Statistics 630 - Assignment 6
(partial solutions)

1. Exer. 3.3.25.

   (a) The authors probably have in mind that you mimic the proofs for the mean and variance of binomial, taking into account that you have two variables to sum over. Here is an alternative explanation. From the definition of the multinomial distribution, you can see that combining categories i and j results in a new category with probability $\theta_i + \theta_j$. It follows that $X_i + X_j$ is the number of "successes" for this combined category and thus has binomial$(n, X_i + X_j)$ distribution. So, on the one hand,

   $$\mathsf{Var}(X_i + X_j) = n(\theta_i + \theta_j)(1 - \theta_i - \theta_j),$$

   and on the other hand,

   $$\mathsf{Var}(X_i+X_j) = \mathsf{Var}(X_i)+\mathsf{Var}(X_j)+2\,\mathsf{Cov}(X_i, X_j) = n\theta_i(1-\theta_i)+n\theta_j(1-\theta_j)+2\,\mathsf{Cov}(X_i, X_j).$$

   You can then solve for $\mathsf{Cov}(X_i, X_j)$. [Why does a negative value make sense? Think about the fact that $X_1 + \cdots + X_k$ is fixed at value n.]

   (b) $\mathsf{Corr}(X_i, X_j)$ has the same value $-\sqrt{\frac{\theta_i\theta_j}{(1-\theta_i)(1-\theta_j)}}$ for any n.

2. (a) Let $X \sim$ beta(a, b). From Exer. 3.2.22 we know that $\mathsf{E}(X) = \frac{a}{a+b}$. By a similar calculation,

   $$\mathsf{E}(X^2) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a+1}(1-x)^{b-1}\,dx = \frac{\Gamma(a+b)}{\Gamma(a+b+2)} \frac{\Gamma(a+2)}{\Gamma(a)} = \frac{(a+1)a}{(a+b+1)(a+b)}.$$

   Then [and, yes, the algebra is messy]

   $$\mathsf{Var}(X) = \mathsf{E}(X^2) - (\mathsf{E}(X))^2 = \cdots = \frac{ab}{(a+b)^2(a+b+1)}.$$

   (b) To make things simpler for ourselves, let $\gamma = \frac{1}{(a_1+a_2+a_3)^2(a_1+a_2+a_3+1)}$. We have

   $$\mathsf{Var}(X_1) = a_1(a_2 + a_3)\gamma, \quad \mathsf{Var}(X_2) = a_2(a_1 + a_3)\gamma, \quad \mathsf{Var}(X_1 + X_2) = (a_1 + a_2)a_3\gamma.$$

   Then, following Problem 1,

   $$\mathsf{Cov}(X_1, X_2) = \frac{\mathsf{Var}(X_1 + X_2) - \mathsf{Var}(X_1) - \mathsf{Var}(X_2)}{2} = -a_1a_2\gamma.$$

3. Exer. 3.4.12. (a)

   $$m_X(s) = \sum_{k=0}^{\infty} \mathrm{e}^{sk}\theta(1-\theta)^k = \theta \sum_{k=0}^{\infty}((1-\theta)\mathrm{e}^s)^k = \frac{\theta}{1-(1-\theta)\mathrm{e}^s}, \quad \text{for } s < -\log(1-\theta).$$

   [Check: $m_X(0) = 1$.]
   (b) $\mathsf{E}(X) = m'_X(0) = \frac{\theta(1-\theta)\mathrm{e}^s}{(1-(1-\theta)\mathrm{e}^s)^2}\Big|_{s=0} = \frac{1-\theta}{\theta}$.
   (c) $\mathsf{E}(X^2) = m''_X(0) = \left(\frac{\theta(1-\theta)\mathrm{e}^s}{(1-(1-\theta)\mathrm{e}^s)^2} + \frac{2\theta(1-\theta)^2\mathrm{e}^{2s}}{(1-(1-\theta)\mathrm{e}^s)^3}\right)\Big|_{s=0} = \frac{1-\theta}{\theta} + \frac{2(1-\theta)^2}{\theta^2}$.
   Hence, $\mathsf{Var}(X) = \mathsf{E}(X^2) - (\mathsf{E}(X))^2 = \frac{1-\theta}{\theta^2}$.

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Exer. 3.4.16. (a)

$$\begin{aligned} m_Y(s) &= \int_{-\infty}^{\infty} \mathrm{e}^{sy} \frac{\mathrm{e}^{-|y|}}{2}\,\mathrm{d}y = \int_{-\infty}^{0} \frac{\mathrm{e}^{(s+1)y}}{2}\,\mathrm{d}y + \int_{0}^{\infty} \frac{\mathrm{e}^{(s-1)y}}{2}\,\mathrm{d}y \\ &= \frac{1}{2}\Big(\frac{1}{s+1} - \frac{1}{s-1}\Big) = \frac{1}{1-s^2}, \quad \text{for } |s| < 1. \end{aligned}$$

[Check: $m_Y(0) = 1$.]

(b) $\mathsf{E}(Y) = m_Y'(0) = \frac{2s}{(1-s^2)^2}\Big|_{s=0} = 0$.

(c) $\mathsf{E}(Y^2) = m_Y''(0) = \Big(\frac{2}{(1-s^2)^2} + \frac{8s^2}{(1-s^2)^3}\Big)\Big|_{s=0} = 2$. Hence, $\mathsf{Var}(X) = \mathsf{E}(X^2) - (\mathsf{E}(X))^2 = 2$.

[In case you are wondering about the point here, finding expectation and variance is not necessarily easier this way – especially if you are starting with the pmf/pdf as in the previous two examples. However, there are cases where you may only know the mgf. Additionally, this is a mathematically useful tool.]

4. Exer. 3.4.20. Suppose $T \sim \text{gamma}(\alpha, \lambda)$. Then, using a change of variable $x = (\lambda - s)t$ and requiring $s < \lambda$,

$$\begin{aligned} m_T(s) &= \int_0^{\infty} \mathrm{e}^{st} \frac{\lambda^\alpha t^{\alpha-1} \mathrm{e}^{-\lambda t}}{\Gamma(\alpha)}\,\mathrm{d}t = \frac{\lambda^\alpha}{(\lambda - s)^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \Big(\frac{\lambda}{\lambda - s}\Big)^{\alpha}, \quad \text{for } s < \lambda. \end{aligned}$$

Exer. 3.4.23. By the previous problem and independence of the X_i's, we can compute

$$m_Y(s) = \prod_{i=1}^{n} m_{X_i}(s) = \prod_{i=1}^{n} \Big(\frac{\lambda}{\lambda - s}\Big)^{\alpha_i} = \Big(\frac{\lambda}{\lambda - s}\Big)^{\alpha_1 + \cdots + \alpha_n},$$

which is the $\text{gamma}(\alpha_1 + \cdots + \alpha_n, \lambda)$ mgf. Therefore, $Y \sim \text{gamma}(\alpha_1 + \cdots + \alpha_n, \lambda)$. [Note: <u>same</u> second parameter λ for each X_i.]

5. Exer. 3.5.4. (a–d) Using indicators, $\mathsf{E}(X \mid Y = y) = I_{\{2\}}(y) - \frac{2}{3} I_{\{3\}}(y) - 2I_{\{7\}}(y) + 6I_{\{13\}}(y)$.
(e) $\mathsf{E}(X \mid Y) = I_{\{2\}}(Y) - \frac{2}{3} I_{\{3\}}(Y) - 2I_{\{7\}}(Y) + 6I_{\{13\}}(Y)$, which is a random variable taking 4 possible values $(1, -\frac{2}{3}, -2, 6)$.

Exer. 3.5.16. First, $\mathsf{E}(X \mid Y = y) = \frac{\alpha}{y}$ (mean of the $\text{gamma}(\alpha, y)$ distribution). Thus $\mathsf{E}(X \mid Y) = \frac{\alpha}{Y}$. Now let $W = \frac{1}{Y} \sim \text{exponential}(\lambda)$, which has mean $\frac{1}{\lambda}$. Then

$$\mathsf{E}(\mathsf{E}(X \mid Y)) = \mathsf{E}(\alpha W) = \frac{\alpha}{\lambda}.$$

6. $\mathsf{E}(T) = \frac{1}{\lambda}$ and $\mathsf{Var}(T) = \frac{1}{\lambda^2}$. Also, $\mathsf{E}(U|T) = \frac{T}{2}$ and $\mathsf{Var}(U|T) = \frac{T^2}{12}$. Therefore,

$$\mathsf{E}(U) = \mathsf{E}(\mathsf{E}(U|T)) = \mathsf{E}\Big(\frac{T}{2}\Big) = \frac{1}{2\lambda}$$

and

$$\mathsf{Var}(U) = \mathsf{E}(\mathsf{Var}(U|T)) + \mathsf{Var}(\mathsf{E}(U|T)) = \mathsf{E}\Big(\frac{T^2}{12}\Big) + \mathsf{Var}\Big(\frac{T}{2}\Big) = \frac{2/\lambda^2}{12} + \frac{1/\lambda^2}{4} = \frac{5}{12\lambda^2}.$$

[You can also find $\mathsf{E}(U^2) = \mathsf{E}(\mathsf{E}(U^2|T)) = \mathsf{E}\Big(\frac{T^2}{3}\Big)$, and go from there.]

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7. Exer. 3.6.10. (a) $\mathsf{E}(W) = \frac{3}{4}$, $\mathsf{Var}(W) = \frac{3}{5} - (\frac{3}{4})^2 = \frac{3}{80}$.
   (b) By Chebychev, $\mathsf{P}(|W - \mathsf{E}(W)| \geq \frac{1}{4}) \leq \frac{3/80}{(1/4)^2} = \frac{3}{5}$.
   (c) Exactly, in this case, $\mathsf{P}(|W - \mathsf{E}(W)| \geq \frac{1}{4}) = \mathsf{P}(W \leq \frac{1}{2}) + \mathsf{P}(W \geq 1) = \frac{1}{8}$.

8. Exer. 4.2.10. Let $m = \frac{1+2^2+\cdots+6^2}{6} = \frac{91}{6}$. Then $\mathsf{E}(Z_n) = \frac{91n}{6}$ and, by WLLN, $\frac{1}{n}Z_n \to \frac{91}{6}$ in probability.