is 4.0. Plot the two prior distributions for those cases, on the same graph, and compare them.

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(c) Suppose $n = 20$ and the sample mean is 5.1. Plot the two posterior distributions for those cases and compare them. Find the two posterior means and compare them.

(d) Find and compute the 90% credible intervals for each of the cases above.

7. Consider the hypotheses $H_0 : \theta = 0$ and $H_1 : \theta = 1$ which are to be compared with statistic X having one of the following distributions:

x	1	2	3	4	5	6
$f_0(x)$	0.02	0.13	0.35	0.35	0.13	0.02
$f_1(x)$	0.01	0.04	0.10	0.18	0.27	0.40

(a) If the prior probabilities are $\mathsf{P}(H_0) = \mathsf{P}(H_1)$, which outcomes favor H_0?

(b) If the prior probabilities are $\mathsf{P}(H_0) = 3\mathsf{P}(H_1)$, which outcomes favor H_0?

(c) Compute the likelihood ratio for each possible value (x) of X and order the values of x according to the LR.

(d) What is the likelihood ratio test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$ at level $\alpha = 0.15$? What is the likelihood ratio test at level $\alpha = 0.02$?

(e) Find the power for each of the tests found in part (d).