Statistics 630 - Assignment 5
(partial solutions)

2. Exer. 2.9.14. Consider the special case with $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 1$. The convolution to get the pdf for Z requires "completing the square" in the exponential function. This goes as follows.

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_Y(z-x) f_X(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}((z-x)^2+x^2)}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-(x-z/2)^2}\mathrm{e}^{-z^2/4}\,\mathrm{d}x = \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-z^2/4} \times \int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\mathrm{e}^{-(x-z/2)^2}\,\mathrm{d}x \\ &= \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-z^2/4}, \end{aligned}$$

since the integrand in the next-to-last expression is a normal$(z/2, 1/2)$ pdf which integrates to 1. From the result we see that $Z \sim$ normal$(0, 2)$.

[The general case is argued in a similar manner except the algebra is messier.]

3. Exer. 3.1.2. (a) $\mathsf{E}(X) = \frac{47}{7}$. (b) $\mathsf{E}(Y) = \frac{11}{7}$. (c) $\mathsf{E}(3X + 7Y) = 3\frac{47}{7} + 7\frac{11}{7}$. (d) $\mathsf{E}(X^2) = \frac{331}{7}$. (e) $\mathsf{E}(Y^2) = \frac{41}{7}$. (f) $\mathsf{E}(XY) = \frac{67}{7}$.

4. Exer. 3.1.6. $\mathsf{E}(Y + Z) = 100(0.3) + 7$.

5. Exer. 3.1.23. Since $X_i \sim$ binomial(n, θ_i) (see Example 2.8.5 in the book), we know $\mathsf{E}(X_i) = n\theta_i$. [For some perspective, the multinomial random vector $(X_1, \ldots, X_k)$ consists of the counts in each of k categories from a random sample of n individuals. The "raw data" are the n independent *categorical* responses; $(X_1, \ldots, X_k)$ is merely a summary. The example in this problem has just $k = 3$ categories, but the result is the same no matter how many categories there are.]

6. Exer. 3.2.2. (a) $\mathsf{E}(X) = \frac{2}{3}$. (b) $\mathsf{E}(Y) = \frac{46}{63}$. (c) $\mathsf{E}(3X+7Y) = 3\frac{2}{3} + 7\frac{46}{63} = \frac{64}{9}$. (d) $\mathsf{E}(X^2) = \frac{23}{45}$. (e) $\mathsf{E}(Y^2) = \frac{7}{12}$. (f) $\mathsf{E}(XY) = \frac{10}{21}$.

7. Exer. 3.2.6. $11\mathsf{E}(X) + 14\mathsf{E}(Y) + 3 = 11(-10.5) + 14(-8) + 3$.

8. Exer. 3.2.19. Use a change of variable $y = 1 + x$.

$$\mathsf{E}(X) = \int_0^{\infty} x\,\frac{\alpha}{(1+x)^{\alpha+1}}\,\mathrm{d}x = \alpha\int_1^{\infty}(y^{-\alpha} - y^{-\alpha-1})\,\mathrm{d}y = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1, \\ \infty & \text{if } \alpha \le 1. \end{cases}$$

Exer. 3.2.22. This uses property (2.4.7) in the book.

$$\begin{aligned} \mathsf{E}(X) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x\,x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta}. \end{aligned}$$

9. Exer. 3.3.2. (b) $\mathsf{Cov}(X, Y) = -\frac{48}{49}$. (c) $\mathsf{Var}(X) = \frac{108}{49}$, $\mathsf{Var}(Y) = \frac{166}{49}$. (d) $\mathsf{Corr}(X, Y) = -0.35849$.

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10. Exer. 3.2.8. (Recall that Y is also a gamma(1,9) random variable.) $\mathsf{E}(Y+Z) = \frac{1}{9} + \frac{5}{4}$.
    Exer. 3.2.20. Use $\mathsf{Var}(X) = \mathsf{E}(X^2) - \mathsf{E}(X)^2$ and simplify. And so $\mathsf{Var}(Y+Z) = \frac{1}{9} + \frac{5}{4^2}$.

11. Exer. 3.3.6. By independence, $\mathsf{Cov}(X, Z) = 0$. And by the properties of covariance, $\mathsf{Cov}(X+Y, Z) = \mathsf{Cov}(X, Z) + \mathsf{Cov}(Y, Z) = \mathsf{Cov}(Y, Z)$.

12. Since the random variables are all uncorrelated (each pair has covariance = 0), $\mathsf{Var}(X+Z) = \sigma_X^2 + \sigma_Z^2$, $\mathsf{Var}(Y+Z) = \sigma_Y^2 + \sigma_Z^2$, and $\mathsf{Cov}(X+Z, Y+Z) = \mathsf{Cov}(X, Y) + \mathsf{Cov}(X, Z) + \mathsf{Cov}(Y, Z) + \mathsf{Cov}(Z, Z) = \sigma_Z^2$. Thus, $\mathsf{Corr}(X+Z, Y+Z) = \frac{\sigma_Z^2}{\sqrt{(\sigma_X^2 + \sigma_Z^2)(\sigma_Y^2 + \sigma_Z^2)}}$.