AlphaGeometry - Solving olympiad geometry problems with AI

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***Abstract*—AlphaGeometry is the state-of-the-art proof-generative artificial intelligence for geometry problems. Developed by Google DeepMind, AlphaGeomtry has been able to solve correctly as many geometry-related problems as a gold medalist of the International Mathematical Olympics, whereas previous approaches were able to correctly solve half as many problems. One important advantage that AlphaGeometry has over other models for proof-generative artificial intelligence is the fact that it produces proofs that are easy for a human to read and check as well as receiving the problem in an easy-to-write prompt. AlphaGeometry is also an important advancement in the artificial intelligence paradigm it belongs to, NeuroSymbolic Artificial Intelligence. This report aims to explain briefly how AlphaGeometry works, why it is important for Artificial Intelligence and Mathematics and which problems it can solve.**

***Index Terms*—AlphaGeometry, artificial inteligence, neuro symbolic ai, geometry problems, mathematical proofs.**

I. Introduction

In general, mathematical proofs have been of great interest to Artificial Intelligence, not only checking their correctness but also being able to generate a correct proof for a theorem or proposition given a set of hypotheses and (maybe) a possible conclusion. In particular, geometrical problems have been known to be harder for a computer to understand and solve, not only because of the abstractness of the concepts and their interpolations but also due to the added complication of needing to translate from (maybe) visual geometrical hypotheses and statements to something the computer would understand.

Neuro Symbolic Artificial Intelligence aims to combine the power of artificial intelligence, with a set of rules, constraints or instructions that the artificial intelligence needs to follow.

AlphaGeometry produces geometrical proofs in a clever way combining a Large Language Model and a Symbolic Engine. The Language model generates new possible pathways for the Symbolic Engine to explore. The Symbolic Engine gets new statements that can be deduced from the available hypotheses. The proofs that AlphaGeometry generates are easy to read and check by a human, not only displaying the statements that are deduced from the existing ones but also stating the auxiliary constructions provided by the Large Language Model.

The advancement of Alpha Geometry in the area of proof-generating artificial intelligence is huge since it can be the starting point for other models in different mathematical areas as well as expanding the already existing model into one that can solve more complex and different geometrical problems from the ones that AlphaGeometry can already manage.

II. Neuro-Symbolic AI

Neuro Symbolic AI is a paradigm of AI that combines Neural Networks and Symbolic AI, and is what AlphaGeometry is based on. The first is the core of deep learning and is used primarily for perception and intuition based on data driven aproaches to learn from a large amount of data. The latter is concerned with structured knowledge, logic, reasoning and rules.

This paradigm aims to create a joint AI that is not only good at data-driven approaches, but is also efficient, reliable and thrustworthy which are properties of symbolic AI.

The combination of these components can be implemented in a multiple of different ways. Some of them are:

- Symbolic Neural Symbolic: Words or subwords from Symbolic AI are the input and output of a neural network such as a large Language Model.
- Symbolic[Neural]: Symbolic techniques are used to involve neural techniques.
- Neural—Symbolic: Neural Arquitecture interpret perceptual data as symbols and symbolic arquitecture is used to reason relationships between that data.
- Neural Symbolic $\rightarrow$ Neural: Symbolic Reasoning is used to create training data that is later used to train a Deep Learning Model.
- Neual_{Symbolic}: A neural network is generated from Symbolic rules.
- Neural[Symbolic]: The neural model is allowed to directly call a Symbolic reasoning engine to perform an action or evaluate a state.

More implementations exist, as these ones do not include, for instance, multi-agent systems.

The applications of this paradigm are large in number, as this paradigm can be used to monitor and regulate the use of AI in driverless-cars, robotics, decision-making AI, among others.

III. State of the Art before AlphaGeometry

Automated proof of Euclidean Geometry theorems presents significant challenges. One of the main ones is the fact that

the proofs produced by the machine are not always readable by humans. Furthermore, due to the fact that these kinds of theorems are mainly used in Mathematical Olympiads, there is a shortage of training data. The approaches used to prove these theorems prior to AlphaGeometry can be divided into two categories.

First, there are algebraic approaches, that is, those that treat points with a coordinate system and use methods such as Gröbner Basis or Wu's method to manipulate the resulting polynomial equations. While it is guaranteed that these methods can prove any true theorem, they are also computationally very expensive, so they cannot always perform the proof in a reasonable time. Furthermore, because these calculations are very different from the way humans solve geometry problems, the proofs are not understandable to humans either. Secondly, there are synthetic approaches, that is, those that are based on the construction of auxiliary points and geometric reasoning such as similarity and congruence of triangles. Such systems produce proofs that are understandable to a human, however, there is no theoretical guarantee that they can prove any true theorem. Before AlphaGeometry, the performance of systems based on these approaches was not so good, only managing to solve 10 of the last 30 recent IMO Geometry problems.

IV. AlphaGeometry at a glance

Google DeepMind's AlphaGeometry is a clear example of an implementation of the Neuro-Symbolic AI paradigm. The model seamlessly combines a Large Language Model AI using Deep Neural Networks and Transformers, and a Symbolic Engine.

The Symbolic Engine is composed of two different components, the Deductive Engine and the Algebraic Rules Module. The Deductive Engine (DD) specializes in getting new statements that are implied or obtained from the problem's premises and from previously generated statements. (This would be obtaining q, from the logic operation $p \rightarrow q$, where p are the problem's premises or previously obtained statements). The Algebraic Rules Module (AR) specializes in getting angle ratios and distance chasing, among others. In general, AR gets results from algeraic constructions based on the available data.

The Large Language Model used in AlphaGeometry is used to generate new constructions for the problem that may give new paths for the Symbolic Engine to get statements that might result in the problem's proof. The Language Model is used whenever the Symbolic Engine cannot get new statements from the available data.

All statements and constructions are organized in a directed graph, representing each implication between statements and constructions. When the proof is obtained, AlphaGeometry performs graph pruning, breadth-first search and also does a traceback for algebraic deduction, among other processes to get the proof that does not use statements, constructions, relationships that did not result in the proof as well as maintainig the correctness of the implications and constructions.

One of the greatest impacts of AlphaGeometry is the fact that the generated proof is then returned in a easy human-readable way. This makes it easier to check for correctness.

V. AlphaGeometry in Practice: An Example Reviewed

In this section, we explore a practical example of how the AlphaGeometry system approaches proving a geometric theorem. The setup involves triangle ABC, with point D on line AC and point E as an arbitrary point not on line AC. The challenge is to prove specific angle equalities and relationships among these points.

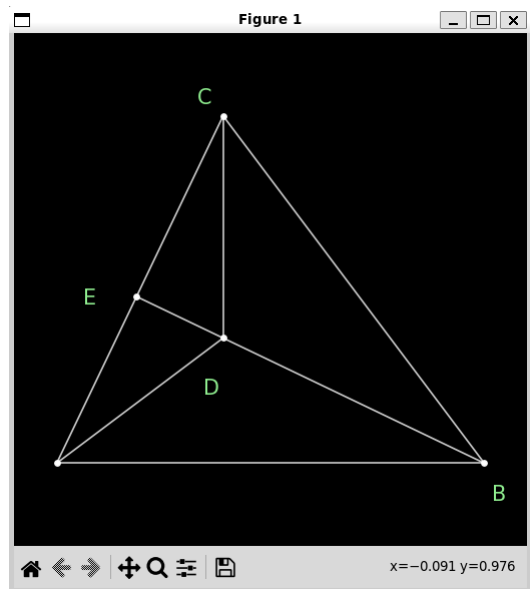

Fig. 1. Example of AlphaGeometry's construction

A. Theorem Premises

The problem begins with the following configurations:

- Points A, B, C, and D are given, with D located on line segment AC.
- Line segment BD is perpendicular to line AC.
- CD is perpendicular to line AB.

B. Auxiliary Constructions

AlphaGeometry employs a sophisticated method of constructing auxiliary points and lines to simplify and enable the proof of complex geometrical theorems. For this example, the construction of auxiliary point E plays a pivotal role in establishing necessary geometric relations and proofs:

- **Introduction of Point E:** AlphaGeometry strategically places point E to facilitate essential geometric inferences. In the configuration, E is chosen such that it lies on a line with points D and B. This collinearity is not arbitrary but is selected based on the neural model's prediction, which suggests that aligning these points would create geometrically significant relationships.

- **Strategic Placement:** By placing E on the line extended from D through B, the system can use the known relationships of perpendicular lines (BD and AC) to infer new angle relationships. The alignment of points C, E, and A is similarly strategic, enabling the system to leverage entire line segments and their properties in the proof process.

C. Proof Steps

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 * From theorem premises:
A B C D : Points
BD ⊥ AC [00]
CD ⊥ AB [01]

 * Auxiliary Constructions:
E : Points
D,B,E are collinear [02]
C,E,A are collinear [03]

 * Proof steps:
001. D,B,E are collinear [02] & C,A,E are collinear [03] & B
D ⊥ AC [00] ⇒ ∠DEC = ∠AEB [04]
002. D,B,E are collinear [02] & C,A,E are collinear [03] & B
D ⊥ AC [00] ⇒ ∠CEB = ∠DEA [05]
003. C,A,E are collinear [03] & BD ⊥ AC [00] ⇒ DB ⊥ CE [06]
004. CD ⊥ AB [01] & DB ⊥ CE [06] ⇒ ∠ABD = ∠DCE [07]
005. C,A,E are collinear [03] & D,B,E are collinear [02] & ∠
∠ABD = ∠DCE [07] ⇒ ∠DCE = ∠ABE [08]
006. ∠DEC = ∠AEB [04] & ∠DCE = ∠ABE [08] (Similar Triangles)
⇒ DE:EA = CE:BE [09]
007. DE:EA = CE:BE [09] & ∠CEB = ∠DEA [05] (Similar Triangle
s)⇒ ∠CBE = ∠DAE [10]
008. DE:EA = CE:BE [09] & ∠CEB = ∠DEA [05] (Similar Triangle
s)⇒ ∠ECB = ∠EDA [11]
009. ∠CBE = ∠DAE [10] & D,B,E are collinear [02] & C,E,A are
 collinear [03] & ∠ECB = ∠EDA [11] ⇒ AD ⊥ BC
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Fig. 2. Example of AlphaGeometry's reasoning

1) Utilizing the collinearity of D, B, E, and the perpendicularity of lines, AlphaGeometry deduces:

$$\angle DEC = \angle AEB$$

2) Considering the properties of triangles and angle chasing, it follows that:

$$\angle CEB = \angle DEA$$

3) From the symmetry and the properties of the lines and points:

$$\angle ABD = \angle DCE$$

4) These angles imply that triangles ABD and EDC are similar by AA (Angle-Angle) similarity, leading to further geometric properties and ratios being deduced.

D. Conclusion of Alpha's Geometry solution

The successful proof that the constructed line through points A, D, and E is perpendicular to line BC exemplifies the innovative and effective approach of AlphaGeometry. This system combines neural language model predictions with traditional symbolic logic, resulting in a powerful tool that solves complex geometry problems efficiently. AlphaGeometry represents a significant advancement in the field of automated theorem proving, offering a novel solution that bridges the gap between advanced artificial intelligence technologies and classic mathematical reasoning. The ability of AlphaGeometry to derive and verify intricate geometric properties autonomously not only underscores its potential as a pioneering tool in educational and research settings but also highlights its capability to handle tasks that traditionally required deep human expertise. This exemplar not only demonstrates the practical application of cutting-edge AI in academic domains but also sets a new standard for the future of automated reasoning and theorem proving.

VI. Conclusions

Automatic theorem proving is an active area of research within Machine Learning. In the particular case of geometry, the aim is to produce proofs that are not only correct, but also easily readable by a human. In this sense, AlphaGeometry presents a great improvement over previous models, as they produced algebra-based tests that were computationally expensive and also too long to be read by a person. The fact that the proofs produced by Alpha Geometry are readable by a human has the great advantage of allowing an AI to make decisions based on a series of principles (in this case, the Euclidean axioms of geometry). This paradigm is known as the neurosymbolic paradigm and has application in very distant fields such as Autonomous Driving and "legal AI". The way this was achieved is by dividing the neural network into two parts, one that suggests new constructions and the other that verifies, based on these new constructions, whether it is possible to demonstrate the requested toerema. In this way, the neural network is prevented from having "allucinations" like most large language models.

The approach that was used with AlphaGeometry can be adapted to automatically prove theorems in areas such as algebra, number theory, analysis, among others. These areas present more difficulties than geometry because the set of "possible constructions" is much broader. However, it is a matter of time before AI can prove advanced theorems in more areas of mathematics than geometry.

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