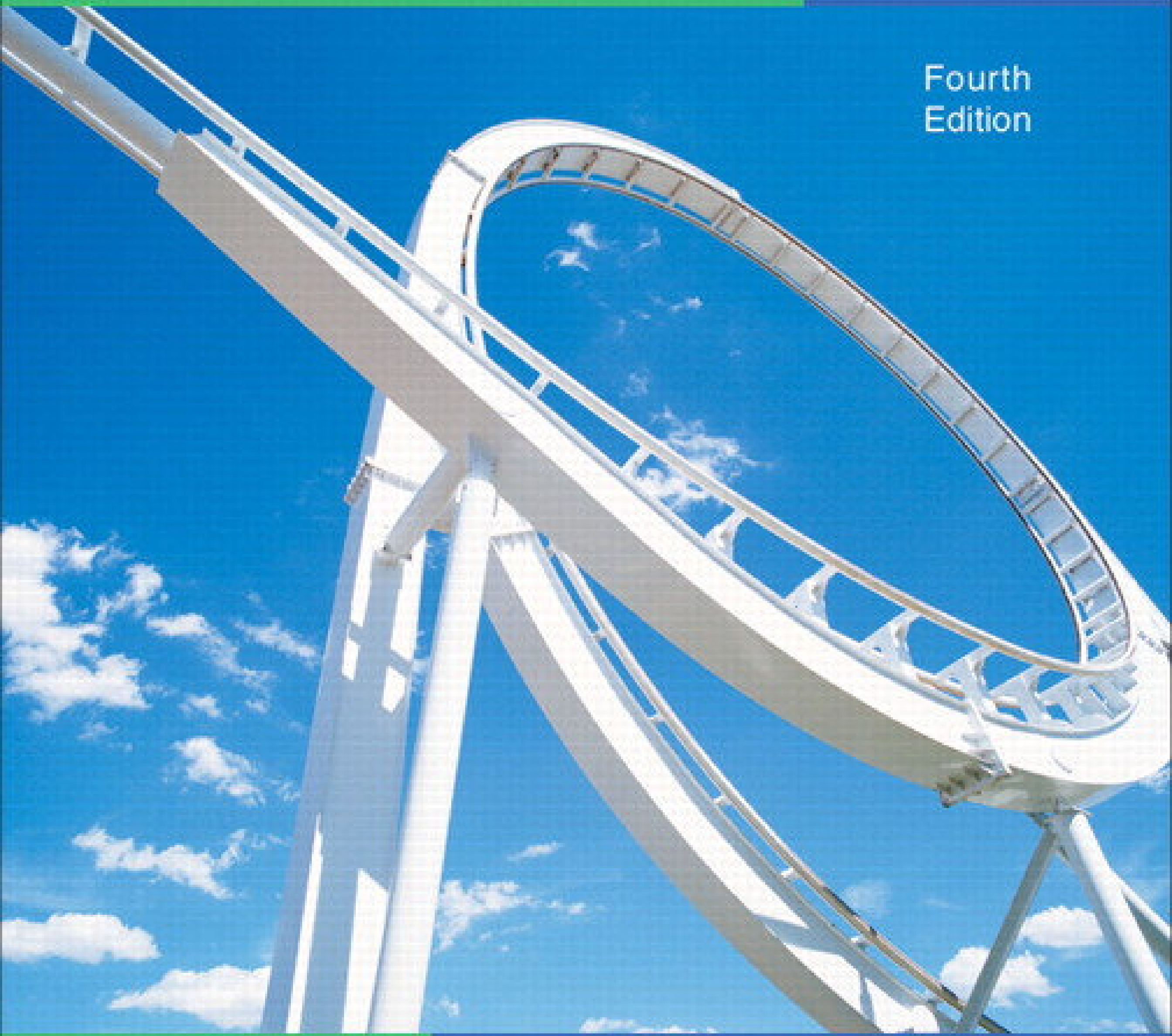
Calculus for Engineers



Donald Trim

INSTRUCTOR'S SOLUTIONS MANUAL

Calculus for Engineers

Fourth Edition

Donald Trim
University of Manitoba

PEARSON
Prentice
Hall

Toronto

CONTENTS

This manual contains detailed solutions to all exercises in the text. We would appreciate being made aware of any errors.

CHAPTER 1

EXERCISES 1.2

1. The solution is $x = 2/3$.

2. The solution is $x = -5/14$.

3. Since $x^2 + 2x - 3 = (x+3)(x-1)$, solutions are $x = -3$ and $x = 1$.

4. Since $12x^2 + 11x - 5 = (3x-1)(4x+5)$, solutions are $x = 1/3$ and $x = -5/4$.

5. Since the discriminant $5^2 - 4(2)(10) = -55$ is negative, the equation has no real solutions.

6. Quadratic formula 1.5 gives $x = \dfrac{-10 \pm \sqrt{100 - 4(-4)(9)}}{-8} = \dfrac{-10 \pm \sqrt{244}}{-8} = \dfrac{5 \pm \sqrt{61}}{4}$.

7. Since $x^2 + 8x + 16 = (x+4)^2$, the solution is $x = -4$ with multiplicity 2.

8. Since $4x^2 - 36x + 81 = (2x-9)^2$, the solution is $x = 9/2$ with multiplicity 2.

9. Quadratic formula 1.5 gives $x = \dfrac{-5 \pm \sqrt{25 - 4(2)(-10)}}{4} = \dfrac{-5 \pm \sqrt{105}}{4}$.

10. Since the discriminant $(-8)^2 - 4(4)(9) = -80$ is negative, the equation has no real solutions.

11. Since $x^3 - 3x^2 + 3x - 1 = (x-1)^3$, the solution is $x = 1$ with multiplicity 3.

12. Possible rational solutions are ± 1, $\pm 1/2$, $\pm 1/4$, $\pm 1/8$. We find that $x = -1/2$ is a solution. We factor $2x + 1$ from the cubic,

$$8x^3 + 12x^2 + 6x + 1 = (2x+1)(4x^2 + 4x + 1) = (2x+1)(2x+1)^2 = (2x+1)^3.$$

The only solution is $x = -1/2$ with multiplicity 3.

13. Possible rational solutions are $x = \pm 1, \pm 2, \pm 5, \pm 10$. We find that $x = 2$ is a solution. We factor $x - 2$ from the cubic,

$$x^3 - 2x^2 + 5x - 10 = (x-2)(x^2+5) = 0.$$

The other two solutions are complex.

14. Possible rational solutions are $x = \pm 1, \pm 3, \pm 9$, but it should also be clear that no positive value of x can satisfy the equation. We find that $x = -1$ is a solution. We factor $x + 1$ from the cubic,

$$x^3 + 4x^2 + 12x + 9 = (x+1)(x^2 + 3x + 9) = 0.$$

Since the discriminant of the quadratic is negative, the other two solutions are complex.

15. Possible rational solutions are $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32, \pm 64$, but it should also be clear that no positive value of x can satisfy the equation. We find that $x = -4$ is a solution. We factor $x + 4$ from the cubic,

$$x^3 + 12x^2 + 48x + 64 = (x+4)(x^2 + 8x + 16) = (x+4)(x+4)^2 = (x+4)^3.$$

The only solution is $x = -4$ with multiplicity 3.

16. Possible rational solutions are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$. We find that $x = -3$ is a solution. We factor $x + 3$ from the quartic,

$$x^4 + 7x^3 + 9x^2 - 21x - 36 = (x+3)(x^3 + 4x^2 - 3x - 12).$$

Possible rational zeros of the cubic are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. We find that $x = -4$ is a zero. We factor $x + 4$ from the cubic,

$$x^4 + 7x^3 + 9x^2 - 21x - 36 = (x+3)(x+4)(x^2 - 3).$$

The solutions are $x = -3, -4, \pm\sqrt{3}$.

17. Since $x^4 - 16 = (x^2+4)(x^2-4) = (x^2+4)(x+2)(x-2)$, the real solutions are $x = \pm 2$.

18. Possible rational solutions are ± 1, ± 3, ± 5, ± 15, $\pm 1/2$, $\pm 3/2$, $\pm 5/2$, $\pm 15/2$. We find that $x = -5$ is a solution. We factor $x + 5$ from the quartic,

$$2x^4 + 9x^3 - 6x^2 - 8x - 15 = (x + 5)(2x^3 - x^2 - x - 3).$$

Possible rational zeros of the cubic are ± 1, ± 3, $\pm 1/2$, $\pm 3/2$. We find that $x = 3/2$ is a zero. We factor $2x - 3$ from the cubic,

$$2x^4 + 9x^3 - 6x^2 - 8x - 15 = (x + 5)(2x - 3)(x^2 + x + 1).$$

Since the quadratic has a negative discriminant, the only real solutions are $x = -5$ and $x = 3/2$.

19. Possible rational solutions are ± 1, ± 3, ± 9, $\pm 1/2$, $\pm 3/2$, $\pm 9/2$, $\pm 1/3$, $\pm 1/6$. We find that $x = -1/2$ is a solution. We factor $2x + 1$ from the quartic,

$$6x^4 + x^3 + 53x^2 + 9x - 9 = (2x + 1)(3x^3 - x^2 + 27x - 9).$$

Possible rational zeros of the cubic are ± 1, ± 3, ± 9, $\pm 1/3$. We find that $x = 1/3$ is a zero. We factor $3x - 1$ from the cubic,

$$6x^4 + x^3 + 53x^2 + 9x - 9 = (2x + 1)(3x - 1)(x^2 + 9).$$

Since the quadratic has complex zeros, the only real solutions are $x = -1/2$ and $x = 1/3$.

20. No real numbers can satisfy this equation.

21. Possible rational solutions are ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 9, ± 12, ± 18, ± 24, ± 36, ± 72. We find that $x = 24$ is a solution. We factor $x - 24$ from the cubic,

$$x^3 - 23x^2 - 21x - 72 = (x - 24)(x^2 + x + 3).$$

Since the quadratic has a negative discriminant, the only real solution is $x = 24$.

22. Possible rational solutions are ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 9, $\pm 12, \pm 16$, ± 18, ± 24, ± 32, ± 36, ± 48, ± 64, ± 72, ± 96, ± 144, ± 192, ± 192, ± 288, ± 576. We find that $x = -4$ is a solution. We factor $x + 4$ from the quartic,

$$x^4 - 4x^3 - 44x^2 + 96x + 576 = (x + 4)(x^3 - 8x^2 - 12x + 144).$$

Possible rational zeros of the cubic are ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 9, $\pm 12, \pm 16$, ± 18, ± 24, ± 36, ± 48, ± 72, ± 144. We find that $x = -4$ is a zero. We factor $x + 4$ from the cubic,

$$x^4 - 4x^3 - 44x^2 + 96x + 576 = (x + 4)(x + 4)(x^2 - 12x + 36) = (x + 4)^2(x - 6)^2.$$

Thus, $x = -4$ and $x = 6$ are solutions, each of multiplicity 2.

23. Since $3x^4 + x^3 + 5x^2 = x^2(3x^2 + x + 5)$, and the quadratic has a negative discriminant, the only real solution is $x = 0$ with multiplicity 2.

24. Possible rational solutions are ± 1, ± 2, ± 4, ± 5, ± 10, ± 20, $\pm 1/2$, $\pm 5/2, \pm 1/3$, $\pm 2/3$, $\pm 4/3$, $\pm 5/3$, $\pm 10/3$, $\pm 20/3$, $\pm 1/6$, $\pm 5/6$. We find that $x = 5/6$ is a zero. We factor $6x - 5$ from the cubic,

$$6x^3 + x^2 + 19x - 20 = (6x - 5)(x^2 + x + 4).$$

Since the quadratic has a negative discriminant, $x = 5/6$ is the only real solution.

25. Possible rational zeros are ± 1, ± 3, ± 5, ± 9, ± 15, ± 45. We find that $x = -5$ is a solution. We factor $x + 5$ from the polynomial,

$$x^5 + 5x^4 - 9x - 45 = (x + 5)(x^4 - 9) = (x + 5)(x^2 + 3)(x^2 - 3).$$

Solutions are $x = -5$ and $x = \pm\sqrt{3}$.

26. Possible rational solutions are ± 1, ± 2, ± 3, ± 4, ± 5, ± 6, ± 8, $\pm 10, \pm 12$, ± 15, ± 20, ± 24, ± 30, ± 40, ± 60, ± 120. We find that $x = 1$ is a solution. We factor $x - 1$ from the polynomial,

$$x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = (x-1)(x^4 - 14x^3 + 71x^2 - 154x + 120).$$

We use the same set of rational possibilities for the quartic. We find that $x = 2$ is a zero. When we factor $x - 2$ from the quartic,

$$x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = (x-1)(x-2)(x^3 - 12x^2 + 47x - 60).$$

Possible rational zeros of the cubic are ± 1, ± 2, ± 3, ± 4, ± 5, ± 6, $\pm 10, \pm 12$, ± 15, ± 20, ± 30, ± 60. We find that $x = 3$ is a zero. We factor $x - 3$ from the cubic,

$$\begin{aligned} x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 &= (x-1)(x-2)(x-3)(x^2 - 9x + 20) \\ &= (x-1)(x-2)(x-3)(x-4)(x-5). \end{aligned}$$

Solutions are therefore $x = 1$, 2, 3, 4, 5.

27. Possible rational solutions are ± 1, ± 2, ± 4, $\pm 1/2$, $\pm 1/4$, but it should also be clear that no positive value of x can satisfy the equation. We find that $x = -1/2$ is a solution. We factor $2x + 1$ from the quartic,

$$4x^4 + 4x^3 + 17x^2 + 16x + 4 = (2x+1)(2x^3 + x^2 + 8x + 4).$$

Possible rational zeros of the cubic are ± 1, ± 2, ± 4, $\pm 1/2$, and once again we reject the positive values. We find that $x = -1/2$ is a zero. We factor $2x + 1$ from the cubic,

$$4x^4 + 4x^3 + 17x^2 + 16x + 4 = (2x+1)(2x+1)(x^2 + 4).$$

Thus, $x = -1/2$ is the only real solution and it has multiplicity 2.

28. Possible rational solutions are ± 1, ± 2, ± 4, $\pm 1/5$, $\pm 2/5$, $\pm 4/5$, $\pm 1/25$, $\pm 2/25$, $\pm 4/25$. We find that $x = 2/5$ is a solution. We factor $5x - 2$ from the quartic,

$$25x^4 - 120x^3 + 109x^2 - 36x + 4 = (5x-2)(5x^3 - 22x^2 + 13x - 2).$$

Possible rational zeros of the cubic are ± 1, ± 2, $\pm 1/5$, $\pm 2/5$. We find that $x = 2/5$ is a zero. We factor $5x - 2$ from the cubic,

$$25x^4 - 120x^3 + 109x^2 - 36x + 4 = (5x-2)(5x-2)(x^2 - 4x + 1).$$

Thus, $x = 2/5$ is a real solution with multiplicity 2 and the quadratic formula gives the remaining two solutions

$$x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

29. Possible rational solutions are ± 1, ± 2, ± 4, ± 5, ± 8, ± 10, ± 20, $\pm 25, \pm 40$, ± 50, ± 100, ± 200. We find that $x = 1$ is a zero. We factor $x - 1$ from the polynomial,

$$x^5 + 9x^4 + 47x^3 + 125x^2 + 18x - 200 = (x-1)(x^4 + 10x^3 + 57x^2 + 182x + 200).$$

We use the same rational numbers for the quartic, but reject the positive values. We find that $x = -2$ is a zero. We factor $x + 2$ from the quartic,

$$x^5 + 9x^4 + 47x^3 + 125x^2 + 18x - 200 = (x-1)(x+2)(x^3 + 8x^2 + 41x + 100).$$

For zeros of the cubic we use -1, -2, -4, -5, -10, -20, -25, -50, -100. We find that $x = -4$ is a zero. When we factor it out,

$$x^5 + 9x^4 + 47x^3 + 125x^2 + 18x - 200 = (x-1)(x+2)(x+4)(x^2 + 4x + 25).$$

Since the quadratic has a negative discriminant, the real solutions are $x = -4$, -2, 1.

30. Possible rational solutions are ±1, ±2, ±3, ±4, ±6, ±8, ±9, ±12, ±16, ±18, ±24, ±27, ±36, ±48, ±54, ±72, ±81, ±108, ±144, ±162, ±216, ±324, ±432, ±648, ±1296. We find that $x = 3$ is a solution. When we factor $x - 3$ from the polynomial,

$$x^6 + 16x^4 - 81x^2 - 1296 = (x-3)(x^5 + 3x^4 + 25x^3 + 75x^2 + 144x + 432).$$

When we note that no positive value can satisfy the fifth degree polynomial, possible rational zeros are -1, -2, -3, -4, -6, -8, -9, -12, -16, -18, -24, -27, -36, -48, -54, -72, -108, -144, -216, -432. We find that $x = -3$ is a zero. When we factor $x + 3$ from the polynomial,

$$x^6 + 16x^4 - 81x^2 - 1296 = (x-3)(x+3)(x^4 + 25x^2 + 144).$$

Since there can be no real zeros of the quartic, the real solutions are $x = \pm3$.

31. $2(x+5)(x-1)$

32. $2[x - (3+\sqrt{65})/4][x - (3-\sqrt{65})/4]$

33. $(x+2)(x-3)(x-10)$

34. $24(x+5/3)(x-1/4)(x-1/2)$

35. $(x+1)(x-1)^3$

36. $16(x-1/2)^2(x+1/2)^2$

37. One polynomial is $(x+1/3)(x-4/5)(x-3)(x-4)^3$. This polynomial could be multiplied by any constant.

38. According to the rational root theorem, possible rational zeros must be divisors of a_0 divided by divisors of 1. This means that possible rational zeros are divisors of a_0.

EXERCISES 1.3

1. With formula 1.10, the distance is $\sqrt{2^2 + 1^2} = \sqrt{5}$.

2. With formula 1.10, the distance is $\sqrt{6^2 + (-3)^2} = 3\sqrt{5}$.

3. With formula 1.10, the distance is $\sqrt{(-2)^2 + (-6)^2} = 2\sqrt{10}$.

4. With formula 1.10, the distance is $\sqrt{(-7)^2 + (-3)^2} = \sqrt{58}$.

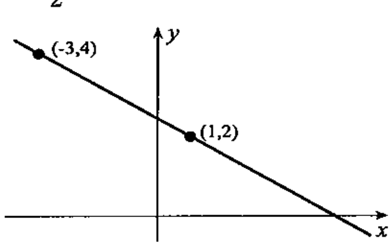
5. With formula 1.11, the midpoint is $\left(\dfrac{1+3}{2}, \dfrac{3+4}{2}\right) = \left(2, \dfrac{7}{2}\right)$.

6. With formula 1.11, the midpoint is $\left(\dfrac{-2+4}{2}, \dfrac{1-2}{2}\right) = \left(1, -\dfrac{1}{2}\right)$.

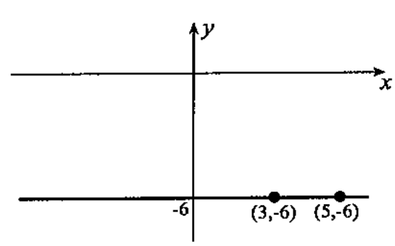
7. With formula 1.11, the midpoint is $\left(\dfrac{-1-3}{2}, \dfrac{-2-8}{2}\right) = (-2, -5)$.

8. With formula 1.11, the midpoint is $\left(\dfrac{3-4}{2}, \dfrac{2-1}{2}\right) = \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

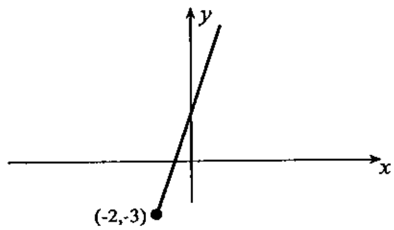
9. With slope $-2/4 = -1/2$, equation 1.13 gives $y - 2 = -\dfrac{1}{2}(x-1) \implies x + 2y = 5$.



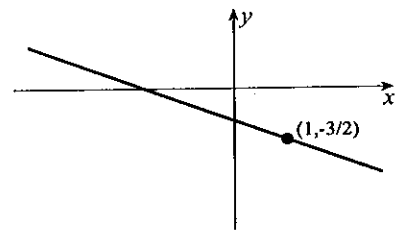
10. The line is horizontal. Its equation is $y = -6$.



11. With formula 1.13, the equation is
$y + 3 = 3(x + 2) \implies y = 3x + 3.$



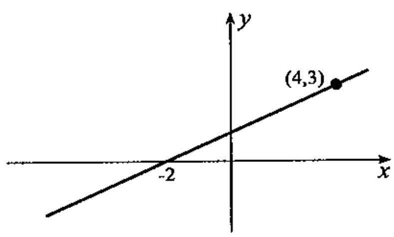
12. With formula 1.13, the equation is
$y + \dfrac{3}{2} = -\dfrac{1}{2}(x - 1) \implies x + 2y + 2 = 0.$



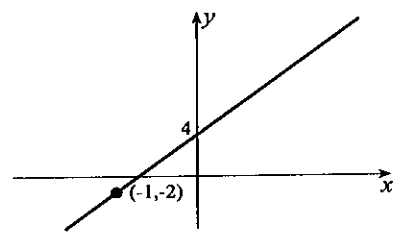
13. The equation of the y-axis is $x = 0$.

14. The equation of the x-axis is $y = 0$.

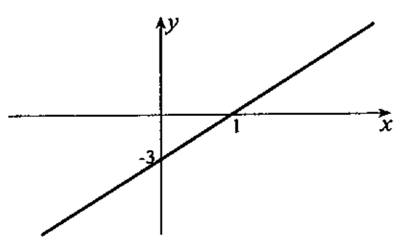
15. With $m = (3 - 0)/(4 + 2) = 1/2$, equation 1.13 gives $y = (1/2)(x + 2) \implies 2y = x + 2$.



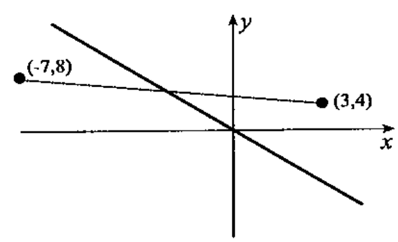
16. With $m = (4 + 2)/(0 + 1) = 6$, equation 1.13 gives $y - 4 = 6(x - 0) \implies y = 6x + 4$.



17. With slope $(0 + 3)/(1 - 0) = 3$, equation 1.13 gives $y - 0 = 3(x - 1) \implies y = 3x - 3$.



18. The midpoint of the line segment is
$$\left(\frac{3-7}{2}, \frac{4+8}{2}\right) = (-2, 6).$$
With $m = (6 - 0)/(-2 - 0) = -3$, equation 1.13 gives $y - 0 = -3(x - 0) \implies y = -3x$.



19. Since slopes -1 and 1 are negative reciprocals, the lines are perpendicular.

20. Since slopes of both lines are $-1/3$, they are parallel.

21. Since slopes of both lines are $1/3$, they are parallel.

22. Since slopes $-2/3$ and $3/2$ are negative reciprocals, the lines are perpendicular.

23. Since slopes are 3 and $-1/2$, the lines are neither parallel nor perpendicular.

24. Since slopes are 1 and $-2/3$, the lines are neither parallel nor perpendicular.

25. The lines are perpendicular.

26. Since slopes are -1 and 3, the lines are neither parallel nor perpendicular.

27. When we subtract the equations, $y + 2y = 0 + 3 \implies y = 1$. The point of intersection is $(-1, 1)$.

28. The point of intersection is $(1, 2)$.

29. When we subtract 3 times the second equation from the first, $4y + 18y = 6 - 9 \implies y = -3/22$. The point of intersection is $(24/11, -3/22)$.

30. When we substitute $y = 2x + 6$ into the second equation, $x = (2x + 6) + 4 \implies x = -10$. The point of intersection is $(-10, -14)$.

31. When we substitute $x = 2 - 2y/3$ into the second equation, $2(2 - 2y/3) - y/4 = 15 \Longrightarrow y = -132/19$. The point of intersection is $(126/19, -132/19)$.

32. If we multiply the first equation by 5 and add the result to the second equation, we obtain $73x = 37 \Longrightarrow x = 37/73$. The point of intersection is $(37/73, 153/146)$.

33. Formula 1.16 gives $\dfrac{|3+4-1|}{\sqrt{1+1}} = \dfrac{6}{\sqrt{2}}$.

34. Formula 1.16 gives $\dfrac{|1-6-3|}{\sqrt{1+4}} = \dfrac{8}{\sqrt{5}}$.

35. Formula 1.16 gives $\dfrac{|5-1-4|}{\sqrt{1+1}} = 0$.

36. Formula 1.16 gives $\dfrac{|3+1|}{\sqrt{1+1}} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$.

37. The distance is 7.

38. Formula 1.16 gives $\dfrac{|60+4+3|}{\sqrt{225+4}} = \dfrac{67}{\sqrt{229}}$.

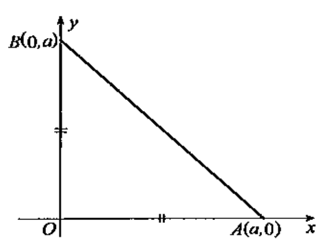
39. Since the point of intersection of the given lines is $(3,1)$, and the slope of $x + 2y = 15$ is $-1/2$, the required equation is $y - 1 = -(1/2)(x-3) \Longrightarrow x + 2y = 5$.

40. Since the slope of $x - y = 4$ is 1, and the point of intersection of $2x + 3y = 3$ and $x - y = 4$ is $(3,-1)$, the required equation is $y + 1 = -1(x-3) \Longrightarrow x + y = 2$.

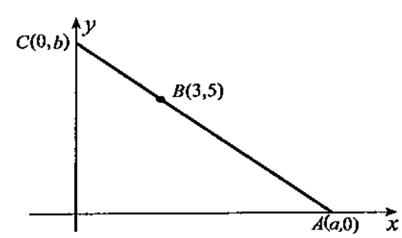
41. Since the slope of the line through $(1,2)$ and $(-3,0)$ is $2/4$, the required equation is $y - 6 = (1/2)(x - 5) \Longrightarrow 2y = x + 7$.

42. Since the slope of the line through $(-3,4)$ and $(1,-2)$ is $(4+2)/(-3-1) = -3/2$, the equation of the required line is $y + 2 = (2/3)(x+3) \Longrightarrow 2x = 3y$.

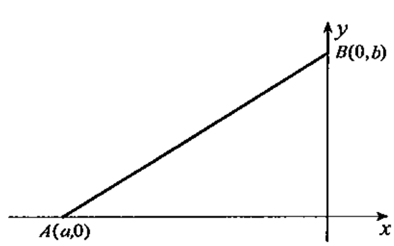
43. Since $\|OA\| = \|OB\|$, coordinates of A and B are $(a,0)$ and $(0,a)$, respectively. Since the area of $\Delta\, OAB$ is 8, it follows that $(1/2)a^2 = 8 \Longrightarrow a = 4$. The equation of line AB is $y = -(x-4) \Longrightarrow x + y = 4$.



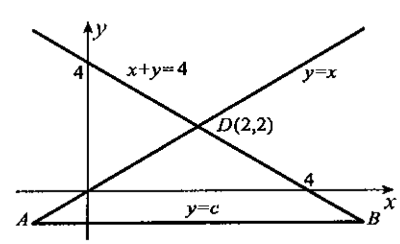
44. When we equate slopes of line segments AC and AB, $\quad -\dfrac{b}{a} = \dfrac{5}{3-a} \Longrightarrow b = \dfrac{5a}{a-3}$.
Combine this with the fact that the area of $\Delta\, OAB = (1/2)ab = 30$, and we obtain $a = 6$ and $b = 10$. The equation of the line is $y = -(5/3)(x-6) \Longrightarrow 5x + 3y = 30$.



45. Since the slope of line segment AB is 2, and its length is 3, it follows that $-b/a = 2$ and $\sqrt{a^2+b^2} = 3$. These can be solved for $a = -3/\sqrt{5}$ and $b = 6/\sqrt{5}$. The equation of AB is $y = 2x + 6/\sqrt{5}$.

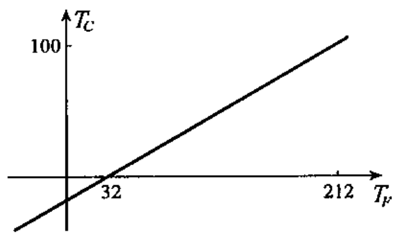


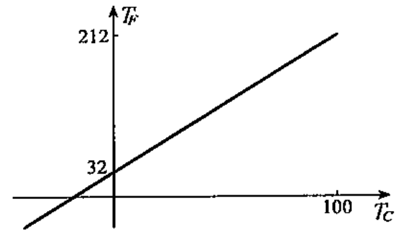
46. If the equation is $y = c$, then $A = (c,c)$ and $B = (4-c, c)$. Since triangle ABD has area 9, it follows that $9 = \dfrac{1}{2}(2-c)(4-2c)$, solutions of which are $5, -1$. The required equation is therefore $y = -1$.



47. (a) The conversion equation is $T_C = 5(T_F - 32)/9$.
(b) The conversion equation is $T_F = 9T_C/5 + 32$.
(c) They are one and the same line if we plot T_F along the horizontal axis and T_C along the vertical axis (left figure below). If we plot T_F along the vertical axis and T_C along the horizontal axis in part

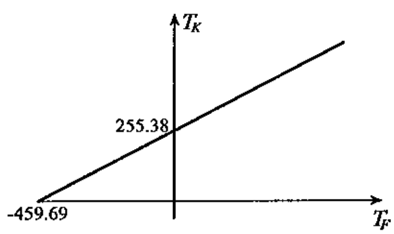
(b), we obtain the right figure below.

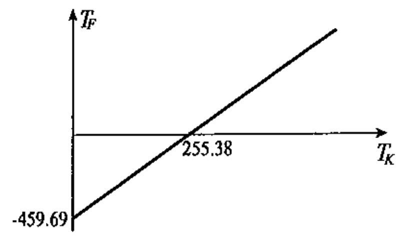




(d) If we set $T_F = T_C$ in the equation from part (a), we obtain $T_C = 5(T_C - 32)/9 \Longrightarrow 4T_C = -160 \Longrightarrow T_C = -40$.

48. (a) The conversion equation is $T_K = 5(T_F - 32)/9 + 273.16$.
(b) The conversion equation is $T_F = 9(T_K - 273.16)/5 + 32$.
(c) They are one and the same line if we plot T_F along the horizontal axis and T_K along the vertical axis (left figure below). If we plot T_F along the vertical axis and T_K along the horizontal axis in part (b), we obtain the right figure below.



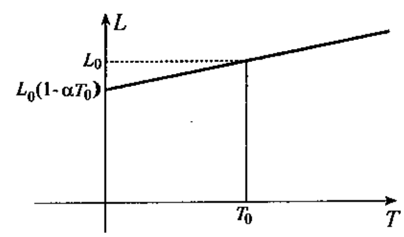


(d) If we set $T_F = T_K$ in the equation from part (a), we obtain $T_K = 5(T_K - 32)/9 + 273.16 \Longrightarrow 4T_K = 2298.44 \Longrightarrow T_K = 574.61$.

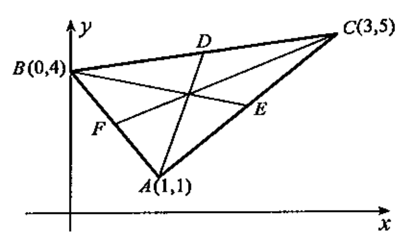
49. (a) If the temperature is changed to T, the change in the length of the bar is $\alpha L_0(T - T_0)$, and therefore its length is

$$L = L_0 + \alpha L_0(T - T_0) = L_0[1 + \alpha(T - T_0)].$$

(b) The ends will be in contact when the rails have length $L = 10.003$ m. This occurs when $10.003 = 10[1 + 1.17 \times 10^{-5}(T - 20)] \Longrightarrow T = 45.6$.

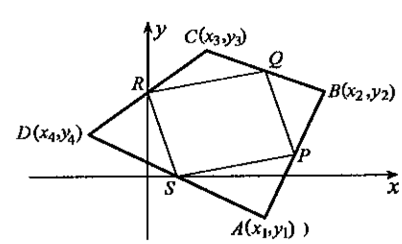


50. Coordinates of E, the midpoint of side AC, are $(2, 3)$. Since the slope of BE is $-1/2$, the equation of median BE is $y - 4 = -(1/2)x \Longrightarrow x + 2y = 8$. Similarly, equations for medians AD and CF are $7x - y = 6$ and $y = x + 2$, respectively. Line segments AD and CF intersect in the point $(4/3, 10/3)$, and this point also satisfies $x + 2y = 8$. Thus, the three medians intersect at the point $(4/3, 10/3)$.



51. Coordinates of P, Q, R, and S are $P((x_1 + x_2)/2, (y_1 + y_2)/2)$, $Q((x_2 + x_3)/2, (y_2 + y_3)/2)$, $R((x_3 + x_4)/2, (y_3 + y_4)/2)$, and $S((x_1 + x_4)/2, (y_1 + y_4)/2)$. Slopes of the line segments PS, RQ, PQ, and RS are, respectively,

$$\frac{(y_2 - y_4)/2}{(x_2 - x_4)/2}, \quad \frac{(y_2 - y_4)/2}{(x_2 - x_4)/2}, \quad \frac{(y_3 - y_1)/2}{(x_3 - x_1)/2}, \quad \frac{(y_3 - y_1)/2}{(x_3 - x_1)/2}.$$



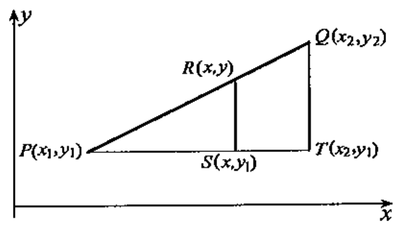
Since PS and RQ are parallel, as are PQ and RS, $PQRS$ is a parallelogram.

52. The midpoint of the line segment is $(1,-1)$, and its slope is $-3/2$. The equation of the perpendicular bisector is $y+1=(2/3)(x-1) \implies 2x=3y+5$.

53. If coordinates of the point are (x,y), then $(x-1)^2+(y-2)^2=(x+1)^2+(y-4)^2$ and $(x-1)^2+(y-2)^2=(x+3)^2+(y-1)^2$. When expanded and simplified, these reduce to $x-y=-3$ and $8x+2y=-5$, the solution of which is $(-11/10, 19/10)$.

54. Suppose two nonvertical, parallel lines have slopes m_1 and m_2. Their equations can be expressed in the form $y=m_1x+b_1$ and $y=m_2x+b_2$. To find their point of intersection we set $m_1x+b_1=m_2x+b_2$. Since the lines do not intersect, there must be no solution of this equation. This happens only if $m_1=m_2$; that is, the lines have the same slope. Conversely, if two nonvertical lines have the same slope, their equations must be of the form $y=mx+b_1$ and $y=mx+b_2$, where $b_1 \neq b_2$. When we attempt to find their point of intersection by setting $mx+b_1=mx+b_2$, we find that $b_1=b_2$, a contradiction. In other words, the lines do not intersect.

55. Because triangles PTQ and PSR are similar, ratios of corresponding sides are equal,



$$\frac{\|PQ\|}{\|PR\|}=\frac{\|PT\|}{\|PS\|}=\frac{x_2-x_1}{x-x_1}.$$

If we subtract 1 from each side of this equation,

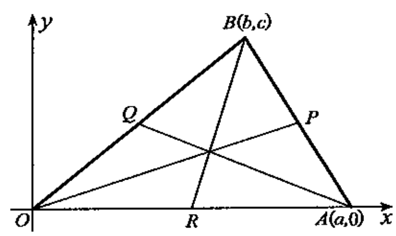
$$\frac{\|PQ\|}{\|PR\|}-1=\frac{x_2-x_1}{x-x_1}-1$$

$$\implies \quad \frac{\|PQ\|-\|PR\|}{\|PR\|}=\frac{x_2-x}{x-x_1} \quad \implies \quad \frac{r_2}{r_1}=\frac{x_2-x}{x-x_1}.$$

Thus, $r_2x-r_2x_1=r_1x_2-r_1x \implies x=\dfrac{r_1x_2+r_2x_1}{r_1+r_2}$. A similar proof gives the corresponding formula for the y-coordinate of R.

56. No. Definition 1.1 defines parallelism only for different lines.

57. We choose a coordinate system with the origin at one vertex of the triangle, and the positive x-axis along one side. The coordinates of the vertices are then $O(0,0)$, $A(a,0)$, and $B(b,c)$. Using equation 1.11, coordinates of the midpoints of the sides are



$$P\left(\frac{a+b}{2},\frac{c}{2}\right), \quad Q\left(\frac{b}{2},\frac{c}{2}\right), \quad R\left(\frac{a}{2},0\right).$$

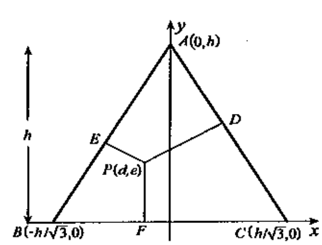
The sum of the squares of the lengths of the medians is

$$\begin{aligned}\|OP\|^2+\|AQ\|^2+\|BR\|^2&=\left[\left(\frac{a+b}{2}\right)^2+\left(\frac{c}{2}\right)^2\right]+\left[\left(a-\frac{b}{2}\right)^2+\left(-\frac{c}{2}\right)^2\right]+\left[\left(b-\frac{a}{2}\right)^2+c^2\right]\\&=\frac{3}{2}(a^2+b^2+c^2-ab).\end{aligned}$$

Three-quarters of the sum of the squares of the lengths of the sides is

$$\frac{3}{4}\left(\|OA\|^2+\|AB\|^2+\|OB\|^2\right)=\frac{3}{4}\left\{a^2+[(b-a)^2+c^2]+(b^2+c^2)\right\}=\frac{3}{2}(a^2+b^2+c^2-ab).$$

58. If we choose the coordinate system in the figure, equations of AC and AB are $y=-\sqrt{3}(x-h/\sqrt{3}) \implies \sqrt{3}x+y-h=0$ and $y=\sqrt{3}(x+h/\sqrt{3}) \implies \sqrt{3}x-y+h=0$. If $P(d,e)$ is any point interior to the triangle, we can use formula 1.16 to find the sum of the distances from P to the three sides



$$\|PF\|+\|PD\|+\|PE\|$$

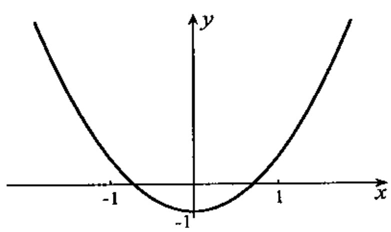
$$= e + \frac{|\sqrt{3}d + e - h|}{\sqrt{3+1}} + \frac{|\sqrt{3}d - e + h|}{\sqrt{3+1}} = e + \frac{1}{2}|\sqrt{3}d + e - h| + \frac{1}{2}|\sqrt{3}d - e + h|.$$

Because $P(d, e)$ is below and to the left of line AC, it follows that $\sqrt{3}d + e - h < 0$. Because P is below and to the right of AB, $\sqrt{3}d - e + h > 0$. Hence,

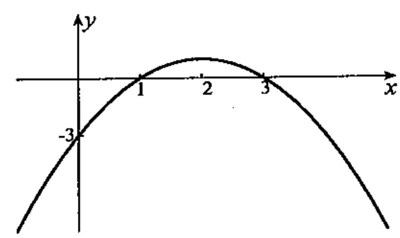
$$\|PF\| + \|PD\| + \|PE\| = e + \frac{1}{2}(-\sqrt{3}d - e + h) + \frac{1}{2}(\sqrt{3}d - e + h) = h.$$

EXERCISES 1.4A

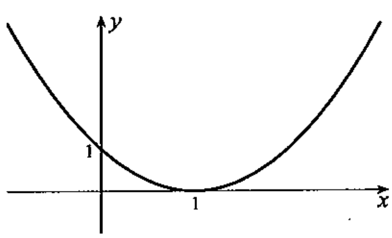
1. This is the parabola $y = 2x^2$ shifted 1 unit downward.



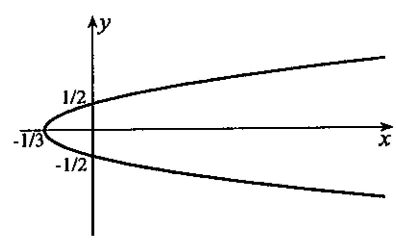
2. Factored in the form $y = -(x-3)(x-1)$, the x-intercepts of the parabola are 1 and 3. Its maximum is at $x = 2$.



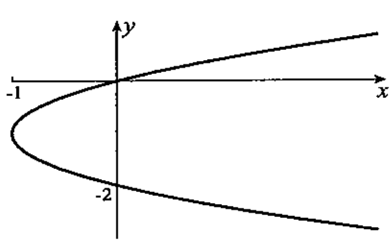
3. Factored in the form $y = (x-1)^2$, the parabola has its minimum at $(1, 0)$.



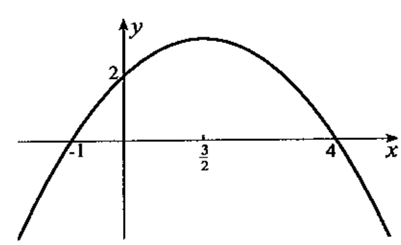
4. This is the parabola $x = 4y^2/3$ shifted 1/3 unit to the left.



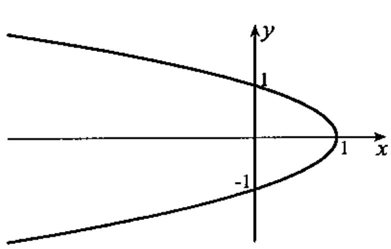
5. Factored in the form $x = y(2+y)$, the parabola opens to the right with y-intercepts 0 and -2.



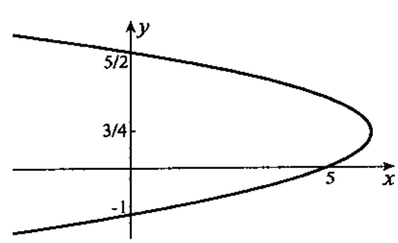
6. Factored in the form $y = -(x+1)(x-4)/2$, x-intercepts of the parabola are -1 and 4. Its maximum is at $x = 3/2$.



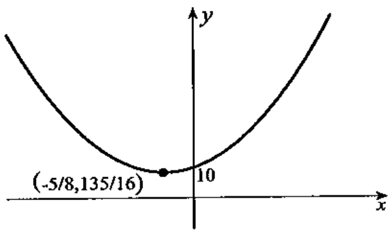
7. The parabola $x = 1 - y^2$ opens left with x-intercept 1 and y-intercepts ± 1.



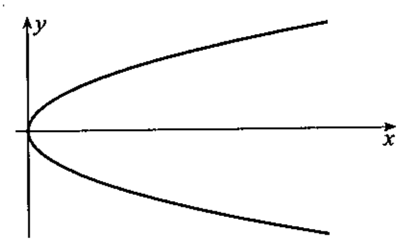
8. Factored in the form $x = -(2y-5)(y+1)$, y-intercepts of the parabola are -1 and $5/2$. Its maximum in the x-direction occurs for $y = 3/4$.



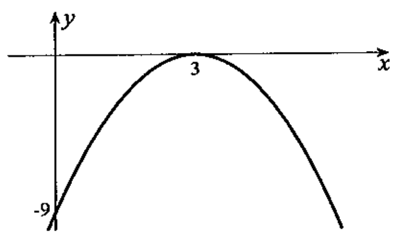
9. Expressed in the form $y = 4(x^2 + 5x/4) + 10$
$= 4(x + 5/8)^2 + 10 - 25/16$
$= 4(x + 5/8)^2 + 135/16$, the parabola
opens upward with minimum at $(-5/8, 135/16)$.



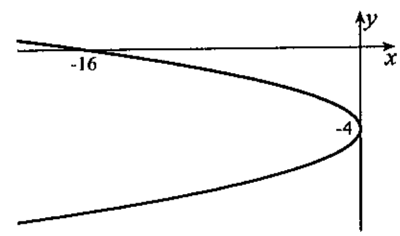
10. This parabola opens to the right.



11. Expressed in the form $y = -(x^2 - 6x + 9)$
$= -(x - 3)^2$, the parabola opens downward
from the point $(3, 0)$.



12. This parabola opens to the left
and touches the y-axis at $y = -4$.



13. (a) The y-intercept is -5, and x-intercepts are $\dfrac{2 \pm \sqrt{4 + 20}}{2} = 1 \pm \sqrt{6}$.
(b) The x-intercept is 4, and from $x = 4(y - 1)^2$, the y-intercept is 1.

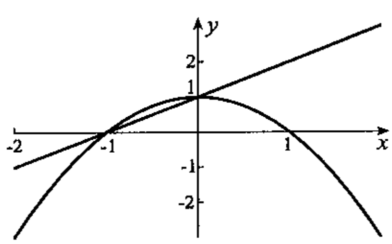
14. The equation must be of the form $y = ax^2 + 1$. Since $(2, 3)$ is on the parabola, $3 = 4a + 1 \implies a = 1/2$.

15. The equation must be of the form $x = 2 + ay^2$. Since $(0, 4)$ is on the parabola, $0 = 2 + 16a \implies a = -1/8$.

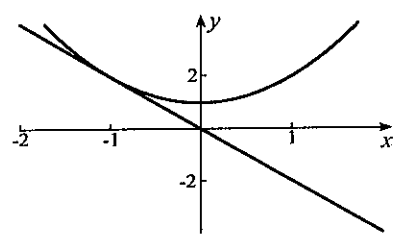
16. Since the parabola crosses the x-axis at $x = -1$ and $x = 3$, it must be of the form $y = a(x + 1)(x - 3)$. Since $(0, -1)$ is on the parabola, $-1 = a(1)(-3) \implies a = 1/3$.

17. Since the parabola touches the x-axis at $x = 1$, it must be of the form $y = a(x - 1)^2$. Since $(0, 2)$ is on the parabola, $2 = a(-1)^2 \implies a = 2$.

18. We set $x + 1 = 1 - x^2 \implies 0 = x^2 + x$
$= x(x + 1) \implies x = 0, -1$. Points of
intersection are $(-1, 0)$ and $(0, 1)$.

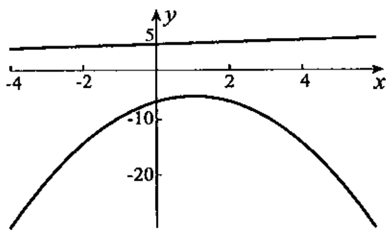


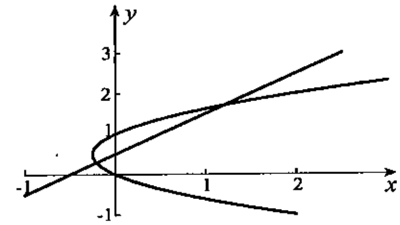
19. We set $-2x = 1 + x^2 \implies 0 = x^2 + 2x + 1$
$= (x + 1)^2 \implies x = -1$. The only point
of intersection is $(-1, 2)$.



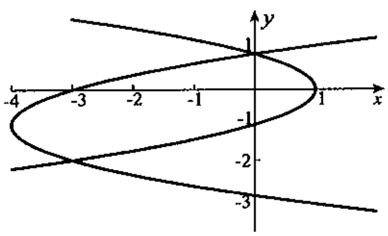
20. We set $2x - x^2 - 6 = 5 + x/5 \implies$
$5x^2 - 9x + 55 = 0$. With equation 1.5,
$x = \left(9 \pm \sqrt{81 - 4(5)(55)}\right)/10$. Since
$81 - 4(5)(55) < 0$, there are no points
of intersection.

21. We set $y(y - 1) = y - 1/2 \implies 2y^2 - 4y + 1 = 0$.
With equation 1.5, $y = (4 \pm \sqrt{16 - 8})/4$
$= (2 \pm \sqrt{2})/2$. Points of intersection are
$\left((1 \pm \sqrt{2})/2, (2 \pm \sqrt{2})/2\right)$.

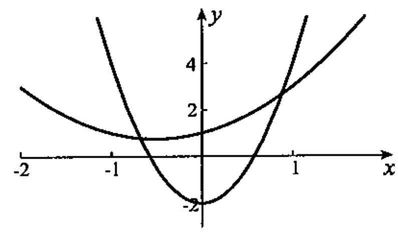




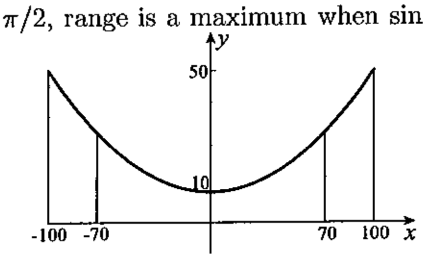
22. We set $-y^2+1=y^2+2y-3$ $\Longrightarrow 0=2y^2+2y-4=2(y+2)(y-1)$ $\Longrightarrow y=1,-2$. Points of intersection are $(0,1)$ and $(-3,-2)$.



23. We set $6x^2-2=x^2+x+1 \Longrightarrow 5x^2-x-3=0$. Equation 1.5 gives $x=(1\pm\sqrt{1+60})/10$ $=(1\pm\sqrt{61})/10$. Points of intersection are $\left((1\pm\sqrt{61})/10,(43\pm3\sqrt{61})/25\right)$.



24. The range of the shell is $R=(v^2/9.81)\sin 2\theta$. Since $0\le\theta\le\pi/2$, range is a maximum when $\sin 2\theta =$ $1 \Longrightarrow 2\theta=\pi/2 \Longrightarrow \theta=\pi/4$ radians.



25. With coordinates as shown, the equation of the parabola takes the form $y=ax^2+10$. Since the point $(100,50)$ is on the parabola, $50=(10\,000)a+10 \Longrightarrow a=1/250$. When $x=70$, we obtain $y=(70)^2/250+10=148/5$ m for the length of the supporting rod.

26. We set $5x=(x-2)^4+4 \Longrightarrow (x-2)^4-5x+4=0 \Longrightarrow x^4-8x^3+24x^2-37x+20=0$. Possible rational solutions are ±1, ±2, ±4, ±5, ±10, ±20. One solution is $x=1$, so that

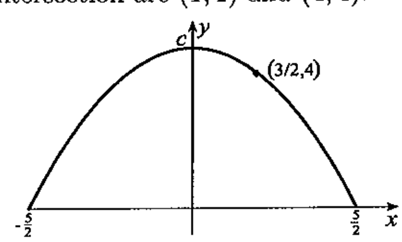
$$x^4-8x^3+24x^2-37x+20=(x-1)(x^3-7x^2+17x-20)=0.$$

We find that $x=4$ is a zero of the cubic, so that

$$x^4-8x^3+24x^2-37x+20=(x-1)(x-4)(x^2-3x+5).$$

Since $x^2-3x+5=0$ has no real solutions, the only points of intersection are $(1,1)$ and $(4,4)$.

27. With the coordinate system shown, the equation of the parabola takes the form $y=c(25/4-x^2)$. Since the point $(3/2,4)$ is on the parabola, $4=c(25/4-9/4) \Longrightarrow$ $c=1$. The arch is therefore $25/4$ units high.



28. If the parabola is to pass through these points, then

$$2=a+b+c, \qquad 10=9a-3b+c, \qquad 4=9a+3b+c.$$

The solution of these equations is $a=1/2$, $b=-1$, $c=5/2$.

29. (a) Since resistances at temperatures 0°, 100°, and 700° are 10.000, 13.946, and 24.172,

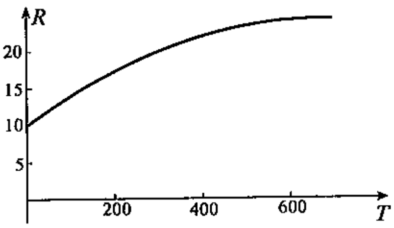
$$10.000=R_0(1), \qquad 13.946=R_0(1+100a+10\,000b), \qquad 24.172=R_0(1+700a+490\,000b).$$

The solution of these equations is $R_0=10.000$,

$a = 0.004\,266\,2$, and $b = -3.202\,4 \times 10^{-6}$.
(b) The parabola is plotted to the right.
(c) The resistance is 20 ohms when

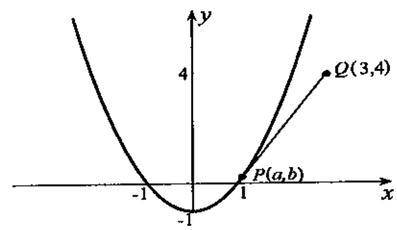
$$20 = 10(1 + aT + bT^2) \implies bT^2 + aT - 1 = 0.$$

Solutions of this equation are $T = \dfrac{-a \pm \sqrt{a^2 + 4b}}{2b}$.
Since only the positive solution is acceptable, $T = (-a + \sqrt{a^2 + 4b})/(2b) \approx 304$.



30. If $P(a, b)$ is the point at which the rope meets the parabola, the equation of the line PQ is $y - 4 = [(b - 4)/(a - 3)](x - 3)$. The x-coordinates of points of intersection of this line with the parabola are given by the equation

$$x^2 - 1 = \left(\frac{b-4}{a-3}\right)(x - 3) + 4$$
$$\implies$$
$$x^2 - \left(\frac{b-4}{a-3}\right)x + 3\left(\frac{b-4}{a-3}\right) - 5 = 0.$$



Since the line is to meet the parabola in only one point, the discriminant of this quadratic must be zero

$$\left(\frac{b-4}{a-3}\right)^2 - 4\left[3\left(\frac{b-4}{a-3}\right) - 5\right] = 0 \implies (b-4)^2 - 12(b-4)(a-3) + 20(a-3)^2 = 0.$$

Since $P(a, b)$ is on the parabola, $b = a^2 - 1$, and when we substitute this into the above equation,

$$0 = (a^2 - 5)^2 - 12(a^2 - 5)(a - 3) + 20(a - 3)^2 = a^4 - 12a^3 + 46a^2 - 60a + 25.$$

Possible rational solutions of this equation are ± 1, ± 5, ± 25. We find that $a = 1$ is a solution. When it is factored from the quartic,

$$0 = a^4 - 12a^3 + 46a^2 - 60a + 25 = (a - 1)(a^3 - 11a^2 + 35a - 25).$$

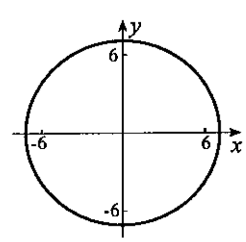
Once again $a = 1$ is a zero of the cubic, so that

$$0 = a^4 - 12a^3 + 46a^2 - 60a + 25 = (a - 1)(a - 1)(a^2 - 10a + 25) = (a - 1)^2(a - 5)^2.$$

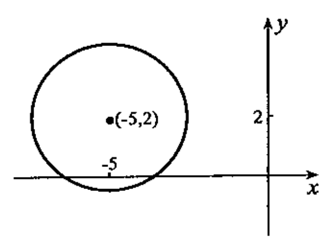
Clearly, $a = 5$ is inadmissible, and the required point is $(1, 0)$.

EXERCISES 1.4B

1. The circle is centred at the origin with radius $5\sqrt{2}$.

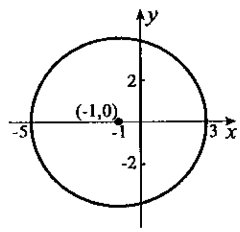


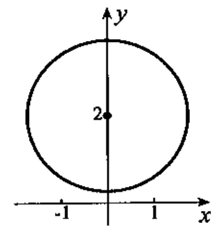
2. The centre of the circle is $(-5, 2)$ and its radius is $\sqrt{6}$.



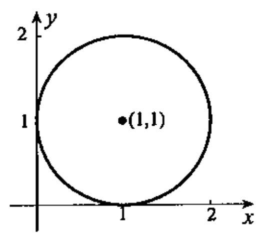
3. When we complete the square on the x-terms, $(x + 1)^2 + y^2 = 16$. The centre of the circle is $(-1, 0)$ and its radius is 4.

4. When we complete the square on the y-terms, $x^2 + (y - 2)^2 = 3$. The centre of the circle is $(0, 2)$ and its radius is $\sqrt{3}$.

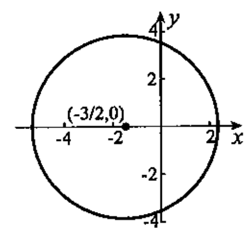




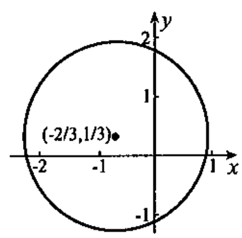
5. When we complete squares on x- and y-terms, $(x-1)^2+(y-1)^2=1$. The centre of the circle is $(1,1)$ and its radius is 1.



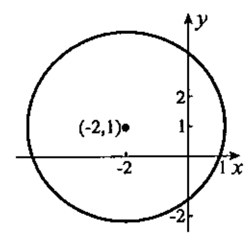
6. When we complete the square on the x-terms, $(x+3/2)^2+y^2=59/4$. The centre of the circle is $(-3/2,0)$ and its radius is $\sqrt{59}/2$.



7. When we complete squares on x- and y-terms, $(x+2/3)^2+(y-1/3)^2=23/9$. The centre of the circle is $(-2/3,1/3)$ and its radius is $\sqrt{23}/3$.



8. When we complete squares on x- and y-terms, $(x+2)^2+(y-1)^2=10$. The centre of the circle is $(-2,1)$ and its radius is $\sqrt{10}$.



9. When we complete squares on x- and y-terms, $(x-1)^2+(y-2)^2=0$. The only point satisfying this equation is $(1,2)$.

10. When we complete squares on x- and y-terms, $(x+3)^2+(y+3/2)^2=-35/4$. No point can satisfy this equation.

11. With centre $(0,0)$ and radius 2, the equation of the circle is $x^2+y^2=4$.

12. Since the centre is $(1,0)$ and its radius is 1, the equation of the circle is $(x-1)^2+y^2=1$.

13. Since the centre is $(3,4)$ and its radius is 2, the equation of the circle is $(x-3)^2+(y-4)^2=4$.

14. The figure indicates that the centre of the circle is $(3/2,-3/2)$. Its radius is then $\sqrt{(3/2)^2+(-3/2)^2}=3/\sqrt{2}$. The equation of the circle is therefore $(x-3/2)^2+(y+3/2)^2=9/2$.

15. If we take the equation of the circle in form 1.22, and substitute each of the points $(3,0)$, $(2,7)$, and $(-5,6)$,

$$\begin{aligned} 9+0+3f+e&=0, \\ 4+49+2f+7g+e&=0, \\ 25+36-5f+6g+e&=0, \end{aligned} \quad \Longrightarrow \quad \begin{aligned} 3f+e&=-9, \\ 2f+7g+e&=-53, \\ -5f+6g+e&=-61. \end{aligned}$$

The solution of these equations is $f=2$, $g=-6$, and $e=-15$. The equation of the circle is $x^2+y^2+2x-6y-15=0$.

16. If we take the equation for the circle in form 1.22, and substitute each of the points $(1,3)$, $(5,1)$ and $(2,-2)$,

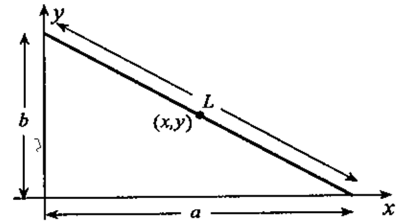
$$\begin{aligned} 1+9+f+3g+e&=0, & \qquad & f+3g+e=-10,\\ 25+1+5f+g+e&=0, & \Longrightarrow \qquad & 5f+g+e=-26,\\ 4+4+2f-2g+e&=0, & & 2f-2g+e=-8. \end{aligned}$$

The solution of these equations is $f=-14/3$, $g=-4/3$, and $e=-4/3$. The equation of the circle is $x^2+y^2-14x/3-4y/3-4/3=0$ or $3x^2+3y^2-14x-4y-4=0$.

17. If a and b are as shown in the figure, coordinates of the midpoint of the ladder are $x=a/2$ and $y=b/2$. Since a and b always satisfy the equation $a^2+b^2=L^2$, it follows that

$$(2x)^2+(2y)^2=L^2 \quad \Longrightarrow \quad x^2+y^2=L^2/4.$$

Hence, the midpoint follows a quarter circle with centre at the foot of the wall and radius $L/2$.



18. If we take the equation of the circle in the form $(x-h)^2+(y-k)^2=r^2$, and substitute the two points

$$(3-h)^2+(4-k)^2=r^2, \qquad (1-h)^2+(-10-k)^2=r^2.$$

When these equations are subtracted, the result is

$$0=(3-h)^2-(1-h)^2+(4-k)^2-(-10-k)^2 \quad \Longrightarrow \quad h+7k=-19.$$

(a) When the centre of the circle is on the line $2x+3y+16=0$, we must also have $2h+3k+16=0$. When these equations are solved, $h=-5$ and $k=-2$. The radius of the circle is $r=\sqrt{(3+5)^2+(4+2)^2}=10$.
(b) When the centre is on the line $x+7y+19=0$, we must have $h+7k+19=0$. But this is the same equation obtained from the two points. In other words, there is an infinity of circles with centres on the line $x+7y+19=0$ passing through the points $(3,4)$ and $(1,-10)$. Any equation of the form $(x+7k+19)^2+(y-k)^2=r^2$, where $r^2=(3+7k+19)^2+(4-k)^2=50k^2+300k+500$.

19. We set $x^2+2x+(3x+2)^2=4 \Longrightarrow 0=10x^2+14x=2x(5x+7)$. Thus, $x=0$ and $x=-7/5$, and the points are $(0,2)$ and $(-7/5,-11/5)$.

20. We set $x^2+(1-2x)^2-4(1-2x)+1=0 \Longrightarrow 5x^2+4x-2=0$. Solutions are $x=(-4\pm\sqrt{16+40})/10=(-2\pm\sqrt{14})/5$. Intersection points are $\big((-2\pm\sqrt{14})/5,(9\mp 2\sqrt{14})/5\big)$.

21. We set $x^2+(3x^2+4)^2=9 \Longrightarrow 9x^4+25x^2+7=0$. But this is impossible since $9x^4$ and $25x^2$ are both nonnegative. There are no points of intersection.

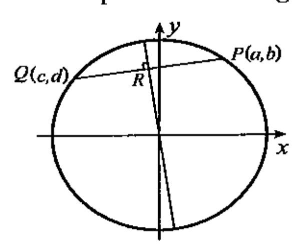
22. We set $(x+3)^2+16(x+1)=25 \Longrightarrow 0=x^2+22x=x(x+22)$. Thus, $x=0$ and $x=-22$. From $x=0$ we obtain the points $(0,\pm 4)$, while $x=-22$ yields no points.

23. When we complete squares on x- and y-terms, $\left(x+\dfrac{f}{2}\right)^2+\left(y+\dfrac{g}{2}\right)^2=\dfrac{f^2}{4}+\dfrac{g^2}{4}-e=\dfrac{1}{4}(f^2+g^2-4e)$.
If $f^2+g^2-4e>0$, this equation represents a circle with centre $(-f/2,-g/2)$ and radius $\sqrt{f^2+g^2-4e}/2$. If $f^2+g^2-4e=0$, only the point $(-f/2,-g/2)$ satisfies the equation. If $f^2+g^2-4e<0$, no points can satisfy the equation.

24. If we choose a coordinate system with origin at the centre of the circle, the equation of the circle is $x^2+y^2=r^2$. Let $P(a,b)$ and $Q(c,d)$ be any two points on the circle. The midpoint of line segment PQ has coordinates $\big((a+c)/2,(b+d)/2\big)$. Since the slope of the perpendicular bisector of PQ is $-(c-a)/(d-b)$, the equation of the perpendicular bisector is

$$y-\frac{b+d}{2}=-\left(\frac{c-a}{d-b}\right)\left(x-\frac{a+c}{2}\right).$$

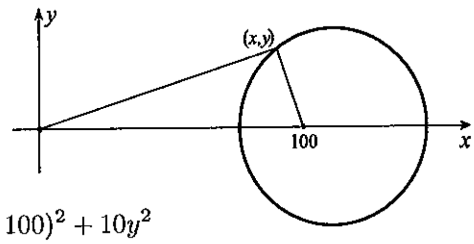
This line passes through the origin if and only if



$$-\frac{b+d}{2} = -\left(\frac{c-a}{d-b}\right)\left(-\frac{a+c}{2}\right) \iff -(b+d)(d-b) = (c-a)(a+c) \iff a^2+b^2 = c^2+d^2.$$

Since P and Q are on the circle, it follows that $a^2+b^2=r^2$ and $c^2+d^2=r^2$, and therefore $a^2+b^2=c^2+d^2$.

25. If I is the brightness of the source at $(100,0)$, then the amount of light received at point (x,y) from this light is $A_1 = \dfrac{kI}{(x-100)^2+y^2}$, where k is a constant of proportionality. The amount of light received from the source at the origin is $A_2 = \dfrac{k(10I)}{x^2+y^2}$.



For $A_1 = A_2$,

$$\frac{kI}{(x-100)^2+y^2} = \frac{10kI}{x^2+y^2} \implies x^2+y^2 = 10(x-100)^2+10y^2$$

$$\implies 9x^2+9y^2-2000x+10^5=0 \implies \left(x-\frac{1000}{9}\right)^2 + y^2 = \frac{-10^5}{9}+\frac{10^6}{81} = \frac{10^5}{81}.$$

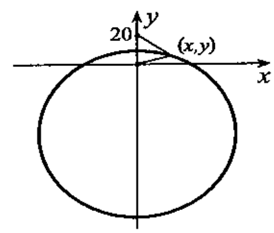
We have a circle centred at $(1000/9, 0)$ and radius $100\sqrt{10}/9$.

26. If L is the loudness of the speaker at $(0,20)$, then the amount of sound received at point (x,y) from this speaker is

$$A_1 = \frac{kL}{x^2+(y-20)^2},$$

where k is a constant of proportionality. The amount of sound received from the speaker at the origin is

$$A_2 = \frac{k(0.7L)}{x^2+y^2}.$$



For $A_1 = A_2$, $\quad \dfrac{kL}{x^2+(y-20)^2} = \dfrac{7kL}{10(x^2+y^2)} \implies 10(x^2+y^2) = 7[x^2+(y-20)^2]$. This gives

$$3x^2+3y^2+280y = 2800 \implies x^2+\left(y+\frac{140}{3}\right)^2 = \frac{2800}{3}+\frac{140^2}{9} = \frac{28\,000}{9}.$$

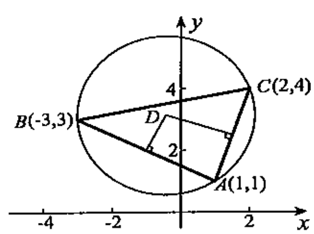
We have a circle centred at $(0, -140/3)$ and radius $20\sqrt{70}/3$.

27. (a) The centre of the circle is the intersection of the perpendicular bisectors of AB and AC (see Exercise 24). These perpendicular bisectors have equations

$$y-2 = -\frac{4}{-2}(x+1), \quad y-\frac{5}{2} = -\frac{1}{3}\left(x-\frac{3}{2}\right).$$

The solution of these equations is $D(-3/7, 22/7)$. The radius of the circumcircle is therefore

$$\|AD\| = \sqrt{(10/7)^2+(-15/7)^2} = 5\sqrt{13}/7.$$



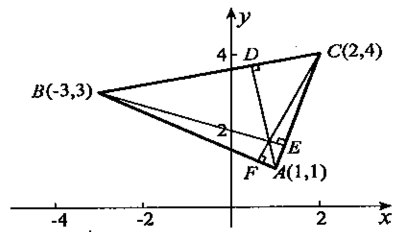
The equation of the circumcircle is $\left(x+\dfrac{3}{7}\right)^2 + \left(y-\dfrac{22}{7}\right)^2 = \dfrac{325}{49}$.

(b) We take the equation of the circle in the form $(x-h)^2+(y-k)^2=r^2$. Since $(1,1)$, $(-3,3)$ and $(2,4)$ are points on the circle,

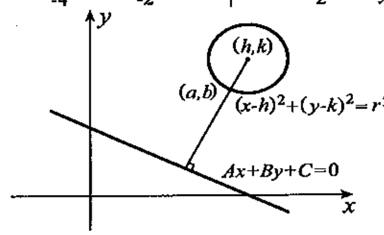
$$(1-h)^2+(1-k)^2=r^2, \qquad (-3-h)^2+(3-k)^2=r^2, \qquad (2-h)^2+(4-k)^2=r^2.$$

The solution of this system is $h=-3/7$, $k=22/7$, and $r=5\sqrt{13}/7$, giving the same equation as in part (a).

28. Since the slope of BC is $1/5$, the equation of altitude AD is $y-1=-5(x-1) \implies 5x+y=6$. Since the slope of AC is 3, the equation of altitude BE is $y-3=-(1/3)(x+3) \implies x+3y=6$. The point of intersection of these altitudes is $(6/7, 12/7)$. The equation of altitude CF is $y-4=2(x-2)$, and the point $(6/7, 12/7)$ satisfies this equation. Hence, the altitudes intersect at $(6/7, 12/7)$.



29. The shortest distance occurs for the point (a,b) on the circle where the line joining (h,k) and (a,b) is perpendicular to the line. In terms of slopes, this condition requires $(b-k)/(a-h)=B/A \implies b=k+B(a-h)/A$. Since (a,b) is on the circle, $(a-h)^2+(b-k)^2=r^2$. These equations imply that



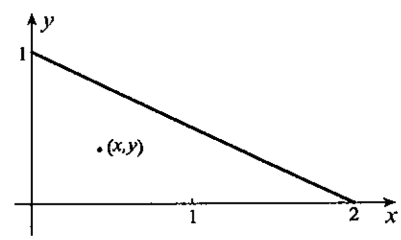
$$(a-h)^2+\left[k+\frac{B}{A}(a-h)-k\right]^2=r^2 \implies (a-h)^2\left(1+\frac{B^2}{A^2}\right)=r^2 \implies a=h\pm\frac{rA}{\sqrt{A^2+B^2}}.$$

Corresponding y-coordinates are $b=k\pm\dfrac{rB}{\sqrt{A^2+B^2}}$. The diagram indicates why there are two points. Using distance formula 1.16, the minimum distance is the smaller of the numbers

$$\frac{\left|A\left(h\pm rA/\sqrt{A^2+B^2}\right)+B\left(k\pm rB/\sqrt{A^2+B^2}\right)+C\right|}{\sqrt{A^2+B^2}}=\frac{\left|(Ah+Bk+C)\pm r(A^2+B^2)/\sqrt{A^2+B^2}\right|}{\sqrt{A^2+B^2}}$$
$$=\frac{\left|(Ah+Bk+C)\pm r\sqrt{A^2+B^2}\right|}{\sqrt{A^2+B^2}}.$$

30. If (x,y) is the incentre, then it must be equidistant from the three sides of the triangle, $y=0$, $x=0$, and $x+2y-2=0$. When we equate these distances, using formula 1.16, the result is

$$|y|=|x|=\frac{|x+2y-2|}{\sqrt{1^2+2^2}}=\frac{|x+2y-2|}{\sqrt{5}}.$$



The equation $|y|=|x|$ implies that $y=\pm x$. We combine this with the two possibilities $x+2y-2>0$ and $x+2y-2<0$, for four possible cases. With $y=x$ and $x+2y-2<0$, we obtain $x=-(x+2x-2)/\sqrt{5} \implies x=(3-\sqrt{5})/2$. This gives the incentre $\left((3-\sqrt{5})/2, (3-\sqrt{5})/2\right)$. With $y=x$ and $x+2y-2>0$, we obtain $x=(x+2x-2)/\sqrt{5} \implies x=(3+\sqrt{5})/2$. This gives the point $\left((3+\sqrt{5})/2, (3+\sqrt{5})/2\right)$. With $y=-x$, we obtain the two additional points $\left((1+\sqrt{5})/2, -(1+\sqrt{5})/2\right)$ and $\left((1-\sqrt{5})/2, (\sqrt{5}-1)/2\right)$. Three circles can be drawn to touch the sides of the triangle when they are extended. The additional three points are the centres of these circles, but they are outside the triangle.

31. The amount of sound received at point (x,y) from the source at (x_1,y_1) is $A_1=kI_1/[(x-x_1)^2+(y-y_1)^2]$, where k is a constant of proportionality. The amount of sound received from the source at (x_2,y_2) is $A_2=kI_2/[(x-x_2)^2+(y-y_2)^2]$. For $A_1=A_2$,

$$\frac{kI_1}{(x-x_1)^2+(y-y_1)^2}=\frac{kI_2}{(x-x_2)^2+(y-y_2)^2}.$$

If we set $\alpha=I_1/I_2$, then

$$\alpha[(x-x_2)^2+(y-y_2)^2]=(x-x_1)^2+(y-y_1)^2$$

$$\implies (\alpha-1)x^2+(\alpha-1)y^2+2(x_1-\alpha x_2)x+2(y_1-\alpha y_2)y=x_1^2-\alpha x_2^2+y_1^2-\alpha y_2^2$$

$$\Longrightarrow \quad \left(x+\frac{x_1-\alpha x_2}{\alpha-1}\right)^2+\left(y+\frac{y_1-\alpha y_2}{\alpha-1}\right)^2=\frac{x_1^2-\alpha x_2^2}{\alpha-1}+\frac{y_1^2-\alpha y_2^2}{\alpha-1}+\left(\frac{x_1-\alpha x_2}{\alpha-1}\right)^2+\left(\frac{y_1-\alpha y_2}{\alpha-1}\right)^2.$$

This is a circle if the right side is positive. Consider

$$\begin{aligned}\frac{x_1^2-\alpha x_2^2}{\alpha-1}+\left(\frac{x_1-\alpha x_2}{\alpha-1}\right)^2 &= \frac{1}{(\alpha-1)^2}[(\alpha-1)(x_1^2-\alpha x_2^2)+(x_1-\alpha x_2)^2]\\ &= \frac{1}{(\alpha-1)^2}[\alpha(x_1^2-\alpha x_2^2)-(x_1^2-\alpha x_2^2)+(x_1^2-2\alpha x_1x_2+\alpha^2x_2^2)]\\ &= \frac{\alpha}{(\alpha-1)^2}(x_1-x_2)^2.\end{aligned}$$

With a similar result for the y-terms, we can write that

$$\left(x+\frac{x_1-\alpha x_2}{\alpha-1}\right)^2+\left(y+\frac{y_1-\alpha y_2}{\alpha-1}\right)^2=\frac{\alpha}{(\alpha-1)^2}[(x_1-x_2)^2+(y_1-y_2)^2].$$

We have a circle centre $\left(\dfrac{x_1-\alpha x_2}{1-\alpha},\dfrac{y_1-\alpha y_2}{1-\alpha}\right)$ and radius $\sqrt{\dfrac{\alpha}{(\alpha-1)^2}[(x_1-x_2)^2+(y_1-y_2)^2]}$. The equation of the line through (x_1,y_1) and (x_2,y_2) is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \Longrightarrow \quad y=\frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1.$$

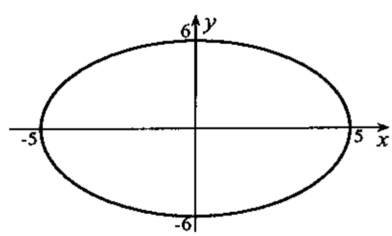
If we substitute $x=(x_1-\alpha x_2)/(1-\alpha)$ into the right side, we obtain

$$\begin{aligned}y &= \frac{y_2-y_1}{x_2-x_1}\left(\frac{x_1-\alpha x_2}{1-\alpha}-x_1\right)+y_1=\frac{y_2-y_1}{x_2-x_1}\left(\frac{x_1-\alpha x_2-x_1+\alpha x_1}{1-\alpha}\right)+y_1\\ &= \frac{y_2-y_1}{x_2-x_1}\left[\frac{\alpha(x_1-x_2)}{1-\alpha}\right]+y_1=(y_1-y_2)\left(\frac{\alpha}{1-\alpha}\right)+y_1\\ &= \frac{\alpha y_1-\alpha y_2+y_1-\alpha y_1}{1-\alpha}=\frac{y_1-\alpha y_2}{1-\alpha},\end{aligned}$$

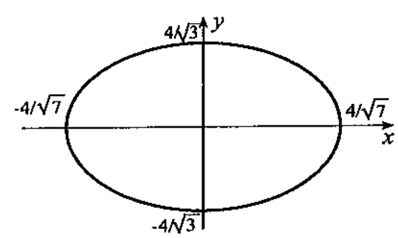
and this is the y-coordinate of the centre of the circle. Hence, the centre of the circle lies on the line through (x_1,y_1) and (x_2,y_2).

EXERCISES 1.4C

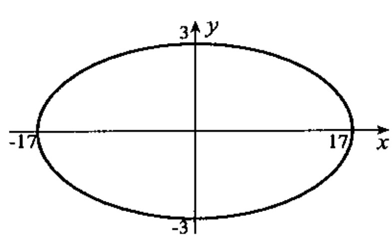
1. The ellipse is centred at the origin. The x- and y-intercepts are ± 5 and ± 6.



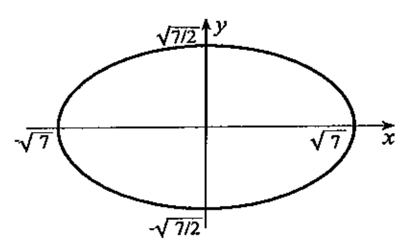
2. The ellipse is centred at the origin. Its x- and y-intercepts are $x=\pm 4/\sqrt{7}$ and $y=\pm 4/\sqrt{3}$.



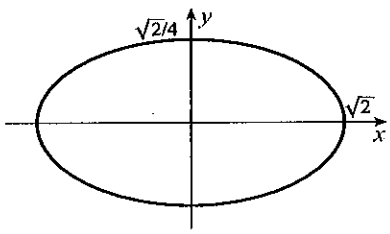
3. The ellipse is centred at the origin. Its x- and y-intercepts are $x=\pm 17$ and $y=\pm 3$.



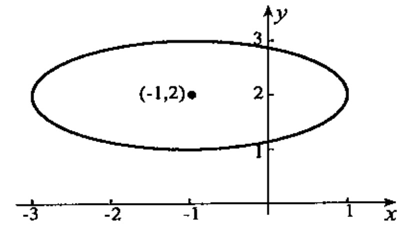
4. The ellipse is centred at the origin. Its x- and y-intercepts are $x=\pm\sqrt{7}$ and $y=\pm\sqrt{7/2}$.



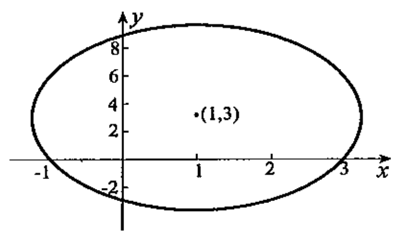
5. The ellipse is centred at the origin. Its x- and y-intercepts are $x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}/4$.



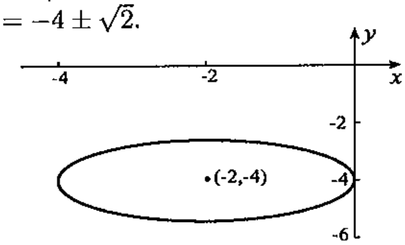
6. When we complete squares on x- and y-terms, $(x+1)^2/4 + (y-2)^2 = 1$. The centre of the ellipse is $(-1, 2)$. It intersects the line $y = 2$ when $x = -3$ and $x = 1$, and intersects the line $x = -1$ when $y = 1$ and $y = 3$.



7. When we complete squares on x- and y-terms,
$$\frac{(x-1)^2}{44/9} + \frac{(y-3)^2}{44} = 1.$$
The centre of the ellipse is $(1, 3)$.
It intersects the line $y = 3$ when $x = 1 \pm 2\sqrt{11}/3$, and intersects the line $x = 1$ when $y = 3 \pm 2\sqrt{11}$.

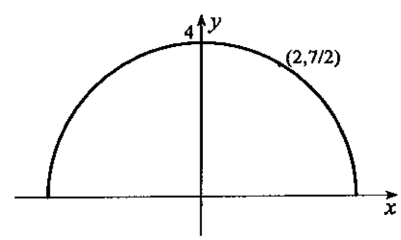


8. When we complete squares on x- and y-terms,
$$\frac{(x+2)^2}{4} + \frac{(y+4)^2}{2} = 1.$$
The centre of the ellipse is $(-2, -4)$.
It intersects the line $y = -4$ when $x = -4$ and $x = 0$, and intersects the line $x = -2$ when $y = -4 \pm \sqrt{2}$.

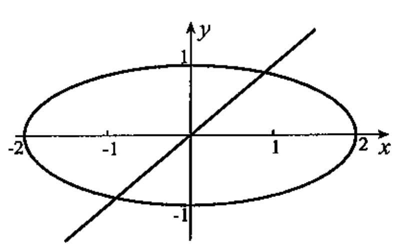


9. If an ellipse of form $x^2/a^2 + y^2/b^2 = 1$ is to pass through the points $(-2, 4)$ and $(3, 1)$, then
$$\frac{4}{a^2} + \frac{16}{b^2} = 1, \qquad \frac{9}{a^2} + \frac{1}{b^2} = 1.$$
These can be solved for $a^2 = 28/3$ and $b^2 = 28$.

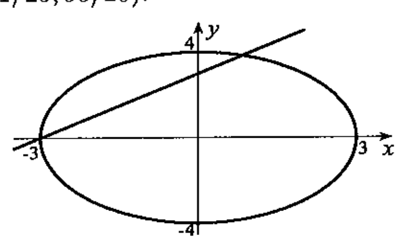
10. With the coordinate system shown, the equation of the ellipse must be of the form $x^2/a^2 + y^2/b^2 = 1$. Since $(0, 4)$ and $(2, 7/2)$ are points on the ellipse,
$$\frac{16}{b^2} = 1, \qquad \frac{4}{a^2} + \frac{49/4}{b^2} = 1.$$
These imply that $a^2 = 256/15$ and $b^2 = 16$. The width of the arch is therefore $2a = 32/\sqrt{15}$.



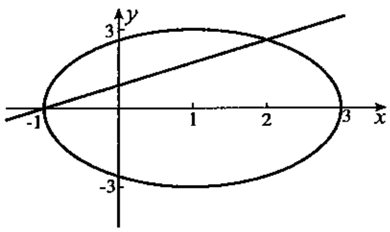
11. We set $x^2 + 4x^2 = 4 \implies x = \pm 2/\sqrt{5}$. Points of intersection are therefore $(\pm 2/\sqrt{5}, \pm 2/\sqrt{5})$.



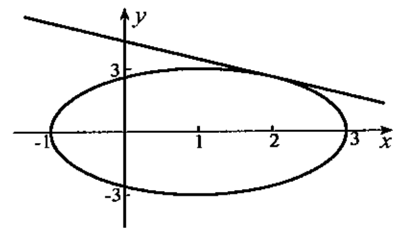
12. If we substitute $y = x + 3$ into the equation of the ellipse, $16x^2 + 9(x+3)^2 = 144$, from which $0 = 25x^2 + 54x - 63 = (x+3)(25x - 21)$. Points of intersection of the curves are $(-3, 0)$ and $(21/25, 96/25)$.



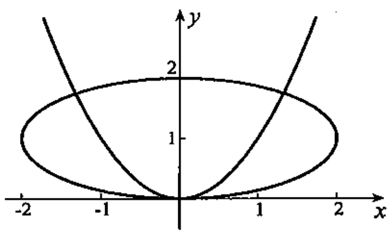
13. If we substitute $y = \sqrt{3}(x+1)/2$ into the equation of the ellipse, $9x^2 - 18x + 4\left[\frac{\sqrt{3}(x+1)}{2}\right]^2 = 27$, from which $0 = 12x^2 - 12x - 24 = 12(x-2)(x+1)$. Points of intersesction are therefore $(2, 3\sqrt{3}/2)$ and $(-1, 0)$.



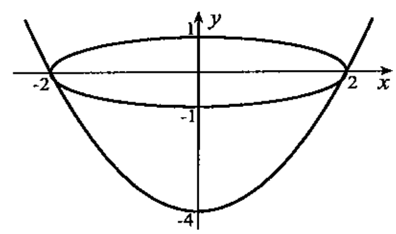
14. If we put $y = \sqrt{3}(5-x)/2$ into the equation of the ellipse, $9x^2 - 18x + 4\left[\frac{\sqrt{3}}{2}(5-x)\right]^2 = 27$, from which $0 = 12x^2 - 48x + 48 = 12(x-2)^2$. The line and ellipse therefore touch at the point $(2, 3\sqrt{3}/2)$.



15. We set $0 = x^2 + 4x^4 - 8x^2 = x^2(4x^2 - 7)$. Points of intersection are $(0, 0)$ and $(\pm\sqrt{7}/2, 7/4)$.



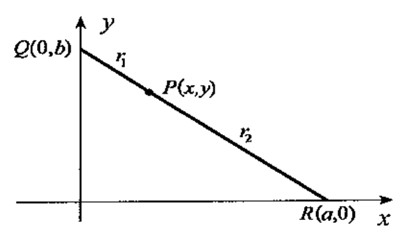
16. If we substitute $y = x^2 - 4$ into the equation of the ellipse, $x^2 + 4(x^2-4)^2 = 4$, from which $0 = 4x^4 - 31x^2 + 60 = (x^2 - 4)(4x^2 - 15)$. Solutions are $x = \pm 2$, and $\pm\sqrt{15}/2$. Points intersection are $(\pm\sqrt{15}/2, -1/4)$ and $(\pm 2, 0)$.



17. If a and b are as shown in the figure to the right, then $L^2 = a^2 + b^2$. Let $P(x, y)$ be the coordinates of the point on the ladder such that the ratio of the lengths $\|PQ\|$ and $\|PR\|$, is r_1/r_2. According to Exercise 55 in Section 1.3,

$$x = \frac{r_1 a}{r_1 + r_2}, \quad y = \frac{r_2 b}{r_1 + r_2}.$$

If we solve these for a and b and substitute

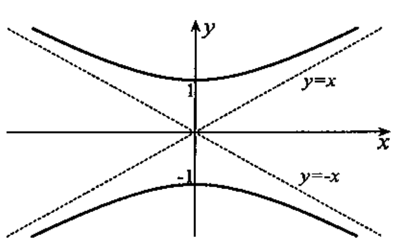


into the equation involving L, we obtain the equation of an ellipse

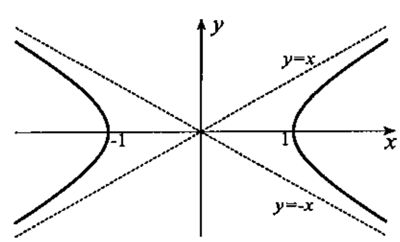
$$\left(\frac{r_1 + r_2}{r_1}x\right)^2 + \left(\frac{r_1 + r_2}{r_2}y\right)^2 = L^2 \implies \frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = \frac{L^2}{(r_1 + r_2)^2}.$$

EXERCISES 1.4D

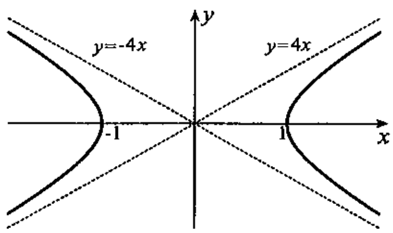
1. Asymptotes for the hyperbola are $y = \pm x$, intersecting at the origin. It intersects the y-axis at $y = \pm 1$.



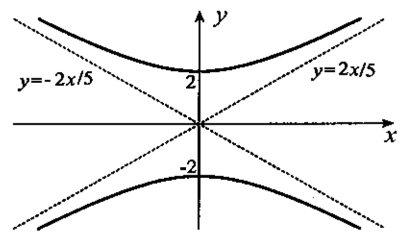
2. Asymptotes for the hyperbola are $y = \pm x$, intersecting at the origin. The hyperbola intersects the x-axis at $x = \pm 1$.



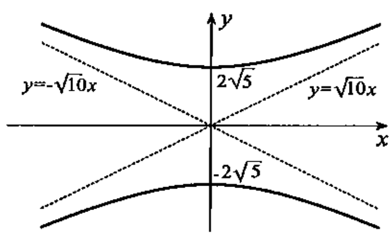
3. Asymptotes for the hyperbola are $y = \pm 4x$, intersecting at the origin. It intersects the x-axis at $x = \pm 1$.



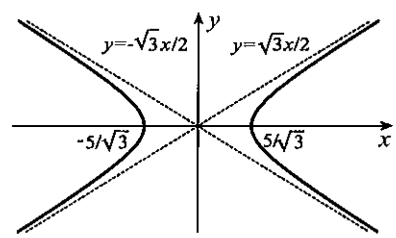
4. Asymptotes for the hyperbola are $y = \pm 2x/5$, intersecting at the origin. The hyperbola intersects the y-axis at $y = \pm 2$.



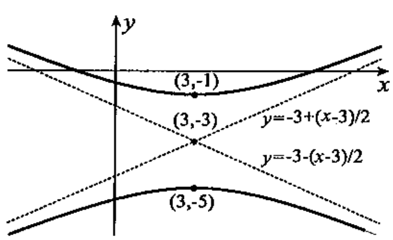
5. Asymptotes for the hyperbola are $y = \pm\sqrt{10}x$, intersecting at the origin. It intersects the y-axis at $y = \pm 2\sqrt{5}$.



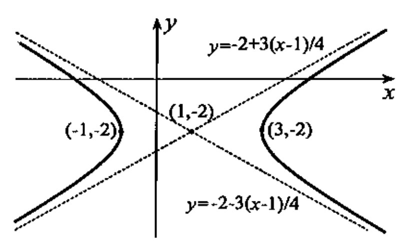
6. Asymptotes for the hyperbola are $y = \pm\sqrt{3}x/2$, intersecting at the origin. The hyperbola intersects the x-axis at $x = \pm 5/\sqrt{3}$.



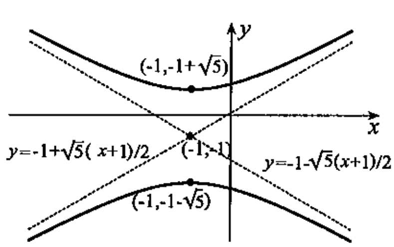
7. When we complete squares on x- and y-terms, the equation becomes $\dfrac{(y+3)^2}{4} - \dfrac{(x-3)^2}{16} = 1$. Asymptotes are $y = -3 \pm (x-3)/2$ intersecting in the point $(3, -3)$. The hyperbola cuts the line $x = 3$ when $y = -1$ and $y = -5$.



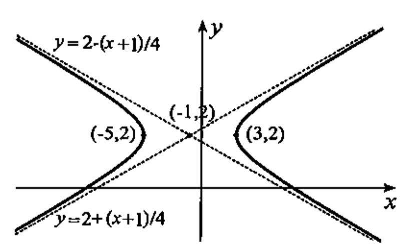
8. When we complete squares on x- and y-terms, the equation becomes $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{9/4} = 1$. Asymptotes are $y = -2 \pm 3(x-1)/4$ intersecting in the point $(1, -2)$. The hyperbola cuts the line $y = -2$ when $x = -1$ and $x = 3$.



9. When we complete squares on x- and y-terms, the equation becomes $\dfrac{(y+1)^2}{5} - \dfrac{(x+1)^2}{4} = 1$. Asymptotes are $y = -1 \pm \sqrt{5}(x+1)/2$ intersecting in the point $(-1, -1)$. The hyperbola cuts the line $x = -1$ when $y = -1 \pm \sqrt{5}$.



10. When we complete squares on x- and y-terms, the equation becomes $\dfrac{(x+1)^2}{16} - (y-2)^2 = 1$. Asymptotes are $y = 2 \pm (x+1)/4$ intersecting in the point $(-1, 2)$. The hyperbola cuts the line $y = 2$ when $x = -5$ and $x = 3$.

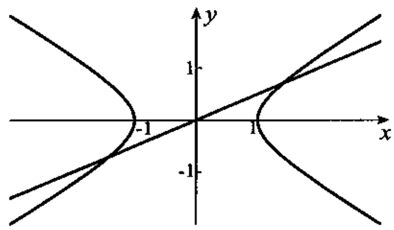


11. If we take the equation of the hyperbola in the form $x^2/a^2 - y^2/b^2 = 1$ with asymptotes $y = \pm bx/a$, then $b/a = 4$. Since the point $(1, 2)$ is on the hyperbola,

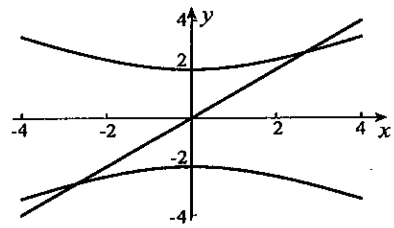
$$\frac{1}{a^2} - \frac{4}{b^2} = 1 \quad \Longrightarrow \quad \frac{1}{a^2} - \frac{4}{16a^2} = 1 \quad \Longrightarrow \quad a^2 = \frac{3}{4}.$$

Thus, $b^2 = 16a^2 = 12$, and the equation of the hyperbola is $16x^2 - y^2 = 12$.

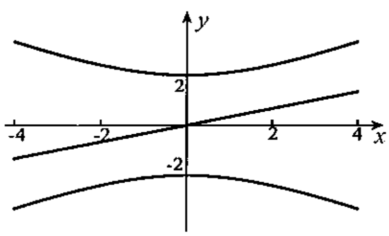
12. If we substitute $x = 2y$ into the equation of the hyperbola, $(2y)^2 - 2y^2 = 1 \Longrightarrow y = \pm 1/\sqrt{2}$. Points of intersection are therefore $\pm(\sqrt{2}, 1/\sqrt{2})$.



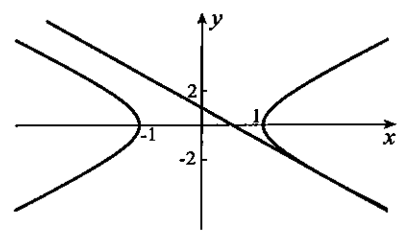
13. We set $9x^2 - 4x^2 = 36 \Longrightarrow x = \pm 6/\sqrt{5}$. Points of intersection are therefore $\pm(6/\sqrt{5}, 6/\sqrt{5})$.



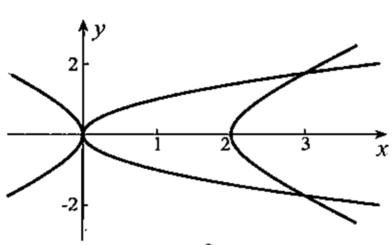
14. The figure suggests that the line and hyperbola do not intersect. This can be verified algebraically. If we substitute $x = 3y$ into the equation of the hyperbola, $36 = 9y^2 - 4(3y)^2 = -27y^2$, an impossibility.



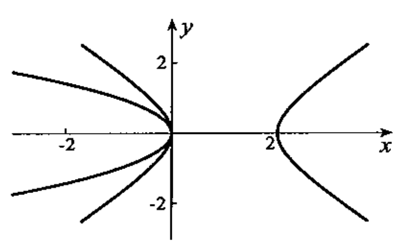
15. If we substitute $y = 1 - 2x$ into the equation of the hyperbola, $3x^2 - (1 - 2x)^2 = 3$ from which $0 = -x^2 + 4x - 4 = -(x - 2)^2$. The only point of intersection is $(2, -3)$.



16. If we substitute $x = y^2$ into the equation of the hyperbola, $0 = (y^2)^2 - 2(y^2) - y^2 = y^2(y^2 - 3)$. Thus, $y = 0$, $\pm\sqrt{3}$, and these give the points of intersection $(0, 0)$ and $(3, \pm\sqrt{3})$.



17. If we substitute $x = -y^2$ into the equation of the hyperbola, $0 = (-y^2)^2 - 2(-y^2) - y^2 = y^2(y^2 + 1)$. The only point of intersection is $(0, 0)$.

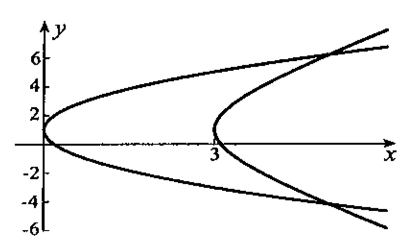


18. If we substitute $(y - 1)^2 = 27x/5$ into the equation of the hyperbola, $36 = 9(x - 1)^2 - 4\left(\dfrac{27x}{5}\right)$.

This equation reduces to

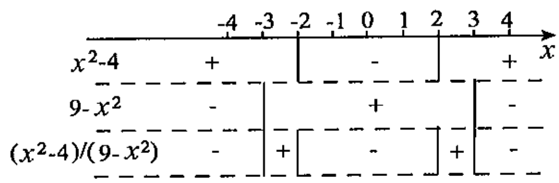
$$0 = 5x^2 - 22x - 15 = (x - 5)(5x + 3).$$

Thus, $x = 5$ or $x = -3/5$. Since x cannot be negative (the equation of the parabola demands this), the solution $x = 5$ leads to the two points $(5, 1 \pm 3\sqrt{3})$.

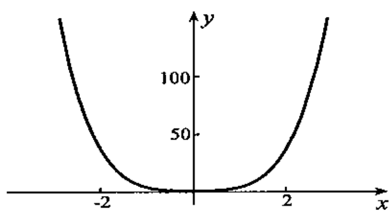


EXERCISES 1.5

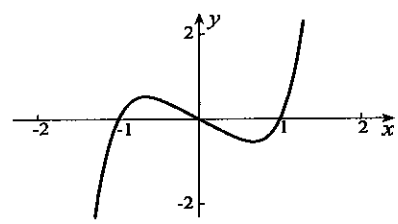
1. We require $9 - x^2 \geq 0 \implies x^2 \leq 9 \implies -3 \leq x \leq 3$.
2. Since $x - 2$ cannot be 0, the function is defined for all $x \neq 2$.
3. The function is defined for all $x \neq 0$.
4. Since $x^2 - 4$ must be positive, x must be greater than 2 or less than -2. Hence, $|x| > 2$.
5. Since $4 - x^2$ must be positive, x^2 must be less than 4. This requires $-2 < x < 2$. We must also eliminate $x = 0$.
6. The first two lines of the diagram indicate when the expressions $x^2 - 4$ and $9 - x^2$ are positive and negative. The third line combines these to give the sign of $(x^2 - 4)/(9 - x^2)$. It indicates that x must be in one of the intervals $-3 < x \leq -2$ or $2 \leq x < 3$; that is, $2 \leq |x| < 3$.



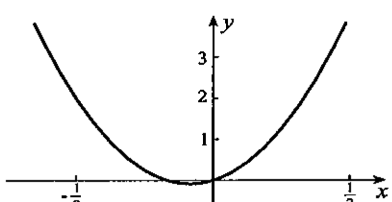
7. Since $-x^2 + 6x - 9 = -(x^2 - 6x + 9) = -(x - 3)^2$, the function is only defined for $x = 3$.
8. Since $x^3 - x^2 = x^2(x - 1) \geq 0$ for $x = 0$ and $x \geq 1$, the largest domain consists of the point $x = 0$ and the interval $x \geq 1$.
9. Since $f(-x) = 1 + (-x)^2 + 2(-x)^4$
$= 1 + x^2 + 2x^4 = f(x)$, the function is even.



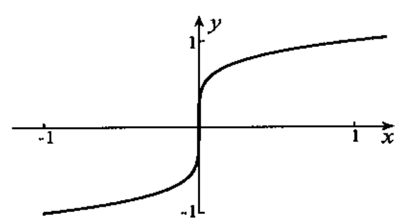
10. Since $f(-x) = (-x)^5 - (-x) = -x^5 + x$
$= -(x^5 - x) = -f(x)$, the function is odd.



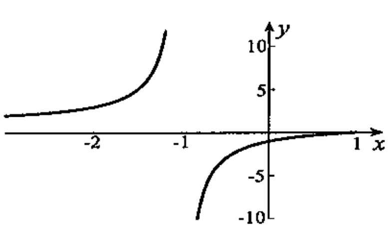
11. Since $f(-x) = 12(-x)^2 + 2(-x) = 12x^2 - 2x$, and this is neither $f(x)$ nor $-f(x)$, the function is neither even nor odd.



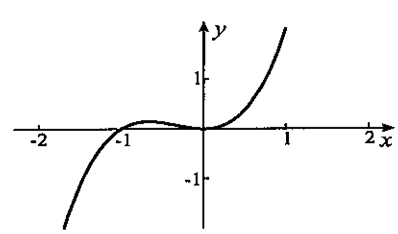
12. Since $f(-x) = (-x)^{1/5} = -x^{1/5} = -f(x)$, the function is odd.



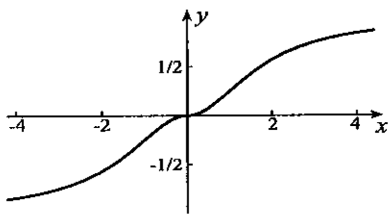
13. Since $f(-x) = \dfrac{-x-1}{-x+1} = \dfrac{x+1}{x-1}$, and this is neither $f(x)$ nor $-f(x)$, the function is neither even nor odd.



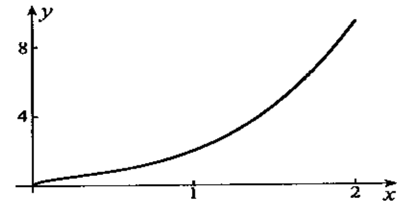
14. Since $f(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2$, and this is neither $f(x)$ nor $-f(x)$, the function is neither even nor odd.



15. Since $f(-x) = \dfrac{(-x)|-x|}{3+(-x)^2} = -\dfrac{x|x|}{3+x^2} = -f(x)$, the function is odd.

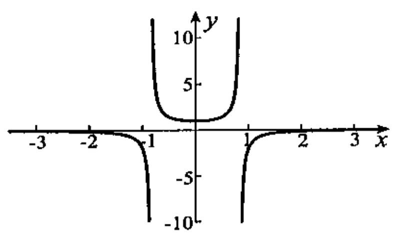


16. Since the function is not defined for $x < 0$, it cannot be even or odd.



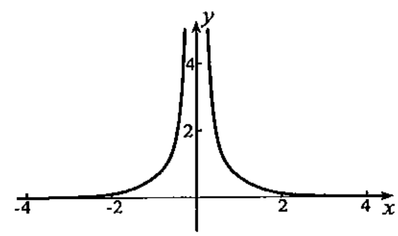
17. Since $f(-x) = \dfrac{(-x)^2+1}{1-2(-x)^4} = \dfrac{x^2+1}{1-2x^4} = f(x)$,

the function is even.



18. Since $f(-x) = \dfrac{(-x)^2+\sqrt{(-x)^4+1}}{(-x)^6+3(-x)^2}$

$= \dfrac{x^2+\sqrt{x^4+1}}{x^6+3x^2} = f(x)$, the function is even.



19. Even and odd parts of this function are $f_e(x) = 3x^2$ and $f_o(x) = x^3 - 2x$.

20. Even and odd parts of this function are

$$f_e(x) = \frac{1}{2}\left(\frac{x-2}{x+5} + \frac{-x-2}{-x+5}\right) = \frac{x^2+10}{x^2-25}, \qquad f_o(x) = \frac{1}{2}\left(\frac{x-2}{x+5} - \frac{-x-2}{-x+5}\right) = \frac{-7x}{x^2-25}.$$

21. Since this function is even, its even part is itself and its odd part is zero.

22. Since this function is odd, its odd part is itself and its even part is zero.

23. Even and odd parts of this function are

$$f_e(x) = \frac{1}{2}\left(\frac{2x}{3+5x} + \frac{-2x}{3-5x}\right) = \frac{-10x^2}{9-25x^2}, \qquad f_o(x) = \frac{1}{2}\left(\frac{2x}{3+5x} - \frac{-2x}{3-5x}\right) = \frac{6x}{9-25x^2}.$$

24. Since this function is only defined for $x \geq 1$, it does not have even and odd parts.

25. When we set $x = 0$ in equation 1.30b, $f(0) = -f(0) \implies f(0) = 0$.

26. (a) Let $f(x)$ and $g(x)$ be any two even functions and set $h(x) = f(x)g(x)$. Then,

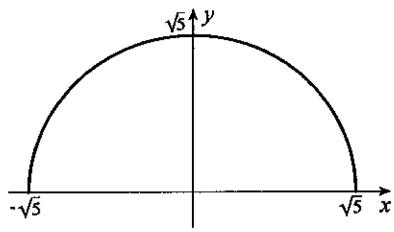
$$h(-x) = f(-x)g(-x) = [f(x)][g(x)] = f(x)g(x) = h(x).$$

Thus, $h(x)$ is an even function. The proof for two odd functions is similar.
(b) When $f(x)$ is even and $g(x)$ is odd, and $h(x) = f(x)g(x)$,

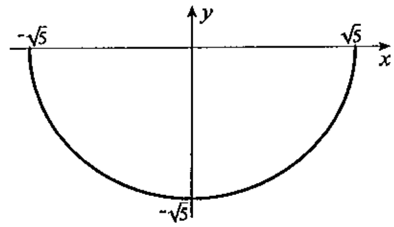
$$h(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -h(x).$$

Thus, $h(x)$ is an odd function.

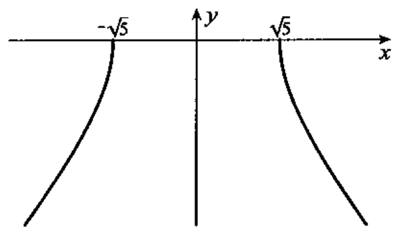
27. This is a semicircle.



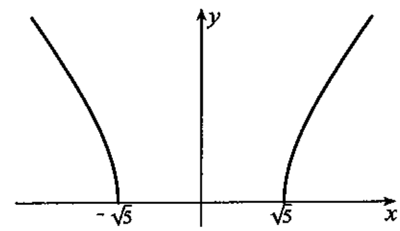
28. This is a semicircle.



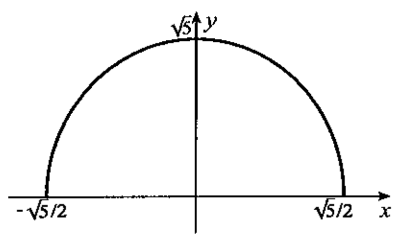
29. This is the lower half of the hyperbola $x^2 - y^2 = 5$.



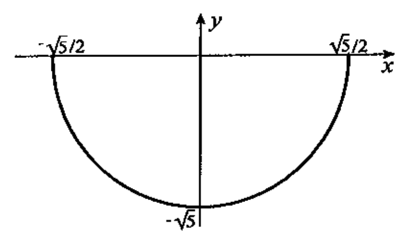
30. This is the upper half of the hyperbola $x^2 - y^2 = 5$.



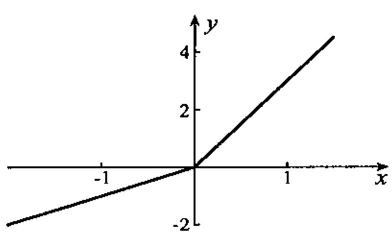
31. This is the upper half of the ellipse $4x^2 + y^2 = 5$.



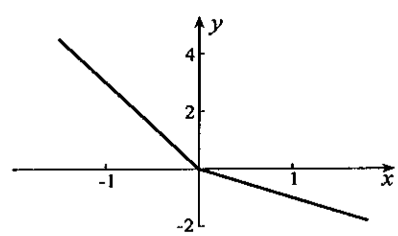
32. This is the lower half of the ellipse $4x^2 + y^2 = 5$.



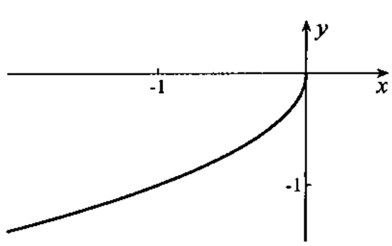
33. When $x \geq 0$, $y = x + 2x = 3x$; and when $x < 0$, $y = -x + 2x = x$.



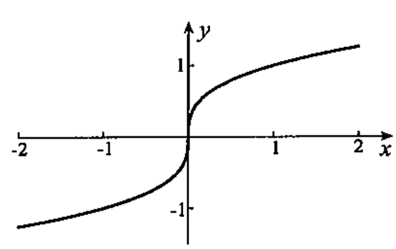
34. When $x \geq 0$, $y = x - 2x = -x$; and when $x < 0$, $y = -x - 2x = -3x$.



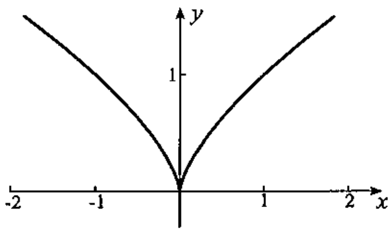
35. This is the lower half of the parabola $x = -y^2$



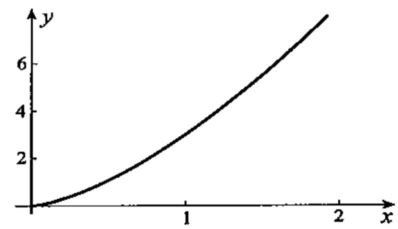
36. We rewrite the equation of the curve in the form $x = y^3$.



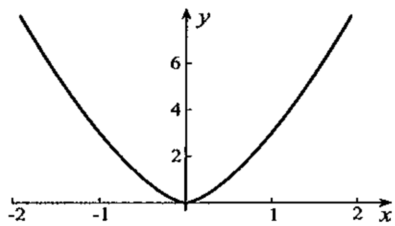
37. The curve is shown below.



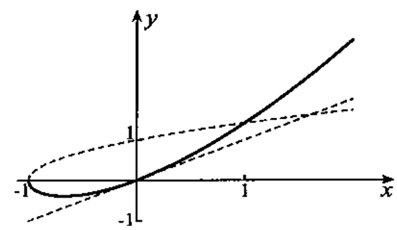
38. This function is only defined for $x \geq 0$.



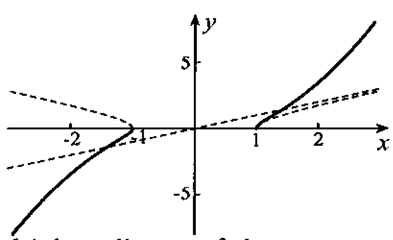
39. The curve is shown below.



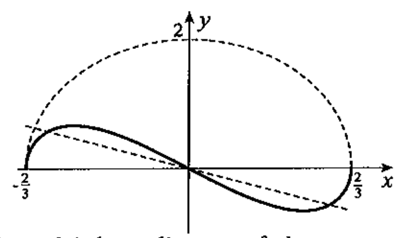
40. We multiply ordinates of the curves $y = x$ and $y = \sqrt{x+1}$, the top half of a parabola



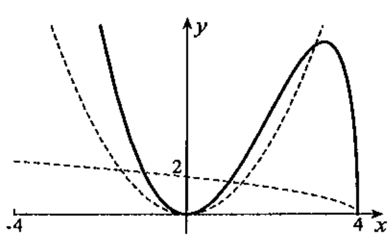
41. We multiply ordinates of the curves $y = x$ and $y = \sqrt{x^2 - 1}$, the top half of a hyperbola.



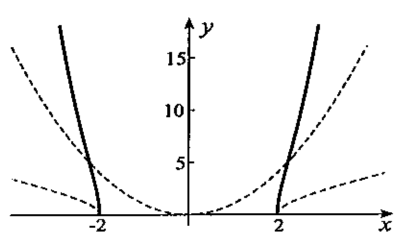
42. We multiply ordinates of the curves $y = -x$ and $y = \sqrt{4 - 9x^2}$, the top half of an ellipse.



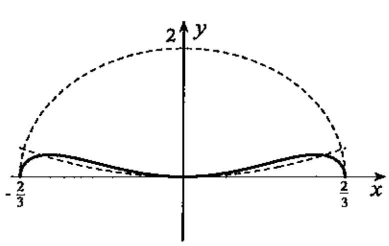
43. We multiply ordinates of the curves $y = x^2$ and $y = \sqrt{4 - x}$, the top half of a parabola.



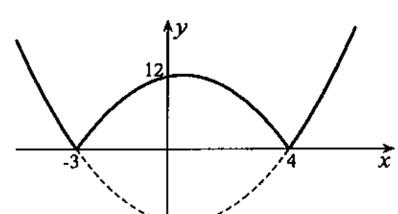
44. We multiply ordinates of the curves $y = x^2$ and $y = \sqrt{x^2 - 4}$, the top half of a hyperbola.



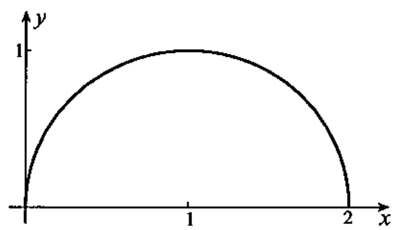
45. We multiply ordinates of the curves $y = x^2$ and $y = \sqrt{4 - 9x^2}$, the top half of an ellipse.



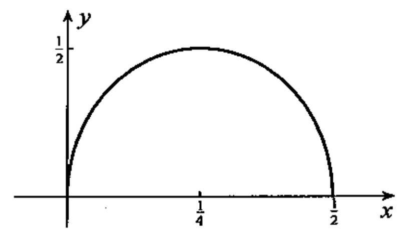
46. We draw the parabola $y = x^2 - x - 12$ $= (x-4)(x+3)$ and turn that part below the x-axis, upside down.



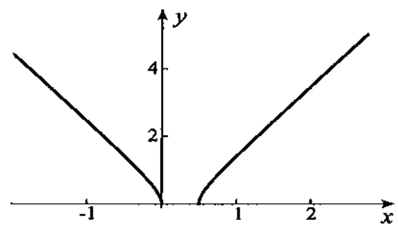
47. This is the top half of the circle
$(x-1)^2 + y^2 = 1.$



48. This is the top half of the ellipse
$16(x-1/4)^2 + 4y^2 = 1.$

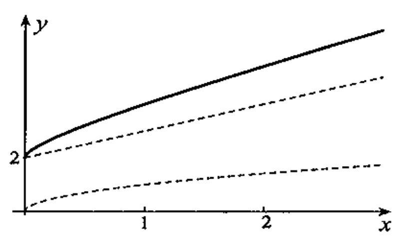


49. This is the top half of the hyperbola
$16(x-1/4)^2 - 4y^2 = 1.$

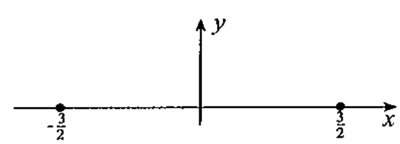


50. If we set $y = \sqrt{2x - x^2 - 4}$, square the equation, and complete the square on the x-terms, the result is $(x-1)^2 + y^2 = -3$. No real values of x and y can satisfy this equation.

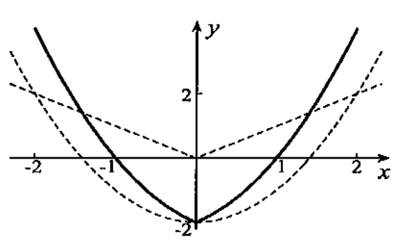
51. Addition of ordinates of the curves $y = x + 2$ and $y = \sqrt{x}$ gives the curve.



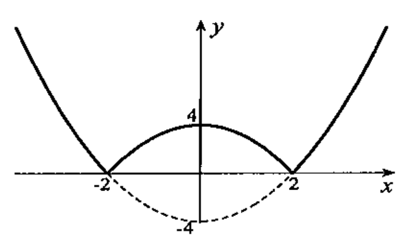
52. The square root $\sqrt{9 - 4x^2}$ is defined only for $|x| \le 3/2$, whereas the square root $\sqrt{4x^2 - 9}$ is defined for $|x| \ge 3/2$. The only points common to these intervals are $x = \pm 3/2$. The graph therefore consists of the two points $(\pm 3/2, 0)$



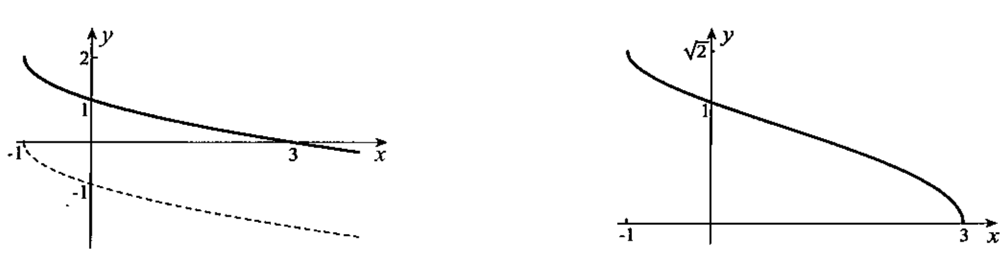
53. Addition of ordinates of the curves $y = x^2 - 2$ and $y = |x|$ gives the curve.



54. Since $\sqrt{(x^2-4)} = |x^2 - 4|$, we draw the parabola $y = x^2 - 4$, and then turn that part of the parabola between ± 2 upside down.

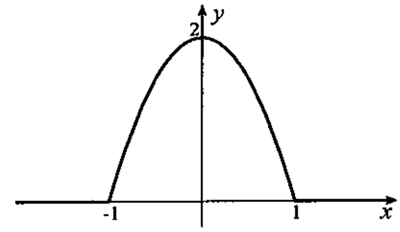


55. We first draw the lower half of the parabola $x = y^2 - 1$, shift it upwards 2 units (left figure below), and then take square roots of ordinates (right figure below).



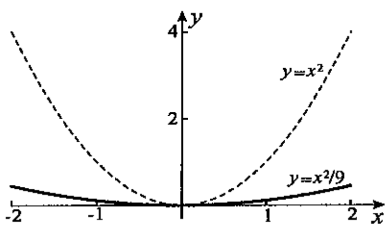
56. Since $\sqrt{(x^2-1)^2}-(x^2-1)=|x^2-1|-(x^2-1)$, it follows that

$$f(x)=\begin{cases}(x^2-1)-(x^2-1) & |x|>1\\ -(x^2-1)-(x^2-1) & |x|\le 1\end{cases}$$
$$=\begin{cases}0 & |x|>1\\ 2(1-x^2) & |x|\le 1.\end{cases}$$

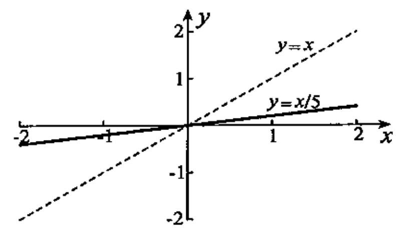


57. Every horizontal line that intersects the curve does so exactly once.

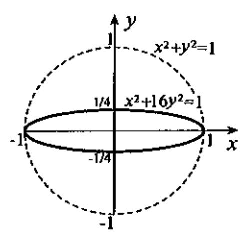
58. $y=x^2/9$ is a stretch of $y=x^2$ by a factor of 3 in the x-direction. It is also a compression by a factor of 9 in the y-direction.



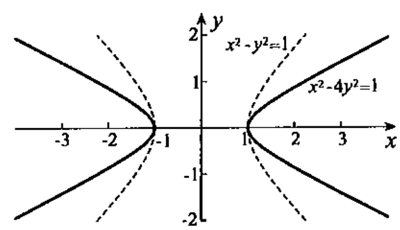
59. $y=x/5$ is a stretch of $y=x$ by a factor of 5 in the x-direction. It is also a compression of $y=x$ by a factor of 5 in the y-direction.



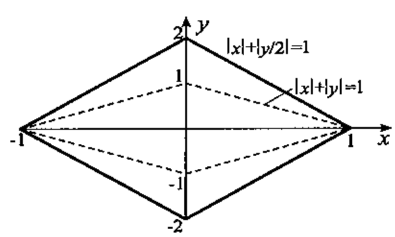
60. $x^2+16y^2=1$ is a compression of $x^2+y^2=1$ by a factor of 4 in the y-direction.



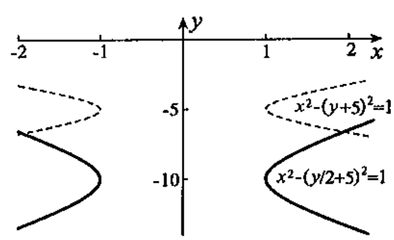
61. $x^2-4y^2=1$ is a compression of $x^2-y^2=1$ by a factor of 2 in the y-direction.



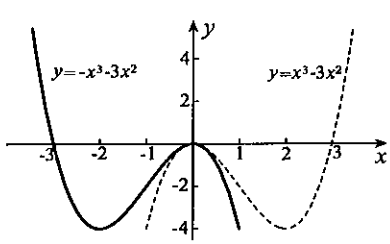
62. $|x|+|y/2|=1$ is a stretch of $|x|+|y|=1$ by a factor of 2 in the y-direction.



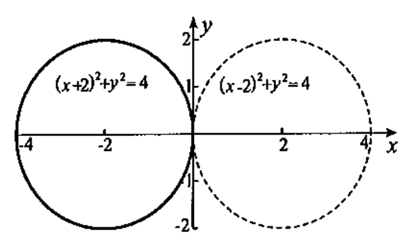
63. $x^2-(y/2+5)^2=1$ is a stretch of $x^2-(y+5)^2=1$ by a factor of 2 in the y-direction.



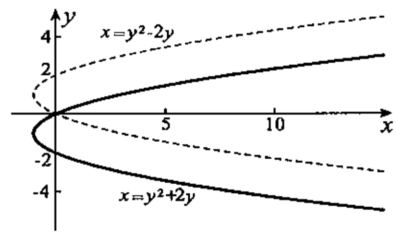
64. $y=-x^3-3x^2$ is $y=x^3-3x^2$ reflected in the y-axis.



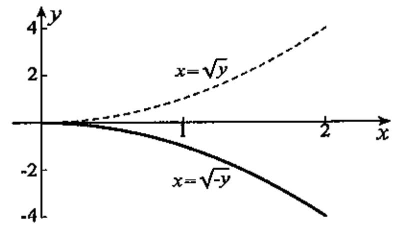
65. $(x+2)^2+y^2=4$ is $(x-2)^2+y^2=4$ reflected in the y-axis.



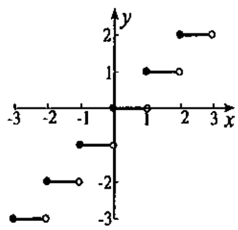
66. $x = y^2 + 2y$ is $x = y^2 - 2y$ reflected in the x-axis.



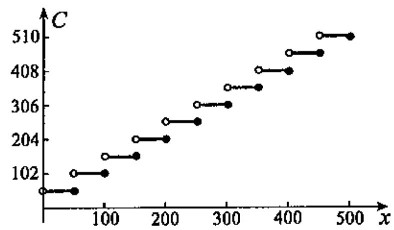
67. $x = \sqrt{-y}$ is $x = \sqrt{y}$ reflected in the x-aixs.



68. (a)



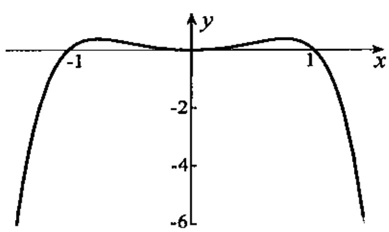
(b)



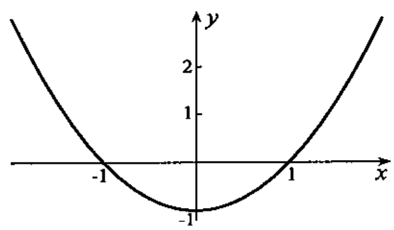
(c) Let $C(x)$ be the cost for mailing an item of x grams $(0 \le x \le 500)$. When $x/50$ is an integer, $C(x) = 51(x/50)$. When $x/50$ is not an integer,

$$C(x) = 51\left(1 + \text{integer part of } \frac{x}{50}\right) = 51\left(1 + \lfloor x/50 \rfloor\right) = 51\lfloor 1 + x/50 \rfloor.$$

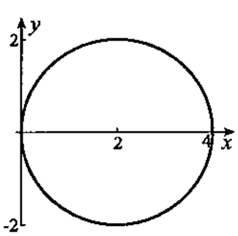
69. This defines a function.



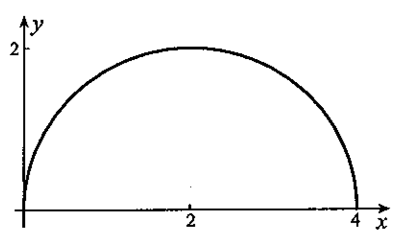
70. This defines a function.



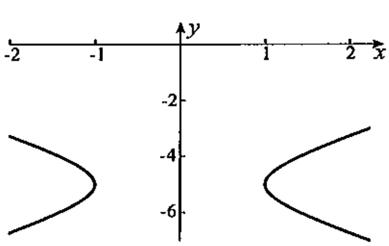
71. This does not define a function.



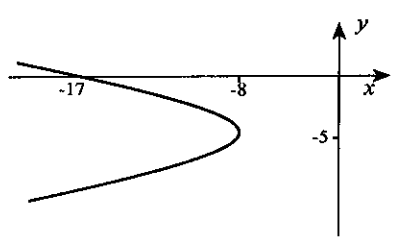
72. This defines a function.



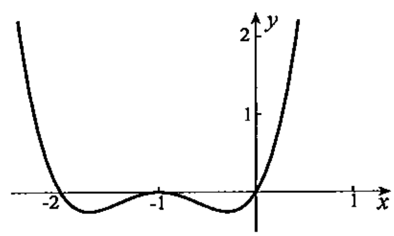
73. This does not define a function.



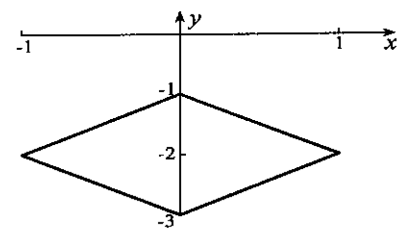
74. This does not define a function.



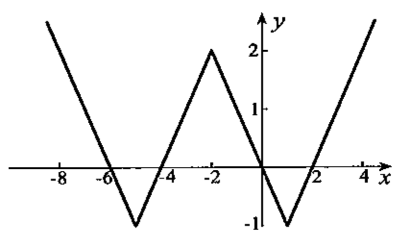
75. This defines a function.



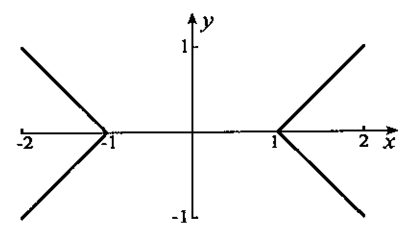
76. This does not define a function.



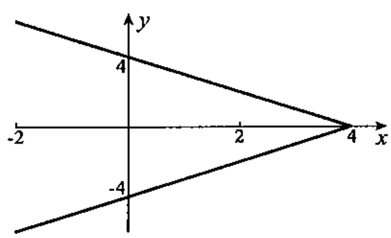
77. This defines a function.



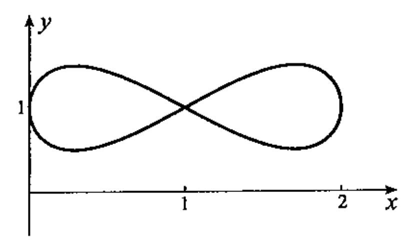
78. This does not define a function.



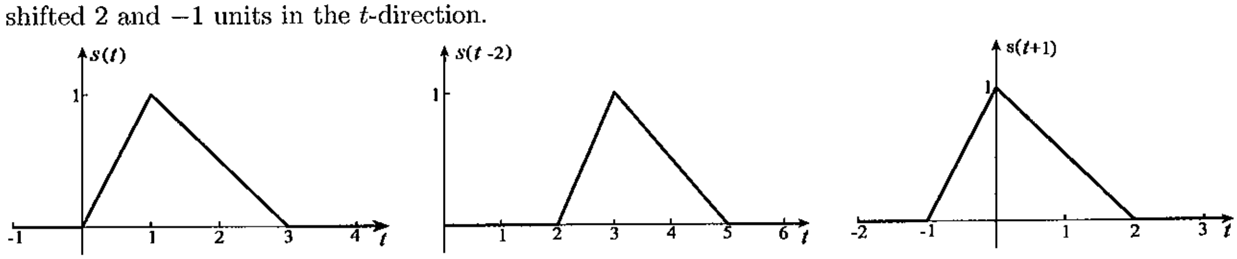
79. This does not define a function.



80. This does not define a function.

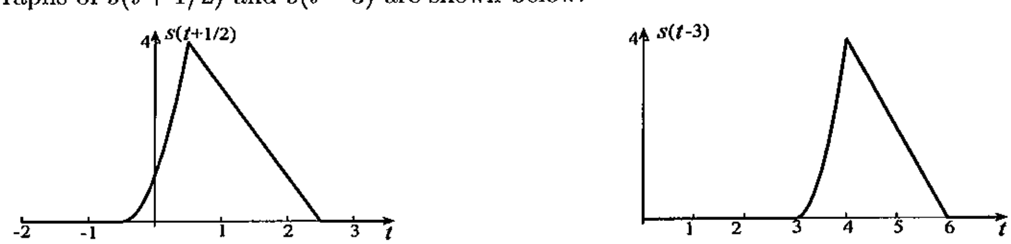


81. Graphs of $s(t-2)$ and $s(t+1)$ in the middle and right figures below are those of $s(t)$ in the left figure shifted 2 and -1 units in the t-direction.



82. (a) Since the equation of the parabola is $s = 4t^2$ and that of the line is $s = -2(t-3)$, the algebraic definition of the signal is $s(t) = \begin{cases} 0, & t < 0 \\ 4t^2, & 0 \le t \le 1 \\ 2(3-t), & 1 < t \le 3 \\ 0, & t > 3 \end{cases}$.

(b) Graphs of $s(t+1/2)$ and $s(t-3)$ are shown below.



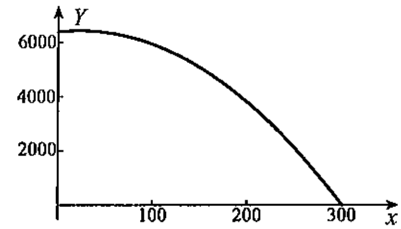
(c) The algebraic representations can be obtained by replacing t by $t+1/2$ and $t-3$ in $s(t)$,

$$s(t+1/2) = \begin{cases} 0, & t+1/2<0 \\ 4(t+1/2)^2, & 0 \le t+1/2 \le 1 \\ 2(3-t-1/2), & 1 < t+1/2 \le 3 \\ 0, & t+1/2>3 \end{cases} = \begin{cases} 0, & t<-1/2 \\ 4(t+1/2)^2, & -1/2 \le t \le 1/2 \\ 5-2t, & 1/2 < t \le 5/2 \\ 0, & t>5/2 \end{cases};$$

$$s(t-3) = \begin{cases} 0, & t-3<0 \\ 4(t-3)^2, & 0 \le t-3 \le 1 \\ 2(3-t+3), & 1 < t-3 \le 3 \\ 0, & t-3>3 \end{cases} = \begin{cases} 0, & t<3 \\ 4(t-3)^2, & 3 \le t \le 4 \\ 2(6-t), & 4 < t \le 6 \\ 0, & t>6 \end{cases}.$$

83. If x additional trees are planted, the number of trees is $255+x$, and the yield per tree is $25-x/12$. The total yield for this number of additional trees is

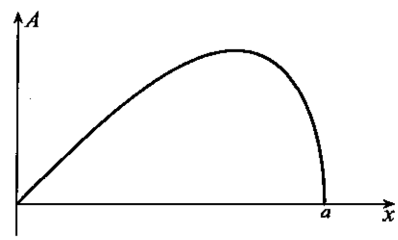
$$Y = f(x) = (255+x)\left(25-\frac{x}{12}\right) = \frac{1}{12}(255+x)(300-x).$$



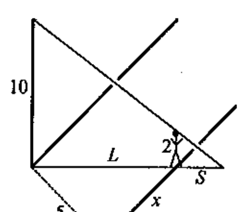
To keep Y nonnegative, and not cut down any trees, we restrict the domain of the function to $0 \le x \le 300$. The maximum on this parabola is halfway between its intercepts; that is, at $x = (300-255)/2 = 45/2$. Thus, 22 more trees should be planted. (We do not suggest 23 trees, since an extra tree would have to be purchased.)

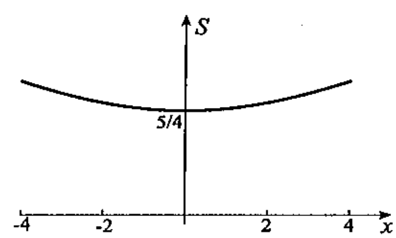
84. The area of the rectangle shown is $A = 4xy$. When we solve the equation of the ellipse for the positive value of y, the result is $y = (b/a)\sqrt{a^2-x^2}$. The area of the rectangle can therefore be expressed in the form

$$A = f(x) = \frac{4bx}{a}\sqrt{a^2-x^2}, \quad 0 \le x \le a.$$



85. From similar vertical triangles in the left figure below, $\frac{10}{2} = \frac{L+S}{S}$, and this equation can be solved for $L = 4S$. Since $L^2 = x^2+25$, it follows that $16S^2 = x^2+25$, or, $S = f(x) = \frac{\sqrt{x^2+25}}{4}$. To draw a graph of this function, we return to the equation $16S^2 - x^2 = 25$. The graph is the upper half of this hyperbola (right figure below).

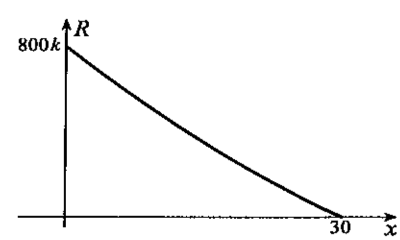




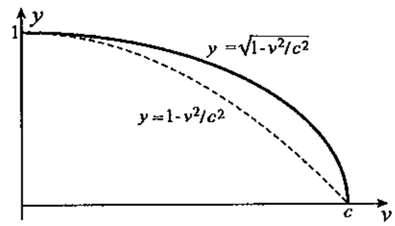
86. The amounts of A and B used to produce an amount x of C are $2x/3$ and $x/3$, respectively. It follows that

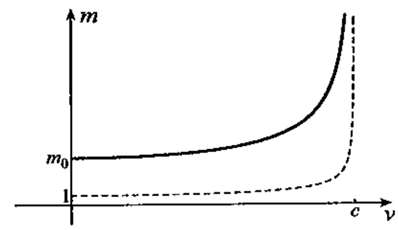
$$R = k\left(20-\frac{2x}{3}\right)\left(40-\frac{x}{3}\right) = \frac{2k}{9}(30-x)(120-x), \quad 0 \le x \le 30.$$

The rate is a maximum at $x = 0$.



87. To draw a graph of this function we begin with a graph of the parabola $y = 1 - v^2/c^2$ in the left figure below. It is shown only for nonnegative v since speed is never negative. To draw $y = \sqrt{1 - v^2/c^2}$ in the same figure, we take square root of ordinates on the parabola. To obtain $y = 1/\sqrt{1 - v^2/c^2}$ in the right figure, we divide ordinates of $y = \sqrt{1 - v^2/c^2}$ into 1. The final graph is a result of multiplying all ordinates by the constant value m_0.



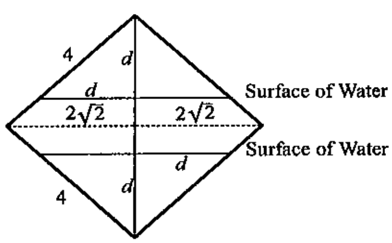


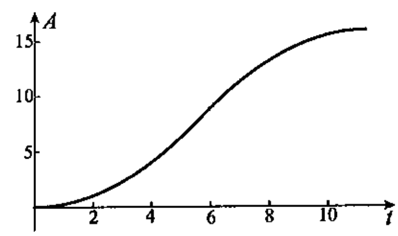
The graph suggests that as the speed of an object approaches the speed of light, its mass becomes indefinitely large. Consequently, it is impossible to accelerate an object with positive rest mass to the speed of light.

88. The depth d of the lowest corner of the square below the surface of the water is given by $d = t/2$. For $0 \le t \le 4\sqrt{2}$ (when the lower half of the square is being submerged), the area submerged at time t is $A = d^2 = (t/2)^2 = t^2/4$ (left figure below). For $4\sqrt{2} < t \le 8\sqrt{2}$ (when the top half is being submerged),

$$A = 16 - \left(4\sqrt{2} - \frac{t}{2}\right)^2 = -16 + 4\sqrt{2}t - \frac{t^2}{4}.$$

A graph of the function $A(t)$ is shown in the right figure below.





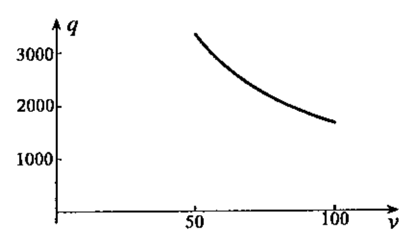
89. If the distance between cars is $d = 3v^2/500$, then the number of cars per kilometre is

$$\frac{1}{d}(1000) = \frac{500}{3v^2}(1000) = \frac{500\,000}{3v^2}.$$

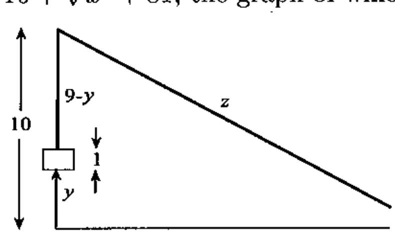
It follows that the number of cars passing any given point per hour is

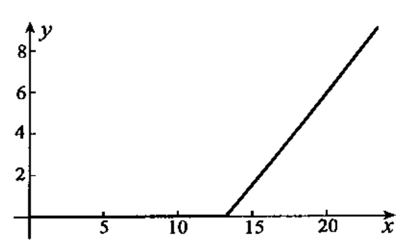
$$q = \left(\frac{500\,000}{3v^2}\right)v = \frac{500\,000}{3v}.$$

The graph indicates that q is maximized for $v = 50$.

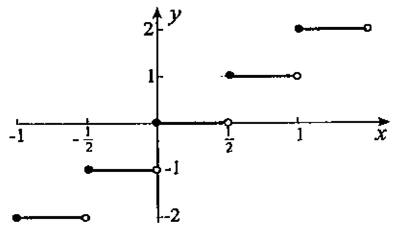


90. The rope becomes taut when $x = \sqrt{256 - 81} = 5\sqrt{7}$ and the box reaches the pulley when $x = \sqrt{625 - 81} = 4\sqrt{34}$. Between these values of x, the left figure below indicates that $z^2 = x^2 + 81$ and $z + (9 - y) = 25$. When we eliminate z between these equations, the result is $x^2 + 81 = (16 + y)^2 \implies y = -16 + \sqrt{x^2 + 81}$, the graph of which is shown in the right figure below.

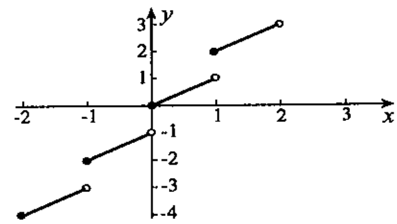




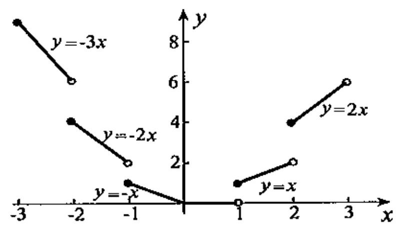
91.



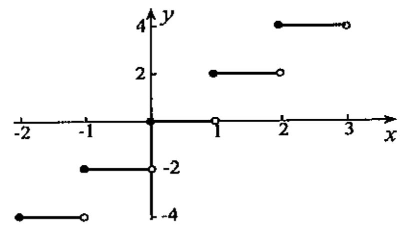
92.



93.



94.



95. To disprove this statement, it is sufficient to give an example for which it is invalid. Suppose that $f(x) = x$, $g(x) = x$, and $x = 1.8$. Then

$$\lfloor f(1.8) + g(1.8) \rfloor = \lfloor 1.8 + 1.8 \rfloor = \lfloor 3.6 \rfloor = 3,$$

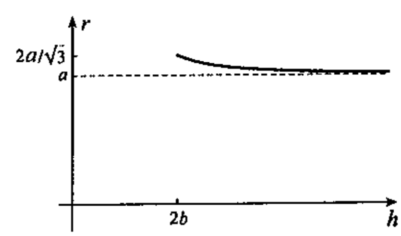
whereas

$$\lfloor f(1.8) \rfloor + \lfloor g(1.8) \rfloor = \lfloor 1.8 \rfloor + \lfloor 1.8 \rfloor = 1 + 1 = 2.$$

96. If we solve the equation for r in terms of h, the result is $r = \dfrac{ah}{\sqrt{h^2 - b^2}}$. When $h = 2b$, $r = 2a/\sqrt{3}$. By writing the function in the form

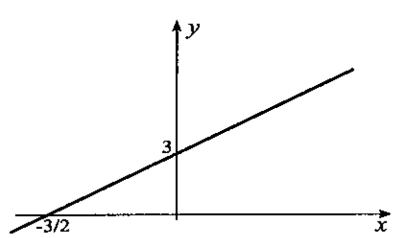
$$r = \frac{ah}{\sqrt{h^2 - b^2}} = \frac{a}{\sqrt{1 - \frac{b^2}{h^2}}},$$

we see that as h increases, r decreases, and for large h, r is very close to a.

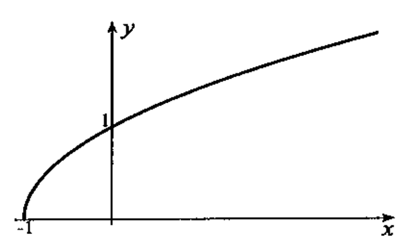


EXERCISES 1.6

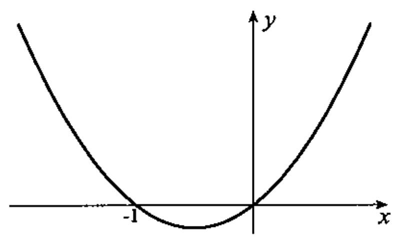
1. Since the function is increasing for all x, it has an inverse. When we solve $y = 2x + 3$ for x, we obtain $x = (y - 3)/2$, and therefore the inverse function is $f^{-1}(x) = (x - 3)/2$.



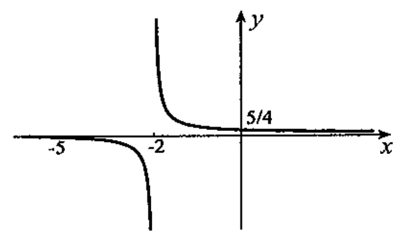
2. Since the function is increasing for $x \geq -1$, it has an inverse. When we solve $y = \sqrt{x + 1}$, for x, we obtain $x = y^2 - 1$, and therefore the inverse function is $f^{-1}(x) = x^2 - 1$, $x \geq 0$.



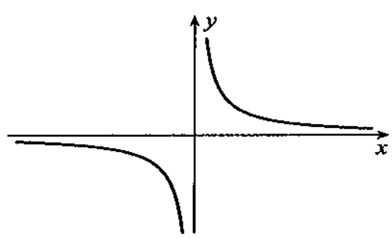
3. Since horizontal lines $y = c > -1/4$ intersect the graph in two points, there is no inverse function.



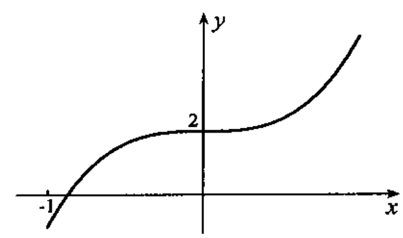
4. Since horizontal lines intersect the graph of $f(x)$ in only one point, the function has an inverse. When we solve $y = (x+5)/(2x+4)$ for x, we obtain $x = (5-4y)/(2y-1)$. The inverse function is $f^{-1}(x) = (5-4x)/(2x-1)$.



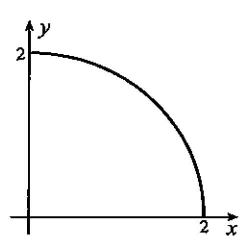
5. Since horizontal lines intersect the graph of $f(x)$ in only one point, the function has an inverse. When we solve $y = 1/x$ for x, we obtain $x = 1/y$. The inverse function is $f^{-1}(x) = 1/x$.



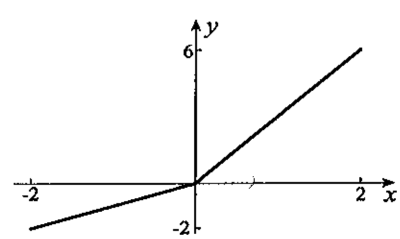
6. Since $f(x)$ is increasing for all x, the function has an inverse. Because $x = [(y-2)/3]^{1/3}$, the inverse function is $f^{-1}(x) = [(x-2)/3]^{1/3}$.



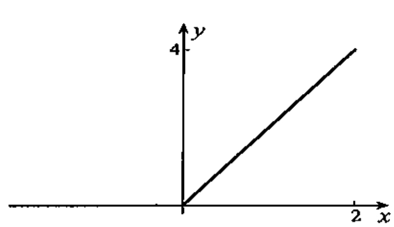
7. Since the function is decreasing for $0 \le x \le 2$, it has an inverse. When we solve $y = \sqrt{4-x^2}$ for x, we obtain $x = \sqrt{4-y^2}$. The inverse function is $f^{-1}(x) = \sqrt{4-x^2}$.



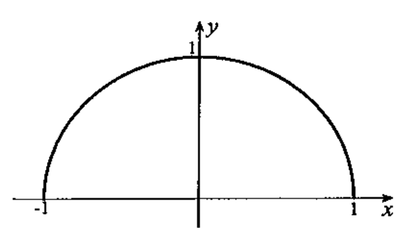
8. Because $f(x)$ is increasing for all x, the function has an inverse. Graphically, the inverse function is $f^{-1}(x) = x$ for $x < 0$ and $x/3$ for $x \ge 0$.



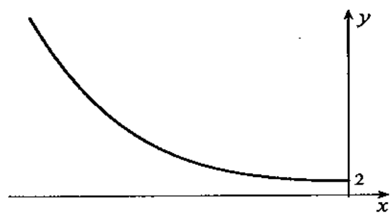
9. Since the line $y = 0$ intersects the graph in an infinity of points, there is no inverse function.



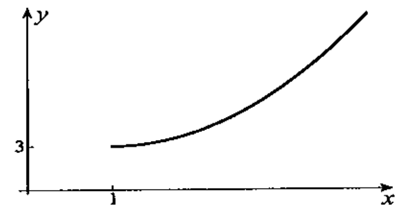
10. Since horizontal lines $y = c$, $0 \le c < 1$, intersect the graph in two points, there is no inverse function.



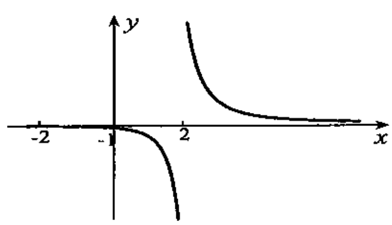
11. Since the function is decreasing for $x \leq 0$, the function has an inverse. If we set $y = x^4 + 2x^2 + 2$, then $(x^2)^2 + 2x^2 + 2 - y = 0$, from which $x^2 = \left[-2 \pm \sqrt{4 - 4(2-y)}\right]/2 = -1 \pm \sqrt{y-1}$. Since x^2 must be positive, we choose the positive root, in which case $x = -\sqrt{-1 + \sqrt{y-1}}$. The inverse function is $f^{-1}(x) = -\sqrt{-1+\sqrt{x-1}}, \quad x \geq 2$.



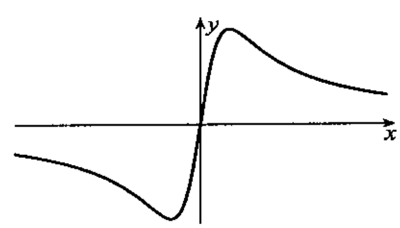
12. Because $f(x)$ is increasing for $x \geq 1$, it has an inverse function. If we write $x^2 - 2x + (4-y) = 0$, then $x = \left[2 \pm \sqrt{4 - 4(4-y)}\right]/2 = 1 \pm \sqrt{y-3}$. Since $x > 1$, we must choose the positive sign, and the inverse function is $f^{-1}(x) = 1 + \sqrt{x-3}$, $x \geq 3$.



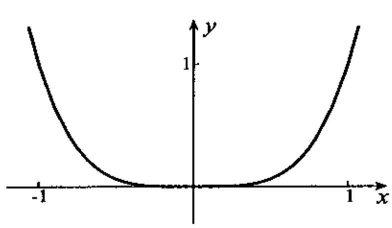
13. Since horizontal lines intersect the graph of $f(x)$ in only one point, the function has an inverse. When we set $y = [(x+2)/(x-2)]^3$, then $(x+2)/(x-2) = y^{1/3}$, or, $x + 2 = y^{1/3}(x-2)$, from which $x = 2(y^{1/3}+1)/(y^{1/3}-1)$. The inverse function is $f^{-1}(x) = 2(x^{1/3}+1)/(x^{1/3}-1)$.



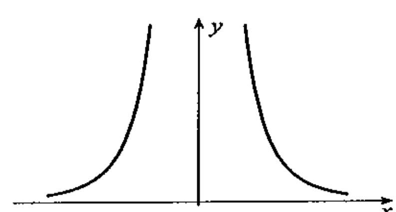
14. Since almost all horizontal lines that intersect the graph do so twice, there is no inverse function.



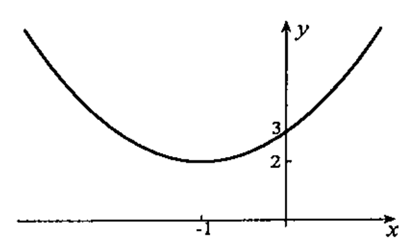
15. Since horizontal lines $y = c > 0$ intersect the graph of the function twice, the function does not have an inverse. On the intervals $x \leq 0$ and $x \geq 0$, it does have inverses. For $x \geq 0$, the inverse function is $f^{-1}(x) = x^{1/4}$, and for $x \leq 0$, the inverse is $f^{-1}(x) = -x^{1/4}$.



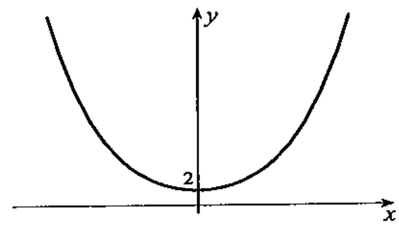
16. Since horizontal lines $y = c > 0$ intersect the graph of the function twice, the function does not have an inverse. On the intervals $x < 0$ and $x > 0$, it does have inverses. For $x > 0$, the inverse function is $f^{-1}(x) = 1/x^{1/4}$, and for $x < 0$, the inverse is $f^{-1}(x) = -1/x^{1/4}$.



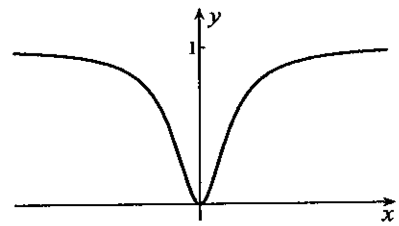
17. Since horizontal lines $y = c > 2$ intersect the graph twice, the function does not have an inverse. On the intervals $x \leq -1$ and $x \geq -1$, it does have inverses. We set $y = x^2 + 2x + 3$, or, $x^2 + 2x + (3-y) = 0$, and solve for $x = \left[-2 \pm \sqrt{4 - 4(3-y)}\right]/2 = -1 \pm \sqrt{y-2}$. For $x \leq -1$, the inverse function is $f^{-1}(x) = -1 - \sqrt{x-2}$, and when $x \geq -1$, the inverse function is $f^{-1}(x) = -1 + \sqrt{x-2}$.



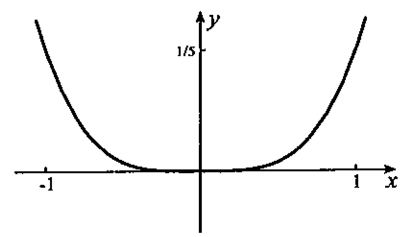
18. Because horizontal lines $y = c > 2$ intersect the graph of the function twice, the function does not have an inverse. On the intervals $x \le 0$ and $x \ge 0$, it does have inverses. To find them, we set $y = x^4 + 4x^2 + 2 = (x^2 + 2)^2 - 2$, and solve for $x^2 + 2 = \pm\sqrt{y+2}$, but only the positive result is acceptable. Hence $x^2 + 2 = \sqrt{y+2}$, and therefore $x = \pm\sqrt{\sqrt{y+2} - 2}$. For $x \ge 0$, the inverse function is $f^{-1}(x) = \sqrt{\sqrt{x+2} - 2}$, and for $x \le 0$, the inverse is $f^{-1}(x) = -\sqrt{\sqrt{x+2} - 2}$.



19. Because horizontal lines between $y = 0$ and $y = 1$ intersect the graph twice, the function does not have an inverse. On the intervals $x \le 0$ and $x \ge 0$, it does have inverses. To find them, we set $y = x^2/(x^2 + 4)$, from which $x^2 = 4y/(1 - y)$. Square roots give $x = \pm 2\sqrt{y/(1-y)}$. For $x \le 0$, the inverse function is $f^{-1}(x) = -2\sqrt{x/(1-x)}$, and for $x \ge 0$, the inverse is $f^{-1}(x) = 2\sqrt{x/(1-x)}$.

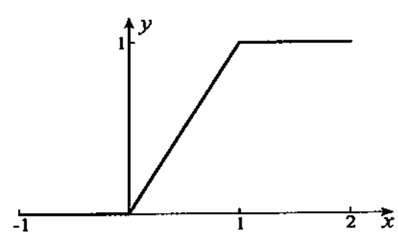


20. Because horizontal lines $y = c > 0$ intersect the graph of the function twice, the function does not have an inverse. On the intervals $x \le 0$ and $x \ge 0$, it does have inverses. To find them, we set $y = x^4/(x^2 + 4)$, from which $x^4 - yx^2 - 4y = 0$. This quadratic in x^2 has solutions $x^2 = (y \pm \sqrt{y^2 + 16y})/2$, but only the positive root is acceptable. Thus, $x = \pm\sqrt{(y + \sqrt{y^2 + 16y})/2}$. The inverse function for $x \ge 0$ is $f^{-1}(x) = \sqrt{(x + \sqrt{x^2 + 16x})/2}$, and for $x \le 0$ the inverse is $f^{-1}(x) = -\sqrt{(x + \sqrt{x^2 + 16x})/2}$.

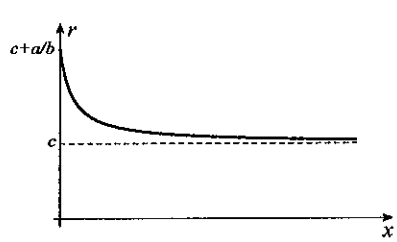


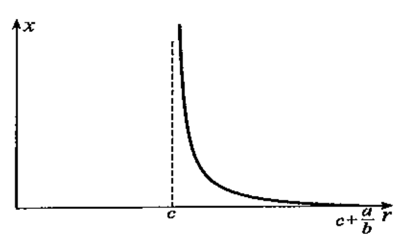
21. An example is the function $f(x) = 1$. On no interval is $f(x)$ one-to-one.

22. The function $f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$ (shown to the right) is one-to-one on $0 \le x \le 1$, but on no other interval.

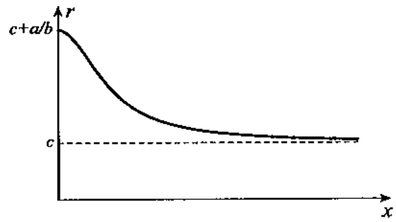


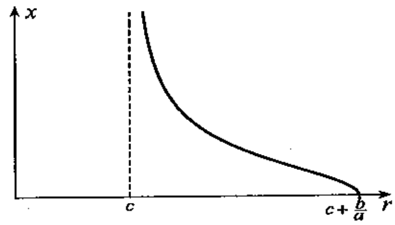
23. (a) From the given equation, $a/(x + b) = r - c$, from which $x + b = a/(r - c)$. The inverse function is $f^{-1}(r) = a/(r - c) - b$. The function and its inverse are shown below.



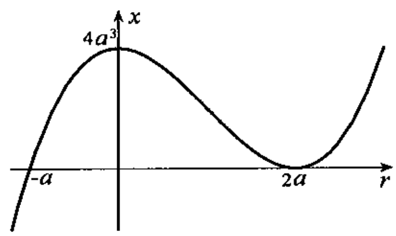


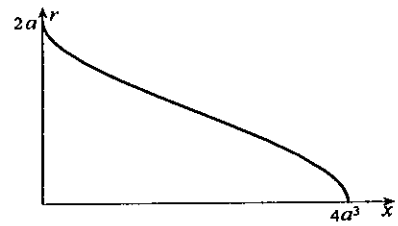
(b) From the given equation, $a/(x^2 + b) = r - c$, from which $x^2 + b = a/(r - c)$, or, $x = \pm\sqrt{a/(r-c) - b}$. The inverse function is $f^{-1}(r) = \sqrt{a/(r-c) - b}$. The function and its inverse are shown below.





24. The demand function factors as $f(r) = (r+a)(r-2a)^2$, and therefore its graph must be somewhat as shown in the left figure below. It is decreasing from $r = 0$ to $r = 2a$ for the following reasons. First $f(0) = 4a^3$. Secondly, when written in the form $f(r) = 4a^3 - r^2(3a - r)$, we see that because $r^2(3a - r) > 0$ for $0 < r < 2a$, we must have $f(r) < 4a^3$ on this interval. The function therefore has an inverse function for $0 < r < 2a$. The domain of the inverse function is the range of $f(r)$, namely, $0 < x < 4a^3$. We can sketch its graph (right figure below) by reflecting the graph of $x = f(r)$ in the line $x = r$.





25. The graph shows that $f(x)$ has an inverse. To solve $y = x^2/(1+x)^2$ for x, we set

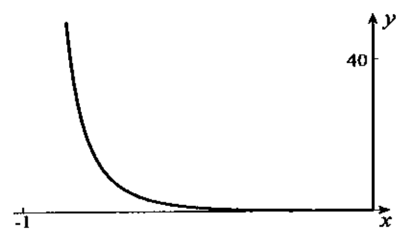
$$x^2 = (1+x)^2 y = (1 + 2x + x^2)y$$

$$\Longrightarrow$$

$$(y-1)x^2 + (2y)x + y = 0.$$

Solutions of this quadratic equation are

$$x = \frac{-2y \pm \sqrt{4y^2 - 4y(y-1)}}{2(y-1)} = \frac{-y \pm \sqrt{y}}{y-1}.$$



Since x must be negative for all $y > 0$, we choose $x = \dfrac{-y + \sqrt{y}}{y-1} = \dfrac{\sqrt{y}(1 - \sqrt{y})}{(\sqrt{y}+1)(\sqrt{y}-1)} = \dfrac{-\sqrt{y}}{\sqrt{y}+1}$. The inverse function is therefore $f^{-1}(x) = \dfrac{-\sqrt{x}}{\sqrt{x}+1}$.

EXERCISES 1.7

In questions 1–10 we multiply the degree measure by $\pi/180$ to find the radian measure of the angle. The answers are:

1. $\pi/6$	**2.** $\pi/3$	**3.** $3\pi/4$	**4.** $-\pi/2$	**5.** $-5\pi/3$
6. $17\pi/4$	**7.** $2\pi/5$	**8.** $-32\pi/45$	**9.** $107\pi/60$	**10.** $-213\pi/180$

In questions 11–20 we multiply the radian measure by $180/\pi$ to find the degree measure of the angle. The answers are:

11. $60°$	**12.** $-225°$	**13.** $270°$	**14.** $1440°$	**15.** $-150°$
16. $180/\pi°$	**17.** $-540/\pi°$	**18.** $450/\pi°$	**19.** $-206.3°$	**20.** $1980/\pi°$

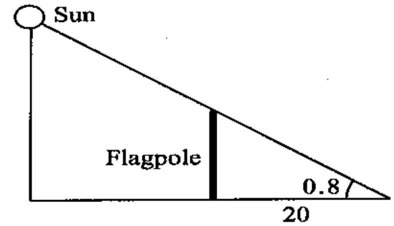
21. In each case, we divide the length of the arc by the radius 4 of the circle. The angles are (a) 1/2 radian (b) 7/4 radians (c) 4/5 radian.

22. Since the height from the top of the transit to the top of the building is $30 \tan 1.30$, the height of the building is $2 + 30 \tan(1.30) = 110.1$ m.

23. The height of the smaller building is $100 \tan(3/5) = 68.4$ m. Since the vertical distance from the top of the smaller building to the top of the taller building is $100 \tan(11/10)$, the height of the taller building is $68.4 + 100 \tan(11/10) = 2.65 \times 10^2$ m.

24. From the figure to the right, the height of the flagpole is

$20 \tan 0.8 = 20.6$ m.



25. The cosine law gives the length a of the third side of the triangle,

$a^2 = 2^2 + 3^2 - 2(2)(3)\cos(\pi/3) = 7 \implies a = \sqrt{7}.$

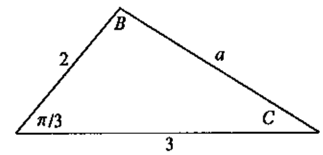
The sine law gives the angles,

$\dfrac{\sin B}{3} = \dfrac{\sin(\pi/3)}{a} \implies \sin B = \dfrac{3(\sqrt{3}/2)}{\sqrt{7}},$

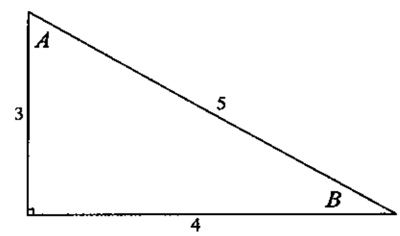
from which $B = 1.38$ radians;

$\dfrac{\sin C}{2} = \dfrac{\sin(\pi/3)}{a} \implies \sin C = \dfrac{2(\sqrt{3}/2)}{\sqrt{7}},$

from which $C = 0.714$ radians.



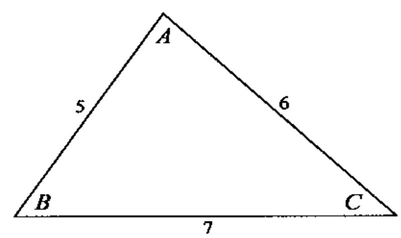
26. This is a right angle-angled triangle. Since $\sin A = 4/5$, it follows that $A = 0.927$ radians. From $\sin B = 3/5$, we obtain $B = 0.644$ radians.



27. The cosine law for the triangle gives

$7^2 = 5^2 + 6^2 - 2(5)(6)\cos A \implies \cos A = \dfrac{61-49}{60} = \dfrac{1}{5},$

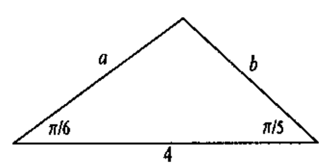
from which $A = 1.37$ radians. A similar calculation gives $B = 0.997$ radians. Then $C = \pi - A - B = 0.775$ radians.



28. The remaining angle is $\pi - \pi/6 - \pi/5 = 19\pi/30$ radians. The sine law gives the remaining two sides,

$\dfrac{a}{\sin(\pi/5)} = \dfrac{4}{\sin(19\pi/30)} \implies a = \dfrac{4\sin(\pi/5)}{\sin(19\pi/30)} = 2.57.$

$\dfrac{b}{\sin(\pi/6)} = \dfrac{4}{\sin(19\pi/30)} \implies b = \dfrac{4\sin(\pi/6)}{\sin(19\pi/30)} = 2.19.$



29. To prove 1.44a we proceed as follows (the proof of 1.44b is similar):

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$
$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

30. To verify 1.45a, we set $B = A$ in 1.43a,

$$\sin(A+A) = \sin A\cos A + \cos A\sin A \quad\implies\quad \sin 2A = 2\sin A\cos A.$$

To verify 1.46a, set $B = A$ in 1.43c. Then use the fact that $\sin^2 A + \cos^2 A = 1$ to derive 1.46b,c. To verify 1.47, set $B = A$ in 1.44.

31. To derive 1.48b, we add 1.43a,b,

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B \quad \Longrightarrow \quad \sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)].$$

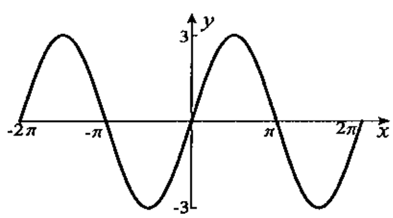
Verifications of 1.48a,c are similar.

32. If we set $X=A+B$ and $Y=A-B$, and solve for A and B, results are $A=(X+Y)/2$ and $B=(X-Y)/2$. If we substitute these into 1.48b,

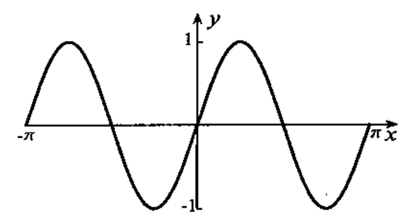
$$\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)=\frac{1}{2}(\sin X+\sin Y) \quad \Longrightarrow \quad \sin X+\sin Y=2\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right).$$

Proofs of 1.49b,c,d are similar.

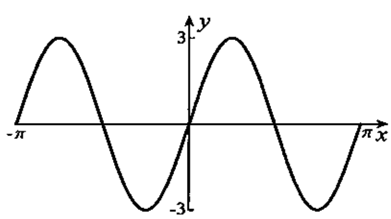
33. The amplitude is 3.



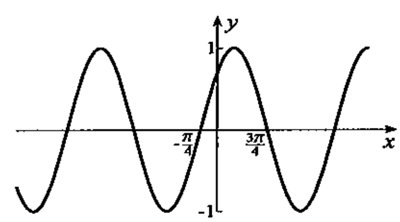
34. The period is π.



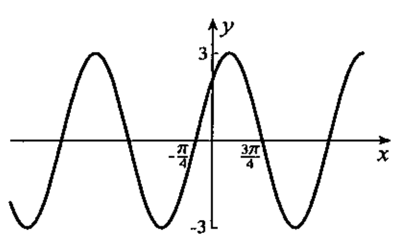
35. The amplitude is 3 and the period is π.



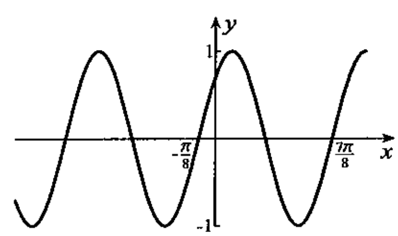
36. The sine curve is shifted $\pi/4$ units to the left.



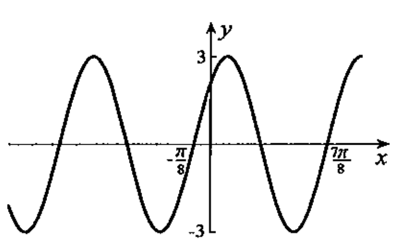
37. The amplitude is 3 and the curve is shifted $\pi/4$ units to the left.



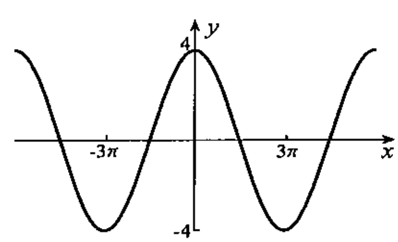
38. The period is π and the curve is shifted $\pi/8$ units to the left.



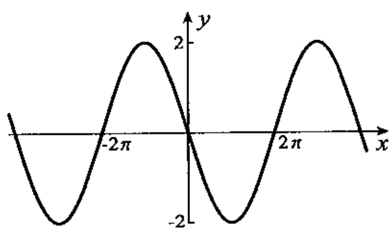
39. The period is π, the amplitude is 3, and the curve is shifted $\pi/8$ units to the left.



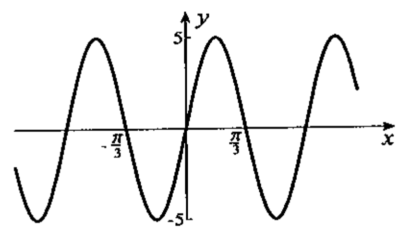
40. The amplitude is 4 and the period is 6π.



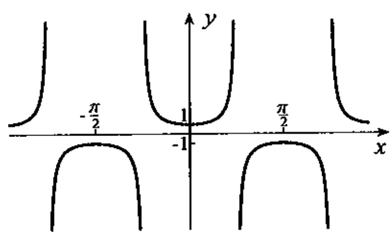
41. The amplitude is 2, the period is 4π, and the curve is shifted 2π units to the right.



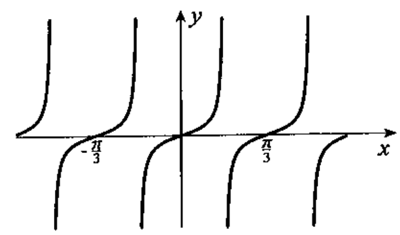
42. The amplitude is 5, the period is $2\pi/3$, and the curve is shifted $\pi/6$ units to the right.



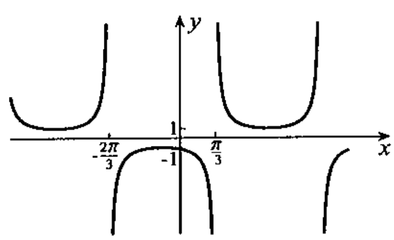
43. The period is π.



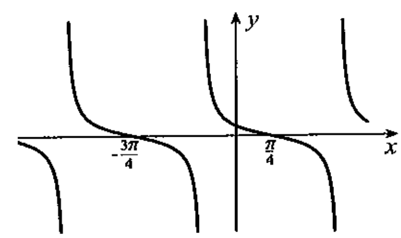
44. The period is $\pi/3$.



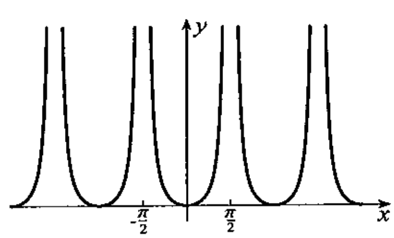
45. The cosecant curve is shifted $\pi/3$ units to the right.



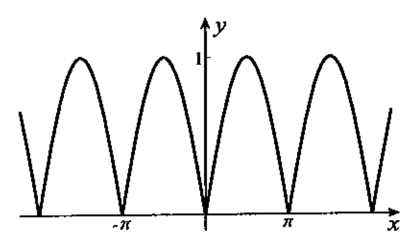
46. The cotangent curve is shifted $\pi/4$ units to the left.



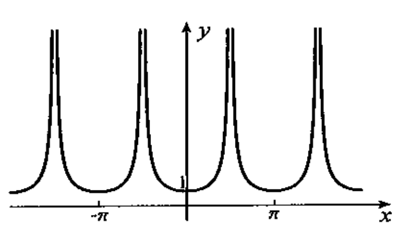
47. We square ordinates of the tangent curve.



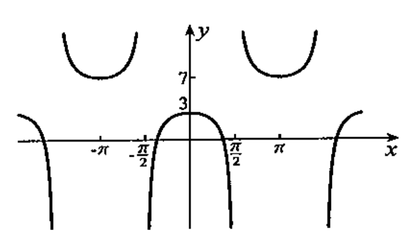
48. Since $f(x) = |\sin x|$, we invert that part of the sine curve below the x-axis.



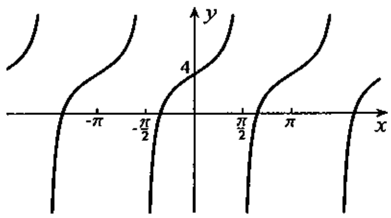
49. Since $f(x) = |\sec x|$, we invert that part of the secant curve below the x-axis.



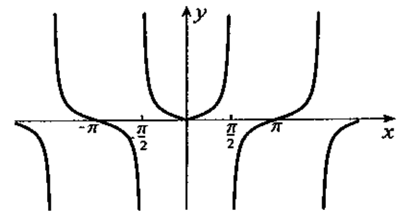
50. We invert the secant curve, double ordinates, and shift the curve vertically 5 units.



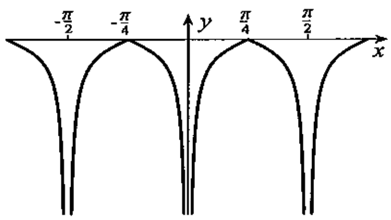
51. We double ordinates of the tangent curve and shift vertically 4 units.



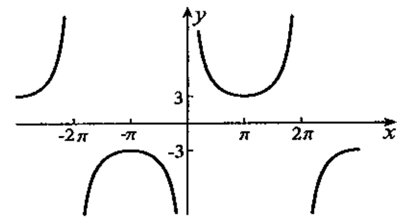
52. We reflect that part of the tangent curve to the right of the y-axis in the y-axis.



53. We invert that part of the cotangent curve above the x-axis and change the period.



54. The period is 4π and ordinates are multiplied by 3.



55. (a) $R = 42.69$ m (b) $R = 42.78$ m (c) For $R = 42.78$ when $\theta = \pi/4$ and $h = 2$, we must have

$$42.78 = \frac{v^2}{9.81\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2} + \frac{2(9.81)(2)}{v^2}}\right) = \frac{v^2}{9.81(2)}\left(1 + \sqrt{1 + \frac{8(9.81)}{v^2}}\right).$$

Therefore, $\dfrac{2(9.81)(42.78)}{v^2} - 1 = \sqrt{1 + \dfrac{8(9.81)}{v^2}}$. Squaring gives

$$1 - \frac{4(9.81)(42.78)}{v^2} + \frac{4(9.81)^2(42.78)^2}{v^4} = 1 + \frac{8(9.81)}{v^2},$$

from which

$$-4(9.81)(42.78)v^2 + 4(9.81)^2(42.78)^2 = 8(9.81)v^2 \implies v = \sqrt{\frac{4(9.81)^2(42.78)^2}{4(9.81)(42.78) + 8(9.81)}} = 20.02 \text{ m/s}.$$

56. One angle satisfying the equation is $x = \pi/3$ radians. All solutions can be expressed in the form $x = \pi/3 + 2n\pi, 2\pi/3 + 2n\pi$, where n is an integer. Following the lead of Example 1.38, these angles can be combined in the form

$$\frac{\pi}{2} \pm \frac{\pi}{6} + 2n\pi = \frac{(4n+1)\pi}{2} \pm \frac{\pi}{6}, \quad n \text{ an integer.}$$

57. There are no solutions of this equation.

58. One angle satisfying the equation is $x = 2\pi/3$ radians. All solutions can be expressed in the form $x = \pm 2\pi/3 + 2n\pi$, where n is an integer.

59. One angle satisfying the equation is $x = \pi/6$ radians. All solutions can be expressed in the form $x = \pi/6 + n\pi$, where n is an integer.

60. If we divide by $\cos x$, then $\tan x = 1$. One solution of this equation is $x = \pi/4$ radians. All solutions can be expressed in the form $x = \pi/4 + n\pi$, where n is an integer.

61. This equation implies that $\cos x = \pm 1/\sqrt{2}$. One solution of $\cos x = 1/\sqrt{2}$ is $\pi/4$ radians and one solution of $\cos x = -1/\sqrt{2}$ is $3\pi/4$ radians. All solutions can be expressed in the form $x = \pi/4 + n\pi/2$, where n is an integer.

62. One solution of this equation for $2x$ is $2x = 3\pi/4$. All solutions can be expressed in the form

$$2x = \pm\frac{3\pi}{4} + 2n\pi \qquad \Longrightarrow \qquad x = \pm\frac{3\pi}{8} + n\pi, \quad n \text{ an integer.}$$

63. One solution of this equation for $3x$ is $3x = -\pi/3$. All solutions can be expressed in the form

$$3x = -\frac{\pi}{3} + n\pi \qquad \Longrightarrow \qquad x = -\frac{\pi}{9} + \frac{n\pi}{3}, \quad n \text{ an integer.}$$

64. One solution of the equation $\sin 3x = 1/2$ for $3x$ is $3x = \pi/6$. All solutions can be expressed in the form

$$3x = \frac{\pi}{2} \pm \frac{\pi}{3} + 2n\pi \qquad \Longrightarrow \qquad x = \frac{\pi}{6} \pm \frac{\pi}{9} + \frac{2n\pi}{3}, \quad n \text{ an integer.}$$

65. One solution of this equation for $4x$ is $4x = 3\pi/4$. All solutions can be expressed in the form

$$4x = \pm\frac{3\pi}{4} + 2n\pi \qquad \Longrightarrow \qquad x = \pm\frac{3\pi}{16} + \frac{n\pi}{2}, \quad n \text{ an integer.}$$

66. If $\sin 2x = \sin x$, then $2\sin x \cos x = \sin x \Longrightarrow (2\cos x - 1)\sin x = 0$, and therefore either $\sin x = 0$ or $\cos x = 1/2$. Solutions of the former are $x = n\pi$, where n is an integer, and solutions of the latter are $x = \pm\pi/3 + 2n\pi$, where n is an integer.

67. This a quadratic equation in $\sin x$ that can be factored $(\sin x - 2)(\sin x + 1) = 0$. Either $\sin x = 2$ or $\sin x = -1$. The first of these is impossible, and solutions of the second are $x = -\pi/2 + 2n\pi$, where n is an integer.

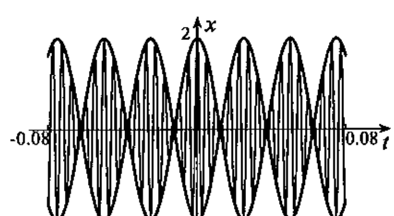
68. This equation implies that $\cot x = \pm 1/\sqrt{3}$. One solution of $\cot x = 1/\sqrt{3}$ is $\pi/3$ radians and one solution of $\cot x = -1/\sqrt{3}$ is $-\pi/3$ radians. All solutions can be expressed in the form $x = \pm\pi/3 + n\pi$, where n is an integer.

69. If we square the equation, $\sin^2 x + 2\sin x \cos x + \cos^2 x = 1 \Longrightarrow \sin x \cos x = 0$. Solutions of this equation are $x = n\pi$ and $x = (2n+1)\pi/2$, where n is an integer. But only $x = 2n\pi$ and $x = (4n+1)\pi/2$ satisfy the original equation.

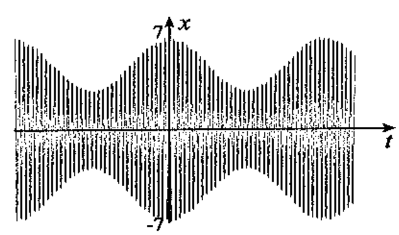
70. (a) Using identity 1.49c,

$$x(t) = \cos(440\pi t) + \cos(360\pi t) = 2\cos(400\pi t)\cos(40\pi t).$$

(b)



71. (a)

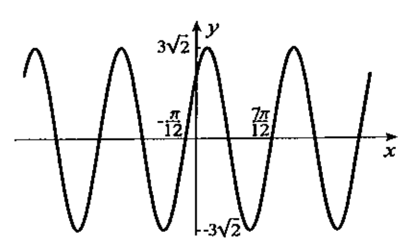


(b) The graph in part (a) indicates minimum and maximum amplitudes of 3 and 7.

72. If we set $f(x) = 3\sin 3x + 3\cos 3x = A\sin(3x + \phi)$, and expand the right side,

$$3\sin 3x + 3\cos 3x = A(\sin 3x \cos\phi + \cos 3x \sin\phi).$$

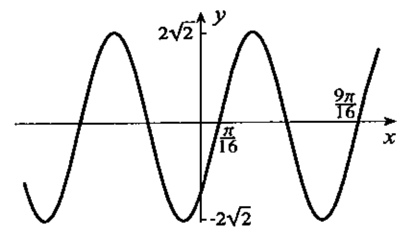
This will be true for all x if we set $A\cos\phi = 3$ and $A\sin\phi = 3$. When these are squared and added, $3^2 + 3^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = 3\sqrt{2}$, then $\sin\phi = 1/\sqrt{2}$ and $\cos\phi = 1/\sqrt{2}$. These are satisfied by $\phi = \pi/4$. The amplitude is $3\sqrt{2}$, the period is $2\pi/3$, and the phase shift is $-\pi/12$.



73. If we set $f(x) = 2\sin 4x - 2\cos 4x = A\sin(4x+\phi)$, and expand the right side,

$2\sin 4x - 2\cos 4x = A(\sin 4x\,\cos\phi + \cos 4x\,\sin\phi)$.

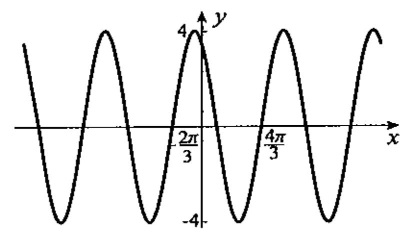
This will be true for all x if we set $A\cos\phi = 2$ and $A\sin\phi = -2$. When these are squared and added, $2^2 + (-2)^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = 2\sqrt{2}$, then $\sin\phi = -1/\sqrt{2}$ and $\cos\phi = 1/\sqrt{2}$. These are satisfied by $\phi = -\pi/4$. The amplitude is $2\sqrt{2}$, the period is $\pi/2$, and the phase shift is $\pi/16$.



74. If we set $f(x) = -2\sin x + 2\sqrt{3}\cos x = A\sin(x+\phi)$, and expand the right side,

$-2\sin x + 2\sqrt{3}\cos x = A(\sin x\,\cos\phi + \cos x\,\sin\phi)$.

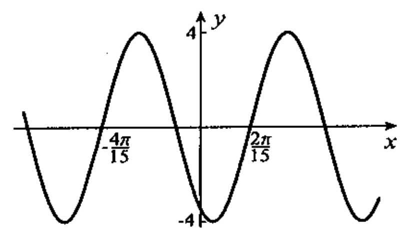
This will be true for all x if we set $A\cos\phi = -2$ and $A\sin\phi = 2\sqrt{3}$. When these are squared and added, $(-2)^2 + (2\sqrt{3})^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = 4$, then $\sin\phi = \sqrt{3}/2$ and $\cos\phi = -1/2$. These are satisfied by $\phi = 2\pi/3$. The amplitude is 4, the period is 2π, and the phase shift is $-2\pi/3$.



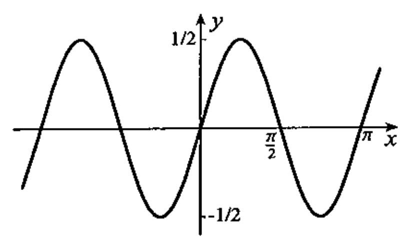
75. If we set $f(x) = -2\sin 5x - 2\sqrt{3}\cos 5x = A\sin(5x+\phi)$, and expand the right side,

$-2\sin 5x - 2\sqrt{3}\cos 5x = A(\sin 5x\,\cos\phi + \cos 5x\,\sin\phi)$.

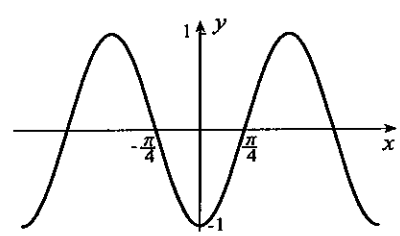
This will be true for all x if we set $A\cos\phi = -2$ and $A\sin\phi = -2\sqrt{3}$. When these are squared and added, $(-2)^2 + (-2\sqrt{3})^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = 4$, then $\sin\phi = -\sqrt{3}/2$ and $\cos\phi = -1/2$. These are satisfied by $\phi = -2\pi/3$. The amplitude is 4, the period is $2\pi/5$, and the phase shift is $2\pi/15$.



76. This is simply done using equation 1.45, $f(x) = (1/2)\sin 2x$. The amplitude is 1/2, the period is π, and the phase shift is 0.



77. This can be done using equation 1.46a, $f(x) = -\cos 2x = -\sin(\pi/2 - 2x) = \sin(2x - \pi/2)$. The amplitude is 1, the period is π, and the phase shift is $\pi/4$.



78. We expand $\cos 3x$,

$$\begin{aligned}\cos 3x = \cos(2x+x) &= \cos 2x\,\cos x - \sin 2x\,\sin x\\ &= (2\cos^2 x - 1)\cos x - (2\sin x\,\cos x)\sin x = 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x)\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

79. We expand $\sin 4x$,

$$\sin 4x = 2\sin 2x\,\cos 2x = 2(2\sin x\,\cos x)(2\cos^2 x - 1) = 8\cos^3 x\,\sin x - 4\cos x\,\sin x.$$

80. We expand $\tan 3x$,

$$\tan 3x = \tan(2x+x) = \frac{\tan 2x + \tan x}{1-\tan 2x\,\tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x}+\tan x}{1-\dfrac{2\tan^2 x}{1-\tan^2 x}}$$
$$= \frac{2\tan x + \tan x - \tan^3 x}{1-\tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}.$$

81. We use double-angle formulas on the right side,

$$\frac{\sin x}{1+\cos x} = \frac{2\sin(x/2)\cos(x/2)}{1+[2\cos^2(x/2)-1]} = \frac{\sin(x/2)}{\cos(x/2)} = \tan(x/2).$$

82. We expand the right side, $\quad \tan\left(x+\dfrac{\pi}{4}\right) = \dfrac{\tan x + \tan(\pi/4)}{1-\tan x\,\tan(\pi/4)} = \dfrac{\tan x + 1}{1-\tan x}.$

83. If we set $A\cos\omega x + B\sin\omega x = R\sin(\omega x+\phi)$, and expand the right side using identity 1.43a,

$$A\cos\omega x + B\sin\omega x = R(\sin\omega x\cos\phi + \cos\omega x\sin\phi).$$

This is satisfied if we set $A = R\sin\phi$ and $B = R\cos\phi$. When these are squared and added, $A^2+B^2 = R^2$, from which we choose $R = \sqrt{A^2+B^2}$. With this, ϕ is defined by $\sin\phi = A/\sqrt{A^2+B^2}$ and $\cos\phi = B/\sqrt{A^2+B^2}$.

84. No. The two equations define the quadrant for ϕ, but the equation $\tan\phi = A/B$ does not.

85. If we set $2\sin 2x\cos 2x = \cos 2x$, then either $\cos 2x = 0$ or $2\sin 2x = 1$. The first implies that $2x = (2n+1)\pi/2 \Longrightarrow x = (2n+1)\pi/4$, where n is an integer. The second implies that $2x = \pi/6+2n\pi,\ 5\pi/6+2n\pi$, where n is an integer. The only solutions between 0 and 2 are $\pi/12$, $\pi/4$, and $5\pi/12$.

86. If we use identity 1.49c, $0 = \cos x + \cos 3x = 2\cos 2x\cos x$. Hence, $\cos 2x = 0$ or $\cos x = 0$. Solutions of the first are defined by $2x = (2n+1)\pi/2 \Longrightarrow x = (2n+1)\pi/4$, where n is an integer. Solutions of $\cos x = 0$ are $(2n+1)\pi/2$. The only solutions between 0 and 2 are $\pi/4$ and $\pi/2$.

87. If we use identity 1.49b on the left side of $\sin 4x - \sin 2x = \cos 3x$, then $\cos 3x = 2\cos 3x\sin x$. This implies that $\cos 3x = 0$ or $\sin x = 1/2$. The first gives $3x = (2n+1)\pi/2 \Longrightarrow x = (2n+1)\pi/6$, where n is an integer. The second gives $x = \pi/6+2n\pi, 5\pi/6+2n\pi$. Since the solutions $\pi/6+2n\pi$ are contained in the set $(2n+1)\pi/6$, the full set of solutions is $x = (2n+1)\pi/6$, $(12n+5)\pi/6$. The only solutions between 0 and 2 are $\pi/6$ and $\pi/2$.

88. If we square the equation,

$$\sin^2 x + 2\sin x\cos x + \cos^2 x = 3\sin^2 x\cos^2 x \quad\Longrightarrow\quad 3\sin^2 x\cos^2 x - 2\sin x\cos x - 1 = 0$$
$$\Longrightarrow\quad (3\sin x\cos x + 1)(\sin x\cos x - 1) = 0 \quad\Longrightarrow\quad \sin 2x = -2/3 \quad\text{or}\quad \sin 2x = 2.$$

The second of these is impossible. From the first,

$$2x = \begin{cases} -0.7297+2n\pi \\ -2.412+2n\pi \end{cases} \quad\Longrightarrow\quad x = \begin{cases} -0.365+n\pi \\ -1.21+n\pi \end{cases},$$

where n is an integer. Only $x = -0.365+(2n+1)\pi$, $-1.21+2n\pi$ satisfy the original equation. None of the solutions are between 0 and 2.

89. If A, B, and C are the angles in a triangle, then $A+B+C = \pi$. If we take tangents of both sides of this equation,

$$\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C} = 0 \quad\Longrightarrow\quad \tan(A+B)+\tan C = 0.$$

If we expand $\tan(A+B)$,

$$\frac{\tan A + \tan B}{1-\tan A\tan B} = -\tan C \quad\Longrightarrow\quad \tan A + \tan B = -\tan C + \tan A\tan B\tan C,$$

and this gives the required result.

EXERCISES 1.8

1. $\operatorname{Tan}^{-1}(-1/3) = -0.322$

2. $\operatorname{Sin}^{-1}(1/4) = 0.253$

3. $\operatorname{Sec}^{-1}(\sqrt{3}) = 0.955$

4. $\operatorname{Csc}^{-1}(-2/\sqrt{3}) = -2\pi/3$

5. $\operatorname{Cot}^{-1}(1) = \pi/4$

6. $\operatorname{Cos}^{-1}(3/2)$ does not exist since the domain of $\operatorname{Cos}^{-1}x$ is $-1 \le x \le 1$.

7. $\operatorname{Sin}^{-1}(\pi/2)$ does not exist since the domain of $\operatorname{Sin}^{-1}x$ is $-1 \le x \le 1$.

8. $\operatorname{Tan}^{-1}(-1) = -\pi/4$

9. $\sin(\operatorname{Tan}^{-1}\sqrt{3}) = \sin(\pi/3) = \sqrt{3}/2$

10. $\tan\left(\operatorname{Sin}^{-1}3\right)$ does not exist since the domain of $\operatorname{Sin}^{-1}x$ is $-1 \le x \le 1$.

11. $\operatorname{Sin}^{-1}[\tan(1/6)] = \operatorname{Sin}^{-1}0.16823 = 0.169$

12. $\operatorname{Tan}^{-1}[\sin(1/6)] = \operatorname{Tan}^{-1}[0.165\,896] = 0.164$

13. $\sec[\operatorname{Cos}^{-1}(1/2)] = \sec(\pi/3) = 2$

14. $\operatorname{Sin}^{-1}[\sin(3\pi/4)] = \operatorname{Sin}^{-1}[1/\sqrt{2}] = \pi/4$

15. $\sin[\operatorname{Sin}^{-1}(1/\sqrt{2})] = 1/\sqrt{2}$

16. $\operatorname{Sin}^{-1}\left[\cos\left(\operatorname{Sec}^{-1}(-\sqrt{2})\right)\right] = \operatorname{Sin}^{-1}\left[\cos(-3\pi/4)\right] = \operatorname{Sin}^{-1}\left[-1/\sqrt{2}\right] = -\pi/4$

17. Since one solution is $x = \operatorname{Sin}^{-1}(1/3) = 0.340$, all solutions are $0.340 + 2n\pi$ and $\pi - 0.340 + 2n\pi$, where n is an integer. They can also be represented more compactly in the form

$$\frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.340\right) + 2n\pi = (4n+1)\pi/2 \pm 1.23.$$

18. From the solution $x = \operatorname{Tan}^{-1}(-1.2) = -0.876$, we obtain $x = n\pi - 0.876$, where n is an integer.

19. One solution for $2x$ is $2x = \operatorname{Cos}^{-1}(1/3) = 1.23$. All solutions are given by

$$2x = \pm 1.23 + 2n\pi \qquad \Longrightarrow \qquad x = \pm 0.615 + n\pi, \quad n \text{ an integer.}$$

20. One solution of $\cot 4x = -2.2$ for $4x$ is $4x = \operatorname{Cot}^{-1}(-2.2) = 2.715$. All solutions are given by

$$4x = 2.715 + n\pi \qquad \Longrightarrow \qquad x = .679 + \frac{n\pi}{4}, \quad n \text{ an integer.}$$

21. One solution of $\sin(1-x) = 0.7$ for $1-x$ is $1-x = \operatorname{Sin}^{-1}(0.7) = 0.7754$. All solutions are given by

$$1 - x = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.7754\right) + 2n\pi \qquad \Longrightarrow \qquad x = 1 - \frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.7754\right) - 2n\pi = -0.571 \pm 0.795 - 2n\pi,$$

where n is an integer.

22. One solution of $\tan 3x = -3.2/3$ for $3x$ is $3x = \operatorname{Tan}^{-1}(-3.2/3) = -0.8176$. All solutions are given by

$$3x = -0.8176 + n\pi \qquad \Longrightarrow \qquad x = -0.273 + \frac{n\pi}{3}, \quad n \text{ an integer.}$$

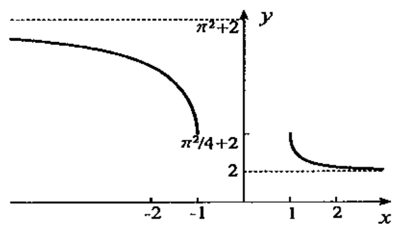
23. When we set $0 = 4\sin^2 x - 2(1 - \sin^2 x) - 1 = 6\sin^2 x - 3 = 3(2\sin^2 x - 1)$, it follows that $\sin x = \pm 1/\sqrt{2}$. Solutions are $x = \pm\pi/4 + n\pi = (4n \pm 1)\pi/4$, where n is an integer.

24. Since $1 = 4\sin^2 x + 2(1 - \sin^2 x)$, we require $2\sin^2 x = -1$, an impossibility.

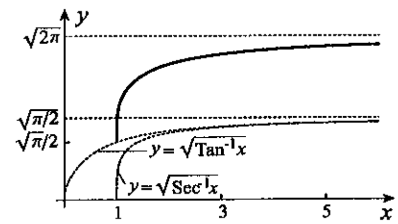
25. From this quadratic, $\cos x = \left(3 \pm \sqrt{9-4}\right)/2 = \left(3 \pm \sqrt{5}\right)/2$. We must choose $\cos x = \left(3 - \sqrt{5}\right)/2$. From the solution $x = \operatorname{Cos}^{-1}[(3 - \sqrt{5})/2] = 1.179$, all solutions are $x = \pm 1.179 + 2n\pi$, where n is an integer.

26. This quadratic equation in $\sin x$ has solutions $\sin x = \left(3 \pm \sqrt{9+20}\right)/2 = \left(3 \pm \sqrt{29}\right)/2$. Because neither of these numbers is between -1 and 1, there are no solutions to the equation.

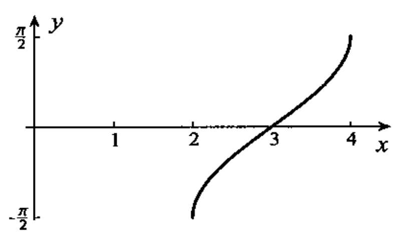
27. We square ordinates of $y = \mathrm{Csc}^{-1}x$, and then shift vertically 2 units.



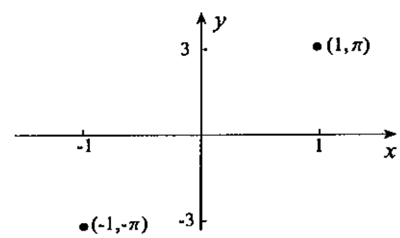
28. We add ordinates of $y = \sqrt{\mathrm{Tan}^{-1}x}$ and $y = \sqrt{\mathrm{Sec}^{-1}x}$.



29. We shift $y = \mathrm{Sin}^{-1}x$ to the right 3 units.



30. Since the domain of $\mathrm{Sin}^{-1}x$ is $|x| \le 1$, and that of $\mathrm{Csc}^{-1}x$ is $|x| \ge 1$, the function is defined only for $x = \pm 1$. Its graph is therefore two points.



31. Using equation 1.59, $\tan\phi = 1 \Longrightarrow \phi = \pi/4$ radians.

32. Using equation 1.59, $\tan\phi = -1/2 \Longrightarrow \phi = 2.68$ radians.

33. Using equation 1.59, $\tan\phi = 3/2 \Longrightarrow \phi = 0.983$ radians.

34. Using equation 1.59, $\tan\phi = 3 \Longrightarrow \phi = 1.25$ radians.

35. Since the line is vertical, $\phi = \pi/2$ radians.

36. Since the line is horizontal, $\phi = 0$.

37. Since slopes of the lines are -1 and 1, the lines are perpendicular.

38. Since slopes of both lines are $-1/3$, the lines are parallel.

39. Since slopes of both lines are $1/3$, the lines are parallel.

40. Since slopes of the lines are $-2/3$ and $3/2$, the lines are perpendicular.

41. Since slopes of the lines are 3 and $-1/2$, formula 1.60 gives

$$\theta = \mathrm{Tan}^{-1}\left|\frac{3+1/2}{1+3(-1/2)}\right| = 1.43 \text{ radians.}$$

42. Since slopes of the lines are 1 and $-2/3$, formula 1.60 gives

$$\theta = \mathrm{Tan}^{-1}\left|\frac{1+2/3}{1+(-2/3)}\right| = 1.37 \text{ radians.}$$

43. The lines are perpendicular.

44. Since slopes of the lines are -1 and 3, formula 1.60 gives

$$\theta = \mathrm{Tan}^{-1}\left|\frac{-1-3}{1-3}\right| = 1.11 \text{ radians.}$$

45. The sine law applied to triangle OAB gives $(\sin\phi)/l = (\sin\theta)/L \Longrightarrow \sin\phi = (l/L)\sin\theta$. Thus, $\phi = \mathrm{Sin}^{-1}[(l/L)\sin\theta]$.

46. Since $0 = \tan^2 x(\sin x + 1) - 3(\sin x + 1) = (\tan^2 x - 3)(\sin x + 1)$, it follows that $\tan x = \pm\sqrt{3}$ or $\sin x = -1$. The solutions of these equations are $x = \pm\pi/3 + n\pi$ and $x = -\pi/2 + 2n\pi$, where n is an integer, but only $x = n\pi \pm \pi/3$ satisfy the original equation.

47. If we square the equation, $\sin^2 x + 2\sin x\cos x + \cos^2 x = 1 \implies \sin x\cos x = 0$. Solutions of this equation are $x = n\pi$ and $x = (2n+1)\pi/2$, where n is an integer. But only $x = 2n\pi$ and $x = (4n+1)\pi/2$ satisfy the original equation.

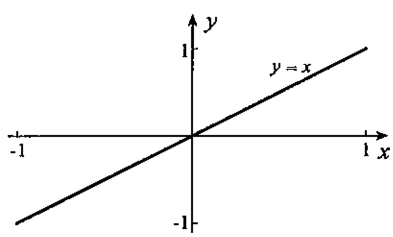
48. This equation implies that $\sin x = \pm 3\pi/4 + 2n\pi$, where n is an integer. But for no n are these values between between ± 1. Hence, there are no solutions.

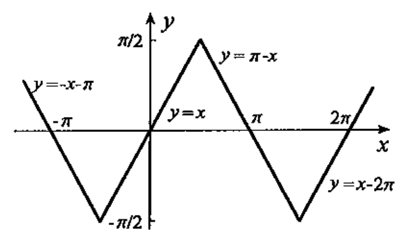
49. This equation implies that $\mathrm{Sin}^{-1}x = 2n\pi \pm \pi/3$, where n is an integer. Since values of $\mathrm{Sin}^{-1}x$ must lie between $-\pi/2$ and $\pi/2$, n must be zero. Thus, $x = \pm\sqrt{3}/2$.

50. This equation implies that $\tan x = \pm 3\pi/4 + 2n\pi$. Therefore, $x = \mathrm{Tan}^{-1}(2n\pi \pm 3\pi/4) + m\pi$ where m and n are integers.

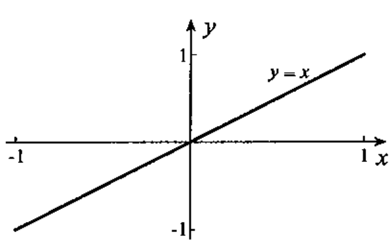
51. From the equation, $\tan(x^2 + 4) = \cos(2\pi - 5) = \cos 5$. This implies that $x^2 + 4 = n\pi + \mathrm{Tan}^{-1}(\cos 5)$, from which $x = \pm\sqrt{n\pi + \mathrm{Tan}^{-1}(\cos 5) - 4} = \pm\sqrt{n\pi - 3.724}$, where $n \geq 2$ is an integer.

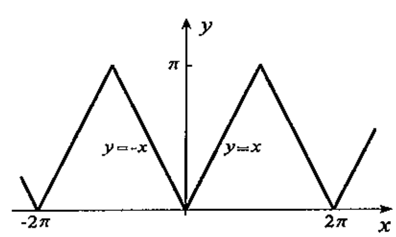
52. (a) Since $\mathrm{Sin}^{-1}x$ is defined only for $-1 \leq x \leq 1$, and on this interval, the sine function is the inverse of $\mathrm{Sin}^{-1}x$, it follows that $f(x) = x$. Its graph is shown to the left below.
(b) On the interval $-\pi/2 \leq x \leq \pi/2$, $\mathrm{Sin}^{-1}x$ is the inverse of $\sin x$ and therefore $f(x) = x$ on this interval. For $\pi/2 \leq x \leq 3\pi/2$, $f(x) = \pi - x$. These define $f(x)$ on the interval $-\pi/2 \leq x \leq 3\pi/2$ of length 2π. Since $\sin x$ is 2π-periodic, so also is $f(x)$, and the graph is shown to the right below.





53. (a) Since $\mathrm{Cos}^{-1}x$ is defined only for $-1 \leq x \leq 1$, and on this interval, the cosine function is the inverse of $\mathrm{Cos}^{-1}x$, it follows that $f(x) = x$. Its graph is shown to the left below.
(b) On the interval $0 \leq x \leq \pi$, $\mathrm{Cos}^{-1}x$ is the inverse of $\cos x$, and therefore $f(x) = x$ on this interval. For $-\pi \leq x \leq 0$, $f(x) = -x$. These define $f(x)$ on the interval $-\pi \leq x \leq \pi$ of length 2π. Since $\cos x$ is 2π-periodic, so also is $f(x)$, and the graph is shown to the right below.





54. If we expand $R\sin(2x + \phi)$ according to 1.43a and equate it to $f(x)$, we obtain

$$R\sin 2x\cos\phi + R\cos 2x\sin\phi = 4\sin 2x + \cos 2x.$$

This equation is satisfied for all x if R and ϕ satisfy $R\cos\phi = 4$ and $R\sin\phi = 1$. When these are squared and added, the result is $R^2 = 17$. Consequently, $R = \sqrt{17}$, and

$$\cos\phi = \frac{4}{\sqrt{17}}, \qquad \sin\phi = \frac{1}{\sqrt{17}}.$$

The only angle in the range $0 < \phi < \pi$ satisfying these is $\phi = 0.245$ radians. Thus, $f(x)$ can be expressed in the form $\sqrt{17}\sin(2x + 0.245)$.

55. If we expand $R\cos(3x+\phi)$ according to 1.43c and equate it to $f(x)$, we obtain

$$R\cos 3x\cos\phi - R\sin 3x\sin\phi = -2\sin 3x + 4\cos 3x.$$

This equation is satisfied for all x if R and ϕ satisfy $R\cos\phi = 4$ and $R\sin\phi = 2$. When these are squared and added, the result is $R^2 = 20$. Consequently, $R = 2\sqrt{5}$, and

$$\cos\phi = \frac{2}{\sqrt{5}}, \qquad \sin\phi = \frac{1}{\sqrt{5}}.$$

The only angle in the range $0 < \phi < \pi$ satisfying these is $\phi = 0.464$ radians. Thus, $f(x)$ can be expressed in the form $2\sqrt{5}\cos(3x + 0.464)$.

56. If we expand $R\sin(2x+\phi)$ according to 1.43a and equate it to $f(x)$, we obtain

$$R\sin 2x\cos\phi + R\cos 2x\sin\phi = -2\sin 2x + 4\cos 2x.$$

This equation is satisfied for all x if R and ϕ satisfy $R\cos\phi = -2$ and $R\sin\phi = 4$. When these are squared and added, the result is $R^2 = 20$. Consequently, $R = 2\sqrt{5}$, and

$$\cos\phi = \frac{-1}{\sqrt{5}}, \qquad \sin\phi = \frac{2}{\sqrt{5}}.$$

The only angle in the range $0 < \phi < \pi$ satisfying these is $\phi = 2.03$ radians. Thus, $f(x)$ can be expressed in the form $2\sqrt{5}\sin(2x + 2.03)$.

57. If we expand $R\cos(3x+\phi)$ according to 1.43c and equate it to $f(x)$, we obtain

$$R\cos 3x\cos\phi - R\sin 3x\sin\phi = -4\sin 3x + 5\cos 3x.$$

This equation is satisfied for all x if R and ϕ satisfy $R\cos\phi = 5$ and $R\sin\phi = 4$. When these are squared and added, the result is $R^2 = 41$. Consequently, $R = \sqrt{41}$, and

$$\cos\phi = \frac{5}{\sqrt{41}}, \qquad \sin\phi = \frac{4}{\sqrt{41}}.$$

The only angle in the range $0 < \phi < \pi$ satisfying these is $\phi = 0.675$ radians. Thus, $f(x)$ can be expressed in the form $\sqrt{41}\cos(3x + 0.675)$.

58. We set
$$\begin{aligned} x(t) &= A\sin(\omega t+\phi) = A(\sin\omega t\cos\phi + \cos\omega t\sin\phi) = f(t) + g(t) \\ &= 4[\cos\omega t\cos(2\pi/3) - \sin\omega t\sin(2\pi/3)] + 3[\sin\omega t\cos(\pi/3) + \cos\omega t\sin(\pi/3)] \\ &= \left(-2\sqrt{3} + \frac{3}{2}\right)\sin\omega t + \left(-2 + \frac{3\sqrt{3}}{2}\right)\cos\omega t. \end{aligned}$$

This will be true if we choose A and ϕ to satisfy $A\cos\phi = \dfrac{3}{2} - 2\sqrt{3}$ and $A\sin\phi = \dfrac{3\sqrt{3}}{2} - 2$. When these are squared and added, the result is

$$A^2 = \left(\frac{3}{2} - 2\sqrt{3}\right)^2 + \left(\frac{3\sqrt{3}}{2} - 2\right)^2 = 25 - 12\sqrt{3} \quad\Longrightarrow\quad A = \sqrt{25 - 12\sqrt{3}}.$$

Hence, $\cos\phi = \dfrac{3/2 - 2\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}$ and $\sin\phi = \dfrac{3\sqrt{3}/2 - 2}{\sqrt{25 - 12\sqrt{3}}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = 2.846$ radians.

59. We set
$$\begin{aligned} x(t) &= A\cos(\omega t+\phi) = A(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = f(t) + g(t) \\ &= 4[\cos\omega t\cos(2\pi/3) - \sin\omega t\sin(2\pi/3)] + 3[\sin\omega t\cos(\pi/3) + \cos\omega t\sin(\pi/3)] \\ &= \left(-2\sqrt{3} + \frac{3}{2}\right)\sin\omega t + \left(-2 + \frac{3\sqrt{3}}{2}\right)\cos\omega t. \end{aligned}$$

This will be true if we choose A and ϕ to satisfy $A\cos\phi = \frac{3\sqrt{3}}{2} - 2$ and $-A\sin\phi = \frac{3}{2} - 2\sqrt{3}$. When these are squared and added, the result is

$$A^2 = \left(\frac{3\sqrt{3}}{2} - 2\right)^2 + \left(\frac{3}{2} - 2\sqrt{3}\right)^2 = 25 - 12\sqrt{3} \implies A = \sqrt{25 - 12\sqrt{3}}.$$

Hence, $\cos\phi = \frac{3\sqrt{3}/2 - 2}{\sqrt{25 - 12\sqrt{3}}}$ and $\sin\phi = \frac{2\sqrt{3} - 3/2}{\sqrt{25 - 12\sqrt{3}}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = 1.275$ radians.

60. We set $x(t) = A\cos(\omega t + \phi) = A(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = f(t) + g(t)$

$$= 2(\sin\omega t\cos 4 + \cos\omega t\sin 4) + 3(\sin\omega t\cos 1 + \cos\omega t\sin 1)$$
$$= (2\cos 4 + 3\cos 1)\sin\omega t + (2\sin 4 + 3\sin 1)\cos\omega t.$$

This will be true if we choose A and ϕ to satisfy $A\cos\phi = 2\sin 4 + 3\sin 1$ and $-A\sin\phi = 2\cos 4 + 3\cos 1$. When these are squared and added, the result is

$$A^2 = (2\sin 4 + 3\sin 1)^2 + (2\cos 4 + 3\cos 1)^2 \implies A = \sqrt{13 + 12\cos 3}.$$

Hence, $\cos\phi = \frac{2\sin 4 + 3\sin 1}{\sqrt{13 + 12\cos 3}}$ and $\sin\phi = -\frac{2\cos 4 + 3\cos 1}{\sqrt{13 + 12\cos 3}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = -0.301$ radians.

61. We set $x(t) = A\sin(\omega t + \phi) = A(\sin\omega t\cos\phi + \cos\omega t\sin\phi) = f(t) + g(t)$

$$= 2(\sin\omega t\cos 4 + \cos\omega t\sin 4) + 3(\sin\omega t\cos 1 + \cos\omega t\sin 1)$$
$$= (2\cos 4 + 3\cos 1)\sin\omega t + (2\sin 4 + 3\sin 1)\cos\omega t.$$

This will be true if we choose A and ϕ to satisfy $A\cos\phi = 2\cos 4 + 3\cos 1$ and $A\sin\phi = 2\sin 4 + 3\sin 1$. When these are squared and added, the result is

$$A^2 = (2\cos 4 + 3\cos 1)^2 + (2\sin 4 + 3\sin 1)^2 \implies A = \sqrt{13 + 12\cos 3}.$$

Hence, $\cos\phi = \frac{2\cos 4 + 3\cos 1}{\sqrt{13 + 12\cos 3}}$ and $\sin\phi = \frac{2\sin 4 + 3\sin 1}{\sqrt{13 + 12\cos 3}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = 1.270$ radians.

62. We set $x(t) = A\sin(\omega t + \phi) = A(\sin\omega t\cos\phi + \cos\omega t\sin\phi) = f(t) + g(t) + h(t)$

$$= 5\sin\omega t + 4[\cos\omega t\cos(\pi/3) - \sin\omega t\sin(\pi/3)] + 2[\sin\omega t\cos(\pi/4) + \cos\omega t\sin(\pi/4)]$$
$$= (5 - 2\sqrt{3} + \sqrt{2})\sin\omega t + (2 + \sqrt{2})\cos\omega t.$$

This will be true if we choose A and ϕ to satisfy $A\cos\phi = 5 - 2\sqrt{3} + \sqrt{2}$ and $A\sin\phi = 2 + \sqrt{2}$. When these are squared and added, the result is

$$A^2 = (5 - 2\sqrt{3} + \sqrt{2})^2 + (2 + \sqrt{2})^2 \implies A = \sqrt{45 + 14\sqrt{2} - 20\sqrt{3} - 4\sqrt{6}}.$$

Hence, $\cos\phi = \frac{5 - 2\sqrt{3} + \sqrt{2}}{\sqrt{45 + 14\sqrt{2} - 20\sqrt{3} - 4\sqrt{6}}}$ and $\sin\phi = \frac{2 + \sqrt{2}}{\sqrt{45 + 14\sqrt{2} - 20\sqrt{3} - 4\sqrt{6}}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = 0.858$ radians.

63. We set $x(t) = A\cos(\omega t + \phi) = A(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = f(t) + g(t) + h(t)$

$$= 5\sin\omega t + 4[\cos\omega t\cos(\pi/3) - \sin\omega t\sin(\pi/3)] + 2[\sin\omega t\cos(\pi/4) + \cos\omega t\sin(\pi/4)]$$
$$= (5 - 2\sqrt{3} + \sqrt{2})\sin\omega t + (2 + \sqrt{2})\cos\omega t.$$

This will be true if we choose A and ϕ to satisfy $A\cos\phi = 2 + \sqrt{2}$ and $-A\sin\phi = 5 - 2\sqrt{3} + \sqrt{2}$. When these are squared and added, the result is

$$A^2 = (2 + \sqrt{2})^2 + (5 - 2\sqrt{3} + \sqrt{2})^2 \implies A = \sqrt{45 + 14\sqrt{2} - 20\sqrt{3} - 4\sqrt{6}}.$$

Hence, $\cos\phi = \dfrac{2+\sqrt{2}}{\sqrt{45+14\sqrt{2}-20\sqrt{3}-4\sqrt{6}}}$ and $\sin\phi = -\dfrac{5-2\sqrt{3}+\sqrt{2}}{\sqrt{45+14\sqrt{2}-20\sqrt{3}-4\sqrt{6}}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = -0.713$ radians.

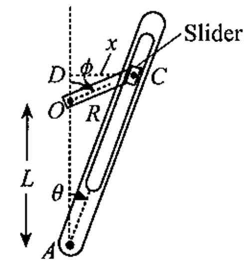
64. If we let x be the distance from slider C to line OA, then

$x = R\sin\phi$ and $x = (L + R\cos\phi)\tan\theta$.

Equating these gives

$R\sin\phi = (L + R\cos\phi)\tan\theta$,

from which $\tan\theta = \dfrac{R\sin\phi}{L + R\cos\phi}$.



Thus, $\theta = \text{Tan}^{-1}\left(\dfrac{R\sin\phi}{L+R\cos\phi}\right)$.

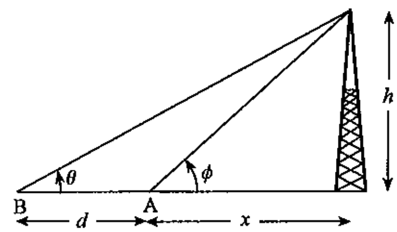
65. If x is the distance from A to the foot of the tower, then

$\tan\theta = \dfrac{h}{d+x}$ and $\tan\phi = \dfrac{h}{x}$.

From these,

$x = h\cot\theta - d$ and $x = h\cot\phi$.

Equating these gives



$$h\cot\theta - d = h\cot\phi \quad\Longrightarrow\quad \cot\theta = \cot\phi + \frac{d}{h}.$$

Since θ is an acute angle, we can write that

$$\theta = \text{Cot}^{-1}\left(\cot\phi + \frac{d}{h}\right).$$

66. The equation

$$\theta = \theta_0\,\cos\omega t + \frac{v_0}{\omega L}\sin\omega t = R\,\sin(\omega t + \phi) = R\,\sin\omega t\,\cos\phi + R\,\cos\omega t\,\sin\phi$$

is satisfied if R and ϕ satisfy

$$R\cos\phi = \frac{v_0}{\omega L} \qquad\text{and}\qquad R\sin\phi = \theta_0.$$

When these are squared and added, $R^2 = \theta_0^2 + v_0^2/(\omega^2L^2)$. If we choose $R = \sqrt{\theta_0^2 + v_0^2/(\omega^2L^2)}$, then

$$\cos\phi = \frac{v_0}{\omega L\sqrt{\theta_0^2+v_0^2/(\omega^2L^2)}} = \frac{v_0}{\sqrt{v_0^2+\omega^2L^2\theta_0^2}}, \qquad \sin\phi = \frac{\theta_0}{\sqrt{\theta_0^2+v_0^2/(\omega^2L^2)}} = \frac{\omega L\theta_0}{\sqrt{v_0^2+\omega^2L^2\theta_0^2}}.$$

Because $v_0 > 0$, it follows that $\cos\phi > 0$, and we may take $-\pi/2 < \phi < \pi/2$. Since $\sin\phi$ has the same sign as θ_0, angle ϕ is in $0 < \phi < \pi/2$ when $\theta_0 > 0$, and is in $-\pi/2 < \phi < 0$ when $\theta_0 < 0$. Now

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\omega L\theta_0}{\sqrt{v_0^2+\omega^2L^2\theta_0^2}}\,\frac{\sqrt{v_0^2+\omega^2L^2\theta_0^2}}{v_0} = \frac{\omega L\theta_0}{v_0}.$$

We can write $\phi = \text{Tan}^{-1}(\omega L\theta_0/v_0)$ since principal values are between $-\pi/2$ and 0 when $\theta_0 < 0$, and between 0 and $\pi/2$ when $\theta_0 > 0$.

67. When $v_0 < 0$, then $\cos\phi < 0$ and ϕ is an angle in the second or third quadrant. If $\theta_0 > 0$, then $\sin\phi > 0$, and ϕ is in the first or second quadrant. Hence, ϕ must be in the second quadrant. Since $\tan\phi < 0$, the formula for ϕ is $\phi = \pi + \text{Tan}^{-1}(\omega L\theta_0/v_0)$. On the other hand, if $\theta_0 < 0$, then $\sin\phi < 0$, and ϕ is in the third or fourth quadrant. Hence, ϕ must be in the third quadrant. Since $\tan\phi > 0$, the formula for ϕ is $\phi = -\pi + \text{Tan}^{-1}(\omega L\theta_0/v_0)$.

68. The equation

$$y = y_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t = R\sin(\omega t + \phi) = R\sin\omega t\cos\phi + R\cos\omega t\sin\phi$$

is satisfied if R and ϕ satisfy

$$R\cos\phi = \frac{v_0}{\omega}, \qquad R\sin\phi = y_0.$$

When these are squared and added, $R^2 = y_0^2 + v_0^2/\omega^2$. If we choose $R = \sqrt{y_0^2 + v_0^2/\omega^2}$, then

$$\cos\phi = \frac{v_0}{\omega\sqrt{y_0^2 + v_0^2/\omega^2}} = \frac{v_0}{\sqrt{v_0^2 + \omega^2 y_0^2}}, \qquad \sin\phi = \frac{y_0}{\sqrt{y_0^2 + v_0^2/\omega^2}} = \frac{\omega y_0}{\sqrt{v_0^2 + \omega^2 y_0^2}}.$$

Because $v_0 > 0$, it follows that $\cos\phi > 0$, and we may take $-\pi/2 < \phi < \pi/2$. Since $\sin\phi$ has the same sign as y_0, angle ϕ is in $0 < \phi < \pi/2$ when $y_0 > 0$, and is in $-\pi/2 < \phi < 0$ when $y_0 < 0$. Now

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\omega y_0}{\sqrt{v_0^2 + \omega^2 y_0^2}}\frac{\sqrt{v_0^2 + \omega^2 y_0^2}}{v_0} = \frac{\omega y_0}{v_0}.$$

We can write $\phi = \text{Tan}^{-1}(\omega y_0/v_0)$ since principal values are between $-\pi/2$ and 0 when $y_0 < 0$, and between 0 and $\pi/2$ when $y_0 > 0$.

69. When $v_0 < 0$, then $\cos\phi < 0$ and ϕ is an angle in the second or third quadrant. If $y_0 > 0$, then $\sin\phi > 0$, and ϕ is in the first or second quadrant. Hence, ϕ must be in the second quadrant. Since $\tan\phi < 0$, the formula for ϕ is $\phi = \pi + \text{Tan}^{-1}(\omega y_0/v_0)$. On the other hand, if $y_0 < 0$, then $\sin\phi < 0$, and ϕ is in the third or fourth quadrant. Hence, ϕ must be in the third quadrant. Since $\tan\phi > 0$, the formula for ϕ is $\phi = -\pi + \text{Tan}^{-1}(\omega y_0/v_0)$.

70. If we expand $A\cos(\omega t - \phi)$ and equate it to $f(x)$, we obtain

$$A\cos\omega t\cos\phi + A\sin\omega t\sin\phi = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\left[R\cos\omega t + \left(\omega L - \frac{1}{\omega C}\right)\sin\omega t\right].$$

This equation is satisfied for all t if A and ϕ satisfy

$$A\cos\phi = \frac{E_0 R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \qquad A\sin\phi = \frac{E_0\left(\omega L - \frac{1}{\omega C}\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

When these are squared and added, the result is $A^2 = E_0^2$. Consequently, $A = E_0$, and

$$\cos\phi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \qquad \sin\phi = \frac{\omega L - \frac{1}{\omega C}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

Because $R > 0$, it follows that $\cos\phi > 0$, and this is consistent with the demand that $-\pi/2 < \phi < \pi/2$. We could use the equation in $\sin\phi$ to define ϕ, or we can also write that $\tan\phi = [\omega L - 1/(\omega C)]/R$ so that $\phi = \text{Tan}^{-1}\{[\omega L - 1/(\omega C)]/R\}$.

71. If $5\cos\omega t = A(\cos\omega t\cos\pi/6 + \sin\omega t\sin\pi/6) + 5(\cos\omega t\cos\phi - \sin\omega t\sin\phi)$, then

$$5 = \frac{\sqrt{3}A}{2} + 5\cos\phi \qquad \text{and} \qquad 0 = \frac{A}{2} - 5\sin\phi.$$

These imply that $(5\cos\phi)^2 + (5\sin\phi)^2 = (5 - \sqrt{3}A/2)^2 + (A/2)^2$, from which $25 = 25 - 5\sqrt{3}A + 3A^2/4 + A^2/4 \Longrightarrow A = 0$ or $A = 5\sqrt{3}$. Since A must be positive, we choose $A = 5\sqrt{3}$, in which case

$$\cos\phi = -\frac{1}{2} \qquad \text{and} \qquad \sin\phi = \frac{\sqrt{3}}{2}.$$

These require $\phi = 2\pi/3 + 2n\pi$, where n is an integer.

72. If $5\cos\omega t = A(\cos\omega t\cos 1 - \sin\omega t\sin 1) + 5(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$, then

$$5 = A\cos 1 + 5\sin\phi \qquad \text{and} \qquad 0 = -A\sin 1 + 5\cos\phi.$$

These imply that $(5\sin\phi)^2 + (5\cos\phi)^2 = (5 - A\cos 1)^2 + (A\sin 1)^2$, from which $25 = 25 - 10A\cos 1 + A^2 \Longrightarrow A = 0$ or $A = 10\cos 1$. Since A must be positive, we choose $A = 10\cos 1$, in which case

$$\sin\phi = 1 - 2\cos^2 1 = -\cos 2 \qquad \text{and} \qquad \cos\phi = 2\sin 1\cos 1 = \sin 2.$$

From the second of these, we may write $\cos\phi = \cos(\pi/2 - 2)$. We conclude that $\phi = \pm(\pi/2 - 2) + 2n\pi$, where n is an integer. But, $\sin(\pi/2 - 2 + 2n\pi) = \sin(\pi/2 - 2) = \cos 2$, which is not true. Hence, we must take $\phi = -(\pi/2 - 2) + 2n\pi = 2 + (4n-1)\pi/2$.

73. When $x \ge 1$, we set $y = \mathrm{Csc}^{-1}x$, in which case $0 < y \le \pi/2$. It follows that $x = \csc y$, and

$$\frac{1}{x} = \frac{1}{\csc y} = \sin y.$$

If we apply the inverse sine function to both sides of this equation, the result is

$$\mathrm{Sin}^{-1}\left(\frac{1}{x}\right) = \mathrm{Sin}^{-1}(\sin y) = y,$$

because y is in the principal value range of the inverse sine function. Hence, when $x \ge 1$,

$$\mathrm{Csc}^{-1}x = \mathrm{Sin}^{-1}\left(\frac{1}{x}\right).$$

When $x \le -1$, we again set $y = \mathrm{Csc}^{-1}x$, and obtain $\quad \mathrm{Sin}^{-1}\left(\dfrac{1}{x}\right) = \mathrm{Sin}^{-1}(\sin y)$.

But in this case the right side is not equal to y, because $-\pi < y \le -\pi/2$. To remedy this, we note that when $-\pi < y \le -\pi/2$, we may write $\sin y = \sin(-\pi - y)$. Since $-\pi - y$ is in the principal range for the inverse sine function $(-\pi/2 \le -\pi - y < 0)$, it follows that

$$\mathrm{Sin}^{-1}\left(\frac{1}{x}\right) = \mathrm{Sin}^{-1}(\sin y) = \mathrm{Sin}^{-1}[\sin(-\pi - y)] = -\pi - y = -\pi - \mathrm{Csc}^{-1}x.$$

Thus, $\mathrm{Csc}^{-1}x = -\pi - \mathrm{Sin}^{-1}(1/x)$.

74. When $0 \le x \le 1$, we set $y = \mathrm{Sin}^{-1}x$, in which case $0 \le y \le \pi/2$. It follows that $x = \sin y$, and because $\sin^2 y + \cos^2 y = 1$, we have

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

Since y is an angle in the first quadrant, its cosine must be nonnegative, and therefore $\cos y = \sqrt{1 - x^2}$. When we apply the inverse cosine function to both sides of this equation, we obtain $\mathrm{Cos}^{-1}(\cos y) = y = \mathrm{Cos}^{-1}\sqrt{1 - x^2}$. When $-1 \le x < 0$, we continue to set $y = \mathrm{Sin}^{-1}x$, and once again obtain $\cos y = \sqrt{1 - x^2}$, because $-\pi/2 \le y < 0$. Application of the inverse cosine function gives $\mathrm{Cos}^{-1}(\cos y) = \mathrm{Cos}^{-1}\sqrt{1 - x^2}$, but the left side is not equal to y because y is not in the principal value range of the inverse cosine function. This is easily adjusted by noting that with $-\pi/2 \le y < 0$, we have $\cos y = \cos(-y)$. Hence,

$$\mathrm{Cos}^{-1}(\cos y) = \mathrm{Cos}^{-1}[\cos(-y)] = -y = \mathrm{Cos}^{-1}\sqrt{1 - x^2};$$

that is, $y = -\mathrm{Cos}^{-1}\sqrt{1 - x^2}$.

75. When $x \ge 1$, we set $y = \mathrm{Sec}^{-1}x$, in which case $0 \le y < \pi/2$. It follows that $x = \sec y$, and

$$\frac{1}{x} = \frac{1}{\sec y} = \cos y.$$

If we apply the inverse cosine function to both sides of this equation, the result is

$$\text{Cos}^{-1}\left(\frac{1}{x}\right) = \text{Cos}^{-1}(\cos y) = y,$$

because y is in the principal value range of the inverse cosine function. Hence, when $x \geq 1$,

$$\text{Sec}^{-1}x = \text{Cos}^{-1}\left(\frac{1}{x}\right).$$

When $x \leq -1$, we again set $y = \text{Sec}^{-1}x$, and obtain $\text{Cos}^{-1}\left(\frac{1}{x}\right) = \text{Cos}^{-1}(\cos y)$.

But in this case the right side is not equal to y, because $-\pi \leq y < -\pi/2$. To remedy this, we note that when $-\pi \leq y < -\pi/2$, we may write $\cos y = \cos(-y)$. Since $-y$ is in the principal range for the inverse cosine function ($\pi/2 < -y \leq \pi$), it follows that

$$\text{Cos}^{-1}\left(\frac{1}{x}\right) = \text{Cos}^{-1}(\cos y) = \text{Cos}^{-1}[\cos(-y)] = -y = -\text{Sec}^{-1}x;$$

that is, $\text{Sec}^{-1}x = -\text{Cos}^{-1}(1/x)$.

76. When $x > 0$, we set $y = \text{Cot}^{-1}x$, in which case $0 < y < \pi/2$. It follows that $x = \cot y$, and

$$\frac{1}{x} = \frac{1}{\cot y} = \tan y.$$

If we apply the inverse tangent function to both sides of this equation, the result is

$$\text{Tan}^{-1}\left(\frac{1}{x}\right) = \text{Tan}^{-1}(\tan y) = y,$$

because y is in the principal value range of the inverse tangent function. Hence, when $x > 0$, we can say that $\text{Cot}^{-1}x = \text{Tan}^{-1}(1/x)$.

When $x < 0$, we again set $y = \text{Cot}^{-1}x$, and obtain

$$\text{Tan}^{-1}\left(\frac{1}{x}\right) = \text{Tan}^{-1}(\tan y).$$

But in this case the right side is not equal to y, because $\pi/2 < y < \pi$. To remedy this, we note that $\tan y = \tan(y - \pi)$, and when $\pi/2 < y < \pi$, $y - \pi$ is in the principal value range for the inverse tangent function. It follows that

$$\text{Tan}^{-1}\left(\frac{1}{x}\right) = \text{Tan}^{-1}(\tan y) = \text{Tan}^{-1}[\tan(y - \pi)] = y - \pi = \text{Cot}^{-1}x - \pi.$$

Thus, $\text{Cot}^{-1}x = \pi + \text{Tan}^{-1}(1/x)$.

77. When $x \geq 1$, we set $y = \text{Sec}^{-1}x$, in which case $0 \leq y < \pi/2$. It follows that $x = \sec y$, and because $1 + \tan^2 y = \sec^2 y$, we have $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$. Since y is an angle in the first quadrant, its tangent is positive, and therefore $\tan y = \sqrt{x^2 - 1}$. When we apply the inverse tangent function to both sides of this equation, we obtain $\text{Tan}^{-1}(\tan y) = y = \text{Tan}^{-1}\sqrt{x^2 - 1}$; that is, $\text{Sec}^{-1}x = \text{Tan}^{-1}\sqrt{x^2 - 1}$.

When $x \leq -1$, we again set $y = \text{Sec}^{-1}x$, and obtain $\tan y = \sqrt{x^2 - 1}$, because $-\pi \leq y < -\pi/2$. Application of the inverse tangent function gives $\text{Tan}^{-1}(\tan y) = \text{Tan}^{-1}\sqrt{x^2 - 1}$, but the left side is not equal to y because y is not in the principal value range of the inverse tangent function. This is easily adjusted by noting that $\tan y = \tan(\pi + y)$, and when $-\pi \leq y < -\pi/2$, $\pi + y$ is in the principal value range of the inverse tangent function. Hence $\text{Tan}^{-1}(\tan y) = \text{Tan}^{-1}[\tan(\pi + y)] = \pi + y = \text{Tan}^{-1}\sqrt{x^2 - 1}$; that is, $y = -\pi + \text{Tan}^{-1}\sqrt{x^2 - 1}$.

78. When $x \geq 1$, we set $y = \text{Csc}^{-1}x$, in which case $0 < y \leq \pi/2$. It follows that $x = \csc y$, and because $1 + \cot^2 y = \csc^2 y$, we have $\cot y = \pm\sqrt{\csc^2 y - 1} = \pm\sqrt{x^2 - 1}$. Since y is an angle in the first quadrant, its cotangent is positive, and therefore $\cot y = \sqrt{x^2 - 1}$. When we apply the inverse cotangent function to both sides of this equation, we obtain $\mathrm{Cot}^{-1}(\cot y) = y = \mathrm{Cot}^{-1}\sqrt{x^2-1}$; that is, $\mathrm{Csc}^{-1}x = \mathrm{Cot}^{-1}\sqrt{x^2-1}$.

When $x \le -1$, we again set $y = \mathrm{Csc}^{-1}x$, and obtain $\cot y = \sqrt{x^2-1}$, because $-\pi < y \le -\pi/2$. Application of the inverse cotangent function gives $\mathrm{Cot}^{-1}(\cot y) = \mathrm{Cot}^{-1}\sqrt{x^2-1}$, but the left side is not equal to y because y is not in the principal value range of the inverse cotangent function. This is easily adjusted by noting that $\cot y = \cot(\pi + y)$, and when $-\pi < y \le -\pi/2$, $\pi + y$ is in the principal value range of the inverse cotangent function. Hence $\mathrm{Cot}^{-1}(\cot y) = \mathrm{Cot}^{-1}[\cot(\pi+y)] = \pi + y = \mathrm{Cot}^{-1}\sqrt{x^2-1}$; that is, $y = -\pi + \mathrm{Cot}^{-1}\sqrt{x^2-1}$.

79. If we set $y = 2\,\mathrm{Tan}^{-1}\sqrt{\dfrac{1+x}{1-x}}$, then $\dfrac{1+x}{1-x} = \tan^2(y/2)$. When we solve this equation for x, the result is

$$x = \frac{\tan^2(y/2)-1}{\tan^2(y/2)+1} = \frac{\tan^2(y/2)-1}{\sec^2(y/2)} = \sin^2(y/2) - \cos^2(y/2) = -\cos y.$$

Because $\cos y = -x$, it follows that $y = \mathrm{Cos}^{-1}(-x) = \pi - \mathrm{Cos}^{-1}x$, and the proof is complete.

EXERCISES 1.9

1. If $\log_{10}(2+x) = -1$, then $2 + x = 10^{-1} \Longrightarrow x = -2 + 1/10 = -19/10$.

2. If $10^{3x} = 5$, then $3x = \log_{10} 5 \Longrightarrow x = (1/3)\log_{10} 5$.

3. If $\log_{10}(x^2 + 2x + 1) = 1$, then $(x+1)^2 = 10^1 \Longrightarrow x = -1 \pm \sqrt{10}$.

4. If $\ln(x^2 + 2x + 10) = 1$, then $x^2 + 2x + 10 = e^1 \Longrightarrow x = \left[-2 \pm \sqrt{4 - 4(10-e)}\right]/2$. These are not real.

5. If $10^{5-x^2} = 100 = 10^2$, then $5 - x^2 = 2 \Longrightarrow x = \pm\sqrt{3}$.

6. If $10^{1-x^2} = 100 = 10^2$, then $1 - x^2 = 2$. This equation does not have real solutions.

7. We write $1 = \log_{10}[(x-3)x]$, and take exponentials, $10 = (x-3)x \Longrightarrow 0 = x^2 - 3x - 10 = (x-5)(x+2) \Longrightarrow x = 5, -2$. Only $x = 5$ satisfies the original equation.

8. We write $1 = \log_{10}[(3-x)x]$, and take exponentials, $10 = (3-x)x \Longrightarrow 0 = x^2 - 3x + 10$. This equation has no real solutions.

9. If we take exponentials, $x(x-3) = 10 \Longrightarrow 0 = x^2 - 3x - 10 = (x-5)(x+2) \Longrightarrow x = 5, -2$.

10. We write $\log_{10}[x^2(x-1)] = 2 \Longrightarrow x^2(x-1) = 10^2 = 100 \Longrightarrow 0 = x^3 - x^2 - 100 = (x-5)(x^2 + 4x + 20) \Longrightarrow x = 5$.

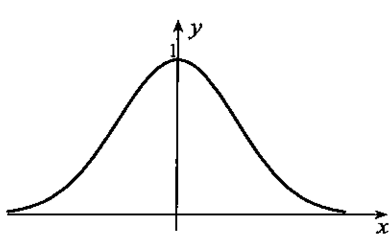
11. We write $\log_a[x(x+2)] = 2 \Longrightarrow x(x+2) = a^2 \Longrightarrow x^2 + 2x - a^2 = 0$. Solutions of this quadratic equation are $x = (-2 \pm \sqrt{4 + 4a^2})/2 = -1 \pm \sqrt{1 + a^2}$. Since x must be positive, only $x = -1 + \sqrt{1 + a^2}$ is acceptable.

12. We take exponentials to obtain $x(x+2) = a^2 \Longrightarrow x^2 + 2x - a^2 = 0$. Solutions of this quadratic equation are $x = (-2 \pm \sqrt{4 + 4a^2})/2 = -1 \pm \sqrt{1 + a^2}$.

13. Taking exponentials gives $\log_{10}\left(\dfrac{x+3}{200x}\right) + 4 = 10^{-1} = 1/10$. Taking exponentials again gives

$$\frac{x+3}{200x} = 10^{-39/10} \Longrightarrow x + 3 = 200(10^{-39/10})x \Longrightarrow x = \frac{3}{200(10^{-39/10}) - 1}.$$

14.



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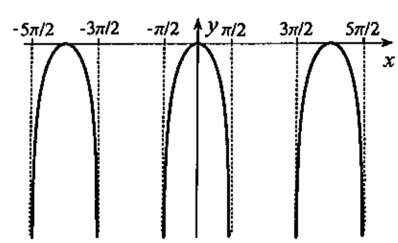

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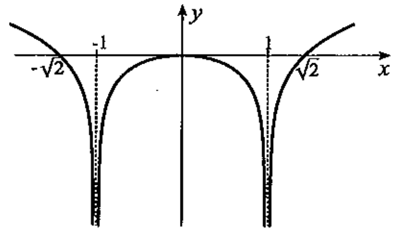

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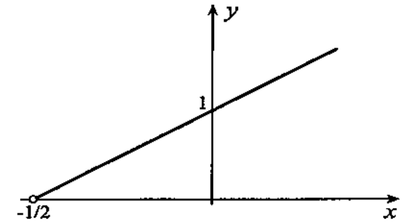

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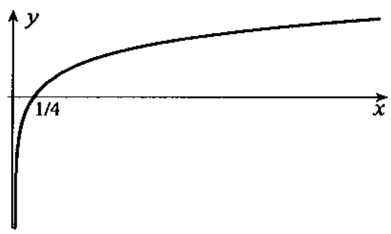

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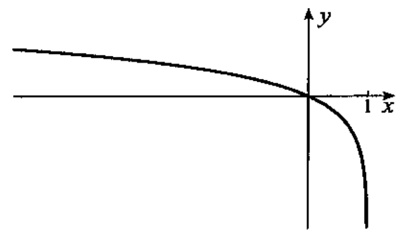

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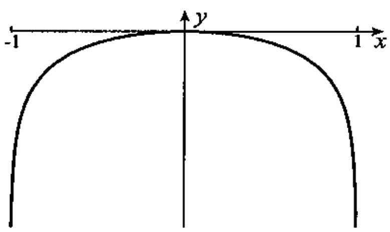

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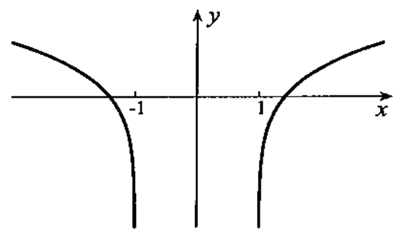

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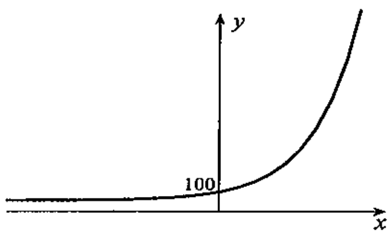

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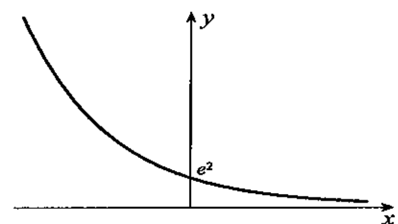

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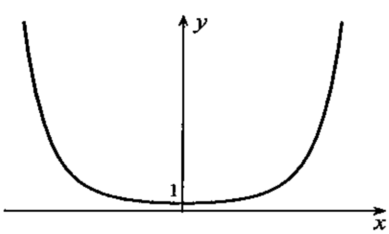

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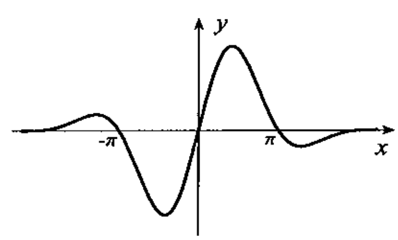

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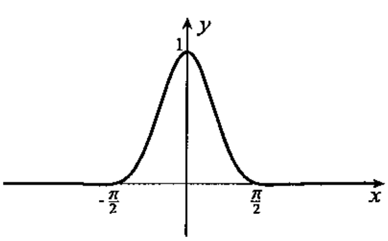

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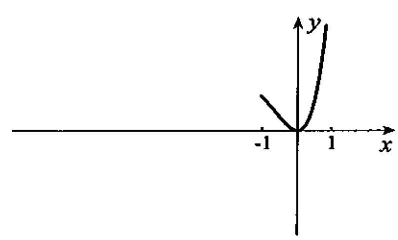
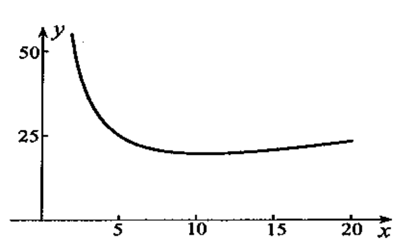


28. The graph shows that the minimum is between 10 and 11. Values $y(10) = 19.562$ and $y(11) = 19.557$ indicate that the minimum occurs for $x = 11$.



29. $f(-x)=\dfrac{1-e^{-1/(-x)}}{1+e^{-1/(-x)}}=\dfrac{1-e^{1/x}}{1+e^{1/x}}$. If we divide numerator and denominator by $e^{1/x}$, we obtain $f(-x)=\dfrac{e^{-1/x}-1}{e^{-1/x}+1}=-\left(\dfrac{1-e^{-1/x}}{1+e^{-1/x}}\right)=-f(x)$. Thus, $f(x)$ is an odd function.

30. The graph of $f(x)$ is symmetric about the y-axis, whereas the graph of $g(x)$ exists only for $x>0$. They are identical to the right of the y-axis.

31. (a) The amount after the first year is $A_0(1.035)$. The amount after the second year is $A_0(1.035)^2$. Continuation leads to the formula $A_0(1.035)^t$ for the amount of timber after t years.
(b) Timber doubles when $2A_0=A_0(1.035)^t$. If we divide by A_0 and take logarithms to any base, say 10, $\log_{10}2=t\log_{10}1.035 \implies t=\log_{10}2/\log_{10}1.035=20.1$ years.

32. After the first year its value is $20\,000(3/4)$. After two years, the value is $20\,000(3/4)^2$, and after t years, it is $20\,000(3/4)^t$.

33. If we take 56th roots of both sides of the equation,

$$10^{-6/56}=1-2.08\times10^{-6}y \implies y=\frac{1-10^{-3/28}}{2.08\times10^{-6}}=1.05\times10^5.$$

34. If y is the logarithm of x to base a, $y=\log_a x$, then $x=a^y$. It follows that $x=(1/a)^{-y}$, and this implies that $-y$ is the logarithm of x to base $1/a$.

35. If we set $z=\log_a(x_1/x_2)$, then $a^z=\dfrac{x_1}{x_2}=\dfrac{a^{\log_a x_1}}{a^{\log_a x_2}}=a^{\log_a x_1-\log_a x_2}$. Thus, $\log_a x_1-\log_a x_2=z=\log_a(x_1/x_2)$. If we set $z=\log_a x_1^{x_2}$, then $a^z=x_1^{x_2}=(a^{\log_a x_1})^{x_2}=a^{x_2\log_a x_1}$. Thus, $x_2\log_a x_1=z=\log_a x_1^{x_2}$.

36. No. Both x_1 and x_2 must be positive.

37. (a) If we exponentiate both sides of $R=\log_{10}(I/I_0)$, we obtain $I/I_0=10^R \implies I=I_0\,10^R$.
(b) Richter scale readings are $\log_{10}(1.20\times10^6)=6.08$ and $\log_{10}(6.20\times10^4)=4.79$.

38. (a) After one interest period the accumulated value is $P[1+i/(100n)]$. After two interest periods, it is $P[1+i/(100n)]^2$. Continuing, the accumulated value after t years, or nt interest periods is $P[1+i/(100n)]^{nt}$.
(b) When $A=2P$, $i=8$ and $n=2$,

$$2P=P\left(1+\frac{8}{200}\right)^{2t}=P\left(\frac{26}{25}\right)^{2t} \implies 2t\log_{10}(26/25)=\log_{10}2 \implies t=\frac{\log_{10}2}{2\log_{10}(26/25)}=8.84.$$

Thus, money doubles in 9 years.
(c) If we write that $A=P\left(1+\dfrac{i}{100n}\right)^{nt}=P\left[\left(1+\dfrac{i}{100n}\right)^{100n/i}\right]^{it/100}$, and note that $\left(1+\dfrac{i}{100n}\right)^{100n/i}$ gets closer and closer to e as n gets larger and larger, we conclude that as n gets larger and larger, A approaches $P\,e^{it/100}$.
(d) The accumulated value is $P=1000\,e^{6(10)/100}=1822.12$. For the accumulated value at 6% compounded only once each year, we obtain $1000(1.06)^{10}=1790.85$.

39. (a) If the voltage is V_0/e at time τ, then

$$\frac{V_0}{e}=V_0e^{-\tau/(RC)}.$$

Division by V_0 and logarithms give

$$-1=-\frac{\tau}{RC} \implies \tau=RC.$$

(b) The voltage at time $t+\tau$ is $V_0e^{-(t+\tau)/(RC)}=V_0e^{-t/(RC)}e^{-\tau/(RC)}=Ve^{-\tau/\tau}=V/e$.

40. (a) If the current is i_0/e at time τ, then

$$\frac{i_0}{e} = i_0 e^{-R\tau/L}.$$

Division by i_0 and logarithms give

$$-1 = -\frac{R\tau}{L} \implies \tau = \frac{L}{R}.$$

(b) The current at time $t+\tau$ is $i_0 e^{-R(t+\tau)/L} = i_0 e^{-Rt/L} e^{-R\tau/L} = ie^{-\tau/\tau} = i/e$.

41. If we multiply by a^{2x}, we obtain a quadratic equation in a^{2x}, namely, $0 = 3(a^{2x})^2 - 10(a^{2x}) + 3 = (a^{2x} - 3)(3a^{2x} - 1)$. Consequently, $a^{2x} = 3$ or $a^{2x} = 1/3$, and these give $x = \pm(1/2)\log_a 3$.

42. If we write $2^x + 2^{2x} = 2^{3x}$, and divide by 2^x, then

$$1 + 2^x = 2^{2x} \implies (2^x)^2 - 2^x - 1 = 0 \implies 2^x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since 2^x must be positive, we choose only $(1+\sqrt{5})/2$, and take logarithms,

$$x \log_{10} 2 = \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \implies x = \frac{\log_{10}\left(\frac{1+\sqrt{5}}{2}\right)}{\log_{10} 2} = 0.694.$$

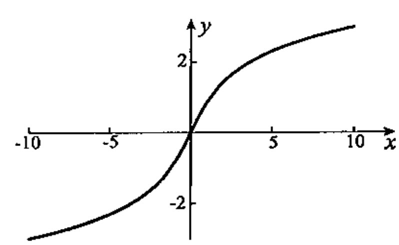
43. If we take logarithms to base 10,

$$(x+4)\log_{10} 3 = (x-1)\log_{10} 7 \implies x = \frac{4\log_{10} 3 + \log_{10} 7}{\log_{10} 7 - \log_{10} 3} = 7.48.$$

44. If we set $y = \log_x 2$, then $2 = x^y$. But, then $y = \log_{2x} 8$ also, and this implies that $8 = (2x)^y = 2^y x^y = 2^y(2) = 2^{y+1}$. Consequently, $y + 1 = 3$ or $y = 2$. Thus, $2 = x^2$, and since x must be positive, $x = \sqrt{2}$.

45. Since

$$\begin{aligned} f(-x) &= \ln\left(-x + \sqrt{(-x)^2+1}\right) \\ &= \ln\left[\left(\sqrt{x^2+1} - x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right] \\ &= \ln\left(\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right) \\ &= -\ln\left(\sqrt{x^2+1}+x\right) = -f(x), \end{aligned}$$



the function is odd. A plot is shown to the right.

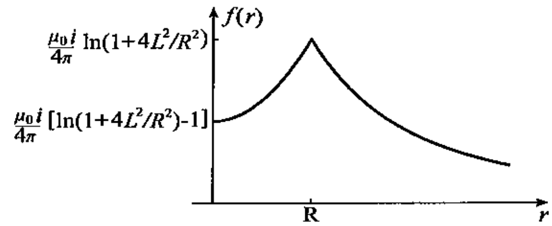
46. Repair costs for the second year are $50(1.2)$, for the third year, $50(1.2)^2$, and so on. If $R(t)$ represents repair costs for t years, then $R(t) = 50 + 50(1.2) + \cdots + 50(1.2)^{t-1}$. If we multiply this by 1.2, then $1.2R(t) = 50(1.2) + 50(1.2)^2 + \cdots + 50(1.2)^t$, and

$$1.2R(t) - R(t) = 50(1.2)^t - 50 \implies R(t) = \frac{50(1.2)^t - 50}{1.2 - 1} = 250[(1.2)^t - 1].$$

Thus, the average yearly cost asociated with owning the car for t years is

$$C(t) = \frac{1}{t}\left\{20\,000\left[1 - \left(\frac{3}{4}\right)^t\right] + 250\left[\left(\frac{6}{5}\right)^t - 1\right]\right\}.$$

47. (a) For $0 \le r \le R$, the graph is a parabola. For $r > R$, the graph decreases as r increases, and gets closer and closer to the r-axis.



(b) For $f(0) = f(r)$,
$$\frac{\mu_0 i}{4\pi}\left[\ln\left(1+\frac{4L^2}{R^2}\right)-1\right] = \frac{\mu_0 i}{4\pi}\ln\left(1+\frac{4L^2}{r^2}\right).$$
This implies that
$$\frac{1}{e}\left(1+\frac{4L^2}{R^2}\right) = 1+\frac{4L^2}{r^2} \implies \frac{4L^2}{r^2} = \frac{1}{e}\left(1+\frac{4L^2}{R^2}\right)-1 \implies r = \frac{2LR}{\sqrt{(R^2+4L^2)/e-R^2}}.$$

48. If we multiply by e^{2x},
$$(e^{2x})^2 - 2y(e^{2x}) - 1 = 0 \implies e^{2x} = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}.$$
Since e^{2x} must be positive, $e^{2x} = y + \sqrt{y^2+1} \implies x = (1/2)\ln(y+\sqrt{y^2+1})$.

49. If we multiply by e^x, $\quad (e^x)^2 - 2y(e^x) + 1 = 0 \implies e^x = \dfrac{2y \pm \sqrt{4y^2-4}}{2} = y \pm \sqrt{y^2-1}.$
Thus, $x = \ln(y \pm \sqrt{y^2-1})$.

50. If we cross multiply,
$$y(e^x+e^{-x}) = e^x - e^{-x} \implies e^x(1-y) = e^{-x}(1+y) \implies e^{2x} = \frac{1+y}{1-y} \implies x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right).$$

EXERCISES 1.10

1. $3\cosh 1 = \dfrac{3}{2}(e+e^{-1}) = 4.63$

2. $\sinh(\pi/2) = \dfrac{e^{\pi/2}-e^{-\pi/2}}{2} = 2.30$

3. $\tanh\sqrt{1-\sin 3} = \dfrac{e^{\sqrt{1-\sin 3}} - e^{-\sqrt{1-\sin 3}}}{e^{\sqrt{1-\sin 3}} + e^{-\sqrt{1-\sin 3}}} = 0.729$

4. $\mathrm{Sin}^{-1}(\mathrm{sech}10) = \mathrm{Sin}^{-1}\left(\dfrac{2}{e^{10}+e^{-10}}\right) = 9.08 \times 10^{-5}$

5. Since $2\,\mathrm{csch}1 = \dfrac{4}{e-e^{-1}} = 1.70 > 1$, there is no value for $\mathrm{Cos}^{-1}(2\,\mathrm{csch}1)$.

6. $\coth(\sinh 5) = \dfrac{e^{\sinh 5}+e^{-\sinh 5}}{e^{\sinh 5}-e^{-\sinh 5}} = 1.00$

7. $\sqrt{\ln|\sinh(-3)|} = \sqrt{\ln|(e^{-3}-e^3)/2|} = 1.52$

8. $\mathrm{sech}[\sec(\pi/3)] = \mathrm{sech}2 = \dfrac{2}{e^2+e^{-2}} = 0.266$

9. $e^{-2\cosh e} = e^{-(e^e+e^{-e})} = 2.45 \times 10^{-7}$

10. $\sinh[\mathrm{Cot}^{-1}(-3\pi/10)] = \sinh 2.3266 = \dfrac{e^{2.3266}-e^{-2.3266}}{2} = 5.07$

11. We verify representatives of these identities:
$$\tanh(A+B) = \frac{\sinh(A+B)}{\cosh(A+B)} = \frac{\sinh A\cosh B + \cosh A\sinh B}{\cosh A\cosh B + \sinh A\sinh B}$$
$$= \frac{\dfrac{\sinh A\cosh B}{\cosh A\cosh B} + \dfrac{\cosh A\sinh B}{\cosh A\cosh B}}{\dfrac{\cosh A\cosh B}{\cosh A\cosh B} + \dfrac{\sinh A\sinh B}{\cosh A\cosh B}} = \frac{\tanh A + \tanh B}{1+\tanh A\tanh B}$$
$$\sinh 2A = \frac{e^{2A}-e^{-2A}}{2} = 2\left(\frac{e^A-e^{-A}}{2}\right)\left(\frac{e^A+e^{-A}}{2}\right) = 2\sinh A\cosh A$$

$$\cosh 2A = \frac{e^{2A}+e^{-2A}}{2} = \left(\frac{e^A+e^{-A}}{2}\right)^2 + \left(\frac{e^A-e^{-A}}{2}\right)^2 = \cosh^2 A + \sinh^2 A$$

Adding the two equations in 1.77a gives $\sinh(A+B)+\sinh(A-B) = 2\sinh A\cosh B$, and this is 1.77j. If we set $X = A+B$ and $Y = A-B$, then $A = (X+Y)/2$ and $B = (X-Y)/2$. Substitution of these into 1.77j gives

$$\sinh\left(\frac{X+Y}{2}\right)\cosh\left(\frac{X-Y}{2}\right) = \frac{1}{2}\sinh X + \frac{1}{2}\sinh Y.$$

This is 1.77l with X and Y replacing A and B.

12. If we write the equations in the form

$$A(\cos kL - \cosh kL) = -B(\sin kL - \sinh kL), \quad A(\cos kL + \cosh kL) = -B(\sin kL + \sinh kL),$$

and divide one by the other,

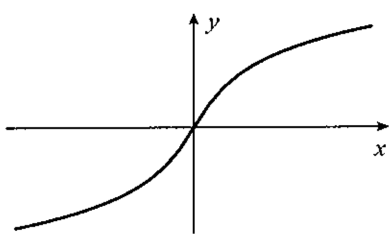
$$\frac{\cos kL - \cosh kL}{\cos kL + \cosh kL} = \frac{\sin kL - \sinh kL}{\sin kL + \sinh kL}.$$

Hence,

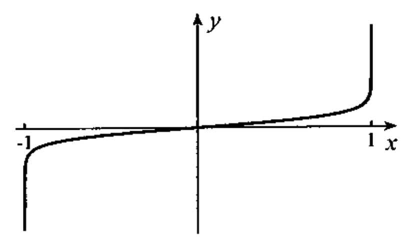
$$(\cos kL - \cosh kL)(\sin kL + \sinh kL) = (\cos kL + \cosh kL)(\sin kL - \sinh kL)$$

or, $2\cos kL\sinh kL = 2\sin kL\cosh kL$. Division by $2\cos kL\cosh kL$ gives $\tanh kL = \tan kL$.

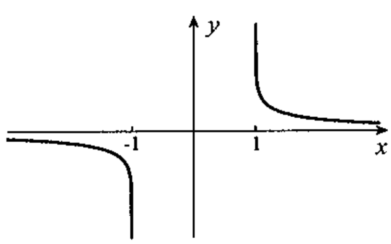
13. (a) $f(x) = \mathrm{Sinh}^{-1}x$



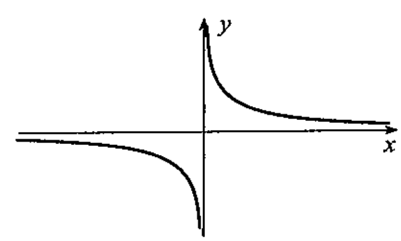
$f(x) = \mathrm{Tanh}^{-1}x$



$f(x) = \mathrm{Coth}^{-1}x$

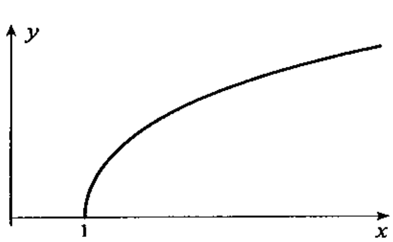


$f(x) = \mathrm{Csch}^{-1}x$

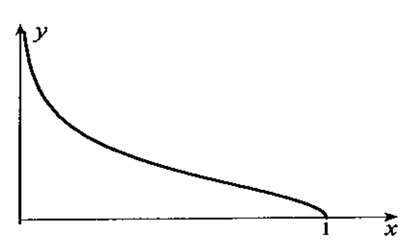


(b) These curves do not pass the horizontal line test for existence of inverse functions.

$f(x) = \mathrm{Cosh}^{-1}x$



$f(x) = \mathrm{Sech}^{-1}x$



(c) If we set $y = \mathrm{Sinh}^{-1}x$, then $x = \sinh y = (e^y - e^{-y})/2$. Multiplication by e^y gives

$$(e^y)^2 - 2x(e^y) - 1 = 0 \quad \Longrightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Since e^y must be positive, we must take $e^y = x + \sqrt{x^2+1} \Longrightarrow y = \ln\left(x + \sqrt{x^2+1}\right)$. Derivations of the other two results are similar.

14. (a) Since $\ln\left(\dfrac{\sqrt{mg/\beta} - v}{\sqrt{mg/\beta} + v}\right) = -2\sqrt{\dfrac{mg}{\beta}}\left(\dfrac{\beta t}{m}\right) = -2\sqrt{\dfrac{\beta g}{m}}t$, exponentiation gives $\dfrac{\sqrt{mg/\beta} - v}{\sqrt{mg/\beta} + v} = e^{-2\sqrt{\beta g/m}t}$, and therefore $\sqrt{mg/\beta} - v = (\sqrt{mg/\beta} + v)e^{-2\sqrt{\beta g/m}t}$. When we solve this equation for v, the result is

$$v = \frac{\sqrt{mg/\beta} - \sqrt{mg/\beta}e^{-2\sqrt{\beta g/m}t}}{1 + e^{-2\sqrt{\beta g/m}t}} = \sqrt{\frac{mg}{\beta}}\left(\frac{1 - e^{-2\sqrt{\beta g/m}t}}{1 + e^{-2\sqrt{\beta g/m}t}}\right).$$

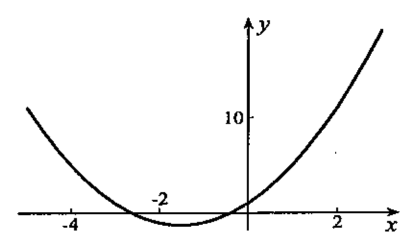
Multiplication of numerator and denominator by $e^{\sqrt{\beta g/m}t}$ gives

$$v = \sqrt{\frac{mg}{\beta}}\left(\frac{e^{\sqrt{\beta g/m}t} - e^{-\sqrt{\beta g/m}t}}{e^{\sqrt{\beta g/m}t} + e^{-\sqrt{\beta g/m}t}}\right) = \sqrt{\frac{mg}{\beta}}\tanh\left(\sqrt{\frac{\beta g}{m}}t\right).$$

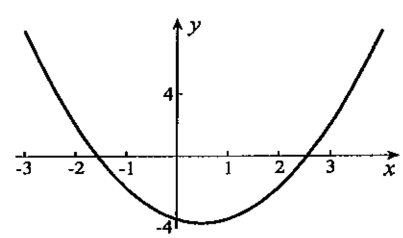
(b) Since the hyperbolic tangent function gets closer and closer to 1 as its argument gets large, it follows that the limiting velocity is $\sqrt{mg/\beta}$.

EXERCISES 1.11

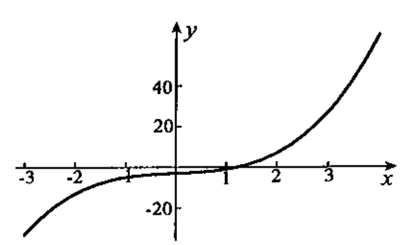
1. The plot shows two roots. My electronic device gives $-2.618\,034\,0$ and $-0.381\,966\,0$ for roots of $f(x) = x^2 + 3x + 1 = 0$. To verify that $-2.618\,034$ is accurate to six decimals, we calculate $f(-2.618\,033\,5) = -1.1 \times 10^{-6}$ and $f(-2.618\,034\,5) = 1.1 \times 10^{-6}$. A similar calculation verifies the accuracy of $-0.381\,966$.



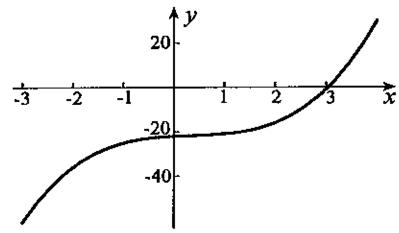
2. The plot shows two roots. My electronic device gives $-1.561\,552\,8$ and $2.561\,552\,8$ for roots of $f(x) = x^2 - x - 4 = 0$. To verify that $2.561\,553$ is accurate to six decimals, we calculate $f(2.561\,552\,5) = -1.3 \times 10^{-6}$ and $f(2.561\,553\,5) = 2.8 \times 10^{-6}$. A similar calculation verifies the accuracy of $-1.561\,553$.



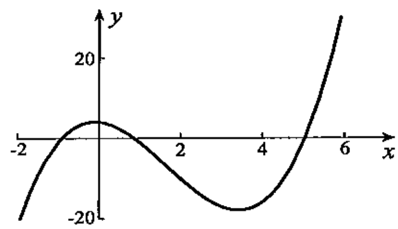
3. The plot shows one root. My electronic device gives $1.213\,411\,7$ for the root of $f(x) = x^3 + x - 3 = 0$. To verify that $1.213\,412$ is accurate to six decimals, we calculate $f(1.213\,411\,5) = -8.8 \times 10^{-7}$ and $f(1.213\,412\,5) = 4.5 \times 10^{-6}$.



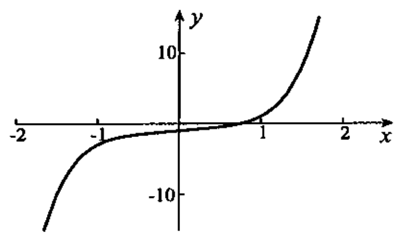
4. The plot shows one root. My electronic device gives 3.044 723 1 for the root of $f(x) = x^3 - x^2 + x - 22 = 0$. To verify that 3.044 723 is accurate to six decimals, we calculate $f(3.044\,722\,5) = -1.5 \times 10^{-5}$ and $f(3.044\,723\,5) = 8.0 \times 10^{-6}$.



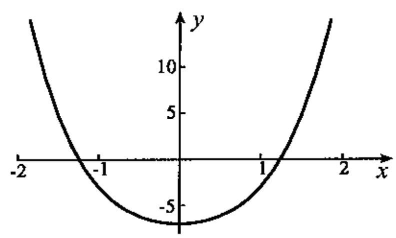
5. The plot shows three roots. My electronic device gives $-0.911\,503\,3$, 0.870 538 7 and 5.040 964 6 for the roots of $f(x) = x^3 - 5x^2 - x + 4 = 0$. To verify that $-0.911\,503$ is accurate to six decimals, we calculate $f(-0.911\,503\,5) = -2.0 \times 10^{-6}$ and $f(-0.911\,502\,5) = 8.6 \times 10^{-6}$. The other roots are 0.870 539 and 5.040 965.



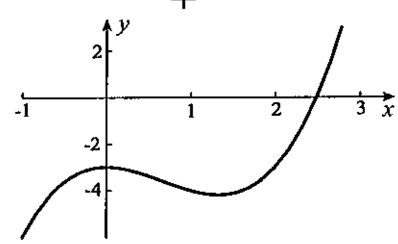
6. The plot shows one root. My electronic device gives 0.754 877 7 for the root of $f(x) = x^5 + x - 1 = 0$. To verify that 0.754 878 is accurate to six decimals, we calculate $f(0.754\,877\,5) = -4.4 \times 10^{-7}$ and $f(0.754\,878\,5) = 2.2 \times 10^{-6}$.



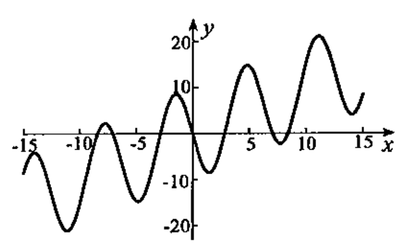
7. The plot shows two roots. My electronic device gives $\pm 1.241\,523\,8$ for the roots of $f(x) = x^4 + 3x^2 - 7 = 0$. To verify that 1.241 524 is accurate to six decimals, we calculate $f(1.241\,523\,5) = -4.0 \times 10^{-6}$ and $f(1.241\,524\,5) = 1.1 \times 10^{-5}$. Symmetry verifies the other root.



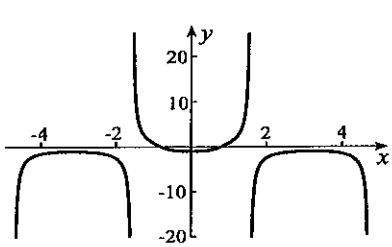
8. We rewrite the equation in the form $f(x) = x^3 - 2x^2 - 3 = 0$, the graph of which is shown to the right. My electronic calculator gives the root 2.485 584 0. To verify that 2.485 584 is accurate to six decimals, we calculate $f(2.485\,583\,5) = -4.3 \times 10^{-6}$ and $f(2.485\,584\,5) = 4.3 \times 10^{-6}$.



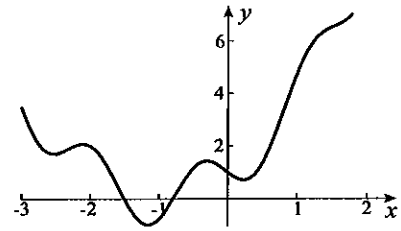
9. The plot shows seven roots, one of which is $x = 0$. My electronic device gives the other six as $\pm 2.852\,341\,9$, $\pm 7.068\,174\,4$, and $\pm 8.423\,204\,0$. To verify that 2.852 342 is a root of $f(x) = x - 10 \sin x$ accurate to six decimals, we evaluate $f(2.852\,341\,5) = -4.2 \times 10^{-6}$ and $f(2.852\,342\,5) = 6.4 \times 10^{-6}$. Verification that $\pm 7.068\,174$ and $\pm 8.423\,204$ are roots accurate to six decimals is similar.



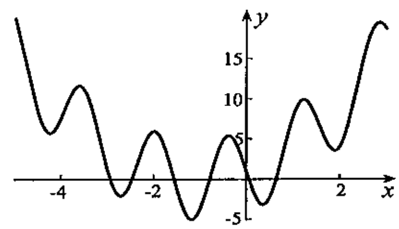
10. The plot shows two roots. My electronic device gives $\pm 0.795\,323\,9$ for the roots of $f(x) = \sec x - 2/(1 + x^4) = 0$. To verify that 0.795 324 is accurate to six decimals, we calculate $f(0.795\,323\,5) = -1.4 \times 10^{-6}$ and $f(0.795\,324\,5) = 2.1 \times 10^{-6}$.



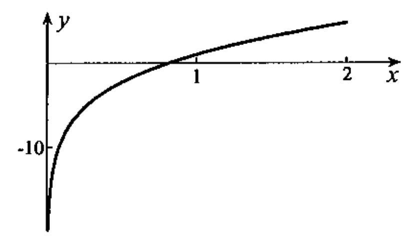
11. The plot shows two roots. My electronic device gives $-1.506\,052\,7$ and $-0.795\,823\,2$ for the roots of $f(x) = (x+1)^2 - \sin 4x = 0$. To verify that $-1.506\,053$ is accurate to six decimals, we calculate $f(-1.506\,052\,5) = -1.0 \times 10^{-6}$ and $f(-1.506\,053\,5) = 3.8 \times 10^{-6}$. Verification that $-0.795\,823$ is accurate to six decimals is similar.



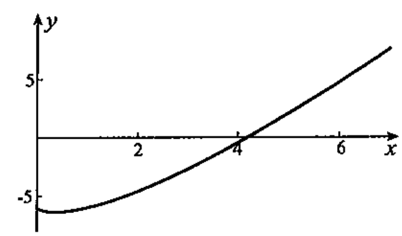
12. The plot shows six roots. My electronic device gives $-2.931\,137\,1$, $-2.467\,517\,5$, $-1.555\,365\,0$, $-0.787\,652\,8$, $0.056\,257\,6$ and $0.642\,850\,7$ for the roots of $f(x) = (x+1)^2 - 5\sin 4x = 0$. To verify that $-2.931\,137$ is accurate to six decimals, we calculate $f(-2.931\,136\,5) = -1.1 \times 10^{-5}$ and $f(-2.931\,137\,5) = 6.7 \times 10^{-6}$. Verification that $-2.467\,518$, $-1.555\,365$, $-0.787\,653$, $0.056\,258$, and $0.642\,851$ are accurate to six decimals is similar.



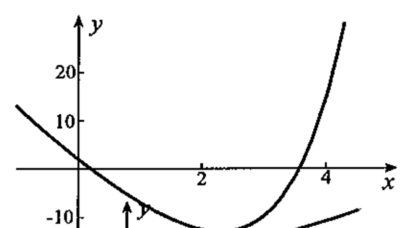
13. The plot shows one root. My electronic device gives $0.815\,553\,4$ for the root of $f(x) = x + 4\ln x = 0$. To verify that $0.815\,553$ is accurate to six decimals, we calculate $f(0.815\,552\,5) = -5.4 \times 10^{-6}$ and $f(0.815\,553\,5) = 4.8 \times 10^{-7}$.



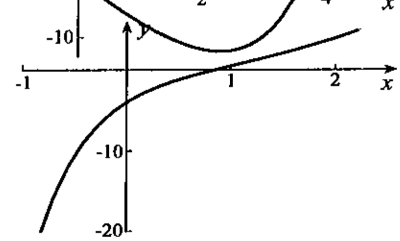
14. The plot shows one root. My electronic device gives $4.188\,760\,1$ for the root of $f(x) = x\ln x - 6 = 0$. To verify that $4.188\,760$ is accurate to six decimals, we calculate $f(4.188\,759\,5) = -1.5 \times 10^{-6}$ and $f(4.188\,760\,5) = 9.3 \times 10^{-7}$.



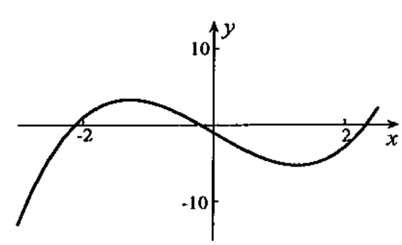
15. The plot shows two roots. My electronic device gives $0.204\,183\,6$ and $3.576\,065\,3$ for the roots of $f(x) = e^x + e^{-x} - 10x = 0$. To verify that $0.204\,184$ is accurate to six decimals, we calculate $f(0.204\,183\,5) = 9.5 \times 10^{-7}$ and $f(0.204\,184\,5) = -8.6 \times 10^{-6}$. The other root to six decimals is $3.576\,065$.



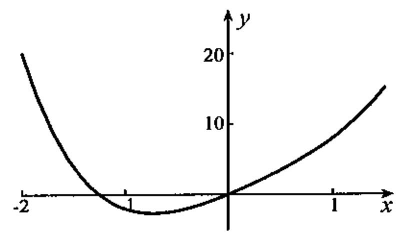
16. The plot shows one root. My electronic device gives $0.852\,605\,5$ for the root of $f(x) = x^2 - 4e^{-2x} = 0$. To verify that $0.852\,606$ is accurate to six decimals, we calculate $f(0.852\,605\,5) = -6.4 \times 10^{-9}$ and $f(0.852\,606\,5) = 3.2 \times 10^{-6}$.



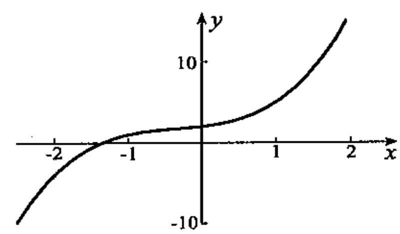
17. The plot shows three roots. My electronic device gives $-2.128\,4$, $-0.201\,6$, and $2.330\,1$ for the roots of $f(x) = x^3 - 5x - 1 = 0$. To verify that 2.330 has error no greater than 10^{-3}, we calculate $f(2.329) = -1.2 \times 10^{-2}$ and $f(2.331) = 1.1 \times 10^{-2}$. Verification that -2.128 and -0.202 have the same accuracy is similar.



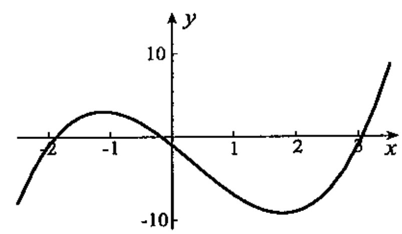
18. The plot shows two roots, one of which is $x = 0$. My electronic device gives $-1.248\,30$ for the other root of $f(x) = x^4 - x^3 + 2x^2 + 6x = 0$. To verify that $-1.248\,3$ has error no greater than 10^{-4}, we calculate $f(-1.248\,2) = -1.1 \times 10^{-3}$ and $f(-1.248\,4) = 1.2 \times 10^{-3}$.



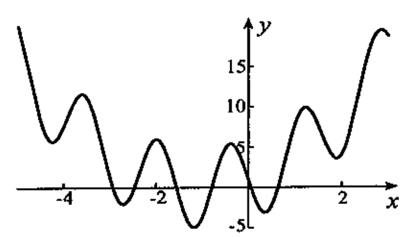
19. We rewrite the equation in the form $f(x) = x^3 + x^2 + x + 2 = 0$. The plot shows one root. My electronic device gives $-1.353\,210$ for the root. To verify that $-1.353\,21$ has error no greater than 10^{-5}, we calculate $f(-1.353\,20) = 3.8 \times 10^{-5}$ and $f(-1.353\,22) = -3.8 \times 10^{-5}$.



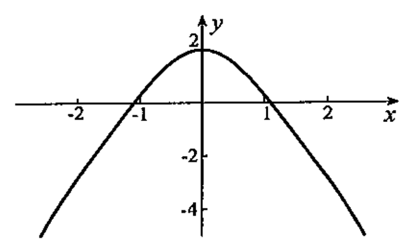
20. We rewrite the equation in the form $f(x) = x^3 - x^2 - 6x - 1 = 0$. The plot shows three roots. My electronic device gives $-1.892\,0$, $-0.172\,5$, and $3.064\,4$ for the roots. To verify that -1.892 has error no greater than 10^{-3}, we calculate $f(-1.893) = -8.9 \times 10^{-3}$ and $f(-1.891) = 8.1 \times 10^{-3}$. Verification that -0.172 and 3.064 also have the required accuracy is similar.



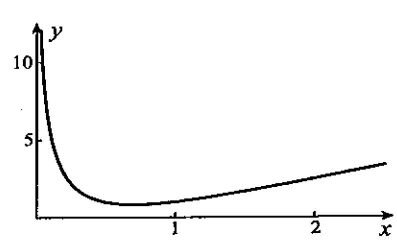
21. The plot shows six roots. My electronic device gives $-2.931\,1$, $-2.467\,5$, $-1.555\,4$, $-0.787\,7$, $0.056\,3$ and $0.642\,9$ for the roots of $f(x) = (x+1)^2 - 5\sin 4x = 0$. To verify that -2.931 has error no greater than 10^{-3}, we calculate $f(-2.932) = 1.5 \times 10^{-2}$ and $f(-2.930) = -2.0 \times 10^{-2}$. Verification that -2.468, -1.555, -0.788, 0.056, and 0.643 have error no greater than 10^{-3} is similar.



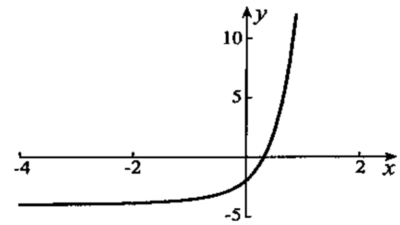
22. The plot shows two roots. My electronic device gives $\pm 1.098\,59$ for the roots of $f(x) = \cos^2 x - x^2 + 1 = 0$. To verify that $1.098\,6$ has error no greater than 10^{-4}, we calculate $f(1.098\,5) = 2.6 \times 10^{-4}$ and $f(1.098\,7) = -3.4 \times 10^{-4}$.



23. The plot shows no solutions.



24. The plot shows one root. My electronic device gives 0.321 21 for the root of $f(x) = e^{3x} + e^x - 4 = 0$. To verify that 0.321 2 has error no greater than 10^{-4}, we calculate $f(0.321\,1) = -1.0 \times 10^{-3}$ and $f(0.321\,3) = 8.2 \times 10^{-4}$.



25. To find x-coordinates of the points of intersection, we set $x^3 = x + 5$. My electronic device gives $x = 1.904\,161$ as the only solution of $f(x) = x^3 - x - 5 = 0$. The values $f(1.904\,160\,5) = -3.5 \times 10^{-6}$ and $f(1.904\,161\,5) = 6.3 \times 10^{-6}$ confirm six-decimal accuracy. Both equations $y = x^3$ and $y = x + 5$ give the same four decimals $y = 6.9042$. The point of intersection is therefore $(1.9042, 6.9042)$.

26. To find x-coordinates of the points of intersection, we set $(x+1)^2 = x^3 - 4x$. My electronic device gives $x = -1.891\,954$, -0.172480, and $3.064\,435$ as solutions of $f(x) = x^3 - 4x - (x+1)^2 = 0$. The values $f(-1.891\,954\,5) = -5.1 \times 10^{-7}$ and $f(-1.891\,953\,5) = 8.0 \times 10^{-6}$ confirm six-decimal accuracy of the first. Both equations $y = x^3 - 4x$ and $y = (x+1)^2$ give the same four decimals $y = 0.7956$. A point of intersection is therefore $(-1.8920, 0.7956)$. Similar procedures lead to the other points of intersection, $(-0.1725, 0.6848)$ and $(3.0644, 16.1596)$.

27. To find x-coordinates of the points of intersection, we set $x^4 - 20 = x^3 - 2x^2$. My electronic device gives $x = -1.726\,688$, and $2.130\,189$ as solutions of $f(x) = x^4 - x^3 + 2x^2 - 20 = 0$. The values $f(2.130\,188\,5) = -7.8 \times 10^{-6}$ and $f(2.130\,189\,5) = 2.6 \times 10^{-5}$ confirm six-decimal accuracy of the second. Both equations $y = x^4 - 20$ and $y = x^3 - 2x^2$ give the same four decimals $y = 0.5908$. A point of intersection is therefore $(2.1302, 0.5908)$. A similar procedure leads to the other point of intersection $(-1.7267, -11.1109)$.

28. To find x-coordinates of the points of intersection, we set $x/(x+1) = x^2 + 2$. My electronic device gives $x = -1.353\,210$ as the only solution of $f(x) = x^3 + x^2 + x + 2 = 0$. The values $f(-1.353\,210\,5) = -2.0 \times 10^{-6}$ and $f(-1.353\,209\,5) = 1.8 \times 10^{-6}$ confirm six-decimal accuracy. Both equations $y = x/(x+1)$ and $y = x^2+2$ give the same four decimals $y = 3.8312$. The point of intersection is therefore $(-1.3532, 3.8312)$.

29. My electronic device gives $x = 3.926\,602$ as the smallest positive solution of $f(x) = \tan x - (e^x - e^{-x})/(e^x + e^{-x}) = 0$. The values $f(3.926\,601\,5) = -1.6 \times 10^{-6}$ and $f(3.926\,602\,5) = 3.8 \times 10^{-7}$ confirm six-decimal accuracy. When divided by 20π, the smallest frequency is 0.0625. A similar procedure for the second frequency gives 0.1125.

30. (a) My electronic device gives $t = 3.833$ as the solution of $f(t) = 1181(1 - e^{-t/10}) - 98.1t = 0$. Since $y(3.825) = 0.14$ and $y(3.835) = -0.03$, it follows that to 2 decimals $t = 3.83$ s.

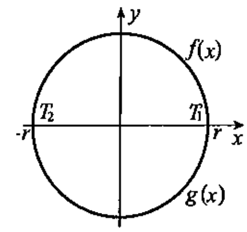
(b) When we set $0 = y = 20t - 4.905t^2$, the positive solution is 4.08 s.

31. My electronic device gives $z = 0.012\,957$ as the solution of $f(z) = 2Pz - e^{Lz} + e^{-Lz} = 0$ when $P = 80$ and $L = 70$. Since $f(0.012\,956\,5) = 2.3 \times 10^{-5}$ and $f(0.0129\,957\,5) = -1.9 \times 10^{-5}$, we can say that the solution is $z = 0.012\,957$ to six decimals. This gives $T = \rho g/(2z) = 189.3$.

32. To simplify calculations, we set $z = c/\lambda$. Then, z must satisfy the equation $f(z) = (5 - z)e^z - 5 = 0$. My electronic device gives $z = 4.965\,114\,232$. With this approximation for z, we obtain $\lambda = c/z = 0.000\,028\,974$. For a seven decimal answer, we use $g(\lambda) = (5\lambda - c)e^{c/\lambda} - 5\lambda$ to calculate $g(0.000\,028\,95) = -1.7 \times 10^{-5}$ and $g(0.000\,029\,05) = 5.1 \times 10^{-5}$. Thus, to 7 decimals, $\lambda = 0.000\,029\,0$.

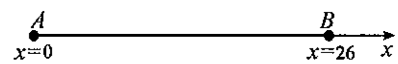
33. Consider the function $g(x) = f(x) - x$. Since the range of $f(x)$ is $a \le x \le b$, it follows that $f(a) \ge a$ and $f(b) \le b$. Consequently, $g(a) = f(a) - a \ge 0$ and $g(b) = f(b) - b \le 0$. If $f(a) = a$, then $x = a$ is a solution of $f(x) = x$. If $f(b) = b$, then $x = b$ is a solution. When $f(a) \ne a$ and $f(b) \ne b$, the Zero Intermediate Value Theorem implies that there is at least one solution of $g(x) = 0$ in the interval $a < x < b$. This gives a solution of $f(x) = x$.

34. Suppose we let T_1 and T_2 be the temperatures at the points $(r, 0)$ and $(-r, 0)$. Then, $F(r) = f(r) - g(-r) = T_1 - T_2$ and $F(-r) = f(-r) - g(r) = T_2 - T_1$. If $T_1 = T_2$, then temperatures are the same at the points $(r, 0)$ and $(-r, 0)$. Otherwise, values of $F(x)$ have opposite signs at $x = r$ and $x = -r$.



This implies that there is a value of x between $-r$ and r at which $F(x) = 0$. At this value, $f(x) = g(-x)$, and these give equal temperatures at points opposite each other on the equator.

35. Let $f_1(t)$ be the position of the runner on Saturday at any time t on the course taking time $t = 0$ at 7:00 a.m. Choose $x = 0$ at A and $x = 26$ at B, so that $f_1(0) = 0$ and $f_1(T_1) = 26$, where T_1 is the time to finish the marathon on Saturday. Similarly, let $f_2(t)$ be the position of the runner on Sunday with $f_2(0) = 26$ and $f_2(T_2) = 0$, where T_2 is her finish time on Sunday. Consider the function $f(t) = f_1(t) - f_2(t)$. If T is the smaller of T_1 and T_2, then, $f(0) = f_1(0) - f_2(0) = -26$ and $f(T) = f_1(T) - f_2(T) > 0$. Consequently, there is a value of t between $t = 0$ and $t = T$ at which $f(t) = 0$, and at this time $f_1(t) = f_2(t)$; that is, the runner is at this position at the same times on the two days.



36. (a) Suppose $f(x)$ is continuous on an interval I. Let c and d, where $d > c$, be any two points in the range of $f(x)$ and e be any number between c and d. There exist values a and b in I such that $d = f(b)$ and $c = f(a)$. If we define a function $F(x) = f(x) - e$, then

$$F(a)F(b) = [f(a) - e][f(b) - e] = (c - e)(d - e) < 0.$$

Since $F(x)$ is continuous, the zero intermediate value theorem guarantees the existence of a number z between a and b for which $0 = F(z) = f(z) - e$. This means that e is in the range of $f(x)$. Hence, the range of $f(x)$ is an interval.

(b) Not necessarily. For example, $f(x) = 1$ maps every interval on the x-axis onto a single point.

REVIEW EXERCISES

1. Possible rational solutions are ± 1, ± 2, ± 4. We find that $x = 2$ is a solution. We factor $x - 2$ from the cubic,

$$x^3 - x^2 - 4 = (x - 2)(x^2 + x + 2).$$

Since the discriminant of the quadratic is negative, the only real solution is $x = 2$.

2. Possible rational solutions are ± 1, ± 3, ± 9, ± 27, $\pm 1/2$, $\pm 3/2$, $\pm 9/2$, $\pm 27/2$. We find that $x = 3$ is a solution. We factor $x - 3$ from the cubic,

$$2x^3 - 9x^2 + 27 = (x - 3)(2x^2 - 3x - 9) = (x - 3)(2x + 3)(x - 3) = (x - 3)^2(2x + 3).$$

Solutions are $x = 3$ with multiplicity 2 and $x = -3/2$.

3. Possible rational solutions are ± 1, ± 5, $\pm 1/2$, $\pm 5/2$. We find that $x = 1$ is a solution. We factor $x - 1$ from the quartic,

$$2x^4 - x^3 - 9x^2 + 13x - 5 = (x - 1)(2x^3 + x^2 - 8x + 5).$$

Possible rational zeros of the cubic are the same. We find that $x = 1$ is a zero, and factor $x - 1$ from the cubic,

$$2x^4 - x^3 - 9x^2 + 13x - 5 = (x - 1)^2(2x^2 + 3x - 5) = (x - 1)^2(x - 1)(2x + 5) = (x - 1)^3(2x + 5).$$

Solutions are $x = 1$ with multiplicity 3 and $x = -5/2$.

4. The list of possible rational solutions here is formidable. Perhaps we can be a little ingenious. The fact that $36x^4$ and 225 are perfect squares suggests investigating whether the poynomial is the square of a quadratic expression. A little experimentation reveals that

$$36x^4 + 12x^3 - 179x^2 - 30x + 225 = (6x^2 + x - 15)^2 = (3x + 5)^2(2x - 3)^2.$$

Solutions are therefore $x = -5/3$ and $x = 3/2$ each of multiplicity 2.

5. The distance between the points is $\sqrt{(4+1)^2 + (2-3)^2} = \sqrt{26}$. The midpoint of the line segment is $\big((4-1)/2, (3+2)/2\big) = (3/2, 5/2)$.

6. The distance between the points is $\sqrt{(2+3)^2 + (1+4)^2} = 5\sqrt{2}$. The midpoint of the line segment is $\big((2-3)/2, (1-4)/2\big) = (-1/2, -3/2)$.

7. Since the slope of the line is $1/2$, its equation is $y - 3 = (1/2)(x - 2)$ or $2y = x + 4$.

8. Since the slope of the line joining $(-2, 1)$ and the origin is $-1/2$, and the midpoint of the line segment joining $(1, 3)$ and $(-1, 5)$ is $(0, 4)$, the equation of the required line is $y - 4 = 2(x - 0)$, or, $y = 2x + 4$.

9. If we set $4y - 11 = \sqrt{y^2 + 9}$, we see that the solution is $y = 4$. The point of intersection is $(5, 4)$. Since the slope of the required line is -4, its equation is $y - 4 = -4(x - 5)$ or $4x + y = 24$.

10. If we substitute $y = x^2$ into the second equation, $5x = 6 - x^4$, or $x^4 + 5x - 6 = 0$. Possible rational solutions are ± 1, ± 2, ± 3, ± 6. We find that $x = 1$ is a solution and factor $x - 1$ from the quartic

$$x^4 + 5x - 6 = (x - 1)(x^3 + x^2 + x + 6) = 0.$$

We now see that $x = -2$ is a zero of the cubic so that

$$x^4 + 5x - 6 = (x - 1)(x + 2)(x^2 - x + 3) = 0.$$

The curves therefore intersect at the points $(-2, 4)$ and $(1, 1)$. The equation of the line joining these points is $y - 1 = -(x - 1) \Longrightarrow x + y = 2$.

11. This function is defined for all reals.

12. For $x^2 - 5$ to be nonnegative, $|x| \geq \sqrt{5}$.

13. Since $x^2 + 3x + 2 = (x + 1)(x + 2)$, the function is defined for all x except $x = -1$ and $x = -2$.

14. Since $x^3 + 2x^2 + x = x(x + 1)^2$, we cannot set $x = 0$ or $x = -1$.

15. This function is defined for all reals.

16. $x \geq 0$

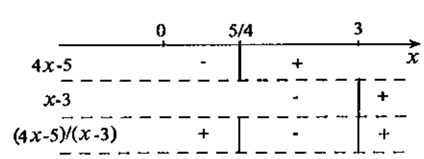
17. Since $x^2 + 4x - 6 = 0$ when $x = \big(-4 \pm \sqrt{16 + 24}\big)/2 = -2 \pm \sqrt{10}$, and is negative between these values, we must restrict x to the intervals $x \leq -2 - \sqrt{10}$ and $x \geq -2 + \sqrt{10}$. These can be combined into $|x + 2| \geq \sqrt{10}$.

18. Since $2x^2 + 4x - 5 = 0$ when $x = \big(-4 \pm \sqrt{16 + 40}\big)/4 = -1 \pm \sqrt{14}/2$, and is negative between these values, we must restrict x to the intervals $x < -1 - \sqrt{14}/2$ and $x > -1 + \sqrt{14}/2$. These can be combined into $|x + 1| > \sqrt{14}/2$.

19. If we express the function in the form

$$f(x) = \sqrt{\frac{4x - 5}{x - 3}},$$

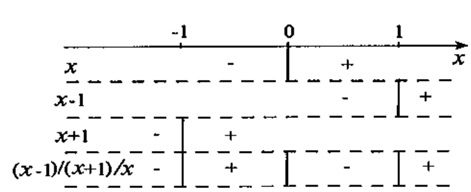
the sign diagram to the right indicates that the function is defined for $x \leq 5/4$ and $x > 3$.



20. We require

$$0 \leq x - \frac{1}{x} = \frac{x^2 - 1}{x} = \frac{(x - 1)(x + 1)}{x}.$$

The sign diagram indicates that this occurs for $-1 \leq x < 0$, and $x \geq 1$.

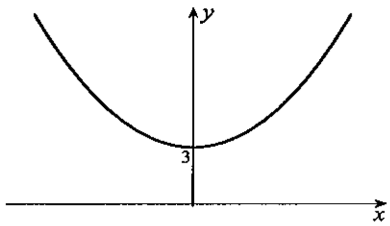


21. Straight line **22.** Parabola **23.** None of these **24.** Ellipse **25.** Hyperbola **26.** None of these **27.** Circle **28.** Parabola **29.** Ellipse **30.** Circle **31.** Hyperbola **32.** Hyperbola

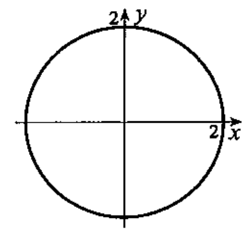
33. With formula 1.16, the distance is $\left|\dfrac{2(1)-(-3)+3}{\sqrt{2^2+(-1)^2}}\right| = \dfrac{8}{\sqrt{5}}$.

34. With formula 1.16, the distance is $\left|\dfrac{(-2)+3(-5)-4}{\sqrt{1^2+3^2}}\right| = \dfrac{21}{\sqrt{10}}$.

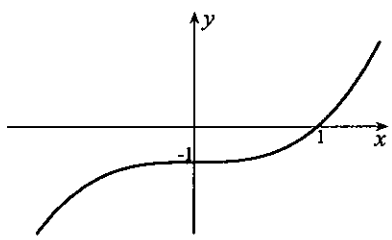
35. This is the parabola $y = 2x^2$ shifted upward 3 units.



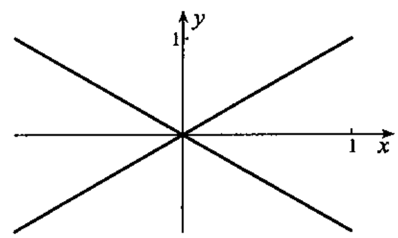
36. This is the circle $x^2 + y^2 = 4$ with centre $(0, 0)$ and radius 2.



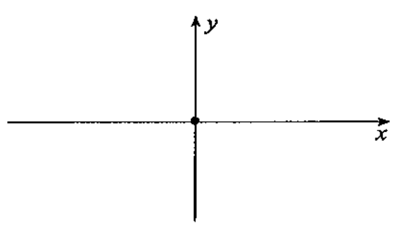
37. This is the cubic $y = x^3$ shifted downward 1 unit.



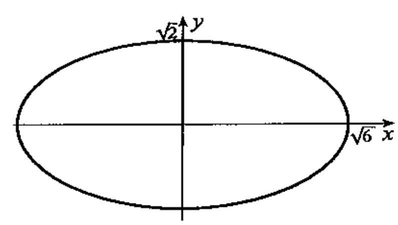
38. This equation describes two straight lines $y = \pm x$.



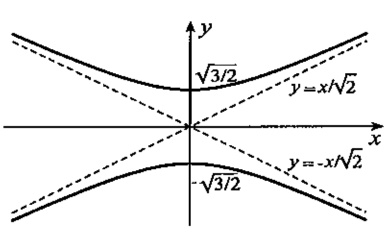
39. Only the point $(0, 0)$ satisfies this equation.



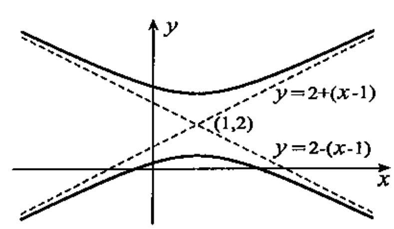
40. This is an ellipse with x-intercepts $\pm\sqrt{6}$ and y-intercepts $\pm\sqrt{2}$.



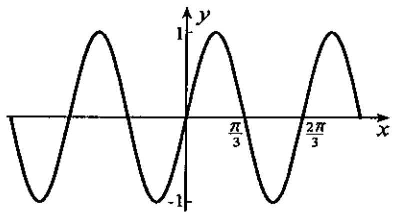
41. This is a hyperbola with y-intercepts $\pm\sqrt{3/2}$ and asymptotes $y = \pm x/\sqrt{2}$.



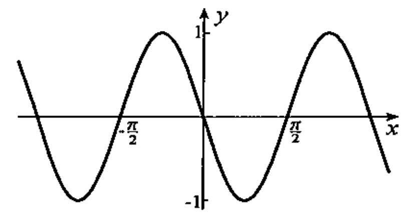
42. When we complete squares on x- and y-terms, $(y-2)^2 - (x-1)^2 = 2$. Asymptotes for this hyperbola are $y = 2 \pm (x - 1)$ intersecting at $(1, 2)$.



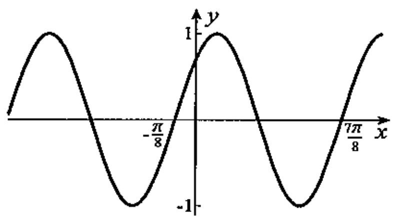
43. The period is $2\pi/3$.



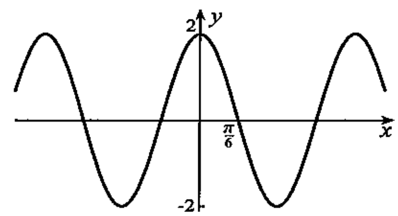
44. When we expand the cosine function, $y = -\sin 2x$.



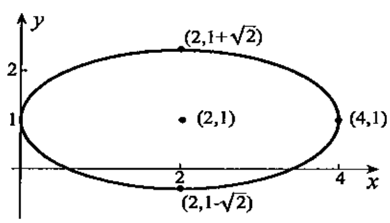
45. The period is π, and the phase shift is $\pi/8$.



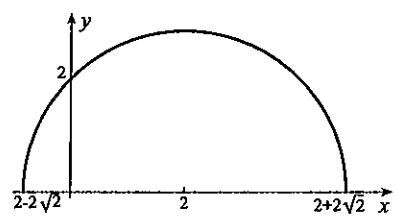
46. When we expand the sine function, $y = 2\cos 3x$.



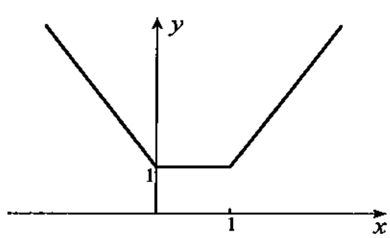
47. Completion of squares on x- and y-terms gives $(x-2)^2 + 2(y-1)^2 = 4$. This is an ellipse.



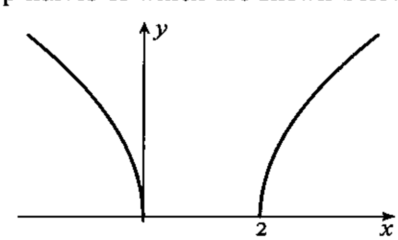
48. If we square the equation $y^2 = -x^2 + 4x + 4$, and then complete the square on the x-terms, $(x-2)^2 + y^2 = 8$. This is a circle with centre $(2,0)$ and radius $2\sqrt{2}$. The original equation describes the top half of the circle.



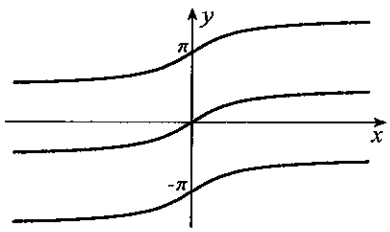
49. We add ordinates of $y = |x|$ and $y = |x-1|$.



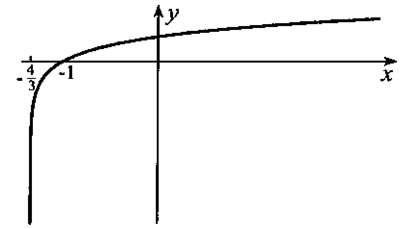
50. When we square the equation, $y^2 = |x-1| - 1$, or, $y^2 + 1 = |x-1|$. Thus, $x - 1 = \pm(y^2+1)$, or, $x = 1 \pm (y^2+1)$ $= -y^2$, $y^2 + 2$. This is two parabolas, the top halves of which are shown below.



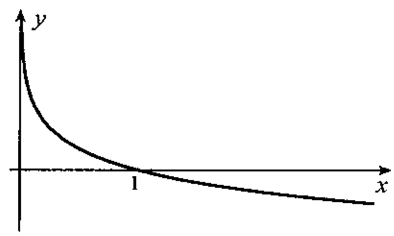
51. We interchange axes in Figure 1.90c.



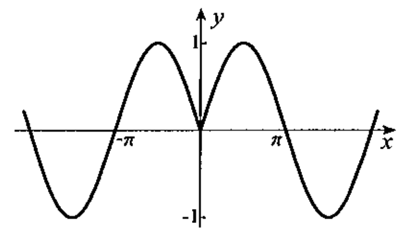
52. We must have $x > -4/3$. The x-intercept is -1.



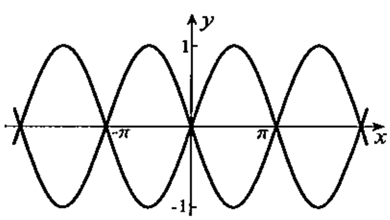
53. We interchange axes in Figure 1.115 and reflected in the x-axis.



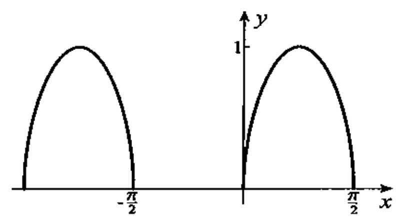
54. We graph this even function by reflecting that part of $y = \sin x$ to the right of the y-axis in the y-axis.



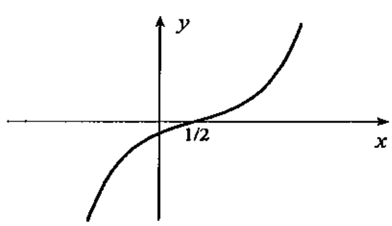
55. We draw $y = |\sin x|$ and then take its refection in the x-axis.



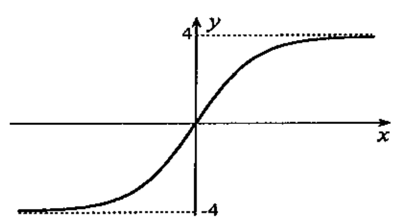
56. We take square roots of ordinates of $y = \sin 2x$.



57. The graph has the shape of $\sinh x$ shifted 1/2 unit to the right.



58. The graph has the shape of $\tanh x$ with asymptotes $y = \pm 4$.



59. With slopes $-1/2$ and 3, equation 1.60 gives $\theta = \mathrm{Tan}^{-1}\left|\dfrac{3 + 1/2}{1 - 3/2}\right| = 1.43$ radians.

60. With slopes $1/4$ and $-2/3$, equation 1.60 gives $\theta = \mathrm{Tan}^{-1}\left|\dfrac{1/4 + 2/3}{1 - 1/6}\right| = 0.833$ radians.

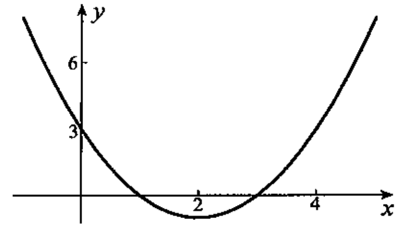
61. $f(x) = 1$

62. Since $(x+1)(2-x)$ is nonnegative only for $-1 \le x \le 2$, the function $\sqrt{(x+1)(2-x)}$ is only defined for these values of x.

63. $f(x) = 1/(x^2 - 1)$

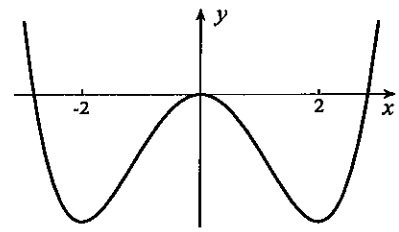
64. Since $\sqrt{-x}$ is defined only for $x \le 0$, a function is $1 + \sqrt{-x}$.

65. The graph shows that $f(x)$ has an inverse on the intervals $x \leq 2$ and $x \geq 2$. If we set $y = x^2 - 4x + 3 = (x-2)^2 - 1$, then $(x-2)^2 = y + 1$, from which $x = 2 \pm \sqrt{y+1}$. The equation $x = 2 - \sqrt{y+1}$ describes the left half of the curve, and $x = 2 + \sqrt{y+1}$ describes the right half. For $x \leq 2$ then, the inverse function is $f^{-1}(x) = 2 - \sqrt{x+1}$, and for $x \geq 2$, it is $f^{-1}(x) = 2 + \sqrt{x+1}$.



66. The graph shows that $f(x)$ has an inverse on the intervals $x \leq -2$, $-2 \leq x \leq 0$, $0 \leq x \leq 2$, and $x \geq 2$. If we set $y = x^4 - 8x^2$, then $(x^2)^2 - 8(x^2) - y = 0$. The quadratic formula gives

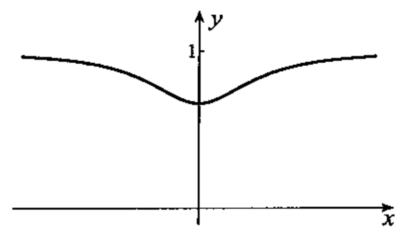
$$x^2 = \frac{8 \pm \sqrt{64+4y}}{2} = 4 \pm \sqrt{16+y},$$



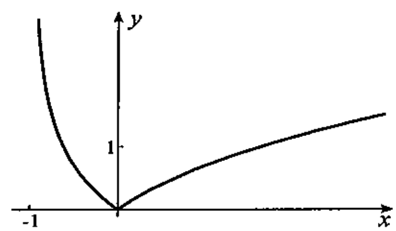
from which $x = \pm\sqrt{4 \pm \sqrt{16+y}}$. Since $x = -\sqrt{4+\sqrt{16+y}}$, $x = -\sqrt{4-\sqrt{16-y}}$, $x = \sqrt{4-\sqrt{16-y}}$, and $x = \sqrt{4+\sqrt{16+y}}$ describe parts of the graph on each of the above intervals, respectively, it follows that inverse functions on these intervals are:

$$f^{-1}(x) = -\sqrt{4+\sqrt{16+x}} \text{ for } x \leq -2; \quad f^{-1}(x) = -\sqrt{4-\sqrt{16-x}} \text{ for } -2 \leq x \leq 0;$$
$$f^{-1}(x) = \sqrt{4-\sqrt{16-x}} \text{ for } 0 \leq x \leq 2; \quad f^{-1}(x) = \sqrt{4+\sqrt{16+x}} \text{ for } x \geq 2.$$

67. The graph shows that $f(x)$ has an inverse on the intervals $x \leq 0$ and $x \geq 0$. If we set $y = (x^2+2)/(x^2+3)$, then $(x^2+3)y = x^2+2$, from which $(y-1)x^2 = 2 - 3y$. Thus $x = \pm\sqrt{(2-3y)/(y-1)}$. The inverse function for $x \leq 0$ is $f^{-1}(x) = -\sqrt{(2-3x)/(x-1)}$, and that for $x \geq 0$ is $f^{-1}(x) = \sqrt{(2-3x)/(x-1)}$.



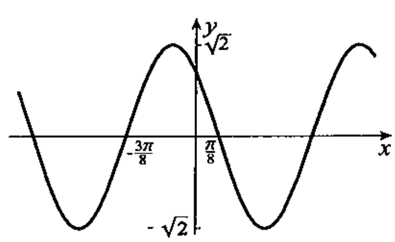
68. The graph shows that $f(x)$ has an inverse for $-1 < x \leq 0$ and for $x \geq 0$. If we set $y = \sqrt{x^2/(x+1)}$, then $x^2 = (x+1)y^2$, from which $x^2 - y^2x - y^2 = 0$. The quadratic formula gives $x = (y^2 \pm \sqrt{y^4+4y^2})/2$. Since $x = (y^2 - \sqrt{y^4+4y^2})/2$ describes the graph for $-1 < x \leq 0$, and $x = (y^2 + \sqrt{y^4+4y^2})/2$ describes the graph for $x \geq 0$, inverse functions are $f^{-1}(x) = (x^2 - \sqrt{x^4+4x^2})/2$ for $-1 < x \leq 0$, and $f^{-1}(x) = (x^2 + \sqrt{x^4+4x^2})/2$ for $x \geq 0$.



69. If we set $f(x) = \cos 2x - \sin 2x = A \sin(2x + \phi)$, and expand the right side,

$\cos 2x - \sin 2x = A(\sin 2x \cos\phi + \cos 2x \sin\phi)$.

This will be true for all x if we set $A\cos\phi = -1$ and $A \sin\phi = 1$. When these are squared and added, $2 = (-1)^2 + 1^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = \sqrt{2}$, then $\sin\phi = 1/\sqrt{2}$ and $\cos\phi = -1/\sqrt{2}$. These are satisfied by $\phi = 3\pi/4$. The amplitude is $\sqrt{2}$, the period is π, and the phase shift is $-3\pi/8$. Angles for which $f(x) = 0$ are defined by $0 = \sqrt{2}\sin(2x + 3\pi/4) \Longrightarrow 2x + 3\pi/4 = n\pi$, where n is an integer. These give $x = -3\pi/8 + n\pi/2$. The second smallest positive solution is $x = 5\pi/8$, which we could also have seen from the graph.



70. Since $2\sin 2x$ has period π, and $3\cos 3x$ has period $2\pi/3$, the function $f(x)$ has period 2π.

71. This quadratic in $\cos x$ can be factored $0 = \cos^2 x + 5\cos x - 6 = (\cos x + 6)(\cos x - 1)$. Thus, either $\cos x = -6$ or $\cos x = 1$. The former is impossible, and solutions of the latter are $x = 2n\pi$, n an integer.

72. One solution for $2x$ of $\sin 2x = 1/4$ is $2x = \mathrm{Sin}^{-1}(1/4) = 0.253$. All solutions are given by

$$2x = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.253\right) + 2n\pi \qquad \Longrightarrow \qquad x = \frac{\pi}{4} \pm 0.659 + n\pi, \quad n \text{ an integer.}$$

73. We can rewrite the equation in the form $\sin(x+1) = \pm 1/\sqrt{3}$. One solution of $\sin(x+1) = 1/\sqrt{3}$ for $x+1$ is $x + 1 = \mathrm{Sin}^{-1}(1/\sqrt{3}) = 0.6155$. All solutions are given by

$$x + 1 = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.6155\right) + 2n\pi \qquad \Longrightarrow \qquad x = -1 + \frac{\pi}{2} \pm 0.955 + 2n\pi, \quad n \text{ an integer.}$$

From $\sin(x+1) = -1/\sqrt{3}$, we obtain the solutions

$$x = -1 - \frac{\pi}{2} \pm 0.955 + 2n\pi, \quad n \text{ an integer.}$$

74. Since $5 - 2\pi$ is in the principal value range for the inverse tangent function, we may take tangents of both sides of the equation,

$$3x + 2 = \tan(5 - 2\pi) \qquad \Longrightarrow \qquad x = -\frac{2}{3} + \frac{1}{3}\tan(5 - 2\pi) = -1.79.$$

75. If $\cos 2x = \sin x$, then $0 = \sin x - (1 - 2\sin^2 x) = 2\sin^2 x + \sin x - 1 = (2\sin x - 1)(\sin x + 1)$. Thus, either $\sin x = -1$, the solutions of which are $x = 2n\pi - \pi/2 = (4n-1)\pi/2$, where n is an integer, or, $\sin x = 1/2$. Solutions of this equation are $x = \pi/6 + 2n\pi$, $5\pi/6 + 2n\pi$. These can be represented more compactly in the form $x = 2n\pi + \pi/2 \pm \pi/3 = (4n+1)\pi/2 \pm \pi/3$.

76. If we write $\ln[\sin x(1 + \sin x)] = \ln(3/2)$, and exponentiate both sides to base e, we obtain $\sin x(1 + \sin x) = 3/2$, or, $2\sin^2 x + 2\sin x - 3 = 0$. Thus, $\sin x = \dfrac{-2 \pm \sqrt{4 + 24}}{4} = \dfrac{-1 \pm \sqrt{7}}{2}$. Since $\sin x$ must be nonnegative in the original equation, we take $\sin x = (\sqrt{7} - 1)/2$. Solutions of this equation are

$$x = \{\mathrm{Sin}^{-1}[(\sqrt{7} - 1)/2] + 2n\pi, \quad \pi - \mathrm{Sin}^{-1}[(\sqrt{7} - 1)/2] + 2n\pi\} = \{0.966 + 2n\pi, \quad 2.175 + 2n\pi\}.$$

These can be expressed in the form $x = \dfrac{\pi}{2} \pm \left(\dfrac{\pi}{2} - 0.966\right) + 2n\pi = \left(\dfrac{4n+1}{2}\right)\pi \pm 0.604$.

77. If $3\,\mathrm{Sin}^{-1}(e^{x+2}) = 2$, then $e^{x+2} = \sin(2/3)$, from which $x = -2 + \ln[\sin(2/3)] = -2.48$.

78. If $3\sin(e^{x+2}) = 2$, then $\sin(e^{x+2}) = 2/3$. This implies that

$$e^{x+2} = \{\mathrm{Sin}^{-1}(2/3) + 2n\pi, \quad \pi - \mathrm{Sin}^{-1}(2/3) + 2n\pi\} = \{0.73 + 2n\pi, \quad \pi - 0.73 + 2n\pi\}.$$

These values can be represented more compactly as

$$e^{x+2} = \frac{\pi}{2} \pm 0.84 + 2n\pi = \left(\frac{4n+1}{2}\right)\pi \pm 0.84.$$

Because e^{x+2} must be positive, n must be a nonnegative integer. Finally then,

$$x = \ln\left[\left(\frac{4n+1}{2}\right)\pi \pm 0.84\right] - 2, \text{ where } n \geq 0.$$

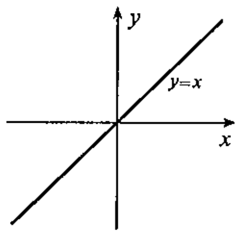
79. Since $\mathrm{Tan}^{-1}(1/4) = 0.245$, it follows that $x\cosh 2 = 0.245 + n\pi$, were n is an integer. Hence, $x = (0.245 + n\pi)/\cosh 2 = 0.065 + 0.27n\pi$.

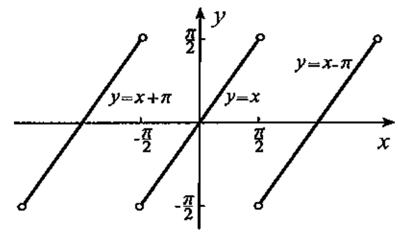
80. If $4 = \sinh x = (e^x - e^{-x})/2$, then multiplication by e^x gives $e^{2x} - 8e^x - 1 = 0$. Thus,

$$e^x = \frac{8 \pm \sqrt{64 + 4}}{2} = 4 \pm \sqrt{17}.$$

Since e^x must be positive, $e^x = 4 + \sqrt{17}$, and $x = \ln(4 + \sqrt{17}) = 2.09$.

81. Since $\tan x$ is the inverse of $\mathrm{Tan}^{-1}x$, for all x, it follows that $f(x) = x$. Its graph is shown to the left below.
(b) On the interval $-\pi/2 \le x \le \pi/2$, the function $\mathrm{Tan}^{-1}x$ is the inverse $\tan x$. For these values of x then, $f(x) = x$. Since $\tan x$ is π-periodic, so also is $f(x)$, and the graph is shown to the right below.





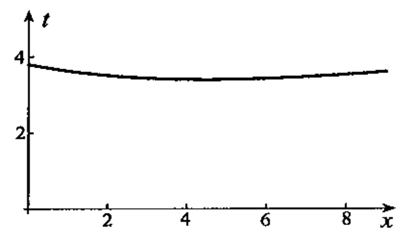
82. To solve the equation $x(t) = 0$, we divide by $\cos 4t$,

$$\tan 4t = -\frac{1}{10} \qquad \Longrightarrow \qquad 4t = \mathrm{Tan}^{-1}(-0.1) + n\pi = -0.09967 + n\pi \qquad \Longrightarrow \qquad t = \frac{1}{4}(-0.09967 + n\pi),$$

where n is an integer. The smallest positive solution is 0.760 (when $n = 1$).

83. Travel time on water is distance on water $\sqrt{x^2+36}$ divided by rowing speed; travel time on land is distance on land $9 - x$ divided by walking speed. Total travel time is therefore

$$t = f(x) = \frac{\sqrt{x^2+36}}{3} + \frac{9-x}{5}.$$



84. My electronic device gives 1.5260 for the only root of $f(x) = x^3 - 2x^2 + 4x - 5 = 0$. To verify that 1.526 is accurate to three decimals, we calculate $f(1.5255) = -2.2 \times 10^{-3}$ and $f(1.5265) = 2.6 \times 10^{-3}$.

85. My electronic device gives 1.4096 and -0.6367 for roots of $f(x) = x^2 - 1 - \sin x = 0$. To verify that 1.410 is accurate to three decimals, we calculate $f(1.4095) = -3.3 \times 10^{-4}$ and $f(1.4105) = 2.3 \times 10^{-3}$. A similar calculation confirms -0.637 as the other root.

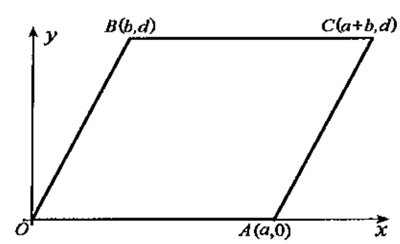
86. My electronic device gives -11.61869, -0.87380, and 0.49249 for roots of $f(x) = x^3 + 12x^2 + 4x - 5 = 0$. To verify that 0.4925 has error less than 10^{-4}, we calculate $f(0.4924) = -1.5 \times 10^{-3}$ and $f(0.4926) = 1.8 \times 10^{-3}$. A similar calculation confirms -11.6187 and -0.8738 as the other roots.

87. My electronic device gives -4.93852, -3.69799, -0.04161, and 2.84222 for roots of $f(x) = x^2 - 1 - 24\sin x = 0$. To verify that 2.8422 has error less than 10^{-4}, we calculate $f(2.8421) = -3.3 \times 10^{-3}$ and $f(2.8423) = 2.4 \times 10^{-3}$. A similar calculation confirms -4.9385, -3.6980, and -0.0416 as the other roots.

88. Let us take a coordinate system as shown, in which case $a^2 = b^2 + d^2$. Slopes of AB and OC are $d/(b-a)$ and $d/(b+a)$. The product of these is

$$\left(\frac{d}{b-a}\right)\left(\frac{d}{b+a}\right) = \frac{d^2}{b^2-a^2} = -1.$$

Hence the diagonals are perpendicular.



CHAPTER 2

EXERCISES 2.1

1. $\lim_{x\to 7}\frac{x^2-5}{x+2}=\frac{49-5}{7+2}=\frac{44}{9}$

2. $\lim_{x\to -2}\frac{x^3+8}{x+5}=\frac{0}{3}=0$

3. $\lim_{x\to -5}\frac{x^2+3x+2}{x^2+25}=\frac{25-15+2}{25+25}=\frac{6}{25}$

4. $\lim_{x\to 0}\frac{x^2+3x}{3x^2-2x}=\lim_{x\to 0}\frac{x(x+3)}{x(3x-2)}=\lim_{x\to 0}\frac{x+3}{3x-2}=\frac{3}{-2}=-\frac{3}{2}$

5. $\lim_{x\to 3^+}\frac{2x-3}{x^2-5}=\frac{6-3}{9-5}=\frac{3}{4}$

6. $\lim_{x\to 2^-}\frac{2x-4}{3x+2}=\frac{0}{8}=0$

7. $\lim_{x\to 0^-}\frac{x^4+5x^3}{3x^4-x^3}=\lim_{x\to 0^-}\frac{x^3(x+5)}{x^3(3x-1)}=\lim_{x\to 0^-}\frac{x+5}{3x-1}=\frac{5}{-1}=-5$

8. $\lim_{x\to 2^+}\frac{x^2+2x+4}{x-3}=\frac{4+4+4}{-1}=-12$

9. $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2}(x+2)=4$

10. $\lim_{x\to 3^+}\frac{x^2-9}{x-3}=\lim_{x\to 3^+}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3^+}(x+3)=6$

11. $\lim_{x\to 5^-}\frac{x^2-25}{x-5}=\lim_{x\to 5^-}\frac{(x+5)(x-5)}{x-5}=\lim_{x\to 5^-}(x+5)=10$

12. $\lim_{x\to 3}\frac{x^2-2x-3}{3-x}=\lim_{x\to 3}\frac{(x-3)(x+1)}{3-x}=\lim_{x\to 3}[-(x+1)]=-4$

13. $\lim_{x\to 2}\frac{x^2-4x+4}{x-2}=\lim_{x\to 2}\frac{(x-2)^2}{x-2}=\lim_{x\to 2}(x-2)=0$

14. $\lim_{x\to 2}\frac{x^3-6x^2+12x-8}{x^2-4x+4}=\lim_{x\to 2}\frac{(x-2)^3}{(x-2)^2}=\lim_{x\to 2}(x-2)=0$

15. $\lim_{x\to 1}\frac{x^3-6x^2+11x-6}{x^2-3x+2}=\lim_{x\to 1}\frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)}=\lim_{x\to 1}(x-3)=-2$

16. $\lim_{x\to 2}\frac{x^3-6x^2+11x-6}{x^2-3x+2}=\lim_{x\to 2}\frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)}=\lim_{x\to 2}(x-3)=-1$

17. $\lim_{x\to 3^+}\frac{x^3-6x^2+11x-6}{x^2-3x+2}=\frac{27-54+33-6}{9-9+2}=0$

18. $\lim_{x\to 3^-}\frac{x^3-6x^2+11x-6}{x^2-3x+2}=\frac{27-54+33-6}{9-9+2}=0$

19. $\lim_{x\to 0}\frac{x^3-6x^2+11x-6}{x^2-3x+2}=\frac{-6}{2}=-3$

20. $\lim_{x\to -1}\frac{12x+5}{x^2-2x+1}=\frac{-7}{4}=-\frac{7}{4}$

21. $\lim_{x\to 1}\sqrt{\frac{2-x}{2+x}}=\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}$

22. $\lim_{x\to 5}\frac{\sqrt{1-x^2}}{3x+2}$ does not exist

23. $\lim_{x\to 0}\frac{\tan x}{\sin x}=\lim_{x\to 0}\frac{\sin x}{\sin x\,\cos x}=\lim_{x\to 0}\frac{1}{\cos x}=1$

24. $\lim_{x\to \pi/4}\frac{\sin x}{\tan x}=\frac{1/\sqrt{2}}{1}=\frac{1}{\sqrt{2}}$

25. $\lim_{x\to 0}\frac{\sin 4x}{\sin 2x}=\lim_{x\to 0}\frac{2\sin 2x\,\cos 2x}{\sin 2x}=\lim_{x\to 0}(2\cos 2x)=2$

26. $\lim_{x\to 0^+}\frac{\sin 6x}{\sin 3x}=\lim_{x\to 0^+}\frac{2\sin 3x\,\cos 3x}{\sin 3x}=\lim_{x\to 0^+}(2\cos 3x)=2$

27. $\lim_{x\to 0^+}\frac{\sin 2x}{\tan x}=\lim_{x\to 0^+}\frac{2\sin x\,\cos x}{\sin x\,/\cos x}=\lim_{x\to 0^+}(2\cos^2 x)=2$

28. $\lim_{x\to 2}\frac{x-2}{\sqrt{x}-\sqrt{2}}=\lim_{x\to 2}\frac{(\sqrt{x}+\sqrt{2})(\sqrt{x}-\sqrt{2})}{\sqrt{x}-\sqrt{2}}=\lim_{x\to 2}(\sqrt{x}+\sqrt{2})=2\sqrt{2}$

29. $$\lim_{x\to 0}\frac{\sqrt{1-x}-\sqrt{1+x}}{x}=\lim_{x\to 0}\left(\frac{\sqrt{1-x}-\sqrt{1+x}}{x}\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\right)$$
$$=\lim_{x\to 0}\frac{(1-x)-(1+x)}{x(\sqrt{1-x}+\sqrt{1+x})}=\lim_{x\to 0}\frac{-2}{\sqrt{1-x}+\sqrt{1+x}}=\frac{-2}{1+1}=-1$$

30. $$\lim_{x\to 5^+}\frac{|x^2-25|}{x^2-25}=\lim_{x\to 5^+}\frac{x^2-25}{x^2-25}=\lim_{x\to 5^+}(1)=1$$

31. $$\lim_{x\to 5^-}\frac{|x^2-25|}{x^2-25}=\lim_{x\to 5^-}\frac{-(x^2-25)}{x^2-25}=-1$$

32. Since $\lim_{x\to 5^+}\frac{|x^2-25|}{x^2-25}=1$ (Exercise 30) and $\lim_{x\to 5^-}\frac{|x^2-25|}{x^2-25}=\lim_{x\to 5^-}\frac{-(x^2-25)}{x^2-25}=\lim_{x\to 5^-}(-1)=-1$, it follows that the given limit does not exist.

33. $$\lim_{x\to 0^+}\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x}}=\lim_{x\to 0^+}\left(\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x}}\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right)=\lim_{x\to 0^+}\frac{x+2-2}{\sqrt{x}(\sqrt{x+2}+\sqrt{2})}$$
$$=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{x+2}+\sqrt{2}}=0$$

34. $$\lim_{x\to 0}\frac{1-\sqrt{x^2+1}}{2x^2}=\lim_{x\to 0}\left[\frac{1-\sqrt{x^2+1}}{2x^2}\frac{1+\sqrt{x^2+1}}{1+\sqrt{x^2+1}}\right]=\lim_{x\to 0}\frac{1-(x^2+1)}{2x^2\left(1+\sqrt{x^2+1}\right)}$$
$$=\lim_{x\to 0}\frac{-1}{2\left(1+\sqrt{x^2+1}\right)}=\frac{-1}{2(2)}=-\frac{1}{4}$$

35. $$\lim_{x\to -2}\frac{x+2}{\sqrt{-x}-\sqrt{2}}=\lim_{x\to -2}\left(\frac{x+2}{\sqrt{-x}-\sqrt{2}}\frac{\sqrt{-x}+\sqrt{2}}{\sqrt{-x}+\sqrt{2}}\right)=\lim_{x\to -2}\frac{(x+2)(\sqrt{-x}+\sqrt{2})}{-x-2}$$
$$=\lim_{x\to -2}[-(\sqrt{-x}+\sqrt{2})]=-2\sqrt{2}$$

36. $$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to 0}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$
$$=\lim_{x\to 0}\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=\lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}=\frac{2}{1+1}=1$$

37. $$\lim_{x\to -2^+}\frac{\sqrt{x+3}-\sqrt{-x-1}}{\sqrt{x+2}}=\lim_{x\to -2^+}\left(\frac{\sqrt{x+3}-\sqrt{-x-1}}{\sqrt{x+2}}\frac{\sqrt{x+3}+\sqrt{-x-1}}{\sqrt{x+3}+\sqrt{-x-1}}\right)$$
$$=\lim_{x\to -2^+}\frac{(x+3)-(-x-1)}{\sqrt{x+2}(\sqrt{x+3}+\sqrt{-x-1})}=\lim_{x\to -2^+}\frac{2(x+2)}{\sqrt{x+2}(\sqrt{x+3}+\sqrt{-x-1})}$$
$$=\lim_{x\to -2^+}\frac{2\sqrt{x+2}}{\sqrt{x+3}+\sqrt{-x-1}}=0$$

38. $$\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}=\lim_{x\to 0}\left[\frac{x}{\sqrt{x+4}-2}\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right]=\lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{(x+4)-4}=\lim_{x\to 0}\left(\sqrt{x+4}+2\right)=4$$

39. $$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+x}-\sqrt{2-x}}=\lim_{x\to 0}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+x}-\sqrt{2-x}}\frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}+\sqrt{2-x}}\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$
$$=\lim_{x\to 0}\frac{(1+x-1+x)(\sqrt{2+x}+\sqrt{2-x})}{(2+x-2+x)(\sqrt{1+x}+\sqrt{1-x})}=\lim_{x\to 0}\frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{1+x}+\sqrt{1-x}}=\sqrt{2}$$

40. $$\lim_{x\to 0}\frac{\sqrt{x+1}-\sqrt{2x+1}}{\sqrt{3x+4}-\sqrt{2x+4}}=\lim_{x\to 0}\left(\frac{\sqrt{x+1}-\sqrt{2x+1}}{\sqrt{3x+4}-\sqrt{2x+4}}\frac{\sqrt{x+1}+\sqrt{2x+1}}{\sqrt{x+1}+\sqrt{2x+1}}\frac{\sqrt{3x+4}+\sqrt{2x+4}}{\sqrt{3x+4}+\sqrt{2x+4}}\right)$$
$$=\lim_{x\to 0}\frac{(x+1-2x-1)(\sqrt{3x+4}+\sqrt{2x+4})}{(3x+4-2x-4)(\sqrt{x+1}+\sqrt{2x+1})}$$
$$=\lim_{x\to 0}\left(-\frac{\sqrt{3x+4}+\sqrt{2x+4}}{\sqrt{x+1}+\sqrt{2x+1}}\right)=-2$$

41. $$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}=\lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x-1}\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right)=\lim_{x\to 1}\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}=\lim_{x\to 1}\frac{1}{\sqrt{x+3}+2}=\frac{1}{4}$$

42. $\displaystyle \lim_{x\to a}\frac{x^2-a^2}{x-a}=\lim_{x\to a}\frac{(x+a)(x-a)}{x-a}=\lim_{x\to a}(x+a)=2a$

43. $\displaystyle \lim_{x\to a}\frac{x^3-a^3}{x-a}=\lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{x-a}=\lim_{x\to a}(x^2+ax+a^2)=3a^2$

44. $\displaystyle \lim_{x\to -a}\frac{x+a}{x^2+ax-x-a}=\lim_{x\to -a}\frac{x+a}{(x+a)(x-1)}=\lim_{x\to -a}\frac{1}{x-1}=-\frac{1}{a+1}$

45. $\displaystyle \lim_{x\to 0}\frac{\sin 2ax}{\sin ax}=\lim_{x\to 0}\frac{2\sin ax\cos ax}{\sin ax}=\lim_{x\to 0}(2\cos ax)=2$

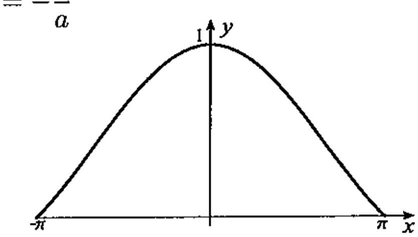
46. $\displaystyle \lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$

47.
$$\begin{aligned}\lim_{x\to 0^+}\frac{\sqrt{x+a}-\sqrt{a}}{\sqrt{x}}&=\lim_{x\to 0^+}\left[\frac{\sqrt{x+a}-\sqrt{a}}{\sqrt{x}}\,\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right]\\&=\lim_{x\to 0^+}\frac{x+a-a}{\sqrt{x}\left(\sqrt{x+a}+\sqrt{a}\right)}=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{x+a}+\sqrt{a}}=0\end{aligned}$$

48.
$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}&=\lim_{x\to 0}\left[\frac{\sqrt{a+x}-\sqrt{a-x}}{x}\,\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}\right]\\&=\lim_{x\to 0}\frac{(a+x)-(a-x)}{x\left(\sqrt{a+x}+\sqrt{a-x}\right)}=\lim_{x\to 0}\frac{2}{\sqrt{a+x}+\sqrt{a-x}}=\frac{2}{\sqrt{a}+\sqrt{a}}=\frac{1}{\sqrt{a}}\end{aligned}$$

49.
$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x^2+a^2}-\sqrt{2x^2+a^2}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}&=\lim_{x\to 0}\left[\frac{\sqrt{x^2+a^2}-\sqrt{2x^2+a^2}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}\,\frac{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}\,\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{\sqrt{3x^2+4}+\sqrt{2x^2+4}}\right]\\&=\lim_{x\to 0}\left[\frac{(x^2+a^2-2x^2-a^2)\left(\sqrt{3x^2+4}+\sqrt{2x^2+4}\right)}{(3x^2+4-2x^2-4)\left(\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}\right)}\right]\\&=\lim_{x\to 0}\left[-\frac{\sqrt{3x^2+4}+\sqrt{2x^2+4}}{\sqrt{x^2+a^2}+\sqrt{2x^2+a^2}}\right]=-\frac{2}{a}\end{aligned}$$

50. Although the plot does not show it, there should be a hole in the graph at $x=0$. The plot suggests that the limit is 1.



51. Since $-1\le\sin(1/x)\le 1$ for all x, it follows that $-|x|\le|x|\sin(1/x)\le|x|$. Since $-|x|$ is always nonpositive and $|x|$ is always nonnegative, we can say that $-|x|\le x\sin(1/x)\le|x|$. Because $\lim_{x\to 0}(-|x|)=\lim_{x\to 0}|x|=0$, the squeeze theorem implies that $\lim_{x\to 0}x\sin(1/x)=0$ also.

52. Since $-1\le\cos(3/x)\le 1$ for all x, it follows that $-x^4\le x^4\cos(3/x)\le x^4$. Because $\lim_{x\to 0}(-x^4)=\lim_{x\to 0}x^4=0$, the squeeze theorem implies that $\lim_{x\to 0}x^4\cos(3/x)=0$ also.

53. Left and right limits exist, but the "full" limit does not exist.

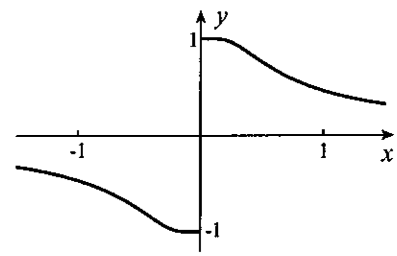
54. This statement is false. For example, $x^2<2x^2$ for all $x\ne 0$, but $\lim_{x\to 0}x^2=\lim_{x\to 0}2x^2=0$.

55. If we use the stated result, then,

$$\begin{aligned}\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}&=\lim_{h\to 0}\frac{1}{h}\{(x+h-x)[(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}]\}\\&=\lim_{h\to 0}[(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}]=nx^{n-1}.\end{aligned}$$

56.
$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}&=\lim_{h\to 0}\left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right]=\lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\&=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\end{aligned}$$

57. A graph of the function is shown to the right. It indicates that the left-hand limit is -1, the right-hand limit is 1, and because these limits are different, the limit as $x \to 0$ does not exist. The function has no value at $x = 0$.

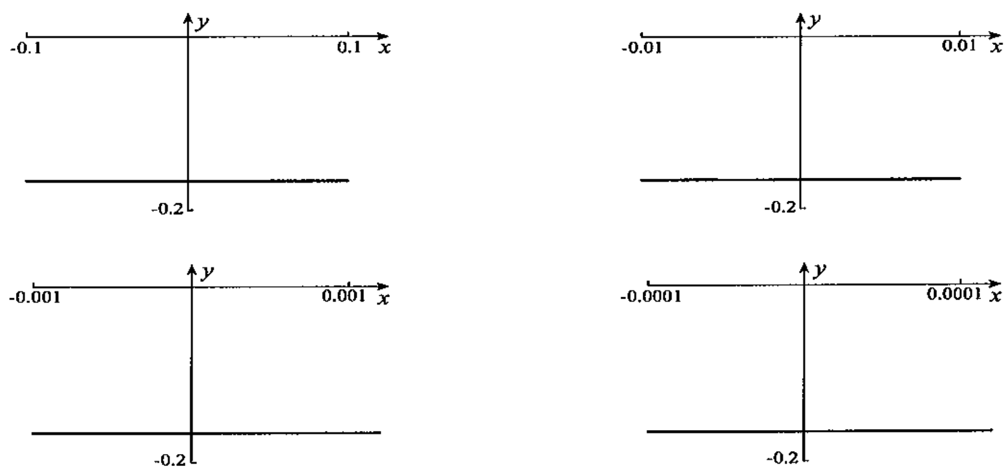


58. (a) Our calculator gave

x	0.1	0.01	0.001	0.0001	0.00001	0.000001	0.0000001
$(\sin x - x)/x^3$	-0.16658	-0.16667	-0.16667	-0.1667	-0.17	0.0	0.0

It would appear that the limit is 0.

(b) Plots of the function on the suggested intervals are shown below.



They suggest that the limit is approximately -0.17.

59. The only way for this limit to exist is for $\lim_{x\to a} g(x) = L$.

60. If we set $z = -x$, then $\displaystyle\lim_{x\to -a} f(x) = \lim_{z\to a} f(-z) = \lim_{z\to a} f(z) = L$.

61. If we set $z = -x$, then $\displaystyle\lim_{x\to -a^-} f(x) = \lim_{z\to a^+} f(-z) = \lim_{z\to a^+} f(z) = L$.

62. This limit cannot be determined.

63. If we set $z = -x$, then $\displaystyle\lim_{x\to -a} f(x) = \lim_{z\to a} f(-z) = \lim_{z\to a} [-f(z)] = -\lim_{z\to a} f(z) = -L$.

64. If we set $z = -x$, then $\displaystyle\lim_{x\to -a^-} f(x) = \lim_{z\to a^+} f(-z) = \lim_{z\to a^+} [-f(z)] = -\lim_{z\to a^+} f(z) = -L$.

65. There is not enough information to find $\lim_{x\to -a^+} f(x)$.

66. The limit is 0 if $F = 0$; it does not exist if $F \neq 0$.

67. (a) Since $\displaystyle\lim_{x\to 0^+} e^{-1/x} = 0$, it follows that $\displaystyle\lim_{x\to 0^+} \frac{a + ce^{-1/x}}{b + de^{-1/x}} = \frac{a}{b}$.

(b) If we multiply numerator and denominator by $e^{1/x}$ and use the fact that $\displaystyle\lim_{x\to 0^-} e^{1/x} = 0$, we obtain

$$\lim_{x\to 0^-} \frac{a + ce^{-1/x}}{b + de^{-1/x}} = \lim_{x\to 0^-} \frac{ae^{1/x} + c}{be^{1/x} + d} = \frac{c}{d}.$$

(c) The limit does not exist since left and right limits are not the same, unless $a/b = c/d$, in which case the limit is a/b.

EXERCISES 2.2

1. $\lim_{x\to2^+}\frac{1}{x-2}=\infty$

2. $\lim_{x\to2^-}\frac{1}{x-2}=-\infty$

3. $\lim_{x\to2}\frac{1}{x-2}$ does not exist since $\lim_{x\to2^-}\frac{1}{x-2}=-\infty$ and $\lim_{x\to2^+}\frac{1}{x-2}=\infty$

4. $\lim_{x\to2^+}\frac{1}{(x-2)^2}=\infty$

5. $\lim_{x\to2^-}\frac{1}{(x-2)^2}=\infty$

6. $\lim_{x\to2}\frac{1}{(x-2)^2}=\infty$

7. $\lim_{x\to1}\frac{5x}{(x-1)^3}$ does not exist since $\lim_{x\to1^-}\frac{5x}{(x-1)^3}=-\infty$ and $\lim_{x\to1^+}\frac{5x}{(x-1)^3}=\infty$

8. $\lim_{x\to1/2}\frac{6x^2+7x-5}{2x-1}=\lim_{x\to1/2}\frac{(3x+5)(2x-1)}{2x-1}=\lim_{x\to1/2}(3x+5)=\frac{13}{2}$

9. $\lim_{x\to1}\frac{2x+3}{x^2-2x+1}=\lim_{x\to1}\frac{2x+3}{(x-1)^2}=\infty$

10. $\lim_{x\to2}\frac{x-2}{x^2-4x+4}=\lim_{x\to2}\frac{x-2}{(x-2)^2}=\lim_{x\to2}\frac{1}{x-2}$
    Since $\lim_{x\to2^+}\frac{1}{x-2}=\infty$ and $\lim_{x\to2^-}\frac{1}{x-2}=-\infty$, the given limit does not exist.

11. $\lim_{x\to0}\csc x$ does not exist since $\lim_{x\to0^-}\csc x=-\infty$ and $\lim_{x\to0^+}\csc x=\infty$.

12. $\lim_{x\to\pi/4}\sec(x-\pi/4)=1$

13. This limit does not exist since $\lim_{x\to3\pi/4^-}\sec(x-\pi/4)=\infty$ and $\lim_{x\to3\pi/4^+}\sec(x-\pi/4)=-\infty$.

14. $\lim_{x\to0^+}\cot x=\infty$

15. $\lim_{x\to\pi/2^+}\tan x=-\infty$

16. $\lim_{x\to\pi/2^-}\tan x=\infty$

17. $\lim_{x\to1}\frac{x^2-2x+1}{x^3-3x^2+3x-1}=\lim_{x\to1}\frac{(x-1)^2}{(x-1)^3}=\lim_{x\to1}\frac{1}{x-1}$ which does not exist since $\lim_{x\to1^-}\frac{1}{x-1}=-\infty$ and $\lim_{x\to1^+}\frac{1}{x-1}=\infty$

18. $\lim_{x\to0}\frac{\sqrt{1+x}-1}{x^2}=\lim_{x\to0}\left[\frac{\sqrt{1+x}-1}{x^2}\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right]=\lim_{x\to0}\frac{(1+x)-1}{x^2\left(\sqrt{1+x}+1\right)}=\lim_{x\to0}\frac{1}{x\left(\sqrt{1+x}+1\right)}$
    Since $\lim_{x\to0^+}\frac{1}{x\left(\sqrt{x+1}+1\right)}=\infty$ and $\lim_{x\to0^-}\frac{1}{x\left(\sqrt{x+1}+1\right)}=-\infty$, the given limit does not exist.

19. $\lim_{x\to0}\frac{2x}{1-\sqrt{x^2+1}}=\lim_{x\to0}\left(\frac{2x}{1-\sqrt{x^2+1}}\frac{1+\sqrt{x^2+1}}{1+\sqrt{x^2+1}}\right)=\lim_{x\to0}\frac{2x(1+\sqrt{x^2+1})}{1-(x^2+1)}=\lim_{x\to0}\frac{-2(1+\sqrt{x^2+1})}{x}$
    This limit does not exist since $\lim_{x\to0^-}\frac{-2(1+\sqrt{x^2+1})}{x}=\infty$ and $\lim_{x\to0^+}\frac{-2(1+\sqrt{x^2+1})}{x}=-\infty$.

20. Since $\lim_{x\to4^+}\frac{|4-x|}{x^2-8x+16}=\lim_{x\to4^+}\frac{x-4}{(x-4)^2}=\lim_{x\to4^+}\frac{1}{x-4}=\infty$
    and $\lim_{x\to4^-}\frac{|4-x|}{x^2-8x+16}=\lim_{x\to4^-}\frac{4-x}{(x-4)^2}=\lim_{x\to4^-}\frac{-1}{x-4}=\infty$, the given limit does not exist.

21. $\lim_{x\to0^+}\ln(4x)=-\infty$

22. $\lim_{x\to1}\frac{1}{\ln|x-1|}=0$

23. $\lim_{x\to0}e^{1/x}$ does not exist since $\lim_{x\to0^+}e^{1/x}=\infty$.

24. $\lim_{x\to0}e^{1/|x|}=\infty$

25. $\lim_{x\to a^+}\frac{x-a}{x^2-2ax+a^2}=\lim_{x\to a^+}\frac{x-a}{(x-a)^2}=\lim_{x\to a^+}\frac{1}{x-a}=\infty$

26. $\lim\limits_{x\to a}\dfrac{|x-a|}{x^2-2ax+a^2}=\lim\limits_{x\to a}\dfrac{|x-a|}{(x-a)^2}=\lim\limits_{x\to a}\dfrac{1}{|x-a|}=\infty$

27. $$\lim_{x\to 0^-}\frac{\sqrt{a+x}-\sqrt{a}}{x^2}=\lim_{x\to 0^-}\left[\frac{\sqrt{a+x}-\sqrt{a}}{x^2}\frac{\sqrt{a+x}+\sqrt{a}}{\sqrt{a+x}+\sqrt{a}}\right]$$
$$=\lim_{x\to 0^-}\frac{a+x-a}{x^2\left(\sqrt{a+x}+\sqrt{a}\right)}=\lim_{x\to 0^-}\frac{1}{x\left(\sqrt{a+x}+\sqrt{a}\right)}=-\infty$$

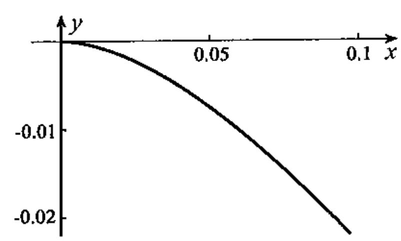
28. Since $\lim\limits_{x\to -a^-}e^{1/(|x|-a)}=\infty$ and $\lim\limits_{x\to -a^+}e^{1/(|x|-a)}=0$, the limit does not exist.

29. (a) The table suggests that the limit is 0.

x	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
x^2	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$\ln x$	-2.30	-4.61	-6.91	-9.21	-11.5
$x^2\ln x$	-2.30×10^{-2}	-4.61×10^{-4}	-6.91×10^{-6}	-9.21×10^{-8}	-1.15×10^{-9}

x	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
x^2	10^{-12}	10^{-14}	10^{-16}	10^{-18}	10^{-20}
$\ln x$	-13.8	-16.1	-18.4	-20.7	-23.0
$x^2\ln x$	-1.38×10^{-11}	-1.61×10^{-13}	-1.84×10^{-15}	-2.07×10^{-17}	-2.30×10^{-19}

(b) The graph of $x^2\ln x$ to the right also suggests that the limit is 0.



30.

x	1.0	0.1	0.05	0.01	0.005
x^{10}	1	10^{-10}	9.77×10^{-14}	10^{-20}	9.77×10^{-24}
$e^{1/x}$	2.72	2.20×10^{4}	4.85×10^{8}	2.69×10^{43}	7.23×10^{86}
$x^{10}e^{1/x}$	2.72	2.20×10^{-6}	4.74×10^{-5}	2.69×10^{23}	7.06×10^{63}

Thus, $\lim\limits_{x\to 0^+}x^{10}e^{1/x}=\infty$.

EXERCISES 2.3

1. $\lim\limits_{x\to\infty}\dfrac{x+1}{2x-1}=\lim\limits_{x\to\infty}\dfrac{1+\frac{1}{x}}{2-\frac{1}{x}}=\dfrac{1}{2}$

2. $\lim\limits_{x\to\infty}\dfrac{1-x}{3+2x}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}-1}{\frac{3}{x}+2}=-\dfrac{1}{2}$

3. $\lim\limits_{x\to\infty}\dfrac{x^2+1}{2x^3+5}=\lim\limits_{x\to\infty}\dfrac{1+\frac{1}{x^2}}{2x+\frac{5}{x^2}}=0$

4. $\lim\limits_{x\to\infty}\dfrac{1-4x^3}{3+2x-x^2}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x^2}-4x}{\frac{3}{x^2}+\frac{2}{x}-1}=\infty$

5. $\lim\limits_{x\to-\infty}\dfrac{2+x-x^2}{3+4x^2}=\lim\limits_{x\to-\infty}\dfrac{\frac{2}{x^2}+\frac{1}{x}-1}{\frac{3}{x^2}+4}=-\dfrac{1}{4}$

6. $\lim\limits_{x\to-\infty}\dfrac{x^3-2x^2}{3x^3+4x^2}=\lim\limits_{x\to-\infty}\dfrac{1-\frac{2}{x}}{3+\frac{4}{x}}=\dfrac{1}{3}$

7. $\lim\limits_{x\to-\infty}\dfrac{x^3-2x^2+x+1}{x^4+3x}=\lim\limits_{x\to-\infty}\dfrac{1-\frac{2}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{x+\frac{3}{x^2}}=0$

8. $\lim\limits_{x\to-\infty}\dfrac{x^3-2x^2+x+1}{x^2-x+1}=\lim\limits_{x\to-\infty}\dfrac{x-2+\frac{1}{x}+\frac{1}{x^2}}{1-\frac{1}{x}+\frac{1}{x^2}}=-\infty$

9. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{2x+1}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x^2}}}{2+\frac{1}{x}}=\frac{1}{2}$

10. $\displaystyle\lim_{x\to\infty}\frac{3x-1}{\sqrt{5+4x^2}}=\lim_{x\to\infty}\frac{3-\frac{1}{x}}{\sqrt{\frac{5}{x^2}+4}}=\frac{3}{2}$

11. This limit does not exist because the function is not defined for $x<-1/\sqrt{2}$.

12. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{1-2x}}{x+2}=\lim_{x\to-\infty}\frac{\frac{\sqrt{1-2x}}{\sqrt{-x}}}{\frac{x+2}{\sqrt{-x}}}=\lim_{x\to-\infty}\frac{\sqrt{-\frac{1}{x}+2}}{-\sqrt{-x}+\frac{2}{\sqrt{-x}}}=0$

13. $\displaystyle\lim_{x\to\infty}\sqrt{\frac{2+x}{x-2}}=\lim_{x\to\infty}\sqrt{\frac{\frac{2}{x}+1}{1-\frac{2}{x}}}=1$

14. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{3+x}}{\sqrt{x}}=\lim_{x\to\infty}\sqrt{\frac{3}{x}+1}=1$

15. $\displaystyle\lim_{x\to\infty}(x^2-x^3)=\lim_{x\to\infty}x^2(1-x)=-\infty$

16. $\displaystyle\lim_{x\to\infty}\left(x+\frac{1}{x}\right)=\infty$

17. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x+5}}=\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{1+\frac{5}{x}}}=\infty$

18. $\displaystyle\lim_{x\to-\infty}\frac{x^2}{\sqrt{3-x}}=\lim_{x\to-\infty}\frac{\frac{x^2}{\sqrt{-x}}}{\frac{\sqrt{3-x}}{\sqrt{-x}}}=\lim_{x\to-\infty}\frac{-x\sqrt{-x}}{\sqrt{1-\frac{3}{x}}}=\infty$

19. $\displaystyle\lim_{x\to-\infty}\frac{x}{\sqrt[3]{4+x^3}}=\lim_{x\to-\infty}\frac{1}{\sqrt[3]{\frac{4}{x^3}+1}}=1$

20. $\displaystyle\lim_{x\to\infty}\frac{3x}{\sqrt[3]{2+4x^3}}=\lim_{x\to\infty}\frac{3}{\sqrt[3]{\frac{2}{x^3}+4}}=\frac{3}{\sqrt[3]{4}}$

21. $\displaystyle\lim_{x\to\infty}\frac{1}{2x}\cos x=0$

22. $\displaystyle\lim_{x\to-\infty}\frac{1}{2x}\cos x=0$

23. $\displaystyle\lim_{x\to\infty}\frac{\sin 4x}{x^2}=0$

24. $\displaystyle\lim_{x\to\infty}\frac{\sin^2 x}{x}=0$

25. $\displaystyle\lim_{x\to-\infty}\tan x$ does not exist

26. $\displaystyle\lim_{x\to\infty}\frac{1}{x}\tan x$ does not exist

27. $\displaystyle\lim_{x\to\infty}(\sqrt{x^2+1}-x)=\lim_{x\to\infty}\left[(\sqrt{x^2+1}-x)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right]=\lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}=0$

28. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+4}-x\right)=\lim_{x\to\infty}\left[\left(\sqrt{x^2+4}-x\right)\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}+x}\right]=\lim_{x\to\infty}\left[\frac{(x^2+4)-x^2}{\sqrt{x^2+4}+x}\right]=0$

29. $\displaystyle\lim_{x\to\infty}(\sqrt{2x^2+1}-x)=\lim_{x\to\infty}\left[(\sqrt{2x^2+1}-x)\frac{\sqrt{2x^2+1}+x}{\sqrt{2x^2+1}+x}\right]=\lim_{x\to\infty}\frac{2x^2+1-x^2}{\sqrt{2x^2+1}+x}$

$$=\lim_{x\to\infty}\frac{x^2+1}{\sqrt{2x^2+1}+x}=\lim_{x\to\infty}\frac{x+\frac{1}{x}}{\sqrt{2+\frac{1}{x^2}}+1}=\infty$$

30. $\displaystyle\lim_{x\to-\infty}\left(\sqrt{2x^2+1}-x\right)=\infty$

31. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{3x^2+2}}{x+4}=\lim_{x\to\infty}\frac{\sqrt{3+\frac{2}{x^2}}}{1+\frac{4}{x}}=\sqrt{3}$

32. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{4x^2+7}}{2x+3}=\lim_{x\to\infty}\frac{\sqrt{4+\frac{7}{x^2}}}{2+\frac{3}{x}}=1$

33. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{3x^2+2}}{x+4}=\lim_{x\to-\infty}\frac{\frac{1}{x}\sqrt{3x^2+2}}{1+\frac{4}{x}}=\lim_{x\to-\infty}\frac{-\sqrt{3+\frac{2}{x^2}}}{1+\frac{4}{x}}=-\sqrt{3}$

34. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{4x^2+7}}{2x+3}=\lim_{x\to-\infty}\frac{\frac{\sqrt{4x^2+7}}{x}}{\frac{2x+3}{x}}=\lim_{x\to-\infty}\frac{-\sqrt{4+\frac{7}{x^2}}}{2+\frac{3}{x}}=-1$

35. $\lim_{x\to\infty}(\sqrt{x^2+4}-\sqrt{x^2-1}) = \lim_{x\to\infty}\left[(\sqrt{x^2+4}-\sqrt{x^2-1})\frac{\sqrt{x^2+4}+\sqrt{x^2-1}}{\sqrt{x^2+4}+\sqrt{x^2-1}}\right]$

$$= \lim_{x\to\infty}\frac{(x^2+4)-(x^2-1)}{\sqrt{x^2+4}+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{5}{\sqrt{x^2+4}+\sqrt{x^2-1}} = 0$$

36. $\lim_{x\to\infty}(\sqrt[3]{1+x}-\sqrt[3]{x}) = \lim_{x\to\infty}\left\{\left[(1+x)^{1/3}-x^{1/3}\right]\frac{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}\right\}$

$$= \lim_{x\to\infty}\left[\frac{(1+x)-x}{(1+x)^{2/3}+(1+x)^{1/3}x^{1/3}+x^{2/3}}\right] = 0$$

37. $\lim_{x\to\infty}(\sqrt{x^2+x}-x) = \lim_{x\to\infty}\left[(\sqrt{x^2+x}-x)\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right] = \lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}$$

38. $\lim_{x\to-\infty}\left(\sqrt{x^2+x}-x\right) = \infty$

39. $\lim_{x\to\infty}\frac{x^2+ax-2}{ax^2+5} = \lim_{x\to\infty}\frac{1+\frac{a}{x}-\frac{2}{x^2}}{a+\frac{5}{x^2}} = \frac{1}{a}$

40. $\lim_{x\to\infty}\frac{x}{\sqrt{ax^2+3x+2}} = \lim_{x\to\infty}\frac{1}{\sqrt{a+\frac{3}{x}+\frac{2}{x^2}}} = \frac{1}{\sqrt{a}}$

41. $\lim_{x\to\infty}(\sqrt{x^2+ax}-x) = \lim_{x\to\infty}\left[(\sqrt{x^2+ax}-x)\frac{\sqrt{x^2+ax}+x}{\sqrt{x^2+ax}+x}\right] = \lim_{x\to\infty}\frac{(x^2+ax)-x^2}{\sqrt{x^2+ax}+x}$

$$= \lim_{x\to\infty}\frac{a}{\sqrt{1+\frac{a}{x}}+1} = \frac{a}{2}$$

42. $\lim_{x\to-\infty}\frac{\sqrt{ax^2+7}}{x-3a} = \lim_{x\to-\infty}\frac{\frac{\sqrt{ax^2+7}}{x}}{\frac{x-3a}{x}} = \lim_{x\to-\infty}\frac{-\sqrt{a+\frac{7}{x^2}}}{1-\frac{3a}{x}} = -\sqrt{a}$

43. A vertical asymptote is $x=-3/4$. Since $\lim_{x\to\pm\infty}\frac{2-x}{3+4x} = -\frac{1}{4}$, the horizontal asymptote is $y=-1/4$. With $f(x)$ expressed in the form $f(x) = -\frac{1}{4}+\frac{11/4}{4x+3}$, we can say that the for large negative x, $f(x) < -1/4$, and for large positive x, $f(x) > -1/4$. Hence, the graph approaches the horizontal asymptote from below as $x\to-\infty$ and from above as $x\to\infty$.

44. A vertical asymptote is $x=5/2$. Since $\lim_{x\to\pm\infty}\frac{x+3}{2x-5} = \frac{1}{2}$, the horizontal asymptote is $y=1/2$. With $f(x)$ expressed in the form $f(x) = \frac{1}{2}+\frac{11/2}{2x-5}$, we can say that the for large negative x, $f(x) < 1/2$, and for large positive x, $f(x) > 1/2$. Hence, the graph approaches the horizontal asymptote from below as $x\to-\infty$ and from above as $x\to\infty$.

45. Since $\lim_{x\to\infty}\frac{3x-1}{\sqrt{5+2x^2}} = \lim_{x\to\infty}\frac{3-1/x}{\sqrt{5/x^2+2}} = \frac{3}{\sqrt{2}}$, $y=3/\sqrt{2}$ is a horizontal asymptote as $x\to\infty$. Since $\lim_{x\to-\infty}\frac{3x-1}{\sqrt{5+2x^2}} = \lim_{x\to-\infty}\frac{3-1/x}{-\sqrt{5/x^2+2}} = -\frac{3}{\sqrt{2}}$, $y=-3/\sqrt{2}$ is a horizontal asymptote as $x\to-\infty$. To determine whether the graph approaches $y=3/\sqrt{2}$ from above or below as $x\to\infty$, we write

$$f(x) = \frac{3x-1}{\sqrt{5+2x^2}} = \sqrt{\frac{(3x-1)^2}{5+2x^2}} = \sqrt{\frac{9x^2-6x+1}{2x^2+5}} = \sqrt{\frac{9}{2}-\frac{6x+43/2}{2x^2+5}}.$$

This shows that $f(x) < 3/\sqrt{2}$ for large x, and the graph therefore approaches the asymptote from below. Similarly, for large negative values of x, we express $f(x)$ in the form $f(x) = -\sqrt{\frac{9}{2}-\frac{6x+43/2}{2x^2+5}}$, and this shows that the graph of $f(x)$ approaches $y=-3/\sqrt{2}$ from below as $x\to-\infty$.

46. A vertical asymptote is $x = -3/2$. Since $\lim_{x\to\infty} \dfrac{\sqrt{5x^2+7}}{2x+3} = \lim_{x\to\infty} \dfrac{\sqrt{5+7/x^2}}{2+3/x} = \dfrac{\sqrt{5}}{2}$, $y = \sqrt{5}/2$ is a horizontal asymptote as $x \to \infty$. Since $\lim_{x\to-\infty} \dfrac{\sqrt{5x^2+7}}{2x+3} = \lim_{x\to-\infty} \dfrac{-\sqrt{5+7/x^2}}{2+3/x} = -\dfrac{\sqrt{5}}{2}$, $y = -\sqrt{5}/2$ is a horizontal asymptote as $x \to -\infty$. To determine whether the graph approaches $y = \sqrt{5}/2$ from above or below as $x \to \infty$, we write

$$f(x) = \frac{\sqrt{5x^2+7}}{2x+3} = \sqrt{\frac{5x^2+7}{(2x+3)^2}} = \sqrt{\frac{5x^2+7}{4x^2+12x+9}} = \sqrt{\frac{5}{4} - \frac{15x+17/4}{4x^2+12x+9}}.$$

This shows that $f(x) < \sqrt{5}/2$ for large x, and the graph therefore approaches the asymptote from below. Similarly, for large negative values of x, we express $f(x)$ in the form $f(x) = -\sqrt{\dfrac{5}{4} - \dfrac{15x+17/4}{4x^2+12x+9}}$, and this shows that the graph of $f(x)$ approaches $y = -\sqrt{5}/2$ from below as $x \to -\infty$.

47. Since $3 + 2x - x^2 = (3-x)(1+x)$, horizontal asymptotes are $x = 3$ and $x = -1$. With $f(x)$ expressed in the form $f(x) = \dfrac{1-4x^3}{3+2x-x^2} = 4x + 8 + \dfrac{28x+23}{x^2-2x-3}$, we see that $y = 4x+8$ is an oblique asymptote that is approached from above as $x \to \infty$, and from below as $x \to -\infty$.

48. Since $x^2 - 3x + 1 = 0$ for $x = (3 \pm \sqrt{9-4})/2 = (3 \pm \sqrt{5})/2$, vertical asymptotes occur at these values of x. With $f(x)$ expressed in the form $f(x) = \dfrac{3x^3+2x-1}{1-3x+x^2} = 3x + 9 + \dfrac{26x-10}{x^2-3x+1}$, we see that $y = 3x+9$ is an oblique asymptote that is approached from above as $x \to \infty$, and from below as $x \to -\infty$.

49. $$\lim_{x\to-\infty} \frac{\sqrt{ax^2+bx+c}}{dx+e} = \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{ax^2+bx+c}}{d+\frac{e}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{a+\frac{b}{x}+\frac{c}{x^2}}}{d+\frac{e}{x}} = -\frac{\sqrt{a}}{d}$$

50. Clearly a and d must both be positive else neither square root is defined for large x. If we rationalize,

$$\begin{aligned}
&\lim_{x\to\infty} \left(\sqrt{ax^2+bx+c} - \sqrt{dx^2+ex+f}\right) \\
&= \lim_{x\to\infty} \left[\left(\sqrt{ax^2+bx+c} - \sqrt{dx^2+ex+f}\right) \frac{\sqrt{ax^2+bx+c} + \sqrt{dx^2+ex+f}}{\sqrt{ax^2+bx+c} + \sqrt{dx^2+ex+f}}\right] \\
&= \lim_{x\to\infty} \left[\frac{(ax^2+bx+c) - (dx^2+ex+f)}{\sqrt{ax^2+bx+c} + \sqrt{dx^2+ex+f}}\right].
\end{aligned}$$

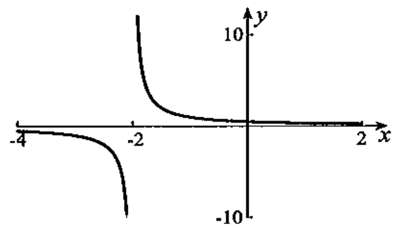
For this limit to exist, we must have $a = d$. When this is the case, the limit becomes

$$\lim_{x\to\infty} \frac{(b-e)x + (c-f)}{\sqrt{ax^2+bx+c} + \sqrt{ax^2+ex+f}} = \lim_{x\to\infty} \frac{(b-e) + \frac{c-f}{x}}{\sqrt{a+\frac{b}{x}+\frac{c}{x^2}} + \sqrt{a+\frac{e}{x}+\frac{f}{x^2}}} = \frac{b-e}{2\sqrt{a}}.$$

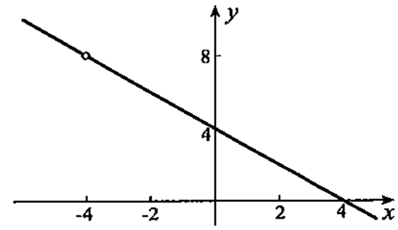
EXERCISES 2.4

1. The function has an infinite discontinuity at $x = -2$.

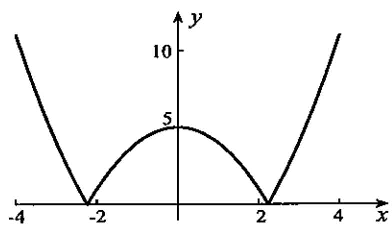
2. For $x \neq -4$, $f(x) = \dfrac{(4-x)(4+x)}{x+4} = 4 - x$. The graph of the function is therefore the straight line $y = 4 - x$ with the point at $x = -4$ deleted. The computer does not show the hole at the removable discontinuity $x = -4$.



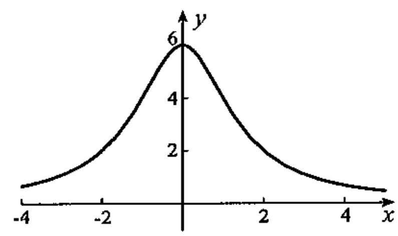
3. The function has no discontinuities.



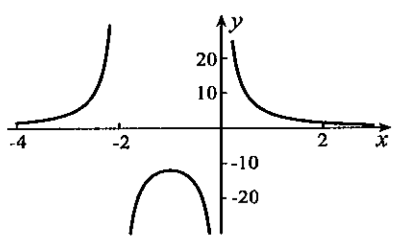
4. The function has no discontinuities.



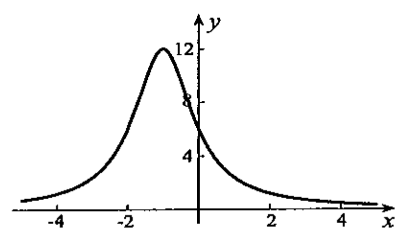
5. The function has infinite discontinuities at $x = 0, -2$.



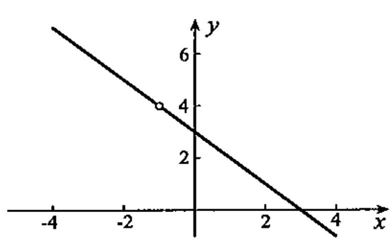
6. The function has no discontinuities.



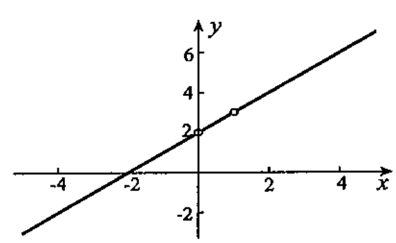
7. For $x \neq -1$, $f(x) = \dfrac{(3-x)(1+x)}{x+1} = 3 - x$. The graph of the function is therefore the straight line $y = 3 - x$ with the point at $x = -1$ deleted. The computer does not show the hole at the removable discontinuity $x = -1$.



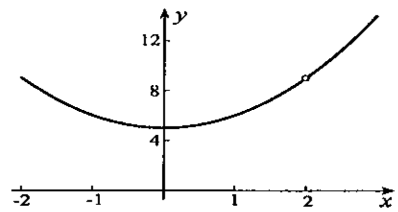
8. For $x \neq 0, 1$, $f(x) = \dfrac{x(x+2)(x-1)}{x(x-1)} = x + 2$. The graph of the function is therefore the straight line $y = x + 2$ with the points at $x = 0, 1$ deleted. The computer does not show holes at the removable discontinuities $x = 0, 1$.

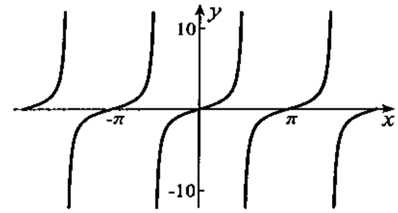


9. For $x \neq 2$, $f(x) = \dfrac{(x-2)(x^2+5)}{x-2} = x^2 + 5$. The graph of the function is therefore the parabola $y = x^2 + 5$ with the point at $x = 2$ deleted. The computer does not show the hole at the removable discontinuity $x = 2$.

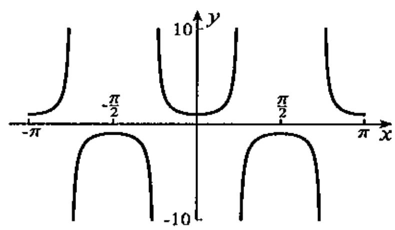


10. The tangent function has infinite discontinuities at $x = (2n+1)\pi/2$, where n is an integer.

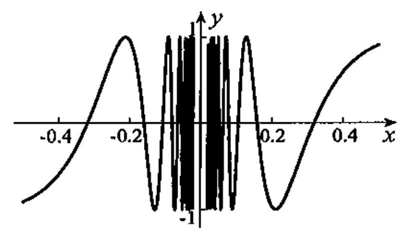




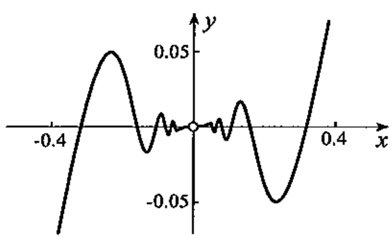
11. The function has infinite discontinuities when $2x = \dfrac{(2n+1)\pi}{2} \Longrightarrow x = \dfrac{(2n+1)\pi}{4}$, where n is an integer.



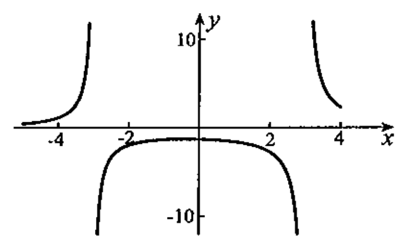
12. The function is discontinuous at $x = 0$. The discontinuity is not removable, jump, or infinite.



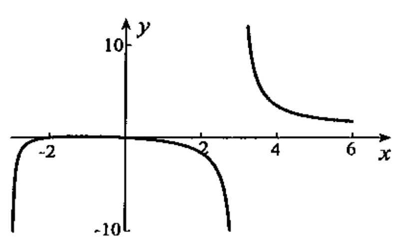
13. The function has a removable discontinuity at $x = 0$. The computer does not show the hole at $x = 0$.



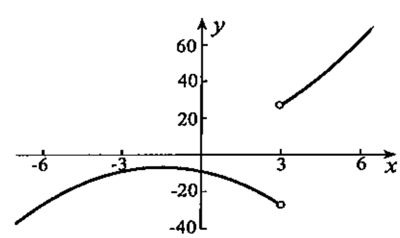
14. The function has infinite discontinuities at $x = \pm 3$.



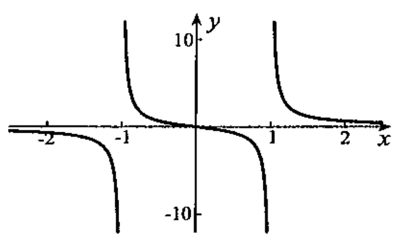
15. The function has infinite discontinuities at $x = \pm 3$.



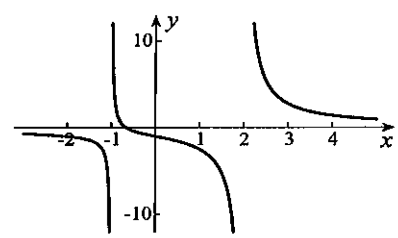
16. The function has a jump discontinuity at $x = 3$. The computer does not show the empty circles at $x = 3$.



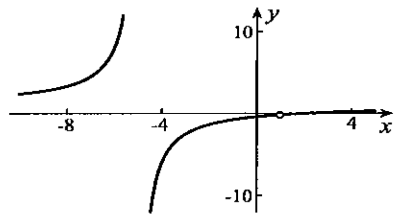
17. The function has infinite discontinuities at $x = \pm 1$.



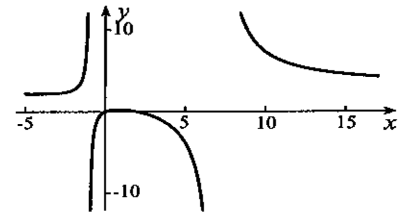
18. The function has infinite discontinuities at $x = -1, 2$.



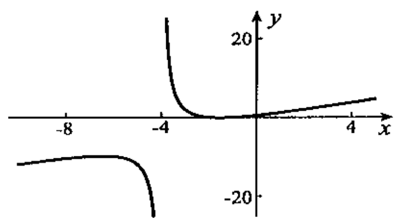
19. For $x \neq 1$, $f(x) = \dfrac{(x-1)(x-2)}{(x-1)(x+5)} = \dfrac{x-2}{x+5}$. The graph of the function is therefore the curve $y = (x-2)/(x+5)$ with the point at $x = 1$ deleted. The computer does not show the hole at the removable discontinuity $x = 1$. There is also an infinite discontinuity at $x = -5$.



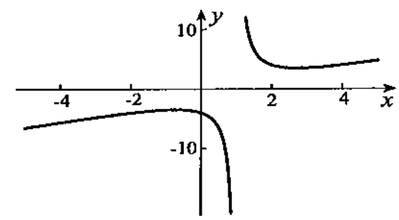
20. The function has infinite discontinuities at $x = -1, 7$.



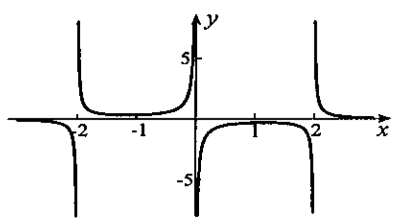
21. The function has an infinite discontinuity at $x = -4$.



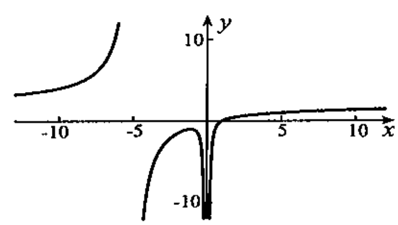
22. The function has an infinite discontinuity at $x = 1$.



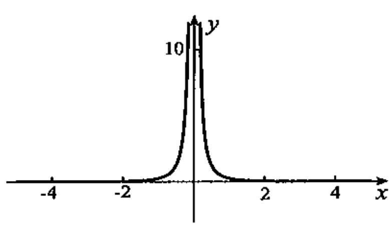
23. The function has infinite discontinuities at $x = 0, \pm 2$.



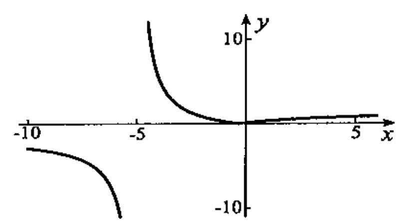
24. The function has infinite discontinuities at $x = 0, -5$.



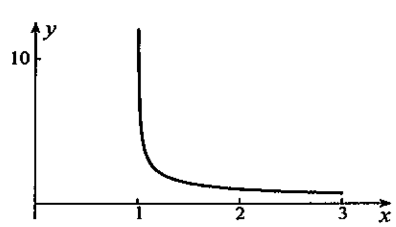
25. The function has an infinite discontinuity at $x = 0$.



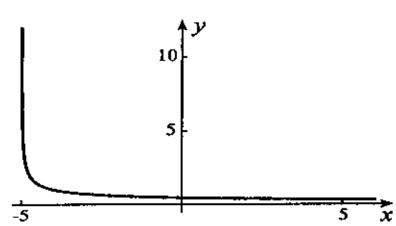
26. The function has an infinite discontinuity at $x = -5$.



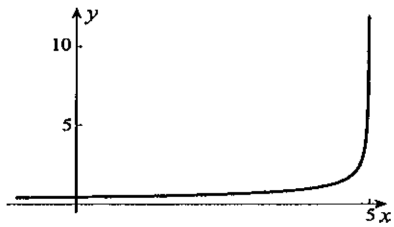
27. The function is continuous for $x > 1$.



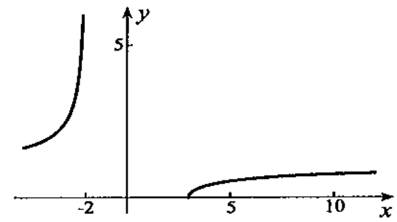
28. The function is continuous for $x > -5$.



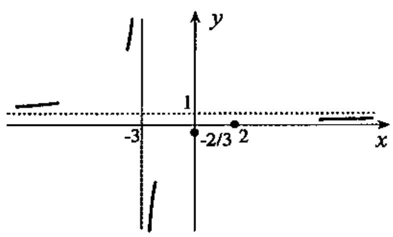
29. The function is continuous for $x < 5$.

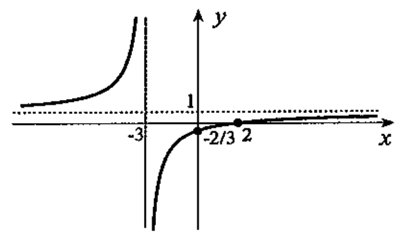


30. The function is continuous for $x < -2$ and $x \geq 3$.

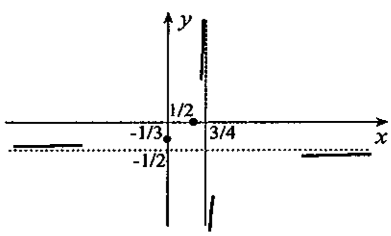


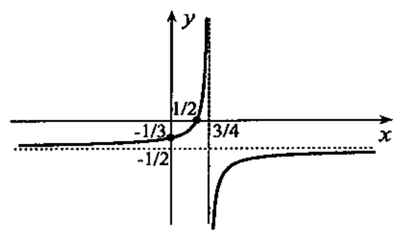
31. Right- and left-limits as $x \to -3$ and $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.



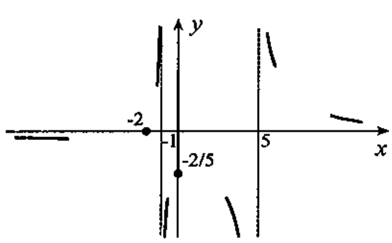


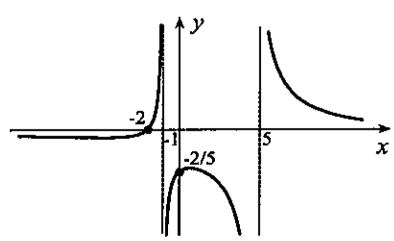
32. Right- and left-limits as $x \to 3/4$ and $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.



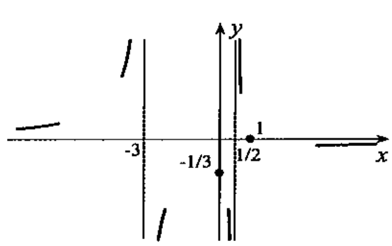


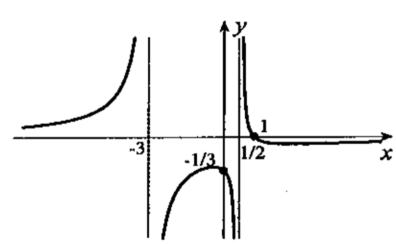
33. First we factor the denominator $f(x) = \dfrac{x+2}{(x-5)(x+1)}$. Right- and left-limits as $x \to -1$ and $x \to 5$, and limits as $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.



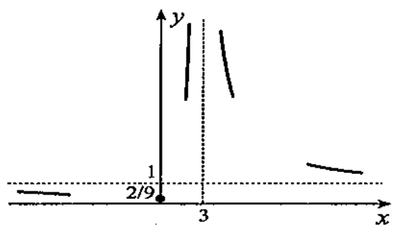


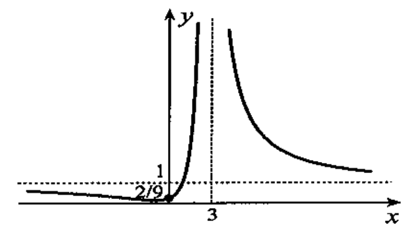
34. First we factor the denominator $f(x) = \dfrac{1-x}{(2x-1)(x+3)}$. Right- and left-limits as $x \to -3$ and $x \to 1/2$, and limits as $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.



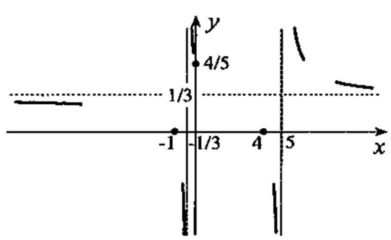


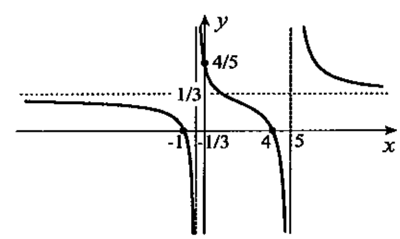
35. First we factor the denominator $f(x) = \dfrac{x^2+x+2}{(x-3)^2}$. Right- and left-limits as $x \to 3$ lead to the vertical asymptote in the left drawing below. To take limits as $x \to \pm\infty$, we use long division to write $f(x)$ in the form $f(x) = 1 + \dfrac{7x-7}{x^2-6x+9}$. Limits as $x \to \pm\infty$ lead to the horizontal asymptote in the same drawing. With a y-intercept equal to 2/9 and no x-intercept, we finish the graph as shown to the right.



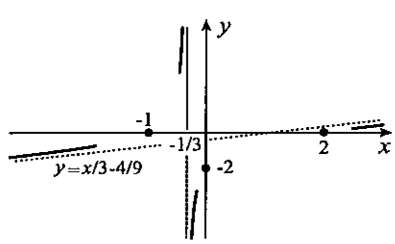


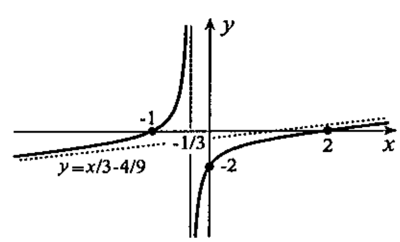
36. First we factor numerator and denominator $f(x) = \dfrac{(x-4)(x+1)}{(3x+1)(x-5)}$. Right- and left-limits as $x \to -1/3$ and $x \to 5$ lead to the vertical asymptotes in the left drawing below. To take limits as $x \to \pm\infty$, we use long division to write $f(x)$ in the form $f(x) = \dfrac{1}{3} + \dfrac{5x/3 - 7/3}{3x^2-14x-5}$. Limits as $x \to \pm\infty$ lead to the horizontal asymptote in the same drawing. With x- and y-intercepts, we finish the graph as shown to the right.



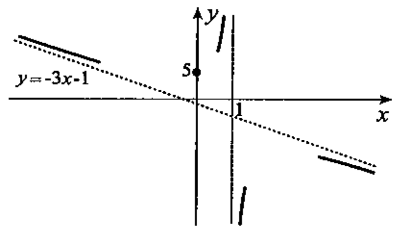


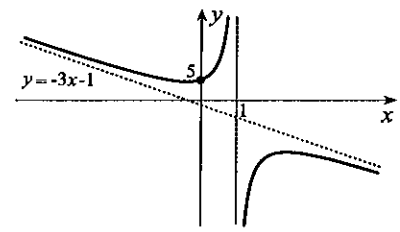
37. First we factor the numerator $f(x) = \dfrac{(x-2)(x+1)}{3x+1}$. Right- and left-limits as $x \to -1/3$ lead to the vertical asymptote in the left drawing below. The graph has an oblique asymptote that we can identify with long division, $f(x) = \dfrac{x}{3} - \dfrac{4}{9} - \dfrac{14/9}{3x+1}$. The line $y = x/3 - 4/9$ is the oblique asymptote. With x-intercepts at -1 and 2, and y-intercept equal to -2, we finish the graph as shown to the right.



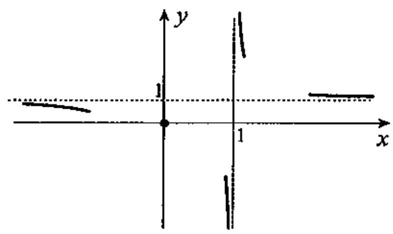


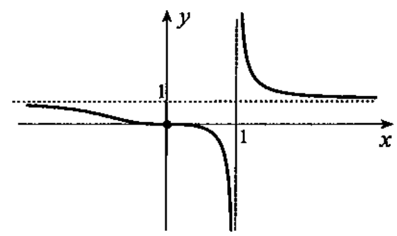
38. Right- and left-limits as $x \to 1$ lead to the vertical asymptote in the left drawing below. The graph has an oblique asymptote that we can identify with long division, $f(x) = -3x - 1 + \dfrac{6}{1-x}$. The line $y = -3x - 1$ is the oblique asymptote. With no x-intercepts, and y-intercept equal to 5, we finish the graph as shown to the right.



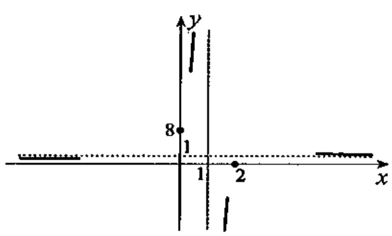


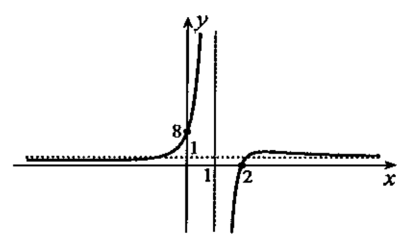
39. First we factor the denominator $f(x) = \dfrac{x^3}{(x-1)(x^2+x+1)}$. Right- and left-limits as $x \to 1$, and limits as $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts both at the origin, we finish the graph as shown to the right.



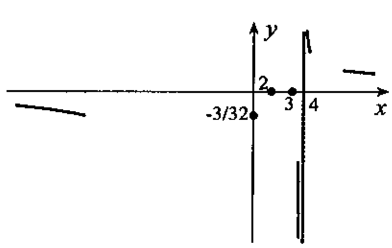


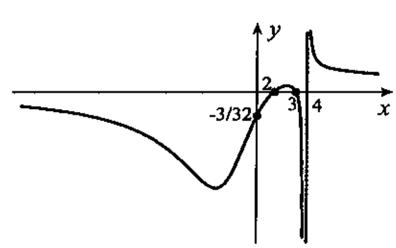
40. First we factor numerator and denominator $f(x) = \dfrac{(x-2)(x^2+x+4)}{(x-1)^3}$. Right- and left-limits as $x \to 1$ lead to the vertical asymptote in the left drawing below. To take limits as $x \to \pm\infty$, we use long division to write $f(x)$ in the form $f(x) = 1 + \dfrac{2x^2-x-7}{x^3-3x^2+3x-1}$. Limits as $x \to \pm\infty$ lead to the horizontal asymptote in the same drawing. With x- and y-intercepts, we finish the graph as shown to the right.



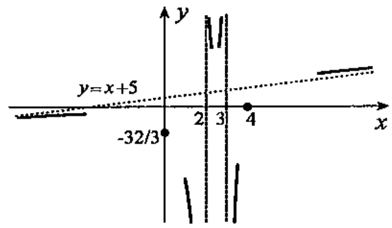


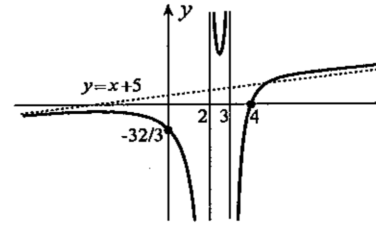
41. First we factor numerator and denominator $f(x) = \dfrac{(x-2)(x-3)}{(x-4)(x^2+4x+16)}$. Right- and left-limits as $x \to 4$, and limits as $x \to \pm\infty$ lead to the vertical and horizontal asymptotes in the left drawing below. With x- and y-intercepts, we finish the graph as shown to the right.





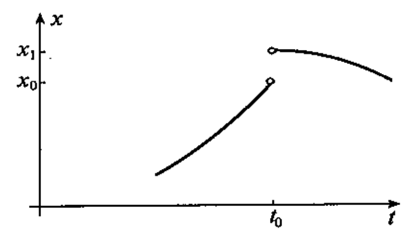
42. First we factor the numerator and denominator $f(x) = \dfrac{(x-4)(x^2+4x+16)}{(x-2)(x-3)}$. Right- and left-limits as $x \to 2$ and $x \to 3$ lead to the vertical asymptotes in the left drawing below. The graph as an oblique asymptote that we can identify with long division, $f(x) = x + 5 + \dfrac{19x-94}{x^2-5x+6}$. The line $y = x+5$ is the oblique asymptote. With x-intercept 4, and y-intercept equal to $-32/3$, we finish the graph as shown to the right.



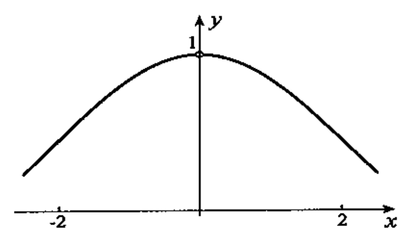


43. Yes, since $f(0) = 0 = \lim_{x\to 0} f(x)$ (see Exercise 51 in Exercises 2.1).

44. No. If it were to have a discontinuity at $t = t_0$ as in the figure to the right, the particle would disappear at position x_0 and reappear instantaneously at position x_1.



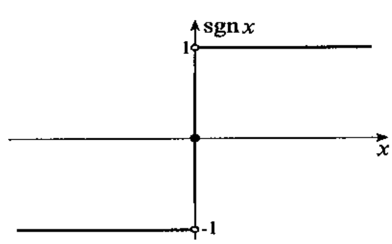
45. The graph of the function to the right was generated on a computer but the hole was added manually. It indicates that $\lim_{x\to 0} x^{-1}\sin x = 1$, and therefore the discontinuity is removable.

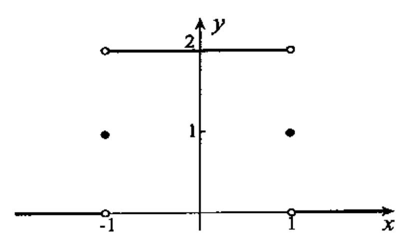


46. If we set $h = x - a$ in the left side of 2.4a, then $f(a) = \lim\limits_{x\to a} f(x) = \lim\limits_{h+a\to a} f(a+h) = \lim\limits_{h\to 0} f(a+h)$.

47. (a) The graph indicates that the function is discontinous at $x = 0$.

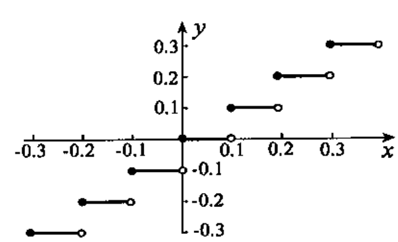
(b) The function is discontinuous at $x = \pm 1$.





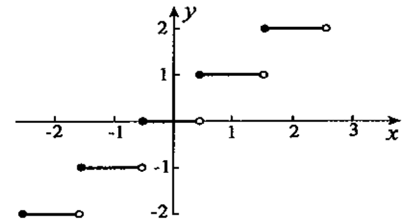
48. (a) The function is discontinuous at $x = n/10$, where n is an integer.
(b) Let a positive number x be denoted by $z.abc\cdots$ where z is the integer part, and a, b, and c are the first three decimals. Then

$$f(z.abc\cdots) = (1/10)\lfloor za.bc\cdots\rfloor = \tfrac{1}{10}(za) = z.a.$$



49. The function $\lfloor 100x+1\rfloor/100$. For example, if $x = -2.357$, then $\lfloor 100(-2.357)+1\rfloor/100 = \lfloor -234.7\rfloor/100 = -235/100 = -2.35$.

50. (a) The function is discontinuous at $x = n + 1/2$, where n is an integer.
(b) Let a positive number x be denoted by $z.a\cdots$ where z is the integer part, and a is the first decimal. Suppose that a is equal to 0, 1, 2, 3, or 4. The integer part of $x + 1/2$ is z and the first decimal in the number $x + 1/2$ is 5, 6, 7, 8, or 9, and therefore $f(x) = \lfloor x + 1/2 \rfloor = z$. On the other hand, suppose that a is equal to 5, 6, 7, 8, or 9. Then the integer part of $x + 1/2$ is $z + 1$, and its first decimal is 0, 1, 2, 3, or 4. Hence, $f(x) = \lfloor x + 1/2 \rfloor = z + 1$.



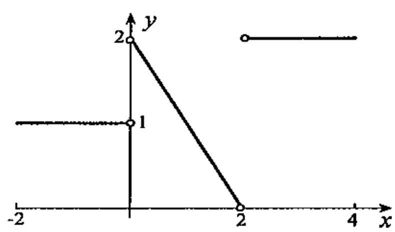
51. (a) $\lfloor 10x + 1/2 \rfloor/10$ (b) $\lfloor 100x + 1/2 \rfloor/100$ (c) $\lfloor 10^n x + 1/2 \rfloor/10^n$

52. There are no points at which the function is continuous since $\lim_{x\to a} f(x)$ does not exist for any a.

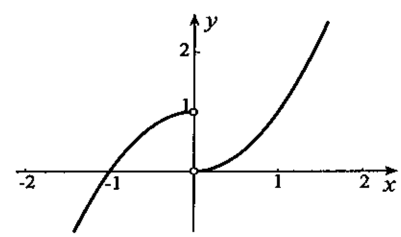
53. The function is continuous only at $x = 0$.

EXERCISES 2.5

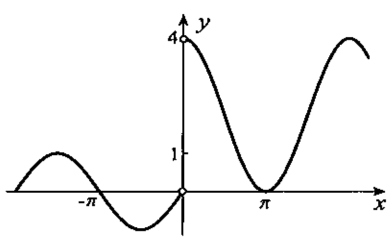
1. $$f(x) = [1 - h(x)] + (2 - x)[h(x) - h(x - 2)] + 2h(x - 2)$$
$$= 1 + (1 - x)h(x) + x\,h(x - 2)$$



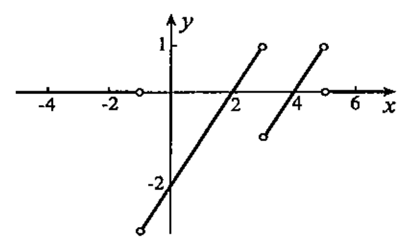
2. $$f(x) = (1 - x^2)[1 - h(x)] + x^2 h(x)$$
$$= 1 - x^2 + (2x^2 - 1)h(x)$$



3. $$f(x) = \sin x[1 - h(x)] + (2 + 2\cos x)h(x)$$
$$= \sin x + (2 + 2\cos x - \sin x)h(x)$$

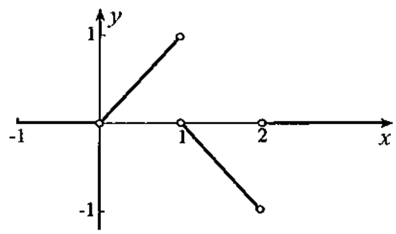


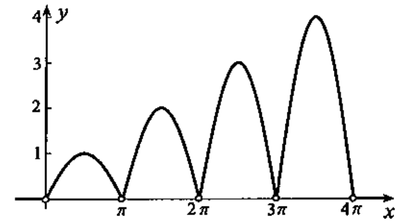
4. $$f(x) = (x - 2)[h(x + 1) - h(x - 3)] + (x - 4)[h(x - 3) - h(x - 5)]$$
$$= (x - 2)h(x + 1) - 2\,h(x - 3) + (4 - x)h(x - 5)$$



5. $$f(x) = x[h(x) - h(x - 1)] + (1 - x)[h(x - 1) - h(x - 2)]$$
$$= x\,h(x) + (1 - 2x)h(x - 1) + (x - 1)h(x - 2)$$

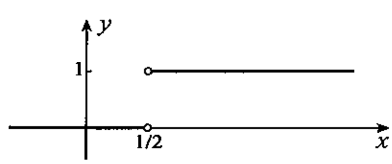
6. $$f(x) = \sin x[h(x) - h(x - \pi)] + 2\sin(x - \pi)[h(x - \pi) - h(x - 2\pi)] + 3\sin(x - 2\pi)[h(x - 2\pi) - h(x - 3\pi)] + 4\sin(x - 3\pi)[h(x - 3\pi) - h(x - 4\pi)]$$
$$= \sin x\,[h(x) - h(x - \pi)] - 2\sin x\,[h(x - \pi) - h(x - 2\pi)] + 3\sin x\,[h(x - 2\pi) - h(x - 3\pi)] - 4\sin x\,[h(x - 3\pi) - h(x - 4\pi)]$$
$$= \sin x\,[h(x) - 3h(x - \pi) + 5h(x - 2\pi) - 7h(x - 3\pi) + 4h(x - 4\pi)]$$





7. $-F[h(t) - h(t - T)]$
8. $10 \sin 4t[h(t - 1) - h(t - 1 - \pi)]$
9. $-50\delta(t - 4)$
10. $[100 + 2(t - 10)][h(t - 10) - h(t - 60)] = (80 + 2t)[h(t - 10) - h(t - 60)]$
11. $60[\delta(t) + \delta(t - 10) + \delta(t - 20) + \delta(t - 30) + \delta(t - 40) + \delta(t - 50) + \delta(t - 60)]$
12. $-(2mg/L)[h(x) - h(x - L/2)]$
13. $-F\delta(x - L/3)$
14. $F_1\delta(x - x_1) - F_2\delta(x - x_2)$
15. $-[3mg/(2L)][h(x - L/3) - h(x - L)]$
16. $h(x - a) - h(x - b) + h(x - c)$
17. $h(t) - h(t - 1) + h(t - 2) - h(t - 3) + h(t - 4) - \cdots$
18. Yes, except at $x = a$ where $h(x - a)h(x - b)$ is undefined whereas $h(x - b) = 0$.

19.



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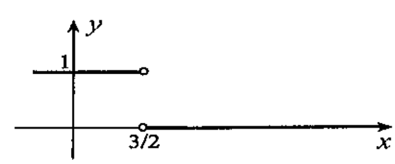

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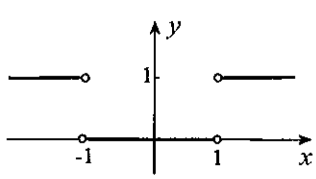

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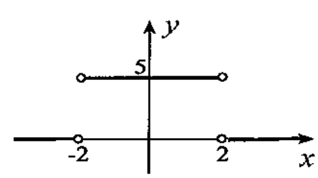

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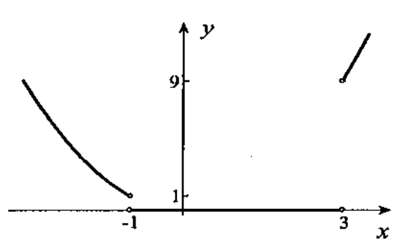

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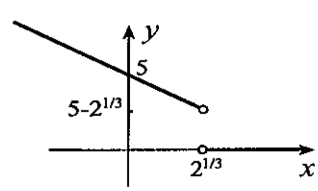


25. The function is

$$[h(t) - h(t - 1)] + 2[h(t - 1) - h(t - 2)] + 3[h(t - 2) - h(t - 3)] + 4h(t - 3)$$
$$= h(t) + h(t - 1) + h(t - 2) + h(t - 3).$$

EXERCISES 2.6

1. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 1 so that $|(x+5)-6| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x-1$, $|x-1| < \epsilon$. Thus, if we choose $0 < |x-1| < \epsilon$, then $|(x+5)-6| < \epsilon$; that is, we can make $x+5$ within ϵ of 6 by choosing x within ϵ of 1.

2. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 2 so that $|(2x-3)-1| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x-2$,

$$|(2x-3)-1| = |2(x-2)| = 2|x-2|.$$

We must now choose x so that $2|x-2| < \epsilon$. But this will be true if $|x-2| < \epsilon/2$. In other words, if we choose x to satisfy $0 < |x-2| < \epsilon/2$, then

$$|(2x-3)-1| = 2|x-2| < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

We have shown that we can make $2x-3$ within ϵ of 1 by choosing x within $\epsilon/2$ of 2.

3. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 0 so that $|(x^2+3)-3| < \epsilon$. To do this, we rewrite the inequality in the form $|x|^2 < \epsilon$. But this will be true if $|x| < \sqrt{\epsilon}$. In other words, if we choose x to satisfy $0 < |x| < \sqrt{\epsilon}$, then

$$|(x^2+3)-3| = |x|^2 < (\sqrt{\epsilon})^2 = \epsilon.$$

We have shown that we can make x^2+3 within ϵ of 3 by choosing x within $\sqrt{\epsilon}$ of 0.

4. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 1 so that $|(x^2+4)-5| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x-1$,

$$|(x^2+4)-5| = |(x-1)^2 + 2(x-1)|.$$

We must now choose x so that

$$|(x-1)^2 + 2(x-1)| < \epsilon.$$

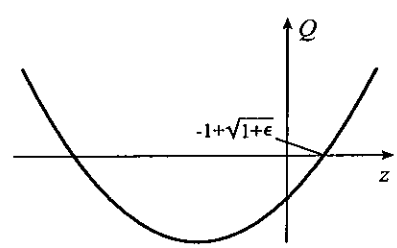
Since $|(x-1)^2 + 2(x-1)| \le |x-1|^2 + 2|x-1|$, the above inequality is satisfied if x is chosen so that

$$|x-1|^2 + 2|x-1| < \epsilon.$$

Suppose we set $z = |x-1|$, and consider the parabola $Q(z) = z^2 + 2z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-2 \pm \sqrt{4+4\epsilon}}{2} = -1 \pm \sqrt{1+\epsilon}.$$

The graph shows that $Q(z) < 0$ whenever $0 < z < -1 + \sqrt{1+\epsilon}$. In other words, if $0 < |x-1| < \sqrt{1+\epsilon} - 1$, then $|x-1|^2 + 2|x-1| < \epsilon$, and therefore $|(x-1)^2 + 2(x-1)| < \epsilon$.



5. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to -2 so that $|(3-x^2)+1| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x+2$,

$$|(3-x^2)+1| = |x^2-4| = |(x+2)^2 - 4(x+2)|.$$

We must now choose x so that

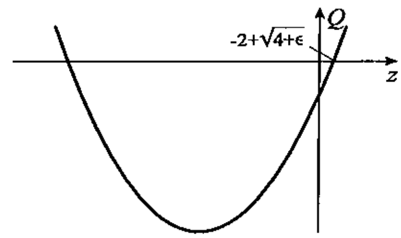
$$|(x+2)^2 - 4(x+2)| < \epsilon.$$

Since $|(x+2)^2 - 4(x+2)| \le |x+2|^2 + 4|x+2|$, the above inequality is satisfied if x is chosen so that

$$|x+2|^2 + 4|x+2| < \epsilon.$$

Suppose we set $z = |x+2|$, and consider the parabola $Q(z) = z^2 + 4z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-4 \pm \sqrt{16+4\epsilon}}{2} = -2 \pm \sqrt{4+\epsilon}.$$



The graph shows that $Q(z) < 0$ whenever $0 < z < -2 + \sqrt{4+\epsilon}$. In other words, if $0 < |x+2| < \sqrt{4+\epsilon} - 2$, then $|x+2|^2 + 4|x+2| < \epsilon$, and therefore $|(x+2)^2 - 4(x+2)| < \epsilon$.

6. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 3 so that $|(x^2 - 7x) + 12| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x - 3$,

$$|(x^2 - 7x) + 12| = |x^2 - 7x + 12| = |(x-3)^2 - (x-3)|.$$

We must now choose x so that

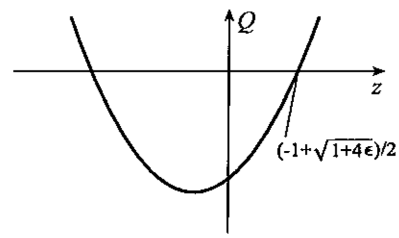
$$|(x-3)^2 - (x-3)| < \epsilon.$$

Since $|(x-3)^2 - (x-3)| \le |x-3|^2 + |x-3|$, the above inequality is satisfied if x is chosen so that

$$|x-3|^2 + |x-3| < \epsilon.$$

Suppose we set $z = |x-3|$, and consider the parabola $Q(z) = z^2 + z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-1 \pm \sqrt{1+4\epsilon}}{2}.$$



The graph shows that $Q(z) < 0$ whenever $0 < z < \left(-1 + \sqrt{1+4\epsilon}\right)/2$. In other words, if $0 < |x-3| < \left(\sqrt{1+4\epsilon} - 1\right)/2$, then $|x-3|^2 + |x-3| < \epsilon$, and therefore $|(x-3)^2 - (x-3)| < \epsilon$.

7. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to -1 so that $|(x^2 - 3x + 4) - 8| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x + 1$,

$$|(x^2 - 3x + 4) - 8| = |x^2 - 3x - 4| = |(x+1)^2 - 5(x+1)|.$$

We must now choose x so that

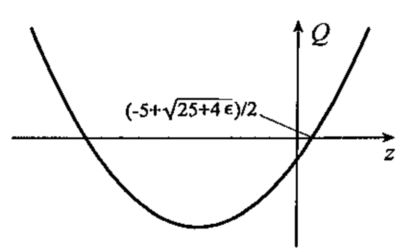
$$|(x+1)^2 - 5(x+1)| < \epsilon.$$

Since $|(x+1)^2 - 5(x+1)| \le |x+1|^2 + 5|x+1|$, the above inequality is satisfied if x is chosen so that

$$|x+1|^2 + 5|x+1| < \epsilon.$$

Suppose we set $z = |x+1|$, and consider the parabola $Q(z) = z^2 + 5z - \epsilon$ in the figure. It crosses the z-axis when

$$z = \frac{-5 \pm \sqrt{25+4\epsilon}}{2}.$$



The graph shows that $Q(z) < 0$ whenever $0 < z < (-5 + \sqrt{25+4\epsilon})/2$. In other words, if $0 < |x+1| < (\sqrt{25+4\epsilon} - 5)/2$, then $|x+1|^2 + 5|x+1| < \epsilon$, and therefore $|(x+1)^2 + 5(x+1)| < \epsilon$.

8. Suppose $\epsilon > 0$ is given. We must show that we can choose x sufficiently close to 1 so that $|(x^2 + 3x + 5) - 9| < \epsilon$. To do this, we rewrite the inequality with the x's in the combination $x - 1$,

$$|(x^2 + 3x + 5) - 9| = |x^2 + 3x - 4| = |(x-1)^2 + 5(x-1)|.$$

We must now choose x so that

$$|(x-1)^2+5(x-1)|<\epsilon.$$

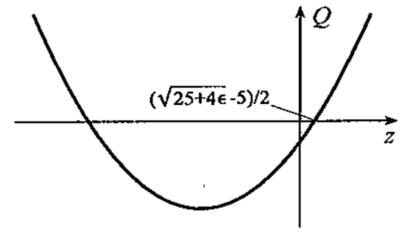
Since $|(x-1)^2+5(x-1)| \le |x-1|^2+5|x-1|$, the above inequality is satisfied if x is chosen so that

$$|x-1|^2+5|x-1|<\epsilon.$$

Suppose we set $z=|x-1|$ and consider the parabola $Q(z)=z^2+5z-\epsilon$ in the figure. It crosses the z-axis when

$$z=\frac{-5\pm\sqrt{25+4\epsilon}}{2}.$$

The graph shows that $Q(z)<0$ whenever $0<z<\left(\sqrt{25+4\epsilon}-5\right)/2$. In other words, if $0<|x-1|<\left(\sqrt{25+4\epsilon}-5\right)/2$, then $|x-1|^2+5|x-1|<\epsilon$, and therefore $|(x-1)^2+5(x-1)|<\epsilon$.



9. Suppose $\epsilon>0$ is given. We must show that we can choose x sufficiently close to 2 so that $|(x+2)/(x-1)-4|<\epsilon$. To do this, we rewrite the inequality with the x's in the combination $x-2$,

$$\left|\frac{x+2}{x-1}-4\right|=\left|\frac{-3x+6}{x-1}\right|=\frac{3|x-2|}{|(x-2)+1|}.$$

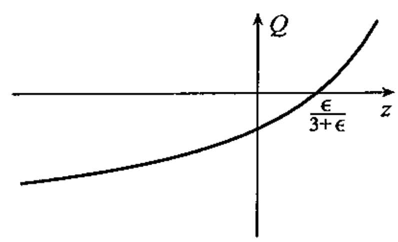
We must now choose x so that

$$\frac{3|x-2|}{|(x-2)+1|}<\epsilon.$$

Since $\dfrac{3|x-2|}{|(x-2)+1|}\le\dfrac{3|x-2|}{1-|x-2|}$, provided $|x-2|<1$, the above inequality is satisfied if x is chosen so that

$$\frac{3|x-2|}{1-|x-2|}<\epsilon.$$

Suppose we set $z=|x-2|$ and consider the curve $Q(z)=3z/(1-z)-\epsilon$ in the figure. It crosses the z-axis when $z=\epsilon/(3+\epsilon)$. The graph shows that $Q(z)<0$ whenever $0<z<\epsilon/(3+\epsilon)$. In other words, if $0<|x-2|<\epsilon/(3+\epsilon)$, then $3|x-2|/(1-|x-2|)<\epsilon$, and therefore $3|(x-2)/|(x-2)+1|<\epsilon$.



10. $\lim_{x\to a^+} f(x)=L$ if given any $\epsilon>0$, there exists a $\delta>0$ such that $|f(x)-L|<\epsilon$ whenever $a<x<a+\delta$.

11. $\lim_{x\to a^-} f(x)=L$ if given any $\epsilon>0$, there exists a $\delta>0$ such that $|f(x)-L|<\epsilon$ whenever $a-\delta<x<a$.

12. $\lim_{x\to\infty} f(x)=L$ if given any $\epsilon>0$, there exists an $X>0$ such that $|f(x)-L|<\epsilon$ whenever $x>X$.

13. $\lim_{x\to-\infty} f(x)=L$ if given any $\epsilon>0$, there exists an $X<0$ such that $|f(x)-L|<\epsilon$ whenever $x<X$.

14. $\lim_{x\to a} f(x)=\infty$ if given any $M>0$, there exists a $\delta>0$ such that $f(x)>M$ whenever $0<|x-a|<\delta$.

15. $\lim_{x\to a} f(x)=-\infty$ if given any $M<0$, there exists a $\delta>0$ such that $f(x)<M$ whenever $0<|x-a|<\delta$.

16. $\lim_{x\to\infty} f(x)=\infty$ if given any $M>0$, there exists an $X>0$ such that $f(x)>M$ whenever $x>X$.

17. $\lim_{x\to\infty} f(x)=-\infty$ if given any $M<0$, there exists an $X>0$ such that $f(x)<M$ whenever $x>X$.

18. $\lim_{x\to-\infty} f(x)=\infty$ if given any $M>0$, there exists an $X<0$ such that $f(x)>M$ whenever $x<X$.

19. $\lim_{x\to-\infty} f(x)=-\infty$ if given any $M<0$, there exists an $X<0$ such that $f(x)<M$ whenever $x<X$.

20. Suppose to the contrary that $f(x)$ has two limits L_1 and L_2 as x approaches a where $L_2 > L_1$ and $L_2 - L_1 = \epsilon$. Since $\lim_{x\to a} f(x) = L_1$, there exists a δ_1 such that $|f(x) - L_1| < \epsilon/3$ when $0 < |x-a| < \delta_1$; that is, when x is in the interval $0 < |x-a| < \delta_1$, the function is within $\epsilon/3$ of L_1. On the other hand, since $\lim_{x\to a} f(x) = L_2$, there exists a δ_2 such that $|f(x) - L_2| < \epsilon/3$ when $0 < |x-a| < \delta_2$; that is, the function is within $\epsilon/3$ of L_2 in the interval $0 < |x-a| < \delta_2$. But this is impossible because L_1 and L_2 are ϵ apart. Consequently, $f(x)$ can have at most one limit as x approaches a.

21. We use the definition in Exercise 14: $\lim_{x\to 1} 1/(x-1)^2 = \infty$ if given any $M > 0$, there exists a $\delta > 0$ such that $1/(x-1)^2 > M$ whenever $0 < |x-1| < \delta$. The inequality $1/(x-1)^2 > M \iff (x-1)^2 < 1/M \iff |x-1| < 1/\sqrt{M}$. Consequently, if we choose $\delta = 1/\sqrt{M}$, then whenever $0 < |x-1| < \delta = 1/\sqrt{M}$, we must have $1/(x-1)^2 > M$.

22. We use the definition in Exercise 15: $\lim_{x\to -2} [-1/(x+2)^2] = -\infty$ if given any number $M < 0$, there exists a $\delta > 0$ such that $-1/(x+2)^2 < M$ whenever $0 < |x+2| < \delta$. The inequality $-1/(x+2)^2 < M \iff (x+2)^2 < -1/M \iff |x+2| < 1/\sqrt{-M}$. Consequently, if we choose $\delta = 1/\sqrt{-M}$, then whenever $0 < |x+2| < \delta = 1/\sqrt{-M}$, we must have $-1/(x+2)^2 < M$.

23. We use the definition in Exercise 16: $\lim_{x\to\infty} (x+5) = \infty$ if given any $M > 0$, there exists an $X > 0$ such that $x + 5 > M$ whenever $x > X$. The inequality $x + 5 > M \iff x > M - 5$. Consequently, if we choose $X = M - 5$, then whenever $x > X$, we must have $x + 5 > M$.

24. We use the definition in Exercise 17: $\lim_{x\to\infty} (5 - x^2) = -\infty$ if given any number $M < 0$, there exists an $X > 0$ such that $5 - x^2 < M$ whenever $x > X$. The inequality $5 - x^2 < M \iff x^2 > 5 - M$, which for positive x implies that $x > \sqrt{5-M}$. Consequently, if we choose $X = \sqrt{5-M}$, then whenever $x > X$, we must have $5 - x^2 < M$.

25. We use the definition in Exercise 12: $\lim_{x\to\infty} (x+2)/(x-1) = 1$ if given any $\epsilon > 0$, there exists an $X > 0$ such that $|(x+2)/(x-1) - 1| < \epsilon$ whenever $x > X$. The inequality $|(x+2)/(x-1) - 1| < \epsilon \iff 3/|x-1| < \epsilon \iff |x-1| > 3/\epsilon$. This inequality is satisfied if $x > 1 + 3/\epsilon$. In other words, if we choose $X = 1 + 3/\epsilon$, then whenever $x > X$, $|(x+2)/(x-1) - 1| < \epsilon$.

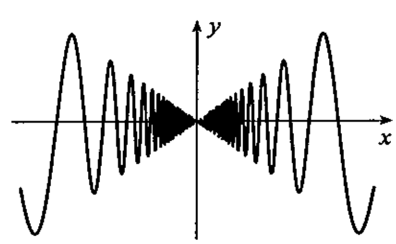
26. We use the definition in Exercise 13: $\lim_{x\to-\infty} (x+2)/(x-1) = 1$ if given any number $\epsilon > 0$, there exists an $X < 0$ such that $|(x+2)/(x-1) - 1| < \epsilon$ whenever $x < X$. The inequality $|(x+2)/(x-1) - 1| < \epsilon \iff 3/|x-1| < \epsilon \iff |x-1| > 3/\epsilon$. This inequality is satisfied if $x < 1 - 3/\epsilon$. In other words, if we choose $X = 1 - 3/\epsilon$, then whenever $x < X$, $|(x+2)/(x-1) - 1| < \epsilon$.

27. We use the definition in Exercise 18: $\lim_{x\to-\infty} (5 - x) = \infty$ if given any $M > 0$, there exists an $X < 0$ such that $5 - x > M$ whenever $x < X$. The inequality $5 - x > M \iff x < 5 - M$. Consequently, if we choose $X = 5 - M$, then whenever $x < X$, we must have $5 - x > M$.

28. We use the definition in Exercise 19: $\lim_{x\to-\infty} (3 + x - x^2) = -\infty$ if given any number $M < 0$, there exists an $X < 0$ such that $3 + x - x^2 < M$ whenever $x < X$. The inequality $3 + x - x^2 < M \iff M > -(x-1/2)^2 + 13/4 \iff (x-1/2)^2 > 13/4 - M \iff |x-1/2| > \sqrt{13/4 - M}$. This is satisfied for negative x if $x - 1/2 < -\sqrt{13/4 - M}$, or, $x < 1/2 - \sqrt{13/4 - M}$. Thus, if we choose $X = 1/2 - \sqrt{13/4 - M}$, then whenever $x < X$, we must have $3 + x - x^2 < M$.

29. Let $\epsilon = L$. Because $\lim_{x\to a} f(x) = L$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x-a| < \delta$. Thus, in the interval $I : 0 < |x-a| < \delta$, we have $-\epsilon < f(x) - L < \epsilon \iff L - \epsilon < f(x) < L + \epsilon$. But with $\epsilon = L$, this implies that in I, $0 < f(x) < 2L$.

30. No. A graph of the function $g(x) = x\sin(1/x)$ is shown to the right. It has limit 0 as $x \to 0$. If values at $x = 1/(n\pi)$ are redefined as 1, then all values of $f(x)$ no longer approach 0 as x approaches 0. Every interval around $x = 0$ contains an infinity of points at which the value of $f(x)$ is equal to 1.



31. Since $\lim_{x\to a} f(x) = F$, there exists a $\delta_1 > 0$ such that

$$|f(x) - F| < \epsilon/2 \qquad \text{whenever } 0 < |x - a| < \delta_1.$$

Since $\lim_{x\to a} g(x) = G$, there exists a $\delta_2 > 0$ such that

$$|g(x) - G| < \epsilon/2 \qquad \text{whenever } 0 < |x - a| < \delta_2.$$

It follows that whenever $0 < |x - a| < \delta$, where δ is the smaller of δ_1 and δ_2,

$$\big|[f(x) + g(x)] - (F + G)\big| = \big|[f(x) - F] + [g(x) - G]\big| \le |f(x) - F| + |g(x) - G| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

32. Since $\lim_{x\to a} f(x) = F$, there exists a $\delta_1 > 0$ such that

$$|f(x) - F| < \epsilon/2 \qquad \text{whenever } 0 < |x - a| < \delta_1.$$

Since $\lim_{x\to a} g(x) = G$, there exists a $\delta_2 > 0$ such that

$$|g(x) - G| < \epsilon/2 \qquad \text{whenever } 0 < |x - a| < \delta_2.$$

It follows that whenever $0 < |x - a| < \delta$, where δ is the smaller of δ_1 and δ_2,

$$\big|[f(x) - g(x)] - (F - G)\big| = \big|[f(x) - F] - [g(x) - G]\big| \le |f(x) - F| + |g(x) - G| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

33. Suppose c is any constant, and $\epsilon > 0$ is any given number. Since $\lim_{x\to a} f(x) = F$, there exists a $\delta > 0$ such that $|f(x) - F| < \epsilon/|c|$ whenever $0 < |x - a| < \delta$. Hence, whenever $0 < |x - a| < \delta$, we can say that

$$|cf(x) - cL| = |c||f(x) - L| < |c|\left(\frac{\epsilon}{|c|}\right) = \epsilon.$$

34. (a) $$\begin{aligned}|f(x)g(x) - FG| &= \big|[f(x)g(x) - f(x)G] + [f(x)G - FG]\big| \\ &\le |f(x)g(x) - f(x)G| + |f(x)G - FG| = |f(x)||g(x) - G| + |G||f(x) - F|\end{aligned}$$

(b) If $\lim_{x\to a} f(x) = F$, it follows that $\lim_{x\to a} |f(x)| = |F|$. There must exist a δ_1 such that whenever $0 < |x - a| < \delta_1$,

$$\big||f(x)| - |F|\big| < 1, \qquad \text{or,} \qquad -1 < |f(x)| - |F| < 1.$$

But then for such x, $|f(x)| < |F| + 1$.
Since $\lim_{x\to a} g(x) = G$, given any $\epsilon > 0$, there exists δ_2 such that when $0 < |x - a| < \delta_2$,

$$|g(x) - G| < \frac{\epsilon}{2(|F| + 1)}.$$

Since $\lim_{x\to a} f(x) = F$, there exists δ_3 such that when $0 < |x - a| < \delta_3$,

$$|f(x) - F| < \frac{\epsilon}{2|G| + 1}.$$

(c) If we set δ equal to the minimum of δ_1, δ_2, and δ_3, then for $0 < |x - a| < \delta$,

$$\begin{aligned}|f(x)g(x) - FG| &\le |f(x)||g(x) - G| + |G||f(x) - F| && \text{(from part (a))} \\ &< |f(x)||g(x) - G| + (|G| + 1/2)|f(x) - F| \\ &< (|F| + 1)\frac{\epsilon}{2(|F| + 1)} + \frac{1}{2}(2|G| + 1)\frac{\epsilon}{2|G| + 1} && \text{(from part (b))} \\ &= \epsilon.\end{aligned}$$

35. (a) $$\begin{aligned}\left|\frac{f(x)}{g(x)} - \frac{F}{G}\right| &= \left|\frac{f(x)G - g(x)F}{g(x)G}\right| = \left|\frac{[f(x)G - FG] + [FG - g(x)F]}{g(x)G}\right| \\ &\le \frac{|f(x)G - FG| + |FG - g(x)F|}{|G||g(x)|} = \frac{|f(x) - F|}{|g(x)|} + \frac{|F||g(x) - G|}{|G||g(x)|}.\end{aligned}$$

(b) Because $\lim_{x\to a} g(x) = G$, it follows that $\lim_{x\to a} |g(x)| = |G|$. Hence with $\epsilon = |G|/2$, there exists a $\delta_1 > 0$ such that whenever $0 < |x-a| < \delta_1$,

$$\big||g(x)| - |G|\big| < \frac{|G|}{2} \quad\Longleftrightarrow\quad -\frac{|G|}{2} < |g(x)| - |G| < \frac{|G|}{2} \quad\Longleftrightarrow\quad \frac{|G|}{2} < |g(x)| < \frac{3|G|}{2}.$$

Because $\lim_{x\to a} f(x) = F$, we can say that for any $\epsilon > 0$, there exists a $\delta_2 > 0$ such that whenever $0 < |x-a| < \delta_2$,

$$|f(x) - F| < \frac{\epsilon|G|}{4}.$$

Because $\lim_{x\to a} g(x) = G$, we can say that for any $\epsilon > 0$, there exists a $\delta_3 > 0$ such that whenever $0 < |x-a| < \delta_3$,

$$|g(x) - G| < \frac{\epsilon|G|^2}{4(|F|+1)}.$$

(c) It now follows that for $0 < |x-a| < \delta$ where δ is the smallest of δ_1, δ_2, and δ_3,

$$\left|\frac{f(x)}{g(x)} - \frac{F}{G}\right| < \frac{\epsilon|G|}{4}\frac{2}{|G|} + \frac{|F|+1}{|G|}\frac{2}{|G|}\frac{\epsilon|G|^2}{4(|F|+1)} = \epsilon.$$

This completes the proof.

REVIEW EXERCISES

1. $\displaystyle\lim_{x\to 1}\frac{x^2-2x}{x+5} = \frac{1-2}{1+5} = -\frac{1}{6}$

2. $\displaystyle\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}\frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1}(x-1) = -2$

3. $\displaystyle\lim_{x\to -2}\frac{x^2+4x+4}{x+3} = 0$

4. $\displaystyle\lim_{x\to\infty}\frac{x+5}{x-3} = \lim_{x\to\infty}\frac{1+\frac{5}{x}}{1-\frac{3}{x}} = 1$

5. $\displaystyle\lim_{x\to -\infty}\frac{x^2+3x+2}{2x^2-5} = \lim_{x\to -\infty}\frac{1+\frac{3}{x}+\frac{2}{x^2}}{2-\frac{5}{x^2}} = \frac{1}{2}$

6. $\displaystyle\lim_{x\to -\infty}\frac{5-x^3}{3+4x^3} = \lim_{x\to -\infty}\frac{\frac{5}{x^3}-1}{\frac{3}{x^3}+4} = -\frac{1}{4}$

7. $\displaystyle\lim_{x\to\infty}\frac{3x^3+2x-5}{x^2+5x} = \lim_{x\to\infty}\frac{3x+\frac{2}{x}-\frac{5}{x^2}}{1+\frac{5}{x}} = \infty$

8. $\displaystyle\lim_{x\to\infty}\frac{4-3x+x^2}{3+5x^3} = \lim_{x\to\infty}\frac{\frac{4}{x^2}-\frac{3}{x}+1}{\frac{3}{x^2}+5x} = 0$

9. $\displaystyle\lim_{x\to 2^+}\frac{x^2-2x}{x^2+2x} = 0$

10. $\displaystyle\lim_{x\to 2^-}\frac{x^2-4x+4}{x-2} = \lim_{x\to 2^-}\frac{(x-2)^2}{x-2} = \lim_{x\to 2^-}(x-2) = 0$

11. $\displaystyle\lim_{x\to 0}\frac{x^2+2x}{3x-2x^2} = \lim_{x\to 0}\frac{x+2}{3-2x} = \frac{2}{3}$

12. $\displaystyle\lim_{x\to 1}\frac{x^2+5x}{(x-1)^2} = \infty$

13. $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x} = 0$

14. $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}$

15. $\displaystyle\lim_{x\to 1/2}\frac{(2-4x)^3}{x(2x-1)^2} = \lim_{x\to 1/2}\frac{8(1-2x)}{x} = 0$

16. $\displaystyle\lim_{x\to\infty}\frac{\cos 5x}{x} = 0$

17. This limit does not exist.

18. $\displaystyle\lim_{x\to -\infty}\frac{\sqrt{3x^2+4}}{2x+5} = \lim_{x\to -\infty}\frac{\frac{\sqrt{3x^2+4}}{x}}{\frac{2x+5}{x}} = \lim_{x\to -\infty}\frac{-\sqrt{3+\frac{4}{x^2}}}{2+\frac{5}{x}} = -\frac{\sqrt{3}}{2}$

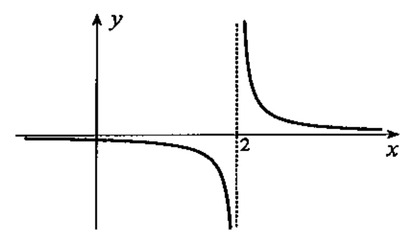
19. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{3x^2+4}}{2x+5}=\lim_{x\to\infty}\frac{\sqrt{3+\frac{4}{x^2}}}{2+\frac{5}{x}}=\frac{\sqrt{3}}{2}$

20. $$\lim_{x\to\infty}\left(\sqrt{2x+1}-\sqrt{3x-1}\right)=\lim_{x\to\infty}\left[\left(\sqrt{2x+1}-\sqrt{3x-1}\right)\frac{\sqrt{2x+1}+\sqrt{3x-1}}{\sqrt{2x+1}+\sqrt{3x-1}}\right]$$

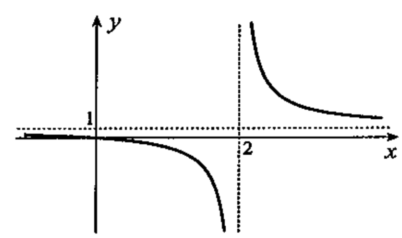
$$=\lim_{x\to\infty}\frac{(2x+1)-(3x-1)}{\sqrt{2x+1}+\sqrt{3x-1}}=\lim_{x\to\infty}\frac{2-x}{\sqrt{2x+1}+\sqrt{3x-1}}$$

$$=\lim_{x\to\infty}\frac{\frac{2-x}{\sqrt{x}}}{\frac{\sqrt{2x+1}+\sqrt{3x-1}}{\sqrt{x}}}=\lim_{x\to\infty}\frac{\frac{2}{\sqrt{x}}-\sqrt{x}}{\sqrt{2+\frac{1}{x}}+\sqrt{3-\frac{1}{x}}}=-\infty$$

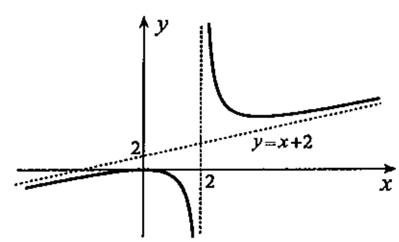
21. The limits $\displaystyle\lim_{x\to 2^+}f(x)=\infty$, $\displaystyle\lim_{x\to 2^-}f(x)=-\infty$, $\displaystyle\lim_{x\to\infty}f(x)=0^+$, and $\displaystyle\lim_{x\to-\infty}f(x)=0^-$ lead to the graph to the right. The discontinuity at $x=2$ is infinite.



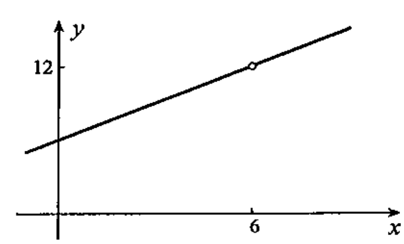
22. The limits $\displaystyle\lim_{x\to 2^+}f(x)=\infty$, $\displaystyle\lim_{x\to 2^-}f(x)=-\infty$, $\displaystyle\lim_{x\to\infty}f(x)=1^+$, and $\displaystyle\lim_{x\to-\infty}f(x)=1^-$ lead to the graph to the right. The discontinuity at $x=2$ is infinite.



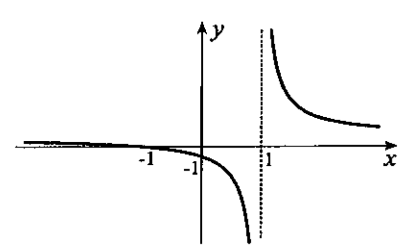
23. We calculate $\displaystyle\lim_{x\to 2^+}f(x)=\infty$ and $\displaystyle\lim_{x\to 2^-}f(x)=-\infty$. From $f(x)=x+2+\dfrac{4}{x-2}$, we obtain the oblique asymptote $y=x+2$, approached from above as $x\to\infty$ and from below as $x\to-\infty$. These give the graph to the right. The discontinuity at $x=2$ is infinite.



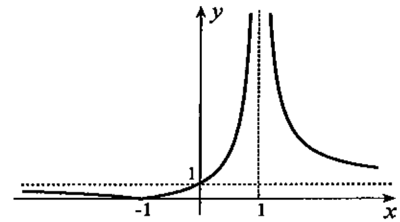
24. For $x\neq 6$, $f(x)=x+6$. Consequently, the graph is a straight line with the point at $x=6$ removed. The discontinuity at $x=6$ is removable.



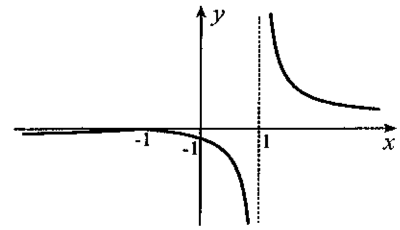
25. The limits $\displaystyle\lim_{x\to 1^+}f(x)=\infty$, $\displaystyle\lim_{x\to 1^-}f(x)=-\infty$, $\displaystyle\lim_{x\to\infty}f(x)=1^+$, and $\displaystyle\lim_{x\to-\infty}f(x)=1^-$ lead to the graph to the right. The discontinuity at $x=1$ is infinite.



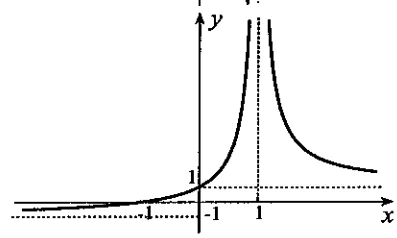
26. The limits $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 1^+$, and $\lim_{x\to-\infty} f(x) = 1^-$ lead to the graph to the right. The discontinuity at $x = 1$ is infinite.



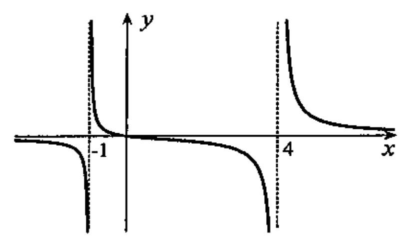
27. The limits $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 1^+$, and $\lim_{x\to-\infty} f(x) = -1^+$ lead to the graph to the right. The discontinuity at $x = 1$ is infinite.



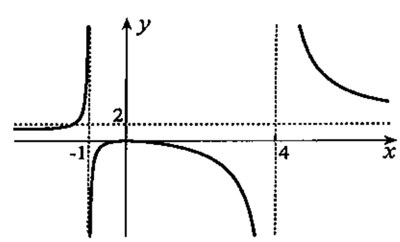
28. The limits $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 1^+$, and $\lim_{x\to-\infty} f(x) = -1^+$ lead to the graph to the right. The discontinuity at $x = 1$ is infinite.



29. With $f(x) = \dfrac{2x}{(x-4)(x+1)}$, we calculate $\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to -1^+} f(x) = \infty$, and $\lim_{x\to -1^-} f(x) = -\infty$. Furthermore, $\lim_{x\to\infty} f(x) = 0^+$ and $\lim_{x\to-\infty} f(x) = 0^-$. The graph is shown to the right. The discontinuities at $x = -1$, 4 are infinite.

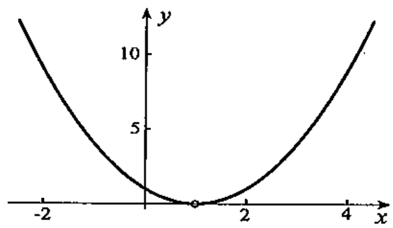


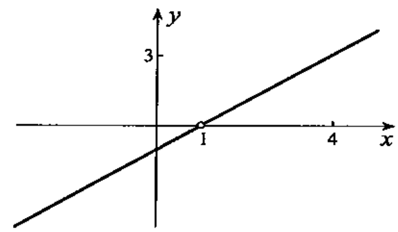
30. With $f(x) = \dfrac{2x^2}{(x-4)(x+1)}$, we calculate $\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to -1^+} f(x) = -\infty$, and $\lim_{x\to -1^-} f(x) = \infty$. Furthermore, with $f(x) = 2 + \dfrac{6x+8}{x^2-3x-4}$, we find $\lim_{x\to\infty} f(x) = 2^+$ and $\lim_{x\to-\infty} f(x) = 2^-$. The graph is shown to the right. The discontinuities at $x = -1$, 4 are infinite.



31. For $x \neq 1$, $f(x) = \dfrac{(x-1)^3}{x-1} = (x-1)^2$. The graph of the function is therefore the parabola $y = (x-1)^2$ with the point at $x = 1$ deleted. The function has a removable discontinuity at $x = 1$.

32. For $x \neq 1$, $f(x) = \dfrac{(x-1)^3}{(x-1)^2} = x - 1$. The graph of the function is therefore the straight line $y = x - 1$ with the point at $x = 1$ deleted. The function has a removable discontinuity at $x = 1$.





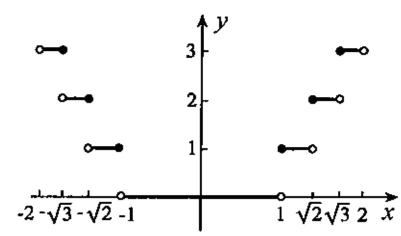
33. The function can be expressed in the form

$$f(x) = x^2[1 - h(x)] + x[h(x) - h(x-4)] + (5-2x)h(x-4) = x^2 + (x - x^2)h(x) + (5-3x)h(x-4).$$

34. The function can be expressed in the form

$$\begin{aligned} f(x) &= (3 + x^3)[1 - h(x+1)] + (x^2+2)[h(x+1) - h(x-2)] + 4h(x-2) \\ &= 3 + x^3 - (x^3 - x^2 + 1)h(x+1) + (2 - x^2)h(x-2). \end{aligned}$$

35. The function is discontinuous at $x = \pm\sqrt{n}$, where $n > 0$ is an integer.



CHAPTER 3

EXERCISES 3.1

1. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h+2)-(x+2)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$

2. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)^2+5]-(3x^2+5)}{h}$

$= \lim_{h\to 0} \frac{3x^2+6xh+3h^2+5-3x^2-5}{h} = \lim_{h\to 0}(6x+3h) = 6x$

3. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[1+2(x+h)-(x+h)^2]-(1+2x-x^2)}{h}$

$= \lim_{h\to 0} \frac{1+2x+2h-x^2-2xh-h^2-1-2x+x^2}{h} = \lim_{h\to 0} \frac{h(2-2x-h)}{h} = 2-2x$

4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3+2(x+h)^2]-(x^3+2x^2)}{h}$

$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3+2x^2+4xh+2h^2-x^3-2x^2}{h} = \lim_{h\to 0} \frac{h(3x^2+3xh+h^2+4x+2h)}{h}$

$= 3x^2+4x$

5. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^4+4(x+h)-12]-(x^4+4x-12)}{h}$

$= \lim_{h\to 0} \frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4+4x+4h-12-x^4-4x+12}{h}$

$= \lim_{h\to 0} \frac{h(4x^3+6x^2h+4xh^2+h^3+4)}{h} = 4x^3+4$

6. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{x+h+4}{x+h-5} - \frac{x+4}{x-5}\right)$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x^2+xh+4x-5x-5h-20)-(x^2+xh-5x+4x+4h-20)}{(x+h-5)(x-5)}\right]$

$= \lim_{h\to 0} \frac{-9}{(x+h-5)(x-5)} = \frac{-9}{(x-5)^2}$

7. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)^2+2}{x+h+3} - \frac{x^2+2}{x+3}\right]$

$= \lim_{h\to 0} \frac{(x+3)(x^2+2xh+h^2+2)-(x^2+2)(x+h+3)}{h(x+h+3)(x+3)} = \lim_{h\to 0} \frac{h(x^2+xh+6x+3h-2)}{h(x+h+3)(x+3)}$

$= \frac{x^2+6x-2}{(x+3)^2}$

8. If we write $f(x) = x^3+2x^2$, then this is the same function as in Exercise 4. Consequently, $f'(x) = 3x^2+4x$.

9. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{3(x+h)-2}{4-(x+h)} - \frac{3x-2}{4-x}\right]$

$= \lim_{h\to 0} \frac{(3x+3h-2)(4-x)-(3x-2)(4-x-h)}{h(4-x-h)(4-x)} = \lim_{h\to 0} \frac{10h}{h(4-x-h)(4-x)} = \frac{10}{(4-x)^2}$

10. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left\{\frac{(x+h)^2-(x+h)+1}{(x+h)^2+(x+h)+1} - \frac{x^2-x+1}{x^2+x+1}\right\}$

$= \lim_{h\to 0} \frac{1}{h}\left\{\frac{(x^2+x+1)[(x+h)^2-(x+h)+1]-(x^2-x+1)[(x+h)^2+(x+h)+1]}{(x^2+x+1)[(x+h)^2+(x+h)+1]}\right\}$

$= \lim_{h\to 0}\left\{\frac{h(2x^2+2xh-2)}{h(x^2+x+1)[(x+h)^2+(x+h)+1]}\right\} = \frac{2x^2-2}{(x^2+x+1)^2}$

11. Since $C = f(r) = 2\pi r$, $\dfrac{dC}{dr} = \lim_{h\to 0} \dfrac{f(r+h) - f(r)}{h} = \lim_{h\to 0} \dfrac{2\pi(r+h) - 2\pi r}{h} = \lim_{h\to 0} \dfrac{2\pi h}{h} = 2\pi.$

12. Since $A = f(r) = \pi r^2$, $\dfrac{dA}{dr} = \lim_{h\to 0} \dfrac{f(r+h) - f(r)}{h} = \lim_{h\to 0} \dfrac{\pi(r+h)^2 - \pi r^2}{h} = \lim_{h\to 0} \dfrac{\pi h(2r+h)}{h} = 2\pi r.$

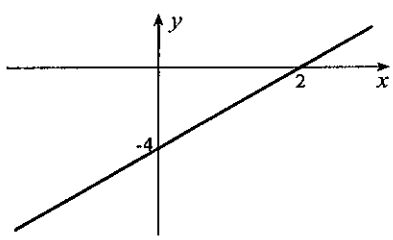
13. Since $A = f(r) = 4\pi r^2$,

$$\frac{dA}{dr} = \lim_{h\to 0} \frac{f(r+h) - f(r)}{h} = \lim_{h\to 0} \frac{4\pi(r+h)^2 - 4\pi r^2}{h} = \lim_{h\to 0} \frac{4\pi h(2r+h)}{h} = 8\pi r.$$

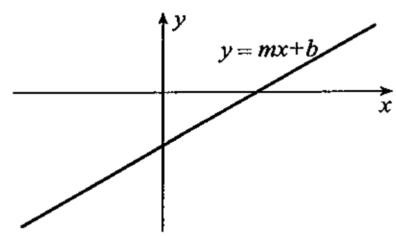
14. Since $V = f(r) = (4/3)\pi r^3$,

$$\frac{dV}{dr} = \lim_{h\to 0} \frac{f(r+h) - f(r)}{h} = \lim_{h\to 0} \frac{(4/3)\pi(r+h)^3 - (4/3)\pi r^3}{h}$$
$$= \frac{4\pi}{3} \lim_{h\to 0} \frac{r^3 + 3r^2h + 3rh^2 + h^3 - r^3}{h} = \frac{4\pi}{3} \lim_{h\to 0} (3r^2 + 3rh + h^2) = 4\pi r^2.$$

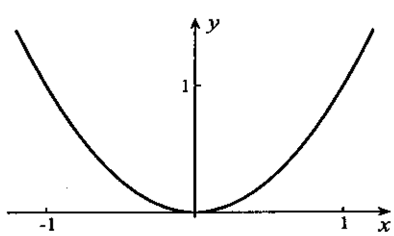
15. Since $f'(x)$ is the slope of the tangent line to the straight line, and the tangent line is the line itself, $f'(x) = 2$.



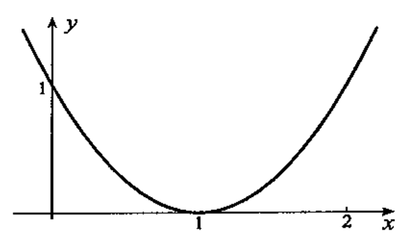
16. Since $f'(x)$ is the slope of the tangent line to the straight line, and the tangent line is the line itself, $f'(x) = m$.



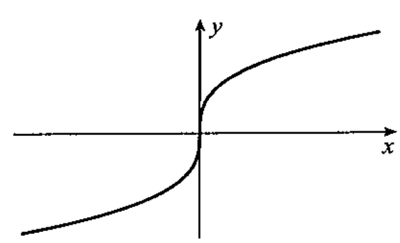
17. Since $f'(0)$ is the slope of the tangent line to the parabola at $x = 0$, it follows that $f'(0) = 0$.



18. Since $f'(1)$ is the slope of the tangent line to the parabola at $x = 1$, it follows that $f'(1) = 0$.



19. Since the tangent to the graph at $(0,0)$ is the y-axis, the derivative does not exist at $x = 0$.



20. Since $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{[(x+h)^2 + 3] - [x^2 + 3]}{h} = \lim_{h\to 0} \dfrac{h(2x+h)}{h} = 2x$, the slope of the tangent line to the parabola at $(1,4)$ is $2(1) = 2$. The equation of the tangent line is $y - 4 = 2(x-1) \Longrightarrow y = 2x + 2$.

21. Since $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{[3 - 2(x+h) - (x+h)^2] - [3 - 2x - x^2]}{h} = \lim_{h\to 0} \dfrac{h(-2 - 2x - h)}{h} = -2 - 2x$, the slope of the tangent line to the parabola at $(4, -21)$ is $-2 - 2(4) = -10$. The equation of the tangent line is $y + 21 = -10(x - 4) \Longrightarrow 10x + y = 19$.

22. Since $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \lim\limits_{h\to 0} \dfrac{1}{h}\left[\dfrac{x^2-(x+h)^2}{x^2(x+h)^2}\right] = \lim\limits_{h\to 0} \dfrac{-(2x+h)}{x^2(x+h)^2} = -\dfrac{2}{x^3}$, the slope of the tangent line at $(2, 1/4)$ is $-2/(2^3) = -1/4$. The equation of the tangent line is $y - 1/4 = -(1/4)(x-2) \Longrightarrow x + 4y = 3$.

23. Since $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{1}{h}\left(\dfrac{x+h+1}{x+h+2} - \dfrac{x+1}{x+2}\right) = \lim\limits_{h\to 0} \dfrac{(x+h+1)(x+2)-(x+h+2)(x+1)}{h(x+h+2)(x+2)}$

$$= \lim_{h\to 0} \frac{h}{h(x+h+2)(x+2)} = \frac{1}{(x+2)^2},$$

the slope of the tangent line at $(0, 1/2)$ is $1/4$. The equation of the tangent line is $y - 1/2 = (1/4)x \Longrightarrow x = 4y - 2$.

24. Since $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \dfrac{h(2x+h)}{h} = 2x$, the slope of the tangent line to the parabola at $(1, 1)$ is 2. The inclination is defined by $\tan\phi = 2$, and is therefore equal to 1.107 radians.

25. Since $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{[(x+h)^3 - 6(x+h)] - (x^3 - 6x)}{h} = \lim\limits_{h\to 0} \dfrac{h(3x^2 + 3xh + h^2 - 6)}{h} = 3x^2 - 6$, the slope of the tangent line to the curve at $(2, -4)$ is 6. The inclination is defined by $\tan\phi = 6$, and is therefore equal to 1.406 radians.

26. According to Exercise 22, the slope of the tangent line at $(2, 1/4)$ is $-1/4$. Since the inclination satisfies $\tan\phi = -1/4$, it follows that $\phi = 2.897$ radians.

27. Since $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{1}{h}\left[\dfrac{1}{x+h+1} - \dfrac{1}{x+1}\right] = \lim\limits_{h\to 0} \dfrac{-h}{h(x+h+1)(x+1)} = -\dfrac{1}{(x+1)^2}$, the slope of the tangent line to the curve at $(0, 1)$ is -1. The inclination is defined by $\tan\phi = -1$, and is therefore equal to $3\pi/4$ radians.

28.
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^8 - x^8}{h} \\ &= \lim_{h\to 0} \frac{(x^8 + 8x^7h + 28x^6h^2 + \cdots + h^8) - x^8}{h} \qquad \text{(using the binomial theorem)} \\ &= \lim_{h\to 0} (8x^7 + 28x^6h + \cdots + h^7) = 8x^7 \end{aligned}$$

29.
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} = \lim_{h\to 0} \frac{(x+h+1)-(x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \frac{1}{2\sqrt{x+1}} \end{aligned}$$

30.
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left\{\frac{1}{(x+h-2)^4} - \frac{1}{(x-2)^4}\right\} \\ &= \lim_{h\to 0} \left\{\frac{(x-2)^4 - [(x-2)^4 + 4(x-2)^3h + 6(x-2)^2h^2 + 4(x-2)h^3 + h^4]}{h(x-2)^4(x+h-2)^4}\right\} \\ &= \lim_{h\to 0} \left\{\frac{-4(x-2)^3 - 6(x-2)^2h - 4(x-2)h^2 - h^3}{(x-2)^4(x+h-2)^4}\right\} = \frac{-4(x-2)^3}{(x-2)^8} = \frac{-4}{(x-2)^5} \end{aligned}$$

31.
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x+h-3}} - \frac{1}{\sqrt{x-3}}\right) \\ &= \lim_{h\to 0} \frac{\sqrt{x-3} - \sqrt{x+h-3}}{h\sqrt{x+h-3}\sqrt{x-3}} \frac{\sqrt{x-3} + \sqrt{x+h-3}}{\sqrt{x-3} + \sqrt{x+h-3}} = \lim_{h\to 0} \frac{(x-3)-(x+h-3)}{h\sqrt{x+h-3}\sqrt{x-3}(\sqrt{x-3} + \sqrt{x+h-3})} \\ &= \frac{-1}{\sqrt{x-3}\sqrt{x-3}(2\sqrt{x-3})} = \frac{-1}{2(x-3)^{3/2}} \end{aligned}$$

32. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)\sqrt{(x+h)+1} - x\sqrt{x+1}}{h}$

$$= \lim_{h\to 0} \left[\frac{(x+h)\sqrt{x+h+1} - x\sqrt{x+1}}{h} \frac{(x+h)\sqrt{x+h+1} + x\sqrt{x+1}}{(x+h)\sqrt{x+h+1} + x\sqrt{x+1}}\right]$$

$$= \lim_{h\to 0} \frac{(x+h)^2(x+h+1) - x^2(x+1)}{h[(x+h)\sqrt{x+h+1} + x\sqrt{x+1}]}$$

$$= \lim_{h\to 0} \frac{x^2 + 2x(x+h+1) + h(x+h+1)}{(x+h)\sqrt{x+h+1} + x\sqrt{x+1}} = \frac{3x^2 + 2x}{2x\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}$$

33. Since $A = \pi r^2$, we have $r = \sqrt{A}/\sqrt{\pi}$. Then,

$$\frac{dr}{dA} = \lim_{h\to 0} \frac{\sqrt{A+h}/\sqrt{\pi} - \sqrt{A}/\sqrt{\pi}}{h} = \frac{1}{\sqrt{\pi}} \lim_{h\to 0} \frac{\sqrt{A+h} - \sqrt{A}}{h} \frac{\sqrt{A+h} + \sqrt{A}}{\sqrt{A+h} + \sqrt{A}}$$

$$= \frac{1}{\sqrt{\pi}} \lim_{h\to 0} \frac{(A+h) - A}{h(\sqrt{A+h} + \sqrt{A})} = \frac{1}{2\sqrt{\pi A}}.$$

34. Since $V = (4/3)\pi r^3$ and $A = 4\pi r^2$, we have $V = f(A) = \dfrac{4\pi}{3}\left(\dfrac{A}{4\pi}\right)^{3/2} = \dfrac{1}{6\sqrt{\pi}} A^{3/2}$. The derivative of this function is

$$\frac{dV}{dA} = \lim_{h\to 0} \frac{f(A+h) - f(A)}{h} = \lim_{h\to 0} \frac{(A+h)^{3/2}/(6\sqrt{\pi}) - A^{3/2}/(6\sqrt{\pi})}{h}$$

$$= \frac{1}{6\sqrt{\pi}} \lim_{h\to 0} \left[\frac{(A+h)^{3/2} - A^{3/2}}{h} \frac{(A+h)^{3/2} + A^{3/2}}{(A+h)^{3/2} + A^{3/2}}\right]$$

$$= \frac{1}{6\sqrt{\pi}} \lim_{h\to 0} \frac{A^3 + 3A^2h + 3Ah^2 + h^3 - A^3}{h[(A+h)^{3/2} + A^{3/2}]} = \frac{1}{6\sqrt{\pi}} \lim_{h\to 0} \frac{3A^2 + 3Ah + h^2}{(A+h)^{3/2} + A^{3/2}}$$

$$= \frac{1}{6\sqrt{\pi}} \frac{3A^2}{2A^{3/2}} = \frac{\sqrt{A}}{4\sqrt{\pi}}.$$

35. The slope of the tangent line to $y = x^2$ at any point is

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = 2x.$$

Hence, the slope of the tangent line to $y = x^2$ at the point of intersection $(1,1)$ is $m_1 = 2$. The slope of $y = \sqrt{x}$ at any point is

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

The slope of the tangent line to $y = \sqrt{x}$ at $(1,1)$ is therefore $m_2 = 1/2$. Using formula 1.60, the angle θ between these tangent lines is $\theta = \mathrm{Tan}^{-1}\left|\dfrac{2 - 1/2}{1 + 2(1/2)}\right| = 0.644$ radians.

36. If $x > 0$, then $f(x) = x$, and $\quad f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h) - x}{h} = \lim\limits_{h\to 0} 1 = 1.$

When $x < 0$, $f(x) = -x$, and $\quad f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{-(x+h) - (-x)}{h} = \lim\limits_{h\to 0} (-1) = -1.$

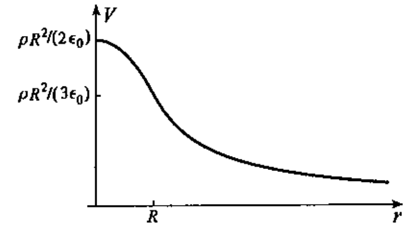
Finally, $\quad f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{|0+h| - 0}{h} = \lim\limits_{h\to 0} \dfrac{|h|}{h}.$

Since this limit does not exist, $f'(0)$ does not exist. These three results are combined in the one formula $f'(x) = |x|/x$.

37. The graph of the function suggests that there is a tangent line when $r = R$. To verify this we calculate

$$f'(R) = \lim_{h\to 0} \frac{f(R+h) - f(R)}{h}.$$

When $h < 0$, we find that



$$\lim_{h\to 0^-} \frac{1}{h}\left\{\frac{\rho}{6\epsilon_0}[3R^2 - (R+h)^2] - \frac{\rho}{6\epsilon_0}(3R^2 - R^2)\right\} = \frac{\rho}{6\epsilon_0}\lim_{h\to 0^-}\frac{-h(2R+h)}{h} = -\frac{\rho R}{3\epsilon_0}.$$

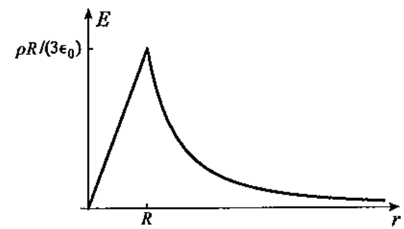
When $h > 0$, we obtain

$$\lim_{h\to 0^+} \frac{1}{h}\left[\frac{R^3\rho}{3\epsilon_0(R+h)} - \frac{R^3\rho}{3\epsilon_0 R}\right] = \frac{R^3\rho}{3\epsilon_0}\lim_{h\to 0^+}\frac{R-(R+h)}{hR(R+h)} = -\frac{R\rho}{3\epsilon_0}.$$

Since these limits are the same, $f'(R) = -\rho R/(3\epsilon_0)$.

38. The graph of the function suggests that $f'(R)$ does not exist since there appears to be no tangent line when $r = R$. To verify this we calculate

$$f'(R) = \lim_{h\to 0} \frac{f(R+h) - f(R)}{h}.$$



When $h < 0$, we find that $\displaystyle\lim_{h\to 0^-} \frac{1}{h}\left[\frac{\rho(R+h)}{3\epsilon_0} - \frac{\rho R}{3\epsilon_0}\right] = \lim_{h\to 0^-}\frac{\rho}{3\epsilon_0} = \frac{\rho}{3\epsilon_0}$. When $h > 0$,

$$\lim_{h\to 0^+} \frac{1}{h}\left[\frac{\rho R^3}{3\epsilon_0(R+h)^2} - \frac{\rho R}{3\epsilon_0}\right] = \frac{\rho R}{3\epsilon_0}\lim_{h\to 0^+}\frac{1}{h}\left[\frac{R^2-(R+h)^2}{(R+h)^2}\right] = \frac{\rho R}{3\epsilon_0}\lim_{h\to 0^+}\frac{-2R-h}{(R+h)^2} = \frac{-2\rho}{3\epsilon_0}.$$

Since these limits are not the same, $f'(R)$ does not exist.

39. $$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^{1/3}-x^{1/3}}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^{1/3}-x^{1/3}}{h}\,\frac{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}} \\ &= \lim_{h\to 0}\frac{(x+h)-x}{h[(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}]} = \frac{1}{3x^{2/3}} \end{aligned}$$

40. By equation 3.3, $\displaystyle f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h} = f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$.
Since $f'(0) = 1$, it follows that $\displaystyle 1 = f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-1}{h}$.
Consequently, $f'(x) = f(x)\cdot 1 = f(x)$.

EXERCISES 3.2

1. $f'(x) = 4x$

2. $f'(x) = 9x^2 + 4$

3. $f'(x) = 20x - 3$

4. $f'(x) = 20x^4 - 30x^2 + 3$

5. $f'(x) = -2x^{-3} = -2/x^3$

6. $f'(x) = 2(-3x^{-4}) = -6/x^4$

7. $f'(x) = 20x^3 - 9x^2 + (-1)x^{-2} = 20x^3 - 9x^2 - 1/x^2$

8. $f'(x) = -(1/2)(-2x^{-3}) + 3(-4x^{-5}) = 1/x^3 - 12/x^5$

9. $f'(x) = 10x^9 - (-10)x^{-11} = 10x^9 + 10/x^{11}$

10. $f'(x) = 20x^3 + \dfrac{1}{4}(-5x^{-6}) = 20x^3 - \dfrac{5}{4x^6}$

11. $f'(x) = -20x^{-5} + (1/4)(5x^4) = -\dfrac{20}{x^5} + \dfrac{5x^4}{4}$

12. $f'(x) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

13. $f'(x) = -6x^{-3} + 2(-1/2)x^{-3/2} = -\dfrac{6}{x^3} - \dfrac{1}{x^{3/2}}$

14. $f'(x) = -(3/2)x^{-5/2} + (3/2)x^{1/2} = -\dfrac{3}{2x^{5/2}} + \dfrac{3\sqrt{x}}{2}$

15. $f'(x) = 2(1/3)x^{-2/3} - 3(2/3)x^{-1/3} = \dfrac{2}{3x^{2/3}} - \dfrac{2}{x^{1/3}}$

16. $f'(x) = \pi(\pi x^{\pi-1}) = \pi^2 x^{\pi-1}$

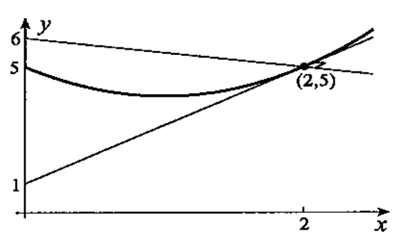
17. Since $f(x) = x^4 + 4x^2 + 4$, we find that $f'(x) = 4x^3 + 8x$.

18. Since $f(x) = 4x - x^{-3}$, we obtain $f'(x) = 4 + 3x^{-4} = 4 + 3/x^4$.

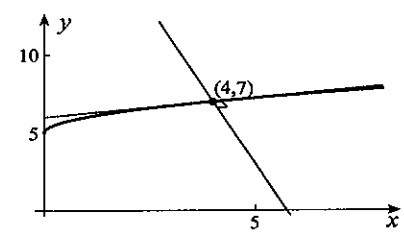
19. $f'(x) = (5/3)x^{2/3} - (2/3)x^{-1/3} = \dfrac{5x^{2/3}}{3} - \dfrac{2}{3x^{1/3}}$

20. Since $f(x) = 8x^3 + 60x^2 + 150x + 125$, we obtain $f'(x) = 24x^2 + 120x + 150 = 6(2x+5)^2$.

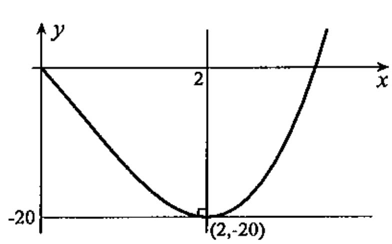
21. Since $f'(x) = 2x - 2$, the slope of the tangent line at $(2,5)$ is $2(2) - 2 = 2$. Equations for the tangent and normal lines are $y - 5 = 2(x-2)$ and $y - 5 = -(1/2)(x-2)$, or, $y = 2x + 1$ and $x + 2y = 12$. The tangent and normal lines do not appear to intersect at right angles because there are different scales along the axes.



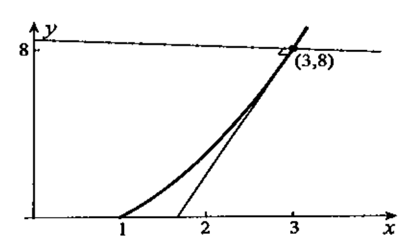
22. Since $f'(x) = (1/2)x^{-1/2}$, the slope of the tangent line at $(4,7)$ is $(1/2)4^{-1/2} = 1/4$. Equations for the tangent and normal lines are $y - 7 = (1/4)(x-4)$ and $y - 7 = -4(x-4)$, or, $4y = x + 24$ and $4x + y = 23$. The tangent and normal lines do not appear to intersect at right angles because there are different scales along the axes.



23. Since $f'(x) = 6x^2 - 6x - 12$, the slope of the tangent line at $(2,-20)$ is $6(4) - 6(2) - 12 = 0$. Equations for the tangent and normal lines are $y = -20$ and $x = 2$.



24. Since $y = x^2 - 1$, $dy/dx = 2x$. The slope of the the tangent line is therefore 6, and equations for the tangent and normal lines are $y - 8 = 6(x-3)$ and $y - 8 = -(1/6)(x-3)$, or, $y = 6x - 10$ and $x + 6y = 51$. The tangent and normal lines do not appear to intersect at right angles because the axes have different scales.



25. Points at which the slope of the tangent line is equal to 2 are defined by

$$2 = \frac{dy}{dx} = x^3 - 2x^2 - 19x + 22 \quad \Longrightarrow \quad 0 = x^3 - 2x^2 - 19x + 20 = (x-1)(x+4)(x-5).$$

The points are $(1, 145/12)$, $(-4, -400/3)$, and $(5, -655/12)$.

26. Since the slope of the tangent line at the point (x_0, y_0) is $2ax_0$, the equation of the tangent line is $y - y_0 = 2ax_0(x - x_0)$. The x-intercept can be found by setting $y = 0$ and solving for x,

$$-y_0 = 2ax_0(x - x_0) \quad \Longrightarrow \quad x = x_0 - \frac{y_0}{2ax_0}.$$

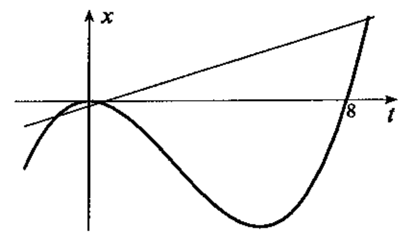
Since $y_0 = ax_0^2$, it follows that the x-intercept is

$$x = x_0 - \frac{ax_0^2}{2ax_0} = x_0 - \frac{x_0}{2} = \frac{x_0}{2},$$

which is halfway between the origin and the point $x = x_0$ on the x-axis.

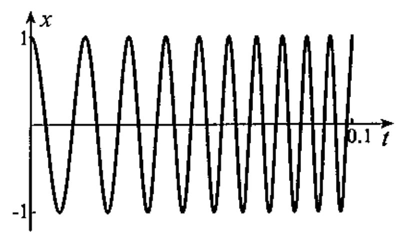
27. The slope of the tangent line to $x = t^3 - 8t^2$ is $f'(t) = 3t^2 - 16t$. For the tangent line to be parallel to $x = 6t - 3$, we set $3t^2 - 16t = 6$ and this implies that

$$t = \frac{16 \pm \sqrt{256 + 72}}{6} = \frac{8 \pm \sqrt{82}}{3}.$$



28. (a) The graph is to the right.

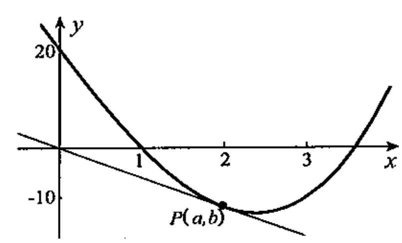
(b) Since $\dfrac{d}{dt}(1000\pi t^2 + 100\pi t) = 2000\pi t + 100\pi$, the frequencies at $t = 0$ and $t = 0.1$ are $100\pi/(2\pi) = 50$ Hz and $300\pi/(2\pi) = 150$ Hz.



29. Since $\dfrac{d}{dt}(\alpha t^2 + \beta t + \phi) = 2\alpha t + \beta$, initial and final frequencies are $(2\alpha t_1 + \beta)/(2\pi)$ and $(2\alpha t_2 + \beta)/(2\pi)$. The change is therefore $(2\alpha t_2 + \beta)/(2\pi) - (2\alpha t_1 + \beta)/(2\pi) = \alpha(t_2 - t_1)/\pi$.

30. If $P(a, b)$ is any point on the curve, the slope of the tangent line at P is $f'(a) = 3a^2 + 2a - 22$. The equation of the tangent line at P is $y - b = (3a^2 + 2a - 22)(x - a)$, and this line will pass through the origin if $-b = (3a^2 + 2a - 22)(-a)$. Since (a, b) is on the curve, its coordinates must satisfy the equation of the curve, $b = a^3 + a^2 - 22a + 20$. When we equate these two expressions for b,

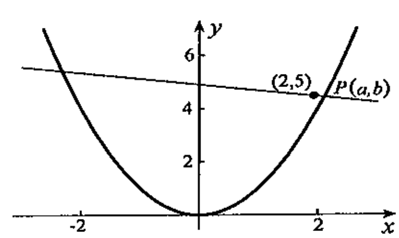
$$a^3 + a^2 - 22a + 20 = 3a^3 + 2a^2 - 22a$$



which simplifies to $0 = 2a^3 + a^2 - 20 = (a - 2)(2a^2 + 5a + 10)$. The only solution is $a = 2$, giving the point $(2, -12)$.

31. If $P(a, b)$ is any point on the curve, the slope of the tangent line at P is $f'(a) = 2a$. The equation of the normal line at P is $y - b = -1/(2a)(x - a)$, and this line will pass through the point $(2, 5)$ if $5 - b = -1/(2a)(2 - a)$. Since (a, b) is on the curve, its coordinates must satisfy the equation of the curve, $b = a^2$. When we equate these two expressions for b,

$$a^2 = 5 + \frac{1}{2a}(2 - a),$$



and this simplifies to $0 = 2a^3 - 9a - 2 = (a + 2)(2a^2 - 4a - 1)$. Solutions are $a = -2, (2 \pm \sqrt{6})/2$. The required points are therefore $(-2, 4)$ and $\big((2 \pm \sqrt{6})/2, (5 \pm 2\sqrt{6})/2\big)$. This could be used to find the shortest distance from the point $(2, 5)$ to the parabola.

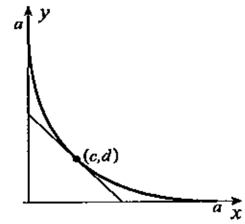
32. The slope of the tangent line to $\sqrt{x}+\sqrt{y}=\sqrt{a}$, or, $y=f(x)=(\sqrt{a}-\sqrt{x})^2=a-2\sqrt{a}\sqrt{x}+x$ at any point (c,d) is $f'(c)=-\dfrac{\sqrt{a}}{\sqrt{c}}+1$.

The equation of the tangent line at this point is

$$y-d=\left(1-\frac{\sqrt{a}}{\sqrt{c}}\right)(x-c),$$

and its x- and y-intercepts are

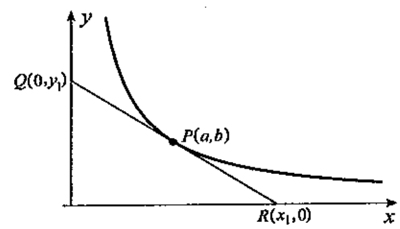
$$c-\frac{d}{1-\sqrt{a}/\sqrt{c}} \quad \text{and} \quad d-c(1-\sqrt{a}/\sqrt{c}).$$



Since (c,d) is on the curve, it follows that $d=(\sqrt{a}-\sqrt{c})^2$, and the sum of the intercepts is

$$\begin{aligned} c-\frac{\sqrt{c}d}{\sqrt{c}-\sqrt{a}}+d-\sqrt{c}(\sqrt{c}-\sqrt{a}) &= c-\frac{\sqrt{c}(\sqrt{a}-\sqrt{c})^2}{\sqrt{c}-\sqrt{a}}+(\sqrt{a}-\sqrt{c})^2-\sqrt{c}(\sqrt{c}-\sqrt{a}) \\ &= c+\sqrt{c}(\sqrt{a}-\sqrt{c})+(\sqrt{a}-\sqrt{c})^2-\sqrt{c}(\sqrt{c}-\sqrt{a}) \\ &= c+2\sqrt{c}(\sqrt{a}-\sqrt{c})+a-2\sqrt{c}\sqrt{a}+c=a. \end{aligned}$$

33. If $P(a,b)$ is any point on the curve, the slope of the tangent line at P is $f'(a)=-1/a^2$. The equation of the tangent line at P is $y-b=(-1/a^2)(x-a)$, and x- and y-intercepts of this line are $x_1=a+a^2b$, $\quad y_1=b+1/a$.

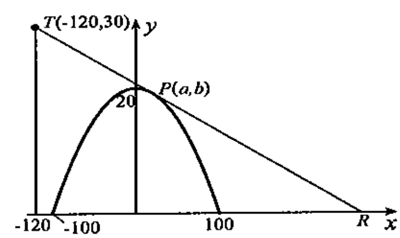


Since $\|PQ\|^2=a^2+(b-y_1)^2=a^2+\left(\dfrac{-1}{a}\right)^2=a^2+\dfrac{1}{a^2}$, and

$$\|PR\|^2=(a-x_1)^2+b^2=(-a^2b)^2+b^2=a^4b^2+b^2=a^4\left(\frac{1}{a^2}\right)+\frac{1}{a^2}=a^2+\frac{1}{a^2},$$

point P does indeed bisect the line segment joining Q and R.

34. The required position occurs at the point R where the tangent line from $T(-120,30)$ to the parabola representing the hill intersects the x-axis. Let the point of tangency be $P(a,b)$. If $y=cx^2+d$ is the equation of the parabola, then using the points $(100,0)$ and $(0,20)$, we obtain $0=100^2c+d$ and $20=c(0)^2+d$. These give $c=-1/500$ and $d=20$, so that the equation of the parabola is $y=f(x)=20-x^2/500$, $-100\le x\le 100$. Since $f'(x)=-x/250$, the slope of the tangent line at $P(a,b)$ is $f'(a)=-a/250$. The slope of this line is also the slope of PT, namely, $(b-30)/(a+120)$, and therefore



$$\frac{b-30}{a+120}=-\frac{a}{250} \quad\Longrightarrow\quad b=30-\frac{a(a+120)}{250}.$$

Since $P(a,b)$ is on the parabola, it also follows that $b=20-a^2/500$. When we equate these expressions for b,

$$30-\frac{a(a+120)}{250}=20-\frac{a^2}{500} \quad\Longrightarrow\quad a^2+240a-5000=0.$$

Of the two solutions $-120\pm 10\sqrt{194}$, only $a=10\sqrt{194}-120$ is positive. The y-coordinate of P is $b=20-(10\sqrt{194}-120)^2/500=(24\sqrt{194}-238)/5$. The equation of the tangent line is therefore $y-b=(-a/250)(x-a)$, and its x-intercept is given by $-b=(-a/250)(x-a)\implies x=250b/a+a$. When we substitute the calculated values for a and b, the result is

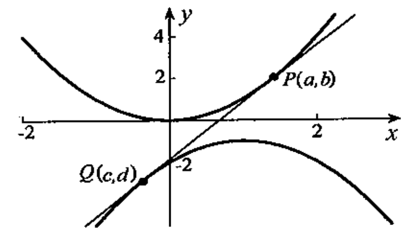
$$x=\frac{250(24\sqrt{194}-238)/5}{10\sqrt{194}-120}+10\sqrt{194}-120=15(4+\sqrt{194})\text{ m}.$$

35. If $f(x) = x^n$, then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \quad \text{and using the given identity}$$
$$= \lim_{h\to 0} \frac{h[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}]}{h} = nx^{n-1}.$$

36. If we denote coordinates of the two points by $P(a,b)$ and $Q(c,d)$, then the fact that P and Q are on the curves requires

$$b = a^2, \quad d = -c^2 + 2c - 2.$$



Since $d(x^2)/dx = 2x$, and $d(-x^2+2x-2)/dx = -2x+2$, and P and Q share a common tangent line, it follows that

$$2a = -2c + 2, \quad \frac{d-b}{c-a} = 2a.$$

To find P and Q, we solve these four equations in a, b, c, and d. If we substitute from the first and second equations into the fourth,

$$a^2 - (-c^2 + 2c - 2) = 2a(a-c) \qquad \Longrightarrow \qquad c^2 - 2c + 2 - a^2 + 2ac = 0.$$

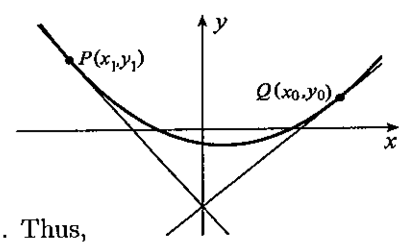
From the third equation, $a = 1 - c$, and therefore $c^2 - 2c + 2 - (1-c)^2 + 2c(1-c) = 0$. This reduces to $2c^2 - 2c - 1 = 0$ with solutions $c = (1 \pm \sqrt{3})/2$. These give the pairs of points $P\big((1+\sqrt{3})/2, (2+\sqrt{3})/2\big)$ and $Q\big((1-\sqrt{3})/2, (-4-\sqrt{3})/2\big)$, and $P\big((1-\sqrt{3})/2, (2-\sqrt{3})/2\big)$ and $Q\big((1+\sqrt{3})/2, (-4+\sqrt{3})/2\big)$.

37. Let $P(x_1, y_1)$ and $Q(x_0, y_0)$ be any two non-vertex points on the parabola. Since $f'(x) = 2ax + b$, equations of the tangent lines at P and Q are respectively

$$y - y_1 = (2ax_1 + b)(x - x_1),$$

and

$$y - y_0 = (2ax_0 + b)(x - x_0).$$



When we solve these for y and equate results,

$$y_1 + (2ax_1 + b)(x - x_1) = y_0 + (2ax_0 + b)(x - x_0),$$

or, $(2ax_0 + b - 2ax_1 - b)x = y_1 - y_0 - x_1(2ax_1 + b) + x_0(2ax_0 + b)$. Thus,

$$x = \frac{y_1 - y_0 - x_1(2ax_1 + b) + x_0(2ax_0 + b)}{2a(x_0 - x_1)}$$
$$= \frac{(ax_1^2 + bx_1 + c) - (ax_0^2 + bx_0 + c) - x_1(2ax_1 + b) + x_0(2ax_0 + b)}{2a(x_0 - x_1)}$$
$$= \frac{a(x_1^2 - x_0^2) + b(x_1 - x_0) - 2a(x_1^2 - x_0^2) - b(x_1 - x_0)}{2a(x_0 - x_1)} = -\frac{a(x_1 + x_0)(x_1 - x_0)}{2a(x_0 - x_1)} = \frac{x_0 + x_1}{2}.$$

Thus, the point of intersection is on the vertical line half way between P and Q.

38. If $x > 0$, then $\dfrac{d}{dx}|x|^n = \dfrac{d}{dx}x^n = nx^{n-1} = n|x|^{n-1}$. On the other hand, if $x < 0$, then

$$\frac{d}{dx}|x|^n = \frac{d}{dx}(-x)^n = (-1)^n \frac{d}{dx}x^n = (-1)^n nx^{n-1} = -n(-x)^{n-1} = -n|x|^{n-1}.$$

Graphs of $y = f(x) = |x|^n$ indicate that $f'(0) = 0$ when $n > 1$. This can also be verified algebraically,

$$f'(0) = \lim_{h\to 0} \frac{|0+h|^n - |0|^n}{h} = \lim_{h\to 0} \frac{|h|^n}{h} = 0 \quad \text{when } n > 1.$$

These three situations are all encompassed by the formula $\dfrac{d}{dx}|x|^n = n|x|^{n-1}\mathrm{sgn}(x)$, where $\mathrm{sgn}(x)$ is the signum function of Exercise 47 in Section 2.4.

39. We use the binomial theorem to write that

$$\frac{d}{dx}(ax+b)^n = \frac{d}{dx}\left[a^n x^n + \binom{n}{1}a^{n-1}x^{n-1}b + \binom{n}{2}a^{n-2}x^{n-2}b^2 + \cdots + \binom{n}{n-1}axb^{n-1} + b^n\right]$$
$$= a^n(nx^{n-1}) + \binom{n}{1}a^{n-1}(n-1)x^{n-2}b + \binom{n}{2}a^{n-2}(n-2)x^{n-3}b^2 + \cdots + \binom{n}{n-1}ab^{n-1}.$$

Now, for $1 \le r \le n-1$, we note that

$$\binom{n}{r}(n-r) = \frac{n!\,(n-r)}{r!\,(n-r)!} = \frac{n(n-1)!\,(n-r)}{r!\,(n-r)(n-r-1)!} = n\binom{n-1}{r}.$$

Consequently,

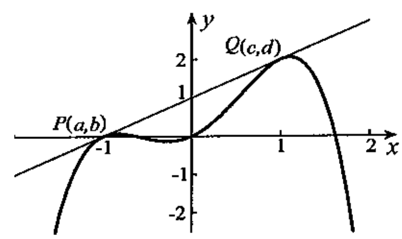
$$\frac{d}{dx}(ax+b)^n = a^n(nx^{n-1}) + n\binom{n-1}{1}a^{n-1}x^{n-2}b + n\binom{n-1}{2}a^{n-2}x^{n-3}b^2 + \cdots + n\binom{n-1}{n-1}ab^{n-1}$$
$$= an\left[a^{n-1}x^{n-1} + \binom{n-1}{1}a^{n-2}x^{n-2}b + \binom{n-1}{2}a^{n-3}x^{n-3}b^2 + \cdots + b^{n-1}\right]$$
$$= an(ax+b)^{n-1}.$$

40. If we denote coordinates of the two points by $P(a,b)$ and $Q(c,d)$, then the fact that P and Q are on the curves requires

$$b = a + 2a^2 - a^4, \quad d = c + 2c^2 - c^4.$$

Since $dy/dx = 1 + 4x - 4x^3$, and P and Q share a common tangent line, it follows that

$$1 + 4a - 4a^3 = 1 + 4c - 4c^3, \quad \frac{d-b}{c-a} = 1 + 4a - 4a^3.$$



To find P and Q, we solve these four equations in a, b, c, and d. The third equation implies that

$$0 = 4a - 4c + 4c^3 - 4a^3 = 4[(a-c) - (a^3 - c^3)] = 4(a-c)(1 - a^2 - ac - c^2).$$

Either $a = c$, which is unacceptable, or $1 = a^2 + ac + c^2$. When we subtract the first two equations, the result is $d - b = c - a + 2(c^2 - a^2) - (c^4 - a^4)$. Substitution of this into the fourth equation gives

$$\frac{c - a + 2(c^2 - a^2) - (c^4 - a^4)}{c - a} = 1 + 4a - 4a^3$$

or,

$$1 + 2(a+c) - (a^3 + a^2c + ac^2 + c^3) = 1 + 4a - 4a^3.$$

We now have two equations in a and c,

$$1 = a^2 + ac + c^2, \quad 0 = 2c - 2a + 4a^3 - (a^3 + a^2c + ac^2 + c^3) = 2c - 2a + 4a^3 - a(a^2 + ac + c^2) - c^3.$$

These imply that $0 = 2c - 2a + 4a^3 - a - c^3 = 2c - 3a + 4a^3 - c(1 - a^2 - ac)$
$= c - 3a + 4a^3 + a^2c + ac^2 = c - 3a + 4a^3 + a^2c + a(1 - a^2 - ac) = c - 2a + 3a^3.$

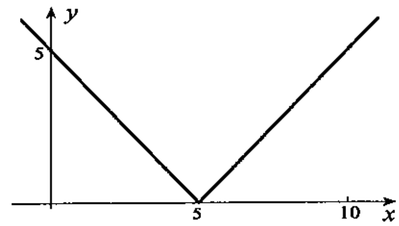
Thus, $c = 2a - 3a^3$, and therefore $1 = a^2 + a(2a - 3a^3) + (2a - 3a^3)^2$, from which

$$0 = 9a^6 - 15a^4 + 7a^2 - 1 = (a-1)(a+1)(3a^2-1)^2.$$

Thus, $a = \pm 1$ or $a = \pm 1/\sqrt{3}$. The first two lead to the points $(-1,0)$ and $(1,2)$. The second pair of values do not lead to distinct points.

EXERCISES 3.3

1. The graph of the function shows that $f(x)$ has left-hand derivative equal to -1 at $x = 5$, and right-hand derivative equal to 1. It therefore has no derivative at $x = 5$.



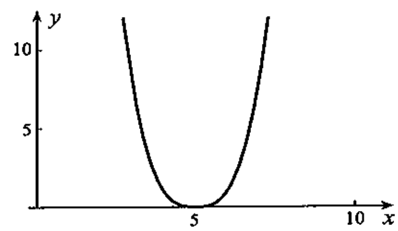
2. Since $f(x)$ is not defined for $x < 0$, it cannot have a left-hand derivative at $x = 0$. Its right-hand derivative at $x = 0$ is defined by

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^{3/2} - 0}{h} = \lim_{h\to 0^+} \sqrt{h} = 0.$$

3. The graph of the function indicates that all three derivatives have value 0 at $x = 5$. We verify this algebraically.

$$\begin{aligned} f'_+(5) &= \lim_{h\to 0^+} \frac{f(5+h) - f(5)}{h} \\ &= \lim_{h\to 0^+} \frac{|5+h-5|^3 - (0)}{h} \\ &= \lim_{h\to 0^+} (h^2) = 0. \end{aligned}$$

Similarly, $f'_-(5) = 0$, and therefore $f'(5) = 0$.

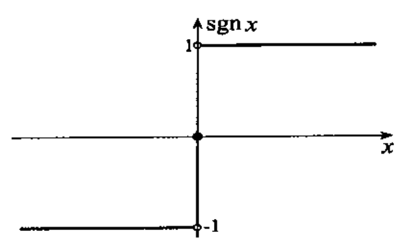


4. The graph of $f(x) = \operatorname{sgn} x$ makes it clear that the function does not have a left-hand derivative at $x = 0$ or a right-hand derivative, and cannot therefore have a derivative. We can also verify this algebraically.

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-1-0}{h} = \infty,$$

and

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{1-0}{h} = \infty.$$



5. Since the function is undefined at $x = 1$, it cannot have a derivative there.

6. The graph of $\lfloor x \rfloor$ in Exercise 68 of Section 1.5 shows that the right derivative is equal to 0, but there is no left derivative, and therefore no derivative.

7. If $h(a)$ is defined as 0, then $h'_-(a) = 0$, but $h'_+(a)$ and $h'(a)$ are still undefined.

8. If $h(a)$ is defined as 1, then $h'_+(a) = 0$, but $h'_-(a)$ and $h'(a)$ are still undefined.

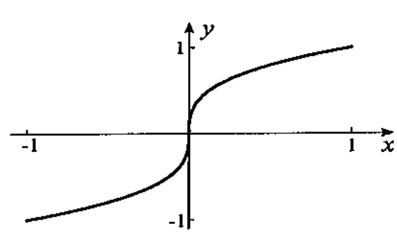
9. If $\operatorname{sgn} x$ has no value at $x = 0$, then it cannot have a left, a right, or a full derivative at $x = 0$.

10. True **11.** False **12.** True **13.** False

14. By equation 3.2,

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^{1/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty. \end{aligned}$$

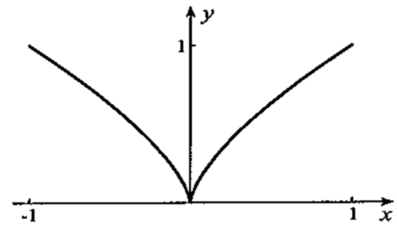
The graph of the function confirms this.



15. By equation 3.2, $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h^{2/3} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{1/3}},$$

and this limit does not exist. The graph of the function confirms this.

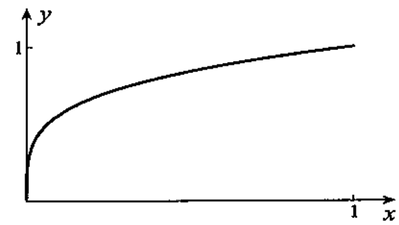


16. Since $f(x)$ is not defined for $x < 0$, it cannot have a derivative at $x = 0$. We can show that it does not have a right-hand derivative at $x = 0$ by calculating

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{h^{1/4} - 0}{h} = \lim_{h \to 0^+} \frac{1}{h^{3/4}} = \infty.$$

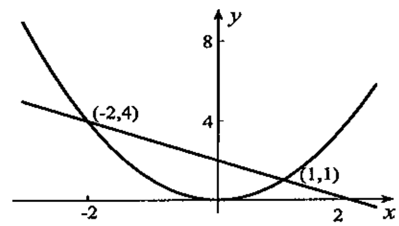
The graph of the function confirms this.



17. To find point(s) of intersection, we set $x^2 = 2 - x$, solutions of which are $x = 1, -2$. These give the points of intersection $(1, 1)$ and $(-2, 4)$. The slope of the tangent line to $y = x^2$ at $(1, 1)$ is 2 and that of $y = 2 - x$ is -1. According to formula 1.60, the angle between the tangent lines at $(1, 1)$ is

$$\theta = \mathrm{Tan}^{-1}\left|\frac{2+1}{1+(2)(-1)}\right| = 1.249 \text{ radians.}$$

A similar calculation for the point $(-2, 4)$ gives the angle 0.540 radians.



18. To find point(s) of intersection, we set $x^2 = 1 - x^2$, solutions of which are $x = \pm 1/\sqrt{2}$. These give the points of intersection $(\pm 1/\sqrt{2}, 1/2)$. Slopes of the curves $y = f(x) = x^2$ and $y = g(x) = 1 - x^2$ at $(1/\sqrt{2}, 1/2)$ are

$$f'(1/\sqrt{2}) = \{2x\}_{|x=1/\sqrt{2}} = \sqrt{2}$$

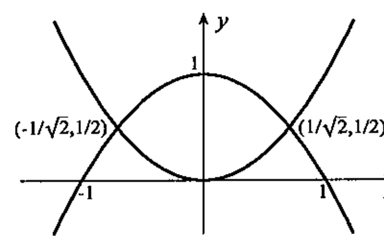
and

$$g'(1/\sqrt{2}) = \{-2x\}_{|x=1/\sqrt{2}} = -\sqrt{2}.$$

Equation 1.60 gives for the angle θ between the curves at this point

$$\theta = \mathrm{Tan}^{-1}\left|\frac{\sqrt{2}+\sqrt{2}}{1+(\sqrt{2})(-\sqrt{2})}\right| = 1.231 \text{ radians.}$$

The same angle is obtained at the other point of intersection.



19. To find point(s) of intersection of the curves we set $2 - x^2 = (1 + x)/2$, solutions of which are $x = -3/2, 1$. Points of intersection are therefore $(1, 1)$ and $(-3/2, -1/4)$. Slopes of the curves $y = f(x) = (x + 1)/2$ and $y = g(x) = 2 - x^2$ at $(1, 1)$ are

$$f'(1) = 1/2 \qquad \text{and} \qquad g'(1) = \{-2x\}_{|x=1} = -2.$$

Since these slopes are negative reciprocals, the curves intersect orthogonally at the point $(1, 1)$. A similar calculation shows that the curves are not orthogonal at $(-3/2, -1/4)$.

20. To find point(s) of intersection of the curves we set $3 - x^2 = (7 + x^2)/4$, solutions of which are $x = \pm 1$. Points of intersection are therefore $(\pm 1, 2)$. Slopes of the curves $y = f(x) = 3 - x^2$ and $y = g(x) = (x^2 + 7)/4$ at $(1, 2)$ are

$$f'(1) = \{-2x\}_{|x=1} = -2 \qquad \text{and} \qquad g'(1) = \{x/2\}_{|x=1} = 1/2.$$

Since these slopes are negative reciprocals, the curves intersect orthogonally at the point $(1, 2)$. The curves are also orthogonal at $(-1, 2)$.

21. Slopes of the curves $y = f(x) = x - 2x^2$ and $y = g(x) = x^3 + 2x$ at the point $(-1, -3)$ are

$$f'(-1) = \{1 - 4x\}_{|x=-1} = 5 \qquad \text{and} \qquad g'(-1) = \{3x^2 + 2\}_{|x=-1} = 5.$$

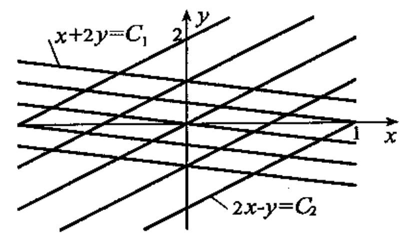
The curves are therefore tangent at the point.

22. Slopes of the curves $y = f(x) = x^3$ and $y = g(x) = x^2 + x - 1$ at the point $(1, 1)$ are

$$f'(1) = \{3x^2\}_{|x=1} = 3 \qquad \text{and} \qquad g'(1) = \{2x + 1\}_{|x=1} = 3.$$

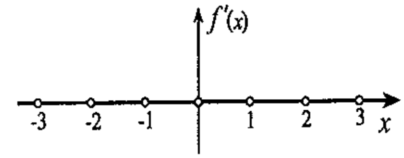
The curves are therefore tangent at the point.

23. (a),(b) Lines are shown to the right.
(c) Yes The slope of every line in the first family is $-1/2$, and the slope of every line in the second family is 2. They do not appear to intersect at right angles because scales are different on the x- and y-axes.



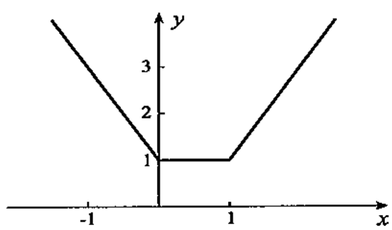
24. Since each straight line in the first family has slope $2/3$ and each line in the second family has slope $-1/k$, the families are orthogonal trajectories if $(2/3)(-1/k) = -1 \Longrightarrow k = 2/3$.

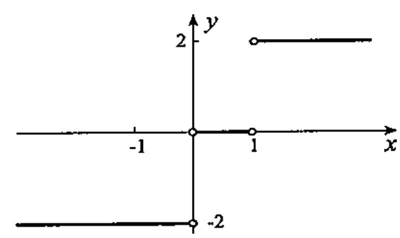
25. $f'(x) = 0$ for all x except for integer values of x for which there is no derivative. The graph is therefore a horizontal line with holes at integer values.



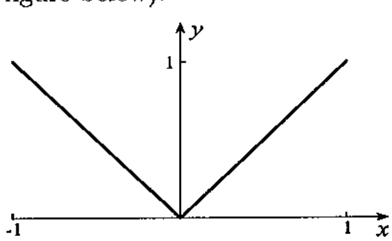
26. According to equation 3.2, $\displaystyle f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h|h| - 0}{h} = \lim_{h\to 0} |h| = 0.$

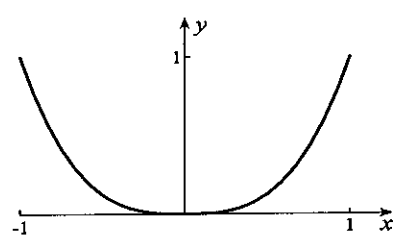
27. Using formula 3.13, $\displaystyle f'(x) = \frac{|x|}{x} + \frac{|x-1|}{x-1}.$



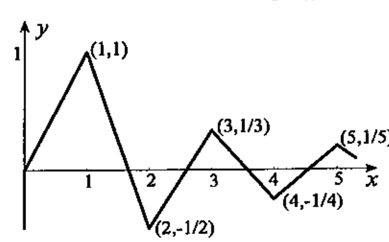


28. Sometimes. The function $f(x) = x$ is differentiable, but $|x|$ does not have a derivative at $x = 0$ (left figure below). On the other hand, $f(x) = x^3$ is differentiable, and $|x^3|$ does have a derivative at $x = 0$ (right figure below).





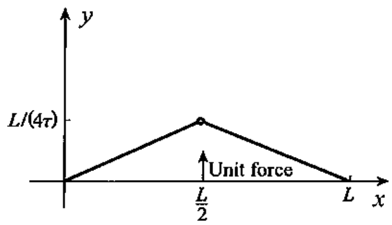
29. No. Consider the function $f(x)$ shown in the figure. It is asymptotic to the x-axis, but because the derivative is undefined at integer values of x, we cannot say that $\lim_{x\to\infty} f'(x) = 0$.

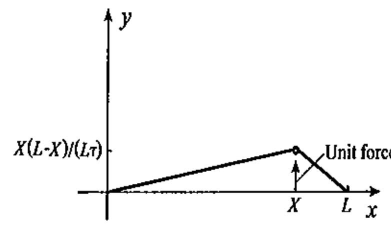


30. (a) For $X = L/2$,

$$G(x; L/2) = \frac{1}{L\tau}[x(L - L/2)h(L/2 - x) + (L/2)(L - x)h(x - L/2)]$$
$$= \frac{1}{\tau}\begin{cases} x/2 & 0 \le x < L/2 \\ (L-x)/2, & L/2 < x \le L \end{cases}.$$

Its graph in the left figure below is symmetric about $x = L/2$ as would be expected.





(b) When $L/2 < X < L$,

$$G(x; X) = \frac{1}{L\tau}\begin{cases} x(L - X), & 0 \le x < X \\ X(L - x), & X < x \le L \end{cases}.$$

This is shown in the right figure above.
(c) Since the functions x, $h(x - X)$, $L - x$, and $h(X - x)$ are continuous for $0 \le x \le L$, except at $x = X$ for the Heaviside functions, it follows that $G(x; X)$ is continuous for all $x \ne X$. Since

$$\lim_{x \to X^+} G(x; X) = \frac{X(L - X)}{L\tau} = \lim_{x \to X^-} G(x; X),$$

the discontinuity at $x = X$ is removable. The graphs in parts (a) and (b) illustrate this.
(d) The jump in the discontinuity of dG/dx at $x = X$ is

$$\lim_{x \to X^+} \frac{dG}{dx} - \lim_{x \to X^-} \frac{dG}{dx} = \frac{1}{L\tau} \lim_{x \to X^+} [(L - X)\,h(X - x) - X\,h(x - X)]$$
$$- \frac{1}{L\tau} \lim_{x \to X^-} [(L - X)\,h(X - x) - X\,h(x - X)]$$
$$= \frac{1}{L\tau}[-X] - \frac{1}{L\tau}[L - X] = -\frac{1}{\tau}.$$

This is the change in the slope of the graph of $G(x; X)$ at X.

31. They are not always the same. For the Heaviside function $h(x - a)$ in Figure 2.35, there is no right-hand derivative $f'_+(a)$ at $x = a$. On the other hand, $\lim_{x \to a^+} f'(x) = 0$.

32. According to equation 3.2, $f'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin\left(\dfrac{1}{h}\right)$. Since this limit does not exist, there is no derivative at $x = 0$.

33. According to equation 3.2, $f'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h^n \sin(1/h) - 0}{h} = \lim_{h \to 0} h^{n-1} \sin\left(\dfrac{1}{h}\right)$. This limit will be 0 when $n > 1$.

34. According to equation 3.2, $f'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \dfrac{f(h) - 0}{h} = \lim_{h \to 0} \dfrac{f(h)}{h}$.
When h is a rational number, $f(h)/h = h^2/h = h$, and when h is irrational, $f(h)/h = 0/h = 0$. It follows that $f'(0) = \lim_{h \to 0} \dfrac{f(h)}{h} = 0$.

EXERCISES 3.4

1. $f'(x) = 2x(x+3) + (x^2+2)(1) = 3x^2 + 6x + 2$

2. $f'(x) = (2-x^2)(2x+4) + (-2x)(x^2+4x+2) = 8 - 12x^2 - 4x^3$

3. $f'(x) = \dfrac{(3x+2)(1) - x(3)}{(3x+2)^2} = \dfrac{2}{(3x+2)^2}$

4. $f'(x) = \dfrac{(4x^2-5)(2x) - x^2(8x)}{(4x^2-5)^2} = \dfrac{-10x}{(4x^2-5)^2}$

5. $f'(x) = \dfrac{(2x-1)(2x) - x^2(2)}{(2x-1)^2} = \dfrac{2x^2-2x}{(2x-1)^2}$

6. $f'(x) = \dfrac{(4x^2+1)(3x^2) - x^3(8x)}{(4x^2+1)^2} = \dfrac{x^2(4x^2+3)}{(4x^2+1)^2}$

7. With $f(x) = x^{3/2} + \sqrt{x}$, we find $f'(x) = (3/2)\sqrt{x} + 1/(2\sqrt{x})$.

8. $f'(x) = \dfrac{(3x+2)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}(3)}{(3x+2)^2} = \dfrac{2-3x}{2\sqrt{x}(3x+2)^2}$

9. $f'(x) = \dfrac{(3x+4)(4x) - (2x^2-5)(3)}{(3x+4)^2} = \dfrac{6x^2+16x+15}{(3x+4)^2}$

10. $f'(x) = \dfrac{(2x^2-1)(1) - (x+5)(4x)}{(2x^2-1)^2} = -\dfrac{2x^2+20x+1}{(2x^2-1)^2}$

11. $f'(x) = \dfrac{(1-3x)(2x+1) - (x^2+x)(-3)}{(1-3x)^2} = \dfrac{1+2x-3x^2}{(1-3x)^2}$

12. $f'(x) = \dfrac{(x^2-5x+1)(2x+2) - (x^2+2x+3)(2x-5)}{(x^2-5x+1)^2} = \dfrac{-7x^2-4x+17}{(x^2-5x+1)^2}$

13. $f'(x) = \dfrac{(x^3-3x^2+2x+5)(0) - (3x^2-6x+2)}{(x^3-3x^2+2x+5)^2} = -\dfrac{3x^2-6x+2}{(x^3-3x^2+2x+5)^2}$

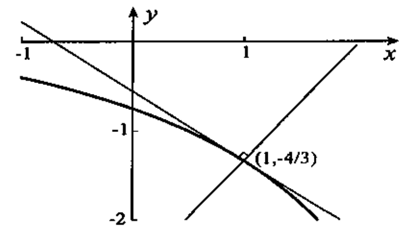
14. If we write the function in the form $f(x) = \dfrac{(x+1)^3+9}{(x+1)^3} = 1 + \dfrac{9}{(x+1)^3}$, its derivative is

$f'(x) = \dfrac{(x+1)^3(0) - 9\dfrac{d}{dx}(x+1)^3}{(x+1)^6}$. Since $\dfrac{d}{dx}(x+1)^3 = \dfrac{d}{dx}(x^3+3x^2+3x+1) = 3x^2+6x+3$ $= 3(x+1)^2$, we obtain $f'(x) = \dfrac{-9(3)(x+1)^2}{(x+1)^6} = \dfrac{-27}{(x+1)^4}$.

15. $f'(x) = \dfrac{(1-\sqrt{x})(1/3)x^{-2/3} - x^{1/3}(-1/2)x^{-1/2}}{(1-\sqrt{x})^2} = \dfrac{2(1-\sqrt{x}) + 3\sqrt{x}}{6x^{2/3}(1-\sqrt{x})^2} = \dfrac{2+\sqrt{x}}{6x^{2/3}(1-\sqrt{x})^2}$

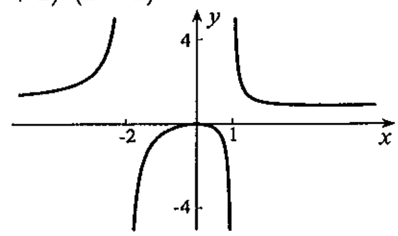
16.
$$f'(x) = \frac{(\sqrt{x}-4)\left(\dfrac{1}{2\sqrt{x}}+2\right) - (\sqrt{x}+2x)\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x}-4)^2}$$
$$= \frac{(\sqrt{x}-4)(1+4\sqrt{x}) - (\sqrt{x}+2x)}{2\sqrt{x}(\sqrt{x}-4)^2} = \frac{2x-16\sqrt{x}-4}{2\sqrt{x}(\sqrt{x}-4)^2} = \frac{x-8\sqrt{x}-2}{\sqrt{x}(\sqrt{x}-4)^2}$$

17. Since $f'(x) = \dfrac{(x-4)(1) - (x+3)(1)}{(x-4)^2} = \dfrac{-7}{(x-4)^2}$, the slope of the tangent line at $(1, -4/3)$ is $f'(1) = -7/9$. Equations for the tangent and normal lines are $y + 4/3 = -(7/9)(x-1)$ and $y + 4/3 = (9/7)(x-1)$, or, $7x + 9y + 5 = 0$ and $27x - 21y = 55$.



18. (a) The plot is shown to the right.

(b) $f'(x) = \dfrac{(x^2+x-2)(2x) - x^2(2x+1)}{(x^2+x-2)^2} = \dfrac{x^2-4x}{(x^2+x-2)^2} = \dfrac{x(x-4)}{(x+2)^2(x-1)^2}$



This expression shows that:

$f'(x) > 0$ for $x < -2$,
$f'(x) > 0$ for $-2 < x < 0$,
$f'(x) < 0$ for $0 < x < 1$,
$f'(x) < 0$ for $1 < x < 4$,
$f'(x) > 0$ for $x > 4$.

We can see the first three of these on the graph, but not the last two. The expression for $f'(x)$ also indicates that $f'(x) = 0$ at $x = 0$ and $x = 4$. This is clear on the graph when $x = 0$, but not when $x = 4$.

19. With $C(x) = a\left(\dfrac{x^2+bx}{x+c}\right)$,

$$C'(x) = a\left[\frac{(x+c)(2x+b)-(x^2+bx)(1)}{(x+c)^2}\right] = a\left[\frac{x^2+2cx+bc}{(x+c)^2}\right] = a\left[\frac{(x^2+2cx+c^2)+(bc-c^2)}{(x+c)^2}\right] = a\left[1+\frac{c(b-c)}{(x+c)^2}\right].$$

20.
$$\begin{aligned}\frac{d}{dx}[f(x)g(x)h(x)] &= f(x)\frac{d}{dx}[g(x)h(x)] + f'(x)[g(x)h(x)]\\ &= f(x)[g(x)h'(x)+g'(x)h(x)] + f'(x)[g(x)h(x)]\\ &= f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)\end{aligned}$$

21. To find point(s) of intersection, we set $x^3 = 2/(1+x^2)$, the only solution of which is $x = 1$. The only point of intersection is therefore $(1,1)$. Slopes of the curves $y = f(x) = x^3$ and $y = g(x) = 2/(1+x^2)$ at $(1,1)$ are

$$f'(1) = \{3x^2\}_{|x=1} = 3 \qquad \text{and} \qquad g'(1) = \left\{\frac{(1+x^2)(0)-2(2x)}{(1+x^2)^2}\right\}_{|x=1} = -1.$$

Angle θ between the curves at $(1,1)$ is given by equation 1.60,

$$\theta = \text{Tan}^{-1}\left|\frac{3+1}{1+(3)(-1)}\right| = 1.107 \text{ radians.}$$

22. To find the point(s) of intersection, we set $2x+2 = x^2/(x-1)$ from which $x = \pm\sqrt{2}$. These give the points of intersection $(\pm\sqrt{2}, 2\pm 2\sqrt{2})$. Slopes of the curves $y = f(x) = 2x+2$ and $y = g(x) = x^2/(x-1)$ at the point $(\sqrt{2}, 2+2\sqrt{2})$ are

$$f'(\sqrt{2}) = 2 \qquad \text{and} \qquad g'(\sqrt{2}) = \left[\frac{(x-1)(2x)-x^2(1)}{(x-1)^2}\right]_{|x=\sqrt{2}} = \frac{2}{1-\sqrt{2}}.$$

Angle θ between the curves at this point is given by equation 1.60,

$$\theta = \text{Tan}^{-1}\left|\frac{2-\left(\dfrac{2}{1-\sqrt{2}}\right)}{1+2\left(\dfrac{2}{1-\sqrt{2}}\right)}\right| = 0.668 \text{ radians.}$$

Slopes of the curves at the other point of intersection $(-\sqrt{2}, 2-2\sqrt{2})$ are

$$f'(-\sqrt{2}) = 2 \qquad \text{and} \qquad g'(-\sqrt{2}) = \left[\frac{(x-1)(2x)-x^2(1)}{(x-1)^2}\right]_{|x=-\sqrt{2}} = \frac{2}{1+\sqrt{2}}.$$

The angle between the curves at this point is

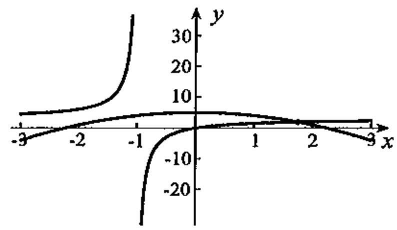
$$\theta = \text{Tan}^{-1}\left|\frac{2-\left(\dfrac{2}{1+\sqrt{2}}\right)}{1+2\left(\dfrac{2}{1+\sqrt{2}}\right)}\right| = 0.415 \text{ radians.}$$

23. Graphs of the curves suggest that they intersect near the point $(2,2)$. To confirm this, we solve

$$5 - x^2 = \frac{3x}{x+1}.$$

A calculator yields the only (real) solution as 1.757 28. The slope of $y = f(x) = 5 - x^2$ at this value of x is

$f'(1.757\,28) = -2(1.757\,28) = -3.514\,56.$



Because the slope of $y = g(x) = 3x/(x+1)$ at any value of x is

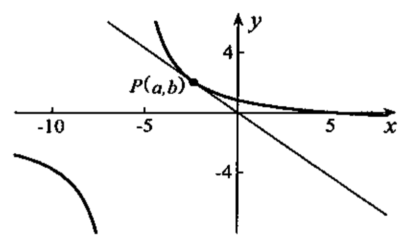
$$g'(x) = \frac{(x+1)(3) - 3x(1)}{(x+1)^2} = \frac{3}{(x+1)^2},$$

slope at $x = 1.757\,28$ is $g'(1.757\,28) = 3/(1.757\,28+1)^2 = 0.394\,602$. Using formula 1.60 with $m_1 = -3.514\,56$ and $m_2 = 0.394\,602$, the angle θ between the curves at their point of intersection is

$$\theta = \text{Tan}^{-1}\left|\frac{-3.514\,56 - 0.394\,602}{1+(-3.514\,56)(0.394\,602)}\right| = 1.47 \text{ radians.}$$

24. The slope of the tangent line to $y = f(x) = (5-x)/(6+x)$ at $P(a,b)$ is

$$f'(a) = \left[\frac{(6+x)(-1)-(5-x)(1)}{(6+x)^2}\right]_{|x=a} = \frac{-11}{(6+a)^2}.$$



The equation of the tangent line at P is

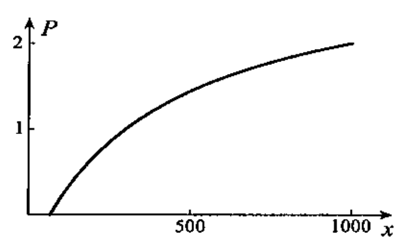
$$y - b = [-11/(6+a)^2](x-a),$$

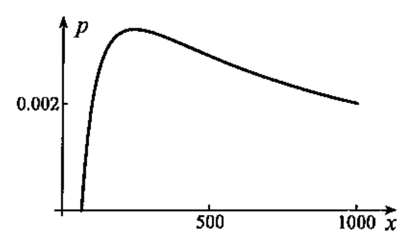
and this line passes through $(0,0)$ if

$$-b = [-11/(6+a)^2(-a).$$

Since $b = (5-a)/(6+a)$, it follows that $\quad -\dfrac{5-a}{6+a} = \dfrac{11a}{(6+a)^2} \implies a^2 - 10a - 30 = 0$. The two solutions of this equation are $a = 5 \pm \sqrt{55}$. The points at which the tangent line passes through $(0,0)$ are therefore $\left(5 \pm \sqrt{55}, \mp\sqrt{55}/(11 \pm \sqrt{55})\right)$.

25. (a),(b) Plots of $P(x)$ and $p(x)$ are shown below.



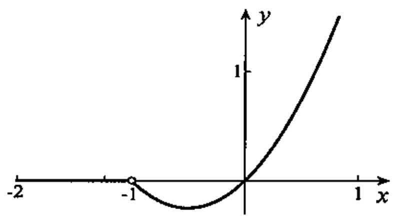


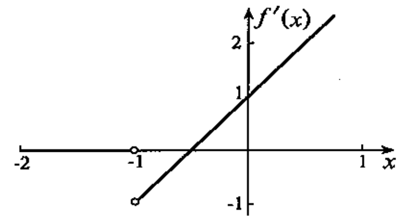
(c) Since average profit $p(x)$ is the slope of the line joining (x, P) to the origin, $p(x)$ is maximized when this line is tangent to the graph. This occurs when

$$\frac{P}{x} = P'(x) \quad \implies \quad \frac{3x-200}{x(x+400)} = \frac{(x+400)(3)-(3x-200)(1)}{(x+400)^2} = \frac{1400}{(x+400)^2}.$$

This equation implies that $1400x = (3x-200)(x+400) \implies 3x^2 - 400x - 80\,000 = 0$. The positive solution of this quadratic is approximately 243.

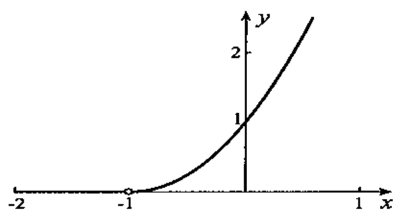
26. (a) $f'(x) = \begin{cases} 0, & x < -1 \\ 2x+1, & x > -1 \end{cases}$

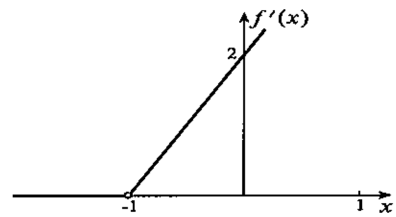




(b) No. The left-hand derivative would be 0 and the right-hand derivative would be -1.

27. (a) $f'(x) = \begin{cases} 0, & x < -1 \\ 2(x+1), & x > -1 \end{cases}$





(b) Yes. $f'(-1) = 0$

EXERCISES 3.5

1. Since $f'(x) = 3x^2 + 20x^3$, $f''(x) = 6x + 60x^2$.

2. Since $f'(x) = 3x^2 - 6x + 2$, $f''(x) = 6x - 6$, and $f'''(x) = 6$.

3. Since $f'(x) = x^3 + 3x + 2 + (x+1)(3x^2+3) = 4x^3 + 3x^2 + 6x + 5$, $f''(x) = 12x^2 + 6x + 6$. Hence, $f''(2) = 12(4) + 6(2) + 6 = 66$.

4. Since $f'(x) = 4x^3 - 6x - 1/x^2$, $f''(x) = 12x^2 - 6 + 2/x^3$, and $f'''(x) = 24x - 6/x^4$. Hence, $f'''(1) = 24(1) - 6/(1)^4 = 18$.

5. Since $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x^{3/2}}$, and $f''(x) = -\dfrac{1}{4x^{3/2}} + \dfrac{3}{4x^{5/2}}$.

6. Since $f'(t) = 3t^2 + 3/t^4$ and $f''(t) = 6t - 12/t^5$, we obtain $f'''(t) = 6 + 60/t^6$.

7. $\dfrac{d^9}{dx^9}(x^{10}) = \dfrac{d^8}{dx^8}(10x^9) = \dfrac{d^7}{dx^7}(10 \cdot 9x^8) = \cdots = 10 \cdot 9 \cdots 2\,x = 10!\,x$

8. Since $f'(u) = \dfrac{(u+1)\left(\dfrac{1}{2\sqrt{u}}\right) - \sqrt{u}(1)}{(u+1)^2} = \dfrac{1-u}{2\sqrt{u}(u+1)^2}$, it follows that

$$f''(u) = \frac{1}{2}\left\{\frac{\sqrt{u}(u+1)^2(-1) - (1-u)\left[\dfrac{1}{2\sqrt{u}}(u+1)^2 + \sqrt{u}\dfrac{d}{du}(u^2+2u+1)\right]}{u(u+1)^4}\right\}$$

$$= \frac{\dfrac{-2u(u+1)^2 - (1-u)[(u+1)^2 + 2u(2u+2)]}{2\sqrt{u}}}{2u(u+1)^4} = \frac{3u^2 - 6u - 1}{4u^{3/2}(u+1)^3}.$$

9. Since $\dfrac{dt}{dx} = \dfrac{(2x-6)(1) - x(2)}{(2x-6)^2} = \dfrac{-6}{(2x-6)^2}$,

$$\frac{d^2t}{dx^2} = -6\left[\frac{(2x-6)^2(0) - \dfrac{d}{dx}(2x-6)^2}{(2x-6)^4}\right] = 6\left[\frac{\dfrac{d}{dx}(4x^2 - 24x + 36)}{(2x-6)^4}\right] = \frac{6(8x-24)}{(2x-6)^4} = \frac{3}{(x-3)^3}.$$

10. Since $f'(x) = \dfrac{(\sqrt{x}+1)(1) - x\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+1)^2} = \dfrac{\sqrt{x}+2}{2(\sqrt{x}+1)^2}$, it follows that

$$f''(x) = \frac{1}{2}\left\{\frac{(\sqrt{x}+1)^2\left(\dfrac{1}{2\sqrt{x}}\right) - (\sqrt{x}+2)\dfrac{d}{dx}(x+2\sqrt{x}+1)}{(\sqrt{x}+1)^4}\right\}$$

$$= \frac{\dfrac{(\sqrt{x}+1)^2 - 2\sqrt{x}(\sqrt{x}+2)\left(1+\frac{1}{\sqrt{x}}\right)}{2\sqrt{x}}}{2(\sqrt{x}+1)^4} = -\frac{\sqrt{x}+3}{4\sqrt{x}(\sqrt{x}+1)^3}.$$

11. (a) $\dfrac{2}{r}\dfrac{dT}{dr} + \dfrac{d^2T}{dr^2} = \dfrac{2}{r}\left(-\dfrac{d}{r^2}\right) + \dfrac{2d}{r^3} = 0$

(b) Since $f(a) = T_a$ and $f(b) = T_b$, it follows that $T_a = c + \dfrac{d}{a}$, $\quad T_b = c + \dfrac{d}{b}$. The solution of these equations is $c = \dfrac{bT_b - aT_a}{b-a}$ and $d = \dfrac{ab(T_a - T_b)}{b-a}$.

12. (a) $\dfrac{d}{dr}\left(r^2\dfrac{dT}{dr}\right) = \dfrac{d}{dr}\left[r^2\left(\dfrac{kr}{3} - \dfrac{c}{r^2}\right)\right] = \dfrac{d}{dr}\left[\dfrac{kr^3}{3} - c\right] = kr^2$

(b) Since $f(a) = T_a$ and $f(b) = T_b$, it follows that $T_a = \dfrac{ka^2}{6} + \dfrac{c}{a} + d$, $\qquad T_b = \dfrac{kb^2}{6} + \dfrac{c}{b} + d$. The solution of these equations is

$$c = \frac{ab}{b-a}\left[(T_a - T_b) + \frac{k}{6}(b^2 - a^2)\right], \qquad d = \frac{bT_b - aT_a}{b-a} - \frac{k}{6}(a^2 + ab + b^2).$$

13. Since $f'(x) = 3ax^2 + 2bx + c$, and $f''(x) = 6ax + 2b$, we must have

$$4 = 3a + 2b + c, \qquad 5 = 12a + 2b.$$

Because $f(1) = 0$ and $f(2) = 4$,

$$0 = a + b + c + d, \qquad 4 = 8a + 4b + 2c + d.$$

The solution of these four equations is $a = 5/4$, $b = -5$, $c = 41/4$, and $d = -13/2$.

14. (a) Using the result in Example 3.19,

$$\frac{d^3}{dx^3}(uv) = \frac{d}{dx}\left(v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right)$$

$$= \frac{dv}{dx}\frac{d^2u}{dx^2} + v\frac{d^3u}{dx^3} + 2\frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{du}{dx}\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}$$

$$= v\frac{d^3u}{dx^3} + 3\frac{dv}{dx}\frac{d^2u}{dx^2} + 3\frac{d^2v}{dx^2}\frac{du}{dx} + u\frac{d^3v}{dx^3}.$$

(b) $\dfrac{d^4}{dx^4}(uv) = v\dfrac{d^4u}{dx^4} + 4\dfrac{dv}{dx}\dfrac{d^3u}{dx^3} + 6\dfrac{d^2v}{dx^2}\dfrac{d^2u}{dx^2} + 4\dfrac{d^3v}{dx^3}\dfrac{du}{dx} + v\dfrac{d^4u}{dx^4}$

15. If $(x^2-1)^n$ were expanded, the result would be a polynomial of degree $2n$. When $2n$ differentiations are performed, the only nonzero term is the $(2n)^{\text{th}}$-derivative of x^{2n}, and this gives $(2n)!$.

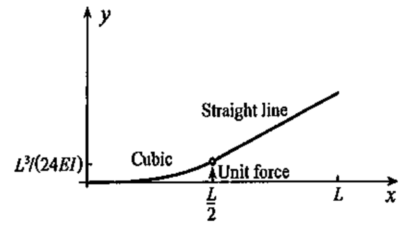
16. (a) When $X = L/2$,

$$G(x; L/2) = \frac{1}{6EI}(x - L/2)^3 h(x - L/2) - \frac{x^3}{6EI} + \frac{Lx^2}{4EI}$$

$$= \frac{1}{12EI}\begin{cases} -2x^3 + 3Lx^2, & 0 \le x < L/2 \\ 2(x-L/2)^3 - 2x^3 + 3Lx^2, & L/2 < x \le L \end{cases}$$

$$= \frac{1}{12EI}\begin{cases} -2x^3 + 3Lx^2, & 0 \le x < L/2 \\ 3xL^2/2 - L^3/4, & L/2 < x \le L \end{cases}.$$

A plot is shown to the right. It has a removable discontinuity at $x = L/2$. It is straight for $L/2 < x \leq L$, as would be expected since no forces act on the board for $x > L/2$.



(b) For $x > X$,

$$
\begin{aligned}
G(x;X) &= \frac{1}{6EI}(x-X)^3 - \frac{x^3}{6EI} + \frac{Xx^2}{2EI} \\
&= \frac{1}{6EI}(x^3 - 3x^2X + 3xX^2 - X^3) - \frac{x^3}{6EI} + \frac{Xx^2}{2EI} = \frac{X^2}{2EI}x - \frac{X^3}{6EI},
\end{aligned}
$$

and this is a straight line.
(c) Since the functions $(x-X)^3$, $h(x-X)$, x^3, and x^2 are continuous for $0 \leq x \leq L$, except at $x = X$ for the Heaviside function, it follows that $G(x;X)$ is continuous for all $x \neq X$. Since

$$
\lim_{x \to X^+} G(x;X) = -\frac{X^3}{6EI} + \frac{X^3}{2EI} = \frac{X^3}{3EI} = \lim_{x \to X^-} G(x;X),
$$

the discontinuity at $x = X$ is removable. Limits of $dG/dx = [(x-X)^2h(x-X) - x^2 + 2Xx]/(2EI)$ and $d^2G/dx^2 = [(x-X)h(x-X) - x + X]/(EI)$, show that they are also continuous except for a removable discontinuity at $x = X$.
(d) Since $d^3G/dx^3 = [h(x-X) - 1]/(EI)$, the jump in d^3G/dx^3 at $x = X$ is

$$
\lim_{x \to X^+} \frac{d^3G}{dx^3} - \lim_{x \to X^-} \frac{d^3G}{dx^3} = 0 - \left(-\frac{1}{EI}\right) = \frac{1}{EI}.
$$

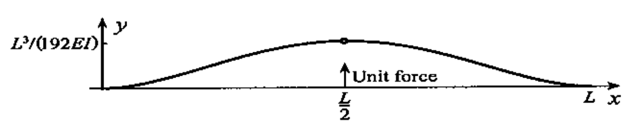
17. (a) When $X = L/2$,

$$
\begin{aligned}
G(x;L/2) &= \frac{1}{6EI}(x-L/2)^3h(x-L/2) + \frac{x^3}{6EIL^3}\left(-L^3 + \frac{3L^3}{4} - \frac{2L^3}{8}\right) \\
&\quad + \frac{x^2}{2EIL^2}\left[\frac{L^3}{8} - 2L\left(\frac{L^2}{4}\right) + L^2\left(\frac{L}{2}\right)\right] \\
&= \frac{1}{48EI}\begin{cases} -4x^3 + 3Lx^2, & 0 \leq x < L/2 \\ 4x^3 - 9Lx^2 + 6L^2x - L^3, & L/2 < x \leq L \end{cases}.
\end{aligned}
$$

It is straightforward to check that values of this function and its derivative are zero at $x = 0$ and $x = L$. We can show that it is symmetric about $x = L/2$ by replacing each x by $L - x$ in that part of $G(x;L/2)$ for $x > L/2$ to show that we get $-4x^3 + 3Lx^2$,

$$
\begin{aligned}
4(L-x)^3 - 9L(L-x)^2 + 6L^2(L-x) - L^3 &= 4(L^3 - 3L^2x + 3Lx^2 - x^3) \\
&\quad - 9L(L^2 - 2Lx + x^2) + 6L^2(L-x) - L^3 \\
&= 3Lx^2 - 4x^3.
\end{aligned}
$$

The graph is shown below with a removable discontinuity at $x = L/2$.



(b) Only the first term of $G(x;X)$ has a discontinuity, and it is at $x = X$. Since

$$
\lim_{x \to X^+} G(x;X) = \lim_{x \to X^-} G(x;X),
$$

the discontinuity at $x = X$ is removable. Limits of

$$\frac{dG}{dx} = \frac{1}{2EI}(x-X)^2h(x-X) + \frac{x^2}{2EIL^3}(-L^3+3LX^2-2X^3) + \frac{x}{EIL^2}(X^3-2LX^2+L^2X)$$

and

$$\frac{d^2G}{dx^2} = \frac{1}{EI}(x-X)h(x-X) + \frac{x}{EIL^3}(-L^3+3LX^2-2X^3) + \frac{1}{EIL^2}(X^3-2LX^2+L^2X)$$

show that they are also continuous except for a removable discontinuity at $x = X$.

(c) Since $\dfrac{d^3G}{dx^3} = \dfrac{1}{EI}h(x-X) + \dfrac{1}{EIL^3}(-L^3+3LX^2-2X^3)$, the jump in d^3G/dx^3 at $x = X$ is

$$\lim_{x\to X^+}\frac{d^3G}{dx^3} - \lim_{x\to X^-}\frac{d^3G}{dx^3} = \frac{1}{EI}.$$

18. When $n = 1$: Left side $= \dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ and

Right side $= \displaystyle\sum_{r=0}^{1}\binom{1}{r}\frac{d^ru}{dx^r}\frac{d^{1-r}v}{dx^{1-r}} = \binom{1}{0}\frac{d^0u}{dx^0}\frac{dv}{dx} + \binom{1}{1}\frac{du}{dx}\frac{d^0v}{dx^0} = u\frac{dv}{dx} + v\frac{du}{dx}.$

The result is therefore valid for $n = 1$. Suppose k is some integer for which the result is valid; that is,

$$\frac{d^k}{dx^k}(uv) = \sum_{r=0}^{k}\binom{k}{r}\frac{d^ru}{dx^r}\frac{d^{k-r}v}{dx^{k-r}}.$$

Then,

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}}(uv) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}(uv)\right] = \sum_{r=0}^{k}\binom{k}{r}\left[\frac{d^{r+1}u}{dx^{r+1}}\frac{d^{k-r}v}{dx^{k-r}} + \frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}}\right]\\
&= \sum_{r=0}^{k}\binom{k}{r}\frac{d^{r+1}u}{dx^{r+1}}\frac{d^{k-r}v}{dx^{k-r}} + \sum_{r=0}^{k}\binom{k}{r}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}}\\
&= \sum_{r=1}^{k+1}\binom{k}{r-1}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}} + \sum_{r=0}^{k}\binom{k}{r}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}}\\
&= \binom{k}{0}\frac{d^0u}{dx^0}\frac{d^{k+1}v}{dx^{k+1}} + \sum_{r=1}^{k}\left[\binom{k}{r-1} + \binom{k}{r}\right]\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}} + \binom{k}{k}\frac{d^{k+1}u}{dx^{k+1}}\frac{d^0v}{dx^0}\\
&= \binom{k+1}{0}\frac{d^0u}{dx^0}\frac{d^{k+1}v}{dx^{k+1}} + \sum_{r=1}^{k}\binom{k+1}{r}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}} + \binom{k+1}{k+1}\frac{d^{k+1}u}{dx^{k+1}}\frac{d^0v}{dx^0}\\
&= \sum_{r=0}^{k+1}\binom{k+1}{r}\frac{d^ru}{dx^r}\frac{d^{k-r+1}v}{dx^{k-r+1}}.
\end{aligned}$$

Since this is the result for $k+1$, the formula is correct for all n by mathematical induction.

EXERCISES 3.6

1. (a) Since x is negative at $t = 1$, the particle is to the left of the origin at this time. Since x is positive at $t = 4$, the particle is to the right of the origin at this time.

(b) Since the slope of the graph is negative at $t = 1/2$, the particle is moving to the left at this time. Since the slope of the graph is positive at $t = 3$, the particle is moving to the right at this time.

(c) Since the slope changes sign three times, the particle changes direction three times.

(d) At $t = 7/2$ the slope of the graph is close to zero, whereas at $t = 9/2$ it is clearly negative. Hence the velocity is greater at $t = 7/2$.

(e) Since speed is the magnitude of velocity, it is greater at $t = 9/2$.

2. The velocity of the particle is $v(t) = \dfrac{2t^3}{3} - \dfrac{32t^2}{5} + \dfrac{50t}{3} - \dfrac{251}{30}$.

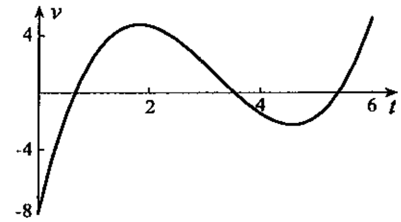
(a) Since $x(1) = -2$, the particle is to the left of the origin at $t = 1$. Since $x(4) = 6$, the particle is to the right of the origin at $t = 4$.

(b) Since $v(1/2) = -31/20$, the particle is moving to the left at $t = 1/2$. Since $v(3) = 61/30$, the particle is moving to the right at $t = 3$.

(c) The graph of $v(t)$ to the right shows that $v(t)$ changes sign three times. Hence, the particle changes direction three times.

(d) Since $v(7/2) = 3/20$ and $v(9/2) = -133/60$, the velocity is greater at $t = 7/2$.

(e) Since $|v(7/2)| = 3/20$ and $|v(9/2)| = 133/60$, the speed is greater at $t = 9/2$.



3. (a) Since x is positive at $t = 1$, the particle is to the right of the origin at this time. Since x is negative at $t = 4$, the particle is to the left of the origin at this time.

(b) Since the slope of the graph is positive at $t = 1/2$, the particle is moving to the right at this time. Since the slope of the graph is negative at $t = 3$, the particle is moving to the left at this time.

(c) Since the slope changes sign twice, the particle changes direction twice.

(d) At $t = 7/2$ the slope of the graph is negative, whereas at $t = 9/2$ it is positive. Hence the velocity is greater at $t = 9/2$.

(e) Since speed is the magnitude of velocity, it would appear to be slightly greater at $t = 9/2$.

4. The velocity of the particle is $v(t) = \dfrac{14t^2}{15} - \dfrac{202t}{45} + \dfrac{132}{45}$.

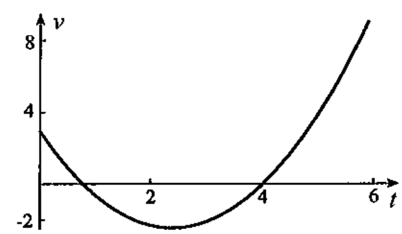
(a) Since $x(1) = 3$, the particle is to the right of the origin at $t = 1$. Since $x(4) = -34/15$, the particle is to the left of the origin at $t = 4$.

(b) Since $v(1/2) = 83/90$, the particle is moving to the right at $t = 1/2$. Since $v(3) = -32/15$, the particle is moving to the left at $t = 3$.

(c) The graph of $v(t)$ to the right shows that $v(t)$ changes sign twice. Hence, the particle changes direction twice.

(d) Since $v(7/2) = -121/90$ and $v(9/2) = 49/30$, the velocity is greater at $t = 9/2$.

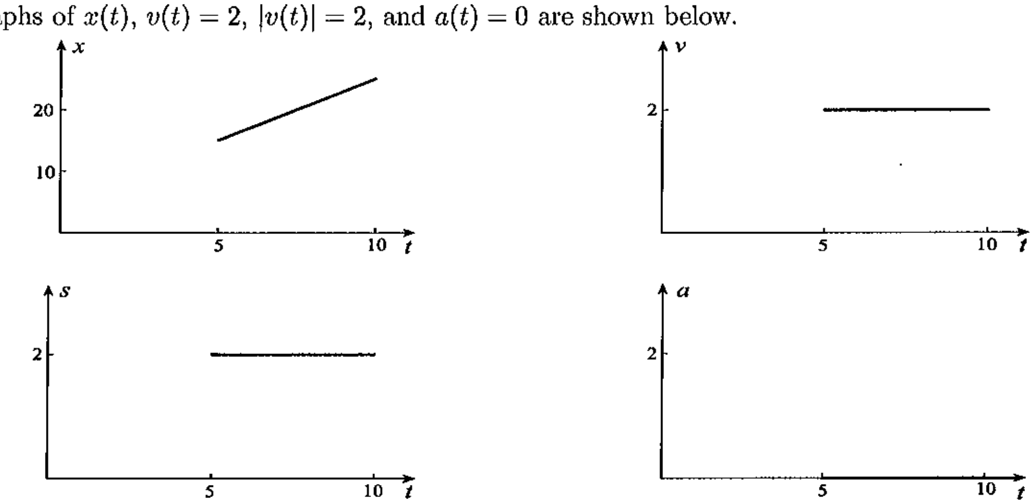
(e) Since $|v(7/2)| = 121/90$ and $|v(9/2)| = 49/30$, the speed is greater at $t = 9/2$.



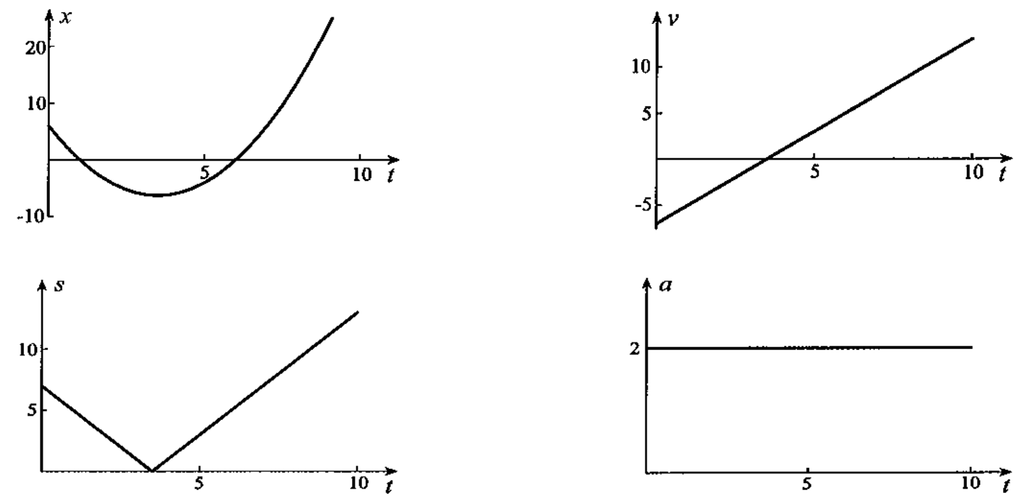
5. If the acceleration of a car changes rapidly (perhaps because the clutch is let out too fast), the car experiences jerky motion. The jerk for the dispacement function in Exercise 2 is $x'''(t) = 4t - 64/5$.

6. The jerk is $x'''(t) = 28/15$.

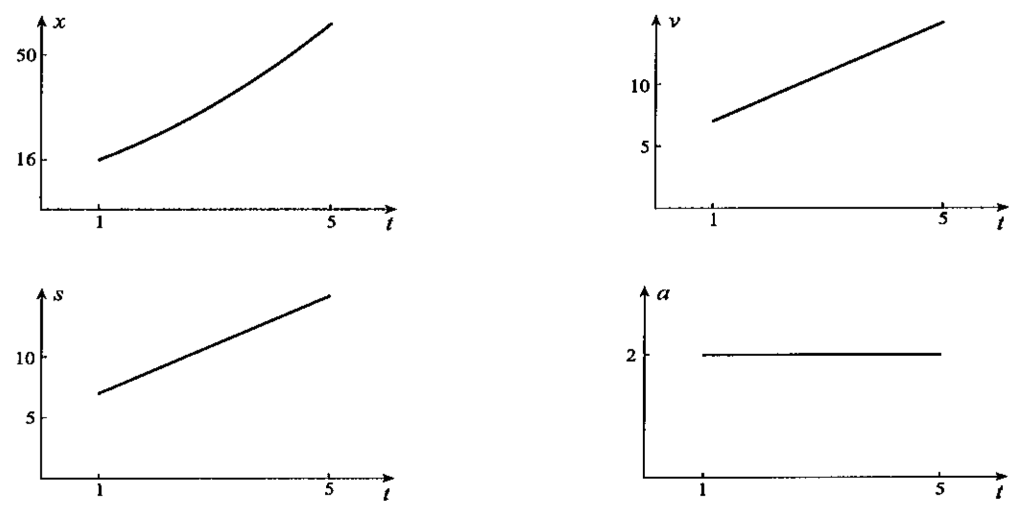
7. Graphs of $x(t)$, $v(t) = 2$, $|v(t)| = 2$, and $a(t) = 0$ are shown below.



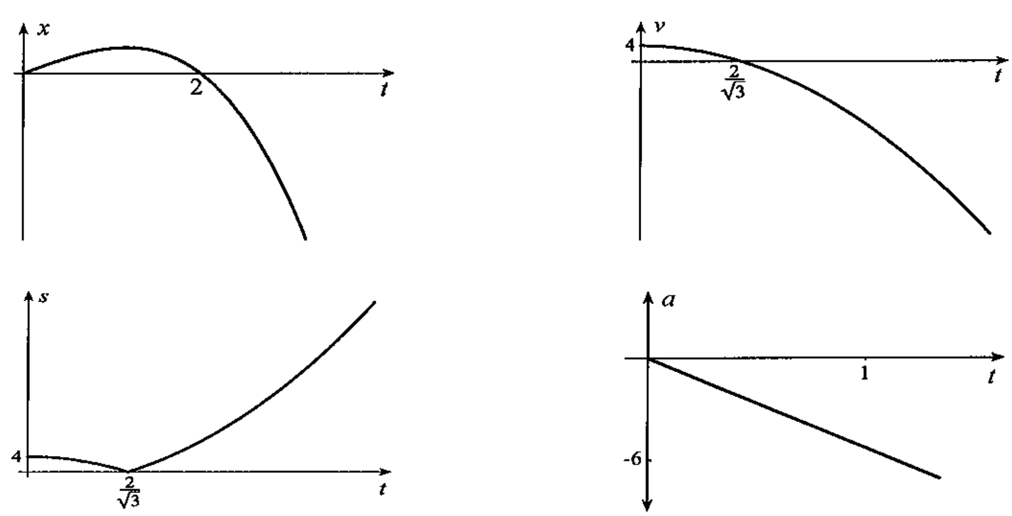
8. Graphs of $x(t)$, $v(t) = 2t - 7$, $|v(t)| = |2t - 7|$, and $a(t) = 2$ are shown below.



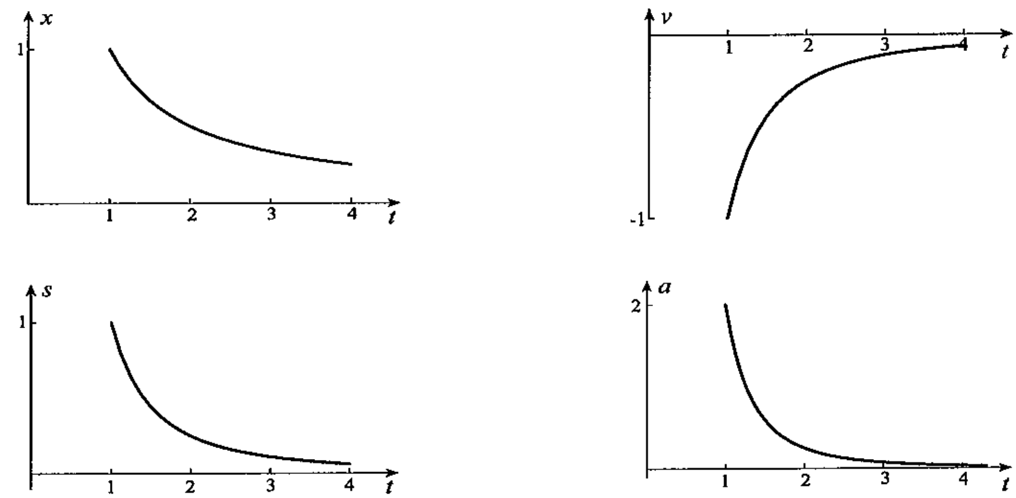
9. Graphs of $x(t)$, $v(t) = 2t + 5$, $|v(t)| = 2t + 5$, and $a(t) = 2$ are shown below.



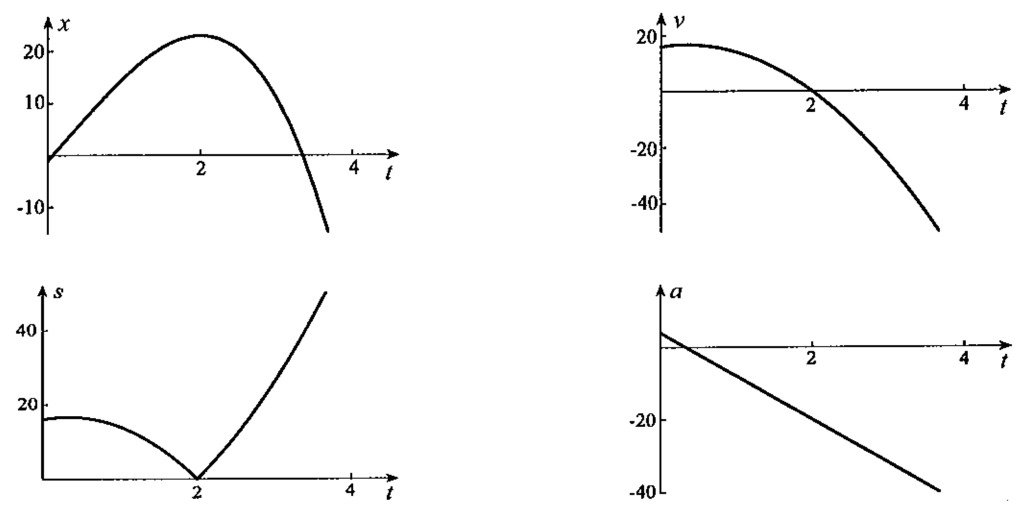
10. Graphs of $x(t)$, $v(t) = 4 - 3t^2$, $|v(t)| = |4 - 3t^2|$, and $a(t) = -6t$ are shown below.



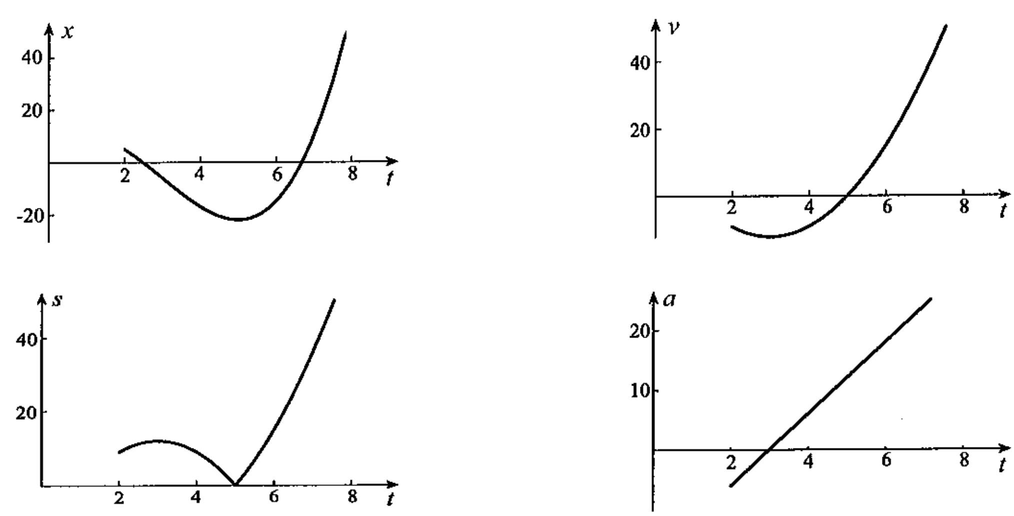
11. Graphs of $x(t)$, $v(t) = -1/t^2$, $|v(t)| = 1/t^2$, and $a(t) = 2/t^3$ are shown below.



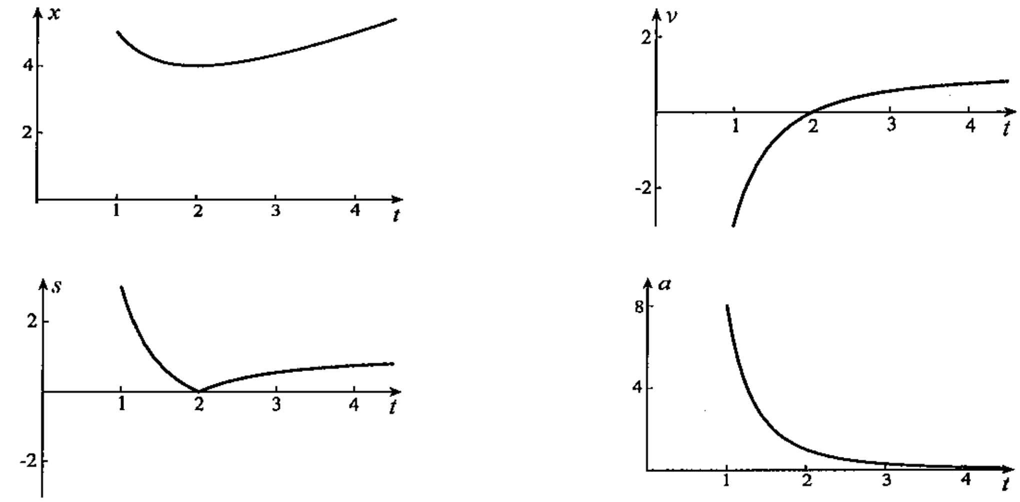
12. Plots of $x(t)$, $v(t) = -6t^2 + 4t + 16$, $|v(t)| = |-6t^2 + 4t + 16|$, and $a(t) = -12t + 4$ are shown below.



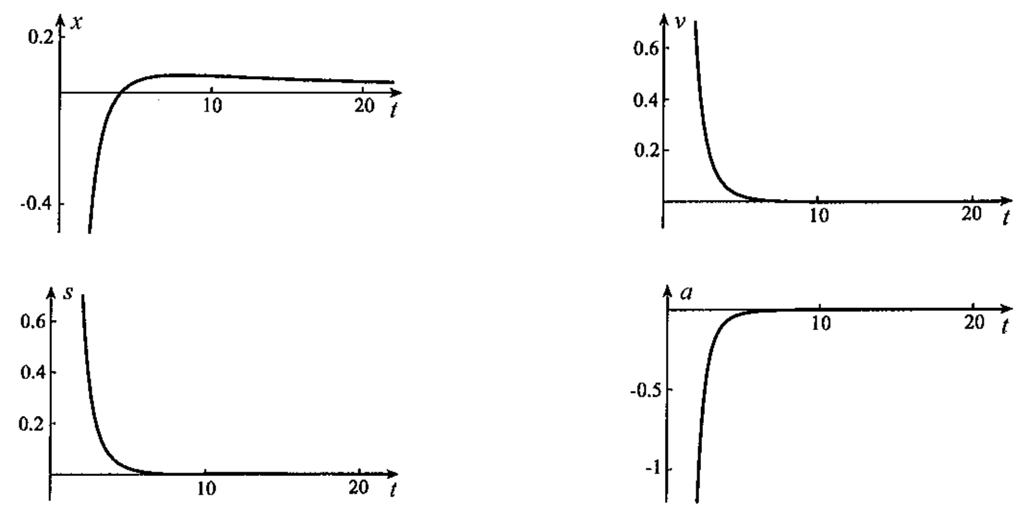
13. Plots of $x(t)$, $v(t) = 3t^2 - 18t + 15$, $|v(t)| = |3t^2 - 18t + 15|$, and $a(t) = 6t - 18$ are shown below.



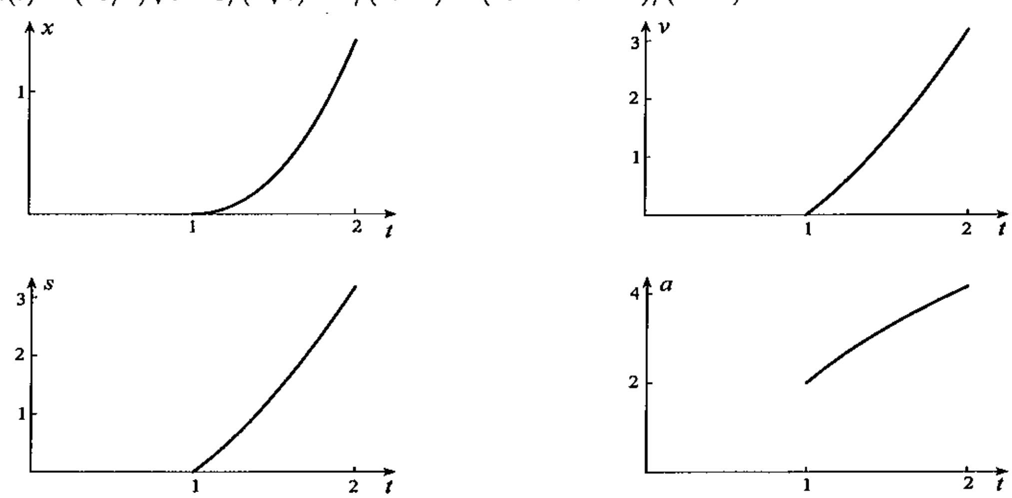
14. Plots of $x(t)$, $v(t) = 1 - 4/t^2$, $|v(t)| = |1 - 4/t^2|$, and $a(t) = 8/t^3$ are shown below.



15. Plots of $x(t)$, $v(t) = -1/t^2 + 8/t^3$, $|v(t)| = |-1/t^2 + 8/t^3|$, and $a(t) = 2/t^3 - 24/t^4$ are shown below.



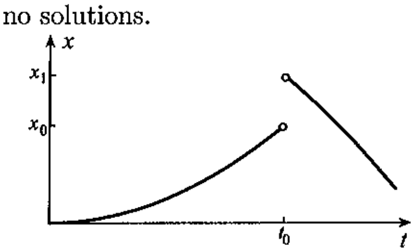
16. Plots of $x(t)$, $v(t) = (5/2)t^{3/2} - 3\sqrt{t} + (1/2)/\sqrt{t} = (5t^2 - 6t + 1)/(2\sqrt{t})$, $|v(t)| = |(5t^2 - 6t + 1)/(2\sqrt{t})|$, and $a(t) = (15/4)\sqrt{t} - 3/(2\sqrt{t}) - 1/(4t^{3/2}) = (15t^2 - 6t - 1)/(4t^{3/2})$ are shown below.



17. We calculate that $v(t) = 3t^2 - 18t + 24 = 3(t-2)(t-4)$ and $a(t) = 6t - 18 = 6(t-3)$.
(a) $x(3) = 19$ m, $v(3) = -3$ m/s, $|v(3)| = 3$ m/s, $a(3) = 0$ m/s^2
(b) The object is at rest when $v(t) = 0$ and this occurs at $t = 2$ s and $t = 4$ s.
(c) Acceleration vanishes at $t = 3$ s.
(d) Since $v(t) > 0$ for $0 \le t < 2$ and $t > 4$, the object is moving to the right for these times. It moves to the left for $2 < t < 4$.
(e) The velocity is 1 m/s if $1 = 3t^2 - 18t + 24 \implies 3t^2 - 18t + 23 = 0$. There are two solutions of this equation $t = (9 \pm 2\sqrt{3})/3$.
(f) The speed is 1 m/s if $|3t^2 - 18t + 24| = 1$. This implies that $3t^2 - 18t + 24 = 1$ or $3t^2 - 18t + 24 = -1$. The first gives the times in part (e). For the second possibility, we solve $3t^2 - 18t + 25 = 0$ for the times $t = (9 \pm \sqrt{6})/3$.
(g) The velocity is 20 m/s if $20 = 3t^2 - 18t + 24 \implies 3t^2 - 18t + 4 = 0$. There are two solutions $t = (9 \pm \sqrt{69})/3$.
(h) The speed is 20 m/s if $|3t^2 - 18t + 24| = 20$. This implies that $3t^2 - 18t + 24 = 20$ or $3t^2 - 18t + 24 = -20$. The first gives the times in part (g). The second quadratic has no solutions.

18. We calulate that $v(t) = 3t^2 - 18t + 15 = 3(t-1)(t-5)$ and $a(t) = 6t - 18 = 6(t-3)$.
(a) $x(3) = -11$ m, $v(3) = -12$ m/s, $|v(3)| = 12$ m/s, $a(3) = 0$ m/s^2
(b) The object is at rest when $v(t) = 0$ and this occurs at $t = 1$ s and $t = 5$ s.
(c) Acceleration vanishes at $t = 3$ s.
(d) Since $v(t) > 0$ for $0 \le t < 1$ and $t > 5$, the object is moving to the right for these times. It moves to the left for $1 < t < 5$.
(e) The velocity is 1 m/s if $1 = 3t^2 - 18t + 15 \implies 3t^2 - 18t + 14 = 0$. There are two solutions $t = (9 \pm \sqrt{39})/3$.
(f) The speed is 1 m/s if $|3t^2 - 18t + 15| = 1$. This implies that $3t^2 - 18t + 15 = 1$ or $3t^2 - 18t + 15 = -1$. The first gives the times in part (e). For the second possibility, we solve $3t^2 - 18t + 16 = 0$ for $t = (9 \pm \sqrt{33})/3$.
(g) The velocity is 20 m/s if $20 = 3t^2 - 18t + 15 \implies 3t^2 - 18t - 5 = 0$. Of the two solutions $t = (9 \pm 4\sqrt{6})/3$, only $t = (9 + 4\sqrt{6})/3$ is positive.
(h) The speed is 20 m/s if $|3t^2 - 18t + 15| = 20$. This implies that $3t^2 - 18t + 15 = 20$ or $3t^2 - 18t + 15 = -20$. The first gives the time in part (g). The second quadratic has no solutions.

19. No. If it were to have a discontinuity at $t = t_0$ as in the figure to the right, the particle would disappear at position x_0 and reappear instantaneously at position x_1. At a removable discontinuity, it reappears at the same position.



20. (a) Its average velocity is

$$\frac{x_2 - x_1}{t_2 - t_1} = \frac{(at_2^2/2 + bt_2 + c) - (at_1^2/2 + bt_1 + c)}{t_2 - t_1} = \frac{(a/2)(t_2^2 - t_1^2) + b(t_2 - t_1)}{t_2 - t_1} = \frac{a}{2}(t_1 + t_2) + b.$$

(b) The instantaneous velocity is equal to the average velocity when

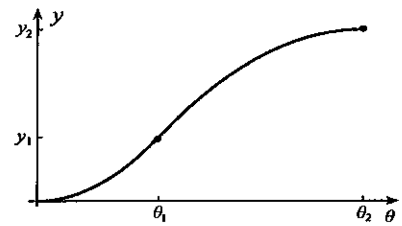
$$at + b = \frac{a}{2}(t_1 + t_2) + b \quad \implies \quad t = \frac{t_1 + t_2}{2}.$$

The position of the object at this time is $x\big((t_1 + t_2)/2\big) = \frac{a}{2}\left(\frac{t_1 + t_2}{2}\right)^2 + b\left(\frac{t_1 + t_2}{2}\right) + c$.
If we subtract this from $(x_1 + x_2)/2$, we obtain

$$\frac{x_1 + x_2}{2} - x\big((t_1 + t_2)/2\big) = \frac{1}{2}\left(\frac{1}{2}at_1^2 + bt_1 + c + \frac{1}{2}at_2^2 + bt_2 + c\right) - \frac{a}{2}\left(\frac{t_1 + t_2}{2}\right)^2 - b\left(\frac{t_1 + t_2}{2}\right) - c$$
$$= \frac{a}{8}(t_1 - t_2)^2.$$

Since this quantity is positive, the object is closer to x_1.

21. Equations of the parabolas must be



$$y(\theta) = \begin{cases} a\theta^2, & 0 \le \theta \le \theta_1 \\ A(\theta-\theta_2)^2 + y_2, & \theta_1 \le \theta \le \theta_2 \end{cases}.$$

For (θ_1, y_1) to be a point on both parabolas,

$y_1 = a\theta_1^2, \qquad y_1 = A(\theta_1 - \theta_2)^2 + y_2.$

For continuity of the slope at (θ_1, y_1), the left-hand derivative of $a\theta^2$ and the right-hand derivative of $A(\theta-\theta_2)^2 + y_2$ must be equal at (θ_1, y_1),

$$2a\theta_1 = 2A(\theta_1 - \theta_2).$$

If we solve the first two equations for a and A, and substitute these into the last equation,

$$2\left(\frac{y_1}{\theta_1^2}\right)\theta_1 = 2\left[\frac{y_1 - y_2}{(\theta_1-\theta_2)^2}\right](\theta_1 - \theta_2) \quad\Longrightarrow\quad y_1(\theta_1 - \theta_2) = \theta_1(y_1 - y_2) \quad\Longrightarrow\quad \frac{\theta_1}{\theta_2} = \frac{y_1}{y_2}.$$

The equation of the line through $(0,0)$ and (θ_2, y_2) is $y = y_2\theta/\theta_2$. Since (θ_1, y_1) satisfies this equation by virtue of the fact that $\theta_1/\theta_2 = y_1/y_2$, (θ_1, y_1) is on the line. Since

$$a = y_1\left(\frac{y_2}{y_1\theta_2}\right)^2 = \frac{y_2^2}{y_1\theta_2^2}, \qquad A = \frac{y_1 - y_2}{\left(\dfrac{y_1\theta_2}{y_2} - \theta_2\right)^2} = \frac{y_2^2}{(y_1-y_2)\theta_2^2},$$

the equation of the curve is

$$y(\theta) = \begin{cases} \dfrac{y_2^2\theta^2}{y_1\theta_2^2}, & 0 \le \theta \le y_1\theta_2/y_2 \\ \dfrac{y_2^2(\theta-\theta_2)^2}{(y_1-y_2)\theta_2^2} + y_2, & y_1\theta_2/y_2 \le \theta \le \theta_2 \end{cases}.$$

22. At point A, we must have

$$y_1 = a\theta_1^2, \qquad y_1 = m\theta_1 + b, \qquad m = 2a\theta_1.$$

From these,

$$a = \frac{y_1}{\theta_1^2}, \qquad m = 2\left(\frac{y_1}{\theta_1^2}\right)\theta_1 = \frac{2y_1}{\theta_1}, \qquad b = y_1 - 2\theta_1\left(\frac{y_1}{\theta_1^2}\right)\theta_1 = -y_1.$$

At point B, we must have

$$y_2 = A(\theta_2 - \theta_3)^2 + y_3, \qquad y_2 = m\theta_2 + b, \qquad m = 2A(\theta_2 - \theta_3).$$

From these,

$$A = \frac{y_2 - y_3}{(\theta_2 - \theta_3)^2}, \qquad m = 2\left[\frac{y_2 - y_3}{(\theta_2-\theta_3)^2}\right](\theta_2 - \theta_3) = \frac{2(y_2 - y_3)}{\theta_2 - \theta_3},$$

and

$$b = y_3 + \frac{(y_2 - y_3)}{(\theta_2-\theta_3)^2}(\theta_2 - \theta_3)^2 - \frac{2(y_2-y_3)\theta_2}{\theta_2 - \theta_3} = \frac{y_2(\theta_2 - \theta_3) - 2(y_2 - y_3)\theta_2}{\theta_2 - \theta_3} = \frac{2y_3\theta_2 - y_2\theta_2 - y_2\theta_3}{\theta_2 - \theta_3}.$$

If we equate the values of b,

$$-y_1 = \frac{2y_3\theta_2 - y_2\theta_2 - y_2\theta_3}{\theta_2 - \theta_3} \quad\Longrightarrow\quad \theta_2 = \frac{(y_1 + y_2)\theta_3}{y_1 - y_2 + 2y_3}.$$

When we equate values of m, $\dfrac{2y_1}{\theta_1} = \dfrac{2(y_2 - y_3)}{\theta_2 - \theta_3}$, from which

$$\theta_1 = \frac{y_1(\theta_2-\theta_3)}{y_2-y_3} = \frac{y_1}{y_2-y_3}\left[\frac{(y_1+y_2)\theta_3}{y_1-y_2+2y_3} - \theta_3\right] = \frac{y_1}{y_2-y_3}\left[\frac{(y_1+y_2)\theta_3 - \theta_3(y_1-y_2+2y_3)}{y_1-y_2+2y_3}\right]$$
$$= \frac{y_1}{y_2-y_3}\left[\frac{2\theta_3(y_2-y_3)}{y_1-y_2+2y_3}\right] = \frac{2y_1\theta_3}{y_1-y_2+2y_3}.$$

These now give

$$a = \frac{y_1}{\theta_1^2} = \frac{y_1(y_1-y_2+2y_3)^2}{4y_1^2\theta_3^2} = \frac{(y_1-y_2+2y_3)^2}{4y_1\theta_3^2},$$
$$m = \frac{2y_1}{\theta_1} = \frac{2y_1(y_1-y_2+2y_3)}{2y_1\theta_3} = \frac{y_1-y_2+2y_3}{\theta_3},$$
$$b = -y_1,$$
$$A = \frac{y_2-y_3}{(\theta_2-\theta_3)^2} = \frac{y_2-y_3}{\left[\dfrac{(y_1+y_2)\theta_3}{y_1-y_2+2y_3} - \theta_3\right]^2} = \frac{(y_2-y_3)(y_1-y_2+2y_3)^2}{[(y_1+y_2)\theta_3 - \theta_3(y_1-y_2+2y_3)]^2}$$
$$= \frac{(y_2-y_3)(y_1-y_2+2y_3)^2}{4\theta_3^2(y_2-y_3)^2} = \frac{(y_1-y_2+2y_3)^2}{4\theta_3^2(y_2-y_3)}.$$

EXERCISES 3.7

1. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(2t - \dfrac{1}{t^2}\right)(2x)$

2. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left[\dfrac{(u+1)(1)-u(1)}{(u+1)^2}\right]\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2\sqrt{x}(u+1)^2}$

3. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (3u^2+2u+1)\left[\dfrac{-1/(2\sqrt{x})}{(\sqrt{x}-4)^2}\right] = -\dfrac{3u^2+2u+1}{2\sqrt{x}(\sqrt{x}-4)^2}$

4. $\dfrac{dy}{dx} = \dfrac{dy}{ds}\dfrac{ds}{dx} = \left[\dfrac{(s^2-2)(1)-s(2s)}{(s^2-2)^2}\right](2x-2) = \dfrac{2(1-x)(s^2+2)}{(s^2-2)^2}$

5. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \left[\dfrac{5v^{3/2}}{2} + \dfrac{3v^{1/2}}{2} + 2v + 1\right]\left[\dfrac{(x^2-1)(1)-x(2x)}{(x^2-1)^2}\right] = \dfrac{-(5v^{3/2}+3\sqrt{v}+4v+2)(x^2+1)}{2(x^2-1)^2}$

6. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left[\dfrac{(t-4)(1)-(t+3)(1)}{(t-4)^2}\right]\left[\dfrac{(x+1)(1)-(x-2)(1)}{(x+1)^2}\right] = \dfrac{-21}{(t-4)^2(x+1)^2}$

7. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left[\dfrac{(t-4)(2t)-(t^2+3)(1)}{(t-4)^2}\right](9x^2+28x+8) = \dfrac{(t^2-8t-3)(9x^2+28x+8)}{(t-4)^2}$

8. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(2u + \dfrac{5}{2}u^{3/2}\right)\left[\dfrac{(x-x^2)(1)-(x+1)(1-2x)}{(x-x^2)^2}\right] = \dfrac{u(4+5\sqrt{u})(x^2+2x-1)}{2(x-x^2)^2}$

9. $f'(x) = (x^3+3)^4 + x(4)(x^3+3)^3(3x^2) = (13x^3+3)(x^3+3)^3$

10. $f'(x) = (1)\sqrt{x+1} + x\left(\dfrac{1}{2\sqrt{x+1}}\right) = \dfrac{3x+2}{2\sqrt{x+1}}$

11. $f'(x) = 2x(2x+1)^2 + x^2(2)(2x+1)(2) = 2x(4x+1)(2x+1)$

12. $f'(x) = \dfrac{\sqrt{2x+1}(1) - x\left(\dfrac{2}{2\sqrt{2x+1}}\right)}{2x+1} = \dfrac{x+1}{(2x+1)^{3/2}}$

13. $f'(x) = 2(x+2)(x^2+3) + (x+2)^2(2x) = 2(x+2)(2x^2+2x+3)$

14. $f'(x) = \dfrac{(3x+5)(2)(2x-1)(2) - (2x-1)^2(3)}{(3x+5)^2} = \dfrac{(2x-1)(6x+23)}{(3x+5)^2}$

15. $f'(x) = \dfrac{(2x-1)^2(3) - (3x+5)(2)(2x-1)(2)}{(2x-1)^4} = -\dfrac{6x+23}{(2x-1)^3}$

16. $f'(x) = 3x^2(2-5x^2)^{1/3} + x^3(1/3)(2-5x^2)^{-2/3}(-10x)$
$= 3x^2(2-5x^2)^{1/3} - \dfrac{10x^4}{3(2-5x^2)^{2/3}} = \dfrac{x^2(18-55x^2)}{3(2-5x^2)^{2/3}}$

17. $f'(x) = \dfrac{(2-5x^2)^{1/3}(3x^2) - x^3(1/3)(2-5x^2)^{-2/3}(-10x)}{(2-5x^2)^{2/3}} = \dfrac{x^2(18-35x^2)}{3(2-5x^2)^{4/3}}$

18. $f'(x) = (x+1)^2(3)(3x+1)^2(3) + 2(x+1)(3x+1)^3 = (3x+1)^2(x+1)[9(x+1) + 2(3x+1)]$
$= (3x+1)^2(x+1)(15x+11)$

19. $f'(x) = \dfrac{(1-\sqrt{x})(1/3)x^{-2/3} - x^{1/3}(-1/2)x^{-1/2}}{(1-\sqrt{x})^2} = \dfrac{2(1-\sqrt{x}) + 3\sqrt{x}}{6x^{2/3}(1-\sqrt{x})^2} = \dfrac{2+\sqrt{x}}{6x^{2/3}(1-\sqrt{x})^2}$

20. $f'(x) = \dfrac{x^2(1/2)(2-3x)^{-1/2}(-3) - \sqrt{2-3x}(2x)}{x^4} = \dfrac{\dfrac{-3x^2}{2\sqrt{2-3x}} - 2x\sqrt{2-3x}}{x^4}$
$= \dfrac{-3x - 4(2-3x)}{2x^3\sqrt{2-3x}} = \dfrac{9x-8}{2x^3\sqrt{2-3x}}$

21. $f'(x) = 4\left(\dfrac{x^3-1}{2x^3+1}\right)^3 \left[\dfrac{(2x^3+1)(3x^2) - (x^3-1)(6x^2)}{(2x^3+1)^2}\right] = \dfrac{36x^2(x^3-1)^3}{(2x^3+1)^5}$

22. $f'(x) = \dfrac{1}{4}\left(\dfrac{2-x}{2+x}\right)^{-3/4}\left[\dfrac{(2+x)(-1) - (2-x)(1)}{(2+x)^2}\right] = \dfrac{1}{4}\left(\dfrac{2+x}{2-x}\right)^{3/4}\left[\dfrac{-4}{(2+x)^2}\right] = \dfrac{-1}{(2-x)^{3/4}(2+x)^{5/4}}$

23. $f'(x) = 3(x^3-2x^2)^2(3x^2-4x)(x^4-2x)^5 + (x^3-2x^2)^3(5)(x^4-2x)^4(4x^3-2)$
$= (x^3-2x^2)^2(x^4-2x)^4[3(3x^2-4x)(x^4-2x) + 5(x^3-2x^2)(4x^3-2)]$
$= x^2(x^3-2x^2)^2(x^4-2x)^4(29x^4 - 52x^3 - 28x + 44)$

24. $f'(x) = 4(x+5)^3\sqrt{1+x^3} + (x+5)^4(1/2)(1+x^3)^{-1/2}(3x^2)$
$= (x+5)^3\left[4\sqrt{1+x^3} + \dfrac{3x^2(x+5)}{2\sqrt{1+x^3}}\right] = (x+5)^3\left[\dfrac{8(1+x^3) + 3x^2(x+5)}{2\sqrt{1+x^3}}\right]$
$= \dfrac{(x+5)^3(11x^3 + 15x^2 + 8)}{2\sqrt{1+x^3}}$

25. $f'(x) = \dfrac{(3+x)^{1/3}(\sqrt{1-x^2} - x^2/\sqrt{1-x^2}) - x\sqrt{1-x^2}(1/3)(3+x)^{-2/3}}{(3+x)^{2/3}}$
$= \dfrac{(3+x)^{1/3}\left(\dfrac{1-2x^2}{\sqrt{1-x^2}}\right) - \dfrac{x\sqrt{1-x^2}}{3(3+x)^{2/3}}}{(3+x)^{2/3}} = \dfrac{3(3+x)(1-2x^2) - x(1-x^2)}{3(3+x)^{4/3}\sqrt{1-x^2}}$
$= \dfrac{9 + 2x - 18x^2 - 5x^3}{3(3+x)^{4/3}\sqrt{1-x^2}}$

26. With the product rule of Exercise 20 in Section 3.4,
$f'(x) = (1)(x+5)^4\sqrt{1+x^3} + x(4)(x+5)^3\sqrt{1+x^3} + x(x+5)^4(1/2)(1+x^3)^{-1/2}(3x^2)$
$= (x+5)^3\left[(x+5)\sqrt{1+x^3} + 4x\sqrt{1+x^3} + \dfrac{3x^3(x+5)}{2\sqrt{1+x^3}}\right]$
$= (x+5)^3\left[\dfrac{2(x+5)(1+x^3) + 8x(1+x^3) + 3x^3(x+5)}{2\sqrt{1+x^3}}\right] = \dfrac{(x+5)^3(13x^4 + 25x^3 + 10x + 10)}{2\sqrt{1+x^3}}$

27. $f'(x) = \dfrac{(x-2)(x+5)^2[2x(x^3+3)^2 + x^2(2)(x^3+3)(3x^2)] - x^2(x^3+3)^2[(x+5)^2 + (x-2)(2)(x+5)]}{(x-2)^2(x+5)^4}$
$= \dfrac{x(x^3+3)[(x-2)(x+5)(8x^3+6) - x(x^3+3)(3x+1)]}{(x-2)^2(x+5)^3}$
$= \dfrac{x(x^3+3)(5x^5 + 23x^4 - 80x^3 - 3x^2 + 15x - 60)}{(x-2)^2(x+5)^3}$

28. $f'(x) = (1)\sqrt{1+x\sqrt{1+x}} + x(1/2)\left(1+x\sqrt{1+x}\right)^{-1/2}[\sqrt{1+x} + x(1/2)(1+x)^{-1/2}]$

$$= \sqrt{1+x\sqrt{1+x}} + \frac{x}{2\sqrt{1+x\sqrt{1+x}}}\left[\frac{2(1+x)+x}{2\sqrt{1+x}}\right]$$

$$= \frac{4\sqrt{1+x}\left(1+x\sqrt{1+x}\right)+3x^2+2x}{4\sqrt{1+x}\sqrt{1+x\sqrt{1+x}}} = \frac{7x^2+6x+4\sqrt{1+x}}{4\sqrt{1+x}\sqrt{1+x\sqrt{1+x}}}$$

29. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(2t - \dfrac{3}{t^4}\right)\left(\dfrac{-x}{\sqrt{4-x^2}}\right)$

30. $\dfrac{dy}{dx} = \dfrac{dy}{ds}\dfrac{ds}{dx} = \left[\dfrac{1}{3}(2s-s^2)^{-2/3}(2-2s)\right]\left[\dfrac{-2x}{(x^2+5)^2}\right] = \dfrac{4x(s-1)}{3(2s-s^2)^{2/3}(x^2+5)^2}$

31. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \left[\dfrac{(v^3-1)(2v)-v^2(3v^2)}{(v^3-1)^2}\right]\left(\sqrt{x^2-1} + \dfrac{x^2}{\sqrt{x^2-1}}\right) = \dfrac{(v^4+2v)(1-2x^2)}{(v^3-1)^2\sqrt{x^2-1}}$

32. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left[\dfrac{(u+5)(1)-u(1)}{(u+5)^2}\right]\left[\dfrac{x(1/2)(x-1)^{-1/2} - \sqrt{x-1}(1)}{x^2}\right]$

$$= \frac{5}{(u+5)^2}\left[\frac{x-2(x-1)}{2x^2\sqrt{x-1}}\right] = \frac{5(2-x)}{2x^2\sqrt{x-1}(u+5)^2}$$

33. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = [4u^3(u^3-2u)^2 + u^4(2)(u^3-2u)(3u^2-2)]\left(\dfrac{1-4x}{2\sqrt{x-2x^2}}\right)$

$$= 2u^3(u^3-2u)[2(u^3-2u)+u(3u^2-2)]\left(\frac{1-4x}{2\sqrt{x-2x^2}}\right) = \frac{u^5(u^2-2)(5u^2-6)(1-4x)}{\sqrt{x-2x^2}}$$

34. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left[1 + \dfrac{1}{2\sqrt{t+\sqrt{t}}}\left(1+\dfrac{1}{2\sqrt{t}}\right)\right]\left[\dfrac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}\right]$

$$= \left[1 + \frac{2\sqrt{t}+1}{4\sqrt{t}\sqrt{t+\sqrt{t}}}\right]\left[\frac{-4x}{(x^2-1)^2}\right]$$

35. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \left\{3\left(\dfrac{v^2+1}{1-v^3}\right)^2\left[\dfrac{(1-v^3)(2v)-(v^2+1)(-3v^2)}{(1-v^3)^2}\right]\right\}\left[\dfrac{-(3x^2+6x)}{(x^3+3x^2+2)^2}\right]$

$$= -\frac{9xv(v^2+1)^2(v^3+3v+2)(x+2)}{(1-v^3)^4(x^3+3x^2+2)^2}$$

36. $\dfrac{dy}{dx} = \dfrac{dy}{dk}\dfrac{dk}{dx} = \left[\dfrac{(1+k+k^2)(1/2)k^{-1/2} - \sqrt{k}(1+2k)}{(1+k+k^2)^2}\right]\left[(x^2+5)^5 + 5x(x^2+5)^4(2x)\right]$

$$= \left[\frac{1+k+k^2-2k(1+2k)}{2\sqrt{k}(1+k+k^2)^2}\right]\left[(x^2+5)^4(x^2+5+10x^2)\right]$$

$$= \frac{(1-k-3k^2)(x^2+5)^4(11x^2+5)}{2\sqrt{k}(1+k+k^2)^2}$$

37. Since y is a function of s, namely, $f[g(s)]$, and s is a function of x, we can write that $\dfrac{dy}{dx} = \dfrac{dy}{ds}\dfrac{ds}{dx}$. But $\dfrac{dy}{ds} = \dfrac{dy}{du}\dfrac{du}{ds}$. Consequently, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{ds}\dfrac{ds}{dx}$.

38. Since F is a function of r, and r is a function of t, the chain rule gives $\dfrac{dF}{dt} = \dfrac{dF}{dr}\dfrac{dr}{dt} = \dfrac{-2Qq}{4\pi\epsilon_0 r^3}\dfrac{dr}{dt}$.
Since $dr/dt = 2$ m/s, we obtain when $r = 2$, $\quad \dfrac{dF}{dt} = -\dfrac{2(3\times10^{-6})(5\times10^{-6})}{4\pi(8.85\times10^{-12})(2)^3}(2) = -0.067$ N/s.

39. Since F is a function of r, and r is a function of t, the chain rule gives $\dfrac{dF}{dt} = \dfrac{dF}{dr}\dfrac{dr}{dt} = \dfrac{-2GMm}{r^3}\dfrac{dr}{dt}$.
Since $dr/dt = -250/9$ m/s when $r = 5$,

$$\frac{dF}{dt} = -\left[\frac{2(6.67\times10^{-11})(5)(4/3)\pi(6.37\times10^6)^3(5.52\times10^3)}{(6.375\times10^6)^3}\right]\left(-\frac{250}{9}\right) = 4.27\times10^{-4}\ \text{N/s}.$$

40. If we use the chain rule on $y = f(x)$, $x = g(y)$, then $\dfrac{dy}{dy} = 1 = \dfrac{dy}{dx}\dfrac{dx}{dy} \implies \dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$.

41. If $f(x)$ is an odd function, then $f(-x) = -f(x)$. To differentiate this equation with respect to x, we set $u = -x$ on the left. Differentiation of $f(u) = -f(x)$ with respect to x then gives $f'(u)\dfrac{du}{dx} = -f'(x) \implies f'(-x)(-1) = -f'(x) \implies f'(-x) = f'(x)$. This shows that $f'(x)$ is an even function. A similar proof shows that $f'(x)$ is odd when $f(x)$ is even.

42. The proof relies on the fact that $\Delta u \neq 0$. If u is a constant function, then $\Delta u \equiv 0$. In addition, even when u is not a constant function, we might have $\Delta u = 0$ for arbitrarily small Δx. A complete proof of the chain rule must account for both these possibilities.

43. Since $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = (2v+1)\left[\dfrac{(x+1)(1) - x(1)}{(x+1)^2}\right] = \dfrac{2v+1}{(x+1)^2}$, it follows that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(x+1)^2(2dv/dx) - (2v+1)(2)(x+1)}{(x+1)^4}\\ &= \frac{2(x+1)^2\left[\dfrac{1}{(x+1)^2}\right] - 2(2v+1)(x+1)}{(x+1)^4} = -\frac{2(2vx+2v+x)}{(x+1)^4}.\end{aligned}$$

44. Since $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left[3(u+1)^2 + \dfrac{1}{u^2}\right]\left[1 + \dfrac{1}{2\sqrt{x+1}}\right]$, it follows that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[3(u+1)^2 + \frac{1}{u^2}\right]\left[1 + \frac{1}{2\sqrt{x+1}}\right] + \left[3(u+1)^2 + \frac{1}{u^2}\right]\frac{d}{dx}\left[1 + \frac{1}{2\sqrt{x+1}}\right]\\ &= \left[6(u+1) - \frac{2}{u^3}\right]\frac{du}{dx}\left[1 + \frac{1}{2\sqrt{x+1}}\right] + \left[3(u+1)^2 + \frac{1}{u^2}\right]\left[\frac{-1}{4(x+1)^{3/2}}\right]\\ &= \left[\frac{6u^3(u+1) - 2}{u^3}\right]\left[1 + \frac{1}{2\sqrt{x+1}}\right]^2 - \frac{3u^2(u+1)^2 + 1}{4u^2(x+1)^{3/2}}.\end{aligned}$$

45. Since $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(\dfrac{1}{2\sqrt{t-1}}\right)[2(x+x^2)(1+2x)] = \dfrac{2x^3+3x^2+x}{\sqrt{t-1}}$, it follows that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\sqrt{t-1}(6x^2+6x+1) - (2x^3+3x^2+x)(1/2)(t-1)^{-1/2}(dt/dx)}{t-1}\\ &= \frac{\sqrt{t-1}(6x^2+6x+1) - (2x^3+3x^2+x)\left(\dfrac{1}{2\sqrt{t-1}}\right)(4x^3+6x^2+2x)}{t-1}\\ &= \frac{(t-1)(6x^2+6x+1) - (2x^3+3x^2+x)^2}{(t-1)^{3/2}}.\end{aligned}$$

46. Since
$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{ds}\frac{ds}{dx} &= \left[\frac{(s+6)(1) - s(1)}{(s+6)^2}\right]\left[\frac{(1+\sqrt{x})(1/2)x^{-1/2} - \sqrt{x}(1/2)x^{-1/2}}{(1+\sqrt{x})^2}\right]\\ &= \left[\frac{6}{(s+6)^2}\right]\left[\frac{1}{2\sqrt{x}(1+\sqrt{x})^2}\right],\end{aligned}$$
it follows that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{3}{(s+6)^2}\right]\left[\frac{1}{\sqrt{x}(1+\sqrt{x})^2}\right] + \left[\frac{3}{(s+6)^2}\right]\frac{d}{dx}\left[\frac{1}{\sqrt{x}(1+\sqrt{x})^2}\right]\\ &= \left[\frac{-6}{(s+6)^3}\frac{ds}{dx}\right]\left[\frac{1}{\sqrt{x}(1+\sqrt{x})^2}\right] + \left[\frac{3}{(s+6)^2}\right]\left[\frac{-1}{2x^{3/2}(1+\sqrt{x})^2} - \frac{2}{\sqrt{x}(1+\sqrt{x})^3 2\sqrt{x}}\right]\end{aligned}$$

$$= \left[\frac{-6}{(s+6)^3\sqrt{x}(1+\sqrt{x})^2}\right]\left[\frac{1}{2\sqrt{x}(1+\sqrt{x})^2}\right] + \left[\frac{3}{(s+6)^2}\right]\left[\frac{-1-\sqrt{x}-2\sqrt{x}}{2x^{3/2}(1+\sqrt{x})^3}\right]$$
$$= \frac{-6\sqrt{x}-3(s+6)(1+\sqrt{x})(1+3\sqrt{x})}{2x^{3/2}(s+6)^3(1+\sqrt{x})^4} = \frac{-3[2\sqrt{x}+(s+6)(1+4\sqrt{x}+3x)]}{2x^{3/2}(1+\sqrt{x})^4(s+6)^3}.$$

47. If we set $u = 2x+3$, then the chain rule gives

$$\frac{d}{dx}f(2x+3) = \frac{d}{du}f(u)\frac{du}{dx} = f'(u)(2) = 2f'(2x+3).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}f(2x+3) = 2[3(2x+3)^2 - 2] = 2(12x^2 + 36x + 25).$$

48. If we set $u = 3 - 4x$, then the chain rule gives

$$\frac{d}{dx}[f(3-4x)]^2 = \frac{d}{du}[f(u)]^2\frac{du}{dx} = 2f(u)f'(u)(-4) = -8f(3-4x)f'(3-4x).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}[f(3-4x)]^2 = -8[(3-4x)^3 - 2(3-4x)][3(3-4x)^2 - 2]$$
$$= 8(4x-3)(16x^2 - 24x + 7)(48x^2 - 72x + 25).$$

49. If we set $u = 1 - x^2$, then the chain rule gives

$$\frac{d}{dx}f(1-x^2) = \frac{d}{du}f(u)\frac{du}{dx} = f'(u)(-2x) = -2xf'(1-x^2).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}f(1-x^2) = -2x[3(1-x^2)^2 - 2] = -2x(1 - 6x^2 + 3x^4).$$

50. If we set $u = x + 1/x$, then

$$\frac{d}{dx}f\left(x+\frac{1}{x}\right) = f'(u)\frac{du}{dx} = f'(u)\left(1-\frac{1}{x^2}\right) = \left(1-\frac{1}{x^2}\right)f'\left(x+\frac{1}{x}\right).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}f\left(x+\frac{1}{x}\right) = \left(1-\frac{1}{x^2}\right)\left[3\left(x+\frac{1}{x}\right)^2 - 2\right] = \frac{(x^2-1)(3x^4+4x^2+3)}{x^4}.$$

51. If we set $u = f(x)$, then

$$\frac{d}{dx}f\big(f(x)\big) = \frac{d}{du}f(u)\frac{du}{dx} = f'(u)f'(x) = f'\big(f(x)\big)f'(x).$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\frac{d}{dx}f\big(f(x)\big) = [3(x^3-2x)^2 - 2](3x^2 - 2) = (3x^2-2)(3x^6 - 12x^4 + 12x^2 - 2).$$

52. If we set $u = 1 - 3x$, then

$$\frac{d}{dx}\sqrt{3-4[f(1-3x)]^2} = \frac{d}{du}\sqrt{3-4[f(u)]^2}\frac{du}{dx} = \frac{1}{2\sqrt{3-4[f(u)]^2}}[-8f(u)f'(u)](-3)$$
$$= \frac{12f(u)f'(u)}{\sqrt{3-4[f(u)]^2}} = \frac{12f(1-3x)f'(1-3x)}{\sqrt{3-4[f(1-3x)]^2}}.$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\begin{aligned}\frac{d}{dx}\sqrt{3 - 4[f(1-3x)]^2} &= \frac{12[(1-3x)^3 - 2(1-3x)][3(1-3x)^2 - 2]}{\sqrt{3 - 4[(1-3x)^3 - 2(1-3x)]^2}} \\ &= \frac{12(1-3x)(9x^2 - 6x - 1)(27x^2 - 18x + 1)}{\sqrt{3 - 4(1-3x)^2(9x^2 - 6x - 1)^2}}.\end{aligned}$$

53. If we set $u = -x$ and $v = x^2$, then

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(-x)}{3 + 2f(x^2)}\right] = \frac{d}{dx}\left[\frac{f(u)}{3 + 2f(v)}\right] &= \frac{[3 + 2f(v)]f'(u)du/dx - f(u)[2f'(v)dv/dx]}{[3 + 2f(v)]^2} \\ &= \frac{[3 + 2f(x^2)]f'(-x)(-1) - f(-x)[2f'(x^2)(2x)]}{[3 + 2f(x^2)]^2} \\ &= -\frac{f'(-x)[3 + 2f(x^2)] + 4xf(-x)f'(x^2)}{[3 + 2f(x^2)]^2}\end{aligned}$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(-x)}{3 + 2f(x^2)}\right] &= -\frac{(3x^2 - 2)[3 + 2(x^6 - 2x^2)] + 4x(-x^3 + 2x)(3x^4 - 2)}{[3 + 2(x^6 - 2x^2)]^2} \\ &= \frac{6x^8 - 20x^6 + 4x^4 - x^2 + 6}{(2x^6 - 4x^2 + 3)^2}\end{aligned}$$

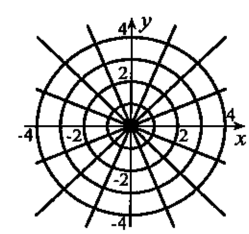
54. If we set $u = x - f(x)$, then

$$\frac{d}{dx}\left[f\big(x - f(x)\big)\right] = f'(u)\frac{du}{dx} = f'(u)[1 - f'(x)] = f'\big(x - f(x)\big)[1 - f'(x)].$$

When $f(u) = u^3 - 2u$, then $f'(u) = 3u^2 - 2$, and

$$\begin{aligned}\frac{d}{dx}\left[f\big(x - f(x)\big)\right] &= \{3[x - f(x)]^2 - 2\}[1 - (3x^2 - 2)] = \{3[x - (x^3 - 2x)]^2 - 2\}(3 - 3x^2) \\ &= \{3[3x - x^3]^2 - 2\}(3 - 3x^2) = [27x^2 - 18x^4 + 3x^6 - 2](3 - 3x^2) \\ &= 3(1 - x^2)(3x^6 - 18x^4 + 27x^2 - 2).\end{aligned}$$

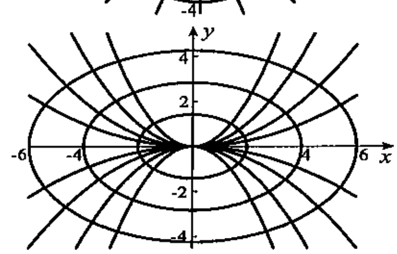
55. Suppose the curves intersect at a point (x, y). The straight line has slope m where $m = y/x$. When we solve $x^2 + y^2 = r^2$ for y, we obtain $y = \pm\sqrt{r^2 - x^2}$. When we differentiate $y = \pm\sqrt{r^2 - x^2}$, we obtain $dy/dx = \mp x/\sqrt{r^2 - x^2} = -x/y$. This is the negative reciprocal of the slope of the line at (x, y).



56. The slope of the tangent line at any point on any of the parabolas in the first family is

$$\frac{dy}{dx} = 2ax = 2\left(\frac{y}{x^2}\right)x = \frac{2y}{x}.$$

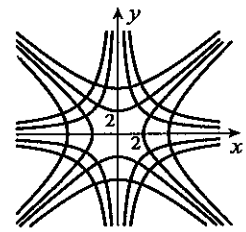
When we solve for y in the family of ellipses, $y = \pm(1/\sqrt{2})\sqrt{c^2 - x^2}$. The slope of the tangent line at any point on any ellipse is



$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{2}}\frac{-2x}{2\sqrt{c^2 - x^2}} = \mp\frac{x}{\sqrt{2}\sqrt{c^2 - x^2}} = \mp\frac{x}{\sqrt{2}(\pm\sqrt{2}y)} = -\frac{x}{2y}.$$

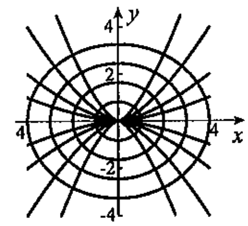
Since these slopes are negative reciprocals, the families are orthogonal.

57. When we solve for y in the first family of hyperbolas, $y = \pm\sqrt{x^2 - C_1}$. The slope of the tangent line at any point on any hyperbola is
$$\frac{dy}{dx} = \pm x/\sqrt{x^2 - C_1} = \frac{\pm x}{(\pm y)} = \frac{x}{y}.$$
The slope of the tangent line at any point on $y = C_2/x$ is
$$\frac{dy}{dx} = -\frac{C_2}{x^2} = -\left(\frac{C_2}{x}\right)\left(\frac{1}{x}\right) = -\frac{y}{x}.$$



Slopes of the families are negative reciprocals, and the families are orthogonal trajectories.

58. When we solve for y in the family of ellipses, $y = \pm\frac{1}{\sqrt{3}}\sqrt{C^2 - 2x^2}$. The slope of the tangent line at any point on any ellipse is
$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{3}}\frac{-4x}{2\sqrt{C^2 - 2x^2}} = \frac{\mp 2x}{\sqrt{3}\sqrt{C^2 - 2x^2}}$$
$$= \frac{\mp 2x}{\sqrt{3}(\pm\sqrt{3}y)} = -\frac{2x}{3y}.$$



If $a > 0$ in the second family, then $y = \sqrt{a}x^{3/2}$, and $\dfrac{dy}{dx} = \dfrac{3}{2}\sqrt{a}x^{1/2} = \dfrac{3}{2}\left(\dfrac{y}{x^{3/2}}\right)x^{1/2} = \dfrac{3y}{2x}$.

If $a < 0$, then $y = \sqrt{-a}(-x)^{3/2}$, and $\dfrac{dy}{dx} = \dfrac{3}{2}\sqrt{-a}(-x)^{1/2}(-1) = -\dfrac{3}{2}\left[\dfrac{y}{(-x)^{3/2}}\right](-x)^{1/2} = \dfrac{3y}{2x}$.

In either case, slopes of the families are negative reciprocals, and the families are orthogonal trajectories.

59. If we differentiate $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ with respect to x,
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\right)\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2}$$
$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}.$$

60. Using the result of Exercise 59, $\dfrac{d^2y}{dx^2} = -\dfrac{12}{x^4}3(u+1)^2 + 6(u+1)\left(3 + \dfrac{4}{x^3}\right)^2$. When $x = 1$, we find that $u = 1$ also, and $\dfrac{d^2y}{dx^2} = -36(2)^2 + 6(2)(7)^2 = 444$.

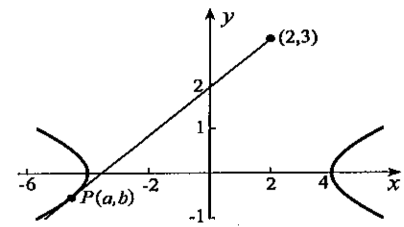
61. If we set $u = x^3$, then $x = u^{1/3}$, and
$$\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = (9x^8 + 6x^5)(1/3)u^{-2/3} = (3x^8 + 2x^5)x^{-2} = 3x^6 + 2x^3.$$

62. If we set $v = x/(x+1)$, then $x = v/(1-v)$, and
$$\frac{dy}{dv} = \frac{dy}{dx}\frac{dx}{dv} = \left[\frac{-x}{\sqrt{1-x^2}}\right]\left[\frac{(1-v)(1) - v(-1)}{(1-v)^2}\right] = \frac{-x}{\sqrt{1-x^2}(1-v)^2}$$
$$= \frac{-x}{\sqrt{1-x^2}\left(1 - \dfrac{x}{x+1}\right)^2} = -\frac{x(x+1)^2}{\sqrt{1-x^2}}.$$

63. If we differentiate the result in Exercise 59 with respect to x,
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\frac{dy}{du} + \frac{d^2u}{dx^2}\frac{d}{dx}\left(\frac{dy}{du}\right) + \frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\left(\frac{du}{dx}\right)^2 + \frac{d^2y}{du^2}\frac{d}{dx}\left(\frac{du}{dx}\right)^2$$
$$= \frac{d^3u}{dx^3}\frac{dy}{du} + \frac{d^2u}{dx^2}\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} + \frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\left(\frac{du}{dx}\right)^2 + \frac{d^2y}{du^2}(2)\left(\frac{du}{dx}\right)\frac{d^2u}{dx^2}$$
$$= \frac{d^3u}{dx^3}\frac{dy}{du} + \frac{d^2u}{dx^2}\frac{d^2y}{du^2}\frac{du}{dx} + \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 2\frac{d^2y}{du^2}\frac{du}{dx}\frac{d^2u}{dx^2} = \frac{d^3u}{dx^3}\frac{dy}{du} + 3\frac{d^2u}{dx^2}\frac{d^2y}{du^2}\frac{du}{dx} + \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3.$$

64. The diagram makes it clear that there are two such points and they both have negative y-coordinates. We therefore take the lower half of the hyperbola in the form $y = f(x) = -\sqrt{x^2 - 16}/4$. The slope of the tangent line to the hyperbola at $P(a, b)$ is

$$f'(a) = \frac{-x}{4\sqrt{x^2 - 16}}_{|x=a} = \frac{-a}{4\sqrt{a^2 - 16}}.$$



Since the tangent line must pass through the point $(2, 3)$ the slope of the tangent line is also given by $(b - 3)/(a - 2)$. Consequently,

$$\frac{b - 3}{a - 2} = \frac{-a}{4\sqrt{a^2 - 16}}.$$

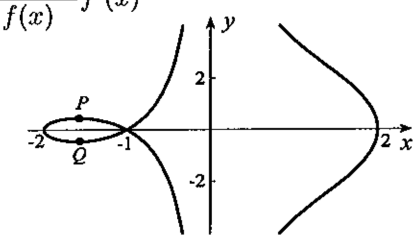
We combine this with $b = -\dfrac{1}{4}\sqrt{a^2 - 16}$, since $P(a, b)$ is on the hyperbola. Substitution from the second into the first gives

$$-\frac{1}{4}\sqrt{a^2 - 16} - 3 = \frac{-a(a - 2)}{4\sqrt{a^2 - 16}}, \qquad \text{or,} \qquad (a^2 - 16) + 12\sqrt{a^2 - 16} = a(a - 2).$$

This equation simplifies to $6\sqrt{a^2 - 16} = 8 - a$, and squaring leads to the quadratic $35a^2 + 16a - 640 = 0$. The two solutions are $a = (-8 \pm 24\sqrt{39})/35$. The y-coordinates of these points are $(-12 \pm \sqrt{39})/35$.

65. $\dfrac{d}{dx}|f(x)|^n = n|f(x)|^{n-1}\dfrac{d}{dx}|f(x)| = n|f(x)|^{n-1}\dfrac{|f(x)|}{f(x)}f'(x) = \dfrac{n|f(x)|^n}{f(x)}f'(x)$

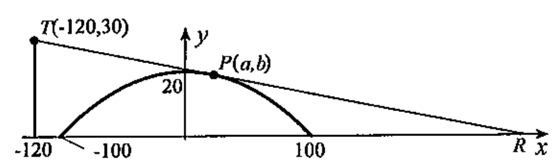
66. The conchoid is shown to the right. There are two points P and Q at which the tangent line is horizontal. To find P the x-coordinate of which is between -2 and -1, we first solve the equation $x^2y^2 = (x + 1)^2(4 - x^2)$ for $y = [(x + 1)/x]\sqrt{4 - x^2}$, and set the derivative equal to zero,



$$0 = \frac{d}{dx}\left[\left(1 + \frac{1}{x}\right)\sqrt{4 - x^2}\right] = \frac{-1}{x^2}\sqrt{4 - x^2} + \left(1 + \frac{1}{x}\right)\frac{-x}{\sqrt{4 - x^2}} = \frac{-(4 - x^2) - x^2(x + 1)}{x^2\sqrt{4 - x^2}} = \frac{-(x^3 + 4)}{x^2\sqrt{4 - x^2}}.$$

The only solution of this equation is $x = -4^{1/3}$. The coordinates of P are therefore $(-4^{1/3}, (4^{1/3} - 1)^{3/2})$. Point Q has coordinates $(-4^{1/3}, -(4^{1/3} - 1)^{3/2})$.

67. The required position occurs at the point R where the tangent line from $T(-120, 30)$ to the arc of the circle representing the hill intersects the x-axis. Let the point of tangency be $P(a, b)$. If $x^2 + (y - k)^2 = r^2$ is the equation of the circle, then using the points $(100, 0)$ and $(0, 20)$, we obtain $100^2 + k^2 = r^2$ and $(20 - k)^2 = r^2$. These give $k = -240$ and $r^2 = 67\,600$, so that the equation of the circular arc is $x^2 + (y + 240)^2 = 67\,600$, or, $y = f(x) = -240 + \sqrt{67\,600 - x^2}$, $-100 \le x \le 100$. Since $f'(x) = (1/2)(67\,600 - x^2)^{-1/2}(-2x) = -x/\sqrt{67\,600 - x^2}$, the slope of the tangent line at $P(a, b)$ is $f'(a) = -a/\sqrt{67\,600 - a^2}$. The slope of this line is also the slope of PT, namely, $(b - 30)/(a + 120)$, and therefore



$$\frac{b - 30}{a + 120} = -\frac{a}{\sqrt{67\,600 - a^2}} \quad \Longrightarrow \quad b = 30 - \frac{a(a + 120)}{\sqrt{67\,600 - a^2}}.$$

Since $P(a, b)$ is on the circular arc, it also follows that $b = -240 + \sqrt{67\,600 - a^2}$. When we equate these expressions for b,

$$30 - \frac{a(a+120)}{\sqrt{67\,600 - a^2}} = -240 + \sqrt{67\,600 - a^2} \quad \Longrightarrow \quad 873a^2 + 162\,240a - 3\,582\,800 = 0.$$

The positive solution is $a = 19.9432$. The y-coordinate of P is $b = 19.2340$. The equation of the tangent line is therefore $y - b = (-a/\sqrt{67\,600 - a^2})(x - a)$, and its x-intercept is given by $-b = (-a/\sqrt{67\,600 - a^2})(x - a) \implies x = (b/a)\sqrt{67\,600 - a^2} + a$. When we substitute the calculated values for a and b, the result is $x \approx 270$ m.

EXERCISES 3.8

1. If we differentiate with respect to x, we find $4y^3\dfrac{dy}{dx} + \dfrac{dy}{dx} = 12x^2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{12x^2}{4y^3+1}$.

2. If we differentiate with respect to x, we find $4x^3 + 2y\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{-4x^3}{3y^2+2y}$.

3. If we differentiate with respect to x, we find $x\dfrac{dy}{dx} + y + 2 = 8y\dfrac{dy}{dx} \Longrightarrow \dfrac{dy}{dx} = \dfrac{y+2}{8y-x}$.

4. Differentiation with respect to x gives $6x^2 - 3y^4 - 12xy^3\dfrac{dy}{dx} + 5y + 5x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = \dfrac{6x^2 - 3y^4 + 5y}{12xy^3 - 5x}$.

5. If we differentiate with respect to x, we find $1 + y^5 + 5xy^4\dfrac{dy}{dx} + 2xy^3 + 3x^2y^2\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{1 + 2xy^3 + y^5}{5xy^4 + 3x^2y^2}$.

6. When we differentiate with respect to x, we obtain $2(x+y)\left(1 + \dfrac{dy}{dx}\right) = 2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{1-x-y}{x+y}$.

7. If we differentiate with respect to x, we find $2x - y - x\dfrac{dy}{dx} - 12y^2\dfrac{dy}{dx} = 2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{2x-y-2}{x+12y^2}$.

8. Differentiation with respect to x gives $\dfrac{1}{2\sqrt{x+y}}\left(1 + \dfrac{dy}{dx}\right) + 2y\dfrac{dy}{dx} = 24x + \dfrac{dy}{dx}$, from which $\left(\dfrac{1}{2\sqrt{x+y}} + 2y - 1\right)\dfrac{dy}{dx} = 24x - \dfrac{1}{2\sqrt{x+y}}$. Thus,
$$\frac{dy}{dx} = \left(\frac{48x\sqrt{x+y} - 1}{2\sqrt{x+y}}\right)\left(\frac{2\sqrt{x+y}}{1 + 2(2y-1)\sqrt{x+y}}\right) = \frac{48x\sqrt{x+y} - 1}{1 + 2(2y-1)\sqrt{x+y}}.$$

9. If we differentiate with respect to x, we find $\dfrac{1}{2\sqrt{1+xy}}\left(y + x\dfrac{dy}{dx}\right) - y - x\dfrac{dy}{dx} = 0$, and therefore
$$\frac{dy}{dx} = \frac{y - \dfrac{y}{2\sqrt{1+xy}}}{-x + \dfrac{x}{2\sqrt{1+xy}}} = \frac{y(2\sqrt{1+xy} - 1)}{x(1 - 2\sqrt{1+xy})} = -\frac{y}{x}.$$

10. If we write $x^2 - xy - y^2 = 4x(x+y)$, or, $3x^2 + 5xy + y^2 = 0$, and differentiate with respect to x, we obtain $6x + 5y + 5x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$. Therefore, $\dfrac{dy}{dx} = -\dfrac{6x+5y}{5x+2y}$.

11. If we differentiate the equation of the curve with respect to x, we find $y^2 + 2xy\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx}_{|(1,1)} = -\dfrac{y^2}{2xy + 3y^2}_{|(1,1)} = -\dfrac{1}{5}$. Equations for the tangent and normal lines are $y - 1 = -(1/5)(x-1)$ and $y - 1 = 5(x-1)$, or, $x + 5y = 6$ and $5x - y = 4$.

12. When we differentiate with respect to x, we find $2x + 3y^2\frac{dy}{dx} + \frac{dy}{dx} = 0$, and therefore $\frac{dy}{dx} = -\frac{2x}{3y^2+1}$. Differentiation of this equation gives

$$\frac{d^2y}{dx^2} = -\frac{(3y^2+1)(2) - 2x\left(6y\frac{dy}{dx}\right)}{(3y^2+1)^2} = -\frac{2(3y^2+1) + 12xy\left(\frac{2x}{3y^2+1}\right)}{(3y^2+1)^2} = -\frac{2(3y^2+1)^2 + 24x^2y}{(3y^2+1)^3}.$$

13. When we differentiate with respect to x, we find $4x - 3y^2\frac{dy}{dx} = -y - x\frac{dy}{dx}$, and therefore $\frac{dy}{dx} = \frac{4x+y}{3y^2-x}$. Differentiation of this equation gives

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(3y^2-x)\left(4+\frac{dy}{dx}\right) - (4x+y)\left(6y\frac{dy}{dx} - 1\right)}{(3y^2-x)^2}\\ &= \frac{(3y^2-x)\left(4+\frac{4x+y}{3y^2-x}\right) - (4x+y)\left[6y\left(\frac{4x+y}{3y^2-x}\right) - 1\right]}{(3y^2-x)^2}\\ &= \frac{(3y^2-x)(12y^2+y) - (4x+y)(3y^2+24xy+x)}{(3y^2-x)^3}\\ &= \frac{36y^4 - 48xy^2 - 2xy - 96x^2y - 4x^2}{(3y^2-x)^3}.\end{aligned}$$

14. If we differentiate with respect to x, we obtain $2y\frac{dy}{dx} + 2\frac{dy}{dx} = 5$, and this equation can be solved for $\frac{dy}{dx} = \frac{5}{2(y+1)}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{-5}{2(y+1)^2}\frac{dy}{dx} = \frac{-5}{2(y+1)^2}\left[\frac{5}{2(y+1)}\right] = \frac{-25}{4(y+1)^3}.$$

15. If we differentiate with respect to x, we obtain $2(x+y)\left(1+\frac{dy}{dx}\right) = 1$, from which $\frac{dy}{dx} = \frac{1}{2(x+y)} - 1$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{-1}{2(x+y)^2}\left(1+\frac{dy}{dx}\right) = \frac{-1}{2(x+y)^2}\left[1 + \frac{1}{2(x+y)} - 1\right] = \frac{-1}{4(x+y)^3}.$$

16. If we differentiate with respect to x, we obtain $3x^2y + x^3\frac{dy}{dx} + y^3 + 3xy^2\frac{dy}{dx} = 0$. Therefore, $\frac{dy}{dx} = -\frac{3x^2y+y^3}{x^3+3xy^2}$. When $x = 1$, we have $y + y^3 = 2$, and the only solution of this equation is $y = 1$. Thus, $\frac{dy}{dx}_{|x=1} = -\frac{3+1}{1+3} = -1$.

17. The equation implies that $x^2 + 2xy + y^2 = x^2 + y^2 \Longrightarrow 2xy = 0 \Longrightarrow y = 0$. Consequently, dy/dx and d^2y/dx^2 are both 0.

18. Differentiation with respect to x leads to $2xy^3 + 3x^2y^2\frac{dy}{dx} + 2 + 4\frac{dy}{dx} = 0$, from which $\frac{dy}{dx} = -\frac{2+2xy^3}{3x^2y^2+4}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = -\frac{(3x^2y^2+4)\left(2y^3 + 6xy^2\frac{dy}{dx}\right) - (2+2xy^3)\left(6xy^2 + 6x^2y\frac{dy}{dx}\right)}{(3x^2y^2+4)^2}$$

$$= -\frac{(3x^2y^2+4)\left[2y^3 - 6xy^2\left(\dfrac{2+2xy^3}{3x^2y^2+4}\right)\right] - (2+2xy^3)\left[6xy^2 - 6x^2y\left(\dfrac{2+2xy^3}{3x^2y^2+4}\right)\right]}{(3x^2y^2+4)^2}$$

$$= \frac{-2y^3(3x^2y^2+4)^2 + 12xy^2(2+2xy^3)(3x^2y^2+4) - 6x^2y(2+2xy^3)^2}{(3x^2y^2+4)^3}$$

19. Differentiation with respect to x gives $y^2 + 2xy\dfrac{dy}{dx} - 6xy - 3x^2\dfrac{dy}{dx} = 1$, from which $\dfrac{dy}{dx} = \dfrac{1+6xy-y^2}{2xy-3x^2}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{(2xy-3x^2)\left(6y + 6x\dfrac{dy}{dx} - 2y\dfrac{dy}{dx}\right) - (1+6xy-y^2)\left(2y + 2x\dfrac{dy}{dx} - 6x\right)}{(2xy-3x^2)^2}$$

$$= \frac{(2xy-3x^2)\left[6y + (6x-2y)\left(\dfrac{1+6xy-y^2}{2xy-3y^2}\right)\right] - (1+6xy-y^2)\left[2y - 6x + 2x\left(\dfrac{1+6xy-y^2}{2xy-3x^2}\right)\right]}{(2xy-3x^2)^2}$$

$$= \frac{6y(2xy-3x^2)^2 + (12x-4y)(2xy-3x^2)(1+6xy-y^2) - 2x(1+6xy-y^2)^2}{(2xy-3x^2)^3}.$$

20. Differentiation with respect to x gives $1 = \dfrac{dy}{dx}\sqrt{1-y^2} + y\dfrac{-y}{\sqrt{1-y^2}}\dfrac{dy}{dx} = \dfrac{1-y^2-y^2}{\sqrt{1-y^2}}\dfrac{dy}{dx} = \dfrac{1-2y^2}{\sqrt{1-y^2}}\dfrac{dy}{dx}$.

Thus, $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{1-2y^2}$. Another differentiation gives

$$\frac{d^2y}{dx^2} = \frac{(1-2y^2)(1/2)(1-y^2)^{-1/2}\left(-2y\dfrac{dy}{dx}\right) - \sqrt{1-y^2}\left(-4y\dfrac{dy}{dx}\right)}{(1-2y^2)^2}$$

$$= \frac{-y(1-2y^2) + 4y(1-y^2)}{\sqrt{1-y^2}(1-2y^2)^2}\frac{dy}{dx} = \frac{3y-2y^3}{\sqrt{1-y^2}(1-2y^2)^2}\frac{\sqrt{1-y^2}}{1-2y^2} = \frac{3y-2y^3}{(1-2y^2)^3}.$$

21. If we differentiate $1 - xy = (4-3y)^2 = 16 - 24y + 9y^2$, then $-y - x\dfrac{dy}{dx} = -24\dfrac{dy}{dx} + 18y\dfrac{dy}{dx}$, from which $\dfrac{dy}{dx} = \dfrac{y}{-x-18y+24}$. When $x = 0$, we find that $y = 1$, and therefore $\dfrac{dy}{dx}_{|x=0} = \dfrac{1}{-18+24} = \dfrac{1}{6}$.

22. When we differentiate with respect to x, we obtain $2xy^3 + 3x^2y^2\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{2xy^3+y}{3x^2y^2+x}$. When $x = 1$, we find that $y^3 + y = 2$, the only solution of which is $y = 1$. Thus, $\dfrac{dy}{dx}_{|x=1} = -\dfrac{2+1}{3+1} = -\dfrac{3}{4}$.

23. If we differentiate with respect to x, then $5y^4\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} - 2\dfrac{dy}{dx} = 0$, from which $\dfrac{dy}{dx} = \dfrac{y}{2-x-5y^4}$. When $x = 2$, we find that $y = 1$, and therefore $\dfrac{dy}{dx}_{|x=2} = \dfrac{1}{2-2-5} = -\dfrac{1}{5}$. Differentiation of the first derivative gives

$$\frac{d^2y}{dx^2} = \frac{(2-x-5y^4)\dfrac{dy}{dx} - y\left(-1 - 20y^3\dfrac{dy}{dx}\right)}{(2-x-5y^4)^2},$$

and when we substitute $x = 2$, $y = 1$, and $dy/dx = -1/5$,

$$\frac{d^2y}{dx^2}_{|x=2} = \frac{(2-2-5)(-1/5) - (1)[-1 - 20(1)^3(-1/5)]}{(2-2-5)^2} = -\frac{2}{25}.$$

24. If we differentiate with respect to x, we find $2x + 2y + 2x\dfrac{dy}{dx} + 6y\dfrac{dy}{dx} = 0$. Therefore, $\dfrac{dy}{dx} = -\dfrac{2x+2y}{2x+6y} = -\dfrac{x+y}{x+3y}$. When $y = 1$, we find that $x^2 + 2x + 1 = 0$, and the only solution of this equation is $x = -1$. Thus, $dy/dx_{|x=-1} = -(-1+1)/(-1+3) = 0$. A second differentiation gives

$$\frac{d^2y}{dx^2} = -\frac{(x+3y)\left(1+\dfrac{dy}{dx}\right) - (x+y)\left(1+3\dfrac{dy}{dx}\right)}{(x+3y)^2},$$

and when we substitute $x = -1$, $y = 1$, and $dy/dx = 0$, $\dfrac{d^2y}{dx^2}_{|x=-1} = -\dfrac{(-1+3)(1)-(-1+1)(1)}{(-1+3)^2} = -\dfrac{1}{2}$.

25. If we differentiate with respect to x, then $y^2 + 2xy\dfrac{dy}{dx} + 2xy + x^2\dfrac{dy}{dx} = 0$, from which $\dfrac{dy}{dx} = -\dfrac{2xy+y^2}{2xy+x^2}$. The slope of the tangent line is 0 when $0 = 2xy + y^2 = y(2x+y)$. Thus, $y = 0$ or $y = -2x$. The first is impossible since x and y must also satisfy $xy^2 + x^2y = 16$. When we substitute $y = -2x$ into this equation, $16 = x(-2x)^2 + x^2(-2x) = 2x^3 \implies x = 2$. Thus, the only point is $(2, -4)$.

26. If we differentiate with respect to x, then $2x + \dfrac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0 \quad \implies \quad \dfrac{dy}{dx} = -3xy^{1/3}$. Since

$$\frac{d^2y}{dx^2} = -3y^{1/3} - xy^{-2/3}\frac{dy}{dx} = -3y^{1/3} - xy^{-2/3}(-3xy^{1/3}) = -3y^{1/3} + 3x^2y^{-1/3} = 3y^{-1/3}(-y^{2/3}+x^2),$$

the second derivative vanishes if $x^2 = y^{2/3}$. When we substitute this into $x^2 + y^{2/3} = 2$, we obtain $x^2 + x^2 = 2$, and therefore $x = \pm 1$. The corresponding y-values are $y = \pm 1$, and the required points are $(1, \pm 1)$ and $(-1, \pm 1)$.

27. If we differentiate the equation with respect to h,

$$-\frac{2(2.4048)^2}{r^3}\frac{dr}{dh} - \frac{2\pi^2}{h^3} = 0 \quad \implies \quad \frac{dr}{dh} = -\left(\frac{\pi}{2.4048}\right)^2\left(\frac{r}{h}\right)^3.$$

28. (a) Since $\dfrac{dy}{dx} = \dfrac{(x+2)(2x+1)-(x^2+x)(1)}{(x+2)^2} = \dfrac{x^2+4x+2}{(x+2)^2}$, it follows that

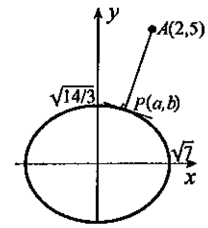
$$\frac{Ey}{Ex} = \frac{x}{\dfrac{x(x+1)}{x+2}}\,\frac{x^2+4x+2}{(x+2)^2} = \frac{x^2+4x+2}{(x+1)(x+2)}.$$

(b) If we differentiate with respect to x, $1 = \dfrac{(3-y)\left(400\dfrac{dy}{dx}\right) - (400y+200)\left(-\dfrac{dy}{dx}\right)}{(3-y)^2} = \dfrac{1400\dfrac{dy}{dx}}{(3-y)^2}$, from which $\dfrac{dy}{dx} = \dfrac{(3-y)^2}{1400}$. Thus, $\dfrac{Ey}{Ex} = \left[\dfrac{400y+200}{y(3-y)}\right]\dfrac{(3-y)^2}{1400} = \dfrac{(2y+1)(3-y)}{7y}$.

29. The elasticity of a function is equal to one if and only if $1 = \dfrac{x}{y}\dfrac{dy}{dx} \implies \dfrac{dy}{dx} = \dfrac{y}{x}$. But y/x is the slope of the line joining the point (x, y) on the graph of the function to the origin. Hence elasticity is equal to one if and only if slope dy/dx of the graph is equal to slope of the line joining the point to the origin.

30. If we differentiate $2x^2+3y^2=14$ with respect to x, $4x+6y\dfrac{dy}{dx}=0$. The slope of the tangent line at P is therefore

$$\frac{dy}{dx}_{|(a,b)}=-\frac{2a}{3b}.$$



Since the slope of AP is $(b-5)/(a-2)$, and this line is perpendicular to the tangent line at P,

$$-\frac{2a}{3b}=-\frac{a-2}{b-5}.$$

When this equation is solved for b, the result is $b=10a/(6-a)$. Because P is on the ellipse, we may write that

$$14=2a^2+3b^2=2a^2+3\left(\frac{10a}{6-a}\right)^2 \implies (14-2a^2)(6-a)^2=300a^2.$$

Our diagram makes it clear that there is one and only one point in the first quadrant which satisfies the requirements, and $a=1$ is a solution of the above equation. Thus, the required point is $(1,2)$.

31. If we differentiate the equation of the hyperbola with respect to x, then $2b^2x-2a^2y(dy/dx)=0$, from which $dy/dx=b^2x/(a^2y)$. The slope of the tangent line at (x_0,y_0) is therefore $b^2x_0/(a^2y_0)$, and the equation of the tangent line is

$$y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0) \implies b^2x_0x-a^2y_0y=b^2x_0^2-a^2y_0^2.$$

Since (x_0,y_0) is on the hyperbola, $b^2x_0^2-a^2y_0^2=a^2b^2$, and the equation of the tangent line simplifies to $b^2x_0x-a^2y_0y=a^2b^2$.

32. If we differentiate the equation of the circle with respect to x,

$$2(x-h)+2(y-k)\frac{dy}{dx}=0 \qquad \implies \qquad \frac{dy}{dx}=-\frac{x-h}{y-k}.$$

Thus,

$$\frac{d^2y}{dx^2}=-\frac{(y-k)(1)-(x-h)dy/dx}{(y-k)^2}=\frac{-1}{(y-k)^2}\left[(y-k)+(x-h)\left(\frac{x-h}{y-k}\right)\right]$$
$$=\frac{-1}{(y-k)^3}\left[(y-k)^2+(x-h)^2\right]=\frac{-r^2}{(y-k)^3}.$$

We now calculate that

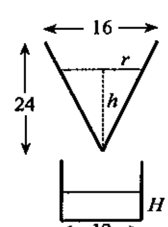
$$\left|\frac{d^2y/dx^2}{[1+(dy/dx)^2]^{3/2}}\right|=\left|\frac{-r^2/(y-k)^3}{\left[1+\left(-\dfrac{x-h}{y-k}\right)^2\right]^{3/2}}\right|=\left|\frac{r^2/(y-k)^3}{\left[\dfrac{(y-k)^2+(x-h)^2}{(y-k)^2}\right]^{3/2}}\right|$$
$$=\left|\frac{r^2/(y-k)^3}{\left[\dfrac{r^2}{(y-k)^2}\right]^{3/2}}\right|=\frac{r^2}{|y-k|^3}\frac{|y-k|^3}{r^3}=\frac{1}{r}.$$

33. Since the amount of solution in the two containers is always the same, call it C, we can write that

$$C=\frac{1}{3}\pi r^2h+\pi(6)^2H.$$

Similar triangles for the cone give $r/h=8/24=1/3$, and therefore

$$C=\frac{1}{3}\pi\left(\frac{h}{3}\right)^2h+36\pi H=\frac{\pi h^3}{27}+36\pi H.$$



If we differentiate this equation with respect to H,

$$\frac{\pi h^2}{9}\frac{dh}{dH}+36\pi=0 \implies \frac{dh}{dH}=-\frac{324}{h^2}.$$

34. The product rule gives $R'(x) = r(x) + xr'(x)$. To obtain $r'(x)$ we differentiate $x = 4a^3 - 3ar^2 + r^3$ with respect to x,

$$1 = -6ar\frac{dr}{dx} + 3r^2\frac{dr}{dx}.$$

Thus, $\dfrac{dr}{dx} = \dfrac{1}{3r^2 - 6ar}$, and $R'(x) = r + \dfrac{x}{3r^2 - 6ar} = \dfrac{3r^3 - 6ar^2 + x}{3r^2 - 6ar}$.

35. If the equation is to be valid for all x, then it must be valid for $x = 0$. Substitution of $x = 0$ yields $a_0 = b_0$. The equation now reads

$$a_1x + a_2x^2 + \cdots + a_nx^n = b_1x + b_2x^2 + \cdots + b_nx^n.$$

If we differentiate this equation with respect to x,

$$a_1 + 2a_2x + \cdots + na_nx^{n-1} = b_1 + 2b_2x + \cdots + nb_nx^{n-1}.$$

If we set $x = 0$, we obtain $a_1 = b_1$. The equation now reads

$$2a_2x + \cdots + na_nx^{n-1} = 2b_2x + \cdots + nb_nx^{n-1}.$$

If we differentiate this equation with respect to x,

$$2a_2 + (3)(2)a_3x + \cdots + n(n-1)a_nx^{n-2} = 2b_2 + (3)(2)b_3x + \cdots + n(n-1)b_nx^{n-1}.$$

If we set $x = 0$, we obtain $a_2 = b_2$. Continued differentiations lead to equality of all coefficients.

36. (a) When we differentiate with respect to x, we obtain $\sqrt{1+2y} + \dfrac{x}{\sqrt{1+2y}}\dfrac{dy}{dx} = 2x - \dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{2x - \sqrt{1+2y}}{\dfrac{x}{\sqrt{1+2y}} + 1}$. Since $y = 0$ when $x = 0$, we obtain $f'(0) = -1/1 = -1$.

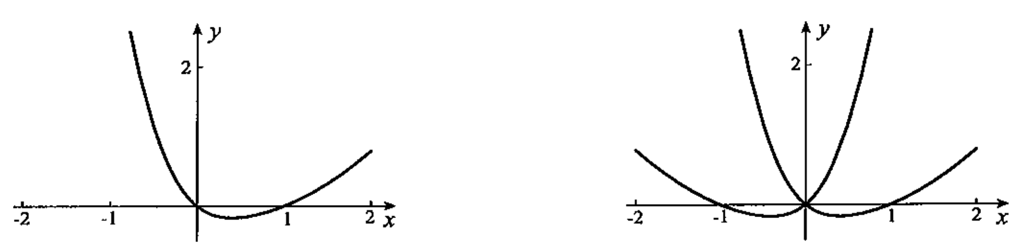
(b) When the equation is squared, $x^2(1+2y) = x^4 - 2x^2y + y^2 \Longrightarrow x^2 + 4x^2y = x^4 + y^2$. If we differentiate this equation with respect to x,

$$2x + 8xy + 4x^2\frac{dy}{dx} = 4x^3 + 2y\frac{dy}{dx} \quad\Longrightarrow\quad \frac{dy}{dx} = \frac{2x + 8xy - 4x^3}{2y - 4x^2} = \frac{x + 4xy - 2x^3}{y - 2x^2}.$$

In this case we cannot simply set $x = 0$ and $y = 0$ to obtain $f'(0)$. To see why $x^2 + 4x^2y = x^4 + y^2$ cannot be used to find $f'(0)$, we write the equation in the form $y^2 - 4x^2y + x^4 - x^2 = 0$. This is a quadratic equation in y with solutions

$$y = \frac{4x^2 \pm \sqrt{16x^4 - 4(x^4 - x^2)}}{2} = 2x^2 \pm \sqrt{3x^4 + x^2}.$$

Thus, the equation $x^2 + 4x^2y = x^4 + y^2$ does not define y as a function of x; there are two values of y satisfying the equation for each value of x. We can also see this graphically. A plot of $x\sqrt{1+2y} = x^2 - y$ is shown in the left figure below. It defines y as a function of x near $x = 0$. A plot of $x^2 + 4x^2y = x^4 + y^2$ in the right figure shows that it does not define y as a function of x near $x = 0$.



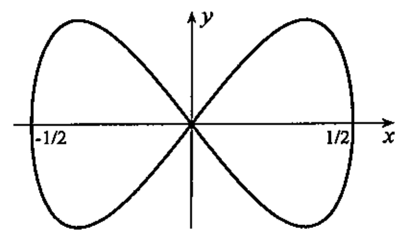
37. The straight lines have slope m where $m = y/x$. Diferentiation of $x^2 + y^2 = r^2$ with respect to x gives $2x + 2y(dy/dx) = 0 \Longrightarrow dy/dx = -x/y$. These slopes are negative reciprocals.

38. The slope of the tangent line at any point on any of the parabolas in the first family is $dy/dx = 2ax = 2x(y/x^2) = 2y/x$. To find the slope of the tangent line at any point on any of the ellipses, we differentiate with respect to x, getting $2x+4y(dy/dx) = 0 \implies dy/dx = -x/(2y)$. Since this is the negative reciprocal of the slope of the parabola, the families are orthogonal trajectories.

39. When we differentiate $x^2 - y^2 = C_1$ with respect to x, we obtain $2x - 2y(dy/dx) = 0 \implies dy/dx = x/y$. When we differentiate $xy = C_2$ with resepct to x, the result is $y + x(dy/dx) = 0 \implies dy/dx = -y/x$. Slopes of the families are negative reciprocals, and the families are orthogonal trajectories.

40. Differentiation of $2x^2 + 3y^2 = C^2$ with respect to x gives $4x + 6y(dy/dx) = 0 \implies dy/dx = -2x/(3y)$. Differentiation of $y^2 = ax^3$ gives $2y\dfrac{dy}{dx} = 3ax^2 \implies \dfrac{dy}{dx} = \dfrac{3ax^2}{2y} = \dfrac{3x^2(y^2/x^3)}{2y} = \dfrac{3y}{2x}$. Since these slopes are negative reciprocals, the families are orthogonal trajectories.

41. (a) If we differentiate $y^2 = x^2 - 4x^4$ with respect to x, the result is $2y(dy/dx) = 2x - 16x^3$, from which $dy/dx = x(1 - 8x^2)/y$.
(b) Since $y = 0$ when $x = 0$, we cannot use the formula in part (a) to calculate dy/dx when $x = 0$. The plot indicates the reason. The curve crosses itself at $(0, 0)$, and therefore the slope is not uniquely defined.

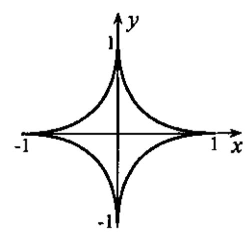


42. (a) If we differentiate with respect to x, we obtain $\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0$, from which $\dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$.

(b) When $x \geq 0$ and $y \geq 0$, the equation of the curve reduces to $\sqrt{x} + \sqrt{y} = 1$. According to the formula in part (a),

$$\lim_{x \to 1^-} \frac{dy}{dx} = -\frac{0}{1} = 0^-, \quad \text{and} \quad \lim_{x \to 0^+} \frac{dy}{dx} = -\infty.$$

This enables us to sketch the first quadrant part of the curve. The remaining parts are obtained by symmetry.



43. There is no derivative because the equation does not define y as a function of x. To see this we complete squares on x- and y-terms, $(x + 2)^2 + (y + 3)^2 = -2$.

44. The chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{\sqrt{u^2-1}(1) - u(1/2)(u^2-1)^{-1/2}(2u)}{u^2-1}\right]\frac{du}{dx} = \frac{u^2-1-u^2}{(u^2-1)^{3/2}}\frac{du}{dx} = \frac{-1}{(u^2-1)^{3/2}}\frac{du}{dx}.$$

To obtain du/dx we differentiate $x^2u^2 + \sqrt{u^2-1} = 4$ with respect to x,

$$2xu^2 + 2x^2u\frac{du}{dx} + \frac{u}{\sqrt{u^2-1}}\frac{du}{dx} = 0 \quad \implies \quad \frac{du}{dx} = \frac{-2xu^2}{2x^2u + \frac{u}{\sqrt{u^2-1}}} = \frac{-2xu\sqrt{u^2-1}}{1 + 2x^2\sqrt{u^2-1}}.$$

Thus, $\dfrac{dy}{dx} = \left[\dfrac{-1}{(u^2-1)^{3/2}}\right]\left[\dfrac{-2xu\sqrt{u^2-1}}{1+2x^2\sqrt{u^2-1}}\right] = \dfrac{2xu}{(u^2-1)\left(1+2x^2\sqrt{u^2-1}\right)}$.

45. If we differentiate the equations with respect to x,

$$4y^3\frac{dy}{dx} + v^3\frac{dy}{dx} + 3yv^2\frac{dv}{dx} = 0, \qquad 2xv + x^2\frac{dv}{dx} + 3v^2 + 6xv\frac{dv}{dx} = 6x^2y + 2x^3\frac{dy}{dx}.$$

When we solve the first for $\dfrac{dv}{dx} = -\left(\dfrac{4y^3+v^3}{3yv^2}\right)\dfrac{dy}{dx}$, and substitute into the second,

$$2xv + 3v^2 - (x^2 + 6xv)\left(\frac{4y^3+v^3}{3yv^2}\right)\frac{dy}{dx} = 6x^2y + 2x^3\frac{dy}{dx} \quad \implies \quad \frac{dy}{dx} = \frac{3yv^2(2xv + 3v^2 - 6x^2y)}{6x^3yv^2 + (x^2+6xv)(4y^3+v^3)}.$$

46. If we differentiate the equation with respect to x,

$$\frac{2xy^3 - 3x^2y^2\dfrac{dy}{dx}}{y^6} - 1 = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{2xy^3 - y^6}{3x^2y^2} = \frac{2xy - y^4}{3x^2}.$$

Consequently, $3x^2\dfrac{dy}{dx} + y^4 = 3x^2\left(\dfrac{2xy - y^4}{3x^2}\right) + y^4 = 2xy.$

47. If we differentiate the equation in the form $\dfrac{2}{y^3} - \dfrac{3}{x^2y^2} = C$ with respect to x, the result is $\dfrac{-6}{y^4}\dfrac{dy}{dx} + \dfrac{6}{(xy)^3}\left(y + x\dfrac{dy}{dx}\right) = 0$. Multiplication by $x^3y^4/6$ gives

$$-x^3\frac{dy}{dx} + y\left(y + x\frac{dy}{dx}\right) = 0 \qquad \Longrightarrow \qquad (xy - x^3)\frac{dy}{dx} + y^2 = 0.$$

48. If $n = a/b$, where a and b are integers, then when $y = x^n = x^{a/b}$, we may set $y^b = x^a$. Differentiation with respect to x gives

$$by^{b-1}\frac{dy}{dx} = ax^{a-1} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{a}{b}\frac{x^{a-1}}{y^{b-1}} = \frac{a}{b}\frac{x^{a-1}}{(x^{a/b})^{b-1}} = \frac{a}{b}x^{a-1-a+a/b} = nx^{n-1}.$$

49. If we differentiate with respect to x, then $y^2 + 2xy\dfrac{dy}{dx} + 2xy + x^2\dfrac{dy}{dx} = 0$, from which $\dfrac{dy}{dx} = -\dfrac{2xy + y^2}{2xy + x^2}$. The slope of the tangent line is 1 when

$$1 = -\frac{2xy + y^2}{2xy + x^2} \quad \Longrightarrow \quad y^2 + 4xy + x^2 = 0 \quad \Longrightarrow \quad (x + y)^2 = -2xy.$$

But the equation $x^2y + xy^2 = 2 \Longrightarrow xy(x + y) = 2 \Longrightarrow (x + y)^2 = 4/(xy)^2$. When these expressions for $(x + y)^2$ are equated, the result is $-2xy = 4/(xy)^2 \Longrightarrow (xy)^3 = -2 \Longrightarrow y = -2^{1/3}/x$. We now substitute this into $xy(x + y) = 2$,

$$x\left(\frac{-2^{1/3}}{x}\right)\left(x - \frac{2^{1/3}}{x}\right) = 2 \Longrightarrow x^2 + 2^{2/3}x - 2^{1/3} = 0 \Longrightarrow x = \frac{-2^{2/3} \pm \sqrt{2^{4/3} + 4(2^{1/3})}}{2} = \frac{-1 \pm \sqrt{3}}{2^{1/3}}.$$

Corresponding y-values are $y = -2^{1/3}\left(\dfrac{2^{1/3}}{-1 \pm \sqrt{3}}\right) = \dfrac{2^{2/3}}{1 \mp \sqrt{3}}$.

50. If we express the first family in the form $y^2(a - x) = x^3$, we may solve for $a = x + x^3/y^2$. Differentiation of this equation with respect to x gives

$$0 = 1 + \frac{3y^2x^2 - 2x^3y\dfrac{dy}{dx}}{y^4} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{3x^2y + y^3}{2x^3}.$$

When we solve the second family for $b = (x^2 + y^2)^2/(2x^2 + y^2)$, and differentiate

$$0 = \frac{(2x^2 + y^2)(2)(x^2 + y^2)\left(2x + 2y\dfrac{dy}{dx}\right) - (x^2 + y^2)^2\left(4x + 2y\dfrac{dy}{dx}\right)}{(2x^2 + y^2)^2}.$$

This implies that $4(2x^2 + y^2)\left(x + y\dfrac{dy}{dx}\right) - 2(x^2 + y^2)\left(2x + y\dfrac{dy}{dx}\right) = 0$, and we can solve for

$$\frac{dy}{dx} = \frac{4x(2x^2 + y^2) - 4x(x^2 + y^2)}{2y(x^2 + y^2) - 4y(2x^2 + y^2)} = \frac{-2x^3}{3x^2y + y^3}.$$

Since these expressions for dy/dx are negative reciprocals, the families are orthogonal trajectories.

51. If we differentiate the equation with respect to x, $\dfrac{2}{3x^{1/3}}+\dfrac{2}{3y^{1/3}}\dfrac{dy}{dx}=0$. When we solve this for dy/dx, we obtain $\dfrac{dy}{dx}=-\dfrac{y^{1/3}}{x^{1/3}}$. The equation of the tangent line at a point (x_0,y_0) is $y-y_0=-\dfrac{y_0^{1/3}}{x_0^{1/3}}(x-x_0)$. The x- and y-intercepts of this line are $x=\dfrac{x_0^{1/3}y_0}{y_0^{1/3}}+x_0=x_0^{1/3}y_0^{2/3}+x_0$ and $y=\dfrac{y_0^{1/3}}{x_0^{1/3}}x_0+y_0=x_0^{2/3}y_0^{1/3}$ $+y_0$. The length L of that part of the tangent line between the coordinate axes is therefore given by

$$\begin{aligned}L^2&=(x_0^{2/3}y_0^{1/3}+y_0)^2+(x_0^{1/3}y_0^{2/3}+x_0)^2=y_0^{2/3}(x_0^{2/3}+y_0^{2/3})^2+x_0^{2/3}(y_0^{2/3}+x_0^{2/3})^2\\&=(x_0^{2/3}+y_0^{2/3})^3=(a^{2/3})^3=a^2.\end{aligned}$$

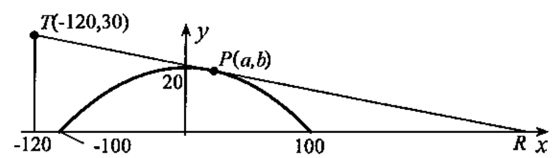
Thus, the length is always equal to a.

52. If we differentiate the equation with respect to x, $3x^2+3y^2(dy/dx)=3ay+3ax(dy/dx)$. When we solve this for dy/dx and equate the result to -1,

$$-1=\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}\quad\Longrightarrow\quad a(y-x)=x^2-y^2=(x-y)(x+y).$$

Thus, $y=x$ or $x+y=-a$. When $y=x$, the equation of the folium implies that $x^3+x^3=3ax^2\implies x=3a/2$. A contradiction is obtained when we put $y=-a-x$ into the equation of the folium. Thus, the only point at which the slope of the tangent line is equal to -1 is $(3a/2,3a/2)$.

53. The required position occurs at the point R where the tangent line from $T(-120,30)$ to the arc of the circle representing the hill intersects the x-axis. Let the point of tangency be $P(a,b)$. If $x^2+(y-k)^2=r^2$ is the equation of the circle, then using the points $(100,0)$ and $(0,20)$, we obtain $100^2+k^2=r^2$ and $(20-k)^2=r^2$. These give $k=-240$ and $r^2=67\,600$, so that the equation of the circular arc is $x^2+(y+240)^2=67\,600$. The slope of the tangent line to this curve is defined by



$$2x+2(y+240)\frac{dy}{dx}=0\implies\frac{dy}{dx}=\frac{-x}{y+240},$$

and hence at P, $\dfrac{dy}{dx}_{|(a,b)}=\dfrac{-a}{b+240}$. The slope of this line is also the slope of PT, namely, $(b-30)/(a+120)$, and therefore

$$\frac{b-30}{a+120}=\frac{-a}{b+240}\quad\Longrightarrow\quad a^2+b^2+210b+120a=7200.$$

Since $P(a,b)$ is on the circular arc, it also follows that $a^2+(b+240)^2=67\,600\implies a^2+b^2+480b=10\,000$. When we solve each of these equations for a^2+b^2 and equate results, we obtain

$$7200-210b-120a=10\,000-480b\quad\Longrightarrow\quad b=\frac{12a+280}{27}.$$

Substituting this into $a^2+b^2+480b=10\,000$ gives

$$a^2+\left(\frac{12a+280}{27}\right)^2+480\left(\frac{12a+280}{27}\right)=10\,000\quad\Longrightarrow\quad 873a^2+162\,240a-3\,582\,800=0.$$

The positive solution is $a=19.9432$. The y-coordinate of P is $b=19.2340$. The equation of the tangent line is therefore $y-b=[-a/(b+240)](x-a)$, and its x-intercept is given by $-b=[-a/(b+240)](x-a)\implies x=b(b+240)/a+a$. When we substitute the calculated values for a and b, the result is $x\approx270$ m.

54. (a) The equation for the ovals must be $c^2 = \sqrt{(x-a)^2+y^2}\sqrt{(x+a)^2+y^2}$. When this equation is squared and rearranged,

$$\begin{aligned} c^4 &= [(x-a)^2+y^2][(x+a)^2+y^2] = (x^2-a^2)^2 + y^2[(x-a)^2+(x+a)^2] + y^4 \\ &= x^4+y^4+a^4-2a^2x^2+2x^2y^2+2a^2y^2 = (x^2+y^2+a^2)^2 - 4a^2x^2. \end{aligned}$$

(b) When the equation for the ovals is differentiated with respect to x,

$$2(x^2+y^2+a^2)\left(2x+2y\frac{dy}{dx}\right) = 8a^2x \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{2a^2x - x(x^2+y^2+a^2)}{y(x^2+y^2+a^2)}.$$

The tangent line is horizontal when $dy/dx = 0$, and this requires $2a^2x = x(x^2+y^2+a^2)$. To satisfy this equation we must set $x = 0$ or $x^2+y^2 = a^2$. When $x = 0$, the equation of the ovals gives $(y^2+a^2)^2 = c^4$, from which $y = \pm\sqrt{c^2-a^2}$. Thus, two points at which the tangent line is horizontal are $(0, \pm\sqrt{c^2-a^2})$.

When $x^2+y^2 = a^2$, the equation of the ovals gives $c^4 = 4a^4 - 4a^2x^2$, from which $x^2 = (4a^4-c^4)/(4a^2)$. If $c \geq \sqrt{2}a$, then no more solutions are obtained. If $c < \sqrt{2}a$, then $x = \pm\sqrt{4a^4-c^4}/(2a)$, and these give the four points $\left(\dfrac{\sqrt{4a^4-c^4}}{2a}, \pm\dfrac{c^2}{2a}\right)$ and $\left(-\dfrac{\sqrt{4a^4-c^4}}{2a}, \pm\dfrac{c^2}{2a}\right)$.

55. When we differentiate the equation of the ellipse with respect to x, the result is $2b^2x + 2a^2y(dy/dx) = 0 \Longrightarrow dy/dx = -b^2x/(a^2y)$. When we differentiate the equation of the hyperbola with respect to x, the result is $2d^2x - 2c^2y(dy/dx) = 0 \Longrightarrow dy/dx = d^2x/(c^2y)$. These curves intersect orthogonally if

$$-1 = \left(-\frac{b^2x}{a^2y}\right)\left(\frac{d^2x}{c^2y}\right) \quad \Longleftrightarrow \quad y = \pm\frac{bdx}{ac}.$$

When we substitute this into the equation of the ellipse,

$$b^2x^2 + \frac{a^2b^2d^2x^2}{a^2c^2} = a^2b^2 \quad \Longleftrightarrow \quad x^2 = \frac{a^2c^2}{c^2+d^2}.$$

Substitution of $y = \pm bdx/(ac)$ into the equation of the hyperbola gives

$$d^2x^2 - \frac{c^2b^2d^2x^2}{a^2c^2} = c^2d^2 \quad \Longleftrightarrow \quad x^2 = \frac{a^2c^2}{a^2-b^2}.$$

Thus, the curves intersect orthogonally if $\dfrac{a^2c^2}{c^2+d^2} = \dfrac{a^2c^2}{a^2-b^2} \Longleftrightarrow c^2+d^2 = a^2-b^2$.

EXERCISES 3.9

1. $\dfrac{dy}{dx} = 2\cos 3x(3) = 6\cos 3x$

2. $\dfrac{dy}{dx} = -\sin x - 4\cos 5x\,(5) = -\sin x - 20\cos 5x$

3. $\dfrac{dy}{dx} = 2\sin x\cos x = \sin 2x$

4. $\dfrac{dy}{dx} = -3\tan^{-4}3x\sec^2 3x\,(3) = -\dfrac{9\sec^2 3x}{\tan^4 3x}$

5. $\dfrac{dy}{dx} = 4\sec^3 10x(\sec 10x\tan 10x)(10) = 40\sec^4 10x\tan 10x$

6. $\dfrac{dy}{dx} = -\csc(4-2x)\cot(4-2x)(-2) = 2\csc(4-2x)\cot(4-2x)$

7. $\dfrac{dy}{dx} = 2\sin(3-2x^2)\cos(3-2x^2)(-4x) = -8x\sin(3-2x^2)\cos(3-2x^2)$

8. $\dfrac{dy}{dx} = \cot x^2 - x\csc^2 x^2(2x) = \cot x^2 - 2x^2\csc^2 x^2$

9. $\dfrac{dy}{dx} = \dfrac{(\cos 5x)(2\cos 2x) - (\sin 2x)(-5\sin 5x)}{(\cos 5x)^2} = \dfrac{2\cos 5x\cos 2x + 5\sin 5x\sin 2x}{\cos^2 5x}$

10. $\dfrac{dy}{dx} = \dfrac{(x+1)(\sin x + x\cos x) - x\sin x\,(1)}{(x+1)^2} = \dfrac{\sin x + x(1+x)\cos x}{(x+1)^2}$

11. $\dfrac{dy}{dx} = 3\sin^2 x\cos x - \sin x$

12. Since $y = \dfrac{1}{2}\sin 4x$, it follows that $\dfrac{dy}{dx} = \dfrac{1}{2}\cos 4x\,(4) = 2\cos 4x$.

13. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{\sin 3x}} 3\cos 3x = \dfrac{3\cos 3x}{2\sqrt{\sin 3x}}$

14. $\dfrac{dy}{dx} = \dfrac{1}{4}(1+\tan^3 x)^{-3/4}(3\tan^2 x\sec^2 x) = \dfrac{3\tan^2 x\sec^2 x}{4(1+\tan^3 x)^{3/4}}$

15. If we differentiate with respect to x, we find $2\cos y\dfrac{dy}{dx} - 3\sin x = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{3\sin x}{2\cos y}$.

16. Differentiation with respect to x gives $\cos y - x\sin y\dfrac{dy}{dx} + y\sin x - \cos x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = \dfrac{\cos y + y\sin x}{\cos x + x\sin y}$.

17. Differentiation with respect to x gives $8\sin x\cos x - 9\cos^2 y\,(-\sin y)\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{8\sin x\cos x}{9\cos^2 y\sin y}$.

18. If we differentiate with respect to x, we find $\sec^2(x+y)\left(1+\dfrac{dy}{dx}\right) = \dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{\sec^2(x+y)}{1-\sec^2(x+y)}$.

19. If we differentiate with respect to x, we obtain $1 + \sec xy\tan xy\left(y + x\dfrac{dy}{dx}\right) = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{1+y\sec xy\tan xy}{x\sec xy\tan xy}$.

20. Differentiation with respect to x gives $3x^2y + x^3\dfrac{dy}{dx} + 2\tan y\sec^2 y\dfrac{dy}{dx} = 3$, and therefore $\dfrac{dy}{dx} = \dfrac{3(1-x^2y)}{x^3 + 2\tan y\sec^2 y}$.

21. Differentiation with respect to x gives $1 = \left(3y^2\csc^3 y - 3y^3\csc^2 y\csc y\cot y\right)\dfrac{dy}{dx}$, from which $\dfrac{dy}{dx} = \dfrac{1}{3y^2\csc^3 y - 3y^3\csc^3 y\cot y}$.

22. $\dfrac{dy}{dx} = -\sin(\tan x)\sec^2 x = -\sec^2 x\sin(\tan x)$

23. $\dfrac{dy}{dx} = 3x^2 - 2x\cos x + x^2\sin x + 2\sin x + 2x\cos x - 2\sin x = 3x^2 + x^2\sin x$

24. Since $y = (\sin^2 x^2 + \cos^2 x^2)(\sin^2 x^2 - \cos^2 x^2) = -(\cos^2 x^2 - \sin^2 x^2) = -\cos 2x^2$, we find that $dy/dx = \sin 2x^2\,(4x) = 4x\sin 2x^2$.

25. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = [3u^2\sec u + u^3\sec u\tan u][\tan(x+1) + x\sec^2(x+1)]$

26. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \left(\dfrac{-\sec v\tan v}{2\sqrt{3-\sec v}}\right)\left(\sec^2\sqrt{x}\dfrac{1}{2\sqrt{x}}\right) = -\dfrac{\sec v\tan v\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{3-\sec v}}$.

27. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(\dfrac{t}{\sqrt{t^2+1}}\right)[\cos(\sin x)\cos x]$

28.
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{1}{3}(1+\sec^3 u)^{-2/3}3\sec^2 u\sec u\tan u\right]\left[\frac{1}{2}(1+\cos x^2)^{-1/2}(-\sin x^2\,(2x))\right]$$
$$= -\frac{x\sin x^2\sec^3 u\tan u}{\sqrt{1+\cos x^2}(1+\sec^3 u)^{2/3}}$$

29. $\dfrac{dy}{dx} = \dfrac{(1+\cos^3 x)(2\sin x\cos x) - \sin^2 x(-3\cos^2 x\sin x)}{(1+\cos^3 x)^2} = \dfrac{\sin x\cos x(2 + 2\cos^3 x + 3\sin^2 x\cos x)}{(1+\cos^3 x)^2}$

30. $\dfrac{dy}{dx} = \dfrac{x^2\sin x[3\tan^2(3x^2-4)\sec^2(3x^2-4)(6x)] - [1+\tan^3(3x^2-4)][2x\sin x + x^2\cos x]}{x^4\sin^2 x}$

$= \dfrac{18x^2\tan^2(3x^2-4)\sec^2(3x^2-4) - (2 + x\cot x)[1+\tan^3(3x^2-4)]}{x^3\sin x}$

31. If we differentiate with respect to x, we obtain $\cos y\dfrac{dy}{dx} = 2x + \dfrac{dy}{dx} \implies \dfrac{dy}{dx} = \dfrac{2x}{\cos y - 1}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{(\cos y - 1)(2) - 2x\left(-\sin y\dfrac{dy}{dx}\right)}{(\cos y-1)^2} = \frac{2(\cos y - 1) + 2x\sin y\left(\dfrac{2x}{\cos y - 1}\right)}{(\cos y - 1)^2} = \frac{2(\cos y-1)^2 + 4x^2\sin y}{(\cos y - 1)^3}.$$

32. If we differentiate with respect to x, we obtain $\sec^2 y\dfrac{dy}{dx} = 1 + x\dfrac{dy}{dx} + y \implies \dfrac{dy}{dx} = \dfrac{1+y}{\sec^2 y - x}$. A second differentiation now gives

$$\frac{d^2y}{dx^2} = \frac{(\sec^2 y - x)(dy/dx) - (1+y)[2\sec^2 y\tan y\,(dy/dx) - 1]}{(\sec^2 y - x)^2}$$

$$= \frac{[\sec^2 y - x - 2(1+y)\sec^2 y\tan y]\left(\dfrac{1+y}{\sec^2 y - x}\right) + (1+y)}{(\sec^2 y - x)^2}$$

$$= \frac{(1+y)[(\sec^2 y - x) - 2(1+y)\sec^2 y\tan y + (\sec^2 y - x)]}{(\sec^2 y - x)^3}$$

$$= \frac{2(1+y)[\sec^2 y - x - (1+y)\sec^2 y\tan y]}{(\sec^2 y - x)^3}.$$

33. $\displaystyle\lim_{x\to0}\frac{\tan x}{x} = \lim_{x\to0}\frac{\sin x}{x}\frac{1}{\cos x} = (1)(1) = 1$

34. $\displaystyle\lim_{x\to0}\frac{1-\cos x}{x} = \lim_{x\to0}\frac{1-[1-2\sin^2(x/2)]}{x} = \lim_{x\to0}\left[\frac{\sin(x/2)}{x/2}\sin(x/2)\right] = (1)(0) = 0$

35. $\displaystyle\lim_{x\to0}\frac{\sin 2x}{x} = \lim_{x\to0}\frac{2\sin x\cos x}{x} = 2\lim_{x\to0}\left[\left(\frac{\sin x}{x}\right)\cos x\right] = 2(1)(1) = 2$

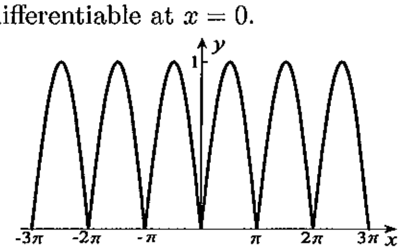
36. $\displaystyle\lim_{x\to\infty}\frac{\sin(2/x)}{\sin(1/x)} = \lim_{x\to\infty}\frac{2\sin(1/x)\cos(1/x)}{\sin(1/x)} = \lim_{x\to\infty}[2\cos(1/x)] = 2$

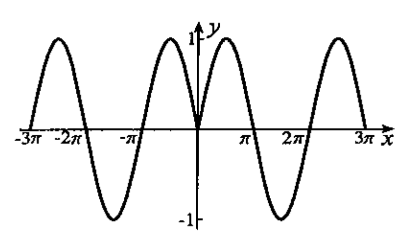
37. $\displaystyle\lim_{x\to0}\frac{(x+1)^2\sin x}{3x^3} = \lim_{x\to0}\frac{(x+1)^2}{3x^2}\frac{\sin x}{x}$. Since $\displaystyle\lim_{x\to0}\frac{(x+1)^2}{3x^2} = \infty$ and $\displaystyle\lim_{x\to0}\frac{\sin x}{x} = 1$, it follows that $\displaystyle\lim_{x\to0}\frac{(x+1)^2\sin x}{3x^3} = \infty$.

38. $\displaystyle\lim_{x\to\pi/2}\frac{\cos x}{(x-\pi/2)^2} = \lim_{x\to\pi/2}\frac{\sin(\pi/2-x)}{(x-\pi/2)^2} = \lim_{x\to\pi/2}\left[\frac{\sin(\pi/2-x)}{\pi/2-x}\frac{1}{\pi/2-x}\right]$

Since $\displaystyle\lim_{x\to\pi/2}\frac{\sin(\pi/2-x)}{\pi/2-x} = 1$ and $\displaystyle\lim_{x\to\pi/2}\frac{1}{\pi/2-x}$ does not exist, the original limit does not exist.

39. Graphs are shown below. $|\sin x|$ is not differentiable at $x = n\pi$, where n is an integer; whereas $\sin|x|$ is not differentiable at $x = 0$.





40. For $x \neq 0$, $g'(x) = \sin\left(\frac{1}{x}\right) + x\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$. The limit of this function as x approaches 0 does not exist.

41. Since $d\theta/dt = -\omega A\,\sin(\omega t + \phi)$, it follows that

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = -\omega^2 A\,\cos(\omega t + \phi) + \omega^2[A\,\cos(\omega t + \phi)] = 0.$$

42. Since $\dfrac{dy}{dt} = A\cos\left(\sqrt{\dfrac{k}{m}}t\right)\sqrt{\dfrac{k}{m}} - B\sin\left(\sqrt{\dfrac{k}{m}}t\right)\sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{k}{m}}\left[A\cos\left(\sqrt{\dfrac{k}{m}}t\right) - B\sin\left(\sqrt{\dfrac{k}{m}}t\right)\right]$, a second differentiation gives

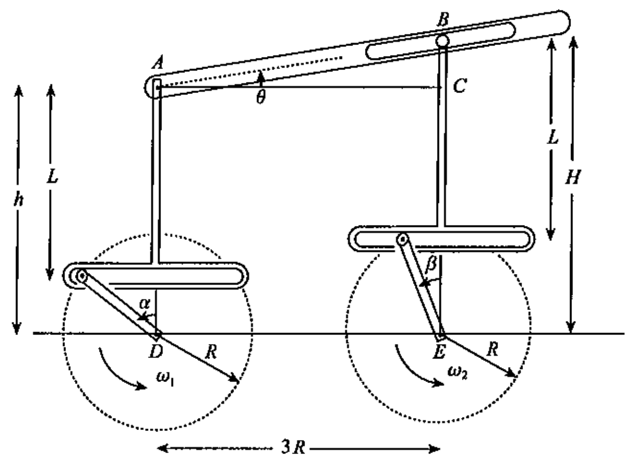
$$\frac{d^2y}{dt^2} = \sqrt{\frac{k}{m}}\left[-A\sin\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}} - B\cos\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}}\right] = -\frac{k}{m}\left[A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)\right].$$

Hence,

$$m\frac{d^2y}{dt^2} + ky = -k\left[A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)\right] + k\left[A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right)\right] = 0.$$

43. From triangle ABC, we can write that $\|BC\| = 3R\tan\theta$. But

$$\begin{aligned}\|BC\| &= \|BE\| - \|CE\| = H - h\\ &= (L + R\cos\beta) - (L + R\cos\alpha)\\ &= R(\cos\beta - \cos\alpha).\end{aligned}$$



Hence, $R(\cos\beta - \cos\alpha) = 3R\tan\theta$. Division by R and differentiation with respect to t gives

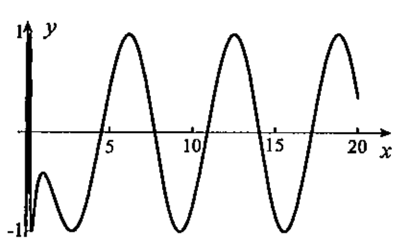
$$-\sin\beta\frac{d\beta}{dt} + \sin\alpha\frac{d\alpha}{dt} = 3\sec^2\theta\frac{d\theta}{dt}$$
$$\Longrightarrow \frac{d\theta}{dt} = \frac{1}{3}\cos^2\theta(\omega_1\sin\alpha - \omega_2\sin\beta).$$

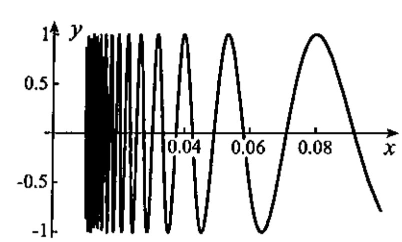
Since $\tan\theta = (\cos\beta - \cos\alpha)/3$, it follows that

$$\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1 + \tan^2\theta} = \frac{1}{1 + (\cos\beta - \cos\alpha)^2/9}.$$

Thus, $\dfrac{d\theta}{dt} = \dfrac{1}{3}\left[\dfrac{9}{9 + (\cos\beta - \cos\alpha)^2}\right](\omega_1\sin\alpha - \omega_2\sin\beta) = \dfrac{3(\omega_1\sin\alpha - \omega_2\sin\beta)}{9 + (\cos\beta - \cos\alpha)^2}$.

44. To get an idea of how many values of x satisfy $f'(x) = 0$ and approximations to them, we plot a graph of $f(x)$. The plot on the interval $0.1 \leq x \leq 20$ in the left figure below suggests regular behaviour of the function for large values of x, but wild oscillations as $x \to 0$. This is consistent with the fact that for large x, the term $1/x$ becomes less and less significant, and $f(x)$ can be approximated by $\cos x$. On the other hand, when x is close to 0, $f(x)$ should behave much like $\sin(1/x)$ in Figure 2.8b. This is illustrated in the right figure below where we have plotted the function on the interval $0.01 \leq x \leq 0.1$.





With these ideas in mind, we now solve $0 = f'(x) = -\left(1 - \frac{1}{x^2}\right)\sin\left(x + \frac{1}{x}\right)$. There are two possibilities:

$$1 - \frac{1}{x^2} = 0 \qquad \text{or} \qquad \sin\left(x + \frac{1}{x}\right) = 0.$$

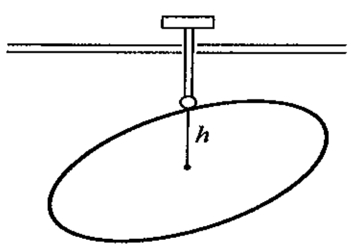
The first gives $x = \pm 1$, and it is clear that the tangent line at $x = 1$ in the left figure is indeed horizontal. The second equation above implies that $x + \frac{1}{x} = n\pi$, where n is an integer. Multiplication by x leads to the quadratic equation

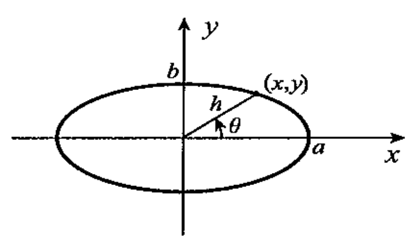
$$x^2 - n\pi x + 1 = 0 \quad \Longrightarrow \quad x = \frac{n\pi \pm \sqrt{n^2\pi^2 - 4}}{2}.$$

Clearly, we must choose $n > 0$ (else $n^2\pi^2 - 4 < 0$ when $n = 0$, and $x < 0$ when $n < 0$). When $n = 1$, the solution is $x = (\pi + \sqrt{\pi^2 - 4})/2 \approx 2.8$. This is the first point to the right of $x = 1$ at which the tangent line is horizontal in the left figure. As n increases, the 4 becomes less and less significant and $x \approx (n\pi \pm n\pi)/2$. When we choose the positive sign, we obtain $x \approx n\pi$, and we can indeed see that points on the graph in the left figure where the tangent line is horizontal do indeed seem to be multiples of π. When we choose the negative sign, we obtain points to the left of $x = 1$ at which the tangent line is horizontal (right figure). For large n, they can be approximated by $1/(n\pi)$, but this is not an easy fact to show.

45. The velocity of the follower is the rate of change of the length h from the end of the follower on the cam to the centre of the ellipse (left figure below). We can calculate the rate of change of h by fixing the ellipse (right figure) and letting the line joining the origin to a point $P(x, y)$ on the ellipse rotate at 600 rpm counterclockwise around the ellipse. The x- and y-coordinates of P can be expressed in terms of h and θ by $x = h\cos\theta$ and $y = h\sin\theta$. We substitute these into the equation $b^2x^2 + a^2y^2 = a^2b^2$ of the ellipse,

$$b^2h^2\cos^2\theta + a^2h^2\sin^2\theta = a^2b^2 \qquad \Longrightarrow \qquad h^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}.$$





Differentiation with respect to t gives

$$2h\frac{dh}{dt} = \frac{-a^2b^2(-2b^2\cos\theta\sin\theta + 2a^2\sin\theta\cos\theta)}{(b^2\cos^2\theta + a^2\sin^2\theta)^2}\frac{d\theta}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = -\frac{a^2b^2(a^2 - b^2)\sin 2\theta}{2h(b^2\cos^2\theta + a^2\sin^2\theta)^2}\frac{d\theta}{dt}.$$

Since the cam rotates at 600 rpm, $\frac{d\theta}{dt} = \frac{2\pi(600)}{60} = 20\pi$ radians per second. Thus,

$$\frac{dh}{dt} = -\frac{a^2b^2(a^2 - b^2)\sin 2\theta}{\dfrac{2ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}(b^2\cos^2\theta + a^2\sin^2\theta)^2}(20\pi) = \frac{10\pi ab(b^2 - a^2)\sin 2\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^{3/2}}.$$

46. According to the extended power rule and formula 3.13,

$$\frac{d}{dx}|\sin x|^n = n|\sin x|^{n-1}\frac{|\sin x|}{\sin x}\cos x$$

valid whenever $x \neq n\pi$. Graphs of $|\sin x|^n$ quickly show that its derivative is zero when $x = n\pi$. Now,

$|\sin x|/\sin x$ is equal to 1 when $\sin x > 0$ and equal to -1 when $\sin x < 0$. It is undefined when $x = n\pi$. On the other hand, when $\mathrm{sgn}(x)$ is the signum function of Exercise 47 in Section 2.4,

$$\mathrm{sgn}(\sin x) = \begin{cases} -1, & \sin x < 0 \\ 0, & \sin x = 0 \\ 1, & \sin x > 0. \end{cases}$$

It follows that we can express the derivative as

$$\frac{d}{dx}|\sin x|^n = n|\sin x|^{n-1}\cos x\,\mathrm{sgn}(\sin x),$$

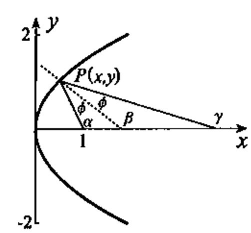
and this is valid for all x.

47. According to equation 3.2, $f'(0) = \lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\dfrac{h^2\sin(1/h)-0}{h} = \lim_{h\to 0} h\sin(1/h) = 0$ (see Example 2.9 in Section 2.1). On the other hand, when $x \neq 0$,

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Since this derivative does not have a limit as $x \to 0$, it follows that $f'(x)$ is not continuous at $x = 0$.

48. Suppose a ray of light from the bulb strikes the reflector at $P(x, y)$ making an angle ϕ with the normal to the reflector at P. Because $\gamma = 2\phi + \alpha$ and $\beta = \alpha + \phi$, it follows that $\gamma = 2\beta - \alpha$, and hence



$$\tan\gamma = \frac{\tan 2\beta - \tan\alpha}{1 + \tan 2\beta\,\tan\alpha} = \frac{\dfrac{2\tan\beta}{1-\tan^2\beta} - \tan\alpha}{1 + \tan 2\beta\,\tan\alpha}.$$

Now the slope dy/dx of the reflector is given by $2y(dy/dx) = 4$, from which $dy/dx = 2/y$, provided $y \neq 0$. Consequently, $\tan\beta = -y/2$ (the slope of the normal line), and

$$\frac{2\tan\beta}{1-\tan^2\beta} - \tan\alpha = \frac{2(-y/2)}{1-(-y/2)^2} - \frac{y}{x-1} = \frac{-y}{1-y^2/4} - \frac{y}{y^2/4-1} = 0.$$

In other words, $\tan\gamma = 0$ which implies that either $\gamma = 0$ or π; that is, rays are reflected parallel to the x-axis. The only ray still in question is that for which $y = 0$. In this case the ray is along the x-axis, and is reflected back along the x-axis.

EXERCISES 3.10

1. $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1-(2x+3)^2}}(2) = \dfrac{-2}{\sqrt{1-(2x+3)^2}}$

2. $\dfrac{dy}{dx} = \dfrac{-1}{1+(x^2+2)^2}(2x) = \dfrac{-2x}{1+(x^2+2)^2}$

3. $\dfrac{dy}{dx} = \dfrac{-1}{(3-4x)\sqrt{(3-4x)^2-1}}(-4) = \dfrac{4}{(3-4x)\sqrt{(3-4x)^2-1}}$

4. $\dfrac{dy}{dx} = \dfrac{1}{1+(2-x^2)^2}(-2x) = \dfrac{-2x}{1+(2-x^2)^2}$

5. $\dfrac{dy}{dx} = \dfrac{1}{(3-2x^2)\sqrt{(3-2x^2)^2-1}}(-4x) = \dfrac{-4x}{(3-2x^2)\sqrt{(3-2x^2)^2-1}}$

6. $\dfrac{dy}{dx} = \mathrm{Csc}^{-1}(x^2+5) - \dfrac{x}{(x^2+5)\sqrt{(x^2+5)^2-1}}(2x) = \mathrm{Csc}^{-1}(x^2+5) - \dfrac{2x^2}{(x^2+5)\sqrt{(x^2+5)^2-1}}$

7. $\dfrac{dy}{dx} = 2x\,\mathrm{Sin}^{-1}(2x) + \dfrac{(x^2+2)}{\sqrt{1-(2x)^2}}(2) = 2x\,\mathrm{Sin}^{-1}(2x) + \dfrac{2(x^2+2)}{\sqrt{1-4x^2}}$

8. $\dfrac{dy}{dx} = \dfrac{1}{1+(x+2)} \dfrac{1}{2\sqrt{x+2}} = \dfrac{1}{2\sqrt{x+2}(x+3)}$

9. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(1-x^2)}} \dfrac{-x}{\sqrt{1-x^2}} = \dfrac{-x}{|x|\sqrt{1-x^2}}$

10. $\dfrac{dy}{dx} = \dfrac{-1}{1+x^2-1} \dfrac{x}{\sqrt{x^2-1}} = \dfrac{-1}{x\sqrt{x^2-1}}$

11. $\dfrac{dy}{dx} = 2\,\mathrm{Tan}^{-1}(x^2)\,\dfrac{2x}{1+x^4} = \dfrac{4x\,\mathrm{Tan}^{-1}(x^2)}{1+x^4}$

12. $\dfrac{dy}{dx} = 2x\,\mathrm{Sec}^{-1}x + \dfrac{x^2}{x\sqrt{x^2-1}} = 2x\,\mathrm{Sec}^{-1}x + \dfrac{x}{\sqrt{x^2-1}}$

13. $\dfrac{dy}{dx} = \sec^2(3\,\mathrm{Sin}^{-1}x)\,\dfrac{3}{\sqrt{1-x^2}} = \dfrac{3\sec^2(3\,\mathrm{Sin}^{-1}x)}{\sqrt{1-x^2}}$

14. $\dfrac{dy}{dx} = \dfrac{-1}{1+\left(\dfrac{1+x}{1-x}\right)^2}\left[\dfrac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right] = \dfrac{-(1-x)^2}{(1-x)^2+(1+x)^2}\left[\dfrac{2}{(1-x)^2}\right] = \dfrac{-2}{2+2x^2} = \dfrac{-1}{1+x^2}$

15. $\dfrac{dy}{dx} = \dfrac{-1}{\dfrac{1}{x}\sqrt{\dfrac{1}{x^2}-1}}\left(\dfrac{-1}{x^2}\right) = \dfrac{1}{x\sqrt{\dfrac{1-x^2}{x^2}}} = \dfrac{|x|}{x\sqrt{1-x^2}}$

16. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(\dfrac{1-x}{1+x}\right)^2}}\left[\dfrac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}\right] = \dfrac{|x+1|}{\sqrt{(1+x)^2-(1-x)^2}}\left[\dfrac{-2}{(x+1)^2}\right]$ Since x must be greater than 0 in order that $-1 \le \dfrac{1-x}{1+x} \le 1$, it follows that $\dfrac{dy}{dx} = \dfrac{(x+1)(-2)}{2\sqrt{x}(x+1)^2} = \dfrac{-1}{\sqrt{x}(x+1)}$.

17. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left[\dfrac{1}{1+(u^2+1/u)^2}\left(2u-\dfrac{1}{u^2}\right)\right][\sec^2(x^2+4)(2x)] = \dfrac{2x(2u^3-1)\sec^2(x^2+4)}{u^6+2u^3+u^2+1}$

18. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(\mathrm{Cos}^{-1}t - \dfrac{t}{\sqrt{1-t^2}}\right)\left(\dfrac{-x}{\sqrt{1-x^2}}\right) = \dfrac{x(t-\sqrt{1-t^2}\,\mathrm{Cos}^{-1}t)}{\sqrt{1-t^2}\sqrt{1-x^2}}$

19. If we differentiate with respect to x, we obtain $y^2\cos x + 2y\dfrac{dy}{dx}\sin x + \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$, from which

$$\frac{dy}{dx} = \frac{\dfrac{1}{1+x^2} - y^2\cos x}{1+2y\sin x}.$$

20. If we differentiate with respect to x, we obtain $\dfrac{1}{\sqrt{1-x^2y^2}}\left(x\dfrac{dy}{dx}+y\right) = 5 + 2\dfrac{dy}{dx}$, and therefore

$$\frac{dy}{dx} = \frac{5 - \dfrac{y}{\sqrt{1-x^2y^2}}}{\dfrac{x}{\sqrt{1-x^2y^2}} - 2} = \frac{5\sqrt{1-x^2y^2} - y}{x - 2\sqrt{1-x^2y^2}}.$$

21. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-1}} - \dfrac{1}{x\sqrt{x^2-1}} = \dfrac{x^2-1}{x\sqrt{x^2-1}} = \dfrac{\sqrt{x^2-1}}{x}$

22. $\dfrac{dy}{dx} = \dfrac{-1/3}{\dfrac{x}{3}\sqrt{\dfrac{x^2}{9}-1}}\left(\dfrac{1}{3}\right) - \dfrac{x^2(1/2)(x^2-9)^{-1/2}(2x) - \sqrt{x^2-9}(2x)}{x^4}$

$$= \frac{-1}{x\sqrt{x^2-9}} - \frac{x^2-2(x^2-9)}{x^3\sqrt{x^2-9}} = \frac{-x^2-(-x^2+18)}{x^3\sqrt{x^2-9}} = \frac{-18}{x^3\sqrt{x^2-9}}$$

23. $\dfrac{dy}{dx} = \mathrm{Cos}^{-1}(x/2) - \dfrac{x/2}{\sqrt{1-x^2/4}} + \dfrac{x}{\sqrt{4-x^2}} = \mathrm{Cos}^{-1}(x/2)$

24. $\dfrac{dy}{dx} = -\dfrac{1}{x^2}\mathrm{Csc}^{-1}(3x) + \dfrac{1}{x}\dfrac{(-1)(3)}{3x\sqrt{9x^2-1}} + \dfrac{\sqrt{9x^2-1}}{x^2} - \dfrac{9x}{x\sqrt{9x^2-1}}$

$= -\dfrac{1}{x^2}\mathrm{Csc}^{-1}(3x) + \dfrac{-1+(9x^2-1)-9x^2}{x^2\sqrt{9x^2-1}} = -\dfrac{1}{x^2}\mathrm{Csc}^{-1}(3x) - \dfrac{2}{x^2\sqrt{9x^2-1}}$

25. $\dfrac{dy}{dx} = 2x\,\mathrm{Sec}^{-1}x + \dfrac{x^2}{x\sqrt{x^2-1}} - \dfrac{x}{\sqrt{x^2-1}} = 2x\,\mathrm{Sec}^{-1}x$

26. $\dfrac{dy}{dx} = (\mathrm{Cos}^{-1}x)^2 - \dfrac{2x\,\mathrm{Cos}^{-1}x}{\sqrt{1-x^2}} - 2 + \dfrac{2x}{\sqrt{1-x^2}} = (\mathrm{Cos}^{-1}x)^2 - 2 + \dfrac{2x}{\sqrt{1-x^2}}(1-\mathrm{Cos}^{-1}x)$

27. $\dfrac{dy}{dx} = 18x\,\mathrm{Cot}^{-1}(3x) - \dfrac{(1+9x^2)(3)}{1+9x^2} + 3 = 18x\,\mathrm{Cot}^{-1}(3x)$

28. $\dfrac{dy}{dx} = \sqrt{4x-x^2} + \dfrac{x-2}{2\sqrt{4x-x^2}}(4-2x) + \dfrac{4}{\sqrt{1-\left(\dfrac{x-2}{2}\right)^2}}\left(\dfrac{1}{2}\right)$

$= \dfrac{(4x-x^2)+(x-2)(2-x)}{\sqrt{4x-x^2}} + \dfrac{4}{\sqrt{4x-x^2}} = \dfrac{4x-x^2-x^2+4x-4+4}{\sqrt{4x-x^2}} = 2\sqrt{4x-x^2}$

29. $\dfrac{dy}{dx} = 4x\,\mathrm{Sin}^{-1}x + \dfrac{2x^2-1}{\sqrt{1-x^2}} + \sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}} = 4x\,\mathrm{Sin}^{-1}x$

30. $\dfrac{dy}{dx} = \dfrac{1}{1+\dfrac{2x^2}{1+x^4}}\left[\dfrac{\sqrt{2}}{\sqrt{1+x^4}} - \dfrac{\sqrt{2}x}{2(1+x^4)^{3/2}}(4x^3)\right]$

$= \dfrac{1+x^4}{1+x^4+2x^2}\left[\dfrac{\sqrt{2}(1+x^4)-2\sqrt{2}x^4}{(1+x^4)^{3/2}}\right] = \dfrac{\sqrt{2}(1-x^4)}{(1+x^2)^2\sqrt{1+x^4}} = \dfrac{\sqrt{2}(1-x^2)}{(1+x^2)\sqrt{1+x^4}}$

31. $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}} - \dfrac{1}{1+x^2-1}\dfrac{x}{\sqrt{x^2-1}} = 0$ Because the derivative always vanishes, the function must be equal to a constant. When $x \geq 1$, we set $\mathrm{Sec}^{-1}x + \mathrm{Cot}^{-1}\sqrt{x^2-1} = C$. Substituting $x = 1$ gives $0 + \pi/2 = C$. Thus, $\mathrm{Sec}^{-1}x + \mathrm{Cot}^{-1}\sqrt{x^2-1} = \pi/2$ when $x \geq 1$. When we set $\mathrm{Sec}^{-1}x + \mathrm{Cot}^{-1}\sqrt{x^2-1} = D$ for $x \leq -1$, and substitute $x = -1$, we obtain $-\pi + \pi/2 = D$. Thus, $\mathrm{Sec}^{-1}x + \mathrm{Cot}^{-1}\sqrt{x^2-1} = -\pi/2$ when $x \leq -1$.

32. (a) If we set $y = \mathrm{Sec}^{-1}x$, then $x = \sec y$, and implicit differentiation gives

$$1 = \sec y\,\tan y\,\frac{dy}{dx} \quad\Longrightarrow\quad \frac{dy}{dx} = \frac{1}{\sec y\,\tan y}.$$

Now, $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$. If $x > 0$, then $0 < y < \pi/2$, and $\tan y = \sqrt{x^2-1}$. If $x < 0$, and principal values are chosen as stated, then $\pi/2 < y < \pi$, and $\tan y = -\sqrt{x^2-1}$. Thus,

$$\frac{dy}{dx} = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}}, & x > 0 \\ \dfrac{-1}{x\sqrt{x^2-1}}, & x < 0 \end{cases} = \frac{1}{|x|\sqrt{x^2-1}}.$$

(b) A similar analysis gives $\dfrac{d}{dx}\mathrm{Csc}^{-1}x = \dfrac{-1}{|x|\sqrt{x^2-1}}$.

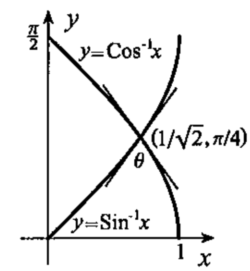
33. The slope of $y = f(x) = \mathrm{Sin}^{-1}x$ at $x = 1/\sqrt{2}$ is $f'(1/\sqrt{2}) = \dfrac{1}{\sqrt{1-(1/\sqrt{2})^2}} = \sqrt{2}$.

The slope of $y = g(x) = \mathrm{Cos}^{-1}x$ at $x = 1/\sqrt{2}$ is

$$g'(1/\sqrt{2}) = \frac{-1}{\sqrt{1-(1/\sqrt{2})^2}} = -\sqrt{2}.$$

According to equation 1.60, the angle θ between the tangent lines is

$$\theta = \mathrm{Tan}^{-1}\left|\frac{\sqrt{2}+\sqrt{2}}{1+(\sqrt{2})(-\sqrt{2})}\right| = 1.23 \text{ radians.}$$



34. To verify 3.37c, we set $y = \mathrm{Tan}^{-1}x$, in which case $x = \tan y$. Differentiation gives $1 = \sec^2 y\,(dy/dx) \Longrightarrow dy/dx = 1/\sec^2 y = 1/(1+\tan^2 y) = 1/(1+x^2)$. Verification of 3.37d is similar. To verify 3.37e, we set $y = \mathrm{Sec}^{-1}x$, in which case $x = \sec y$. Differentiation gives $1 = \sec y \tan y\,(dy/dx) \Longrightarrow dy/dx = 1/(\sec y \tan y)$. Now $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$. When $x > 0$, we obtain $0 < y < \pi/2$, so that $\tan y > 0$. On the other hand, when $x < 0$, principal values give $-\pi < y < -\pi/2$, but $\tan y$ is again positive. Thus, in either case, $\tan y = \sqrt{x^2-1}$, and $dy/dx = 1/(x\sqrt{x^2-1})$. Verification of 3.37f is similar.

35. When the crank rotates at 300 rpm, $\dfrac{d\phi}{dt} = \dfrac{2\pi(300)}{60} = 10\pi$ radians per second. Differentiation of

$$\theta = \mathrm{Tan}^{-1}\left(\frac{R\sin\phi}{L+R\cos\phi}\right),$$

with respect to t gives

$$\frac{d\theta}{dt} = \frac{1}{1+\dfrac{R^2\sin^2\phi}{(L+R\cos\phi)^2}}\left[\frac{(L+R\cos\phi)R\cos\phi\dfrac{d\phi}{dt} - R\sin\phi\left(-R\sin\phi\dfrac{d\phi}{dt}\right)}{(L+R\cos\phi)^2}\right]$$

$$= \frac{(L+R\cos\phi)^2}{(L+R\cos\phi)^2+R^2\sin^2\phi}\left[\frac{R^2+LR\cos\phi}{(L+R\cos\phi)^2}\right](10\pi) = \frac{10\pi(R^2+LR\cos\phi)}{R^2+L^2+2LR\cos\phi}.$$

EXERCISES 3.11

1. $\dfrac{dy}{dx} = 3^{2x}(2)(\ln 3)$

2. $\dfrac{dy}{dx} = \dfrac{1}{3x^2+1}(6x) = \dfrac{6x}{3x^2+1}$

3. $\dfrac{dy}{dx} = \dfrac{1}{2x+1}(2)(\log_{10} e) = \dfrac{2\log_{10} e}{2x+1}$

4. $\dfrac{dy}{dx} = e^{1-2x}(-2) = -2e^{1-2x}$

5. $\dfrac{dy}{dx} = e^{2x} + xe^{2x}(2) = (2x+1)e^{2x}$

6. $\dfrac{dy}{dx} = \ln x + x\left(\dfrac{1}{x}\right) = 1 + \ln x$

7. Since $y = x^2$, we have $\dfrac{dy}{dx} = 2x$.

8. $\dfrac{dy}{dx} = \dfrac{1}{3-4x}(-4)\log_{10} e = \dfrac{-4\log_{10} e}{3-4x}$

9. $\dfrac{dy}{dx} = \dfrac{1}{\sin x}(\cos x) = \cot x$

10. $\dfrac{dy}{dx} = \dfrac{1}{3\cos x}(-3\sin x) = -\tan x$

11. $\dfrac{dy}{dx} = \ln(x+1) + \dfrac{x}{x+1}$

12. $\dfrac{dy}{dx} = 2x + 3x^2e^{4x} + x^3e^{4x}(4) = 2x + (3x^2+4x^3)e^{4x}$

13. $\dfrac{dy}{dx} = \dfrac{(1-x)e^{1-x}(-1) - e^{1-x}(-1)}{(1-x)^2} = \dfrac{xe^{1-x}}{(1-x)^2}$

14. $\dfrac{dy}{dx} = \cos(e^{2x})(2e^{2x}) = 2e^{2x}\cos(e^{2x})$

15. $\dfrac{dy}{dx} = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) = \dfrac{1}{x\ln x}$

16. $\dfrac{dy}{dx} = -2e^{-2x}\sin 3x + e^{-2x}\cos 3x\,(3) = e^{-2x}(3\cos 3x - 2\sin 3x)$

17. $\dfrac{dy}{dx} = \dfrac{1}{x^2e^{4x}}(2xe^{4x} + 4x^2e^{4x}) = \dfrac{2}{x} + 4$

18. $\dfrac{dy}{dx} = \dfrac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \dfrac{4}{(e^x + e^{-x})^2}$

19. $\dfrac{dy}{dx} = -e^{-x}\ln x + e^{-x}(1/x) = e^{-x}(1/x - \ln x)$

20. $\dfrac{dy}{dx} = \sin(\ln x) - \cos(\ln x) + x\left[\dfrac{1}{x}\cos(\ln x) + \dfrac{1}{x}\sin(\ln x)\right] = 2\sin(\ln x)$

21. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = e^{\sin u}(\cos u)[e^{1/x}(-1/x^2)] = -\dfrac{u\,e^{\sin u}\cos u}{x^2}$

22. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = \dfrac{1}{\cos v}(-\sin v)\,2\sin x\,\cos x = -\tan v\,\sin 2x$

23. If we differentiate with respect to x, we obtain $\dfrac{1}{x+y}\left(1 + \dfrac{dy}{dx}\right) = 2xy + x^2\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{2xy - \dfrac{1}{x+y}}{\dfrac{1}{x+y} - x^2} = \frac{2x^2y + 2xy^2 - 1}{1 - x^3 - x^2y}.$$

24. Differentiation with respect to x gives $e^y + xe^y\dfrac{dy}{dx} + 2x\ln y + \dfrac{x^2}{y}\dfrac{dy}{dx} + \sin x\dfrac{dy}{dx} + y\cos x = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{y\cos x + 2x\ln y + e^y}{\sin x + x^2/y + xe^y}$.

25. $\dfrac{dy}{dx} = \dfrac{1}{\sec x + \tan x}(\sec x\,\tan x + \sec^2 x) = \sec x$

26.
$$\begin{aligned}\frac{dy}{dx} &= \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} - \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{x^2+1+x^2}{\sqrt{x^2+1}} - \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{2x^2}{\sqrt{x^2+1}}\end{aligned}$$

27. $\dfrac{dy}{dx} = \dfrac{1}{x + 4 + \sqrt{8x + x^2}}\left(1 + \dfrac{8 + 2x}{2\sqrt{8x + x^2}}\right) = \dfrac{1}{\sqrt{8x + x^2}}$

28. $\dfrac{dy}{dx} = 1 - \dfrac{1}{4(1 + 5e^{4x})}(5e^{4x})(4) = \dfrac{1 + 5e^{4x} - 5e^{4x}}{1 + 5e^{4x}} = \dfrac{1}{1 + 5e^{4x}}$

29. If we differentiate with respect to x, we obtain $e^{xy}\left(y + x\dfrac{dy}{dx}\right) = 2(x+y)\left(1 + \dfrac{dy}{dx}\right)$, from which $\dfrac{dy}{dx} = \dfrac{2x + 2y - y\,e^{xy}}{x\,e^{xy} - 2x - 2y}$.

30. If we differentiate with respect to x, we obtain $e^{1/x}\left(-\dfrac{1}{x^2}\right) + e^{1/y}\left(-\dfrac{1}{y^2}\right)\dfrac{dy}{dx} = -\dfrac{1}{x^2} - \dfrac{1}{y^2}\dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{e^{1/x}(1/x^2) - 1/x^2}{1/y^2 + e^{1/y}(-1/y^2)} = \dfrac{e^{1/x} - 1}{x^2}\dfrac{y^2}{1 - e^{1/y}} = \dfrac{y^2(e^{1/x} - 1)}{x^2(1 - e^{1/y})}$.

31. (a) Since $dT/dr = c/r$, it follows that $\dfrac{d}{dr}\left(r\dfrac{dT}{dr}\right) = \dfrac{d}{dr}(c) = 0$.

(b) If temperatures on inner and outer edges of the cylinders are T_a and T_b, then

$$T_a = c\ln a + d, \qquad T_b = c\ln b + d.$$

These can be solved for $c = \dfrac{T_b - T_a}{\ln(b/a)}$ and $d = \dfrac{T_a\ln b - T_b\ln a}{\ln(b/a)}$.

32. (a) Since $dT/dr = k + c/r$, it follows that $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = \frac{d}{dr}(kr + c) = k$.

(b) If temperatures on inner and outer edges of the insulation are T_a and T_b, then

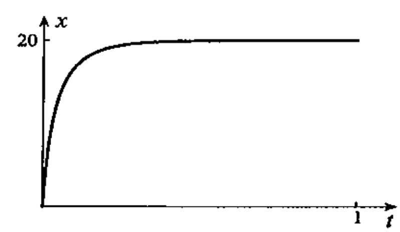
$$T_a = ka + c\ln a + d, \qquad T_b = kb + c\ln b + d.$$

These can be solved for $c = \dfrac{T_b - T_a + k(a-b)}{\ln(b/a)}$ and $d = \dfrac{T_a \ln b - T_b \ln a + k(b \ln a - a \ln b)}{\ln(b/a)}$.

33. (a) If we substitute $x(t)$ into the left and right sides of the differential equation, we obtain

$$\begin{aligned}
\frac{dx}{dt} &= \frac{(3-2e^{-10t})(-60e^{-10t})(-10) - 60(1-e^{-10t})(-2e^{-10t})(-10)}{(3-2e^{-10t})^2}\\
&= \frac{600e^{-10t}(3 - 2e^{-10t} - 2 + 2e^{-10t})}{(3-2e^{-10t})^2} = \frac{600e^{-10t}}{(3-2e^{-10t})^2}\\
(20-x)(30-x) &= \left[20 - \frac{60(1-e^{-10t})}{3-2e^{-10t}}\right]\left[30 - \frac{60(1-e^{-10t})}{3-2e^{-10t}}\right]\\
&= \frac{[60 - 40e^{-10t} - 60(1-e^{-10t})][90 - 60e^{-10t} - 60(1-e^{-10t})]}{(3-2e^{-10t})^2}\\
&= \frac{(20e^{-10t})(30)}{(3-2e^{-10t})^2} = \frac{600e^{-10t}}{(3-2e^{-10t})^2}
\end{aligned}$$

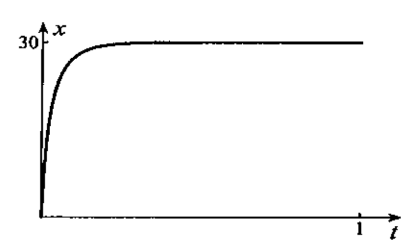
(b) $\lim_{t\to\infty} x(t) = 20$ This is reasonable since 10 g of A will eventually combine with 10 g of B to produce 20 g of C.



34. (a) If we substitute $x(t)$ into the left and right sides of the differential equation, we obtain

$$\begin{aligned}
\frac{dx}{dt} &= \frac{(3-2e^{-15t})(-90e^{-15t})(-15) - 90(1-e^{-15t})(-2e^{-15t})(-15)}{(3-2e^{-15t})^2}\\
&= \frac{1350e^{-15t}(3 - 2e^{-15t} - 2 + 2e^{-15t})}{(3-2e^{-15t})^2} = \frac{1350e^{-15t}}{(3-2e^{-15t})^2}\\
(30-x)(45-x) &= \left[30 - \frac{90(1-e^{-15t})}{3-2e^{-15t}}\right]\left[45 - \frac{90(1-e^{-15t})}{3-2e^{-15t}}\right]\\
&= \frac{[90 - 60e^{-15t} - 90(1-e^{-15t})][135 - 90e^{-15t} - 90(1-e^{-15t})]}{(3-2e^{-15t})^2}\\
&= \frac{(30e^{-15t})(45)}{(3-2e^{-15t})^2} = \frac{1350e^{-15t}}{(3-2e^{-15t})^2}
\end{aligned}$$

(b) $\lim_{t\to\infty} x(t) = 30$ This is reasonable since 10 g of A will eventually combine with 20 g of B to produce 30 g of C.

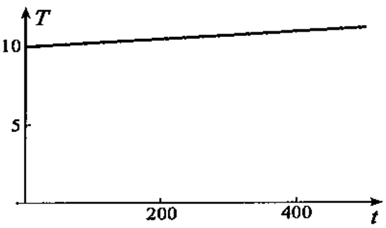


35. (a) The energy balance equation is

$$(4190)(10)\left(\frac{100}{t+1}\right) + 2000 = (4190)(T)\left(\frac{100}{t+1}\right) + (4190)(100)\frac{dT}{dt} \quad\Longrightarrow\quad \frac{dT}{dt} + \frac{T}{t+1} = \frac{10}{t+1} + \frac{2}{419}.$$

(b) If we substitute $T(t)$ into the left side of the differential equation,

$$\begin{aligned}\frac{dT}{dt}+\frac{T}{t+1} &= \left[\frac{(t+1)(4190)-(4190t+4189)(1)}{419(t+1)^2}+\frac{1}{419}\right]\\ &\quad+\frac{1}{t+1}\left[\frac{4190t+4189}{419(t+1)}+\frac{t+1}{419}\right]\\ &= \frac{1}{419(t+1)^2}+\frac{1}{419}+\frac{4190t+4189}{419(t+1)^2}+\frac{1}{419}\\ &= \frac{10}{t+1}+\frac{2}{419}.\end{aligned}$$



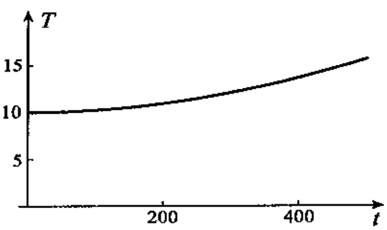
The value of the function at $t=0$ is $T(0)=10$.

36. (a)The energy balance equation is

$$1257+20t=\frac{1257T}{10}+419000\frac{dT}{dt} \quad\Longrightarrow\quad \frac{dT}{dt}+\frac{3T}{10\,000}=\frac{3}{1000}+\frac{t}{20\,950}.$$

(b) If we substitute $T(t)$ into the left side of the differential equation,

$$\begin{aligned}\frac{dT}{dt}+\frac{3T}{10\,000} &= \frac{200}{1257}+\frac{2\,000\,000}{3771}\left(-\frac{3}{10\,000}\right)e^{-3t/10\,000}\\ &\quad+\frac{3}{10\,000}\left(\frac{200t}{1257}-\frac{1\,962\,290}{3771}+\frac{2\,000\,000}{3771}e^{-3t/10\,000}\right)\\ &= \frac{3}{1000}+\frac{t}{20950}.\end{aligned}$$



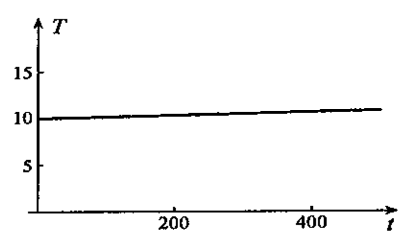
The value of the function at $t=0$ is $T(0)=10$.

37. (a) The energy balance equation is

$$(4190)\left(\frac{3}{100}\right)(10e^{-t})+2000=(4190)(T)\left(\frac{3}{100}\right)+(4190)(100)\frac{dT}{dt} \implies \frac{dT}{dt}+\frac{3T}{10\,000}=\frac{2}{419}+\frac{3}{1000}e^{-t}.$$

(b) If we substitute $T(t)$ into the left side of the differential equation,

$$\begin{aligned}\frac{dT}{dt}+\frac{3T}{10\,000} &= -\frac{74\,240\,000}{12\,566\,229}\left(-\frac{3}{10\,000}\right)e^{-3t/10\,000}+\frac{30}{9997}e^{-t}\\ &\quad+\frac{3}{10\,000}\left(\frac{20\,000}{1257}-\frac{74\,240\,000}{12\,566\,229}e^{-3t/10\,000}-\frac{30}{9997}e^{-t}\right)\\ &= \frac{2}{419}+\frac{3}{1000}e^{-t}.\end{aligned}$$



The value of the function at $t=0$ is $T(0)=10$.

38. We calculate:

$$\frac{dx}{dt}=-e^{-t}(\sin 2t-\cos 2t)+e^{-t}(2\cos 2t+2\sin 2t)=e^{-t}(\sin 2t+3\cos 2t)$$

and

$$\frac{d^2x}{dt^2}=-e^{-t}(\sin 2t+3\cos 2t)+e^{-t}(2\cos 2t-6\sin 2t)=e^{-t}(-7\sin 2t-\cos 2t).$$

Consequently,

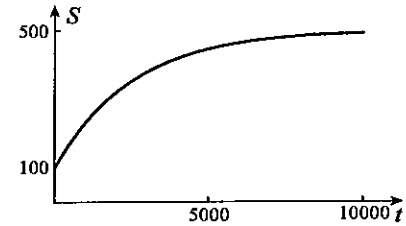
$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^{-t}(-7\sin 2t-\cos 2t)+2e^{-t}(\sin 2t+3\cos 2t)+5e^{-t}(\sin 2t-\cos 2t)=0.$$

In addition, $x(0)=-1$ and $x'(0)=3$.

39. The function satisfies the differential equation since

$$\frac{dS}{dt}+\frac{S}{2500}=\frac{-400}{-2500}e^{-t/2500}+\frac{100}{2500}(5-4e^{-t/2500})=\frac{1}{5}$$

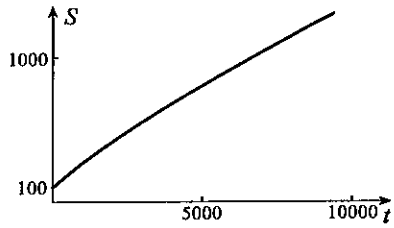
It also satisfies $S(0) = 100$.
The graph is asymptotic to the line $S = 500$.
We would expect this since the concentration of salt in the tank should approach that of the incoming solution, namely, 0.05 kg/L.
In a tank with 10 000 L, this would imply 500 kg of salt.



40. (a) Since solution is added at 4 L/s and removed at 2 L/s, the amount of solution in the tank is $10\,000+2t$. In this case,
$$\frac{dS}{dt} = \frac{1}{5} - \frac{2S}{10\,000 + 2t},$$
subject to the condition that $S(0) = 100$.

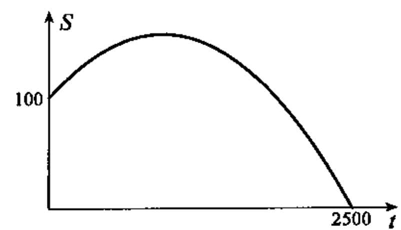
(b) $$\frac{dS}{dt} + \frac{S}{5000+t} = \frac{1}{10} + \frac{2 \times 10^6}{(5000+t)^2} + \frac{1}{5000+t}\left(500 + \frac{t}{10} - \frac{2\times 10^6}{5000+t}\right) = \frac{1}{5}$$

The graph is asymptotic to the line $S = 500 + t/10$.



41. (a) Since solution is added at 4 L/s and removed at 8 L/s, the amount of solution in the tank is $10\,000-4t$.
In this case, $\dfrac{dS}{dt} = \dfrac{1}{5} - \dfrac{8S}{10\,000-4t}$, subject to the condition that $S(0) = 100$.

(b) $$\begin{aligned}\frac{dS}{dt} + \frac{2S}{2500-t} &= \frac{3}{25} - \frac{2t}{15\,625} + \frac{2}{2500-t}\left(100 + \frac{3t}{25} - \frac{t^2}{15\,625}\right)\\ &= \frac{3}{25} - \frac{2t}{15\,625} + \frac{2}{2500-t}\left(\frac{1\,562\,500 + 1875t - t^2}{15\,625}\right)\\ &= \frac{3}{25} - \frac{2t}{15\,625} + \frac{2(625+t)(2500-t)}{15\,625(2500-t)} = \frac{1}{5}\end{aligned}$$

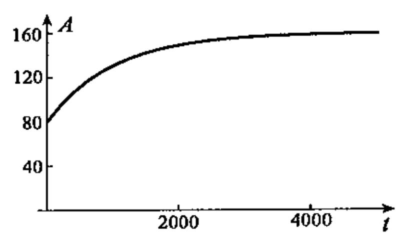


This solution is valid as long as there is solution in the tank. This is for $0 \le t \le 2500$. The parabola has maximum value $625/4$ kg at $t = 1875/2$.

42. (a) Since alcohol enters the vat at the rate of $4/25$ L/s and leaves at the rate of $2(A/2000)$, it follows that $\dfrac{dA}{dt} = \dfrac{4}{25} - \dfrac{A}{1000}$. We must also have $A(0) = 80$.

(b) $$\frac{dA}{dt} + \frac{A}{1000} = \frac{80}{1000}e^{-t/1000} + \frac{1}{1000}(160 - 80e^{-t/1000}) = \frac{4}{25}$$
A plot of the function is shown to the right.
It is asymptotic to the line $A = 160$.

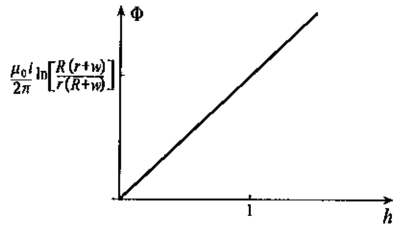
(c) The beer is 5% alcohol when
$$\frac{5}{100}(2000) = 160 - 80e^{-t/1000}.$$
This implies that $e^{-t/1000} = 3/4$, from which $t = 1000 \ln(4/3)$.

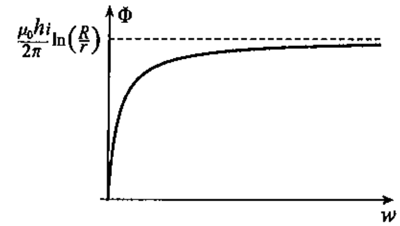


43. First we express M in the form $M = \dfrac{\mu_0 l}{2\pi} \ln\left[\dfrac{(s+w)(s+W)}{s(s+w+W)}\right]$.

Then,
$$\begin{aligned}\frac{dM}{ds} &= \frac{\mu_0 l}{2\pi}\frac{s(s+w+W)}{(s+w)(s+W)}\left[\frac{s(s+w+W)(2s+w+W) - (s+w)(s+W)(2s+w+W)}{s^2(s+w+W)^2}\right]\\ &= \frac{-\mu_0 lwW(2s+w+W)}{2\pi s(s+w)(s+W)(s+w+W)} < 0.\end{aligned}$$

44. The function is linear in h (left figure below). To draw the right graph, we express the function in the form $\Phi = \dfrac{\mu_0 hi}{2\pi}\left[\ln\left(\dfrac{R}{r}\right) + \ln\left(\dfrac{r+w}{R+w}\right)\right]$. It has value 0 at $w = 0$; it is increasing; and $\lim\limits_{w\to\infty} \Phi = \dfrac{\mu_0 hi}{2\pi}\ln\left(\dfrac{R}{r}\right)$.





45. Since $\dfrac{dy}{dx} = A + B\left\{\dfrac{1}{2}\ln\left(\dfrac{x-1}{x+1}\right) + \dfrac{x}{2}\left(\dfrac{x+1}{x-1}\right)\left[\dfrac{(x+1)-(x-1)}{(x+1)^2}\right]\right\} = A + \dfrac{B}{2}\ln\left(\dfrac{x-1}{x+1}\right) + \dfrac{Bx}{x^2-1}$,
and $\dfrac{d^2y}{dx^2} = \dfrac{B}{2}\left(\dfrac{x+1}{x-1}\right)\left[\dfrac{(x+1)-(x-1)}{(x+1)^2}\right] + B\dfrac{(x^2-1)-x(2x)}{(x^2-1)^2} = \dfrac{B}{x^2-1} - \dfrac{B(x^2+1)}{(x^2-1)^2} = \dfrac{-2B}{(x^2-1)^2}$,
it follows that

$$\begin{aligned}(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y &= (1-x^2)\left[\frac{-2B}{(x^2-1)^2}\right] - 2x\left[A + \frac{B}{2}\ln\left(\frac{x-1}{x+1}\right) + \frac{Bx}{x^2-1}\right]\\ &\quad + 2\left[Ax + \frac{Bx}{2}\ln\left(\frac{x-1}{x+1}\right) + B\right]\\ &= \frac{2B}{x^2-1} - 2Ax - Bx\ln\left(\frac{x-1}{x+1}\right) - \frac{2Bx^2}{x^2-1}\\ &\quad + 2Ax + Bx\ln\left(\frac{x-1}{x+1}\right) + 2B\\ &= 0.\end{aligned}$$

46. (a) If we multiply the equation by e^y, then $(e^y)^2 - x\,e^y - 1 = 0 \implies e^y = \dfrac{x \pm \sqrt{x^2+4}}{2}$. Since e^y must be positive, we choose $e^y = \dfrac{x+\sqrt{x^2+4}}{2} \implies y = \ln\left(\dfrac{x+\sqrt{x^2+4}}{2}\right)$. Consequently,

$$\frac{dy}{dx} = \frac{2}{x+\sqrt{x^2+4}}\left(\frac{1}{2}\right)\left(1 + \frac{x}{\sqrt{x^2+4}}\right) = \frac{1}{x+\sqrt{x^2+4}}\left(\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}\right) = \frac{1}{\sqrt{x^2+4}}.$$

(b) If we differentiate implicity with respect to x, then $1 = e^y\dfrac{dy}{dx} + e^{-y}\dfrac{dy}{dx}$, from which $\dfrac{dy}{dx} = \dfrac{1}{e^y + e^{-y}}$. To show that this is the same derivative as in part (a), we note that

$$\begin{aligned}e^y + e^{-y} &= \frac{x+\sqrt{x^2+4}}{2} + \frac{2}{x+\sqrt{x^2+4}} = \frac{x+\sqrt{x^2+4}}{2} + \frac{2}{x+\sqrt{x^2+4}}\left(\frac{x-\sqrt{x^2+4}}{x-\sqrt{x^2+4}}\right)\\ &= \frac{x+\sqrt{x^2+4}}{2} + \frac{2(x-\sqrt{x^2+4})}{x^2-(x^2+4)} = \sqrt{x^2+4}.\end{aligned}$$

This shows that $1/(e^y + e^{-y}) = 1/\sqrt{x^2+4}$.

47. (a) The rate at which energy enters the tank in incoming liquid is $\dot{m}c_pT_0$. The rate at which it leaves in outgoing liquid is $\dot{m}c_pT$. The rate at which energy is used to raise the temperature of the liquid in the tank is $Mc_p(dT/dt)$. Consequently, the energy balance equation becomes

$$\dot{m}c_pT_0 + q = \dot{m}c_pT + Mc_p\frac{dT}{dt}.$$

(b) When $\dot{m}$, T_0 and q are all constants,

$$Mc_p\frac{dT}{dt} + \dot{m}c_pT = Mc_p\left[-\frac{\dot{m}T_0}{M}e^{-\dot{m}t/M} + \left(T_0 + \frac{q}{c_p\dot{m}}\right)\left(\frac{\dot{m}}{M}\right)e^{-\dot{m}t/M}\right]$$

$$+\dot{m}c_p\left[T_0 e^{-\dot{m}t/M}+\left(T_0+\frac{q}{c_p\dot{m}}\right)\left(1-e^{-\dot{m}t/M}\right)\right]$$

$$=e^{-\dot{m}t/M}\left[-\dot{m}c_pT_0+\dot{m}c_p\left(T_0+\frac{q}{c_p\dot{m}}\right)+\dot{m}c_pT_0-\dot{m}c_p\left(T_0+\frac{q}{c_p\dot{m}}\right)\right]$$

$$+\dot{m}c_p\left(T_0+\frac{q}{c_p\dot{m}}\right)$$

$$=q+\dot{m}c_pT_0.$$

48. Implicit differentiation gives $\dfrac{1}{x^2+y^2}\left(2x+2y\dfrac{dy}{dx}\right)=\dfrac{2}{1+(y^2/x^2)}\left(\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{y}{x^2}\right)$, from which

$$\frac{1}{x^2+y^2}\left(x+y\frac{dy}{dx}\right)=\frac{1}{x^2+y^2}\left(x\frac{dy}{dx}-y\right)\quad\Longrightarrow\quad\frac{dy}{dx}=\frac{x+y}{x-y}.$$

A second differentiation now gives

$$\frac{d^2y}{dx^2}=\frac{(x-y)\left(1+\dfrac{dy}{dx}\right)-(x+y)\left(1-\dfrac{dy}{dx}\right)}{(x-y)^2}=\frac{(x-y)\left(1+\dfrac{x+y}{x-y}\right)-(x+y)\left(1-\dfrac{x+y}{x-y}\right)}{(x-y)^2}$$

$$=\frac{2(x^2+y^2)}{(x-y)^3}.$$

49. (a)We divide the discussion into two cases. Since n is rational, we set $n=a/b$, where a and b are relatively prime. For x^n to be defined for $x<0$, b must be odd.
Case 1 (a is even): In this case, $y=x^n=(-x)^n=e^{n\,\ln(-x)}$, and

$$\frac{dy}{dx}=e^{n\,\ln(-x)}\left(\frac{n}{x}\right)=(-x)^n\left(\frac{n}{x}\right)=x^n\left(\frac{n}{x}\right)=nx^{n-1}.$$

Case 11 (a is odd): In this case, $y=-(-x)^n=-e^{n\,\ln(-x)}$, and

$$\frac{dy}{dx}=-e^{n\,\ln(-x)}\left(\frac{n}{x}\right)=-(-x)^n\left(\frac{n}{x}\right)=n(-x)^{n-1}.$$

Because $n-1=a/b-1=(a-b)/b$, and $a-b$ is even, it follows that $(-x)^{n-1}=x^{n-1}$, and $dy/dx=nx^{n-1}$.
(b) Since x^n is not defined for $x<0$ when n is irrational, $f'(0)$ does not exist for irrational n. When n is rational and we set $n=a/b$ as in part (a), x^n is still defined for $x<0$ only when b is odd. In such a case,

$$f'(0)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0}\frac{h^{a/b}}{h}=\lim_{h\to0}h^{a/b-1}=\begin{cases}0, & \text{if } a/b\geq1 \text{ or } a=0\\ 1, & \text{if } a/b=1\\ \text{does not exist}, & \text{if } a/b<1\end{cases}.$$

Thus, $f'(0)$ exists and is equal to zero when $n=0$, or when $n=a/b$, where b is odd and $a/b\geq1$. When $n=1$, $f'(0)=1$, and in all other cases, $f'(0)$ does not exist.

EXERCISES 3.12

1. If we take natural logarithms of $y=x^{-x}$, then $\ln y=-x\,\ln x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx}=-\ln x-x\left(\frac{1}{x}\right)\quad\Longrightarrow\quad\frac{dy}{dx}=-x^{-x}(1+\ln x).$$

2. Natural logarithms of $y=x^{4\cos x}$ give $\ln y=4\cos x\,\ln x$, and differentiation with respect to x yields

$$\frac{1}{y}\frac{dy}{dx}=4\left(-\sin x\,\ln x+\frac{1}{x}\cos x\right)\quad\Longrightarrow\quad\frac{dy}{dx}=4x^{4\cos x}\left(\frac{1}{x}\cos x-\sin x\,\ln x\right).$$

3. If we take natural logarithms of $y = x^{4x}$, then $\ln y = 4x \ln x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = 4\ln x + 4x\left(\frac{1}{x}\right) \quad \Longrightarrow \quad \frac{dy}{dx} = 4x^{4x}(1+\ln x).$$

4. Natural logarithms of $y = (\sin x)^x$ give $\ln y = x\ln(\sin x)$ and differentiation with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + x\frac{\cos x}{\sin x} \quad \Longrightarrow \quad \frac{dy}{dx} = (\sin x)^x[\ln(\sin x) + x\cot x].$$

5. If we take natural logarithms of $y = \left(1+\dfrac{1}{x}\right)^x$, then $\ln y = x\ln\left(1+\dfrac{1}{x}\right)$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right) + x\left(\frac{x}{x+1}\right)\left(\frac{-1}{x^2}\right) \Longrightarrow \frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x\left[\frac{-1}{x+1} + \ln\left(1+\frac{1}{x}\right)\right].$$

6. Natural logarithms of $y = \left(1+\dfrac{1}{x}\right)^{x^2}$ give $\ln y = x^2\ln\left(1+\dfrac{1}{x}\right) = x^2[\ln(x+1) - \ln x]$, and differentiation with respect to x leads to $\dfrac{1}{y}\dfrac{dy}{dx} = 2x[\ln(x+1) - \ln x] + x^2\left(\dfrac{1}{x+1} - \dfrac{1}{x}\right)$. Thus,

$$\frac{dy}{dx} = xy\left[2\ln\left(1+\frac{1}{x}\right) + x\left(\frac{x-(x+1)}{x(x+1)}\right)\right] = x\left(1+\frac{1}{x}\right)^{x^2}\left[2\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right].$$

7. If we take natural logarithms of $y = (1/x)^{1/x}$, then $\ln y = (1/x)\ln(1/x) = -(1/x)\ln x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2}\ln x - \left(\frac{1}{x}\right)\left(\frac{1}{x}\right) \quad \Longrightarrow \quad \frac{dy}{dx} = \left(\frac{1}{x}\right)^{1/x}\left[\frac{1}{x^2}(-1+\ln x)\right] = \left(\frac{1}{x}\right)^{2+1/x}(-1+\ln x).$$

8. If we take natural logarithms of $y = \left(\dfrac{2}{x}\right)^{3/x}$, then $\ln y = \dfrac{3}{x}\ln\left(\dfrac{2}{x}\right) = \dfrac{3}{x}(\ln 2 - \ln x)$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = -\frac{3}{x^2}(\ln 2 - \ln x) + \frac{3}{x}\left(-\frac{1}{x}\right) \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{3y}{x^2}\left[\ln\left(\frac{2}{x}\right) + 1\right] = -\frac{3}{x^2}\left(\frac{2}{x}\right)^{3/x}\left[1 + \ln\left(\frac{2}{x}\right)\right].$$

9. If we take natural logarithms of $y = (\sin x)^{\sin x}$, then $\ln y = \sin x \ln\sin x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \cos x\ln\sin x + \sin x\left(\frac{\cos x}{\sin x}\right) \quad \Longrightarrow \quad \frac{dy}{dx} = (\sin x)^{\sin x}\cos x(1+\ln\sin x).$$

10. When we take natural logarithms of $y = (\ln x)^{\ln x}$, we find $\ln y = \ln x\ln(\ln x)$, and differentiation with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\ln x) + \ln x\left(\frac{1}{x\ln x}\right) \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{y}{x}[\ln(\ln x) + 1] = \frac{1}{x}(\ln x)^{\ln x}[1 + \ln(\ln x)].$$

11. If we take natural logarithms of $y = (x^2+3x^4)^3(x^2+5)^4$, then $\ln y = 3\ln(x^2+3x^4) + 4\ln(x^2+5)$, and differentiation with respect to x gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{3(2x+12x^3)}{x^2+3x^4} + \dfrac{4(2x)}{x^2+5}$. Thus,

$$\begin{aligned}\frac{dy}{dx} &= (x^2+3x^4)^3(x^2+5)^4\left[\frac{3(12x^5+62x^3+10x)+(24x^5+8x^3)}{(x^2+5)(x^2+3x^4)}\right]\\ &= 2x(x^2+3x^4)^2(x^2+5)^3(30x^4+97x^2+15).\end{aligned}$$

12. If we take natural logarithms of $y = \dfrac{\sqrt{x}(1+2x^2)}{\sqrt{1+x^2}}$, then $\ln y = \dfrac{1}{2}\ln x + \ln(1+2x^2) - \dfrac{1}{2}\ln(1+x^2)$. Differentiation with respect to x gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2x} + \dfrac{4x}{1+2x^2} - \dfrac{x}{1+x^2}$, and therefore

$$\frac{dy}{dx} = y\left[\frac{(1+2x^2)(1+x^2)+8x^2(1+x^2)-2x^2(1+2x^2)}{2x(1+2x^2)(1+x^2)}\right] = \frac{6x^4+9x^2+1}{2\sqrt{x}(1+x^2)^{3/2}}.$$

13. If we take natural logarithms of $|y| = |x|\sqrt[3]{1-\sin x}$, then $\ln|y| = \ln|x| + (1/3)\ln(1-\sin x)$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{-\cos x}{3(1-\sin x)}$. Thus,

$$\frac{dy}{dx} = x\sqrt[3]{1-\sin x}\left[\frac{1}{x} - \frac{\cos x}{3(1-\sin x)}\right] = \frac{3-3\sin x - x\cos x}{3(1-\sin x)^{2/3}}.$$

14. If we take natural logarithms of $|y| = |x^2+3x|^3(x^2+5)^4$, then $\ln|y| = 3\ln|x^2+3x| + 4\ln(x^2+5)$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = 3\left(\dfrac{2x+3}{x^2+3x}\right) + 4\left(\dfrac{2x}{x^2+5}\right)$, and therefore

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{3(2x+3)(x^2+5)+8x(x^2+3x)}{(x^2+3x)(x^2+5)}\right] = (x^2+3x)^3(x^2+5)^4\left[\frac{14x^3+33x^2+30x+45}{(x^2+3x)(x^2+5)}\right]\\ &= (x^2+3x)^2(x^2+5)^3(14x^3+33x^2+30x+45).\end{aligned}$$

15. If we take natural logarithms of $|y| = |x|^2e^{4x}$, then $\ln|y| = 2\ln|x| + 4x$. Differentiation with respect to x using formula 3.46 gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + 4 \quad\Longrightarrow\quad \frac{dy}{dx} = x^2e^{4x}\left(\frac{2}{x}+4\right) = 2x(2x+1)e^{4x}.$$

16. When we take natural logarithms of $y = x^{3/2}e^{-2x}$, we have $\ln y = (3/2)\ln x - 2x$, and differentiation with respect to x results in

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{2x} - 2 \quad\Longrightarrow\quad \frac{dy}{dx} = x^{3/2}e^{-2x}\left(\frac{3-4x}{2x}\right) = \frac{1}{2}\sqrt{x}(3-4x)e^{-2x}.$$

17. If we take natural logarithms of $|y| = x^2|\ln x|$, then $\ln|y| = 2\ln x + \ln|\ln x|$. Differentiation with respect to x using formula 3.46 gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{\ln x}\left(\frac{1}{x}\right) \quad\Longrightarrow\quad \frac{dy}{dx} = x^2\ln x\left(\frac{2}{x} + \frac{1}{x\ln x}\right) = x(1+2\ln x).$$

18. Natural logarithms of $|y| = \dfrac{e^x}{|\ln(x-1)|}$ give $\ln|y| = x - \ln|\ln(x-1)|$. Differentiation with respect to x now gives $\dfrac{1}{y}\dfrac{dy}{dx} = 1 - \dfrac{1}{(x-1)\ln(x-1)}$. Thus, $\dfrac{dy}{dx} = \dfrac{e^x}{\ln(x-1)}\left[1 - \dfrac{1}{(x-1)\ln(x-1)}\right]$.

19. If we take natural logarithms of $|y| = |x^3+3|^3|x^2-2x|$, then $\ln|y| = 3\ln|x^3+3| + \ln|x^2-2x|$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = 3\left(\dfrac{3x^2}{x^3+3}\right) + \dfrac{2x-2}{x^2-2x}$, and therefore

$$\frac{dy}{dx} = (x^3+3)^3(x^2-2x)\left[\frac{9x^2(x^2-2x)+(2x-2)(x^3+3)}{(x^3+3)(x^2-2x)}\right] = (x^3+3)^2(11x^4-20x^3+6x-6).$$

20. If we take natural logarithms of $|y| = \dfrac{\sqrt{x}|1-x^2|}{\sqrt{1+x^2}}$, then $\ln|y| = \dfrac{1}{2}\ln x + \ln|1-x^2| - \dfrac{1}{2}\ln(1+x^2)$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2x} - \dfrac{2x}{1-x^2} - \dfrac{x}{1+x^2}$, and therefore

$$\frac{dy}{dx} = \frac{\sqrt{x}(1-x^2)}{\sqrt{1+x^2}}\left[\frac{(1-x^2)(1+x^2) - 4x^2(1+x^2) - 2x^2(1-x^2)}{2x(1-x^2)(1+x^2)}\right] = \frac{1-6x^2-3x^4}{2\sqrt{x}(1+x^2)^{3/2}}.$$

21. Natural logarithms of $|y| = \dfrac{|x^2-1|}{|x|\sqrt{1-4\tan^2 x}}$ give $\ln|y| = \ln|x^2-1| - \ln|x| - \dfrac{1}{2}\ln(1-4\tan^2 x)$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2x}{x^2-1} - \dfrac{1}{x} + \dfrac{8\tan x\sec^2 x}{2(1-4\tan^2 x)}$, and therefore $\dfrac{dy}{dx} = \left(\dfrac{x^2-1}{x\sqrt{1-4\tan^2 x}}\right)\left(\dfrac{2x}{x^2-1} - \dfrac{1}{x} + \dfrac{4\tan x\sec^2 x}{1-4\tan^2 x}\right)$.

22. Natural logarithms of $|y| = |x|^3|x^2-4x|\sqrt{1+x^3}$ give $\ln|y| = 3\ln|x| + \ln|x^2-4x| + \dfrac{1}{2}\ln(1+x^3)$. Differentiation with respect to x using formula 3.46 results in $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{3}{x} + \dfrac{2x-4}{x^2-4x} + \dfrac{3x^2}{2(1+x^3)}$, and therefore

$$\begin{aligned}\frac{dy}{dx} &= x^3(x^2-4x)\sqrt{1+x^3}\left[\frac{6(x^2-4x)(1+x^3) + 2x(2x-4)(1+x^3) + 3x^3(x^2-4x)}{2x(x^2-4x)(1+x^3)}\right]\\ &= \frac{x^3(13x^4-44x^3+10x-32)}{2\sqrt{1+x^3}}.\end{aligned}$$

23. If we take natural logarithms of $|y| = \dfrac{|\sin 3x|^3}{|\tan 2x|^5}$, then $\ln|y| = 3\ln|\sin 3x| - 5\ln|\tan 2x|$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{9\cos 3x}{\sin 3x} - \dfrac{10\sec^2 2x}{\tan 2x}$. Thus,

$$\frac{dy}{dx} = \left(\frac{\sin^3 3x}{\tan^5 2x}\right)(9\cot 3x - 10\sec^2 2x\cot 2x).$$

24. Natural logarithms of $|y| = \dfrac{|\sin 2x||\sec 5x|}{|1-2\cot x|^3}$ give $\ln|y| = \ln|\sin 2x| + \ln|\sec 5x| - 3\ln|1-2\cot x|$. Differentiation with respect to x using 3.46 yields $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2\cos 2x}{\sin 2x} + \dfrac{5\sec 5x\tan 5x}{\sec 5x} - \dfrac{3(2\csc^2 x)}{1-2\cot x}$, from which $\dfrac{dy}{dx} = \dfrac{\sin 2x\sec 5x}{(1-2\cot x)^3}\left[2\cot 2x + 5\tan 5x - \dfrac{6\csc^2 x}{1-2\cot x}\right]$.

25. If we set $y = u^u$ and take natural logarithms, $\ln y = u\ln u$. Differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}\ln u + u\left(\frac{1}{u}\frac{du}{dx}\right) \implies \frac{dy}{dx} = u^u(1+\ln u)\frac{du}{dx}.$$

26. (a) If we take natural logarithms, $\ln x = a\ln r - b(r+c)$. Differentiation with respect to r gives $\dfrac{1}{x}\dfrac{dx}{dr} = \dfrac{a}{r} - b \implies \dfrac{dx}{dr} = \dfrac{x}{r}(a-br)$. Since $r > a/b$, it follows that the derivative $dx/dr < 0$. But if the slope of the tangent line is always negative, x must decrease as r increases. Therefore x increases as r decreases.

(b) If we differentiate $\ln x = a\ln r - b(r+c)$ with respect to x, $\dfrac{1}{x} = \dfrac{a}{r}\dfrac{dr}{dx} - b\dfrac{dr}{dx} = \left(\dfrac{a-br}{r}\right)\dfrac{dr}{dx}$. Thus,

$$\frac{dr}{dx} = \frac{r}{x(a-br)} \implies \frac{Er}{Ex} = \frac{x}{r}\frac{r}{x(a-br)} = \frac{1}{a-br}.$$

EXERCISES 3.13

1. $\dfrac{dy}{dx} = -\text{csch}(2x+3)\coth(2x+3)(2) = -2\,\text{csch}(2x+3)\coth(2x+3)$

2. $\dfrac{dy}{dx} = \sinh(x/2) + x\cosh(x/2)(1/2) = \sinh(x/2) + (x/2)\cosh(x/2)$

3. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1-\text{sech}x}}(\text{sech}x\ \tanh x) = \dfrac{\text{sech}x\ \tanh x}{2\sqrt{1-\text{sech}x}}$

4. $\dfrac{dy}{dx} = \text{sech}^2(\ln x)\left(\dfrac{1}{x}\right)$

5. If we differentiate with respect to x, we obtain $\sinh(x+y)\left(1+\dfrac{dy}{dx}\right) = 2 \Longrightarrow \dfrac{dy}{dx} = \dfrac{2-\sinh(x+y)}{\sinh(x+y)}$.

6. Differentiation with respect to x gives $\dfrac{dy}{dx} - \text{csch}^2 x = \dfrac{1}{2\sqrt{1+y}}\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{\text{csch}^2 x}{1-\dfrac{1}{2\sqrt{1+y}}} = \frac{2\sqrt{1+y}\,\text{csch}^2 x}{2\sqrt{1+y}-1}.$$

7. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (u\sinh u + \cosh u)(e^x - e^{-x}) = 2\sinh x(u\sinh u + \cosh u)$

8. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = [\sec^2(\cosh t)\ \sinh t][-\sin(\tanh x)\text{sech}^2 x] = -\sinh t\,\text{sech}^2 x\ \sec^2(\cosh t)\ \sin(\tanh x)$

9. $\dfrac{dy}{dx} = \dfrac{1}{1+\sinh^2 x}(\cosh x) = \dfrac{\cosh x}{\cosh^2 x} = \text{sech}x$

10. Since $y = (1/2)\ln(\tanh 2x)$,

$$\frac{dy}{dx} = \frac{1}{2\tanh 2x}\text{sech}^2 2x(2) = \frac{1}{\cosh^2 2x}\frac{\cosh 2x}{\sinh 2x} = \frac{1}{\sinh 2x\ \cosh 2x} = \frac{1}{(\sinh 4x)/2} = 2\,\text{csch}4x.$$

11. $\dfrac{d}{dx}\sinh u = \dfrac{d}{du}\left(\dfrac{e^u - e^{-u}}{2}\right)\dfrac{du}{dx} = \dfrac{e^u + e^{-u}}{2}\dfrac{du}{dx} = \cosh u\,\dfrac{du}{dx}$

$\dfrac{d}{dx}\cosh u = \dfrac{d}{du}\left(\dfrac{e^u + e^{-u}}{2}\right)\dfrac{du}{dx} = \dfrac{e^u - e^{-u}}{2}\dfrac{du}{dx} = \sinh u\,\dfrac{du}{dx}$

$\dfrac{d}{du}\tanh u = \dfrac{d}{du}\left(\dfrac{\sinh u}{\cosh u}\right)\dfrac{du}{dx} = \dfrac{\cosh u\ \cosh u - \sinh u\ \sinh u}{\cosh^2 u}\dfrac{du}{dx} = \dfrac{1}{\cosh^2 u}\dfrac{du}{dx} = \text{sech}^2 u\,\dfrac{du}{dx}$

Similarly, $\dfrac{d}{dx}\coth u = -\text{csch}^2 u\,\dfrac{du}{dx}$.

$\dfrac{d}{dx}\text{sech}u = \dfrac{d}{du}\left(\dfrac{1}{\cosh u}\right)\dfrac{du}{dx} = \dfrac{-1}{\cosh^2 u}\sinh u\,\dfrac{du}{dx} = -\dfrac{\sinh u}{\cosh u}\dfrac{1}{\cosh u}\dfrac{du}{dx} = -\text{sech}u\ \tanh u\,\dfrac{du}{dx}$

Similarly, $\dfrac{d}{dx}\text{csch}u = -\text{csch}u\,\coth u\,\dfrac{du}{dx}$.

12. (a)
$$\begin{aligned}
f'(x) &= -Ak\sin kx + Bk\cos kx + Ck\sinh kx + Dk\cosh kx\\
f''(x) &= -Ak^2\cos kx - Bk^2\sin kx + Ck^2\cosh kx + Dk^2\sinh kx\\
f'''(x) &= Ak^3\sin kx - Bk^3\cos kx + Ck^3\sinh kx + Dk^3\cosh kx\\
f''''(x) &= Ak^4\cos kx + Bk^4\sin kx + Ck^4\cosh kx + Dk^4\sinh kx
\end{aligned}$$

Thus, $\dfrac{d^4y}{dx^4} - k^4 y = 0$.

(b) These conditions imply that

$$\begin{aligned}
0 &= f(0) = A + C,\\
0 &= f'(0) = Bk + Dk = k(B + D),\\
0 &= f(L) = A\cos kL + B\sin kL + C\cosh kL + D\sinh kL,\\
0 &= f''(L) = -Ak^2\cos kL - Bk^2\sin kL + Ck^2\cosh kL + Dk^2\sinh kL.
\end{aligned}$$

Thus, $C = -A$, $D = -B$, and

$$\begin{aligned}
0 &= A\cos kL + B\sin kL - A\cosh kL - B\sinh kL = A(\cos kL - \cosh kL) + B(\sin kL - \sinh kL);\\
0 &= -A\cos kL - B\sin kL - A\cosh kL - B\sinh kL = A(\cos kL + \cosh kL) + B(\sin kL + \sinh kL).
\end{aligned}$$

(c) If we write the conditions in part (b) in the form

$$A(\cos kL - \cosh kL) = -B(\sin kL - \sinh kL), \quad A(\cos kL + \cosh kL) = -B(\sin kL + \sinh kL),$$

and divide one by the other,

$$\frac{\cos kL - \cosh kL}{\cos kL + \cosh kL} = \frac{\sin kL - \sinh kL}{\sin kL + \sinh kL}.$$

Hence,

$$(\cos kL - \cosh kL)(\sin kL + \sinh kL) = (\cos kL + \cosh kL)(\sin kL - \sinh kL)$$

or, $2\cos kL\sinh kL = 2\sin kL\cosh kL$. Division by $2\cos kL\cosh kL$ gives $\tanh kL = \tan kL$.

13. If we set $y = \operatorname{Sinh}^{-1}x$, then $x = \sinh y$, and differentiation gives $1 = \cosh y\,\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

If we set $y = \operatorname{Tanh}^{-1}x$, then $x = \tanh y$, and differentiation gives $1 = \operatorname{sech}^2 y\,\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}.$$

If we set $y = \operatorname{Sech}^{-1}x$, then $x = \operatorname{sech} y$, and differentiation gives $1 = -\operatorname{sech} y\,\tanh y\,\dfrac{dy}{dx}$, from which

$$\frac{dy}{dx} = \frac{-1}{\operatorname{sech} y\,\tanh y} = \frac{-1}{\operatorname{sech} y\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}.$$

Derivations of the other derivatives are similar.

EXERCISES 3.14

1. Since $f(x)$ and $f'(x)$ are continuous on $-3 \le x \le 2$, we may apply the mean value theorem on this interval. Equation 3.49 states that $2c + 2 = \dfrac{8 - 3}{2 + 3} = 1$. The only solution is $c = -1/2$.

2. Since $f(x)$ and $f'(x)$ are continuous on $1 \le x \le 3$, we may apply the mean value theorem on this interval. Equation 3.49 states that $3 - 4c = \dfrac{-5 - 5}{3 - 1} = -5$. The only solution is $c = 2$.

3. Since $f(x)$ and $f'(x)$ are continuous on $2 \le x \le 3$, we may apply the mean value theorem on this interval. Equation 3.49 states that $1 = \dfrac{8 - 7}{3 - 2} = 1$. All x's in the interval satisfy this equation.

4. Since $f'(x)$ is not defined at $x = 0$, we cannot apply the mean value theorem on the interval $-1 \le x \le 1$.

5. Since $f(x)$ is continuous on $0 \le x \le 1$, and $f'(x)$ is continuous on $0 < x < 1$, we may apply the mean value theorem on this interval. Equation 3.49 states that $1 = \dfrac{1-0}{1-0} = 1$. All x's in the interval satisfy this equation.

6. Since $f(x)$ and $f'(x)$ are continuous on $-3 \le x \le 2$, we may apply the mean value theorem on this interval. Equation 3.49 states that $3c^2 + 4c - 1 = \dfrac{12+8}{2+3} = 4$. Of the two solutions $c = (-2 \pm \sqrt{19})/3$, only $(-2+\sqrt{19})/3$ is within the interval $-3 \le x \le 2$.

7. Since $f(x)$ and $f'(x)$ are continuous on $-1 \le x \le 2$, we may apply the mean value theorem on this interval. Equation 3.49 states that $3c^2 + 4c - 1 = \dfrac{12-0}{2+1} = 4$. Of the two solutions $c = (-2 \pm \sqrt{19})/3$ of this equation, only $c = (\sqrt{19}-2)/3$ lies in the given interval.

8. Since $f(x)$ and $f'(x)$ are continuous on $2 \le x \le 4$, we may apply the mean value theorem on this interval. Equation 3.49 states that

$$\left[\frac{(x-1)(1)-(x+2)(1)}{(x-1)^2}\right]_{|x=c} = \frac{2-4}{4-2}; \text{ that is,} \qquad \frac{-3}{(c-1)^2} = -1.$$

Of the two solutions $c = 1 \pm \sqrt{3}$ of this equation, only $c = 1+\sqrt{3}$ lies in the given interval.

9. Since $f(x)$ is not defined at $x = -2$, the mean value theorem cannot be applied on the interval $-3 \le x \le 2$.

10. Since $f(x)$ and $f'(x)$ are continuous on $-2 \le x \le 3$, we may apply the mean value theorem on this interval. Equation 3.49 states that

$$\left[\frac{(x+3)(2x)-x^2(1)}{(x+3)^2}\right]_{|x=c} = \frac{3/2-4}{3+2}; \text{ that is,} \qquad \frac{c^2+6c}{(c+3)^2} = -\frac{1}{2}.$$

Of the two solutions $c = -3 \pm \sqrt{6}$ of this equation, only $c = -3+\sqrt{6}$ lies in the given interval.

11. Since $f(x)$ and $f'(x)$ are continuous on $0 \le x \le 2\pi$, we may apply the mean value theorem on this interval. Equation 3.49 states that $\cos c = \dfrac{0-0}{2\pi - 0} = 0$. Solutions of this equation in the given interval are $c = \pi/2, 3\pi/2$.

12. Since $\ln(2x+1)$ and its derivative are continuous on $0 \le x \le 2$, we may apply the mean value theorem. Equation 3.49 states that $\dfrac{2}{2c+1} = \dfrac{\ln 5 - \ln 1}{2} = \dfrac{1}{2}\ln 5$. The solution of this equation is $c = (4-\ln 5)/(2\ln 5)$.

13. Since $f(x)$ and $f'(x)$ are continuous on $-1 \le x \le 1$, we may apply the mean value theorem on this interval. Equation 3.49 states that $-e^{-c} = \dfrac{e^{-1}-e^1}{2}$. The only solution is $c = -\ln[(e^2-1)/(2e)]$.

14. Since $\sec x$ is not defined at $x = \pi/2$, the mean value theorem cannot be applied on the interval $0 \le x \le \pi$.

15. Since $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are continuous on $1 \le x \le 2$, and $g'(x) = 1$ is never zero, we may apply Cauchy's generalized mean value theorem. Equation 3.48 states that $\dfrac{4-1}{2-1} = \dfrac{2c}{1}$. The solution is $c = 3/2$.

16. Since $g'(0) = 0$, Cauchy's generalized mean value theorem cannot be applied on the interval $-1 \le x \le 1$.

17. Since $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are continuous on $0 \le x \le -2$, and $g'(x) = 3x^2+5$ is never zero, we may apply Cauchy's generalized mean value theorem. Equation 3.48 states that

$$\frac{9+1}{22-4} = \frac{2c+3}{3c^2+5} \quad \Longrightarrow \quad 15c^2 - 18c - 2 = 0.$$

Of the solutions $c = (9 \pm \sqrt{111})/15$ of this equation, only $c = (9+\sqrt{111})/15$ lies in the interval $0 \le x \le 2$.

18. Since $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are continuous on $-3 \le x \le -2$, and $g'(x) = -1/(x-1)^2$ is never zero, we may apply Cauchy's generalized mean value theorem. Equation 3.48 states that

$$\frac{2-3/2}{2/3-3/4} = \frac{\dfrac{1}{(c+1)^2}}{\dfrac{-1}{(c-1)^2}} \qquad \Longrightarrow \qquad 6 = \left(\frac{c-1}{c+1}\right)^2.$$

Of the two solutions $c = (1 \pm \sqrt{6})/(1 \mp \sqrt{6})$, only $c = (1+\sqrt{6})/(1-\sqrt{6})$ is in the given interval.

19. The fact that $|f'(x)| \le M$ on $a \le x \le b$ implies that $f'(x)$ exists on $a \le x \le b$ and $f(x)$ is continuous for $a \le x \le b$. Consequently, we may apply the mean value theorem to $f(x)$ on the interval, and state that there exists at least one c for which $f(b) - f(a) = f'(c)(b-a)$. If we take absolute values, $|f(b) - f(a)| = |f'(c)||b-a| \le M|b-a|$.

20. Equation 3.49 for $f(x) = dx^2 + ex + g$ on $a \le x \le b$ states that

$$2dc + e = \frac{(db^2+eb+g)-(da^2+ea+g)}{b-a} = \frac{d(b^2-a^2)+e(b-a)}{b-a} = d(b+a)+e,$$

and therefore $c = (a+b)/2$.

21. Since the derivative of $\sin x$ is always less in absolute value than 1, we can use the result of Exercise 19 to write that $|\sin b - \sin a| \le |b-a|$. The same result is true for the cosine function.

22. If we define a function $F(x) = f(x) - g(x)$, then $F(x)$ is continuous and has a derivative at each point in $a \le x \le b$. We may therefore apply the mean value theorem to $F(x)$ and write $F'(c) = \dfrac{F(b)-F(a)}{b-a}$. Since $F'(x) = f'(x) - g'(x)$, this equation states that there exists a point c between a and b such that $f'(c) - g'(c) = \dfrac{[f(b)-g(b)]-[f(a)-g(a)]}{b-a} = 0$; that is, $f'(c) = g'(c)$.

23. Equation 3.49 states that

$$\begin{aligned} f'(c) = 3dc^2 + 2ec + g = \frac{f(b)-f(a)}{b-a} &= \frac{(db^3+eb^2+gb+h)-(da^3+ea^2+ga+h)}{b-a} \\ &= \frac{d(b^3-a^3)+e(b^2-a^2)+g(b-a)}{b-a} = d(b^2+ab+a^2)+e(b+a)+g. \end{aligned}$$

Thus, c must satisfy a quadratic equation $3dc^2 + 2ec - [d(b^2+ab+a^2)+e(b+a)] = 0$, with solutions

$$c = \frac{-2e \pm \sqrt{4e^2 + 12d[d(b^2+ab+a^2)+e(b+a)]}}{6d} = -\frac{e}{3d} \pm \frac{\sqrt{e^2+3d[d(b^2+ab+a^2)+e(b+a)]}}{3d}.$$

These solutions are equidistant from $-e/(3d)$.

REVIEW EXERCISES

1. $\dfrac{dy}{dx} = 3x^2 - \dfrac{2}{x^3}$

2. $\dfrac{dy}{dx} = 6x + 2 - \dfrac{1}{x^2}$

3. $\dfrac{dy}{dx} = 2 + \dfrac{2}{3x^3} - \dfrac{1}{2x^{3/2}}$

4. $\dfrac{dy}{dx} = \dfrac{1}{3x^{2/3}} - \dfrac{10}{9}x^{2/3}$

5. $\dfrac{dy}{dx} = (x^2+5)^4 + x(4)(x^2+5)^3(2x) = (x^2+5)^3(9x^2+5)$

6. $\dfrac{dy}{dx} = 2(x^2+2)(2x)(x^3-3)^3 + (x^2+2)^2(3)(x^3-3)^2(3x^2) = x(x^2+2)(x^3-3)^2(13x^3+18x-12)$

7. $\dfrac{dy}{dx} = \dfrac{(x^3-5)(6x)-3x^2(3x^2)}{(x^3-5)^2} = \dfrac{-3x(x^3+10)}{(x^3-5)^2}$

8. $\dfrac{dy}{dx} = \dfrac{(x+5)(3)-(3x-2)(1)}{(x+5)^2} = \dfrac{17}{(x+5)^2}$

9. $\dfrac{dy}{dx} = \dfrac{(x^2+2x-1)(2x+2)-(x^2+2x+2)(2x+2)}{(x^2+2x-1)^2} = \dfrac{-6(x+1)}{(x^2+2x-1)^2}$

10. $\dfrac{dy}{dx} = \dfrac{(x^2+5x-2)(4)-4x(2x+5)}{(x^2+5x-2)^2} = \dfrac{-4(x^2+2)}{(x^2+5x-2)^2}$

11. If we differentiate with respect to x, then $y + x\dfrac{dy}{dx} + 9y^2\dfrac{dy}{dx} = 1$, and therefore $\dfrac{dy}{dx} = \dfrac{1-y}{x+9y^2}$.

12. We first write the equation in the form $x^2+y^2 = x^2y$, and then differentiate with respect to x, $2x + 2y\dfrac{dy}{dx} = 2xy + x^2\dfrac{dy}{dx}$. Thus, $\dfrac{dy}{dx} = \dfrac{2xy-2x}{2y-x^2}$.

13. If we differentiate with respect to x, then $2xy^2 + 2x^2y\dfrac{dy}{dx} - 3y\cos x - 3\sin x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = \dfrac{3y\cos x - 2xy^2}{2x^2y - 3\sin x}$.

14. Differentiation with respect to x gives $2xy + x^2\dfrac{dy}{dx} + \dfrac{dy}{dx}\sqrt{1+x} + \dfrac{y}{2\sqrt{1+x}} = 0$. Thus,

$$\frac{dy}{dx} = -\frac{2xy + \dfrac{y}{2\sqrt{1+x}}}{x^2+\sqrt{1+x}} = -\frac{4xy\sqrt{1+x}+y}{2\sqrt{1+x}(x^2+\sqrt{1+x})}.$$

15. $\dfrac{dy}{dx} = 3\tan^2(3x+2)\sec^2(3x+2)(3) = 9\tan^2(3x+2)\sec^2(3x+2)$

16. $\dfrac{dy}{dx} = 2\sec(1-4x)\sec(1-4x)\tan(1-4x)(-4) = -8\sec^2(1-4x)\tan(1-4x)$

17. $\dfrac{dy}{dx} = \dfrac{(\cos 3x)(2\cos 2x) - (\sin 2x)(-3\sin 3x)}{\cos^2 3x} = \dfrac{2\cos 3x\cos 2x + 3\sin 3x\sin 2x}{\cos^2 3x}$

18. $\dfrac{dy}{dx} = \sec(\tan 2x)\tan(\tan 2x)\sec^2 2x(2) = 2\sec^2 2x\sec(\tan 2x)\tan(\tan 2x)$

19. $\dfrac{dy}{dx} = 2x\cos x^2 - x^2\sin x^2(2x) = 2x(\cos x^2 - x^2\sin x^2)$

20. Since $y = \left(\dfrac{1}{2}\sin 2x\right)^2 = \dfrac{1}{4}\sin^2 2x = \dfrac{1}{4}\left(\dfrac{1-\cos 4x}{2}\right) = \dfrac{1}{8}(1-\cos 4x)$, it follows that $\dfrac{dy}{dx} = \dfrac{1}{8}\sin 4x(4) = \dfrac{1}{2}\sin 4x$.

21. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (2u-2)(5/3)(1+2x)^{2/3}(2) = \dfrac{20}{3}(u-1)(1+2x)^{2/3}$

22. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = (1-2\sin 2t)(1+2\sin 2x)$

23. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \left(\dfrac{-3t^2}{2\sqrt{1-t^3}}\right)\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \dfrac{-3xt^2}{2\sqrt{1-t^3}\sqrt{1+x^2}}$

24. $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx} = [\cos^2 v + v(2\cos v)(-\sin v)]\left(\dfrac{-x}{\sqrt{1-x^2}}\right) = \dfrac{x(2v\sin v\cos v - \cos^2 v)}{\sqrt{1-x^2}} = \dfrac{x(v\sin 2v - \cos^2 v)}{v}$

25. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{1+\sqrt{1+x}}}\dfrac{1}{2\sqrt{1+x}} = \dfrac{1}{4\sqrt{1+x}\sqrt{1+\sqrt{1+x}}}$

26. If we differentiate the equation with respect to x, the result is $1 = e^{2y}\left(2\dfrac{dy}{dx}\right) \implies \dfrac{dy}{dx} = \dfrac{1}{2}e^{-2y}$.

27. If write the equation in the form $y = 3x^2/2 + 2x + 3 + 4/x$, then $\dfrac{dy}{dx} = 3x + 2 - \dfrac{4}{x^2}$.

28. $\dfrac{dy}{dx} = 2x\ln(x^2-1) + (x^2+1)\dfrac{2x}{x^2-1}$

29. If we differentiate the equation with respect to x, the result is $\sin y + x\cos y\dfrac{dy}{dx} + 2y + 2x\dfrac{dy}{dx} = 0$, and therefore $\dfrac{dy}{dx} = -\dfrac{2y+\sin y}{2x+x\cos y}$.

30. If we differentiate with respect to x, then $-5\sin(x-y)\left(1-\dfrac{dy}{dx}\right)=0 \Longrightarrow \dfrac{dy}{dx}=1.$

31. $\dfrac{dy}{dx}=\dfrac{-3}{\sqrt{1-(2-3x)^2}}$

32. $\dfrac{dy}{dx}=3\cosh(x^2)(2x)=6x\cosh(x^2)$

33. $$\frac{dy}{dx}=\frac{\dfrac{-\mathrm{Sin}^{-1}x}{\sqrt{1-x^2}}-\dfrac{\mathrm{Cos}^{-1}x}{\sqrt{1-x^2}}}{(\mathrm{Sin}^{-1}x)^2}=-\frac{(\mathrm{Sin}^{-1}x+\mathrm{Cos}^{-1}x)}{\sqrt{1-x^2}(\mathrm{Sin}^{-1}x)^2}$$

34. $$\frac{dy}{dx}=\frac{1}{1+\left(\dfrac{1}{x}+x\right)^2}\left(-\frac{1}{x^2}+1\right)=\frac{1}{1+\dfrac{x^4+2x^2+1}{x^2}}\left(\frac{x^2-1}{x^2}\right)=\frac{x^2-1}{x^4+3x^2+1}$$

35. $\dfrac{dy}{dx}=e^{\cosh x}(\sinh x)$

36. If we differentiate with respect to x, we obtain $\cosh y\,\dfrac{dy}{dx}=\cos x \Longrightarrow \dfrac{dy}{dx}=\dfrac{\cos x}{\cosh y}=\cos x\,\mathrm{sech}y.$

37. If we differentiate with respect to x, we obtain $\dfrac{1}{(x+y)\sqrt{(x+y)^2-1}}\left(1+\dfrac{dy}{dx}\right)=y+x\dfrac{dy}{dx}$, and therefore

$$\frac{dy}{dx}=\frac{\dfrac{1}{(x+y)\sqrt{(x+y)^2-1}}-y}{x-\dfrac{1}{(x+y)\sqrt{(x+y)^2-1}}}=\frac{1-y(x+y)\sqrt{(x+y)^2-1}}{x(x+y)\sqrt{(x+y)^2-1}-1}.$$

38. $$\frac{dy}{dx}=\mathrm{Csc}^{-1}\left(\frac{1}{x^2}\right)-\frac{x}{\dfrac{1}{x^2}\sqrt{\dfrac{1}{x^4}-1}}\left(\frac{-2}{x^3}\right)=\mathrm{Csc}^{-1}\left(\frac{1}{x^2}\right)+\frac{2x^2}{\sqrt{1-x^4}}$$

39. Since $y=e^{2x}\left(\dfrac{e^{2x}+e^{-2x}}{2}\right)=\dfrac{1}{2}(e^{4x}+1)$, it follows that $dy/dx=(1/2)e^{4x}(4)=2\,e^{4x}$.

40. If $\ln[\mathrm{Tan}^{-1}(x+y)]=1/10$, then $\mathrm{Tan}^{-1}(x+y)=e^{1/10}$, from which $x+y=\tan e^{1/10}$ or $y=-x+\tan e^{1/10}$. Consequently, $dy/dx=-1$.

41. If we differentiate the equation with respect to x, the result is

$$1=\frac{1}{2\sqrt{1+x\cot y^2}}\left[\cot y^2-x\csc^2 y^2\left(2y\frac{dy}{dx}\right)\right]\quad\Longrightarrow\quad\frac{dy}{dx}=\frac{\cot y^2-2\sqrt{1+x\cot y^2}}{2xy\csc^2 y^2}=\frac{\cot y^2-2x}{2xy\csc^2 y^2}.$$

42. If we square the equation, $x^2=\dfrac{4+y}{4-y}$. This equation can be solved for $y=\dfrac{4(x^2-1)}{x^2+1}$. Differentiation now gives $\dfrac{dy}{dx}=4\left[\dfrac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}\right]=\dfrac{16x}{(x^2+1)^2}$.

43. $$\frac{dy}{dx}=\frac{1}{2}\left(\frac{4+x^2}{4-x^2}\right)^{-1/2}\left[\frac{(4-x^2)(2x)-(4+x^2)(-2x)}{(4-x^2)^2}\right]=\frac{8x}{\sqrt{4+x^2}(4-x^2)^{3/2}}$$

44. If we take natural logarithms of $|y|=\dfrac{|x|^2\sqrt{1-x}}{|x+5|}$, then $\ln|y|=2\ln|x|+\dfrac{1}{2}\ln(1-x)-\ln|x+5|$. Differentiation with respect to x using formula 3.46 gives $\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{2}{x}-\dfrac{1}{2(1-x)}-\dfrac{1}{x+5}$. Therefore,

$$\frac{dy}{dx}=\frac{x^2\sqrt{1-x}}{x+5}\left[\frac{4(1-x)(x+5)-x(x+5)-2x(1-x)}{2x(1-x)(x+5)}\right]=\frac{x(20-23x-3x^2)}{2\sqrt{1-x}(x+5)^2}.$$

45. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{7-\sqrt{7-\sqrt{x}}}}\dfrac{-1}{2\sqrt{7-\sqrt{x}}}\dfrac{-1}{2\sqrt{x}} = \dfrac{1}{8\sqrt{x}\sqrt{7-\sqrt{x}}\sqrt{7-\sqrt{7-\sqrt{x}}}}$

46. If we write $x^2 - xy = xy + y^2$, or, $y^2 + 2xy - x^2 = 0$, and differentiate with respect to x, we obtain $2y\dfrac{dy}{dx} + 2y + 2x\dfrac{dy}{dx} - 2x = 0$. Thus, $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$.

47. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{1}{2}\left(\dfrac{4+t}{4-t}\right)^{-1/2}\left[\dfrac{(4-t)(1)-(4+t)(-1)}{(4-t)^2}\right](\sec^2 x) = \dfrac{4\sec^2 x}{\sqrt{4+t}(4-t)^{3/2}}$

48. If we use the result of Exercise 40 in Section 3.7,

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\dfrac{(y^3+4y+6)(2y-2)-(y^2-2y)(3y^2+4)}{(y^3+4y+6)^2}} = \frac{(y^3+4y+6)^2}{-y^4+4y^3+4y^2+12y-12}.$$

49. If we write the function in the form $y = \dfrac{(x-2)^3}{(x-2)^2} = x-2$, the derivative is $dy/dx = 1$, provided $x \neq 2$.

50. Since $y = \dfrac{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}{\sqrt{x}-\sqrt{2}} = \sqrt{x}+\sqrt{2}$, it follows that $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, provided $x \neq 2$.

51. If we take natural logarithms of $y = x^{2x}$, then $\ln y = 2x\ln x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = 2\ln x + \frac{2x}{x} \implies \frac{dy}{dx} = 2x^{2x}(1+\ln x).$$

52. If we take natural logarithms, then $\ln y = x\ln\cos x$, and differentiation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \ln(\cos x) + x\left(\frac{-\sin x}{\cos x}\right) \implies \frac{dy}{dx} = (\cos x)^x[\ln(\cos x) - x\tan x].$$

53. $\dfrac{dy}{dx} = \dfrac{(e^x+1)(e^x)-e^x(e^x)}{(e^x+1)^2} = \dfrac{e^x}{(e^x+1)^2}$

54. $\dfrac{dy}{dx} = \dfrac{1}{\log_{10} x}\log_{10} e\dfrac{d}{dx}\log_{10} x = \dfrac{1}{\log_{10} x}\log_{10} e\left(\dfrac{1}{x}\right)\log_{10} e = \dfrac{(\log_{10} e)^2}{x\log_{10} x}$

55. $\dfrac{dy}{dx} = e^x\ln x + e^x\left(\dfrac{1}{x}\right) = \dfrac{e^x(1+x\ln x)}{x}$

56. Differentiation with respect to x gives $1 = e^y\dfrac{dy}{dx} - e^{-y}\dfrac{dy}{dx} \implies \dfrac{dy}{dx} = \dfrac{1}{e^y - e^{-y}}$.

57. Differentiation with respect to x gives $ye^{xy} + xe^{xy}\dfrac{dy}{dx} + xye^{xy}\left(y + x\dfrac{dy}{dx}\right) = 0$, from which $\dfrac{dy}{dx} = -\dfrac{y+xy^2}{x+x^2y} = -\dfrac{y}{x}$.

58. If we differentiate with respect to x, we obtain $2xy + x^2\dfrac{dy}{dx} + \dfrac{1}{x+y}\left(1+\dfrac{dy}{dx}\right) = 1$. Thus,

$$\frac{dy}{dx} = \frac{1-2xy-\dfrac{1}{x+y}}{x^2+\dfrac{1}{x+y}} = \frac{(x+y)(1-2xy)-1}{x^2(x+y)+1}.$$

59. Since $\dfrac{dy}{dx}_{|x=1} = (3x^2+3)_{|x=1} = 6$, equations for the tangent and normal lines are $y-2 = 6(x-1)$ and $y-2 = -(1/6)(x-1)$, or $6x-y=4$ and $x+6y=13$.

60. Since $dy/dx = -1/(x+5)^2$, the slope of the tangent line at $(0,1/5)$ is $-1/25$. Equations for the tangent and normal lines are $y-1/5 = -(1/25)(x-0)$ and $y-1/5 = 25(x-0)$, or $x+25y=5$ and $5y-125x=1$.

61. Since $\dfrac{dy}{dx}_{|x=\pi/2} = (-2\sin 2x)_{|x=\pi/2} = 0$, the tangent and normal lines are $y = -1$ and $x = \pi/2$.

62. Since $\dfrac{dy}{dx} = \dfrac{(2x-5)(2x+3)-(x^2+3x)(2)}{(2x-5)^2} = \dfrac{2x^2-10x-15}{(2x-5)^2}$, the slope of the tangent line at $(1,-4/3)$ is $-23/9$. Equations for the tangent and normal lines are $y+4/3 = -(23/9)(x-1)$ and $y+4/3 = (9/23)(x-1)$, or $23x+9y=11$ and $27x-69y=119$.

63. If we differentiate with respect to x, then $2x - 2y\dfrac{dy}{dx} + 2 - 2\dfrac{dy}{dx} = 3y^2\dfrac{dy}{dx} \Longrightarrow \dfrac{dy}{dx} = \dfrac{2x+2}{3y^2+2y+2}$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \frac{(3y^2+2y+2)(2)-(2x+2)(6y+2)\dfrac{dy}{dx}}{(3y^2+2y+2)^2}$$
$$= \frac{2(3y^2+2y+2)-4(x+1)(3y+1)\left(\dfrac{2x+2}{3y^2+2y+2}\right)}{(3y^2+2y+2)^2} = \frac{2(3y^2+2y+2)^2-8(x+1)^2(3y+1)}{(3y^2+2y+2)^3}.$$

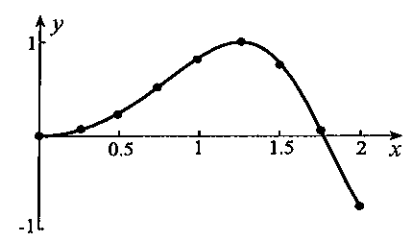
64. If we first expand the equation, we obtain $x^2-5xy+y^2=0$. Differentiation with respect to x now gives $2x-5y-5x\dfrac{dy}{dx}+2y\dfrac{dy}{dx}=0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{2x-5y}{5x-2y}$. A second differentiation yields

$$\frac{d^2y}{dx^2} = \frac{(5x-2y)\left(2-5\dfrac{dy}{dx}\right)-(2x-5y)\left(5-2\dfrac{dy}{dx}\right)}{(5x-2y)^2}$$
$$= \frac{(5x-2y)\left[2-5\left(\dfrac{2x-5y}{5x-2y}\right)\right]-(2x-5y)\left[5-2\left(\dfrac{2x-5y}{5x-2y}\right)\right]}{(5x-2y)^2}$$
$$= \frac{2(5x-2y)^2-5(2x-5y)(5x-2y)-5(2x-5y)(5x-2y)+2(2x-5y)^2}{(5x-2y)^3} = \frac{-42(x^2-5xy+y^2)}{(5x-2y)^3}.$$

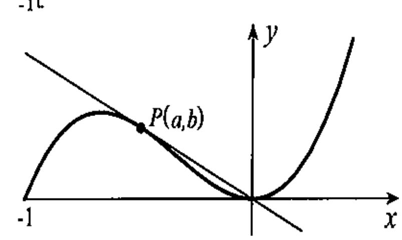
65. If we differentiate with respect to x, then $\cos(x+y)\left(1+\dfrac{dy}{dx}\right) = 1 \Longrightarrow \dfrac{dy}{dx} = \sec(x+y)-1$. A second differentiation gives

$$\frac{d^2y}{dx^2} = \sec(x+y)\,\tan(x+y)\left(1+\frac{dy}{dx}\right) = \sec(x+y)\,\tan(x+y)\,[1+\sec(x+y)-1]$$
$$= \sec^2(x+y)\,\tan(x+y).$$

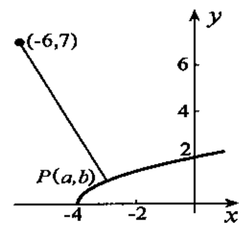
66. The graph to the right was obtained by joining the points shown. The function is not periodic.



67. Let $P(a,b)$ be a required point. If we differentiate the equation of the cubic with respect to x, the result is $dy/dx = 3x^2+2x$. The slope of the tangent line at P is therefore $3a^2+2a$, and the equation of the tangent line is $y-b=(3a^2+2a)(x-a)$. Since this line passes through the origin, a and b must satisfy $-b=(3a^2+2a)(-a)$. Because $P(a,b)$ is on the cubic, a and b must also satisfy $b=a^3+a^2$. When we combine these two equations, we obtain $3a^3+2a^2=a^3+a^2 \Longrightarrow 0=2a^3+a^2=a^2(2a+1)$. Thus, $a=0$, as expected, and $a=-1/2$. The required points are therefore $(0,0)$ and $(-1/2,1/8)$.



68. Let $P(a,b)$ be the required point. If we differentiate the equation of the parabola with respect to x, the result is $1 = 2y(dy/dx)$. The slope of the tangent line at P is therefore $1/(2b)$, and that of the normal line is $-2b$. Since the slope of the normal line at P is also $(b-7)/(a+6)$, it follows that $(b-7)/(a+6) = -2b$. Because $P(a,b)$ is on the parabola, a and b must also satisfy $a = b^2 - 4$. When we substitute this into the above equation, we obtain $(b-7)/(b^2-4+6) = -2b$. Thus, $0 = 2b^3 + 5b - 7 = (b-1)(2b^2+2b+7)$. The required point is therefore $(-3,1)$. The length of the line segment joining $(-6,7)$ and $(-3,1)$ is the shortest distance from $(-6,7)$ to the parabola.



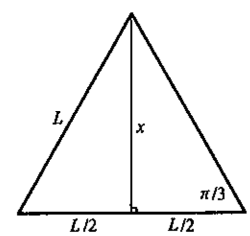
69. (a) There is an infinity of functions defined by the equation.
(b) There are only two continuous functions defined by the equation, $f(x) = \sqrt{1-x^2}$ and $f(x) = -\sqrt{1-x^2}$.

70. Since $x = L\sin(\pi/3) = \sqrt{3}L/2$, the area of the triangle is

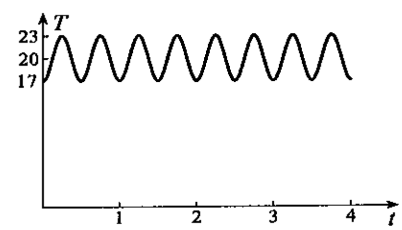
$$A = \frac{xL}{2} = \frac{\sqrt{3}L^2}{4}.$$

Consequently,

$$\frac{dA}{dL} = \frac{\sqrt{3}L}{2}.$$



71. (a) The graph is shown to the right.
(b) Since temperature is rising eight times, the furnace is on eight times.
(c) The rate of change of temperature is $T'(t) = 12\pi\cos(4\pi t - \pi/2)$. The maximum value is 12π degrees per hour.



72. Differentiation with respect to x gives $2(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 2x - 2y\dfrac{dy}{dx}$, and therefore $\dfrac{dy}{dx} = \dfrac{x - 2x(x^2+y^2)}{y + 2y(x^2+y^2)}$. The tangent line is horizontal when $0 = x - 2x(x^2+y^2) = x(1-2x^2-2y^2)$. Since $x = 0$ is not a solution to our problem, we must set $x^2+y^2 = 1/2$. We substitute this into the equation for the lemniscate, obtaining thereby $1/4 = x^2 - y^2$. Addition of this and $1/2 = x^2+y^2$ gives $2x^2 = 3/4$, or, $x = \pm\sqrt{3/8} = \pm\sqrt{6}/4$. These values yield the points $(\sqrt{6}/4, \pm\sqrt{2}/4)$ and $(-\sqrt{6}/4, \pm\sqrt{2}/4)$.

73. Equation 3.49 states that $3c^2 + 3 = \dfrac{232-34}{6-3} = 66$. Of the two solutions $c = \pm\sqrt{21}$, only $c = \sqrt{21}$ lies in the interval $3 \le x \le 6$.

74. For this $f(x)$ and $g(x)$, equation 3.48 gives $\dfrac{6c-2}{3c^2+2} = \dfrac{5-9}{3+3} = -\dfrac{2}{3}$. Of the two solutions $c = (-9\pm\sqrt{93})/6$ of this equation, only $c = (\sqrt{93}-9)/6$ is in the interval $-1 \le x \le 1$.

CHAPTER 4

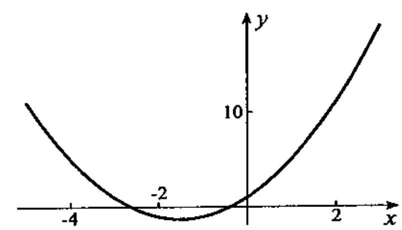
EXERCISES 4.1

1. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}.$$

For the root between -3 and -2 we use an initial approximation $x_1 = -2.5$. The resulting iterations give

$$x_2 = -2.625, \qquad x_3 = -2.618\,0,$$
$$x_4 = -2.618\,034\,0, \quad x_5 = -2.618\,034\,0.$$



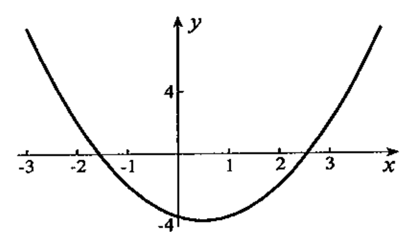
With $f(x) = x^2 + 3x + 1$, we calculate that $f(-2.618\,034\,5) = 1.1 \times 10^{-6}$ and $f(-2.618\,033\,5) = -1.1 \times 10^{-6}$. Hence, to six decimals, a root is $x = -2.618\,034$. A similar procedure for the root between -1 and 0 gives $-0.381\,966$.

2. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^2 - x_n - 4}{2x_n - 1}.$$

For the root between 2 and 3 we use an initial approximation $x_1 = 2.5$. The resulting iterations give

$$x_2 = 2.562\,5, \qquad x_3 = 2.561\,553\,0,$$
$$x_4 = 2.561\,552\,8, \quad x_5 = 2.561\,552\,8.$$



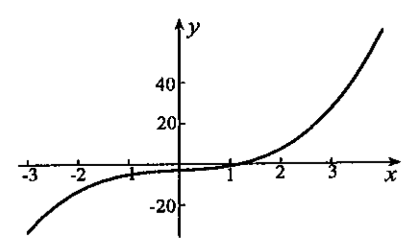
With $f(x) = x^2 - x - 4$, we calculate that $f(2.561\,552\,5) = -1.3 \times 10^{-6}$ and $f(2.561\,553\,5) = 2.8 \times 10^{-6}$. Hence, to six decimals, a root is $x = 2.561\,553$. A similar procedure for the root between -2 and -1 gives $x = -1.561\,553$.

3. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 3}{3x_n^2 + 1}.$$

The only root is between 1 and 2, and we use an initial approximation $x_1 = 1.5$. The resulting iterations give

$$x_2 = 1.258, \qquad x_3 = 1.214\,7,$$
$$x_4 = 1.213\,413, \qquad x_5 = 1.213\,411\,6.$$
$$x_6 = 1.213\,411\,6.$$



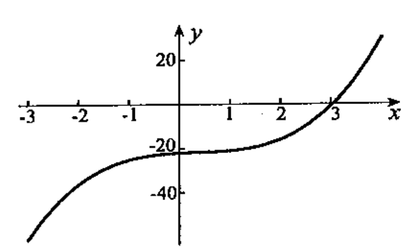
With $f(x) = x^3 + x - 3$, we calculate that $f(1.213\,411\,5) = -8.8 \times 10^{-7}$ and $f(1.213\,412\,5) = 4.5 \times 10^{-6}$. Hence, to six decimals, a root is $x = 1.213\,412$.

4. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n - 22}{3x_n^2 - 2x_n + 1}.$$

The only root of the equation is slightly larger than 3. To find it we use $x_1 = 3.1$. Iteration gives

$$x_2 = 3.045\,789\,3, \quad x_3 = 3.044\,723\,6,$$
$$x_4 = 3.044\,723\,1, \quad x_5 = 3.044\,723\,1.$$



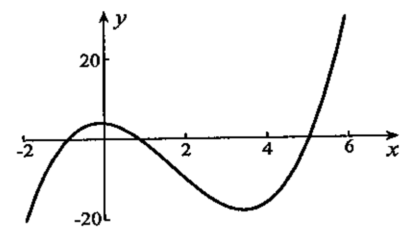
With $f(x) = x^3 - x^2 + x - 22$, we calculate that $f(3.044\,722\,5) = -1.5 \times 10^{-5}$ and $f(3.044\,723\,5) = 8.0 \times 10^{-6}$. Thus, to six decimals, the root is $x = 3.044\,723$.

5. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n^2 - x_n + 4}{3x_n^2 - 10x_n - 1}.$$

For the root slightly larger than 5, we use an initial approximation $x_1 = 5$. The resulting iterations give

$$x_2 = 5.042, \qquad x_3 = 5.040\,965,$$
$$x_4 = 5.040\,964\,6, \quad x_5 = 5.040\,964\,6.$$



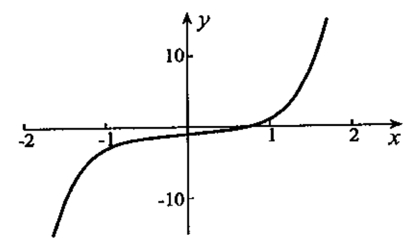
With $f(x) = x^3 - 5x^2 - x + 4$, we calculate that $f(5.040\,964\,5) = -2.4 \times 10^{-6}$ and $f(5.040\,965\,5) = 2.2 \times 10^{-5}$. Hence, to six decimals, a root is $x = 5.040\,965$. A similar procedure for the other roots gives $-0.911\,503$ and $0.870\,539$.

6. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^5 + x_n - 1}{5x_n^4 + 1}.$$

The only root of the equation is between 0 and 1. To find it we use $x_1 = 0.5$. Iteration gives

$$x_2 = 0.857\,142\,9, \quad x_3 = 0.770\,682\,2,$$
$$x_4 = 0.755\,283\,0, \quad x_5 = 0.754\,877\,9,$$
$$x_6 = 0.754\,877\,7, \quad x_7 = 0.754\,877\,7.$$



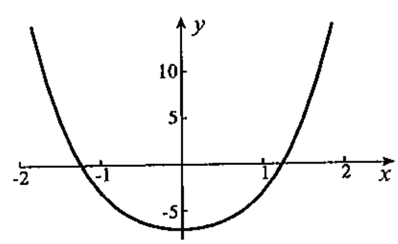
With $f(x) = x^5 + x - 1$, we calculate that $f(0.754\,877\,5) = -4.4 \times 10^{-7}$ and $f(0.754\,878\,5) = 2.2 \times 10^{-6}$. Thus, to six decimals, the root is $x = 0.754\,878$.

7. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^4 + 3x_n^2 - 7}{4x_n^3 + 6x_n}.$$

For the root between 1 and 2 we use an initial approximation $x_1 = 1.2$. The resulting iterations give

$$x_2 = 1.243, \qquad x_3 = 1.241\,152\,6,$$
$$x_4 = 1.241\,523\,8, \quad x_5 = 1.241\,523\,8.$$



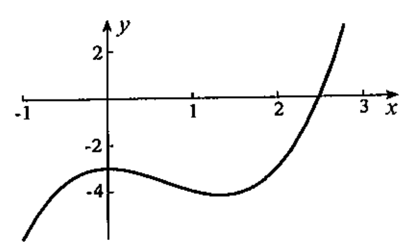
With $f(x) = x^4 + 3x^2 - 7$, we calculate that $f(1.241\,523\,5) = -4.0 \times 10^{-6}$ and $f(1.241\,524\,5) = 1.1 \times 10^{-5}$. Hence, to six decimals, a root is $x = 1.241\,524$. The other root is $-1.241\,524$.

8. The equation can be rearranged into the form $x^3 - 2x^2 - 3 = 0$. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n^2 - 3}{3x_n^2 - 4x_n}.$$

The only root of the equation is between 2 and 3. To find it we use $x_1 = 2.5$. Iteration gives

$$x_2 = 2.485\,714\,3, \quad x_3 = 2.485\,584\,0,$$
$$x_4 = 2.485\,584\,0.$$



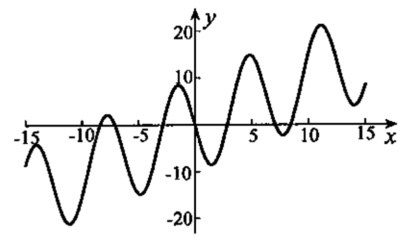
With $f(x) = x^3 - 2x^2 - 3$, we calculate that $f(2.485\,583\,5) = -4.3 \times 10^{-6}$ and $f(2.485\,584\,5) = 4.3 \times 10^{-6}$. Thus, to six decimals, the root is $x = 2.485\,584$.

9. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n - 10\sin x_n}{1 - 10\cos x_n}.$$

For the root between 2 and 3 we use an initial approximation $x_1 = 2.8$. The resulting iterations give

$$x_2 = 2.852\,76, \qquad x_3 = 2.852\,342,$$
$$x_4 = 2.852\,341\,9, \quad x_5 = 2.852\,341\,9.$$



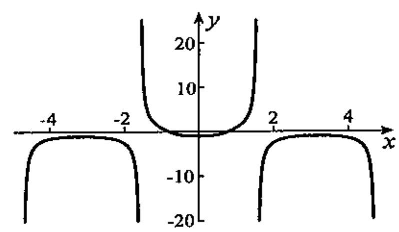
With $f(x) = x - 10\sin x$, we calculate that $f(2.852\,341\,5) = -4.2\times 10^{-6}$ and $f(2.852\,342\,5) = 6.4\times 10^{-6}$. Hence, to six decimals, a root is $x = 2.852\,342$. A similar procedure yields the additional roots 7.068 174 and 8.423 204. Negatives of these three roots also satisfy the equation. So also does $x = 0$.

10. If we rearrange the equation into the form $f(x) = (1 + x^4)\sec x - 2 = 0$, Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{(1 + x_n^4)\sec x_n - 2}{4x_n^3 \sec x_n + (1 + x_n^4)\sec x_n \tan x_n}.$$

For the root near 1 we use $x_1 = 1$. Iteration gives

$$x_2 = 0.870\,777\,3, \quad x_3 = 0.807\,267\,5,$$
$$x_4 = 0.795\,650\,2, \quad x_5 = 0.795\,324\,2,$$
$$x_6 = 0.795\,323\,9, \quad x_7 = 0.795\,323\,9.$$



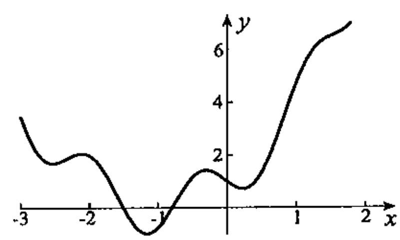
We now calculate that $f(0.795\,323\,5) = -2.0\times 10^{-6}$ and $f(0.795\,324\,5) = 2.9\times 10^{-6}$. Thus, to six decimals, the root is $x = 0.795\,324$. Because of the symmetry of the graph, the other root is $-0.795\,324$.

11. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{(x_n + 1)^2 - \sin 4x_n}{2(x_n + 1) - 4\cos 4x_n}.$$

For the root between -2 and -1 we use an initial approximation $x_1 = -1.5$. The resulting iterations give

$$x_2 = -1.506, \qquad x_3 = -1.506\,05,$$
$$x_4 = -1.506\,052\,7, \quad x_5 = -1.506\,052\,7.$$



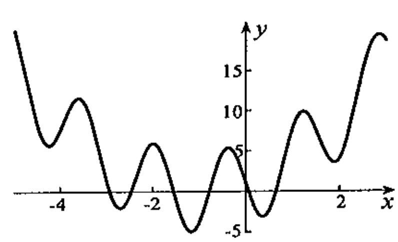
With $f(x) = (x + 1)^2 - \sin 4x$, we calculate that $f(-1.506\,053\,5) = 3.8\times 10^{-6}$ and $f(-1.506\,052\,5) = -1.0\times 10^{-6}$. Hence, to six decimals, a root is $x = -1.506\,053$. A similar procedure gives the other root $-0.795\,823$.

12. The graph suggests that there are 6 roots of the equation. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{(x_n + 1)^2 - 5\sin 4x_n}{2(x_n + 1) - 20\cos 4x_n}.$$

For the smallest positive root we use $x_1 = 0$. Iteration gives

$$x_2 = 0.055\,555\,6, \quad x_3 = 0.056\,257\,3,$$
$$x_4 = 0.056\,257\,6, \quad x_5 = 0.056\,257\,6.$$



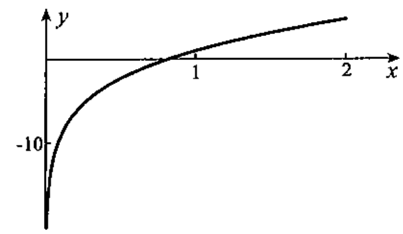
With $f(x) = (x + 1)^2 - 5\sin 4x$, we calculate that $f(0.056\,257\,5) = 1.9\times 10^{-6}$ and $f(0.056\,258\,5) = -1.6\times 10^{-5}$. Thus, to six decimals, the root is $x = 0.056\,258$. Similar procedures lead to the other 5 roots $-2.931\,137$, $-2.467\,518$, $-1.555\,365$, $-0.787\,653$, $0.642\,851$.

13. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n + 4\ln x_n}{1 + 4/x_n}.$$

With $x_1 = 1$ as initial approximation, resulting iterations give

$$x_2 = 0.8, \qquad x_3 = 0.815\,4,$$
$$x_4 = 0.815\,553\,4, \quad x_5 = 0.815\,553\,4.$$



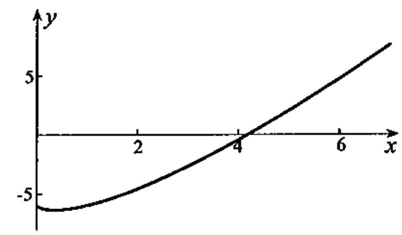
With $f(x) = x + 4\ln x$, we calculate that $f(0.815\,552\,5) = -5.4 \times 10^{-6}$ and $f(0.815\,553\,5) = 4.8 \times 10^{-7}$. Hence, to six decimals, the root is $x = 0.815\,553$.

14. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n \ln x_n - 6}{\ln x_n + 1}.$$

The only root of the equation is between 4 and 5. To find it we use $x_1 = 4.3$. Iteration gives

$$x_2 = 4.189\,350\,5, \quad x_3 = 4.188\,760\,1,$$
$$x_4 = 4.188\,760\,1.$$



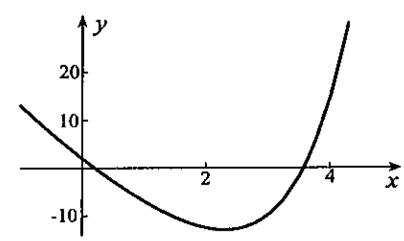
With $f(x) = x\ln x - 6$, we calculate that $f(4.188\,759\,5) = -1.5 \times 10^{-6}$ and $f(4.188\,760\,5) = 9.3 \times 10^{-7}$. Thus, to six decimals, the root is $x = 4.188\,760$.

15. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{e^{x_n} + e^{-x_n} - 10x_n}{e^{x_n} - e^{-x_n} - 10}.$$

For the root between 0 and 1 we use an initial approximation $x_1 = 0.2$. The resulting iterations give

$$x_2 = 0.204\,2, \qquad x_3 = 0.204\,183\,6,$$
$$x_4 = 0.204\,183\,6.$$



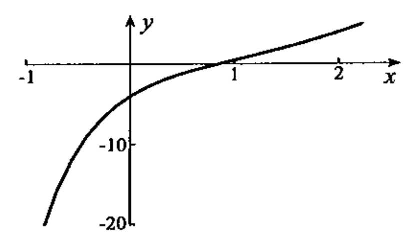
With $f(x) = e^x + e^{-x} - 10x$, we calculate that $f(0.204\,183\,5) = 9.5 \times 10^{-7}$ and $f(0.204\,184\,5) = -8.6 \times 10^{-6}$. Hence, to six decimals, a root is $x = 0.204\,184$. A similar procedure yields the other root 3.576 065.

16. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^2 - 4e^{-2x_n}}{2x_n + 8e^{-2x_n}}.$$

With $x_1 = 0.8$, we obtain

$$x_2 = 0.852\,123\,5, \quad x_3 = 0.852\,605\,5,$$
$$x_4 = 0.852\,605\,5.$$



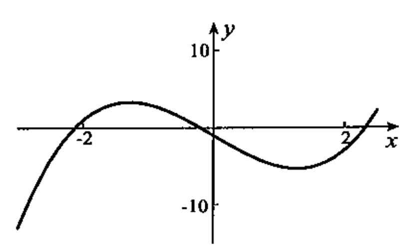
With $f(x) = x^2 - 4e^{-2x}$, we calculate that $f(0.852\,605\,5) = -6.4 \times 10^{-9}$ and $f(0.852\,606\,5) = 3.2 \times 10^{-6}$. Hence, to six decimals, the root is $x = 0.852\,606$.

17. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n - 1}{3x_n^2 - 5}.$$

For the root between 2 and 3 we use an initial approximation $x_1 = 2.3$. The resulting iterations give

$$x_2 = 2.3306, \quad x_3 = 2.3301,$$
$$x_4 = 2.3301.$$



With $f(x) = x^3 - 5x - 1$, we calculate that $f(2.329) = -0.01$ and $f(2.331) = 0.01$. Hence, a root with error less than 10^{-3} is $x = 2.330$. A similar procedure yields the roots -2.128 and -0.202.

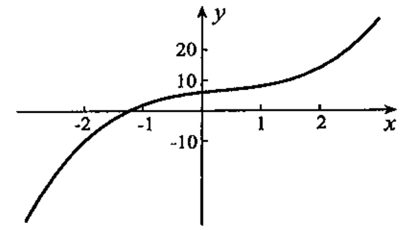
18. Obviously $x = 0$ is a solution of the equation. The remaining solutions must satisfy $f(x) = x^3 - x^2 + 2x + 6 = 0$. The graph indicates only one root between -2 and -1. To find it we use, $x_1 = -1.2$, and

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + 2x_n + 6}{3x_n^2 - 2x_n + 2}.$$

Iteration gives

$x_2 = -1.249\,541, \quad x_3 = -1.248\,299,$

$x_4 = -1.248\,298.$



Since $f(-1.248\,4) = -9.4 \times 10^{-4}$ and $f(-1.248\,2) = 8.9 \times 10^{-4}$, we can say that the root is $x = -1.2483$.

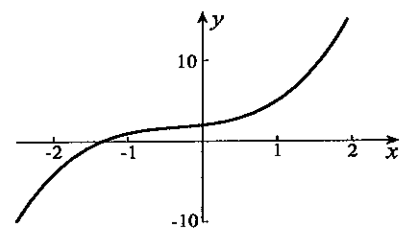
19. We rewrite the equation in the form $x^3 + x^2 + x + 2 = 0$. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 + x_n + 2}{3x_n^2 + 2x_n + 1}.$$

With an initial approximation $x_1 = -1.5$.
The resulting iterations give

$x_2 = -1.368, \qquad x_3 = -1.353\,38,$

$x_4 = -1.353\,210 \quad x_5 = -1.353\,210.$



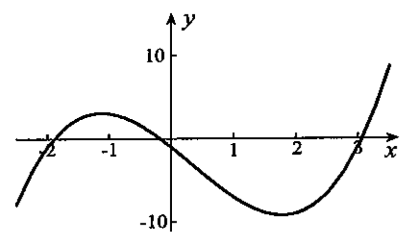
With $f(x) = x^3 + x^2 + x + 2$, we calculate that $f(-1.353\,22) = -3.8 \times 10^{-5}$ and $f(-1.353\,20) = 3.8 \times 10^{-5}$. Hence, the root is $x = -1.353\,21$.

20. We rearrange the equation into the form $f(x) = x^3 - x^2 - 6x - 1 = 0$. The graph suggests three solutions. To find the positive one, we use $x_1 = 3$ and

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - 6x_n - 1}{3x_n^2 - 2x_n - 6}.$$

Iteration gives

$x_2 = 3.066\,67, \quad x_3 = 3.064\,44, \quad x_4 = 3.064\,43.$



Since $f(3.063) = -2.3 \times 10^{-2}$ and $f(3.065) = 9.1 \times 10^{-3}$, the root is $x = 3.064$. A similar procedure leads to the other two roots -1.892 and -0.172.

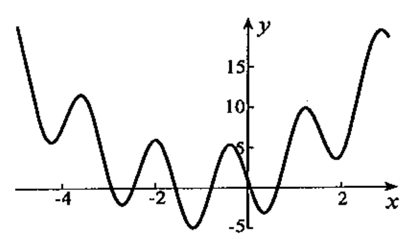
21. Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{(x_n + 1)^2 - 5\sin 4x_n}{2(x_n + 1) - 20\cos 4x_n}.$$

For the root nearest -3, we use an initial approximation $x_1 = -3$. The resulting iterations give

$x_2 = -2.036\,9, \quad x_3 = -2.93\,2,$

$x_4 = -2.931\,1, \quad x_5 = -2.931\,1.$



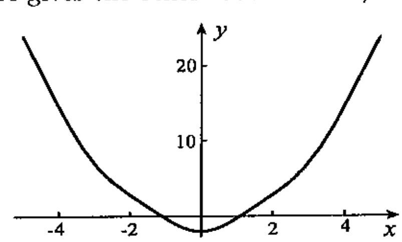
With $f(x) = (x + 1)^2 - 5\sin 4x$, we calculate that $f(-2.932) = 0.01$ and $f(-2.930) = -0.02$. Hence, a root with error less than 10^{-3} is $x = -2.931$. A similar procedure gives the other roots -2.468, -1.555, -0.788, 0.056 and 0.643.

22. The graph indicates two roots. To find the positive one, we use $x_1 = 1$ and

$$x_{n+1} = x_n - \frac{x_n^2 - 1 - \cos^2 x_n}{2x_n + 2\cos x_n \sin x_n}.$$

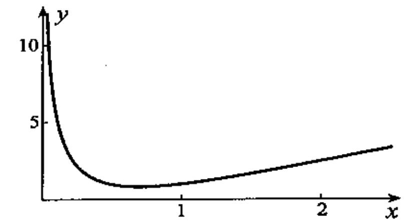
Iteration gives

$x_2 = 1.100\,343, \quad x_3 = 1.098\,587, \quad x_4 = 1.098\,587.$



With $f(x) = x^2 - 1 - \cos^2 x$, we calculate that $f(1.098\,5) = -2.6 \times 10^{-4}$ and $f(1.098\,7) = 3.4 \times 10^{-4}$. Thus, the root is $x = 1.098\,6$. Symmetry gives the other root as $-1.098\,6$.

23. The graph indicates that this equation has no solutions.

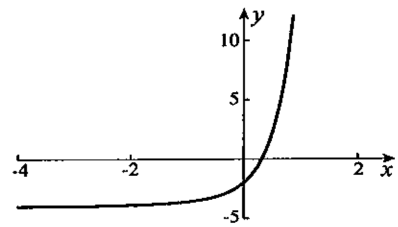


24. The graph indicates a solution of this equation between 0 and 1. To find it we use $x_1 = 0.3$ and

$$x_{n+1} = x_n - \frac{e^{3x_n} + e^{x_n} - 4}{3e^{3x_n} + e^{x_n}}.$$

Iteration gives

$x_2 = 0.321\,83, \quad x_3 = 0.321\,21, \quad x_4 = 0.321\,21.$



With $f(x) = e^{3x} + e^x - 4$, we calculate that $f(0.3211) = -1.0 \times 10^{-3}$ and $f(0.3213) = 8.2 \times 10^{-4}$. Thus, the root is $x = 0.3212$.

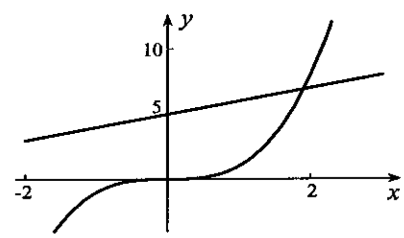
25. The x-coordinate of the point of intersection must satisfy the equation $f(x) = x^3 - x - 5 = 0$. To find it we use $x_1 = 2$ and

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 5}{3x_n^2 - 1}.$$

Iteration gives

$x_2 = 1.909, \qquad x_3 = 1.904\,17,$

$x_4 = 1.904\,161, \quad x_5 = 1.904\,161.$



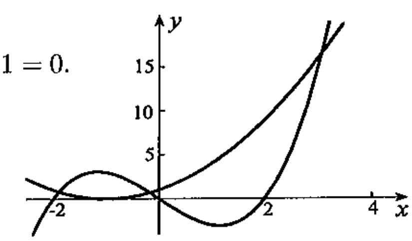
Since $f(1.904\,160\,5) = -3.5 \times 10^{-6}$ and $f(1.904\,161\,5) = 6.3 \times 10^{-6}$, we can say that to six decimals $x = 1.904\,161$. In either of the original equations, $x = 1.904\,161$ yields the same four decimals, $y = 6.904\,2$. The point of intersection is therefore $(1.904\,2, 6.904\,2)$.

26. To find x-coordinates of the points of intersection, we set $(x+1)^2 = x^3 - 4x$, and this reduces to $f(x) = x^3 - x^2 - 6x - 1 = 0$. With $x_1 = 3$, and

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - 6x_n - 1}{3x_n^2 - 2x_n - 6},$$

iteration gives

$x_2 = 3.066\,7, \quad x_3 = 3.064\,435, \quad x_4 = 3.064\,435.$



Since $f(3.064\,434\,5) = -5.4 \times 10^{-7}$ and $f(3.064\,435\,5) = 1.6 \times 10^{-5}$, the root is $x = 3.064\,435$. In either of the original equations, $x = 3.064\,435$ yields the same four decimals, $y = 16.519\,6$. A point of intersection is therefore $(3.064\,4, 16.519\,6)$. Similar procedures lead to the other points of intersection $(-1.892\,0, 0.795\,6)$ and $(-0.172\,5, 0.684\,8)$.

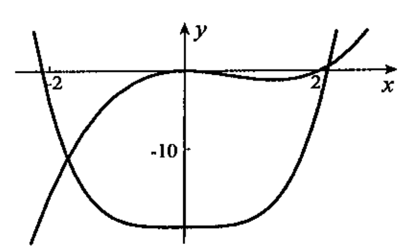
27. The x-coordinates of the points of intersection must satisfy the equation $f(x) = x^4 - x^3 + 2x^2 - 20 = 0$. To find one of them, we use $x_1 = 2$ and

$$x_{n+1} = x_n - \frac{x_n^4 - x_n^3 + 2x_n^2 - 20}{4x_n^3 - 3x_n^2 + 4x_n}.$$

Iteration gives

$x_2 = 2.142\,9, \qquad x_3 = 2.130\,298,$

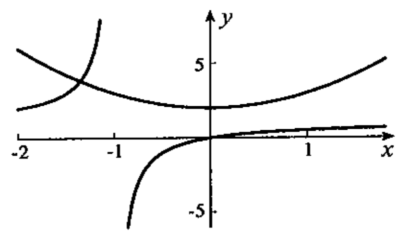
$x_4 = 2.130\,189, \quad x_5 = 2.130\,189.$



Since $f(2.130\,188\,5) = -7.8 \times 10^{-6}$ and $f(2.130\,189\,5) = 2.6 \times 10^{-5}$, we can say that $x = 2.130\,189$. In either of the original equations, $x = 2.130\,189$ yields the same four decimals, $y = 0.590\,8$. A point of intersection is therefore $(2.130\,2, 0.590\,8)$. A similar procedure leads to the other point of intersection $(-1.7267, -11.1109)$.

28. When we equate the two expressions for y, and rearrange the equation, x must satisfy $f(x) = x^3 + x^2 + x + 2 = 0$. The graph indicates that there is only one solution, and to find it we use $x_1 = -1.3$ and

$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 + x_n + 2}{3x_n^2 + 2x_n + 1}.$$



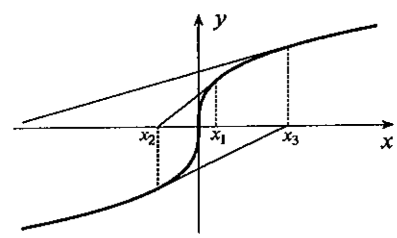
Iteration gives

$$x_2 = -1.368, \qquad x_3 = -1.353\,38,$$
$$x_4 = -1.353\,210, \quad x_5 = -1.353\,210.$$

Since $f(-1.353\,210\,5) = -2.0 \times 10^{-6}$ and $f(-1.353\,209\,5) = 1.8 \times 10^{-6}$, we can say that $x = -1.353\,210$. In either of the original equations, $x = -1.353\,210$ yields the same four decimals, $y = 3.831\,2$. The point of intersection is therefore $(-1.353\,2, 3.831\,2)$.

29. If x_1 is the initial approximation to the solution of $f(x) = x^{1/3} = 0$, then the second approximation as defined by Newton's iterative procedure is

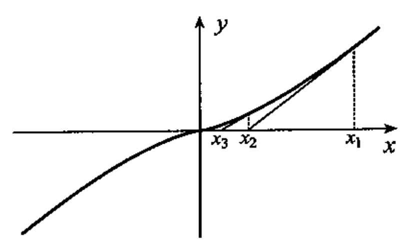
$$x_2 = x_1 - \frac{x_1^{1/3}}{(1/3)x_1^{-2/3}} = x_1 - 3x_1 = -2x_1.$$



What this implies is that every approximation is -2 times the previous one. Approximations therefore do not approach the root $x = 0$. This is illustrated in the graph to the right.

30. If x_1 is the initial approximation to the solution of $f(x) = x^{7/5} = 0$, then the second approximation as defined by Newton's iterative procedure is

$$x_2 = x_1 - \frac{x_1^{7/5}}{(7/5)x_1^{2/5}} = x_1 - \frac{5x_1}{7} = \frac{2}{5}x_1.$$



What this implies is that every approximation is $2/5$ times the previous one. Approximations therefore must approach zero, the root of the equation.
This is illustrated in the graph to the right.

31. (a) For $P = 100\,000$, $i = 5$, $n = 25$, and $m = 12$, $\quad M = 100\,000\left[\frac{5/1200}{1 - (1 + 5/1200)^{-300}}\right] = 584.59.$

(b) For $P = 100\,000$, $n = 25$, $M = 500$, and $m = 12$, i must satisfy $500 = 100\,000\left[\frac{i/1200}{1 - (1 + i/1200)^{-300}}\right]$.
This simplifies to

$$f(i) = \left(1 + \frac{i}{1200}\right)^{300} - \frac{6}{6 - i} = 0.$$

Newton's iterative procedure with an initial approximation of $i_1 = 4$, defines the sequence

$$i_1 = 4, \qquad i_{n+1} = i_n - \frac{(1 + i_n/1200)^{300} - 6/(6 - i_n)}{(1/4)(1 + i_n/1200)^{299} - 6/(6 - i_n)^2}.$$

Iteration gives

$$i_2 = 3.65, \quad i_3 = 3.51, \quad i_4 = 3.49, \quad i_5 = 3.49.$$

Thus, the interest rate is 3.49%.

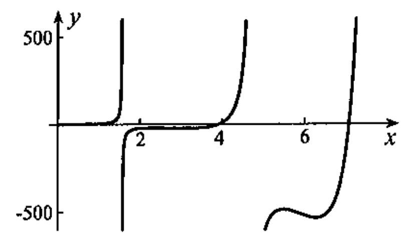
32. If we write the equation in the form $f(x) = e^x(-1+\tan x) + e^{-x}(1+\tan x) = 0$, Newton's iterative procedure defines

$$x_{n+1} = x_n - \frac{e^{x_n}(-1+\tan x_n) + e^{-x_n}(1+\tan x_n)}{e^{x_n}(-1+\tan x_n+\sec^2 x_n) + e^{-x_n}(-1-\tan x_n+\sec^2 x_n)}.$$

For the root just larger than 7, we use an initial approximation $x_1 = 7$. The resulting iterations give

$x_2 = 7.0688,\quad x_3 = 7.068789,$

$x_4 = 7.068583,\quad x_5 = 7.068583.$



Since $f(7.0685825) = -5.8 \times 10^{-4}$ and $f(7.0685835) = 1.8 \times 10^{-3}$, we can say that $x = 7.068583$. When this is divided by 20π, the result to four decimals is 0.1125. Similarly, the smallest frequency is 0.0625.

33. (a) Since $y(3) = 11.8$ and $y(4) = -3.0$, the solution is between 3 and 4. To find it more accurately we use

$$t_1 = 3.8, \qquad t_{n+1} = t_n - \frac{1181(1-e^{-t_n/10}) - 98.1t_n}{118.1e^{-t_n/10} - 98.1}.$$

Iteration gives $t_2 = 3.8334$ and $t_3 = 3.8332$. Since $y(3.825) = 0.14$ and $y(3.835) = -0.03$, it follows that to 2 decimals $t = 3.83$ s.
(b) When we set $0 = y = 20t - 4.905t^2$, the positive solution is 4.08 s.

34. To simplify calculations, we set $z = c/\lambda$. Then, z must satisfy the equation $f(z) = (5-z)e^z - 5 = 0$. Since $f(4) = e^4 - 5$ and $f(5) = -5$, the solution for z is slightly less than 5. To find it more accurately, we use

$$z_1 = 4.9, \qquad z_{n+1} = z_n - \frac{(5-z_n)e^{z_n} - 5}{-e^{z_n} + (5-z_n)e^{z_n}} = z_n - \frac{(5-z_n)e^{z_n} - 5}{(4-z_n)e^{z_n}}.$$

Iteration gives

$$z_2 = 4.969741205, \quad z_3 = 4.965135924, \quad z_4 = 4.965114232, \qquad z_5 = 4.965114232.$$

With this approximation for z, we obtain $\lambda = c/z_5 = 0.000028974$. For a seven decimal answer, we use $g(\lambda) = (5\lambda - c)e^{c/\lambda} - 5\lambda$ to calculate $g(0.00002895) = -1.7 \times 10^{-5}$ and $g(0.00002905) = 5.1 \times 10^{-5}$. Thus, to 7 decimals, $\lambda = 0.0000290$.

35. (i) To find the delay time, we must solve the equation $0.5 = 1 + e^{-2.5\sqrt{11}t}\sin(20t - \pi/2)$, or,

$$g(t) = e^{-2.5\sqrt{11}t}\sin(20t - \pi/2) + 0.5 = 0.$$

Newton's iterative procedure with initial approximation 0.05 defines further approximations by

$$t_1 = 0.05, \quad t_{n+1} = t_n - \frac{e^{-2.5\sqrt{11}t_n}\sin(20t_n - \pi/2) + 0.5}{-2.5\sqrt{11}e^{-2.5\sqrt{11}t_n}\sin(20t_n - \pi/2) + 20e^{-2.5\sqrt{11}t_n}\cos(20t_n - \pi/2)}.$$

Iteration gives $t_2 = 0.0398$, $t_3 = 0.0400$, and $t_4 = 0.0400$. To verify that $t = 0.04$ is the delay time correct to two decimal places we evaluate

$$g(0.035) = -0.07 \qquad \text{and} \qquad g(0.045) = 0.07.$$

(ii) To find the rise time, we must subtract the times when $f(t) = 0.9$ and $f(t) = 0.1$. Following the procedure in (i), we find these times to be 0.0696 and 0.0102 . Rise time is therefore $t = 0.06$.
(iii) Using the above procedures, we find two times at which $f(t) = 1.05$, namely, 0.08 and 0.22. There are three times at which $f(t) = 0.95$, namely, 0.07, 0.26 and 0.34. It follows that $0.95 \le f(t) \le 1.05$ when $t \ge 0.34$; that is, settling time is 0.34.

36. Let $P(x)$ be a cubic polynomial with roots a, b, and c. Then, $P(x) = k(x-a)(x-b)(x-c)$, where k is a constant. Suppose we use Newton's iterative procedure with $x_1 = (a+b)/2$. We require

$$P(x_1) = k\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)\left(\frac{a+b}{2} - c\right) = \frac{k}{8}(b-a)(a-b)(a+b-2c) = -\frac{k}{8}(b-a)^2(a+b-2c),$$

and

$$\begin{aligned} P'(x_1) &= k\left[\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right) + \left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - c\right) + \left(\frac{a+b}{2} - b\right)\left(\frac{a+b}{2} - c\right)\right] \\ &= \frac{k}{4}\left[(b-a)(a-b) + (b-a)(a+b-2c) + (a-b)(a+b-2c)\right] \\ &= -\frac{k}{4}(b-a)^2. \end{aligned}$$

Therefore,

$$x_2 = \frac{a+b}{2} - \frac{-(k/8)(b-a)^2(a+b-2c)}{-(k/4)(b-a)^2} = \frac{a+b}{2} - \frac{1}{2}(a+b-2c) = c.$$

EXERCISES 4.2

1. Since $f'(x) = 2$, the function is increasing for all x.

2. Since $f'(x) = -5$, the function is decreasing for all x.

3. Since $f'(x) = 2x - 3$, it follows that $f'(x) \le 0$ for $x \le 3/2$ and $f'(x) \ge 0$ for $x \ge 3/2$. The function is therefore decreasing for $x \le 3/2$ and increasing for $x \ge 3/2$.

4. Since $f'(x) = -4x + 5$, it follows that $f'(x) \le 0$ for $x \ge 5/4$ and $f'(x) \ge 0$ for $x \le 5/4$. The function is therefore decreasing for $x \ge 5/4$ and increasing for $x \le 5/4$.

5. Since $f'(x) = 6x + 6$, it follows that $f'(x) \le 0$ for $x \le -1$ and $f'(x) \ge 0$ for $x \ge -1$. The function is therefore decreasing for $x \le -1$ and increasing for $x \ge -1$.

6. Since $f'(x) = 2 - 8x$, it follows that $f'(x) \le 0$ for $x \ge 1/4$ and $f'(x) \ge 0$ for $x \le 1/4$. The function is therefore decreasing for $x \ge 1/4$ and increasing for $x \le 1/4$.

7. Since $f'(x) = 6x^2 - 36x + 48 = 6(x-2)(x-4)$, it follows that $f'(x) \le 0$ for $2 \le x \le 4$, and $f'(x) \ge 0$ for $x \le 2$ and $x \ge 4$. The function is therefore decreasing for $2 \le x \le 4$, and increasing for $x \le 2$ and $x \ge 4$.

8. Since $f'(x) = 3x^2 + 12x + 12 = 3(x+2)^2 \ge 0$ for all x, the function is always increasing.

9. Since $f'(x) = 12x^2 - 36x = 12x(x-3)$, it follows that $f'(x) \le 0$ for $0 \le x \le 3$, and $f'(x) \ge 0$ for $x \le 0$ and $x \ge 3$. The function is therefore decreasing for $0 \le x \le 3$, and increasing for $x \le 0$ and $x \ge 3$.

10. Since $f'(x) = -18 - 18x - 6x^2 = -6(3 + 3x + x^2) < 0$ for all x, the function is always decreasing.

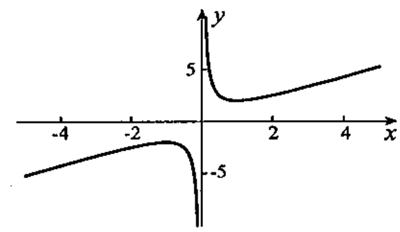
11. Since $f'(x) = 12x^3 + 12x^2 - 24 = 12(x-1)(x^2+2x+2)$, it follows that $f'(x) \le 0$ for $x \le 1$ and $f'(x) \ge 0$ for $x \ge 1$. The function is therefore decreasing for $x \le 1$ and increasing for $x \ge 1$.

12. Since $f'(x) = 12x^3 - 12x^2 + 48x - 48 = 12(x-1)(x^2+4)$, it follows that $f'(x) \le 0$ for $x \le 1$ and $f'(x) \ge 0$ for $x \ge 1$. The function is therefore decreasing for $x \le 1$ and increasing for $x \ge 1$.

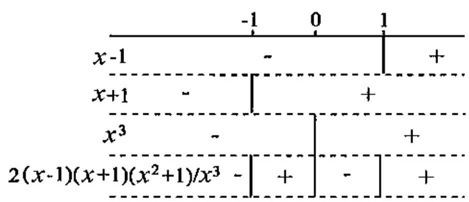
13. Since $f'(x) = 4x^3 - 12x^2 - 16x + 48 = 4(x+2)(x-2)(x-3)$, it follows that $f'(x) \le 0$ for $x \le -2$ and $2 \le x \le 3$, and $f'(x) \ge 0$ for $-2 \le x \le 2$ and $x \ge 3$. The function is therefore decreasing for $x \le -2$ and $2 \le x \le 3$, and increasing for $-2 \le x \le 2$ and $x \ge 3$.

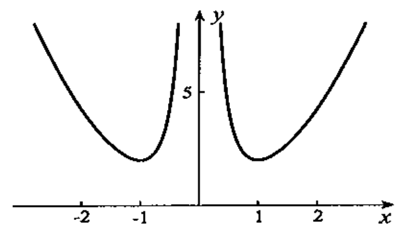
14. Since $f'(x) = 5x^4 - 5 = 5(x-1)(x+1)(x^2+1)$, we find that $f'(x) \le 0$ when $-1 \le x \le 1$, and $f'(x) \ge 0$ when $x \le -1$ and $x \ge 1$. The function is therefore decreasing for $-1 \le x \le 1$, and it is increasing on the intervals $x \le -1$ and $x \ge 1$.

15. Since $f'(x) = 1 - 1/x^2 = (x^2-1)/x^2 = (x-1)(x+1)/x^2$, it follows that $f'(x) \leq 0$ for $-1 \leq x < 0$ and $0 < x \leq 1$, and $f'(x) \geq 0$ for $x \leq -1$ and $x \geq 1$. The function is therefore decreasing for $-1 \leq x < 0$ and $0 < x \leq 1$, and increasing for $x \leq -1$ and $x \geq 1$. The graph corroborates this.

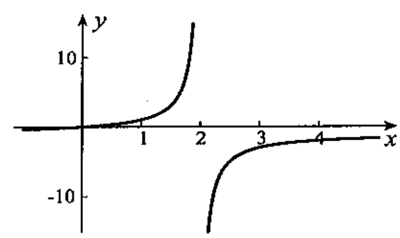


16. The sign diagram below evaluates the sign of $f'(x) = 2x - 2/x^3 = 2(x-1)(x+1)(x^2+1)/x^3$. The function is decreasing on the intervals $x \leq -1$ and $0 < x \leq 1$, and increasing on the intervals $-1 \leq x < 0$ and $x \geq 1$. The graph corroborates this.

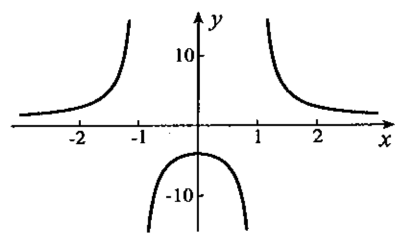




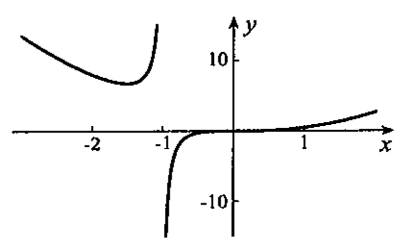
17. Since $f'(x) = \dfrac{(2-x)(1) - x(-1)}{(2-x)^2} = \dfrac{2}{(2-x)^2}$, it follows that $f'(x) \geq 0$ for all $x \neq 2$. The function is therefore increasing for $x < 2$ and $x > 2$. The graph shows that it is not increasing for all x.



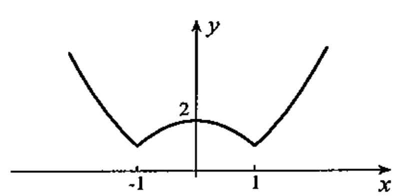
18. Since $f'(x) = \dfrac{(x^2-1)(2x) - (x^2+4)(2x)}{(x^2-1)^2} = \dfrac{-10x}{(x^2-1)^2}$, it follows that $f'(x) \leq 0$ when $0 \leq x < 1$ and $x > 1$, and $f'(x) \geq 0$ when $x < -1$ and $-1 < x \leq 0$. The function is decreasing on the intervals $0 \leq x < 1$ and $x > 1$, and it is increasing on $x < -1$ and $-1 < x \leq 0$. The graph corroborates this.



19. Since $f'(x) = \dfrac{(x+1)(3x^2) - x^3(1)}{(x+1)^2} = \dfrac{x^2(2x+3)}{(x+1)^2}$, it follows that $f'(x) \leq 0$ when $x \leq -3/2$, and $f'(x) \geq 0$ when $-3/2 \leq x < -1$ and $x > -1$. The function is decreasing on the interval $x \leq -3/2$, and it is increasing on $-3/2 \leq x < -1$ and $x > -1$. The graph corroborates this.



20. The graph of $f(x)$ indicates that $f(x)$ is decreasing on the intervals $x \leq -1$ and $0 \leq x \leq 1$, and it is increasing on $-1 \leq x \leq 0$ and $x \geq 1$.



21. Since $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$, we find that $f'(x) \leq 0$ when $x \geq 1$, and $f'(x) \geq 0$ when $x \leq 1$. The function is therefore decreasing for $x \geq 1$, and it is increasing on the interval $x \leq 1$.

22. Since $f'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x}$, it follows that $f'(x) \le 0$ for $x \le 0$ and $x \ge 2$, and $f'(x) \ge 0$ for $0 \le x \le 2$. The function is decreasing on the intervals $x \le 0$ and $x \ge 2$, and it is increasing for $0 \le x \le 2$.

23. Since $f'(x) = 2x/(x^2+5)$, we find that $f'(x) \le 0$ when $x \le 0$, and $f'(x) \ge 0$ when $x \ge 0$. The function is therefore decreasing for $x \le 0$, and it is increasing for $x \ge 0$.

24. Since $f'(x) = \ln x + 1$, we find that $f'(x) \le 0$ when $0 < x \le 1/e$, and $f'(x) \ge 0$ when $x \ge 1/e$. The function is therefore decreasing for $0 < x \le 1/e$ and increasing for $x \ge 1/e$.

25. Since $f(x) = \dfrac{(x+3)(x-3)}{x-3} = x+3$, except for $x = 3$, where $f(x)$ is undefined, the function is increasing for $x < 3$ and $x > 3$.

26. Since $f(x) = \dfrac{(x-1)(x+1)(x+2)}{(2+x)(1-x)} = -x-1$, except for $x = -2$ and $x = 1$, where $f(x)$ is undefined, the function is decreasing for $x < -2$, $-2 < x < 1$, and $x > 1$.

27. If we differentiate implicitly with respect to x, we find

$$1 = -6ar\frac{dr}{dx} + 3r^2\frac{dr}{dx} \quad \Longrightarrow \quad \frac{dr}{dx} = \frac{1}{3r^2 - 6ar} = \frac{1}{3r(r-2a)}.$$

Since this is negative for $0 < r < 2a$, the function is decreasing on this interval.

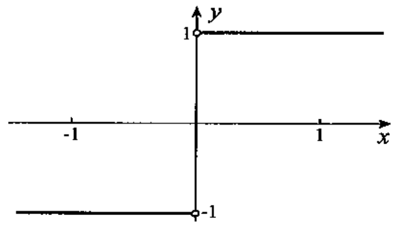
28.

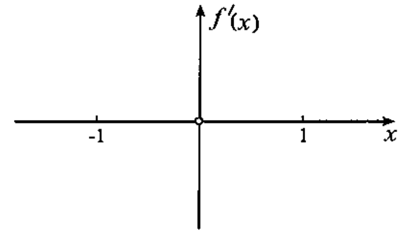
29.

30.

31.

32.





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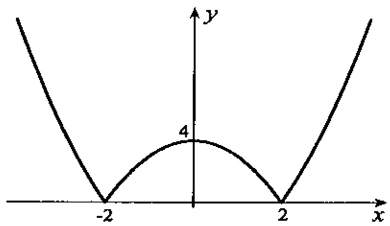
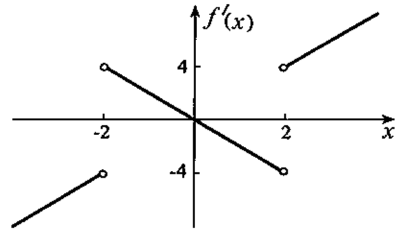
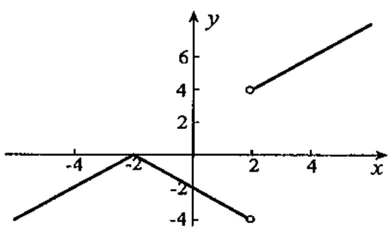
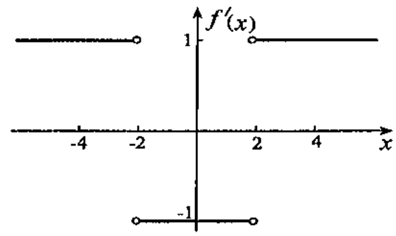




34.





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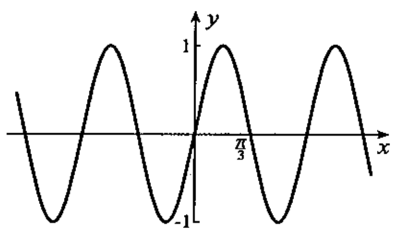
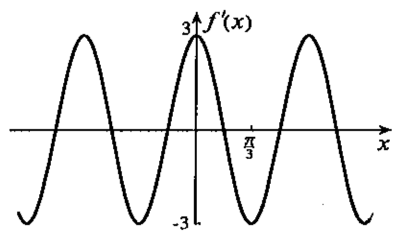
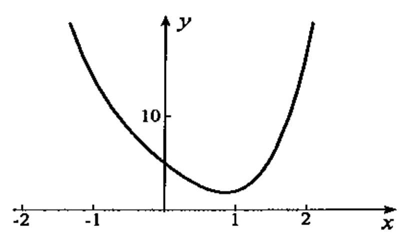




36. We first consider points where $0 = f'(x) = 4x^3 + 4x - 6$. The plot of $f(x)$ indicates that $f'(x)$ has a zero between 0 and 1. To find it more accurately, we use Newton's iterative procedure with $x_1 = 0.7$ and

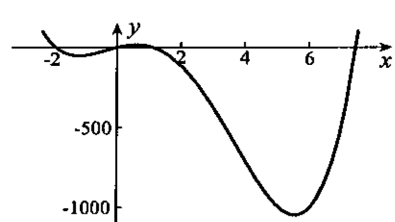
$$x_{n+1} = x_n - \frac{4x_n^3 + 4x_n - 6}{12x_n^2 + 4}.$$



Iteration gives $x_2 = 0.885\,02$, $x_3 = 0.861\,67$, $x_4 = 0.861\,22$, $x_5 = 0.861\,22$. Since $f'(0.861\,15) = -9.6 \times 10^{-4}$ and $f'(0.861\,25) = 3.3 \times 10^{-4}$, it follows that $x = 0.861\,2$ is the solution of $f'(x) = 0$ to four decimals. The graph makes it clear that $f(x)$ is decreasing for $x \le 0.861\,2$ and increasing for $x \ge 0.861\,2$.

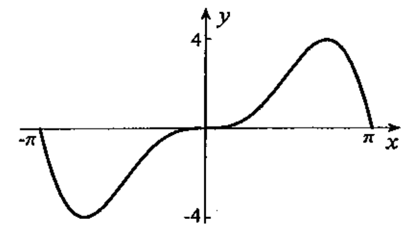
37. We first consider points where $0 = f'(x) = 12x^3 - 60x^2 - 48x + 48 = 12(x^3 - 5x^2 - 4x + 4)$. The plot of $f(x)$ indicates that $f'(x)$ has three zeros. To find the zero between 0 and 1, we use Newton's iterative procedure with $x_1 = 0.6$ and

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n^2 - 4x_n + 4}{3x_n^2 - 10x_n - 4}.$$



Iteration gives $x_2 = 0.601\,79$ and $x_3 = 0.601\,79$. Since $f'(0.601\,75) = 4.6 \times 10^{-3}$ and $f'(0.601\,85) = -6.1 \times 10^{-3}$, it follows that $x = 0.601\,8$ is the solution of $f'(x) = 0$ to four decimals. The other two solutions are $-1.189\,5$ and $5.587\,7$. The graph makes it clear that $f(x)$ is decreasing for $x \le -1.189\,5$ and $0.601\,8 \le x \le 5.587\,7$, and increasing for $-1.189\,5 \le x \le 0.601\,8$ and $x \ge 5.587\,7$.

38. We first consider points where $0 = f'(x) = 2x\sin x + x^2\cos x = x(2\sin x + x\cos x)$. This equation implies that $x = 0$ or $2\sin x + x\cos x = 0$. The graph of $f(x)$ indicates that the second of these has a solution between $x = 2$ and $x = 3$. To find it we use Newton's iterative procedure with

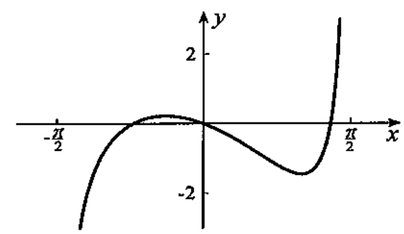


$$x_1 = 2, \qquad x_{n+1} = x_n - \frac{2\sin x_n + x_n\cos x_n}{3\cos x_n - x_n\sin x_n}.$$

Iteration gives $x_2 = 2.321\,58$, $x_3 = 2.289\,13$, $x_4 = 2.288\,93$, and $x_5 = 2.288\,93$. Since $f'(2.288\,85) = 6.7\times10^{-4}$ and $f'(2.288\,95) = -1.7\times10^{-4}$, we can say that a solution of $f'(x) = 0$ to four decimals is $x = 2.288\,9$. The graph makes it clear that so also is $x = -2.288\,9$. We conclude that $f(x)$ is decreasing on the intervals $-\pi \le x \le -2.288\,9$ and $2.288\,9 \le x \le \pi$, and it is increasing on $-2.288\,9 \le x \le 2.288\,9$.

39. We first consider points where $0 = f'(x) = \sec^2 x - 2x - 2$. The graph of $f(x)$ indicates that there are two solutions. To find the solution near $x = 1$, we use Newton's iterative procedure with

$$x_1 = 1, \qquad x_{n+1} = x_n - \frac{\sec^2 x_n - 2x_n - 2}{2\sec^2 x_n\,\tan x_n - 2}.$$



Iteration gives $x_2 = 1.066$, $x_3 = 1.055\,36$, $x_4 = 1.054\,95$, and $x_5 = 1.054\,95$. Since $f'(1.054\,95) = -2.7\times10^{-5}$ and $f'(1.055\,05) = 1.2\times10^{-3}$, we can say that a solution of $f'(x) = 0$ to four decimals is $x = 1.055\,0$. The other solution is $x = -0.407\,1$. We conclude that $f(x)$ is decreasing on the interval $-0.407\,1 \le x \le 1,055\,0$, and it is increasing on the intervals $-\pi/2 < x \le -0.407\,1$ and $1.055\,0 \le x < \pi/2$.

40. For $f(x) = x^{23} + 3x^{15} + 4x + 1$, we find that $f(-1) = -7$ and $f(0) = 1$. By the zero intermediate value theorem, there exists at least one solution of $f(x) = 0$ between $x = -1$ and $x = 0$. Since $f'(x) = 23x^{22} + 45x^{14} + 4 > 0$, the function is increasing for all x. Hence, there can be only one solution of the equation.

41. Suppose that $a > 0$ and $b > 0$ (a similar proof holds when a and b are both negative). Since $f(x) = ax^5 + bx^3 + c$ is continuous, and $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, it follows that the graph of $f(x)$ must cross the x-axis at least once. Since $f'(x) = 5ax^4 + 3bx^2 > 0$, the function is increasing for all x. Hence, there can be only one solution of the equation.

42. Since $f(x) = x^n + ax - 1$ is continuous, and $f(0) = -1$ and $\lim_{x\to\infty} f(x) = \infty$, it follows that the graph of $f(x)$ must cross the x-axis at least once for positive x. Since $f'(x) = nx^{n-1} + a > 0$ for $x > 0$, the function is increasing for $0 \le x < \infty$. Hence, the graph of $f(x)$ can cross the x-axis only once for $x > 0$.

43. Since $f(x) = x^n + x^{n-1} - a$ is continuous, and $f(0) = -a$ and $\lim_{x\to\infty} f(x) = \infty$, it follows that the graph of $f(x)$ must cross the x-axis at least once for positive x. Since $f'(x) = nx^{n-1} + (n-1)x^{n-2} > 0$ for $x > 0$, the function is increasing for $0 \le x < \infty$. Hence, the graph of $f(x)$ can cross the x-axis only once for $x > 0$.

44. If $f(x) = x - \sin x$, then $f(0) = 0$. Since $f'(x) = 1 - \cos x \ge 0$, it follows that the function $f(x)$ is increasing for all $x > 0$. This means that $f(x) > 0$ for all $x > 0$, and therefore $x > \sin x$.

45. The function $f(x) = \cos x - 1 + x^2/2$ has value $f(0) = 0$. Since $f'(x) = -\sin x + x \ge 0$ for $x > 0$ (Exercise 44), it follows that $f(x)$ is increasing for all $x > 0$. This means that $f(x) > 0$ for all $x > 0$, and therefore $\cos x > 1 - x^2/2$.

46. If $f(x) = \sin x - x + x^3/6$, then $f(0) = 0$. Since $f'(x) = \cos x - 1 + x^2/2$, Exercise 45 implies that $f'(x) \ge 0$ for $x > 0$. Consequently, $f(x)$ is increasing for $x > 0$, and $\sin x > x - x^3/6$.

47. The function $f(x) = 1 - x^2/2 + x^4/24 - \cos x$ has value $f(0) = 0$. Since $f'(x) = -x + x^3/6 + \sin x$, Exercise 46 implies that $f'(x) \ge 0$ for $x > 0$. Consequently, $f(x)$ is increasing for all $x > 0$, and $\cos x < 1 - x^2/2 + x^4/24$.

48. If we define a function $f(x) = \dfrac{1}{\sqrt{1+3x}} - 1 + \dfrac{3x}{2}$, then $f(0) = 0$, and

$$f'(x) = \frac{-3}{2(1+3x)^{3/2}} + \frac{3}{2} = \frac{3}{2}\left[1 - \frac{1}{(1+3x)^{3/2}}\right].$$

This derivative is clearly positive for $x > 0$, and therefore the function $f(x)$ is increasing for $x > 0$. Thus, $f(x) > 0$ for $x > 0$, and the required result follows.

49. Not necessarily. The derivative of $f(x)g(x)$ is $f'(x)g(x) + f(x)g'(x)$. Nonnegativity of $f'(x)$ and $g'(x)$ does not guarantee nonnegativity of this derivative.

50. Yes. Since $f(x) > 0$, $f'(x) \geq 0$, $g(x) > 0$, and $g'(x) \geq 0$ on I, it follows that $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \geq 0$ on I also; that is, $f(x)g(x)$ is increasing on I.

51. We prove part (i); the proof of part (ii) is similar. Suppose then that $f'(x) \geq 0$ on an interval $I: a \leq x \leq b$, and let $d_1 < d_2 < \ldots < d_{n-1}$ be the finite number of points in I at which $f'(x) = 0$. Let $d_0 = a$ and $d_n = b$. We prove that $f(x)$ is increasing on each subinterval $I^*: d_{i-1} \leq x \leq d_i$, for $i = 1, \ldots, n$, and therefore it is increasing on I. If x_1 and x_2, where $x_1 > x_2$, are any two points in I^*, then the mean value theorem of Section 3.14 implies the existence of at least one point c in the open interval between d_{i-1} and d_i at which

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \quad \Longrightarrow \quad f(x_1) = f(x_2) + (x_1 - x_2)f'(c).$$

Since $x_1 - x_2 > 0$ and $f'(c) > 0$, it follows that $f(x_1) > f(x_2)$. Hence, $f(x)$ is increasing on I^*.

52. Consider the function $g(x) = x - f(x)$. Since $g'(x) = 1 - f'(x)$, and $f'(x) < 1$ for all x, it follows that $g(x)$ is a decreasing function for all x. Consequently, it can have value 0 at most once; that is, there can exist at most one point x_0 at which $g(x_0) = 0$, and therefore at which $f(x_0) = x_0$.

53. We can express the inequality in the form $\dfrac{\tan b}{b} > \dfrac{\tan a}{a}$. If we define the function $f(x) = \dfrac{\tan x}{x}$, then we must show that $f(x)$ is increasing on the interval $I: 0 < x < \pi/2$. The plot in the left figure below seems to indicate that this is the case, but it is not conclusive. To verify this algebraically, we show that $f'(x) \geq 0$ on I,

$$\begin{aligned} f'(x) &= -\frac{1}{x^2}\tan x + \frac{1}{x}\sec^2 x = \frac{1}{x^2}(x\sec^2 x - \tan x) \\ &= \frac{1}{x^2}\left(\frac{x}{\cos^2 x} - \frac{\sin x}{\cos x}\right) = \frac{1}{x^2\cos^2 x}(x - \sin x\cos x). \end{aligned}$$

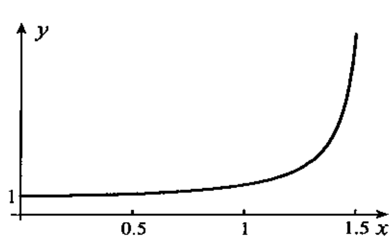
Since $x^2\cos^2 x$ is positive on I, the sign of $f'(x)$ is determined by that of $x - \sin x\cos x$. Therefore, we must show that

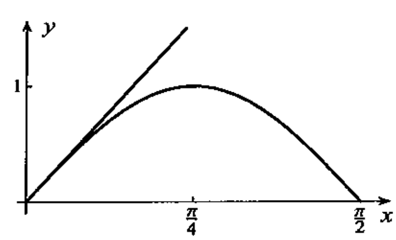
$$0 \leq x - \sin x\cos x.$$

This inequality can be simplified if it is multiplied by 2,

$$0 \leq 2x - 2\sin x\cos x = 2x - \sin 2x \qquad \text{or} \qquad 2x \geq \sin 2x.$$

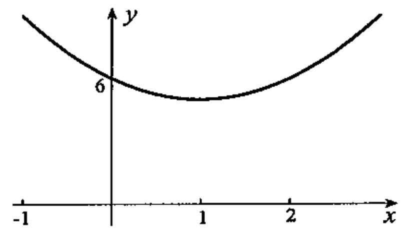
Graphs of the functions $2x$ and $\sin 2x$ are shown in the right figure. Important to this picture are the facts that the line $y = 2x$ and the curve $y = \sin 2x$ both have slope 2 at $x = 0$, but the slope of $y = \sin 2x$ is less than 2 for $x > 0$. Clearly, then, $2x \geq \sin 2x$, and our proof is complete.



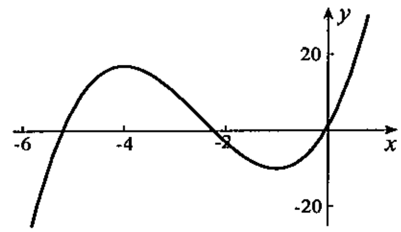


EXERCISES 4.3

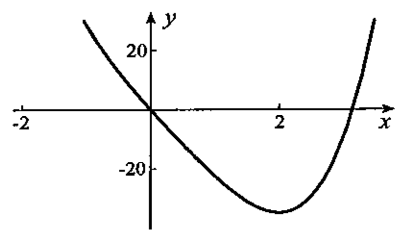
1. Since $f'(x) = 2x - 2$, the only critical point is $x = 1$. Because $f'(x)$ changes from negative to positive as x increases through 1, $x = 1$ gives a relative minimum.



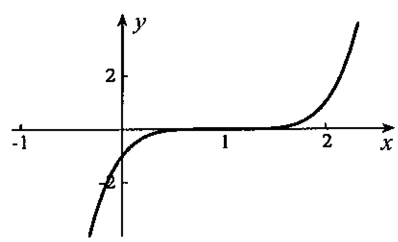
2. Since $f'(x) = 6x^2 + 30x + 24 = 6(x+1)(x+4)$, critical points are $x = -1$ and $x = -4$. Because $f'(x)$ changes from positive to negative as x increases through -4, $x = -4$ yields a relative maximum. Since $f'(x)$ changes from negative to positive as x increases through -1, $x = -1$ gives a relative minimum.



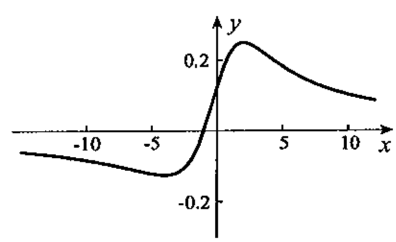
3. Since $f'(x) = 4x^3 - 4x^2 + 4x - 24 = 4(x-2)(x^2+x+3)$, the only critical point is $x = 2$. Because $f'(x)$ changes from negative to positive as x increases through 2, $x = 2$ gives a relative minimum.



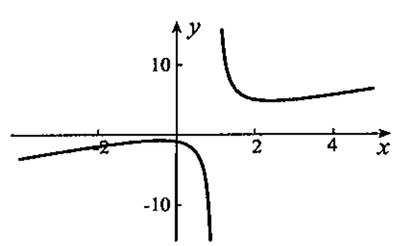
4. Since $f'(x) = 5(x-1)^4$, the only critical point is $x = 1$. Because $f'(x) \geq 0$ for all x, the function is always increasing, and $x = 1$ does not give a relative maximum or minimum.



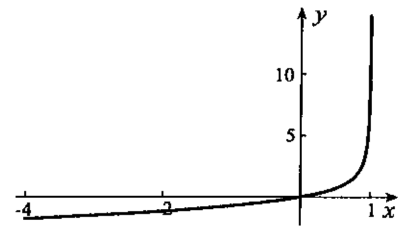
5. Since $f'(x) = \dfrac{(x^2+8)(1) - (x+1)(2x)}{(x^2+8)^2} = \dfrac{-(x+4)(x-2)}{(x^2+8)^2}$, critical points are $x = -4$ and $x = 2$. Since $f'(x)$ changes from negative to positive as x increases through -4, this critical point gives a relative minimum. Because $f'(x)$ changes from positive to negative as x increases through 2, there is a relative maximum at this value.



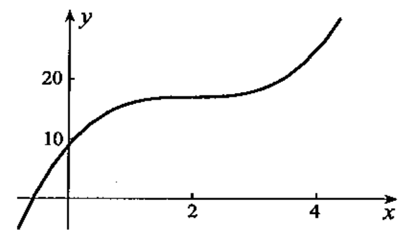
6. Since $f'(x) = \dfrac{(x-1)(2x) - (x^2+1)(1)}{(x-1)^2} = \dfrac{x^2 - 2x - 1}{(x-1)^2}$, critical points are $x = (2 \pm \sqrt{4+4})/2 = 1 \pm \sqrt{2}$. The derivative does not exist at $x = 1$, but this is not a critical point because the function is not defined at $x = 1$. Since $f'(x)$ changes from positive to negative as x increases through $1 - \sqrt{2}$, this critical point gives a relative maximum. Because $f'(x)$ changes from negative to positive as x increases through $1 + \sqrt{2}$, there is a relative minimum at this point.



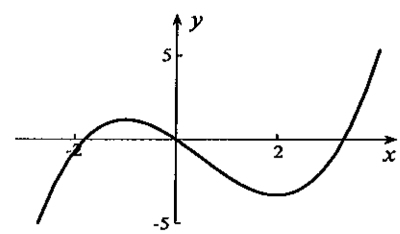
7. Since $f'(x) = \dfrac{\sqrt{1-x} - x(1/2)(1-x)^{-1/2}(-1)}{1-x} = \dfrac{2-x}{2(1-x)^{3/2}}$,
There are no critical points at which $f'(x) = 0$.
The derivative does not exist at $x = 1$, but because $f(1)$ is undefined, $x = 1$ is not critical.



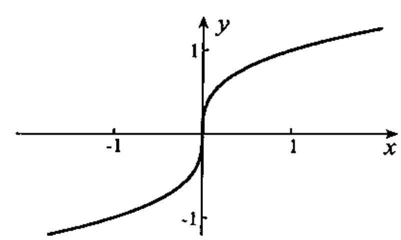
8. Since $f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2$, the only critical point is $x = 2$. Since $f'(x) \geq 0$ for all x, it follows that $f(x)$ is increasing for all x, and there cannot be a relative extremum at $x = 2$.



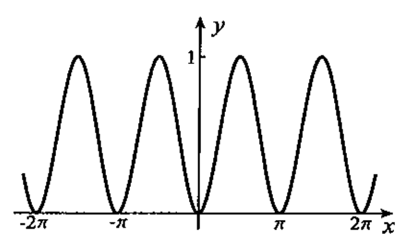
9. Since $f'(x) = x^2 - x - 2 = (x-2)(x+1)$, there are two critical points $x = -1$ and $x = 2$. Since $f'(x)$ changes from positive to negative as x increases through -1, this critical point gives a relative maximum. Because $f'(x)$ changes from negative to positive as x increases through 2, there is a relative minimum at this point.



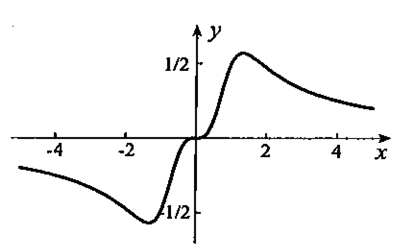
10. Since $f'(x) = (1/3)x^{-2/3}$, and this derivative does not exist at $x = 0$, this is a critical point. Because $f'(x)$ is positive for all $x \neq 0$, it follows that the function is always increasing, and $x = 0$ cannot therefore yield a relative extremum.



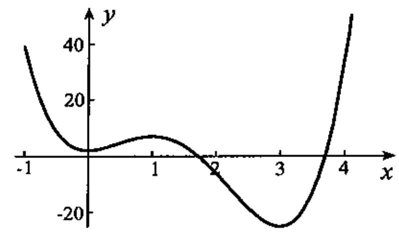
11. Since $f'(x) = 2\sin x \cos x = \sin 2x$, critical points are $x = n\pi/2$, where n is an integer. Because $f'(x)$ changes from negative to positive as x increases through $n\pi$, these critical points gives relative minima. Because $f'(x)$ changes from positive to negative as x increases through $(2n+1)\pi/2$, these critical points give relative maxima.



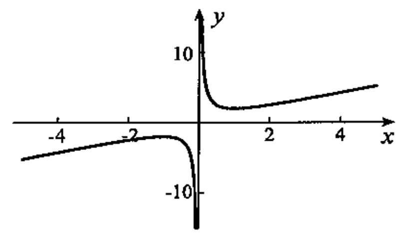
12. Since $f'(x) = \dfrac{(x^4+1)(3x^2) - x^3(4x^3)}{(x^4+1)^2} = \dfrac{x^2(3^{1/4}-x)(3^{1/4}+x)(\sqrt{3}+x^2)}{(x^4+1)^2}$,
critical points occur at $x = 0$ and $x = \pm 3^{1/4}$. Because $f'(x)$ changes from negative to positive as x increases through $-3^{1/4}$, this critical point gives a relative minimum. Since $f'(x)$ changes from positive to negative as x increases through $3^{1/4}$, this critical point yields a relative maximum. The derivative remains positive as x increases through 0. Consequently $f(x)$ is increasing in an interval around $x = 0$, and this point cannot yield a relative extremum.



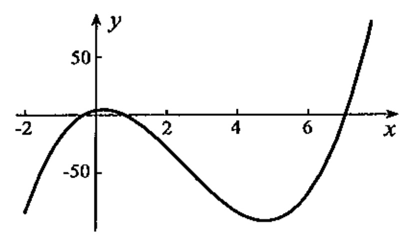
13. Since $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$, critical points are $x = 0, 1, 3$. Because $f'(x)$ changes from negative to positive as x increases through 0 and 3, these critical points give relative minima. Since $f'(x)$ changes from positive to negative as x increases through 1, this critical point yields a relative maximum.



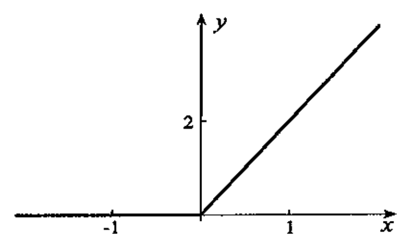
14. Since $f'(x) = 1 - 1/x^2$, the derivative vanishes at $x = \pm 1$. It does not exist at $x = 0$, but this point cannot be critical because $f(0)$ is not defined. Since $f'(x)$ changes from positive to negative as x increases through -1, $x = -1$ gives a relative maximum. The critical point $x = 1$ gives a relative minimum since $f'(x)$ changes from negative to positive as x increases through this point.



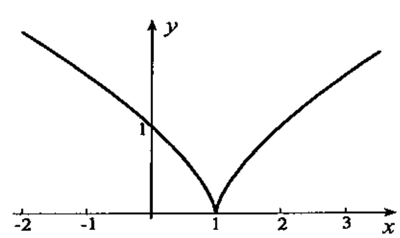
15. Since $f'(x) = 6x^2 - 30x + 6 = 6(x^2 - 5x + 1)$, critical points are $x = (5 \pm \sqrt{25-4})/2 = (5 \pm \sqrt{21})/2$. Because $f'(x)$ changes from positive to negative as x increases through $(5 - \sqrt{21})/2$, this critical point gives a relative maximum. Since $f'(x)$ changes from negative to positive as x increases through $(5 + \sqrt{21})/2$, this critical point yields a relative minimum.



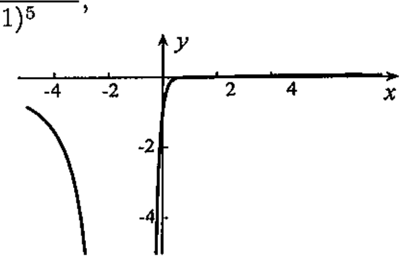
16. The point $x = 0$ is critical since $f'(0)$ does not exist, and it yields a relative minimum. Every point on the negative x-axis is also critical, and each such point yields a relative maximum and a relative minimum (see Definitions 4.3 and 4.4).



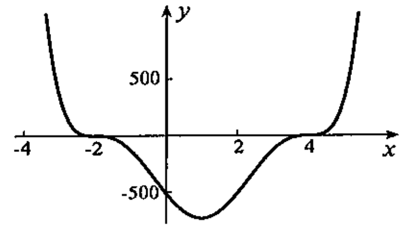
17. Since $f'(x) = (2/3)(x-1)^{-1/3}$, the derivative is never 0, but $x = 1$ is critical since $f'(1)$ does not exist. Because $f'(x)$ changes from negative to positive as x increases through $x = 1$, this critical point gives a relative minimum.



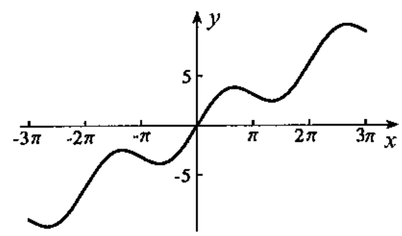
18. Since $f'(x) = \dfrac{(x+1)^4 3(x-1)^2 - (x-1)^3 4(x+1)^3}{(x+1)^8} = \dfrac{(x-1)^2(7-x)}{(x+1)^5}$, critical points are $x = 1$ and $x = 7$. Although $f'(x)$ does not exist at $x = -1$, this point is not critical since $f(-1)$ is not defined. Since $f'(x)$ changes from positive to negative as x increases through 7, $x = 7$ gives a relative maximum. Because $f'(x)$ remains positive as x increases through 1, this point does not give a relative extremum.



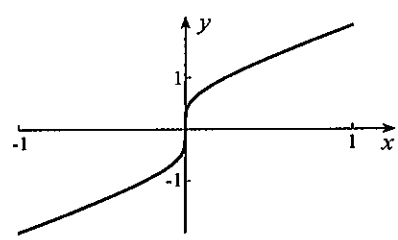
19. Since $f'(x) = 3(x+2)^2(x-4)^3 + 3(x+2)^3(x-4)^2 = 6(x+2)^2(x-4)^2(x-1)$, critical points are $x = -2, 1, 4$. Because $f'(x)$ changes from negative to positive as x increases through 1, this critical point yields a relative minimum. Since $f'(x)$ does not change sign as x passes through $x = -2$ and $x = 4$, the function does not have relative extrema at these points.



20. Since $f'(x) = 1 + 2\cos x$, the derivative vanishes for all values of x satisfying $\cos x = -1/2$. Solutions of this equation are $x = 2\pi/3 + 2n\pi$ and $x = 4\pi/3 + 2n\pi$, where n is an integer. Because $f'(x)$ changes from positive to negative as x increases through the values $2(3n+1)\pi/3$, these critical points give relative maxima. On the other hand, $f'(x)$ changes from negative to positive as x increases through the remaining critical points $x = 2(3n+2)\pi/3$, which therefore give relative minima.



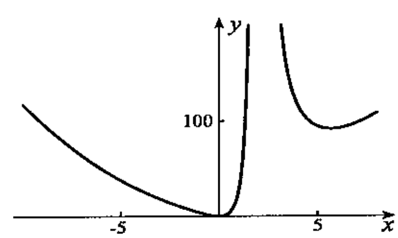
21. Since $f'(x) = (1/5)x^{-4/5} + 1 = \dfrac{5x^{4/5}+1}{5x^{4/5}}$, there is no point at which $f'(x) = 0$. Because $f'(0)$ does not exist, $x = 0$ is critical. Since $f'(x) > 0$ for all x, the function is always increasing, and $x = 0$ cannot give a relative extremum.



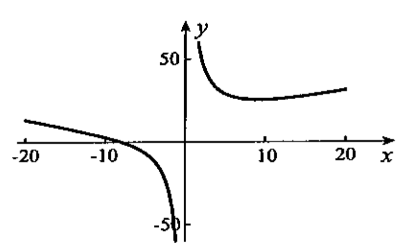
22. For critical points we first solve

$$0 = f'(x) = 2x + \frac{(x-2)^2(50x) - 25x^2(2)(x-2)}{(x-2)^4} = 2x + \frac{50x(x-2-x)}{(x-2)^3} = 2x\left[\frac{(x-2)^3 - 50}{(x-2)^3}\right].$$

Thus, either $x = 0$, or $(x-2)^3 = 50$, and the latter equation requires $x = 2 + 50^{1/3}$. The derivative does not exist at $x = 2$, but neither does $f(2)$, and therefore $x = 2$ is not critical. Since $f'(x)$ changes from negative to positive as x increases through 0, $x = 0$ gives a relative minimum. Because $f'(x)$ changes from negative to positive as x increases through $2 + 50^{1/3}$, this critical point also yields a relative minimum.

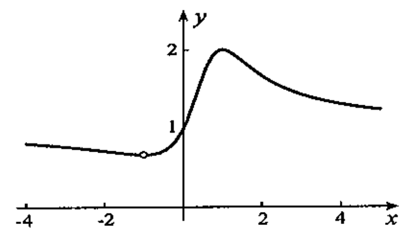


23. For critical points we first solve
$0 = f'(x) = \left(\dfrac{-8}{x^2}\right)\sqrt{x^2+100} + \left(x + \dfrac{8}{x}\right)\dfrac{x}{\sqrt{x^2+100}} =$
$\dfrac{x^3 - 800}{x^2\sqrt{x^2+100}}$. The only solution is $x = 800^{1/3}$. Since $f'(x)$ changes from negative to positive as x increases through this critical point, it yields a relative minimum.

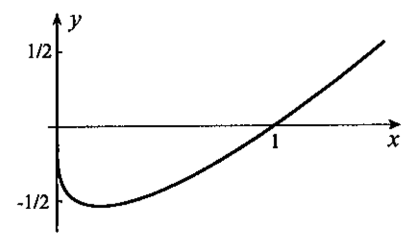


24. Since $f'(x) = \dfrac{(1+x^3)(1+2x+3x^2)-(1+x+x^2+x^3)(3x^2)}{(1+x^3)^2} =$

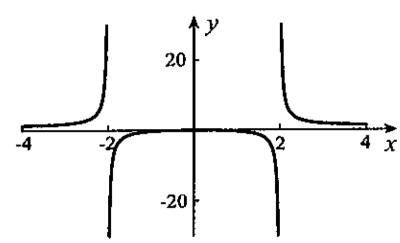
$\dfrac{(1-x)(1+x)^3}{(1+x^3)^2}$, it follows that $x = 1$ is critical. The derivative does not exist at $x = -1$, but this point is not critical since $f(-1)$ is not defined. Because $f'(x)$ changes from positive to negative as x increases through 1, this critical point gives a relative maximum.



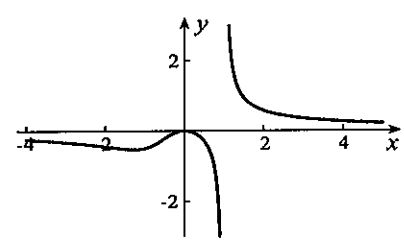
25. Since $f'(x) = \dfrac{5}{4}x^{1/4} - \dfrac{1}{4}x^{-3/4} = \dfrac{5x-1}{4x^{3/4}}$, $x = 1/5$ is a critical point at which $f'(x) = 0$. The right-hand derivative does not exist at $x = 0$, but $f(0) = 0$, so that $x = 0$ is also critical. Because $f'(x)$ changes from negative to positive as x increases through $1/5$, this critical point gives a relative minimum. According to Definition 4.3, $x = 0$ cannot yield a relative maximum.



26. Since $f'(x) = \dfrac{(x^2-4)(2x)-x^2(2x)}{(x^2-4)^2} = \dfrac{-8x}{(x^2-4)^2}$, the only critical point is $x = 0$. The derivative does not exist at $x = \pm 2$, but these are not critical points because $f(\pm 2)$ do not exist. Since $f'(x)$ changes from positive to negative as x increases through $x = 0$, this critical point yields a relative maximum.



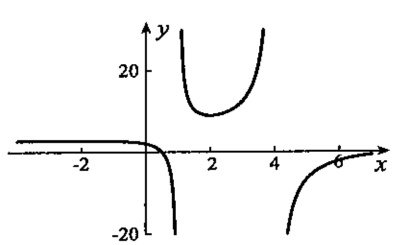
27. Since $f'(x) = \dfrac{(x^3-1)(2x)-x^2(3x^2)}{(x^3-1)^3} = \dfrac{-x(x^3+2)}{(x^3-1)^3}$, $x = 0$ is a critical point, as is $x = -2^{1/3}$. The derivative does not exist at $x = 1$, but this is not a critical point because $f(1)$ does not exist. Because $f'(x)$ changes from positive to negative as x increases through 0, this critical point yields a relative maximum. Since $f'(x)$ changes from negative to positive as x increases through $-2^{1/3}$, this critical point gives a relative minimum.



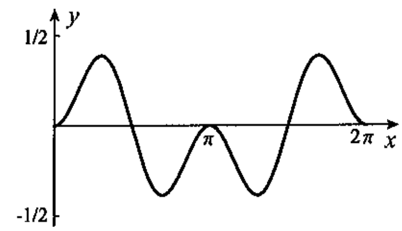
28. With the function written in the form $f(x) = \dfrac{2x^2-17x+8}{x^2-5x+4}$, we set

$0 = f'(x) = \dfrac{(x^2-5x+4)(4x-17)-(2x^2-17x+8)(2x-5)}{(x^2-5x+4)^2} =$

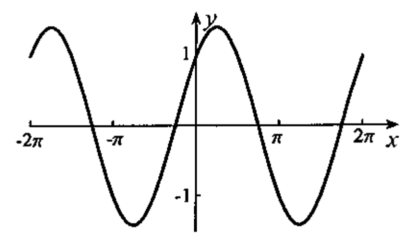
$\dfrac{7(x^2-4)}{(x-1)^2(x-4)^2}$. Critical points are $x = \pm 2$. Since $f'(x)$ changes from positive to negative as x increases through -2, this critical point gives a relative maximum. The critical point $x = 2$ gives a relative minimum as $f'(x)$ changes from negative to positive through this point.



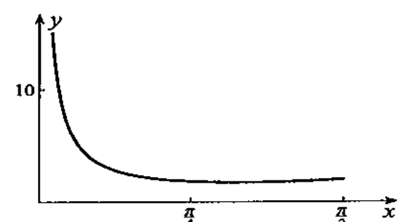
29. Since $f'(x) = 2\sin x\cos^2 x - \sin^3 x =$ $\sin x(2\cos^2 x - \sin^2 x) = \sin x(3\cos^2 x - 1)$, critical points are $x = 0, \pi, 2\pi$ and values of x satisfying $\cos x = \pm 1/\sqrt{3}$. From $\cos x = 1/\sqrt{3}$, we obtain $x = 0.955, 2\pi - 0.955$, and from $\cos x = -1/\sqrt{3}$, we get $x = \pi \pm 0.955$. Since $f'(x)$ changes from positive to negative as x increases through 0.955, π, and $2\pi - 0.955$, these critical points give relative maxima. Since $f'(x)$ changes from negative to positive as x increases through $\pi \pm 0.955$, these critical points give relative minima. End points $x = 0, 2\pi$ cannot give relative minima (see Definition 4.4).



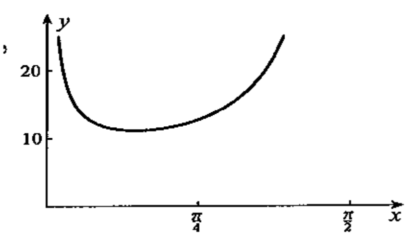
30. Since $f'(x) = \cos x - \sin x$, critical points are $x = \pi/4 + n\pi$, where n is an integer. Since $f'(x)$ changes from positive to negative as x increases through the critical points $\pi/4 + 2n\pi$, these points yield relative maxima. The remaining critical points $5\pi/4 + 2n\pi$ give relative minima as $f'(x)$ changes from negative to positive through these points.



31. Since $f'(x) = -2\csc x\cot x + \csc^2 x = \dfrac{1 - 2\cos x}{\sin^2 x}$, the only critical point is $x = \pi/3$. Because $f'(x)$ changes from negative to positive as x increases through $\pi/3$, the critical point gives a relative minimum.



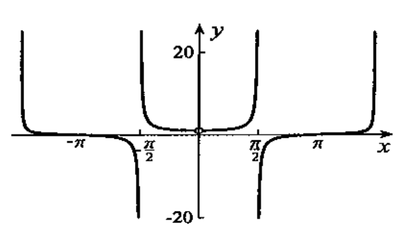
32. Since $f'(x) = -\csc x\cot x + 8\sec x\tan x = \dfrac{\cos x}{\sin^2 x}(8\tan^3 x - 1)$, critical points satisfy $\tan x = 1/2$. The only solution of this equation between 0 and $\pi/2$ is 0.464. Because $f'(x)$ changes from negative to positive as x increases through this critical point, it yields a relative minimum.

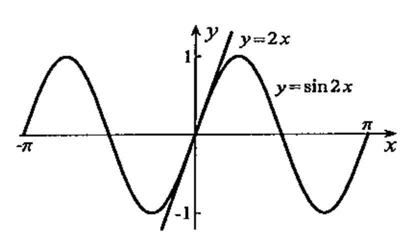


33. The left figure below suggests that the function has no critical points. We can verify this by considering

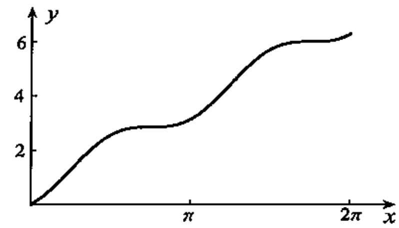
$$0 = f'(x) = \frac{x\sec^2 x - \tan x}{x^2} = \frac{x - \sin x\cos x}{x^2\cos^2 x} = \frac{2x - \sin 2x}{2x^2\cos^2 x}.$$

Graphs of $y = 2x$ and $y = \sin 2x$ in the right figure show $x = 0$ as the only solution of $2x = \sin 2x$. But this cannot be critical because $f(0)$ is not defined.

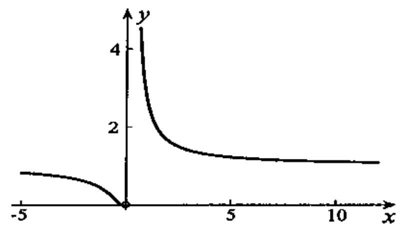




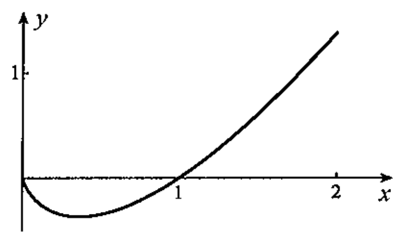
34. Since $f'(x) = 1 + 2\sin x\,\cos x = 1 + \sin 2x$, critical points occur when $\sin 2x = -1$. The only solutions of this equation in the interval $0 < x < 2\pi$ are $x = 3\pi/4$ and $x = 7\pi/4$. Because $f'(x)$ is always nonnegative, the function is increasing, and the critical points do not give relative extrema.



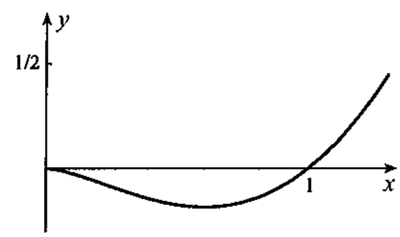
35. The graph indicates that the function has no critical points. We can also see this algebraically since $0 = f'(x) = e^{1/x}(-1/x^2)$ has no solutions.



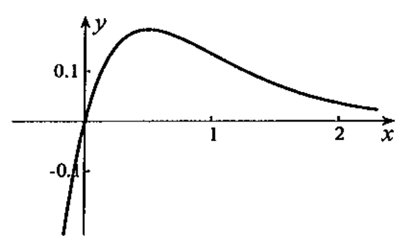
36. The derivative $f'(x) = \ln x + 1$ vanishes when $x = 1/e$. This critical point yields a relative minimum because $f'(x)$ changes from negative to positive as x increases through $1/e$.



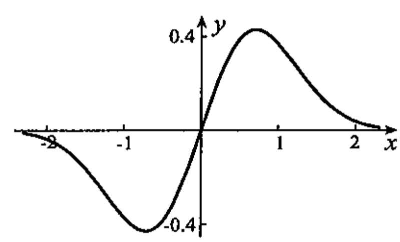
37. The derivative $f'(x) = 2x\ln x + x = x(2\ln x + 1)$ vanishes when $x = 1/\sqrt{e}$. This critical point yields a relative minimum because $f'(x)$ changes from negative to positive as x increases through $1/\sqrt{e}$.



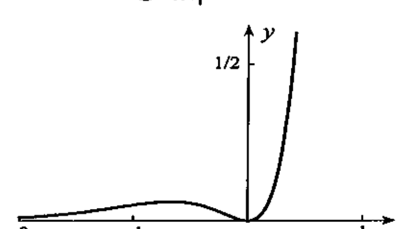
38. Since $f'(x) = e^{-2x} - 2xe^{-2x} = (1-2x)e^{-2x}$, the only critical point is $x = 1/2$. Since $f'(x)$ changes from positive to negative as x increases through $1/2$, the critical point gives a relative maximum.



39. Since $f'(x) = e^{-x^2} - 2x^2\,e^{-x^2} = (1-2x^2)e^{-x^2}$, critical points are $x = \pm 1/\sqrt{2}$. Since $f'(x)$ changes from positive to negative as x increases through $1/\sqrt{2}$, this critical point gives a relative maximum. Since $f'(x)$ changes from negative to positive as x increases through $-1/\sqrt{2}$, this critical point gives a relative minimum.



40. Since $f'(x) = 2xe^{3x} + 3x^2e^{3x} = x(3x+2)e^{3x}$, we have two critical points, $x = 0$ and $x = -2/3$. Because $f'(x)$ changes from negative to positive as x increases through 0, this critical point gives a relative minimum. The critical point $x = -2/3$ gives a relative maximum since $f'(x)$ changes from positive to negative as x increases through this point.

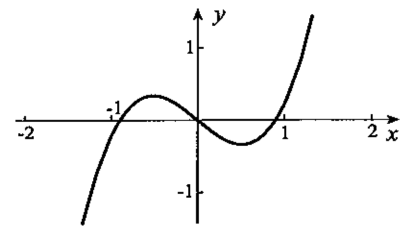


41. For critical points we solve

$$0 = f'(x) = 3x^2 - \frac{1}{1+x^2} = \frac{3x^4+3x^2-1}{1+x^2}.$$

When we set $3x^4+3x^2-1=0$, we obtain

$$x^2 = \frac{-3\pm\sqrt{9+12}}{6} = \frac{-3\pm\sqrt{21}}{6}.$$



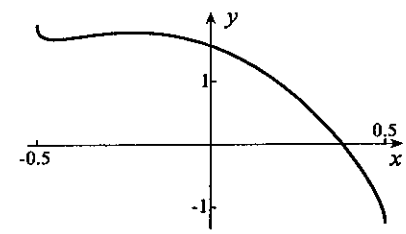
Thus, $x = \pm\sqrt{\dfrac{3+\sqrt{21}}{6}}$. Since $f'(x)$ changes from positive to negative as x increases through the negative value, $x = -\sqrt{(3+\sqrt{21})/6}$ gives a relative maximum. The positive critical point gives a relative minimum since $f'(x)$ changes from negative to positive as increases through it.

42. For critical points we solve

$$0 = f'(x) = \frac{-2}{\sqrt{1-4x^2}} - 10x = \frac{-2(1+5x\sqrt{1-4x^2})}{\sqrt{1-4x^2}}.$$

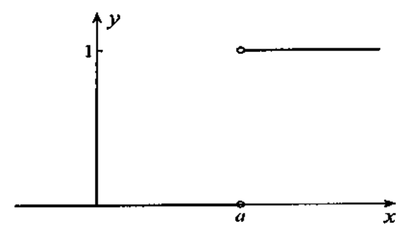
When we set $1+5x\sqrt{1-4x^2}=0$, we obtain

$$25x^2(1-4x^2) = 1$$
$$100x^4 - 25x^2 + 1 = 0$$
$$(20x^2-1)(5x^2-1) = 0$$

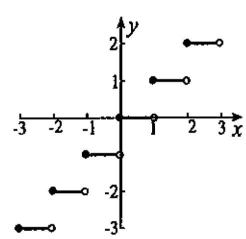


We can only accept the negative solutions of this equation, namely, $x=-1/\sqrt{5}$ and $x=-\sqrt{5}/10$. Since $f'(x)$ changes from negative to positive as x increases through $-1/\sqrt{5}$, this critical point gives a relative minimum. Since $f'(x)$ changes from positive to negative as x increases through $-\sqrt{5}/10$, this critical point gives a relative maximum.

43. Every point except $x=a$ is critical. Each gives both a relative maximum and a relative minimum.



44. Every point is critical. Every integer gives a relative maximum. Every other value of x gives both a relative maximum and a relative minimum.



45. Implicit differentiation gives

$$4x^3 + 3y^2\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-4x^3}{3y^2+5y^4}.$$

The only critical point at which the derivative vanishes is $x=0$. Since the denominator $3y^2+5y^4$ is never negative, the derivative changes from positive to negative as x increases through 0. Hence, $x=0$ gives a relative maximum.

46. Implicit differentiation gives

$$2x + 3y^2\frac{dy}{dx} + \frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-2x}{3y^2+1}.$$

The only critical point at which the derivative vanishes is $x=0$. Since the denominator $3y^2+1$ is always positive, the derivative changes from positive to negative as x increases through 0. Hence, $x=0$ gives a relative maximum.

47. Implicit differentiation gives

$$3x^2y + x^3\frac{dy}{dx} + y^3 + 3xy^2\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = -\frac{3x^2y + y^3}{x^3 + 3xy^2} = -\frac{y(3x^2 + y^2)}{x(x^2 + 3y^2)}.$$

For this to vanish we set $y = 0$, but this is impossible in the original equation $x^3y + xy^3 = 2$.

48. Implicit differentiation gives

$$4y^3\frac{dy}{dx} + y^3 + 3xy^2\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-y^3}{3xy^2 + 4y^3}.$$

For this to vanish we set $y = 0$, but this is not permitted by the original equation $y^4 + xy^3 = 1$.

49. Implicit differentiation gives

$$4x^3y + x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-4x^3y}{x^4 + 5y^4}.$$

For this to vanish we set either $x = 0$ or $y = 0$. The original equation $x^4y + y^5 = 32$ does not permit $y = 0$. When $x = 0$, the value of y is $y = 2$. Since the denominator $x^4 + 5y^4$ is always positive, it follows that the derivative changes from positive to negative as x increases through 0. Hence, $x = 0$ gives a relative maximum.

50. Implicit differentiation gives

$$2xy^4 + 4x^2y^3\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{-2xy^4}{4x^2y^3 + 3y^2}.$$

For this to vanish we set either $x = 0$ or $y = 0$. The original equation $x^2y^4 + y^3 = 1$ does not permit $y = 0$. Since the denominator $4x^2y^3 + 3y^2$ is always positive, the derivative changes from positive to negative as x increases through 0. Hence, $x = 0$ gives a relative maximum.

51. Implicit differentiation gives

$$2x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = -\frac{x + y}{x + 3y}.$$

For this to vanish we set $y = -x$. When this is substituted into the original equation

$$x^2 - 2x^2 + 3x^2 = 2 \qquad \Longrightarrow \qquad 2x^2 = 2 \qquad \Longrightarrow \qquad x = \pm 1.$$

52. Implicit differentiation gives

$$4x^3y + x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 4 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{4 - 4x^3y}{x^4 + 5y^4}.$$

For this to vanish we set $4 - 4x^3y = 0$, from which $y = 1/x^3$. When this is substituted into the original equation

$$\frac{x^4}{x^3} + \frac{1}{x^{15}} = 4x \qquad \Longrightarrow \qquad 3x^{16} = 1 \qquad \Longrightarrow \qquad x = \pm\frac{1}{3^{1/16}}.$$

53. Implicit differentiation gives

$$4x - 3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{4x + y}{3y^2 - x}.$$

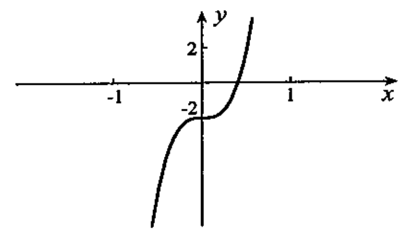
For this to vanish we set $4x + y = 0$, from which $y = -4x$. When this is substituted into the original equation

$$2x^2 - (-4x)^3 + x(-4x) = 4 \qquad \Longrightarrow \qquad 32x^3 - x^2 - 2 = 0.$$

A plot of the function $f(x) = 32x^3 - x^2 - 2$ to the right shows that the only solution of this equation is near 0.5. Newton's iterative procedure can be used to approximate it to four decimals. Iteration of

$$x_1 = 0.5, \quad x_{n+1} = x_n - \frac{32x_n^3 - x_n^2 - 2}{96x_n^2 - 2x_n}$$

leads to the iterates



$$x_2 = 0.42391, \quad x_3 = 0.40819, \quad x_4 = 0.407546, \quad x_5 = 0.407545.$$

To confirm 0.4075 as a four-decimal approximation we evaluate

$$f(0.40745) = -1.4 \times 10^{-3} \qquad \text{and} \qquad f(0.40755) = 7.3 \times 10^{-5}.$$

54. (a) Implicit differentiation gives

$$\frac{dy}{dx} = 2x\sqrt{1-y^2} + x^2\left(\frac{-y}{\sqrt{1-y^2}}\right)\frac{dy}{dx} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{2x\sqrt{1-y^2}}{1 + \dfrac{x^2y}{\sqrt{1-y^2}}} = \frac{2x(1-y^2)}{x^2y + \sqrt{1-y^2}}.$$

Since y cannot be equal to ± 1, the only critical point at which the derivative vanishes is $x = 0$.
(b) If we square the equation, we obtain

$$y^2 = x^4(1-y^2) \qquad \Longrightarrow \qquad y^2(1+x^4) = x^4 \qquad \Longrightarrow \qquad y = \pm\sqrt{\frac{x^4}{1+x^4}}.$$

Since y must be positive, the explicit definition of the function is $y = \dfrac{x^2}{\sqrt{1+x^4}}$. For critical points we solve

$$0 = \frac{dy}{dx} = \frac{\sqrt{1+x^4}(2x) - x^2(1/2)(1+x^4)^{-1/2}(4x^3)}{1+x^4} = \frac{2x(1+x^4) - 2x^5}{(1+x^4)^{3/2}} = \frac{2x}{(1+x^4)^{3/2}}.$$

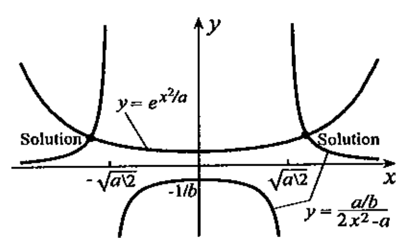
The only solution is $x = 0$.

55. For critical points we solve $\quad 0 = f'(x) = \dfrac{(1+be^{x^2/a})(1) - x(be^{x^2/a})(2x/a)}{(1+be^{x^2/a})^2}$. This implies that

$$0 = 1 + be^{x^2/a} - \frac{2b}{a}x^2e^{x^2/a} = 1 + \frac{b}{a}\left(a - 2x^2\right)e^{x^2/a}$$

$$\Longrightarrow \quad e^{x^2/a} = \frac{a/b}{2x^2 - a}.$$

Graphs of these functions to the right show two solutions, one the negative of the other.



56. (a) Since $f'_+(0) = 0$, it follows that $x = 0$ is critical. The point $x = 1$ is not critical since $f'_-(1) = 2$.
(b) No. End points of intervals cannot be relative extrema (see Definitions 4.3 and 4.4).

57. True The function must be defined at a critical point and derivatives do not exist at points of discontinuity.

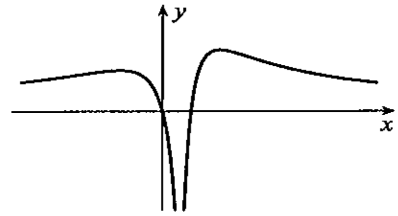
58. False According to Definitions 4.3 and 4.4 they can be neither relative maxima nor relative minima.

59. False It is true that if a function is discontinuous at a point, then it cannot have a derivative there. But, it can be continuous at points where the derivative does not exist. Take, for example, the function $|x|$ at $x = 0$.

60. True All values of x give relative maxima and minima for the function $f(x) = 1$.

61. True This is the first derivative test.

62. False The discontinuous function to the right has two relative maxima but no relative minima.

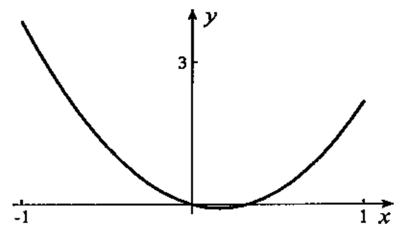


63. True Every x is critical for $f(x) = 1$.

64. False The function in Figure 2.9(b) has an infinite number of such critical points in the interval $0 < x < 1$.

65. Critical points are defined by the equation $0 = f'(x) = 3x^2 - \sin x$. Clearly, $x = 0$ satisfies this equation. The graph of $f'(x)$ shows a second zero near 0.3. To find it, we use Newton's iterative procedure with

$$x_1 = 0.3, \quad x_{n+1} = x_n - \frac{3x_n^2 - \sin x_n}{6x_n - \cos x_n}.$$



Iteration gives $x_2 = 0.330$, $x_3 = 0.3274$, $x_4 = 0.3274$. Since $f'(0.3265) = -9.2 \times 10^{-4}$ and $f(0.3275) = 9.2 \times 10^{-5}$, the critical point is $x = 0.327$.

66. (a) Critical points of $P(V) = \dfrac{RTe^{-a/(RTV)}}{V - b}$ are defined by

$$0 = P'(V) = RT\left[\frac{(V-b)e^{-a/(RTV)}[a/(RTV^2)] - e^{-a/(RTV)}}{(V-b)^2}\right].$$

This implies that

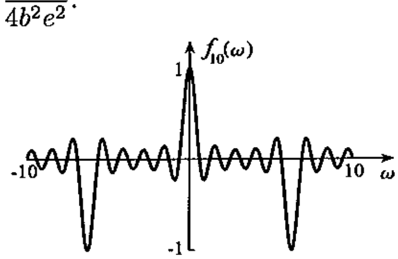
$$0 = \frac{a(V-b)}{RTV^2} - 1 \qquad \Longrightarrow \qquad (RT)V^2 - aV + ab = 0,$$

a quadratic equation with solutions $V = \dfrac{a \pm \sqrt{a^2 - 4abRT}}{2RT}$. These exist provided $a^2 - 4abRT > 0 \Longrightarrow T < a/(4bR)$.

(b) When $T = T_c = a/(4bR)$, there is one critical point, $V = \dfrac{a}{2RT_c} = \dfrac{a}{2R[a/(4bR)]} = 2b$, at which

$$P = \frac{RT_c e^{-\frac{a}{RT_c(2b)}}}{2b - b} = \frac{R\left(\frac{a}{4bR}\right)e^{-2}}{b} = \frac{a}{4b^2e^2}.$$

67. (a) The plot is shown to the right.
(b) $f_{10}(0)$ is not defined even though the plot suggests otherwise. The plot suggests a limit of 1.
(c) The function is even.
(d) The plot would indicate a period of about 12, if there is one. This suggests perhaps a period of 4π. This is confirmed by the calculation



$$f_{10}(\omega + 4\pi) = \frac{\sin[5(\omega + 4\pi)]}{10\sin[(\omega + 4\pi)/2]} = \frac{\sin(5\omega + 20\pi)}{10\sin(\omega/2 + 2\pi)} = \frac{\sin(5\omega)}{10\sin(\omega/2)} = f_{10}(\omega).$$

(e) The smallest positive value of ω at which $f_{10}(\omega)$ has its smallest positive relative maximum is near $x = 2.8$. To find it more accurately, we must solve

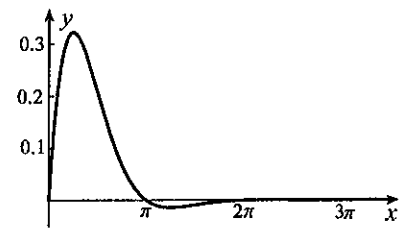
$$0 = f_{10}'(\omega) = \frac{\sin(\omega/2)[5\cos(5\omega)] - \sin(5\omega)[(1/2)\cos(\omega/2)]}{10\sin^2(\omega/2)}.$$

This requires $10\sin(\omega/2)\cos(5\omega) = \sin(5\omega)\cos(\omega/2) \implies g(\omega) = 10\cot(5\omega) - \cot(\omega/2) = 0$. Newton's iterative procedure with initial approximation $\omega_1 = 2.8$ defines the sequence

$$\omega_1 = 2.8, \quad \omega_{n+1} = \omega_n - \frac{10\cot(5\omega_n) - \cot(\omega_n/2)}{-50\csc^2(5\omega_n) + (1/2)\csc^2(\omega_n/2)}.$$

Iteration gives $\omega_2 = 2.824$ and $\omega_3 = 2.824$. Thus, the required value is $\omega \approx 2.824$.
(f) For $f_L(\omega) = 0$, we must have $\sin(\omega L/2) = 0 \implies \omega L/2 = n\pi \implies \omega = 2n\pi/L$, where n is an integer.

68. (a) A plot is shown to the right.
(b) Since $f'(x) = e^{-x}\cos x - e^{-x}\sin x = e^{-x}(\cos x - \sin x)$, critical points are $x = \pi/4 + n\pi$, where $n \geq 0$ is an integer. The graph indicates (as would the first derivative test) that relative maxima occur at $x = \pi/4 + 2n\pi$ and relative minima occur at $x = \pi/4 + (2n+1)\pi$.

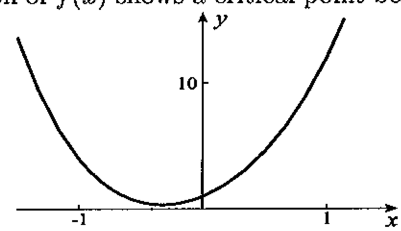


69. Critical points are given by $0 = f'(x) = 4x^3 + 12x + 4$. The graph of $f(x)$ shows a critical point between -1 and 0. To find it, we use Newton's iterative procedure with

$$x_1 = 0, \quad x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{6x_n^2 + 3}.$$

Iteration gives

$$x_2 = -0.33333, \quad x_3 = -0.322\,22,$$
$$x_4 = -0.322\,19, \quad x_5 = -0.322\,19.$$



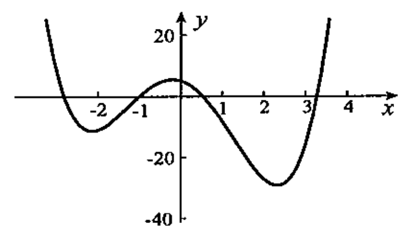
Since $f'(-0.322\,25) = -8.6 \times 10^{-4}$ and $f(-0.322\,15) = 4.7 \times 10^{-4}$, the critical point is $x = -0.322\,2$. Since $f'(x)$ changes from negative to positive as x increases through $-0.322\,2$, the critical point gives a relative minimum.

70. Critical points are defined by $0 = f'(x) = 4x^3 - 20x - 4$. The graph of $f(x)$ indicates that there are three critical points. To find the positive one, we use Newton's iterative procedure with

$$x_1 = 2.2, \quad x_{n+1} = x_n - \frac{x_n^3 - 5x_n - 1}{3x_n^2 - 5}.$$

Iteration gives

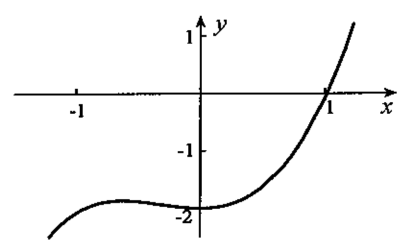
$$x_2 = 2.342\,02, \quad x_3 = 2.330\,15,$$
$$x_4 = 2.330\,06, \quad x_5 = 2.330\,06.$$



Since $f'(2.330\,05) = -3.9 \times 10^{-4}$ and $f'(2.330\,15) = 4.1 \times 10^{-3}$, it follows that $2.330\,1$ is a critical point. Since $f'(x)$ changes from negative to positive as x increases through this value, there is a relative minimum at this critical point. Similar procedures give a relative maximum at $-0.201\,6$ and a relative minimum at $-2.128\,4$.

71. Critical points are defined by $0 = f'(x) = 3x^2 + 2\sin x$. The graph of $f(x)$ indicates that there are two critical points $x = 0$ and one to the left of $x = 0$. There is a relative minimum at $x = 0$. To find the negative critical point, we use Newton's iterative procedure with

$$x_1 = -0.6, \quad x_{n+1} = x_n - \frac{3x_n^2 + 2\sin x_n}{6x_n + 2\cos x_n}.$$

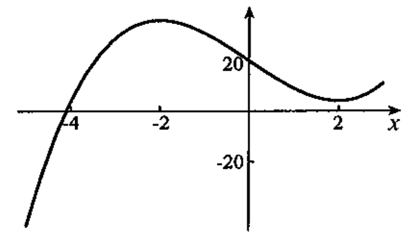


Iteration gives $x_2 = -0.625\,3$, $x_3 = -0.624\,21$, $x_4 = -0.624\,21$. Since $f'(-0.624\,25) = 8.6 \times 10^{-5}$ and $f'(-0.624\,15) = -1.3 \times 10^{-4}$, it follows that $-0.624\,2$ is a critical point. Since $f'(x)$ changes from positive to negative, there is a relative maximum at this critical point.

72. For critical points we solve

$$0 = f'(x) = \frac{(x^2-5x+4)^2(2x) - (x^2-4)(2)(x^2-5x+4)(2x-5)}{(x^2-5x+4)^4} = \frac{-2(x^3-12x+20)}{(x-1)^3(x-4)^3}.$$

The graph of $x^3 - 12x + 20$ to the right indicates that the only critical point is slightly less than -4. To find it we use Newton's iterative procedure with



$$x_1 = -4, \quad x_{n+1} = x_n - \frac{x_n^3 - 12x_n + 20}{3x_n^2 - 12}.$$

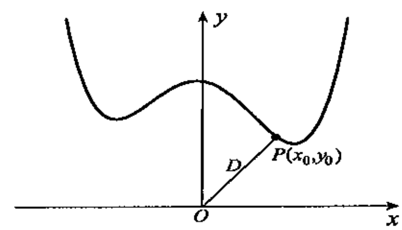
Iteration gives

$x_2 = -4.111, \qquad x_3 = -4.107\,25,$

$x_4 = -4.107\,24, \qquad x_5 = -4.107\,25.$

Since $f'(-4.107\,25) = 7.4 \times 10^{-9}$ and $f'(-4.107\,15) = -1.0 \times 10^{-7}$, the critical point is $-4.107\,2$. It yields a relative maximum.

73. We write that $D = \sqrt{x^2 + [f(x)]^2}$. Relative extrema occur at critical points of D, obtained by solving



$$0 = \frac{dD}{dx} = \frac{2x + 2f(x)f'(x)}{2\sqrt{x^2 + [f(x)]^2}}.$$

If x_0 is a critical point, then

$$2x_0 + 2f(x_0)f'(x_0) = 0 \implies f'(x_0) = \frac{-x_0}{f(x_0)} = -\frac{x_0}{y_0}.$$

But y_0/x_0 is the slope of OP, and therefore slopes of the tangent line at P and the line OP are negative reciprocals; that is, the lines are perpendicular.

74. If we differentiate the equation of the cardioid with respect to x, we obtain $2(x^2+y^2+x)\left(2x + 2y\dfrac{dy}{dx} + 1\right) = 2x + 2y\dfrac{dy}{dx}$. When we solve this for dy/dx and set it equal to 0, we obtain $0 = \dfrac{dy}{dx} = \dfrac{(2x+1)(x^2+y^2+x) - x}{y - 2y(x^2+y^2+x)}$. We now set the numerator equal to 0, and this implies that $x^2 + y^2 + x = x/(2x+1)$. Substitution of this into the equation of the cardioid gives

$$\frac{x^2}{(2x+1)^2} = x^2 + y^2 \qquad \Longrightarrow \qquad \frac{x^2}{(2x+1)^2} = \frac{x}{2x+1} - x.$$

Since $x = 0$ does not lead to a maximum for y, we set $x/(2x+1)^2 = 1/(2x+1) - 1$, and this equation simplifies to $4x^2 + 3x = 0$, with solution $x = -3/4$. The y-coordinate of the point on the cardioid in the second quadrant corresponding to this value of x is $3\sqrt{3}/4$.

75. To find the points on the curve farthest from the origin, we find relative maxima of the function $D = \sqrt{x^2+y^2}$. We do this by finding its critical points,

$$0 = \frac{dD}{dx} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x + 2y\frac{dy}{dx}\right) \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

By differentiating the equation of the bifolium with respect to x, we obtain $2(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 2xy + x^2\dfrac{dy}{dx}$. Substitution of $dy/dx = -x/y$ into this equation gives

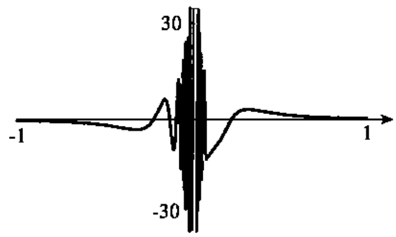
$$4(x^2+y^2)\left[x + y\left(-\frac{x}{y}\right)\right] = 2xy + x^2\left(-\frac{x}{y}\right) \quad \Longrightarrow \quad \frac{x}{y}(2y^2 - x^2) = 0 \quad \Longrightarrow \quad x^2 = 2y^2,$$

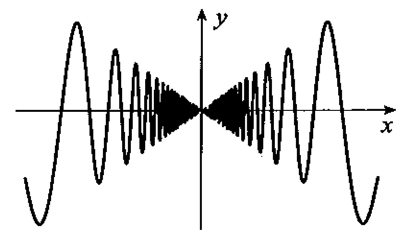
since $x = 0$ gives a relative minimum for D. Substitution of this into the equation of the bifolium now gives $(2y^2 + y^2)^2 = (2y^2)y \Longrightarrow 9y^4 = 2y^3 \Longrightarrow y = 2/9$, since $y = 0$ gives minimum D. Thus, the points farthest from the origin are $(\pm 2\sqrt{2}/9, 2/9)$.

76. (a) By equation 3.2, $f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h \sin(1/h)}{h} = \lim_{h \to 0} \sin(1/h)$, and this limit does not exist (see Example 2.9). Since $f'(0)$ does not exist, but $f(0)$ does, $x = 0$ is a critical point.

(b) The plot of $f'(x) = \sin(1/x) - (1/x)\cos(1/x)$ in the left figure below clearly shows that the derivative does not change sign as x increases through 0; the sign oscillates more and more rapidly the closer x is to zero.

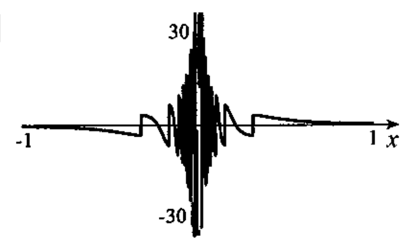
(c) No In every interval around $x = 0$, the function takes on both positive and negative values. (right graph below).





77. (a) By equation 3.2, $f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h \to 0} \dfrac{|h \sin(1/h)|}{h}$, and this limit does not exist. Since the derivative does not exist at $x = 0$, but the function has a value there, $x = 0$ is a critical point.

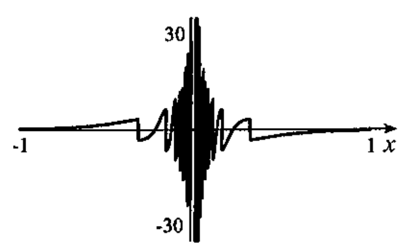
(b) The plot of $f'(x) = \dfrac{|x \sin(1/x)|}{x \sin(1/x)}[\sin(1/x) - (1/x)\cos(1/x)]$ clearly shows that the derivative does not change sign as x increases through 0; the sign oscillates more and more rapidly the closer x is to zero. In addition, there is an increasing number of values of x at which the derivative does not exist, namely $x = 1/(n\pi)$, where n is an integer.



(c) Since function values are always positive for $x \neq 0$, and $f(0) = 0$, it follows that a relative minimum occurs at $x = 0$.

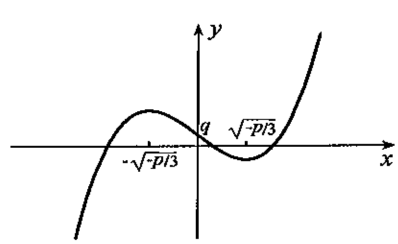
78. (a) By definition 3.2, $f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h \to 0} \dfrac{-|h \sin(1/h)|}{h}$, and this limit does not exist. Since the derivative does not exist at $x = 0$, but the function has a value there, $x = 0$ is a critical point.

(b) The plot of $f'(x) = -\dfrac{|x \sin(1/x)|}{x \sin(1/x)}[\sin(1/x) - (1/x)\cos(1/x)]$ clearly shows that the derivative does not change from positive to negative nor from negative to positive as x increases through 0; its sign oscillates more and more rapidly the closer x is to zero. In addition, there is an increasing number of values of x at which the derivative does not exist, namely $x = 1/(n\pi)$, where n is an integer.



(c) Since function values are always negative for $x \neq 0$, and $f(0) = 0$, it follows that a relative maximum occurs at $x = 0$.

79. If the cubic polynomial $f(x) = x^3 + px + q$ has three distinct zeros, it has a relative maximum and a relative minimum; that is, it has two critical points defined by $0 = f'(x) = 3x^2 + p$. Consequently, p must be negative, and the critical points are $x = \pm\sqrt{-p/3}$. The graph of $f(x)$ must be one of the two situations shown depending on whether q is positive or negative. In either case, $f(\sqrt{-p/3})$ must be negative; that is,

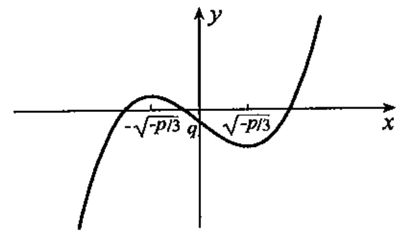


$$0 > \left(\sqrt{\frac{-p}{3}}\right)^3 + p\sqrt{\frac{-p}{3}} + q = \frac{2p}{3}\sqrt{\frac{-p}{3}} + q.$$

Thus, $q < -\frac{2p}{3}\sqrt{\frac{-p}{3}}$. When $q > 0$ the square of this gives $q^2 < \frac{4p^2}{9}\left(\frac{-p}{3}\right) \implies 4p^3 + 27q^2 < 0$.

A similar proof can be given when $q < 0$.

Conversely, if $4p^3 + 27q^2 < 0$, then p must be negative. As a result, $f(x) = x^3 + px + q$ has two critical points $x = \pm\sqrt{-p/3}$. If $q < 0$, the value of $f(x)$ at $x = \sqrt{-p/3}$ must be negative, and that at $x = -\sqrt{-p/3}$ is



$$f(-\sqrt{-p/3}) = \left(-\sqrt{\frac{-p}{3}}\right)^3 + p\left(-\sqrt{\frac{-p}{3}}\right) + q = \sqrt{\frac{-p}{3}}\left(-p + \frac{p}{3}\right) + q = -\frac{2p}{3}\sqrt{\frac{-p}{3}} + q.$$

Because $4p^3 + 27q^2 < 0$, it follows that

$$q^2 < -\frac{4p^3}{27} \implies -q < \sqrt{\frac{-4p^3}{27}} = \sqrt{-\frac{p}{3}\left(\frac{-2p}{3}\right)^2} = -\frac{2p}{3}\sqrt{\frac{-p}{3}} \implies q - \frac{2p}{3}\sqrt{\frac{-p}{3}} > 0.$$

Thus, $f(-\sqrt{-p/3}) > 0$. With a positive relative maximum at $x = -\sqrt{-p/3}$ and a negative relative minimum at $x = \sqrt{-p/3}$, the graph of $y = f(x)$ must cross the x-axis at three distinct points. A similar proof works when $q > 0$.

EXERCISES 4.4

1. Since $f''(x) = 6x - 6 = 6(x-1)$, it follows that $f''(x) \le 0$ for $x \le 1$, and $f''(x) \ge 0$ for $x \ge 1$. Consequently, the graph of the function is concave downward on the interval $x \le 1$, and concave upward on the interval $x \ge 1$. The point $(1, 0)$ that separates these intervals is a point of inflection.

2. Since $f''(x) = 36x^2 + 24x = 12x(3x+2)$, it follows that $f''(x) \le 0$ for $-2/3 \le x \le 0$, and $f''(x) \ge 0$ for $x \le -2/3$ and $x \ge 0$. Consequently, the graph of the function is concave downward on the interval $-2/3 \le x \le 0$, and concave upward on the intervals $x \le -2/3$ and $x \ge 0$. The points $(-2/3, 470/27)$ and $(0, 2)$ that separate these intervals are points of inflection.

3. Since $f''(x) = 2 + 6/x^4$, the second derivative is always positive except at $x = 0$ where it does not exist. The graph is therefore concave upward for $x < 0$ and $x > 0$.

4. Since $f'(x) = \frac{(x^2-1)(2x) - (x^2+4)(2x)}{(x^2-1)^2} = \frac{-10x}{(x^2-1)^2}$, the second derivative vanishes when

$$0 = f''(x) = -10\left[\frac{(x^2-1)^2(1) - x(2)(x^2-1)(2x)}{(x^2-1)^4}\right] = \frac{10(3x^2+1)}{(x^2-1)^3}.$$

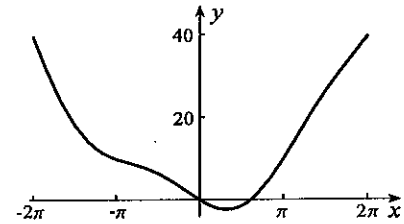
   Since $f''(x) > 0$ for $x < -1$ and for $x > 1$, and $f''(x) < 0$ for $-1 < x < 1$, the graph is concave upward on the intervals $x < -1$ and $x > 1$, and concave downward on $-1 < x < 1$. Since $f(1)$ and $f(-1)$ are not defined, there are no points of inflection.

5. Since $f''(x) = -\cos x$, and the cosine function is positive in quadrants one and four, and negative in quadrants two and three, we can say that the graph is concave downward on the intervals $-2\pi < x \le -3\pi/2$, $-\pi/2 \le x \le \pi/2$, and $3\pi/2 \le x < 2\pi$. It is concave upward on the intervals $-3\pi/2 \le x \le -\pi/2$ and $\pi/2 \le x \le 3\pi/2$. Points $(-3\pi/2, -3\pi/2)$, $(-\pi/2, -\pi/2)$, $(\pi/2, \pi/2)$ and $(3\pi/2, 3\pi/2)$ that separate these intervals are points of inflection.

6. Since $f''(x) = 2 + \sin x$, it follows that $f''(x) > 0$ for all x. Hence the graph is always concave upward.

7. Since $f''(x) = 2 + 2\sin x$, it follows that $f''(x) \ge 0$ for all x. Hence the graph is always concave upward.

8. Since $f''(x) = 2 + 4\sin x$, the second derivative vanishes when $\sin x = -1/2$. The angles in the interval $|x| < 2\pi$ that satisfy this equation are $x = -5\pi/6$, $-\pi/6$, $7\pi/6$, and $11\pi/6$. The graph of the function indicates that the graph is concave upward on the intervals $-2\pi < x \le -5\pi/6$, $-\pi/6 \le x \le 7\pi/6$, and $11\pi/6 \le x < 2\pi$; it is concave downward on $-5\pi/6 \le x \le -\pi/6$ and $7\pi/6 \le x \le 11\pi/6$. The points which separate these intervals are points of inflection, namely, $(-5\pi/6, 2 + 25\pi^2/36)$, $(-\pi/6, 2 + \pi^2/36)$, $(7\pi/6, 2 + 49\pi^2/36)$, and $(11\pi/6, 2 + 121\pi^2/36)$.



9. Since $f''(x) = d/dx(\ln x + 1) = 1/x$, it follows that $f''(x) > 0$ for all $x > 0$. Hence the graph is concave upward for $x > 0$.

10. Since $f'(x) = 2x\ln x + x$, we obtain $f''(x) = 2\ln x + 3$. It follows that $f''(x) = 0$ when $x = e^{-3/2}$. Since $f''(x) \ge 0$ for $x \ge e^{-3/2}$, and $f''(x) \le 0$ for $0 < x \le e^{-3/2}$, it follows that the graph is concave downward on the interval $0 < x \le e^{-3/2}$, and it is concave upward for $x \ge e^{-3/2}$. The point $(e^{-3/2}, -3/(2e^3))$ which separates these intervals is a point of inflection.

11. Since $f'(x) = e^{1/x}(-1/x^2)$, we obtain $f''(x) = e^{1/x}(1/x^4 + 2/x^3) = e^{1/x}(1 + 2x)/x^4$. It follows that $f''(x) = 0$ when $x = -1/2$. Since $f''(x) \ge 0$ for $-1/2 \le x < 0$ and $x > 0$, and $f''(x) \le 0$ for $x \le -1/2$, it follows that the graph is concave upward on the intervals $-1/2 \le x < 0$ and $x > 0$, and it is concave downward for $x \le -1/2$. The point $(-1/2, 1/e^2)$ is a point of inflection.

12. Since $f'(x) = e^{-2x} - 2x\,e^{-2x}$, we find that $f''(x) = -4\,e^{-2x} + 4x\,e^{-2x} = 4(x-1)\,e^{-2x}$. Because $f''(x) \le 0$ for $x \le 1$, and $f''(x) \ge 0$ for $x \ge 1$, the graph is concave downward for $x \le 1$, and concave upward for $x \ge 1$. The point of inflection is $(1, e^{-2})$.

13. Since $f'(x) = 2x\,e^{3x} + 3x^2\,e^{3x}$, we find that $f''(x) = 2\,e^{3x} + 6x\,e^{3x} + 6x\,e^{3x} + 9x^2\,e^{3x} = (2 + 12x + 9x^2)\,e^{3x}$. Because $f''(x) = 0$ for $x = (-12 \pm \sqrt{144 - 72})/18 = (-2 \pm \sqrt{2})/3$, it follows that $f''(x) \ge 0$ for $x \le (-2-\sqrt{2})/3$ and $x \ge (-2+\sqrt{2})/3$, and $f''(x) \le 0$ for $(-2-\sqrt{2})/3 \le x \le (-2+\sqrt{2})/3$. Consequently, the graph is concave upward for $x \le (-2 - \sqrt{2})/3$ and $x \ge (-2 + \sqrt{2})/3$, and concave downward for $(-2 - \sqrt{2})/3 \le x \le (-2 + \sqrt{2})/3$. The points $\big((-2 - \sqrt{2})/3, 0.0426\big)$ and $\big((-2 + \sqrt{2})/3, 0.0212\big)$ that separate these parts of the graph are points of inflection.

14. Since $f'(x) = 2x + e^{-x}$, we obtain $f''(x) = 2 - e^{-x}$. It follows that $f''(x) = 0$ when $x = -\ln 2$. Because $f''(x) \le 0$ when $x \le -\ln 2$, and $f''(x) \ge 0$ when $x \ge -\ln 2$, the graph is concave downward on the interval $x \le -\ln 2$ and concave upward on $x \ge -\ln 2$. The point $(-\ln 2, (\ln 2)^2 - 2)$ separating these intervals is a point of inflection.

15. For critical points we solve $0 = f'(x) = 3x^2 - 6x - 3 = 3(x^2 - 2x - 1)$ for $x = (2 \pm \sqrt{4+4})/2 = 1 \pm \sqrt{2}$. Since $f''(x) = 6x - 6 = 6(x-1)$, we find that $f''(1+\sqrt{2}) = 6\sqrt{2}$ and $f''(1-\sqrt{2}) = -6\sqrt{2}$. Consequently, $x = 1 + \sqrt{2}$ gives a relative minimum and $x = 1 - \sqrt{2}$ gives a relative maximum.

16. Since $f'(x) = 1 - 1/x^2$, critical points occur at $x = \pm 1$. With $f''(x) = 2/x^3$, we calculate that $f''(-1) = -2$ and $f''(1) = 2$. Hence, $x = -1$ yields a relative maximum and $x = 1$ a relative minimum.

17. Since $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$, critical points are $x = 0, 1, 3$. With $f''(x) = 36x^2 - 96x + 36 = 12(3x^2 - 8x + 3)$, we find that $f''(0) = 36$, $f''(1) = -24$, and $f''(3) = 72$. Consequently, $x = 0, 3$ yield relative minima and $x = 1$ gives a relative maximum.

18. Since $f'(x) = (5/4)x^{1/4} - 1/4x^{-3/4} = \dfrac{5x-1}{4x^{3/4}}$, the only critical point at which $f'(x) = 0$ is $x = 1/5$. Because $f''(x) = (5/16)x^{-3/4} + (3/16)x^{-7/4} = \dfrac{5x+3}{16x^{7/4}}$, it follows that $f''(1/5) > 0$, and therefore $x = 1/5$ yields a relative minimum.

19. Since $f'(x) = \ln x + 1$, the only critical point at which $f'(x) = 0$ is $x = 1/e$. With $f''(x) = 1/x$, we find that $f''(1/e) = e$, and $x = 1/e$ gives a relative minimum.

20. Since $f'(x) = 2x\ln x + x = x(2\ln x + 1)$, the only critical point at which $f'(x) = 0$ is $x = 1/\sqrt{e}$. Because $f''(x) = 2\ln x + 3$, it follows that $f''(1/\sqrt{e}) = 2(-1/2) + 3 = 2 > 0$, and the critical point yields a relative minimum.

21. Since $f'(x) = e^{2x} + 2x\,e^{2x} = (1+2x)\,e^{2x}$, the only critical point is $x = -1/2$. With $f''(x) = 2\,e^{2x} + 2(1+2x)\,e^{2x} = 4(1+x)\,e^{2x}$, we find that $f''(-1/2) = 2e^{-1}$, and therefore $x = -1/2$ gives a relative minimum.

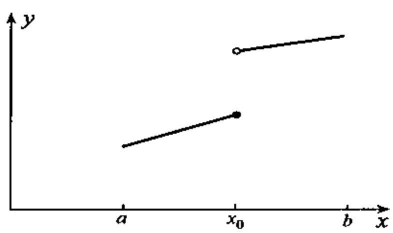
22. Since $f'(x) = 2xe^{-2x} - 2x^2e^{-x} = 2x(1-x)e^{-2x}$, critical points at which $f'(x) = 0$ are $x = 0$ and $x = 1$. Because $f''(x) = (2-4x)e^{-2x} - 2(2x - 2x^2)e^{-2x} = (2 - 8x + 4x^2)e^{-2x}$, it follows that $f''(0) = 2$ and $f''(1) = -2e^{-2}$. Hence, $x = 0$ gives a relative minimum and $x = 1$ gives a relative maximum.

23. If $f''(x) = 0$ on an interval, then $f(x) = Ax + B$ on that interval. The graph is a straight line and the slope is constant. Hence the graph is neither concave upward nor concave downward.

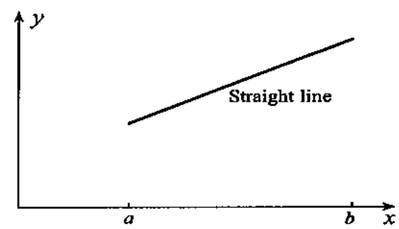
24. For points of inflection we solve $0 = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\sin x + x\cos x) = 2\cos x - x\sin x$. If (x_0, y_0) is a point of inflection, then x_0 satisfies $0 = 2\cos x_0 - x_0 \sin x_0$, and y_0 is given by $y_0 = x_0 \sin x_0$. It follows that $0 = 2\cos x_0 - y_0 \Longrightarrow y_0 = 2\cos x_0$. In other words, (x_0, y_0) is on the curve $y = 2\cos x$.

25. The second derivative of $f(x) = ax^3 + bx^2 + cx + d$ is $f''(x) = 6ax + 2b$, a linear equation with one solution $x = -b/(3a)$. Since $f''(x)$ will change sign as x increases through this value, it will yield a point of inflection.

26.



27.



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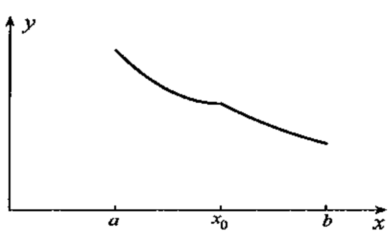

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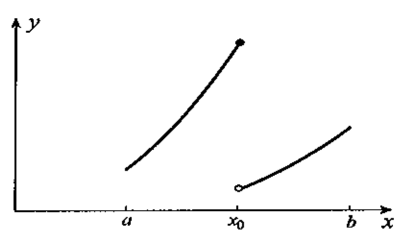

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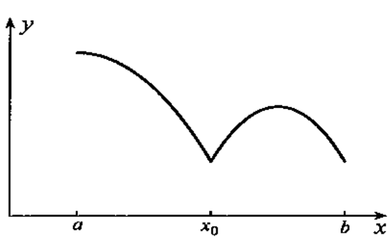

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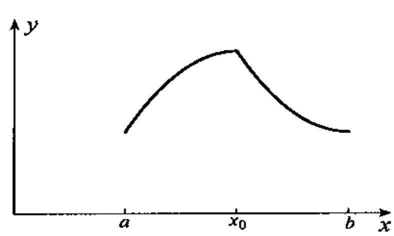


32. According to Example 3.19 in Section 3.5, $[f(x)g(x)]'' = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$. Since the graphs of $f(x)$ and $g(x)$ are concave upward on I, we can state that $f''(x) \ge 0$ and $g''(x) \ge 0$ on I. This does not, however, guarantee that $[f(x)g(x)]'' \ge 0$ on I.

33. According to Example 3.19 in Section 3.5, $[f(x)g(x)]'' = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$. Since the graphs of $f(x)$ and $g(x)$ are concave upward on I, we can state that $f''(x) \ge 0$ and $g''(x) \ge 0$ on I. Since $f(x)$ and $g(x)$ are increasing on I, we can also say that $f'(x) \ge 0$ and $g'(x) \ge 0$ on I. These do not, however, guarantee that $[f(x)g(x)]'' \ge 0$ on I.

34. For a horizontal point of inflection we set first and second derivatives equal to 0,

$$\begin{aligned}
0 = P'(V) &= \frac{-RT}{(V-b)^2} e^{-a/(RT^{3/2}V)} + \frac{RT}{V-b} e^{-a/(RT^{3/2}V)} \left(\frac{a}{RT^{3/2}V^2}\right) \\
&= RTe^{-a/(RT^{3/2}V)} \left[\frac{-1}{(V-b)^2} + \frac{a}{RT^{3/2}V^2(V-b)}\right] \\
&= \frac{RTe^{-a/(RT^{3/2}V)}}{RT^{3/2}V^2(V-b)^2} [a(V-b) - RT^{3/2}V^2],
\end{aligned}$$

$$0 = P''(V) = RTe^{-a/(RT^{3/2}V)}\left(\frac{a}{RT^{3/2}V^2}\right)\left[\frac{-1}{(V-b)^2} + \frac{a}{RT^{3/2}V^2(V-b)}\right]$$
$$+ RTe^{-a/(RT^{3/2}V)}\left[\frac{2}{(V-b)^3} - \frac{2a}{RT^{3/2}V^3(V-b)} - \frac{a}{RT^{3/2}V^2(V-b)^2}\right].$$

Because $P'(V) = 0$, the term in the first set of brackets of $P''(V)$ vanishes, and we therefore set

$$0 = \frac{RTe^{-a/(RT^{3/2}V)}}{RT^{3/2}V^3(V-b)^3}[2RT^{3/2}V^3 - 2a(V-b)^2 - aV(V-b)].$$

From $0 = P'(V)$, we obtain $\quad V - b = \dfrac{RT^{3/2}V^2}{a}, \quad$ which we substitute into $0 = P''(V)$,

$$0 = 2RT^{3/2}V^3 - 2a\left(\frac{R^2T^3V^4}{a^2}\right) - aV\left(\frac{RT^{3/2}V^2}{a}\right) = \frac{RT^{3/2}V^3}{a}(2a - 2RT^{3/2}V - a)$$
$$= \frac{RT^{3/2}V^3}{a}(a - 2RT^{3/2}V).$$

Thus, $a = 2RT^{3/2}V$. It now follows that

$$b = V - \frac{RT^{3/2}V^2}{2RT^{3/2}V} = \frac{V}{2}.$$

We now have a and b in terms of T and V. To replace V with P, we use Dieterici's equation to write

$$P = \frac{RT}{V-b}e^{-a/(RT^{3/2}V)}.$$

When we substitute the expressions for a and b into this equation we obtain

$$P = \frac{RT}{V - V/2}e^{-2RT^{3/2}V/(RT^{3/2}V)} = \frac{2RT}{e^2V} \quad \Longrightarrow \quad V = \frac{2RT}{e^2P}.$$

This then gives

$$a = 2RT^{3/2}\left(\frac{2RT}{e^2P}\right) = \frac{4R^2T^{5/2}}{e^2P}, \qquad b = \frac{RT}{e^2P};$$

that is, expressions for a and b in terms of critical temperature and pressure are

$$a = \frac{4R^2T_c^{5/2}}{e^2P_c}, \qquad b = \frac{RT_c}{e^2P_c}.$$

35. (a) Since $f'(x) = 6x(x^2-1)^2$, critical points are $x = 0$ and $x = \pm 1$. We now calculate $f''(x) = 6(x^2-1)^2 + 24x^2(x^2-1)$ and $f'''(x) = 24x(x^2-1) + 96x^3 - 48x$. Because $f''(0) = 6$, it follows that $x = 0$ yields a relative minimum. Since $f''(\pm 1) = 0$ and $f'''(\pm 1) \neq 0$, it follows that these critical points give horizontal points of inflection.

(b) Since $f'(x) = 2x\sqrt{1-x} - \dfrac{x^2}{2\sqrt{1-x}} = \dfrac{x(4-5x)}{2\sqrt{1-x}}$, critical points at which $f'(x) = 0$ are $x = 0$ and $x = 4/5$. We now calculate

$$f''(x) = \frac{2\sqrt{1-x}(4-10x) - (4x-5x^2)(1-x)^{-1/2}(-1)}{4(1-x)} = \frac{8 - 24x + 15x^2}{4(1-x)^{3/2}}.$$

Because $f''(0) = 2$, and $f''(4/5) = \dfrac{8 - 24(4/5) + 15(4/5)^2}{4(1-4/5)^{3/2}} < 0$, we conclude that $x = 0$ gives a relative minimum and $x = 4/5$ gives a relative maximum.

36. $f'(x) = \dfrac{(x^2+k^2)(-1)-(k-x)(2x)}{(x^2+k^2)^2} = \dfrac{x^2-2kx-k^2}{(x^2+k^2)^2}$ For points of inflection we solve

$$0 = f''(x) = \frac{(x^2+k^2)^2(2x-2k)-(x^2-2kx-k^2)(2)(x^2+k^2)(2x)}{(x^2+k^2)^4},$$

and this simplifies to $0 = \dfrac{-2(x+k)(x^2-4kx+k^2)}{(x^2+k^2)^3}$. Thus, $x=-k$ and $x=(4k\pm\sqrt{16k^2-4k^2})/2 = (2\pm\sqrt{3})k$. Since $f''(x)$ changes sign as x passes through each of these values, each gives a point of inflection, and these points are

$$P\left(-k,\frac{1}{k}\right),\quad Q\left((2+\sqrt{3})k,\frac{1-\sqrt{3}}{4k}\right),\quad R\left((2-\sqrt{3})k,\frac{1+\sqrt{3}}{4k}\right).$$

These points are collinear if the slopes of line segments PR and PQ are equal. This is indeed the case since

$$\text{Slope of } PR = \frac{\dfrac{1+\sqrt{3}}{4k}-\dfrac{1}{k}}{(2-\sqrt{3})k+k} = \frac{1+\sqrt{3}-4}{4k^2(2-\sqrt{3}+1)} = -\frac{1}{4k^2},$$
$$\text{Slope of } PQ = \frac{\dfrac{1-\sqrt{3}}{4k}-\dfrac{1}{k}}{(2+\sqrt{3})k+k} = \frac{1-\sqrt{3}-4}{4k^2(2+\sqrt{3}+1)} = -\frac{1}{4k^2}.$$

37. If the cubic polynomial is $y=f(x)=ax^3+bx^2+cx+d$, then critical points are defined by $0=f'(x)=3ax^2+2bx+c$. Solutions of this equation are

$$x = \frac{-2b\pm\sqrt{4b^2-12ac}}{6a} = \frac{-b\pm\sqrt{b^2-3ac}}{3a},$$

and b^2-3ac must be positive since these values must yield the relative extrema. If we set $x_1 = (-b+\sqrt{b^2-3ac})/(3a)$ and $x_2=(-b-\sqrt{b^2-3ac})/(3a)$, the relative extrema are at $P\big(x_1,f(x_1)\big)$ and $Q\big(x_2,f(x_2)\big)$. For the point of inflection we solve $0=f''(x)=6ax+2b$, and obtain $x=-b/(3a)$. The y-coordinate of the point of inflection is

$$y = a\left(\frac{-b}{3a}\right)^3+b\left(\frac{-b}{3a}\right)^2+c\left(\frac{-b}{3a}\right)+d = \frac{2b^3-9abc+27a^2d}{27a^2}.$$

We now show that the midpoint R of line segment PQ has the same coordinates. The x-coordinate of R is

$$\frac{1}{2}(x_1+x_2) = \frac{1}{2}\left[\frac{-b+\sqrt{b^2-3ac}}{3a}+\frac{-b-\sqrt{b^2-3ac}}{3a}\right] = -\frac{b}{3a}.$$

The y-coordinate of R is

$$\begin{aligned}\frac{1}{2}[f(x_1)+f(x_2)] &= \frac{1}{2}\left[a\left(\frac{-b+\sqrt{b^2-3ac}}{3a}\right)^3+b\left(\frac{-b+\sqrt{b^2-3ac}}{3a}\right)^2+c\left(\frac{-b+\sqrt{b^2-3ac}}{3a}\right)+d\right.\\
&\quad\left.+a\left(\frac{-b-\sqrt{b^2-3ac}}{3a}\right)^3+b\left(\frac{-b-\sqrt{b^2-3ac}}{3a}\right)^2+c\left(\frac{-b-\sqrt{b^2-3ac}}{3a}\right)+d\right]\\
&= \frac{1}{54a^2}\Big[-b^3+3b^2\sqrt{b^2-3ac}-3b(b^2-3ac)+(b^2-3ac)^{3/2}\\
&\qquad -b^3-3b^2\sqrt{b^2-3ac}-3b(b^2-3ac)-(b^2-3ac)^{3/2}\Big]\\
&\quad+\frac{b}{18a^2}\Big[b^2-2b\sqrt{b^2-3ac}+b^2-3ac+b^2+2b\sqrt{b^2-3ac}+b^2-3ac\Big]\end{aligned}$$

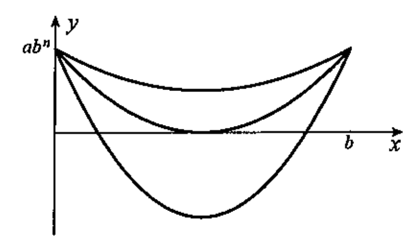
$$+\frac{c}{6a}\left[-b+\sqrt{b^2-3ac}-b-\sqrt{b^2-3ac}\right]+d$$
$$=\frac{1}{54a^2}\left[-8b^3+18abc\right]+\frac{b}{18a^2}\left[4b^2-6ac\right]-\frac{bc}{3a}+d$$
$$=\frac{-4b^3+9abc+6b^3-9abc-9abc+27a^2d}{27a^2}$$
$$=\frac{2b^3-9abc+27a^2d}{27a^2}.$$

38. For critical points of $f(x)$, we solve $0=f'(x)=(1-\cos x)\cos(x-\sin x)$. If we set $1-\cos x=0$, we obtain $x=2n\pi$, where n is an integer. (Other solutions can be obtained by setting $\cos(x-\sin x)=0$, but we can answer the question without looking at this equation.) The second derivative is $f''(x)=-(1-\cos x)^2\sin(x-\sin x)+\sin x\cos(x-\sin x)$, and $f''(2n\pi)=0$. The third derivative is

$$f'''(x)=-(1-\cos x)^3\cos(x-\sin x)-2(1-\cos x)\sin x\sin(x-\sin x)$$
$$+\cos x\cos(x-\sin x)-\sin x(1-\cos x)\sin(x-\sin x).$$

Since $f'''(2n\pi)=1$, Exercise 35 implies that there are horizontal points of inflection when $x=2n\pi$.

39. First we note that $f(0)=f(b)=ab^n>0$. Next, $f'(x)=(n+1)x^n-b^n$ and $f''(x)=(n+1)nx^{n-1}$. Since $f''(x)>0$ for all $x>0$, the graph of $f(x)$ must be concave upward for all $x>0$. This means that it must be one of the three situations shown to the right. There are exactly two solutions of $f(x)=0$ if and only if the relative minimum is negative. The critical point is given by



$$0=f'(x)=(n+1)x^n-b^n \implies x=\frac{b}{(n+1)^{1/n}}.$$

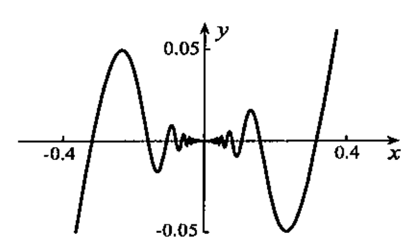
Thus, we require

$$0>f\left(\frac{b}{(n+1)^{1/n}}\right)=\left[\frac{b}{(n+1)^{1/n}}\right]^{n+1}-b^n\left[\frac{b}{(n+1)^{1/n}}\right]+ab^n=b^n\left[\frac{b}{(n+1)^{(n+1)/n}}-\frac{b}{(n+1)^{1/n}}+a\right].$$

This is equivalent to

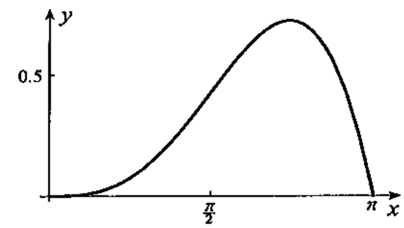
$$a<\frac{b}{(n+1)^{1/n}}-\frac{b}{(n+1)^{(n+1)/n}}=\frac{b}{(n+1)^{(n+1)/n}}(n+1-1)=\frac{bn}{(n+1)^{(n+1)/n}}.$$

40. (a) The graph is shown to the right.
(b) Since $f'(x)$ is not continuous at $x=0$ (Exercise 47 in Section 3.9), it follows that $f''(0)$ cannot exist (see the corollary to Theorem 3.6 in Section 3.3).
(c) Since $f(x)$ takes on positive and negative values in every interval around $x=0$, there cannot be a relative maximum or minimum at $x=0$. Since $f'(x)$ takes on negative and and positive values in every interval around $x=0$, the graph cannot change concavity at $x=0$, and therefore $(0,0)$ cannot be a point of inflection.



41. A graph of the function $f(\theta) = 2\sin\theta - \theta(1+\cos\theta)$ certainly suggests that this is the case. Consider the following verification. The first two derivatives of $f(\theta)$ are

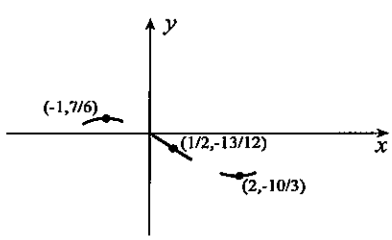
$$f'(\theta) = \theta\sin\theta + \cos\theta - 1,$$
$$f''(\theta) = \theta\cos\theta.$$

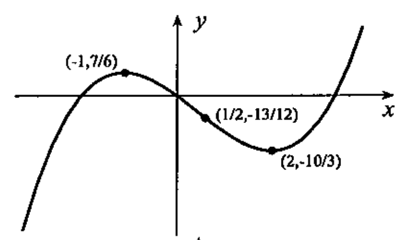


The tangent line at $\theta = 0$ is horizontal. Since $f''(\theta) > 0$ for $0 < \theta < \pi/2$, the graph is concave upward on this interval, and this implies that $f(\theta)$ cannot be equal to zero in this interval. On $\pi/2 < \theta < \pi$, we see that $f''(\theta) < 0$ so that the graph is concave downward on this interval. Consequently, $f(\theta)$ could have at most one zero in this interval. Since $f(\pi) = 0$, this is impossible. Hence, $f(\theta)$ cannot be equal to zero on the interval $0 < \theta < \pi$.

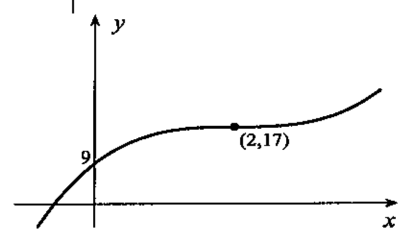
EXERCISES 4.5

1. Since $f'(x) = x^2 - x - 2 = (x-2)(x+1)$, the critical points are $x = -1, 2$. With $f''(x) = 2x - 1$, we find that $f''(-1) = -3$ and $f''(2) = 3$. Consequently, $x = -1$ gives a relative maximum of $f(-1) = 7/6$ and $x = 2$ gives a relative minimum of $f(2) = -10/3$. Since $f''(1/2) = 0$ and $f''(x)$ changes sign as x passes through $1/2$, there is a point of inflection at $(1/2, -13/12)$. This information is shown in the left figure below. We complete the graph of this cubic polynomial as shown in the right figure.

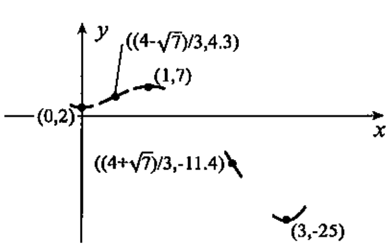


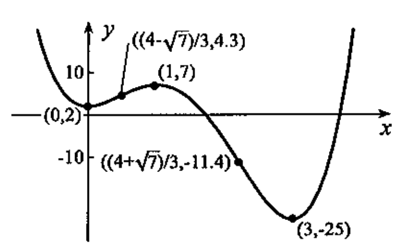


2. Since $f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2$, the only critical point is $x = 2$. Because $f'(x)$ remains positive as x passes through 2, the critical point gives a horizontal point of inflection at $(2, 17)$. Since $f''(x) = 6(x-2)$, there are no other points of inflection. The graph is shown to the right.

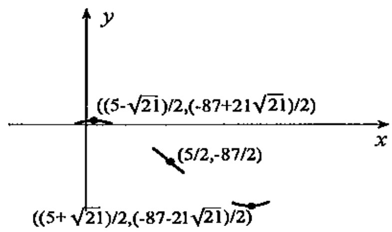


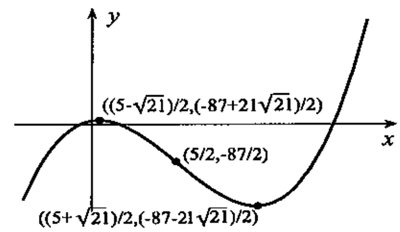
3. Since $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$, the critical points are $x = 0, 1, 3$. With $f''(x) = 36x^2 - 96x + 36 = 12(3x^2 - 8x + 3)$, we find that $f''(0) = 36$, $f''(1) = -24$, and $f''(3) = 72$. Consequently, $x = 0$ and $x = 3$ give relative minima of $f(0) = 2$ and $f(3) = -25$, and $x = 1$ gives a relative maximum of $f(1) = 7$. Since $f''(x) = 0$ when $x = (8 \pm \sqrt{64 - 36})/6 = (4 \pm \sqrt{7})/3$, and $f''(x)$ changes sign as x passes through each of these, there are points of inflection at $\left((4 - \sqrt{7})/3, 4.3\right)$ and $\left((4 + \sqrt{7})/3, -11.4\right)$. This information is shown in the left figure below. We complete the graph of this quartic polynomial as shown in the right figure.



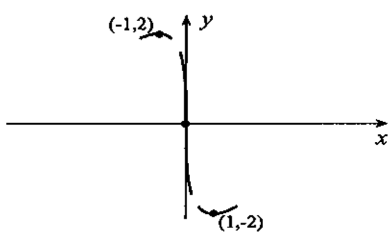


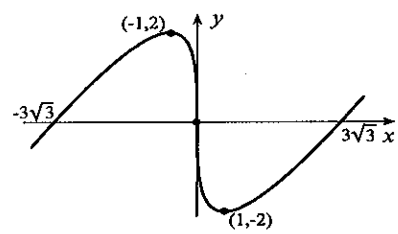
4. Since $f'(x) = 6x^2 - 30x + 6 = 6(x^2 - 5x + 1)$, the critical points are $x = (5 \pm \sqrt{25-4})/2 = (5 \pm \sqrt{21})/2$. Since $f'(x)$ changes from positive to negative as x increases through $(5 - \sqrt{21})/2$, there is a relative maximum at $\left((5-\sqrt{21})/2, (-87+21\sqrt{21})/2\right)$. Similarly, there is a relative minimum at $\left((5+\sqrt{21})/2, (-87-21\sqrt{21})/2\right)$. Since $0 = f''(x) = 12x - 30$ at $x = 5/2$, and $f''(x)$ changes sign as x passes through $5/2$, there is a point of inflection at $(5/2, -87/2)$. This information is shown in the left figure below. We complete the graph of this cubic polynomial as shown in the right figure.





5. For critical points we solve $0 = f'(x) = 1 - x^{-2/3} = \dfrac{x^{2/3}-1}{x^{2/3}}$. Solutions are $x = \pm 1$. Since $f'(0)$ is undefined, but $f(0) = 0$, $x = 0$ is also a critical point. With $f''(x) = (2/3)x^{-5/3}$, we find that $f''(-1) = -2/3$ and $f''(1) = 2/3$. Thus, $x = -1$ yields a relative maximum of $f(-1) = 2$ and $x = 1$ gives a relative minimum of $f(1) = -2$. Because $f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$, it follows that $(0,0)$ must be a point of inflection. Since $\lim_{x\to 0} f'(x) = -\infty$, $(0,0)$ is a vertical point of inflection. This information is shown in the left figure below. The final graph is shown to the right.

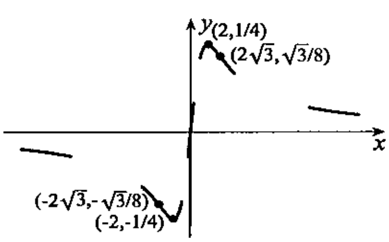


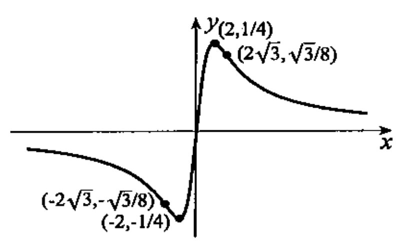


6. For critical points we solve $0 = f'(x) = \dfrac{(x^2+4)(1) - x(2x)}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2}$. Solutions are $x = \pm 2$. We now calculate $f''(x) = \dfrac{(x^2+4)^2(-2x) - (4-x^2)(2)(x^2+4)(2x)}{(x^2+4)^4} = \dfrac{2x(x^2-12)}{(x^2+4)^3}$. Since $f''(-2) = 1/16$ and $f''(2) = -1/16$, there is a relative minimum at $x = -2$ equal to $f(-2) = -1/4$, and a relative maximum at $x = 2$ of $f(2) = 1/4$.

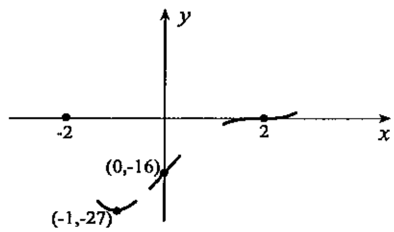
Because $f''(x) = 0$ at $x = 0, \pm 2\sqrt{3}$, and $f''(x)$ changes sign as x passes through each of these, points of inflection occur at $(0,0)$, $(2\sqrt{3}, \sqrt{3}/8)$, and $(-2\sqrt{3}, -\sqrt{3}/8)$.

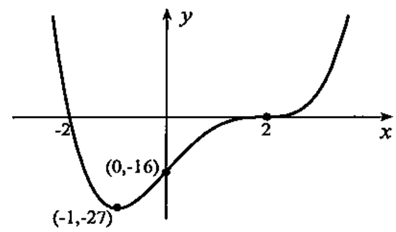
This information is shown in the left diagram below, together with the limits $\lim_{x\to-\infty} f(x) = 0^-$ and $\lim_{x\to\infty} f(x) = 0^+$. The final graph is shown to the right. We could have shortened the analysis by considering only the right half of the graph and using the fact that the function is odd.





7. Since $f'(x) = 3(x-2)^2(x+2)+(x-2)^3 = 4(x-2)^2(x+1)$, the critical points are $x = -1, 2$. Since $f'(x)$ changes from negative to positive as x increases through -1, there is a relative minimum at $(-1, -27)$. The derivative does not change sign at $x = 2$. We now calculate $f''(x) = 8(x-2)(x+1) + 4(x-2)^2 = 12x(x-2)$. Since $f''(x) = 0$ when $x = 0, 2$, and $f''(x)$ changes sign as x passes through these values, there is a point of inflection at $(0, -16)$ and a horizontal point of inflection at $(2, 0)$. This information is shown in the left figure below. The final graph is shown to the right.





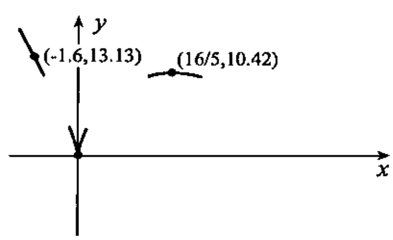
8. For critical points we solve $0 = f'(x) = (2/3)x^{-1/3}(8-x) - x^{2/3} = \dfrac{16-5x}{3x^{1/3}}$. Clearly, $x = 16/5$ is critical, but so also is $x = 0$ because $f'(0)$ does not exist and $f(0) = 0$. We now calculate

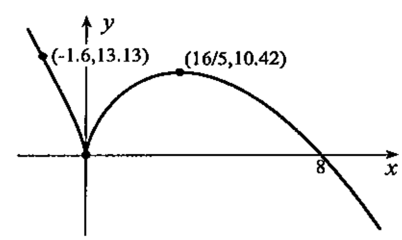
$$f''(x) = \frac{3x^{1/3}(-5) - (16-5x)x^{-2/3}}{9x^{2/3}} = \frac{-2(5x+8)}{9x^{4/3}}.$$

Since $f''(16/5) < 0$, there is a relative maximum of $f(16/5) = 10.42$. Because $f'(x)$ changes from negative to positive as x increases through 0, this critical point gives a relative mimumum of $f(0) = 0$.

Since $f''(-8/5) = 0$, and $f''(x)$ changes sign as x passes through $-8/5$, there is a point of inflection at $(-1.6, 13.13)$.

We have shown this information in the left figure below along with the additional facts that $\lim_{x\to 0^-} f'(x) = -\infty$ and $\lim_{x\to 0^+} f'(x) = \infty$. The final graph is shown to the right.





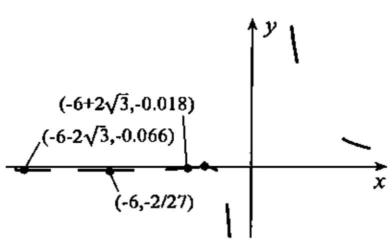
9. For critical points we solve $0 = f'(x) = \dfrac{x^3(2)(x+2) - (x+2)^2(3x^2)}{x^6} = -\dfrac{(x+2)(x+6)}{x^4}$. Solutions are $x = -6, -2$. We now calculate

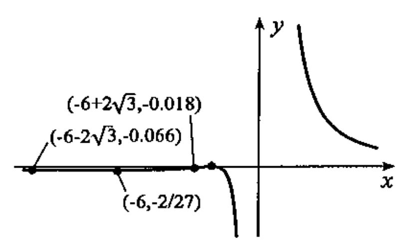
$$f''(x) = -\frac{x^4(2x+8) - (x+2)(x+6)(4x^3)}{x^8} = \frac{2(x^2+12x+24)}{x^5}.$$

Since $f''(-6) > 0$ and $f''(-2) < 0$, there is a relative minimum at $x = -6$ equal to $f(-6) = -2/27$, and a relative maximum at $x = -2$ of $f(-2) = 0$.

Because $f''(x) = 0$ at $x = (-12 \pm \sqrt{144-96})/2 = -6 \pm 2\sqrt{3}$, and $f''(x)$ changes sign as x passes through each of these, points of inflection occur at $(-6-2\sqrt{3}, -0.066)$, $(-6+2\sqrt{3}, -0.018)$.

This information is shown in the left figure below, together with the limits $\lim_{x\to-\infty} f(x) = 0^-$, $\lim_{x\to\infty} f(x) = 0^+$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to 0^+} f(x) = \infty$. The final graph is shown to the right.





10. For critical points we solve

$$0 = f'(x) = 3x^{1/2} - 9 + 6x^{-1/2} = \frac{3(x - 3\sqrt{x} + 2)}{\sqrt{x}} = \frac{3(\sqrt{x} - 1)(\sqrt{x} - 2)}{\sqrt{x}}.$$

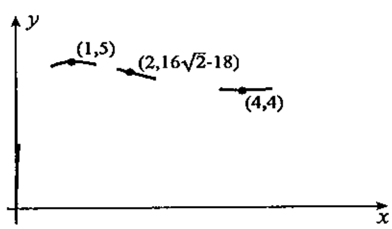
Solutions are $x = 1$ and $x = 4$. The point $x = 0$ is also critical since $f'(0)$ is not defined but $f(0) = 0$. To classify these critical points we calculate

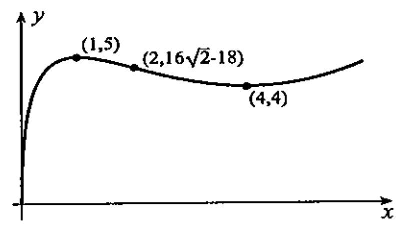
$$f''(x) = \frac{3}{2\sqrt{x}} - \frac{3}{x^{3/2}} = \frac{3(x-2)}{2x^{3/2}}.$$

Since $f''(1) = -3/2$ and $f''(4) = 3/8$, $f(1) = 5$ is a relative maximum and $f(4) = 4$ is a relative minimum. Because $f(x)$ is not defined for $x < 0$, the critical point $x = 0$ does not give a relative extrema. For graphical purposes we note that $\lim_{x \to 0^+} f'(x) = \infty$.

Since $f''(2) = 0$, and $f''(x)$ changes sign as x passes through 2, there is a point of inflection at $(2, 16\sqrt{2} - 18)$.

This information is shown in the left figure below. The final graph is shown to the right.





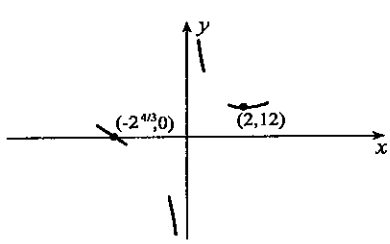
11. For critical points we solve

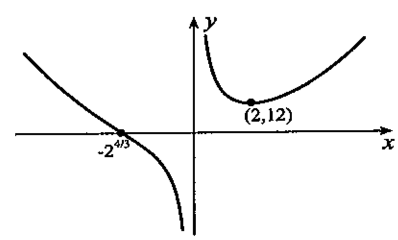
$$0 = f'(x) = \frac{x(3x^2) - (x^3 + 16)(1)}{x^2} = \frac{2(x^3 - 8)}{x^2}.$$

Clearly $x = 2$ is a critical point. To classify this critical point we calculate

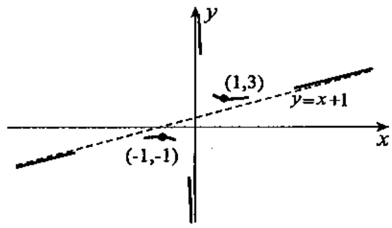
$$f''(x) = 2\left[\frac{x^2(3x^2) - (x^3 - 8)(2x)}{x^4}\right] = \frac{2(x^3 + 16)}{x^3}.$$

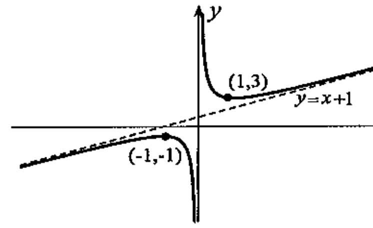
Since $f''(2) = 6$, $f(2) = 12$ is a relative minimum. Since $f''(x) = 0$ when $x = -2^{4/3}$ and $f''(x)$ changes sign as x passes through $-2^{4/3}$, the only point of inflection is $(-2^{4/3}, 0)$. This information is shown in the left figure below. The final graph is shown to the right.



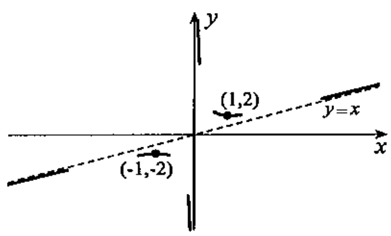


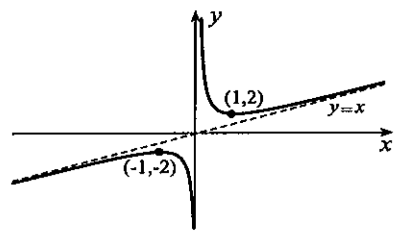
12. The function is discontinuous at $x = 0$. Left- and right-hand limits of the function as $x \to 0$ are $\lim_{x \to 0^-} f(x) = -\infty$ and $\lim_{x \to 0^+} f(x) = \infty$. For critical points we express the function in the form $f(x) = x + 1 + \frac{1}{x}$, and solve $0 = f'(x) = 1 - \frac{1}{x^2}$. Clearly, $x = \pm 1$. The second derivative is $f''(x) = 2/x^3$. Since $f''(1) = 2$, there is a relative minimum at $x = 1$ of $f(1) = 3$. With $f''(-1) = -2$, there is a relative maximum of $f(-2) = -1$. Since $f''(x)$ is never zero, there can be no points of inflection. Finally, we note that $y = x + 1$ is an oblique asymptote. This information is shown in the left figure below. The final graph is shown to the right.





13. The function is discontinuous at $x = 0$. Left- and right-hand limits of the function as $x \to 0$ are $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$. Clearly, $y = x$ is an oblique asymptote. For critical points we solve $0 = f'(x) = 1 - \dfrac{1}{x^2}$. Clearly, $x = \pm 1$. The second derivative is $f''(x) = 2/x^3$. Since $f''(1) = 2$, there is a relative minimum at $x = 1$ of $f(1) = 2$. With $f''(-1) = -2$, there is a relative maximum of $f(-2) = -2$. Since $f''(x)$ is never zero, there can be no points of inflection. This information is shown in the left figure below. The final graph is shown to the right.





14. The function is discontinuous at $x = \pm 2$. We take left-and right-hand limits of $f(x) = \dfrac{x^3}{(x+2)(x-2)}$ as $x \to \pm 2$

$$\lim_{x\to -2^-} f(x) = -\infty, \quad \lim_{x\to -2^+} f(x) = \infty, \quad \lim_{x\to 2^-} f(x) = -\infty, \quad \lim_{x\to 2^+} f(x) = \infty.$$

For critical points we solve

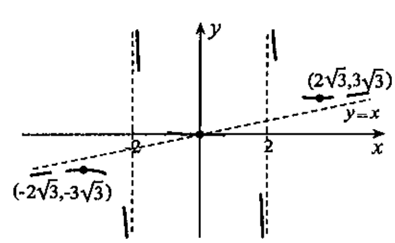
$$0 = f'(x) = \frac{(x^2-4)(3x^2) - x^3(2x)}{(x^2-4)^2} = \frac{x^2(x^2-12)}{(x^2-4)^2}.$$

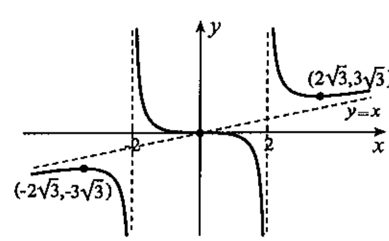
Solutions are $x = 0,\ \pm 2\sqrt{3}$. The second derivative is

$$f''(x) = \frac{(x^2-4)^2(4x^3-24x) - (x^4-12x^2)(2)(x^2-4)(2x)}{(x^2-4)^4} = \frac{8x(x^2+12)}{(x^2-4)^3}.$$

Since $f''(2\sqrt{3}) > 0$, we have a relative minimum $f(2\sqrt{3}) = 3\sqrt{3}$. Similarly, $f''(-2\sqrt{3}) < 0$ indicates that $f(-2\sqrt{3}) = -3\sqrt{3}$ is a relative maximum. Since $f''(x) = 0$ only at $x = 0$, and $f''(x)$ changes sign as x passes through 0, there is a horizontal point of inflection at $(0, 0)$.

By writing $f(x)$ in the form $f(x) = x + \dfrac{4x}{x^2-4}$, we see that the graph is asymptotic to the line $y = x$. This information is shown in the left figure below. The final graph is shown to the right.





15. With $f(x) = \dfrac{2x^2}{(x-2)(x-6)}$, we see that the function is discontinuous at $x = 2$ and $x = 6$. Left- and right-hand limits at these values of x are

$$\lim_{x\to 2^-} f(x) = \infty, \quad \lim_{x\to 2^+} f(x) = -\infty, \quad \lim_{x\to 6^-} f(x) = -\infty, \quad \lim_{x\to 6^+} f(x) = \infty.$$

For critical points, we solve

$$0 = f'(x) = \frac{(x^2-8x+12)(4x) - 2x^2(2x-8)}{(x^2-8x+12)^2} = \frac{16x(3-x)}{(x^2-8x+12)^2}.$$

Solutions are $x = 0, 3$. The second derivative is

$$f''(x) = \frac{(x^2-8x+12)^2(48-32x) - (48x-16x^2)(2)(x^2-8x+12)(2x-8)}{(x^2-8x+12)^4} = \frac{16(2x^3-9x^2+36)}{(x^2-8x+12)^3}.$$

Since $f''(0) > 0$, there is a relative minimum at $x = 0$ of $f(0) = 0$. With $f''(3) < 0$, there is a relative maximum of $f(3) = -6$. For points of inflection, we would solve $2x^3 - 9x^2 + 36 = 0$. With Newton's iterative procedure, and an inital value $x_1 = -2$, iteration of $x_{n+1} = x_n - \dfrac{2x_n^3 - 9x_n^2 + 36}{6x_n^2 - 18x_n}$ gives

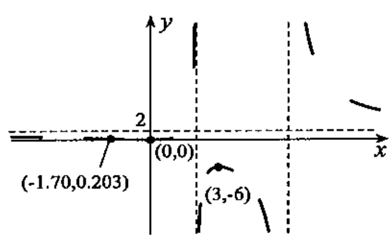
$$x_2 = -1.733, \quad x_3 = -1.704, \quad x_4 = -1.703, \quad x_5 = -1.703.$$

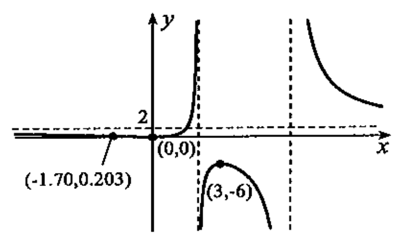
Since $f''(x)$ changes sign as x passes through this solution, there is a point of inflection at $(-1.70, 0.203)$.

By writing $f(x)$ in the form $f(x) = 2 + \dfrac{16x-24}{x^2-8x+12}$, we see that $y = 2$ is a horizontal asymptote, and

$$\lim_{x\to -\infty} f(x) = 2^-, \qquad \lim_{x\to\infty} f(x) = 2^+.$$

This information is shown in the left figure below. The final graph is shown to the right.





16. With $f(x) = \dfrac{x^2+1}{(x-1)(x+1)}$, we see that the function is discontinuous at $x = \pm 1$. Left-and right-hand limits at these values of x are

$$\lim_{x\to -1^-} f(x) = \infty, \quad \lim_{x\to -1^+} f(x) = -\infty, \quad \lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to -1^+} f(x) = \infty.$$

For critical points, we solve

$$0 = f'(x) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

The only solution is $x = 0$. Since $f'(x)$ changes from positive to negative as x increases through 0, there is a relative maximum of $f(0) = -1$. For points of inflection, we solve

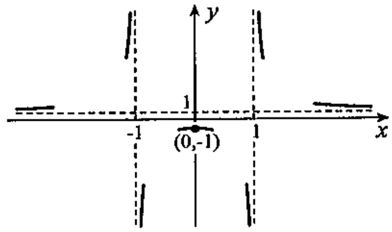
$$0 = f''(x) = \frac{(x^2-1)^2(-4) - (-4x)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}.$$

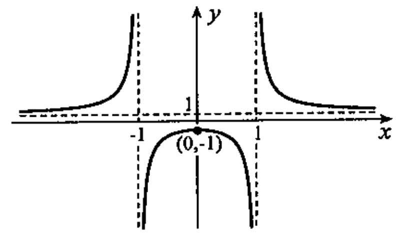
There are no solutions indicating no points of inflection.

By writing $f(x)$ in the form $f(x) = 1 + \dfrac{1}{x^2-1}$, we see that $y = 1$ is a horizontal asymptote, and

$$\lim_{x\to-\infty} f(x) = 1^+, \qquad \lim_{x\to\infty} f(x) = 1^+.$$

This information is shown in the left figure below. The final graph is shown to the right.





17. To analyze the graph near the discontinuity $x = -1$, we calculate

$$\lim_{x\to-1^-} f(x) = -\infty, \qquad \lim_{x\to-1^+} f(x) = -\infty.$$

The x-axis is a horizontal asymptote, and

$$\lim_{x\to-\infty} f(x) = 0^-, \qquad \lim_{x\to\infty} f(x) = 0^+.$$

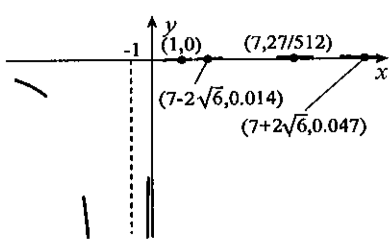
For critical points we solve

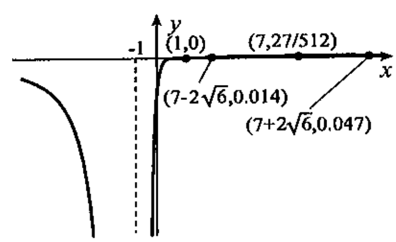
$$0 = f'(x) = \frac{(x+1)^4(3)(x-1)^2 - (x-1)^3(4)(x+1)^3}{(x+1)^8} = \frac{(x-1)^2(7-x)}{(x+1)^5}.$$

Solutions are $x = 1,\ 7$. Since $f'(x)$ changes from positive to negative as x increases through 7, there is a relative maximum at $(7, 27/512)$. The derivative does not change sign at $x = 1$. For points of inflection, we solve

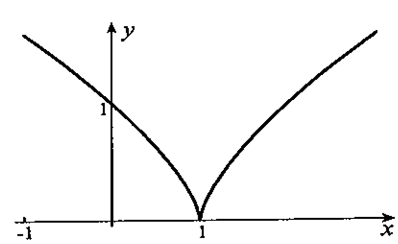
$$0 = f''(x) = \frac{(x+1)^5[2(x-1)(7-x) - (x-1)^2] - (x-1)^2(7-x)(5)(x+1)^4}{(x+1)^{10}} = \frac{2(x-1)(x^2-14x+25)}{(x+1)^6}.$$

Solutions are $x = 1,\ 7 \pm 2\sqrt{6}$. The second derivative changes sign as x passes through each of these. Thus, we have points of inflection at $(1, 0)$ (a horizontal one), and $(7-2\sqrt{6}, 0.014)$ and $(7+2\sqrt{6}, 0.047)$. This information is shown in the left figure below. The final graph is shown to the right.





18. For critical points we solve $f'(x) = (2/3)(x-1)^{-1/3} = 0$. There is no solution, but $x = 1$ is critical since $f(1) = 0$ and $f'(1)$ does not exist. Because $f(x) > 0$ for all $x \neq 1$, it follows that $f(1)$ is a relative minimum. Since $f''(x) = -(2/9)(x-1)^{-4/3}$ never vanishes, there are no points of inflection. The graph is shown to the right.



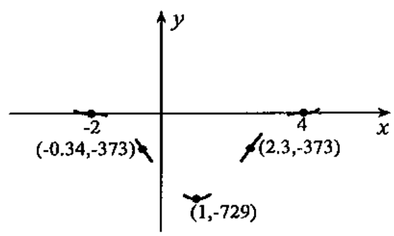
19. For critical points we solve

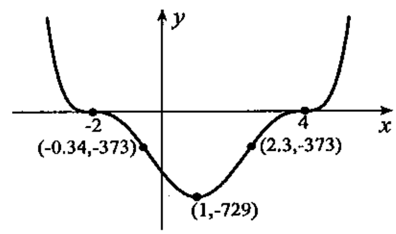
$$0 = f'(x) = 3(x+2)^2(x-4)^3 + 3(x+2)^3(x-4)^2 = 6(x+2)^2(x-4)^2(x-1).$$

Solutions are $x = -2, 1, 4$. The derivative changes from negative to positive as x increases through 1 indicating a relative minimum there of $f(1) = -729$. The derivative does not change sign at $x = -2$ and $x = 4$. For points of inflection we solve

$$\begin{aligned} 0 = f''(x) &= 6[2(x+2)(x-2)^2(x-1) + (x+2)^2(2)(x-4)(x-1) + (x+2)^2(x-4)^2] \\ &= 6(x+2)(x-4)(5x^2 - 10x - 4). \end{aligned}$$

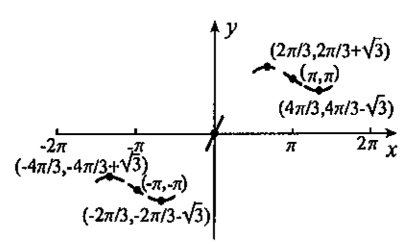
Solutions are $x = -2$, 4, $(5 \pm 3\sqrt{5})/5$. Since $f''(x)$ changes sign as x passes through each of these, there are four points of inflection $(-2, 0)$, $(4, 0)$ (both of which are horizontal), and $((5 + 3\sqrt{5})/5, -373)$ and $((5 - 3\sqrt{5})/5, -373)$. This information is shown in the left figure below. The final graph is shown to the right.

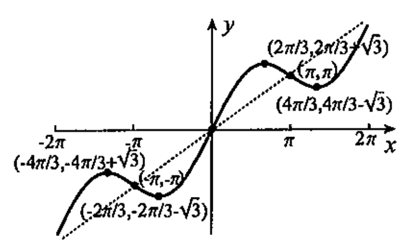




20. For critical points we solve $0 = f'(x) = 1 + 2\cos x$. Solutions are $x = \pm 2\pi/3 + 2n\pi$, where n is an integer. To classify them we calculate $f''(x) = -2\sin x$. Since $f''(2\pi/3 + 2n\pi) < 0$, the critical points $x = 2\pi/3 + 2n\pi$ give relative maxima $f(2\pi/3 + 2n\pi) = \sqrt{3} + 2\pi/3 + 2n\pi$. Similarly, there are relative minima at $(-2\pi/3 + 2n\pi, -\sqrt{3} - 2\pi/3 + 2n\pi)$.

For points of inflection, we solve $0 = f''(x) = -2\sin x$. Solutions are $x = n\pi$, where n is an integer. Since $f''(x)$ changes sign at each of these, they all give points of inflection. This information is shown in the left figure below. The final graph is shown to the right. It oscillates about the line $y = x$.





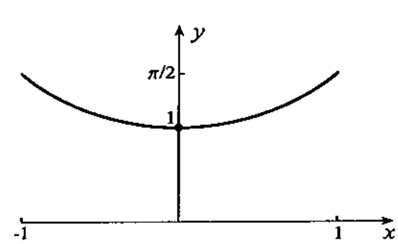
21. For critical points we consider $f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0$. Since this is true for all x in the domain $-1 \le x \le 1$ of the function, the function must be equal to a constant, say $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x = C$. When we set $x = 0$, we obtain $0 + \pi/2 = C$. Thus, the graph is a horizontal line segment joining the points $(-1, \pi/2)$ and $(1, \pi, 2)$. Every point in the interval $-1 < x < 1$ is critical, and each gives a relative maximum and a relative minimum. There are no points of inflection.

22. For critical points we solve

$$\begin{aligned} 0 = f'(x) &= \mathrm{Sin}^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \mathrm{Sin}^{-1}x. \end{aligned}$$

The only solution is $x = 0$. For points of inflection we solve $0 = f''(x) = \dfrac{1}{\sqrt{1-x^2}}$.

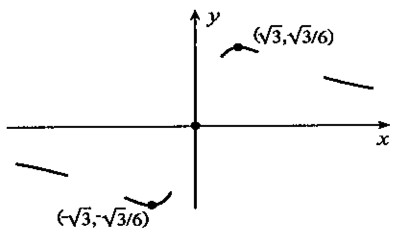
There are no solutions. Because $f''(0) = 1$, $f(x)$ has a relative minimum of $f(0) = 1$.

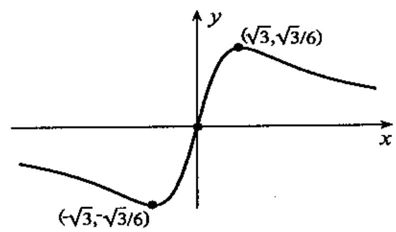


23. For critical points we solve

$$0 = f'(x) = \frac{(x^2+3)(1) - x(2x)}{(x^2+3)^2} = \frac{3-x^2}{(x^2+3)^2}.$$

Solutions are $x = \pm\sqrt{3}$. Since $f'(x)$ changes from negative to positive as x increases through $-\sqrt{3}$, there is a relative minimum at this critical point of $f(-\sqrt{3}) = -\sqrt{3}/6$. Since $f'(x)$ changes from positive to negative as x increases through $\sqrt{3}$, there is a relative maximum at this critical point of $f(\sqrt{3}) = \sqrt{3}/6$. The graph is asymptotic to the x-axis. This information is shown in the left figure below. The final graph is shown to the right.

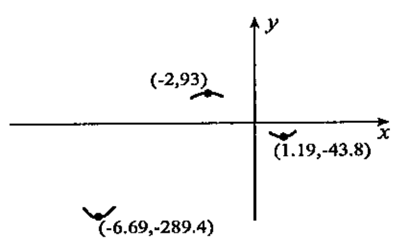


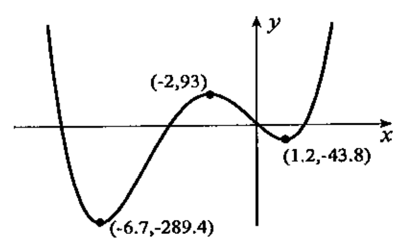


24. For critical points we solve

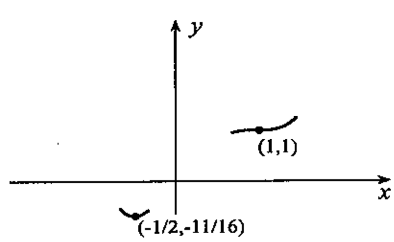
$$0 = f'(x) = 4x^3 + 30x^2 + 12x - 64 = 2(x+2)(2x^2 + 11x - 16).$$

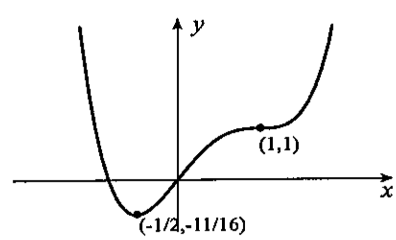
Solutions are $x = -2$ and $x = (-11 \pm \sqrt{121+128})/4 = (-11 \pm \sqrt{249})/4$. Since $f'(x)$ changes from positive to negative as x increases through -2, there is a relative maximum at this critical point of $f(-2) = 93$. Since $f'(x)$ changes from negative to positive as x increases through the remaining critical points, they give relative minima of $f\big((-11-\sqrt{249})/4\big) = f(-6.69) = -289.4$ and $f\big((-11+\sqrt{249})/4\big) = f(1.19) = -43.8$. This information is shown in the left figure below. The final graph is shown to the right.





25. For critical points we solve $0 = f'(x) = 4x^3 - 6x^2 + 2 = 2(x-1)^2(2x+1)$. Solutions of this equation are $x = -1/2, 1$. Since $f'(x)$ changes from negative to positive as x increases through $-1/2$, there is a relative minimum at this critical point of $f(-1/2) = -11/16$. Since $f'(x)$ does not change sign as x passes through 1, there is a horizontal point of inflection at $(1,1)$. This information is shown in the left figure below. The final graph is shown to the right.





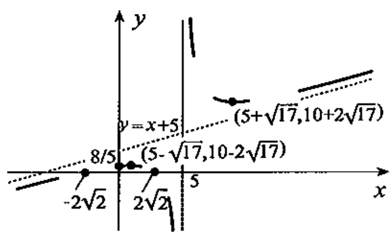
26. The function is discontinuous at $x = 5$, and therefore we calculate

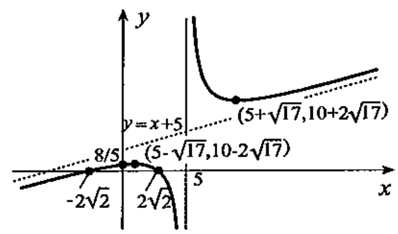
$$\lim_{x\to 5^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 5^+} f(x) = \infty.$$

For critical points we solve

$$0 = f'(x) = \frac{(x-5)(2x) - (x^2-8)(1)}{(x-5)^2} = \frac{x^2 - 10x + 8}{(x-5)^2}.$$

Solutions are $x = \dfrac{10 \pm \sqrt{100-32}}{2} = 5 \pm \sqrt{17}$. Since $f'(x)$ changes from positive to negative as x increases through $5-\sqrt{17}$, there is a relative maximum at this critical point of $f(5-\sqrt{17}) = 10 - 2\sqrt{17}$. Since $f'(x)$ changes from negative to positive as x increases through $5+\sqrt{17}$, there is a relative minimum at this critical point of $f(5+\sqrt{17}) = 10 + 2\sqrt{17}$. Since $f(x) = x + 5 + 17/(x-5)$, the graph has oblique asymptote $y = x + 5$. This information, along with x- and y-intercepts, is shown in the left figure below. The final graph is shown to the right.





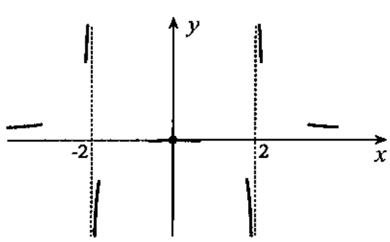
27. The function is discontinuous at $x = \pm 2$, and therefore we calculate

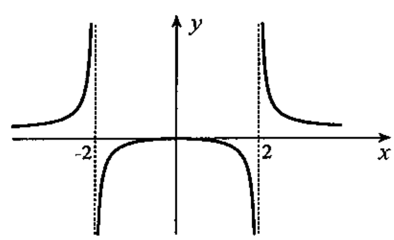
$$\lim_{x\to -2^-} f(x) = \infty, \quad \lim_{x\to -2^+} f(x) = -\infty, \quad \lim_{x\to 2^-} f(x) = -\infty, \quad \lim_{x\to 2^+} f(x) = \infty.$$

For critical points we solve

$$0 = f'(x) = \frac{(x^2-4)(2x) - x^2(2x)}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}.$$

The only solution is $x = 0$. Since $f'(x)$ changes from positive to negative as x increases through the critical point, there is a relative maximum of $f(0) = 0$. The x-axis is a horizontal asymptote; it approaches it from above for $x \to \pm\infty$. This information is shown in the left figure below. The final graph is shown to the right.

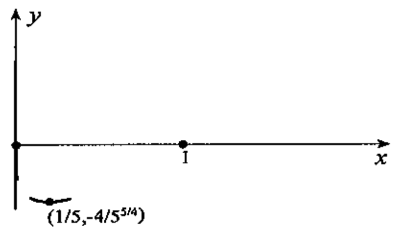


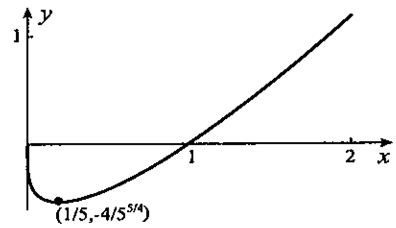


28. For critical points we solve

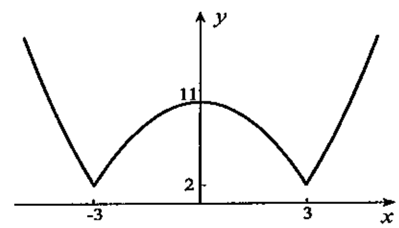
$$0 = f'(x) = \frac{5}{4}x^{1/4} - \frac{1}{4}x^{-3/4} = \frac{5x-1}{4x^{3/4}}.$$

The solution is $x = 1/5$. Since $f'(x)$ changes from negative to positive as x increases through the critical point, there is a relative minimum of $f(1/5) = -4/5^{5/4}$. There is also critical point at $x = 0$, but because the function is not defined for $x < 0$, it cannot yield a relative maximum. We note that $f(0) = 0$ and $\lim_{x\to 0^+} f'(x) = -\infty$. This information, along with the x-intercept $x = 1$, is shown in the left figure below. The final graph is shown to the right.





29. The graph is most easily drawn by taking absolute values of the parabola $y = x^2 - 9$ and shifting it upward 2 units. The derivative is undefined at $x = \pm 3$ where the graph has sharp corners. There are relative minima equal to $f(\pm 3) = 2$ at these points.



30. The function is discontinuous at $x = 1$ and $x = 4$, and we therefore calculate

$$\lim_{x \to 1^-} f(x) = -\infty, \quad \lim_{x \to 1^+} f(x) = \infty, \quad \lim_{x \to 4^-} f(x) = \infty, \quad \lim_{x \to 4^+} f(x) = -\infty.$$

To find critical points we solve

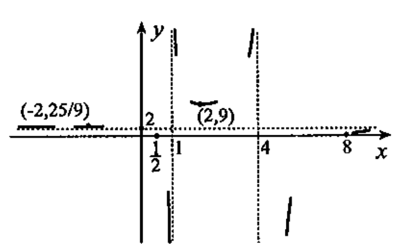
$$0 = f'(x) = \frac{(x^2 - 5x + 4)(4x - 17) - (2x^2 - 17x + 8)(2x - 5)}{(x^2 - 5x + 4)^2} = \frac{7(x - 2)(x + 2)}{(x^2 - 5x + 4)^2}.$$

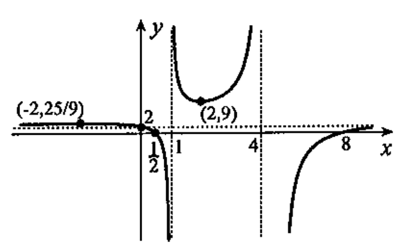
Solutions are $x = -2, 2$. Since $f'(x)$ changes from positive to negative as x increases through -2, there is a relative maximum of $f(-2) = 25/9$. Since $f'(x)$ changes from negative to positive as x increases through 2, there is a relative minimum of $f(2) = 9$.

The graph has horizontal asymptote $y = 2$, and to determine how the asymptote is approached, we express $f(x)$ in the form

$$f(x) = \frac{2x^2 - 17x + 8}{x^2 - 5x + 4} = 2 - \frac{7x}{x^2 - 5x + 4}.$$

This shows that $y = 2$ is approached from below as $x \to \infty$ and from above as $x \to -\infty$. This information is shown in the left figure below. The final graph is shown to the right.

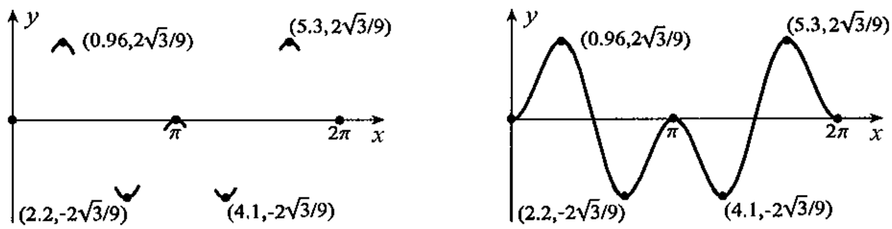




31. For critical points we solve

$$0 = f'(x) = 2\sin x \cos^2 x - \sin^3 x = \sin x(2\cos^2 x - \sin^2 x) = \sin x(3\cos^2 x - 1).$$

Solutions are $x = 0, \pi, 2\pi$ and values of x satisfying $\cos x = \pm 1/\sqrt{3}$. From $\cos x = 1/\sqrt{3}$, we obtain $x = \mathrm{Cos}^{-1}(1/\sqrt{3}) = 0.96$ and $x = 2\pi - 0.96 = 5.3$. From $\cos x = -1/\sqrt{3}$, $x = \mathrm{Cos}^{-1}(-1/\sqrt{3}) = 2.2$ and $x = 2\pi - 2.2 = 4.1$. Being end points, $x = 0, 2\pi$ cannot yield relative extrema. Since $f'(x)$ changes from positive to negative as x increases through π, 0.96 and 5.3, there are a relative maxima of $f(\pi) = 0$, $f(0.96) = f(5.3) = 2\sqrt{3}/9$. Since $f'(x)$ changes from negative to positive as x increases through 2.2 and 4.1, there are relative minima of $f(2.2) = f(4.1) = -2\sqrt{3}/9$. This information is shown in the left figure below. The final graph is shown to the right.

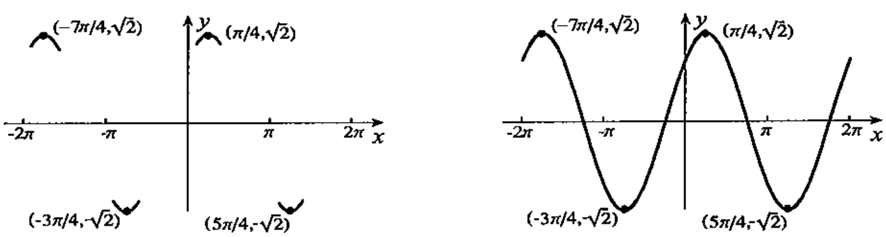


32. For critical points we solve $0 = f'(x) = \cos x - \sin x$. Solutions are $x = \pi/4 + 2n\pi$, where n is an integer. Since $f'(x)$ changes from positive to negative as x increases through $\pi/4 + 2n\pi$, there are a relative maxima of $f(\pi/4 + 2n\pi) = \sqrt{2}$. Since $f'(x)$ changes from negative to positive as x increases through $\pi/4 + (2n+1)\pi$, there are a relative minima of $f(\pi/4 + (2n+1)\pi) = -\sqrt{2}$. This information is shown in the left figure below. The final graph is shown to the right.

The function could also be graphed by expressing it as a general sine function $A\sin(x + \phi)$ (see Example 1.45 in Section 1.8),

$$\sin x + \cos x = A\sin(x + \phi) = A(\sin x \cos\phi + \cos x \sin\phi).$$

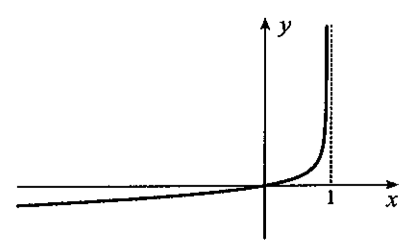
To satisfy this equation we set $A\cos\phi = 1$ and $A\sin\phi = 1$. These are satisfied if we choose $A = \sqrt{2}$ and $\phi = \pi/4$. Thus, $f(x)$ can be expressed in the form $f(x) = \sqrt{2}\sin(x + \pi/4)$.



33. The function is only defined for $x < 1$.
For critical points we solve

$$0 = f'(x) = \frac{\sqrt{1-x} - x(1/2)(1-x)^{-1/2}(-1)}{1-x} = \frac{2-x}{2(1-x)^{3/2}}.$$

The derivative never vanishes for $x < 1$.
To draw the graph we note that $\lim_{x\to 1^-} f(x) = \infty$, the x- and y-intercepts are both 0, and the function is negative when x is negative.



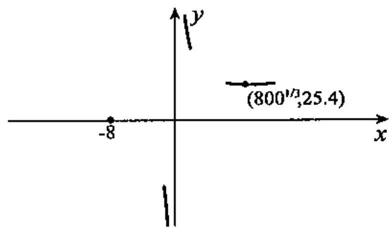
34. The function is discontinuous at $x = 0$, and we therefore calculate

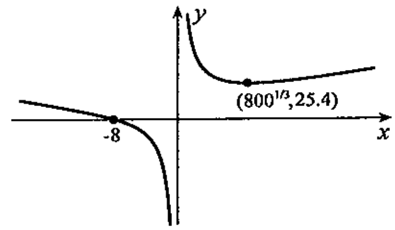
$$\lim_{x\to 0^-} f(x) = -\infty, \qquad \lim_{x\to 0^+} f(x) = \infty.$$

To identify critical points we solve

$$0 = f'(x) = -\frac{8}{x^2}\sqrt{x^2+100} + \left(1 + \frac{8}{x}\right)\frac{x}{\sqrt{x^2+100}} = \frac{x^3 - 800}{x^2\sqrt{x^2+100}}.$$

The only critical point is $800^{1/3}$. Since $f'(x)$ changes from negative to positive as x increases through this critical point, there is a relative minimum of $f(800^{1/3}) = \left(\frac{8 + 800^{1/3}}{800^{1/3}}\right)\sqrt{100 + 800^{2/3}} \approx 25.4$. This information is shown in the left figure below. The final graph is shown to the right.





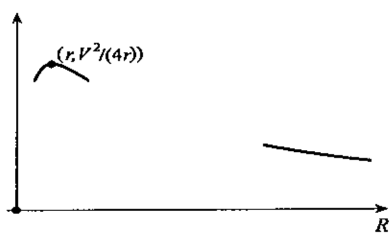
35. (a) $P_R(R) = i^2R = \dfrac{RV^2}{(r+R)^2}$ The graph of this function begins at the origin, is in the first quadrant, and is asymptotic to the R-axis since $\lim_{R\to\infty} P_R(R) = 0^+$. For critical points we solve

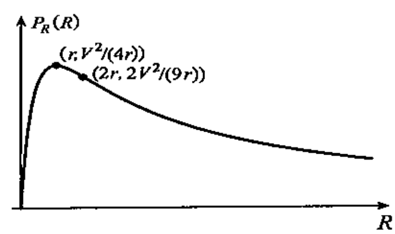
$$0 = P_R'(R) = V^2\left[\frac{(r+R)^2(1) - R(2)(r+R)}{(r+R)^4}\right] = \frac{V^2(r-R)}{(r+R)^3}.$$

Since $P_R'(R)$ changes from positive to negative as R increases through r, there is a relative maximum at $\big(r, V^2/(4r)\big)$. These facts are shown in the left figure below. To finish the graph we solve

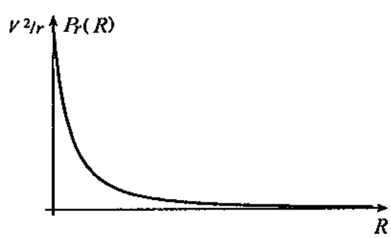
$$0 = P_R''(R) = V^2\left[\frac{(r+R)^3(-1) - (r-R)(3)(r+R)^2}{(r+R)^6}\right] = \frac{2V^2(R-2r)}{(r+R)^4}.$$

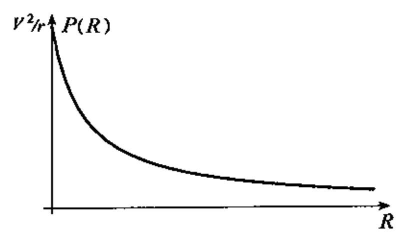
Because $P_R''(R)$ changes sign as R passes through $2r$, there is a point of inflection at $\big(2r, 2V^2/(9r)\big)$. The final graph is to the right.





The function $P_r(R) = V^2r/(r+R)^2$ is decreasing for $R > 0$ and $\lim_{R\to\infty} P_r(R) = 0^+$. Its graph is shown to the left below.



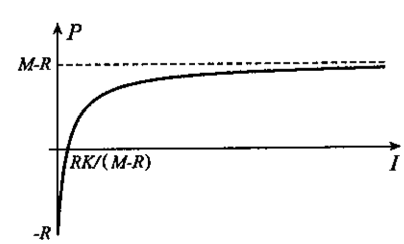


(b) The function $P(R) = V^2/(r+R)$ has the same shape as $P_r(R)$ (right figure above).

36. There are no critical points since

$$\frac{dP}{dI} = \frac{(I+K)M - MI(1)}{(I+K)^2} = \frac{KM}{(I+K)^2}$$

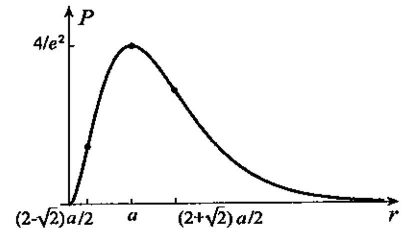
never vanishes. Since $\dfrac{d^2P}{dI^2} = \dfrac{-2KM}{(I+K)^3}$ is always negative, the graph is concave downward. It crosses the I-axis when $0 = \dfrac{MI}{I+K} - R$, and the solution of this equation is $I = RK/(M-R)$. Finally, we have a horizontal asymptote since $P \to (M-R)^-$ as $I \to \infty$.



37. For critical points we solve $0 = P'(r) = \left(\frac{8r}{a^2}\right)e^{-2r/a} + \left(\frac{4r^2}{a^2}\right)e^{-2r/a}\left(\frac{-2}{a}\right) = \frac{8r}{a^3}(a-r)e^{-2r/a}$. Solutions are $r = 0, a$. Since $P'(r)$ changes from positive to negative as r increases through a, there is a relative maximum at $P(a) = 4/e^2$. For points of inflection we solve

$$0 = P''(r) = \frac{8}{a^3}[(a-2r)e^{-2r/a} + (ar - r^2)e^{-2r/a}(-2/a)] = \frac{8}{a^4}(a^2 - 4ar + 2r^2)e^{-2r/a}.$$

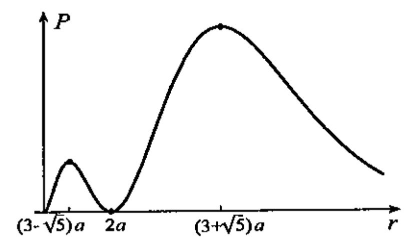
Solutions are $r = (4a \pm \sqrt{16a^2 - 8a^2})/4 = (2 \pm \sqrt{2})a/2$. Since $P''(r)$ changes sign as r passes through these values, there are points of inflection at $\left((2 \pm \sqrt{2})a/2, (6 \pm 4\sqrt{2})e^{-2\mp\sqrt{2}}\right)$. When combined with the fact that $\lim_{r\to\infty} P(r) = 0^+$, the graph to the right is obtained.



38. For critical points we solve

$$0 = P'(r) = \frac{1}{8a^3}\left[2r\left(2-\frac{r}{a}\right)^2 e^{-r/a} + 2r^2\left(2-\frac{r}{a}\right)\left(\frac{-1}{a}\right)e^{-r/a} + r^2\left(2-\frac{r}{a}\right)^2 e^{-r/a}\left(\frac{-1}{a}\right)\right]$$
$$= \frac{r(2a-r)e^{-r/a}}{8a^6}(r^2 - 6ar + 4a^2).$$

Solutions are $r = 0$, $r = 2a$, and $r = (6a \pm \sqrt{36a^2 - 16a^2})/2 = (3 \pm \sqrt{5})a$. Since $P'(r)$ changes from negative to positive as r increases through $2a$, there is a relative minimum of $f(2a) = 0$ at $r = 2a$. Since $P'(r)$ changes from positive to negative as r increases through both $(3 \pm \sqrt{5})a$, they give relative maxima of $P\left((3 \pm \sqrt{5})a\right)$, and these simplify to $(2/a)(9 \pm 4\sqrt{5})e^{-3\mp\sqrt{5}}$. When combined with the fact that $\lim_{r\to\infty} P(r) = 0^+$, the graph to the right is obtained.

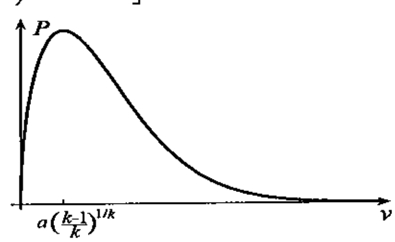


39. For critical points we solve

$$0 = P'(v) = \frac{k}{a}\left[\frac{k-1}{a}\left(\frac{v}{a}\right)^{k-2} e^{-(v/a)^k} + \left(\frac{v}{a}\right)^{k-1} e^{-(v/a)^k}\left(\frac{-k}{a}\right)\left(\frac{v}{a}\right)^{k-1}\right]$$
$$= \frac{k}{a^k}v^{k-2}e^{-(v/a)^k}\left[(k-1) - k\left(\frac{v}{a}\right)^k\right].$$

Thus, $v = a\left(\frac{k-1}{k}\right)^{1/k}$. Since $P'(v)$ changes from a positive quantity to a negative quantity as v increases through this value, there is a relative maximum of

$$\frac{k}{a}\left(\frac{k-1}{k}\right)^{(k-1)/k} e^{-(k-1)/k}.$$

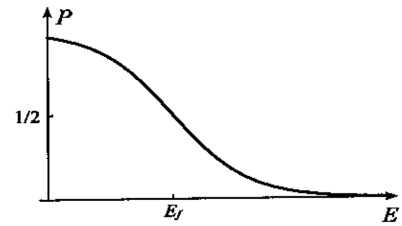


Since $1 < k < 2$, it follows that $\lim_{v\to 0^+} P'(v) = \infty$. Furthermore, with $\lim_{v\to\infty} P(v) = 0$ the graph is shown to the right.

40. For critical points we solve $0 = P'(E) = \frac{-e^{(E-E_f)/(kT)}}{kT[e^{(E-E_f)/(kT)} + 1]^2}$. There are no solutions to this equation. For points of inflection we solve

$$0 = P''(E) = \frac{-1}{kT}\left\{\frac{e^{(E-E_f)/(kT)}}{[e^{(E-E_f)/(kT)} + 1]^2}\left(\frac{1}{kT}\right) - \frac{2e^{(E-E_f)/(kT)}}{[e^{(E-E_f)/(kT)} + 1]^3}e^{(E-E_f)/(kT)}\left(\frac{1}{kT}\right)\right\}$$
$$= \frac{e^{(E-E_f)/(kT)}[e^{(E-E_f)/(kT)} - 1]}{k^2T^2[e^{(E-E_f)/(kT)} + 1]^3}.$$

The only solution is $E = E_f$. Since $P''(E)$ changes sign as E passes through E_f, there is a point of inflection at $(E_f, 1/2)$. The graph is asymptotic to the E-axis.



41. (a) A plot of $f(x) = (4 + e^{10/x})^{-1}$ is shown in the left figure below. Since $f'(x) = \dfrac{-e^{10/x}(-10/x^2)}{(4 + e^{10/x})^2} = \dfrac{10\,e^{10/x}}{x^2(4 + e^{10/x})^2}$, it follows that

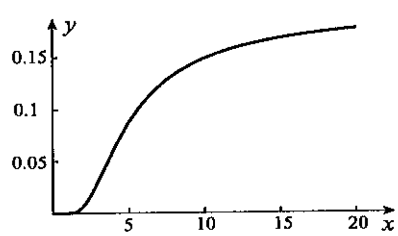
$$\begin{aligned} f''(x) &= \frac{10\,e^{10/x}(-10/x^2)}{x^2(4 + e^{10/x})^2} - \frac{20\,e^{10/x}}{x^3(4 + e^{10/x})^2} - \frac{20\,e^{10/x}e^{10/x}(-10/x^2)}{x^2(4 + e^{10/x})^3} \\ &= \frac{10e^{10/x}[-40 - 10\,e^{10/x} - 2x(4 + e^{10/x}) + 20\,e^{10/x}]}{x^4(4 + e^{10/x})^3} \\ &= \frac{20e^{10/x}[(5 - x)\,e^{10/x} - 4x - 20]}{x^4(4 + e^{10/x})^3}. \end{aligned}$$

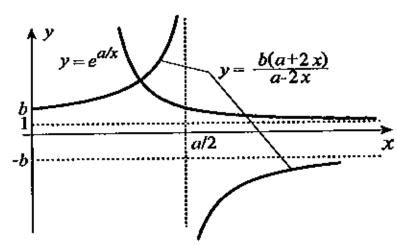
For the point of inflection we solve $0 = (5 - x)\,e^{10/x} - 4x - 20$. Using Newton's iterative procedure, we obtain $x = 3.34$. The point of inflection is therefore $(3.34, 0.042)$.

(b) Since $f'(x) = \dfrac{-e^{a/x}(-a/x^2)}{(b + e^{a/x})^2} = \dfrac{a\,e^{a/x}}{x^2(b + e^{a/x})^2}$, there are no critical points of the function. For points of inflection we solve

$$\begin{aligned} 0 = f''(x) &= \frac{a\,e^{a/x}(-a/x^2)}{x^2(b + e^{a/x})^2} - \frac{2a\,e^{a/x}}{x^3(b + e^{a/x})^2} - \frac{2a\,e^{a/x}e^{a/x}(-a/x^2)}{x^2(b + e^{a/x})^3} \\ &= \frac{a\,e^{a/x}}{x^4(b + e^{a/x})^3}[-a(b + e^{a/x}) - 2x(b + e^{a/x}) + 2a\,e^{a/x}] \\ &= \frac{a\,e^{a/x}}{x^4(b + e^{a/x})^3}[(a - 2x)\,e^{a/x} - 2bx - ab]. \end{aligned}$$

To show that $(a - 2x)\,e^{a/x} - 2bx - ab = 0 \iff e^{a/x} = \dfrac{b(a + 2x)}{a - 2x}$ has exactly one solution, we draw graphs of $e^{a/x}$ and $b(a+2x)/(a-2x)$ as shown in the right figure below. There is one point of intersection of the curves.

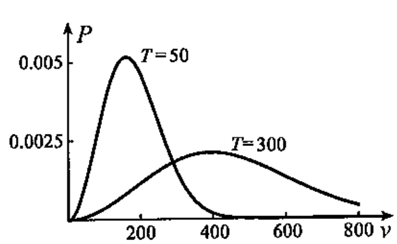




42. (a) The plots are shown to the right.
(b) For critical points we solve

$$\begin{aligned} 0 &= 2v\,e^{-Mv^2/(2RT)} + v^2e^{-Mv^2/(2RT)}\left(\frac{-2Mv}{2RT}\right) \\ &= v\left(2 - \frac{Mv^2}{RT}\right)e^{-Mv^2/(2RT)}. \end{aligned}$$

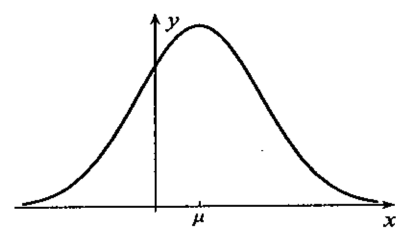
The positive solution is $v = \sqrt{2RT/M}$.
For points of inflection we solve



$$
\begin{aligned}
0 &= \left(2-\frac{Mv^2}{RT}\right)e^{-Mv^2/(2RT)} + v\left(\frac{-2Mv}{RT}\right)e^{-Mv^2/(2RT)} + v\left(2-\frac{Mv^2}{RT}\right)e^{-Mv^2/(2RT)}\left(\frac{-Mv}{RT}\right)\\
&= e^{-Mv^2/(2RT)}\left[2-\frac{Mv^2}{RT}-\frac{2Mv^2}{RT}-\frac{Mv^2}{RT}\left(2-\frac{Mv^2}{RT}\right)\right]\\
&= e^{-Mv^2/(2RT)}\left(\frac{M^2v^4}{R^2T^2}-\frac{5Mv^2}{RT}+2\right).
\end{aligned}
$$

Solutions of this quadratic in $Mv^2/(RT)$ are $\dfrac{Mv^2}{RT} = \dfrac{5 \pm \sqrt{25-8}}{2} \quad \Longrightarrow \quad v = \sqrt{\dfrac{5\pm\sqrt{17}}{2}}\sqrt{\dfrac{RT}{M}}$. Since the second derivative changes sign as v passes through these values, they yield points of inflection.

43. In Example 1.53 of Section 1.9 we sketched the function $f(x) = e^{-ax^2}$. The graph in this exercise has the same shape; it is shifted μ units to the right, and scales are modified.



44. The function can be expressed in the form

$$f(x) = \frac{(x+1)(x^2+1)}{(x+1)(x^2-x+1)} = \begin{cases} \dfrac{x^2+1}{x^2-x+1}, & x \neq -1,\\ \text{undefined}, & x=-1.\end{cases}$$

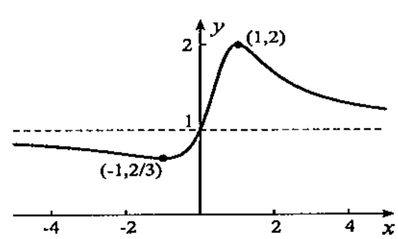
We therefore draw a graph of the function $g(x) = (x^2+1)/(x^2-x+1)$ and then delete the point at $x=-1$. We have a horizontal asymptote for $y = g(x)$ since

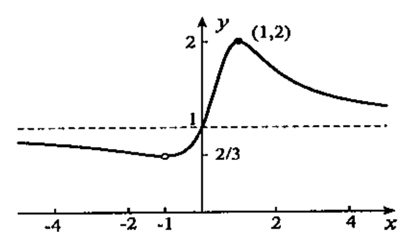
$$\lim_{x\to-\infty}\frac{x^2+1}{x^2-x+1} = 1^- \qquad \text{and} \qquad \lim_{x\to\infty}\frac{x^2+1}{x^2-x+1} = 1^+.$$

For critical points of $g(x)$ we solve

$$0 = g'(x) = \frac{(x^2-x+1)(2x)-(x^2+1)(2x-1)}{(x^2-x+1)^2} = \frac{1-x^2}{(x^2-x+1)^2}.$$

There are two solutions $x = \pm 1$. Since $g'(x)$ changes from negative to positive as x increases through -1, $g(x)$ has a relative minimum $g(-1) = 2/3$. There is a relative maximum $g(1) = 2$ because $g'(x)$ changes from positive to negative as x increases through 1. The graph of $g(x)$ is shown in the left diagram below. That of $f(x)$ is to the right.

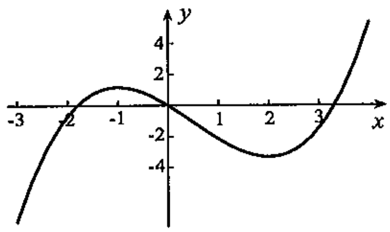


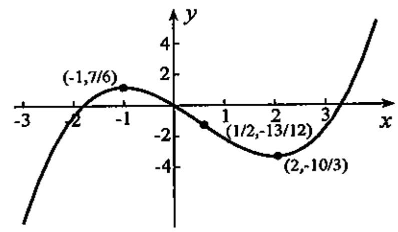


EXERCISES 4.6

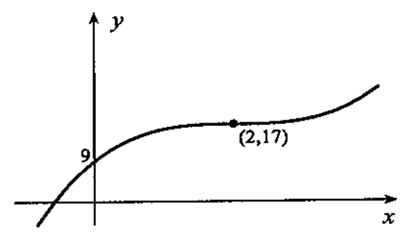
1. The plot in the left figure below indicates one relative maximum, one relative minimum, and one point of inflection. To confirm this, we first find critical points. Since $f'(x) = x^2 - x - 2 = (x-2)(x+1)$, the critical points are $x = -1, 2$. With $f''(x) = 2x - 1$, we find that $f''(-1) = -3$ and $f''(2) = 3$.

Consequently, $x = -1$ gives a relative maximum of $f(-1) = 7/6$ and $x = 2$ gives a relative minimum of $f(2) = -10/3$. Since $f''(1/2) = 0$ and $f''(x)$ changes sign as x passes through $1/2$, there is a point of inflection at $(1/2, -13/12)$. These are shown in the right figure.

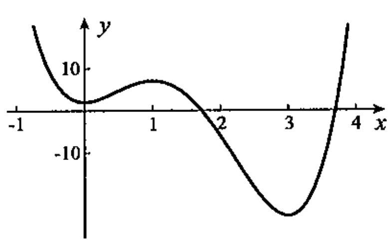


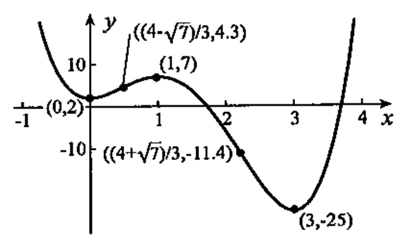


2. The plot suggests a horizontal point of inflection. To confirm this we calculate $f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2$. The only critical point is $x = 2$. Because $f'(x)$ remains positive as x passes through 2, the critical point does indeed give a horizontal point of inflection at $(2, 17)$. Since $f''(x) = 6(x-2)$, there are no other points of inflection.

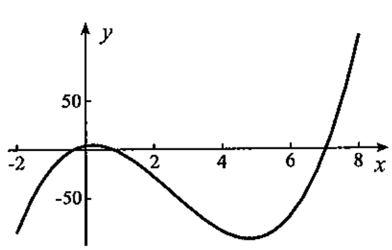


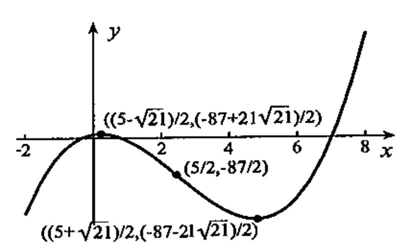
3. The plot in the left figure below indicates one relative maximum, two relative minima, and two points of inflection. To confirm this, we calculate $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$. Critical points are $x = 0, 1, 3$. With $f''(x) = 36x^2 - 96x + 36 = 12(3x^2 - 8x + 3)$, we find that $f''(0) = 36$, $f''(1) = -24$, and $f''(3) = 72$. Consequently, $x = 0$ and $x = 3$ give relative minima of $f(0) = 2$ and $f(3) = -25$, and $x = 1$ gives a relative maximum of $f(1) = 7$. Since $f''(x) = 0$ when $x = (8 \pm \sqrt{64 - 36})/6 = (4 \pm \sqrt{7})/3$, and $f''(x)$ changes sign as x passes through each of these, there are points of inflection at $\big((4 - \sqrt{7})/3, 4.3\big)$ and $\big((4 + \sqrt{7})/3, -11.4\big)$. These are shown in the right figure.



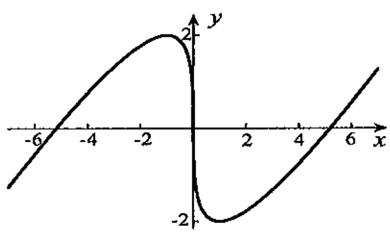


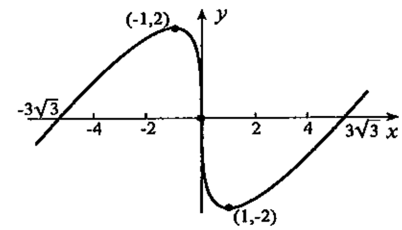
4. The plot in the left figure below indicates one relative maximum, one relative minimum, and one point of inflection. To confirm this, we first find critical points. Since $f'(x) = 6x^2 - 30x + 6 = 6(x^2 - 5x + 1)$, the critical points are $x = (5 \pm \sqrt{25 - 4})/2 = (5 \pm \sqrt{21})/2$. Since $f'(x)$ changes from positive to negative as x increases through $(5 - \sqrt{21})/2$, there is a relative maximum at $\big((5 - \sqrt{21})/2, (-87 + 21\sqrt{21})/2\big)$. Similarly, there is a relative minimum at $\big((5 + \sqrt{21})/2, (-87 - 21\sqrt{21})/2\big)$. Since $0 = f''(x) = 12x - 30$ at $x = 5/2$, and $f''(x)$ changes sign as x passes through $5/2$, there is a point of inflection at $(5/2, -87/2)$. These are shown in the right figure.





5. The plot in the left figure below indicates one relative maximum, one relative minimum, and a vertical point of inflection at $(0,0)$. To confirm this, we first find critical points. Solutions of the equation $0 = f'(x) = 1 - x^{-2/3} = \dfrac{x^{2/3}-1}{x^{2/3}}$ are $x = \pm 1$. Since $f'(0)$ is undefined, but $f(0) = 0$, $x = 0$ is also a critical point. With $f''(x) = (2/3)x^{-5/3}$, we find that $f''(-1) = -2/3$ and $f''(1) = 2/3$. Thus, $x = -1$ yields a relative maximum of $f(-1) = 2$ and $x = 1$ gives a relative minimum of $f(1) = -2$. Because $f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$, it follows that $(0,0)$ must be a point of inflection. Since $\lim_{x\to 0} f'(x) = -\infty$, $(0,0)$ is a vertical point of inflection. These are shown in the right figure.

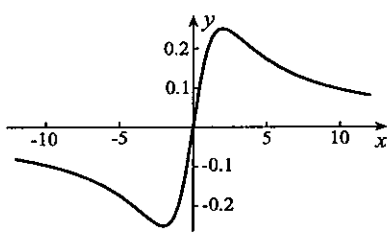


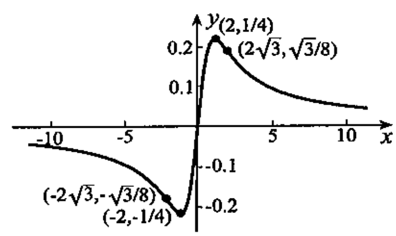


6. The plot in the left figure below indicates one relative maximum, one relative minimum, and three points of inflection. To confirm this, we first find critical points. For critical points we solve the equation $0 = f'(x) = \dfrac{(x^2+4)(1) - x(2x)}{(x^2+4)^2} = \dfrac{4-x^2}{(x^2+4)^2}$. Solutions are $x = \pm 2$. We now calculate the second derivative $f''(x) = \dfrac{(x^2+4)^2(-2x) - (4-x^2)(2)(x^2+4)(2x)}{(x^2+4)^4} = \dfrac{2x(x^2-12)}{(x^2+4)^3}$. Since $f''(-2) = 1/16$ and $f''(2) = -1/16$, there is a relative minimum at $x = -2$ equal to $f(-2) = -1/4$, and a relative maximum at $x = 2$ of $f(2) = 1/4$.

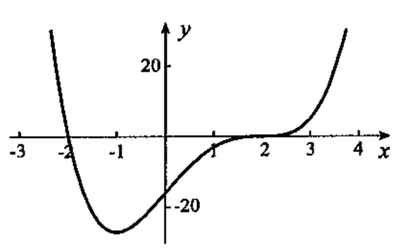
Because $f''(x) = 0$ at $x = 0$, $\pm 2\sqrt{3}$, and $f''(x)$ changes sign as x passes through each of these, points of inflection occur at $(0,0)$, $(2\sqrt{3}, \sqrt{3}/8)$, and $(-2\sqrt{3}, -\sqrt{3}/8)$.

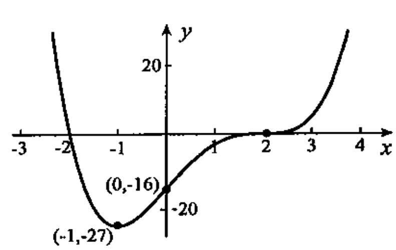
The final graph is shown to the right. We could have shortened the analysis by considering only the right half of the graph and using the fact that the function is odd.





7. The plot in the left figure below indicates one relative minimum and two points of inflection one of which is horizontal. To confirm this, we first find critical points. Since $f'(x) = 3(x-2)^2(x+2) + (x-2)^3 = 4(x-2)^2(x+1)$, the critical points are $x = -1, 2$. Since $f'(x)$ changes from negative to positive as x increases through -1, there is a relative minimum at $(-1, -27)$. The derivative does not change sign at $x = 2$. We now calculate $f''(x) = 8(x-2)(x+1) + 4(x-2)^2 = 12x(x-2)$. Since $f''(x) = 0$ when $x = 0, 2$, and $f''(x)$ changes sign as x passes through these values, there is a point of inflection at $(0, -16)$ and a horizontal point of inflection at $(2, 0)$.





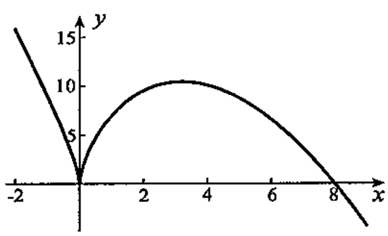
8. The plot in the left figure below indicates one relative minimum one relative maximum, perhaps a point of inflection for negative x, and perhaps a point of inflection at $x = 0$. To decide, we first find critical points. Since $0 = f'(x) = (2/3)x^{-1/3}(8-x) - x^{2/3} = \dfrac{16-5x}{3x^{1/3}}$, $x = 16/5$ is critical, but so also is $x = 0$ because $f'(0)$ does not exist and $f(0) = 0$. We now calculate

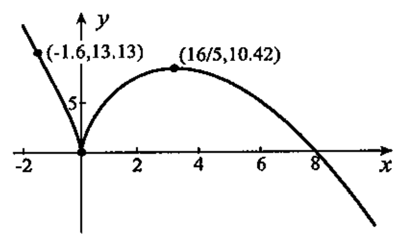
$$f''(x) = \frac{3x^{1/3}(-5) - (16-5x)x^{-2/3}}{9x^{2/3}} = \frac{-2(5x+8)}{9x^{4/3}}.$$

Since $f''(16/5) < 0$, there is a relative maximum of $f(16/5) = 10.42$. Because $f'(x)$ changes from negative to positive as x increases through 0, this critical point gives a relative mimumum of $f(0) = 0$.

Since $f''(-8/5) = 0$, and $f''(x)$ changes sign as x passes through $-8/5$, there is a point of inflection at $(-1.6, 13.13)$. The point $(0,0)$ is not a point of inflection.

The final graph is shown to the right.

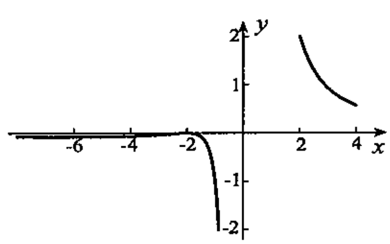


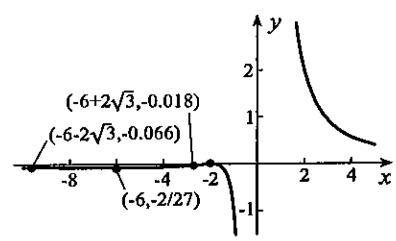


9. The plot in the left figure below indicates a relative maximum at $x = -2$, but the situation to the left of $x = -2$ is not at all clear. To confirm the relative maximum, and and assess the $x < -2$ situation, we begin with critical points. We solve $0 = f'(x) = \dfrac{x^3(2)(x+2) - (x+2)^2(3x^2)}{x^6} = -\dfrac{(x+2)(x+6)}{x^4}$ for $x = -6, -2$. We now calculate $f''(x) = -\dfrac{x^4(2x+8) - (x+2)(x+6)(4x^3)}{x^8} = \dfrac{2(x^2+12x+24)}{x^5}$. Since $f''(-6) > 0$ and $f''(-2) < 0$, there is a relative minimum at $x = -6$ equal to $f(-6) = -2/27$, and a relative maximum at $x = -2$ of $f(-2) = 0$.

Because $f''(x) = 0$ at $x = (-12 \pm \sqrt{144-96})/2 = -6 \pm 2\sqrt{3}$, and $f''(x)$ changes sign as x passes through each of these, points of inflection occur at $(-6-2\sqrt{3}, -0.066)$ and $(-6+2\sqrt{3}, -0.018)$.

The following limits show that the x-axis is the horizontal asymptote and the y-axis is a vertical asymptote, $\lim_{x\to-\infty} f(x) = 0^-$, $\lim_{x\to\infty} f(x) = 0^+$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to 0^+} f(x) = \infty$. The final graph is shown to the right.



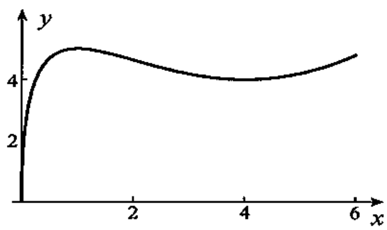
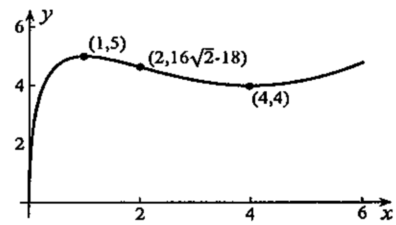


10. The plot in the left figure below indicates one relative minimum one relative maximum, and a point of inflection between them. For critical points we solve

$$0 = f'(x) = 3x^{1/2} - 9 + 6x^{-1/2} = \frac{3(x - 3\sqrt{x} + 2)}{\sqrt{x}} = \frac{3(\sqrt{x}-1)(\sqrt{x}-2)}{\sqrt{x}}.$$

Solutions are $x = 1$ and $x = 4$. The point $x = 0$ is also critical since $f'(0)$ is not defined but $f(0) = 0$. To classify these critical points we calculate $f''(x) = \dfrac{3}{2\sqrt{x}} - \dfrac{3}{x^{3/2}} = \dfrac{3(x-2)}{2x^{3/2}}$. Since $f''(1) = -3/2$ and $f''(4) = 3/8$, $f(1) = 5$ is a relative maximum and $f(4) = 4$ is a relative minimum. Because $f(x)$ is not defined for $x < 0$, the critical point $x = 0$ does not give a relative extrema. We note that $\lim_{x\to 0^+} f'(x) = \infty$.

Since $f''(2) = 0$, and $f''(x)$ changes sign as x passes through 2, there is a point of inflection at $(2, 16\sqrt{2} - 18)$. The final graph is shown to the right.

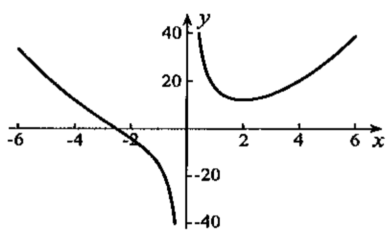
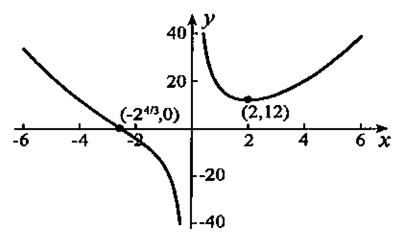
11. The plot in the left figure below indicates a relative minimum at or near $x = 2$, and a point of inflection near $x = -2$. For critical points we solve

$$0 = f'(x) = \frac{x(3x^2) - (x^3 + 16)(1)}{x^2} = \frac{2(x^3 - 8)}{x^2}.$$

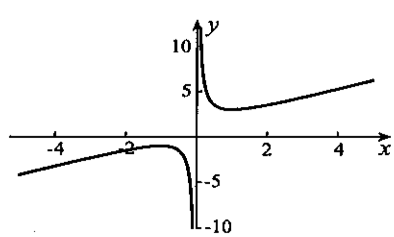
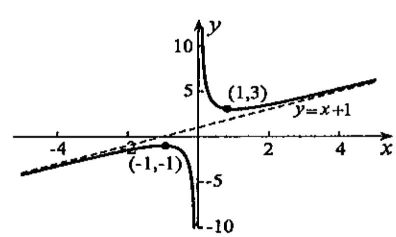
Clearly $x = 2$ is a critical point, and since $f'(x)$ changes from negative to positive as x increases through 2, $f(2) = 12$ is a relative minimum. To determine the point of inflection, we solve

$$0 = f''(x) = 2\left[\frac{x^2(3x^2) - (x^3 - 8)(2x)}{x^4}\right] = \frac{2(x^3 + 16)}{x^3}.$$

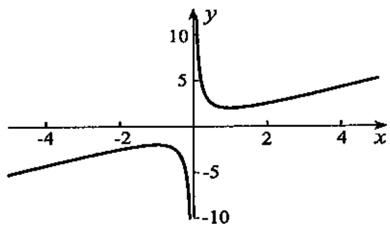
The only solution is $x = -2^{4/3}$. Since $f''(x)$ changes sign as x passes through $-2^{4/3}$, the point of inflection is $(-2^{4/3}, 0)$. The final graph is shown to the right.

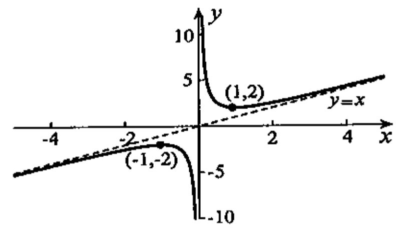



12. The plot in the left figure below indicates one relative minimum, one relative maximum, no points of inflection, and an oblique asymptote. By writing $f(x)$ in the form $f(x) = x + 1 + \dfrac{1}{x}$, we see that the graph is asymptotic to the line $y = x + 1$. Exact locations of the relative extrema can be determined by solving $0 = f'(x) = 1 - \dfrac{1}{x^2}$. Clearly, $x = \pm 1$. The second derivative is $f''(x) = 2/x^3$. Since $f''(1) = 2$, there is a relative minimum at $x = 1$ of $f(1) = 3$. With $f''(-1) = -2$, there is a relative maximum of $f(-1) = -1$. Since $f''(x)$ is never zero, there can be no points of inflection. The final graph is shown to the right.

13. The plot in the left figure below indicates one relative minimum, one relative maximum, no points of inflection, and an oblique asymptote. The graph is asymptotic to the line $y = x$. Exact locations of the relative extrema can be determined by solving $0 = f'(x) = 1 - \dfrac{1}{x^2}$. Clearly, $x = \pm 1$. The second derivative is $f''(x) = 2/x^3$. Since $f''(1) = 2$, there is a relative minimum at $x = 1$ of $f(1) = 2$. With $f''(-1) = -2$, there is a relative maximum of $f(-1) = -2$. Since $f''(x)$ is never zero, there can be no points of inflection. The final graph is shown to the right.





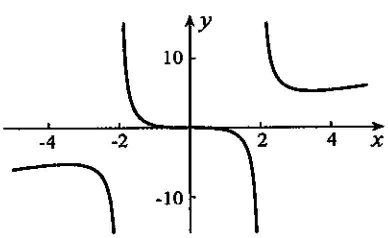
14. The plot in the left figure below indicates one relative minimum, one relative maximum, a horizontal point of inflection at $x = 0$, and an oblique asymptote. By writing $f(x)$ in the form $f(x) = x + \dfrac{4x}{x^2-4}$, we see that the graph is asymptotic to the line $y = x$. There are discontinuities at $x = \pm 2$, which yield vertical asymptotes. To identify the relative extrema, we solve

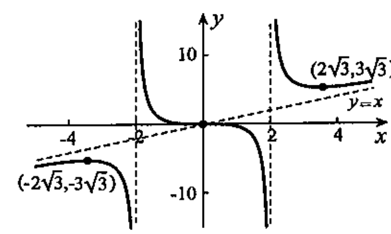
$$0 = f'(x) = \frac{(x^2-4)(3x^2) - x^3(2x)}{(x^2-4)^2} = \frac{x^2(x^2-12)}{(x^2-4)^2}.$$

Solutions are $x = 0,\ \pm 2\sqrt{3}$. The second derivative is

$$f''(x) = \frac{(x^2-4)^2(4x^3-24x) - (x^4-12x^2)(2)(x^2-4)(2x)}{(x^2-4)^4} = \frac{8x(x^2+12)}{(x^2-4)^3}.$$

The fact that $f''(2\sqrt{3}) > 0$ confirms a relative minimum $f(2\sqrt{3}) = 3\sqrt{3}$. Similarly, $f''(-2\sqrt{3}) < 0$ confirms a relative maximum $f(-2\sqrt{3}) = -3\sqrt{3}$. Since $f''(x) = 0$ only at $x = 0$, and $f''(x)$ changes sign as x passes through 0, there is a horizontal point of inflection at $(0,0)$. The final graph is shown to the right.





15. The plot in the left figure below suggests one relative minimum, one relative maximum, one point of inflection, and a horizontal asymptote. The limits $\lim\limits_{x\to\pm\infty} f(x) = 2$, confirm $y = 2$ as a horizontal asymptote. There are discontinuities at $x = 2, 6$, leading to vertical asymptotes. For critical points, we solve

$$0 = f'(x) = \frac{(x^2-8x+12)(4x) - 2x^2(2x-8)}{(x^2-8x+12)^2} = \frac{16x(3-x)}{(x^2-8x+12)^2}.$$

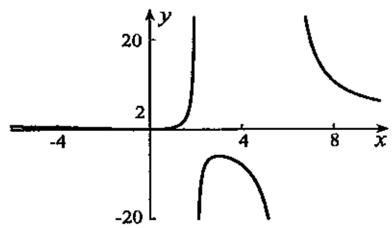
Solutions are $x = 0, 3$. The second derivative is

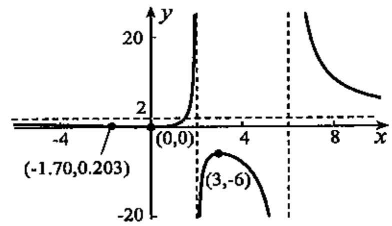
$$f''(x) = \frac{(x^2-8x+12)^2(48-32x) - (48x-16x^2)(2)(x^2-8x+12)(2x-8)}{(x^2-8x+12)^4} = \frac{16(2x^3-9x^2+36)}{(x^2-8x+12)^3}.$$

Since $f''(0) > 0$, there is a relative minimum at $x = 0$ of $f(0) = 0$. With $f''(3) < 0$, there is a relative maximum of $f(3) = -6$. For points of inflection, we would solve $2x^3 - 9x^2 + 36 = 0$. With Newton's iterative procedure, and an inital value $x_1 = -2$, iteration of $x_{n+1} = x_n - \dfrac{2x_n^3 - 9x_n^2 + 36}{6x_n^2 - 18x_n}$ gives

$$x_2 = -1.733, \quad x_3 = -1.704, \quad x_4 = -1.703, \quad x_5 = -1.703.$$

Since $f''(x)$ changes sign as x passes through this solution, there is a point of inflection at $(-1.70, 0.203)$. The final graph is shown to the right.





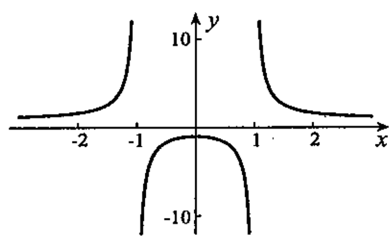
16. The plot in the left figure below suggests one relative maximum, no relative minima, no points of inflection, and a horizontal asymptote. The limits $\lim_{x\to\pm\infty} \dfrac{x^2+1}{x^2-1} = 1^+$ confirm $y = 1$ as the horizontal asymptote. There are discontinuities at $x = \pm 1$, leading to vertical asymptotes. For critical points, we solve

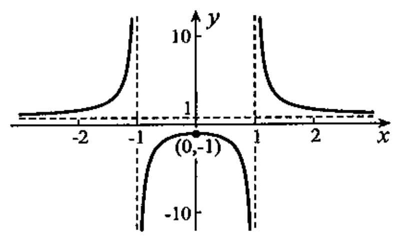
$$0 = f'(x) = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

The only solution is $x = 0$. Since $f'(x)$ changes from positive to negative as x increases through 0, there is a relative maximum of $f(0) = -1$. For points of inflection, we solve

$$0 = f''(x) = \frac{(x^2-1)^2(-4) - (-4x)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}.$$

There are no solutions confirming no points of inflection. The final graph is shown to the right.





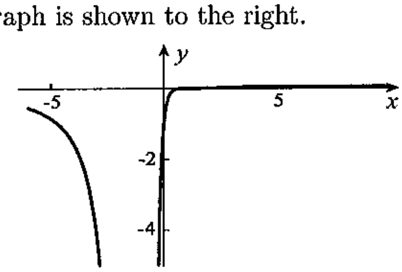
17. The plot in the left figure below appears asymptotic to the x-axis, but what is happening for $x > 0$ is not clear. The horizontal asymptote is confirmed by the limits $\lim_{x\to\infty} f(x) = 0^+$ and $\lim_{x\to-\infty} f(x) = 0^-$. There is a discontinuity at $x = -1$, leading to a vertical asymptote. For critical points we solve

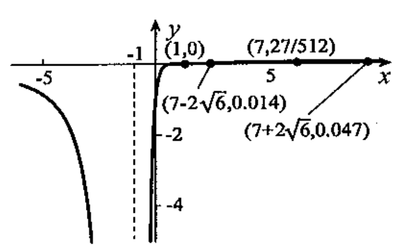
$$0 = f'(x) = \frac{(x+1)^4(3)(x-1)^2 - (x-1)^3(4)(x+1)^3}{(x+1)^8} = \frac{(x-1)^2(7-x)}{(x+1)^5}.$$

Solutions are $x = 1,\ 7$. Since $f'(x)$ changes from positive to negative as x increases through 7, there is a relative maximum at $(7, 27/512)$. The derivative does not change sign at $x = 1$. For points of inflection, we solve

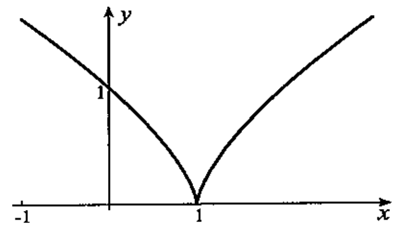
$$0 = f''(x) = \frac{(x+1)^5[2(x-1)(7-x) - (x-1)^2] - (x-1)^2(7-x)(5)(x+1)^4}{(x+1)^{10}} = \frac{2(x-1)(x^2-14x+25)}{(x+1)^6}.$$

Solutions are $x = 1,\ 7 \pm 2\sqrt{6}$. The second derivative changes sign as x passes through each of these. Thus, we have points of inflection at $(1, 0)$ (a horizontal one), and $(7-2\sqrt{6}, 0.014)$ and $(7+2\sqrt{6}, 0.047)$. The final graph is shown to the right.





18. There appears to be a relative minimum at $x = 1$ where the derivative does not exist, and no other significant features to the graph. Because $f(x) > 0$ for all $x \neq 1$, it follows that $f(1)$ is indeed a relative minimum. Since $f'(x) = (2/3)(x-1)^{-1/3}$, we see that $f'(x)$ is not defined at the minimum.



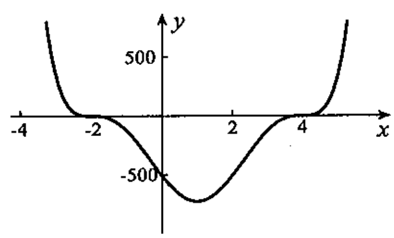
19. The plot in the left figure below suggests one relative minimum, two horizontal points of inflection and two others. For critical points we solve

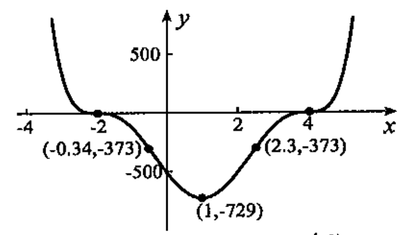
$$0 = f'(x) = 3(x+2)^2(x-4)^3 + 3(x+2)^3(x-4)^2 = 6(x+2)^2(x-4)^2(x-1).$$

Solutions are $x = -2, 1, 4$. The derivative changes from negative to positive as x increases through 1 confirming a relative minimum there of $f(1) = -729$. For points of inflection we solve

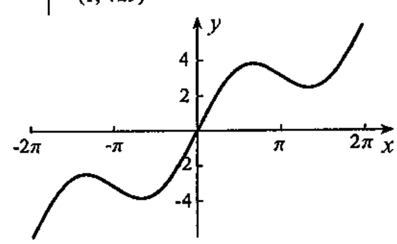
$$\begin{aligned}0 = f''(x) &= 6[2(x+2)(x-2)^2(x-1) + (x+2)^2(2)(x-4)(x-1) + (x+2)^2(x-4)^2]\\ &= 6(x+2)(x-4)(5x^2-10x-4).\end{aligned}$$

Solutions are $x = -2,\ 4,\ (5 \pm 3\sqrt{5})/5$. Since $f''(x)$ changes sign as x passes through each of these, there are four points of inflection $(-2, 0)$, $(4, 0)$ (both of which are horizontal), and $((5+3\sqrt{5})/5, -373)$ and $((5-3\sqrt{5})/5, -373)$.





20. The plot to the right indicates an infinite number of relative extrema and points of inflection. For critical points we solve $0 = f'(x) = 1 + 2\cos x$. Solutions are $x = \pm 2\pi/3 + 2n\pi$, where n is an integer. Relative maxima occur at $(2\pi/3 + 2n\pi, \sqrt{3} + 2\pi/3 + 2n\pi)$, and relative minima at $(-2\pi/3 + 2n\pi, -\sqrt{3} - 2\pi/3 + 2n\pi)$.

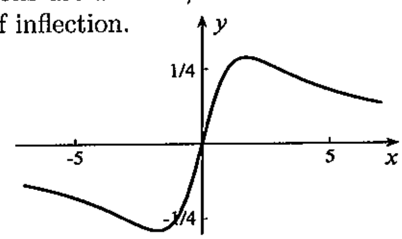


For points of inflection, we solve $0 = f''(x) = -2\sin x$. Solutions are $x = n\pi$, where n is an integer. Since $f''(x)$ changes sign at each of these, they all give points of inflection.

21. For critical points we solve

$$0 = f'(x) = \frac{(x^2+3)(1) - x(2x)}{(x^2+3)^2} = \frac{3-x^2}{(x^2+3)^2}.$$

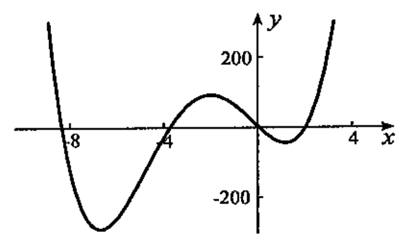
Solutions $x = \pm\sqrt{3}$ give relative extrema at $(\pm\sqrt{3}, \pm\sqrt{3}/6)$.



22. For critical points we solve

$$\begin{aligned}0 = f'(x) &= 4x^3 + 30x^2 + 12x - 64\\ &= 2(x+2)(2x^2+11x-16).\end{aligned}$$

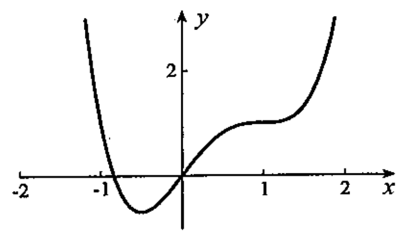
Solutions of this equation are $x = -2$ and $x = (-11 \pm \sqrt{121+128})/4 = (-11 \pm \sqrt{249})/4$. Hence, we have a relative maximum $f(-2) = 93$ and relative minima $f\big((-11-\sqrt{249})/4\big) = f(-6.69) = -289.4$ and $f\big((-11+\sqrt{249})/4\big) = f(1.19) = -43.8$.



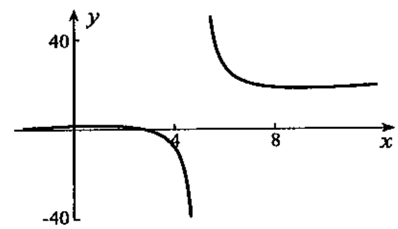
23. For critical points we solve

$$0 = f'(x) = 4x^3 - 6x^2 + 2$$
$$= 2(x-1)^2(2x+1).$$

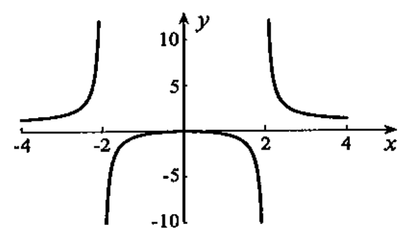
Solutions of this equation are $x = -1/2, 1$. We have a relative minimum $f(-1/2) = -11/16$, but $x = 1$ gives a horizontal point of inflection since $f'(x)$ remains positive as x passes through 1.



24. Since $f(x) = x + 5 + 17/(x-5)$, critical points are given by $0 = f'(x) = 1 - 17/(x-5)^2$. Solutions are $x = 5 \pm \sqrt{17}$. These gives relative extrema equal to $10 \pm 2\sqrt{17}$.



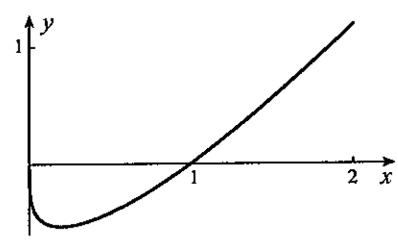
25. Since $f'(x) = \dfrac{(x^2-4)(2x) - x^2(2x)}{(x^2-4)^2} = \dfrac{-8x}{(x^2-4)^2}$, the only critical point is $x = 0$ at which there is a relative maximum of $f(0) = 0$.



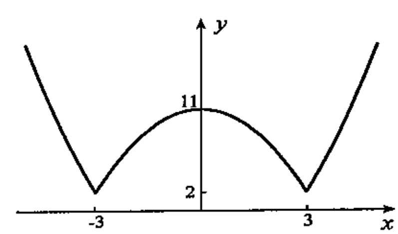
26. For critical points we solve

$$0 = f'(x) = \frac{5}{4}x^{1/4} - \frac{1}{4}x^{-3/4} = \frac{5x-1}{4x^{3/4}}.$$

Thus, $x = 1/5$ at which $f(x)$ has a relative minimum of $f(1/5) = -4/5^{5/4}$. There is also critical point at $x = 0$, but because the function is not defined for $x < 0$, it cannot yield a relative maximum.



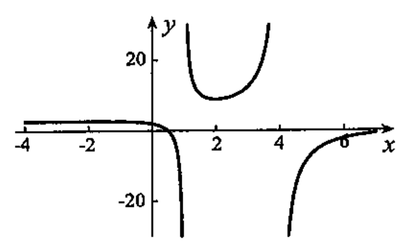
27. The derivative is undefined at $x = \pm 3$ where the graph has sharp corners. There are relative minima equal to $f(\pm 3) = 2$ at these points.



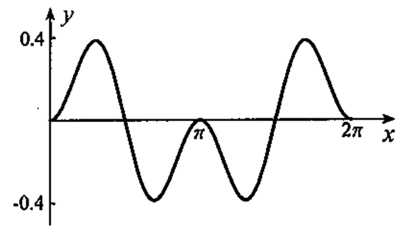
28. To find critical points we solve

$$0 = f'(x)$$
$$= \frac{(x^2-5x+4)(4x-17) - (2x^2-17x+8)(2x-5)}{(x^2-5x+4)^2}$$
$$= \frac{7(x-2)(x+2)}{(x^2-5x+4)^2}.$$

Solutions are $x = -2, 2$. There is a relative maximum of $f(-2) = 25/9$ and a relative minimum of $f(2) = 9$.



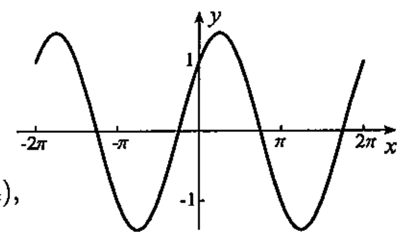
29. Since $f'(x) = 2\sin x\cos^2 x - \sin^3 x = \sin x(2\cos^2 x - \sin^2 x) = \sin x(3\cos^2 x - 1)$, critical points are $x = 0, \pi, 2\pi$ and values of x satisfying $\cos x = \pm 1/\sqrt{3}$. Being end points, $x = 0, 2\pi$ cannot yield relative extrema. Relative maxima occur at π and the values of x satisfying $\cos x = 1/\sqrt{3}$. Values of these maxima are 0 and $(\pm 2/3)(1/\sqrt{3}) = 2\sqrt{3}/9$. Relative minima occur at the critical points satisfying $\cos x = -1/\sqrt{3}$, and the value of the function at these minima is $(\pm 2/3)(-1/\sqrt{3}) = -2\sqrt{3}/9$.



30. This function is most easily analyzed by expressing it as a general sine function $A\sin(x+\phi)$ (see Example 1.45 in Section 1.8),

$$\begin{aligned}\sin x + \cos x &= A\sin(x+\phi)\\ &= A(\sin x\cos\phi + \cos x\sin\phi).\end{aligned}$$

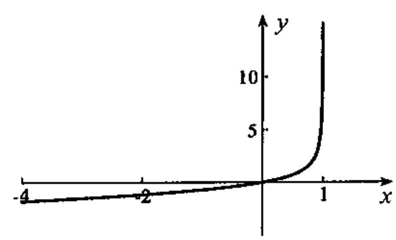
To satisfy this equation we set $A\cos\phi = 1$ and $A\sin\phi = 1$. These are satisfied if we choose $A = \sqrt{2}$ and $\phi = \pi/4$. Thus, $f(x)$ can be expressed in the form $f(x) = \sqrt{2}\sin(x + \pi/4)$, and the relative maxima and minima are $\pm\sqrt{2}$.



31. This function has no relative extrema. We can confirm this with

$$\begin{aligned}f'(x) &= \frac{\sqrt{1-x} - x(1/2)(1-x)^{-1/2}(-1)}{1-x}\\ &= \frac{2-x}{2(1-x)^{3/2}}.\end{aligned}$$

The derivative never vanishes for $x < 1$.

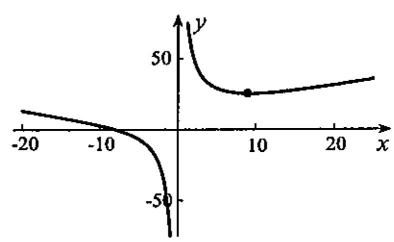


32. To identify critical points we solve

$$\begin{aligned}0 = f'(x) &= -\frac{8}{x^2}\sqrt{x^2+100} + \left(1 + \frac{8}{x}\right)\frac{x}{\sqrt{x^2+100}}\\ &= \frac{x^3 - 800}{x^2\sqrt{x^2+100}}.\end{aligned}$$

The only critical point is $800^{1/3}$ at which there is a relative minimum of

$$f(800^{1/3}) = \left(\frac{8 + 800^{1/3}}{800^{1/3}}\right)\sqrt{100 + 800^{2/3}} \approx 25.4.$$

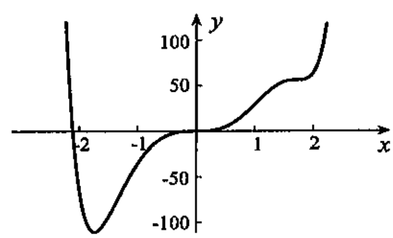


33. (a) A plot of the function is shown to the right.

(b) For critical points we solve

$$\begin{aligned}0 = f'(x) &= 8x^7 - 24x^5 - 40x^4 + 120x^2\\ &= 8x^2(x^2-3)(x^3-5).\end{aligned}$$

Solutions are $x = 0$, $\pm\sqrt{3}$, $5^{1/3}$. Since $f'(x)$ changes from a negative quantity to a positive quantity as x increases through $\pm\sqrt{3}$, these critical points give relative minima. Since $f'(x)$ changes from positive to negative as x increases through $5^{1/3}$, there is a relative maximum there. Finally, because $f''(x) = 56x^6 - 120x^4 - 160x^3 + 240x = 8x(7x^5 - 15x^3 - 20x^2 + 30)$, it follows $f''(0) = 0$, and $f''(x)$ changes sign as x increases through 0. Hence, $x = 0$ yields a horizontal point of inflection.

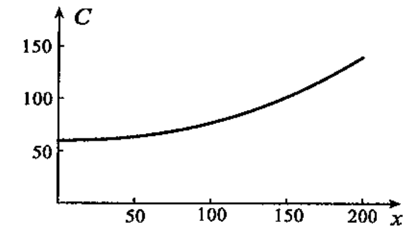


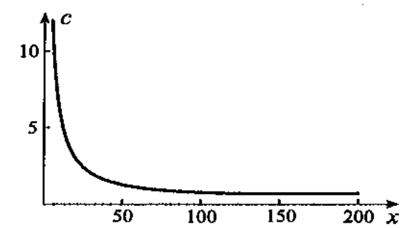
34. (a) A plot is shown below in the left figure below. Derivatives of $C(x)$ are

$$C'(x) = \frac{(x+300)(3x^2+200x)-(x^3+100x^2)}{300(x+300)^2} = \frac{x^3+500x^2+30\,000x}{150(x+300)^2},$$

$$C''(x) = \frac{(x+300)^2(3x^2+1000x+30\,000)-(x^3+500x^2+30\,000x)(2x+600)}{150(x+300)^4}.$$

This simplifies to $C''(x) = \dfrac{x^3+900x^2+270\,000x+9\times 10^6}{150(x+300)^3} > 0$. Hence the graph is concave upward.





(b) A plot is shown in the right figure above.
(c)(i) Critical points of $c(x) = \dfrac{x}{300}\left(\dfrac{x+100}{x+300}\right) + \dfrac{60}{x}$ are given by

$$\begin{aligned} 0 &= \frac{(x+300)(2x+100)-(x^2+100x)(1)}{300(x+300)^2} - \frac{60}{x^2} = \frac{x^2(x^2+600x+30\,000)-18\,000(x+300)^2}{300x^2(x+300)^2} \\ &= \frac{x^2(x^2+600x+90\,000)-60\,000x^2-18\,000(x+300)^2}{300x^2(x+300)^2} = \frac{(x+300)^2(x^2-18\,000)-60\,000x^2}{300x^2(x+300)^2}. \end{aligned}$$

When we equate the numerator to zero, we obtain the required equation.
(ii) The tangent line to $C(x)$ passes through the origin when the slope of the tangent line is equal to $C(x)/x$,

$$\frac{x}{300}\left(\frac{x+100}{x+300}\right) + \frac{60}{x} = \frac{x^3+500x^2+30\,000x}{150(x+300)^2}.$$

If we multiply by $300x(x+300)^2$,

$$\begin{aligned} 0 &= 2x(x^3+500x^2+30\,000x) - x^2(x+100)(x+300) - 18\,000(x+300)^2 \\ &= 2x^2(x^2+500x+30\,000) - x^2(x+300)(x+300) + 200x^2(x+300) - 18\,000(x+300)^2 \\ &= 2x^2(x^2+600x+60\,000) - x^2(x+300)^2 - 18\,000(x+300)^2 \\ &= 2x^2(x^2+600x+90\,000) - 60\,000x^2 - x^2(x+300)^2 - 18\,000(x+300)^2 \\ &= (x+300)^2(x^2-18\,000) - 60\,000x^2. \end{aligned}$$

EXERCISES 4.7

1. For critical points we solve $0 = f'(x) = 3x^2-2x-5 = (3x-5)(x+1)$. Solutions are $x = -1, 5/3$. Since $f(-2) = 2$, $f(-1) = 7$, $f(5/3) = -67/27$, and $f(3) = 7$, absolute minimum and maximum values are $-67/27$ and 7.

2. For critical points we solve $0 = f'(x) = \dfrac{(x+1)(1)-(x-4)(1)}{(x+1)^2} = \dfrac{5}{(x-1)^2}$. Since there are no critical points, we evaluate $f(0) = -4$ and $f(10) = 6/11$. These are the absolute minimum and maximum.

3. For critical points we solve $0 = f'(x) = 1 - \dfrac{1}{x^2}$. Solutions are $x = \pm 1$. Since $f(1/2) = 5/2$, $f(1) = 2$, and $f(5) = 26/5$, absolute minimum and maximum values are 2 and 26/5.

4. For critical points we solve $0 = f'(x) = 1 - 2\cos x$. Solutions on the given interval are $\pi/3$, $5\pi/3$, $7\pi/3$, and $11\pi/3$. Since

$$f(0) = 0, \quad f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}, \quad f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3},$$

$$f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{3} - \sqrt{3}, \quad f\left(\frac{11\pi}{3}\right) = \frac{11\pi}{3} + \sqrt{3}, \quad f(4\pi) = 4\pi,$$

the absolute minimum is $\pi/3 - \sqrt{3}$ and the absolute maximum is $11\pi/3 + \sqrt{3}$.

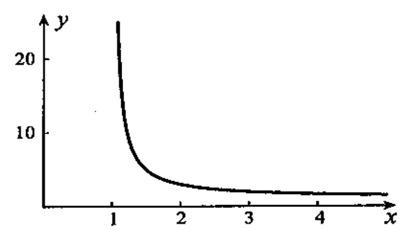
5. For critical points we solve $0 = f'(x) = \sqrt{x+1} + \dfrac{x}{2\sqrt{x+1}} = \dfrac{3x+2}{2\sqrt{x+1}}$. The only solution is $x = -2/3$. Since $f(-1) = 0$, $f(-2/3) = -2\sqrt{3}/9$, and $f(1) = \sqrt{2}$, absolute minimum and maximum are $-2\sqrt{3}/9$ and $\sqrt{2}$.

6. For critical points we solve $0 = f'(x) = \dfrac{-12(2x+2)}{(x^2+2x+2)^2}$, and obtain $x = -1$. Since

$$\lim_{x\to-\infty} f(x) = 0, \qquad f(-1) = 12, \qquad \lim_{x\to 0^-} f(x) = 6,$$

the absolute maximum is 12 but the function does not have an absolute minimum.

7. The graph indicates that the function does not have absolute extrema on this interval.

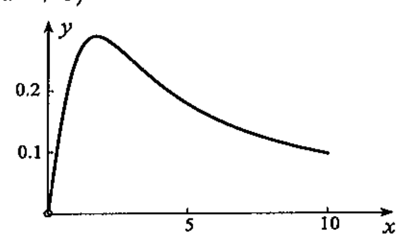


8. For critical points we solve $0 = f'(x) = \dfrac{(x^2+3)(1) - x(2x)}{(x^2+3)^2} = \dfrac{3-x^2}{(x^2+3)^2}$.

Only the critical point $x = \sqrt{3}$ is positive. Since

$$\lim_{x\to 0^+} f(x) = 0, \quad f(\sqrt{3}) = \frac{\sqrt{3}}{6}, \quad \lim_{x\to\infty} f(x) = 0,$$

the absolute maximum is $\sqrt{3}/6$ but the function does not have an absolute minimum.



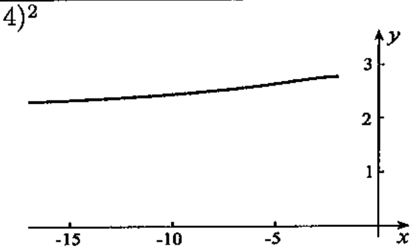
9. For critical points, $0 = f'(x) = \dfrac{(x^2-5x+4)(4x-17) - (2x^2-17x+8)(2x-5)}{(x^2-5x+4)^2}$

$$= \frac{7(x-2)(x+2)}{(x^2-5x+4)^2}.$$

Solutions are $x = -2, 2$. Since

$$f(-2) = \frac{25}{9} \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 2,$$

the absolute maximum is $25/9$, but the function does not have an absolute minimum.



10. We separate the proof into three cases:

CASE 1 – $f(x) = k$, a constant. Then $f'(x) = 0$ for all x, and c can be chosen as any point in the interval.

CASE 2 – $f(x) > f(a)$ for some x in $a < x < b$. Because $f(x)$ is continuous on $a \le x \le b$, it must have an absolute maximum on this interval, and this maximum must occur at a critical point c interior to the interval. Since $f(x)$ is differentiable at every point in $a < x < b$, it follows that the derivative must vanish at the critical point, $f'(c) = 0$.

CASE 3 – $f(x) < f(a)$ for all x in $a < x < b$. In this case the absolute minimum of $f(x)$ must occur at a critical point c between a and b at which $f'(c) = 0$.

11. For critical points of $y(x)$, we solve $0 = 8x^3 - 15Lx^2 + 6L^2x$. Solutions are $x = 0$ and $x = (15L \pm \sqrt{225L^2 - 192L^2})/16 = (15 \pm \sqrt{33})L/16$. Maximum deflection occurs at $x = (15 - \sqrt{33})L/16$.

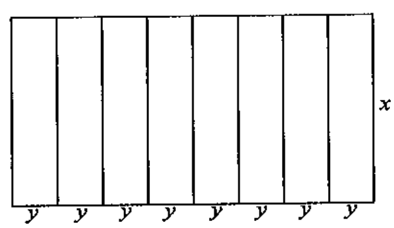
12. If x and y represent the length and width of each plot, then the amount of fencing required is $F = 9x+16y$. Since each plot must have area 9000 m^2, it follows that $xy = 9000$. Thus, $y = 9000/x$, and

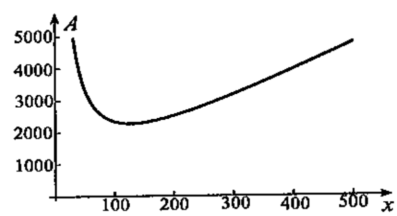
$$F = F(x) = 9x + \frac{144\,000}{x}, \quad x > 0.$$

To find critical points of $F(x)$, we solve $0 = F'(x) = 9 - 144\,000/x^2$. The only positive solution is $x = 40\sqrt{10}$. We now evaluate

$$\lim_{x\to 0^+} F(x) = \infty, \qquad F(40\sqrt{10}) = 720\sqrt{10}, \qquad \lim_{x\to\infty} F(x) = \infty.$$

The minimum amount of fencing is therefore $720\sqrt{10}$ m. The graph of $F(x)$ in the right figure also indicates that $F(x)$ is minimized at its critical point.





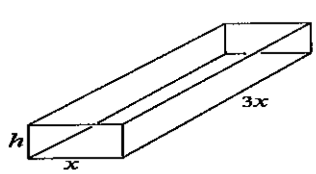
13. If x is the width of the box and h is its height (left figure below), the volume of the box is $V = 3x^2h$. Since the surface area must be 30 m^2, x and h must satisfy $30 = 2(3x^2) + 2(xh) + 2(3xh) = 6x^2 + 8xh$. Hence, $h = (15 - 3x^2)/(4x)$, and

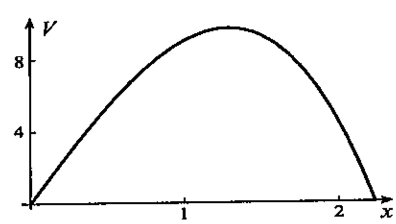
$$V = V(x) = 3x^2\left(\frac{15 - 3x^2}{4x}\right) = \frac{9}{4}(5x - x^3), \quad 0 \le x \le \sqrt{5}.$$

To find critical points of $V(x)$, we solve $0 = V'(x) = (9/4)(5 - 3x^2)$. Solutions are $x = \pm\sqrt{5/3}$. We now evaluate

$$V(0) = 0, \qquad V(\sqrt{15}/3) > 0, \qquad V(\sqrt{5}) = 0.$$

Maximum volume occurs for $x = \sqrt{15}/3$ m, $3x = \sqrt{15}$ m, and $h = \sqrt{15}/2$ m. The graph of $V(x)$ in the right figure below also indicates that $V(x)$ is maximized at its critical point.





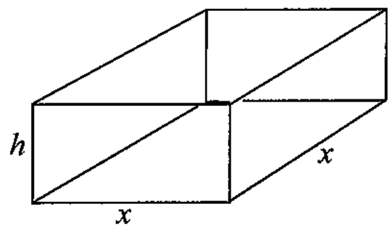
14. The surface area of the box is $A = x^2 + 4xh$. Since the volume of the box must be 6000 L or 6 m^3, x and h must satisfy $6 = x^2h$. Hence, $h = 6/x^2$, and

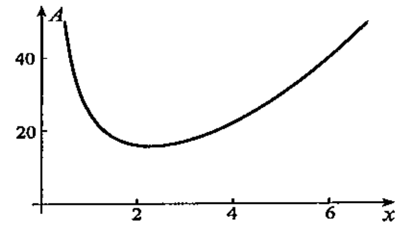
$$A = A(x) = x^2 + 4x\left(\frac{6}{x^2}\right) = x^2 + \frac{24}{x}, \quad x > 0.$$

To find critical points of $A(x)$, we solve $0 = A'(x) = 2x - 24/x^2$. The positive solution is $12^{1/3}$. We now evaluate

$$\lim_{x\to 0^+} A(x) = \infty, \qquad A(12^{1/3}/2) < \infty, \qquad \lim_{x\to\infty} A(x) = \infty.$$

Minimum area therefore occurs when the base of the box is $12^{1/3} \times 12^{1/3}$ m and the height is $6/12^{2/3} = 12^{1/3}/2$ m. The graph of $A(x)$ in the right figure also shows that $A(x)$ is minimized at its critical point.





15. If Y represents the yield of the orchard and x the additional number of trees planted, then

$$Y(x) = (255 + x)(25 - x/12), \quad 0 \le x \le 300.$$

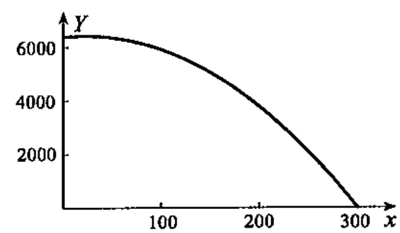
For critical points of this function, we solve

$$0 = Y'(x) = 25 - \frac{255}{12} - \frac{x}{6} \quad \Longrightarrow \quad x = \frac{45}{2}.$$

Since only integer numbers of trees can be planted, we calculate

$$Y(0) = 6375, \qquad Y(22) = 6417.2, \qquad Y(23) = 6417.2, \qquad Y(300) = 0,$$

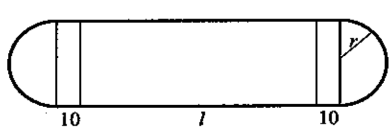
It follows that 22 new trees should be planted (it costing less to plant 22 then 23). The graph of $Y(x)$ to the right also indicates that $Y(x)$ is maximized at its critical point.

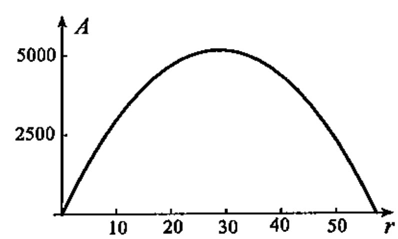


16. The area of the playing field (not including end zones) is $A = 2rl$. Since the perimeter must be 400 m, $400 = 2l + 40 + 2\pi r$. Thus, $l = 180 - \pi r$, and $A = A(r) = 2r(180 - \pi r)$. We must choose $r > 0$, and for l to be positive, we must take $r < 180/\pi$. For critical point of $A(r)$, we solve $0 = A'(r) = 360 - 4\pi r$. The solution is $r = 90/\pi$. We now evaluate

$$\lim_{r \to 0^+} A(r) = 0, \qquad A(90/\pi) > 0, \qquad \lim_{r \to 180/\pi^-} A(r) = 0.$$

Thus, $A(r)$ is maximized when the width of the field is $180/\pi$ m and its length is 90 m. The graph of $A(x)$ in the right figure also indicates that $A(x)$ is maximized at its critical point.





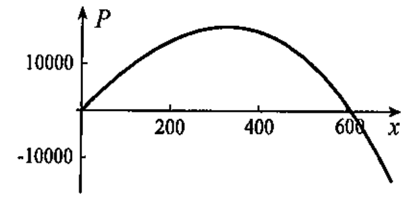
17. Profit on the sale of x objects is $P(x) = x\,r(x) - C(x) = 100x - \dfrac{x^3}{10\,000} - \dfrac{x^2}{10} - 2x - 20$, where $0 \le x \le 1000$. For critical points of this function, we solve

$$0 = P'(x) = 100 - \frac{3x^2}{10\,000} - \frac{x}{5} - 2 \qquad \Longrightarrow \qquad 3x^2 + 2000x - 980\,000 = 0.$$

The positive solution of this equation is $x = 328.3$. Since x must be an integer, we evaluate

$$P(0) = -20, \qquad P(328) = 17\,836.84, \qquad P(329) = 17\,836.77, \qquad P(1000) < 0.$$

The company should produce and sell 328 objects. The graph of $P(x)$ in the figure to the right also indicates that $P(x)$ is maximized at the critical point.

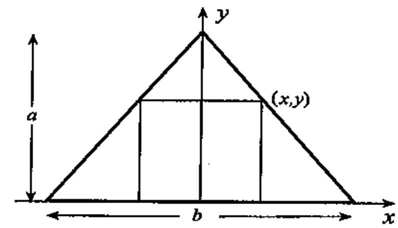


18. The area of the rectangle in the diagram is $A = 2xy$. Since the equation of the line containing the point (x, y) is $y = a - 2ax/b$, we can express A in terms of x as follows

$$A(x) = 2x\left(a - \frac{2ax}{b}\right) = 2a\left(x - \frac{2x^2}{b}\right), \quad 0 \le x \le b/2.$$

For critical points of $A(x)$ we solve

$$0 = A'(x) = 2a\left(1 - \frac{4x}{b}\right) \Longrightarrow x = \frac{b}{4}.$$



Since $A(0) = 0$, $A(b/4) = 2a\left(\frac{b}{4} - \frac{b}{8}\right) = \frac{ab}{4}$, and $A(b/2) = 0$, maximum area is $ab/4$.

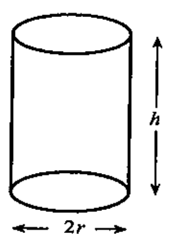
19. If r and h are the radius and height of the can (in centimetres), then the area of metal use to make the can is $A = 2\pi rh + 8r^2$. Because the can must hold $1000\ \text{cm}^3$, it follows that $1000 = \pi r^2 h \Longrightarrow h = 1000/(\pi r^2)$. Thus,

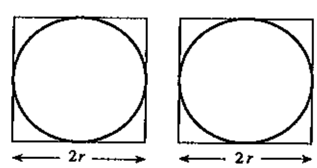
$$A = A(r) = 8r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 8r^2 + \frac{2000}{r}, \quad r > 0.$$

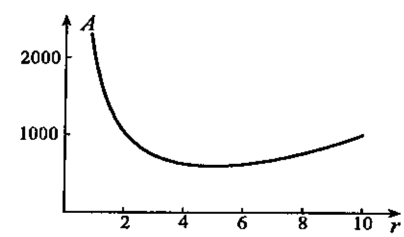
To find critical points of $A(r)$, we solve $0 = A'(r) = 16r - 2000/r^2 \Longrightarrow r = 5$. We now evaluate

$$\lim_{r\to 0+} A(r) = \infty, \qquad A(5) < \infty, \qquad \lim_{r\to\infty} A(r) = \infty.$$

Thus, metal is minimized when $r = 5$ cm and $h = 40/\pi$ cm. The graph of $A(r)$ in the right figure also indicates that $A(r)$ is minimized at its critical point,







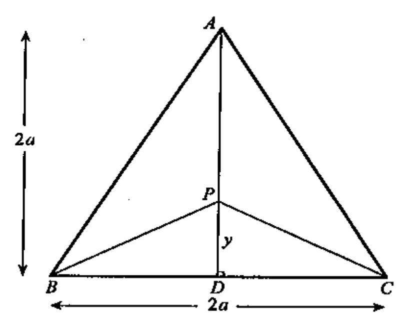
20. When P is at height y, the sum of the lengths of AP, BP, and CP is

$$L = L(y) = (2a - y) + 2\sqrt{a^2 + y^2}, \quad 0 \le y \le 2a.$$

For critical points of $L(y)$ we solve

$$0 = \frac{dL}{dy} = -1 + \frac{2y}{\sqrt{a^2 + y^2}}.$$

This equation can be expressed in the form $2y = \sqrt{a^2 + y^2}$, and squaring gives $4y^2 = a^2 + y^2$, the positive solutions of which is $y = a/\sqrt{3}$. Since



$$L(0) = 2a + 2a = 4a,\ L\left(\frac{a}{\sqrt{3}}\right) = 2a - \frac{a}{\sqrt{3}} + 2\sqrt{a^2 + \frac{a^2}{3}} = a(2 + \sqrt{3}),\ L(2a) = 2\sqrt{a^2 + 4a^2} = 2\sqrt{5}a,$$

minimum length is attained when $y = a/\sqrt{3}$.

21. When the loop is x m from the longer pole, the length of rope is

$$L(x) = \sqrt{x^2+4} + \sqrt{(3-x)^2+1}, \quad 0 \le x \le 3.$$

To find critical points, we solve

$$0 = \frac{dL}{dx} = \frac{x}{\sqrt{x^2+4}} + \frac{x-3}{\sqrt{(3-x)^2+1}} \quad \Longrightarrow \quad x\sqrt{(3-x)^2+1} = -(x-3)\sqrt{x^2+4}.$$

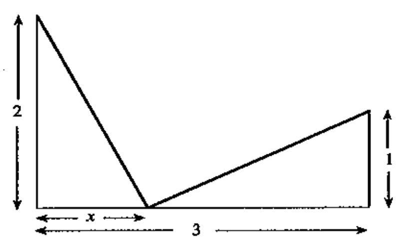
When we square this equation,

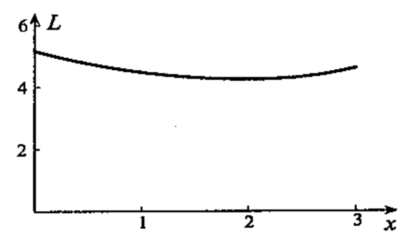
$$x^2(10-6x+x^2) = (x^2-6x+9)(x^2+4) \quad \Longrightarrow \quad 0 = 3x^2-24x+36 = 3(x-2)(x-6).$$

Only the solution $x = 2$ is acceptable. We now evaluate

$$L(0) = 2+\sqrt{10}, \qquad L(2) = 3\sqrt{2}, \qquad L(3) = 1+\sqrt{13}.$$

Thus, the loop should be placed 2 m from the taller pole. The graph of $L(x)$ in the right figure also indicates that it is minimized at its critical point.





22. If length and width of the page are denoted by y and x, then the area of the page is $A = xy$. Since the area of the printed portion of the page must be 150 cm^2, we must have $150 = (x-5)(y-7.5)$. This equation can be solved for $x = (10y+225)/(2y-15)$, and therefore

$$A = A(y) = \frac{10y^2+225y}{2y-15}, \quad y > \frac{15}{2}.$$

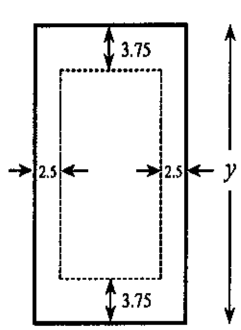
To find critical points, we solve

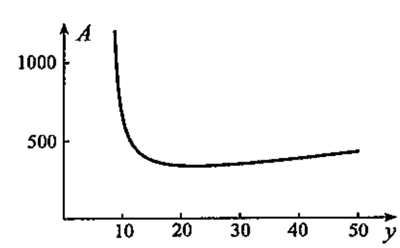
$$0 = A'(y) = \frac{(2y-15)(20y+225)-(10y^2+225y)(2)}{(2y-15)^2}.$$

When we equate the numerator to 0, we obtain $0 = 20y^2-300y-3375 = 5(2y-45)(2y+15) \Longrightarrow y = 45/2$. We now evaluate

$$\lim_{y\to 0^+} A(y) = \infty, \qquad A(45/2) < \infty, \qquad \lim_{y\to\infty} A(y) = \infty.$$

Hence, area is smallest when $y = 45/2$ cm and $x = 15$ cm. The graph of $A(y)$ in the right figure also indicates that $A(y)$ is minimized at its critical point.





23. If $Q(x,y)$ is any point on the hyperbola, the length of line PQ is

$$L = L(x) = \sqrt{(x-4)^2 + y^2} = \sqrt{(x-4)^2 + x^2 + 9} = \sqrt{2x^2 - 8x + 25}, \quad 0 \le x < \infty.$$

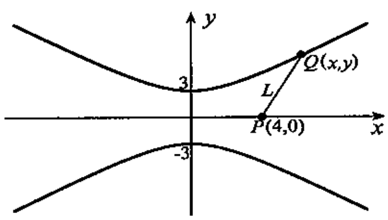
To find critical points, we solve

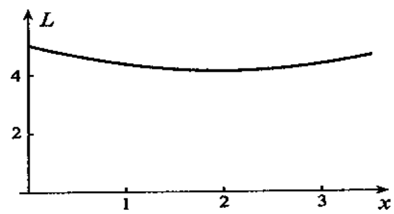
$$0 = \frac{dL}{dx} = \frac{4x - 8}{2\sqrt{2x^2 - 8x + 25}}.$$

The only solution is $x = 2$. We now evaluate

$$L(0) = 5, \qquad L(2) = \sqrt{17}, \qquad \lim_{x \to \infty} L(x) = \infty.$$

Thus, the points on the hyperbola closest to $(4,0)$ are $(2, \pm\sqrt{13})$. The graph of $L(x)$ to the right also shows that the minimum of $L(x)$ occurs at its critical point.





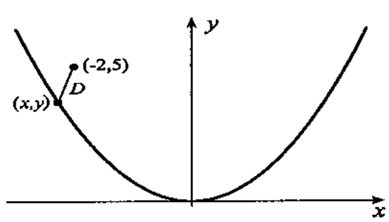
24. If (x,y) is any point on the parabola, then the distance D from $(-2,5)$ to (x,y) is given by $D^2 = (x+2)^2 + (y-5)^2 = (x+2)^2 + (x^2-5)^2$. To minimize D we minimize D^2 on the interval $x \le 0$. For critical points, we solve

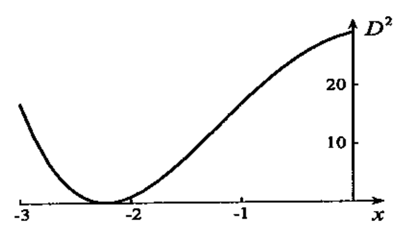
$$0 = \frac{d}{dx}D^2 = 2(x+2) + 2(x^2-5)(2x) = 2(2x^3 - 9x + 2) = 2(x-2)(2x^2 + 4x - 1).$$

The only negative solution is $x = -1 - \sqrt{6}/2$. We now evaluate

$$\lim_{x \to -\infty} D^2(x) = \infty, \qquad D^2(-1 - \sqrt{6}/2) = 0.05, \qquad D^2(0) = 29.$$

The point closest to $(-2,5)$ is $(-1-\sqrt{6}/2, 5/2 + \sqrt{6})$. The graph of D^2 to the right also shows that D^2 is minimized at its critical point.





25. The illumination at height h above the table is

$$I = I(h) = \frac{k \cos\theta}{d^2} = \frac{kh}{d^3} = \frac{kh}{(r^2 + h^2)^{3/2}}, \quad h \ge 0,$$

where k is a constant. For critical points we solve

$$0 = \frac{dI}{dh} = k\left[\frac{(r^2 + h^2)^{3/2} - h(3/2)\sqrt{r^2 + h^2}(2h)}{(r^2 + h^2)^3}\right] = \frac{k(r^2 - 2h^2)}{(r^2 + h^2)^{5/2}}.$$

The positive solution is $h = r/\sqrt{2}$. Since $\quad I(0) = 0, \quad I(r/\sqrt{2}) = \dfrac{2\sqrt{3}k}{9r^2}, \quad \lim_{h \to \infty} I(h) = 0, \quad$ maximum illumination occurs when $h = r/\sqrt{2}$.

26. The area of the triangle is $A = xy/2$. Since slopes of the line segments joining $(0, y)$ to $(x, 0)$ and $(2, 5)$ to $(x, 0)$ must be the same, we have $\dfrac{y-0}{0-x} = \dfrac{0-5}{x-2} \Longrightarrow y = \dfrac{5x}{x-2}$. Thus,

$$A = A(x) = \frac{5x^2}{2(x-2)}, \quad x > 2.$$

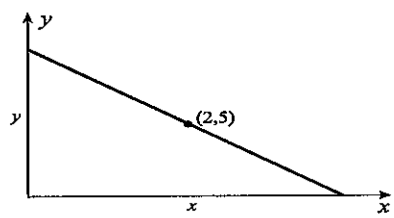
For critical points, we solve

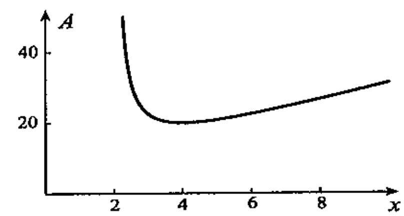
$$0 = A'(x) = \frac{5}{2}\left[\frac{(x-2)(2x) - x^2(1)}{(x-2)^2}\right] = \frac{5x(x-4)}{2(x-2)^2}.$$

We now evaluate

$$\lim_{x \to 2^+} A(x) = \infty, \qquad A(4) < \infty, \qquad \lim_{x \to \infty} A(x) = \infty.$$

Area is minimized when $x = 4$ and $y = 10$. The graph of $A(x)$ in the right figure also indicates that $A(x)$ is minimized at its critical point.





27. Maximum flow occurs when area $ABCD$ is a maximum. If we denote the area by $a(\theta)$, then

$$\begin{aligned} a(\theta) &= \|BE\|\left(\frac{\|AD\| + \|BC\|}{2}\right) = \frac{1}{6}\sin\theta(2\|AE\| + 2\|BC\|) = \frac{1}{3}\sin\theta\left(\frac{1}{3}\cos\theta + \frac{1}{3}\right) \\ &= \frac{1}{9}\sin\theta(1 + \cos\theta), \quad 0 \le \theta \le \pi/2. \end{aligned}$$

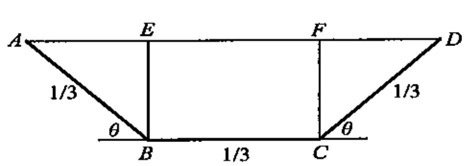
To find critical points, we solve

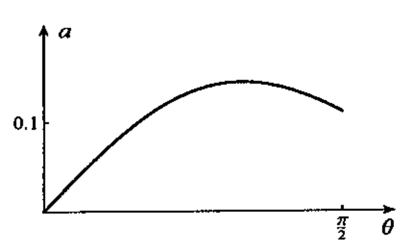
$$\begin{aligned} 0 = \frac{da}{d\theta} &= \frac{1}{9}\cos\theta(1 + \cos\theta) + \frac{1}{9}\sin\theta(-\sin\theta) = \frac{1}{9}(\cos\theta + \cos^2\theta - \sin^2\theta) \\ &= \frac{1}{9}(\cos\theta + 2\cos^2\theta - 1) = \frac{1}{9}(2\cos\theta - 1)(\cos\theta + 1). \end{aligned}$$

The only solution in the interval $0 \le \theta \le \pi/2$ is $\theta = \pi/3$. We now evaluate

$$a(0) = 0, \qquad a(\pi/3) = \frac{\sqrt{3}}{12}, \qquad a(\pi/2) = \frac{1}{9}.$$

Thus, maximum flow occurs when $\theta = \pi/3$. The graph of $a(\theta)$ in the right figure also indicates that $a(\theta)$ is maximized at its critical point.





28. The manufacturer's yearly costs for x employees is $C = 20\,000x + 365y$. When 1000 automobiles are to be produced, we must have $1000 = x^{2/5}y^{3/5} \Longrightarrow y = 10^5/x^{2/3}$, and therefore

$$C(x) = 20\,000x + \frac{365 \times 10^5}{x^{2/3}}, \quad x > 0.$$

For critical points of $C(x)$, we set

$$0 = C'(x) = 20\,000 + 365 \times 10^5 \left(\frac{-2}{3}\right) x^{-5/3} \quad \Longrightarrow \quad x = 70.97.$$

Since x must be an integer, we evalulate

$$\lim_{x\to 0^+} C(x) = \infty, \quad C(70) = 3.5490 \times 10^6, \quad C(71) = 3.5487 \times 10^6, \quad \lim_{x\to\infty} C(x) = \infty.$$

It follows that $C(x)$ is minimized for $x = 71$.

29. Since critical points of $Q(p)$ are the same as those of $Q^2(p)$, we solve

$$0 = \frac{dQ^2}{dp} = \frac{2A^2\gamma p_0\rho_0}{\gamma - 1}\left[\frac{2}{\gamma}\left(\frac{p}{p_0}\right)^{2/\gamma-1}\left(\frac{1}{p_0}\right) - \left(\frac{\gamma+1}{\gamma}\right)\left(\frac{p}{p_0}\right)^{1/\gamma}\left(\frac{1}{p_0}\right)\right].$$

Consequently,

$$2\left(\frac{p}{p_0}\right)^{2/\gamma-1} = (\gamma+1)\left(\frac{p}{p_0}\right)^{1/\gamma} \Longrightarrow \left(\frac{p}{p_0}\right)^{1/\gamma-1} = \frac{\gamma+1}{2} \Longrightarrow p = p_0\left(\frac{\gamma+1}{2}\right)^{\gamma/(1-\gamma)} = p_0\left(\frac{2}{1+\gamma}\right)^{\gamma/(\gamma-1)}.$$

Since $Q(0) = 0 = Q(p_0)$, it follows that this critical point must yield a maximum for Q, and is therefore the critical pressure p_c.

30. The printer will choose the number of set types that will minimize production costs. If x set types are used, then the cost for the set types themselves is $2x$ dollars. With this number of set types, the press prints $1000x$ cards per hour. In order to produce 200 000 cards, the press must therefore run for $200\,000/(1000x)$ hours at a cost of $[10][200\,000/(1000x)]$ dollars. The total cost of producing the cards when x set types are used is therefore given by

$$C(x) = 2x + \frac{2000}{x}, \quad 1 \le x \le 40.$$

To find critical points of $C(x)$, we set

$$0 = C'(x) = 2 - \frac{2000}{x^2}.$$

The only solution of this equation in the interval $1 \le x \le 40$ is $x = 10\sqrt{10} = 31.6$. Since x must be an integer, we evaluate

$$C(1) = 2002, \quad C(31) = 126.52, \quad C(32) = 126.50, \quad C(40) = 130.$$

The printer should therefore use 32 set types.

31. The x and y-coordinates of S_2 and S_1, in kilometres, as functions of time are, respectively, $x = 8t$ and $y = 20 - 6t$. The distance between the ships is therefore

$$D(t) = \sqrt{x^2 + y^2} = \sqrt{64t^2 + (20 - 6t)^2} = \sqrt{100t^2 - 240t + 400}, \quad t \ge 0.$$

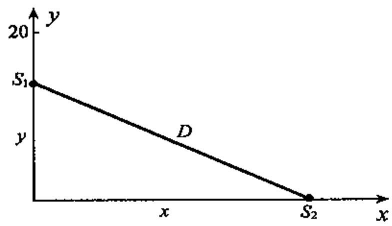
For critical points, we solve

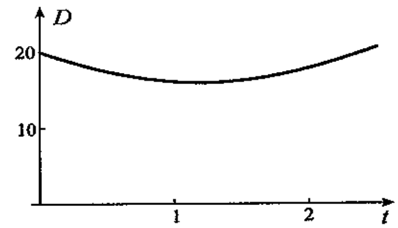
$$0 = \frac{dD}{dt} = \frac{200t - 240}{2\sqrt{100t^2 - 240t + 400}}.$$

The solution is $t = 6/5$. We now evaluate

$$D(0) = 20, \qquad D(6/5) = 16, \qquad \lim_{t\to\infty} D(t) = \infty.$$

The ships are closest together at 1:12 p.m. The graph of $D(t)$ in the right figure also indicates that $D(t)$ is minimized at its critical point.





32. (a) If the courier heads to point R, his travel time is

$$T(x) = \frac{\sqrt{x^2+36}}{14} + \frac{3-x}{50}, \quad 0 \le x \le 3.$$

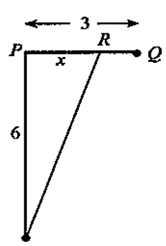
To find critical points, we solve

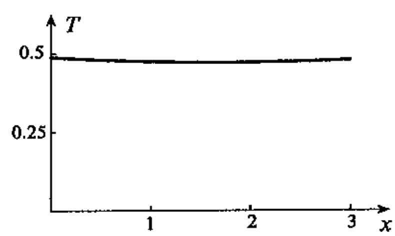
$$0 = T'(x) = \frac{x}{14\sqrt{x^2+36}} - \frac{1}{50}.$$

This equation can be expressed in the form $25x = 7\sqrt{x^2+36}$, and squaring gives $625x^2 = 49(x^2+36)$. The positive solution of this equation is $x = 7/4$.
We now evaluate

$$T(0) = 0.49, \qquad T(7/4) = 0.47, \qquad T(3) = 0.48.$$

Thus, travel time is minimized when the courier heads to the point on the road 7/4 km from P. The graph of $T(x)$ function in the right figure also indicates that $T(x)$ is minimized at its critical point.





(b) In this case travel time is given by the formula

$$T(x) = \frac{\sqrt{x^2+36}}{14} + \frac{1-x}{50}, \quad 0 \le x \le 1.$$

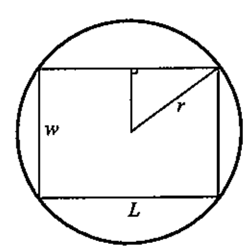
The derivative of this function is identical to that in part (a), but in this case the point $x = 7/4$ must be rejected. Since $T(0) = 6/14 + 1/50 = 0.4486$ and $T(1) = \sqrt{37}/14 = 0.4345$, the courier should head directly to Q.

33. If the length and width of an inscribed rectangle are L and w, then its area is $A = Lw$. Since $r^2 = (L/2)^2 + (w/2)^2$, it follows that
$L = \sqrt{4r^2 - w^2}$, and

$$A(w) = w\sqrt{4r^2 - w^2}, \quad 0 \le w \le 2r.$$

For critical points of $A(w)$, we solve

$$0 = \frac{dA}{dw} = \sqrt{4r^2 - w^2} + \frac{w(-2w)}{2\sqrt{4r^2 - w^2}} = \frac{4r^2 - 2w^2}{\sqrt{4r^2 - w^2}}.$$



The positive solution is $w = \sqrt{2}r$. Since

$$A(0) = 0, \qquad A(\sqrt{2}r) = \sqrt{2}r\sqrt{4r^2 - 2r^2} = 2r^2, \qquad A(2r) = 0,$$

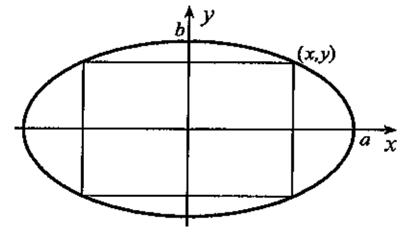
the area of the largest rectangle is $2r^2$.

34. The area of the rectangle is $A = 4xy$. When we solve the equation of the ellipse for the positive value of y, the result is $y = (b/a)\sqrt{a^2 - x^2}$. The area of the rectangle can therefore be expressed in the form

$$A = A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2}, \quad 0 \le x \le a.$$

For critical points of $A(x)$ we solve

$$0 = A'(x) = \frac{4b}{a}\left(\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right).$$



This equation can be expressed in the form $\sqrt{a^2 - x^2} = \dfrac{x^2}{\sqrt{a^2 - x^2}}$, from which $a^2 - x^2 = x^2$. The positive solution is $x = a/\sqrt{2}$. Since

$$A(0) = 0, \qquad A\left(\frac{a}{\sqrt{2}}\right) > 0, \qquad A(a) = 0,$$

area is maximized when the length of the rectangle in the x-direction is $\sqrt{2}a$ and that in the y-direction is $\sqrt{2}b$.

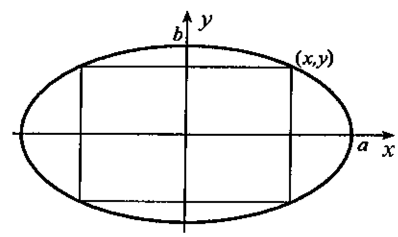
35. The perimeter of the rectangle shown is

$$P(x) = 4x + 4y = 4x + \frac{4b}{a}\sqrt{a^2 - x^2}, \; 0 \le x \le a.$$

For critical points of this function, we solve

$$0 = \frac{dP}{dx} = 4 + \frac{-4bx}{a\sqrt{a^2 - x^2}} = \frac{4a\sqrt{a^2 - x^2} - 4bx}{a\sqrt{a^2 - x^2}}.$$

This implies that $a^2(a^2 - x^2) = b^2x^2$, from which $x = a^2/\sqrt{a^2 + b^2}$. Since



$$P(0) = 4b, \quad P\left(\frac{a^2}{\sqrt{a^2 + b^2}}\right) = \frac{4a^2}{\sqrt{a^2 + b^2}} + \frac{4b}{a}\sqrt{a^2 - \frac{a^4}{a^2 + b^2}} = 4\sqrt{a^2 + b^2}, \quad P(a) = 4a,$$

maximum perimeter occurs for $x = a^2/\sqrt{a^2 + b^2}$.

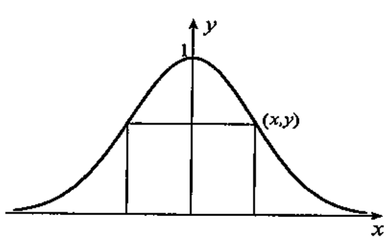
36. The area of the rectangle is $A = 2xy$. Since $y = e^{-x^2}$, area can be expressed as $A = A(x) = 2xe^{-x^2}$, $x \ge 0$. To find critical points, we solve

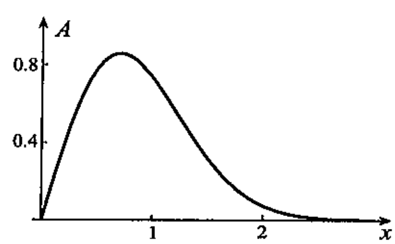
$$0 = A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2(1 - 2x^2)e^{-x^2}.$$

The positive solution of this equation is $x = 1/\sqrt{2}$. We now evaluate

$$A(0) = 0, \qquad T(1/\sqrt{2}) = \sqrt{2/e}, \qquad \lim_{x\to\infty} A(x) = 0.$$

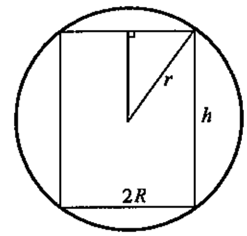
Maximum area is therefore $\sqrt{2/e}$. The graph of $A(x)$ also indicates that it is maximized at its critical point.





37. The volume of the cylinder, the cross section of which is shown, is $V = \pi R^2 h$. Since $r^2 = R^2 + h^2/4$,

$$V(h) = \pi\left(r^2 - \frac{h^2}{4}\right)h, \quad 0 \le h \le 2r.$$



For critical points of $V(h)$, we solve

$$0 = \frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right).$$

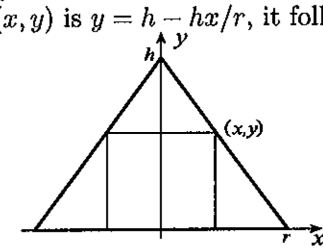
The positive solution is $h = 2r/\sqrt{3}$. Since

$$V(0) = 0, \qquad V(2r/\sqrt{3}) = \pi\left(r^2 - \frac{r^2}{3}\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}, \qquad V(2r) = 0,$$

the largest cylinder has volume $4\pi r^3/(3\sqrt{3})$.

38. We have shown cross-sections of the cone and an inscribed cylinder. The volume of the cylinder is $V = \pi x^2 y$. Since the equation of the line containing the point (x, y) is $y = h - hx/r$, it follows that

$$V = \pi x^2\left(h - \frac{hx}{r}\right) = \pi h\left(x^2 - \frac{x^3}{r}\right), \quad 0 \le x \le r.$$



For critical points of $V(x)$ we solve

$$0 = \frac{dV}{dx} = \pi h\left(2x - \frac{3x^2}{r}\right).$$

The positive solution is $x = 2r/3$. Since

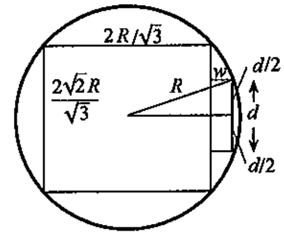
$$V(0) = 0, \qquad V\left(\frac{2r}{3}\right) = \pi h\left(\frac{4r^2}{9} - \frac{8r^3}{27r}\right) = \frac{4\pi h r^2}{27}, \qquad V(r) = 0,$$

maximum volume for the cylinder is $4\pi h r^2/27$.

39. The strength of the beams is $S = kwd^2$. Since $\left(\frac{d}{2}\right)^2 + \left(w + \frac{R}{\sqrt{3}}\right)^2 = R^2$,

it follows that

$$d^2 = 4R^2 - 4\left(w + \frac{R}{\sqrt{3}}\right)^2,$$



and

$$S = kw\left[4R^2 - 4\left(w + \frac{R}{\sqrt{3}}\right)^2\right], \; 0 \le w \le R - \frac{R}{\sqrt{3}}.$$

For critical points,

$$0 = \frac{dS}{dw} = k\left[4R^2 - 4\left(w + \frac{R}{\sqrt{3}}\right)^2\right] + kw\left[-8\left(w + \frac{R}{\sqrt{3}}\right)\right] = -\frac{4k}{3}(9w^2 + 4\sqrt{3}Rw - 2R^2).$$

Consequently, $w = (-4\sqrt{3}R \pm \sqrt{48R^2 + 72R^2})/18$, but only $w = (\sqrt{30} - 2\sqrt{3})R/9$ is acceptable. Since $S(0) = S(R - R/\sqrt{3}) = 0$, the strongest beams occur when widths are $w = (\sqrt{30} - 2\sqrt{3})R/9$ cm and depths are

$$d = \sqrt{4R^2 - 4\left[\frac{(\sqrt{30} - 2\sqrt{3})R}{9} + \frac{R}{\sqrt{3}}\right]^2} = \frac{2\sqrt{48 - 6\sqrt{10}}R}{9} \text{ cm}.$$

40. For critical points we solve

$$0 = \frac{dR}{d\theta} = \frac{v^2}{g}\left(-\sin^2\theta + \cos^2\theta - \sin\theta\sqrt{\sin^2\theta + \frac{2gh}{v^2}} + \frac{\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + 2gh/v^2}}\right).$$

From this equation,

$$(\sin^2\theta - \cos^2\theta)\sqrt{\sin^2\theta + \frac{2gh}{v^2}} = \sin\theta\cos^2\theta - \sin\theta\left(\sin^2\theta + \frac{2gh}{v^2}\right)$$
$$= \sin\theta(\cos^2\theta - \sin^2\theta) - \frac{2gh}{v^2}\sin\theta.$$

Hence, $\quad -\cos 2\theta\sqrt{\sin^2\theta + \frac{2gh}{v^2}} = \sin\theta\cos 2\theta - \frac{2gh}{v^2}\sin\theta$. Squaring this gives

$$\cos^2 2\theta\left(\sin^2\theta + \frac{2gh}{v^2}\right) = \sin^2\theta\cos^2 2\theta - \frac{4gh}{v^2}\sin^2\theta\cos 2\theta + \frac{4g^2h^2}{v^4}\sin^2\theta,$$

from which

$$\cos^2 2\theta = \frac{2gh}{v^2}\sin^2\theta - 2\sin^2\theta\cos 2\theta = \left(\frac{2gh}{v^2} - 2\cos 2\theta\right)\left(\frac{1-\cos 2\theta}{2}\right)$$
$$= \frac{gh}{v^2} - \cos 2\theta - \frac{gh}{v^2}\cos 2\theta + \cos^2 2\theta.$$

Thus, $\cos 2\theta = \dfrac{gh/v^2}{1 + gh/v^2} = \dfrac{gh}{v^2 + gh}$. The only solution of this equation in the interval $0 < \theta < \pi/2$ is $\theta = \dfrac{1}{2}\mathrm{Cos}^{-1}\left(\dfrac{gh}{v^2 + gh}\right)$. It is geometrically clear that there is an angle between $\theta = 0$ and $\theta = \pi/2$ that maximizes R, and since only one critical point has been obtained, it must maximize R. For $v = 13.7$ m/s and $h = 2.25$ m, $\theta = (1/2)\mathrm{Cos}^{-1}\{9.81(2.25)/[13.7^2 + 9.81(2.25)]\} = 0.733$ radians.

41. The longest beam that can be transported around corner C is the shortest of all line segments that touch the walls at A and B and pass through C. If L is the length of AB, then

$$L(\theta) = \|AC\| + \|BC\| = 6\csc\theta + 3\sec\theta, \quad 0 < \theta < \pi/2.$$

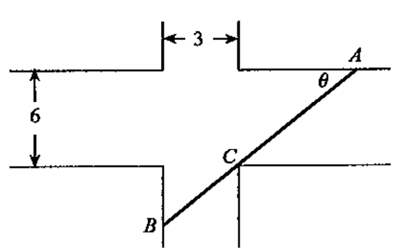
For critical points, we solve

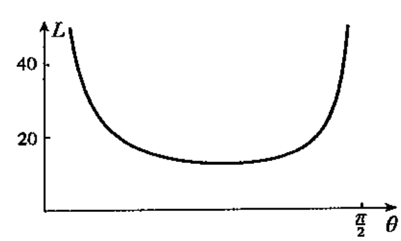
$$0 = L'(\theta) = -6\csc\theta\cot\theta + 3\sec\theta\tan\theta \quad\Longrightarrow\quad \frac{6\cos\theta}{\sin^2\theta} = \frac{3\sin\theta}{\cos^2\theta}.$$

This equation implies that $\tan\theta = 2^{1/3}$. The acute angle satisfying this equation is $\theta = 0.90$ radians. We now evaluate

$$\lim_{\theta\to 0^+} L(\theta) = \infty, \qquad L(0.90) = 12.5, \qquad \lim_{\theta\to\pi/2^-} L(\theta) = \infty.$$

The longest beam that can be transported around the corner is 12.5 m. The graph of $L(\theta)$ also indicates that it is minimized at its critical point.





42. Length L of the beam is the sum of the lengths L_1 and L_2. Now,

$$L_1 = \|FD\|\sec\theta = \sec\theta\left(3 - \frac{1}{3}\sin\theta\right), \quad \text{and} \quad L_2 = \|GE\|\csc\theta = \csc\theta\left(6 - \frac{1}{3}\cos\theta\right).$$

Hence, $\qquad L = L_1 + L_2 = \dfrac{1}{3}\left[\sec\theta(9 - \sin\theta) + \csc\theta(18 - \cos\theta)\right],$

and this function is defined for $0 < \theta < \pi/2$. The longest beam that can be transported around the corner is represented by the minimum value of $L(\theta)$. To find critical points, we solve

$$0 = \frac{dL}{d\theta} = \frac{1}{3}\left[\sec\theta(-\cos\theta) + \sec\theta\,\tan\theta(9 - \sin\theta) + \csc\theta(\sin\theta) - \csc\theta\,\cot\theta(18 - \cos\theta)\right]$$

$$= \frac{1}{3}\left[-1 + \frac{9\sin\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} + 1 - \frac{18\cos\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}\right] = \frac{1}{3}\left[\frac{9\sin\theta - \sin^2\theta}{\cos^2\theta} - \frac{18\cos\theta - \cos^2\theta}{\sin^2\theta}\right].$$

Critical points therefore satisfy

$$\begin{aligned} 0 = f(\theta) &= 9\sin^3\theta - \sin^4\theta - 18\cos^3\theta + \cos^4\theta \\ &= 9\sin^3\theta - 18\cos^3\theta + (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \\ &= 9\sin^3\theta - 18\cos^3\theta + 2\cos^2\theta - 1. \end{aligned}$$

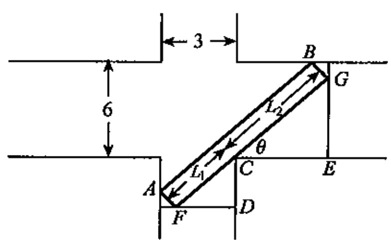
To solve this equation we use Newton's iterative procedure with

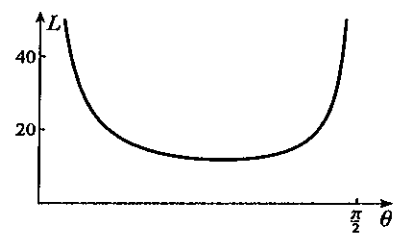
$$\theta_1 = 0.9, \quad \theta_{n+1} = \theta_n - \frac{9\sin^3\theta_n - 18\cos^3\theta_n + 2\cos^2\theta_n - 1}{27\sin^2\theta_n\cos\theta_n + 54\cos^2\theta_n\sin\theta_n - 4\cos\theta_n\sin\theta_n}.$$

Iteration gives $\theta_2 = 0.9091$ and $\theta_3 = 0.9091$. We now evaluate

$$\lim_{\theta\to 0^+} L(\theta) = \infty, \qquad L(0.90) = 11.8, \qquad \lim_{\theta\to\pi/2^-} L(\theta) = \infty.$$

Thus, the length of the longest beam is 11.8 m. The graph of $L(\theta)$ in the right figure also shows that it is minimized at its critical point.



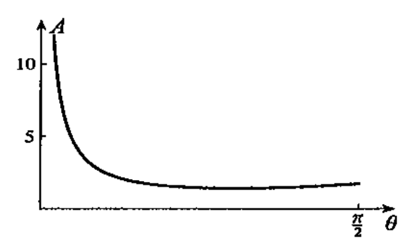


43. With x and y fixed, we minimize $f(\theta) = \sqrt{3}\csc\theta - \cot\theta$. Critical points of $f(\theta)$ are given by

$$0 = f'(\theta) = -\sqrt{3}\csc\theta\,\cot\theta + \csc^2\theta = \frac{1 - \sqrt{3}\cos\theta}{\sin^2\theta}.$$

The angle in the first quadrant for which $\cos\theta = 1/\sqrt{3}$ is $\theta = 0.96$ radians. We now evaluate

$$\lim_{\theta\to 0^+} L(\theta) = \infty, \qquad L(0.96) = 1.41, \qquad L(\pi/2) = \sqrt{3}.$$



Thus, $f(\theta)$ is minimized for $\theta = 0.96$ radians. The graph of $f(\theta)$ also indicates that it is minimized at its critical point.

44. For critical points we solve $\quad 0 = f'(x) = \dfrac{(x^2+c)(1) - x(2x)}{(x^2+c)^2} = \dfrac{c - x^2}{(x^2+c)^2}$. The only solution in $0 \le x \le c$ is $x = \sqrt{c}$. We now calculate

$$f(0) = 0, \qquad f(\sqrt{c}) = \frac{\sqrt{c}}{c+c} = \frac{1}{2\sqrt{c}}, \qquad f(c) = \frac{c}{c^2+c} = \frac{1}{c+1}.$$

Since $c > 0$, we can say that $1/(c+1) \le 1/(2\sqrt{c})$ if and only if

$$c + 1 \ge 2\sqrt{c} \iff c - 2\sqrt{c} + 1 \ge 0 \iff (\sqrt{c} - 1)^2 \ge 0,$$

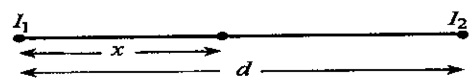
and this is always valid. Hence, the absolute maximum and minimum values of $f(x)$ on $0 \le x \le c$ are $1/(2\sqrt{c})$ and 0.

45. (a) If we substitute $n = t^\alpha/\beta$ into the formula for $C(t)$, we obtain $C(t) = \dfrac{p}{t} + \dfrac{r}{\beta}t^{\alpha-1}$, $t > 0$. For critical points of $C(t)$ we solve $0 = -\dfrac{p}{t^2} + \dfrac{r}{\beta}(\alpha-1)t^{\alpha-2} = \dfrac{1}{\beta t^2}[-p\beta + r(\alpha-1)t^\alpha]$. The solution of this equation is $[p\beta/(r\alpha - r)]^{1/\alpha}$. Since $\lim_{t\to 0^+} C(t) = \infty$, $C\left(\left(\dfrac{p\beta}{r\alpha - r}\right)^{1/\alpha}\right) < \infty$, $\lim_{t\to\infty} C(t) = \infty$, the machine should be replaced in $[p\beta/(r\alpha - r)]^{1/\alpha}$ years.

(b) When $\alpha = 1$, then $C(t) = \dfrac{p}{t} + \dfrac{r}{\beta}$. Since this is a decreasing function for $t > 0$, and is asymptotic to the line $C = r/\beta$, it follows that the machine should be kept as long as possible. When $\alpha < 1$, then $C(t) = \dfrac{p}{t} + \dfrac{r}{\beta t^{1-\alpha}}$. Once again this is a decreasing function for $t > 0$, and is asymptotic to the t-axis. The machine should therefore be kept as long as possible.

46. The illumination L at a point which is a distance x from the source with intensity I_1 is

$$L = L(x) = \frac{kI_1}{x^2} + \frac{kI_2}{(d-x)^2}, \quad 0 < x < d,$$



where k is a constant. For critical points of this function we solve

$$0 = \frac{dL}{dx} = -\frac{2kI_1}{x^3} + \frac{2kI_2}{(d-x)^3} \Longrightarrow I_2x^3 = I_1(d-x)^3 \Longrightarrow \left(\frac{d-x}{x}\right)^3 = \frac{I_2}{I_1}.$$

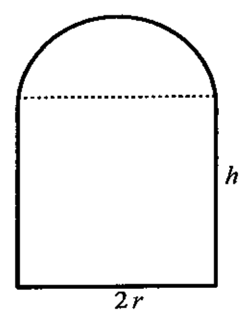
The solution of this equation is $x = dI_1^{1/3}/(I_1^{1/3} + I_2^{1/3})$. Since

$$\lim_{x\to 0^+} L(x) = \infty \qquad \text{and} \qquad \lim_{x\to d^-} L(x) = \infty,$$

illumination is minimized at the critical point $dI_1^{1/3}/(I_1^{1/3} + I_2^{1/3})$ units from the I_1 source.

47. If k is the amount of light per unit area transmitted by the clear glass, then the total amount of light admitted by the window is

$$A = k(2rh) + \frac{k}{2}\left(\frac{1}{2}\pi r^2\right) = \frac{k(8rh + \pi r^2)}{4}.$$



If P represents the fixed perimeter of the window, then $P = \pi r + 2h + 2r$. Thus, $h = (P - \pi r - 2r)/2$, and

$$A(r) = \frac{k}{4}\left[8r\left(\frac{P - \pi r - 2r}{2}\right) + \pi r^2\right] = \frac{k}{4}(4rP - 3\pi r^2 - 8r^2).$$

We must take $r \ge 0$, and for $h \ge 0$, we must also take $P - \pi r - 2r \ge 0$ or $r \le P/(\pi + 2)$. For critical points of $A(r)$, we solve

$$0 = \frac{dA}{dr} = \frac{k}{4}(4P - 6\pi r - 16r).$$

The solution is $r = 2P/(3\pi + 8)$. Since

$$A(0) = 0, \quad A\left(\frac{2P}{3\pi+8}\right) = \frac{kP^2}{3\pi+8}, \quad A\left(\frac{P}{\pi+2}\right) = \frac{k\pi P^2}{4(\pi+2)^2},$$

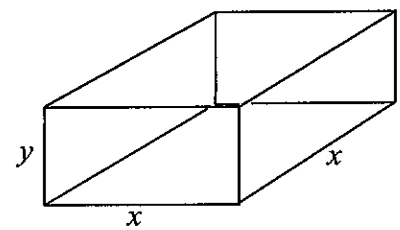
A is maximized for $r = 2P/(3\pi + 8)$. For this r, $h = P(\pi+4)/(6\pi+16)$, and the ratio of h to $2r$ is

$$\frac{h}{2r} = \left[\frac{P(\pi+4)}{2(3\pi+8)}\right]\left(\frac{3\pi+8}{4P}\right) = \frac{\pi+4}{8}.$$

48. (a) The cost in pennies for the material to construct a tank with base x m and height y m is

$$C = 125(x^2 + 4xy) + 475x^2.$$

Since the tank must hold 100 m^3, x and y must satisfy $100 = x^2y$. Thus, $y = 100/x^2$, and



$$C = C(x) = 125x^2 + 500x\left(\frac{100}{x^2}\right) + 475x^2$$
$$= 600x^2 + \frac{50\,000}{x}, \quad x > 0.$$

For critical points of $C(x)$ we solve

$$0 = 1200x - \frac{50\,000}{x^2}.$$

The solution is $x = 5/3^{1/3}$. Since $\lim_{x\to 0^+} C(x) = \infty$ and $\lim_{x\to\infty} C(x) = \infty$, it follows that cost is minimized when $x = 5/3^{1/3}$ m and $y = 4 \cdot 3^{2/3}$ m.

(b) When welding is taken into account the cost is

$$C(x) = 125(x^2 + 4xy) + 475x^2 + 750(8x + 4y) = 600x^2 + \frac{50\,000}{x} + 750\left(8x + \frac{400}{x^2}\right)$$
$$= 600x^2 + \frac{50\,000}{x} + 6000x + \frac{300\,000}{x^2}, \quad x > 0.$$

For critical points of this function we solve

$$0 = 1200x - \frac{50\,000}{x^2} + 6000 - \frac{600\,000}{x^3},$$

and this equation simplifies to $3x^4 + 15x^3 - 125x - 1500 = 0$. The positive solution of this equation, $x = 4.19$, can be found using Newton's iterative procedure (see Example 4.1 in Section 4.1). Since limits of $C(x)$ are once again "infinity" as $x \to 0^+$ and $x \to \infty$, cost is minimized for $x = 4.19$ m and $y = 5.70$ m.

49. For critical points of $f(x)$, we solve

$$0 = f'(x) = \frac{n!}{m!\,(n-m)!}[mx^{m-1}(1-x)^{n-m} - (n-m)x^m(1-x)^{n-m-1}]$$
$$= \frac{n!x^{m-1}(1-x)^{n-m-1}(m-nx)}{m!\,(n-m)!}.$$

Solutions are $x = 0, m/n, 1$. Since $f(0) = 0$, $f(m/n) > 0$, and $f(1) = 0$, it follows that $x = m/n$ maximizes $f(x)$.

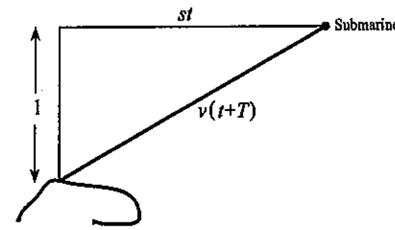
50. Total cost for the pipeline is $C(x) = c_1\sqrt{x^2 + y_1^2} + c_2\sqrt{(x_2 - x)^2 + y_2^2}$, $0 \le x \le x_2$. For critical points of this function we solve $0 = C'(x) = \dfrac{c_1 x}{\sqrt{x^2 + y_1^2}} + \dfrac{c_2(x_2 - x)(-1)}{\sqrt{(x_2 - x)^2 + y_2^2}}$. Since $\sin\theta_1 = x/\sqrt{x^2 + y_1^2}$ and $\sin\theta_2 = (x_2 - x)/\sqrt{(x_2 - x)^2 + y_2^2}$, it follows that at the critical point $0 = c_1 \sin\theta_1 - c_2 \sin\theta_2$. We can show that the critical point yields a minimum by calculating

$$C''(x) = \frac{c_1}{\sqrt{x^2 + y_1^2}} - \frac{c_1 x^2}{(x^2 + y_1^2)^{3/2}} + \frac{c_2}{\sqrt{(x_2 - x)^2 + y_2^2}} - \frac{c_2(x_2 - x)^2}{[(x_2 - x)^2 + y_2^2]^{3/2}}$$
$$= \frac{c_1 y_1^2}{(x^2 + y_1^2)^{3/2}} + \frac{c_2 y_2^2}{[(x_2 - x)^2 + y_2^2]^{3/2}}.$$

Since $C''(x)$ is positive for $0 \le x \le x_2$, the graph of $C(x)$ is concave upward. Consequently, the critical point must minimize $C(x)$.

51. Let T be the number of hours before midnight that the raft must leave the island in order to intercept the submarine t hours after midnight. Then, $1 + s^2t^2 = v^2(t+T)^2$, and this can be solved for

$$T(t) = \frac{\sqrt{1+s^2t^2}}{v} - t, \quad t \ge 0.$$



For critical points of this function, we solve

$$0 = \frac{dT}{dt} = \frac{2s^2t}{2v\sqrt{1+s^2t^2}} - 1 = \frac{s^2t - v\sqrt{1+s^2t^2}}{v\sqrt{1+s^2t^2}}.$$

When we square $s^2t = v\sqrt{1+s^2t^2}$, we obtain $s^4t^2 = v^2(1+s^2t^2)$, from which $t = \dfrac{v}{s\sqrt{s^2-v^2}}$. To finish the problem, we calculate that $T(0) = 1/v$,

$$T\left(\frac{v}{s\sqrt{s^2-v^2}}\right) = \frac{\sqrt{1+\dfrac{s^2v^2}{s^2(s^2-v^2)}}}{v} - \frac{v}{s\sqrt{s^2-v^2}} = \frac{s^2}{vs\sqrt{s^2-v^2}} - \frac{v}{s\sqrt{s^2-v^2}} = \frac{\sqrt{s^2-v^2}}{vs},$$

and $\lim\limits_{t\to\infty} T(t) = \lim\limits_{t\to\infty} \dfrac{\sqrt{1+s^2t^2}-vt}{v} = \infty$, (since $s > v$). Since $\sqrt{s^2-v^2}/(vs) < 1/v$, the raft should leave $v/(s\sqrt{s^2-v^2})$ hours before midnight.

52. The volume of the box is $V = lwh$. Dimensions on the cardboard make it clear that $2l + 2w = 2$, and therefore $w = 1-l$. The fact that the outer flaps must meet in the centre requires $l = 2(1/2)(1-h) = 1-h$. Hence, $w = 1-(1-h) = h$ and

$$V = V(h) = (1-h)hh = h^2 - h^3, \quad 0 \le h \le 1.$$

For critical points we solve $0 = V'(h) = 2h - 3h^2 = h(2-3h)$. Since $V(0) = 0$, $V(2/3) > 0$, and $V(1) = 0$, maximum volume occurs when $h = w = 2/3$ m and $l = 1/3$ m. Since the sum of the spaces between inner flaps on top and bottom is $2w + 2l - 2 = 2/3$, the inner flaps are $1/3$ m apart.

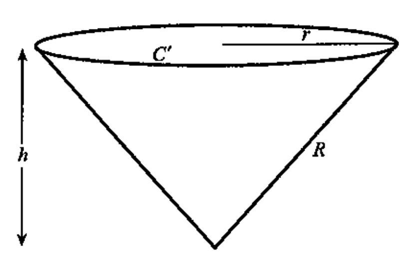
53. If C_f is the fuel cost per hour, then $C_f = ks^3$ where s is the speed of the ship and k is a constant. Since $C_f = B$ when $s = b$, we obtain $k = B/b^3$. Thus, $C_f = Bs^3/b^3$, and the total cost per hour for running the ship is $C^* = Bs^3/b^3 + A$. If the length of the trip is D km, then the time to make the trip at speed s is D/s, and the total cost of the trip is

$$C = C(s) = (C^*)\left(\frac{D}{s}\right) = \frac{D}{s}\left(\frac{Bs^3}{b^3} + A\right) = D\left(\frac{Bs^2}{b^3} + \frac{A}{s}\right), \quad s > 0.$$

For critical points we solve $0 = C'(s) = D(2Bs/b^3 - A/s^2)$, the solution of which is $s = \left(Ab^3/(2B)\right)^{1/3}$ Since $\lim_{s\to 0^+} C(s) = \infty$ and $\lim_{s\to\infty} C(s) = \infty$, it follows that $C(s)$ is minimized for $s = \left(Ab^3/(2B)\right)^{1/3}$ km/hr.

54. The volume of the drinking cup is $V = \pi r^2 h/3$. Since $R^2 = r^2 + h^2$, we may write

$$V = V(h) = \frac{1}{3}\pi(R^2 - h^2)h.$$



To obtain the domain of this function, we relate θ and h by noting that the length of the arc joining A and B is the same as that of C':

$$R(2\pi - \theta) = 2\pi r = 2\pi\sqrt{R^2 - h^2}.$$

This equation implies that $h = 0$ when $\theta = 0$, and $h = R$ when $\theta = 2\pi$. The appropriate domain for $V(h)$ is therefore $0 \le h \le R$. For critical points of the function we solve

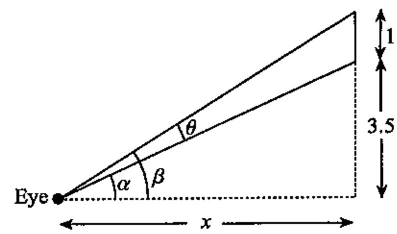
$$0 = V'(h) = \frac{1}{3}\pi(R^2 - 3h^2).$$

The positive solution is $h = R/\sqrt{3}$. Since $V(0) = 0 = V(R)$, it follows that V is maximized for $h = R/\sqrt{3}$. For this h, $R(2\pi - \theta) = 2\pi\sqrt{R^2 - R^2/3}$, and the solution of this equation is $\theta = 2\pi(1 - \sqrt{6}/3)$.

55. The letters appear tallest when the angle θ that they subtend at the eye is greatest. Since

$\theta = \beta - \alpha = \mathrm{Tan}^{-1}\left(\frac{9/2}{x}\right) - \mathrm{Tan}^{-1}\left(\frac{7/2}{x}\right)$,

$0 < x < \infty$, critical points of $\theta(x)$ are given by



$$0 = \frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{9}{2x}\right)^2}\left(-\frac{9}{2x^2}\right) - \frac{1}{1 + \left(\frac{7}{2x}\right)^2}\left(-\frac{7}{2x^2}\right) = \frac{-18}{4x^2 + 81} + \frac{14}{4x^2 + 49}.$$

Equivalently, $18(4x^2 + 49) = 14(4x^2 + 81)$, which has only one positive solution, $x = 3\sqrt{7}/2$. Since $\theta(x)$ approaches zero as x approaches zero and as x becomes very large, it follows that θ is maximized when the motorist is $3\sqrt{7}/2$ m from the sign.

56. The thrust to speed ratio is

$$g(v) = \frac{F}{v} = \frac{1}{2}\rho A v\left(0.000182 + \frac{4w^2}{6.5\pi\rho^2 A^2 v^4}\right) = \frac{1}{2}\rho A\left(0.000182v + \frac{4w^2}{6.5\pi\rho^2 A^2 v^3}\right).$$

For critical points of this function we solve

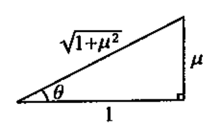
$$0 = g'(v) = \frac{1}{2}\rho A\left(0.000182 - \frac{12w^2}{6.5\pi\rho^2 A^2 v^4}\right) \qquad \Longrightarrow \qquad v = \left[\frac{12w^2}{0.000182(6.5)\pi\rho^2 A^2}\right]^{1/4}.$$

Since $g(v)$ becomes infinite as $v \to 0$ and $v \to \infty$, it follows that this value of v must minimize $g(v)$.
(a) At sea level, when $A = 1600$, $w = 150,000$, and $\rho = 0.0238$, the speed is 323 mph.
(b) At 30,000 feet, when $A = 1600$, $w = 150,000$, and $\rho = (0.375)(0.0238)$, the speed is 527 mph.

57. For critical points of $F(\theta)$ we solve

$0 = F'(\theta) = \dfrac{-\mu mg}{(\cos\theta + \mu\sin\theta)^2}(-\sin\theta + \mu\cos\theta).$

Thus, $0 = -\sin\theta + \mu\cos\theta$, or, $\tan\theta = \mu$, and this implies that $\theta = \mathrm{Tan}^{-1}\mu$. Since



$$F(0) = \mu mg, \qquad F(\mathrm{Tan}^{-1}\mu) = \frac{\mu mg}{\dfrac{1}{\sqrt{1+\mu^2}} + \dfrac{\mu^2}{\sqrt{1+\mu^2}}} = \frac{\mu mg}{\sqrt{1+\mu^2}}, \qquad F(\pi/2) = mg,$$

it follows that F is minimized for $\theta = \mathrm{Tan}^{-1}\mu$.

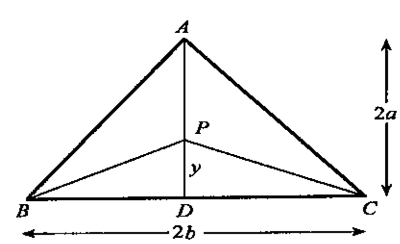
58. From the figure to the right, the sum of the distance from P to the vertices is

$L(y) = (2a - y) + 2\sqrt{b^2 + y^2}, \quad 0 \le y \le 2a.$

For critical points of this function, we solve

$0 = L'(y) = -1 + \dfrac{2y}{\sqrt{b^2 + y^2}}.$

The solution is $y = b/\sqrt{3}$. To show that this value minimizes $L(y)$, we evaluate



$$L(0) = 2a + 2b, \qquad L(b/\sqrt{3}) = 2a - \frac{b}{\sqrt{3}} + 2\sqrt{b^2 + \frac{b^2}{3}} = 2a + \sqrt{3}b, \qquad L(2a) = 2\sqrt{b^2 + 4a^2}.$$

Certainly the second of these is less than the first. Since the second and third are both positive, the second is less than the third if, and only if,

$$(2a + \sqrt{3}b)^2 < 4(b^2 + 4a^2) \qquad \Longleftrightarrow \qquad 4a^2 + 4\sqrt{3}ab + 3b^2 < 4b^2 + 16a^2.$$

But this is equivalent to

$$0 < 12a^2 - 4\sqrt{3}ab + b^2 = (2\sqrt{3}a - b)^2,$$

which is obviously true. Thus, $y = b/\sqrt{3}$ does indeed minimize L.

59. Suppose a trip of length d km is to be driven. Since the time for the trip at speed v is d/v, the wages earned by the driver are wd/v, and the cost for gas is $pd/f(v)$; that is, company costs are

$$C(v) = \frac{wd}{v} + \frac{pd}{a - bv}, \quad 80 \le v \le 100.$$

For critical points,

$$0 = C'(v) = -\frac{wd}{v^2} + \frac{bpd}{(a - bv)^2}.$$

This gives

$$\frac{bp}{(a - bv)^2} = \frac{w}{v^2} \quad \Longrightarrow \quad \left(\frac{a - bv}{v}\right)^2 = \frac{bp}{w} \quad \Longrightarrow \quad \frac{a - bv}{v} = \sqrt{\frac{bp}{w}} \quad \Longrightarrow \quad v = \frac{a}{b + \sqrt{bp/w}}.$$

This must minimize $C(v)$ since it is the only positive critical point and $C(v)$ becomes infinite as $v \to 0$ and $v \to a/b^-$.

60. The distance D from (x_1, y_1) to any point $P(x, y)$ on the line $Ax + By + C = 0$ is given by

$$\begin{aligned} D^2 &= (x - x_1)^2 + (y - y_1)^2 = (x - x_1)^2 + \left(-\frac{C}{B} - \frac{Ax}{B} - y_1\right)^2 \qquad \text{(provided } B \neq 0) \\ &= (x - x_1)^2 + \frac{(C + Ax + By_1)^2}{B^2}, \quad -\infty < x < \infty. \end{aligned}$$

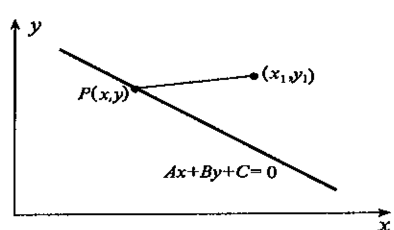
To minimize D we minimize D^2. For critical points of D^2 we solve

$$0 = 2(x - x_1) + 2\frac{(C + Ax + By_1)}{B^2}A.$$

The solution of this equation is

$$x = \frac{B^2 x_1 - AC - ABy_1}{A^2 + B^2}.$$

Since D^2 becomes infinite as $x \to \pm\infty$, it follows that D^2 is minimized for this value of x, and the minimum distance is

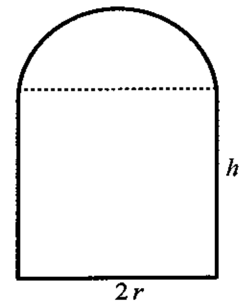


$$\begin{aligned} &\sqrt{\left(\frac{B^2x_1 - AC - ABy_1}{A^2 + B^2} - x_1\right)^2 + \frac{1}{B^2}\left(C + \frac{AB^2x_1 - A^2C - A^2By_1}{A^2 + B^2} + By_1\right)^2} \\ &= \sqrt{A^2\left(\frac{Ax_1 + By_1 + C}{A^2 + B^2}\right)^2 + B^2\left(\frac{Ax_1 + By_1 + C}{A^2 + B^2}\right)^2} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}. \end{aligned}$$

If $B = 0$, then the line $Ax + C = 0$ is vertical and the minimum distance is $|C/A + x_1|$. But this is predicted by the above formula when $B = 0$. Consequently, the formula is correct for any line whatsoever.

61. If k is the amount of light per unit area admitted by the clear glass, then the total amount of light admitted by the entire window is

$$L = k(2rh) + \frac{pk}{2}(\pi r^2) = \frac{k}{2}(4rh + \pi pr^2).$$



The total cost of the window plus frame is

$$A = a(2rh) + b(\pi r^2/2) + c(\pi r)$$

from which $h = \dfrac{A - \pi r^2 b/2 - \pi rc}{2ar}$. Thus,

$$L = L(r) = \frac{k}{2}\left[\frac{2}{a}\left(A - \frac{\pi r^2 b}{2} - \pi rc\right) + \pi pr^2\right].$$

Clearly, $r > 0$, and for $h \geq 0$, we must choose $A - \pi r^2 b/2 - \pi rc \geq 0$. This condition implies that $r \leq r_m = (-c + \sqrt{c^2 + 2Ab/\pi})/b$. For critical points of $L(r)$ we solve

$$0 = L'(r) = \frac{k}{2}\left(-\frac{2\pi br}{a} - \frac{2\pi c}{a} + 2\pi pr\right) \implies r = \frac{c}{ap - b}.$$

We now calculate that

$$\lim_{r \to 0^+} L(r) = \frac{kA}{a}, \quad L\left(\frac{c}{ap-b}\right) = \frac{kA}{a} - \frac{k\pi c^2}{2a(ap-b)}, \quad L(r_m) = \frac{k\pi p}{2b^2}\left(-c + \sqrt{c^2 + \frac{2Ab}{\pi}}\right)^2.$$

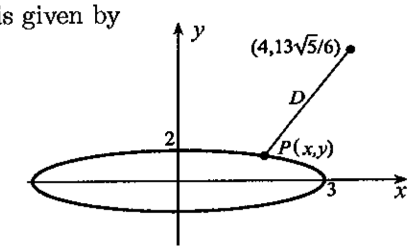
If $p < b/a$, then the critical point $c/(ap - b) < 0$, and is therefore inadmissible. If $p > b/a$, then $c/(ap-b) > 0$, but clearly in this case the limit of L as $r \to 0^+$ is greater than L evaluated at $c/(ap-b)$. Thus, in either case, maximum $L(r)$ is achieved at one of the ends of the interval. The difference between the values of L at the end points is

$$\begin{aligned}\lim_{r \to 0^+} L(r) - L(r_m) &= \frac{kA}{a} - \frac{k\pi p}{2b^2}\left(-c + \sqrt{c^2 + \frac{2Ab}{\pi}}\right)^2 \\ &= \frac{k}{b}\left[A\left(\frac{b}{a} - p\right) + \frac{c\pi p}{b}\left(\sqrt{c^2 + \frac{2Ab}{\pi}} - c\right)\right].\end{aligned}$$

If $p < b/a$, then this difference is positive, implying that r should be chosen as small as possible. When $p > b/a$, no conclusion can yet be drawn. This difference could be negative or positive.

62. The distance D from $(4, 13\sqrt{5}/6)$ to any point $P(x, y)$ on the ellipse is given by $D^2 = (x-4)^2 + \left(y - \dfrac{13\sqrt{5}}{6}\right)^2$, where x and y must satisfy $4x^2 + 9y^2 = 36$. For critical points of D^2, we solve

$$0 = 2(x-4) + 2\left(y - \frac{13\sqrt{5}}{6}\right)\frac{dy}{dx}.$$

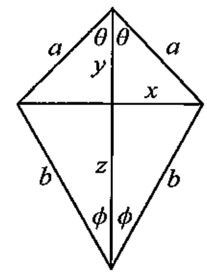


We obtain dy/dx by differentiating the equation of the ellipse, $8x + 18y\,dy/dx = 0$. We solve this for dy/dx and substitute into the equation defining critical points, $0 = x - 4 + \left(y - \dfrac{13\sqrt{5}}{6}\right)\left(-\dfrac{4x}{9y}\right)$. This equation can be solved for $y = \dfrac{26\sqrt{5}x}{3(36-5x)}$. When this is substituted into the equation of the ellipse,

$$36 = 4x^2 + 9\left[\frac{26\sqrt{5}x}{3(36-5x)}\right]^2,$$

and this equation simplifies to $(36-5x)^2(36-4x^2) - 3380x^2 = 0$. The only solution between 0 and 3 is $x = 2$. The corresponding y-coordinate of the point is $y = 2\sqrt{5}/3$. Since $D^2(0) = 24.09$, $D^2(2) = 15.25$, and $D^2(3) = 24.47$, the point $(2, 2\sqrt{5}/3)$ is indeed closest to $(4, 13\sqrt{5}/6)$.

63. The area of the kite is



$$\begin{aligned}A &= xy + xz\\ &= (a\sin\theta)(a\cos\theta) + (b\sin\phi)(b\cos\phi)\\ &= \frac{1}{2}(a^2\sin 2\theta + b^2\sin 2\phi).\end{aligned}$$

Angles θ and ϕ are related by the sine law,

$$\frac{\sin\theta}{b} = \frac{\sin\phi}{a}.$$

For critical points of A, we solve

$$0 = \frac{dA}{d\theta} = \frac{1}{2}\left(2a^2\cos 2\theta + 2b^2\cos 2\phi\frac{d\phi}{d\theta}\right), \qquad \text{where} \quad a\cos\theta = b\cos\phi\frac{d\phi}{d\theta}.$$

For critical points then

$$0 = a^2\cos 2\theta + b^2\cos 2\phi\left(\frac{a\cos\theta}{b\cos\phi}\right) \quad\Longrightarrow\quad \frac{a\cos 2\theta}{\cos\theta} = -\frac{b\cos 2\phi}{\cos\phi}.$$

If we divide this result by $a\sin\theta = b\sin\phi$, we obtain

$$\frac{a\cos 2\theta}{a\cos\theta\sin\theta} = -\frac{b\cos 2\phi}{b\cos\phi\sin\phi} \quad\Longrightarrow\quad \tan 2\theta = -\tan 2\phi.$$

Thus, $2\theta = -2\phi + n\pi$, or, $\theta = -\phi + n\pi/2$, where n is an integer. Since θ and ϕ are both acute angles, n must be 1, and substitution of this into the sine law gives $a\sin\theta = b\sin(\pi/2 - \theta) = b\cos\theta$. Finally, then, the critical point is the acute angle for which $\tan\theta = b/a$. When $\theta = 0$, ϕ must also be zero, and in this case $A = 0$. When $\theta = \pi/2$, $\sin\phi = a/b$, and

$$A = b^2\left(\frac{a}{b}\right)\sqrt{1 - a^2/b^2} = a\sqrt{b^2 - a^2}.$$

When $\tan\theta = b/a$, we obtain

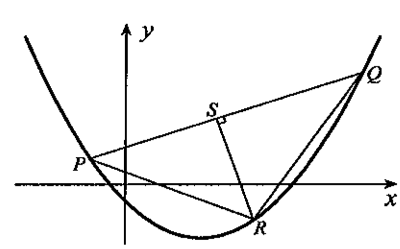
$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}},\ \cos\theta = \frac{a}{\sqrt{a^2+b^2}},\ \sin\phi = \frac{a}{\sqrt{a^2+b^2}},\ \cos\phi = \frac{b}{\sqrt{a^2+b^2}},$$

and

$$A = a^2\left(\frac{ab}{a^2+b^2}\right) + b^2\left(\frac{ab}{a^2+b^2}\right) = ab.$$

Since $ab > \sqrt{a^2 - b^2}$, it follows that A is maximized for θ defined by $\tan\theta = b/a$.

64. The area of triangle PQR is $F = \|RS\|\|PQ\|/2$. If we use formula 1.16 for length $\|RS\|$, then



$$F(x) = \frac{\|PQ\|}{2\sqrt{A^2+B^2}}|Ax + B(ax^2 + bx + c) + C|,$$

where $\|PQ\|$ is constant. For critical points of this function we solve

$$0 = F'(x) = \frac{\|PQ\|}{2\sqrt{A^2+B^2}}(A + 2Bax + bB)\frac{|Ax + B(ax^2+bx+c) + C|}{Ax + B(ax^2+bx+c) + C}$$

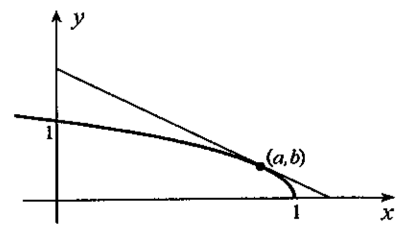
(see formula 3.13). The only solution of this equation is $x = -(A + bB)/(2aB)$. Now, there must be a value of x between P and Q which maximizes F since coordinates of these points lead to degenerate triangles with zero area. Hence, this critical point maximizes the area.

65. Since the slope of the tangent line to the curve at a point (a, b) is $-(1/2)/\sqrt{1-a}$, the equation of the tangent line at the point is

$$y - b = \frac{-1}{2\sqrt{1-a}}(x-a).$$

The x- and y-intercepts of this line are

$$x = 2b\sqrt{1-a} + a, \quad y = b + \frac{a}{2\sqrt{1-a}}.$$



The area of the triangle formed by the tangent line at (a, b) is therefore

$$A = \frac{1}{2}\left(b + \frac{a}{2\sqrt{1-a}}\right)(2b\sqrt{1-a}+a) = \frac{1}{2}\left(\sqrt{1-a} + \frac{a}{2\sqrt{1-a}}\right)[2(1-a)+a] = \frac{(2-a)^2}{4\sqrt{1-a}},$$

defined for $-\infty < a < 1$. For critical points of $A(a)$, we solve

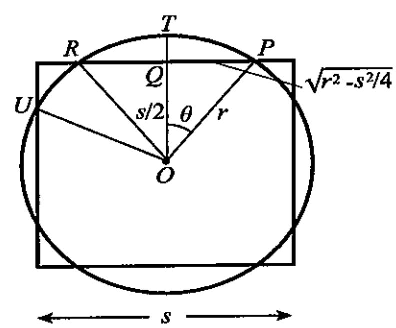
$$0 = \frac{dA}{da} = \frac{1}{4}\left[\frac{\sqrt{1-a}(2)(2-a)(-1) - (2-a)^2(1/2)(1-a)^{-1/2}(-1)}{1-a}\right]$$
$$= \frac{(2-a)[-4(1-a)+(2-a)]}{8(1-a)^{3/2}} = \frac{(2-a)(3a-2)}{8(1-a)^{3/2}}.$$

The only acceptable solution is $a = 2/3$. Since area becomes infinite as $a \to -\infty$ and $a \to 1^-$, it follows that area is minimized when $a = 2/3$. The minimum area is $A(2/3) = 4\sqrt{3}/9$.

66. Since $b > a + c$, irrigating land produces more profit than leaving it unirrigated. It follows that the length r of the arm should be at least $s/2$. When $r > s/2$, however, uncultivated land is being irrigated as well, with no return, and clearly r should be no larger than $s/\sqrt{2}$, the distance to the corner of the field. The profit for an arm of length r is

$$P = b(A_1) + a(A_2) - c(\pi r^2),$$

where A_1 is the area of irrigated land and A_2 is the area of unirrigated land.



Area A_1 is πr^2, less four times the area of the segment PRT of the circle,

$$A_1 = \pi r^2 - 4\left[\frac{1}{2}r^2(2\theta) - \frac{s}{2}\sqrt{r^2 - s^2/4}\right] = \pi r^2 - 4r^2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) + s\sqrt{4r^2 - s^2}.$$

Area A_2 is the area of the field, less four times the area of sector ORU, less eight times the area of triangle OQR,

$$A_2 = s^2 - 4\left[\frac{1}{2}r^2\left(\frac{\pi}{2} - 2\theta\right)\right] - 8\left[\frac{s}{4}\sqrt{r^2 - s^2/4}\right] = s^2 - 2r^2\left[\frac{\pi}{2} - 2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right)\right] - s\sqrt{4r^2 - s^2}$$
$$= s^2 - \pi r^2 + 4r^2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) - s\sqrt{4r^2 - s^2}.$$

Profit when the arm has length r is

$$P(r) = b\left[\pi r^2 - 4r^2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) + s\sqrt{4r^2 - s^2}\right] + a\left[s^2 - \pi r^2 + 4r^2\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) - s\sqrt{4r^2 - s^2}\right] - c(\pi r^2)$$
$$= \pi r^2(b - a - c) + as^2 + 4r^2(a - b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) + (b - a)s\sqrt{4r^2 - s^2},$$

defined for $s/2 \le r \le s/\sqrt{2}$. For critical points of the function we solve

$$
\begin{aligned}
0 = \frac{dP}{dr} &= 2\pi r(b-a-c) + 8r(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) - \frac{4r^2(a-b)}{\sqrt{1-s^2/(4r^2)}}\left(\frac{-s}{2r^2}\right) + \frac{(b-a)s(8r)}{2\sqrt{4r^2-s^2}} \\
&= 2\pi r(b-a-c) + 8r(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) + \frac{4rs(a-b)}{\sqrt{4r^2-s^2}} + \frac{4rs(b-a)}{\sqrt{4r^2-s^2}} \\
&= 2\pi r(b-a-c) + 8r(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right).
\end{aligned}
$$

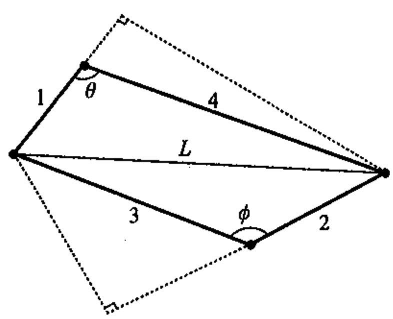
Since r cannot be zero, we set

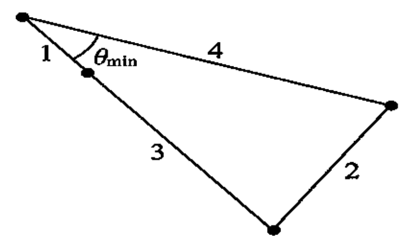
$$0 = \pi(b-a-c) + 4(a-b)\mathrm{Cos}^{-1}\left(\frac{s}{2r}\right) \qquad \Longrightarrow \qquad r = \frac{s}{2}\sec\left[\frac{\pi(b-a-c)}{4(b-a)}\right].$$

Since profit decreases as $r \to (s/2)^+$ and $r \to (s/\sqrt{2})^-$, it follows that this value of r must maximize profit.

67. If we divide the quadrilateral into two triangles as shown in the left figure below, the area of the quadrilateral is

$$A = \frac{1}{2}(1)[4\sin(\pi-\theta)] + \frac{1}{2}(2)[3\sin(\pi-\phi)] = 2\sin\theta + 3\sin\phi.$$





By the cosine law,

$$L^2 = 1^2 + 4^2 - 2(1)(4)\cos\theta = 2^2 + 3^2 - 2(2)(3)\cos\phi \implies 12\cos\phi - 8\cos\theta + 4 = 0 \implies \cos\phi = \frac{2\cos\theta - 1}{3}.$$

We can now express A in terms of θ,

$$A(\theta) = 2\sin\theta + 3\sqrt{1-\cos^2\phi} = 2\sin\theta + 3\sqrt{1-\left(\frac{2\cos\theta-1}{3}\right)^2} = 2\sin\theta + 2\sqrt{2+\cos\theta-\cos^2\theta}.$$

Because $1 + 4 = 5 = 2 + 3$, the maximum value for θ is π. The minimum value of θ is shown in the configuration in the right figure above. Using the cosine law, we find that $4 = 16 + 16 - 2(4)(4)\cos\theta_{\min}$, and this gives $\theta_{\min} = \mathrm{Cos}^{-1}(7/8)$. For critical points of $A(\theta)$, we solve

$$0 = \frac{dA}{d\theta} = 2\cos\theta + \frac{-\sin\theta + 2\sin\theta\cos\theta}{\sqrt{2+\cos\theta-\cos^2\theta}}.$$

When we transpose the first term and square the equation, we obtain

$$4\cos^2\theta(2+\cos\theta-\cos^2\theta) = \sin^2\theta(1-4\cos\theta+4\cos^2\theta).$$

When we replace $\sin^2\theta$ with $1-\cos^2\theta$, the equation simplies to

$$0 = 5\cos^2\theta + 4\cos\theta - 1 = (5\cos\theta-1)(\cos\theta+1).$$

Thus, $\theta = \pi$ or $\theta = \mathrm{Cos}^{-1}(1/5)$. Since

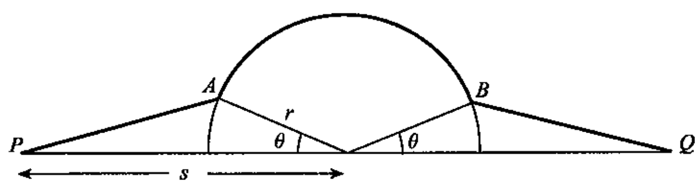
$$A(\mathrm{Cos}^{-1}(7/8)) = 2\sqrt{1-\frac{49}{64}} + 2\sqrt{2+\frac{7}{8}-\frac{49}{64}} = \sqrt{15}, \quad A(\mathrm{Cos}^{-1}(1/5)) = 2\sqrt{1-\frac{1}{25}} + 2\sqrt{2+\frac{1}{5}-\frac{1}{25}} = 2\sqrt{6},$$

and $A(\pi) = 2\sqrt{2-1-1} = 0$, it follows that maximum area is $2\sqrt{6}$.

68. If we take speed in the bush as v, then it is $2v$ on the beach. The time to walk from P to Q by the path in the figure below is

$$T = \frac{\text{Length of arc}AB}{2v} + \frac{2\|AP\|}{v} = \frac{r(\pi - 2\theta)}{2v} + \frac{2}{v}\sqrt{r^2+s^2-2rs\cos\theta}.$$

Obviously the smallest value of θ is 0. The largest value occurs when PA is tangent to the circle and this occurs when $\theta = \mathrm{Cos}^{-1}(r/s)$.



For critical values of $T(\theta)$, we solve

$$0 = \frac{dT}{d\theta} = -\frac{r}{v} + \frac{2}{v}\left(\frac{2rs\sin\theta}{2\sqrt{r^2+s^2-2rs\cos\theta}}\right).$$

When we transpose the first term and square both sides of the equation, we obtain

$$r^2+s^2-2rs\cos\theta = 4s^2\sin^2\theta = 4s^2(1-\cos^2\theta) \quad\Longrightarrow\quad 4s^2\cos^2\theta - 2rs\cos\theta + (r^2-3s^2) = 0.$$

Solutions of this quadratic equation are

$$\cos\theta = \frac{2rs \pm \sqrt{4r^2s^2 - 16s^2(r^2-3s^2)}}{8s^2} = \frac{r \pm \sqrt{12s^2-3r^2}}{4s}.$$

Since $\cos\theta$ must be positive, the only critical point is $\theta = \mathrm{Cos}^{-1}\left(\dfrac{r+\sqrt{12s^2-3r^2}}{4s}\right)$. We could show that this value of θ minimizes time by evaluating $T(\theta)$ at this value and at $\theta = 0$ and $\theta = \mathrm{Cos}^{-1}(7/8)$. It is difficult to see which is the smallest. As an alternative, we calculate the second derivative of $T(\theta)$,

$$\begin{aligned}
\frac{d^2T}{d\theta^2} &= \frac{2rs}{v}\left[\frac{\cos\theta}{\sqrt{r^2+s^2-2rs\cos\theta}} + \frac{(-1/2)\sin\theta(2rs\sin\theta)}{(r^2+s^2-2rs\cos\theta)^{3/2}}\right] \\
&= \frac{2rs}{v(r^2+s^2-2rs\cos\theta)^{3/2}}\left[(r^2+s^2-2rs\cos\theta)\cos\theta - rs\sin^2\theta\right] \\
&= \frac{2rs}{v(r^2+s^2-2rs\cos\theta)^{3/2}}\left[(r^2+s^2)\cos\theta - 2rs\cos^2\theta - rs(1-\cos^2\theta)\right] \\
&= \frac{2rs}{v(r^2+s^2-2rs\cos\theta)^{3/2}}\left[-rs\cos^2\theta + (r^2+s^2)\cos\theta - rs\right] \\
&= \frac{-2r^2s^2}{v(r^2+s^2-2rs\cos\theta)^{3/2}}\left(\cos\theta - \frac{r}{s}\right)\left(\cos\theta - \frac{s}{r}\right).
\end{aligned}$$

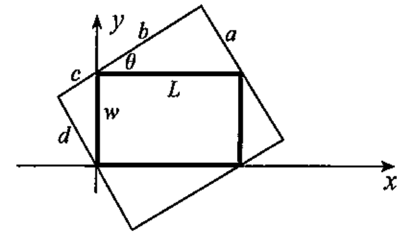
Since $r < s$, the last factor is always negative. In addition, because $\cos\theta > r/s$, the middle factor is positive. It follows that $d^2T/d\theta^2$ is always positive, and therefore a graph of $T(\theta)$ would be concave upward. The critical point must minimize travel time. You should therefore head to point P that creates the critical angle.

69. The area of the rectangle shown to the right is

$$A = Lw + ab + cd.$$

Since $a = L\sin\theta$, $b = L\cos\theta$, $c = w\sin\theta$, and $d = w\cos\theta$, we can express A in terms of θ,

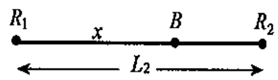
$$A = Lw + L^2\sin\theta\cos\theta + w^2\sin\theta\cos\theta$$
$$= Lw + \left(\frac{L^2}{2} + \frac{w^2}{2}\right)\sin 2\theta,\ 0 < \theta < \frac{\pi}{2}.$$

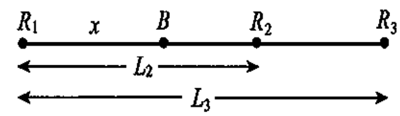


This function is a maximum when $\sin 2\theta = 1$. Hence, maximum area is

$$Lw + L^2/2 + w^2/2 = (L+w)^2/2.$$

70. Consider first the case when $n = 2$ (left figure below). If the blue stake is at a distance x from one of the red stakes, then the sum of the lengths of the two ropes is $L = x + (L_2 - x) = L_2$ (provided the blue stake is placed between the red ones). In this case then, the blue stake can be placed anywhere between the red one and L will always be the same.





Consider now the case when $n = 3$ (right figure above). If the blue stake is at a distance x from the left most red stake, then the total length of the three ropes is

$$L = L(x) = x + |x - L_2| + |x - L_3|, \quad 0 \le x \le L_3.$$

The derivative of this function is

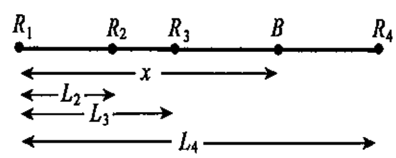
$$L'(x) = 1 + \frac{|x - L_2|}{x - L_2} + \frac{|x - L_3|}{x - L_3}$$

except at $x = L_2$ and $x = L_3$ where the derivative does not exist. Because the only values for the quotient terms are ± 1, it follows that at no point can $L'(x) = 0$. The only critical points are $x = L_2, L_3$. Since

$$L(0) = L_2 + L_3, \quad L(L_2) = L_2 + |L_2 - L_3| = L_3, \quad L(L_3) = L_3 + |L_3 - L_2|,$$

it follows that $L(x)$ is minimized when the blue stake is placed at the position of the intermediate stake. Now consider the case when $n = 4$.
If we proceed as in the $n = 3$ case, the total length of the four ropes is

$$L = L(x) = x + |x - L_2| + |x - L_3| + |x - L_4|, \quad 0 \le x \le L_4.$$



The derivative of this function (except at $x = L_2, L_3, L_4$) is

$$L'(x) = 1 + \frac{|x - L_2|}{x - L_2} + \frac{|x - L_3|}{x - L_3} + \frac{|x - L_4|}{x - L_4}.$$

This derivative is equal to zero for all x in the interval $L_2 < x < L_3$. Since

$$\begin{aligned}
L(0) &= L_2 + L_3 + L_4,\\
L(L_2) &= L_2 + |L_2 - L_3| + |L_2 - L_4| = L_4 + L_3 - L_2,\\
L(L_3) &= L_3 + |L_3 - L_2| + |L_3 - L_4| = L_4 + L_3 - L_2,\\
L(L_4) &= L_4 + |L_4 - L_2| + |L_4 - L_3| = L_4 + (2L_4 - L_3) - L_2,\\
L(L_2 < x < L_3) &= x + (x - L_2) + (L_3 - x) + (L_4 - x) = L_4 + L_3 - L_2,
\end{aligned}$$

it follows that the minimum value is $L_4 + L_3 - L_2$ and this is achieved at every position between R_2 and R_3, inclusively.

In general, for n red stakes at positions L_i,

$$L(x) = x + \sum_{i=2}^{n} |x - L_i|, \qquad 0 \le x \le L_n.$$

Critical points of this function are defined by

$$0 = L'(x) = 1 + \sum_{i=2}^{n} \frac{|x - L_i|}{x - L_i}.$$

If n is odd, it is impossible for $L'(x)$ to be equal to zero, and therefore $L(x)$ is minimized at one of the positions of the n red stakes. Case $n = 3$ suggests that the optimun stake is the middle one. If n is even, then $L'(x)$ is zero for all values of x in the middle interval. Cases $n = 2$ and $n = 4$ suggest that the value of $L(x)$ is the same for all values of x in the middle interval, and $L(x)$ has a larger value at any other red stake position. The optimum position is therefore anywhere in the middle interval.

71. (a) Since $L_1 = \|AE\| - \|DE\| = X - \|CE\| \cot\theta = X - Y\cot\theta$, and $L_2 = \|CE\| \csc\theta = Y\csc\theta$,

$$R = f(\theta) = \frac{k}{r_1^4}(X - Y\cot\theta) + \frac{k}{r_2^4} Y\csc\theta.$$

(b) For critical points of $f(\theta)$ we solve

$$0 = f'(\theta) = \frac{k}{r_1^4}(Y\csc^2\theta) - \frac{k}{r_2^4} Y\csc\theta\cot\theta = \frac{kY}{\sin^2\theta}\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right).$$

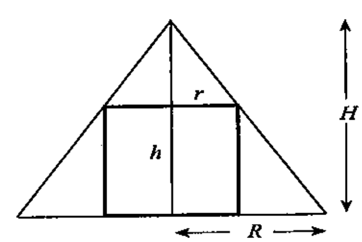
Thus, $\cos\theta = r_2^4/r_1^4$, and there is only one solution $\overline{\theta}$ of this equation in the range $0 < \theta < \pi/2$.
(c) Since $f'(\theta)$ changes from negative to positive as θ increases through $\overline{\theta}$, this critical point gives a relative minimum.
(d) When $f(\theta)$ is evaluated at $\overline{\theta}$,

$$f(\overline{\theta}) = \frac{kX}{r_1^4} + kY\left(\frac{\csc\overline{\theta}}{r_2^4} - \frac{\cot\overline{\theta}}{r_1^4}\right) = \frac{kX}{r_1^4} + kY\left(\frac{r_1^4}{r_2^4\sqrt{r_1^8 - r_2^8}} - \frac{r_2^4}{r_1^4\sqrt{r_1^8 - r_2^8}}\right)$$
$$= \frac{kX}{r_1^4} + kY\frac{\sqrt{r_1^8 - r_2^8}}{r_1^4 r_2^4} = \frac{kX}{r_1^4} + \frac{kY}{r_2^4}\sqrt{1 - \left(\frac{r_2}{r_1}\right)^8}.$$

Since $f(\pi/2) = kX/r_1^4 + kY/r_2^4$, it follows that $f(\overline{\theta}) < f(\pi/2)$.
(e) The smallest value of θ is defined by $\tan\theta = Y/X$. If we call this angle θ_m, then $\overline{\theta}$ yields an absolute minimum if $\theta_m < \overline{\theta}$. If, however, $\theta_m > \overline{\theta}$, then $f(\theta)$ is minimized for $\theta = \theta_m$.

72. If r and h are the radius and height of the cylinder (figure to the right), then its surface area is $A = 2\pi r^2 + 2\pi rh$. Similar triangles give $r/R = (H - h)/H$, from which $h = H(R - r)/R$. Thus,

$$A(r) = 2\pi r^2 + \frac{2\pi r H(R - r)}{R}, \quad 0 \le r \le R.$$



For critical points we solve

$$0 = \frac{dA}{dr} = 4\pi r + \frac{2\pi H}{R}(R - 2r) \qquad \Longrightarrow \qquad r = \frac{HR}{2H - 2R}.$$

This is positive only if $H > R$ so that we consider two cases. Case 1: $H > R$ In this case, we evaluate $A(0) = 0$, $A(R) = 2\pi R^2$, and

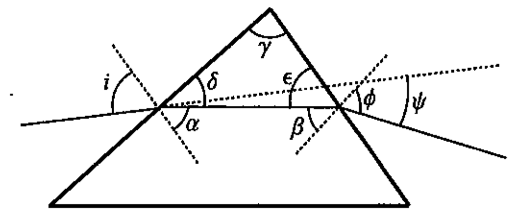
$$A\left(\frac{HR}{2H - 2R}\right) = 2\pi\left[\frac{H^2R^2}{4(H - R)^2}\right] + \frac{2\pi H}{R}\left[\frac{HR}{2(H - R)}\right]\left[R - \frac{HR}{2(H - R)}\right]$$
$$= \frac{\pi H^2R^2}{2(H - R)^2} + \frac{\pi H^2[2R(H - R) - HR]}{2(H - R)^2} = \frac{\pi H^2R^2 + \pi H^2(RH - 2R^2)}{2(H - R)^2}$$
$$= \frac{\pi RH^3 - \pi H^2R^2}{2(H - R)^2} = \frac{\pi H^2R}{2(H - R)}.$$

This will be maximum surface area if

$$0 < \frac{\pi H^2 R}{2(H-R)} - 2\pi R^2 = \frac{\pi R}{2(H-R)}[H^2 - 4R(H-R)] = \frac{\pi R(H-2R)^2}{2(H-R)}.$$

This is obviously true, and therefore surface area is maximized for the critical value of r.
Case 2: $H \le R$ In this case there is no critical point. Since $A(0) = 0$ and $A(R) = 2\pi R^2$, surface area is maximized when $r = R$. In this case, the cylinder has no height.

73. (a) If we introduce angles δ, ϵ, and γ in the figure to the right, then $\gamma + \delta + \epsilon = \pi$, $\alpha + \delta = \pi/2$, and $\beta + \epsilon = \pi/2$. From these,



$$\begin{aligned}\gamma &= \pi - (\delta + \epsilon)\\ &= \pi - (\pi/2 - \alpha + \pi/2 - \beta) = \alpha + \beta,\end{aligned}$$

and

$$\begin{aligned}\psi &= (i - \alpha) + (\phi - \beta)\\ &= i + \phi - (\alpha + \beta) = i + \phi - \gamma.\end{aligned}$$

Angle γ is known. It remains to find ϕ explicitly in terms of i. First,

$$\sin\phi = n\sin\beta = n\sin(\gamma - \alpha) = n(\sin\gamma\cos\alpha - \cos\gamma\sin\alpha).$$

Now, $\sin\alpha = (1/n)\sin i$, and because α is an acute angle,

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \frac{\sin^2 i}{n^2}}.$$

Thus,

$$\sin\phi = n\left(\sin\gamma\sqrt{1 - \frac{\sin^2 i}{n^2}} - \cos\gamma\frac{\sin i}{n}\right) = \sin\gamma\sqrt{n^2 - \sin^2 i} - \cos\gamma\,\sin i.$$

Since ϕ is between 0 and $\pi/2$, we may write that

$$\psi(i) = i - \gamma + \phi = i - \gamma + \mathrm{Sin}^{-1}\left(\sin\gamma\sqrt{n^2 - \sin^2 i} - \cos\gamma\,\sin i\right).$$

(b) Since $\psi = i + \phi - \gamma$, where $\sin\phi = \sin\gamma\sqrt{n^2 - \sin^2 i} - \cos\gamma\,\sin i$, differentiation gives $d\psi/di = 1 + d\phi/di$, where

$$\cos\phi\frac{d\phi}{di} = \frac{\sin\gamma}{2\sqrt{n^2 - \sin^2 i}}(-2\,\sin i\,\cos i) - \cos\gamma\,\cos i.$$

Thus,

$$\frac{d\psi}{di} = 1 + \frac{-\sin i\,\cos i\,\sin\gamma - \cos\gamma\,\cos i\,\sqrt{n^2 - \sin^2 i}}{\cos\phi\,\sqrt{n^2 - \sin^2 i}},$$

and for critical points of $\psi(i)$, we solve

$$\begin{aligned}0 &= \cos\phi\,\sqrt{n^2 - \sin^2 i} - \sin i\,\cos i\,\sin\gamma - \cos\gamma\,\cos i\,\sqrt{n^2 - \sin^2 i}\\ &= (\cos\phi - \cos\gamma\,\cos i)\frac{\sin\phi + \cos\gamma\,\sin i}{\sin\gamma} - \sin i\,\cos i\,\sin\gamma.\end{aligned}$$

From this equation,

$$\begin{aligned}0 &= \cos\phi\,\sin\phi - \cos\gamma\,\cos i\,\sin\phi + \cos\phi\,\cos\gamma\,\sin i - \cos^2\gamma\,\cos i\,\sin i - \sin i\,\cos i\,\sin^2\gamma\\ &= \frac{1}{2}\sin 2\phi + \cos\gamma(\sin i\,\cos\phi - \cos i\,\sin\phi) - \frac{1}{2}\sin 2i\end{aligned}$$

$$= \frac{1}{2}(\sin 2\phi - \sin 2i) + \cos\gamma \sin(i - \phi)$$
$$= \frac{1}{2}[2\cos(\phi + i)\sin(\phi - i)] + \cos\gamma\sin(i - \phi)$$
$$= \sin(\phi - i)[\cos(\phi + i) - \cos\gamma]$$
$$= \sin(\phi - i)\{-2\sin[(\phi + i + \gamma)/2]\sin[(\phi + i - \gamma)/2]\}.$$

To satisfy this equation, we must have one of the following situations:

$$i = \phi + n\pi; \qquad \phi + i + \gamma = 2n\pi; \qquad \phi + i - \gamma = 2n\pi.$$

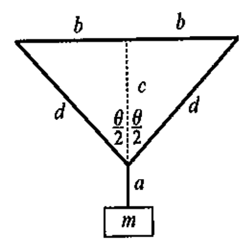
Since the last two are impossible, we conclude that the minimum value ψ_m for ψ occurs when $i = \phi$. For this value, $\psi_m = 2i - \gamma$, and

$$\sin i = \sin\gamma\sqrt{n^2 - \sin^2 i} - \cos\gamma \sin i.$$

When we solve this equation for n, the result is

$$n = \sqrt{\frac{2\sin^2 i(1 + \cos\gamma)}{\sin^2\gamma}} = \sqrt{\frac{2\sin^2 i[1 + 2\cos^2(\gamma/2) - 1]}{[2\sin(\gamma/2)\cos(\gamma/2)]^2}}$$
$$= \frac{\sin i}{\sin(\gamma/2)} = \frac{\sin[(\psi_m + \gamma)/2]}{\sin(\gamma/2)}.$$

74. The distance from m to the line through the rings is $L = a + c$. But, $c = b\cot(\theta/2)$. Furthermore, if k is the length of the rope, then $k = a + 2d + 2b$, or, $a = k - 2b - 2d = k - 2b - 2b\csc(\theta/2)$. Thus, $L = f(\theta) = k - 2b - 2b\csc(\theta/2) + b\cot(\theta/2)$. The minimum value for θ is $2\,\mathrm{Sin}^{-1}[2b/(k - 2b)]$ occurring when $a = 0$, and its maximum value is π. For critical points of $f(\theta)$, we solve



$$0 = f'(\theta) = b\csc(\theta/2)\cot(\theta/2) - \frac{b}{2}\csc^2(\theta/2) = \frac{b}{2}\csc(\theta/2)[2\cot(\theta/2) - \csc(\theta/2)].$$

This equation imples that $2\cos(\theta/2) = 1$, and therefore $\theta = 2\pi/3$. We now evaluate

$$f(2\pi/3) = k - 2b - 2b(2/\sqrt{3}) + b(1/\sqrt{3}) = k - (2 + \sqrt{3})b,$$
$$f(\pi) = k - 2b - 2b = k - 4b,$$
$$f\left[2\,\mathrm{Sin}^{-1}\left(\frac{2b}{k - 2b}\right)\right] = k - 2b - 2b\left(\frac{k - 2b}{2b}\right) + b\left(\frac{\sqrt{k^2 - 4kb}}{2b}\right) = \frac{\sqrt{k^2 - 4kb}}{2}.$$

Certainly $f(2\pi/3) > f(\pi)$. Furthermore, $f(2\pi/3)$ is greater than $f[2\,\mathrm{Sin}^{-1}[2b/(k - 2b)]]$ if and only if $[k - (2 + \sqrt{3})b]^2 > (1/4)(k^2 - 4kb)$, or,

$$0 < 4[k - (2 + \sqrt{3})b]^2 - (k^2 - 4kb) = 4[k^2 - 2(2 + \sqrt{3})bk + (7 + 4\sqrt{3})b^2] - k^2 + 4kb$$
$$= 3k^2 - 4(3 + 2\sqrt{3})bk + 4(7 + 4\sqrt{3})b^2 = [\sqrt{3}k - 2(2 + \sqrt{3})b]^2,$$

which is obviously valid. Thus, $\theta = 2\pi/3$ maximizes $f(\theta)$. Note, however, that this is the case provided the rope is sufficiently long that $\theta = 2\pi/3$ is indeed a possible configuration. This is true if a is greater than zero when $\theta = 2\pi/3$; that is, if

$$0 < k - 2b - 2b(2/\sqrt{3}) = k - 2b(1 + 2/\sqrt{3}) \quad\Longrightarrow\quad k > \frac{2}{\sqrt{3}}(\sqrt{3} + 2)b.$$

If k is less than this value, $f(\theta)$ is largest for the largest value of θ.

75. If O is the centre of the circular arc, then the area of the pasture is the area of sector $OBCD$ less the area of triangle OBD (left figure below),

$$A = \frac{1}{2}r^2(\theta) - [r\sin(\theta/2)][r\cos(\theta/2)] = \frac{r^2}{2}(\theta - \sin\theta).$$

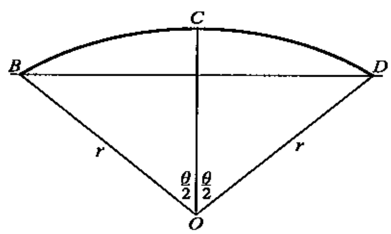
Since $1000 = r\theta$, it follows that $r = 1000/\theta$, and

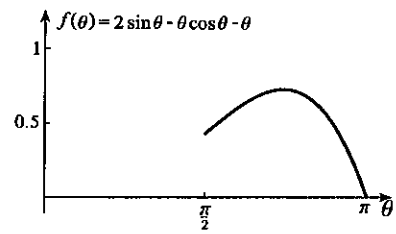
$$A(\theta) = \left(\frac{1000}{\theta}\right)^2\left(\frac{1}{2}\right)(\theta - \sin\theta), \quad \pi/2 \le \theta \le \pi.$$

For critical points of $A(\theta)$ we solve

$$0 = \frac{\theta^2(1-\cos\theta) - (\theta - \sin\theta)(2\theta)}{\theta^4} = \frac{-\theta - \theta\cos\theta + 2\sin\theta}{\theta^3}.$$

There is no solution to the equation obtained by setting the numerator equal to zero in the interval $\pi/2 < \theta < \pi$ (right figure below). Since $A(\pi/2) \approx 115\,668$ and $A(\pi) = 500\,000/\pi > 115\,668$, maximum area is $500\,000/\pi$ m^2 when it is a semicircle.





76. Triangle ABC in the left diagram below is right-angled at C and therefore $y^2 = x^2 + c^2$. Since triangle ADC is also right-angled, $c^2 = a^2 + (c - z)^2$, and this equation can be solved for $c = (a^2 + z^2)/(2z)$. Finally, from triangle BCE, $x^2 = z^2 + (a - x)^2$, and this equation can be solved for $x = (a^2 + z^2)/(2a)$. We may write therefore that

$$y^2 = \left(\frac{a^2 + z^2}{2a}\right)^2 + \left(\frac{a^2 + z^2}{2z}\right)^2 = (a^2 + z^2)^2\left(\frac{1}{4a^2} + \frac{1}{4z^2}\right) = \frac{(a^2 + z^2)^3}{4a^2z^2}.$$

Since a is constant, we have expressed y in terms of z. The smallest value for z occurs when the fold begins in the lower left corner (see the middle diagram below). In this case $b^2 = a^2 + (b - z)^2$, and this equation can be solved for $z = b - \sqrt{b^2 - a^2}$. The largest value for z is a shown in the right diagram. We therefore minimize the function

$$y^2 = f(z) = \frac{(z^2 + a^2)^3}{4a^2z^2}, \qquad b - \sqrt{b^2 - a^2} \le z \le a.$$

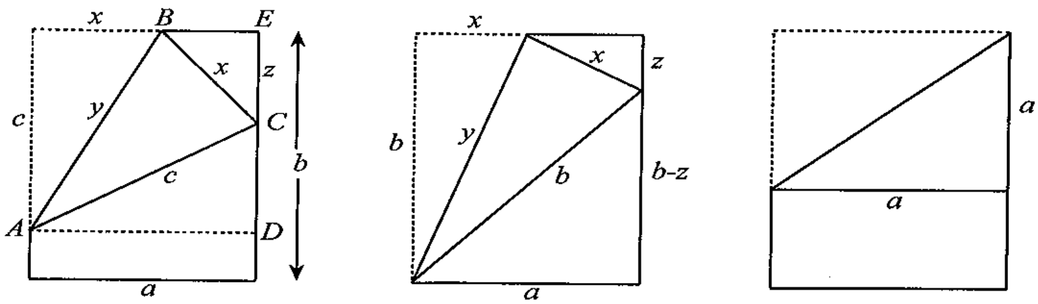
For critical points we solve

$$0 = f'(z) = \frac{1}{4a^2}\left[\frac{z^2(3)(z^2 + a^2)^2(2z) - (z^2 + a^2)^3(2z)}{z^4}\right] = \frac{(z^2 + a^2)^2(2z^2 - a^2)}{2a^2z^3}.$$

The only critical point is $z = a/\sqrt{2}$. We now calculate

$$f(a) = \frac{(a^2 + a^2)^3}{4a^2a^2} = 2a^2 \quad \text{and} \quad f\left(\frac{a}{\sqrt{2}}\right) = \frac{\left(\frac{a^2}{2} + a^2\right)^3}{4a^2\left(\frac{a^2}{2}\right)} = \frac{27a^2}{16}.$$

We should also calculate $f(z)$ at $z = b - \sqrt{b^2 - a^2}$, but the middle diagram makes it clear that $f(z)$ cannot have a minimum value in this situation. Hence the minimum length of the fold is $3\sqrt{3}a/4$ when $z = a/\sqrt{2}$ and $x = 3a/4$.



77. The area of triangle PQR is $\|PQ\|\|RS\|/2$ where $\|RS\| = \dfrac{|y - mx - c|}{\sqrt{1+m^2}}$. Thus,

$$A = \frac{\|PQ\|}{2\sqrt{1+m^2}}|y - mx - c|.$$

It is not necessary to divide the discussion into two parts depending on whether R is above or below the line and find intervals on which the functions are defined. It is clear that the maximum occurs at a critical point. To find critical points we solve

$$0 = \frac{dA}{dx} = \frac{\|PQ\|}{2\sqrt{1+m^2}}\frac{|y-mx-c|}{y-mx-c}\left(\frac{dy}{dx} - m\right).$$

Thus, we must have $dy/dx = m$ at critical points. Differentiation of the equation of the ellipse gives

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

When we combine this with $dy/dx = m$, we obtain $-b^2x/(a^2y) = m \Longrightarrow y = -b^2x/(a^2m)$. Substitution of this into the equation of the ellipse gives

$$\frac{x^2}{a^2} + \frac{1}{b^2}\left(\frac{b^4x^2}{a^4m^2}\right) = 1 \quad \Longrightarrow \quad x = \frac{\pm a^2m}{\sqrt{a^2m^2+b^2}} \quad \text{and} \quad y = \frac{\pm b^2}{\sqrt{a^2m^2+b^2}}.$$

The equation $y = -b^2x/(a^2m)$ requires x and y to have opposite signs when $m > 0$ and the same signs when $m < 0$. In either case, critical points are $\pm\left(\dfrac{a^2m}{\sqrt{a^2m^2+b^2}}, \dfrac{-b^2}{\sqrt{a^2m^2+b^2}}\right)$. The diagrams makes it clear that when $m > 0$, the sign of x should be chosen opposite to that of c, and when $m < 0$, the sign of x should be the same as that of c. If $c = 0$, either choice gives the same area.

78. This is essentially the same problem as Exercise 75.

EXERCISES 4.8

1. The velocity and acceleration are

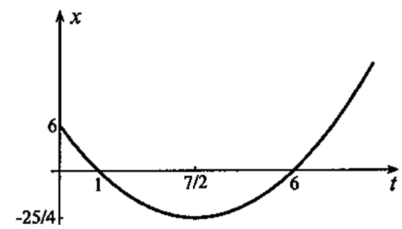
$$v(t) = 2 \text{ m/s},$$
$$a(t) = 0 \text{ m/s}^2.$$

The object begins 15 m to the right of the origin, and moves to the right with constant velocity 2 m/s forever.

2. The velocity and acceleration are

$$v(t) = 2t - 7 \text{ m/s},$$
$$a(t) = 2 \text{ m/s}^2.$$

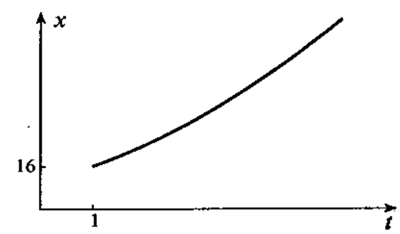
At time $t = 0$, the object is at $x = 6$ m and moving to the left with speed 7 m/s. It continues to move to the left until $t = 7/2$ s when it stops at $x = -25/4$ m. It then moves to the right with ever increasing speed.



3. The velocity and acceleration are

$$v(t) = 2t + 5 \text{ m/s},$$
$$a(t) = 2 \text{ m/s}^2.$$

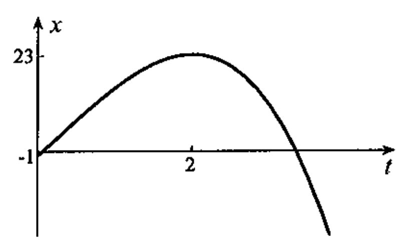
At time $t = 1$, the object is at $x = 16$ m and moving to the right with speed 7 m/s. It continues to move to the right with ever increasing speed.



4. The velocity and acceleration are

$$v(t) = -6t^2 + 4t + 16 = -2(3t + 4)(t - 2) \text{ m/s},$$
$$a(t) = -12t + 4 = -4(3t - 1) \text{ m/s}^2.$$

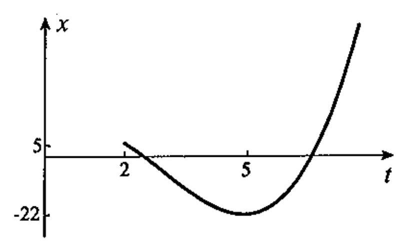
The object begins at time $t = 0$ at position $x = -1$ m with speed 16 m/s to the right. Its acceleration at this time is 4 m/s^2 so that it is picking up speed. At time $t = 1/3$ s, acceleration is zero and thereafter acceleration is negative. This means that its velocity decreases for $t \geq 1/3$ s. At time $t = 2$ s (and $x = 23$ m), the object's velocity is zero. Thereafter it moves to the left with increasing speed.



5. The velocity and acceleration are

$$v(t) = 3t^2 - 18t + 15 = 3(t - 1)(t - 5) \text{ m/s},$$
$$a(t) = 6t - 18 = 6(t - 3) \text{ m/s}^2.$$

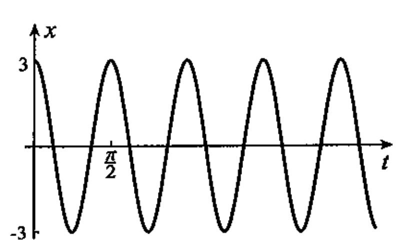
The object begins at time $t = 2$ at position $x = 5$ m with speed 9 m/s to the left. Its acceleration at this time is -6 m/s^2 so that it is speeding up. At time $t = 3$ s, acceleration is zero and thereafter acceleration is positive. This means that its velocity increases for $t \geq 3$ s. At time $t = 5$ s (and $x = -22$ m), the object's velocity is zero. Thereafter it moves to the right with increasing speed.



6. The velocity and acceleration are

$$v(t) = -12 \sin 4t \text{ m/s},$$
$$a(t) = -48 \cos 4t \text{ m/s}^2.$$

The object moves back and forth along the x-axis between $x = \pm 3$ m. Its velocity is zero in the turns, and its acceleration is equal to zero each time it passes through $x = 0$. The period of each oscillation is $\pi/2$ s.

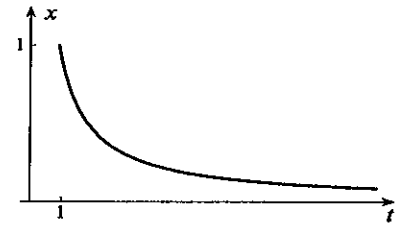


7. The velocity and acceleration are

$v(t) = -1/t^2$ m/s,

$a(t) = 2/t^3$ m/s^2.

The object is at position $x = 1$ m and moving to the left with speed 1 m/s at time $t = 1$. It continues to move to the left forever with ever decreasing speed gradually approaching the origin.

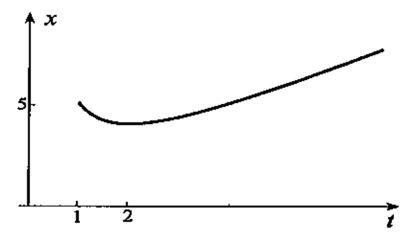


8. The velocity and acceleration are

$v(t) = 1 - 4/t^2$ m/s,

$a(t) = 8/t^3$ m/s^2.

At time $t = 1$, the object is at position $x = 5$ m and moving to the left with speed 3 m/s. Its acceleration is always to the right resulting in an instantaneous stop at $t = 2$ s at $x = 4$ m. The object then moves to the right thereafter with increasing speed. For large t, the velocity of the object approaches 1 m/s, and its acceleration approaches 0.

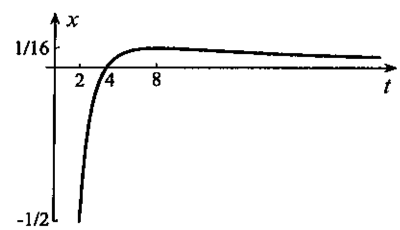


9. The velocity and acceleration are

$v(t) = -1/t^2 + 8/t^3 = (8-t)/t^3$ m/s,

$a(t) = 2/t^3 - 24/t^4 = 2(t-12)/t^4$ m/s^2.

At time $t = 2$, the object is at position $x = -1/2$ m and moving to the right with speed 3/4 m/s. Its acceleration is negative for $2 \le t \le 12$ resulting in an instantaneous stop at $t = 8$ s at $x = 1/16$ m. The object then moves to the left thereafter picking up speed until $t = 12$ s. For large $t > 12$, its speed decreases as it gradually approaches the origin.

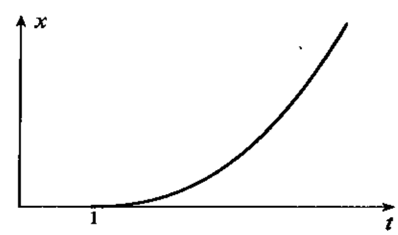


10. The velocity and acceleration are

$$v(t) = \frac{5}{2}t^{3/2} - 3\sqrt{t} + \frac{1}{2\sqrt{t}} = \frac{(5t-1)(t-1)}{2\sqrt{t}} \text{ m/s},$$

$$a(t) = \frac{15}{4}\sqrt{t} - \frac{3}{2\sqrt{t}} - \frac{1}{4t^{3/2}} = \frac{15t^2 - 6t - 1}{4t^{3/2}} \text{ m/s}^2.$$

The object starts at the origin with zero velocity, but with positive acceleration. It therefore moves to the right, continuing to do so forever with ever increasing speed.



11. A plot of the displacement function is shown to the right. We need its velocity and acceleration:

$v(t) = 3t^2 - 18t + 24 = 3(t-2)(t-4)$ m/s,

$a(t) = 6t - 18 = 6(t-3)$ m/s^2.

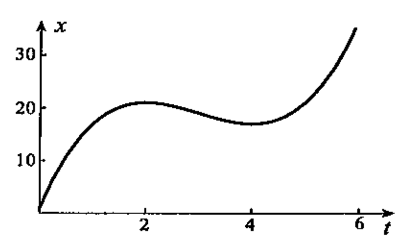
(a) Since $v(1) = 9$ and $a(1) = -12$, the object is slowing down, its speed is decreasing.

(b) Critical points of $v(t)$ occur when acceleration vanishes, namely, at $t = 3$. Since $v(0) = 24$, $v(3) = -3$, and $v(6) = 24$, maximum and minimum velocities are 24 m/s and -3 m/s.

(c) Maximum speed is 24 m/s and minimum speed is 0.

(d) Since acceleration is linear in t, maximum and minimum accelerations occur at $t = 0$ and $t = 6$. They are ± 18 m/s^2.

(e) Since $a'(t) = 6 > 0$, the acceleration is always increasing.



12. A plot of the displacement function is shown to the right. We need its velocity and acceleration:

$v(t) = -15 + 18t - 3t^2 = -3(t-1)(t-5)$ m/s,

$a(t) = 18 - 6t = -6(t-3)$ m/s^2.

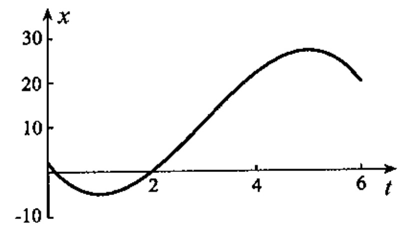
(a) Since $v(1) = 0$, the object is stopped at $t = 1$, it's speed is neither increasing nor decreasing.

(b) Critical points of $v(t)$ occur when acceleration vanishes, namely, at $t = 3$. Since $v(0) = -15$, $v(3) = 12$, and $v(6) = -15$, maximum and minimum velocities are 12 m/s and -15 m/s.

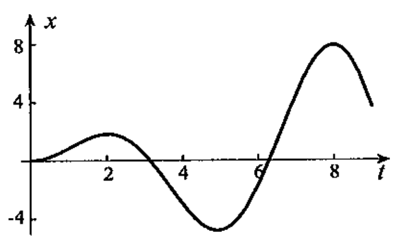
(c) Maximum speed is 15 m/s and minimum speed is 0.

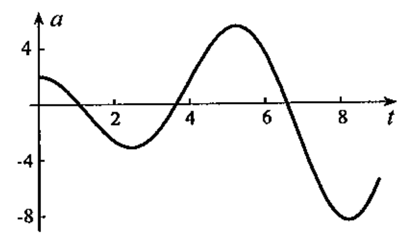
(d) Since acceleration is linear in t, maximum and minimum acceleration occur at $t = 0$ and $t = 6$. They are ± 18 m/s^2.

(e) Since $a'(t) = -6 < 0$, the acceleration is never increasing.



13. (a) A plot is shown in the left figure below.





(b) Velocity vanishes when $0 = v(t) = \sin t + t\cos t$. This occurs for $t = 0$ and three additional times. To find the next time, we use Newton's iterative procedure with $t_1 = 2$ and

$$t_{n+1} = t_n - \frac{\sin t_n + t_n \cos t_n}{2\cos t_n - t_n \sin t_n}.$$

Iteration gives $t_2 = 2.029$, $t_3 = 2.0288$, and $t_4 = 2.0288$. Since $v(2.0285) = 7.0 \times 10^{-4}$ and $v(2.0295) = -2.0 \times 10^{-3}$, to three decimals, the velocity is zero at $t = 2.029$ s. Similar procedures give the remaining times $t = 4.913$ s and $t = 7.979$ s.

(c) Acceleration is equal to 1 when $1 = a(t) = 2\cos t - t\sin t$. The right plot above for $a(t)$ indicates that this occurs three times. To find the first time, we use Newton's iterative procedure with $t_1 = 0.7$ and

$$t_{n+1} = t_n - \frac{2\cos t_n - t_n \sin t_n - 1}{-3\sin t_n - t_n \cos t_n}.$$

Iteration gives $t_2 = 0.7314$ and $t_3 = 0.7314$. Since $a(0.7305) - 1 = 2.3 \times 10^{-3}$ and $a(0.7315) - 1 = -2.9 \times 10^{-4}$, we can say that acceleration is 1 at $t = 0.731$ (accurate to three decimals). Similar procedures give the additional times $t = 3.853$ and $t = 6.436$.

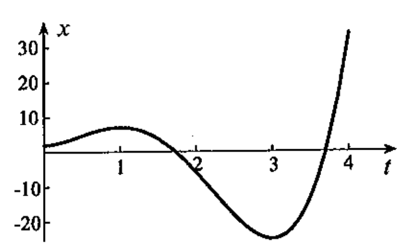
14. A plot of the displacement function is shown to the right. We need the velocity and acceleration functions:

$v(t) = 12t^3 - 48t^2 + 36t = 12t(t-1)(t-3)$ m/s,

$a(t) = 36t^2 - 96t + 36 = 12(3t^2 - 8t + 3)$ m/s^2.

(a) Acceleration is zero for

$$t = \frac{8 \pm \sqrt{64-36}}{6} = \frac{4 \pm \sqrt{7}}{3}.$$



Since $a(t) \ge 0$ for $0 \le t \le (4-\sqrt{7})/3$ and $(4+\sqrt{7})/3 \le t \le 4$, velocity is increasing on these intervals, and it is decreasing for $(4-\sqrt{7})/3 \le t \le (4+\sqrt{7})/3$.

(b) Speed is increasing on the intervals $0 \le t \le (4-\sqrt{7})/3$, $1 \le t \le (4+\sqrt{7})/3$, and $3 \le t \le 4$. It is decreasing for $(4-\sqrt{7})/3 \le t \le 1$ and $(4+\sqrt{7})/3 \le t \le 3$.

(c) Critical points of velocity occur where acceleration is zero, namely $t = (4 \pm \sqrt{7})/3$. Since

$$v(0) = 0, \quad v((4-\sqrt{7})/3) = \frac{8(-10+7\sqrt{7})}{9}, \quad v((4+\sqrt{7})/3) = \frac{-8(10+7\sqrt{7})}{9}, \quad v(4) = 144,$$

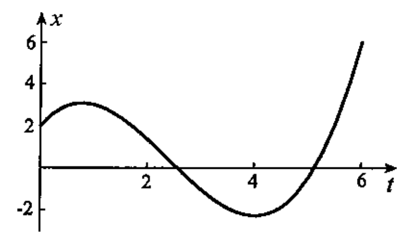
velocity is a maximum at $t = 4$ and a minimum at $t = (4+\sqrt{7})/3$.
(d) Speed is a maximum at $t = 4$ and a minimum at $t = 0, 1, 3$ where it is zero.
(e) The graph makes it clear that maximum distance from the origin is $x(4) = 34$.
(f) To answer this we should maximize $x(t) - 5$. Critical points of this function are $t = 1, 3$. Since

$$x(0) - 5 = -3, \quad x(1) - 5 = 2, \quad x(3) - 5 = -30, \quad x(4) - 5 = 29,$$

maximum distance from $x = 5$ is 30 at $t = 3$.

15. A plot of the displacement function is shown to the right. We need the velocity and acceleration functions:

$$v(t) = 14t^2/15 - 202t/45 + 132/45$$
$$= 2(21t^2 - 101t + 66)/45 \text{ m/s},$$
$$a(t) = 28t/15 - 202/45 \text{ m/s}^2.$$



(a) Acceleration is zero for $t = 101/42$, and velocity is decreasing for $0 \le t \le 101/42$, and increasing for $101/42 \le t \le 6$.
(b) Velocity is equal to zero when

$$t = \frac{101 \pm \sqrt{101^2 - 84(66)}}{42} = \frac{101 \pm \sqrt{4657}}{42}.$$

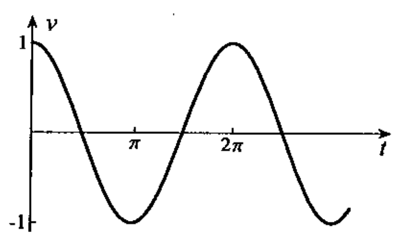
Speed is therefore increasing on the intervals $(101 - \sqrt{4657})/42 \le t \le 101/42$ and $(101 + \sqrt{4657})/42 \le t \le 6$. It is decreasing for $0 \le t \le (101 - \sqrt{4657})/42$ and $101/42 \le t \le (101 + \sqrt{4657})/42$.
(c) Critical points of velocity occur where acceleration is zero, namely $t = 101/42$. Since

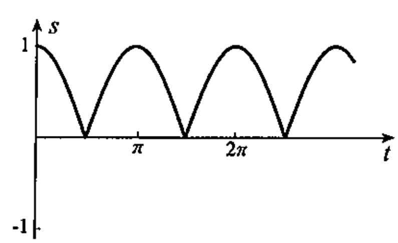
$$v(0) = \frac{132}{45}, \quad v(101/42) = -\frac{4657}{1890}, \quad v(6) = \frac{48}{5},$$

velocity is a maximum at $t = 6$ and a minimum at $t = 101/42$.
(d) Speed is a maximum at $t = 6$ and a minimum at $t = (101 \pm \sqrt{4657})/42$ when it is zero.
(e) The graph makes it clear that maximum distance from the origin is $x(6) = 6$.
(f) The graph makes it clear that maximum distance from $x = 5$ is $5 - x((101 + \sqrt{4657})/42) = 7.27$.

16. A horizontal point of inflection.

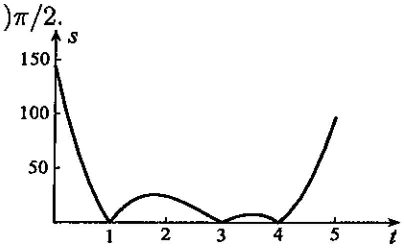
17. Not always. Suppose the displacement function is $x(t) = \sin t$, $t \ge 0$. Graphs of velocity and speed functions are shown below.





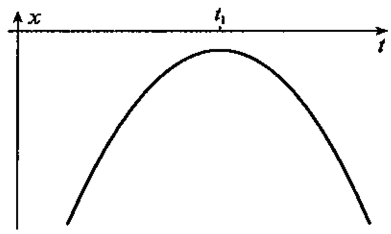
Critical points of the velocity graph are $t = n\pi$, where $n \ge 0$ is an integer. On the other hand, the speed graph has critical points at these points and also at $t = (2n+1)\pi/2$.

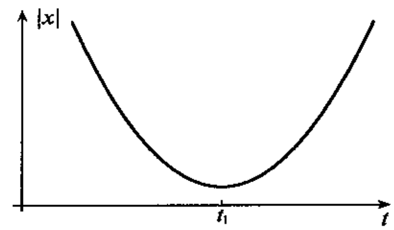
18. The speed graph to the right indicates maximum speed at $t = 0$ of $|v(0)| = 144$ m/s.



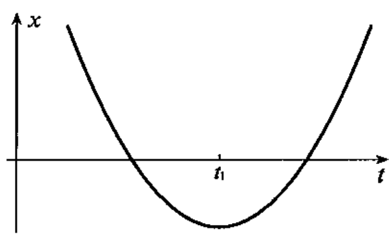
19. If we regard velocity as a function of position, and apply the chain rule to the situation, $v = v(x)$, $x = x(t)$, then $a = \dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = \dfrac{dv}{dx}v$.

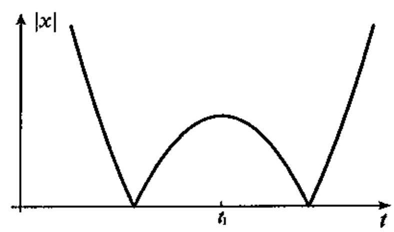
20. This is not always true. Consider the position function shown to the left below together with its absolute value to the right. Position has relative maximum at t_1, whereas its absolute value has relative maximum at t_2.



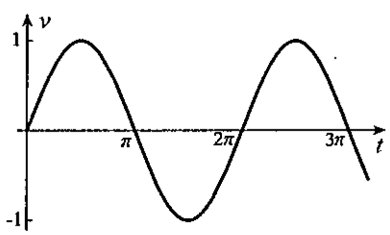


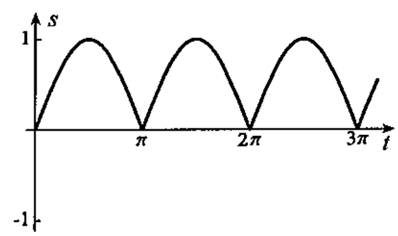
21. This is not always true. Consider the position function shown to the left below together with its absolute value to the right. Position has relative minimum at t_1, whereas its absolute value has relative maximum at t_1.



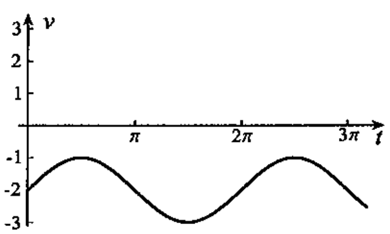


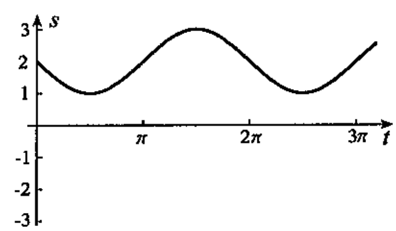
22. This is not always true. Consider the velocity function $v(t) = \sin t$, $t \geq 0$. Graphs of the velocity and speed functions are shown below. Velocity has relative minima at $t = 3\pi/2 + 2n\pi$, where n is a nonnegative integer, whereas speed has relative maxima at these times.





23. This is not always true. Consider the velocity function $v(t) = -2 + \sin t$, $t \geq 0$. Graphs of the velocity and speed functions are shown below. Velocity has relative maxima at $t = \pi/2 + 2n\pi$, where n is a nonnegative integer, whereas speed has relative minima at these times.





24. (a) Since ϕ lies in the interval $-\pi/2 < \phi < \pi/2$, we can solve $e = L\sin\phi - r\sin\theta$ for

$$\sin\phi = \frac{e + r\sin\theta}{L} \qquad \Longrightarrow \qquad \cos\phi = \sqrt{1 - \left(\frac{e + r\sin\theta}{L}\right)^2}.$$

Hence,

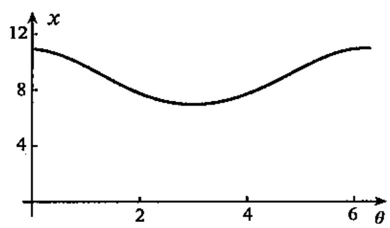
$$x = r\cos\theta + L\sqrt{1 - \left(\frac{e + r\sin\theta}{L}\right)^2} = r\cos\theta + \sqrt{L^2 - (e + r\sin\theta)^2}.$$

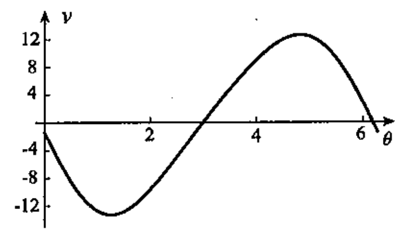
(b) When $L = 9$, $r = 2$, and $e = 1$,

$$x = 2\cos\theta + \sqrt{81 - (1 + 2\sin\theta)^2},$$

a plot of which is shown in the left figure below.
(c) Since the length of the stroke is the difference between maximum and minimum values of x, the graph suggests a stroke length of about 4 cm. The formula in Example 4.28 gives the stroke length

$$s = \sqrt{(9+2)^2 - 1^2} - \sqrt{(9-2)^2 - 1^2} = 2\sqrt{30} - 4\sqrt{3} = 4.03.$$





(d) Differentiation of x with respect to t gives

$$v = \frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = \omega\left[-r\sin\theta + \frac{-2(e + r\sin\theta)r\cos\theta}{2\sqrt{L^2 - (e + r\sin\theta)^2}}\right]$$
$$= -\omega r\left[\sin\theta + \frac{\cos\theta(e + r\sin\theta)}{\sqrt{L^2 - (e + r\sin\theta)^2}}\right].$$

If we substitute for $\sin\phi$ and $\cos\phi$ in the velocity formula of Example 4.31,

$$v = \frac{-\omega r(\sin\theta\cos\phi + \cos\theta\sin\phi)}{\cos\phi} = -\omega r\left[\sin\theta + \frac{\cos\theta\left(\dfrac{e + r\sin\theta}{L}\right)}{\sqrt{1 - \left(\dfrac{e + r\sin\theta}{L}\right)^2}}\right]$$
$$= -\omega r\left[\sin\theta + \frac{\cos\theta(e + r\sin\theta)}{\sqrt{L^2 - (e + r\sin\theta)^2}}\right],$$

the above result.
(e) With $\omega = 2\pi$, $L = 9$, $r = 2$, and $e = 1$,

$$v = -4\pi\left[\sin\theta + \frac{\cos\theta(1 + 2\sin\theta)}{\sqrt{81 - (1 + 2\sin\theta)^2}}\right],$$

a plot of which is shown in the right figure above. The velocity would appear to be zero when the displacement graph is at its highest and lowest points. These are the inner and outer dead positions in Example 4.28. Maximum and minimum velocities are approximately ± 13 cm/s.

25. If we let $y = f(x) = ax^3 + bx^2 + cx + d$ denote the cubic polynomial for the landing curve (left figure below), then d must be zero for the curve to pass through $(0, 0)$. For a smooth touchdown, we also demand that

$$0 = f'(0) = (3ax^2 + 2bx + c)_{|x=0} = c.$$

Suppose we denote horizontal distance from $x = 0$ to the point at which descent commences by L. Then descent begins at the point (L, h), and

$$h = f(L) = aL^3 + bL^2 \qquad \text{and} \qquad 0 = f'(L) = 3aL^2 + 2bL.$$

These equations can be solved for $a = -2h/L^3$ and $b = 3h/L^2$, so that the glide path has equation

$$y = f(x) = -\frac{2hx^3}{L^3} + \frac{3hx^2}{L^2}.$$

Since a constant horizontal speed U must be maintained, we can say that $dx/dt = -U$ (the negative sign is necessary because $dx/dt < 0$). It follows that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \left(-\frac{6hx^2}{L^3} + \frac{6hx}{L^2}\right)(-U).$$

Vertical acceleration is therefore

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dx}{dt} = \left(\frac{-12hx}{L^3} + \frac{6h}{L^2}\right)(-U)^2 = \frac{6hU^2}{L^2}\left(1 - \frac{2x}{L}\right).$$

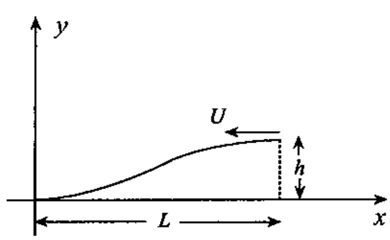
The right figure below shows that acceleration is positive for $0 < x < L/2$ and negative for $L/2 < x < L$. In addition, in magnitude, it is largest at $x = 0$ and $x = L$, where

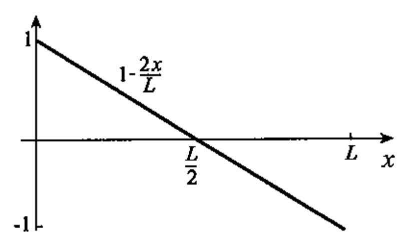
$$\left|\frac{d^2y}{dt^2}\right| = \frac{6hU^2}{L^2}.$$

Consequently, we must have

$$\frac{6hU^2}{L^2} \le k \qquad \Longrightarrow \qquad L \ge \sqrt{\frac{6hU^2}{k}};$$

that is, the plane must begin descent when it is at least $\sqrt{6hU^2/k}$ metres from touchdown.•



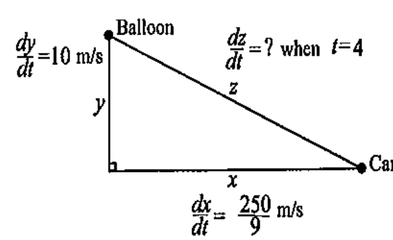


EXERCISES 4.9

1. The distance z between the balloon and the car is given by $z^2 = x^2 + y^2$. Differentiation of this equation with respect to time t gives

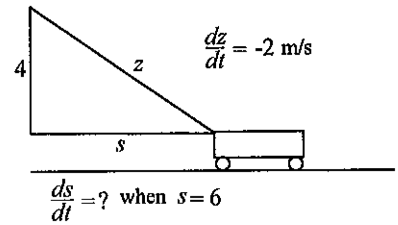
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

When $t = 4$, we know that $x = 1000/9$, $y = 40$, and $z = \sqrt{(1000/9)^2 + (40)^2} = (40/9)\sqrt{706}$. Consequently, at this time



$$\frac{40}{9}\sqrt{706}\frac{dz}{dt} = \frac{1000}{9}\left(\frac{250}{9}\right) + 40(10) \quad \Longrightarrow \quad \frac{dz}{dt} = \frac{9}{40\sqrt{706}}\left(\frac{250\,000}{81} + 400\right) = 29.5.$$

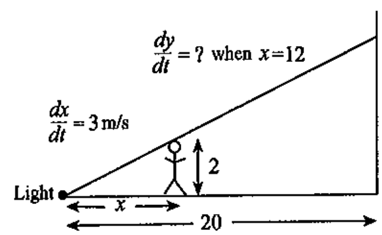
The balloon and child are separating at 29.5 m/s.



2. For the right-angled triangle, we may write $z^2 = s^2 + 16$. Differentiation with respect to time gives $2z\frac{dz}{dt} = 2s\frac{ds}{dt}$. When $s = 6$, we find that $z = 2\sqrt{13}$, and

$$(2\sqrt{13})(-2) = (6)\frac{ds}{dt}.$$

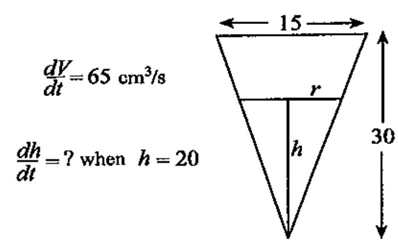
Thus, $ds/dt = -2\sqrt{13}/3$. The cart is therefore moving to the left at $2\sqrt{13}/3$ m/s.



3. By similar triangles, $y/20 = 2/x \implies y = 40/x$. If we differentiate this with respect to time t, $\frac{dy}{dt} = -\frac{40}{x^2}\frac{dx}{dt}$. When $x = 12$, we obtain

$$\frac{dy}{dt} = -\frac{40}{144}(3) = -\frac{5}{6}.$$

The length of the shadow is therefore decreasing at the rate of 5/6 m/s.



4. When the depth of liquid is h, the volume of liquid in the funnel is $V = \frac{1}{3}\pi r^2 h$. Similar triangles give $\frac{r}{h} = \frac{15/2}{30}$. Thus, $r = h/4$, and

$$V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3.$$

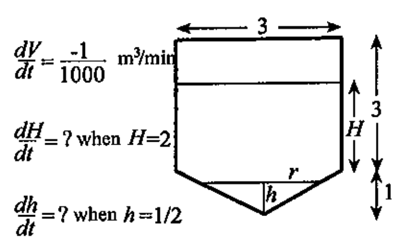
Differentiation with respect to time gives

$$\frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt}.$$

Since the net rate of change of the volume of liquid in the funnel is 65 cm^3/s, we can say that when $h = 20$,

$$65 = \frac{\pi}{16}(20)^2\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{65 \cdot 16}{400\pi} = \frac{13}{5\pi}.$$

The liquid level is therefore rising at $13/(5\pi)$ cm/s.



5. (a) When the water level is in the cylinder, the volume of water in the tank is $V = V_{\text{cone}} + \pi(3/2)^2 H$ where V_{cone}, the volume of water in the cone is constant. Differentiation with respect to time gives $\frac{dV}{dt} = \frac{9\pi}{4}\frac{dH}{dt}$. Consequently, $\frac{dH}{dt} = \frac{4}{9\pi}\left(\frac{-1}{1000}\right) = \frac{-1}{2250\pi}$. The water level is dropping at $1/(2250\pi)$ metres per minute.

(b) When the water level is in the cone, the volume of water in the tank is $V = (1/3)\pi r^2 h$. By similar triangles, $r/h = (3/2)/1 \implies r = 3h/2$. Hence, $V = (1/3)\pi(3h/2)^2 h = 3\pi h^3/4$. When we differentiate this equation with respect to time, $dV/dt = (9\pi h^2/4)dh/dt$, and when $h = 1/2$, we obtain

$$-\frac{1}{1000} = \frac{9\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{16}{9\pi}\left(\frac{-1}{1000}\right) = \frac{-2}{1125\pi}.$$

The water level is dropping at $2/(1125\pi)$ metres per minute.

6. The area of the triangle is

$$A = \frac{1}{2}xy = \frac{1}{2}x(x^2 + x + 4).$$

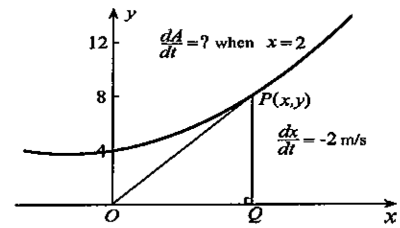
If we differentiate with respect to time,

$$\frac{dA}{dt} = \frac{1}{2}\left(3x^2\frac{dx}{dt} + 2x\frac{dx}{dt} + 4\frac{dx}{dt}\right) = \frac{1}{2}(3x^2 + 2x + 4)\frac{dx}{dt}.$$

When $x = 2$, we obtain

$$\frac{dA}{dt} = \frac{1}{2}[3(2)^2 + 2(2) + 4](-2) = -20.$$

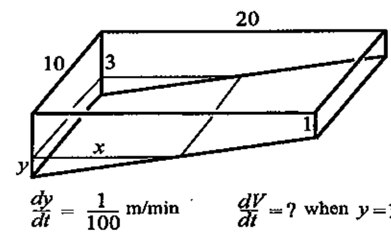
The area is therefore decreasing at 20 m^2/s.



7. The volume of water in the pool is given by $V = (1/2)xy(10) = 5xy$. Similar triangles gives $y/x = 2/20 \Longrightarrow x = 10y$. Hence, $V = 5y(10y) = 50y^2$. Differentiation with respect to time gives $dV/dt = (100y)dy/dt$. Consequently, when $y = 1$,

$$\frac{dV}{dt} = 100(1)\left(\frac{1}{100}\right) = 1.$$

The pool is being filled at 1 cubic metre per minute.



8. If P and V are pressure and volume of the gas, Boyle's law states that $P = k/V$, where k is a constant. Differentiation with respect to time gives

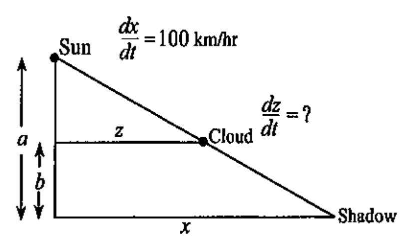
$$\frac{dP}{dt} = -\frac{k}{V^2}\frac{dV}{dt}.$$

Since $V = 1/100$ m^3 when $P = 50$ N/m^2, it follows that $50 = k(100)$, which implies that $k = 1/2$. At this instant,

$$\frac{dP}{dt} = -\frac{1/2}{(1/100)^2}\left(\frac{1}{2000}\right) = -2.5.$$

Pressure is decreasing at 2.5 $N/m^2/s$.

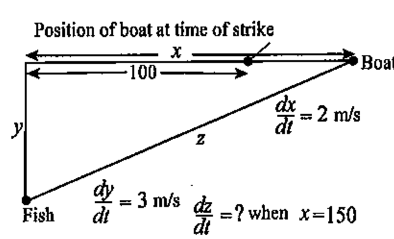
9. If b is the height of the clouds and a is the height of the sun, then similar triangles give $\dfrac{z}{a-b} = \dfrac{x}{a}$, from which $z = \left(\dfrac{a-b}{a}\right)x$. Differentiation with respect to time gives $\dfrac{dz}{dt} = \left(\dfrac{a-b}{a}\right)\dfrac{dx}{dt}$. Since b is so much smaller than a, we can say that $a - b \approx a$, and therefore $dz/dt \approx dx/dt = 100$ km/hr.



10. The amount of line between reel and fish is $z^2 = x^2 + y^2$. Differentiation of this equation with respect to time gives $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$. It takes 25 s for the boat to travel the 50 m from the instant the fish struck, and during this time the fish dives 75 m. The amount of line line between fish and reel at this instant is $\sqrt{150^2 + 75^2} = 75\sqrt{5}$. Consequently,

$$75\sqrt{5}\frac{dz}{dt} = (150)(2) + 75(3) \quad \Longrightarrow \quad \frac{dz}{dt} = \frac{525}{75\sqrt{5}} = \frac{7}{\sqrt{5}}.$$

The line is therefore being played out at $7/\sqrt{5}$ m/s.



11. Differentiation of $PV^{7/5} = k$ with respect to time gives

$$V^{7/5}\frac{dP}{dt} + \frac{7}{5}PV^{2/5}\frac{dV}{dt} = 0.$$

At the instant in question, $V = 1/10$ m^3, $P = 4 \times 10^5$ N/m^2, and $dV/dt = -1/1000$ m^3/s, so that

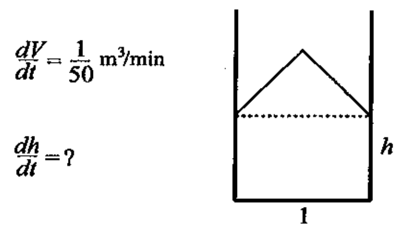
$$\left(\frac{1}{10}\right)^{7/5}\frac{dP}{dt} + \frac{7}{5}(4 \times 10^5)\left(\frac{1}{10}\right)^{2/5}\left(\frac{-1}{1000}\right) = 0 \quad \Longrightarrow \quad \frac{dP}{dt} = 5600.$$

The pressure is increasing at 5600 N/m^2/s.

12. (a) When sand completely covers the bottom of the cylinder, the volume of sand is

$$V = C + \pi\left(\frac{1}{2}\right)^2 h = C + \frac{\pi h}{4},$$

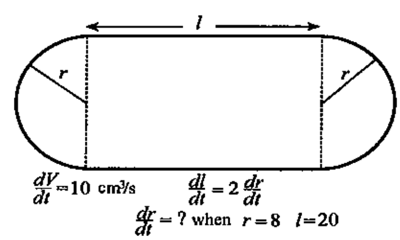
where C is the volume of sand in the cone. Since C is constant, differentiation of this equation with respect to time gives $\dfrac{dV}{dt} = \dfrac{\pi}{4}\dfrac{dh}{dt}$.



Since $dV/dt = 1/50$, it follows that $dh/dt = 2/(25\pi)$, and the top of the pile is rising at $2/(25\pi)$ m/min.
(b) Since the height of the cone is constant, sand also travels along the side of the cylinder at $2/(25\pi)$ m/min.

13. The volume of the balloon is $V = \dfrac{4}{3}\pi r^3 + \pi r^2 l$.
Differentiation with respect to time gives

$$\begin{aligned}\frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt} + 2\pi r l\frac{dr}{dt} + \pi r^2\frac{dl}{dt}\\ &= 4\pi r^2\frac{dr}{dt} + 2\pi r l\frac{dr}{dt} + 2\pi r^2\frac{dr}{dt}\\ &= (6\pi r^2 + 2\pi r l)\frac{dr}{dt}.\end{aligned}$$



When $r = 8$ and $l = 20$, $\quad 10 = [6\pi(64) + 2\pi(8)(20)]\dfrac{dr}{dt} \implies \dfrac{dr}{dt} = \dfrac{5}{352\pi}$. The radius is increasing at $5/(352\pi)$ cm/s.

14. (a) When Car 2 is on that part of the racetrack between D and B, its x- and y-coordinates must satisfy the equation $(x-50)^2 + y^2 = 2500$, the equation of a circle of radius 50 centred at the point $(50, 0)$. If we differentiate this equation with respect to time,

$$2(x-50)\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Longrightarrow \quad \frac{dy}{dt} = -\frac{x-50}{y}\frac{dx}{dt}.$$

Since the rate of change of the x-coordinate of Car 2 is the same as that of Car 1, namely 10 m/s, it follows that $dx/dt = 10$, and

$$\frac{dy}{dt} = \frac{10(50-x)}{y} \text{ m/s}.$$

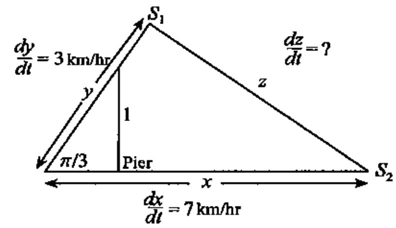
(b) When the car is at E, its x-coordinate is 75 and its y-coordinate is $-\sqrt{2500 - (75-50)^2} = -25\sqrt{3}$. At this point,

$$\frac{dy}{dt} = \frac{10(50-75)}{-25\sqrt{3}} = \frac{10}{\sqrt{3}} \text{ m/s}.$$

(c) As Car 2 approaches B, its x-coordinate approaches 100 and its y-coordinate approaches 0. This means that the rate of change of its y-coordinate is $\displaystyle\lim_{x\to 100}\frac{10(50-x)}{y} = \infty$. Car 1 therefore suffers the most damage.

15. When the ships are at the positions S_1 and S_2, the distance z between them is given by the cosine law,

$$z^2 = x^2 + y^2 - 2xy\cos(\pi/3) = x^2 + y^2 - xy.$$



Differentiation with respect to time gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - y\frac{dx}{dt} - x\frac{dy}{dt}.$$

At the instant in question, $x = 3/4 + \cot(\pi/3) = 3/4 + 1/\sqrt{3}$, $y = \csc(\pi/3) = 2/\sqrt{3}$, and $z = \sqrt{(3/4 + 1/\sqrt{3})^2 + (2/\sqrt{3})^2 - (3/4 + 1/\sqrt{3})(2/\sqrt{3})} = 5/4$.
At this instant,

$$2\left(\frac{5}{4}\right)\frac{dz}{dt} = 2\left(\frac{3}{4} + \frac{1}{\sqrt{3}}\right)(7) + 2\left(\frac{2}{\sqrt{3}}\right)(3) - \left(\frac{2}{\sqrt{3}}\right)(7) - \left(\frac{3}{4} + \frac{1}{\sqrt{3}}\right)(3) \implies \frac{dz}{dt} = \frac{3(11 + 4\sqrt{3})}{10}.$$

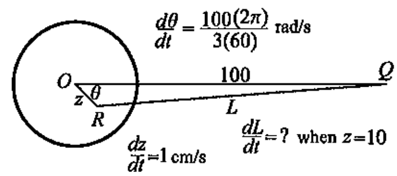
The ships are separating at $3(11 + 4\sqrt{3})/10$ km/hr.

16. The cosine law applied to triangle ORQ gives $L^2 = z^2 + 100^2 - 200z\cos\theta$. Differentiation of this equation with respect to time gives

$$2L\frac{dL}{dt} = 2z\frac{dz}{dt} - 200\cos\theta\frac{dz}{dt} + 200z\sin\theta\frac{d\theta}{dt}.$$

When $z = 10$ and $\theta = \pi/4$, we obtain

$$L = \sqrt{100 + 100^2 - 200(10)(1/\sqrt{2})} = 93.197567.$$

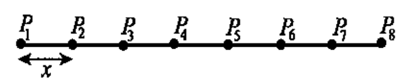


Substitution of these values into the equation involving dL/dt gives

$$2(93.197567)\frac{dL}{dt} = 2(10)(1) - 200\left(\frac{1}{\sqrt{2}}\right)(1) + 200(10)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{10\pi}{9}\right),$$

and this equation can be solved for $dL/dt = 25.83$. The distance is therefore increasing at 25.83 cm/s.

17. We assume that the first person P_1 on the whip stays on the same spot and the distance x between skaters is constant. The fourth person P_4 travels on a circle of radius $3x$, and therefore distance travelled by this skater along the arc of the circle is given by $3x\theta$, where θ is the angle through which the skater has turned. Similarly, distance travelled by the seventh skater P_7 is given $6x\theta$. Since the rate of change of θ is the same for the skaters, it follows that the seventh skater travels twice as fast the the fourth skater.

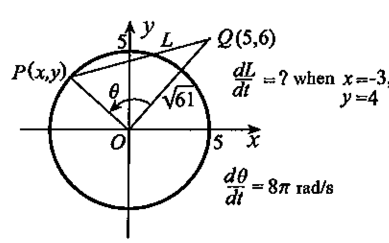


18. The cosine law on triangle OPQ gives

$$L^2 = 61 + 25 - 2(5)\sqrt{61}\cos\theta = 86 - 10\sqrt{61}\cos\theta.$$

Differentiation of this equation with respect to time gives

$$2L\frac{dL}{dt} = 10\sqrt{61}\sin\theta\frac{d\theta}{dt}.$$

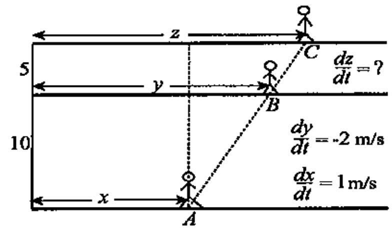


When the particle is at $P(-3,4)$, the length of PQ is $L = \sqrt{(5+3)^2 + (6-4)^2} = 2\sqrt{17}$, and the equation $68 = 86 - 10\sqrt{61}\cos\theta$ gives $\cos\theta = 9/(5\sqrt{61})$. It follows that $\sin\theta = \sqrt{1 - 81/(25 \cdot 61)} = 38/(5\sqrt{61})$. Substitution of these into the equation for dL/dt gives

$$2(2\sqrt{17})\frac{dL}{dt} = 10\sqrt{61}\left(\frac{38}{5\sqrt{61}}\right)(8\pi).$$

When this is solved for dL/dt, the result is 115.8 cm/s.

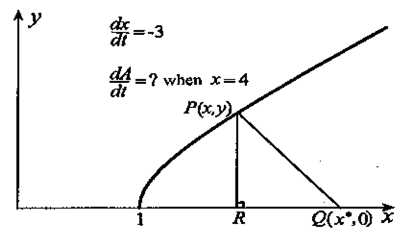
19. If x, y, and z are distances from some fixed vertical line to A, B, and C, respectively, similar triangles give $\dfrac{y-x}{10} = \dfrac{z-x}{15} \implies z = \dfrac{3y-x}{2}$. Differentiation with respect to time gives $dz/dt = (1/2)(3dy/dt - dx/dt) = (1/2)(-6-1) = -7/2$. Thus, C walks to the left at 7/2 m/s.



20. (a) The slope of the tangent line to the hyperbola can be obtained by differentiating $x^2 - y^2 = 1$ with respect to x, $2x - 2y(dy/dx) = 0 \implies dy/dx = x/y$. The slope of the normal line at P is therefore $-y/x$. Since the slope of this line is also $(y-0)/(x-x^*)$, it follows that

$$-\frac{y}{x} = \frac{y}{x-x^*}.$$

When we cross multiply and solve for x^*, we obtain the required result that $x^* = 2x$.



(b) The area of triangle PQR is

$$A = \frac{1}{2}\|PR\|\|RQ\| = \frac{1}{2}y(x^*-x) = \frac{1}{2}\sqrt{x^2-1}(2x-x) = \frac{1}{2}x\sqrt{x^2-1}.$$

Differentiation with respect to time gives

$$\frac{dA}{dt} = \frac{1}{2}\left[\sqrt{x^2-1}\frac{dx}{dt} + x\left(\frac{x}{\sqrt{x^2-1}}\frac{dx}{dt}\right)\right] = \frac{2x^2-1}{2\sqrt{x^2-1}}\frac{dx}{dt}.$$

When $x = 4$, we obtain $\dfrac{dA}{dt} = \dfrac{2(4)^2-1}{2\sqrt{16-1}}(-3) = -\dfrac{31\sqrt{15}}{10}$. The area is therefore decreasing at $31\sqrt{15}/10$ units of area per unit of time.

21. The total volume V of solution in the filter and cylinder is constant, and therefore $V = \pi(6)^2H + (1/3)\pi r^2 h$. Similar triangles give $r/h = 8/24$, so that

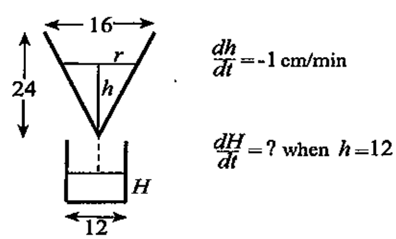
$$V = 36\pi H + \frac{\pi}{3}\left(\frac{h}{3}\right)^2 h = 36\pi H + \frac{\pi h^3}{27}.$$

Differentiation with respect to time gives

$$0 = 36\pi\frac{dH}{dt} + \frac{\pi h^2}{9}\frac{dh}{dt}.$$

When $h = 12$,

$$0 = 36\pi\frac{dH}{dt} + \frac{\pi(12)^2}{9}(-1) \implies \frac{dH}{dt} = \frac{4}{9}.$$

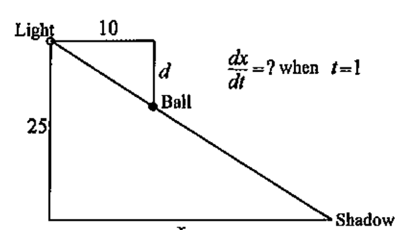


The water level is rising at 4/9 centimetres per minute in the cylinder.

22. Similar triangles give $\dfrac{x}{25} = \dfrac{10}{d} \implies x = \dfrac{250}{d} = \dfrac{250}{4.905t^2}$.

Differentiation of this gives $\dfrac{dx}{dt} = -\dfrac{500}{4.905t^3}$.

At $t = 1$ s, we obtain $dx/dt = -500/4.905 = -102$. The shadow is therefore moving at 102 m/s.



23. The distance D from $(1,2)$ to any point (x,y) on the parabola is given by

$$\begin{aligned} D^2 &= (x-1)^2 + (y-2)^2 \\ &= (x-1)^2 + (x^2-3x-2)^2. \end{aligned}$$

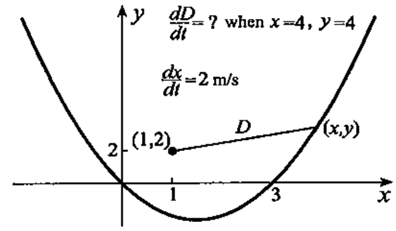
Differentiation with respect to time gives

$$2D\frac{dD}{dt} = 2(x-1)\frac{dx}{dt} + 2(x^2-3x-2)(2x-3)\frac{dx}{dt}.$$

At the point $(4,4)$, $D = \sqrt{(4-1)^2+(4-2)^2} = \sqrt{13}$, and

$$2\sqrt{13}\frac{dD}{dt} = 2(3)(2) + 2(2)(5)(2) \implies \frac{dD}{dt} = 2\sqrt{13}.$$

The distance is increasing at $2\sqrt{13}$ m/s.



24. When we solve the equation of the curve for y in terms of x, we obtain $y = -x \pm 4\sqrt{x}$. Since we are concerned with that part of the curve which contains the point $(4,4)$, we must choose $y = 4\sqrt{x} - x$. Distance D from any point (x,y) on the curve to $(1,2)$ is given by $D^2 = (x-1)^2 + (y-2)^2 = (x-1)^2 + (4\sqrt{x}-x-2)^2$. Differentiation with respect to time gives

$$2D\frac{dD}{dt} = 2(x-1)\frac{dx}{dt} + 2(4\sqrt{x}-x-2)\left(\frac{2}{\sqrt{x}}-1\right)\frac{dx}{dt}.$$

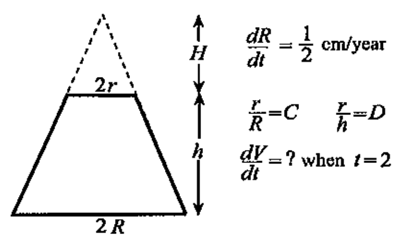
When $(x,y) = (4,4)$, we find $D = \sqrt{(4-1)^2+(4-2)^2} = \sqrt{13}$, and

$$2\sqrt{13}\frac{dD}{dt} = 2(3)(2) + 2(8-4-2)(1-1)(2) \quad\implies\quad \frac{dD}{dt} = \frac{6}{\sqrt{13}}.$$

The distance is changing at $6/\sqrt{13}$ m/s.

25. (a) By similar triangles, $H/r = (H+h)/R \implies H = rh/(R-r)$.
The volume of the trunk is the difference in the volumes of two cones,

$$\begin{aligned} V &= \frac{1}{3}\pi R^2(H+h) - \frac{1}{3}\pi r^2 H = \frac{1}{3}\pi(R^2-r^2)H + \frac{1}{3}\pi R^2 h \\ &= \frac{1}{3}\pi(R^2-r^2)\left(\frac{rh}{R-r}\right) + \frac{1}{3}\pi R^2 h \\ &= \frac{\pi h}{3(R-r)}[r(R^2-r^2) + R^2(R-r)] \\ &= \frac{\pi h}{3}(R^2+rR+r^2). \end{aligned}$$



(b) If we differentiate this equation with respect to time,

$$\frac{dV}{dt} = \frac{\pi}{3}(R^2+rR+r^2)\frac{dh}{dt} + \frac{\pi h}{3}\left(2R\frac{dR}{dt} + r\frac{dR}{dt} + R\frac{dr}{dt} + 2r\frac{dr}{dt}\right).$$

But $r/R = C$ and $r/h = D$, where C and D are constants. At the present time $r = 10$, $R = 50$, and $h = 3000$, and therefore $C = 10/50 = 1/5$, and $D = 10/3000 = 1/300$. Differentiation of $r = R/5$ and $h = 300r$ with respect to t gives

$$\frac{dr}{dt} = \frac{1}{5}\frac{dR}{dt} = \frac{1}{10}, \qquad \frac{dh}{dt} = 300\frac{dr}{dt} = 30.$$

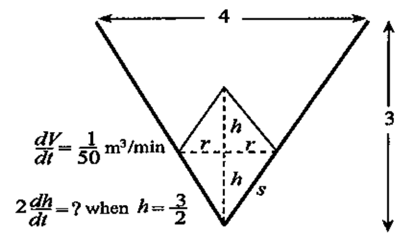
In two years, $r = 10 + 2/10 = 51/5$, $h = 3000 + 60 = 3060$, $R = 50 + 2/2 = 51$, and

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{3}\left[51^2 + \left(\frac{51}{5}\right)(51) + \left(\frac{51}{5}\right)^2\right](30) \\ &\quad + \frac{\pi}{3}(3060)\left[2(51)\left(\frac{1}{2}\right) + \left(\frac{51}{5}\right)\left(\frac{1}{2}\right) + 51\left(\frac{1}{10}\right) + 2\left(\frac{51}{5}\right)\left(\frac{1}{10}\right)\right] = 3.0 \times 10^5. \end{aligned}$$

The volume is increasing at 0.30 cubic metres per year.

26. (a) When the height of each cone is h, the volume of sand in the container is $V = 2\pi r^2 h/3$. Similar triangles require $r/h = 2/3$, and therefore

$$V = \frac{2}{3}\pi\left(\frac{2h}{3}\right)^2 h = \frac{8}{27}\pi h^3.$$



Differentiation with respect to time gives

$$\frac{dV}{dt} = \frac{8}{9}\pi h^2 \frac{dh}{dt}.$$

When $h = 3/2$, this result yields

$$\frac{1}{50} = \frac{8}{9}\pi\left(\frac{3}{2}\right)^2 \frac{dh}{dt}.$$

Thus, $dh/dt = 1/(100\pi)$, and the top of the pile is rising at $1/(50\pi)$ metres per minute.
(b) Since $s^2 = h^2 + r^2 = h^2 + (2h/3)^2 = 13h^2/9$, differentiation gives $2s(ds/dt) = (26/9)h(dh/dt)$. When $h = 3/2$, $s = \sqrt{13(3/2)^2/9} = \sqrt{13}/2$, and

$$2\left(\frac{\sqrt{13}}{2}\right)\frac{ds}{dt} = \frac{26}{9}\left(\frac{3}{2}\right)\left(\frac{1}{100\pi}\right) \implies \frac{ds}{dt} = \frac{\sqrt{13}}{300\pi}.$$

The sand is rising along the side of the container at $\sqrt{13}/(300\pi)$ metres per minute.

27. If we denote the length of rod AC by q, then $q^2 = y^2 + (k-x)^2$. Differentiation of this equation with respect to time gives

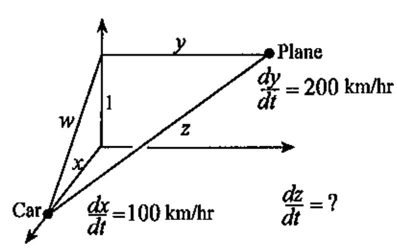
$$0 = 2y\frac{dy}{dt} - 2(k-x)\frac{dx}{dt} \implies \frac{dy}{dt} = \frac{k-x}{y}\frac{dx}{dt}.$$

But from Consulting Project 6, $dx/dt = 4\pi x l \sin\theta/(l\cos\theta - x)$, and therefore

$$\frac{dy}{dt} = \frac{k-x}{y}\left(\frac{4\pi x l \sin\theta}{l\cos\theta - x}\right) \text{ m/s}.$$

28. If we differentiate $z^2 = y^2 + w^2 = y^2 + (1 + x^2)$, with respect to time t,

$$2z\frac{dz}{dt} = 2y\frac{dy}{dt} + 2x\frac{dx}{dt}.$$



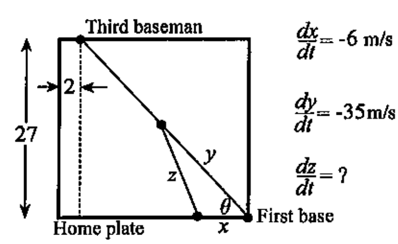
At the instant in question, $x = 2 + 100/60 = 11/3$ km, $y = 200/60 = 10/3$ km, and $z^2 = 100/9 + 1 + 121/9 = 230/9$ km. Hence,

$$\frac{\sqrt{230}}{3}\frac{dz}{dt} = \frac{10}{3}(200) + \frac{11}{3}(100),$$

from which $dz/dt = 3100/\sqrt{230}$ km/hr.

29. By the cosine law, $z^2 = x^2 + y^2 - 2xy\cos\theta$. If we differentiate this equation with respect to time t,

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - 2\cos\theta\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right).$$



At the instant in question, $x = 27/4$, $y = \sqrt{25^2 + 27^2}/2 = \sqrt{1354}/2$, $\cos\theta = 25/\sqrt{25^2 + 27^2} = 25/\sqrt{1354}$, and

$$z^2 = \left(\frac{27}{4}\right)^2 + \frac{1354}{4} - 2\left(\frac{27}{4}\right)\left(\frac{\sqrt{1354}}{2}\right)\left(\frac{25}{\sqrt{1354}}\right) = \frac{3445}{16}.$$

Hence,

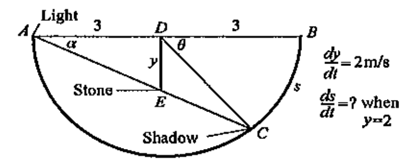
$$\frac{\sqrt{3445}}{4}\frac{dz}{dt} = \frac{27}{4}(-6) + \frac{\sqrt{1354}}{2}(-35) - \frac{25}{\sqrt{1354}}\left[\frac{27}{4}(-35) + \frac{\sqrt{1354}}{2}(-6)\right] \implies \frac{dz}{dt} = -30.6 \text{ m/s}.$$

30. As a point on the chain travels around the front sprocket, the distance S that it travels is $S = R\theta$, where θ is the angle through which it turns. Similarly, a point on the rear sprocket travels through a distance $s = r\phi$ where ϕ is the angle through which it turns. Since the chain is inextensible, it follows that $S = s \Longrightarrow R\theta = r\phi$. Since the stone rotates through the same angle ϕ as the point on the chain on the rear sprocket, the distance the stone travels as it rotates through angle ϕ is $\overline{S} = \overline{R}\phi$. Consequently, $\overline{S} = \overline{R}R\theta/r$, and differentiation of this equation with respect to time gives

$$\frac{d\overline{S}}{dt} = \frac{\overline{R}R}{r}\frac{d\theta}{dt} = \frac{\overline{R}R}{r}(2\pi).$$

The stone therefore travels at $2\pi\overline{R}R/r$ m/s when it leaves the tire.

31. We must find an equation relating y, distance the stone has fallen, and s, the distance the shadow of the stone has moved along the side of the pool. Since arc length along a circle is the product of the radius of the circle and the angle θ that it subtends at the centre of the circle, we know that $s = 3\theta$. We now relate y and θ. Since triangle ADC is isosceles, angles DAC and DCA are equal. Hence, $s = 3(2\alpha) = 6\alpha$. From triangle ADE, we may write $y = 3\tan\alpha$. Combine this with $s = 6\alpha$, and we finally relate s and y:



$$y = 3\tan\left(\frac{s}{6}\right).$$

Differentiation with respect to t gives

$$\frac{dy}{dt} = 3\sec^2\left(\frac{s}{6}\right)\left(\frac{1}{6}\frac{ds}{dt}\right).$$

When the stone is 1 m from the bottom of the tank, $y = 2$ m, and therefore $2 = 3\tan\left(\frac{s}{6}\right)$. It follows that at this instant

$$\sec^2\left(\frac{s}{6}\right) = 1 + \tan^2\left(\frac{s}{6}\right) = 1 + \left(\frac{2}{3}\right)^2 = \frac{13}{9}.$$

Substitution of this into the equation that relates dy/dt and ds/dt, along with $dy/dt = 2$, yields

$$2 = 3\left(\frac{13}{9}\right)\left(\frac{1}{6}\frac{ds}{dt}\right).$$

Hence, $ds/dt = 36/13$, and the shadow is moving at $36/13$ m/s.

32. If we differentiate $A^2 = s(s-a)(s-b)(s-c)$ with respect to time,

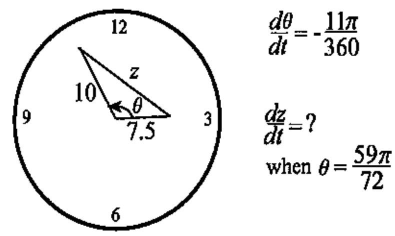
$$2A\frac{dA}{dt} = (s-a)(s-b)(s-c)\frac{ds}{dt} + s(s-b)(s-c)\left(\frac{ds}{dt} - \frac{da}{dt}\right) + s(s-a)(s-c)\left(\frac{ds}{dt} - \frac{db}{dt}\right) + s(s-a)(s-b)\left(\frac{ds}{dt} - \frac{dc}{dt}\right).$$

Furthermore, when we differentiate $s = (a+b+c)/2$, $\quad \dfrac{ds}{dt} = \dfrac{1}{2}\left(\dfrac{da}{dt} + \dfrac{db}{dt} + \dfrac{dc}{dt}\right)$.
Since $da/dt = db/dt = dc/dt = 1$, it follows that $ds/dt = 3/2$. When $a = 3$, $b = 4$, and $c = 5$, we obtain $s = 6$, and $A = \sqrt{6(6-3)(6-4)(6-5)} = 6$. For these values,

$$2(6)\frac{dA}{dt} = (3)(2)(1)\left(\frac{3}{2}\right) + 6(2)(1)\left(\frac{3}{2} - 1\right) + 6(3)(1)\left(\frac{3}{2} - 1\right) + 6(3)(2)\left(\frac{3}{2} - 1\right).$$

This implies that $dA/dt = 7/2$ square centimetres per minute.

33. (a) Since the minute hand sweeps out 2π radians each hour, or $\pi/30$ radians every minute, and the hour hand sweeps out $\pi/360$ radians each minute, the rate of change of the angle between them is $\pm(\pi/30 - \pi/360) = \pm 11\pi/360$ radians per minute, the plus or minus depending on the whether the hands are approaching each other or separating from each other.



(b) When the cosine law is applied to the triangle to the right,

$$z^2 = 100 + \frac{225}{4} - 150\cos\theta.$$

Differentiation with respect to time gives

$$2z\frac{dz}{dt} = 150\sin\theta\frac{d\theta}{dt}.$$

In this case, $d\theta/dt = -11\pi/360$. At 3:00, $\cos\theta = 0$, and therefore $z = \sqrt{100 + 225/4} = 25/2$. At this time then

$$2\left(\frac{25}{2}\right)\frac{dz}{dt} = 150(1)\left(\frac{-11\pi}{360}\right) \quad \Longrightarrow \quad \frac{dz}{dt} = -11\pi/60 \text{ cm/min}.$$

(c) At 8:05 the hands are separating at $d\theta/dt = 11\pi/360$. At this time, $\theta = 59\pi/72$, and $z = \sqrt{100 + 225/4 - 150\cos(59\pi/72)} = 16.81543$. When these are substituted into the equation for dz/dt in part (b),

$$2(16.81543)\frac{dz}{dt} = 150\sin\left(\frac{59\pi}{72}\right)\left(\frac{11\pi}{360}\right) \quad \Longrightarrow \quad \frac{dz}{dt} = 0.23 \text{ cm/min}.$$

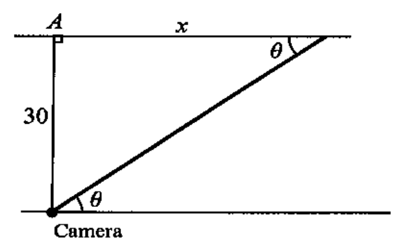
34. (a) As the runner approaches A, $x = 30\cot\theta$. Differentiation with respect to time t gives

$$\frac{dx}{dt} = -30\csc^2\theta\,\frac{d\theta}{dt}.$$

When the runner is at A, $\theta = \pi/2$, and

$$-4 = -30(1)\frac{d\theta}{dt}.$$

Hence, $d\theta/dt = 2/15$ radians per second.

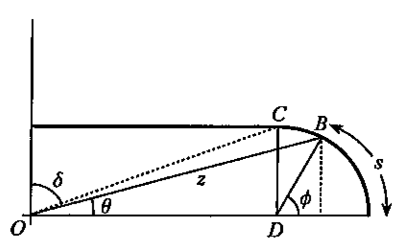


(b) At point C where the straightaway begins, $\delta = \mathrm{Tan}^{-1}(50/30) = 1.03 < \pi/3$; that is, B is on the curved portion of the track. Since length s along the curved portion is given by $s = 30\phi$,

$$\frac{ds}{dt} = 30\frac{d\phi}{dt}.$$

Thus, $d\phi/dt = 2/15$ rad/s. By the cosine law,

$$\begin{aligned} z^2 &= \|OD\|^2 + \|BD\|^2 - 2\|OD\|\|BD\|\cos(\pi - \phi) \\ &= \|OD\|^2 + \|BD\|^2 + 2\|OD\|\|BD\|\cos\phi, \end{aligned}$$



and therefore

$$2z\frac{dz}{dt} = -2(50)(30)\sin\phi\,\frac{d\phi}{dt}.$$

This gives $\dfrac{dz}{dt} = \dfrac{-1500\sin\phi}{z}\left(\dfrac{2}{15}\right) = \dfrac{-200\sin\phi}{z}$. By the sine law, $\dfrac{\sin\theta}{\|BD\|} = \dfrac{\sin(\pi-\phi)}{z} = \dfrac{\sin\phi}{z}$, or, $z\sin\theta = 30\sin\phi$. Differentiation with respect to t gives $\sin\theta\,\dfrac{dz}{dt} + z\cos\theta\,\dfrac{d\theta}{dt} = 30\cos\phi\,\dfrac{d\phi}{dt}$, and therefore at B,

$$\frac{d\theta}{dt} = \frac{1}{z\cos\theta}\left[30\cos\phi\left(\frac{2}{15}\right) + \frac{200\sin\phi\sin\theta}{z}\right] = \frac{2}{\sqrt{3}z}\left(4\cos\phi + \frac{100\sin\phi}{z}\right) = \frac{8}{\sqrt{3}z^2}(z\cos\phi + 25\sin\phi).$$

When $\theta = \pi/6$, we have $z = 60\sin\phi$ and $z^2 = 2500 + 900 + 3000\cos\phi$. These give $3600\sin^2\phi = 3400 + 3000\sqrt{1-\sin^2\phi}$, from which $324\sin^4\phi - 387\sin^2\phi + 64 = 0$. The solution of this for ϕ is $\phi = 1.5087$, and this gives $z = 60\sin(1.5087) = 59.884$. Thus,

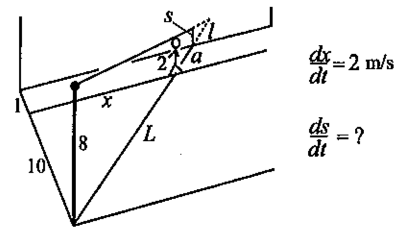
$$\frac{d\theta}{dt} = \frac{8}{\sqrt{3}(59.884)^2}[59.884\cos 1.5087 + 25\sin 1.5087] = 0.0369 \text{ rad/s}.$$

35. From similar vertical triangles,

$$\frac{s}{l} = \frac{8}{L+l+a} \quad \text{and} \quad \frac{2}{l+a} = \frac{s}{l}.$$

The second implies that

$$2l = s(l+a) \Longrightarrow l = \frac{as}{2-s}.$$



Substitution into the first gives

$$8\left(\frac{as}{2-s}\right) = s\left(L + \frac{as}{2-s} + a\right) \Longrightarrow 8as = s(L+a)(2-s) + as^2 \Longrightarrow a = \frac{sL(2-s)}{8s - s^2 - s(2-s)} = \frac{L(2-s)}{6}.$$

From horizontal right-angled triangles, we have $(L+a)^2 = (x + \sqrt{a^2-1})^2 + 121$ and $L^2 = x^2 + 100$. If we replace a in the first of these

$$\left[L + \frac{L(2-s)}{6}\right]^2 = \left[x + \sqrt{\frac{L^2(2-s)^2}{36} - 1}\right]^2 + 121 \Longrightarrow \frac{L^2}{36}(8-s)^2 = \left[x + \frac{\sqrt{L^2(2-s)^2 - 36}}{6}\right]^2 + 121.$$

If we now replace L^2 with $x^2 + 100$, and multiply by 36,

$$\begin{aligned}(x^2+100)(8-s)^2 &= [6x + \sqrt{(x^2+100)(2-s)^2 - 36}]^2 + 121(36)\\ &= 36x^2 + 12x\sqrt{(x^2+100)(2-s)^2-36} + (x^2+100)(2-s)^2 - 36 + 121(36).\end{aligned}$$

Consequently,

$$\begin{aligned}36x^2 + 12x\sqrt{(x^2+100)(2-s)^2-36} + 120(36) &= (x^2+100)(64 - 16s + s^2 - 4 + 4s - s^2)\\ &= (x^2+100)(60-12s).\end{aligned}$$

Division by 12 gives

$$3x^2 + x\sqrt{(x^2+100)(2-s)^2-36} + 360 = (x^2+100)(5-s) = 5x^2 + 500 - x^2s - 100s,$$

from which

$$x\sqrt{(x^2+100)(2-s)^2-36} = 2x^2 + 140 - x^2s - 100s = x^2(2-s) + 20(7-5s).$$

If we square both sides,

$$x^2(x^2+100)(2-s)^2 - 36x^2 = x^4(2-s)^2 + 40x^2(2-s)(7-5s) + 400(7-5s)^2,$$

from which

$$\begin{aligned}0 &= x^2(560 - 680s + 200s^2 - 400 + 400s - 100s^2 + 36) + 400(7-5s)^2\\ &= x^2(196 - 280s + 100s^2) + 400(7-5s)^2 = 4(x^2+100)(7-5s)^2.\end{aligned}$$

This implies that $s = 7/5$ for all x. In other words, the height of the shadow on the wall never changes, and therefore $ds/dt = 0$.

EXERCISES 4.10

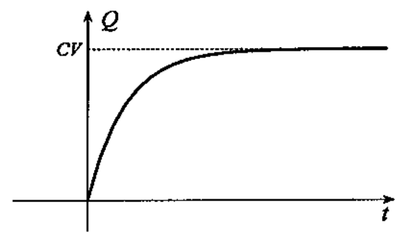
1. (a) When we substitute the function into the left side of the equation,

$$R\frac{dQ}{dt}+\frac{Q}{C}=R\left[D\,e^{-t/(RC)}\left(\frac{-1}{RC}\right)\right]+\frac{1}{C}\left[D\,e^{-t/(RC)}+CV\right]=V.$$

(b) With $Q(0)=Q_0$, we find that
$Q_0=D+CV \implies D=Q_0-CV,$
and therefore

$$Q(t)=(Q_0-CV)e^{-t/(RC)}+CV$$
$$=CV[1-e^{-t/(RC)}]+Q_0\,e^{-t/(RC)}.$$

(c) The graph when $Q_0=0$ is shown to the right.



2. If we substitute for i into the left side of the equation,

$$R\frac{di}{dt}+\frac{i}{C}=R\left[Ae^{-t/(RC)}\left(\frac{-1}{RC}\right)+\frac{\omega V_0}{Z}\cos(\omega t-\phi)\right]+\frac{A}{C}e^{-t/(RC)}+\frac{V_0}{CZ}\sin(\omega t-\phi)$$
$$=\frac{\omega RV_0}{Z}(\cos\omega t\,\cos\phi+\sin\omega t\,\sin\phi)+\frac{V_0}{CZ}(\sin\omega t\,\cos\phi-\cos\omega t\,\sin\phi).$$

Because $\tan\phi=-1/(\omega CR)$, it follows that

$$\sin\phi=\frac{-1}{\sqrt{1+\omega^2C^2R^2}}=\frac{-1}{\omega C\sqrt{R^2+1/(\omega^2C^2)}}=\frac{-1}{\omega CZ},$$
$$\cos\phi=\frac{\omega CR}{\sqrt{1+\omega^2C^2R^2}}=\frac{\omega CR}{\omega CZ}=\frac{R}{Z},$$

provided we choose ϕ in the fourth quadrant. With these,

$$R\frac{di}{dt}+\frac{i}{C}=\frac{\omega RV_0}{Z}\left[\frac{R}{Z}\cos\omega t-\frac{1}{\omega CZ}\sin\omega t\right]+\frac{V_0}{CZ}\left[\frac{R}{Z}\sin\omega t+\frac{1}{\omega CZ}\cos\omega t\right]$$
$$=\frac{V_0}{Z^2}\left(\omega R^2+\frac{1}{\omega C^2}\right)\cos\omega t=\frac{V_0\omega}{Z^2}\left(R^2+\frac{1}{\omega^2C^2}\right)\cos\omega t=V_0\omega\cos\omega t=\frac{dV}{dt}.$$

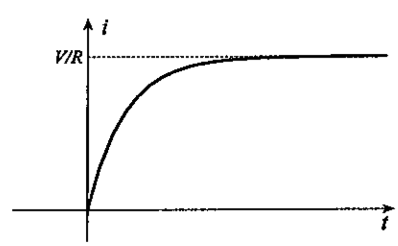
3. (a) When we substitute the function into the left side of the equation,

$$L\frac{di}{dt}+Ri=L\left[D\,e^{-Rt/L}\left(\frac{-R}{L}\right)\right]+R\left(D\,e^{-Rt/L}+\frac{V}{R}\right)=V.$$

(b) With $i(0)=0$, we find that
$0=D+V/R \implies D=-V/R$, and therefore

$$i(t)=\left(\frac{-V}{R}\right)e^{-Rt/L}+\frac{V}{R}=\frac{V}{R}(1-e^{-Rt/L}).$$

The graph is shown to the right.



4. If we substitute for i into the left side of the equation,

$$L\frac{di}{dt}+Ri=L\left[Ae^{-Rt/L}(-R/L)+\frac{\omega V_0}{Z}\cos(\omega t-\phi)\right]+RAe^{-Rt/L}+\frac{RV_0}{Z}\sin(\omega t-\phi)$$
$$=\frac{\omega LV_0}{Z}(\cos\omega t\,\cos\phi+\sin\omega t\,\sin\phi)+\frac{RV_0}{Z}(\sin\omega t\,\cos\phi-\cos\omega t\,\sin\phi).$$

Because $\tan\phi = \omega L/R$, it follows that

$$\sin\phi = \frac{\omega L}{\sqrt{R^2+\omega^2L^2}} = \frac{\omega L}{Z}, \qquad \cos\phi = \frac{R}{\sqrt{R^2+\omega^2L^2}} = \frac{R}{Z},$$

provided we choose ϕ in the first quadrant. With these,

$$\begin{aligned} L\frac{di}{dt} + Ri &= \frac{\omega L V_0}{Z}\left(\frac{R}{Z}\cos\omega t + \frac{\omega L}{Z}\sin\omega t\right) + \frac{RV_0}{Z}\left(\frac{R}{Z}\sin\omega t - \frac{\omega L}{Z}\cos\omega t\right) \\ &= \frac{V_0}{Z^2}(\omega^2L^2+R^2)\sin\omega t = V_0\sin\omega t. \end{aligned}$$

5. (a) When we substitute the function into the left side of the equation,

$$\begin{aligned} L\frac{d^2Q}{dt^2} + \frac{Q}{C} &= L\left(-\frac{D}{LC}\cos\frac{t}{\sqrt{LC}} - \frac{E}{LC}\sin\frac{t}{\sqrt{LC}} + \frac{A\omega}{\omega L - \frac{1}{\omega C}}\sin\omega t\right) \\ &\quad + \frac{1}{C}\left(D\cos\frac{t}{\sqrt{LC}} + E\sin\frac{t}{\sqrt{LC}} - \frac{A/\omega}{\omega L - \frac{1}{\omega C}}\sin\omega t\right) \\ &= \left(\frac{\omega L - \frac{1}{\omega C}}{\omega L - \frac{1}{\omega C}}\right)A\sin\omega t = A\sin\omega t = V. \end{aligned}$$

(b) Using the facts that $Q(0) = Q_0$ and $i(0) = 0$, we obtain

$$Q_0 = D, \qquad 0 = \frac{E}{\sqrt{LC}} - \frac{A}{\omega L - \frac{1}{\omega C}}.$$

Thus, $E = \dfrac{A\sqrt{LC}}{\omega L - \frac{1}{\omega C}}$, and this gives the required function.

6. Since $\dfrac{di}{dt} = -\dfrac{A}{\sqrt{LC}}\sin\left(\dfrac{t}{\sqrt{LC}}\right) + \dfrac{B}{\sqrt{LC}}\cos\left(\dfrac{t}{\sqrt{LC}}\right)$, it follows that

$$L\frac{d^2i}{dt^2} + \frac{i}{C} = L\left[-\frac{A}{LC}\cos\left(\frac{t}{\sqrt{LC}}\right) - \frac{B}{LC}\sin\left(\frac{t}{\sqrt{LC}}\right)\right] + \frac{1}{C}\left[A\cos\left(\frac{t}{\sqrt{LC}}\right) + B\sin\left(\frac{t}{\sqrt{LC}}\right)\right] = 0.$$

7. (a) When we substitute the function into the left side of the equation,

$$\begin{aligned} R\frac{dQ}{dt} + \frac{Q}{C} &= R\left\{\frac{CA\omega}{1+\omega^2R^2C^2}\cos\omega t + \frac{\omega RC^2A}{1+\omega^2R^2C^2}\left[e^{-t/(RC)}\left(\frac{-1}{RC}\right) + \omega\sin\omega t\right]\right\} \\ &\quad + \frac{1}{C}\left\{\frac{CA}{1+\omega^2R^2C^2}\sin\omega t + \frac{\omega RC^2A}{1+\omega^2R^2C^2}[e^{-t/(RC)} - \cos\omega t]\right\} \\ &= \left(\frac{\omega^2R^2C^2A + A}{1+\omega^2R^2C^2}\right)\sin\omega t = A\sin\omega t. \end{aligned}$$

(b) We express $Q(t)$ in the form

$$Q(t) = \frac{\omega RC^2A}{1+\omega^2R^2C^2}e^{-t/(RC)} + \frac{CA}{1+\omega^2R^2C^2}(\sin\omega t - \omega RC\cos\omega t),$$

and set

$$\sin\omega t - \omega RC\cos\omega t = B\cos(\omega t - \phi) = B(\cos\omega t\cos\phi + \sin\omega t\sin\phi).$$

This equation is satisfied if B and ϕ are chosen to satisfy

$$1 = B\sin\phi, \qquad -\omega RC = B\cos\phi.$$

These imply that $B^2 = 1 + \omega^2R^2C^2$ and $\tan\phi = -1/(\omega CR)$, and therefore

$$Q(t) = \frac{\omega RC^2 A}{1+\omega^2R^2C^2}e^{-t/(RC)} + \frac{CA}{1+\omega^2R^2C^2}\sqrt{1+\omega^2R^2C^2}\cos(\omega t-\phi)$$
$$= \frac{\omega RC^2 A}{1+\omega^2R^2C^2}e^{-t/(RC)} + \frac{A/\omega}{\sqrt{R^2+1/(\omega^2C^2)}}\cos(\omega t-\phi).$$

8. (a) When we substitute the function into the left side of the equation,

$$L\frac{di}{dt}+Ri = L\left\{\frac{\omega LA}{R^2+\omega^2L^2}\left[e^{-Rt/L}\left(\frac{-R}{L}\right)+\omega\sin\omega t\right]+\frac{RA\omega}{R^2+\omega^2L^2}\cos\omega t\right\}$$
$$+R\left[\frac{\omega LA}{R^2+\omega^2L^2}(e^{-Rt/L}-\cos\omega t)+\frac{RA}{R^2+\omega^2L^2}\sin\omega t\right]$$
$$=\left(\frac{\omega^2L^2A+R^2A}{R^2+\omega^2L^2}\right)\sin\omega t = A\sin\omega t = V.$$

(b) We express $i(t)$ in the form

$$i(t) = \frac{\omega LA}{R^2+\omega^2L^2}e^{-Rt/L}+\frac{A}{R^2+\omega^2L^2}(R\sin\omega t-\omega L\cos\omega t),$$

and set

$$R\sin\omega t-\omega L\cos\omega t = B\sin(\omega t-\phi) = B(\sin\omega t\cos\phi-\cos\omega t\sin\phi).$$

This equation is satisfied if B and ϕ are chosen to satisfy

$$R = B\cos\phi,\qquad \omega L = B\sin\phi.$$

These imply that $B^2 = R^2+\omega^2L^2$ and $\tan\phi = \omega L/R$, and therefore

$$i(t) = \frac{\omega LA}{R^2+\omega^2L^2}e^{-Rt/L}+\frac{A}{R^2+\omega^2L^2}\sqrt{R^2+\omega^2L^2}\sin(\omega t-\phi)$$
$$= \frac{\omega LA}{R^2+\omega^2L^2}e^{-Rt/L}+\frac{A}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t-\phi)$$

9. If we substitute for i into the left side of the equation,

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C} = \frac{1}{C}\left\{e^{-Rt/(2L)}[D\cos\nu t+E\sin\nu t]+\frac{A}{Z}\sin(\omega t-\phi)\right\}$$
$$+R\left\{-\frac{R}{2L}e^{-Rt/(2L)}[D\cos\nu t+E\sin\nu t]+e^{-Rt/(2L)}[-D\nu\sin\nu t+E\nu\cos\nu t]+\frac{\omega A}{Z}\cos(\omega t-\phi)\right\}$$
$$+L\left\{\frac{R^2}{4L^2}e^{-Rt/(2L)}[D\cos\nu t+E\sin\nu t]-\frac{R}{L}e^{-Rt/(2L)}[-D\nu\sin\nu t+E\nu\cos\nu t]\right.$$
$$\left.+e^{-Rt/(2L)}[-D\nu^2\cos\nu t-E\nu^2\sin\nu t]-\frac{\omega^2A}{Z}\sin(\omega t-\phi)\right\}$$
$$= e^{-Rt/(2L)}\cos\nu t\left[\frac{D}{C}-\frac{DR^2}{2L}+E\nu R+\frac{DR^2}{4L}-E\nu R-DL\nu^2\right]$$
$$+e^{-Rt/(2L)}\sin\nu t\left[\frac{E}{C}-\frac{ER^2}{2L}-D\nu R+\frac{ER^2}{4L}+D\nu R-EL\nu^2\right]$$
$$+A\left(\frac{1}{CZ}-\frac{\omega^2L}{Z}\right)\sin(\omega t-\phi)+\frac{\omega AR}{Z}\cos(\omega t-\phi)$$
$$= D\,e^{-Rt/(2L)}\cos\nu t\left[\frac{1}{C}-\frac{R^2}{4L}-L\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)\right]$$
$$+E\,e^{-Rt/(2L)}\sin\nu t\left[\frac{1}{C}-\frac{R^2}{4L}-L\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)\right]$$
$$+\frac{\omega A}{Z}\left(\frac{1}{\omega C}-\omega L\right)[\sin\omega t\cos\phi-\cos\omega t\sin\phi]+\frac{\omega AR}{Z}[\cos\omega t\cos\phi+\sin\omega t\sin\phi].$$

Because $\tan\phi = \dfrac{\omega L - 1/(\omega C)}{R}$, it follows that

$$\sin\phi = \frac{\omega L - 1/(\omega C)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{\omega L - 1/(\omega C)}{Z}, \qquad \cos\phi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{R}{Z}.$$

With these,

$$\begin{aligned} L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} &= \frac{\omega A}{Z}\left[\frac{1}{\omega C} - \omega L\right]\left[\frac{R}{Z}\sin\omega t - \frac{\omega L - 1/(\omega C)}{Z}\cos\omega t\right] \\ &\quad + \frac{\omega AR}{Z}\left[\frac{R}{Z}\cos\omega t + \frac{\omega L - 1/(\omega C)}{Z}\sin\omega t\right] \\ &= \frac{\omega A}{Z^2}\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]\cos\omega t = \omega A\cos\omega t = \frac{dV}{dt}. \end{aligned}$$

10. (a) If we set

$$R\,\cos\omega t + \left(\omega L - \frac{1}{\omega C}\right)\sin\omega t = A\cos(\omega t - \phi) = A(\cos\omega t\,\cos\phi + \sin\omega t\,\sin\phi),$$

then, $R = A\,\cos\phi$ and $\omega L - \dfrac{1}{\omega C} = A\,\sin\phi$. These imply that $A = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$, and therefore

$$\begin{aligned} i(t) &= \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\cos(\omega t - \phi) \\ &= \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos(\omega t + \phi). \end{aligned}$$

To maximize the amplitude, we minimize the denominator. This occurs when $\omega L - \dfrac{1}{\omega C} = 0$, and this implies that $\omega = 1/\sqrt{LC}$.

(b) Critical points of $i(t)$ are given by

$$0 = \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left[-R\omega\,\sin\omega t + \omega\left(\omega L - \frac{1}{\omega C}\right)\cos\omega t\right].$$

If $\bar{t}$ denotes a solution of this equation, then $\tan\omega\bar{t} = \dfrac{\omega L - \frac{1}{\omega C}}{R}$. This implies that

$$\sin\omega\bar{t} = \pm\frac{\omega L - 1/(\omega C)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \qquad \cos\omega\bar{t} = \pm\frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

Consequently,

$$i(\bar{t}) = \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left[\frac{\pm R^2}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \pm \frac{\left(\omega L - 1/(\omega C)\right)^2}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\right] = \frac{\pm V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

The value of ω that makes this a maximum is $1/\sqrt{LC}$.

EXERCISES 4.11

Although many of the limits in these exercises can be done without L'Hôpital's rule, we shall demonstrate use of this rule whenever it is applicable.

1. $\lim\limits_{x\to 0}\dfrac{x^2+3x}{x^3+5x^2}=\lim\limits_{x\to 0}\dfrac{2x+3}{3x^2+10x}=\pm\infty$, depending on whether $x\to 0^+$ or $x\to 0^-$

2. $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=\lim\limits_{x\to 3}\dfrac{2x}{1}=6$

3. $\lim\limits_{x\to -\infty}\dfrac{x^3+3x-2}{x^2+5x+1}=\lim\limits_{x\to -\infty}\dfrac{3x^2+3}{2x+5}=\lim\limits_{x\to -\infty}\dfrac{6x}{2}=-\infty$

4. $\lim\limits_{x\to\infty}\dfrac{2x^2+3x}{5x^3+4}=\lim\limits_{x\to\infty}\dfrac{4x+3}{15x^2}=\lim\limits_{x\to\infty}\dfrac{4}{30x}=0$

5. $\lim\limits_{x\to 5}\dfrac{x^2-10x+25}{x^3-125}=\lim\limits_{x\to 5}\dfrac{2x-10}{3x^2}=0$

6. L'Hôpital's rule is not appliable. $\lim\limits_{x\to 1}\dfrac{1}{(x-1)^2}=\infty$

7. L'Hôpital's rule does not work on this limit. Instead we divide numerator and denominator by x,

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{2x+5}=\lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x^2}}}{2+\frac{5}{x}}=\frac{1}{2}.$$

8. L'Hôpital's rule is not applicable. $\lim\limits_{x\to -\infty}\dfrac{\sin x}{x}=0$

9. $\lim\limits_{x\to\infty}\dfrac{\sin(2/x)}{\sin(1/x)}=\lim\limits_{x\to\infty}\dfrac{(-2/x^2)\cos(2/x)}{(-1/x^2)\cos(1/x)}=\lim\limits_{x\to\infty}\dfrac{2\cos(2/x)}{\cos(1/x)}=2$

10. $\lim\limits_{x\to\pi/2}\dfrac{\cos x}{(x-\pi/2)^2}=\lim\limits_{x\to\pi/2}\dfrac{-\sin x}{2(x-\pi/2)}=\pm\infty$ depending on whether x approaches $\pi/2$ from left or right

11. $\lim\limits_{x\to 1^+}\dfrac{(1-1/x)^3}{\sqrt{x-1}}=\lim\limits_{x\to 1^+}\dfrac{3(1-1/x)^2(1/x^2)}{\dfrac{1}{2\sqrt{x-1}}}=\lim\limits_{x\to 1^+}\dfrac{6\sqrt{x-1}}{x^2}\left(1-\dfrac{1}{x}\right)^2=0$

12. $\lim\limits_{x\to\infty}\dfrac{\sin(1/x)}{1/x^2}=\lim\limits_{x\to\infty}\dfrac{-(1/x^2)\cos(1/x)}{-2/x^3}=\lim\limits_{x\to\infty}\dfrac{x}{2}\cos\left(\dfrac{1}{x}\right)=\infty$

13. $\lim\limits_{x\to 9^-}\dfrac{\sqrt{x}-3}{\sqrt{9-x}}=\lim\limits_{x\to 9^-}\dfrac{1/(2\sqrt{x})}{-1/(2\sqrt{9-x})}=\lim\limits_{x\to 9^-}\dfrac{\sqrt{9-x}}{-\sqrt{x}}=0$

14. $\lim\limits_{x\to 0}\dfrac{\sqrt{5+x}-\sqrt{5-x}}{x}=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{2\sqrt{5+x}}+\dfrac{1}{2\sqrt{5-x}}}{1}=\dfrac{1}{\sqrt{5}}$

15. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}=\lim\limits_{x\to 0}\dfrac{\sin x}{6x}=\lim\limits_{x\to 0}\dfrac{\cos x}{6}=\dfrac{1}{6}$

16. $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=\lim\limits_{x\to a}\dfrac{nx^{n-1}}{1}=na^{n-1}$

17. $\lim\limits_{x\to 0}\dfrac{(1-\cos x)^2}{3x^2}=\lim\limits_{x\to 0}\dfrac{2\sin x(1-\cos x)}{6x}=\lim\limits_{x\to 0}\dfrac{2\cos x-2\cos^2 x+2\sin^2 x}{6}=0$

18. $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=\lim\limits_{x\to 0}\dfrac{\sec^2 x}{1}=1$

19. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\tan 2x}=\lim\limits_{x\to 0}\dfrac{3\cos 3x}{2\sec^2 2x}=\dfrac{3}{2}$

20. $\lim\limits_{x\to 1}\dfrac{\left(1-\sqrt{2-x}\right)^{3/2}}{x-1}=\lim\limits_{x\to 1}\dfrac{\dfrac{3}{2}\left(1-\sqrt{2-x}\right)^{1/2}\left(\dfrac{1}{2\sqrt{2-x}}\right)}{1}=0$

21. $\lim_{x\to 0} \dfrac{\sqrt{x+1}-\sqrt{2x+1}}{\sqrt{3x+4}-\sqrt{2x+4}} = \lim_{x\to 0} \dfrac{\dfrac{1}{2\sqrt{x+1}} - \dfrac{1}{\sqrt{2x+1}}}{\dfrac{3}{2\sqrt{3x+4}} - \dfrac{1}{\sqrt{2x+4}}} = -2$

22. $\lim_{x\to 0} \dfrac{(1-\cos x)^2}{3x^4} = \lim_{x\to 0} \dfrac{2(1-\cos x)\sin x}{12x^3} = \lim_{x\to 0} \dfrac{2\sin x - \sin 2x}{12x^3} = \lim_{x\to 0} \dfrac{2\cos x - 2\cos 2x}{36x^2}$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{72x} = \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{72} = \frac{1}{12}$$

23. $\lim_{x\to\infty} x \sin\left(\dfrac{1}{x}\right) = \lim_{x\to\infty} \dfrac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \dfrac{(-1/x^2)\cos(1/x)}{-1/x^2} = 1$

24. It is easier to factor the numerator in this limit than to apply L'Hôpital's rule,

$$\lim_{x\to 2} \frac{(x-2)^{10}}{(\sqrt{x}-\sqrt{2})^{10}} = \lim_{x\to 2} \frac{(\sqrt{x}+\sqrt{2})^{10}(\sqrt{x}-\sqrt{2})^{10}}{(\sqrt{x}-\sqrt{2})^{10}} = \lim_{x\to 2} (\sqrt{x}+\sqrt{2})^{10} = 2^{15}.$$

25. $\lim_{x\to 0} \left[\dfrac{4}{x^2} - \dfrac{2}{1-\cos x}\right] = \lim_{x\to 0} \left[\dfrac{4 - 4\cos x - 2x^2}{x^2(1-\cos x)}\right] = \lim_{x\to 0} \dfrac{4\sin x - 4x}{2x(1-\cos x) + x^2 \sin x}$

$$= \lim_{x\to 0} \frac{4\cos x - 4}{2(1-\cos x) + 4x\sin x + x^2\cos x} = \lim_{x\to 0} \frac{-4\sin x}{6\sin x + 6x\cos x - x^2 \sin x}$$

$$= \lim_{x\to 0} \frac{-4\cos x}{12\cos x - 8x\sin x - x^2\cos x} = -\frac{1}{3}$$

26. L'Hôpital's rule is not applicable. $\lim_{x\to\infty} xe^x = \infty$

27. $\lim_{x\to\infty} x^2 e^{-4x} = \lim_{x\to\infty} \dfrac{x^2}{e^{4x}} = \lim_{x\to\infty} \dfrac{2x}{4e^{4x}} = \lim_{x\to\infty} \dfrac{2}{16 e^{4x}} = 0$

28. $\lim_{x\to-\infty} x\sin\left(\dfrac{4}{x}\right) = \lim_{x\to-\infty} \dfrac{\sin(4/x)}{1/x} = \lim_{x\to-\infty} \dfrac{(-4/x^2)\cos(4/x)}{-1/x^2} = \lim_{x\to-\infty} 4\cos\left(\dfrac{4}{x}\right) = 4$

29. $\lim_{x\to 0} x\cot x = \lim_{x\to 0} \dfrac{x}{\tan x} = \lim_{x\to 0} \dfrac{1}{\sec^2 x} = 1$ **30.** $\lim_{x\to 0} \csc x(1-\cos x) = \lim_{x\to 0} \dfrac{1-\cos x}{\sin x} = \lim_{x\to 0} \dfrac{\sin x}{\cos x} = 0$

31. If we set $L = \lim_{x\to 0^+} (\sin x)^x$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to 0^+} (\sin x)^x\right] = \lim_{x\to 0^+} [\ln(\sin x)^x] = \lim_{x\to 0^+} [x\ln(\sin x)] = \lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x}$$

$$= \lim_{x\to 0^+} \frac{\cos x/\sin x}{-1/x^2} = -\lim_{x\to 0^+} \frac{x^2\cos x}{\sin x} = -\lim_{x\to 0^+} \frac{2x\cos x - x^2 \sin x}{\cos x} = 0.$$

Thus, $L = \lim_{x\to 0^+} (\sin x)^x = e^0 = 1$.

32. If we set $L = \lim_{x\to 0^+} x^{\sin x}$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to 0^+} x^{\sin x}\right] = \lim_{x\to 0^+} [\ln(x^{\sin x})] = \lim_{x\to 0^+} [\sin x \ln x] = \lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-\csc x\cot x} = \lim_{x\to 0^+} \frac{-\sin x\tan x}{x} = \lim_{x\to 0^+} \frac{-(\sin x\sec^2 x + \cos x\tan x)}{1} = 0.$$

Thus, $L = \lim_{x\to 0^+} x^{\sin x} = e^0 = 1$.

33. If we set $L = \lim_{x\to\infty}\left(\dfrac{x+5}{x+3}\right)^x$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^x\right] = \lim_{x\to\infty}\left[\ln\left(\frac{x+5}{x+3}\right)^x\right] = \lim_{x\to\infty}\left[x\ln\left(\frac{x+5}{x+3}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(\dfrac{x+5}{x+3}\right)}{1/x}$$

$$= \lim_{x\to\infty}\frac{\left(\dfrac{x+3}{x+5}\right)\left[\dfrac{(x+3)-(x+5)}{(x+3)^2}\right]}{-1/x^2} = \lim_{x\to\infty}\frac{2x^2}{(x+3)(x+5)} = \lim_{x\to\infty}\frac{4x}{2x+8} = \lim_{x\to\infty}\frac{4}{2} = 2.$$

Thus, $L = \lim_{x\to\infty}\left(\dfrac{x+5}{x+3}\right)^x = e^2$.

34. If we set $L = \lim_{x\to 0}(1+x)^{\cot x}$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to 0}(1+x)^{\cot x}\right] = \lim_{x\to 0}[\cot x\,\ln(1+x)] = \lim_{x\to 0}\frac{\ln(1+x)}{\tan x} = \lim_{x\to 0}\frac{\dfrac{1}{1+x}}{\sec^2 x} = 1.$$

Thus, $L = \lim_{x\to 0}(1+x)^{\cot x} = e$.

35. If we set $L = \lim_{x\to\infty} x^{1/x}$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to\infty} x^{1/x}\right] = \lim_{x\to\infty}\left[\ln(x^{1/x})\right] = \lim_{x\to\infty}\left(\frac{1}{x}\ln x\right) = \lim_{x\to\infty}\frac{1/x}{1} = 0.$$

Thus, $L = \lim_{x\to\infty} x^{1/x} = e^0 = 1$.

36. If we set $L = \lim_{x\to 0^+}|\ln x|^{\sin x}$, and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to 0^+}|\ln x|^{\sin x}\right] = \lim_{x\to 0^+}\ln\left[|\ln x|^{\sin x}\right] = \lim_{x\to 0^+}(\sin x\,\ln|\ln x|) = \lim_{x\to 0^+}\frac{\ln|\ln x|}{\csc x}$$

$$= \lim_{x\to 0^+}\frac{\dfrac{1}{|\ln x|}\dfrac{1}{x}\dfrac{|\ln x|}{\ln x}}{-\csc x\,\cot x} = \lim_{x\to 0^+}\frac{-\sin x\,\tan x}{x\ln x}.$$

To verify that we can use L'Hôpital's rule again, we note that

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

Thus, $\ln L = \lim_{x\to 0^+}\dfrac{-(\sin x\,\sec^2 x + \cos x\,\tan x)}{x/x + \ln x} = 0$, and $L = \lim_{x\to 0^+}|\ln x|^{\sin x} = e^0 = 1$.

37. $\lim_{x\to 0^+} x\,e^{1/x} = \lim_{x\to 0^+}\dfrac{e^{1/x}}{1/x} = \lim_{x\to 0^+}\dfrac{(-1/x^2)e^{1/x}}{-1/x^2} = \infty$

38. This limit does not exist since $\lim_{x\to 0^+}(\tan x - \csc x) = -\infty$ and $\lim_{x\to 0^-}(\tan x - \csc x) = \infty$.

39. $\lim_{x\to 0}(\csc x - \cot x) = \lim_{x\to 0}\left(\dfrac{1-\cos x}{\sin x}\right) = \lim_{x\to 0}\dfrac{\sin x}{\cos x} = 0$

40. $\lim_{x\to 1}\left(\dfrac{x}{\ln x} - \dfrac{1}{x\ln x}\right) = \lim_{x\to 1}\dfrac{x^2-1}{x\ln x} = \lim_{x\to 1}\dfrac{2x}{\ln x + 1} = 2$

41. $\lim_{x\to 1}\left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right) = \lim_{x\to 1}\dfrac{x\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1}\dfrac{\ln x + 1 - 1}{\ln x + (x-1)/x} = \lim_{x\to 1}\dfrac{1/x}{1/x + 1/x^2} = \dfrac{1}{2}$

42. $\displaystyle \lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}=\lim_{x\to 0}\frac{(1-\cos 2x)/2-x^2}{x^2(1-\cos 2x)/2}$

$$=\lim_{x\to 0}\frac{1-\cos 2x-2x^2}{x^2-x^2\cos 2x}=\lim_{x\to 0}\frac{2\sin 2x-4x}{2x+2x^2\sin 2x-2x\cos 2x}$$

$$=\lim_{x\to 0}\frac{4\cos 2x-4}{2+8x\sin 2x+4x^2\cos 2x-2\cos 2x}$$

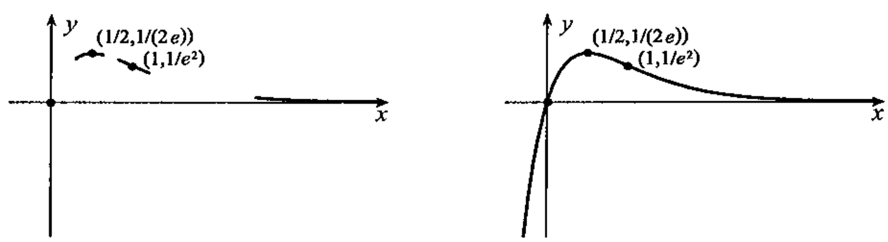
$$=\lim_{x\to 0}\frac{-8\sin 2x}{12\sin 2x+24x\cos 2x-8x^2\sin 2x}$$

$$=\lim_{x\to 0}\frac{-16\cos 2x}{48\cos 2x-64x\sin 2x-16x^2\cos 2x}=-\frac{1}{3}$$

43. For critical points, we solve $0=f'(x)=e^{-2x}-2x\,e^{-2x}=(1-2x)e^{-2x}$. The only solution is $x=1/2$. Since $f'(x)$ changes from positive to negative as x increases through $1/2$, this critical point gives a relative maximum of $f(1/2)=1/(2e)$. For points of inflection, we solve $0=f''(x)=-2e^{-2x}-2(1-2x)e^{-2x}=4(x-1)e^{-2x}$. The solution is $x=1$. Since $f''(x)$ changes sign as x passses through 1, there is a point of inflection at $(1,1/e^2)$. Since

$$\lim_{x\to\infty}x\,e^{-2x}=\lim_{x\to\infty}\frac{x}{e^{2x}}=\lim_{x\to\infty}\frac{1}{2e^{2x}}=0^+,$$

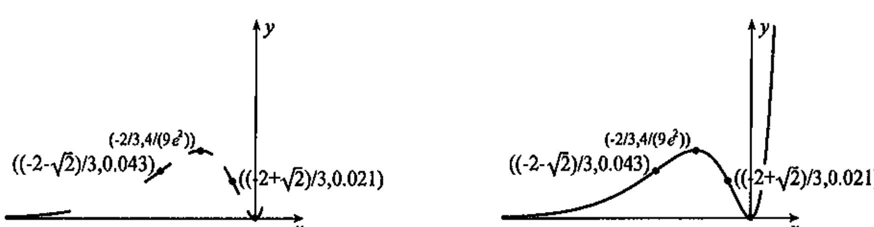
the x-axis is a horizontal asymptote. This information, shown in the left figure below, leads to the final graph in the right figure.



44. For critical points, we solve $0=f'(x)=2x\,e^{3x}+3x^2\,e^{3x}=x(2+3x)e^{3x}$. Solutions are $x=0$ and $x=-2/3$. Since $f'(x)$ changes from positive to negative as x increases through $-2/3$, this critical point gives a relative maximum of $f(-2/3)=4/(9e^2)$. There is a relative minimum of $f(0)=0$ at $x=0$ because $f'(x)$ changes from negative to positive as x increases through this value. For points of inflection, we solve $0=f''(x)=(2+6x)e^{3x}+3x(2+3x)e^{3x}=(9x^2+12x+2)e^{3x}$. Since $f''(x)$ changes sign as x passes through the solutions $x=(-2\pm\sqrt{2})/3$, there are points of inflection at $((-2-\sqrt{2})/3,0.043)$ and $((-2+\sqrt{2})/3,0.021)$. Since

$$\lim_{x\to-\infty}x^2\,e^{3x}=\lim_{x\to-\infty}\frac{x^2}{e^{-3x}}=\lim_{x\to-\infty}\frac{2x}{-3e^{-3x}}=\lim_{x\to-\infty}\frac{2}{9e^{-3x}}=0^+,$$

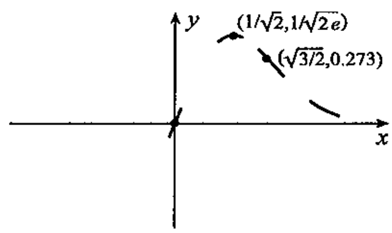
the x-axis is a horizontal asymptote. This information, shown in the left figure below, leads to the final graph in the right figure.

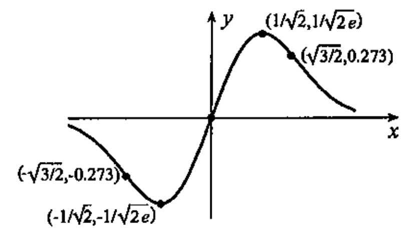


45. The function is odd so that we need only draw the graph for $x \geq 0$. For critical points, we solve $0 = f'(x) = e^{-x^2} - 2x^2\, e^{-x^2} = (1 - 2x^2)e^{-x^2}$. The positive solution is $x = 1/\sqrt{2}$. Since $f'(x)$ changes from positive to negative as x increases through $1/\sqrt{2}$, this critical point gives a relative maximum of $f(1/\sqrt{2}) = 1/\sqrt{2e}$. For points of inflection, we solve $0 = f''(x) = -4xe^{-x^2} - 2x(1 - 2x^2)e^{-x^2} = 2x(2x^2 - 3)e^{-x^2}$. Since $f''(x)$ changes sign as x passes through the solution $x = 0$ and the positive solution $x = \sqrt{3/2}$, they give points of inflection $(0, 0)$ and $(\sqrt{3/2}, 0.273)$. Since

$$\lim_{x\to\infty} x\, e^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0^+,$$

the x-axis is a horizontal asymptote. This information, shown in the left figure below, leads to the final graph in the right figure.





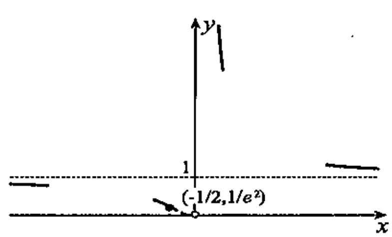
46. The following limits show that $y = 1$ is a horizontal asymptote, and the y-axis is a vertical asymptote,

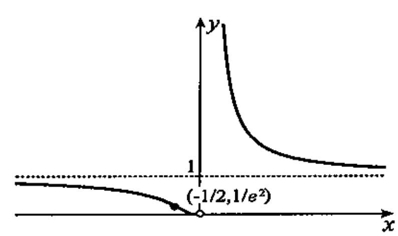
$$\lim_{x\to-\infty} e^{1/x} = 1^-, \quad \lim_{x\to\infty} e^{1/x} = 1^+, \quad \lim_{x\to 0^+} e^{1/x} = \infty.$$

Since $f'(x) = -(1/x^2)e^{1/x}$, the function has no critical points, but we notice that

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \frac{-1/x^2}{e^{-1/x}} = \lim_{x\to 0^-} \frac{2/x^3}{(1/x^2)e^{-1/x}} = \lim_{x\to 0^-} \frac{2/x}{e^{-1/x}} = \lim_{x\to 0^-} \frac{-2/x^2}{(1/x^2)e^{-1/x}} = \lim_{x\to 0^-} (-2e^{1/x}) = 0^-.$$

We can locate points of inflection by solving $0 = f''(x) = \dfrac{2}{x^3}e^{1/x} + \dfrac{1}{x^4}e^{1/x} = \dfrac{1}{x^4}(2x + 1)e^{1/x}$. Since the only solution is $x = -1/2$, and $f''(x)$ changes sign as x passes through this value, a point of inflection is $(-1/2, 1/e^2)$. This information, shown in the left figure below, leads to the final graph in the right figure.

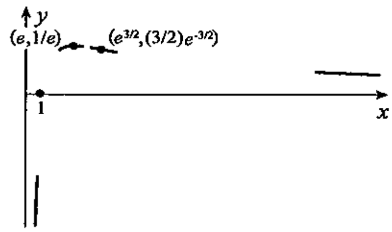


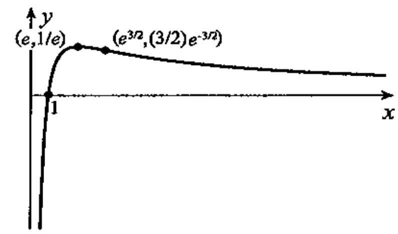


47. The following limits show that $y = 0$ is a horizontal asymptote, and the y-axis is a vertical asymptote,

$$\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty, \qquad \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0^+.$$

For critical points we solve $0 = f'(x) = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$. The only solution is $x = e$. Since $f'(x)$ changes from positive to negative as x increases through e, this critical point gives a relative maximum of $f(e) = 1/e$. For points of inflection, we solve $0 = f''(x) = \dfrac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \dfrac{-3 + 2\ln x}{x^3}$. Since $f''(x)$ changes sign as x passes through the solution $x = e^{3/2}$, there is a point of inflection at $(e^{3/2}, (3/2)/e^{3/2})$. This information, shown in the left figure below, leads to the final graph in the right figure.





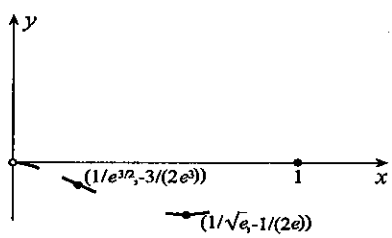
48. For critical points, we solve $0 = f'(x) = 2x\ln x + x^2/x = x(2\ln x + 1)$. The only solution is $x = 1/\sqrt{e}$. Since $f''(x) = 2\ln x + 2x/x + 1 = 2\ln x + 3$, it follows that $f''(1/\sqrt{e}) = 2$. The critical point therefore gives a relative minimum of $f(1/\sqrt{e}) = -1/(2e)$. Since $f''(e^{-3/2}) = 0$, and $f''(x)$ changes sign as x passes through $e^{-3/2}$, there is a point of inflection at $\left(e^{-3/2}, -3/(2e^3)\right)$. We use L'Hôpital's rule to show that the graph approaches the origin,

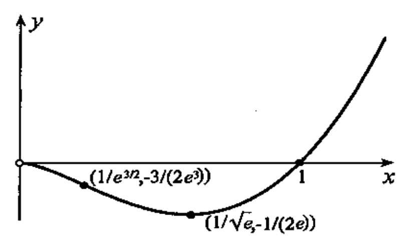
$$\lim_{x\to 0^+} x^2\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+}\left(-\frac{x^2}{2}\right) = 0^-.$$

The slope of the graph also approaches zero as $x \to 0^+$,

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} [x(2\ln x + 1)] = \lim_{x\to 0^+} \frac{2\ln x + 1}{1/x} = \lim_{x\to 0^+} \frac{2/x}{-1/x^2} = \lim_{x\to 0^+} (-2x) = 0^-.$$

This information, shown in the left figure below, leads to the final graph in the right figure.





49. We evaluate the following limits to examine the graph near the discontinuity at $x = 0$:

$$\lim_{x\to 0^+} x\,e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \infty, \qquad \lim_{x\to 0^-} x\,e^{1/x} = 0^-.$$

For critical points we solve $0 = f'(x) = e^{1/x} + x\,e^{1/x}(-1/x^2) = [(x-1)/x]e^{1/x}$. The only solution is $x = 1$. Since $f'(x)$ changes from negative to positive as x increases through $x = 1$, this critical point gives a relative minimum of $f(1) = e$. To determine the slope of the graph as $x \to 0^-$, we consider

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-}\left(\frac{x-1}{x}\right)e^{1/x} = \lim_{x\to 0^-} \frac{x-1}{x\,e^{-1/x}}.$$

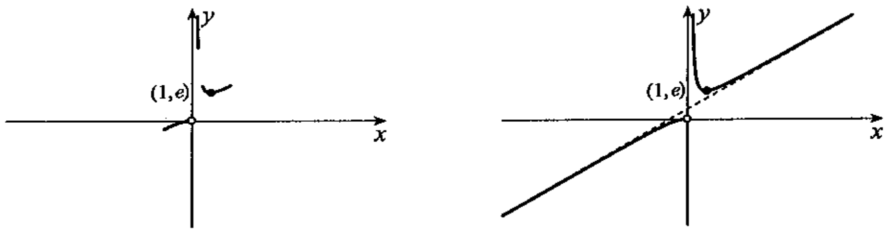
Since

$$\lim_{x\to 0^-} x\,e^{-1/x} = \lim_{x\to 0^-} \frac{e^{-1/x}}{1/x} = \lim_{x\to 0^-} \frac{e^{-1/x}(1/x^2)}{-1/x^2} = -\infty,$$

it follows that $\lim_{x\to 0^-} f'(x) = 0^+$. For points of inflection, we solve

$$0 = f''(x) = \frac{1}{x^2}e^{1/x} + \left(\frac{x-1}{x}\right)e^{1/x}\left(\frac{-1}{x^2}\right) = \frac{1}{x^3}e^{1/x}.$$

There are no points of inflection. This information, shown in the left figure below, leads to the final graph in the right figure. The line $y = x + 1$ is an oblique asymptote, but we do not yet have the tools to show this.



50. The limits

$$\lim_{x\to 0^+}\frac{x^2}{\ln x} = 0^-, \qquad \lim_{x\to 1^-}\frac{x^2}{\ln x} = -\infty, \qquad \lim_{x\to 1^+}\frac{x^2}{\ln x} = \infty,$$

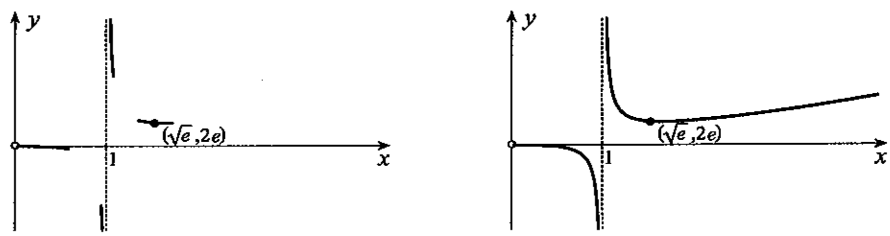
show that the curve approaches the origin as $x \to 0^+$, and that the line $x = 1$ is a vertical asymptote. For critical points, we solve

$$0 = f'(x) = \frac{2x}{\ln x} - \frac{x^2}{(\ln x)^2}\left(\frac{1}{x}\right) = \frac{x}{(\ln x)^2}(2\ln x - 1).$$

Thus, $x = \sqrt{e}$, and this gives a relative minimum of $f(\sqrt{e}) = 2e$ ($f'(x)$ changing from negative to positive as x increases through $\sqrt{e}$). The following limit shows that the slope of the graph approaches zero as $x \to 0^+$,

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}\left[\frac{2x}{\ln x} - \frac{x}{(\ln x)^2}\right] = 0^-.$$

This information, shown in the left figure below, leads to the final graph in the right figure.



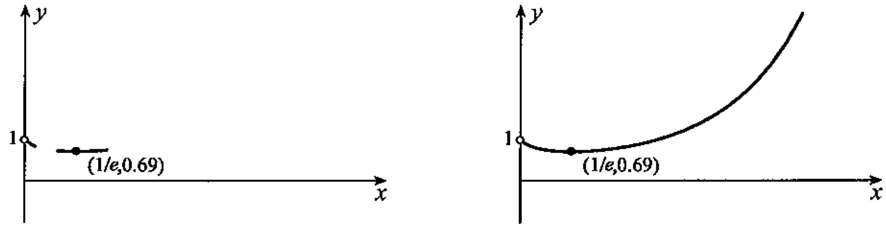
51. To find critical points of the function, we set $y = x^x$ and take logarithms, $\ln y = x\ln x$. Implicit differentiation gives

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x \quad \Longrightarrow \quad \frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x).$$

Since $f'(x) = 0$ when $x = 1/e$ and $f'(x)$ changes from negative to positive as x increases through $1/e$, there is a relative minimum of $f(1/e) \approx 0.69$. The function is undefined for $x = 0$, and we therefore set $L = \lim_{x\to 0^+} x^x$ and take natural logarithms,

$$\ln L = \ln \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

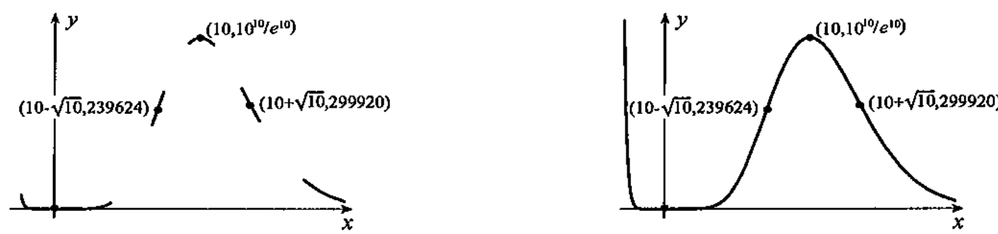
Thus, $L = e^0 = 1$, so that the graph approaches 1 as $x \to 0^+$. To find the slope of the graph as $x \to 0^+$ we note that $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} x^x(1 + \ln x) = -\infty$. This information, shown in the left figure below, leads to the final graph in the right figure.



52. For critical points we solve $0 = f'(x) = 10x^9e^{-x} - x^{10}e^{-x} = x^9(10-x)e^{-x}$. The solutions are $x = 0$ and $x = 10$. Since $f'(x)$ changes from positive to negative as x increases through 10, this critical point gives a relative maximum of $f(10) = 10^{10}/e^{10}$. We have a relative minimum at $f(0) = 0$ since $f'(x)$ changes from negative to positive as x increases through 0. For points of inflection, we solve

$$0 = f''(x) = (90x^8 - 10x^9)e^{-x} - (10x^9 - x^{10})e^{-x} = x^8(x^2 - 20x^9 + 90)e^{-x}.$$

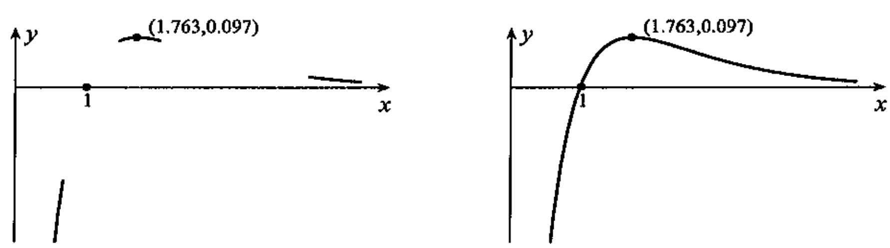
Solutions are $x = \dfrac{20 \pm \sqrt{400-360}}{2} = 10 \pm \sqrt{10}$. Since $f''(x)$ changes sign as x passes through these values, there are points of inflection at $(10-\sqrt{10}, 239\,624)$ and $(10+\sqrt{10}, 299\,920)$. Repeated applications of L'Hôpital's rule show that $\lim_{x\to\infty} x^{10}e^{-x} = 0^+$. This information, shown in the left figure below, leads to the final graph in the right figure.



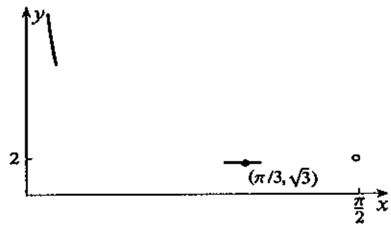
53. The limit $\lim_{x\to 0^+} e^{-x}\ln x = -\infty$ shows that the y-axis is a vertical asymptote. For critical points we solve $0 = f'(x) = -e^{-x}\ln x + e^{-x}(1/x) = (1/x)e^{-x}(1 - x\ln x)$. To find the only solution of this equation, we use Newton's iterative procedure with $x_1 = 2$, $\quad x_{n+1} = x_n - \dfrac{1 - x_n \ln x_n}{-1 - \ln x_n}$. Iteration gives $x_2 = 1.77$, $x_3 = 1.763$, and $x_4 = 1.763$. It is straightforward to verify that this is the root to 3 decimals. Since $f'(x)$ changes from positive to negative as x increases through 1.763, there is a relative maximum of $f(1.763) = 0.097$. L'Hôpital's rule shows that the x-axis is a horizontal asymptote,

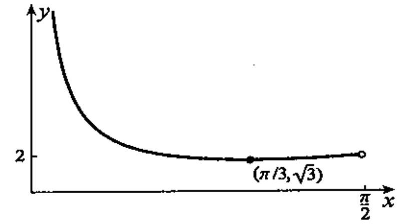
$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty}\frac{\ln x}{e^x} = \lim_{x\to\infty}\frac{1/x}{e^x} = 0^+.$$

This information, shown in the left figure below, leads to the final graph in the right figure.



54. For critical points, we solve $0 = f'(x) = -2\csc x\cot x + \csc^2 x = \csc^2 x(1 - 2\cos x)$. The only critical point is $x = \pi/3$. Since $f'(x)$ changes from negative to positive as x increases through $\pi/3$, there is a relative minimum of $f(\pi/3) = \sqrt{3}$ at this critical point. The limit $\lim_{x\to 0^+}(2\csc x - \cot x) = \lim_{x\to 0^+}\frac{2-\cos x}{\sin x} = \infty$ indicates that the y-axis is a vertical asymptote. This information, shown in the left figure below, leads to the final graph in the right figure.





55. If we set $L = \lim_{x\to\infty}\left(\frac{x+a}{x+b}\right)^{cx}$, and take natural logarithms, then

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{x+a}{x+b}\right)^{cx}\right] = \lim_{x\to\infty}\left[cx\ln\left(\frac{x+a}{x+b}\right)\right] = c\lim_{x\to\infty}\left[\frac{\ln\left(\frac{x+a}{x+b}\right)}{1/x}\right]$$

$$= c\lim_{x\to\infty}\left\{\frac{\frac{x+b}{x+a}\left[\frac{(x+b)-(x+a)}{(x+b)^2}\right]}{-1/x^2}\right\} = c\lim_{x\to\infty}\frac{(a-b)x^2}{(x+a)(x+b)}$$

$$= c\lim_{x\to\infty}\frac{2(a-b)x}{2x+a+b} = c\lim_{x\to\infty}\frac{2(a-b)}{2} = c(a-b), \quad \text{and this implies that } L = e^{c(a-b)}.$$

56. If we set $L = \lim_{x\to\infty}(x-\ln x)$, and take exponentials on both sides of the equation, $e^L = e^{\lim_{x\to\infty}(x-\ln x)}$. If we interchange the limit and exponentiation operations,

$$e^L = \lim_{x\to\infty} e^{x-\ln x} = \lim_{x\to\infty}\frac{e^x}{x} = \lim_{x\to\infty}\frac{e^x}{1} = \infty.$$

It follows therefore that $L = \lim_{x\to\infty}(x-\ln x) = \infty$ also.

57. $$\lim_{E\to 0^+}\left(\frac{e^E+e^{-E}}{e^E-e^{-E}}-\frac{1}{E}\right) = \lim_{E\to 0^+}\frac{E(e^E+e^{-E})-e^E+e^{-E}}{E(e^E-e^{-E})} = \lim_{E\to 0^+}\frac{e^E+e^{-E}+E(e^E-e^{-E})-e^E-e^{-E}}{e^E-e^{-E}+E(e^E+e^{-E})}$$

$$= \lim_{E\to 0^+}\frac{E(e^E-e^{-E})}{e^E-e^{-E}+E(e^E+e^{-E})} = \lim_{E\to 0^+}\frac{e^E-e^{-E}+E(e^E+e^{-E})}{e^E+e^{-E}+e^E+e^{-E}+E(e^E-e^{-E})}$$

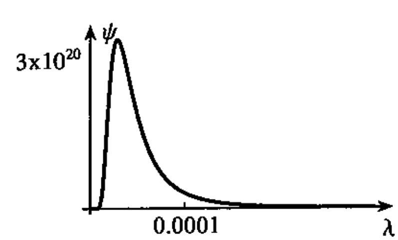
$$= 0$$

58. (a) $$\lim_{\lambda\to 0^+}\psi(\lambda) = \lim_{\lambda\to 0^+}\frac{-5k\lambda^{-6}}{e^{c/\lambda}(-c/\lambda^2)} = \lim_{\lambda\to 0^+}\frac{5k\lambda^{-4}}{ce^{c/\lambda}} = \lim_{\lambda\to 0^+}\frac{5k(-4)\lambda^{-5}}{ce^{c/\lambda}(-c/\lambda^2)}$$

$$= \lim_{\lambda\to 0^+}\frac{20k\lambda^{-3}}{c^2e^{c/\lambda}} = \cdots = \lim_{\lambda\to 0^+}\frac{120k}{c^5e^{c/\lambda}} = 0$$

$$\lim_{\lambda\to\infty}\psi(\lambda) = \lim_{\lambda\to\infty}\frac{-5k\lambda^{-6}}{e^{c/\lambda}(-c/\lambda^2)} = \lim_{\lambda\to\infty}\frac{5k}{c\lambda^4e^{c/\lambda}} = 0$$

(b) For critical points of $\psi(\lambda)$ we solve $0 = \psi'(\lambda) = \frac{-k}{[\lambda^5(e^{c/\lambda}-1)]^2}\left[5\lambda^4(e^{c/\lambda}-1)+\lambda^5e^{c/\lambda}(-c/\lambda^2)\right]$,
and therefore $0 = \lambda^3[5\lambda(e^{c/\lambda}-1)-ce^{c/\lambda}]$.
Since $\lambda \neq 0$, we must set $(5\lambda - c)e^{c/\lambda} = 5\lambda$.
This equation was solved using Newton's iterative procedure in Exercise 34 of Section 4.1. The critical point is $\lambda = 0.000\,029\,0$.
(c) The limits in (a) together with the fact that $\psi(\lambda)$ is always positive for $\lambda > 0$ implies that the critical point must give a relative maximum. The graph of the function is shown to the right.



59. $f(\theta) = \sin\theta \left[\dfrac{\sin\left(\frac{\pi}{2}\cos\theta - \frac{\pi}{2}\right)}{\cos\theta - 1} + \dfrac{\sin\left(\frac{\pi}{2}\cos\theta + \frac{\pi}{2}\right)}{\cos\theta + 1}\right] = \sin\theta\left[\dfrac{-\cos\left(\frac{\pi}{2}\cos\theta\right)}{\cos\theta - 1} + \dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\cos\theta + 1}\right]$

$$= \sin\theta\,\cos\left(\frac{\pi}{2}\cos\theta\right)\left(\frac{-\cos\theta - 1 + \cos\theta - 1}{\cos^2\theta - 1}\right) = \sin\theta\,\cos\left(\frac{\pi}{2}\cos\theta\right)\left(\frac{-2}{-\sin^2\theta}\right) = \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

With L'Hôpital's rule, $\lim_{\theta\to 0} f(\theta) = \lim_{\theta\to 0} \dfrac{-2\sin\left(\frac{\pi}{2}\cos\theta\right)\left(-\frac{\pi}{2}\sin\theta\right)}{\cos\theta} = 0.$

60. By L'Hôpital's rule,

$$5 = \lim_{x\to 0} \frac{ae^{ax} - b + (1 + 2cx)\sin(x + cx^2)}{6x^2 + 10x}.$$

Since the limit of the numerator is $a - b$ and that of the denominator is 0, the only way this limit can be 5 is for $a = b$. In this case,

$$5 = \lim_{x\to 0} \frac{ae^{ax} - a + (1 + 2cx)\sin(x + cx^2)}{6x^2 + 10x}$$
$$= \lim_{x\to 0} \frac{a^2e^{ax} + 2c\sin(x + cx^2) + (1 + 2cx)^2\cos(x + cx^2)}{12x + 10} = \frac{a^2 + 1}{10}.$$

Thus, $a = \pm 7 = b$, and c is arbitrary.

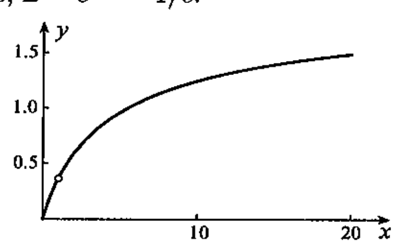
61. (a) A plot on the interval $0 \le x \le 20$ is shown in the left figure below. We added the hole at $x = 1$ since the function is undefined there. Suppose we let

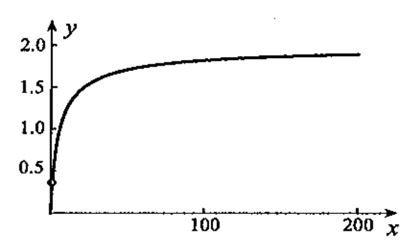
$$L = \lim_{x\to 1} x\left(\frac{2}{x+1}\right)^{(x+1)/(x-1)} = \lim_{x\to 1}\left(\frac{2}{x+1}\right)^{(x+1)/(x-1)},$$

provided the latter limit exists. To evaluate it we take natural logarithms and use L/Hôpital's rule,

$$\ln L = \ln\left[\lim_{x\to 1}\left(\frac{2}{x+1}\right)^{(x+1)/(x-1)}\right] = \lim_{x\to 1}\left[\left(\frac{x+1}{x-1}\right)\ln\left(\frac{2}{x+1}\right)\right] = \lim_{x\to 1}\frac{\ln\left(\dfrac{2}{x+1}\right)}{\dfrac{x-1}{x+1}}$$
$$= \lim_{x\to 1}\left[\frac{\left(\dfrac{x+1}{2}\right)\left(\dfrac{-2}{(x+1)^2}\right)}{\dfrac{(x+1)-(x-1)}{(x+1)^2}}\right] = \lim_{x\to 1}\left[-\frac{x+1}{2}\right] = -1.$$

Hence, $L = e^{-1} = 1/e$.





(b) A plot on the interval $0 \le x \le 200$ is shown in the right figure above. To evaluate the limit of $f(x)$ as $x \to \infty$, we write

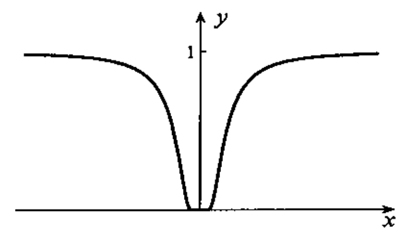
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\left(\frac{2}{x+1}\right)^{1+2/(x-1)} = \lim_{x\to\infty}\left(\frac{2x}{x+1}\right)\left(\frac{2}{x+1}\right)^{2/(x-1)} = 2\lim_{x\to\infty}\left(\frac{2}{x+1}\right)^{2/(x-1)},$$

provided this limit exists. If we set it equal to L and take natural logarithms,

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{2}{x+1}\right)^{2/(x-1)}\right] = \lim_{x\to\infty}\left[\frac{2}{x-1}\ln\left(\frac{2}{x+1}\right)\right] = 2\lim_{x\to\infty}\left[\frac{\left(\frac{x+1}{2}\right)\left(\frac{-2}{(x+1)^2}\right)}{1}\right] = 0.$$

Hence $L = e^0 = 1$, and it follows that $\lim\limits_{x\to\infty} f(x) = 2$.

62. (a) Limits as $x \to 0^+$ and $x \to \infty$, together with symmetry about the y-axis, give the graph to the right.



(b) If we can show that the right-hand limit is zero, then the left-hand limit must also be zero. Suppose we set $L = \lim_{x\to 0^+} x^{-n}e^{-1/x^2}$ and take natural logarithms,

$$\ln L = \ln\left(\lim_{x\to 0^+}\frac{e^{-1/x^2}}{x^n}\right) = -\lim_{x\to 0^+}\left(\frac{1}{x^2} + n\ln x\right) = -\lim_{x\to 0^+}\left(\frac{1+nx^2\ln x}{x^2}\right).$$

Since

$$\lim_{x\to 0^+} x^2\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x^2} = \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}(-x^2/2) = 0,$$

it follows that $\ln L = -\lim\limits_{x\to 0^+}\left(\dfrac{1+nx^2\ln x}{x^2}\right) = -\infty$. Consequently, $L = 0$.

(c) $f'(0) = \lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0}\dfrac{e^{-1/h^2}}{h} = 0$ (by part (b))

Suppose that k is some integer for which $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{h\to 0}\frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h\to 0}\frac{f^{(k)}(h)}{h}.$$

Now, any number of differentiations of $f(x) = e^{-1/x^2}$ gives rise to terms of the form $(A/x^n)\,e^{-1/x^2}$, where n is a positive integer and A is a constant. It follows then that $f^{(k)}(h)/h$ must consist of terms of the form $(A/h^n)e^{-1/h^2}$ which have limit zero as $h \to 0$. Thus, $f^{(k+1)}(0) = 0$, and by mathematical induction, $f^{(n)}(0) = 0$ for all $n \geq 1$.

EXERCISES 4.12

1. $dy = f'(x)\,dx = (2x+3)\,dx$

2. $dy = f'(x)\,dx = \left[\dfrac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right]dx = \dfrac{-2}{(x-1)^2}dx$

3. $dy = f'(x)\,dx = \dfrac{2x-2}{2\sqrt{x^2-2x}}dx = \dfrac{x-1}{\sqrt{x^2-2x}}dx$

4. $dy = f'(x)\,dx = [2x\cos(x^2+2) + \sin x]\,dx$

5. $dy = f'(x)\,dx = [(1/3)x^{-2/3} - (5/3)x^{2/3}]dx$

6. $dy = f'(x)\,dx = [3x^2\sqrt{3-4x^2} + (1/2)x^3(3-4x^2)^{-1/2}(-8x)]\,dx$
$$= \frac{3x^2(3-4x^2)-4x^4}{\sqrt{3-4x^2}}dx = \frac{x^2(9-16x^2)}{\sqrt{3-4x^2}}dx$$

7. $dy = f'(x)\,dx = (2x\sin x + x^2\cos x)dx$

8. Since $f(x) = \dfrac{(x-1)^3+6}{(x-1)^2} = x-1+\dfrac{6}{(x-1)^2}$, $\quad dy = f'(x)dx = \left[1 - \dfrac{12}{(x-1)^3}\right]dx.$

9. $dy = f'(x)\,dx = [(1/2)(1+\sqrt{1-x})^{-1/2}(1/2)(1-x)^{-1/2}(-1)]dx = \dfrac{-1}{4\sqrt{1-x}\sqrt{1+\sqrt{1-x}}}dx$

10. $dy = f'(x)\,dx = \left[\dfrac{(x^3+5x)(3x^2-4x)-(x^3-2x^2)(3x^2+5)}{(x^3+5x)^2}\right]dx$

$$= \frac{2x^4+10x^3-10x^2}{(x^3+5x)^2}dx = \frac{2(x^2+5x-5)}{(x^2+5)^2}dx$$

11. Approximate percentage changes are

$$\frac{100\,dM}{M} = \frac{100(m\,dv)}{mv} = \frac{100\,dv}{v} = 1, \qquad \frac{100\,dK}{K} = \frac{100(mv\,dv)}{mv^2/2} = 2\left(\frac{100\,dv}{v}\right) = 2.$$

12. The approximate percentage change in F is

$$100\frac{dF}{F} = \frac{100}{F}\left(-\frac{2GmM}{r^3}dr\right) = -\frac{200GmM}{r^3}dr\left(\frac{r^2}{GmM}\right) = -2\left(100\frac{dr}{r}\right) = -2(2) = -4.$$

13. The approximate percentage change in R is

$$100\frac{dR}{R} = \frac{100}{R}\left(\frac{2v^2\cos 2\theta}{9.81}d\theta\right) = \frac{200v^2\cos 2\theta}{9.81}d\theta\left(\frac{9.81}{v^2\sin 2\theta}\right)$$

$$= 200\cot 2\theta\,d\theta = 2\theta\cot 2\theta\left(100\frac{d\theta}{\theta}\right) = 2\theta\cot 2\theta,$$

since the change in θ is 1%. When $\theta = \pi/3$, this becomes $100\dfrac{dR}{R} = 2\left(\dfrac{\pi}{3}\right)\left(-\dfrac{1}{\sqrt{3}}\right) = -\dfrac{2\sqrt{3}\pi}{9}$.

14. The approximate percentage change in H is

$$100\frac{dH}{H} = \frac{100}{H}\left(\frac{2v^2\sin\theta\cos\theta}{19.62}d\theta\right) = \frac{200v^2\sin\theta\cos\theta}{19.62}d\theta\left(\frac{19.62}{v^2\sin^2\theta}\right)$$

$$= 200\cot\theta\,d\theta = 2\theta\cot\theta\left(100\frac{d\theta}{\theta}\right) = 4\theta\cot\theta,$$

since the change in θ is 2%. When $\theta = \pi/3$, this becomes

$$100\frac{dH}{H} = 4\left(\frac{\pi}{3}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}\pi}{9}.$$

15. Since $V = kP^{-5/7}$ for some constant k, the approximate percentage change in V is

$$\frac{100\,dV}{V} = \frac{100(-5/7)kP^{-12/7}\,dP}{kP^{-5/7}} = -\frac{5}{7}\left(\frac{100\,dP}{P}\right) = -\frac{10}{7}.$$

16. The differential of F is $dF = -\dfrac{2GmM}{r^3}dr$. If we set $dF = -0.01m$ for a decrease from $9.81m$ to $9.80m$, then

$$-0.01m = -\frac{2GmM}{r^3}dr.$$

But when $r = 6.37\times 10^6$, we know that $F = 9.81m$ so that

$$9.81m = \frac{GmM}{(6.37\times 10^6)^2}.$$

Consequently,

$$-0.01m = \frac{-2}{(6.37\times 10^6)^3}(9.81m)(6.37\times 10^6)^2\,dr,$$

and this equation implies that $dr = 3.25\times 10^3$. Thus, at a height of 3.25 km, the gravitational force of attraction decreases to $9.80m$ N.

17. The approximate percentage change in F is

$$\begin{aligned}100\frac{dF}{F} &= \frac{100}{F}\left[\frac{-9.81\mu m(-\sin\theta+\mu\cos\theta)}{(\cos\theta+\mu\sin\theta)^2}d\theta\right]\\ &= \left[\frac{9.81(100)\mu m(\sin\theta-\mu\cos\theta)}{(\cos\theta+\mu\sin\theta)^2}\right]d\theta\left(\frac{\cos\theta+\mu\sin\theta}{9.81\mu m}\right)\\ &= \frac{100(\sin\theta-\mu\cos\theta)}{\cos\theta+\mu\sin\theta}d\theta = \frac{\theta(\sin\theta-\mu\cos\theta)}{\cos\theta+\mu\sin\theta}\left(100\frac{d\theta}{\theta}\right)\\ &= \frac{2\theta(\sin\theta-\mu\cos\theta)}{\cos\theta+\mu\sin\theta},\end{aligned}$$

since the change in θ is 2%. When $\theta = \pi/4$, this becomes

$$100\frac{dF}{F} = \frac{2(\pi/4)(1/\sqrt{2}-\mu/\sqrt{2})}{1/\sqrt{2}+\mu/\sqrt{2}} = \frac{\pi(1-\mu)}{2(\mu+1)}.$$

18. (a) The approximate percentage error in V due to an $a\%$ error in r is

$$100\frac{dV}{V} = \frac{100}{V}(2\pi rh\,dr) = \frac{200\pi rh}{\pi r^2h}dr = 2\left(100\frac{dr}{r}\right) = 2a.$$

(b) The approximate percentage error in V due to a $b\%$ error in h is

$$100\frac{dV}{V} = \frac{100}{V}(\pi r^2\,dh) = \frac{100\pi r^2}{\pi r^2h}dh = 100\frac{dh}{h} = b.$$

(c) The maximum approximate percentage error in V due to errors $a\%$ in r and $b\%$ in h is

$$100\frac{\text{Maximum change in V}}{V} = \frac{100}{\pi r^2h}(2\pi rh\,dr+\pi r^2\,dh) = 2\left(100\frac{dr}{r}\right)+\left(100\frac{dh}{h}\right) = 2a+b.$$

19. The approximate percentage error in y is $\quad 100\frac{dy}{y} = \frac{100}{x^n}nx^{n-1}\,dx = n\left(100\frac{dx}{x}\right) = na.$

20. (a) The approximate percentage error in z due to an $a\%$ error in x is

$$100\frac{dz}{z} = \frac{100}{x^ny^m}(nx^{n-1}y^m\,dx) = n\left(100\frac{dx}{x}\right) = na.$$

(b) The approximate percentage error in z due to a $b\%$ error in y is

$$100\frac{dz}{z} = \frac{100}{x^ny^m}(my^{m-1}x^n\,dx) = m\left(100\frac{dy}{y}\right) = mb.$$

(c) The maximum approximate percentage error in z due to errors $a\%$ in x and $b\%$ in y is

$$100\frac{\text{Maximum change in z}}{z} = \frac{100}{x^ny^m}(nx^{n-1}y^m\,dx+my^{m-1}x^n\,dy) = n\left(100\frac{dx}{x}\right)+m\left(100\frac{dy}{y}\right) = na+mb.$$

21. (a) The approximate percentage error in z due to an $a\%$ error in x is

$$100\frac{dz}{z} = \frac{100}{x^n/y^m}\left(\frac{nx^{n-1}}{y^m}dx\right) = n\left(100\frac{dx}{x}\right) = na.$$

(b) The approximate percentage error in z due to a $b\%$ error in y is

$$100\frac{dz}{z} = \frac{100}{x^n/y^m}\left(\frac{-mx^n}{y^{m+1}}dx\right) = -m\left(100\frac{dy}{y}\right) = -mb.$$

(c) The maximum approximate percentage error in z due to errors $a\%$ in x and $b\%$ in y is

$$100\frac{\text{Maximum change in z}}{z} = \frac{100}{x^n/y^m}\left(\frac{nx^{n-1}}{y^m}dx - \frac{mx^n}{y^{m+1}}dy\right)$$
$$= n\left(100\frac{dx}{x}\right) - m\left(100\frac{dy}{y}\right).$$

For maximum error we take dx as positive and dy as negative in which case

$$100\frac{\text{Maximum change in z}}{z} = n\left(100\frac{dx}{x}\right) + m\left(100\frac{-dy}{y}\right) = na + mb.$$

22. The approximate percentage change in n is

$$100\frac{dn}{n} = \frac{100}{n}\frac{(1/2)\cos[(\psi_m+\gamma)/2]}{\sin(\gamma/2)}d\psi_m$$
$$= \frac{50\cos[(\psi_m+\gamma)/2]}{\sin(\gamma/2)}d\psi_m\left\{\frac{\sin(\gamma/2)}{\sin[(\psi_m+\gamma)/2]}\right\} = \frac{\psi_m}{2}\cot[(\psi_m+\gamma)/2]\left(100\frac{d\psi_m}{\psi_m}\right).$$

Since the error in ψ_m is 1% when it is $\pi/6$ and $\gamma = \pi/3$,

$$100\frac{dn}{n} = \frac{\pi/6}{2}\cot[(\pi/6+\pi/3)/2](1) = \frac{\pi}{12}.$$

23. The approximate percentage change in n is

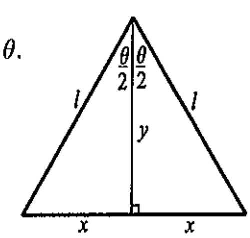
$$100\frac{dn}{n} = \frac{100}{n}\left\{\frac{\sin(\gamma/2)(1/2)\cos[(\psi_m+\gamma)/2] - \sin[(\psi_m+\gamma)/2](1/2)\cos(\gamma/2)}{\sin^2(\gamma/2)}\right\}d\gamma$$
$$= 50\left[\frac{-\sin(\psi_m/2)}{\sin^2(\gamma/2)}\right]d\gamma\left\{\frac{\sin(\gamma/2)}{\sin[(\psi_m+\gamma)/2]}\right\} = \frac{-\gamma\sin(\psi_m/2)}{2\sin(\gamma/2)\sin[(\psi_m+\gamma)/2]}\left(100\frac{d\gamma}{\gamma}\right).$$

Since the error in γ is 1% when $\psi_m = \pi/6$ and $\gamma = \pi/3$,

$$100\frac{dn}{n} = -\frac{(\pi/3)\sin(\pi/12)}{2\sin(\pi/6)\sin[(\pi/6+\pi/3)/2]}(1) = -\frac{\sqrt{2}\pi}{3}\sin(\pi/12).$$

REVIEW EXERCISES

1. (a) The area of the triangle is $A = 2\left(\frac{1}{2}xy\right) = \left(l\sin\frac{\theta}{2}\right)\left(l\cos\frac{\theta}{2}\right) = \frac{l^2}{2}\sin\theta.$



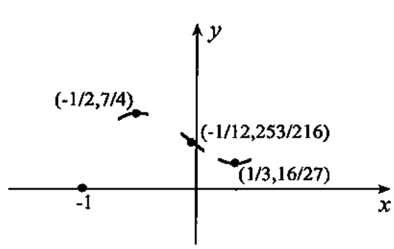
(b) If we differentiate with respect to t,

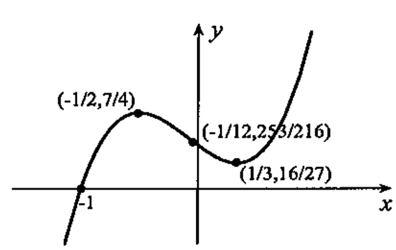
$$\frac{dA}{dt} = \frac{l^2}{2}\cos\theta\frac{d\theta}{dt} = \frac{l^2}{4}\cos\theta.$$

When $\theta = 0, \pi/2, \pi$, this rate has values, respectively, $l^2/4$, 0, $-l^2/4$.

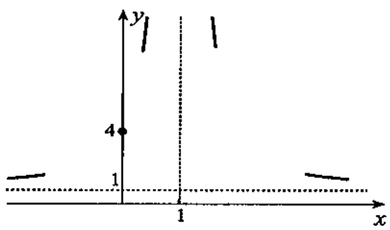
(c) $|dA/dt|$ is largest when $\theta = 0, \pi$ and smallest when $\theta = \pi/2$.

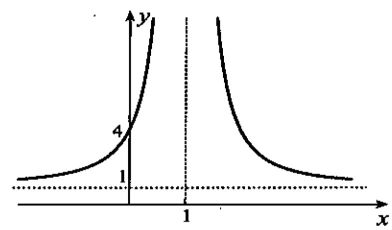
2. (a) For critical points we solve $0 = f'(x) = 12x^2 + 2x - 2 = 2(3x-1)(2x+1)$. Solutions are $x = 1/3$ and $x = -1/2$. Since $f''(x) = 24x + 2$, it follows that $f''(1/3) = 10$ and $f''(-1/2) = -10$. Consequently, $x = 1/3$ gives a relative minimum of $f(1/3) = 16/27$, and $x = -1/2$ gives a relative maximum of $f(-1/2) = 7/4$. Since $f''(-1/12) = 0$ and $f''(x)$ changes sign as x passes through $-1/12$, there is a point of inflection at $(-1/12, 253/216)$. This information, shown in the left figure below, leads to the final graph in the right figure.





(b) With the function written as $f(x) = \dfrac{(x^2 - 2x + 1) + 3}{(x-1)^2} = 1 + \dfrac{3}{(x-1)^2}$, critical points are given by $0 = f'(x) = -6/(x-1)^3$. There are no solutions and hence no relative extrema. Since the second derivative $f''(x) = 18/(x-1)^4$ never vanishes, there are no points of inflection. Limits as $x \to \pm\infty$ and right- and left-limits at $x = 1$ give the information in the left figure below. The final graph is to the right.





3. If x and y are any two positive numbers with sum $x + y = c$, their product is $P = P(x) = xy = x(c - x)$, $0 < x < c$. For critical points of $P(x)$, we solve $0 = P'(x) = c - 2x$. The solution is $x = c/2$. Since

$$\lim_{x\to 0^+} P(x) = 0, \qquad P(c/2) = \frac{c^2}{4}, \qquad \lim_{x\to c^-} P(x) = 0,$$

the maximum value of $P(x)$ is $c^2/4$, occurring when $x = y = c/2$.

4. If x and y are any two positive numbers with sum $x + y = c$, their product is $P = P(x) = xy = x(c - x)$, $0 < x < c$. For critical points of $P(x)$, we solve $0 = P'(x) = c - 2x$. The solution is $x = c/2$. Since

$$\lim_{x\to 0^+} P(x) = 0, \qquad P(c/2) = \frac{c^2}{4}, \qquad \lim_{x\to c^-} P(x) = 0,$$

it follows that $P(x)$ has no absolute minimum on the interval $0 < x < c$. The product can be made arbitrarily close to 0 by choosing x or y sufficiently close to 0.

5. If x and y are any two positive numbers with product $xy = c$, their sum is $S = S(x) = x + y = x + (c/x)$, $x > 0$. For critical points of $S(x)$ we solve $0 = S'(x) = 1 - c/x^2$. The positive solution is $x = \sqrt{c}$. Since

$$\lim_{x\to 0^+} S(x) = \infty, \qquad S(\sqrt{c}) = 2\sqrt{c}, \qquad \lim_{x\to\infty} S(x) = \infty,$$

the minimum value of $S(x)$ is $2\sqrt{c}$, occurring when $x = y = \sqrt{c}$.

6. If x and y are any two positive numbers with product $xy = c$, their sum is $S = S(x) = x + y = x + (c/x)$, $x > 0$. For critical points of $S(x)$ we solve $0 = S'(x) = 1 - c/x^2$. The positive solution is $x = \sqrt{c}$. Since

$$\lim_{x\to 0^+} S(x) = \infty, \qquad S(\sqrt{c}) = 2\sqrt{c}, \qquad \lim_{x\to\infty} S(x) = \infty,$$

it follows that $S(x)$ has no absolute maximum on the interval $x > 0$. The sum can be made arbitrarily large by choosing x or y sufficiently close to 0.

7. If we differentiate the cosine law $\ell^2 = 16 + 9 - 2(4)(3)\cos\theta = 25 - 24\cos\theta$ with respect to time t, we obtain $2\ell\dfrac{d\ell}{dt} = 24\sin\theta\dfrac{d\theta}{dt}$. When $\ell = 4$, we find that $16 = 25 - 24\cos\theta \implies \cos\theta = 3/8$. It follows that $\sin\theta = \sqrt{1 - 9/64} = \sqrt{55}/8$, and at this instant,

$$2(4)(-1) = 24\left(\frac{\sqrt{55}}{8}\right)\frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{-8}{3\sqrt{55}}.$$

The angle is therefore decreasing at $8/(3\sqrt{55})$ radians per minute.

8. $\displaystyle\lim_{x\to 0}\frac{3x^2 + 2x^3}{3x^3 - 2x^2} = \lim_{x\to 0}\frac{6x + 6x^2}{9x^2 - 4x} = \lim_{x\to 0}\frac{6 + 12x}{18x - 4} = -\frac{3}{2}$

9. $\displaystyle\lim_{x\to\infty}\frac{\sin 3x}{2x} = 0$

10. $\displaystyle\lim_{x\to 4}\frac{x^2 - 16}{x - 4} = \lim_{x\to 4}\frac{2x}{1} = 8$

11. $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{2x} = \lim_{x\to 0}\frac{3\cos 3x}{2} = \frac{3}{2}$ **12.** $\displaystyle\lim_{x\to -\infty}\frac{\sin x^2}{2x} = 0$

13. $\displaystyle\lim_{x\to 2^+}\frac{\sqrt{x-2}}{\sqrt{x}-\sqrt{2}} = \lim_{x\to 2^+}\frac{1/(2\sqrt{x-2})}{1/(2\sqrt{x})} = \lim_{x\to 2^+}\frac{\sqrt{x}}{\sqrt{x-2}} = \infty$

14. $\displaystyle\lim_{x\to\infty} x^2e^{-3x} = \lim_{x\to\infty}\frac{x^2}{e^{3x}} = \lim_{x\to\infty}\frac{2x}{3e^{3x}} = \lim_{x\to\infty}\frac{2}{9e^{3x}} = 0$

15. If we set $L = \lim_{x\to 0^+} x^{2x}$ and take natural logarithms, then

$$\ln L = \ln\left(\lim_{x\to 0^+} x^{2x}\right) = \lim_{x\to 0^+}(2x\ln x) = 2\lim_{x\to 0^+}\frac{\ln x}{1/x} = 2\lim_{x\to 0^+}\frac{1/x}{-1/x^2} = 0.$$

Hence, $L = e^0 = 1$.

16. $\displaystyle\lim_{x\to 0^+} x^4\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x^4} = \lim_{x\to 0^+}\frac{1/x}{-4/x^5} = \lim_{x\to 0^+}\left(\frac{-x^4}{4}\right) = 0$

17. $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{\tan 3x} = \lim_{x\to 0}\frac{2\cos 2x}{3\sec^2 3x} = \frac{2}{3}$

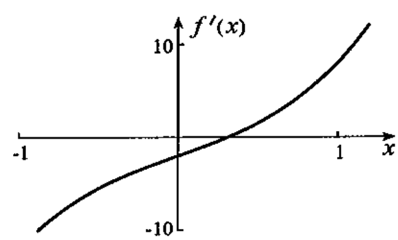
18. If we set $L = \displaystyle\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$ and take natural logarithms,

$$\begin{aligned}\ln L &= \lim_{x\to\infty} x\ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{x+1}{x-1}\right)}{1/x} = \lim_{x\to\infty}\frac{\left(\frac{x-1}{x+1}\right)\left[\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right]}{-1/x^2}\\ &= \lim_{x\to\infty}\frac{2x^2}{x^2-1} = \lim_{x\to\infty}\frac{4x}{2x} = 2.\end{aligned}$$

Thus, $L = e^2$.

19. $\displaystyle\lim_{x\to-\infty} x\,e^x = \lim_{x\to-\infty}\frac{x}{e^{-x}} = \lim_{x\to-\infty}\frac{1}{-e^{-x}} = 0$

20. (a) For critical points we solve
$0 = f'(x) = 4x^3 + 6x - 2 = 2(2x^3 + 3x - 1)$.
The graph of $f'(x)$ to the right indicates that there is a critical point between $x = 0$ and $x = 1$. To find it with Newton's iterative procdure, we use $x_1 = 1/3$ and



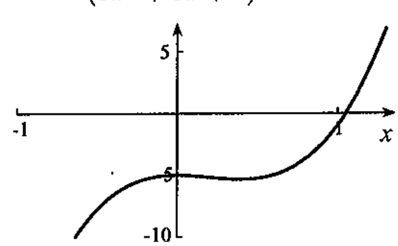
$$x_{n+1} = x_n - \frac{2x_n^3 + 3x_n - 1}{6x_n^2 + 3}.$$

Iteration gives $x_2 = 0.313\,131\,3$, $x_3 = 0.312\,908\,4$, $x_4 = 0.312\,908\,4$. Since $f'(0.312\,907\,5) = -6.5\times 10^{-6}$ and $f'(0.312\,908\,5) = 6.4\times 10^{-7}$, the critical point is 0.312 908.

(b) For critical points we solve

$$0 = f'(x) = \frac{(3x^3+5x+1)(3x^2) - (x^3+1)(9x^2+5)}{(3x^3+5x+1)^2} = \frac{10x^3 - 6x^2 - 5}{(3x^3+5x+1)^2}.$$

Thus, critical points are defined by the equation $10x^3 - 6x^2 - 5 = 0$. The graph of the function $10x^3 - 6x^2 - 5$ to the right indicates only one critical point just larger than 1. To find it we use $x_1 = 1$ and



$$x_{n+1} = x_n - \frac{10x_n^3 - 6x_n^2 - 5}{30x_n^2 - 12x_n}.$$

Iteration gives $x_2 = 1.055\,555\,6$, $x_3 = 1.051\,904\,7$, $x_4 = 1.051\,888\,1$, $x_5 = 1.051\,888\,1$. Since $f'(1.051\,887\,5) = -1.3\times 10^{-7}$ and $f'(1.051\,888\,5) = 8.5\times 10^{-8}$, the critical point is 1.051 888.

21. A plot of the displacement function is shown to the right. The velocity and acceleartion are

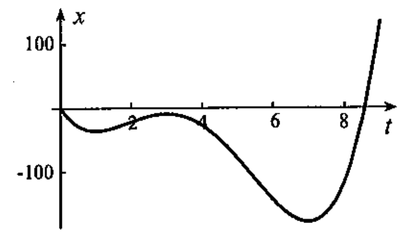
$v(t) = 4t^3 - 44t^2 + 124t - 84,$

$a(t) = 12t^2 - 88t + 124.$

The velocity is zero when

$0 = 4(t-1)(t-3)(t-7) \implies t = 1, 3, 7.$

The acceleration is zero when $0 = 4(3t^2 - 22t + 31)$ and solutions of this equation are $t = (11 \pm 2\sqrt{7})/3$.



22. The velocity at t_0 abruptly changes from a positive quantity to a negative quantity. This could be caused by a collision with a large object.

23. No. If the acceleration is constant, then the second derivative can never be zero (unless it is always zero in which case the graph is a straight line).

24. If squares of side length x are cut from the corners, the resulting box has volume

$V = x(l-2x)^2 = 4x^3 - 4lx^2 + l^2x,\ 0 \le x \le l/2.$

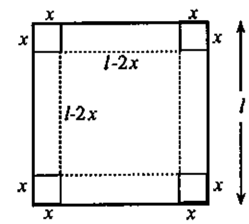
Critical points of V are given by

$0 = V'(x) = 12x^2 - 8lx + l^2 = (2x - l)(6x - l).$

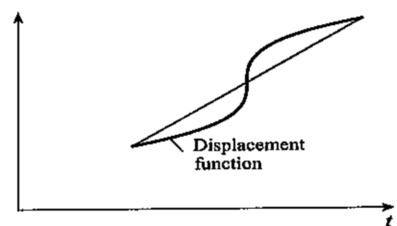
Thus, $x = l/2$ or $x = l/6$. Since

$$V(0) = 0, \quad V(l/6) = \frac{2l^3}{27}, \quad V(l/2) = 0,$$

maximum volume is $2l^3/27$.



25. (a) Yes The average velocity is the slope of the line joining the end points of the graph of the displacement function. Mean Value Theorem 3.19 guarantees at least one value of t for which the slope of the tangent line is equal to the slope of the line joining the end points. The slope of the tangent line is the instantaneous velocity.
(b) No



26. (a) We take the following limits at the discontinuities $x = \pm 1$:

$$\lim_{x\to -1^-} f(x) = -\infty, \quad \lim_{x\to -1^+} f(x) = \infty, \quad \lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = \infty.$$

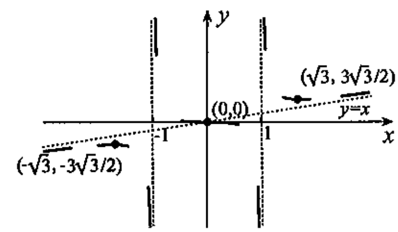
Critical points are given by

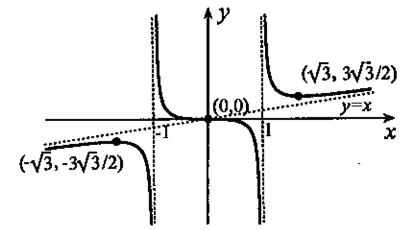
$$0 = f'(x) = \frac{(x^2-1)(3x^2) - x^3(2x)}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2}.$$

Solutions are $x = 0, \pm\sqrt{3}$. Since $f'(x)$ changes from a positive quantity to a negative quantity as x increases through $-\sqrt{3}$, there is a relative maximum at $x = -\sqrt{3}$ of $-3\sqrt{3}/2$. Since $f'(x)$ changes from a negative quantity to a positive quantity as x increases through $\sqrt{3}$, there is a relative minimum at $x = \sqrt{3}$ of $3\sqrt{3}/2$. Because $f'(x)$ does not change sign at $x = 0$, it gives a horizontal point of inflection $(0, 0)$. To verify that no other points of inflection occur, we calculate

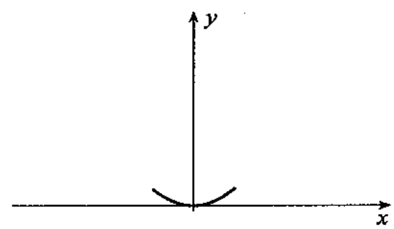
$$f''(x) = \frac{(x^2-1)^2(4x^3-6x) - (x^4-3x^2)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{x(2x^2+6)}{(x^2-1)^3}.$$

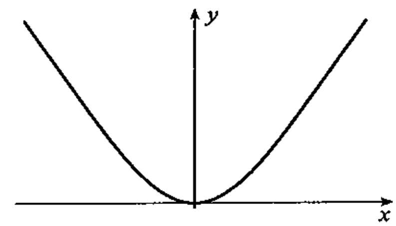
Since $f''(x)$ is 0 only at $x = 0$, there are no other points of inflection. By writing $f(x)$ in the form $f(x) = x + x/(x^2-1)$, we see that $y = x$ is an oblique asymptote for the graph. This information, shown in the left figure below, leads to the final graph in the right figure.





(b) For critical points we consider $0 = f'(x) = 2x + 2\sin x\,\cos x = 2x + \sin 2x$. The only solution of this equation is $x = 0$. For points of inflection we solve $0 = f''(x) = 2 + 2\cos 2x$. Solutions of this equation are $x = (2n+1)\pi/2$, where n is an integer. Since $f''(x)$ does not change sign as x passes through these points, the graph has no points of inflection. In addition, since $f''(0) = 4$, there is a relative minimum at $x = 0$ of $f(0) = 0$. This information, little as it is, shown in the left figure below, leads to the final graph in the right figure.





27. Not necessarily It depends on what side c is doing.

28. If the price is raised x dollars per ticket, then the expected numbers of $10 + x$, $9 + x$, and $8 + x$ dollar tickets the team expects to sell are respectively

$$10\,000(1 - x/10), \qquad 20\,000(1 - x/10), \qquad 30\,000(1 - x/10).$$

Total revenue at the new prices is therefore

$$\begin{aligned} R(x) &= 10\,000(1 - x/10)(10 + x) + 20\,000(1 - x/10)(9 + x) + 30\,000(1 - x/10)(8 + x) \\ &= 10\,000(1 - x/10)(6x + 52) = 2000(10 - x)(3x + 26). \end{aligned}$$

We must take $x \geq 0$ and x cannot be greater than 10, else no tickets will be sold. For critical points of $R(x)$ we solve $0 = R'(x) = 2000(-6x + 4)$. Thus, $x = 2/3$. Since $R''(x) = -12\,000$, the graph of the function $R(x)$ is always concave downward. This means that $x = 2/3$ must yield an absolute maximum. The price increase should be 67 cents.

29. If the price is raised x dollars per ticket, then the expected numbers of $10 + x$, $9 + x$, and $8 + x$ dollar tickets the team expects to sell are respectively

$$10\,000(1 - x/10), \qquad 20\,000(1 - x/10), \qquad 30\,000(1 - x/10).$$

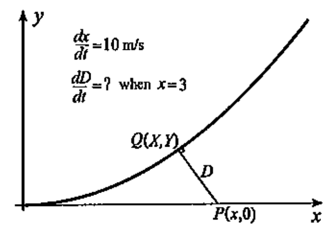
Total revenue at the new prices is therefore

$$\begin{aligned} R(x) &= 10\,000(1 - x/10)(10.5 + x) + 20\,000(1 - x/10)(9.5 + x) + 30\,000(1 - x/10)(8.5 + x) \\ &= 1000(10 - x)(6x + 55). \end{aligned}$$

We must take $x \geq 0$ and x cannot be greater than 10, else no tickets will be sold. For critical points, we solve $0 = R'(x) = 1000(-12x + 5)$. Thus, $x = 5/12$. Since $R''(x) = -12\,000$, the graph of the function $R(x)$ is always concave downward. This means that $x = 5/12$ must yield an absolute maximum. The price increase should be 42 cents.

30. When $\|PQ\|$ is the shortest distance from P to the parabola $y = x^2$, line PQ is perpendicular to the tangent line to $y = x^2$ at $Q(X, Y)$. It follows therefore that

$$2X = -\frac{1}{\dfrac{Y-0}{X-x}} = \frac{x-X}{Y}.$$



We combine this with $Y = X^2$ to obtain

$$2X(X^2) = x - X,$$

or $2X^3 + X - x = 0$. This equation defines the x-coordinate X of Q in terms of the x-coordinate x of P. The distance D from P to $y = x^2$ is then given by

$$D^2 = (x-X)^2 + Y^2 = (x-X)^2 + X^4 = (2X^3)^2 + X^4 = 4X^6 + X^4.$$

Differentiation of this equation with respect to time gives

$$2D\frac{dD}{dt} = 24X^5\frac{dX}{dt} + 4X^3\frac{dX}{dt}.$$

But differentiation of $2X^3 + X - x = 0$ gives

$$6X^2\frac{dX}{dt} + \frac{dX}{dt} - \frac{dx}{dt} = 0.$$

When $x = 3$, X is defined by $2X^3 + X - 3 = 0$, and the only solution of this equation is $X = 1$. At this instant then

$$6(1)^2\frac{dX}{dt} + \frac{dX}{dt} - 10 = 0,$$

from which $dX/dt = 10/7$. Since $D = \sqrt{(3-1)^2 + 1^2} = \sqrt{5}$ at this instant,

$$2\sqrt{5}\frac{dD}{dt} = 24(1)^5\left(\frac{10}{7}\right) + 4(1)^3\left(\frac{10}{7}\right),$$

and therefore $dD/dt = 4\sqrt{5}$. The distance is therefore increasing at $4\sqrt{5}$ m/s.

31. The travel time for the cow is

$$t(x) = \frac{\|AB\| + \|BC\|}{2} + \frac{1}{30} = \frac{\sqrt{x^2 + 9/16} + \sqrt{(1-x)^2 + 1}}{2} + \frac{1}{30}, \quad 0 \le x \le 1.$$

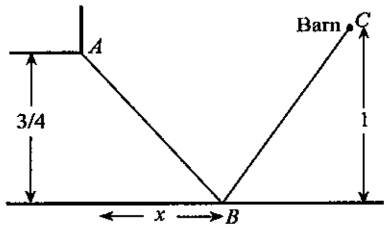
For critical points we solve

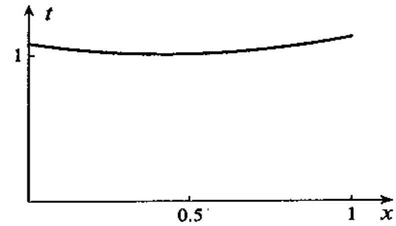
$$0 = \frac{dt}{dx} = \frac{1}{2}\left(\frac{x}{\sqrt{x^2 + 9/16}} + \frac{x-1}{\sqrt{(1-x)^2 + 1}}\right).$$

If we tanspose one of the terms, and square, $\dfrac{x^2}{x^2 + 9/16} = \dfrac{(x-1)^2}{(1-x)^2 + 1}$. When we cross multiply, the equation simplifies to $7x^2 + 18x - 9 = 0$ with solutions $x = 3/7$ and $x = -3$. We now evaluate

$$t(0) = 1.115, \qquad t(3/7) = 1.041, \qquad t(1) = 1.158.$$

Minimum time is therefore $t(3/7) = 62.5$ minutes. The graph of $t(x)$ in the right figure also indicates that it is minimized at its critical point.





When the cow walks twice as fast, travel time is $T(x) = \sqrt{x^2 + 9/16} + \sqrt{(1-x)^2 + 1} + 1/30$. This function has the same critical points as $t(x)$, and therefore minimum travel time again occurs for $x = 3/7$. Minimum time is $T(3/7) \approx 32.2$ minutes.

32. The farmer's losses when x hectares of corn and y hectares of potatoes are planted are $L = pax^2 + qby^2$. Since $x + y = 100$,

$$L = L(x) = pax^2 + qb(100 - x)^2, \quad 0 \le x \le 100.$$

For critical points of $L(x)$ we solve $0 = L'(x) = 2pax - 2qb(100 - x)$. The solution is $x = x_c = 100qb/(pa + qb)$. Now

$$L(0) = 10\,000qb, \qquad L(x_c) = \frac{10\,000abpq}{pa + qb}, \qquad L(100) = 10\,000pa.$$

If we write

$$\frac{1}{L(0)} = \frac{10^{-4}}{qb}, \qquad \frac{1}{L(x_c)} = 10^{-4}\left(\frac{1}{qb} + \frac{1}{pa}\right), \qquad \frac{1}{L(100)} = \frac{10^{-4}}{pa},$$

it is clear that $1/L(x_c)$ is greater than $1/L(0)$ and $1/L(100)$. Consequently, $L(x_c)$ is less than $L(0)$ and $L(100)$, and $L(x)$ is minimized for $x = 100qb/(pa + qb)$ hectares and $y = 100pa/(pa + qb)$ hectares.

Notice that if a increases while p, q, and b remain constant, our results suggest that more and more potatoes should be planted. Conversely large b implies planting more corn. On the other hand, if a, b, and q remain constant but p increases, the farmer should plant more potatoes. The reason is that with a large area in corn, he will suffer a substantial loss of money.

CHAPTER 5

EXERCISES 5.1

1. $\int (x^3 - 2x)\,dx = \frac{x^4}{4} - x^2 + C$

2. $\int (x^4 + 3x^2 + 5x)\,dx = \frac{x^5}{5} + x^3 + \frac{5x^2}{2} + C$

3. $\int (2x^3 - 3x^2 + 6x + 6)\,dx = \frac{x^4}{2} - x^3 + 3x^2 + 6x + C$

4. $\int \sin x\,dx = -\cos x + C$

5. $\int 3\cos x\,dx = 3\sin x + C$

6. $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

7. $\int \left(x^{10} - \frac{1}{x^3}\right) dx = \frac{x^{11}}{11} + \frac{1}{2x^2} + C$

8. $\int \left(\frac{1}{x^2} - \frac{2}{x^4}\right) dx = -\frac{1}{x} + \frac{2}{3x^3} + C$

9. $\int (x^{3/2} - x^{2/7})\,dx = \frac{2}{5}x^{5/2} - \frac{7}{9}x^{9/7} + C$

10. $\int \left(\frac{1}{x^2} + \frac{1}{2\sqrt{x}}\right) dx = -\frac{1}{x} + \sqrt{x} + C$

11. $\int \left(\frac{4}{x^{3/2}} + 2x^{1/3}\right) dx = -\frac{8}{\sqrt{x}} + \frac{3}{2}x^{4/3} + C$

12. $\int \left(-\frac{1}{2x^2} + 3x^3\right) dx = \frac{1}{2x} + \frac{3x^4}{4} + C$

13. $\int \frac{1}{x^\pi}dx = \frac{1}{(1-\pi)x^{\pi-1}} + C$

14. $\int (2\sqrt{x} + 3x^{3/2} - 5x^{5/2})\,dx = \frac{4}{3}x^{3/2} + \frac{6}{5}x^{5/2} - \frac{10}{7}x^{7/2} + C$

15. $\int x^2(x^2 - 3)\,dx = \int (x^4 - 3x^2)\,dx = \frac{x^5}{5} - x^3 + C$

16. $\int \sqrt{x}(x+1)\,dx = \int (x^{3/2} + \sqrt{x})\,dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + C$

17. $\int \left(\frac{x-2}{x^3}\right) dx = \int \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx = -\frac{1}{x} + \frac{1}{x^2} + C$

18. $\int x^2(1+x^2)^2\,dx = \int (x^2 + 2x^4 + x^6)\,dx = \frac{x^3}{3} + \frac{2x^5}{5} + \frac{x^7}{7} + C$

19. $\int (x^2+1)^3\,dx = \int (x^6 + 3x^4 + 3x^2 + 1)\,dx = \frac{x^7}{7} + \frac{3x^5}{5} + x^3 + x + C$

20. $\int \frac{(x-1)^2}{\sqrt{x}}dx = \int \left(x^{3/2} - 2\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \frac{2}{5}x^{5/2} - \frac{4}{3}x^{3/2} + 2\sqrt{x} + C$

21. If we take the indefinite integral of $f'(x) = x^2 - 3x + 2$ with respect to x, we obtain $f(x) = x^3/3 - 3x^2/2 + 2x + C$. Since $f(2) = 1$, it follows that $1 = 8/3 - 6 + 4 + C \implies C = 1/3$, and therefore $y = x^3/3 - 3x^2/2 + 2x + 1/3$.

22. If we take the indefinite integral of $f'(x) = 2x^3 + 4x$ with respect to x, we obtain $f(x) = x^4/2 + 2x^2 + C$. Since $f(0) = 5$, it follows that $5 = C$, and therefore $y = x^4/2 + 2x^2 + 5$.

23. If we take the indefinite integral of $f'(x) = -2x^4 + 3x^2 + 6$ with respect to x, we obtain $f(x) = -2x^5/5 + x^3 + 6x + C$. Since $f(1) = 0$, it follows that $0 = -2/5 + 1 + 6 + C \implies C = -33/5$, and therefore $y = -2x^5/5 + x^3 + 6x - 33/5$.

24. If we take the indefinite integral of $f'(x) = 2 - 4x + 8x^7$ with respect to x, we obtain $f(x) = 2x - 2x^2 + x^8 + C$. Since $f(1) = 1$, it follows that $1 = 2 - 2 + 1 + C$. Thus, $C = 0$, and $y = 2x - 2x^2 + x^8$.

25. If the equation of the curve is $y = f(x)$, then $f''(x) = 6x^2$. Integration gives $f(x) = x^4/2 + Cx + D$. Since $f(0) = 2$ and $f(-1) = 3$, we have $2 = D$ and $3 = 1/2 - C + D$. Thus, $y = f(x) = x^4/2 - x/2 + 2$.

26. Integration of $f''(x) = -5x$ with respect to x gives $f'(x) = -5x^2/2 + C$. Because $f(2) = 3$ is a relative maximum, it follows that $f'(2) = 0$. Thus, $0 = -5(2) + C$, from which $C = 10$, and $f'(x) = -5x^2/2 + 10$. Another integration now gives $f(x) = -5x^3/6 + 10x + D$. Since $f(2) = 3$, we find that $3 = -20/3 + 20 + D$. Thus, $D = -31/3$, and $f(x) = -5x^3/6 + 10x - 31/3$.

27. No For a relative minimum to occur at $x = 2$, the second derivative there should by positive. But this is impossible if the second derivative is equal to $-5x$.

In Exercises 28–67 we use the following tabular setup to summarize calculations. The first column is an initial proposal. The second column is the derivative of this proposal. The last column is the final answer.

	Initial proposal	Derivative of proposal	Final answer
28.	$(x+2)^{3/2}$	$\frac{3}{2}(x+2)^{1/2}$	$\frac{2}{3}(x+2)^{3/2}+C$
29.	$(x+5)^{5/2}$	$\frac{5}{2}(x+5)^{3/2}$	$\frac{2}{5}(x+5)^{5/2}+C$
30.	$(2-x)^{3/2}$	$\frac{3}{2}(2-x)^{1/2}(-1)$	$-\frac{2}{3}(2-x)^{3/2}+C$
31.	$\sqrt{4x+3}$	$\frac{1}{2}(4x+3)^{-1/2}(4)$	$\frac{1}{2}\sqrt{4x+3}+C$
32.	$(2x-3)^{5/2}$	$\frac{5}{2}(2x-3)^{3/2}(2)$	$\frac{1}{5}(2x-3)^{5/2}+C$
33.	$(3x+1)^6$	$6(3x+1)^5(3)$	$\frac{1}{18}(3x+1)^6+C$
34.	$(1-2x)^8$	$8(1-2x)^7(-2)$	$-\frac{1}{16}(1-2x)^8+C$
35.	$\frac{1}{x+4}$	$\frac{-1}{(x+4)^2}$	$\frac{-1}{x+4}+C$
36.	$\frac{1}{(1+3x)^5}$	$\frac{-5}{(1+3x)^6}(3)$	$\frac{-1}{15(1+3x)^5}+C$
37.	$(x^2+1)^4$	$4(x^2+1)^3(2x)$	$\frac{1}{8}(x^2+1)^4+C$
38.	$(2+3x^3)^8$	$8(2+3x^3)^7(9x^2)$	$\frac{1}{72}(2+3x^3)^8+C$
39.	$\frac{1}{2+x^2}$	$\frac{-2x}{(2+x^2)^2}$	$\frac{-1}{2(2+x^2)}+C$
40.	$\sin 2x$	$2\cos 2x$	$\frac{1}{2}\sin 2x+C$
41.	$\cos^3 x$	$3\cos^2 x(-\sin x)$	$-\frac{1}{3}\cos^3 x+C$
42.	$\sin^2 2x$	$2\sin 2x\cos 2x(2)$	$\frac{3}{4}\sin^2 2x+C$
43.	$\sec 12x$	$\sec 12x\tan 12x\,(12)$	$\frac{1}{12}\sec 12x+C$
44.	$\cot 4x$	$-4\csc^2 4x$	$-\frac{1}{4}\cot 4x+C$
45.	e^{4x}	$4e^{4x}$	$\frac{1}{4}e^{4x}+C$
46.	e^{-x^2}	$e^{-x^2}(-2x)$	$-\frac{1}{2}e^{-x^2}+C$
47.	$e^{3/x}$	$e^{3/x}\left(\frac{-3}{x^2}\right)$	$-\frac{1}{3}e^{3/x}+C$
48.	e^{4x-3}	$e^{4x-3}(4)$	$\frac{1}{4}e^{4x-3}+C$
49.	$\ln\lvert 3x+2\rvert$	$\frac{3}{3x+2}$	$\frac{1}{3}\ln\lvert 3x+2\rvert+C$
50.	$\ln\lvert 7-5x\rvert$	$\frac{1}{7-5x}(-5)$	$-\frac{2}{5}\ln\lvert 7-5x\rvert+C$

51.	$\ln\|1-x^2\|$	$\dfrac{-2x}{1-x^2}$	$-\dfrac{1}{2}\ln\|1-x^2\|+C$
52.	$\ln\|1-4x^3\|$	$\dfrac{1}{1-4x^3}(-12x^2)$	$-\dfrac{1}{4}\ln\|1-4x^3\|+C$
53.	2^x	$2^x(\ln 2)$	$\dfrac{2^x}{\ln 2}+C=(\log_2 e)\,2^x+C$
54.	3^{2x}	$3^{2x}(2)\ln 3$	$\dfrac{1}{2\ln 3}3^{2x}=\dfrac{1}{2}(\log_3 e)3^{2x}+C$
55.	$\ln\|e^x+1\|$	$\dfrac{e^x}{e^x+1}$	$\ln(e^x+1)+C$
56.	$(1+\cos x)^5$	$5(1+\cos x)^4(-\sin x)$	$-\dfrac{1}{5}(1+\cos x)^5+C$
57.	$\dfrac{1}{\sin^2 x}$	$\dfrac{-2\,\cos x}{\sin^3 x}$	$\dfrac{-1}{2\,\sin^2 x}+C$
58.	$(1+e^{2x})^4$	$4(1+e^{2x})^3(2e^{2x})$	$\dfrac{1}{8}(1+e^{2x})^4+C$
59.	$\dfrac{1}{\tan x}$	$\dfrac{-\sec^2 x}{\tan^2 x}$	$\dfrac{-1}{\tan x}+C=-\cot x+C$
60.	$\mathrm{Sin}^{-1}2x$	$\dfrac{2}{\sqrt{1-4x^2}}$	$\dfrac{1}{2}\mathrm{Sin}^{-1}2x+C$
61.	$\mathrm{Tan}^{-1}3x$	$\dfrac{3}{1+9x^2}$	$\dfrac{1}{3}\mathrm{Tan}^{-1}3x+C$
62.	$\mathrm{Sec}^{-1}\sqrt{3}x$	$\dfrac{\sqrt{3}}{\sqrt{3}x\sqrt{3x^2-1}}$	$\mathrm{Sec}^{-1}\sqrt{3}x+C$
63.	$\ln(1+5x^2)$	$\dfrac{10x}{1+5x^2}$	$\dfrac{3}{10}\ln(1+5x^2)+C$
64.	$\sinh 4x$	$4\cosh 4x$	$\dfrac{1}{4}\sinh 4x+C$
65.	$\cosh 3x^2$	$6x\,\sinh 3x^2$	$\dfrac{1}{6}\cosh 3x^2+C$
66.	$\mathrm{sech}2x$	$-2\mathrm{sech}2x\tanh 2x$	$-\dfrac{1}{2}\mathrm{sech}2x+C$
67.	$\coth 4x^3$	$-12x^2\mathrm{csch}^2 4x^3$	$-\dfrac{1}{12}\coth 4x^3+C$

68. Integration with respect to x gives $y=\dfrac{x^4}{4}+\dfrac{1}{x}+C$.

69. Since $\dfrac{d}{dx}(3-4x)^{3/2}=(3/2)\sqrt{3-4x}(-4)$, it follows that $y=-(1/6)(3-4x)^{3/2}+C$.

70. Since $\dfrac{d}{dx}\dfrac{1}{(3x+5)^{1/2}}=\dfrac{-1/2}{(3x+5)^{3/2}}(3)$, it follows that $y=\dfrac{-2}{3(3x+5)^{1/2}}+C$.

71. Since $\dfrac{d}{dx}(2x^3+4)^5=5(2x^3+4)^4(6x^2)$, it follows that $y=(1/30)(2x^3+4)^5+C$.

72. Since $\dfrac{d}{dx}\dfrac{1}{2+3x^4}=\dfrac{-1}{(2+3x^4)^2}(12x^3)$, it follows that $y=\dfrac{-1}{12(2+3x^4)}+C$.

73. Since $\dfrac{d}{dx}\cos^3 x=3\,\cos^2 x\,(-\sin x)$, it follows that $y=-\cos x-(1/3)\cos^3 x+C$.

74. Indefinite integrals on the intervals $x<0$ and $x>0$ give $y=f(x)=\begin{cases}-\dfrac{1}{x}+C, & x<0\\ -\dfrac{1}{x}+D, & x>0.\end{cases}$

The conditions $f(-1)=-2$ and $f(1)=1$ require $C=-3$ and $D=2$.

75. (a) Indefinite integrals on the intervals $x < 0$ and $x > 0$ give

$$\int \operatorname{sgn}(x)\,dx = \begin{cases} -x + C, & x < 0 \\ x + D, & x > 0. \end{cases}$$

(b) Graphs of the indefinite integrals of $\operatorname{sgn}(x)$ (as determined in part (a)) are shown in the left figure below. If $\operatorname{sgn}(x)$ has an antiderivative $F(x)$ on an interval containing $x = 0$, then $F(x)$ has a derivative on this interval, including at $x = 0$. By Theorem 3.6, $F(x)$ is continuous at $x = 0$, in which case its graph must be as shown in the right figure. But this function does not have a derivative at $x = 0$.

EXERCISES 5.2

1. Integration of $dv/dt = a(t) = t + 2$ gives $v(t) = t^2/2 + 2t + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = dx/dt = t^2/2 + 2t$. Integration now gives $x(t) = t^3/6 + t^2 + D$. The condition $x(0) = 0$ requires $D = 0$, and therefore $x(t) = t^3/6 + t^2$.

2. Integration of $dv/dt = a(t) = 6 - 2t$ gives $v(t) = 6t - t^2 + C$. Since $v(0) = 5$, it follows that $C = 5$, and $v(t) = dx/dt = 6t - t^2 + 5$. Integration now gives $x(t) = 3t^2 - t^3/3 + 5t + D$. The condition $x(0) = 0$ requires $D = 0$, and therefore $x(t) = 3t^2 - t^3/3 + 5t$.

3. Integration of $dv/dt = a(t) = 6 - 2t$ gives $v(t) = 6t - t^2 + C$. Since $v(0) = 5$, it follows that $C = 5$, and $v(t) = dx/dt = 6t - t^2 + 5$. Integration now gives $x(t) = 3t^2 - t^3/3 + 5t + D$. The condition $x(0) = 0$ requires $D = 0$, and therefore $x(t) = 3t^2 - t^3/3 + 5t$.

4. Integration of $dv/dt = a(t) = 120t - 12t^2$ gives $v(t) = 60t^2 - 4t^3 + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = dx/dt = 60t^2 - 4t^3$. Integration now gives $x(t) = 20t^3 - t^4 + D$. The condition $x(0) = 4$ requires $D = 4$, and therefore $x(t) = 20t^3 - t^4 + 4$.

5. Integration of $dv/dt = a(t) = t^2 + 1$ gives $v(t) = t^3/3 + t + C$. Since $v(0) = -1$, it follows that $C = -1$, and $v(t) = dx/dt = t^3/3 + t - 1$. Integration now gives $x(t) = t^4/12 + t^2/2 - t + D$. The condition $x(0) = 1$ requires $D = 1$, and therefore $x(t) = t^4/12 + t^2/2 - t + 1$.

6. Integration of $dv/dt = a(t) = t^2+5t+4$ gives $v(t) = t^3/3+5t^2/2+4t+C$. Since $v(0) = -2$, it follows that $C = -2$, and $v(t) = dx/dt = t^3/3+5t^2/2+4t-2$. Integration now gives $x(t) = t^4/12+5t^3/6+2t^2-2t+D$. The condition $x(0) = -3$ requires $D = -3$, and therefore $x(t) = t^4/12 + 5t^3/6 + 2t^2 - 2t - 3$.

7. Integration of $dv/dt = a(t) = \cos t$ gives $v(t) = \sin t + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = dx/dt = \sin t$. Integration now gives $x(t) = -\cos t + D$. The condition $x(0) = 0$ requires $D = 1$, and therefore $x(t) = 1 - \cos t$.

8. Integration of $dv/dt = 3\sin t$ gives $v(t) = -3\cos t + C$. Since $v(0) = 1$, it follows that $1 = -3 + C$. Thus, $C = 4$ and $v(t) = dx/dt = 4 - 3\cos t$. Integration now gives $x(t) = 4t - 3\sin t + D$. The condition $x(0) = 4$ requires $D = 4$, and therefore $x(t) = 4t + 4 - 3\sin t$.

9. (a) Since $a(t) = 6t - 9$, we obtain $a(5) = 21$ m/s^2.
(b) Integration gives $x(t) = t^3 - 9t^2/2 + 6t + C$. Since $x(0) = 1$, it follows that $C = 1$ and $x(t) = t^3 - 9t^2/2 + 6t + 1$. Thus, $x(2) = 3$ m.
(c) Since $v(5/4) = -9/16$ and $a(5/4) = -3/2$, the object is speeding up.
(d) For critical points of $x(t)$ we solve $0 = v(t) = 3(t-1)(t-2)$ for $t = 1, 2$. Since $x(0) = 1$, $x(1) = 7/2$, $x(2) = 3$, and $\lim_{t\to\infty} x = \infty$, the closest the object is to the origin is 1 m.

10. (a) Integration of $dv/dt = 6t - 2$ gives $v(t) = 3t^2 - 2t + C$. Since $v(0) = -3$, we find that $C = -3$ and $v(t) = 3t^2 - 2t - 3$. Integration now gives $x(t) = t^3 - t^2 - 3t + D$. The condition $x(0) = 1$ requires $D = 1$, and therefore $x(t) = t^3 - t^2 - 3t + 1$.
(b) The velocity is zero when $3t^2 - 2t - 3 = 0$, a quadratic equation with solutions $t = (2 \pm \sqrt{4+36})/6 = (1 \pm \sqrt{10})/3$. Only the solution $t = (1 + \sqrt{10})/3$ is positive.

11. (a) Integration of $dv/dt = 6t - 15$ gives $v(t) = 3t^2 - 15t + C$. Since $v(2) = 6$, it follows that $6 = 12 - 30 + C \implies C = 24$. Thus, $v(1) = 3 - 15 + 24 = 12$ m/s.
(b) Integration of $dx/dt = 3t^2 - 15t + 24$ gives $x(t) = t^3 - 15t^2/2 + 24t + D$. Since $x(0) = 10$, we obtain $D = 10$, and $x(t) = t^3 - 15t^2/2 + 24t + 10$.
(c) For critical points of $x(t)$, we solve $0 = v(t) = 3(t^2 - 5t + 8)$. Since there are no solutions of this equation, we evaluate $x(0) = 10$ and $\lim_{t\to\infty} x = \infty$. The closest distance is 10 m.

12. (a) Integration of $dv/dt = 3 - t/5$ gives $v(t) = 3t - t^2/10 + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = 3t - t^2/10$. Integration now gives $x(t) = 3t^2/2 - t^3/30 + D$. If we choose a positive x-axis in the direction of motion of the car with $x = 0$ at $t = 0$, then $x(0) = 0$. This condition requires $D = 0$, and therefore $x(t) = 3t^2/2 - t^3/30$. The position of the car after 10 s is $x(10) = 3(100)/2 - (1000)/30 = 350/3$ m.
(b) For $t > 10$, the acceleration is $a(t) = -2$. Integration of this yields $v(t) = -2t + E$. Because $v(10) = 3(10) - 100/10 = 20$, it follows that $20 = -2(10) + E$. Hence, $E = 40$, and $v(t) = 40 - 2t$. Integration now gives $x(t) = 40t - t^2 + F$. Because $x(10) = 350/3$, it follows that $350/3 = 40(10) - 100 + F$. Thus, $F = -550/3$, and $x(t) = 40t - t^2 - 550/3$. The car comes to a stop when $0 = v(t) = 40 - 2t$, and this implies that $t = 20$. The position of the car at this time is $x(20) = 40(20) - (20)^2 - 550/3 = 650/3$ m.

13. Integration of $v(t) = 180 - 18t$ gives $x(t) = 180t - 9t^2 + C$. If we choose $x = 0$ at the position at which the plane touches the ground, then $C = 0$ and $x(t) = 180t - 9t^2$. Since the speed of the plane is zero in 10 seconds, the distance that it moves after touching the ground is $x(10) = 180(10) - 9(10^2) = 900$ m.

14. We choose y as positive upward with $y = 0$ and $t = 0$ at the point and instant of projection. The acceleration of the stone is $a = -9.81$. Integration gives $v(t) = -9.81t + C$. Since $v(0) = 10$, it follows that $C = 10$, and $v(t) = -9.81t + 10$. Integration now gives $y(t) = -4.905t^2 + 10t + D$. The condition $y(0) = 0$ requires $D = 0$, and therefore $y(t) = -4.905t^2 + 10t$. At the peak height of the stone, $0 = v(t) = -9.81t + 10$, and this occurs when $t = 10/9.81$. The height of the stone at this time is $y(10/9.81) = -4.905(10/9.81)^2 + 10(10/9.81) = 5.1$ m.

15. We choose y as positive downward with $y = 0$ and $t = 0$ at the point and instant the stone is dropped. The acceleration of the stone is $a = 9.81$. Integration gives $v(t) = 9.81t + C$. Since $v(0) = 0$, it follows that $C = 0$, and $v(t) = 9.81t$. Integration now gives $y(t) = 4.905t^2 + D$. The condition $y(0) = 0$ requires $D = 0$, and therefore $y(t) = 4.905t^2$. The time that it takes the stone to drop 25 m is given by $25 = 4.905t^2 \implies t = \sqrt{25/4.905}$. Thus, the stone should be dropped $5/\sqrt{4.905}$ s before the wood reaches the appropriate spot.

16. We choose y as positive upward with $y = 0$ and $t = 0$ at the point and instant the ball is thrown. The acceleration of the ball is $a = -9.81$. Integration gives $v(t) = -9.81t + C$. If v_0 is the initial speed of the ball, then $C = v_0$, and $v(t) = v_0 - 9.81t$. Integration gives $y(t) = v_0 t - 4.905t^2 + D$. The condition $y(0) = 0$ requires $D = 0$, and therefore $y(t) = v_0 t - 4.905t^2$. For the ball just to reach your friend, we must have $v = 0$ when $y = 20$:

$$0 = v_0 - 9.81t, \qquad 20 = v_0 t - 4.905t^2.$$

The first implies that $t = v_0/9.81$, and this can be substituted into the second,

$$20 = v_0\left(\frac{v_0}{9.81}\right) - 4.905\left(\frac{v_0}{9.81}\right)^2.$$

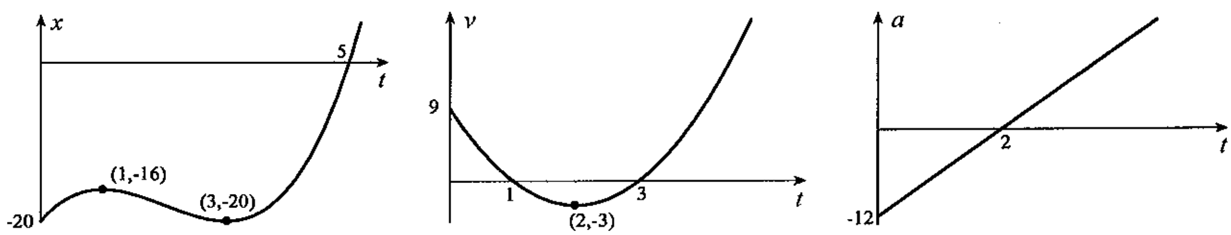
The positive solution of this equation is $v_0 = 19.8$ m/s.

17. Suppose the car is travelling to the right and we choose x positive to the right with $x = 0$ and $t = 0$ at the point and instant the brakes are applied. Let a represent the constant acceleration that will stop the car just as it touches the tree. Integration of $dv/dt = a$ gives $v(t) = at + C$. Since $v(0) = 20$, we have $C = 20$ and $v(t) = at + 20$. A second integration gives $x(t) = at^2/2 + 20t + D$. Since $x(0) = 0$, it follows that $D = 0$ and $x(t) = at^2/2 + 20t$. Since the velocity of the car is zero at the tree when $x = 50$, we can say that at the tree,

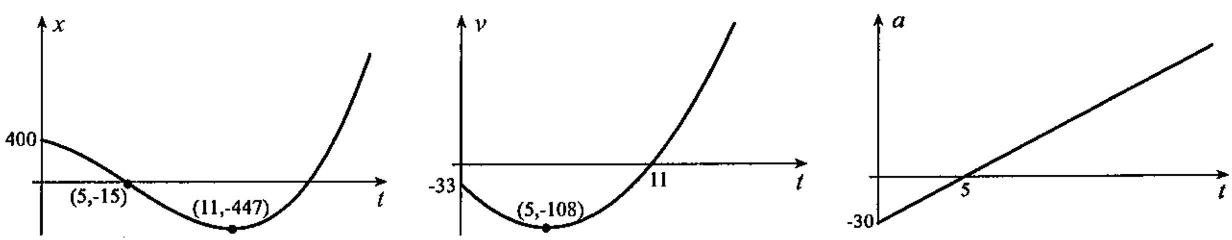
$$50 = \frac{1}{2}at^2 + 20t, \qquad 0 = at + 20.$$

When these are solved for a and t, the result is $a = -4$. Consequently, if the acceleration is less than -4 m/s^2, the car stops before striking the tree.

18. The velocity and acceleration are $v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$ and $a(t) = 6t - 12 = 6(t-2)$. The graph of $x(t)$ has critical points at $t = 1$ and $t = 3$, the first giving a relative maximum of $x(1) = -16$, and the second a relative minimum of $x(3) = -20$. The graph has a point of inflection at $(2, -18)$. The graph of $v(t)$ has zeros at $t = 1$ and $t = 3$, the critical points of $x(t)$. The graph of $a(t)$ has a zero at $t = 2$, the point of inflection for $x(t)$.



19. Integration of $dv/dt = 6t - 30$ gives $v(t) = 3t^2 - 30t + C$. Since $v(0) = -33$, it follows that $C = -33$ and $v(t) = 3t^2 - 30t - 33 = 3(t-11)(t+1)$. A second integration gives $x(t) = t^3 - 15t^2 - 33t + D$. Since $x(0) = 400$, we find that $D = 400$, and $x(t) = t^3 - 15t^2 - 33t + 400$. The graph of $x(t)$ has a critical point at $t = 11$, giving a relative minimum of $x(11) = -447$. There is a point of inflection at $(5, -15)$. The zero $t = 11$ of $v(t)$ is the critical point of $x(t)$. The zero $t = 5$ of $a(t)$ gives the point of inflection for $x(t)$.



20. We choose $x = 0$ and $t = 0$ at the point and instant the brakes are applied. If we assume that the acceleration of the car is -9.81 m/s^2, and determine the initial speed v_0 which produces a skid mark of 9 m, then v_0 is the maximum possible speed. In other words, we are testifying for the defence. Integration of $a = -9.81$ gives $v(t) = -9.81t + C$. The condition $v(0) = v_0$ requires $C = v_0$, and therefore $v(t) = -9.81t + v_0$. Another integration yields $x(t) = -4.905t^2 + v_0 t + D$. Since $x(0) = 0$, it follows that $D = 0$. Because $x = 9$ when $v = 0$,

$$0 = -9.81t + v_0, \qquad 9 = -4.905t^2 + v_0 t.$$

The first requires $t = v_0/9.81$, and when this is substituted into the second,

$$9 = -4.905\left(\frac{v_0}{9.81}\right)^2 + v_0\left(\frac{v_0}{9.81}\right).$$

The positive solution of this equation is $v_0 = 13.3$ m/s or 47.8 km/hr.

21. (a) Suppose we let x measure distance in the direction of motion of the car, taking $x=0$ and $t=0$ at the point and instant the brakes are applied. Integration of $dv/dt=-5$ gives $v(t)=-5t+C$. Since $v(0)=250/9$, we find $C=250/9$, and $v(t)=-5t+250/9$. A second integration gives $x(t)=-5t^2/2+250t/9+D$. With $x(0)=0$, we obtain $D=0$, and $x(t)=-5t^2/2+250t/9$. The car comes to a stop when $0=v=-5t+250/9 \Longrightarrow t=50/9$, and at this instant $x(50/9)=-(5/2)(50/9)^2+(250/9)(50/9)=6250/81$ m.
(b) With $v(0)=125/9$, the velocity is $v(t)=-5t+125/9$, and distance travelled is $x(t)=-5t^2/2+125t/9$. The car stops when $t=25/9$, at which time $x(25/9)=-(5/2)(25/9)^2+(125/9)(25/9)=3125/162$ m.
(c) The ratio of these distances is 4.
(d) Stopping times from the instant the driver takes his foot from the accelerator are

$$\frac{6250}{81}+\left(\frac{3}{4}\right)\left(\frac{250}{9}\right)=\frac{15875}{162} \quad \text{and} \quad \frac{3125}{162}+\left(\frac{3}{4}\right)\left(\frac{125}{9}\right)=\frac{9625}{324}.$$

The ratio of these distances is 3.3.

22. We choose y as positive downward with $y=0$ and $t=0$ at the point and instant the stone is dropped. The acceleration of the stone is $a=9.81$. Integration gives $v(t)=9.81t+C$. Since $v(0)=0$, it follows that $C=0$, and $v(t)=9.81t$. Integration gives $y(t)=4.905t^2+D$. The condition $y(0)=0$ requires $D=0$, and therefore $y(t)=4.905t^2$. If d is the distance from the top of the well to the surface of the water and T is the time it takes the stone to fall this distance, then $d=4.905T^2 \Longrightarrow T=\sqrt{d/4.905}$. Since the time taken for the sound to travel the distance d is $d/340$, it follows that

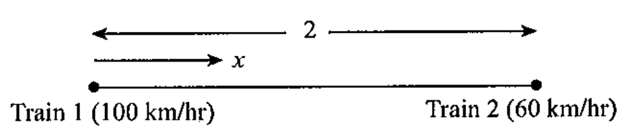
$$\sqrt{\frac{d}{4.905}}+\frac{d}{340}=3.1 \quad \Longrightarrow \quad \sqrt{\frac{d}{4.905}}=3.1-\frac{d}{340}.$$

Squaring gives

$$\frac{d}{4.905}=(3.1)^2-\frac{3.1d}{170}+\frac{d^2}{340^2} \quad \Longrightarrow \quad \frac{d^2}{340^2}-\left(\frac{3.1}{170}+\frac{1}{4.905}\right)d+(3.1)^2=0.$$

Solutions of this quadratic are $d=\dfrac{(3.1/170+1/4.905)\pm\sqrt{(3.1/170+1/4.905)^2-4(3.1)^2/340^2}}{2/340^2}$. Only the solution $d=43.3$ satisfies the original equation. Hence the depth of the well is 43.3 m.

23. (a)



The acceleration of Train 1 is $a_1=-1/4$. Thus, $v_1=-t/4+C$. Since $v_1(0)=250/9$, we have $C=250/9$, and $v_1=-t/4+250/9$. Hence, $x_1=-t^2/8+250t/9+D$. Because $x_1(0)=0$, it follows that $D=0$, and $x_1(t)=-t^2/8+250t/9$. A similar calculation for Train 2 with $a_2=1/4$, $v_2(0)=-50/3$, and $x_2(0)=2000$ gives $v_2=t/4-50/3$, and $x_2(t)=t^2/8-50t/3+2000$. These expressions for x_1 and x_2 are valid until each train stops or a collision takes place. Train 1 stops when $t=1000/9$ and therefore at position

$$x_1=-\frac{1}{8}\left(\frac{1000}{9}\right)^2+\frac{250}{9}\left(\frac{1000}{9}\right)=1543.2 \text{ m}.$$

Train 2 stops when $t=200/3$, and therefore at position

$$x_2=\frac{1}{8}\left(\frac{200}{3}\right)^2-\frac{50}{3}\left(\frac{200}{3}\right)+2000=1444.4 \text{ m}.$$

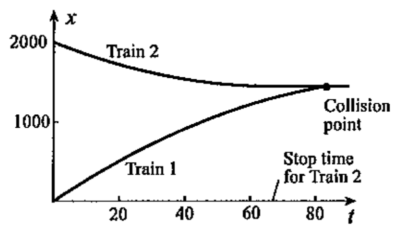
Since Train 2 would stop to the left of Train 1, a collision occurs.

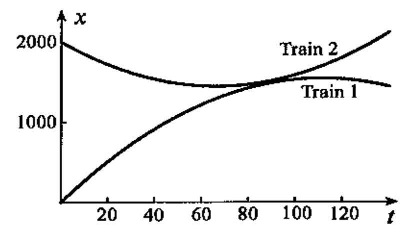
(b) In this case the expressions for x_1 and x_2 are for all $t \geq 0$ since the trains reverse directions after stopping (unless a collision occurs). A collision occurs if and only if $x_1 = x_2$; that is,

$$-\frac{t^2}{8} + \frac{250t}{9} = \frac{t^2}{8} - \frac{50t}{3} + 2000 \quad \Longrightarrow \quad \frac{t^2}{4} - \frac{400t}{9} + 2000 = 0.$$

Since this equation has no real solutions, a collision does not occur.

(c)





24. We take y as positive downward from the top of the building with $t = 0$ at the instant the bearing is dropped. Let h be the distance from the top of the building to the top of the window and H be the distance from the bottom of the window to the sidewalk. Integration of $dv/dt = 9.81$ gives $v(t) = 9.81t + C$. Since $v(0) = 0$, we find that $C = 0$ and $v(t) = 9.81t$. A second integration gives $y(t) = 4.905t^2 + D$. With $y(0) = 0$, we obtain $D = 0$, and $y(t) = 4.905t^2$. If T is the time taken to reach the top of the window on the way down, then

$$h = 4.905T^2, \qquad h + 1 = 4.905(T + 1/8)^2, \qquad h + 1 + H = 4.905(T + 9/8)^2.$$

When the first of these is subtracted from the second, and the resulting equation is solved for T, we obtain $T = 0.753$. Substitution of this into the third gives $h + 1 + H = 17.3$. Thus, the building is 17.3 metres high.

25. (a) We take y as positive upward with $y = 0$ on the ground, and $t = 0$ at the instant the ball is thrown upward. The acceleration of the ball is $a_b = -9.81$. Integration gives $v_b(t) = -9.81t + C$. Since $v_b(0) = 30$, we have $C = 30$, and $v_b(t) = -9.81t + 30$. Integration gives $y_b(t) = -4.905t^2 + 30t + D$. The condition $y_b(0) = 30$ requires $D = 30$, and therefore $y_b(t) = -4.905t^2 + 30t + 30$. The ball reaches peak height when $v_b = 0 \Longrightarrow t = 30/9.81$. The height of the ball at this time is $y_b = -4.905(30/9.81)^2 + 30(30/9.81) + 30 = 75.9$ m.

(b) Since the height of the elevator floor at time t is $y_e = 10t + 28$, the elevator catches the ball when

$$y_b = y_e \quad \Longrightarrow \quad -4.905t^2 + 30t + 30 = 10t + 28 \quad \Longrightarrow \quad -4.905t^2 + 20t + 2 = 0.$$

The positive solution is $t = 4.18$ s.

26. Let us choose a different coordinate system to that of Example 5.6 by taking y as positive upward with $y = 0$ and $t = 0$ at the point and instant of projection of the first stone. The acceleration of stone 1 is

$$a_1 = -9.81 = \frac{dv_1}{dt},$$

from which $v_1(t) = -9.81t + C$. Since $v_1(0) = 25$, it follows that $25 = C$, and $v_1(t) = -9.81t + 25$, $t \geq 0$. Thus,

$$\frac{dy_1}{dt} = -9.81t + 25,$$

from which we have $y_1(t) = -4.905t^2 + 25t + D$. Since $y_1(0) = 0$, we find that $0 = D$, and $y_1(t) = -4.905t^2 + 25t$, $t \geq 0$.

The acceleration of stone 2 is also -9.81; hence we have

$$a_2 = -9.81 = \frac{dv_2}{dt},$$

from which $v_2(t) = -9.81t + E$. Because $v_2(1) = 20$, we must have $20 = -9.81(1) + E$, or $E = 29.81$. Thus, $v_2(t) = -9.81t + 29.81$, $t \geq 1$. Consequently,

$$\frac{dy_2}{dt} = -9.81t + 29.81,$$

from which $y_2(t) = -4.905t^2 + 29.81t + F$. Since $y_2(1) = 0$, it follows that $0 = -4.905(1)^2 + 29.81(1) + F$, or $F = -24.905$. Thus, $y_2(t) = -4.905t^2 + 29.81t - 24.905$, $t \geq 1$. The stones will pass each other if y_1 and y_2 are ever equal for the same time t; that is, if

$$-4.905t^2 + 25t = -4.905t^2 + 29.81t - 24.905.$$

The solution of this equation is

$$t = \frac{24.905}{4.81} = 5.2.$$

Because the first stone does not strike the base of the cliff for 7.7 s (Example 5.6), it follows that the stones do indeed pass each other 5.2 s after the first stone is projected.

27. $$F = m_0 \left\{ \frac{\sqrt{1-(v^2/c^2)}(dv/dt) - v(1/2)[1-(v^2/c^2)]^{-1/2}(-2v/c^2)(dv/dt)}{1-(v^2/c^2)} \right\}$$
$$= m_0 \frac{1-(v^2/c^2)+(v^2/c^2)}{[1-(v^2/c^2)]^{3/2}} \frac{dv}{dt} = \frac{m_0 a}{[1-(v^2/c^2)]^{3/2}}.$$

Newton's second law $F = ma = m_0 a$ indicates that the force necessary to impart an acceleration a to a mass is independent of the velocity of m. The relativistic formula states that F depends on v as well as a. In addition, notice that as v approaches c, the speed of light, forces necessary to give accelerations become extremely large.

28. Flow rate when distances are a fraction of the safe distance is

$$r(v) = \frac{v}{l + k\left(vT - \dfrac{v^2}{2a}\right)}.$$

The critical point of this function is defined by

$$0 = \frac{\left[l + k\left(vT - \dfrac{v^2}{2a}\right)\right](1) - v\left[k\left(T - \dfrac{v}{a}\right)\right]}{\left[l + k\left(vT - \dfrac{v^2}{2a}\right)\right]^2} \qquad \Longrightarrow \qquad 0 = \frac{1}{2a}(2al + kv^2).$$

Thus, $v = \sqrt{-2al/k}$.

29. (a) We choose y positive upward with $y = 0$ and $t = 0$ at the point and instant the first stone is released. For stone 1, $a_1 = -g$, from which $v_1 = -gt + C$. Since $v_1(0) = v_0'$, it follows that $C = v_0'$, and $v_1 = -gt + v_0'$. A second integration gives $x_1 = -gt^2/2 + v_0't + D$. Since $x_1(0) = 0$, we find that $D = 0$, and $x_1 = -gt^2/2 + v_0't$, $t \geq 0$. For stone 2, $a_2 = -g$, from which $v_2 = -gt + E$. With $v_2(t_0) = v_0''$, we obtain $v_0'' = -gt_0 + E$, and therefore $v_2 = -g(t - t_0) + v_0''$. Integration now gives $x_2 = -g(t - t_0)^2/2 + v_0''t + F$. The initial condition $x_2(t_0) = 0$ gives $0 = v_0''t_0 + F$, and therefore $x_2 = -g(t - t_0)^2/2 + v_0''(t - t_0)$, $t \geq t_0$. The stones pass each other for $t \geq t_0$ if

$$-\frac{gt^2}{2} + v_0't = -\frac{g(t-t_0)^2}{2} + v_0''(t - t_0),$$

and the solution of this equation for t is

$$t = \frac{(gt_0 + 2v_0'')t_0}{2(gt_0 + v_0'' - v_0')}.$$

Since the numerator is positive, it follows that t will be positive if $gt_0 > v_0' - v_0''$. In addition, they will pass one another during the motion if and only if this value of t is greater than t_0; that is,

$$\frac{(gt_0 + 2v_0'')t_0}{2(gt_0 + v_0'' - v_0')} > t_0,$$

and this condition reduces to $v_0' > gt_0/2$.
(b) The times required for the stones to commence their downward trajectories are

$$t_1 = \frac{v_0'}{g} \qquad \text{and} \qquad t_2 = \frac{v_0''}{g} + t_0.$$

Stone 1 will commence downward first therefore if

$$\frac{v_0'}{g} < \frac{v_0''}{g} + t_0,$$

and this is equivalent to $gt_0 > v_0' - v_0''$.
(c) Stone 1 reaches its projection point again at $t = 2v_0'/g$, and this will occur after Stone 2 is released if and only if $2v_0'/g > t_0$, or $v_0' > gt_0/2$.

30. During the acceleration stage, $a = dv/dt = 3$ so that $v = 3t + C$. If we choose time $t = 0$ as the vehicle leaves a speed bump with velocity 2.5 m/3, then $C = 2.5$ and $v = 3t + 2.5$. If x measures displacement, then $x = 3t^2/2 + 2.5t + D$. If we take $x = 0$ at the speed bump, then $D = 0$ and $x = 3t^2/2 + 2.5t$. Since speed is to be 10 m/s after the acceleration stage, which we suppose takes T seconds, $10 = 3T + 2.5 \Longrightarrow T = 2.5$ s. The position of the vehicle at this time is $x = 3(2.5)^2/2 + 2.5(2.5) = 15.625$ m. During the deceleration stage, $a = dv/dt = -7$ so that $v = -7t + E$. Since speed is 10 m/s when $t = 2.5$ s, $10 = -7(2.5) + E \Longrightarrow E = 27.5$, and $v = -7t + 27.5$. Displacement during this stage is $x = -7t^2/2 + 27.5t + F$. Since $x = 15.625$ when $t = 2.5$, $15.625 = -7(2.5)^2/2 + 27.5(2.5) + F \Longrightarrow F = -31.25$. Since speed at the end of this stage is to be 2.5 m/s at the second bump, we can find when this occurs by solving $2.5 = -7t + 27.5 \Longrightarrow t = 25/7$. The displacement of the vehicle at this time is $x = -7(25/7)^2/2 + 27.5(25/7) - 31.25 = 22.3$. This is the distance in metres between speed bumps.

31. We choose x as positive to the right (the direction of motion) with $x = 0$ and $t = 0$ at position and instant motion commences. It t_1 is the time at which acceleration ends, then $T - t_1$ is the time at which deceleration begins. If a is the acceleration of the car during the time interval $0 < t < t_1$, then its velocity and position during this time interval are $v = at$ and $x = at^2/2$. Velocity and position of the car at t_1 are $V = at_1$ and $x_1 = at_1^2/2$. During the time interal $t_1 < t < T - t_1$, position of the car is $x = at_1^2/2 + V(t - t_1)$. Position at time $T - t_1$ is $at_1^2/2 + V(T - 2t_1)$. Since acceleration of the car during the time interval $T - t_1 < t < T$ is $-a$, its velocity is $v = -at + C$. Because $v(T - t_1) = V$, it follows that $V = -a(T - t_1) + C \Longrightarrow C = V + a(T - t_1)$ and $v = -at + V + a(T - t_1)$. Position of the car is $x = -at^2/2 + Vt + a(T - t_1)t + E$. Since $x(T - t_1) = at_1^2/2 + V(T - 2t_1)$, it follows that

$$\frac{at_1^2}{2} + V(T - 2t_1) = -\frac{a}{2}(T - t_1)^2 + V(T - t_1) + a(T - t_1)(T - t_1) + E \Longrightarrow E = \frac{at_1^2}{2} - Vt_1 - \frac{a}{2}(T - t_1)^2.$$

The position of the car at time T is

$$D = x(T) = -\frac{aT^2}{2} + VT + a(T - t_1)T + \frac{at_1^2}{2} - Vt_1 - \frac{a}{2}(T - t_1)^2 = V(T - t_1) \Longrightarrow t_1 = T - \frac{D}{V}.$$

Consequently, the length of time at speed V is $T - 2t_1 = T - 2\left(T - \dfrac{D}{V}\right) = \dfrac{2D}{V} - T$.

EXERCISES 5.3

In Exercises 1–7, it is not necessary to use a substitution; these integrations can be done by adjusting constants.

1. $\displaystyle\int (5x + 14)^9\,dx = \frac{1}{50}(5x + 14)^{10} + C$

2. $\displaystyle\int \sqrt{1 - 2x}\,dx = -\frac{1}{3}(1 - 2x)^{3/2} + C$

3. $\displaystyle\int \frac{1}{(3y-12)^{1/4}}dy = \frac{4}{9}(3y-12)^{3/4} + C$

4. $\displaystyle\int \frac{5}{(5-42x)^{1/4}}dx = \frac{-(5)(4)}{3(42)}(5-42x)^{3/4} + C = -\frac{10}{63}(5-42x)^{3/4} + C$

5. $\displaystyle\int x^2(3x^3+10)^4\,dx = \frac{1}{45}(3x^3+10)^5 + C$

6. $\displaystyle\int \frac{x}{(x^2+4)^2}dx = \frac{-1}{2(x^2+4)} + C$

7. $\displaystyle\int \sin^4 x\,\cos x\,dx = \frac{1}{5}\sin^5 x + C$

8. If we set $u = x-2$, then $du = dx$, and

$$\begin{aligned}\int \frac{x^2}{(x-2)^4}dx &= \int \frac{(u+2)^2}{u^4}du = \int\left(\frac{1}{u^2}+\frac{4}{u^3}+\frac{4}{u^4}\right)du\\ &= -\frac{1}{u}-\frac{2}{u^2}-\frac{4}{3u^3}+C = \frac{-1}{x-2}-\frac{2}{(x-2)^2}-\frac{4}{3(x-2)^3}+C.\end{aligned}$$

9. If we set $u = 1-3z$, then $du = -3\,dz$, and

$$\begin{aligned}\int z\sqrt{1-3z}\,dz &= \int\left(\frac{1-u}{3}\right)\sqrt{u}\left(\frac{du}{-3}\right) = \frac{1}{9}\int (u^{3/2}-\sqrt{u})\,du\\ &= \frac{1}{9}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C = \frac{2}{45}(1-3z)^{5/2}-\frac{2}{27}(1-3z)^{3/2}+C.\end{aligned}$$

10. If we set $u = 2x+3$, then $du = 2\,dx$, and

$$\begin{aligned}\int \frac{x}{\sqrt{2x+3}}dx &= \int \frac{(u-3)/2}{\sqrt{u}}\left(\frac{du}{2}\right) = \frac{1}{4}\int\left(\sqrt{u}-\frac{3}{\sqrt{u}}\right)du\\ &= \frac{1}{4}\left(\frac{2}{3}u^{3/2}-6\sqrt{u}\right)+C = \frac{1}{6}(2x+3)^{3/2}-\frac{3}{2}\sqrt{2x+3}+C.\end{aligned}$$

11. $\displaystyle\int \frac{1+\sqrt{x}}{\sqrt{x}}dx = \int\left(\frac{1}{\sqrt{x}}+1\right)dx = 2\sqrt{x}+x+C$

12. If we set $u = s^2+5$, then $du = 2s\,ds$, and

$$\begin{aligned}\int s^3\sqrt{s^2+5}\,ds &= \int s^2\sqrt{s^2+5}\,s\,ds = \int (u-5)\sqrt{u}\left(\frac{du}{2}\right) = \frac{1}{2}\int (u^{3/2}-5\sqrt{u})\,du\\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2}-\frac{10}{3}u^{3/2}\right)+C = \frac{1}{5}(s^2+5)^{5/2}-\frac{5}{3}(s^2+5)^{3/2}+C.\end{aligned}$$

13. If we set $u = \sin x$, then $du = \cos x\,dx$, and

$$\begin{aligned}\int \sin^2 x\,\cos^3 x\,dx &= \int \sin^2 x(1-\sin^2 x)\cos x\,dx = \int u^2(1-u^2)\,du\\ &= \frac{u^3}{3}-\frac{u^5}{5}+C = \frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C.\end{aligned}$$

14. If we set $u = 1-\cos x$, then $du = \sin x\,dx$, and

$$\int \sqrt{1-\cos x}\,\sin x\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2}+C = \frac{2}{3}(1-\cos x)^{3/2}+C.$$

15. If we set $u = 3-x^2$, then $du = -2x\,dx$, and

$$\begin{aligned}\int \frac{x^3}{(3-x^2)^3}dx &= \int \frac{x^2}{(3-x^2)^3}x\,dx = \int \frac{3-u}{u^3}\left(\frac{du}{-2}\right) = \frac{1}{2}\int\left(\frac{1}{u^2}-\frac{3}{u^3}\right)du\\ &= \frac{1}{2}\left(-\frac{1}{u}+\frac{3}{2u^2}\right)+C = \frac{-1}{2(3-x^2)}+\frac{3}{4(3-x^2)^2}+C.\end{aligned}$$

16. If we set $u = y - 4$, then $du = dy$, and

$$\int y^2\sqrt{y-4}\,dy = \int (u+4)^2\sqrt{u}\,du = \int (u^{5/2} + 8u^{3/2} + 16\sqrt{u})\,du = \frac{2}{7}u^{7/2} + \frac{16}{5}u^{5/2} + \frac{32}{3}u^{3/2} + C$$
$$= \frac{2}{7}(y-4)^{7/2} + \frac{16}{5}(y-4)^{5/2} + \frac{32}{3}(y-4)^{3/2} + C.$$

17. If we set $y = \sqrt{u}$, then $dy = \dfrac{1}{2\sqrt{u}}du$, and

$$\int \frac{(1+\sqrt{u})^{1/2}}{\sqrt{u}}du = \int (1+y)^{1/2}\,2\,dy = \frac{4}{3}(1+y)^{3/2} + C = \frac{4}{3}(1+\sqrt{u})^{3/2} + C.$$

18. If we set $u = 3x^3 - 5$, then $du = 9x^2\,dx$, and

$$\int x^8(3x^3-5)^6\,dx = \int (x^3)^2(3x^3-5)^6x^2\,dx = \int \left(\frac{u+5}{3}\right)^2 u^6\left(\frac{du}{9}\right) = \frac{1}{81}\int (u^8 + 10u^7 + 25u^6)\,du$$
$$= \frac{1}{81}\left(\frac{u^9}{9} + \frac{5u^8}{4} + \frac{25u^7}{7}\right) + C = \frac{1}{729}(3x^3-5)^9 + \frac{5}{324}(3x^3-5)^8 + \frac{25}{567}(3x^3-5)^7 + C.$$

19. $\displaystyle\int \frac{1+z^{1/4}}{\sqrt{z}}dz = \int\left(\frac{1}{\sqrt{z}} + \frac{1}{z^{1/4}}\right)dz = 2\sqrt{z} + \frac{4z^{3/4}}{3} + C$

20. $\displaystyle\int \frac{x+1}{(x^2+2x+2)^{1/3}}dx = \frac{3}{4}(x^2+2x+2)^{2/3} + C$

21. $\displaystyle\int \frac{(x-1)(x+2)}{\sqrt{x}}dx = \int\left(x^{3/2} + \sqrt{x} - \frac{2}{\sqrt{x}}\right)dx = \frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3} - 4\sqrt{x} + C$

22. If we set $u = 3 - 4\sin x$, then $du = -4\cos x\,dx$, and

$$\int \frac{\cos^3 x}{(3-4\sin x)^4}dx = \int \frac{(1-\sin^2 x)\cos x}{(3-4\sin x)^4}dx = \int \frac{1-[(3-u)/4]^2}{u^4}\left(-\frac{du}{4}\right) = \frac{1}{64}\int \frac{-7-6u+u^2}{u^4}du$$
$$= \frac{1}{64}\int\left(\frac{1}{u^2} - \frac{6}{u^3} - \frac{7}{u^4}\right)du = \frac{1}{64}\left(-\frac{1}{u} + \frac{3}{u^2} + \frac{7}{3u^3}\right) + C$$
$$= \frac{-1}{64(3-4\sin x)} + \frac{3}{64(3-4\sin x)^2} + \frac{7}{192(3-4\sin x)^3} + C.$$

23. If we set $u = 1 + \sin 4t$, then $du = 4\cos 4t\,dt$, and

$$\int \sqrt{1+\sin 4t}\,\cos^3 4t\,dt = \int \sqrt{1+\sin 4t}\,(1-\sin^2 4t)\cos 4t\,dt$$
$$= \int \sqrt{u}[1-(u-1)^2]\left(\frac{du}{4}\right) = \frac{1}{4}\int (2u^{3/2} - u^{5/2})\,du$$
$$= \frac{1}{4}\left(\frac{4u^{5/2}}{5} - \frac{2u^{7/2}}{7}\right) + C = \frac{1}{5}(1+\sin 4t)^{5/2} - \frac{1}{14}(1+\sin 4t)^{7/2} + C.$$

24. If we set $u = 1 + \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}dx$, and

$$\int \sqrt{1+\sqrt{x}}\,dx = \int \sqrt{u}\,2(u-1)\,du = 2\int (u^{3/2} - \sqrt{u})\,du$$
$$= 2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + C.$$

25. $\displaystyle\int \tan^2 x\,\sec^2 x\,dx = \frac{1}{3}\tan^3 x + C$

26. $\displaystyle\int \tan x\,\sec^2 x\,dx = \frac{1}{2}\tan^2 x + C$

27. $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}dx = \frac{1}{2}\ln(e^{2x}+1) + C$

28. If we set $u = \ln x$, then $du = \dfrac{1}{x}dx$, and $\displaystyle\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$

29. $\displaystyle\int \frac{1}{x\ln x}dx = \ln|\ln x| + C$

30. If we set $u = \ln(x^2+1)$, then $du = \dfrac{2x}{x^2+1}dx$, and

$$\int \frac{x}{(x^2+1)[\ln(x^2+1)]^2}dx = \int \frac{1}{u^2}\left(\frac{du}{2}\right) = -\frac{1}{2u} + C = \frac{-1}{2\ln(x^2+1)} + C.$$

31. (a) If we set $u = \sin x$, then $du = \cos x\,dx$, and

$$\begin{aligned}\int \sin^3 x\,\cos^3 x\,dx &= \int \sin^3 x(1-\sin^2 x)\cos x\,dx = \int u^3(1-u^2)\,du\\ &= \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.\end{aligned}$$

(b) If we set $u = \cos x$, then $du = -\sin x\,dx$, and

$$\begin{aligned}\int \sin^3 x\,\cos^3 x\,dx &= \int (1-\cos^2 x)\cos^3 x\,\sin x\,dx = \int (1-u^2)u^3\,(-du)\\ &= -\frac{u^4}{4} + \frac{u^6}{6} + C = -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + C.\end{aligned}$$

(c) $$\begin{aligned}\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C &= \frac{1}{4}(1-\cos^2 x)^2 - \frac{1}{6}(1-\cos^2 x)^3 + C\\ &= \left(\frac{1}{4}-\frac{1}{6}\right) + \left(-\frac{1}{2}+\frac{1}{2}\right)\cos^2 x + \left(\frac{1}{4}-\frac{1}{2}\right)\cos^4 x + \frac{1}{6}\cos^6 x + C\\ &= -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + D\end{aligned}$$

32. If $x \ge 0$, then $\displaystyle\int \sqrt{\frac{x^2}{1+x}}dx = \int \frac{x}{\sqrt{1+x}}dx$. If we set $u = 1+x$, then $du = dx$, and

$$\int \sqrt{\frac{x^2}{1+x}}dx = \int \frac{u-1}{\sqrt{u}}du = \int \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du = \frac{2}{3}u^{3/2} - 2\sqrt{u} + C = \frac{2}{3}(1+x)^{3/2} - 2\sqrt{1+x} + C.$$

If $-1 < x < 0$, then $\displaystyle\int \sqrt{\frac{x^2}{1+x}}dx = \int \frac{-x}{\sqrt{1+x}}dx = -\frac{2}{3}(1+x)^{3/2} + 2\sqrt{1+x} + C.$

33. If we set $u = 2/x$, then $du = -(2/x^2)dx$. For $4x - x^2 = x(4-x)$ to be nonnegative, x must be in the interval $0 \le x \le 4$. It follows that u is positive, and

$$\int \frac{\sqrt{4x-x^2}}{x^3}dx = \int \frac{\sqrt{\dfrac{8}{u} - \dfrac{4}{u^2}}}{2/u}\left(\frac{du}{-2}\right) = -\frac{1}{2}\int \sqrt{2u-1}\,du = -\frac{1}{6}(2u-1)^{3/2} + C = -\frac{1}{6}\left(\frac{4}{x} - 1\right)^{3/2} + C.$$

34. If we set $u = 1/x$, then $du = -\dfrac{1}{x^2}dx$. For $x - x^2 = x(1-x)$ to be nonnegative, x must be in the interval $0 \le x \le 1$. It follows that u is positive, and

$$\int \frac{\sqrt{x-x^2}}{x^4}dx = \int \frac{\sqrt{\dfrac{1}{u} - \dfrac{1}{u^2}}}{\dfrac{1}{u^2}}(-du) = -\int u\sqrt{u-1}\,du.$$

We now set $v = u - 1$, in which case $dv = du$, and

$$\int \frac{\sqrt{x-x^2}}{x^4}\,dx = -\int (v+1)\sqrt{v}\,dv = -\int (v^{3/2}+\sqrt{v})\,dv = -\left(\frac{2}{5}v^{5/2}+\frac{2}{3}v^{3/2}\right)+C$$
$$= -\frac{2}{5}(u-1)^{5/2}-\frac{2}{3}(u-1)^{3/2}+C = -\frac{2}{5}\left(\frac{1}{x}-1\right)^{5/2}-\frac{2}{3}\left(\frac{1}{x}-1\right)^{3/2}+C.$$

35. Since $u^2=\dfrac{5+x}{1-x} \implies u^2(1-x)=5+x \implies x=\dfrac{u^2-5}{u^2+1}$, we obtain
$dx=\dfrac{(u^2+1)(2u)-(u^2-5)(2u)}{(u^2+1)^2}du=\dfrac{12u}{(u^2+1)^2}du$. Since

$$5-4x-x^2=5-4\left(\frac{u^2-5}{u^2+1}\right)-\left(\frac{u^2-5}{u^2+1}\right)^2=\frac{5(u^2+1)^2-4(u^2+1)(u^2-5)-(u^2-5)^2}{(u^2+1)^2}=\frac{36u^2}{(u^2+1)^2},$$

$$\int \frac{x}{(5-4x-x^2)^{3/2}}dx=\int \frac{\dfrac{u^2-5}{u^2+1}}{\left[\dfrac{36u^2}{(u^2+1)^2}\right]^{3/2}}\frac{12u}{(u^2+1)^2}du=\int \frac{12u(u^2-5)(u^2+1)^3}{216u^3(u^2+1)^3}du$$
$$=\frac{1}{18}\int \frac{u^2-5}{u^2}du=\frac{1}{18}\left(u+\frac{5}{u}\right)+C=\frac{1}{18}\left(\frac{u^2+5}{u}\right)+C=\frac{1}{18}\left(\frac{\dfrac{5+x}{1-x}+5}{\sqrt{\dfrac{5+x}{1-x}}}\right)+C$$
$$=\frac{1}{18}\left[\frac{5+x+5-5x}{(1-x)\sqrt{\dfrac{5+x}{1-x}}}\right]+C=\frac{5-2x}{9\sqrt{5-4x-x^2}}+C.$$

36. Since $u^2=\dfrac{1-x}{1+x} \implies u^2(1+x)=1-x \implies x=\dfrac{1-u^2}{1+u^2}$, we obtain
$dx=\dfrac{(1+u^2)(-2u)-(1-u^2)(2u)}{(1+u^2)^2}du=\dfrac{-4u}{(1+u^2)^2}du$, and

$$\int \frac{1}{3(1-x^2)-(5+4x)\sqrt{1-x^2}}dx=\int \frac{1}{3(1-x)(1+x)-(5+4x)\sqrt{(1-x)(1+x)}}dx$$
$$=\int \frac{-4u/(1+u^2)^2}{3\left(1-\dfrac{1-u^2}{1+u^2}\right)\left(1+\dfrac{1-u^2}{1+u^2}\right)-\left(5+\dfrac{4-4u^2}{1+u^2}\right)\sqrt{\left(1-\dfrac{1-u^2}{1+u^2}\right)\left(1+\dfrac{1-u^2}{1+u^2}\right)}}du$$
$$=\int \frac{-4u}{\dfrac{12u^2}{(1+u^2)^2}-\dfrac{9+u^2}{1+u^2}\sqrt{\dfrac{4u^2}{(1+u^2)^2}}}\frac{1}{(1+u^2)^2}du$$
$$=\int \frac{2}{(u-3)^2}du=-\frac{2}{u-3}+C=\frac{-2}{\sqrt{\dfrac{1-x}{1+x}}-3}+C.$$

37. If $u-x=\sqrt{x^2+x+4}$, then $u^2-2ux+x^2=x^2+x+4 \implies x=\dfrac{u^2-4}{1+2u}$, from which
$dx=\dfrac{(1+2u)(2u)-(u^2-4)(2)}{(1+2u)^2}du=\dfrac{2u^2+2u+8}{(1+2u)^2}du$. Then,

$$\int \sqrt{x^2+x+4}\,dx=\int (u-x)\,dx=\int \left(u-\frac{u^2-4}{1+2u}\right)\frac{2u^2+2u+8}{(1+2u)^2}du=\int \frac{2(u^2+u+4)^2}{(1+2u)^3}du,$$

and the integrand is a rational function of u.

38. If $(x+1)u = \sqrt{4+3x-x^2}$, then $(x+1)^2u^2 = 4+3x-x^2 = (4-x)(1+x)$, or, $(x+1)u^2 = 4-x$. When this equation is solved for x, the result is $x = \dfrac{4-u^2}{1+u^2}$, and therefore

$$dx = \frac{(1+u^2)(-2u)-(4-u^2)(2u)}{(1+u^2)^2}du = \frac{-10u}{(1+u^2)^2}du.$$

Since $\sqrt{4+3x-x^2} = (x+1)u = u\left(\dfrac{4-u^2}{1+u^2}+1\right) = \dfrac{5u}{1+u^2}$,

$$\int \frac{1}{\sqrt{4+3x-x^2}}dx = \int \frac{1+u^2}{5u}\frac{-10u}{(1+u^2)^2}du = -2\int \frac{1}{1+u^2}du,$$

and the integrand is a rational function of u.

EXERCISES 5.4

1. Deflections must satisfy equation 5.8 with $F(x) = -9.81m/L$ subject to conditions $y(0) = y''(0) = 0$ and $y(L) = y''(L) = 0$. Integration of the differential equation gives $y(x) = [-9.81mx^4/(24L) + Ax^3 + Bx^2 + Cx + D]/(EI)$. The boundary conditions require

$$0 = EI\,y(0) = D, \quad 0 = EI\,y''(0) = 2B,$$

$$0 = EI\,y(L) = \frac{-9.81mL^3}{24} + AL^3 + BL^2 + CL + D, \quad 0 = EI\,y''(L) = \frac{-9.81mL}{2} + 6AL + 2B.$$

Solutions are $A = 9.81m/12$, $B = 0$, $C = -9.81mL^2/24$, and $D = 0$, so that

$$y(x) = \frac{1}{EI}\left(-\frac{9.81mx^4}{24L} + \frac{9.81mx^3}{12} - \frac{9.81mL^2x}{24}\right) = -\frac{9.81m}{24EIL}(x^4 - 2Lx^3 + L^3x).$$

2. Deflections must satisfy equation 5.8 with $F(x) = -9.81m/L$ subject to conditions $y(0) = y'(0) = 0$ and $y(L) = y'(L) = 0$. Integration of the differential equation gives $y(x) = [-9.81mx^4/(24L) + Ax^3 + Bx^2 + Cx + D]/(EI)$. The boundary conditions require

$$0 = EI\,y(0) = D, \quad 0 = EI\,y'(0) = C,$$

$$0 = EI\,y(L) = \frac{-9.81mL^3}{24} + AL^3 + BL^2 + CL + D, \quad 0 = EI\,y'(L) = \frac{-9.81mL^2}{6} + 3AL^2 + 2BL + C.$$

Solutions are $A = 9.81m/12$, $B = -9.81mL/24$, $C = 0$, and $D = 0$, so that

$$y(x) = \frac{1}{EI}\left(-\frac{9.81mx^4}{24L} + \frac{9.81mx^3}{12} - \frac{9.81mLx^2}{24}\right) = -\frac{9.81m}{24EIL}(x^4 - 2Lx^3 + L^2x^2).$$

3. Deflections must satisfy equation 5.8 with $F(x) = -9.81m/L$ subject to conditions $y(0) = y'(0) = 0$ and $y''(L) = y'''(L) = 0$. Integration of the differential equation gives $y(x) = [-9.81mx^4/(24L) + Ax^3 + Bx^2 + Cx + D]/(EI)$. The boundary conditions require

$$0 = EI\,y(0) = D, \quad 0 = EI\,y'(0) = C,$$

$$0 = EI\,y''(L) = \frac{-9.81mL}{2} + 6AL + 2B, \quad 0 = EI\,y'''(L) = -9.81m + 6A.$$

Solutions are $A = 9.81m/6$, $B = -9.81mL/4$, $C = 0$, and $D = 0$, so that

$$y(x) = \frac{1}{EI}\left(-\frac{9.81mx^4}{24L} + \frac{9.81mx^3}{6} - \frac{9.81mLx^2}{4}\right) = -\frac{9.81m}{24EIL}(x^4 - 4Lx^3 + 6L^2x^2).$$

4. Deflections must satisfy equation 5.8 with $F(x) = -9.81m/L$ subject to conditions $y(0) = y'(0) = 0$ and $y(L) = y''(L) = 0$. Integration of the differential equation gives $y(x) = [-9.81mx^4/(24L) + Ax^3 + Bx^2 + Cx + D]/(EI)$. The boundary conditions require

$$0 = EI\,y(0) = D, \quad 0 = EI\,y'(0) = C,$$

$$0 = EI\,y(L) = \frac{-9.81mL^3}{24} + AL^3 + BL^2 + CL + D, \quad 0 = EI\,y''(L) = \frac{-9.81mL}{2} + 6AL + 2B.$$

Solutions are $A = 5(9.81)m/48$, $B = -9.81mL/16$, $C = 0$, and $D = 0$, so that

$$y(x) = \frac{1}{EI}\left(-\frac{9.81mx^4}{24L} + \frac{5(9.81)mx^3}{48} - \frac{9.81mLx^2}{16}\right) = -\frac{9.81m}{48EIL}(2x^4 - 5Lx^3 + 3L^2x^2).$$

5. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = -\dfrac{F}{EI}\delta(x - L/2)$ subject to the boundary conditions $y(0) = y'(0) = 0 = y(L) = y'(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3h(x - L/2) + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy'(0) = C,$$

$$0 = EIy(L) = -\frac{F}{6}\left(\frac{L}{2}\right)^3 + AL^3 + BL^2 + CL + D, \quad 0 = EIy'(L) = -\frac{F}{2}\left(\frac{L}{2}\right)^2 + 3AL^2 + 2BL + C.$$

These give $A = \dfrac{F}{12}$, $B = \dfrac{-FL}{16}$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3h(x - L/2) + \frac{Fx^3}{12} - \frac{FLx^2}{16}\right] = \frac{-F}{48EI}\left[8(x - L/2)^3h(x - L/2) - 4x^3 + 3Lx^2\right].$$

6. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = -\dfrac{F}{EI}\delta(x - L/2)$ subject to the boundary conditions $y(0) = y'(0) = 0 = y''(L) = y'''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3h(x - L/2) + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D,\ 0 = EIy'(0) = C,\ 0 = EIy''(L) = -F\left(\frac{L}{2}\right) + 6AL + 2B,\ 0 = EIy'''(L) = -F + 6A.$$

These give $A = \dfrac{F}{6}$, $B = \dfrac{-FL}{4}$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3h(x - L/2) + \frac{Fx^3}{6} - \frac{FLx^2}{4}\right] = \frac{-F}{12EI}\left[2F(x - L/2)^3h(x - L/2) - 2x^3 + 3Lx^2\right].$$

For $x > L/2$,

$$y = \frac{-F}{12EI}\left[2F(x - L/2)^3 - 2x^3 + 3Lx^2\right] = \frac{FL^2}{48EI}(L - 6x),$$

the equation of a straight line.

7. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = -\dfrac{F}{EI}\delta(x - L/2)$ subject to the boundary conditions $y(0) = y''(0) = 0 = y(L) = y''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy''(0) = 2B,$$

$$0 = EIy(L) = -\frac{F}{6}\left(\frac{L}{2}\right)^3 + AL^3 + BL^2 + CL + D, \qquad 0 = EIy''(L) = -F\left(\frac{L}{2}\right) + 6AL + 2B.$$

These give $A = \dfrac{F}{12}$, $C = \dfrac{-FL^2}{16}$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) + \frac{Fx^3}{12} - \frac{FL^2x}{16}\right] = \frac{-F}{48EI}\left[8F(x - L/2)^3 h(x - L/2) - 4x^3 + 3L^2x\right].$$

8. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\left[-F\delta(x - L/2) - \dfrac{mg}{L}\right]$ subject to the boundary conditions $y(0) = y'(0) = 0 = y(L) = y'(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) - \frac{mgx^4}{24L} + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy'(0) = C,$$

$$0 = EIy(L) = -\frac{F}{6}\left(\frac{L}{2}\right)^3 - \frac{mgL^3}{24} + AL^3 + BL^2 + CL + D,$$

$$0 = EIy'(L) = -\frac{F}{2}\left(\frac{L}{2}\right)^2 - \frac{mgL^2}{6} + 3AL^2 + 2BL + C.$$

These give $A = (F + mg)/12$, $B = -L(3F + 2mg)/48$, and therefore

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) - \frac{mgx^4}{24L} + \frac{(F + mg)x^3}{12} - \frac{L(3F + 2mg)x^2}{48}\right].$$

9. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\left[-F\delta(x - L/2) - \dfrac{mg}{L}\right]$ subject to the boundary conditions $y(0) = y'(0) = 0 = y''(L) = y'''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) - \frac{mgx^4}{24L} + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy'(0) = C,$$

$$0 = EIy''(L) = -F\left(\frac{L}{2}\right) - \frac{mgL}{2} + 6AL + 2B, \; 0 = EIy'''(L) = -F - mg + 6A.$$

These give $A = (F + mg)/6$ and $B = -L(F + mg)/4$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x - L/2)^3 h(x - L/2) - \frac{mgx^4}{24L} + \frac{(F + mg)x^3}{6} - \frac{L(F + mg)x^2}{4}\right].$$

10. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\left[-F\delta(x-L/2) - \dfrac{mg}{L}\right]$ subject to the boundary conditions $y(0) = y''(0) = 0 = y(L) = y''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x-L/2)^3h(x-L/2) - \frac{mgx^4}{24L} + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D, \qquad 0 = EIy''(0) = 2B,$$
$$0 = EIy(L) = -\frac{F}{6}\left(\frac{L}{2}\right)^3 - \frac{mgL^3}{24} + AL^3 + BL^2 + CL + D,$$
$$0 = EIy''(L) = -F\left(\frac{L}{2}\right) - \frac{mgL}{2} + 6AL + 2B.$$

These give $A = (F+mg)/12$, and $C = -L^2(3F+2mg)/48$, and hence

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x-L/2)^3h(x-L/2) - \frac{mgx^4}{24L} + \frac{(F+mg)x^3}{12} - \frac{L^2(3F+2mg)x}{48}\right].$$

11. (a) According to Exercise 10,

$$y(x) = \frac{1}{10^6}\left[-\frac{1500}{6}(x-2)^3h(x-2) - \frac{1000x^4}{24(4)} + \frac{(1500+1000)x^3}{12} - \frac{16(4500+2000)x}{48}\right]$$
$$= \frac{1}{10^6}\left[-250(x-2)^3h(x-2) - \frac{125x^4}{12} + \frac{625x^3}{3} - \frac{6500x}{3}\right].$$

(b) Maximum deflection occurs at the centre of the beam,

$$y(2) = \frac{1}{10^6}\left[-\frac{125(2)^4}{12} + \frac{625(2)^3}{3} - \frac{6500(2)}{3}\right] = -0.00283.$$

Since maximum deflection cannot exceed $4/360 = 0.011$ m, the beam is acceptable.

12. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\{-98.1 - 98.1[h(x-5) - h(x-10)]\}$ $= \dfrac{-98.1}{EI}[1 + h(x-5)]$ since $h(x-10) = 0$ if $0 < x < 10$, subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times using equation 5.11 gives

$$y(x) = \frac{-4.0875}{EI}[x^4 + (x-5)^4h(x-5) + Ax^3 + Bx^2 + Cx + D].$$

The boundary conditions require

$$0 = EIy(0) = -4.0875D, \qquad 0 = EIy'(0) = -4.0875C,$$
$$0 = EIy''(10) = -4.0875[12(10)^2 + 12(5)^2 + 6A(10) + 2B],$$
$$0 = EIy'''(10) = -4.0875[24(10) + 24(5) + 6A].$$

These give $A = -60$ and $B = 1050$, and therefore

$$y(x) = -\frac{4.0875}{EI}[x^4 + (x-5)^4h(x-5) - 60x^3 + 1050x^2].$$

Deflection at $x = 10$ is greater in this case.

13. Deflections must satisfy the differential equation $\dfrac{d^4y}{dx^4} = \dfrac{1}{EI}\{-98.1 - 98.1[h(x-5/2) - h(x-15/2)]\}$, subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times using equation 5.11 gives

$$y(x) = \frac{-4.0875}{EI}[x^4 + (x-5/2)^4 h(x-5/2) - (x-15/2)^4 h(x-15/2) + Ax^3 + Bx^2 + Cx + D].$$

The boundary conditions require

$$0 = EIy(0) = -4.0875D \qquad 0 = EIy'(0) = -4.0875C,$$
$$0 = EIy''(10) = -4.0875[12(10)^2 + 12(15/2)^2 - 12(5/2)^2 + 6A(10) + 2B],$$
$$0 = EIy'''(10) = -4.0875[24(10) + 24(15/2) - 24(5/2) + 6A].$$

These gives $A = -60$ and $B = 900$, and therefore

$$y(x) = \frac{-4.0875}{EI}[x^4 + (x-5/2)^4 h(x-5/2) - (x-15/2)^4 h(x-15/2) - 60x^4 + 900x^2].$$

Deflection at $x = 10$ in this case is greater than in Figure 5.9 but less than that in Exercise 12.

14. When the concentrated force is placed at x_0 just to the left of the end of the beam, the differential equation for displacements is $\dfrac{d^4y}{dx^4} = -\dfrac{F}{EI}\delta(x-x_0)$, subject to the boundary conditions $y(0) = y'(0) = 0 = y''(L) = y'''(L)$. Four integrations of the differential equation give

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x-x_0)^3 h(x-x_0) + Ax^3 + Bx^2 + Cx + D\right].$$

The boundary conditions require

$$0 = EIy(0) = D,\ 0 = EIy'(0) = C,\ 0 = EIy''(L) = -F(L-x_0) + 6AL + 2B,\ 0 = EIy'''(L) = -F + 6A.$$

These give $A = \dfrac{F}{6}$, $B = \dfrac{-Fx_0}{2}$, and hence $y(x) = \dfrac{1}{EI}\left[-\dfrac{F}{6}(x-x_0)^3 h(x-x_0) + \dfrac{Fx^3}{6} - \dfrac{Fx_0x^2}{2}\right]$. If we now take the limit as $x_0 \to L^-$, we obtain

$$y(x) = \frac{1}{EI}\left[-\frac{F}{6}(x-L)^3 h(x-L) + \frac{Fx^3}{6} - \frac{FLx^2}{2}\right] = \frac{Fx^2(x-3L)}{6EI}.$$

EXERCISES 5.5

1. If $N(t)$ is the number of bacteria at time t, then the fact that they increase at a rate proportional to N can be expressed as $dN/dt = kN$, where k is a constant. This is a separable differential equation $\dfrac{dN}{N} = k\,dt$. Solutions are defined implicitly by

$$\ln|N| = kt + C \qquad \Longrightarrow \qquad |N| = e^{kt+C} \qquad \Longrightarrow \qquad N = De^{kt}, \qquad (D = e^C).$$

If N_0 is the number of bacteria at time $t = 0$, then $N_0 = D$, and $N(t) = N_0e^{kt}$. Since $N(2) = (5/4)N_0$, it follows that $(5/4)N_0 = N_0e^{2k}$. Hence, $k = (1/2)\ln(5/4)$. If T is the time when the number of bacteria doubles, then $2N_0 = N_0e^{kT}$. This equation can be solved for $T = k^{-1}\ln 2 = \dfrac{2}{\ln(5/4)}\ln 2 = 6.21$ hours.

2. If $N(t)$ is the number of bacteria at time t, then the fact that they increase at a rate proportional to N can be expressed as $dN/dt = kN$, where k is a constant. This is a separable differential equation $\dfrac{dN}{N} = k\,dt$. Solutions are defined implicitly by

$$\ln|N| = kt + C \qquad \Longrightarrow \qquad |N| = e^{kt+C} \qquad \Longrightarrow \qquad N = De^{kt}, \qquad (D = e^C).$$

If N_0 is the number of bacteria at time $t = 0$, then $N_0 = D$, and $N(t) = N_0 e^{kt}$. Since $N(3) = 2N_0$, it follows that $2N_0 = N_0 e^{3k}$. Hence, $k = (\ln 2)/3$. If T is the time when the number of bacteria triples, then $3N_0 = N_0 e^{kT}$. This equation can be solved for $T = k^{-1} \ln 3 = \dfrac{3}{\ln 2} \ln 3 = 4.75$ hours.

3. The amount of radioactive material in a sample at any time t is $A = A_0 e^{kt}$ where A_0 is the amount at time $t = 0$. If $A(15) = A_0/2$, then $A_0/2 = A_0 e^{15k}$, from which $k = -(1/15)\ln 2$. If T is the time for 90% of the sample to decay, then, $A_0/10 = A_0 e^{kT}$. This can be solved for $T = -k^{-1} \ln 10 = \dfrac{15}{\ln 2} \ln 10 = 49.83$ days.

4. The amount of radioactive material in a sample at any time t is $A = A_0 e^{kt}$ where A_0 is the amount at time $t = 0$. If $A(3) = 0.9A_0$, then $0.9A_0 = A_0 e^{3k}$, from which $k = \ln(0.9)/3$. The half life T of the material is the time at which $A = A_0/2$; that is, $A_0/2 = A_0 e^{kT}$. This can be solved for $T = -k^{-1} \ln 2 = -\dfrac{3}{\ln(0.9)} \ln 2 = 19.74$ seconds.

5. Since half the amount decreases during each half-life, only 1/16 or 6.25% remains after 4 half-lives.

6. If $A(t)$ is the amount of drug in the body as a function of time t, then $\dfrac{dA}{dt} = kA$, $k < 0$ a constant. This differential equation is separable, $\dfrac{dA}{A} = k\,dt$. Solutions are defined implicitly by $\ln A = kt + C$. Exponentiation yields $A = e^{kt+C} = De^{kt}$, where $D = e^C$. If we choose time $t = 0$ when the original amount is A_0, then $A_0 = D$, and therefore $A = A_0 e^{kt}$. Since $A(1) = 0.95A_0$, we have $0.95A_0 = A_0 e^k$. This equation implies that $k = \ln(0.95)$. The amount of drug in the body will be $A_0/2$ when $A_0/2 = A_0 e^{kt}$, and the solution of this equation for t is $t = -\dfrac{1}{k} \ln 2 = -\dfrac{\ln 2}{\ln 0.95} = 13.51$ hours.

7. (a) Let V be the volume of sugar remaining at any given time t, and x be the length of the side of the cube at this time. Since dissolving occurs at a rate proportional to the surface area of the remaining cube,

$$\frac{dV}{dt} = k(6x^2), \quad k < 0 \text{ a constant.}$$

Since $V = x^3$, it follows that $6kx^2 = \dfrac{d}{dt}(x^3) = 3x^2 \dfrac{dx}{dt}$. Thus, $\dfrac{dx}{dt} = 2k$, and integration of this equation gives $x = 2kt + C$. Since $x(0) = 1$, constant C must be 1, and $x = 2kt + 1$. The cube has completely dissolved when $0 = x = 2kt + 1$, and therefore when $t = -1/(2k)$.
(b) When dissolving is proportional to the amount of sugar remaining,

$$\frac{dV}{dt} = kV, \quad k < 0 \text{ a constant.}$$

This is differential equation 5.19 with solution $V = Ce^{kt}$. Since $V(0) = 1$ cubic centimetre, it follows that $C = 1$, and $V = e^{kt}$. The cube completely dissolves when $0 = V = e^{kt}$, but this occurs only after an infinitely long time.

8. (a) Since sugar particles dissolve independently, the time taken for the sugar to dissolve is the time taken for each spherical particle to dissolve. Since dissolving occurs at a rate proportional to the surface area of the particle,

$$\frac{dV}{dt} = k(4\pi r^2), \quad k < 0 \text{ a constant.}$$

Since $V = (4/3)\pi r^3$, it also follows that $4k\pi r^2 = \dfrac{d}{dt}\left(\dfrac{4\pi r^3}{3}\right) = 4\pi r^2 \dfrac{dr}{dt}$. Thus, $\dfrac{dr}{dt} = k$, and integration gives $r = kt + C$. The initial condition $r(0) = r_0$ implies that $C = r_0$, and therefore $r = kt + r_0$. The sugar is completely dissolved when $0 = r = kt + r_0 \implies t = -r_0/k$.
(b) In this case, $dV/dt = kV$. The solution of this differential equation is $V = Ce^{kt}$. The initial condition $V(0) = (4/3)\pi r_0^3 = V_0$ implies that $C = V_0$, and $V = V_0 e^{kt}$. The sugar is completely dissolved when $V = 0$, but this does not occur in finite time.

9. According to the discussion on carbon dating, the amount of C^{14} in the fossil at time t after the creature's death is $A = A_0 e^{kt}$ where A_0 is the amount present at death and $k = -\ln 2/5550$. If T is the time at which 1.51% of A_0 remains, then $0.0151 A_0 = A_0 e^{kT}$. The solution of this equation is $T = \ln(0.0151)/k = 33\,574$ years.

10. According to the discussion on carbon dating, the amount of C^{14} in the fossil at time t after the creature's death is $A = A_0 e^{kt}$ where A_0 is the amount present at death and $k = -\ln 2/5550$. When $t = 100\,000$, the percentage of C^{14} present is

$$\frac{100A}{A_0} = \frac{100A_0}{A_0} e^{100\,000k} = 100e^{100\,000(-\ln 2)/5550} = 3.8 \times 10^{-4}\%.$$

11. If $A(t)$ represents the amount of drug in the body at time t (in hours), then $\dfrac{dA}{dt} = kA$, where $k < 0$ is a constant. Separation of variables gives $\dfrac{1}{A}\,dA = k\,dt$, and therefore $\ln|A| = kt + C$, or, $A = De^{kt}$. If A_0 is the size of the original dose injected at time $t = 0$, then $A_0 = D$, and $A = A_0 e^{kt}$. Since $A(1) = 0.95A_0$, it follows that $0.95A_0 = A_0 e^k$. Thus, $k = \ln(0.95)$, and $A = A_0 e^{t\ln(0.95)}$. The dose decreases to $A_0/2$ when $A_0/2 = A_0 e^{t\ln(0.95)}$, the solution of which is $t = -\ln 2/\ln(0.95) = 13.51$ h.

12. The rate of change dC/dt of the amount of glucose in the blood is equal to the rate at which it is added less the rate at which it is used up,

$$\frac{dC}{dt} = R - kC, \quad k > 0 \text{ a constant.}$$

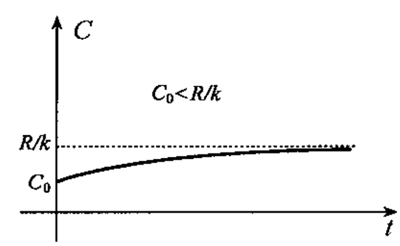
This is separable, $\dfrac{dC}{R - kC} = dt$, and solutions are therefore defined implicitly by $-\dfrac{1}{k}\ln|R - kC| = t + D$. Thus, $|R - kC| = e^{-k(t+D)}$, from which $R - kC = Ee^{-kt}$, where $E = \pm e^{-kD}$ is a constant. We can now solve for

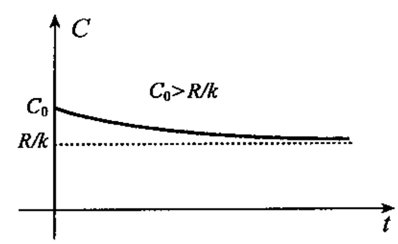
$$C(t) = \frac{1}{k}(R - Ee^{-kt}).$$

Since $C(0) = C_0$, it follows that $C_0 = (R - E)/k$, and this implies that $E = R - kC_0$. Thus,

$$C(t) = \frac{1}{k}\left[R - (R - kC_0)e^{-kt}\right] = \frac{R}{k}(1 - e^{-kt}) + C_0 e^{-kt}.$$

Graphs in the cases $C_0 < R/k$ and $C_0 > R/k$ are shown below.





13. If a quantity decreases at a rate proportional to its present amount, then the amount A present at any given time is given by $A = A_0 e^{kt}$, where A_0 is the amount at time $t = 0$ and k is a negative constant. The percentage decrease in a time interval of length h beginning at time t is

$$100\left[\frac{A(t) - A(t+h)}{A(t)}\right] = 100\left[\frac{A_0e^{kt} - A_0e^{k(t+h)}}{A_0e^{kt}}\right] = 100(1 - e^{kh}).$$

Since this quantity is independent of t, the percentage decrease is the same at any time.

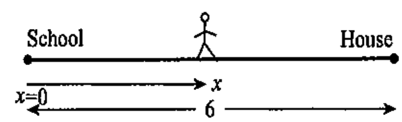
14. Separation of variables leads to $\dfrac{1}{T-20}\,dT = k\,dt$, and therefore $\ln|T-20| = kt + C$. The absolute values may be dropped since $T \geq 20$. Exponentiation then gives $T = 20 + De^{kt}$. Since $T(0) = 90$, we find that $D = 70$, and therefore $T = 20 + 70e^{kt}$. Because $T(40) = 60$, it follows that $60 - 20 = 70e^{40k}$, and $k = (1/40)\ln(4/7)$. Hence,

$$T = 20 + 70e^{(1/40)\ln(4/7)t} = 20 + 70e^{-0.01399t}.$$

15. If $T(t)$ represents the temperature of the mercury in the thermometer as a function of time t, then according to Newton's law of cooling, $\dfrac{dT}{dt} = k(T+20)$, where $k < 0$ is a constant. Separation of variables leads to $\dfrac{1}{T+20}\,dT = k\,dt$, and therefore $\ln|T+20| = kt + C$. When we solve for T, the result is $T(t) = -20 + De^{kt}$. If we choose time $t = 0$ when $T = 25$, then $25 = -20 + D$. Thus, $D = 45$, and $T(t) = -20 + 45e^{kt}$. Because $T(4) = 0$, it follows that $0 = -20 + 45e^{4k}$, from which $k = (1/4)\ln(20/45)$. The temperature is $-19°$ C when $-19 = -20 + 45e^{kt} \implies t = (1/k)\ln(1/45) = 18.8$ minutes.

16. When the boy is x km from school, his velocity is

$$\frac{dx}{dt} = kx^2, \qquad k = \text{a constant}.$$



We separate variables, $\dfrac{dx}{x^2} = k\,dt$, in which case solutions are defined implicitly by

$$-\frac{1}{x} = kt + C \quad \implies \quad x = \frac{-1}{kt + C}.$$

If we choose time $t = 0$ when $x = 6$, then $6 = -1/C$. Thus, $x = -1/(kt - 1/6) = 6/(1 - 6kt)$. Since $x(1) = 3$, it follows that $3 = 6/(1 - 6k) \implies k = -1/6$, and $x(t) = 6/(1+t)$ km. The boy reaches school when $x = 0$, but this does not happen in finite time.

17. If $S(t)$ represents the number of grams of sugar in the tank at any given time, then dS/dt is equal to the rate at which sugar is added to the tank less the rate at which it leaves the tank. It is being added at (1/5)(10)=2 grams per minute. Since the amount of solution in the tank is always 100 litres, it follows that the rate at which sugar leaves the tank is $(1/5)S/100 = S/500$. Consequently,

$$\frac{dS}{dt} = 2 - \frac{S}{500} \qquad \implies \qquad \frac{1}{1000 - S}dS = \frac{1}{500}dt,$$

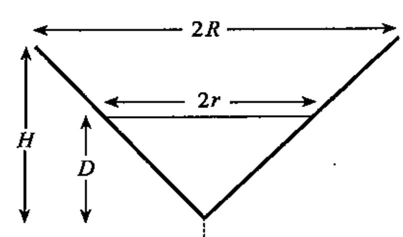
a separated differential equation. Solutions are defined implicitly by

$$-\ln|1000 - S| = \frac{t}{500} + C \quad \implies \quad 1000 - S = \pm e^{-t/500 - C} \quad \implies \quad S = 1000 + De^{-t/500}, \quad (D = \pm e^{-C}).$$

Since $S(0) = 4000$, it follows that $4000 = 1000 + D \implies D = 3000$. Hence, the number of grams of sugar in the tank is $S = 1000 + 3000e^{-t/500}$.

18. The volume of water in the tank is

$$V = \frac{1}{3}\pi r^2 D.$$



Because $r/D = R/H$, it follows that

$$V = \frac{1}{3}\pi\left(\frac{RD}{H}\right)^2 D = \frac{\pi R^2}{3H^2}D^3.$$

Thus, $\dfrac{dV}{dt} = \dfrac{\pi R^2 D^2}{H^2}\dfrac{dD}{dt}$.

But the rate at which water exits through the hole is Av. In other words,

$$\frac{\pi R^2 D^2}{H^2}\frac{dD}{dt} = -Av = -Ac\sqrt{2gD}.$$

We separate variables, $D^{3/2}\,dD = -\dfrac{\sqrt{2g}AcH^2}{\pi R^2}\,dt$, in which case solutions are defined implicitly by

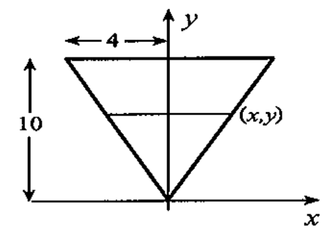
$\frac{2}{5}D^{5/2} = -\frac{\sqrt{2g}AcH^2}{\pi R^2}t + C$. If we choose time $t = 0$ when the tank is full ($D = H$), then $C = (2/5)H^{5/2}$, and

$$\frac{2}{5}D^{5/2} = -\frac{\sqrt{2g}AcH^2}{\pi R^2}t + \frac{2}{5}H^{5/2}.$$

The tank empties when $D = 0$, and this occurs when $t = \frac{2}{5}H^{5/2}\frac{\pi R^2}{\sqrt{2g}AcH^2} = \frac{\pi R^2}{5cA}\sqrt{\frac{2H}{g}}$.

19. When the depth of water is y, the volume is $V = (\pi/3)x^2y$. By similar triangles, $y/x = 10/4$, so that $V = (\pi/3)x^2(5x/2) = (5\pi/6)x^3$. Differentiation with respect to time t gives

$$\frac{dV}{dt} = \frac{5\pi x^2}{2}\frac{dx}{dt}.$$

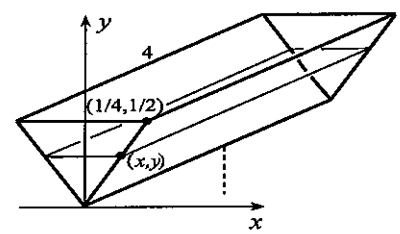


Because water evaporates at a rate proportional to the surface area, we can say that $\frac{dV}{dt} = k(\pi x^2)$, where k is a constant. Consequently,

$$\frac{5\pi x^2}{2}\frac{dx}{dt} = k\pi x^2 \quad\Longrightarrow\quad \frac{dx}{dt} = \frac{2k}{5}.$$

Integration gives $x(t) = 2kt/5 + C$. If we take $t = 0$ when the container is full, then $x(0) = 4$, and this implies that $4 = C$. Thus, $x(t) = 2kt/5 + 4$. Since $x(5) = 18/5$, it follows that $18/5 = 2k(5)/5 + 4 \Longrightarrow k = -1/5$. Thus, $x(t) = 4 - 2t/25$. The water has all evaporated when $x = 0$, and this occurs when $4 - 2t/25 = 0 \Longrightarrow t = 50$ days.

20. When the depth of water in the trough is y, the volume of water is $V = 4(xy) = 4xy$. Similar triangles require $y/x = (1/2)/(1/4) \Longrightarrow y = 2x$. Thus, $V = 4y(y/2) = 2y^2$. Differentiation of this equation with respect to time t gives $dV/dt = 4y(dy/dt)$. Because water exits through a hole of area 10^{-4} m^2 with speed $\sqrt{gy/2}$, it follows that $dV/dt = -10^{-4}\sqrt{gy/2}$.



Hence, $4y\frac{dy}{dt} = -\frac{\sqrt{gy/2}}{10^4} \Longrightarrow \sqrt{y}\,dy = -\frac{\sqrt{g}}{10^4(4\sqrt{2})}\,dt$. Solutions of this separated equation are defined implicitly by $\frac{2}{3}y^{3/2} = -\frac{\sqrt{g}\,t}{10^4(4\sqrt{2})} + C$. If we choose $t = 0$ when the trough is full, then $y(0) = 1/2$, and this implies that $(2/3)(1/2)^{3/2} = C$. Thus, $\frac{2}{3}y^{3/2} = \frac{-\sqrt{g}\,t}{10^4(4\sqrt{2})} + \frac{1}{3\sqrt{2}}$. The tank empties when $y = 0$, and the time at which this occurs is $t = \frac{4 \times 10^4}{3\sqrt{g}} = 4257$ seconds, or 70.95 minutes.

21. If we replace d^2x/dt^2 by $v(dv/dx)$, the differential equation becomes $v\frac{dv}{dx} = -\frac{k}{M}x$, and this can be separated, $v\,dv = -\frac{k}{M}x\,dx$. Solutions are defined implicitly by $v^2/2 = -kx^2/(2M) + C$. Since $v = v_0$ when $x = 0$, it follows that $v_0^2/2 = C$, and

$$\frac{v^2}{2} = -\frac{kx^2}{2M} + \frac{v_0^2}{2} \quad\Longrightarrow\quad v = \pm\sqrt{v_0^2 - kx^2/M}.$$

22. If we set (see Exercise 21) $\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$, then $mv\frac{dv}{dr} = -\frac{GmM}{r^2}$, or $v\,dv = -\frac{GM}{r^2}\,dr$. Solutions of this separated differential equation are defined implicitly by $\frac{v^2}{2} = \frac{GM}{r} + C$. Since $v = 0$ when $r = R + h$, where R is the radius of the earth, it follows that $0 = GM/(R + h) + C$, and therefore

$$\frac{v^2}{2} = \frac{GM}{r} - \frac{GM}{R+h}.$$

The velocity of m when it strikes the earth ($r = R$) is defined by

$$\frac{v^2}{2} = \frac{GM}{R} - \frac{GM}{R+h} = \frac{GMh}{R(R+h)} \quad \Longrightarrow \quad v = -\sqrt{\frac{2GMh}{R(R+h)}}.$$

Maximum attainable speed occurs when h becomes infinite; that is,

$$|v_{\max}| = \lim_{h\to\infty} \sqrt{\frac{2GMh}{R(R+h)}} = \sqrt{\frac{2GM}{R}}.$$

23. If V and A are the volume and area of the disk of ice, then $\dfrac{dV}{dt} = kA$ where $k < 0$ is a constant. Since the ratio of the radius R to the thickness T of the disk remains constant, and is 2 at time $t = 0$ when the disk begins to melt, we can say that $R = 2T$ for all t. Since $V = \pi R^2 T/2 = \pi(2T)^2 T/2 = 2\pi T^3$ and $A = \pi R^2 + 2RT + \pi RT = \pi(2T)^2 + 2(2T)T + \pi(2T)T = 2(3\pi + 2)T^2$, it follows that

$$\frac{d}{dt}(2\pi T^3) = 2k(3\pi+2)T^2 \quad \Longrightarrow \quad 6\pi T^2 \frac{dT}{dt} = 2k(3\pi+2)T^2 \quad \Longrightarrow \quad \frac{dT}{dt} = \frac{k}{3\pi}(3\pi+2).$$

This diferential equation must be solved subject to the initial condition $T(0) = 1$. Integration gives $T(t) = k(3\pi+2)t/(3\pi) + C$. The initial condition requires $1 = C$. Because $T(10) = 1/4$, it follows that

$$\frac{1}{4} = \frac{k(3\pi+2)(10)}{3\pi} + 1 \quad \Longrightarrow \quad k = -\frac{9\pi}{40(3\pi+2)}.$$

Hence, $T(t) = -3t/40 + 1$. The disk is totally melted when $0 = T(t) = -3t/40 + 1 \Longrightarrow t = 40/3$ minutes.

24. If V and A represent volume and area of the mothball at any time, then the assumption that evaporation is proportional to surface area is represented by the equation

$$\frac{dV}{dt} = kA,$$

where k is a constant. We have four variables in the problem: t, r, A, and V. By substituting $V = 4\pi r^3/3$ and $A = 4\pi r^2$, we eliminate V and A:

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 k \quad \Longrightarrow \quad 4\pi r^2 \frac{dr}{dt} = 4\pi r^2 k \quad \Longrightarrow \quad \frac{dr}{dt} = k.$$

Solutions of this differential equation are $r = kt + C$. Using the conditions $r(0) = R$ and $r(1) = R/2$, we find $C = R$ and $k = -R/2$. Consequently,

$$r(t) = R - \frac{Rt}{2} = R\left(1 - \frac{t}{2}\right).$$

The mothball completely disappears when $r = 0$, and this occurs when $t = 2$ years.

25. If we let r be the radius of the raindrop, m its mass, and y the distance that it has fallen, then $\dfrac{dm}{dt} = k(4\pi r^2)\left(\dfrac{dy}{dt}\right)$. Since $m = 4\pi r^3 \rho/3$, where ρ is the density of water,

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3 \rho\right) = 4\pi k r^2 \frac{dy}{dt} \quad \Longrightarrow \quad 4\pi r^2 \rho \frac{dr}{dt} = 4\pi k r^2 \frac{dy}{dt} \quad \Longrightarrow \quad \frac{dr}{dy} = \frac{k}{\rho}.$$

Integration of this differential equation gives $r(y) = ky/\rho + C$. If we choose $y = 0$ at the position when the raindrop is initially formed ($r = 0$), then $r(0) = 0$, and this implies that $C = 0$. Thus, $r = ky/\rho$.

26. The modified Torricelli law in equation 5.22 implies that water exits through the hole with horizontal speed $v_x = c\sqrt{2g(H-h)}$, where $0 < c < 1$ is a constant, and $g = 9.81$ is the acceleration due to gravity. Suppose we follow a droplet of water on its journey to the ground if it exits at time $t = 0$. Because the only force acting on it is gravity, its vertical acceleration is $a_y = -g$, from which $v_y = -gt + C$. Since the initial velocity of the droplet is horizontal, $v_y(0) = 0$, and this implies that $C = 0$. Integration of $dy/dt = -gt$ gives $y = -gt^2/2 + D$. Since $y(0) = h$, it follows that $h = D$, and $y = -gt^2/2 + h$. The horizontal acceleration of the droplet is zero so that its horizontal velocity must always be $v_x = c\sqrt{2g(H-h)}$. Since this is dx/dt, we integrate to get $x = ct\sqrt{2g(H-h)} + E$. Because $x(0) = 0$, we obtain $E = 0$, and $x = ct\sqrt{2g(H-h)}$. The droplet hits the ground when

$$0 = y = -\frac{gt^2}{2} + h \quad \Longrightarrow \quad t = \sqrt{\frac{2h}{g}}.$$

The x-coordinate of the point at which the droplet hits the ground is therefore

$$x = c\sqrt{\frac{2h}{g}}\sqrt{2g(H-h)} = 2c\sqrt{h(H-h)}, \quad 0 \le h \le H.$$

We must find the value of h that maximizes this function. For critical points we solve

$$0 = \frac{dx}{dh} = \frac{c}{\sqrt{h(H-h)}}(H-2h) \quad \Longrightarrow \quad h = \frac{H}{2}.$$

Since $x(0) = x(H) = 0$, it follows that x is maximized when $h = H/2$.

27. (a) Two integrations of the differential equation give $(EI)y = \dfrac{A}{6}x^3 - \dfrac{mg}{24}x^4 + Cx + D$. Because $f(0) = f(L/2) = 0$,

$$0 = D, \qquad 0 = \frac{A}{6}\left(\frac{L}{2}\right)^3 - \frac{mg}{24}\left(\frac{L}{2}\right)^4 + C\left(\frac{L}{2}\right) + D.$$

These imply that $C = (L^2/192)(mgL - 8A)$, and therefore

$$y = \frac{1}{EI}\left[\frac{A}{6}x^3 - \frac{mg}{24}x^4 + \frac{L^2}{192}(mgL - 8A)x\right].$$

(b) The diagram makes it clear that dy/dx must be 0 at $x = L/2$; that is,

$$0 = \frac{A}{2}\left(\frac{L}{2}\right)^2 - \frac{mg}{6}\left(\frac{L}{2}\right)^3 + \frac{L^2}{192}(mgL - 8A) \quad \Longrightarrow \quad A = \frac{3mgL}{16}.$$

28. If we multiply the differential equation by r^2, it can be written in the form

$$0 = r^2\frac{d^2T}{dr^2} + 2r\frac{dT}{dr} = \frac{d}{dr}\left(r^2\frac{dT}{dr}\right).$$

Integration gives $r^2\dfrac{dT}{dr} = C$, where C is a constant, from which $dT/dr = C/r^2$. A second integration now gives $T = -C/r + D$. Since $T(1) = 10$ and $T(2) = 20$, it follows that $10 = -C + D$ and $20 = -C/2 + D$. These imply that $C = 20$ and $D = 30$, so that $T(r) = 30 - 20/r$.

29. Let $C(t)$ be the volume of CO_2 in the room at any time t. Then dC/dt is the rate of change of the volume of CO_2 in the room. It is equal to the rate at which CO_2 enters less the rate at which it exits. It enters at a rate of 0.0025 m^3/min. Since the concentration of CO_2 in the room at any time is $C(t)/100$, the exit rate is $5C(t)/100 = 0.05C(t)$. Consequently,

$$\frac{dC}{dt} = 0.0025 - 0.05C = \frac{1}{400} - \frac{C}{20} = \frac{1-20C}{400}.$$

It follows that $\dfrac{dC}{1-20C} = \dfrac{dt}{400}$, a separated equation with solutions defined implicitly by

$$-\frac{1}{20}\ln|1-20C| = \frac{t}{400} + D \quad \Longrightarrow \quad \ln|1-20C| = -\frac{t}{20} + E,$$

where $E = -20D$. Exponentiation gives

$$|1-20C| = e^E e^{-t/20} \quad \Longrightarrow \quad 1-20C = \pm e^E e^{-t/20} \quad \Longrightarrow \quad C = \frac{1}{20}(1 \pm e^E e^{-t/20}) = \frac{1}{20}(1+Fe^{-t/20}),$$

where $F = \pm e^E$. The fact that $C(0) = 1/10$ requires

$$\frac{1}{10} = \frac{1}{20}(1+F) \quad \Longrightarrow \quad F = 1 \quad \Longrightarrow \quad C(t) = \frac{1}{20}(1+e^{-t/20}).$$

The limit of this function as $t \to \infty$ is $1/20$ m^3.

30. (a) We can separate the differential equation $\dfrac{1}{\rho^{2-\delta}}d\rho = -\dfrac{1}{k\delta}dh$. Solutions are defined implicitly by

$$\frac{1}{(\delta-1)\rho^{1-\delta}} = -\frac{h}{k\delta} + C.$$

Because $\rho = \rho_0$ when $h = 0$, $\dfrac{1}{(\delta-1)\rho_0^{1-\delta}} = C$, and therefore

$$\frac{1}{(\delta-1)\rho^{1-\delta}} = -\frac{h}{k\delta} + \frac{1}{(\delta-1)\rho_0^{1-\delta}}.$$

This can be written $\rho^{\delta-1} = -\dfrac{h}{k}\left(\dfrac{\delta-1}{\delta}\right) + \rho_0^{\delta-1}$.

(b) If $P = k\rho^\delta$, then $\rho^{\delta-1} = \left(\rho^\delta\right)^{(\delta-1)/\delta} = \left(\dfrac{P}{k}\right)^{(\delta-1)/\delta}$. Because $P = P_0$ when $\rho = \rho_0$, it follows that $\rho_0^{\delta-1} = \left(\dfrac{P_0}{k}\right)^{(\delta-1)/\delta}$. When these are substituted into the result of part (a),

$$\left(\frac{P}{k}\right)^{1-1/\delta} = -\frac{h}{k}\left(\frac{\delta-1}{\delta}\right) + \left(\frac{P_0}{k}\right)^{1-1/\delta},$$

or,

$$P^{1-1/\delta} = P_0^{1-1/\delta} - \frac{h}{k}\left(1-\frac{1}{\delta}\right)k^{1-1/\delta} = P_0^{1-1/\delta} - h\left(1-\frac{1}{\delta}\right)k^{-1/\delta}.$$

But $\rho_0^{\delta-1} = (P_0/k)^{(\delta-1)/\delta}$ implies that $k^{-1/\delta} = \rho_0 P_0^{-1/\delta}$, and therefore

$$P^{1-1/\delta} = P_0^{1-1/\delta} - h\left(1-\frac{1}{\delta}\right)\rho_0 P_0^{-1/\delta}.$$

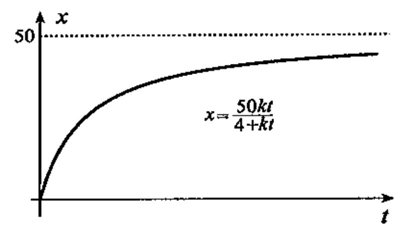
(c) If we define the effective height of the atmosphere when $P = 0$, then this occurs for

$$0 = P_0^{1-1/\delta} - h\left(1-\frac{1}{\delta}\right)\rho_0 P_0^{-1/\delta} \quad \Longrightarrow \quad h = \frac{\delta P_0}{(\delta-1)\rho_0}.$$

31. If $x(t)$ represents the number of grams of dissolved chemical at time t, then

$$\frac{dx}{dt} = k(50 - x)\left(\frac{25}{100} - \frac{x}{200}\right) = \frac{k}{200}(50 - x)^2,$$

where k is a constant. This equation can be separated, $\frac{1}{(50-x)^2}\,dx = \frac{k}{200}\,dt$, and solutions are defined implicitly by $\frac{1}{50-x} = \frac{kt}{200} + C$. Since $x(0) = 0$, it follows that $1/50 = C$, and $\frac{1}{50-x} = \frac{kt}{200} + \frac{1}{50}$. When we solve this equation for x, we obtain $x(t) = \frac{50kt}{4+kt}$ g.



32. (a) If $x(t)$ represents the amount of C in the mixture at time t, then

$$\frac{dx}{dt} = k\left(20 - \frac{2x}{3}\right)\left(10 - \frac{x}{3}\right) = \frac{2k}{9}(30 - x)^2.$$

We can separate this equation, $\frac{dx}{(30-x)^2} = \frac{2k\,dt}{9} \implies \frac{1}{30-x} = \frac{2kt}{9} + C$. For $x(0) = 0$, we must have $1/30 = C$, and therefore $\frac{1}{30-x} = \frac{2kt}{9} + \frac{1}{30} \implies x(t) = \frac{600kt}{20kt+3}$.

REVIEW EXERCISES

1. $\int (3x^3 - 4x^2 + 5)\,dx = \frac{3x^4}{4} - \frac{4x^3}{3} + 5x + C$

2. $\int \left(\frac{1}{x^5} + 2x - \frac{1}{x^3}\right) dx = -\frac{1}{4x^4} + x^2 + \frac{1}{2x^2} + C$

3. $\int (2x^2 - 3x + 7x^6)\,dx = \frac{2x^3}{3} - \frac{3x^2}{2} + x^7 + C$

4. $\int \left(\frac{1}{x^2} - 2\sqrt{x}\right) dx = -\frac{1}{x} - \frac{4}{3}x^{3/2} + C$

5. $\int \sqrt{x-2}\,dx = \frac{2}{3}(x-2)^{3/2} + C$

6. $\int x(1+3x^2)^4\,dx = \frac{1}{30}(1+3x^2)^5 + C$

7. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \frac{2}{3}x^{3/2} - 2\sqrt{x} + C$

8. $\int \left(\frac{x^2+5}{\sqrt{x}}\right) dx = \int \left(x^{3/2} + \frac{5}{\sqrt{x}}\right) dx = \frac{2}{5}x^{5/2} + 10\sqrt{x} + C$

9. $\int \frac{1}{(x+5)^4} dx = \frac{-1}{3(x+5)^3} + C$

10. $\int \left(\frac{\sqrt{x}}{x^2} - \frac{15}{\sqrt{x}}\right) dx = \int \left(\frac{1}{x^{3/2}} - \frac{15}{\sqrt{x}}\right) dx = -\frac{2}{\sqrt{x}} - 30\sqrt{x} + C$

11. $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x + C$

12. $\int x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{3/2} + C$

13. $\int x\cos x^2\,dx = \frac{1}{2}\sin x^2 + C$

14. $\int x^2(1-2x^2)^2\,dx = \int (x^2 - 4x^4 + 4x^6)\,dx = \frac{x^3}{3} - \frac{4x^5}{5} + \frac{4x^7}{7} + C$

15. If we set $u = 1 + x$, then $du = dx$, and

$$\begin{aligned}\int x\sqrt{1+x}\,dx &= \int (u-1)\sqrt{u}\,du = \int (u^{3/2} - \sqrt{u})\,du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C.\end{aligned}$$

16. If we set $u = 2 - x$, then $du = -dx$, and

$$\int \frac{x}{\sqrt{2-x}}dx = \int \frac{2-u}{\sqrt{u}}(-du) = \int \left(\sqrt{u} - \frac{2}{\sqrt{u}}\right) du = \frac{2}{3}u^{3/2} - 4\sqrt{u} + C = \frac{2}{3}(2-x)^{3/2} - 4\sqrt{2-x} + C.$$

17. $\displaystyle\int \frac{1}{(1+x)^2}dx = \frac{-1}{1+x} + C$

18. $\displaystyle\int (2+\sqrt{x})^2\,dx = \int (4 + 4\sqrt{x} + x)\,dx = 4x + \frac{8}{3}x^{3/2} + \frac{x^2}{2} + C$

19. If we set $u = \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}dx$, and

$$\int \frac{1}{\sqrt{x}(2+\sqrt{x})^2}dx = \int \frac{1}{(2+u)^2}(2\,du) = 2\left(\frac{-1}{2+u}\right) + C = \frac{-2}{2+\sqrt{x}} + C.$$

20. $\displaystyle\int \sin^4 x \cos x\,dx = \frac{1}{5}\sin^5 x + C$

21. $\displaystyle\int e^{3-5x}\,dx = -\frac{1}{5}e^{3-5x} + C$

22. $\displaystyle\int xe^{-4x^2}\,dx = -\frac{1}{8}e^{-4x^2} + C$

23. $\displaystyle\int \frac{e^x - 1}{e^{2x}}dx = \int (e^{-x} - e^{-2x})\,dx = -e^{-x} + \frac{1}{2}e^{-2x} + C$

24. If we set $u = \ln x$, then $du = \dfrac{1}{x}dx$, and $\displaystyle\int \frac{1}{5x\ln x}dx = \frac{1}{5}\int \frac{1}{u}du = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\ln x| + C.$

25. If we set $u = 2x^2$ and $du = 4x\,dx$, then

$$\int \frac{x}{\sqrt{1-4x^4}}dx = \int \frac{1}{\sqrt{1-u^2}}\left(\frac{du}{4}\right) = \frac{1}{4}\mathrm{Sin}^{-1}u + C = \frac{1}{4}\mathrm{Sin}^{-1}(2x^2) + C.$$

26. If we set $u = \sqrt{7}x$, then $du = \sqrt{7}\,dx$, and

$$\int \frac{3}{1+7x^2}dx = \int \frac{3}{1+u^2}\left(\frac{du}{\sqrt{7}}\right) = \frac{3}{\sqrt{7}}\mathrm{Tan}^{-1}u + C = \frac{3}{\sqrt{7}}\mathrm{Tan}^{-1}\sqrt{7}x + C.$$

27. $\displaystyle\int x\cosh 5x^2\,dx = \frac{1}{10}\sinh 5x^2 + C$

28. $\displaystyle\int \mathrm{sech}^2 5x\,dx = \frac{1}{5}\tanh 5x + C$

29. If $N(t)$ is the number of bacteria in the culture, then N increases at a rate proportional to N (see Exercise 1 in Section 5.5);

$$\frac{dN}{dt} = kN \qquad \Longrightarrow \qquad \frac{dN}{N} = k\,dt.$$

This is a separated differential equation with solutions defined implicitly by

$$\ln N = kt + C \qquad \Longrightarrow \qquad N = e^{kt+C} = De^{kt}, \quad \text{where } D = e^C.$$

If N_0 is the original number of bacteria when $t = 0$, say, then $N_0 = D$. Thus, $N(t) = N_0e^{kt}$. Since the number of bacteria triples in 3 days, $3N_0 = N_0e^{3k}$, and this implies that $k = (1/3)\ln 3$. The number of bacteria quadruples when

$$4N_0 = N_0e^{kt} \qquad \Longrightarrow \qquad t = \frac{1}{k}\ln 4 = \frac{\ln 4}{(1/3)\ln 3} = 3.8 \text{ days}.$$

30. If $T(t)$ represents the temperature of the water as a function of time t, then according to Newton's law of cooling,

$$\frac{dT}{dt} = k(T+20) \qquad \Longrightarrow \qquad \frac{dT}{T+20} = k\,dt, \quad \text{where } k < 0 \text{ is a constant.}$$

This is a separated differential equation with solutions defined implicitly by

$$\ln|T+20| = kt + C \qquad \Longrightarrow \qquad T + 20 = e^{kt+C} \qquad \Longrightarrow \qquad T = -20 + De^{kt}$$

where $D = e^C$. If we choose time $t = 0$ when $T = 70$, then $70 = -20 + D$. Thus, $D = 90$, and $T(t) = -20+90e^{kt}$. Because $T(10) = 50$, it follows that $50 = -20+90e^{4k}$, from which $k = (1/4)\ln(7/9)$. This solution would only be valid until the time at which the water reaches temperature zero and freezes.

31. If $f''(x) = x^2+1$, then $f'(x) = x^3/3+x+C$. Since $f'(1) = 4$, it follows that $4 = 1/3+1+C \Longrightarrow C = 8/3$, and $f'(x) = x^3/3 + x + 8/3$. Integration gives $f(x) = x^4/12 + x^2/2 + 8x/3 + D$. Since $f(1) = 1$, we obtain $1 = 1/12 + 1/2 + 8/3 + D \Longrightarrow D = -9/4$, and $y = f(x) = x^4/12 + x^2/2 + 8x/3 - 9/4$.

32. If we integrate $f''(x) = 12x^2$, the result is $f'(x) = 4x^3+C$. A second integration gives $f(x) = x^4+Cx+D$. Since $f(1) = 4$ and $f(-1) = -3$, it follows that

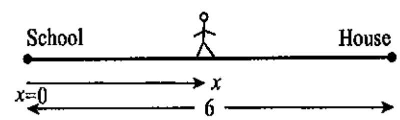
$$4 = (1)^4 + C(1) + D \qquad \text{and} \qquad -3 = (-1)^4 + C(-1) + D.$$

Solutions of these equations are $C = 7/2$ and $D = -1/2$. The equation of the required curve is therefore $y = x^4 + 7x/2 - 1/2$.

33. Integration of $f''(x) = 24x^2 + 6x$ gives $f'(x) = 8x^3 + 3x^2 + C$. Since $f'(1) = 4$, it follows that $4 = 8+3+C \Longrightarrow C = -7$. A second integration of $f'(x) = 8x^3+3x^2-7$ gives $f(x) = 2x^4+x^3-7x+D$. Since $f(1) = 8$, we must have $8 = 2 + 1 - 7 + D \Longrightarrow D = 12$. Thus, $f(x) = 2x^4 + x^3 - 7x + 12$.

34. When the boy is x km from school, his velocity is

$$\frac{dx}{dt} = k\sqrt{x}, \qquad k = \text{a constant.}$$



If we separate $\dfrac{dx}{\sqrt{x}} = k\,dt$, then solutions are defined implicitly by

$$2\sqrt{x} = kt + C.$$

If we choose time $t = 0$ when $x = 6$, then $2\sqrt{6} = C$. Thus, $2\sqrt{x} = kt + 2\sqrt{6}$. Since $x(1) = 3$, it follows that $2\sqrt{3} = k + 2\sqrt{6}$, from which $k = 2(\sqrt{3} - \sqrt{6})$, and $2\sqrt{x} = 2(\sqrt{3} - \sqrt{6})t + 2\sqrt{6}$. Thus, $x = [(\sqrt{3}-\sqrt{6})t+\sqrt{6}]^2$ km. The boy reaches school when $x = 0$, and this occurs when $t = \sqrt{6}/(\sqrt{6}-\sqrt{3})$ hours.

35. If we take y as positive upward with $y = 0$ and $t = 0$ at the point and instant the ball is released, then the acceleration of the ball is $dv/dt = -9.81$. Integration gives $v = -9.81t + C$. Since $v(0) = 30$, we obtain $C = 30$, and $v = -9.81t + 30$. A second integration gives $y = -4.905t^2 + 30t + D$. Since $y(0) = 0$, we find $D = 0$, and $y = -4.905t^2 + 30t$. The ball reaches its peak height when $0 = v = -9.81t + 30 \Longrightarrow t = 30/9.81$. The height of the ball at this time is $-4.905(30/9.81)^2 + 30(30/9.81) = 45.9$ m.

36. We choose y as positive downward with $y = 0$ and $t = 0$ at the instant the stone is released. The acceleration of the stone is $dv/dt = 9.81$. Inegration gives $v(t) = 9.81t+C$. If we denote by v_0 the initial speed of the stone, then $v(0) = v_0$, and this implies that $C = v_0$. Thus, $dy/dt = 9.81t + v_0$. Integration now yields $y(t) = 4.905t^2 + v_0t + D$. The condition $y(0) = 0$ requires $D = 0$. Because $y(2.2) = 50$, it follows that $50 = 4.905(2.2)^2 + v_0(2.2)$, and this implies that $v_0 = 11.9$ m/s.

37. If we set $u = 1 + \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}dx$, and

$$\begin{aligned}\int \frac{1}{\sqrt{1+\sqrt{x}}}dx &= \int \frac{1}{\sqrt{u}}2(u-1)\,du = 2\int\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du \\ &= 2\left(\frac{2u^{3/2}}{3} - 2\sqrt{u}\right) + C = \frac{4}{3}(1+\sqrt{x})^{3/2} - 4\sqrt{1+\sqrt{x}} + C.\end{aligned}$$

38. If we set $u = \sqrt{1+x}$, then $du = \dfrac{1}{2\sqrt{1+x}}dx$, and

$$\begin{aligned}\int \frac{x}{\sqrt{1+x}+1}dx &= \int \frac{u^2-1}{u+1}2u\,du = 2\int(u^2-u)\,du = 2\left(\frac{u^3}{3} - \frac{u^2}{2}\right) + C \\ &= \frac{2}{3}(1+x)^{3/2} - (1+x) + C = \frac{2}{3}(1+x)^{3/2} - x + D.\end{aligned}$$

39. By adjusting constants, $\displaystyle\int \frac{\sin x}{\sqrt{4+3\cos x}}dx = -\frac{2}{3}\sqrt{4+3\cos x} + C.$

40. If we set $u = 3-2x^3$, then $du = -6x^2\,dx$, and

$$\int x^8(3-2x^3)^6\,dx = \int (x^3)^2(3-2x^3)^6x^2\,dx = \int\left(\frac{3-u}{2}\right)^2 u^6\left(-\frac{du}{6}\right) = \frac{1}{24}\int(-9u^6+6u^7-u^8)\,du$$
$$= \frac{1}{24}\left(-\frac{9u^7}{7}+\frac{3u^8}{4}-\frac{u^9}{9}\right)+C = \frac{-3}{56}(3-2x^3)^7+\frac{1}{32}(3-2x^3)^8-\frac{1}{216}(3-2x^3)^9+C.$$

41. $\displaystyle\int\frac{(2+x)^4}{x^6}dx = \int\left(\frac{16}{x^6}+\frac{32}{x^5}+\frac{24}{x^4}+\frac{8}{x^3}+\frac{1}{x^2}\right)dx = \frac{-16}{5x^5}-\frac{8}{x^4}-\frac{8}{x^3}-\frac{4}{x^2}-\frac{1}{x}+C$

42. If we set $u=\sin x$, then $du = \cos x\,dx$, and

$$\int\sin^3 x\cos^3 x\,dx = \int\sin^3 x(1-\sin^2 x)\cos x\,dx = \int u^3(1-u^2)\,du$$
$$= \int(u^3-u^5)\,du = \frac{u^4}{4}-\frac{u^6}{6}+C = \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.$$

43. If we set $u=\ln x$, then $du = (1/x)\,dx$, and

$$\int\frac{1}{x\sqrt{1+3\ln x}}dx = \int\frac{1}{\sqrt{1+3u}}du = \frac{2}{3}\sqrt{1+3u}+C = \frac{2}{3}\sqrt{1+3\ln x}+C.$$

44. If we set $u=\cos x$, then $du = -\sin x\,dx$, and

$$\int\tan x\,dx = \int\frac{\sin x}{\cos x}dx = \int\frac{1}{u}(-du) = -\ln|u|+C = -\ln|\cos x|+C.$$

45. Integration of $a(t) = \begin{cases} 2t^2/25, & 0\le t\le 5\\ 4-2t/5, & 5<t\le 10\\ 0, & 10<t\le 15\end{cases}$ gives $v(t) = \begin{cases} 2t^3/75+C, & 0\le t\le 5\\ 4t-t^2/5+D, & 5<t\le 10\\ E, & 10<t\le 15.\end{cases}$

Since $v(0)=0$, it follows that $C=0$. To obtain D and E we demand that velocity be continuous at $t=5$ and $t=10$. This requires

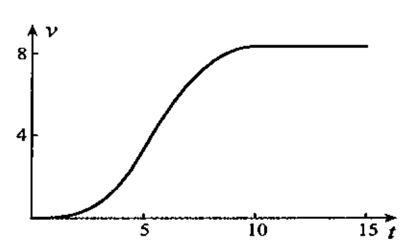
$$\frac{2(5)^3}{75} = 4(5)-\frac{25}{5}+D, \qquad 4(10)-\frac{100}{5}+D = E.$$

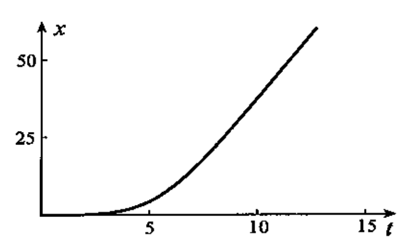
These give $D=-35/3$ and $E=25/3$, so that $v(t) = \begin{cases} 2t^3/75, & 0\le t\le 5\\ 4t-t^2/5-35/3, & 5<t\le 10\\ 25/3, & 10<t\le 15.\end{cases}$ A second integration gives $x(t) = \begin{cases} t^4/150+F, & 0\le t\le 5\\ 2t^2-t^3/15-35t/3+G, & 5<t\le 10\\ 25t/3+H, & 10<t\le 15.\end{cases}$ Since $x(0)=0$, we find that $F=0$.

To obtain G and H we demand that displacement be continuous at $t=5$ and $t=10$. This requires

$$\frac{(5)^4}{150} = 2(5)^2-\frac{5^3}{15}-\frac{35(5)}{3}+G, \qquad 2(10)^2-\frac{10^3}{15}-\frac{35(10)}{3}+G = \frac{25(10)}{3}+H.$$

These give $G=125/6$ and $H=-275/6$. These functions are graphed below.



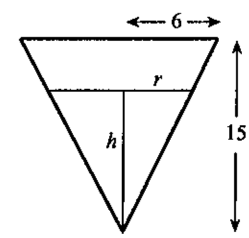


46. Since the slope of $y = f(x)$ at any point is dy/dx, it follows that $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2$. If we write $\dfrac{dy}{y^2} = \dfrac{dx}{2x^2}$, then solutions are defined implicitly by $-\dfrac{1}{y} = -\dfrac{1}{2x} + C$. Since the curve is to pass through $(1,1)$, it follows that $-1/1 = -1/2 + C$. Thus $C = -1/2$, and $-\dfrac{1}{y} = -\dfrac{1}{2x} - \dfrac{1}{2} \Longrightarrow y = 2x/(1+x)$.

47. The slope of each of these curves is $dy/dx = 3x^2$. For a curve $y = f(x)$ to intersect each of the cubics at right angles, its slope at each point must be $f'(x) = -1/(3x^2)$. Integration gives $f(x) = 1/(3x) + D$. For that curve through the point $(1,1)$, we must have $1 = 1/3 + D$. Hence, the required curve is $y = 1/(3x) + 2/3$.

48. If let $V(t)$ be the volume of water in the cone (figure to the right), then

$$\frac{dV}{dt} = k(\pi r^2).$$



But $V = (1/3)\pi r^2 h$, and from similar triangles, $r/h = 6/15$. Hence,

$$V = \frac{1}{3}\pi r^2\left(\frac{5r}{2}\right) = \frac{5\pi r^3}{6}.$$

If we substitute this into the differential equation, we obtain

$$\frac{5\pi}{6}(3r^2)\frac{dr}{dt} = k\pi r^2 \qquad \Longrightarrow \qquad \frac{dr}{dt} = \frac{2k}{5}.$$

Integration gives $r = 2kt/5 + C$. If we take $t = 0$ when the cone is full, $6 = C$. Thus, $r = 2kt/5 + 6$. Since the water level drops 1 cm in 6 days, and $h = 5r/2$, it follows that

$$14 = \frac{5}{2}\left[\frac{2k(6)}{5} + 6\right] \qquad \Longrightarrow \qquad k = -\frac{1}{6}.$$

The radius of the surface of the water is therefore $r(t) = -t/15 + 6$. Half the water has evaporated when

$$\frac{1}{3}\pi r^2 h = \frac{1}{2}\left[\frac{1}{3}\pi(6)^2(15)\right].$$

Since $h = 5r/2$, this equation becomes $r^2(5r/2) = 270 \Longrightarrow r = 108^{1/3}$. We can now find out how long this takes,

$$108^{1/3} = -\frac{t}{15} + 6 \qquad \Longrightarrow \qquad t = 15(6 - 108^{1/3}) = 18.6 \text{ days}.$$

49. Deflections of the beam must satisfy the differential equation

$$\frac{d^4y}{dx^4} = -\frac{9.81M}{5EI}[h(x) - h(x-5)] = k[1 - h(x-5)], \quad \text{where } k = -9.81M/(5EI),$$

(since $h(x) = 1$ for $0 < x < 10$), subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times gives
$y(x) = k\left[\dfrac{x^4}{24} - \dfrac{(x-5)^4}{24}h(x-5) + Ax^3 + Bx^2 + Cx + D\right]$. The boundary conditions require

$$0 = y(0) = k(D), \qquad 0 = y'(0) = k(C),$$

$$0 = y''(10) = k\left[\frac{10^2}{2} - \frac{5^2}{2} + 60A + 2B\right], \qquad 0 = y'''(10) = k(10 - 5 + 6A).$$

These give $A = -5/6$ and $B = 25/4$, and therefore the function describing deflections of the beam is

$$y(x) = -\frac{9.81M}{5EI}\left[\frac{x^4}{24} - \frac{(x-5)^4}{24}h(x-5) - \frac{5x^3}{6} + \frac{25x^2}{4}\right].$$

For $x > 5$, $y(x) = -\frac{9.81M}{5EI}\left[\frac{x^4}{24} - \frac{(x-5)^4}{24} - \frac{5x^3}{6} + \frac{25x^2}{4}\right] = -\frac{9.81M}{5EI}\left(\frac{125x}{6} - \frac{625}{24}\right)$. Since the equation is linear for $x > 5$, the beam is straight on this interval. This is to be expected since there is no load on the beam for $x > 5$.

50. Deflections of the beam must satisfy the differential equation

$$\frac{d^4y}{dx^4} = -\frac{9.81M}{5EI}[h(x-5) - h(x-10)] = k\,h(x-5), \quad \text{where } k = -9.81M/(5EI),$$

(since $h(x-10) = 0$ for $0 < x < 10$), subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times gives
$y(x) = k\left[\frac{(x-5)^4}{24}h(x-5) + Ax^3 + Bx^2 + Cx + D\right]$. The boundary conditions require

$$0 = y(0) = k(D), \qquad 0 = y'(0) = k(C),$$

$$0 = y''(10) = k\left[\frac{5^2}{2} + 6A(10) + 2B\right], \qquad 0 = y'''(10) = k(5 + 6A).$$

These gives $A = -5/6$ and $B = 75/4$, and therefore the function describing deflections of the beam is

$$y(x) = -\frac{9.81M}{5EI}\left[\frac{(x-5)^4}{24}h(x-5) - \frac{5x^3}{6} + \frac{75x^2}{4}\right].$$

For $x < 5$, $y(x) = -\frac{9.81M}{5EI}\left[-\frac{5x^3}{6} + \frac{75x^2}{4}\right]$. Since this equation is not linear, the beam is not straight on this interval, nor should it be.

51. Deflections of the beam must satisfy the differential equation

$$\frac{d^4y}{dx^4} = -\frac{9.81M}{5EI}[h(x-5/2) - h(x-15/2)] = k[h(x-5/2) - h(x-15/2)],$$

where $k = -9.81M/(5EI)$, subject to the boundary conditions $y(0) = y'(0) = 0 = y''(10) = y'''(10)$. Integration of the differential equation four times gives
$y(x) = k\left[\frac{(x-5/2)^4}{24}h(x-5/2) - \frac{(x-15/2)^4}{24}h(x-15/2) + Ax^3 + Bx^2 + Cx + D\right]$. The boundary conditions require

$$0 = y(0) = k(D), \qquad 0 = y'(0) = k(C),$$

$$0 = y''(10) = k\left[\frac{(15/2)^2}{2} - \frac{(5/2)^2}{2} + 6A(10) + 2B\right], \qquad 0 = y'''(10) = k(15/2 - 5/2 + 6A).$$

These give $A = -5/6$ and $B = 25/2$, and therefore the function describing deflections of the beam is

$$y(x) = -\frac{9.81M}{5EI}\left[\frac{(x-5/2)^4}{24}h(x-5/2) - \frac{(x-15/2)^4}{24}h(x-15/2) - \frac{5x^3}{6} + \frac{25x^2}{2}\right].$$

For $x < 5/2$, $y(x) = -\frac{9.81M}{5EI}\left[-\frac{5x^3}{6} + \frac{25x^2}{2}\right]$, and for $x > 15/2$,

$$y(x) = -\frac{9.81M}{5EI}\left[\frac{(x-5/2)^4}{24} - \frac{(x-15/2)^4}{24} - \frac{5x^3}{6} + \frac{25x^2}{2}\right] = -\frac{9.81M}{5EI}\left[\frac{1625x}{24} - \frac{3125}{24}\right].$$

Since the equation is linear for $x > 15/2$, the beam is straight on this interval. This is to be expected since there is no load on the beam for $x > 15/2$. It is not straight for $x < 5/2$.

CHAPTER 6

EXERCISES 6.1

In trying to find the pattern by which terms are formed it is often beneficial to write the value of the index of summation above each term. We follow this suggestion when appropriate.

1. $\overset{k=1}{2\cdot 3}+\overset{k=2}{3\cdot 4}+\overset{k=3}{4\cdot 5}+\overset{k=4}{5\cdot 6}+\cdots+\overset{k=?}{99\cdot 100}=\sum_{k=1}^{98}(k+1)(k+2)$

2. $\overset{k=1}{\frac{1}{2}}+\overset{k=2}{\frac{2}{4}}+\overset{k=3}{\frac{3}{8}}+\overset{k=4}{\frac{4}{16}}+\overset{k=5}{\frac{5}{32}}+\cdots+\overset{k=?}{\frac{10}{1024}}=\sum_{k=1}^{10}\frac{k}{2^k}$

3. $\overset{k=1}{\frac{16}{14+15}}+\overset{k=2}{\frac{17}{15+16}}+\overset{k=3}{\frac{18}{16+17}}+\cdots+\overset{k=?}{\frac{199}{197+198}}=\sum_{k=1}^{184}\frac{k+15}{(k+13)+(k+14)}=\sum_{k=1}^{184}\frac{k+15}{2k+27}$

4. Since each term is the square root of a positive integer, we write

$$1+\sqrt{2}+\sqrt{3}+2+\sqrt{5}+\sqrt{6}+\sqrt{7}+\sqrt{8}+3+\cdots+121=\sum_{k=1}^{14641}\sqrt{k}.$$

5. $\overset{k=1}{1}+\overset{k=2}{\frac{1}{2}}+\overset{k=3}{\frac{1}{2\cdot 3}}+\overset{k=4}{\frac{1}{2\cdot 3\cdot 4}}+\overset{k=5}{\frac{1}{2\cdot 3\cdot 4\cdot 5}}+\cdots+\overset{k=?}{\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot 16}}=\sum_{k=1}^{16}\frac{1}{k!}$

6. Disregarding the sign changes, each term is an integer. To have even terms positive and odd terms negative, we use the fact that $(-1)^k$ is 1 when k is an even integer and -1 when k is an odd integer. In other words, $\quad -2+3-4+5-6+7-8+\cdots-1020=\sum_{k=1}^{1019}(-1)^k(k+1).$

7. $\overset{k=1}{\frac{2\cdot 3}{1\cdot 4}}+\overset{k=2}{\frac{6\cdot 7}{5\cdot 8}}+\overset{k=3}{\frac{10\cdot 11}{9\cdot 12}}+\overset{k=4}{\frac{14\cdot 15}{13\cdot 16}}++\overset{k=?}{\frac{414\cdot 415}{413\cdot 416}}=\sum_{k=1}^{104}\frac{(4k-2)(4k-1)}{(4k-3)(4k)}$

8. $\overset{k=1}{\frac{\tan 1}{2}}+\overset{k=2}{\frac{\tan 2}{1+2^2}}+\overset{k=3}{\frac{\tan 3}{1+3^2}}+\overset{k=4}{\frac{\tan 4}{1+4^2}}+\cdots+\overset{k=?}{\frac{\tan 225}{1+225^2}}=\sum_{k=1}^{225}\frac{\tan k}{1+k^2}$

9. $\overset{k=1}{4^3}+\overset{k=2}{5^2}+\overset{k=3}{6}+\overset{k=4}{1}+\overset{k=5}{\frac{1}{8}}+\cdots+\overset{k=?}{\frac{1}{25^{18}}}=\sum_{k=1}^{22}(k+3)^{4-k}$

10. $0.9+0.99+0.999+\cdots+0.999\,999\,999=\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+\cdots+\frac{999\,999\,999}{1\,000\,000\,000}=\sum_{k=1}^{9}\frac{10^k-1}{10^k}$

11. If we set $i=n+3$, then values of i corresponding to $n=1$ and $n=24$ are $i=4$ and $i=27$. Thus,

$$\sum_{n=1}^{24}\frac{n^2}{2n+1}=\sum_{i=4}^{27}\frac{(i-3)^2}{2(i-3)+1}=\sum_{i=4}^{27}\frac{i^2-6i+9}{2i-5}.$$

12. If we set $m=k-2$, then values of m corresponding to $k=2$ and $k=101$ are $m=0$ and $m=99$. Thus,

$$\sum_{k=2}^{101}\frac{3k-k^2}{\sqrt{k+5}}=\sum_{m=0}^{99}\frac{3(m+2)-(m+2)^2}{\sqrt{(m+2)+5}}=\sum_{m=0}^{99}\frac{2-m-m^2}{\sqrt{m+7}}.$$

13. If we set $j=n-4$, then values of j corresponding to $n=5$ and $n=20$ are $j=1$ and $j=16$. Thus,

$$\sum_{n=5}^{20}(-1)^n\frac{2^n}{n^2+1}=\sum_{j=1}^{16}(-1)^{j+4}\frac{2^{j+4}}{(j+4)^2+1}=\sum_{j=1}^{16}16(-1)^j\frac{2^j}{j^2+8j+17}.$$

14. If we set $m = i + 2$, then values of m corresponding to $i = 0$ and $i = 37$ are $m = 2$ and $m = 39$. Thus,

$$\sum_{i=0}^{37} \frac{3^{3i}}{i!} = \sum_{m=2}^{39} \frac{3^{3(m-2)}}{(m-2)!} = \sum_{m=2}^{39} \frac{3^{3m}}{729(m-2)!}.$$

15. If we set $n = r - 5$, then values of n corresponding to $r = 15$ and $r = 225$ are $n = 10$ and $n = 220$. Thus,

$$\sum_{r=15}^{225} \frac{1}{r^2 - 10r} = \sum_{n=10}^{220} \frac{1}{(n+5)^2 - 10(n+5)} = \sum_{n=10}^{220} \frac{1}{n^2 - 25}.$$

16. $\displaystyle\sum_{n=1}^{12} (3n+2) = 3\sum_{n=1}^{12} n + 2\sum_{n=1}^{12} 1 = 3\left[\frac{(12)(13)}{2}\right] + 2(12) = 258$

17. $\displaystyle\sum_{j=1}^{21} (2j^2 + 3j) = 2\sum_{j=1}^{21} j^2 + 3\sum_{j=1}^{21} j = 2\left[\frac{21(22)(43)}{6}\right] + 3\left[\frac{21(22)}{2}\right] = 7315$

18.
$$\begin{aligned}\sum_{m=1}^{n} (4m-2)^2 &= \sum_{m=1}^{n} (16m^2 - 16m + 4) = 16\sum_{m=1}^{n} m^2 - 16\sum_{m=1}^{n} m + 4\sum_{m=1}^{n} 1\\ &= 16\left[\frac{n(n+1)(2n+1)}{6}\right] - 16\left[\frac{n(n+1)}{2}\right] + 4n = \frac{4n(4n^2-1)}{3}\end{aligned}$$

19.
$$\begin{aligned}\sum_{k=2}^{29} (k^3 - 3k^2) &= \sum_{k=1}^{29} (k^3 - 3k^2) + 2\\ &= \sum_{k=1}^{29} k^3 - 3\sum_{k=1}^{29} k^2 + 2 = \left[\frac{(29)^2(30)^2}{4}\right] - 3\left[\frac{29(30)(59)}{6}\right] + 2 = 163\,562\end{aligned}$$

20.
$$\begin{aligned}\sum_{n=1}^{25} (n+5)(n-4) &= \sum_{n=1}^{25} (n^2 + n - 20) = \sum_{n=1}^{25} n^2 + \sum_{n=1}^{25} n - 20\sum_{n=1}^{25} 1\\ &= \frac{(25)(26)(51)}{6} + \frac{25(26)}{2} - 20(25) = 5350\end{aligned}$$

21.
$$\begin{aligned}\sum_{i=1}^{n} i(i-3)^2 &= \sum_{i=1}^{n} (i^3 - 6i^2 + 9i) = \sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 9\sum_{i=1}^{n} i\\ &= \frac{n^2(n+1)^2}{4} - 6\left[\frac{n(n+1)(2n+1)}{6}\right] + 9\left[\frac{n(n+1)}{2}\right] = \frac{n(n+1)}{4}[n(n+1) - 4(2n+1) + 18]\\ &= \frac{n(n+1)(n^2 - 7n + 14)}{4}\end{aligned}$$

22. $\displaystyle\sum_{n=10}^{24} (n^2 - 5) = \sum_{n=1}^{24} n^2 - \sum_{n=1}^{9} n^2 - 5(15) = \frac{24(25)(49)}{6} - \frac{9(10)(19)}{6} - 75 = 4540$

23.
$$\begin{aligned}\sum_{i=7}^{17} (i^3 - 3i^2) &= \sum_{i=1}^{17} i^3 - \sum_{i=1}^{6} i^3 - 3\sum_{i=1}^{17} i^2 + 3\sum_{i=1}^{6} i^2\\ &= \frac{17^2 \cdot 18^2}{4} - \frac{6^2 \cdot 7^2}{4} - 3\left[\frac{17(18)(35)}{6}\right] + 3\left[\frac{6(7)(13)}{6}\right] = 17\,886\end{aligned}$$

24.
$$\begin{aligned}\sum_{k=5}^{n} (k+3)(k+4) &= \sum_{k=1}^{n} (k^2 + 7k + 12) - \sum_{k=1}^{4} (k^2 + 7k + 12)\\ &= \sum_{k=1}^{n} k^2 + 7\sum_{k=1}^{n} k + 12n - \sum_{k=1}^{4} k^2 - 7\sum_{k=1}^{4} k - 12(4)\\ &= \frac{n(n+1)(2n+1)}{6} + 7\left[\frac{n(n+1)}{2}\right] + 12n - 30 - 70 - 48 = \frac{n^3 + 12n^2 + 47n - 444}{3}\end{aligned}$$

25. $$\sum_{i=n}^{2n}(i^2+2i-3)=\sum_{1=1}^{2n}(i^2+2i-3)-\sum_{i=1}^{n-1}(i^2+2i-3)$$
$$=\frac{2n(2n+1)(4n+1)}{6}+2\left[\frac{2n(2n+1)}{2}\right]-3(2n)$$
$$-\frac{(n-1)(n)(2n-1)}{6}-2\left[\frac{(n-1)n}{2}\right]+3(n-1)$$
$$=\frac{14n^3+33n^2+n-18}{6}$$

26. $$\sum_{i=m}^{n}[f(i)+g(i)]=[f(m)+g(m)]+[f(m+1)+g(m+1)]+\cdots+[f(n)+g(n)]$$
$$=[f(m)+f(m+1)+\cdots+f(n)]+[g(m)+g(m+1)+\cdots+g(n)]=\sum_{i=m}^{n}f(i)+\sum_{i=m}^{n}g(i)$$

$$\sum_{i=m}^{n}cf(i)=[c(f(m)]+[cf(m+1)]+\cdots+[cf(n)]=c[f(m)+f(m+1)+\cdots+f(n)]=c\sum_{i=m}^{n}f(i)$$

27. $$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=1-\frac{1}{n+1}=\frac{n}{n+1}$$

28. $$\sum_{i=1}^{n}[f(i)-f(i-1)]=[f(1)-f(0)]+[f(2)-f(1)]+[f(3)-f(2)]+\cdots$$
$$+[f(n-1)-f(n-2)]+[f(n)-f(n-1)]$$
$$=f(n)-f(0)$$

29. Since $i^4-(i-1)^4=4i^3-6i^2+4i-1$ is valid for any integer i whatsoever,we may add the identity from $i=1$ to $i=n$,

$$\sum_{i=1}^{n}[i^4-(i-1)^4]=\sum_{i=1}^{n}(4i^3-6i^2+4i-1).$$

When we write the left-hand side in full, we find

$$\sum_{i=1}^{n}[i^4-(i-1)^4]=[1^4-0^4]+[2^4-1^4]+[3^4-2^4]+\cdots[n^4-(n-1)^4].$$

Most of these terms cancel one another, leaving only n^4; that is, $\sum_{i=1}^{n}[i^4-(i-1)^4]=n^4$. Thus,

$$n^4=\sum_{i=1}^{n}(4i^3-6i^2+4i-1)=4\sum_{i=1}^{n}i^3-6\sum_{i=1}^{n}i^2+4\sum_{i=1}^{n}i-\sum_{i=1}^{n}1$$
$$=4\sum_{i=1}^{n}i^3-6\left[\frac{n(n+1)(2n+1)}{6}\right]+4\left[\frac{n(n+1)}{2}\right]-n.$$

We can solve this equation for $\sum_{i=1}^{n}i^3$,

$$\sum_{i=1}^{n}i^3=\frac{1}{4}\left[n^4+n(n+1)(2n+1)-2n(n+1)+n\right]=\frac{n^2(n+1)^2}{4}.$$

30. No For example if $n = 2$, then the left side is $\sum_{i=1}^{2}[f(i)g(i)] = f(1)g(1) + f(2)g(2)$, whereas the right side is $\left[\sum_{i=1}^{2} f(i)\right]\left[\sum_{i=1}^{2} g(i)\right] = [f(1) + f(2)][g(1) + g(2)]$.

31. (a) $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{i=1}^{n} ar^{i-1}$

(b) If we multiply S_n by r and subtract it from S_n, we obtain

$$S_n - rS_n = [a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}] - [ar + ar^2 + ar^3 + \cdots + ar^n].$$

Since all but two terms cancel on the right, this simplifies to

$$(1-r)S_n = a - ar^n \quad \Longrightarrow \quad S_n = \frac{a(1-r^n)}{1-r}, \quad \text{provided } r \neq 1.$$

32. $\dfrac{1}{8} + \dfrac{1}{16} + \cdots + \dfrac{1}{1\,048\,576} = \dfrac{(1/8)[1-(1/2)^{18}]}{1-1/2} = \dfrac{1-2^{-18}}{4}$

33. $1 - \dfrac{1}{3} + \dfrac{1}{9} + \cdots - \dfrac{1}{19\,683} = \dfrac{1-(-1/3)^{10}]}{1+1/3} = \dfrac{3(1-3^{-10})}{4}$

34. $40(0.99) + 40(0.99)^2 + \cdots + 40(0.99)^{15} = \dfrac{40(0.99)[1-(0.99)^{15}]}{1-0.99} = 3960[1-(0.99)^{15}] = 554.2$

35. $\sqrt{0.99} + 0.99 + (0.99)^{3/2} + \cdots + (0.99)^{10} = \dfrac{\sqrt{0.99}[1-(\sqrt{0.99})^{20}]}{1-\sqrt{0.99}} = 18.98$

36. $\left|\sum_{i=1}^{n} f(i)\right| = |f(1) + f(2) + \cdots + f(n)| \leq |f(1)| + |f(2)| + \cdots + |f(n)| = \sum_{i=1}^{n} |f(i)|$

37. Without the signs, the terms can be represented by $1/2^k$ from $k = 0$ to $k = 12$. To obtain the correct signs, we use $\sqrt{2}\sin[(2k+1)\pi/4]$. In sigma notation then, the summation is represented by $\sum_{k=0}^{12} \dfrac{\sqrt{2}\sin[(2k+1)\pi/4]}{2^k}$.

EXERCISES 6.3

1. Since $f(x) = x$ is continuous for $0 \leq x \leq 1$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $1/n$, we use the points $x_i = i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = i/n$. Equation 6.10 now gives

$$\int_0^1 x\,dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} x_i^* \Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 1/n$, the norm of the partition is $\|\Delta x_i\| = 1/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\int_0^1 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right] = \frac{1}{2}.$$

2. Since $f(x) = 3x$ is continuous for $0 \leq x \leq 2$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use $x_i = 2i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = 2i/n$. Equation 6.10 now gives

$$\int_0^2 3x\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} 3x_i^*\Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 2/n$, the norm of the partition is $\|\Delta x_i\| = 2/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\int_0^2 3x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} 3\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{12}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{12}{n^2}\left[\frac{n(n+1)}{2}\right] = 6.$$

3. Since $f(x) = 3x+2$ is continuous for $0 \le x \le 1$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $1/n$, we use the points $x_i = i/n$, $i = 0,\dots,n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = i/n$. Equation 6.10 now gives

$$\int_0^1 (3x+2)\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} (3x_i^*+2)\Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 1/n$, the norm of the partition is $\|\Delta x_i\| = 1/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\begin{aligned}\int_0^1 (3x+1)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[3\left(\frac{i}{n}\right)+2\right]\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}(3i+2n)\\ &= \lim_{n\to\infty}\frac{1}{n^2}\left(3\sum_{i=1}^{n} i + 2n^2\right) = \lim_{n\to\infty}\frac{1}{n^2}\left[\frac{3n(n+1)}{2}+2n^2\right] = \frac{7}{2}.\end{aligned}$$

4. Since $f(x) = x^3$ is continuous for $0 \le x \le 2$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use $x_i = 2i/n$, $i = 0,\dots,n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = 2i/n$. Equation 6.10 gives

$$\int_0^2 x^3\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} (x_i^*)^3\Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 2/n$, the norm of the partition is $\|\Delta x_i\| = 2/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\int_0^2 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{16}{n^4}\sum_{i=1}^{n} i^3 = \lim_{n\to\infty}\frac{16}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = 4.$$

5. Since $f(x) = x^2+2x$ is continuous for $1 \le x \le 2$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $1/n$, we use the points $x_i = 1+i/n$, $i = 0,\dots,n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = 1+i/n$. Equation 6.10 now gives

$$\int_1^2 (x^2+2x)\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n} [(x_i^*)^2+2x_i^*]\Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 1/n$, the norm of the partition is $\|\Delta x_i\| = 1/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\begin{aligned}\int_1^2 (x^2+2x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{i}{n}\right)^2+2\left(1+\frac{i}{n}\right)\right]\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}(3n^2+4ni+i^2)\\ &= \lim_{n\to\infty}\frac{1}{n^3}\left(3n^3+4n\sum_{i=1}^{n} i+\sum_{i=1}^{n} i^2\right) = \lim_{n\to\infty}\frac{1}{n^3}\left[3n^3+\frac{4n^2(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}\right]\\ &= 3+2+\frac{1}{3} = \frac{16}{3}.\end{aligned}$$

6. Since $f(x) = 1 - x$ is continuous for $-1 \le x \le 0$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $1/n$, we use the points $x_i = -1 + i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = -1 + i/n$. Equation 6.10 now gives

$$\int_{-1}^{0} (-x+1)\,dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} (-x_i^* + 1)\Delta x_i = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} \left(1 - \frac{i}{n} + 1\right)\left(\frac{1}{n}\right).$$

Since all subintervals have equal length $\Delta x_i = 1/n$, the norm of the partition is $\|\Delta x_i\| = 1/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\int_{-1}^{0} (-x+1)\,dx = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} (2n - i) = \lim_{n\to\infty} \frac{1}{n^2}\left[2n^2 - \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \frac{3n-1}{2n} = \frac{3}{2}.$$

7. Since $f(x) = x^2$ is continuous for $-1 \le x \le 1$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use the points $x_i = -1 + 2i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = -1 + 2i/n$. Equation 6.10 now gives

$$\int_{-1}^{1} x^2\,dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} (x_i^*)^2 \Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 2/n$, the norm of the partition is $\|\Delta x_i\| = 2/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\begin{aligned}\int_{-1}^{1} x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-1 + \frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n^3} \sum_{i=1}^{n} (n^2 - 4ni + 4i^2)\\ &= \lim_{n\to\infty} \frac{2}{n^3}\left[n^3 - \frac{4n^2(n+1)}{2} + \frac{4n(n+1)(2n+1)}{6}\right] = 2\left(1 - 2 + \frac{4}{3}\right) = \frac{2}{3}.\end{aligned}$$

8. Since $f(x) = x^3$ is continuous for $-1 \le x \le 1$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use the points $x_i = -1 + 2i/n$, $i = 0, \ldots, n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = -1 + 2i/n$. Equation 6.10 now gives

$$\int_{-1}^{1} x^3\,dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} (x_i^*)^3 \Delta x_i.$$

Since all subintervals have equal length $\Delta x_i = 2/n$, the norm of the partition is $\|\Delta x_i\| = 2/n$, and taking the limit as $\|\Delta x_i\| \to 0$ is tantamount to letting $n \to \infty$. Thus,

$$\begin{aligned}\int_{-1}^{1} x^3\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n^4} \sum_{i=1}^{n} (-n^3 + 6n^2 i - 12ni^2 + 8i^3)\\ &= \lim_{n\to\infty} \frac{2}{n^4}\left[-n^4 + \frac{6n^3(n+1)}{2} - \frac{12n^2(n+1)(2n+1)}{6} + \frac{8n^2(n+1)^2}{4}\right]\\ &= 2(-1 + 3 - 4 + 2) = 0.\end{aligned}$$

9. The integrand x^{15} is an odd function. If we set up a partition with points symmetrically placed about $x = 0$, then terms in summation 6.10 cancel, and the value of the integral is zero.

10. (a) For n equal subdivisions of the interval $0 \le x \le 1$, we use the points $x_i = i/n$ where $i = 0, \cdots, n$. With star points chosen as $x_i^* = x_i = i/n$, equation 6.10 gives

$$\int_0^1 2^x dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^n 2^{x_i^*} \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n 2^{i/n} \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n 2^{i/n}.$$

(b) The formula in question 31(b) of Exercises 6.1 allows us to evaluate the sum in closed form,

$$\int_0^1 2^x dx = \lim_{n\to\infty} \left\{ \frac{1}{n} \frac{2^{1/n}\left[1-(2^{1/n})^n\right]}{1-2^{1/n}} \right\} = \lim_{n\to\infty} \frac{2^{1/n}}{n(2^{1/n}-1)}.$$

(c) The limit of the numerator is 1. The denominator is of the indeterminate form $0\cdot\infty$, and we therefore use L'Hôpital's rule to evaluate

$$\lim_{n\to\infty} [n(2^{1/n}-1)] = \lim_{n\to\infty} \frac{2^{1/n}-1}{1/n} = \lim_{n\to\infty} \frac{2^{1/n}(-1/n^2)\ln 2}{-1/n^2} = \lim_{n\to\infty} (2^{1/n} \ln 2) = \ln 2.$$

Hence, $\displaystyle\int_0^1 2^x dx = \frac{1}{\ln 2} = \log_2 e.$

11. For n equal subdivisions of the interval $1 \le x \le 3$, we use the points $x_i = 1 + 2i/n$ where $i = 0, \cdots, n$. With star points chosen as $x_i^* = x_i = 1 + 2i/n$, equation 6.10 gives

$$\int_1^3 e^x\, dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^n e^{x_i^*} \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n e^{1+2i/n} \left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2e}{n} \sum_{i=1}^n e^{2i/n}.$$

The formula in Exercise 31(b) of Exercises 6.1 allows us to evaluate the sum in closed form,

$$\int_1^3 e^x\, dx = \lim_{n\to\infty} \left\{ \frac{2e}{n} \frac{e^{2/n}\left[1-(e^{2/n})^n\right]}{1-e^{2/n}} \right\} = 2(e-e^3) \lim_{n\to\infty} \frac{e^{2/n}}{n(1-e^{2/n})}.$$

The limit of the numerator is 1. The denominator is of the indeterminate form $0\cdot\infty$, and we therefore use L'Hôpital's rule to evaluate

$$\lim_{n\to\infty} [n(1-e^{2/n})] = \lim_{n\to\infty} \frac{1-e^{2/n}}{1/n} = \lim_{n\to\infty} \frac{-e^{2/n}(-2/n^2)}{-1/n^2} = \lim_{n\to\infty} (-2\,e^{2/n}) = -2.$$

Hence, $\displaystyle\int_1^3 e^x\, dx = \frac{2(e-e^3)}{-2} = e^3 - e.$

12. For n equal subdivisions of the interval $0 \le x \le \pi$, we use the points $x_i = i\pi/n$ where $i = 0, \cdots, n$. With star points chosen as $x_i^* = x_i$, equation 6.10 gives

$$\begin{aligned}
\int_0^\pi \sin x\, dx &= \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^n \sin x_i^* \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n \sin\left(\frac{i\pi}{n}\right)\left(\frac{\pi}{n}\right) = \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^n \sin\left(\frac{i\pi}{n}\right) \\
&= \lim_{n\to\infty} \frac{\pi}{n} \left[\frac{\sin\dfrac{(n+1)\pi}{2n} \sin\dfrac{n}{2}\left(\dfrac{\pi}{n}\right)}{\sin\dfrac{\pi}{2n}} \right] \qquad \text{(by given formula)} \\
&= \left[\lim_{n\to\infty} \frac{\dfrac{\pi}{n}}{\sin\dfrac{\pi}{2n}} \right] \left[\lim_{n\to\infty} \sin\frac{(n+1)\pi}{2n} \right] \qquad \text{(provided both limits exist)} \\
&= \left[\lim_{n\to\infty} \frac{-\pi/n^2}{\dfrac{-\pi}{2n^2}\cos\dfrac{\pi}{2n}} \right] \left[\sin\frac{\pi}{2} \right] \qquad \text{(using L'Hôpital's rule)} \\
&= 2.
\end{aligned}$$

13. For n equal subdivisions of the interval $0 \le x \le \pi/2$, we use the points $x_i = i\pi/(2n)$ where $i = 0, \cdots, n$. With star-points chosen as $x_i^* = x_i$, equation 6.10 gives

$$\int_0^{\pi/2} \cos x\, dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^n \cos x_i^* \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^n \cos\left(\frac{i\pi}{2n}\right)\left(\frac{\pi}{2n}\right) = \lim_{n\to\infty} \frac{\pi}{2n} \sum_{i=1}^n \cos\left(\frac{i\pi}{2n}\right)$$

$$= \lim_{n\to\infty} \frac{\pi}{2n}\left[\frac{\cos\dfrac{(n+1)\pi}{4n}\sin\dfrac{\pi}{4}}{\sin\dfrac{\pi}{4n}}\right] \qquad \text{(by given formula)}$$

$$= \frac{1}{\sqrt{2}}\left[\lim_{n\to\infty} \frac{\dfrac{\pi}{2n}}{\sin\dfrac{\pi}{4n}}\right]\left[\lim_{n\to\infty} \cos\frac{(n+1)\pi}{4n}\right] \qquad \text{(provided both limits exist)}$$

$$= \frac{1}{\sqrt{2}}\left[\lim_{n\to\infty} \frac{-\pi/(2n^2)}{\dfrac{-\pi}{4n^2}\cos\dfrac{\pi}{4n}}\right]\left[\cos\frac{\pi}{4}\right] \qquad \text{(using L'Hôpital's rule)}$$

$$= \frac{1}{\sqrt{2}}(2)\left(\frac{1}{\sqrt{2}}\right) = 1.$$

14. (a) With the suggested choice for x_i and x_i^*,

$$\int_a^b x^k\, dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^n (x_i^*)^k \Delta x_i = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^n (ah^i)^k (ah^i - ah^{i-1})$$

$$= \lim_{\|\Delta x_i\|\to 0} a^{k+1} \sum_{i=1}^n h^{ik}(h^i - h^{i-1}) = a^{k+1} \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^n h^{ik} h^{i-1}(h-1)$$

$$= a^{k+1} \lim_{\|\Delta x_i\|\to 0} \left(\frac{h-1}{h}\right) \sum_{i=1}^n (h^{k+1})^i.$$

Now, $\Delta x_i = ah^i - ah^{i-1} = ah^{i-1}(h-1)$, where $h = (b/a)^{1/n}$. Since $h > 1$, this is largest when i is largest; that is, $\|\Delta x_i\| = \Delta x_n = ah^{n-1}(h-1)$. This can be made to approach 0 by letting $n \to \infty$. Thus,

$$\int_a^b x^k\, dx = a^{k+1} \lim_{n\to\infty} \left(\frac{h-1}{h}\right) \sum_{i=1}^n (h^{k+1})^i.$$

(b) Since the sum on the right is a finite geometric series with $a = r = h^{k+1}$,

$$\int_a^b x^k\, dx = a^{k+1} \lim_{n\to\infty} \left(\frac{h-1}{h}\right)\left\{\frac{h^{k+1}[1-(h^{k+1})^n]}{1-h^{k+1}}\right\} = a^{k+1} \lim_{n\to\infty} (h-1)\left\{\frac{h^k[1-h^{n(k+1)}]}{1-h^{k+1}}\right\}.$$

We now set $h = (b/a)^{1/n}$,

$$\int_a^b x^k\, dx = a^{k+1} \lim_{n\to\infty} \left[\left(\frac{b}{a}\right)^{1/n} - 1\right]\left\{\frac{\left(\frac{b}{a}\right)^{k/n}\left[1-\left(\frac{b}{a}\right)^{k+1}\right]}{1-\left(\frac{b}{a}\right)^{(k+1)/n}}\right\}$$

$$= a^{k+1}\left(\frac{a^{k+1}-b^{k+1}}{a^{k+1}}\right) \lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{k/n}\left[\left(\frac{b}{a}\right)^{1/n}-1\right]}{1-\left(\frac{b}{a}\right)^{(k+1)/n}}$$

$$= (b^{k+1} - a^{k+1}) \lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{k/n}\left[\left(\frac{b}{a}\right)^{1/n}-1\right]}{\left(\frac{b}{a}\right)^{(k+1)/n}-1}.$$

(c) As $n \to \infty$, the factor $(b/a)^{k/n} \to 1$. The remainder of the limit is of the indeterminate form 0/0 and we therefore use L'Hôpital's rule to calculate that

$$\lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{1/n} - 1}{\left(\frac{b}{a}\right)^{(k+1)/n} - 1} = \lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{1/n}(-1/n^2)\ln(b/a)}{\left(\frac{b}{a}\right)^{(k+1)/n}[-(k+1)/n^2]\ln(b/a)}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{b}{a}\right)^{1/n}}{\left(\frac{b}{a}\right)^{(k+1)/n}(k+1)} = \frac{1}{k+1}.$$

Finally then, $\displaystyle\int_a^b x^k\,dx = (b^{k+1} - a^{k+1})\left(\frac{1}{k+1}\right)$.

15. Let the interval $a \le x \le b$ be subdivided into n subintervals in any manner whatsoever. If we choose star-points in each subinterval as rational numbers, then the limit of the summation in equation 6.10 will be $b - a$. On the other hand, if star-points are chosen as irrational numbers, the limit of the summation will be zero. Since the limit depends on the choice of star-points, the definite integral does not exist.

EXERCISES 6.4

1. $\displaystyle\int_3^4 (x^3+3)\,dx = \left\{\frac{x^4}{4} + 3x\right\}_3^4 = (64+12) - \left(\frac{81}{4} + 9\right) = \frac{187}{4}$

2. $\displaystyle\int_1^3 (x^2 - 2x + 3)\,dx = \left\{\frac{x^3}{3} - x^2 + 3x\right\}_1^3 = (9 - 9 + 9) - \left(\frac{1}{3} - 1 + 3\right) = \frac{20}{3}$

3. $\displaystyle\int_{-1}^1 (4x^3 + 2x)\,dx = \left\{x^4 + x^2\right\}_{-1}^1 = (1+1) - (1+1) = 0$

4. $\displaystyle\int_{-3}^{-1} \frac{1}{x^2}dx = \left\{-\frac{1}{x}\right\}_{-3}^{-1} = (1) - \left(\frac{1}{3}\right) = \frac{2}{3}$

5. $\displaystyle\int_4^2 \left(x^2 + \frac{3}{x^3}\right)dx = \left\{\frac{x^3}{3} - \frac{3}{2x^2}\right\}_4^2 = \left(\frac{8}{3} - \frac{3}{8}\right) - \left(\frac{64}{3} - \frac{3}{32}\right) = -\frac{1819}{96}$

6. $\displaystyle\int_0^{\pi/2} \sin x\,dx = \{-\cos x\}_0^{\pi/2} = (0) - (-1) = 1$

7. $\displaystyle\int_{-1}^1 (x^2 - 1 - x^4)\,dx = \left\{\frac{x^3}{3} - x - \frac{x^5}{5}\right\}_{-1}^1 = \left(\frac{1}{3} - 1 - \frac{1}{5}\right) - \left(-\frac{1}{3} + 1 + \frac{1}{5}\right) = -\frac{26}{15}$

8. $\displaystyle\int_{-1}^{-2} \left(\frac{1}{x^2} - 2x\right)dx = \left\{-\frac{1}{x} - x^2\right\}_{-1}^{-2} = \left(\frac{1}{2} - 4\right) - (1 - 1) = -\frac{7}{2}$

9. $\displaystyle\int_1^2 (x^4 + 3x^2 + 2)\,dx = \left\{\frac{x^5}{5} + x^3 + 2x\right\}_1^2 = \left(\frac{32}{5} + 8 + 4\right) - \left(\frac{1}{5} + 1 + 2\right) = \frac{76}{5}$

10. $\displaystyle\int_0^1 x(x^2+1)\,dx = \int_0^1 (x^3 + x)\,dx = \left\{\frac{x^4}{4} + \frac{x^2}{2}\right\}_0^1 = \left(\frac{1}{4} + \frac{1}{2}\right) - 0 = \frac{3}{4}$

11. $\displaystyle\int_0^1 x^2(x^2+1)^2\,dx = \int_0^1 (x^6 + 2x^4 + x^2)\,dx = \left\{\frac{x^7}{7} + \frac{2x^5}{5} + \frac{x^3}{3}\right\}_0^1 = \left(\frac{1}{7} + \frac{2}{5} + \frac{1}{3}\right) - (0) = \frac{92}{105}$

12. $\displaystyle\int_0^{2\pi} \cos 2x\,dx = \left\{\frac{1}{2}\sin 2x\right\}_0^{2\pi} = (0) - (0) = 0$

13. $\displaystyle\int_1^3 \frac{x^2+3}{x^2}dx = \int_1^3 \left(1 + \frac{3}{x^2}\right)dx = \left\{x - \frac{3}{x}\right\}_1^3 = (3-1) - (1-3) = 4$

14. $\displaystyle\int_0^1 (x^{2.2} - x^\pi)\,dx = \left\{\frac{x^{3.2}}{3.2} - \frac{x^{\pi+1}}{\pi+1}\right\}_0^1 = \left(\frac{1}{3.2} - \frac{1}{\pi+1}\right) - 0 = \frac{5}{16} - \frac{1}{\pi+1}$

15. $\int_{-1}^{1} x^2(x^3 - x)\,dx = \int_{-1}^{1} (x^5 - x^3)\,dx = \left\{\frac{x^6}{6} - \frac{x^4}{4}\right\}_{-1}^{1} = \left(\frac{1}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{4}\right) = 0$

16. $\int_{3}^{4} \frac{(x^2-1)^2}{x^2}dx = \int_{3}^{4}\left(x^2 - 2 + \frac{1}{x^2}\right)dx = \left\{\frac{x^3}{3} - 2x - \frac{1}{x}\right\}_{3}^{4} = \left(\frac{64}{3} - 8 - \frac{1}{4}\right) - \left(9 - 6 - \frac{1}{3}\right) = \frac{125}{12}$

17. $\int_{1}^{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = \left\{\frac{2x^{3/2}}{3} - 2\sqrt{x}\right\}_{1}^{2} = \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right) = \frac{2(2-\sqrt{2})}{3}$

18. $\int_{-2}^{3}(x-1)^3\,dx = \left\{\frac{1}{4}(x-1)^4\right\}_{-2}^{3} = \frac{1}{4}(16) - \frac{1}{4}(81) = -\frac{65}{4}$

19. $\int_{2}^{4}\frac{(x^2-1)(x^2+1)}{x^2}dx = \int_{2}^{4}\left(x^2 - \frac{1}{x^2}\right)dx = \left\{\frac{x^3}{3} + \frac{1}{x}\right\}_{2}^{4} = \left(\frac{64}{3} + \frac{1}{4}\right) - \left(\frac{8}{3} + \frac{1}{2}\right) = \frac{221}{12}$

20. $\int_{0}^{\pi/4} 3\cos x\,dx = \{3\sin x\}_{0}^{\pi/4} = \frac{3}{\sqrt{2}} - 0 = \frac{3}{\sqrt{2}}$ **21.** $\int_{0}^{\pi/4}\sec^2 x\,dx = \left\{\tan x\right\}_{0}^{\pi/4} = 1$

22. $\int_{\pi/2}^{\pi}\sin x\,\cos x\,dx = \left\{\frac{1}{2}\sin^2 x\right\}_{\pi/2}^{\pi} = \frac{1}{2}(0) - \frac{1}{2}(1) = -\frac{1}{2}$

23. This integral does not exist since the integrand is undefined at $x = \pi/3$.

24. $\int_{-\pi/4}^{\pi/4}\sec x\,\tan x\,dx = \left\{\sec x\right\}_{-\pi/4}^{\pi/4} = (\sqrt{2}) - (\sqrt{2}) = 0$

25. $\int_{0}^{2} 2^x\,dx = \left\{\frac{2^x}{\ln 2}\right\}_{0}^{2} = \frac{4}{\ln 2} - \frac{1}{\ln 2} = \frac{3}{\ln 2}$ **26.** $\int_{-1}^{2} e^x\,dx = \left\{e^x\right\}_{-1}^{2} = e^2 - e^{-1}$

27. $\int_{0}^{1} e^{3x}\,dx = \left\{\frac{e^{3x}}{3}\right\}_{0}^{1} = \frac{e^3}{3} - \frac{1}{3} = \frac{e^3 - 1}{3}$ **28.** $\int_{-3}^{-2}\frac{1}{x}dx = \{\ln|x|\}_{-3}^{-2} = (\ln 2) - (\ln 3) = \ln(2/3)$

29. $\int_{1}^{3}\frac{(x+1)^2}{x}dx = \int_{1}^{3}\left(x + 2 + \frac{1}{x}\right)dx = \left\{\frac{x^2}{2} + 2x + \ln|x|\right\}_{1}^{3} = \left(\frac{9}{2} + 6 + \ln 3\right) - \left(\frac{1}{2} + 2\right) = 8 + \ln 3$

30. $\int_{0}^{1} 3^{4x}\,dx = \left\{\frac{1}{4}3^{4x}\log_3 e\right\}_{0}^{1} = \frac{1}{4}\log_3 e(3^4 - 3^0) = 20\log_3 e$

31. $\int_{0}^{5}|x|\,dx = \int_{0}^{5} x\,dx = \left\{\frac{x^2}{2}\right\}_{0}^{5} = \frac{25}{2}$

32. Since $|x+1|$ is positive between $x = 0$ and $x = 4$,

$$\int_{0}^{4} x|x+1|\,dx = \int_{0}^{4} x(x+1)\,dx = \int_{0}^{4}(x^2 + x)\,dx = \left\{\frac{x^3}{3} + \frac{x^2}{2}\right\}_{0}^{4} = \left(\frac{64}{3} + 8\right) - 0 = \frac{88}{3}.$$

33. $\int_{-5}^{5}|x|\,dx = \int_{-5}^{0} -x\,dx + \int_{0}^{5} x\,dx = \left\{-\frac{x^2}{2}\right\}_{-5}^{0} + \left\{\frac{x^2}{2}\right\}_{0}^{5} = \frac{25}{2} + \frac{25}{2} = 25$

34. Because $x + 1$ changes sign at $x = -1$, we divide the integral into two parts,

$$\begin{aligned}\int_{-2}^{1} x|x+1|\,dx &= \int_{-2}^{-1} x(-x-1)\,dx + \int_{-1}^{1} x(x+1)\,dx = \left\{-\frac{x^3}{3} - \frac{x^2}{2}\right\}_{-2}^{-1} + \left\{\frac{x^3}{3} + \frac{x^2}{2}\right\}_{-1}^{1}\\ &= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{8}{3} - 2\right) + \left(\frac{1}{3} + \frac{1}{2}\right) - \left(-\frac{1}{3} + \frac{1}{2}\right) = -\frac{1}{6}.\end{aligned}$$

35. $\int_{-1/2}^{1/2}\frac{1}{\sqrt{1-x^2}}dx = \{\mathrm{Sin}^{-1}x\}_{-1/2}^{1/2} = \mathrm{Sin}^{-1}(1/2) - \mathrm{Sin}^{-1}(-1/2) = \frac{\pi}{3}$

36. $\int_{-}^{1}\frac{1}{1+x^2}dx = \{\mathrm{Tan}^{-1}x\}_{-1}^{1} = \mathrm{Tan}^{-1}1 - \mathrm{Tan}^{-1}(-1) = \frac{\pi}{2}$

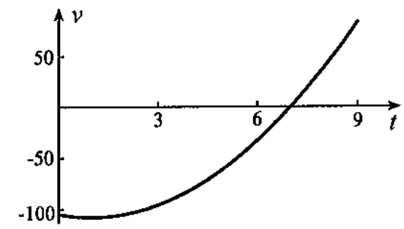
37. $\displaystyle\int_2^3 \frac{1}{x\sqrt{x^2-1}}dx = \left\{\mathrm{Sec}^{-1}x\right\}_2^3 = \mathrm{Sec}^{-1}3 - \mathrm{Sec}^{-1}(2) = \mathrm{Sec}^{-1}3 - \frac{\pi}{3}$

38. $\displaystyle\int_0^1 \cosh 2x\,dx = \left\{\frac{1}{2}\sinh 2x\right\}_0^1 = \frac{1}{2}\sinh 2$

39. This integral does not exist since $\mathrm{csch}x$ is not defined at $x = 0$.

40. $\displaystyle\int_0^{1/2} \frac{1}{1+4x^2}dx = \left\{\frac{1}{2}\mathrm{Tan}^{-1}2x\right\}_0^{1/2} = \frac{1}{2}\mathrm{Tan}^{-1}(1) = \frac{\pi}{8}$

41.
$$\begin{aligned}\int_0^9 v(t)\,dt &= \int_0^9 (3t^2 - 6t - 105)\,dt \\ &= \left\{t^3 - 3t^2 - 105t\right\}_0^9 \\ &= 729 - 243 - 945 = -459\end{aligned}$$

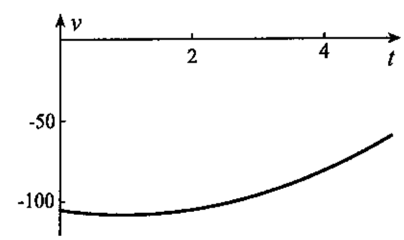


This is the displacement of the particle at $t = 9$ relative to its position at $t = 0$. Since $v(t) = 3(t-7)(t+5)$ changes sign at $t = 7$,

$$\begin{aligned}\int_0^9 |v(t)|\,dt &= \int_0^7 (-3t^2 + 6t + 105)\,dt + \int_7^9 (3t^2 - 6t - 105)\,dt \\ &= \left\{-t^3 + 3t^2 + 105t\right\}_0^7 + \left\{t^3 - 3t^2 - 105t\right\}_7^9 \\ &= (-343 + 147 + 735) + (729 - 243 - 945) - (343 - 147 - 735) = 619.\end{aligned}$$

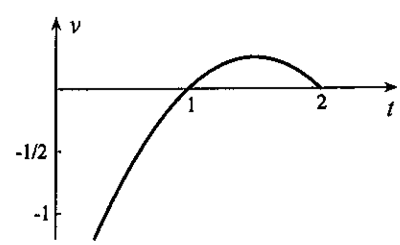
This is the distance travelled by the particle between $t = 0$ and $t = 9$.

42.
$$\begin{aligned}\int_0^5 v(t)\,dt &= \int_0^5 (3t^2 - 6t - 105)\,dt = \left\{t^3 - 3t^2 - 105t\right\}_0^5 \\ &= 125 - 75 - 525 = -475\end{aligned}$$



This is the displacement of the particle at $t = 5$ relative to its position at $t = 0$. Since $v(t) = 3(t-7)(t+5)$ is negative for $0 \le t \le 5$, the integral of $|v(t)|$ is 475. This is the distance travelled by the particle between $t = 0$ and $t = 5$.

43.
$$\begin{aligned}\int_0^2 v(t)\,dt &= \int_0^2 (-t^2 + 3t - 2)\,dt \\ &= \left\{-\frac{t^3}{3} + \frac{3t^2}{2} - 2t\right\}_0^2 \\ &= -\frac{8}{3} + 6 - 4 = -\frac{2}{3}\end{aligned}$$

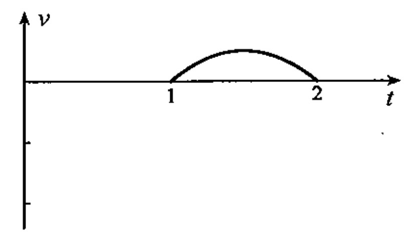


This is the displacement of the particle at $t = 2$ relative to its position at $t = 0$. Since $v(t) = -(t-1)(t-2)$ changes sign at $t = 1$,

$$\begin{aligned}\int_0^2 |v(t)|\,dt &= \int_0^1 (t^2 - 3t + 2)\,dt + \int_1^2 (-t^2 + 3t - 2)\,dt = \left\{\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right\}_0^1 + \left\{-\frac{t^3}{3} + \frac{3t^2}{2} - 2t\right\}_1^2 \\ &= \left(\frac{1}{3} - \frac{3}{2} + 2\right) + \left(-\frac{8}{3} + 6 - 4\right) - \left(-\frac{1}{3} + \frac{3}{2} - 2\right) = 1.\end{aligned}$$

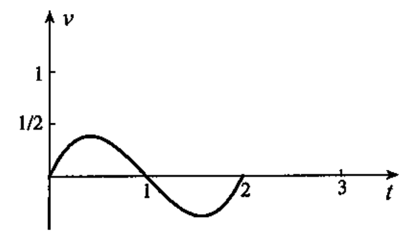
This is the distance travelled by the particle between $t = 0$ and $t = 2$.

44. $$\int_1^2 v(t)\,dt = \int_1^2 (-t^2+3t-2)\,dt = \left\{-\frac{t^3}{3}+\frac{3t^2}{2}-2t\right\}_1^2 = \left(-\frac{8}{3}+6-4\right)-\left(-\frac{1}{3}+\frac{3}{2}-2\right)=\frac{1}{6}$$



This is the displacement of the particle at $t=2$ relative to its position at $t=1$. Since $v(t)=-(t-1)(t-2)$ is positive between $t=1$ and $t=2$, the integral of $|v(t)|$ is also $1/6$. It is the distance travelled by the particle between $t=1$ and $t=2$.

45. $$\int_0^2 v(t)\,dt = \int_0^2 (t^3-3t^2+2t)\,dt = \left\{\frac{t^4}{4}-t^3+t^2\right\}_0^2 = 4-8+4=0$$

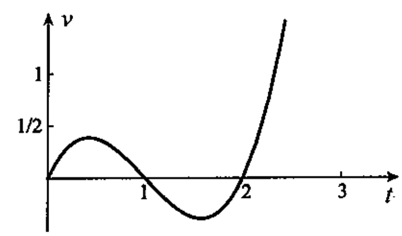


This means that the position of the particle is the same at $t=0$ and $t=2$. Since $v(t)=t(t-1)(t-2)$ changes sign at $t=1$,

$$\int_0^2 |v(t)|\,dt = \int_0^1 (t^3-3t^2+2t)\,dt + \int_1^2 (-t^3+3t^2-2t)\,dt = \left\{\frac{t^4}{4}-t^3+t^2\right\}_0^1 + \left\{-\frac{t^4}{4}+t^3-t^2\right\}_1^2$$
$$= \left(\frac{1}{4}-1+1\right)+(-4+8-4)-\left(-\frac{1}{4}+1-1\right)=\frac{1}{2}.$$

This the distance travelled by the particle between $t=0$ and $t=2$.

46. $$\int_0^3 v(t)\,dt = \int_0^3 (t^3-3t^2+2t)\,dt = \left\{\frac{t^4}{4}-t^3+t^2\right\}_0^3 = \frac{81}{4}-27+9=\frac{9}{4}$$



This is the displacement of the particle at $t=3$ relative to its position at $t=0$. Since $v(t)=t(t-1)(t-2)$ changes sign at $t=1$ and $t=2$,

$$\int_0^3 |v(t)|\,dt = \int_0^1 (t^3-3t^2+2t)\,dt + \int_1^2 (-t^3+3t^2-2t)\,dt + \int_2^3 (t^3-3t^2+2t)\,dt$$
$$= \left\{\frac{t^4}{4}-t^3+t^2\right\}_0^1 + \left\{-\frac{t^4}{4}+t^3-t^2\right\}_1^2 + \left\{\frac{t^4}{4}-t^3+t^2\right\}_2^3$$
$$= \left(\frac{1}{4}-1+1\right)+(-4+8-4)-\left(-\frac{1}{4}+1-1\right)+\left(\frac{81}{4}-27+9\right)-(4-8+4)=\frac{11}{4}.$$

This is the distance travelled by the particle between $t=0$ and $t=3$.

47. The integrand is clearly nonnegative for $0\le x\le \pi/4$. For these values of x, the largest value of $\sin x$ is $1/\sqrt{2}$ and the smallest value of $1+x^2$ is 1. It follows that $\sin x/(1+x^2)$ cannot be larger than $1/\sqrt{2}$. By inequality 6.15 then

$$0\left(\frac{\pi}{4}\right)\le \int_0^{\pi/4}\frac{\sin x}{1+x^2}\,dx \le \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) \quad\Longrightarrow\quad 0\le \int_0^{\pi/4}\frac{\sin x}{1+x^2}dx\le \frac{\sqrt{2}\pi}{8}.$$

48. The integrand is clearly nonnegative for $0 \le x \le \pi/2$. For these values of x, the largest value of $\sin x$ is 1 and the smallest value of $1+x$ is 1. It follows that $\sin x/(1+x)$ cannot be larger than 1. By inequality 6.15 then

$$0\left(\frac{\pi}{2}\right) \le \int_0^{\pi/2} \frac{\sin x}{1+x}\,dx \le 1\left(\frac{\pi}{2}\right) \quad \Longrightarrow \quad 0 \le \int_0^{\pi/2} \frac{\sin x}{1+x}dx \le \frac{\pi}{2}.$$

49. The integrand cannot be negative for $0 \le x \le \pi$, and for these values of x, the largest value of $\sin x$ is 1 and the smallest value of $2+x^2$ is 2. It follows that the integrand cannot be larger than 1/2, and

$$0(\pi) \le \int_0^{\pi} \frac{\sin x}{2+x^2}dx \le \frac{1}{2}\pi \quad \Longrightarrow \quad 0 \le \int_0^{\pi} \frac{\sin x}{2+x^2}dx \le \frac{\pi}{2}.$$

50. The integrand is nonnegative for $\pi/4 \le x \le \pi/2$. For these values of x, the largest value of $\sin 2x$ is 1 and the smallest value of $10+x^2$ is $10+\pi^2/16$. It follows that $\sin 2x/(10+x^2)$ cannot be larger than $1/(10+\pi^2/16)$. By inequality 6.15 then

$$0\left(\frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/2} \frac{\sin 2x}{10+x^2}\,dx \le \frac{1}{10+\pi^2/16}\left(\frac{\pi}{4}\right) \quad \Longrightarrow \quad 0 \le \int_{\pi/4}^{\pi/2} \frac{\sin 2x}{10+x^2}dx \le \frac{4\pi}{160+\pi^2}.$$

51. Because $1 \le 1+4x^4 \le 5$ and $\cos 1 \le \cos x^2 \le 1$ for $0 \le x \le 1$, it follows that $\cos 1 \le (1+4x^4)\cos x^2 \le 5$, and

$$(\cos 1)1 \le \int_0^1 (1+4x^4)\cos x^2\,dx \le 5(1) \quad \Longrightarrow \quad \cos 1 \le \int_0^1 (1+4x^4)\cos x^2\,dx \le 5.$$

52. Because $\sqrt{5} \le \sqrt{4+x^3} \le \sqrt{31}$ for $1 \le x \le 3$, it follows that $\quad 2\sqrt{5} \le \displaystyle\int_1^3 \sqrt{4+x^3}\,dx \le 2\sqrt{31}.$

EXERCISES 6.5

1. By equation 6.19, $\displaystyle\frac{d}{dx}\int_0^x (3t^2+t)\,dt = 3x^2+x.$

2. By equation 6.19, $\displaystyle\frac{d}{dx}\int_1^x \frac{1}{\sqrt{t^2+1}}dt = \frac{1}{\sqrt{x^2+1}}.$

3. By reversing limits, $\displaystyle\frac{d}{dx}\int_x^2 \sin(t^2)\,dt = -\frac{d}{dx}\int_2^x \sin(t^2)\,dt = -\sin(x^2).$

4. By reversing limits, $\displaystyle\frac{d}{dx}\int_x^{-1} t^3\cos t\,dt = -\frac{d}{dx}\int_{-1}^x t^3\cos t\,dt = -x^3\cos x.$

5. We set $u = 3x$ and use the chain rule,

$$\frac{d}{dx}\int_0^{3x} (2t-t^4)^2\,dt = \left[\frac{d}{du}\int_0^u (2t-t^4)^2\,dt\right]\frac{du}{dx} = (2u-u^4)^2(3) = 3(6x-81x^4)^2 = 27x^2(2-27x^3)^2.$$

6. We set $u = 2x$ and use the chain rule,

$$\frac{d}{dx}\int_1^{2x} \sqrt{t+1}\,dt = \left[\frac{d}{du}\int_1^u \sqrt{t+1}\,dt\right]\frac{du}{dx} = \sqrt{u+1}(2) = 2\sqrt{2x+1}.$$

7. We set $u = 3x^2$ and use the chain rule,

$$\frac{d}{dx}\int_4^{3x^2} \sin(3t+4)\,dt = \left[\frac{d}{du}\int_4^u \sin(3t+4)\,dt\right]\frac{du}{dx} = \sin(3u+4)\,(6x) = 6x\,\sin(9x^2+4).$$

8. We set $u = 5x+4$ and use the chain rule,

$$\frac{d}{dx}\int_{-2}^{5x+4} \sqrt{t^3+1}\,dt = \left[\frac{d}{du}\int_{-2}^u \sqrt{t^3+1}\,dt\right]\frac{du}{dx} = \sqrt{u^3+1}\,(5) = 5\sqrt{(5x+4)^3+1}.$$

9. When a is any number between x and $2x$, we may write

$$\begin{aligned}\int_x^{2x}(3\sqrt{t}-2t)\,dt &= \int_x^a(3\sqrt{t}-2t)\,dt+\int_a^{2x}(3\sqrt{t}-2t)\,dt\\ &= -\int_a^x(3\sqrt{t}-2t)\,dt+\int_a^{2x}(3\sqrt{t}-2t)\,dt.\end{aligned}$$

In the second integral we set $u=2x$ and use the chain rule,

$$\begin{aligned}\frac{d}{dx}\int_x^{2x}(3\sqrt{t}-2t)\,dt &= -(3\sqrt{x}-2x)+\left[\frac{d}{du}\int_a^u(3\sqrt{t}-2t)\,dt\right]\frac{du}{dx}\\ &= -3\sqrt{x}+2x+(3\sqrt{u}-2u)(2)\\ &= -3\sqrt{x}+2x+6\sqrt{2x}-8x=3(2\sqrt{2}-1)\sqrt{x}-6x.\end{aligned}$$

10. When a is any number between $4x$ and $4x+4$, we may write

$$\begin{aligned}\int_{4x}^{4x+4}\left(t^3-\frac{1}{\sqrt{t}}\right)dt &= \int_{4x}^{a}\left(t^3-\frac{1}{\sqrt{t}}\right)dt+\int_a^{4x+4}\left(t^3-\frac{1}{\sqrt{t}}\right)dt\\ &= -\int_a^{4x}\left(t^3-\frac{1}{\sqrt{t}}\right)dt+\int_a^{4x+4}\left(t^3-\frac{1}{\sqrt{t}}\right)dt.\end{aligned}$$

In these integrals we set $u=4x$ and $v=4x+4$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{4x}^{4x+4}\left(t^3-\frac{1}{\sqrt{t}}\right)dt &= -\left[\frac{d}{du}\int_a^u\left(t^3-\frac{1}{\sqrt{t}}\right)dt\right]\frac{du}{dx}+\left[\frac{d}{dv}\int_a^v\left(t^3-\frac{1}{\sqrt{t}}\right)dt\right]\frac{dv}{dx}\\ &= -\left[u^3-\frac{1}{\sqrt{u}}\right](4)+\left[v^3-\frac{1}{\sqrt{v}}\right](4)\\ &= -4\left[(4x)^3-\frac{1}{\sqrt{4x}}\right]+4\left[(4x+4)^3-\frac{1}{\sqrt{4x+4}}\right]\\ &= -256x^3+\frac{2}{\sqrt{x}}+256(x+1)^3-\frac{2}{\sqrt{x+1}}\\ &= 256(3x^2+3x+1)+\frac{2}{\sqrt{x}}-\frac{2}{\sqrt{x+1}}.\end{aligned}$$

11. When a is any number between $-2x$ and x, we may write

$$\int_{-2x}^{x}\tan(3t+1)\,dt=\int_{-2x}^{a}\tan(3t+1)\,dt+\int_a^x\tan(3t+1)\,dt=-\int_a^{-2x}\tan(3t+1)\,dt+\int_a^x\tan(3t+1)\,dt.$$

In the first integral we set $u=-2x$ and use the chain rule,

$$\begin{aligned}\frac{d}{dx}\int_{-2x}^{x}\tan(3t+1)\,dt &= -\left[\frac{d}{du}\int_a^u\tan(3t+1)\,dt\right]\frac{du}{dx}+\tan(3x+1)\\ &= -\tan(3u+1)(-2)+\tan(3x+1)\\ &= 2\tan(1-6x)+\tan(3x+1).\end{aligned}$$

12. When a is a number between $-x^2$ and $-2x^2$, we may write

$$\begin{aligned}\int_{-x^2}^{-2x^2}\sec(1-t)\,dt &= \int_{-x^2}^{a}\sec(1-t)\,dt+\int_a^{-2x^2}\sec(1-t)\,dt\\ &= -\int_a^{-x^2}\sec(1-t)\,dt+\int_a^{-2x^2}\sec(1-t)\,dt.\end{aligned}$$

In these integrals we set $u=-x^2$ and $v=-2x^2$ respectively, and use chain rules,

$$\frac{d}{dx}\int_{-x^2}^{-2x^2}\sec(1-t)\,dt = -\left[\frac{d}{du}\int_a^u \sec(1-t)\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_a^v \sec(1-t)\,dt\right]\frac{dv}{dx}$$
$$= -\sec(1-u)(-2x) + \sec(1-v)(-4x)$$
$$= 2x\sec(1+x^2) - 4x\sec(1+2x^2).$$

13. We set $u = \sin x$ and use the chain rule,

$$\frac{d}{dx}\int_0^{\sin x}\cos(t^2)\,dt = \left[\frac{d}{du}\int_0^u \cos(t^2)\,dt\right]\frac{du}{dx} = \cos(u^2)(\cos x) = \cos x\,\cos(\sin^2 x).$$

14. When a is a number between $\cos x$ and $\sin x$, we may write

$$\int_{\cos x}^{\sin x}\frac{1}{\sqrt{t+1}}dt = \int_{\cos x}^{a}\frac{1}{\sqrt{t+1}}dt + \int_a^{\sin x}\frac{1}{\sqrt{t+1}}dt = -\int_a^{\cos x}\frac{1}{\sqrt{t+1}}dt + \int_a^{\sin x}\frac{1}{\sqrt{t+1}}dt.$$

In these integrals we set $u = \cos x$ and $v = \sin x$ respectively, and use chain rules,

$$\frac{d}{dx}\int_{\cos x}^{\sin x}\frac{1}{\sqrt{t+1}}dt = -\left[\frac{d}{du}\int_a^u \frac{1}{\sqrt{t+1}}dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_a^v \frac{1}{\sqrt{t+1}}dt\right]\frac{dv}{dx}$$
$$= -\frac{1}{\sqrt{u+1}}(-\sin x) + \frac{1}{\sqrt{v+1}}(\cos x) = \frac{\sin x}{\sqrt{\cos x + 1}} + \frac{\cos x}{\sqrt{\sin x + 1}}.$$

15. We set $u = 2\sqrt{x}$ and use the chain rule,

$$\frac{d}{dx}\int_0^{2\sqrt{x}}\sqrt{t}\,dt = \left[\frac{d}{du}\int_0^u \sqrt{t}\,dt\right]\frac{du}{dx} = \sqrt{u}\left(\frac{1}{\sqrt{x}}\right) = \frac{\sqrt{2}}{x^{1/4}}.$$

16. If a is a number between $\sqrt{x}$ and $2\sqrt{x}$, we may write

$$\int_{\sqrt{x}}^{2\sqrt{x}}\sqrt{t}\,dt = \int_{\sqrt{x}}^{a}\sqrt{t}\,dt + \int_a^{2\sqrt{x}}\sqrt{t}\,dt = -\int_a^{\sqrt{x}}\sqrt{t}\,dt + \int_a^{2\sqrt{x}}\sqrt{t}\,dt.$$

In these integrals we set $u = \sqrt{x}$ and $v = 2\sqrt{x}$ respectively, and use chain rules,

$$\frac{d}{dx}\int_{\sqrt{x}}^{2\sqrt{x}}\sqrt{t}\,dt = -\left[\frac{d}{du}\int_a^u \sqrt{t}\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_a^v \sqrt{t}\,dt\right]\frac{dv}{dx}$$
$$= -\sqrt{u}\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{v}\left(\frac{1}{\sqrt{x}}\right) = -\frac{x^{1/4}}{2\sqrt{x}} + \frac{\sqrt{2}x^{1/4}}{\sqrt{x}} = \frac{2\sqrt{2}-1}{2x^{1/4}}.$$

17. We set $u = x^2$ and use the chain rule,

$$\frac{d}{dx}\int_1^{x^2} t^2 e^{4t}\,dt = \left[\frac{d}{du}\int_1^u t^2 e^{4t}\,dt\right]\frac{du}{dx} = u^2 e^{4u}(2x) = 2x^5 e^{4x^2}.$$

18. $\displaystyle \frac{d}{dx}\int_x^2 \ln(t^2+1)\,dt = -\frac{d}{dx}\int_2^x \ln(t^2+1)\,dt = -\ln(x^2+1)$

19. If a is a number between x and $2x$, we may write

$$\int_x^{2x} t\,\ln t\,dt = \int_x^a t\,\ln t\,dt + \int_a^{2x} t\,\ln t\,dt = -\int_a^x t\,\ln t\,dt + \int_a^{2x} t\,\ln t\,dt.$$

In the second integral we set $u = 2x$ and use the chain rule,

$$\frac{d}{dx}\int_x^{2x} t\,\ln t\,dt = -x\,\ln x + \left[\frac{d}{du}\int_a^u t\,\ln t\,dt\right]\frac{du}{dx}$$
$$= -x\,\ln x + u\,\ln u\,(2) = -x\,\ln x + 4x\,\ln(2x) = (4\,\ln 2)x + 3x\,\ln x.$$

20. If a is a number between $-2x$ and $3x$, then

$$\int_{-2x}^{3x} e^{-4t^2}\,dt = \int_{-2x}^{a} e^{-4t^2}\,dt + \int_{a}^{3x} e^{-4t^2}\,dt = -\int_{a}^{-2x} e^{-4t^2}\,dt + \int_{a}^{3x} e^{-4t^2}\,dt.$$

In these integrals we set $u = -2x$ and $v = 3x$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{-2x}^{3x} e^{-4t^2}\,dt &= -\left[\frac{d}{du}\int_{a}^{u} e^{-4t^2}\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_{a}^{v} e^{-4t^2}\,dt\right]\frac{dv}{dx}\\ &= -e^{-4u^2}(-2) + e^{-4v^2}(3) = 2\,e^{-16x^2} + 3\,e^{-36x^2}.\end{aligned}$$

21. If c is a number between $a(x)$ and $b(x)$, then

$$\int_{a(x)}^{b(x)} f(t)\,dt = \int_{a(x)}^{c} f(t)\,dt + \int_{c}^{b(x)} f(t)\,dt = -\int_{c}^{a(x)} f(t)\,dt + \int_{c}^{b(x)} f(t)\,dt.$$

In these integrals we set $u = a(x)$ and $v = b(x)$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt &= -\left[\frac{d}{du}\int_{c}^{u} f(t)\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_{c}^{v} f(t)\,dt\right]\frac{dv}{dx}\\ &= -f(u)\,a'(x) + f(v)\,b'(x) = f[b(x)]\frac{db}{dx} - f[a(x)]\frac{da}{dx}.\end{aligned}$$

Equation 6.19 is the special case when $a(x)$ is a constant and $b(x) = x$.

22. $\displaystyle \frac{d}{dx}\int_{-2}^{5x+4} \sqrt{t^3+1}\,dt = \sqrt{(5x+4)^3+1}(5)$

23. $$\begin{aligned}\frac{d}{dx}\int_{4x}^{4x+4}\left(t^3 - \frac{1}{\sqrt{t}}\right)dt &= \left[(4x+4)^3 - \frac{1}{\sqrt{4x+4}}\right](4) - \left[(4x)^3 - \frac{1}{\sqrt{4x}}\right](4)\\ &= 256(x+1)^3 - \frac{2}{\sqrt{x+1}} - 256x^3 + \frac{2}{\sqrt{x}} = 256(3x^2+3x+1) + \frac{2}{\sqrt{x}} - \frac{2}{\sqrt{x+1}}\end{aligned}$$

24. $$\begin{aligned}\frac{d}{dx}\int_{-x^2}^{-2x^2} \sec(1-t)\,dt &= [\sec(1+2x^2)](-4x) - [\sec(1+x^2)](-2x)\\ &= 2x\sec(1+x^2) - 4x\sec(1+2x^2)\end{aligned}$$

25. $\displaystyle \frac{d}{dx}\int_{\cos x}^{\sin x} \frac{1}{\sqrt{t+1}}dt = \frac{\cos x}{\sqrt{\sin x+1}} - \frac{-\sin x}{\sqrt{\cos x+1}}$

26. $\displaystyle \frac{d}{dx}\int_{\sqrt{x}}^{2\sqrt{x}} \sqrt{t}\,dt = \sqrt{2\sqrt{x}}\left(\frac{1}{\sqrt{x}}\right) - \sqrt{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{2\sqrt{2}-1}{2x^{1/4}}$

27. $\displaystyle \frac{d}{dx}\int_{x}^{2} \ln(t^2+1)\,dt = -\ln(x^2+1)(1) = -\ln(x^2+1)$

28. $\displaystyle \frac{d}{dx}\int_{-2x}^{3x} e^{-4t^2}\,dt = e^{-4(3x)^2}(3) - e^{-4(-2x)^2}(-2) = 3\,e^{-36x^2} + 2\,e^{-16x^2}$

EXERCISES 6.6

1. The average value is $\displaystyle \frac{1}{2}\int_0^2 (x^2-2x)\,dx = \frac{1}{2}\left\{\frac{x^3}{3} - x^2\right\}_0^2 = \frac{1}{2}\left(\frac{8}{3} - 4\right) = -\frac{2}{3}$.
2. Since the integrand is an odd function, its average value over the interval $-1 \le x \le 1$ is 0.
3. The average value is $\displaystyle \int_0^1 (x^3-x)\,dx = \left\{\frac{x^4}{4} - \frac{x^2}{2}\right\}_0^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$.
4. The average value is $\displaystyle \frac{1}{2-1}\int_1^2 x^4\,dx = \left\{\frac{x^5}{5}\right\}_1^2 = \frac{1}{5}(32-1) = \frac{31}{5}$.

5. The average value is $\int_0^1 \sqrt{x+1}\,dx = \left\{\frac{2}{3}(x+1)^{3/2}\right\}_0^1 = \frac{2}{3}(2\sqrt{2}-1)$.

6. The average value is $\frac{1}{2}\int_{-1}^1 \sqrt{x+1}\,dx = \frac{1}{2}\left\{\frac{2}{3}(x+1)^{3/2}\right\}_{-1}^1 = \frac{2\sqrt{2}}{3}$.

7. The average value is $\int_0^1 (x^4-1)\,dx = \left\{\frac{x^5}{5}-x\right\}_0^1 = \frac{1}{5}-1 = -\frac{4}{5}$.

8. The average value is $\frac{1}{2}\int_0^2 (x^4-1)\,dx = \frac{1}{2}\left\{\frac{x^5}{5}-x\right\}_0^2 = \frac{1}{2}\left(\frac{32}{5}-2\right) = \frac{11}{5}$.

9. The average value is $\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\,dx = \frac{1}{\pi}\left\{\sin x\right\}_{-\pi/2}^{\pi/2} = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$.

10. The average value is $\frac{1}{\pi/2}\int_0^{\pi/2}\cos x\,dx = \frac{2}{\pi}\{\sin x\}_0^{\pi/2} = \frac{2}{\pi}$.

11. The average value is

$$\frac{1}{4}\int_{-2}^2 |x|\,dx = \frac{1}{4}\int_{-2}^0 (-x)\,dx + \frac{1}{4}\int_0^2 x\,dx = \frac{1}{4}\left\{-\frac{x^2}{2}\right\}_{-2}^0 + \frac{1}{4}\left\{\frac{x^2}{2}\right\}_0^2 = \frac{1}{4}(2)+\frac{1}{4}(2) = 1.$$

12. The average value is $\frac{1}{2}\int_0^2 |x|\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left\{\frac{x^2}{2}\right\}_0^2 = 1$.

13. The average value is

$$\begin{aligned}\frac{1}{3}\int_0^3 |x^2-4|\,dx &= \frac{1}{3}\int_0^2 (4-x^2)\,dx + \frac{1}{3}\int_2^3 (x^2-4)\,dx\\ &= \frac{1}{3}\left\{4x-\frac{x^3}{3}\right\}_0^2 + \frac{1}{3}\left\{\frac{x^3}{3}-4x\right\}_2^3 = \frac{1}{3}\left(8-\frac{8}{3}\right)+\frac{1}{3}\left(9-12-\frac{8}{3}+8\right) = \frac{23}{9}.\end{aligned}$$

14. The average value is

$$\begin{aligned}\frac{1}{6}\int_{-3}^3 |x^2-4|\,dx &= \frac{1}{6}\left[\int_{-3}^{-2}(x^2-4)\,dx + \int_{-2}^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx\right]\\ &= \frac{1}{6}\left[\left\{\frac{x^3}{3}-4x\right\}_{-3}^{-2} + \left\{4x-\frac{x^3}{3}\right\}_{-2}^2 + \left\{\frac{x^3}{3}-4x\right\}_2^3\right]\\ &= \frac{1}{6}\left[\left(-\frac{8}{3}+8\right)-(-9+12)+\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)+(9-12)-\left(\frac{8}{3}-8\right)\right]\\ &= \frac{23}{9}.\end{aligned}$$

15. Since the function is odd, its average value on $-1 \le x \le 1$ is zero.

16. The average value is $\frac{1}{4}\int_{-1}^3 \operatorname{sgn} x\,dx = \frac{1}{4}\left[\int_{-1}^0 (-1)\,dx + \int_0^3 (1)\,dx\right] = \frac{1}{4}\left[\{-x\}_{-1}^0 + \{x\}_0^3\right] = \frac{1}{2}$.

17. The average value is $\frac{1}{2}\int_0^2 h(x-1)\,dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}\left\{x\right\}_1^2 = \frac{1}{2}$.

18. The average value is $\frac{1}{2}\int_0^2 h(x-4)\,dx = \frac{1}{2}\int_0^2 (0)\,dx = 0$.

19. The average value is $\frac{1}{3}\int_0^3 \lfloor x\rfloor\,dx = \frac{1}{3}\left[\int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx\right] = \frac{1}{3}(0+1+2) = 1$.

20. The average value is

$$\frac{1}{3.5}\int_0^{3.5}\lfloor x\rfloor\,dx = \frac{1}{3.5}\left[\int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^{3.5} 3\,dx\right] = \frac{1}{3.5}\left[0+1+2+\frac{3}{2}\right] = \frac{9}{7}.$$

21. If $v(t)$ is the velocity function, its average value for $t_1 \le t \le t_2$ is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}\Big\{x(t)\Big\}_{t_1}^{t_2} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{x_2 - x_1}{t_2 - t_1}.$$

22. The average value is $\displaystyle\frac{1}{R}\int_0^R c(R^2 - r^2)\,dr = \frac{c}{R}\left\{R^2 r - \frac{r^3}{3}\right\}_0^R = \frac{c}{R}\left(R^3 - \frac{R^3}{3}\right) = \frac{2cR^2}{3}$.

23. According to 6.31, $\displaystyle 2(2c - c^2) = \int_0^2 (2x - x^2)\,dx = \left\{x^2 - \frac{x^3}{3}\right\}_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$.
Value of c satisfying $6(2c - c^2) = 4 \implies 3c^2 - 6c + 2 = 0$ are $c = (6 \pm \sqrt{36 - 24})/6 = (3 \pm \sqrt{3})/3$.

24. According to 6.31, $\displaystyle 4(c^3 - 8c) = \int_{-2}^2 (x^3 - 8x)\,dx = \left\{\frac{x^4}{4} - 4x^2\right\}_{-2}^2 = (4 - 16) - (4 - 16) = 0$.

The only value of c between -2 and 2 that satisfies this equation is $c = 0$.

25. According to 6.31, $\displaystyle \frac{\pi}{2}\cos c = \int_0^{\pi/2} \cos x\,dx = \Big\{\sin x\Big\}_0^{\pi/2} = 1$.
The only value of c between 0 and $\pi/2$ that satisfies this equation is $c = 0.881$ radians.

26. According to 6.31, $\displaystyle \pi\cos c = \int_0^{\pi} \cos x\,dx = \{\sin x\}_0^{\pi} = 0$. The only value of c between 0 and π which satisfies this equation is $c = \pi/2$.

27. According to 6.31, $\displaystyle 2\sqrt{c+1} = \int_1^3 \sqrt{x+1}\,dx = \left\{\frac{2}{3}(x+1)^{3/2}\right\}_1^3 = \frac{4(4 - \sqrt{2})}{3}$.

The solution of this equation is $\displaystyle c = -1 + \frac{4}{9}(4 - \sqrt{2})^2 = \frac{63 - 32\sqrt{2}}{9}$.

28. From 6.31, $\displaystyle 1(c^2)(c+1) = \int_0^1 x^2(x+1)\,dx = \left\{\frac{x^4}{4} + \frac{x^3}{3}\right\}_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$.

Thus, c must satisfy $g(c) = 12c^3 + 12c^2 - 7 = 0$. There is only one solution of this equation between 0 and 1, which we can find by Newton's iterative procedure,

$$c_1 = 0.5, \qquad c_{n+1} = c_n - \frac{12c_n^3 + 12c_n^2 - 7}{36c_n^2 + 24c_n}.$$

Iteration gives $c_2 = 0.619$, $c_3 = 0.603\,50$, $c_4 = 0.603\,20$, $c_5 = 0.603\,20$. Since $g(0.603\,15) = -0.0015$ and $g(0.603\,25) = 0.0013$, we can say that to 4 decimals, $c = 0.6032$.

29. From 6.31, $\displaystyle 2c\sqrt{c^2+1} = \int_0^2 x\sqrt{x^2+1}\,dx = \left\{\frac{1}{3}(x^2+1)^{3/2}\right\}_0^2 = \frac{1}{3}(5\sqrt{5} - 1)$.
Thus, c must satisfy $6c\sqrt{c^2+1} = 5\sqrt{5} - 1$ or $g(c) = 18c^4 + 18c^2 + 5\sqrt{5} - 63 = 0$. There is only one solution of this equation between 0 and 2, which we can find by Newton's iterative procedure,

$$c_1 = 1, \qquad c_{n+1} = c_n - \frac{18c_n^4 + 18c_n^2 + 5\sqrt{5} - 63}{72c_n^3 + 36c_n}.$$

Iteration gives $c_2 = 1.146$, $c_3 = 1.1268$, $c_4 = 1.1264$, $c_5 = 1.1264$. Since $g(1.1255) = -0.13$ and $g(1.1265) = 0.0090$, we can say that to 3 decimals, $c = 1.126$.

30. From 6.31, $\displaystyle 1\left(\frac{1}{c^2} + \frac{1}{c^3}\right) = \int_1^2 \left(\frac{1}{x^2} + \frac{1}{x^3}\right)dx = \left\{-\frac{1}{x} - \frac{1}{2x^2}\right\}_1^2 = \left(-\frac{1}{2} - \frac{1}{8}\right) - \left(-1 - \frac{1}{2}\right) = \frac{7}{8}$.

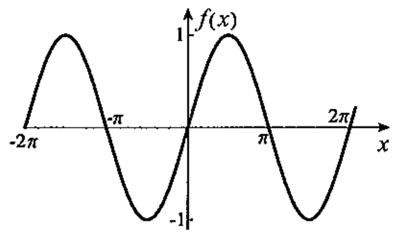
Thus, c must satisfy $g(c) = 7c^3 - 8c - 8 = 0$. There is only one solution of this equation between 1 and 2, which we can find by Newton's iterative procedure,

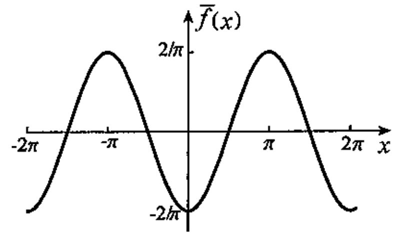
$$c_1 = 1.5, \qquad c_{n+1} = c_n - \frac{7c_n^3 - 8c_n - 8}{21c_n^2 - 8}.$$

Iteration gives $c_2 = 1.408$, $c_3 = 1.3998$, $c_4 = 1.3998$. Since $g(1.3995) = -0.0086$ and $g(1.4005) = 0.025$, we can say that to 3 decimals, $c = 1.400$.

31. The moving average is

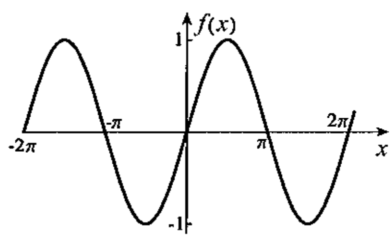
$$\overline{f}(x) = \frac{1}{\pi}\int_{x-\pi}^{x} \sin t\,dt = \frac{1}{\pi}\left\{-\cos t\right\}_{x-\pi}^{x} = \frac{1}{\pi}[\cos(x-\pi) - \cos x] = \frac{1}{\pi}(-\cos x - \cos x) = -\frac{2}{\pi}\cos x.$$

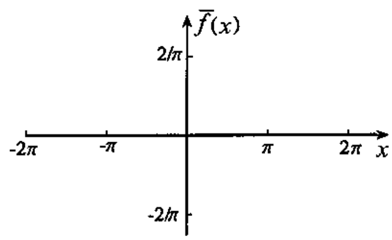




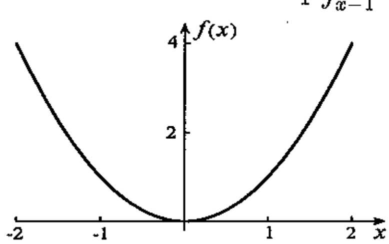
32. The moving average is

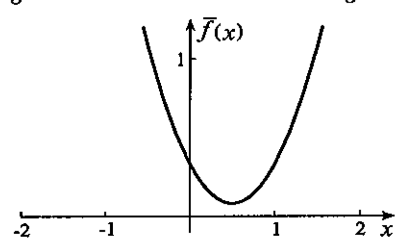
$$\overline{f}(x) = \frac{1}{2\pi}\int_{x-2\pi}^{x} \sin t\,dt = \frac{1}{2\pi}\left\{-\cos t\right\}_{x-2\pi}^{x} = \frac{1}{2\pi}[\cos(x-2\pi) - \cos x] = \frac{1}{2\pi}(\cos x - \cos x) = 0.$$



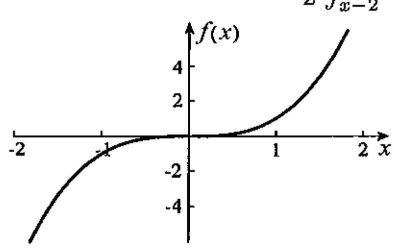


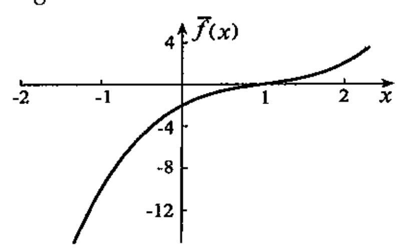
33. The moving average is $\overline{f}(x) = \frac{1}{1}\int_{x-1}^{x} t^2\,dt = \left\{\frac{t^3}{3}\right\}_{x-1}^{x} = \frac{1}{3}[x^3 - (x-1)^3] = x^2 - x + \frac{1}{3}$.





34. The moving average is $\overline{f}(x) = \frac{1}{2}\int_{x-2}^{x} t^3\,dt = \frac{1}{2}\left\{\frac{t^4}{4}\right\}_{x-2}^{x} = \frac{1}{8}[x^4 - (x-2)^4] = x^3 - 3x^2 + 4x - 2$.

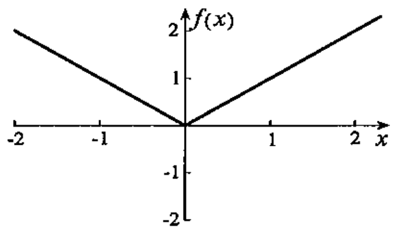


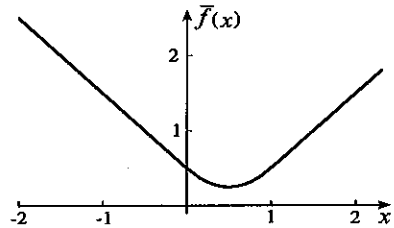


35. The moving average is $\overline{f}(x) = \frac{1}{1}\int_{x-1}^{x} |t|\,dt$. When $x \le 0$, $\overline{f}(x) = \int_{x-1}^{x} -t\,dt = \left\{-\frac{t^2}{2}\right\}_{x-1}^{x} = \frac{1}{2} - x$.

When $0 < x < 1$, $\overline{f}(x) = \int_{x-1}^{0} -t\,dt + \int_{0}^{x} t\,dt = \left\{-\frac{t^2}{2}\right\}_{x-1}^{0} + \left\{\frac{t^2}{2}\right\}_{0}^{x} = x^2 - x + \frac{1}{2}$.

When $x \ge 1$, $\overline{f}(x) = \int_{x-1}^{x} t\,dt = \left\{\frac{t^2}{2}\right\}_{x-1}^{x} = x - \frac{1}{2}$.





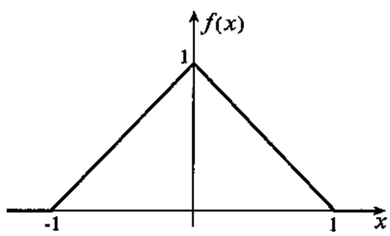
36. The moving average is $\dfrac{1}{1}\displaystyle\int_{x-1}^{x} f(t)\,dt.$ When $x \le -1$, $\overline{f}(x) = 0$.

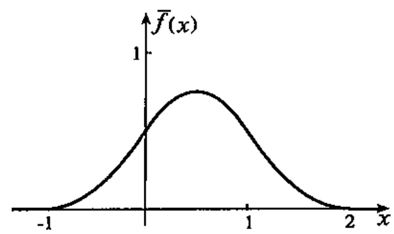
When $-1 < x < 0$, $\overline{f}(x) = \displaystyle\int_{-1}^{x} (1+t)\,dt = \left\{t + \frac{t^2}{2}\right\}_{-1}^{x} = \frac{x^2}{2} + x + \frac{1}{2}.$

When $0 \le x < 1$, $\overline{f}(x) = \displaystyle\int_{x-1}^{0} (1+t)\,dt + \int_{0}^{x} (1-t)\,dt = \left\{t + \frac{t^2}{2}\right\}_{x-1}^{0} + \left\{t - \frac{t^2}{2}\right\}_{0}^{x} = -x^2 + x + \frac{1}{2}.$

When $1 \le x < 2$, $\overline{f}(x) = \displaystyle\int_{x-1}^{1} (1-t)\,dt = \left\{t - \frac{t^2}{2}\right\}_{x-1}^{1} = \frac{x^2}{2} - 2x + 2.$

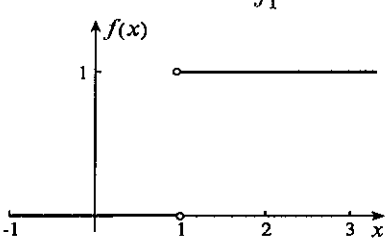
When $x \ge 2$, $\overline{f}(x) = 0$.

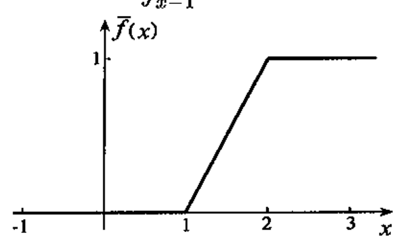




37. The moving average is $\dfrac{1}{1}\displaystyle\int_{x-1}^{x} h(t-1)\,dt.$ When $x < 1$, $\overline{f}(x) = 0$.

When $1 \le x < 2$, $\overline{f}(x) = \displaystyle\int_{1}^{x} 1\,dt = x - 1;$ when $x \ge 2$, $\overline{f}(x) = \displaystyle\int_{x-1}^{x} 1\,dt = 1.$

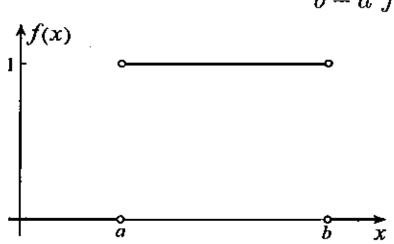


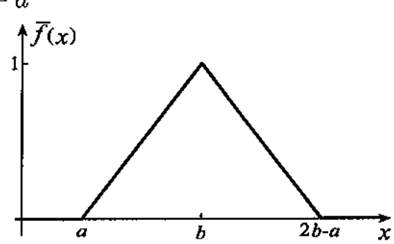


38. The moving average is $\overline{f}(x) = \dfrac{1}{b-a}\displaystyle\int_{x-b+a}^{x} [h(x-a) - h(x-b)]\,dx.$ When $x < a$, $\overline{f}(x) = 0$.

When $a \le x < b$, $\overline{f}(x) = \dfrac{1}{b-a}\displaystyle\int_{a}^{x} 1\,dt = \frac{x-a}{b-a}.$

When $b \le x < 2b - a$, $\overline{f}(x) = \dfrac{1}{b-a}\displaystyle\int_{x-b+a}^{b} 1\,dt = \frac{2b-x-a}{b-a}.$ When $x \ge 2b - a$, $\overline{f}(x) = 0$.





EXERCISES 6.7

1. $\displaystyle\int_1^2 x(3x^2-2)^4\,dx = \left\{\frac{1}{30}(3x^2-2)^5\right\}_1^2 = \frac{33\,333}{10}$

2. If we set $u = 1-z$, then $du = -dz$, and

$$\int_0^1 z\sqrt{1-z}\,dz = \int_1^0 (1-u)\sqrt{u}(-du) = \int_1^0 (u^{3/2}-\sqrt{u})\,du = \left\{\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right\}_1^0 = -\left(\frac{2}{5}-\frac{2}{3}\right) = \frac{4}{15}.$$

3. If we set $u = x+3$, then $du = dx$, and

$$\int_{-1}^0 \frac{x}{\sqrt{x+3}}dx = \int_2^3 \frac{u-3}{\sqrt{u}}du = \int_2^3 \left(\sqrt{u} - \frac{3}{\sqrt{u}}\right) du$$
$$= \left\{\frac{2u^{3/2}}{3} - 6\sqrt{u}\right\}_2^3 = (2\sqrt{3}-6\sqrt{3}) - \left(\frac{4\sqrt{2}}{3} - 6\sqrt{2}\right) = \frac{14\sqrt{2}}{3} - 4\sqrt{3}.$$

4. $\displaystyle\int_{\pi/4}^{\pi/3} \cos^5 x \sin x\,dx = \left\{-\frac{1}{6}\cos^6 x\right\}_{\pi/4}^{\pi/3} = -\frac{1}{6}\left(\frac{1}{2}\right)^6 + \frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^6 = \frac{7}{384}$

5. If we set $u = 9-x^2$, then $du = -2x\,dx$, and

$$\int_1^3 x^3\sqrt{9-x^2}\,dx = \int_1^3 x^2\sqrt{9-x^2}\,x\,dx = \int_8^0 (9-u)\sqrt{u}\left(\frac{du}{-2}\right) = \frac{1}{2}\int_0^8 (9\sqrt{u} - u^{3/2})\,du$$
$$= \frac{1}{2}\left\{6u^{3/2} - \frac{2u^{5/2}}{5}\right\}_0^8 = \frac{1}{2}\left(96\sqrt{2} - \frac{256\sqrt{2}}{5}\right) = \frac{112\sqrt{2}}{5}.$$

6. This definite integral does not exist because the integrand is not defined for $-2\sqrt{3} \le x \le 2\sqrt{3}$.

7. If we set $u = y-4$, then $du = dy$, and

$$\int_4^5 y^2\sqrt{y-4}\,dy = \int_0^1 (u+4)^2\sqrt{u}\,du = \int_0^1 (u^{5/2} + 8u^{3/2} + 16\sqrt{u})\,du$$
$$= \left\{\frac{2u^{7/2}}{7} + \frac{16u^{5/2}}{5} + \frac{32u^{3/2}}{3}\right\}_0^1 = \frac{2}{7} + \frac{16}{5} + \frac{32}{3} = \frac{1486}{105}.$$

8. If we set $u = 1+x$, then $du = dx$, and

$$\int_{1/2}^1 \sqrt{\frac{x^2}{1+x}}dx = \int_{1/2}^1 \frac{x}{\sqrt{1+x}}dx = \int_{3/2}^2 \frac{u-1}{\sqrt{u}}du = \int_{3/2}^2 \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du$$
$$= \left\{\frac{2}{3}u^{3/2} - 2\sqrt{u}\right\}_{3/2}^2 = \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\sqrt{\frac{3}{2}} - 2\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}} - \frac{2\sqrt{2}}{3}.$$

9. If we set $v = 1+\sqrt{u}$, then $dv = \dfrac{du}{2\sqrt{u}}$, and

$$\int_1^4 \frac{\sqrt{1+\sqrt{u}}}{\sqrt{u}}du = \int_2^3 \sqrt{v}\,(2\,dv) = \left\{\frac{4v^{3/2}}{3}\right\}_2^3 = 4\sqrt{3} - \frac{8\sqrt{2}}{3} = \frac{4(3\sqrt{3}-2\sqrt{2})}{3}.$$

10. If we set $u = x^2+2x+2$, then $du = (2x+2)\,dx$, and

$$\int_{-2}^1 \frac{x+1}{(x^2+2x+2)^{1/3}}dx = \int_2^5 \frac{1}{u^{1/3}}\left(\frac{du}{2}\right) = \frac{1}{2}\left\{\frac{3}{2}u^{2/3}\right\}_2^5 = \frac{3}{4}\left(5^{2/3} - 2^{2/3}\right).$$

11. If we set $u = x - 2$, then $du = dx$, and

$$\int_3^4 \frac{x^2}{(x-2)^4}dx = \int_1^2 \frac{(u+2)^2}{u^4}du = \int_1^2 \left(\frac{1}{u^2} + \frac{4}{u^3} + \frac{4}{u^4}\right) du$$
$$= \left\{-\frac{1}{u} - \frac{2}{u^2} - \frac{4}{3u^3}\right\}_1^2 = \left(-\frac{1}{2} - \frac{1}{2} - \frac{1}{6}\right) - \left(-1 - 2 - \frac{4}{3}\right) = \frac{19}{6}.$$

12. If we set $u = 2 + 3\sin x$, then $du = 3\cos x\, dx$, and

$$\int_0^{\pi/6} \sqrt{2 + 3\sin x}\cos x\, dx = \int_2^{7/2} \sqrt{u}\left(\frac{du}{3}\right) = \frac{1}{3}\left\{\frac{2}{3}u^{3/2}\right\}_2^{7/2} = \frac{2}{9}\left(\frac{7\sqrt{7}}{2\sqrt{2}} - 2\sqrt{2}\right) = \frac{\sqrt{2}}{18}(7\sqrt{7} - 8).$$

13. If we set $u = 1 + \cos x$, then $du = -\sin x\, dx$, and

$$\int_{\pi/4}^{\pi/2} \frac{\sin^3 x}{(1+\cos x)^4}dx = \int_{\pi/4}^{\pi/2} \frac{1 - \cos^2 x}{(1+\cos x)^4}\sin x\, dx = \int_{1+1/\sqrt{2}}^{1} \frac{1 - (u-1)^2}{u^4}(-du)$$
$$= \int_{1+1/\sqrt{2}}^{1}\left(\frac{1}{u^2} - \frac{2}{u^3}\right) du = \left\{-\frac{1}{u} + \frac{1}{u^2}\right\}_{1+1/\sqrt{2}}^{1}$$
$$= (-1+1) - \left[\frac{-1}{1 + 1/\sqrt{2}} + \frac{1}{(1+1/\sqrt{2})^2}\right] = \frac{\sqrt{2}}{\sqrt{2}+1} - \frac{2}{3 + 2\sqrt{2}}$$
$$= \sqrt{2}(\sqrt{2} - 1) - 2(3 - 2\sqrt{2}) = 3\sqrt{2} - 4.$$

14. $\displaystyle\int_1^4 \frac{(x+1)(x-1)}{\sqrt{x}}dx = \int_1^4 \left(x^{3/2} - \frac{1}{\sqrt{x}}\right) dx = \left\{\frac{2}{5}x^{5/2} - 2\sqrt{x}\right\}_1^4 = \left(\frac{64}{5} - 4\right) - \left(\frac{2}{5} - 2\right) = \frac{52}{5}$

15. If we set $u = 1 + \sqrt{x}$, then $du = \dfrac{dx}{2\sqrt{x}}$, and

$$\int_4^9 \sqrt{1 + \sqrt{x}}\, dx = \int_3^4 \sqrt{u}\, 2(u-1)\, du = 2\int_3^4 (u^{3/2} - \sqrt{u})\, du = 2\left\{\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right\}_3^4$$
$$= 2\left(\frac{64}{5} - \frac{16}{3} - \frac{18\sqrt{3}}{5} + 2\sqrt{3}\right) = \frac{16(14 - 3\sqrt{3})}{15}.$$

16. If we set $u = 1 + x$, then $du = dx$, and

$$\int_{-1/2}^{1} \sqrt{\frac{x^2}{1+x}}dx = \int_{-1/2}^{0} \frac{-x}{\sqrt{1+x}}dx + \int_0^1 \frac{x}{\sqrt{1+x}}dx = \int_{1/2}^{1} \frac{-(u-1)}{\sqrt{u}}du + \int_1^2 \frac{u-1}{\sqrt{u}}du$$
$$= \int_{1/2}^{1}\left(\frac{1}{\sqrt{u}} - \sqrt{u}\right) du + \int_1^2 \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du = \left\{2\sqrt{u} - \frac{2}{3}u^{3/2}\right\}_{1/2}^{1} + \left\{\frac{2}{3}u^{3/2} - 2\sqrt{u}\right\}_1^2$$
$$= \left(2 - \frac{2}{3}\right) - \left(\sqrt{2} - \frac{1}{3\sqrt{2}}\right) + \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right) = \frac{1}{6}(16 - 9\sqrt{2}).$$

17. If we set $u = x + 2$, then $du = dx$, and

$$\int_{-1}^{1} \frac{|x|}{(x+2)^3}dx = \int_{-1}^{0} \frac{-x}{(x+2)^3}dx + \int_0^1 \frac{x}{(x+2)^3}dx = \int_1^2 \frac{-(u-2)}{u^3}du + \int_2^3 \frac{u-2}{u^3}du$$
$$= \int_1^2 \left(\frac{2}{u^3} - \frac{1}{u^2}\right) du + \int_2^3 \left(\frac{1}{u^2} - \frac{2}{u^3}\right) du = \left\{-\frac{1}{u^2} + \frac{1}{u}\right\}_1^2 + \left\{-\frac{1}{u} + \frac{1}{u^2}\right\}_2^3$$
$$= \left(-\frac{1}{4} + \frac{1}{2} + 1 - 1\right) + \left(-\frac{1}{3} + \frac{1}{9} + \frac{1}{2} - \frac{1}{4}\right) = \frac{5}{18}.$$

18. If we set $u = x + 2$, then $du = dx$, and

$$\int_{-1}^{1}\left|\frac{x}{(x+2)^3}\right|dx = \int_{-1}^{0}\frac{-x}{(x+2)^3}dx + \int_{0}^{1}\frac{x}{(x+2)^3}dx = \int_{1}^{2}\left(\frac{2-u}{u^3}\right)du + \int_{2}^{3}\left(\frac{u-2}{u^3}\right)du$$
$$= \int_{1}^{2}\left(\frac{2}{u^3}-\frac{1}{u^2}\right)du + \int_{2}^{3}\left(\frac{1}{u^2}-\frac{2}{u^3}\right)du = \left\{-\frac{1}{u^2}+\frac{1}{u}\right\}_{1}^{2} + \left\{-\frac{1}{u}+\frac{1}{u^2}\right\}_{2}^{3}$$
$$= \left(-\frac{1}{4}+\frac{1}{2}\right) - (-1+1) + \left(-\frac{1}{3}+\frac{1}{9}\right) - \left(-\frac{1}{2}+\frac{1}{4}\right) = \frac{5}{18}.$$

19. $\displaystyle\int_0^1 x^2 e^{x^3}\,dx = \left\{\frac{e^{x^3}}{3}\right\}_0^1 = \frac{e-1}{3}$

20. If we set $u = \ln x$, then $du = \dfrac{1}{x}dx$, and $\displaystyle\int_1^2 \frac{(\ln x)^2}{x}dx = \int_0^{\ln 2} u^2\,du = \left\{\frac{u^3}{3}\right\}_0^{\ln 2} = \frac{1}{3}(\ln 2)^3.$

21. $\displaystyle\int_2^4 \frac{1}{x\ln x}dx = \Big\{\ln|\ln x|\Big\}_2^4 = \ln(\ln 4) - \ln(\ln 2)$

22. If we set $u = 4 + 3\tan x$, then $du = 3\sec^2 x\,dx$, and

$$\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{\sqrt{4+3\tan x}}dx = \int_1^7 \frac{1}{\sqrt{u}}\left(\frac{du}{3}\right) = \frac{1}{3}\Big\{2\sqrt{u}\Big\}_1^7 = \frac{2(\sqrt{7}-1)}{3}.$$

23. If $f(x)$ is odd, and we set $u = -x$ in the first of the integrals on the right in the following equation,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = \int_a^0 f(-u)(-du) + \int_0^a f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0.$$

The proof is similar in the even case.

24. We can rewrite the integral as the sum of three integrals:

$$\int_a^{a+p} f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^p f(x)\,dx + \int_p^{a+p} f(x)\,dx.$$

In the last integral we change variables according to $u = x - p$. Then $du = dx$, and

$$\int_a^{a+p} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^p f(x)\,dx + \int_0^a f(u+p)\,du$$
$$= -\int_0^a f(x)\,dx + \int_0^p f(x)\,dx + \int_0^a f(u)\,du = \int_0^p f(x)\,dx.$$

25. If we set $u = \dfrac{1}{\sqrt{x}}$, then $du = -\dfrac{1}{2x^{3/2}}dx$, and

$$\int_1^3 \frac{1}{x^{3/2}\sqrt{4-x}}dx = \int_1^{1/\sqrt{3}} \frac{1}{\sqrt{4-1/u^2}}(-2du) = -2\int_1^{1/\sqrt{3}} \frac{u}{\sqrt{4u^2-1}}du$$
$$= -2\left\{\frac{1}{4}\sqrt{4u^2-1}\right\}_1^{1/\sqrt{3}} = -\frac{1}{2}\sqrt{4/3-1} + \frac{1}{2}\sqrt{3} = \frac{1}{\sqrt{3}}.$$

26. If we set $u = 1/x$, then $du = -dx/x^2$, and

$$\int_{-6}^{-1}\frac{\sqrt{x^2-6x}}{x^4}dx = \int_{-1/6}^{-1}\frac{\sqrt{\dfrac{1}{u^2}-\dfrac{6}{u}}}{1/u^4}\left(-\frac{du}{u^2}\right) = -\int_{-1/6}^{-1} u^2\sqrt{\frac{1-6u}{u^2}}du = \int_{-1/6}^{-1} u\sqrt{1-6u}\,du.$$

We now set $v = 1 - 6u$, in which case $dv = -6\,du$, and

$$\int_{-1/6}^{-1} \frac{\sqrt{x^2-6x}}{x^4}dx = \int_2^7 \left(\frac{1-v}{6}\right)\sqrt{v}\left(\frac{dv}{-6}\right) = \frac{1}{36}\int_2^7 (v^{3/2}-\sqrt{v})\,dv$$
$$= \frac{1}{36}\left\{\frac{2v^{5/2}}{5} - \frac{2v^{3/2}}{3}\right\}_2^7 = \frac{1}{36}\left(\frac{98\sqrt{7}}{5} - \frac{14\sqrt{7}}{3} - \frac{8\sqrt{2}}{5} + \frac{4\sqrt{2}}{3}\right) = \frac{56\sqrt{7}-\sqrt{2}}{135}.$$

27. If we set $u^2 = \dfrac{1-x}{5+x}$, then $(5+x)u^2 = 1-x$, and $x = \dfrac{1-5u^2}{1+u^2}$.

Thus, $dx = \dfrac{(1+u^2)(-10u)-(1-5u^2)(2u)}{(1+u^2)^2}du = \dfrac{-12u}{(1+u^2)^2}du$, and

$$\int_{-4}^{0} \frac{x}{(5-4x-x^2)^{3/2}}dx = \int_{-4}^{0}\frac{x}{[(5+x)(1-x)]^{3/2}}dx$$
$$= \int_{\sqrt{5}}^{1/\sqrt{5}} \frac{(1-5u^2)/(1+u^2)}{\left[\left(5+\dfrac{1-5u^2}{1+u^2}\right)\left(1-\dfrac{1-5u^2}{1+u^2}\right)\right]^{3/2}}\frac{-12u}{(1+u^2)^2}du$$
$$= \int_{\sqrt{5}}^{1/\sqrt{5}} \frac{-12u(1-5u^2)}{\left[\dfrac{36u^2}{(u^2+1)^2}\right]^{3/2}}\frac{1}{(1+u^2)^3}du = \int_{\sqrt{5}}^{1/\sqrt{5}} -\frac{1-5u^2}{18u^2}du$$
$$= \frac{1}{18}\int_{1/\sqrt{5}}^{\sqrt{5}}\left(\frac{1}{u^2}-5\right)du = \frac{1}{18}\left\{-\frac{1}{u}-5u\right\}_{1/\sqrt{5}}^{\sqrt{5}}$$
$$= \frac{1}{18}\left(-\frac{1}{\sqrt{5}}-5\sqrt{5}\right) - \frac{1}{18}\left(-\sqrt{5}-\sqrt{5}\right) = -\frac{8\sqrt{5}}{45}.$$

28. If we set $\psi = \pi\cos\theta$, then $d\psi = -\pi\sin\theta\,d\theta$, and

$$\int_0^{\pi} \frac{\cos^2[(\pi/2)\cos\theta]}{\sin\theta}d\theta = \int_{\pi}^{-\pi}\frac{\cos^2(\psi/2)}{\sin\theta}\left(\frac{-d\psi}{\pi\sin\theta}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos^2(\psi/2)}{1-\psi^2/\pi^2}d\psi.$$

Since $\cos^2(\psi/2) = (1+\cos\psi)/2$, it follows that

$$\int_0^{\pi} \frac{\cos^2[(\pi/2)\cos\theta]}{\sin\theta}d\theta = \pi\int_{-\pi}^{\pi}\frac{(1+\cos\psi)/2}{\pi^2-\psi^2}d\psi = \frac{\pi}{2}\int_{-\pi}^{\pi}\frac{1+\cos\psi}{(\pi+\psi)(\pi-\psi)}d\psi.$$

We now set $\phi = \psi+\pi$, and $d\phi = d\psi$,

$$\int_0^{\pi} \frac{\cos^2[(\pi/2)\cos\theta]}{\sin\theta}d\theta = \frac{\pi}{2}\int_0^{2\pi}\frac{1+\cos(\phi-\pi)}{\phi(2\pi-\phi)}d\phi = \frac{\pi}{2}\int_0^{2\pi}\frac{1-\cos\phi}{\phi(2\pi-\phi)}d\phi.$$

REVIEW EXERCISES

1. $\displaystyle\int_0^3 (x^2+3x-2)\,dx = \left\{\frac{x^3}{3}+\frac{3x^2}{2}-2x\right\}_0^3 = 9+\frac{27}{2}-6 = \frac{33}{2}$

2. $\displaystyle\int_{-1}^1 (x^2-x^4)\,dx = \left\{\frac{x^3}{3}-\frac{x^5}{5}\right\}_{-1}^1 = \left(\frac{1}{3}-\frac{1}{5}\right)-\left(-\frac{1}{3}+\frac{1}{5}\right) = \frac{4}{15}$

3. $\displaystyle\int_{-1}^1 (x^3-3x)\,dx = \left\{\frac{x^4}{4}-\frac{3x^2}{2}\right\}_{-1}^1 = \left(\frac{1}{4}-\frac{3}{2}\right)-\left(\frac{1}{4}-\frac{3}{2}\right) = 0$

4. $\displaystyle\int_0^2 (x^2-2x)\,dx = \left\{\frac{x^3}{3}-x^2\right\}_0^2 = \frac{8}{3}-4 = -\frac{4}{3}$

5. $\displaystyle\int_1^2 (x+1)^2\,dx = \left\{\frac{(x+1)^3}{3}\right\}_1^2 = \frac{1}{3}(27-8) = \frac{19}{3}$

6. $\displaystyle\int_{-3}^{-2} \frac{1}{x^2}dx = \left\{-\frac{1}{x}\right\}_{-3}^{-2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

7. $\displaystyle\int_4^9 \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) dx = \left\{2\sqrt{x} - \frac{2x^{3/2}}{3}\right\}_4^9 = (6-18) - \left(4 - \frac{16}{3}\right) = -\frac{32}{3}$

8. $\displaystyle\int_0^\pi \cos x\,dx = \left\{\sin x\right\}_0^\pi = 0$

9. $\displaystyle\int_{-1}^1 x(x+1)^2\,dx = \left\{\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right\}_{-1}^1 = \left(\frac{1}{4} + \frac{2}{3} + \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \frac{4}{3}$

10. $\displaystyle\int_1^2 x^2(x^2+3)\,dx = \left\{\frac{x^5}{5} + x^3\right\}_1^2 = \left(\frac{32}{5} + 8\right) - \left(\frac{1}{5} + 1\right) = \frac{66}{5}$

11. $\displaystyle\int_0^3 \sqrt{x+1}\,dx = \left\{\frac{2(x+1)^{3/2}}{3}\right\}_0^3 = \frac{2}{3}(8-1) = \frac{14}{3}$

12. $\displaystyle\int_1^5 x\sqrt{x^2-1}\,dx = \left\{\frac{1}{3}(x^2-1)^{3/2}\right\}_1^5 = \frac{1}{3}(24)^{3/2} = 16\sqrt{6}$

13. $\displaystyle\int_1^4 \left(\frac{\sqrt{x}+1}{\sqrt{x}}\right) dx = \left\{x + 2\sqrt{x}\right\}_1^4 = (4+4) - (1+2) = 5$

14. If we set $u = x+1$, then $du = dx$, and

$$\int_{-1}^0 x\sqrt{x+1}\,dx = \int_0^1 (u-1)\sqrt{u}\,du = \int_0^1 (u^{3/2} - \sqrt{u})\,du = \left\{\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right\}_0^1 = \frac{2}{5} - \frac{2}{3} = -\frac{4}{15}.$$

15. If we set $u = x+1$, then $du = dx$, and

$$\begin{aligned}\int_1^2 \frac{x^2+1}{(x+1)^4}dx &= \int_2^3 \frac{(u-1)^2+1}{u^4}du = \int_2^3 \left(\frac{1}{u^2} - \frac{2}{u^3} + \frac{2}{u^4}\right) du \\ &= \left\{-\frac{1}{u} + \frac{1}{u^2} - \frac{2}{3u^3}\right\}_2^3 = \left(-\frac{1}{3} + \frac{1}{9} - \frac{2}{81}\right) - \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{12}\right) = \frac{7}{81}.\end{aligned}$$

16. If we set $u = 2-x$, then $du = -dx$, and

$$\begin{aligned}\int_{-4}^{-2} x^2\sqrt{2-x}\,dx &= \int_6^4 (2-u)^2\sqrt{u}(-du) = \int_4^6 (4\sqrt{u} - 4u^{3/2} + u^{5/2})\,du = \left\{\frac{8}{3}u^{3/2} - \frac{8}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right\}_4^6 \\ &= \left(16\sqrt{6} - \frac{8(36)\sqrt{6}}{5} + \frac{12(36)\sqrt{6}}{7}\right) - \left(\frac{64}{3} - \frac{256}{5} + \frac{256}{7}\right) = \frac{2112\sqrt{6} - 704}{105}.\end{aligned}$$

17. If we set $u = 1 + \sin x$, then $du = \cos x\,dx$, and

$$\int_0^{\pi/4} \frac{\cos x}{(1+\sin x)^2}dx = \int_1^{1+1/\sqrt{2}} \frac{1}{u^2}du = \left\{-\frac{1}{u}\right\}_1^{1+1/\sqrt{2}} = \frac{-1}{1+1/\sqrt{2}} + 1 = \sqrt{2} - 1.$$

18. $\displaystyle\int_2^3 x(1+2x^2)^4\,dx = \left\{\frac{1}{20}(1+2x^2)^5\right\}_2^3 = \frac{1}{20}(19^5 - 9^5) = 120\,852.5$

19. If we set $u = x^{1/3}$, then $du = \dfrac{dx}{3x^{2/3}}$, and

$$\int_1^8 \frac{(1+x^{1/3})^2}{x^{2/3}}dx = \int_1^2 (1+u)^2(3\,du) = \left\{(1+u)^3\right\}_1^2 = 27 - 8 = 19.$$

20. $\displaystyle\int_{-4}^{4} |x+2|\,dx = \int_{-4}^{-2} -(x+2)\,dx + \int_{-2}^{4} (x+2)\,dx = -\left\{\frac{x^2}{2}+2x\right\}_{-4}^{-2} + \left\{\frac{x^2}{2}+2x\right\}_{-2}^{4}$
$= -(2-4)+(8-8)+(8+8)-(2-4) = 20$

21. (a) Since $f(x)=x-5$ is continuous for $0 \le x \le 2$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $2/n$, we use the points $x_i = 2i/n$, $i=0,\ldots,n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = 2i/n$. Equation 6.10 now gives

$$\int_0^2 (x-5)\,dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^{n} (x_i^* - 5)\Delta x_i = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}-5\right)$$
$$= \lim_{n\to\infty}\frac{2}{n^2}\sum_{i=1}^{n}(2i-5n) = \lim_{n\to\infty}\frac{2}{n^2}\left[\frac{2n(n+1)}{2} - 5n^2\right] = 2-10 = -8.$$

(b) Since $f(x)=x^2+3$ is continuous for $0 \le x \le 3$, the definite integral exists, and we may choose any partition and star-points in its evaluation. For n equal subdivisions of length $3/n$, we use the points $x_i = 3i/n$, $i=0,\ldots,n$. If we choose the right end of each subinterval as star-point, then $x_i^* = x_i = 3i/n$. Equation 6.10 now gives

$$\int_0^3 (x^2+3)\,dx = \lim_{\|\Delta x_i\|\to 0}\sum_{i=1}^{n}[(x_i^*)^2+3]\Delta x_i = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2+3\right]$$
$$= \lim_{n\to\infty}\frac{9}{n^3}\sum_{i=1}^{n}(3i^2+n^2) = \lim_{n\to\infty}\frac{9}{n^3}\left[\frac{3n(n+1)(2n+1)}{6}+n^3\right] = 9+9 = 18.$$

22. (a) Since x^2+3x is nonnegative for $0 \le x \le 4$, it is impossible for the limit summation of equation 6.10 to give a negative number.
(b) Since $1/x$ is negative for $-3 \le x \le -2$, it is impossible for the limit summation of 6.10 to give a positive number.

23. The average value is $\displaystyle\frac{1}{1}\int_0^1 \sqrt{x+4}\,dx = \left\{\frac{2(x+4)^{3/2}}{3}\right\}_0^1 = \frac{2}{3}(5\sqrt{5}-8).$

24. The averge value is $\displaystyle\frac{1}{1}\int_{-2}^{-1}\left(\frac{1}{x^2}-x\right)dx = \left\{-\frac{1}{x}-\frac{x^2}{2}\right\}_{-2}^{-1} = \left(1-\frac{1}{2}\right)-\left(\frac{1}{2}-2\right) = 2.$

25. If we set $u=x+1$, and $du=dx$, the average value is

$$\int_0^1 x\sqrt{x+1}\,dx = \int_1^2 (u-1)\sqrt{u}\,du = \left\{\frac{2u^{5/2}}{5}-\frac{2u^{3/2}}{3}\right\}_1^2 = \left(\frac{8\sqrt{2}}{5}-\frac{4\sqrt{2}}{3}\right)-\left(\frac{2}{5}-\frac{2}{3}\right) = \frac{4(\sqrt{2}+1)}{15}.$$

26. If we set $u=\sin x$ and $du=\cos x\,dx$, the average value is

$$\frac{1}{\pi/2}\int_0^{\pi/2}\cos^3 x\,\sin^2 x\,dx = \frac{2}{\pi}\int_0^{\pi/2}(1-\sin^2 x)\sin^2 x\,\cos x\,dx = \frac{2}{\pi}\int_0^1 (1-u^2)u^2\,du$$
$$= \frac{2}{\pi}\left\{\frac{u^3}{3}-\frac{u^5}{5}\right\}_0^1 = \frac{2}{\pi}\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{4}{15\pi}.$$

27. By equation 6.19, $\displaystyle\frac{d}{dx}\int_1^x t\sqrt{t^3+1}\,dt = x\sqrt{x^3+1}$

28. When we reverse the limits, $\displaystyle\frac{d}{dx}\int_x^{-3} t^2(t+1)^3\,dt = -\frac{d}{dx}\int_{-3}^{x} t^2(t+1)^3\,dt = -x^2(x+1)^3.$

29. We set $u=x^2$ and use the chain rule,

$$\frac{d}{dx}\int_1^{x^2}\sqrt{t^2+1}\,dt = \left[\frac{d}{du}\int_1^u \sqrt{t^2+1}\,dt\right]\frac{du}{dx} = \sqrt{u^2+1}(2x) = 2x\sqrt{x^4+1}.$$

30. When we reverse the limits, and set $u = 2x$,

$$\frac{d}{dx}\int_{2x}^{4} t\cos t\,dt = -\frac{d}{dx}\int_{4}^{2x} t\cos t\,dt = -\left[\frac{d}{du}\int_{4}^{u} t\cos t\,dt\right]\frac{du}{dx} = -u\cos u(2) = -4x\cos 2x.$$

31. When a is any number between $2x+3$ and $1-x$, we may write

$$\int_{2x+3}^{1-x}\frac{1}{t^2+1}dt = \int_{2x+3}^{a}\frac{1}{t^2+1}dt + \int_{a}^{1-x}\frac{1}{t^2+1}dt = -\int_{a}^{2x+3}\frac{1}{t^2+1}dt + \int_{a}^{1-x}\frac{1}{t^2+1}dt.$$

In these integrals we set $u = 2x+3$ and $v = 1-x$ respectively, and use chain rules,

$$\begin{aligned}\frac{d}{dx}\int_{2x+3}^{1-x}\frac{1}{t^2+1}dt &= -\left[\frac{d}{du}\int_{a}^{u}\frac{1}{t^2+1}dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_{a}^{v}\frac{1}{t^2+1}dt\right]\frac{dv}{dx}\\ &= \frac{-1}{u^2+1}(2) + \frac{1}{v^2+1}(-1) = -\frac{2}{(2x+3)^2+1} - \frac{1}{(1-x)^2+1}.\end{aligned}$$

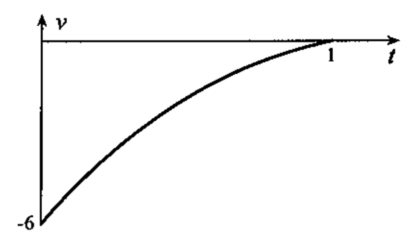
32. When a is a number between $-x^2$ and x^2,

$$\int_{-x^2}^{x^2}\sin^2 t\,dt = \int_{-x^2}^{a}\sin^2 t\,dt + \int_{a}^{x^2}\sin^2 t\,dt = -\int_{a}^{-x^2}\sin^2 t\,dt + \int_{a}^{x^2}\sin^2 t\,dt.$$

In these integrals we set $u = -x^2$ and $v = x^2$ respectively, and use chain rules,

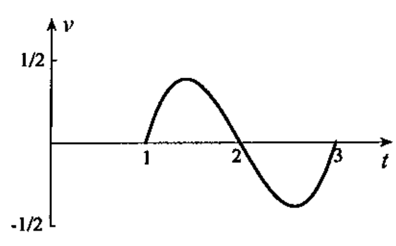
$$\begin{aligned}\frac{d}{dx}\int_{-x^2}^{x^2}\sin^2 t\,dt &= -\left[\frac{d}{du}\int_{a}^{u}\sin^2 t\,dt\right]\frac{du}{dx} + \left[\frac{d}{dv}\int_{a}^{v}\sin^2 t\,dt\right]\frac{dv}{dx}\\ &= -\sin^2 u(-2x) + \sin^2 v(2x) = 2x[\sin^2(-x^2) + \sin^2(x^2)] = 4x\sin^2 x^2.\end{aligned}$$

33.
$$\begin{aligned}\int_0^1 v(t)\,dt &= \int_0^1 (t^3 - 6t^2 + 11t - 6)\,dt\\ &= \left\{\frac{t^4}{4} - 2t^3 + \frac{11t^2}{2} - 6t\right\}_0^1\\ &= \frac{1}{4} - 2 + \frac{11}{2} - 6 = -\frac{9}{4}\end{aligned}$$



This is the displacement of the particle at $t = 1$ relative to its position at $t = 0$. Since $v(t) = (t-1)(t-2)(t-3)$ is always negative between $t = 0$ and $t = 1$, it follows that the integral of $|v(t)|$ is equal to $9/4$. This is the distance travelled by the particle between $t = 0$ and $t = 1$.

34.
$$\begin{aligned}\int_1^3 v(t)\,dt &= \int_1^3 (t^3 - 6t^2 + 11t - 6)\,dt\\ &= \left\{\frac{t^4}{4} - 2t^3 + \frac{11t^2}{2} - 6t\right\}_1^3\\ &= \left(\frac{81}{4} - 54 + \frac{99}{2} - 18\right)\\ &\quad - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right) = 0\end{aligned}$$



This means that the particle is at the same position at times $t = 1$ and $t = 3$. Since $v(t) = (t-1)(t-2)(t-3)$ changes sign at $t = 2$,

$$\begin{aligned}\int_1^3 |v(t)|\,dt &= \int_1^2 (t^3 - 6t^2 + 11t - 6)\,dt + \int_2^3 (-t^3 + 6t^2 - 11t + 6)\,dt\\ &= \left\{\frac{t^4}{4} - 2t^3 + \frac{11t^2}{2} - 6t\right\}_1^2 + \left\{-\frac{t^4}{4} + 2t^3 - \frac{11t^2}{2} + 6t\right\}_2^3\end{aligned}$$

$$= (4 - 16 + 22 - 12) - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6\right)$$
$$+ \left(-\frac{81}{4} + 54 - \frac{99}{2} + 18\right) - (-4 + 16 - 22 + 12)$$
$$= \frac{1}{2}.$$

This is the distance travelled by the particle between $t = 1$ and $t = 3$.

35. If we set $u = x^2 + 1$, then $du = 2x\,dx$, and

$$\int_0^1 \frac{x^3}{(x^2+1)^{3/2}}dx = \int_0^1 \frac{x^2}{(x^2+1)^{3/2}}x\,dx = \int_1^2 \frac{u-1}{u^{3/2}}\left(\frac{du}{2}\right) = \frac{1}{2}\int_1^2 \left(\frac{1}{\sqrt{u}} - \frac{1}{u^{3/2}}\right) du$$
$$= \frac{1}{2}\left\{2\sqrt{u} + \frac{2}{\sqrt{u}}\right\}_1^2 = \frac{1}{2}\left(2\sqrt{2} + \frac{2}{\sqrt{2}} - 2 - 2\right) = \frac{3\sqrt{2}-4}{2}.$$

36. If we set $u = 1 + \sin x$, then $du = \cos x\,dx$, and

$$\int_0^{\pi/6} \frac{\cos^3 x}{\sqrt{1+\sin x}}dx = \int_0^{\pi/6} \frac{1-\sin^2 x}{\sqrt{1+\sin x}}\cos x\,dx = \int_1^{3/2} \frac{1-(u-1)^2}{\sqrt{u}}du = \int_1^{3/2} (2\sqrt{u} - u^{3/2})\,du$$
$$= \left\{\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right\}_1^{3/2} = \left(2\sqrt{\frac{3}{2}} - \frac{9}{10}\sqrt{\frac{3}{2}}\right) - \left(\frac{4}{3} - \frac{2}{5}\right) = \frac{33\sqrt{6}-56}{60}.$$

37. Because the integrand is an odd function, its integral over the interval $-1 \le x \le 1$ must be equal to 0.

38. If we set $u = x + 3$, then $du = dx$, and

$$\int_{-1}^2 \left|\frac{x}{\sqrt{3+x}}\right| du = \int_2^5 \left|\frac{u-3}{\sqrt{u}}\right| du = \int_2^3 \frac{3-u}{\sqrt{u}}du + \int_3^5 \frac{u-3}{\sqrt{u}}du$$
$$= \left\{6\sqrt{u} - \frac{2}{3}u^{3/2}\right\}_2^3 + \left\{\frac{2}{3}u^{3/2} - 6\sqrt{u}\right\}_3^5$$
$$= (6\sqrt{3} - 2\sqrt{3}) - \left(6\sqrt{2} - \frac{4\sqrt{2}}{3}\right) + \left(\frac{10\sqrt{5}}{3} - 6\sqrt{5}\right) - (2\sqrt{3} - 6\sqrt{3})$$
$$= \frac{1}{3}(24\sqrt{3} - 14\sqrt{2} - 8\sqrt{5}).$$

39. $\displaystyle\int_{-1}^2 x^2(4-x^3)^5\,dx = \left\{\frac{(4-x^3)^6}{-18}\right\}_{-1}^2 = \frac{4^6 - 5^6}{-18} = \frac{1281}{2}$

40. If we set $u = x^3 + 2x^2 + x$, then $du = (3x^2 + 4x + 1)\,dx$, and

$$\int_1^5 \frac{6x^2+8x+2}{\sqrt{x^3+2x^2+x}}dx = \int_4^{180} \frac{1}{\sqrt{u}}\,2du = \left\{4\sqrt{u}\right\}_4^{180} = 4(\sqrt{180} - 2) = 24\sqrt{5} - 8.$$

41. $\displaystyle\int_1^2 \frac{x-25}{\sqrt{x}-5}dx = \int_1^2 (\sqrt{x}+5)\,dx = \left\{\frac{2x^{3/2}}{3} + 5x\right\}_1^2 = \left(\frac{4\sqrt{2}}{3} + 10\right) - \left(\frac{2}{3} + 5\right) = \frac{4\sqrt{2}+13}{3}$

42. $\displaystyle\int_0^1 \frac{1}{\sqrt{2+x}+\sqrt{x}}dx = \int_0^1 \frac{1}{\sqrt{2+x}+\sqrt{x}}\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}dx = \int_0^1 \frac{\sqrt{2+x}-\sqrt{x}}{2+x-x}dx$

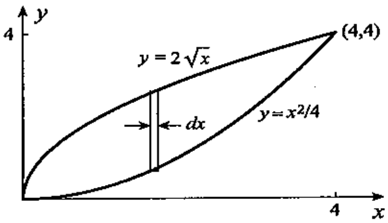
$$= \frac{1}{2}\left\{\frac{2}{3}(2+x)^{3/2} - \frac{2}{3}x^{3/2}\right\}_0^1 = \frac{1}{2}\left(2\sqrt{3} - \frac{2}{3}\right) - \frac{1}{2}\left(\frac{4\sqrt{2}}{3}\right) = \frac{1}{3}(3\sqrt{3} - 1 - 2\sqrt{2})$$

CHAPTER 7

EXERCISES 7.1

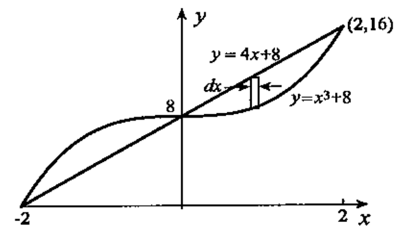
1. $A = \int_0^4 (2\sqrt{x} - x^2/4)\,dx$

$= \left\{ \frac{4x^{3/2}}{3} - \frac{x^3}{12} \right\}_0^4 = \frac{16}{3}$



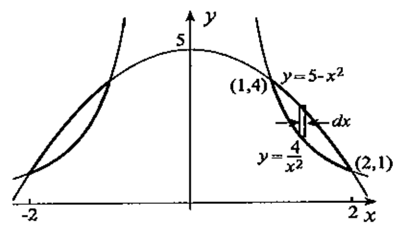
2. $A = 2\int_0^2 [(4x+8) - (x^3+8)]\,dx$

$= 2\left\{ 2x^2 - \frac{x^4}{4} \right\}_0^2 = 8$



3. $A = 2\int_1^2 \left[(5 - x^2) - \frac{4}{x^2} \right] dx$

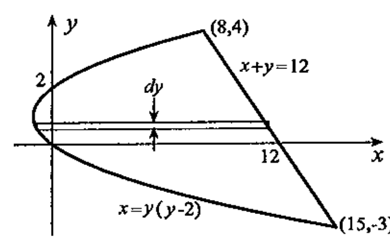
$= 2\left\{ 5x - \frac{x^3}{3} + \frac{4}{x} \right\}_1^2 = \frac{4}{3}$



4. $A = \int_{-3}^4 [(12 - y) - y(y-2)]\,dy$

$= \int_{-3}^4 (12 + y - y^2)\,dy$

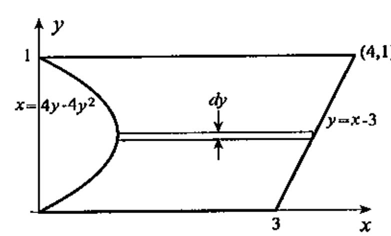
$= \left\{ 12y + \frac{y^2}{2} - \frac{y^3}{3} \right\}_{-3}^4 = \frac{343}{6}$



5. $A = \int_0^1 [(y+3) - (4y - 4y^2)]\,dy$

$= \int_0^1 (3 - 3y + 4y^2)\,dy$

$= \left\{ 3y - \frac{3y^2}{2} + \frac{4y^3}{3} \right\}_0^1 = \frac{17}{6}$

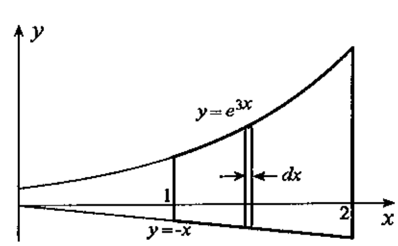


6. $A = \int_1^2 (e^{3x} + x)\,dx$

$= \left\{ \frac{e^{3x}}{3} + \frac{x^2}{2} \right\}_1^2$

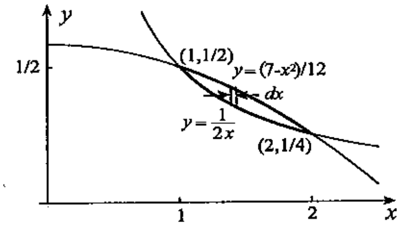
$= \left(\frac{e^6}{3} + 2 \right) - \left(\frac{e^3}{3} + \frac{1}{2} \right)$

$= \frac{1}{3}(e^6 - e^3) + \frac{3}{2}$



7. $A = \int_1^2 \left(\frac{7 - x^2}{12} - \frac{1}{2x} \right) dx$

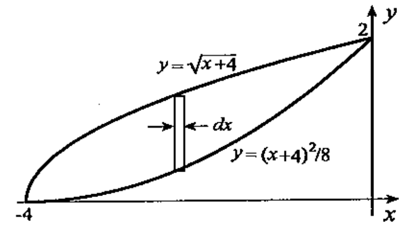
$= \left\{ \frac{7x}{12} - \frac{x^3}{36} - \frac{1}{2} \ln|x| \right\}_1^2 = \frac{7 - 9 \ln 2}{18}$



8. $A = \int_{-4}^0 \left[\sqrt{x+4} - \frac{(x+4)^2}{8} \right] dx$

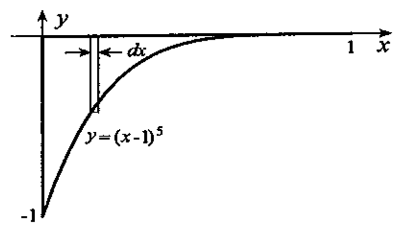
$= \left\{ \frac{2}{3}(x+4)^{3/2} - \frac{1}{24}(x+4)^3 \right\}_{-4}^0$

$= \frac{8}{3}$

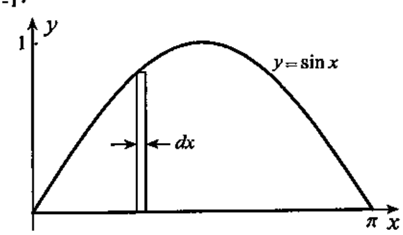


9. $A = \int_0^1 -(x-1)^5 \, dx$

$= \left\{ -\frac{(x-1)^6}{6} \right\}_0^1 = \frac{1}{6}$

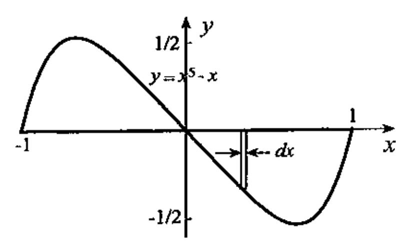


10. $A = \int_0^\pi \sin x \, dx = \{ -\cos x \}_0^\pi = 2$



11. $A = 2 \int_0^1 (-x^5 + x) \, dx$

$= 2 \left\{ -\frac{x^6}{6} + \frac{x^2}{2} \right\}_0^1 = \frac{2}{3}$

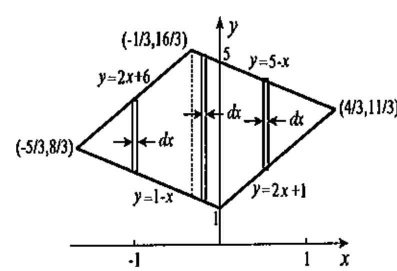


12. $A = \int_{-5/3}^{-1/3} [(2x+6) - (1-x)] \, dx + \int_{-1/3}^0 [(5-x) - (1-x)] \, dx$

$+ \int_0^{4/3} [(5-x) - (2x+1)] \, dx$

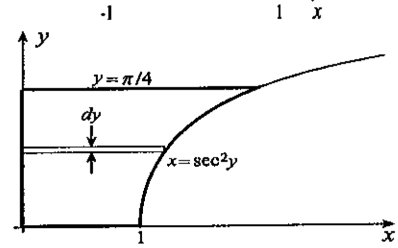
$= \int_{-5/3}^{-1/3} (3x+5) \, dx + \int_{-1/3}^0 4 \, dx + \int_0^{4/3} (4 - 3x) \, dx$

$= \left\{ \frac{3x^2}{2} + 5x \right\}_{-5/3}^{-1/3} + \{4x\}_{-1/3}^0 + \left\{ 4x - \frac{3x^2}{2} \right\}_0^{4/3} = \frac{20}{3}$



13. $A = \int_0^{\pi/4} \sec^2 y \, dy$

$= \left\{ \tan y \right\}_0^{\pi/4} = 1$

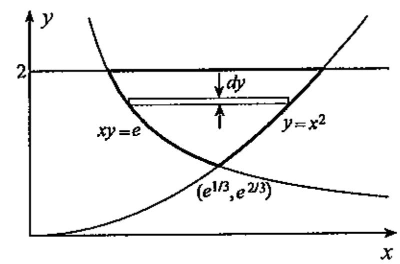


14. $A = \int_{e^{2/3}}^{2} \left(\sqrt{y} - \frac{e}{y}\right) dy$

$$= \left\{\frac{2}{3}y^{3/2} - e\ln|y|\right\}\Bigg|_{e^{2/3}}^{2}$$

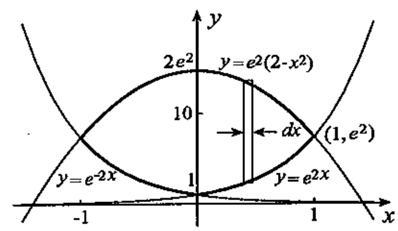
$$= \left(\frac{4\sqrt{2}}{3} - e\ln 2\right) - \left(\frac{2e}{3} - \frac{2e}{3}\right)$$

$$= \frac{4\sqrt{2}}{3} - e\ln 2$$



15. $A = 2\int_0^1 [e^2(2-x^2) - e^{2x}]\,dx$

$$= 2\left\{2e^2x - \frac{e^2x^3}{3} - \frac{e^{2x}}{2}\right\}\Bigg|_0^1 = \frac{7e^2+3}{3}$$

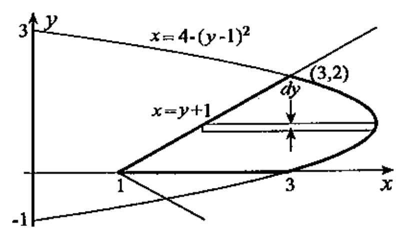


16. $A = \int_0^2 [4 - (y-1)^2 - (y+1)]\,dy$

$$= \int_0^2 [3 - y - (y-1)^2]\,dy$$

$$= \left\{3y - \frac{y^2}{2} - \frac{(y-1)^3}{3}\right\}\Bigg|_0^2$$

$$= \left(6 - 2 - \frac{1}{3}\right) - \left(\frac{1}{3}\right) = \frac{10}{3}$$



17. (a) The area is $A = \int_1^3 (16 - x^2)\,dx = \left\{16x - \frac{x^3}{3}\right\}\Bigg|_1^3 = \frac{70}{3}$.

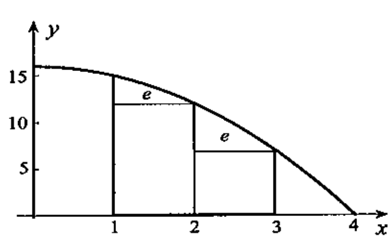
(b) $A_2 = f(2)(1) + f(3)(1) = 12 + 7 = 19$ The error is $70/3 - 19 = 13/3$. It is the areas marked with e in the left figure.

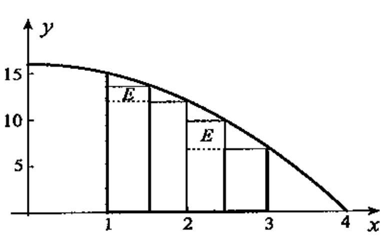
(c) $A_4 = f(3/2)(1/2) + f(2)(1/2) + f(5/2)(1/2) + f(3)(1/2) = \frac{1}{2}\left(\frac{55}{4} + 12 + \frac{39}{4} + 7\right) = \frac{85}{4}$

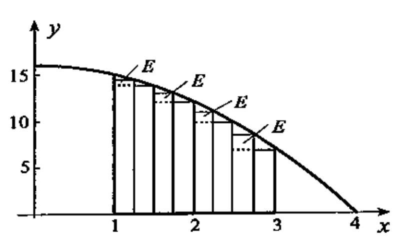
The error is $70/3 - 85/4 = 25/12$. The extra precision is the addition of the two rectangles marked with an E in the middle figure.

(d) $A_8 = \frac{1}{4}[f(5/4) + f(3/2) + f(7/4) + f(2) + f(9/4) + f(5/2) + f(11/4) + f(3)] = \frac{357}{16}$

The error is $70/3 - 357/16 = 49/48$. The extra precision is the addition of the four rectangles marked with an E in the right figure.







18. (a) The area is $A = \int_1^3 (x^3+1)\,dx = \left\{\frac{x^4}{4}+x\right\}_1^3 = 22.$

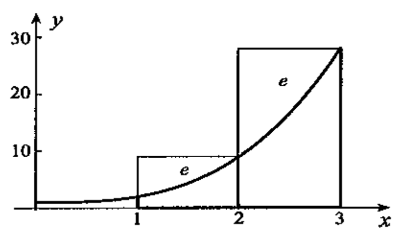
(b) $A_2 = f(2)(1) + f(3)(1) = 9 + 28 = 37$ The error is $37 - 22 = 15$. It is the areas marked with e in the left figure.

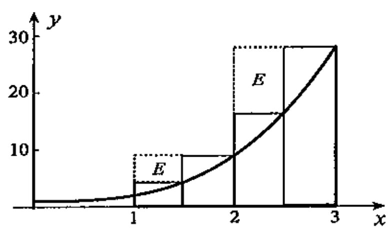
(c) $A_4 = f(3/2)(1/2) + f(2)(1/2) + f(5/2)(1/2) + f(3)(1/2) = \frac{1}{2}\left(\frac{35}{8} + 9 + \frac{133}{8} + 28\right) = 29$

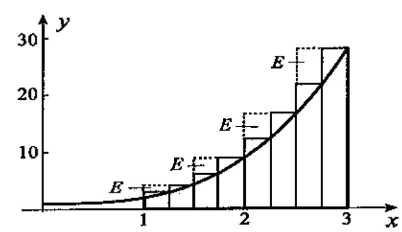
The error is $29 - 22 = 7$. The extra precision is the deletion of the two rectangles marked with an E in the middle figure.

(d) $A_8 = \frac{1}{4}[f(5/4) + f(3/2) + f(7/4) + f(2) + f(9/4) + f(5/2) + f(11/4) + f(3)] = \frac{203}{8}$

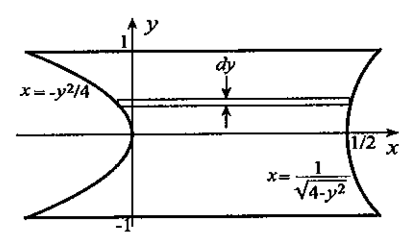
The error is $203/8 - 22 = 27/8$. The extra precision is the deletion of the four rectangles marked with an E in the right figure.



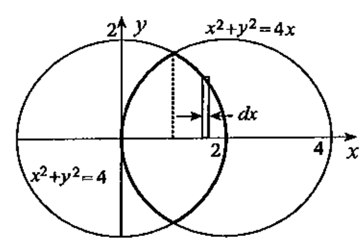




19. $A = 2\int_0^1 \left(\frac{1}{\sqrt{4-y^2}} + \frac{y^2}{4}\right) dy$

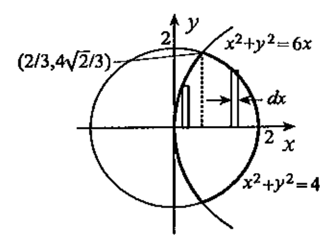


20. $A = 4\int_1^2 \sqrt{4-x^2}\,dx$

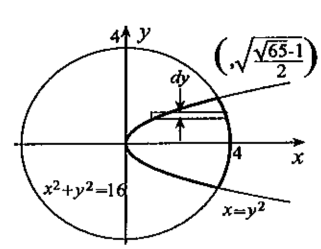


21. $A = 2\int_0^{2/3} \sqrt{6x - x^2}\,dx$

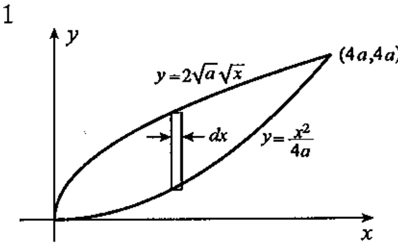
$+2\int_{2/3}^2 \sqrt{4-x^2}\,dx$



22. $A = 2\int_0^{\sqrt{(\sqrt{65}-1)/2}} (\sqrt{16-y^2} - y^2)\,dy$



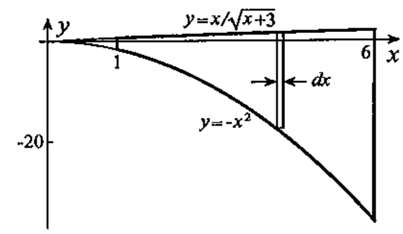
23. $$A = \int_0^{4a} \left(2\sqrt{a}\,\sqrt{x} - \frac{x^2}{4a} \right) dx$$
$$= \left\{ \frac{4\sqrt{a}\,x^{3/2}}{3} - \frac{x^3}{12a} \right\}_0^{4a} = \frac{16a^2}{3}$$



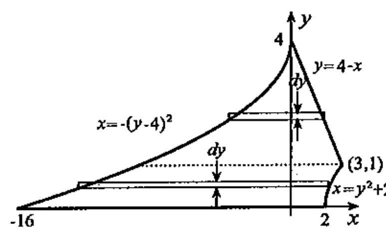
24. $$A = \int_1^6 \left(\frac{x}{\sqrt{x+3}} + x^2 \right) dx$$

If we set $u = x+3$ in the first term, then $du = dx$, and

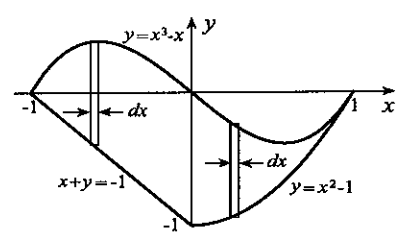
$$A = \int_4^9 \frac{u-3}{\sqrt{u}} du + \left\{ \frac{x^3}{3} \right\}_1^6$$
$$= \left\{ \frac{2}{3}u^{3/2} - 6\sqrt{u} \right\}_4^9 + \frac{215}{3} = \frac{235}{3}$$



25. $$A = \int_0^1 [(y^2+2) + (y-4)^2]\, dy$$
$$+ \int_1^4 [(4-y) + (y-4)^2]\, dy$$
$$= \left\{ \frac{y^3}{3} + 2y + \frac{1}{3}(y-4)^3 \right\}_0^1$$
$$+ \left\{ 4y - \frac{y^2}{2} + \frac{1}{3}(y-4)^3 \right\}_1^4 = \frac{169}{6}$$



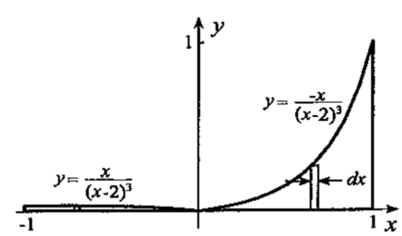
26. $$A = \int_{-1}^0 [(x^3 - x) - (-x-1)]\, dx$$
$$+ \int_0^1 [(x^3 - x) - (x^2 - 1)]\, dx$$
$$= \left\{ \frac{x^4}{4} + x \right\}_{-1}^0 + \left\{ \frac{x^4}{4} - \frac{x^2}{2} - \frac{x^3}{3} + x \right\}_0^1$$
$$= \frac{7}{6}$$



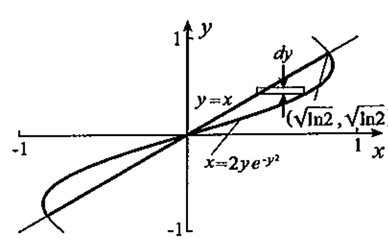
27. $$A = \int_{-1}^0 \frac{x}{(x-2)^3} dx + \int_0^1 \frac{-x}{(x-2)^3} dx$$

If we set $u = x-2$ and $du = dx$ in these integrals,

$$A = \int_{-3}^{-2} \frac{u+2}{u^3} du - \int_{-2}^{-1} \frac{u+2}{u^3} du$$
$$= \left\{ -\frac{1}{u} - \frac{1}{u^2} \right\}_{-3}^{-2} - \left\{ -\frac{1}{u} - \frac{1}{u^2} \right\}_{-2}^{-1}$$
$$= \frac{5}{18}.$$



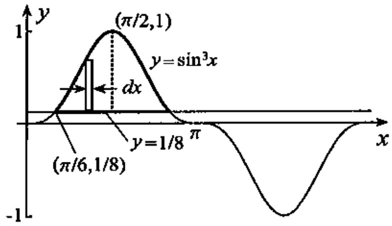
28. $$A = 2\int_0^{\sqrt{\ln 2}} (2ye^{-y^2} - y)\, dy$$
$$= 2\left\{ -e^{-y^2} - \frac{y^2}{2} \right\}_0^{\sqrt{\ln 2}}$$
$$= 2\left(-e^{-\ln 2} - \frac{\ln 2}{2} \right) - 2(-1)$$
$$= 1 - \ln 2$$



29. $A = 2\int_{\pi/6}^{\pi/2} (\sin^3 x - 1/8)\, dx$

$= 2\int_{\pi/6}^{\pi/2} [\sin x(1-\cos^2 x) - 1/8]\, dx$

$= 2\left\{-\cos x + \dfrac{1}{3}\cos^3 x - \dfrac{x}{8}\right\}\Bigg|_{\pi/6}^{\pi/2} = \dfrac{9\sqrt{3}-\pi}{12}$

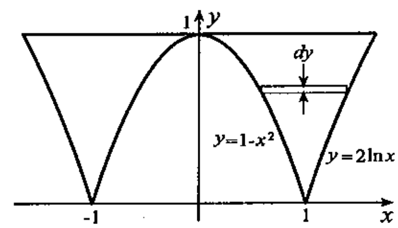


30. $A = 2\int_0^1 (e^{y/2} - \sqrt{1-y})\, dy$

$= 2\left\{2e^{y/2} + \dfrac{2}{3}(1-y)^{3/2}\right\}\Bigg|_0^1$

$= 2(2\sqrt{e}) - 2\left(2 + \dfrac{2}{3}\right)$

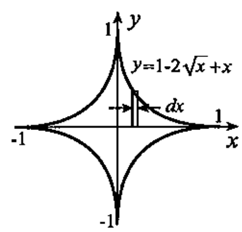
$= 4\sqrt{e} - \dfrac{16}{3}$



31. $A = 4\int_0^1 (1 - 2\sqrt{x} + x)\, dx$

$= 4\left\{x - \dfrac{4x^{3/2}}{3} + \dfrac{x^2}{2}\right\}\Bigg|_0^1$

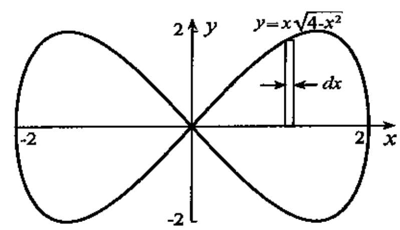
$= \dfrac{2}{3}$



32. $A = 4\int_0^2 x\sqrt{4-x^2}\, dx$

$= 4\left\{-\dfrac{1}{3}(4-x^2)^{3/2}\right\}\Bigg|_0^2$

$= \dfrac{4}{3}(4)^{3/2} = \dfrac{32}{3}$



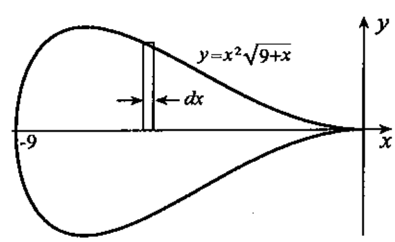
33. $A = 2\int_{-9}^{0} x^2\sqrt{9+x}\, dx$

If we set $u = 9 + x$ and $du = dx$, then

$A = 2\int_0^9 (u-9)^2\sqrt{u}\, du$

$= 2\int_0^9 (u^{5/2} - 18u^{3/2} + 81\sqrt{u})\, du$

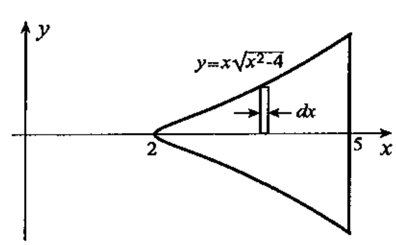
$= 2\left\{\dfrac{2u^{7/2}}{7} - \dfrac{36u^{5/2}}{5} + \dfrac{162u^{3/2}}{3}\right\}\Bigg|_0^9 = \dfrac{23\,328}{35}.$



34. $A = 2\int_2^5 x\sqrt{x^2-4}\, dx$

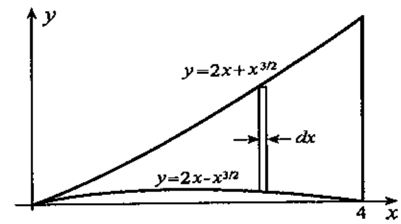
$= 2\left\{\dfrac{1}{3}(x^2-4)^{3/2}\right\}\Bigg|_2^5$

$= \dfrac{2}{3}(21^{3/2}) = 14\sqrt{21}$



35. $A = \int_0^4 [(2x + x^{3/2}) - (2x - x^{3/2})]\, dx$

$= 2\int_0^4 x^{3/2}\, dx = 2\left\{\frac{2x^{5/2}}{5}\right\}_0^4 = \frac{128}{5}$



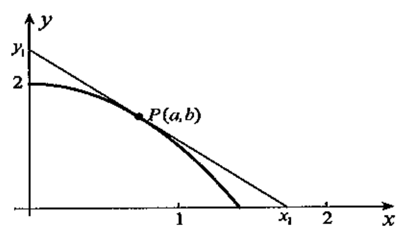
36. The equation of the tangent line at any point $P(a,b)$ on the first quadrant part of the parabola is $y - b = -2a(x-a)$ (left figure below). The x- and y-intercepts of this line are $x_1 = a + b/(2a)$, and $y_1 = b + 2a^2$. The area of the triangle is $A = \frac{1}{2}x_1y_1 = \frac{1}{2}\left(a + \frac{b}{2a}\right)(b + 2a^2)$. Since $b = 2 - a^2$, we can express A in the form

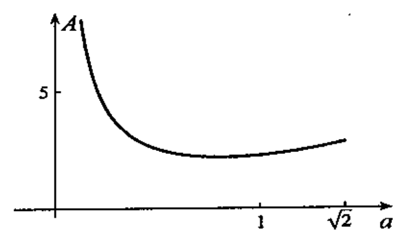
$$A = \frac{1}{2}\left(a + \frac{2-a^2}{2a}\right)(2 - a^2 + 2a^2) = \frac{1}{4a}(2+a^2)^2, \quad 0 < a \le \sqrt{2}.$$

The plot of this function in the right figure indicates that its minimum occurs at the critical point. To find it we solve

$$0 = \frac{dA}{da} = \frac{1}{4}\left[\frac{a(2)(2+a^2)(2a) - (2+a^2)^2}{a^2}\right] = \frac{(2+a^2)(3a^2-2)}{4a^2}.$$

The only positive solution is $a = \sqrt{2/3}$. Hence the required point is $(\sqrt{2/3}, 4/3)$.





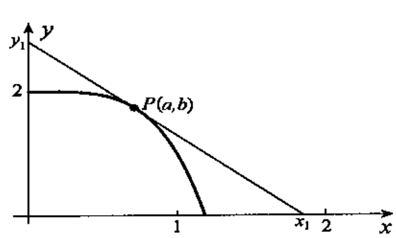
37. The equation of the tangent line at any point $P(a,b)$ on the first quadrant part of the curve is $y - b = -4a^3(x-a)$ (left figure below). The x- and y-intercepts of this line are $x_1 = a + b/(4a^3)$, and $y_1 = b + 4a^4$. The area of the triangle is $A = \frac{1}{2}x_1y_1 = \frac{1}{2}\left(a + \frac{b}{4a^3}\right)(b + 4a^4)$. Since $b = 2 - a^4$, we can express A in the form

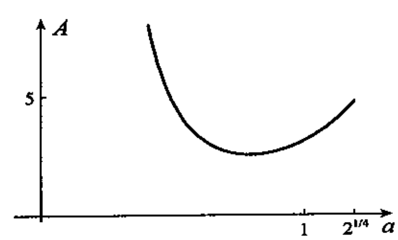
$$A = \frac{1}{2}\left(a + \frac{2-a^4}{4a^3}\right)(2 - a^4 + 4a^4) = \frac{1}{8a^3}(2+3a^4)^2, \quad 0 < a \le 2^{1/4}.$$

The plot of this function in the right figure indicates that its minimum occurs at the critical point. To find it we solve

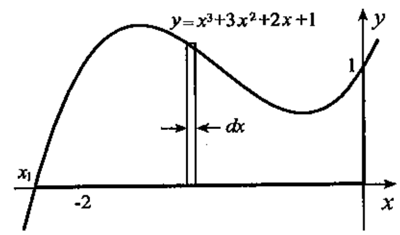
$$0 = \frac{dA}{da} = \frac{1}{8}\left[\frac{a^3(24a^3)(2+3a^4) - 3a^2(2+3a^4)^2}{a^6}\right] = \frac{3(2+3a^4)(5a^4-2)}{8a^4}.$$

The only positive solution is $a = (2/5)^{1/4}$. Hence the required point is $((2/5)^{1/4}, 8/5)$.





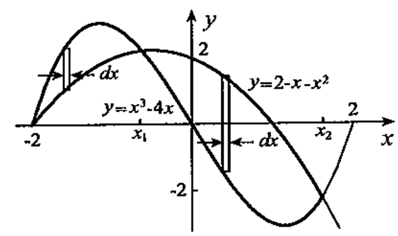
38. Newton's iterative procedure gives the x-intercept of the cubic as $x_1 = -2.324\,718$. Hence,



$$A = \int_{x_1}^{0} (x^3 + 3x^2 + 2x + 1)\,dx$$
$$= \left\{\frac{x^4}{4} + x^3 + x^2 + x\right\}_{x_1}^{0}$$
$$= 2.182.$$

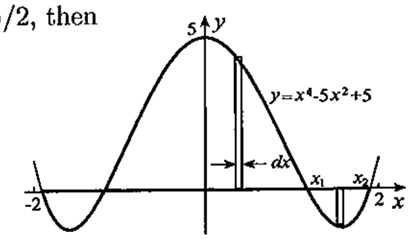
39. The x-coordinates of points of intersection of the curves are defined by the equation $x^3 + x^2 - 3x - 2 = 0$. Newton's iterative procedure gives $x_1 = -0.618\,034$ and $x_2 = 1.618\,034$ as solutions of this equation. Hence,

$$A = \int_{-2}^{x_1} [(x^3 - 4x) - (2 - x - x^2)]\,dx$$
$$+ \int_{x_1}^{x_2} [(2 - x - x^2) - (x^3 - 4x)]\,dx$$
$$= \left\{\frac{x^4}{4} + \frac{x^3}{3} - \frac{3x^2}{2} - 2x\right\}_{-2}^{x_1}$$
$$+ \left\{-\frac{x^4}{4} - \frac{x^3}{3} + \frac{3x^2}{2} + 2x\right\}_{x_1}^{x_2} = 5.946.$$



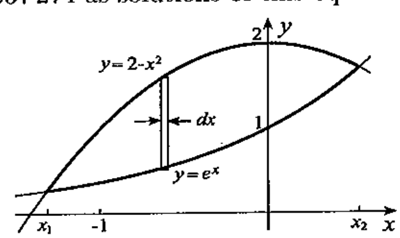
40. Since $x^4 - 5x^2 + 5$ is a quadratic in x^2, x-intercepts can be found exactly; they are $\pm\sqrt{(5 + \sqrt{5})/2}$ and $\pm\sqrt{(5 - \sqrt{5})/2}$. If we set $x_1 = \sqrt{(5 - \sqrt{5})/2}$, and $x_2 = \sqrt{(5 + \sqrt{5})/2}$, then

$$A = 2\int_{0}^{x_1} (x^4 - 5x^2 + 5)\,dx$$
$$+2\int_{x_1}^{x_2} (-x^4 + 5x^2 - 5)\,dx$$
$$= 2\left\{\frac{x^5}{5} - \frac{5x^3}{3} + 5x\right\}_{0}^{x_1} + 2\left\{-\frac{x^5}{5} + \frac{5x^3}{3} - 5x\right\}_{x_1}^{x_2} = 8.436.$$



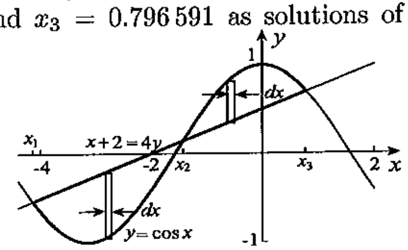
41. The x-coordinates of points of intersection of the curves are defined by the equation $e^x = 2 - x^2$. Newton's method gives $x_1 = -1.315\,974$, and $x_2 = 0.537\,274$ as solutions of this equation. Thus,

$$A = \int_{x_1}^{x_2} (2 - x^2 - e^x)\,dx$$
$$= \left\{2x - \frac{x^3}{3} - e^x\right\}_{x_1}^{x_2}$$
$$= 1.452.$$



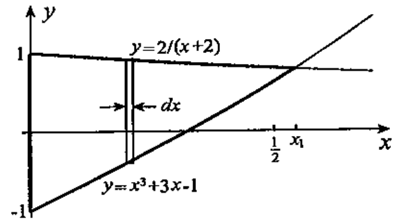
42. The x-coordinates of points of intersection of the curves are defined by the equation $\cos x = (x + 2)/4$. Newton's method gives $x_1 = -4.146\,081$, $x_2 = -1.427\,069$, and $x_3 = 0.796\,591$ as solutions of this equation. Thus,

$$A = \int_{x_1}^{x_2} \left[\frac{1}{4}(x + 2) - \cos x\right] dx + \int_{x_2}^{x_3} \left[\cos x - \frac{1}{4}(x + 2)\right] dx$$
$$= \left\{\frac{1}{8}(x + 2)^2 - \sin x\right\}_{x_1}^{x_2} + \left\{\sin x - \frac{1}{8}(x + 2)^2\right\}_{x_2}^{x_3}$$
$$= 2.067.$$



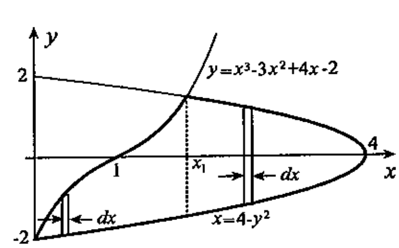
43. The x-coordinate of the point of intersection of the curves is defined by the equation $2/(x+2) = x^3 + 3x - 1 \Longrightarrow x^4 + 2x^3 + 3x^2 + 5x - 4 = 0$. Newton's method gives $x_1 = 0.542\,373$ as the solution of this equation. Thus,

$$A = \int_0^{x_1} \left(\frac{2}{x+2} - x^3 - 3x + 1 \right) dx$$
$$= \left\{ 2 \ln|x+2| - \frac{x^4}{4} - \frac{3x^2}{2} + x \right\}_0^{x_1}$$
$$= 0.559.$$



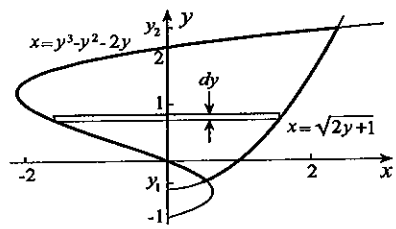
44. Points of intersection of the curves have y-coordinates satisfying $y = (4-y^2)^3 - 3(4-y^2)^2 + 4(4-y^2) - 2$, and this equation simplifies to $y^6 - 9y^4 + 28y^2 + y - 30 = 0$. The negative solution is $y = -2$ and the positive solution can be obtained by Newton's method as $y = 1.466\,078$. The equation of the parabola gives the corresponding x-coordinate as $x_1 = 1.850\,616$. The area is

$$A = \int_0^{x_1} (x^3 - 3x^2 + 4x - 2 + \sqrt{4-x})\, dx$$
$$+2 \int_{x_1}^4 \sqrt{4-x}\, dx$$
$$= \left\{ \frac{x^4}{4} - x^3 + 2x^2 - 2x - \frac{2}{3}(4-x)^{3/2} \right\}_0^{x_1}$$
$$+2 \left\{ -\frac{2}{3}(4-x)^{3/2} \right\}_{x_1}^4 = 7.177.$$



45. Points of intersection of the curves have y-coordinates satisfying $y^3 - y^2 - 2y = \sqrt{2y+1}$, or, $(y^3 - y^2 - 2y)^2 = 2y + 1$. The solutions can be obtained by Newton's method as $y_1 = -0.354\,740$ and $y_2 = 2.310\,040$. The area is

$$A = \int_{y_1}^{y_2} [\sqrt{2y+1} - (y^3 - y^2 - 2y)]\, dy$$
$$= \left\{ \frac{1}{3}(2y+1)^{3/2} - \frac{y^4}{4} + \frac{y^3}{3} + y^2 \right\}_{y_1}^{y_2} = 6.608.$$



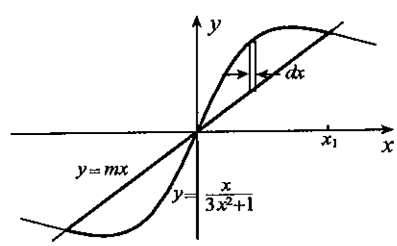
46. For points of intersection of the curves, we solve

$$mx = \frac{x}{3x^2+1}$$

This equation simplifies to $3mx^3 + mx = x$, one solution of which is $x = 0$. Other solutions must satisfy $3mx^2 + m - 1 = 0$, from which

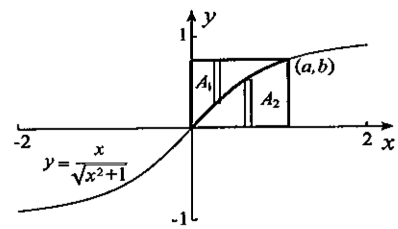
$$x = \pm\sqrt{\frac{1-m}{3m}}.$$

Consequently, an area is defined when $0 < m < 1$. If we set $x_1 = \sqrt{(1-m)/(3m)}$, the required area is



$$A = 2\int_0^{x_1} \left(\frac{x}{3x^2+1} - mx \right) dx = 2\left\{ \frac{1}{6}\ln(3x^2+1) - \frac{mx^2}{2} \right\}_0^{x_1} = \frac{1}{3}\ln(3x_1^2+1) - mx_1^2$$
$$= \frac{1}{3}\ln\left[3\left(\frac{1-m}{3m}\right)+1\right] - m\left(\frac{1-m}{3m}\right) = \frac{1}{3}\ln\left[\frac{1-m+m}{m}\right] + \frac{m-1}{3} = -\frac{1}{3}\ln m + \frac{m-1}{3}.$$

47. For $A_2 = 2A_1$,



$$\int_0^a \frac{x}{\sqrt{x^2+1}}dx = 2\int_0^a \left(b - \frac{x}{\sqrt{x^2+1}}\right) dx$$
$$\Longrightarrow$$
$$\left\{\sqrt{x^2+1}\right\}_0^a = 2\left\{bx - \sqrt{x^2+1}\right\}_0^a$$
$$\Longrightarrow$$
$$\sqrt{a^2+1} - 1 = 2(ab - \sqrt{a^2+1} + 1).$$

Since (a,b) is on the curve, $b = a/\sqrt{a^2+1}$, and the above equation can be expressed in the form

$$3\sqrt{a^2+1} = \frac{2a^2}{\sqrt{a^2+1}} + 3 \quad\Longrightarrow\quad 3(a^2+1) = 2a^2 + 3\sqrt{a^2+1} \quad\Longrightarrow\quad a^2 + 3 = 3\sqrt{a^2+1}.$$

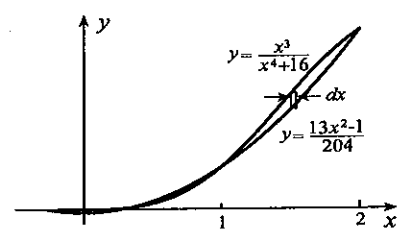
When this is squared, it simplifies to $a^4 - 3a^2 = 0$ with solutions $a = 0, \pm\sqrt{3}$. Thus, there are three points $(0,0)$ and $(\pm\sqrt{3}, \pm\sqrt{3}/2)$.

48. If x_{i-1} and x_i are the ends of the i^{th} rectangle, then $\Delta x_i = 2/2^n = 1/2^{n-1}$, and $x_i = 1 + i/2^{n-1}$. It follows that

$$A_{2^n} = \sum_{i=1}^{2^n} f(x_i)\Delta x_i = \sum_{i=1}^{2^n}\left[16 - \left(1 + \frac{i}{2^{n-1}}\right)^2\right]\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^{n-1}}\sum_{i=1}^{2^n}\left[15 - \frac{i}{2^{n-2}} - \frac{i^2}{2^{2n-2}}\right]$$
$$= \frac{1}{2^{n-1}}\left[15(2^n) - \frac{1}{2^{n-2}}\frac{2^n(2^n+1)}{2} - \frac{1}{2^{2n-2}}\frac{2^n(2^n+1)(2^{n+1}+1)}{6}\right] \quad \text{(see equations 6.3,6.4)}$$
$$= 30 - \left(4 + \frac{1}{2^{n-2}}\right) - \frac{1}{6}\left(1 + \frac{1}{2^n}\right)\left(16 + \frac{1}{2^{n-3}}\right) = \frac{70}{3} - \frac{1}{2^{n-2}} - \frac{1}{6}\left(\frac{3}{2^{n-3}} + \frac{1}{2^{2n-3}}\right).$$

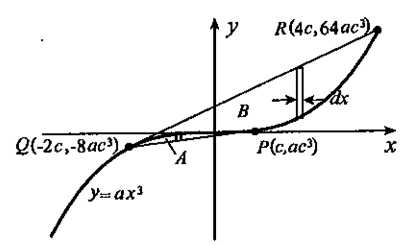
Since $A = 70/3$, we may write $A_{2^n} = A - \left[\frac{1}{2^{n-2}} + \frac{1}{6}\left(\frac{3}{2^{n-3}} + \frac{1}{2^{2n-3}}\right)\right]$, and clearly, $\lim_{n\to\infty} A_{2^n} = A$.

49. The figure indicates that there are indeed three areas bounded by the curves. The curves intersect at $x = -0.249, 0.340, 1, 2$. The largest of the three areas is



$$A = \int_1^2 \left(\frac{x^3}{x^4+16} - \frac{13x^2-1}{204}\right) dx$$
$$= \left\{\frac{1}{4}\ln|x^4+16| - \frac{13x^3}{612} + \frac{x}{204}\right\}_1^2 = 0.014.$$

50. If the coordinates of P are (c, ac^3), then the tangent line at P has equation



$$y - ac^3 = 3ac^2(x - c),$$

and the x-coordinate of Q is defined by

$$ax^3 = ac^3 + 3ac^2(x - c).$$

The solution of this equation is $x = -2c$. The equation of the tangent line at Q is $y + 8ac^3 = 12ac^2(x + 2c)$, and the intersection of this line with $y = ax^3$ is the point $R(4c, 64ac^3)$. Now,

$$A = \int_{-2c}^{c} [ax^3 - ac^3 - 3ac^2(x - c)]\,dx = \left\{\frac{ax^4}{4} - \frac{3ac^2x^2}{2} + 2ac^3x\right\}_{-2c}^{c} = \frac{27}{4}ac^4,$$

and

$$B = \int_{-2c}^{4c} [-8ac^3 + 12ac^2(x + 2c) - ax^3]\,dx = \left\{16ac^3x + 6ac^2x^2 - \frac{ax^4}{4}\right\}_{-2c}^{4c} = 108ac^4.$$

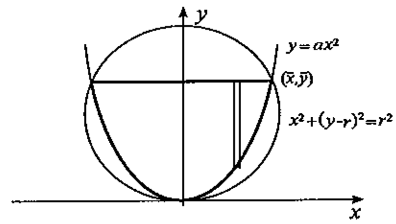
Thus, $B = 16A$.

51. To find the points of intersection of the curves, we solve

$$\frac{y}{a}+(y-r)^2=r^2 \implies \frac{y}{a}+y^2-2ry+r^2=r^2.$$

Solutions are $y=0$ and $y=2r-1/a$. The x-coordinate for the second of these is $x=\sqrt{2r/a-1/a^2}$. Let us denote the point of intersection by

$$(\overline{x},\overline{y})=\left(\sqrt{\frac{2r}{a}-\frac{1}{a^2}},2r-\frac{1}{a}\right).$$



The area inside the parabola and below the line $y=\overline{y}$ is

$$\begin{aligned}A&=2\int_0^{\overline{x}}(\overline{y}-ax^2)\,dx=2\left\{\overline{y}x-\frac{ax^3}{3}\right\}_0^{\overline{x}}=2\left(\overline{x}\overline{y}-\frac{a\overline{x}^3}{3}\right)=2\overline{x}\left(a\overline{x}^2-\frac{a\overline{x}^2}{3}\right)\\&=\frac{4a\overline{x}^3}{3}=\frac{4a}{3}\left(\frac{2r}{a}-\frac{1}{a^2}\right)^{3/2}=\frac{4}{3a^2}(2ar-1)^{3/2}.\end{aligned}$$

The domain of this function $A(a)$ is $\frac{1}{2r}\le a<\infty$. For critical points we solve

$$\begin{aligned}0=\frac{dA}{da}&=\frac{4}{3}\left[-\frac{2}{a^3}(2ar-1)^{3/2}+\frac{1}{a^2}\left(\frac{3}{2}\right)(2ar-1)^{1/2}(2r)\right]\\&=\frac{4\sqrt{2ar-1}}{3a^3}[-2(2ar-1)+3ar]=\frac{4\sqrt{2ar-1}(2-ar)}{3a^3}.\end{aligned}$$

Solutions are $a=2/r$ and $a=1/(2r)$. Since

$$A\left(\frac{1}{2r}\right)=0,\qquad A\left(\frac{2}{r}\right)>0,\qquad \lim_{a\to\infty}A(a)=0,$$

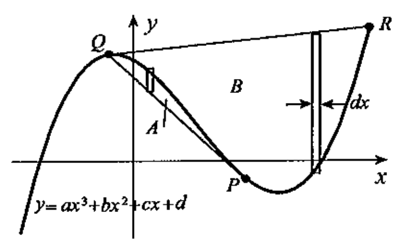
it follows that area is maximized when $a=2/r$.

52. If the coordinates of P are (e,ae^3+be^2+ce+d), the tangent line at P has equation

$$y-(ae^3+be^2+ce+d)=(3ae^2+2be+c)(x-e).$$

The intersection point Q of this line with the cubic has x-coordinate $x=-(b+2ae)/a$. The equation of the tangent line at Q is

$$\begin{aligned}y-&\left(-\frac{2b^2e}{a}-8be^2-8ae^3-\frac{bc}{a}-2ce+d\right)\\&=\left(\frac{b^2}{a}+8be+12ae^2+c\right)\left(x+\frac{b}{a}+2e\right).\end{aligned}$$



and the intersection of this line with the cubic has x-coordinate $x=(b+4ae)/a$. Now,

$$\begin{aligned}A&=\int_{-(b+2ae)/a}^{e}[(ax^3+bx^2+cx+d)-(ae^3+be^2+ce+d)-(3ae^2+2be+c)(x-e)]\,dx\\&=\left\{\frac{ax^4}{4}+\frac{bx^3}{3}-(ae^3+be^2)x-\frac{(3ae^2+2be)(x-e)^2}{2}\right\}_{-(b+2ae)/2}^{e}\\&=\frac{27ae^4}{4}+9be^3+\frac{b^4}{12a^3}+\frac{b^3e}{a^2}+\frac{9b^2e^2}{2a}.\end{aligned}$$

$$B=\int_{-(b+2ae)/a}^{(b+4ae)/a}\Big[-\frac{2b^2e}{a}-8be^2-8ae^3-\frac{bc}{a}-2ce+d$$

$$+\left(\frac{b^2}{a}+8be+12ae^2+c\right)\left(x+\frac{b}{a}+2e\right)-(ax^3+bx^2+cx+d)\Big]\,dx$$

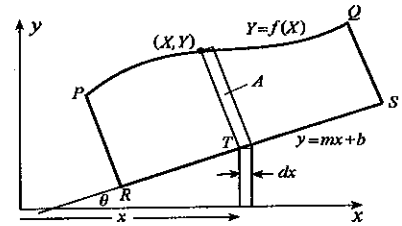
$$=\int_{-(b+2ae)/a}^{(b+4ae)/a}\left[20be^2+16ae^3+\frac{b^3}{a^2}+\frac{8b^2e}{a}+x\left(\frac{b^2}{a}+8be+12ae^2\right)-ax^3-bx^2\right]dx$$

$$=\left\{\left(20be^2+16ae^3+\frac{b^3}{a^2}+\frac{8b^2e}{a}\right)x+\left(\frac{b^2}{a}+8be+12ae^2\right)\frac{x^2}{2}-\frac{ax^4}{4}-\frac{bx^3}{3}\right\}_{-(b+2ae)/a}^{(b+4ae)/a}$$

$$=108ae^4+144be^3+\frac{4b^4}{3a^3}+\frac{16b^3e}{a^2}+\frac{72b^2e^2}{a}=16A.$$

53. The width of the rectangle A shown is $\sec\theta\,dx=\sqrt{1+\tan^2\theta}\,dx=\sqrt{1+m^2}\,dx$. Using formula 1.16, the length of the rectangle is the distance from (X,Y) to $y=mx+b$,

$$\frac{|Y-mX-b|}{\sqrt{1+m^2}}=\frac{|f(X)-mX-b|}{\sqrt{1+m^2}}.$$



Hence the required area is

$$\text{Area}=\int_{x_R}^{x_S}\frac{|f(X)-mX-b|}{\sqrt{1+m^2}}\sqrt{1+m^2}\,dx$$

$$=\int_{x_R}^{x_S}|f(X)-mX-b|\,dx.$$

We now need to relate x and X. Since the slope of the line joining $(X,Y)=(X,f(X))$ and T is $-1/m$,

$$\frac{Y-mx-b}{X-x}=-\frac{1}{m}\quad\Longrightarrow\quad mf(X)-m^2x-bm=x-X\quad\Longrightarrow\quad x=\frac{1}{1+m^2}[X+mf(X)-bm].$$

Instead of trying to solve this for X in terms of x, we treat it as a change of variables in the area integral. Then $dx=\dfrac{1}{1+m^2}[1+mf'(X)]\,dX$. Since $X=x_P$ when $x=x_R$ and $X=x_Q$ when $x=x_S$,

$$\text{Area}=\int_{x_P}^{x_Q}|f(X)-mX-b|\frac{1}{1+m^2}[1+mf'(X)]\,dX.$$

We now replace X with x, and note that $f(X)>mX+b$,

$$\text{Area}=\frac{1}{1+m^2}\int_{x_P}^{x_Q}[f(x)-mx-b][1+mf'(x)]\,dx.$$

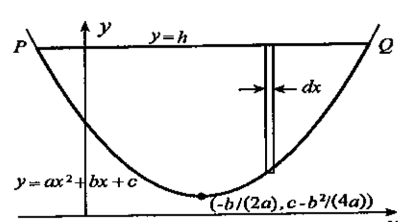
54. If x_P and x_Q are x-coordinates of P and Q, the area is

$$A=\int_{x_P}^{x_Q}(h-ax^2-bx-c)\,dx$$

$$=\left\{(h-c)x-\frac{ax^3}{3}-\frac{bx^2}{2}\right\}_{x_P}^{x_Q}$$

$$=(h-c)(x_Q-x_P)-\frac{a}{3}(x_Q^3-x_P^3)-\frac{b}{2}(x_Q^2-x_P^2)$$

$$=(x_Q-x_P)\left[(h-c)-\frac{a}{3}(x_Q^2+x_Qx_P+x_P^2)-\frac{b}{2}(x_Q+x_P)\right].$$



To find the x-coordinates of P and Q we solve

$$h=ax^2+bx+c\quad\Longrightarrow\quad ax^2+bx+(c-h)=0\quad\Longrightarrow\quad x=\frac{-b\pm\sqrt{b^2-4a(c-h)}}{2a}.$$

Thus, $x_P=\dfrac{-b-\sqrt{b^2-4a(c-h)}}{2a}$ and $x_Q=\dfrac{-b+\sqrt{b^2-4a(c-h)}}{2a}$. Hence,

$$x_P^2 + x_P x_Q + x_Q^2 = \frac{1}{4a^2}[b^2 + 2b\sqrt{b^2 - 4a(c-h)} + b^2 - 4a(c-h) + b^2 - b^2 + 4a(c-h) + b^2 - 2b\sqrt{b^2 - 4a(c-h)} + b^2 - 4a(c-h)]$$
$$= \frac{1}{a^2}[b^2 - a(c-h)],$$

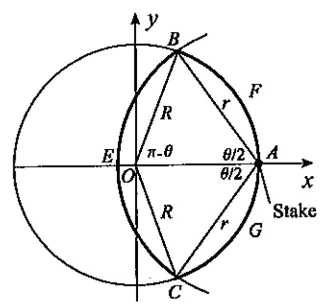
and

$$A = (x_Q - x_P)\left\{(h-c) - \frac{1}{3a}[b^2 - a(c-h)] - \frac{b}{2}\left(-\frac{b}{a}\right)\right\}$$
$$= (x_Q - x_P)\left[(h-c) - \frac{b^2}{3a} - \frac{1}{3}(h-c) + \frac{b^2}{2a}\right]$$
$$= (x_Q - x_P)\left[\frac{2}{3}(h-c) + \frac{b^2}{6a}\right] = \frac{2}{3}(x_Q - x_P)\left(h - c + \frac{b^2}{4a}\right),$$

where $h - c + b^2/(4a)$ is the distance from the vertex to the line $y = h$.

55. If the length of the rope is r, then the cow can graze on half the pasture if

$$\frac{1}{2}\pi R^2 = \text{Area } ABF + \text{Area } ACG + \text{Area } ABEC = 2(\text{Area } ABF) + \text{Area } ABEC.$$



Using the formula $R^2(\theta - \sin\theta)/2$ for the area of a sector of a circle subtended by angle θ,

$$\frac{1}{2}\pi R^2 = 2(R^2/2)[\pi - \theta - \sin(\pi - \theta)] + \frac{1}{2}r^2\theta,$$

and this equation can be written in the form

$$r^2\theta = R^2(2\theta + 2\sin\theta - \pi).$$

But from the sine law applied to triangle ABO, $\dfrac{\sin(\pi - \theta)}{r} = \dfrac{\sin(\theta/2)}{R}$, from which

$$r = \frac{R\sin\theta}{\sin(\theta/2)} = \frac{2R\sin(\theta/2)\cos(\theta/2)}{\sin(\theta/2)} = 2R\cos(\theta/2).$$

Thus, $4R^2\,\theta\cos^2(\theta/2) = R^2(2\theta + 2\sin\theta - \pi)$, or,

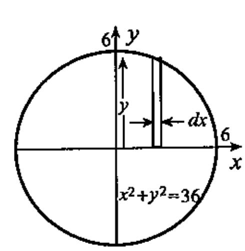
$$\pi = 2\theta + 2\sin\theta - 4\theta\left(\frac{1 + \cos\theta}{2}\right) = 2(\sin\theta - \theta\cos\theta).$$

Newton's interative procedure can be used to solve this equation for $\theta = 1.906$ radians, and therefore $r = 2R\cos(1.906/2) = 1.158R$.

EXERCISES 7.2

1. $V = 2\int_0^6 \pi y^2\,dx = 2\pi\int_0^6 (36 - x^2)\,dx$

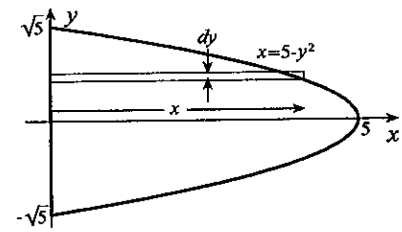
$$= 2\pi\left\{36x - \frac{x^3}{3}\right\}_0^6$$
$$= 288\pi$$



2. $V = 2\int_0^{\sqrt{5}} \pi x^2\,dy = 2\pi \int_0^{\sqrt{5}} (5-y^2)^2\,dy$

$= 2\pi \int_0^{\sqrt{5}} (25 - 10y^2 + y^4)\,dy$

$= 2\pi \left\{ 25y - \frac{10y^3}{3} + \frac{y^5}{5} \right\}_0^{\sqrt{5}} = \frac{80\sqrt{5}\pi}{3}$

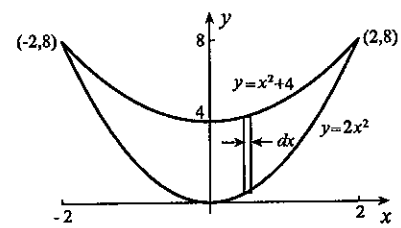


3. $V = 2\int_0^2 [\pi(x^2+4)^2 - \pi(2x^2)^2]\,dx$

$= 2\pi \int_0^2 (16 + 8x^2 - 3x^4)\,dx$

$= 2\pi \left\{ 16x + \frac{8x^3}{3} - \frac{3x^5}{5} \right\}_0^2$

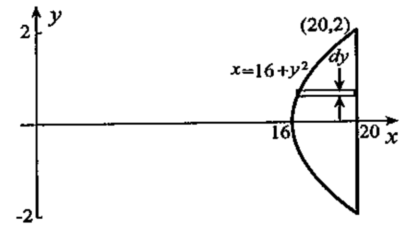
$= \frac{1024\pi}{15}$



4. $V = 2\int_0^2 [\pi(20)^2 - \pi(16+y^2)^2]\,dy$

$= 2\pi \int_0^2 (144 - 32y^2 - y^4)\,dy$

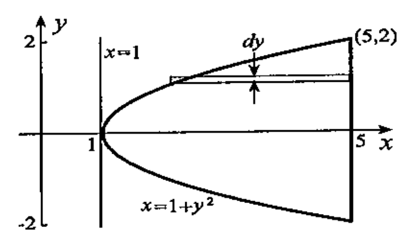
$= 2\pi \left\{ 144y - \frac{32y^3}{3} - \frac{y^5}{5} \right\}_0^2 = \frac{5888\pi}{15}$



5. $V = 2\int_0^2 [\pi(5-1)^2 - \pi(1+y^2-1)^2]\,dy$

$= 2\pi \int_0^2 (16 - y^4)\,dy$

$= 2\pi \left\{ 16y - \frac{y^5}{5} \right\}_0^2 = \frac{256\pi}{5}$

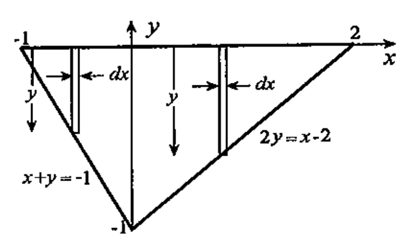


6. $V = \int_{-1}^0 \pi(-y)^2\,dx + \int_0^2 \pi(-y)^2\,dx$

$= \pi \int_{-1}^0 (1+x)^2\,dx + \pi \int_0^2 \frac{1}{4}(2-x)^2\,dx$

$= \pi \left\{ \frac{1}{3}(1+x)^3 \right\}_{-1}^0 + \pi \left\{ -\frac{1}{12}(2-x)^3 \right\}_0^2$

$= \pi$

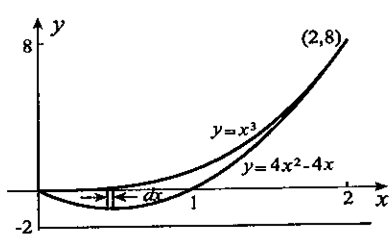


7. $V = \int_0^2 [\pi(x^3+2)^2 - \pi(4x^2-4x+2)^2]\,dx$

$= \pi \int_0^2 (x^6 - 16x^4 + 36x^3 - 32x^2 + 16x)\,dx$

$= \pi \left\{ \frac{x^7}{7} - \frac{16x^5}{5} + 9x^4 - \frac{32x^3}{3} + 8x^2 \right\}_0^2$

$= \frac{688\pi}{105}$

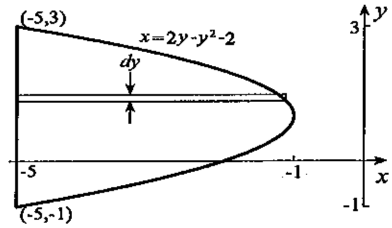


8. $V = \int_{-1}^{3} [\pi(5)^2 - \pi(-2y + y^2 + 2)^2]\, dy$

$$= \pi \int_{-1}^{3} (21 + 8y - 8y^2 + 4y^3 - y^4)\, dy$$

$$= \pi \left\{ 21y + 4y^2 - \frac{8y^3}{3} + y^4 - \frac{y^5}{5} \right\}_{-1}^{3}$$

$$= \frac{1088\pi}{15}$$

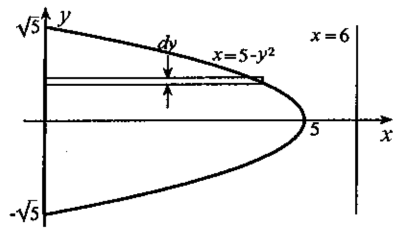


9. $V = 2 \int_{0}^{\sqrt{5}} [\pi(6)^2 - \pi(6 - 5 + y^2)^2]\, dy$

$$= 2\pi \int_{0}^{\sqrt{5}} (35 - 2y^2 - y^4)\, dy$$

$$= 2\pi \left\{ 35y - \frac{2y^3}{3} - \frac{y^5}{5} \right\}_{0}^{\sqrt{5}}$$

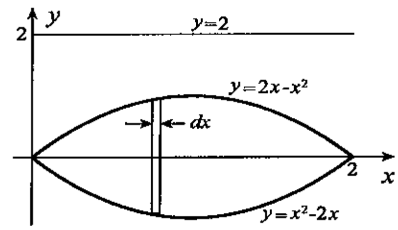
$$= \frac{160\sqrt{5}\pi}{3}$$



10. $V = \int_{0}^{2} [\pi(2 - x^2 + 2x)^2 - \pi(2 - 2x + x^2)^2]\, dx$

$$= 8\pi \int_{0}^{2} (2x - x^2)\, dx$$

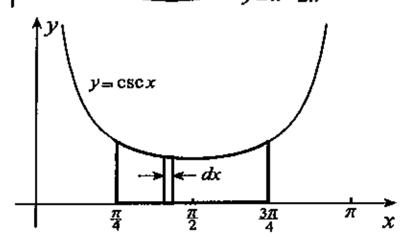
$$= 8\pi \left\{ x^2 - \frac{x^3}{3} \right\}_{0}^{2} = \frac{32\pi}{3}$$



11. $V = 2 \int_{\pi/4}^{\pi/2} \pi \csc^2 x\, dx$

$$= 2\pi \left\{ -\cot x \right\}_{\pi/4}^{\pi/2}$$

$$= 2\pi$$



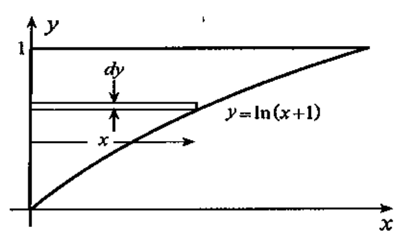
12. $V = \int_{0}^{1} \pi x^2\, dy = \pi \int_{0}^{1} (e^y - 1)^2\, dy$

$$= \pi \int_{0}^{1} (e^{2y} - 2e^y + 1)\, dy$$

$$= \pi \left\{ \frac{e^{2y}}{2} - 2e^y + y \right\}_{0}^{1}$$

$$= \pi \left(\frac{e^2}{2} - 2e + 1 - \frac{1}{2} + 2 \right)$$

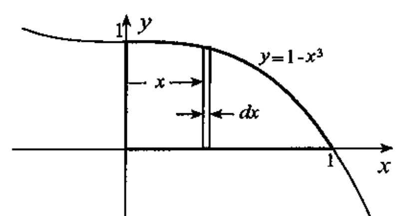
$$= \frac{\pi}{2}(e^2 - 4e + 5)$$



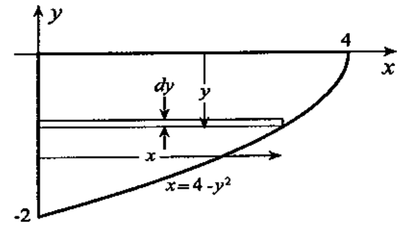
13. $V = \int_{0}^{1} 2\pi x(1 - x^3)\, dx$

$$= 2\pi \left\{ \frac{x^2}{2} - \frac{x^5}{5} \right\}_{0}^{1}$$

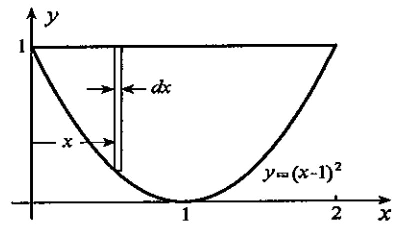
$$= \frac{3\pi}{5}$$



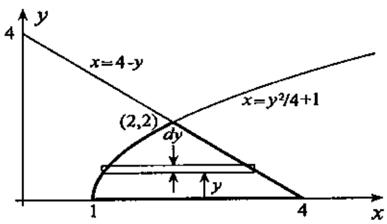
14. $$V = \int_{-2}^{0} 2\pi(-y)x\,dy$$
$$= -2\pi \int_{-2}^{0} y(4-y^2)\,dy$$
$$= -2\pi \left\{2y^2 - \frac{y^4}{4}\right\}_{-2}^{0} = 8\pi$$



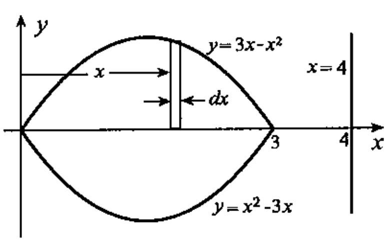
15. $$V = \int_0^2 2\pi x[1-(x-1)^2]\,dx$$
$$= 2\pi \int_0^2 (2x^2 - x^3)\,dx$$
$$= 2\pi \left\{\frac{2x^3}{3} - \frac{x^4}{4}\right\}_0^2$$
$$= \frac{8\pi}{3}$$



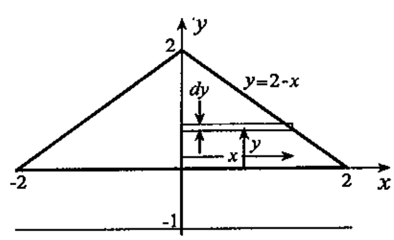
16. $$V = \int_0^2 2\pi y\left(4 - y - \frac{y^2}{4} - 1\right)dy$$
$$= \frac{\pi}{2}\int_0^2 (-y^3 - 4y^2 + 12y)\,dy$$
$$= \frac{\pi}{2}\left\{-\frac{y^4}{4} - \frac{4y^3}{3} + 6y^2\right\}_0^2 = \frac{14\pi}{3}$$



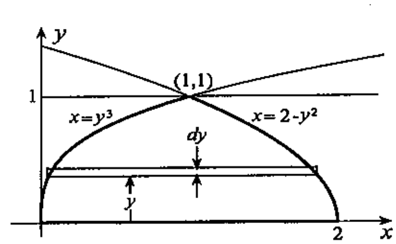
17. $$V = 2\int_0^3 2\pi(4-x)(3x-x^2)\,dx$$
$$= 4\pi\int_0^3 (12x - 7x^2 + x^3)\,dx$$
$$= 4\pi\left\{6x^2 - \frac{7x^3}{3} + \frac{x^4}{4}\right\}_0^3$$
$$= 45\pi$$



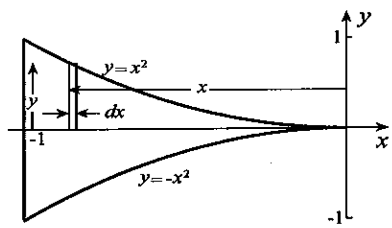
18. $$V = 2\int_0^2 2\pi(y+1)x\,dy$$
$$= 4\pi\int_0^2 (y+1)(2-y)\,dy$$
$$= 4\pi\int_0^2 (2+y-y^2)\,dy$$
$$= 4\pi\left\{2y + \frac{y^2}{2} - \frac{y^3}{3}\right\}_0^2 = \frac{40\pi}{3}$$



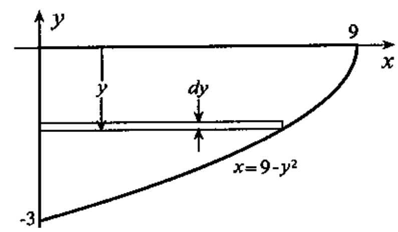
19. $$V = \int_0^1 2\pi(1-y)(2-y^2-y^3)\,dy$$
$$= 2\pi\int_0^1 (2 - 2y - y^2 + y^4)\,dy$$
$$= 2\pi\left\{2y - y^2 - \frac{y^3}{3} + \frac{y^5}{5}\right\}_0^1$$
$$= \frac{26\pi}{15}$$



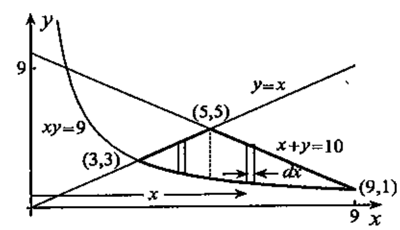
20. $$V = 2\int_{-1}^{0} 2\pi(x+1)y\,dx$$
$$= 4\pi\int_{-1}^{0}(x+1)x^2\,dx$$
$$= 4\pi\int_{-1}^{0}(x^3+x^2)\,dx$$
$$= 4\pi\left\{\frac{x^4}{4}+\frac{x^3}{3}\right\}_{-1}^{0} = \frac{\pi}{3}$$



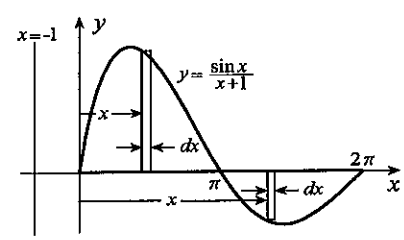
21. $$V = \int_{-3}^{0} 2\pi(-y)(9-y^2)\,dy$$
$$= 2\pi\int_{-3}^{0}(y^3-9y)\,dy$$
$$= 2\pi\left\{\frac{y^4}{4}-\frac{9y^2}{2}\right\}_{-3}^{0}$$
$$= \frac{81\pi}{2}$$



22. $$V = \int_{3}^{5} 2\pi x\left(x-\frac{9}{x}\right)dx + \int_{5}^{9} 2\pi x\left(10-x-\frac{9}{x}\right)dx$$
$$= 2\pi\left\{\frac{x^3}{3}-9x\right\}_{3}^{5} + 2\pi\left\{5x^2-\frac{x^3}{3}-9x\right\}_{5}^{9}$$
$$= \frac{344\pi}{3}$$



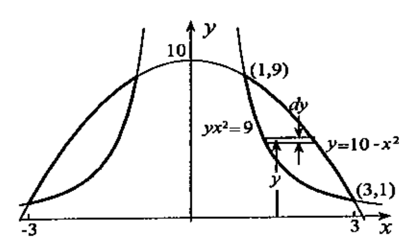
23. $$V = \int_{0}^{\pi} 2\pi(x+1)\left(\frac{\sin x}{x+1}\right)dx + \int_{\pi}^{2\pi} 2\pi(x+1)\left(-\frac{\sin x}{x+1}\right)dx$$
$$= 2\pi\Big\{-\cos x\Big\}_{0}^{\pi} + 2\pi\Big\{\cos x\Big\}_{\pi}^{2\pi}$$
$$= 8\pi$$



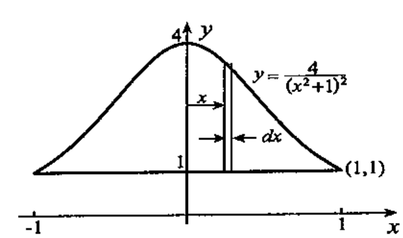
24. $$V = 2\int_{1}^{9} 2\pi y\left(\sqrt{10-y}-\frac{3}{\sqrt{y}}\right)dy$$

If we set $u = 10-y$ and $du = -dy$ in the first term,

$$V = 4\pi\int_{9}^{1}(10-u)\sqrt{u}(-du) + 4\pi\left\{-2y^{3/2}\right\}_{1}^{9}$$
$$= 4\pi\left\{\frac{2}{5}u^{5/2}-\frac{20}{3}u^{3/2}\right\}_{9}^{1} - 8\pi(27-1)$$
$$= \frac{1472\pi}{15}.$$



25. $$V = \int_{0}^{1} 2\pi x\left[\frac{4}{(x^2+1)^2}-1\right]dx$$
$$= 2\pi\left\{\frac{-2}{x^2+1}-\frac{x^2}{2}\right\}_{0}^{1}$$
$$= \pi$$

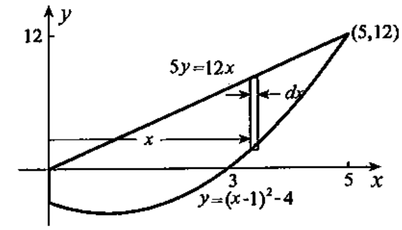


26. $V = \int_0^5 2\pi x \left[\frac{12x}{5} - (x-1)^2 + 4\right] dx$

$= \frac{2\pi}{5} \int_0^5 (-5x^3 + 22x^2 + 15x)\, dx$

$= \frac{2\pi}{5} \left\{-\frac{5x^4}{4} + \frac{22x^3}{3} + \frac{15x^2}{2}\right\}_0^5$

$= \frac{775\pi}{6}$



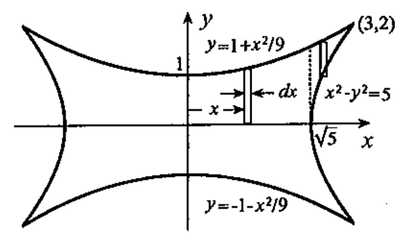
27. $V = 2\int_0^{\sqrt{5}} 2\pi x \left(1 + \frac{x^2}{9}\right) dx$

$+2\int_{\sqrt{5}}^3 2\pi x \left(1 + \frac{x^2}{9} - \sqrt{x^2-5}\right) dx$

$= 4\pi \left\{\frac{x^2}{2} + \frac{x^4}{36}\right\}_0^{\sqrt{5}}$

$+4\pi \left\{\frac{x^2}{2} + \frac{x^4}{36} - \frac{1}{3}(x^2-5)^{3/2}\right\}_{\sqrt{5}}^3$

$= \frac{49\pi}{3}$



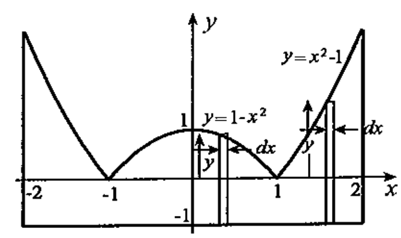
28. $V = 2\int_0^1 \pi(y+1)^2\, dx + 2\int_1^2 \pi(y+1)^2\, dx$

$= 2\pi \int_0^1 (1 - x^2 + 1)^2\, dx + 2\pi \int_1^2 (x^2 - 1 + 1)^2\, dx$

$= 2\pi \int_0^1 (4 - 4x^2 + x^4)\, dx + 2\pi \int_1^2 x^4\, dx$

$= 2\pi \left\{4x - \frac{4x^3}{3} + \frac{x^5}{5}\right\}_0^1 + 2\pi \left\{\frac{x^5}{5}\right\}_1^2$

$= \frac{272\pi}{15}$

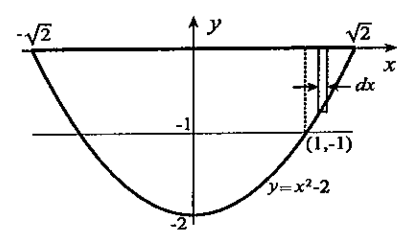


29. To eliminate duplications, we consider only the area above $y = -1$.

$V = 2\pi(1)^2(1) + 2\int_1^{\sqrt{2}} [\pi(1)^2 - \pi(x^2 - 2 + 1)^2]\, dx$

$= 2\pi + 2\pi \int_1^{\sqrt{2}} (2x^2 - x^4)\, dx$

$= 2\pi + 2\pi \left\{\frac{2x^3}{3} - \frac{x^5}{5}\right\}_1^{\sqrt{2}} = \frac{16\pi(\sqrt{2}+1)}{15}$

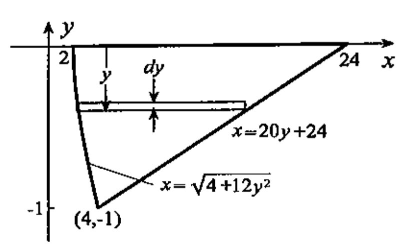


30. $V = \int_{-1}^0 2\pi(-y)(20y + 24 - \sqrt{4 + 12y^2})\, dy$

$= -2\pi \int_{-1}^0 \left(20y^2 + 24y - y\sqrt{4+12y^2}\right)\, dy$

$= -2\pi \left\{\frac{20y^3}{3} + 12y^2 - \frac{1}{36}(4 + 12y^2)^{3/2}\right\}_{-1}^0$

$= \frac{68\pi}{9}$



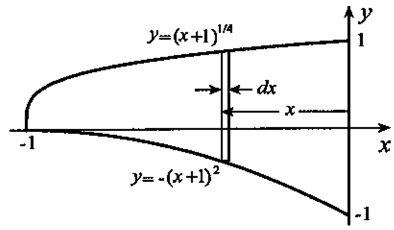
31. $V = \int_{-1}^{0} 2\pi(-x)[(x+1)^{1/4} + (x+1)^2]\,dx$

If we set $u = x+1$ and $du = dx$ in the term involving $(x+1)^{1/4}$,

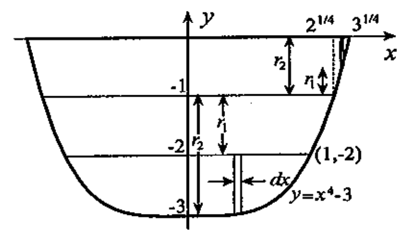
$$V = -2\pi \int_0^1 (u-1)u^{1/4}\,du - 2\pi \int_{-1}^{0} (x^3 + 2x^2 + x)\,dx$$

$$= -2\pi \left\{ \frac{4u^{9/4}}{9} - \frac{4u^{5/4}}{5} \right\}_0^1 - 2\pi \left\{ \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} \right\}_{-1}^{0}$$

$$= \frac{79\pi}{90}$$



32. To eliminate duplications we reject that part of the area between $y = -2$ and $y = -1$. We rotate that part of the area to the right of the y-axis and double the result. The rectangle $0 \le x \le 2^{1/4}$, $-1 \le y \le 0$ gives a cylinder, and the other two parts of the area require integrations,



$$V = 2[\pi(1)^2(2^{1/4})] + 2\int_{2^{1/4}}^{3^{1/4}} (\pi r_2^2 - \pi r_1^2)\,dx + 2\int_0^1 (\pi r_2^2 - \pi r_1^2)\,dx$$

$$= 2^{5/4}\pi + 2\pi \int_{2^{1/4}}^{3^{1/4}} [(1)^2 - (x^4 - 3 + 1)^2]\,dx + 2\pi \int_0^1 [(-1 - x^4 + 3)^2 - 1^2]\,dx$$

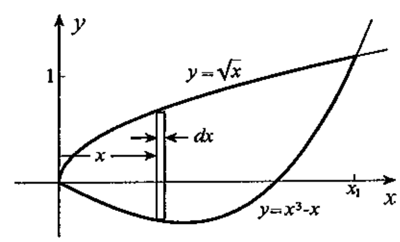
$$= 2^{5/4}\pi + 2\pi \int_{2^{1/4}}^{3^{1/4}} (-3 + 4x^4 - x^8)\,dx + 2\pi \int_0^1 (3 - 4x^4 + x^8)\,dx$$

$$= 2^{5/4}\pi + 2\pi \left\{ -3x + \frac{4x^5}{5} - \frac{x^9}{9} \right\}_{2^{1/4}}^{3^{1/4}} + 2\pi \left\{ 3x - \frac{4x^5}{5} + \frac{x^9}{9} \right\}_0^1 = \frac{16\pi}{45}(13 + 2^{17/4} - 3^{9/4}).$$

33. Newton's method can be used to solve the equation $x^3 - x = \sqrt{x}$ for the x-coordinate of the point of intersection of the curves. The result is $x_1 = 1.362\,599$. The volume of the solid of revolution is

$$V = \int_0^{x_1} 2\pi x(\sqrt{x} - x^3 + x)\,dx$$

$$= 2\pi \left\{ \frac{2x^{5/2}}{5} - \frac{x^5}{5} + \frac{x^3}{3} \right\}_0^{x_1} = 4.843.$$



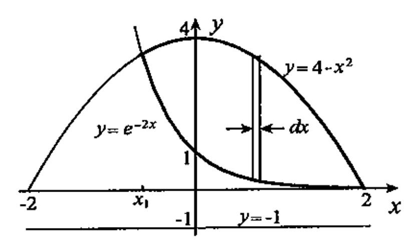
34. Newton's method can be used to solve the equation $e^{-2x} + x^2 - 4 = 0$ for x-coordinates of the points of intersection of the two curves. The results are $x_1 = -0.639\,263$ and $x_2 = 1.995\,373$. The volume of the solid of revolution is

$$V = \int_{x_1}^{x_2} [\pi(4 - x^2 + 1)^2 - \pi(e^{-2x} + 1)^2]\,dx$$

$$= \pi \int_{x_1}^{x_2} (24 - 10x^2 + x^4 - e^{-4x} - 2e^{-2x})\,dx$$

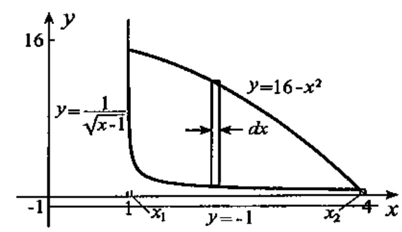
$$= \pi \left\{ 24x - \frac{10x^3}{3} + \frac{x^5}{5} + \frac{e^{-4x}}{4} + e^{-2x} \right\}_{x_1}^{x_2}$$

$$= 111.303.$$



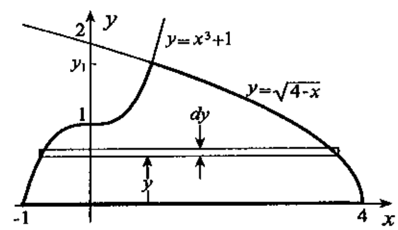
35. Newton's method can be used to solve the equation $16 - x^2 = 1/\sqrt{x-1}$ for x-coordinates of the points of intersection of the two curves. The results are $x_1 = 1.004\,450$ and $x_2 = 3.926\,248$. The volume of the solid of revolution is

$$\begin{aligned} V & \int_{x_1}^{x_2} \left[\pi(16 - x^2 + 1)^2 - \pi\left(\frac{1}{\sqrt{x-1}} + 1\right)^2\right] dx \\ &= \pi \int_{x_1}^{x_2} \left(288 - 34x^2 + x^4 - \frac{1}{x-1} - \frac{2}{\sqrt{x-1}}\right) dx \\ &= \pi \left\{288x - \frac{34x^3}{3} + \frac{x^5}{5} - \ln|x-1| - 4\sqrt{x-1}\right\}_{x_1}^{x_2} \\ &= 1069.241. \end{aligned}$$



36. The y-coordinate of the point of intersection of the curves can be obtained by solving the equation $y = (4 - y^2)^3 + 1$. Newton's iterative procedure leads to the solution $y_1 = 1.757\,401\,58$. The volume of the solid of revolution is

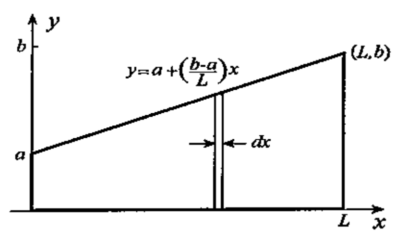
$$\begin{aligned} V &= \int_0^{y_1} 2\pi y[(4 - y^2) - (y-1)^{1/3}]\, dy \\ &= 2\pi \int_0^{y_1} [4y - y^3 - y(y-1)^{1/3}]\, dy. \end{aligned}$$



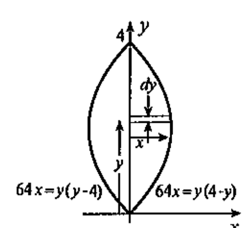
We set $u = y - 1$ and $du = dy$ in the last term,

$$\begin{aligned} V &= 2\pi \left\{2y^2 - \frac{y^4}{4}\right\}_0^{y_1} - 2\pi \int_{-1}^{y_1 - 1} (u+1)u^{1/3}\, du = 2\pi \left\{2y^2 - \frac{y^4}{4}\right\}_0^{y_1} - 2\pi \left\{\frac{3}{7}u^{7/3} + \frac{3}{4}u^{4/3}\right\}_{-1}^{y_1 - 1} \\ &= 2\pi \left\{2y^2 - \frac{y^4}{4} - \frac{3}{7}(y-1)^{7/3} - \frac{3}{4}(y-1)^{4/3}\right\}_0^{y_1} = 21.186. \end{aligned}$$

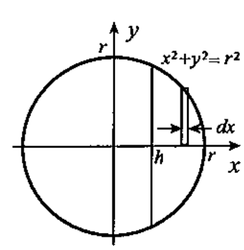
37.
$$\begin{aligned} V &= \int_0^L \pi \left[a + \left(\frac{b-a}{L}\right)x\right]^2 dx \\ &= \pi \left\{\frac{L}{3(b-a)}\left[a + \left(\frac{b-a}{L}\right)x\right]^3\right\}_0^L \\ &= \frac{\pi L(a^2 + ab + b^2)}{3} \end{aligned}$$



38.
$$\begin{aligned} V &= 2\int_0^4 2\pi y x\, dy = 4\pi \int_0^4 y\left[\frac{y(4-y)}{64}\right] dy \\ &= \frac{\pi}{16}\left\{\frac{4y^3}{3} - \frac{y^4}{4}\right\}_0^4 = \frac{4\pi}{3} \end{aligned}$$



39. (a)
$$\begin{aligned} V &= \int_h^r \pi(r^2 - x^2)\, dx \\ &= \pi \left\{r^2 x - \frac{x^3}{3}\right\}_h^r \\ &= \pi \left(r^3 - \frac{r^3}{3} - r^2 h + \frac{h^3}{3}\right) \\ &= \frac{\pi}{3}(r - h)^2(2r + h) \end{aligned}$$



(b) For the volume in part (a) to be one-third that of the sphere,

$$\frac{\pi}{3}(r-h)^2(2r+h) = \frac{1}{3}\left(\frac{4}{3}\pi r^3\right) \implies 3h^3 - 9r^2h + 2r^3 = 0 \implies 3\left(\frac{h}{r}\right)^3 - 9\left(\frac{h}{r}\right) + 2 = 0.$$

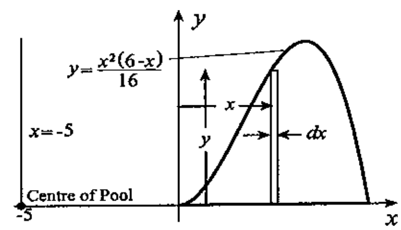
The ratio $z = h/r$ must satisfy the cubic $3z^3 - 9z + 2 = 0$. Newton's iterative procedure yields the solution $z = 0.2261$.

40. (a) For the points $(0,0)$, $(4,2)$, and $(6,0)$ to be on the cubic, a, b, c, and d must satisfy

$$0 = d, \quad 2 = 64a + 16b + 4c + d, \quad 0 = 216a + 36b + 6c + d.$$

In addition, because $y'(4) = 0$, we must have $0 = 48a + 8b + c$. These four equations imply that $a = -1/16$, $b = 3/8$, $c = d = 0$, and therefore $y = -x^3/16 + 3x^2/8 = x^2(6-x)/16$.
(b) The amount of fill required is

$$\begin{aligned} V &= \int_0^6 2\pi(x+5)y\,dx \\ &= 2\pi\int_0^6 (x+5)\frac{x^2(6-x)}{16}dx \\ &= \frac{\pi}{8}\int_0^6 (30x^2 + x^3 - x^4)\,dx \\ &= \frac{\pi}{8}\left\{10x^3 + \frac{x^4}{4} - \frac{x^5}{5}\right\}_0^6 = \frac{1161\pi}{10}\ \text{m}^3. \end{aligned}$$

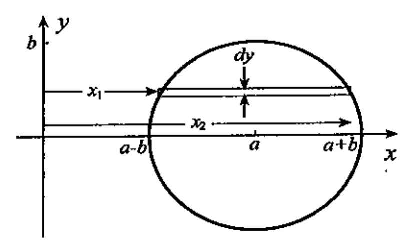


41. With horizontal rectangles, washers give

$$V = 2\int_0^b (\pi x_2^2 - \pi x_1^2)\,dy.$$

By solving the equation $(x-a)^2 + y^2 = b^2$ for x, we obtain $x_2 = a + \sqrt{b^2 - y^2}$ and $x_1 = a - \sqrt{b^2 - y^2}$. Thus,

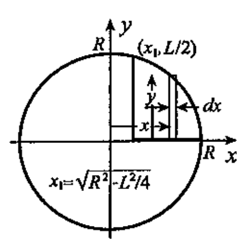
$$\begin{aligned} V &= 2\pi\int_0^b [(a + \sqrt{b^2-y^2})^2 - (a - \sqrt{b^2-y^2})^2]\,dy \\ &= 2\pi\int_0^b 4a\sqrt{b^2-y^2}\,dy = 8\pi a\int_0^b \sqrt{b^2-y^2}\,dy \end{aligned}$$



This integral, less the constant out front, is the area of one-quarter of the circle. It follows then that $V = 8\pi a(1/4)\pi(b^2) = 2\pi^2 ab^2$.

42. The volume remaining is

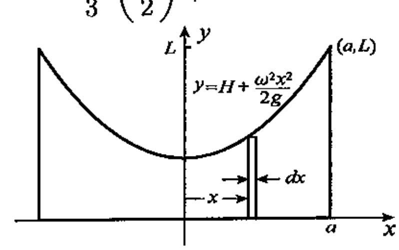
$$\begin{aligned} V &= 2\int_{x_1}^R 2\pi xy\,dx = 4\pi\int_{x_1}^R x\sqrt{R^2-x^2}\,dx \\ &= 4\pi\left\{-\frac{1}{3}(R^2-x^2)^{3/2}\right\}_{x_1}^R \\ &= \frac{4\pi}{3}(R^2 - x_1^2)^{3/2}. \end{aligned}$$



Since $x_1 = \sqrt{R^2 - L^2/4}$, the volume is $V = \dfrac{4\pi}{3}\left[R^2 - R^2 + \dfrac{L^2}{4}\right]^{3/2} = \dfrac{4\pi}{3}\left(\dfrac{L}{2}\right)^3$, and this is the volume of a sphere with radius $L/2$.

43. The volume occupied by the water as it reaches the top is

$$\begin{aligned} V &= \int_0^a 2\pi x\left(H + \frac{\omega^2x^2}{2g}\right)dx = 2\pi\left\{\frac{Hx^2}{2} + \frac{\omega^2x^4}{8g}\right\}_0^a \\ &= \pi a^2\left(H + \frac{\omega^2a^2}{4g}\right). \end{aligned}$$



Because the volume of water in the pail does not change, and it was half full originally,

$$\pi a^2\left(\frac{L}{2}\right)=\pi a^2\left(H+\frac{\omega^2a^2}{4g}\right)\quad\Longrightarrow\quad L=2H+\frac{\omega^2a^2}{2g}.$$

When the water just reaches the top, the point (a,L) is on the parabola, and therefore

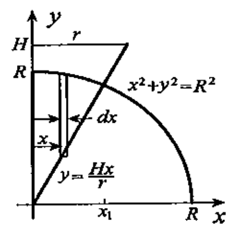
$$L=H+\frac{\omega^2a^2}{2g}\quad\Longrightarrow\quad H=L-\frac{\omega^2a^2}{2g}.$$

Thus, $L=2\left(L-\dfrac{\omega^2a^2}{2g}\right)+\dfrac{\omega^2a^2}{2g}$, and when this equation is solved for ω, the result is $\omega=\sqrt{2gL}/a$.

44. The volume can be obtained by rotating the area bounded by the circle $x^2+y^2=R^2$, the line $y=Hx/r$, and the y-axis about the y-axis. The x-coordinate of the point of intersection of the line and circle, call it x_1, must satisfy

$$x^2+\frac{H^2x^2}{r^2}=R^2.$$

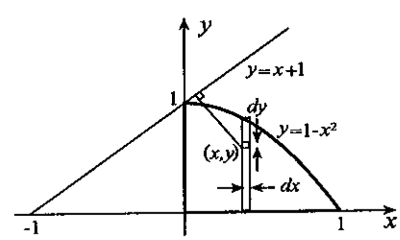
The solution of this equation is $x_1=rR/\sqrt{r^2+H^2}$.



The required volume is

$$\begin{aligned}V&=\int_0^{x_1}2\pi x\left(\sqrt{R^2-x^2}-\frac{Hx}{r}\right)dx=2\pi\left\{-\frac{1}{3}(R^2-x^2)^{3/2}-\frac{H}{r}\frac{x^3}{3}\right\}_0^{x_1}\\&=-\frac{2\pi}{3}\left[(R^2-x_1^2)^{3/2}+\frac{Hx_1^3}{r}-R^3\right]=-\frac{2\pi}{3}\left[\left(R^2-\frac{r^2R^2}{r^2+H^2}\right)^{3/2}+\frac{H}{r}\frac{r^3R^3}{(r^2+H^2)^{3/2}}-R^3\right]\\&=-\frac{2\pi}{3}\left[\frac{R^3H^3}{(r^2+H^2)^{3/2}}+\frac{Hr^2R^3}{(r^2+H^2)^{3/2}}-R^3\right]=\frac{2\pi R^3}{3}\left[1-\frac{H(r^2+H^2)}{(r^2+H^2)^{3/2}}\right]\\&=\frac{2\pi R^3}{3}\left[1-\frac{H}{\sqrt{r^2+H^2}}\right].\end{aligned}$$

45. If we divide the vertical rectangle of width dx into smaller rectangles of width dy, the perpendicular distance from this small rectangle to the line $y=x+1$ is given by distance formula 1.16 as $|x-y+1|/\sqrt{2}=(x-y+1)/\sqrt{2}$. When the small rectangle is rotated around $y=x+1$, it produces a ring with volume approximately equal to $2\pi[(x-y+1)/\sqrt{2}]\,dx\,dy$. By adding over all the small rectangles in the vertical rectangle, we obtain the volume resulting from rotating the vertical rectangle around the line as

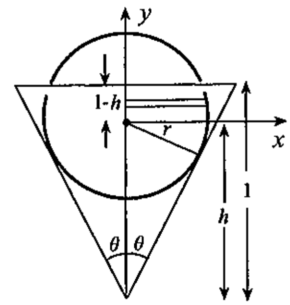


$$\int_0^{1-x^2}\frac{2\pi(x-y+1)}{\sqrt{2}}dx\,dy=\sqrt{2}\,\pi\,dx\left\{-\frac{(x-y+1)^2}{2}\right\}_0^{1-x^2}=\frac{\pi}{\sqrt{2}}(1+2x-2x^3-x^4)\,dx.$$

The volume of the solid of revolution is therefore

$$V=\int_0^1\frac{\pi}{\sqrt{2}}(1+2x-2x^3-x^4)\,dx=\frac{\pi}{\sqrt{2}}\left\{x+x^2-\frac{x^4}{2}-\frac{x^5}{5}\right\}_0^1=\frac{13\sqrt{2}\pi}{20}.$$

46. Let h be the height of the centre of a sphere lying within the cone above the vertex of the cone. The resulting radius of the sphere is $r = h\sin\theta$. We now calculate the volume of the sphere inside the cone using volumes of solids of revolution,

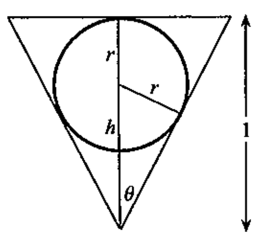


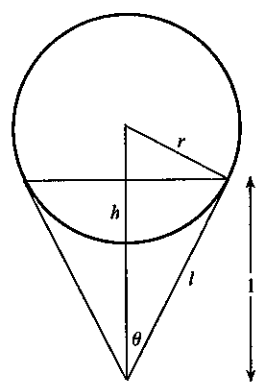
$$V = \int_{-r}^{1-h} \pi(r^2 - y^2)\,dy = \pi\left\{r^2 y - \frac{y^3}{3}\right\}_{-r}^{1-h}$$
$$= \pi\left[r^2(1-h) - \frac{1}{3}(1-h)^3 + r^3 - \frac{r^3}{3}\right].$$

When we substitute $r = h\sin\theta$, we obtain the function to be maximized

$$V(h) = \pi\left[h^2\sin^2\theta(1-h) - \frac{1}{3}(1-h)^3 + \frac{2h^3}{3}\sin^3\theta\right].$$

Certainly h must be nonnegative, but we can do better for a lower bound. Once the sphere is completely within the cone (left figure below), the volume decreases as h decreases. Consequently, the smallest value of h to use is that when the top of the sphere is level with the top of the cone. This occurs when $h + r = 1$ so that $h = 1 - r = 1 - h\sin\theta \Longrightarrow h = 1/(1+\sin\theta)$. The maximum value of h occurs when the sphere is tangent to the cone at its top edge (right figure below). For this situation $h = l\sec\theta = \sec^2\theta$. Thus, the domain of $V(h)$ is $1/(1+\sin\theta) \le h \le \sec^2\theta$.





For critical points of $V(h)$, we solve

$$0 = V'(h) = \pi[\sin^2\theta(2h - 3h^2) + (1-h)^2 + 2h^2\sin^3\theta].$$

This implies that

$$h^2(1 - 3\sin^2\theta + 2\sin^3\theta) + h(2\sin^2\theta - 2) + 1 = 0.$$

Solutions of this quadratic are

$$h = \frac{2 - 2\sin^2\theta \pm \sqrt{(2\sin^2\theta - 2)^2 - 4(1 - 3\sin^2\theta + 2\sin^3\theta)}}{2(1 - 3\sin^2\theta + 2\sin^3\theta)}.$$

Now,

$$\begin{aligned}(2\sin^2\theta - 2)^2 - 4(1 - 3\sin^2\theta + 2\sin^3\theta) &= 4\sin^4\theta - 8\sin^2\theta + 4 - 4 + 12\sin^2\theta - 8\sin^3\theta \\ &= 4\sin^2\theta(\sin^2\theta - 2\sin\theta + 1) \\ &= 4\sin^2\theta(1 - \sin\theta)^2,\end{aligned}$$

and

$$2\sin^3\theta - 3\sin^2\theta + 1 = (\sin\theta - 1)(2\sin^2\theta - \sin\theta - 1)$$
$$= (1 - \sin\theta)(1 + \sin\theta - 2\sin^2\theta)$$
$$= (1 - \sin\theta)(1 - \sin\theta)(1 + 2\sin\theta).$$

Thus, critical points are

$$h = \frac{2(1-\sin\theta)(1+\sin\theta) \pm 2\sin\theta(1-\sin\theta)}{2(1-\sin\theta)^2(1+2\sin\theta)}$$
$$= \frac{1}{1-\sin\theta}, \quad \frac{1}{(1-\sin\theta)(1+2\sin\theta)}.$$

Since $\dfrac{1}{1-\sin\theta} > \dfrac{1}{1-\sin^2\theta} = \dfrac{1}{\cos^2\theta} = \sec^2\theta$, the first critical point is not in the domain of $V(h)$. The second critical point, call it $\tilde{h}$, is in the domain of $V(h)$ because

$$\frac{1}{1+\sin\theta} \le \frac{1}{1+\sin\theta-2\sin^2\theta} = \frac{1}{(1-\sin\theta)(1+2\sin\theta)} \le \frac{1}{(1-\sin\theta)(1+\sin\theta)} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

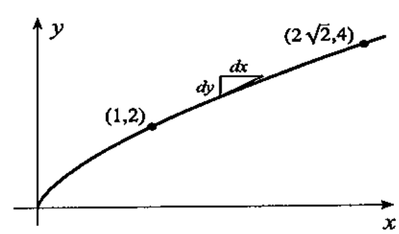
To verify that $\tilde{h}$ gives an absolute maximum, we could evaluate V at $\tilde{h}$ and at the end points of its domain of definition. It is simpler, however, to notice that $V(h)$ is a cubic polynomial in h, with two critical points, $\tilde{h}$ being the smaller one. If the coefficient of h^3 in $V(h)$ is positive, then $\tilde{h}$ must yield a relative maximum that is also the absolute maximum. The coefficient of h^3 is

$$\pi\left(-\sin^2\theta + \frac{1}{3} + \frac{2}{3}\sin^3\theta\right) = \frac{\pi}{3}(2\sin^3\theta - 3\sin^2\theta + 1) = \frac{\pi}{3}(1-\sin\theta)^2(1+2\sin\theta) > 0.$$

Thus, $\tilde{h}$ maximizes V, and the radius of the sphere is $r = \tilde{h}\sin\theta = \dfrac{\sin\theta}{(1-\sin\theta)(1+2\sin\theta)}$.

EXERCISES 7.3

1. Small lengths along the curve are approximated by

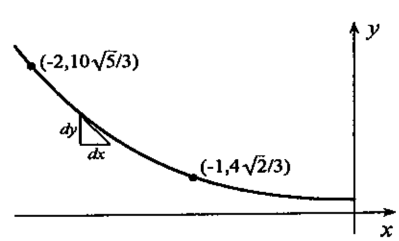


$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1+\left(\frac{4x^{-1/3}}{3}\right)^2}\,dx = \frac{\sqrt{9x^{2/3}+16}}{3x^{1/3}}dx.$$

Total length of the curve is

$$L = \int_1^{2\sqrt{2}} \frac{\sqrt{9x^{2/3}+16}}{3x^{1/3}}dx = \frac{1}{3}\left\{\frac{(9x^{2/3}+16)^{3/2}}{9}\right\}_1^{2\sqrt{2}} = \frac{34\sqrt{34}-125}{27}.$$

2. Small lengths along the curve are approximated by

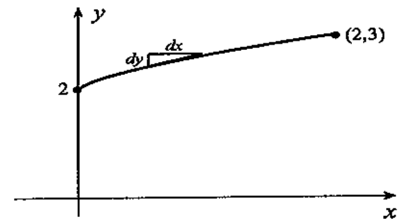


$$\sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1+[2x(x^2+1)^{1/2}]^2}\,dx = \sqrt{1+4x^2(x^2+1)}\,dx$$
$$= (2x^2+1)\,dx.$$

Total length of the curve is

$$L = \int_{-2}^{-1}(2x^2+1)\,dx = \left\{\frac{2x^3}{3}+x\right\}_{-2}^{-1} = \frac{17}{3}.$$

3. Small lengths along the curve are approximated by

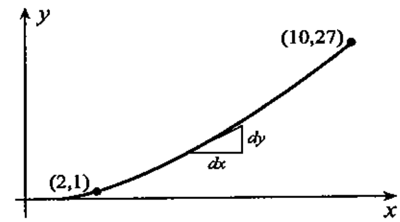


$$\begin{aligned}\sqrt{(dx)^2+(dy)^2} &= \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\ &= \sqrt{1+(3\sqrt{y-2})^2}dy = \sqrt{9y-17}\,dy.\end{aligned}$$

Total length of the curve is

$$L = \int_2^3 \sqrt{9y-17}\,dy = \left\{\frac{2}{27}(9y-17)^{3/2}\right\}_2^3 = \frac{2(10\sqrt{10}-1)}{27}.$$

4. We approximate small lengths along the curve by

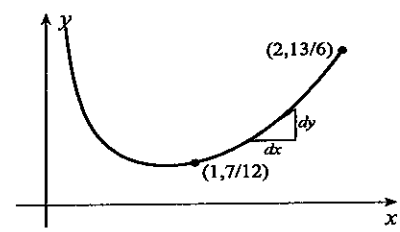


$$\begin{aligned}\sqrt{(dx)^2+(dy)^2} &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1+\left(\frac{3}{2}\sqrt{x-1}\right)^2}\,dx = \frac{1}{2}\sqrt{9x-5}\,dx.\end{aligned}$$

Total length of the curve is

$$L = \int_2^{10} \frac{1}{2}\sqrt{9x-5}\,dx = \frac{1}{2}\left\{\frac{2}{27}(9x-5)^{3/2}\right\}_2^{10} = \frac{85^{3/2}-13^{3/2}}{27}.$$

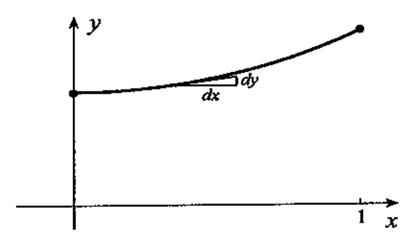
5. Small lengths along the curve are approximated by



$$\begin{aligned}\sqrt{(dx)^2+(dy)^2} &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1+\left(\frac{3x^2}{4}-\frac{1}{3x^2}\right)^2}\,dx = \sqrt{1+\left(\frac{9x^4}{16}-\frac{1}{2}+\frac{1}{9x^4}\right)}dx \\ &= \sqrt{\left(\frac{3x^2}{4}+\frac{1}{3x^2}\right)^2}\,dx = \left(\frac{3x^2}{4}+\frac{1}{3x^2}\right)dx.\end{aligned}$$

Total length of the curve is $L = \int_1^2 \left(\frac{3x^2}{4}+\frac{1}{3x^2}\right)dx = \left\{\frac{x^3}{4}-\frac{1}{3x}\right\}_1^2 = \frac{23}{12}$.

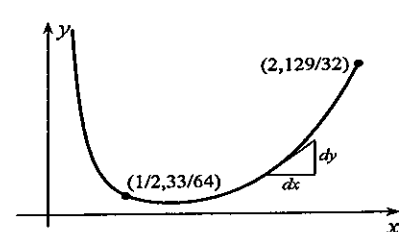
6. Small lengths along the curve are approximated by



$$\begin{aligned}\sqrt{(dx)^2+(dy)^2} &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}\,dx = \sqrt{\frac{e^{2x}+2+e^{-2x}}{4}}dx \\ &= \frac{1}{2}(e^x+e^{-x})\,dx.\end{aligned}$$

Total length of the curve is $L = \int_0^1 \frac{1}{2}(e^x+e^{-x})\,dx = \frac{1}{2}\left\{e^x-e^{-x}\right\}_0^1 = \frac{1}{2}(e-e^{-1})$.

7. Small lengths along the curve are approximated by



$$\begin{aligned}\sqrt{(dx)^2+(dy)^2} &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1+\left(x^3-\frac{1}{4x^3}\right)^2}\,dx = \sqrt{1+\left(x^6-\frac{1}{2}+\frac{1}{16x^6}\right)}dx \\ &\sqrt{\left(x^3+\frac{1}{4x^3}\right)^2}\,dx = \left(x^3+\frac{1}{4x^3}\right)dx.\end{aligned}$$

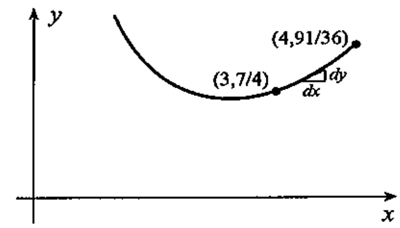
Total length of the curve is $L = \int_{1/2}^{2} \left(x^3 + \frac{1}{4x^3}\right) dx = \left\{\frac{x^4}{4} - \frac{1}{8x^2}\right\}_{1/2}^{2} = \frac{285}{64}.$

8. We approximate small lengths along the curve by

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + \left(\frac{x^2}{12} - \frac{3}{x^2}\right)^2}\, dx = \sqrt{1 + \frac{x^4}{144} - \frac{1}{2} + \frac{9}{x^4}}\, dx$$

$$= \sqrt{\left(\frac{x^2}{12} + \frac{3}{x^2}\right)^2}\, dx = \left(\frac{x^2}{12} + \frac{3}{x^2}\right) dx.$$



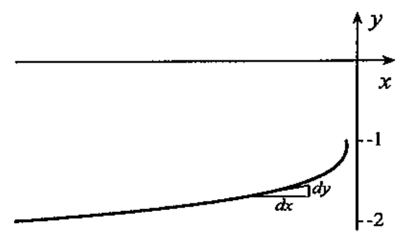
Total length of the curve is $L = \int_3^4 \left(\frac{x^2}{12} + \frac{3}{x^2}\right) dx = \left\{\frac{x^3}{36} - \frac{3}{x}\right\}_3^4 = \frac{23}{18}.$

9. We approximate small lengths along the curve by

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \sqrt{1 + \left(\frac{7y^6}{20} - \frac{5}{7y^6}\right)^2}\, dy = \sqrt{1 + \frac{49y^{12}}{400} - \frac{1}{2} + \frac{25}{49y^{12}}}\, dy$$

$$= \sqrt{\left(\frac{7y^6}{20} + \frac{5}{7y^6}\right)^2}\, dy = \left(\frac{7y^6}{20} + \frac{5}{7y^6}\right) dy.$$



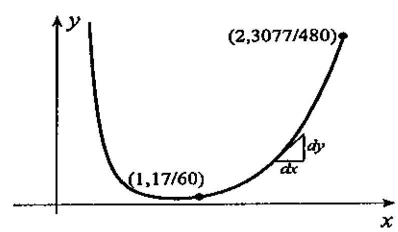
Total length of the curve is $L = \int_{-2}^{-1} \left(\frac{7y^6}{20} + \frac{5}{7y^6}\right) dy = \left\{\frac{y^7}{20} - \frac{1}{7y^5}\right\}_{-2}^{-1} = \frac{7267}{1120}.$

10. We approximate small lengths along the curve by

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + \left(x^4 - \frac{1}{4x^4}\right)^2}\, dx = \sqrt{1 + x^8 - \frac{1}{2} + \frac{1}{16x^8}}\, dx$$

$$= \sqrt{\left(x^4 + \frac{1}{4x^4}\right)^2}\, dx = \left(x^4 + \frac{1}{4x^4}\right) dx.$$



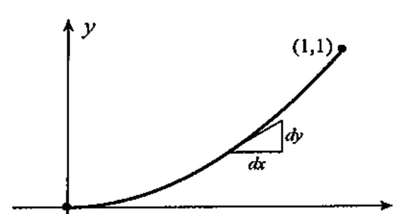
Total length of the curve is $L = \int_1^2 \left(x^4 + \frac{1}{4x^4}\right) dx = \left\{\frac{x^5}{5} - \frac{1}{12x^3}\right\}_1^2 = \frac{3011}{480}.$

11. With small lengths along the curve approximated by

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + 4x^2}\, dx,$$

the length of the curve is given by

$$L = \int_0^1 \sqrt{1 + 4x^2}\, dx.$$

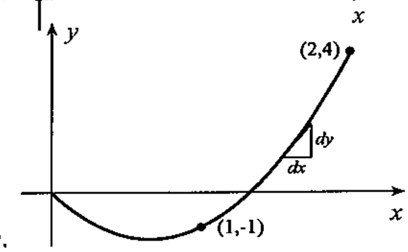


12. We approximate small lengths along the curve by

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + (6x - 4)^2}\, dx = \sqrt{36x^2 - 48x + 17}\, dx.$$

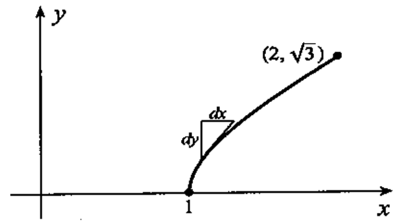
Total length of the curve is given by $L = \int_1^2 \sqrt{36x^2 - 48x + 17}\, dx.$



13. We approximate small lengths along the curve by

$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\sqrt{1+\left(\frac{x}{\sqrt{x^2-1}}\right)^2}\,dx=\sqrt{\frac{2x^2-1}{x^2-1}}\,dx.$$

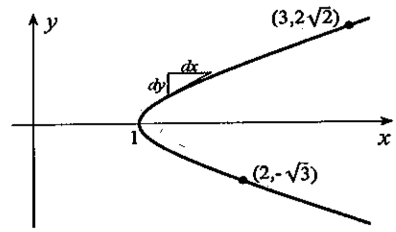
Total length of the curve is given by $L=\int_1^2\sqrt{\frac{2x^2-1}{x^2-1}}dx$.



14. We approximate small lengths along the curve by

$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$
$$=\sqrt{1+\left(\frac{y}{\sqrt{1+y^2}}\right)^2}\,dy=\sqrt{\frac{1+2y^2}{1+y^2}}\,dy.$$

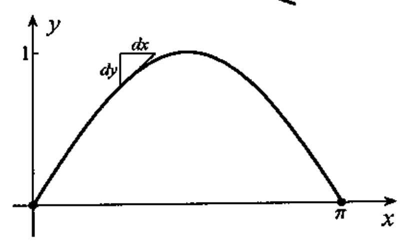
Total length of the curve is given by $L=\int_{-\sqrt{3}}^{2\sqrt{2}}\sqrt{\frac{1+2y^2}{1+y^2}}dy$.



15. We approximate small lengths along the curve by

$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\cos^2 x}\,dx.$$

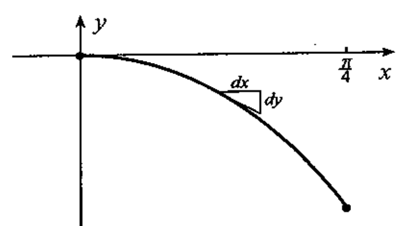
Total length of the curve is given by $L=\int_0^{\pi}\sqrt{1+\cos^2 x}\,dx$.



16. We approximate small lengths along the curve by

$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2}\,dx=\sec x\,dx.$$

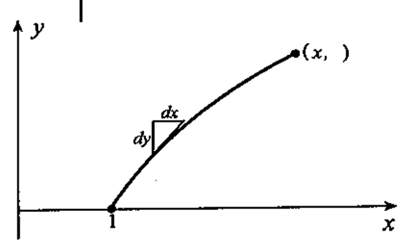
Total length of the curve is given by $L=\int_0^{\pi/4}\sec x\,dx$.



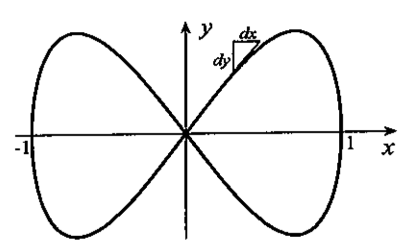
17. We approximate small lengths along the curve by

$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=\sqrt{1+(1/x)^2}\,dx=\frac{\sqrt{x^2+1}}{x}dx.$$

The length of the curve from $(1,0)$ to any point with x-coordinate equal to x is $L=\int_1^x\frac{\sqrt{t^2+1}}{t}dt$.



18. Differentiation of $8y^2=x^2-x^4$ with respect to x gives $16y\frac{dy}{dx}=2x-4x^3$, and from this equation $\frac{dy}{dx}=\frac{x-2x^3}{8y}$. Small lengths along that portion of the curve in the first quadrant are approximated by



$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\left(\frac{x-2x^3}{8y}\right)^2}\,dx=\frac{\sqrt{64y^2+x^2-4x^4+4x^6}}{8y}\,dx$$

$$= \frac{\sqrt{8x^2 - 8x^4 + x^2 - 4x^4 + 4x^6}}{8y}\,dx = \frac{3x - 2x^3}{2\sqrt{2}x\sqrt{1 - x^2}}\,dx = \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}\,dx.$$

Total length of the curve is four times that in the first quadrant,

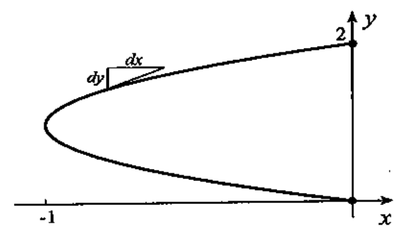
$$L = 4\int_0^1 \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}dx = \sqrt{2}\int_0^1 \frac{3 - 2x^2}{\sqrt{1 - x^2}}dx.$$

19. We approximate small lengths along the curve by

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$
$$= \sqrt{1 + (2y - 2)^2}dy = \sqrt{4y^2 - 8y + 5}\,dy.$$

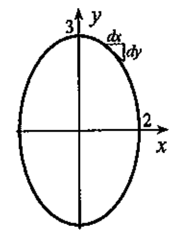
Total length of the curve is given by

$$L = \int_0^2 \sqrt{4y^2 - 8y + 5}\,dy.$$



20. We approximate small lengths along that part of the ellipse in the first quadrant by

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1 + \left(\frac{-3x}{2\sqrt{4 - x^2}}\right)^2}\,dx = \sqrt{\frac{16 + 5x^2}{4(4 - x^2)}}\,dx.$$

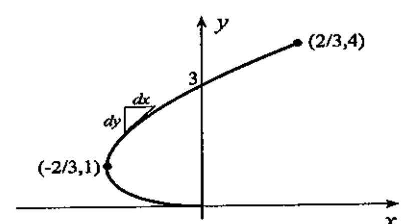


Total length of the curve is four times that in the first quadrant

$$L = 4\int_0^2 \sqrt{\frac{16 + 5x^2}{4(4 - x^2)}}\,dx = 2\int_0^2 \sqrt{\frac{16 + 5x^2}{4 - x^2}}\,dx.$$

21. We approximate small lengths along the curve by

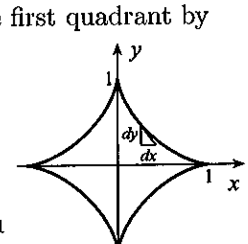
$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$
$$= \sqrt{1 + \left(\frac{\sqrt{y}}{2} - \frac{1}{2\sqrt{y}}\right)^2}\,dy = \sqrt{1 + \left(\frac{y}{4} - \frac{1}{2} + \frac{1}{4y}\right)}dy$$
$$= \sqrt{\left(\frac{\sqrt{y}}{2} + \frac{1}{2\sqrt{y}}\right)^2}\,dy = \left(\frac{\sqrt{y}}{2} + \frac{1}{2\sqrt{y}}\right)dy$$



Total length of the curve is $L = \int_1^4 \left(\frac{\sqrt{y}}{2} + \frac{1}{2\sqrt{y}}\right)dy = \left\{\frac{y^{3/2}}{3} + \sqrt{y}\right\}_1^4 = \frac{10}{3}$.

22. Differentiation of $x^{2/3} + y^{2/3} = 1$ with respect to x gives $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$, and therefore $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$. We approximate small lengths along the curve in the first quadrant by

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \sqrt{1 + \left(\frac{y}{x}\right)^{2/3}}\,dx = \frac{\sqrt{x^{2/3} + y^{2/3}}}{x^{1/3}}dx = x^{-1/3}\,dx.$$



Total length of the curve is therefore $L = 4\int_0^1 x^{-1/3}\,dx = 4\left\{\frac{3}{2}x^{2/3}\right\}_0^1 = 6.$

23. Small lengths along the curve are approximated by

$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\left(x^n-\frac{1}{4x^n}\right)^2}\,dx$$
$$=\sqrt{1+\left(x^{2n}-\frac{1}{2}+\frac{1}{16x^{2n}}\right)}\,dx=\sqrt{\left(x^n+\frac{1}{4x^n}\right)^2}\,dx=\left(x^n+\frac{1}{4x^n}\right)dx.$$

The length of the curve is therefore

$$L=\int_a^b\left(x^n+\frac{1}{4x^n}\right)dx=\left\{\frac{x^{n+1}}{n+1}-\frac{1}{4(n-1)x^{n-1}}\right\}_a^b=\frac{b^{n+1}-a^{n+1}}{n+1}+\frac{a^{1-n}-b^{1-n}}{4(n-1)}.$$

24. Small lengths along the curve are approximated by

$$\sqrt{(dx)^2+(dy)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\left[\frac{(2n+1)x^{2n}}{4(2n-1)}-\frac{(2n-1)}{(2n+1)x^{2n}}\right]^2}\,dx$$
$$=\sqrt{1+\frac{(2n+1)^2x^{4n}}{16(2n-1)^2}-\frac{1}{2}+\frac{(2n-1)^2}{(2n+1)^2x^{4n}}}\,dx$$
$$=\sqrt{\left[\frac{(2n+1)x^{2n}}{4(2n-1)}+\frac{2n-1}{(2n+1)x^{2n}}\right]^2}\,dx=\left[\frac{(2n+1)x^{2n}}{4(2n-1)}+\frac{2n-1}{(2n+1)x^{2n}}\right]dx.$$

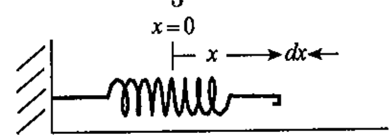
The length of the curve is therefore

$$L=\int_a^b\left[\frac{(2n+1)x^{2n}}{4(2n-1)}+\frac{2n-1}{(2n+1)x^{2n}}\right]dx=\left\{\frac{x^{2n+1}}{4(2n-1)}-\frac{1}{(2n+1)x^{2n-1}}\right\}_a^b$$
$$=\frac{b^{2n+1}-a^{2n+1}}{4(2n-1)}+\frac{1}{2n+1}\left(\frac{1}{a^{2n-1}}-\frac{1}{b^{2n-1}}\right).$$

EXERCISES 7.4

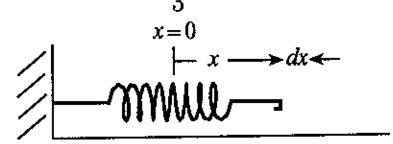
1. Let the spring be stretched in the positive x-direction and $x=0$ correspond to the free end of the spring in the unstretched position (figure below). The restoring force of the spring is $F_s=-kx$. Since $F_s=-10$ N when $x=0.03$ m, it follows that $-10=-0.03k$, and therefore $k=1000/3$ N/m. The force required to counteract the spring force when the spring is stretched an amount x is $F(x)=1000x/3$. The work done by this force in stretching the spring a further distance dx is $\frac{1000x}{3}dx$ J. The total work in stretching the spring from 5 cm to 7 cm is

$$W=\int_{0.05}^{0.07}\frac{1000}{3}x\,dx=\frac{1000}{3}\left\{\frac{x^2}{2}\right\}_{0.05}^{0.07}=\frac{2}{5}\text{ J}.$$

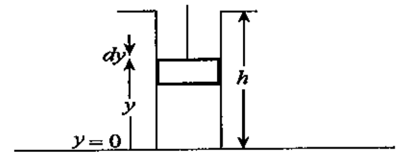


2. Let the spring be stretched in the positive x-direction and $x=0$ correspond to the free end of the spring in the unstretched position (figure below). The restoring force of the spring is $F_s=-kx$. Since $F_s=-10$ N when $x=0.03$ m, it follows that $-10=-0.03k$, and therefore $k=1000/3$ N/m. The force required to counteract the spring force when the spring is stretched an amount x is $F(x)=1000x/3$. The work done by this force in stretching the spring a further distance dx is $\frac{1000x}{3}dx$ J. The total work in stretching the spring from 7 cm to 9 cm is

$$W=\int_{0.07}^{0.09}\frac{1000}{3}x\,dx=\frac{1000}{3}\left\{\frac{x^2}{2}\right\}_{0.07}^{0.09}=\frac{8}{15}\text{ J}.$$

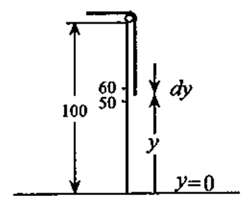


3. When the cage is y m from the bottom of the shaft, the force of gravity on the cage and that part of the cable lifting the cage is $-9.81[M + m(h - y)]$ N. The force required to counter gravity is therefore $9.81[M + m(h - y)]$ N. The work done in raising the cable a further distance dy is $9.81[M + m(h - y)]\,dy$ J. The total work to raise the cage from the bottom of the shaft is



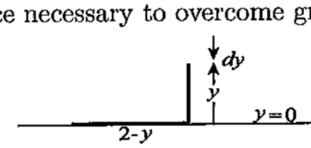
$$W = \int_0^h 9.81[M + m(h - y)]\,dy = 9.81\left\{My - \frac{m}{2}(h - y)^2\right\}_0^h = 9.81(Mh + mh^2/2) \text{ J}.$$

4. When the lower end of the cable is y m above the ground, the force of gravity on that part of the cable hanging from the building is $-9.81(2)(100 - y)$ N. The force required to counter gravity is therefore $19.62(100 - y)$ N. The work done by this force in raising the cable a further distance dy is $19.62(100 - y)\,dy$ J. The total work to raise the cable the 10 m is



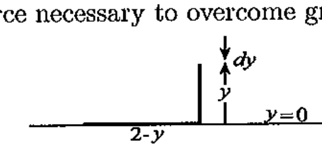
$$W = \int_{50}^{60} 19.62(100 - y)\,dy = 19.62\left\{100y - \frac{y^2}{2}\right\}_{50}^{60} = 8829 \text{ J}.$$

5. When the end of the chain has been lifted a distance y, the force necessary to overcome gravity and hold the chain in this position is $9.81(10)y$ N. The work done by this force in lifting the end of the chain an additional amount dy is $98.1y\,dy$ J. The total work to lift the end of the chain 2 m is



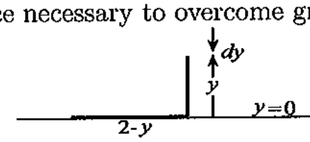
therefore $W = \displaystyle\int_0^2 98.1y\,dy = 98.1\left\{\frac{y^2}{2}\right\}_0^2 = 196.2$ J.

6. When the end of the chain has been lifted a distance y, the force necessary to overcome gravity and hold the chain in this position is $9.81(10)y$ N. The work done by this force in lifting the end of the chain an additional amount dy is $98.1y\,dy$ J. The total work to lift the end of the chain 1 m is



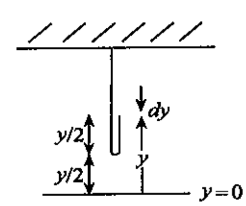
therefore $W = \displaystyle\int_0^1 98.1y\,dy = 98.1\left\{\frac{y^2}{2}\right\}_0^1 = 49.05$ J.

7. When the end of the chain has been lifted a distance y, the force necessary to overcome gravity and hold the chain in this position is $9.81(10)y$ N. The work done by this force in lifting the end of the chain an additional amount dy is $98.1y\,dy$ J. The total work to lift the end of the chain 4 m is



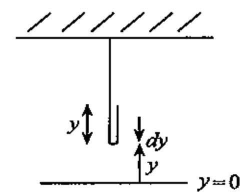
therefore $W = \displaystyle\int_0^2 98.1y\,dy + \int_2^4 (9.81)(20)\,dy = 98.1\left\{\frac{y^2}{2}\right\}_0^2 + 196.2(2) = 588.6$ J.

8. (a) When the lower end has been lifted a distance y, the force necessary to overcome gravity is $9.81(3)(y/2)$ N. The work done by this force in lifting the lower end of the chain an additional amount dy is $(29.43y/2)\,dy$ J. The total work done in lifting the end the 5 m is



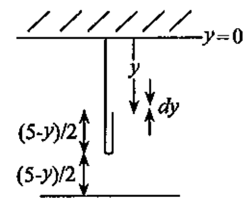
$$W = \int_0^5 \frac{29.43y}{2}\,dy = \frac{29.43}{2}\left\{\frac{y^2}{2}\right\}_0^5 = 183.9 \text{ J}.$$

(b) In this case the force necessary to counter gravity at the position shown is $9.81(3)y$ N. To move the bend in the cable up a distance dy, the end of the cable must be lifted a distance $2dy$, requiring $29.43y(2dy)$ J of work. Hence, the total work done is



$$W = \int_0^{2.5} 58.86y\,dy = 58.86\left\{\frac{y^2}{2}\right\}_0^{2.5} = 183.9 \text{ J.}$$

(c) When the chain is at the position shown, the force to overcome gravity is $F(y) = -9.81(3)(5-y)/2$ N. The work to raise the chain through a displacement dy which is in the negative y-direction is $[-29.43(5-y)/2]dy$. The total work to raise the chain is therefore



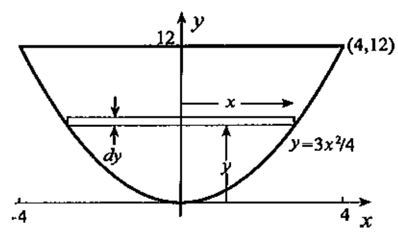
$$W = \int_5^0 -\frac{29.43}{2}(5-y)\,dy = -\frac{29.43}{2}\left\{-\frac{1}{2}(5-y)^2\right\}_5^0 = 183.9 \text{ J.}$$

9. The force of gravity on the disc of water shown is

$$F_g = -9.81(1000)\pi x^2\,dy = -9810\pi\left(\frac{4y}{3}\right)dy \text{ N.}$$

The work that an equal and opposite force would do in raising this disc to the top of the tank is

$$(12-y)9810\pi\left(\frac{4y}{3}\right)dy \text{ J.}$$



The total work to empty the tank is therefore

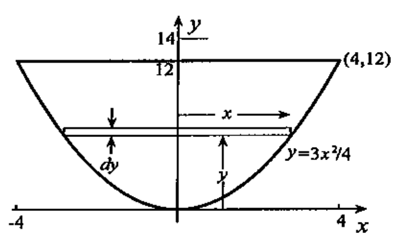
$$W = \int_0^{12}(12-y)9810\pi\left(\frac{4y}{3}\right)dy = \frac{4(9810)\pi}{3}\left\{6y^2 - \frac{y^3}{3}\right\}_0^{12} = 1.18\times 10^7 \text{ J.}$$

10. The force of gravity on the disc of water shown is

$$F_g = -9.81(1000)\pi x^2\,dy = -9810\pi\left(\frac{4y}{3}\right)dy \text{ N.}$$

The work that an equal and opposite force would do in raising this disc to a level 2 m above the top of the tank is

$$(14-y)9810\pi\left(\frac{4y}{3}\right)dy \text{ J.}$$



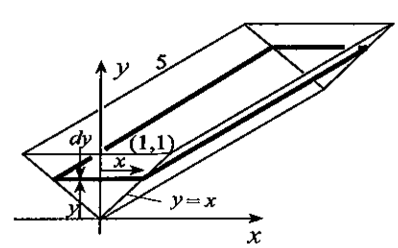
The total work to empty the tank is therefore

$$W = \int_0^{12}(14-y)9810\pi\left(\frac{4y}{3}\right)dy = \frac{4(9810)\pi}{3}\left\{7y^2 - \frac{y^3}{3}\right\}_0^{12} = 1.78\times 10^7 \text{ J.}$$

11. The force of gravity on a slab of water dy m thick is

$$-9.81(1000)2x(5)dy = -98\,100y\,dy \text{ N.}$$

To lift this slab to the top of the trough requires $(1-y)(98\,100)y\,dy$ J of work. Hence the work required to empty the trough is



$$W = \int_0^1 98\,100y(1-y)\,dy$$

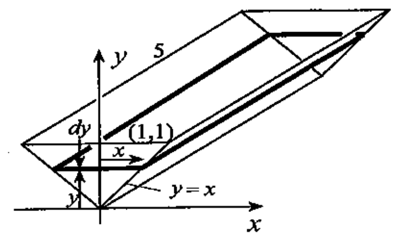
$$= 98\,100\left\{\frac{y^2}{2} - \frac{y^3}{3}\right\}_0^1 = 16\,350 \text{ J.}$$

12. The force of gravity on a slab of water dy m thick is

$$-9.81(1000)2x(5)dy = -98\,100y\,dy \text{ N}.$$

To lift this slab to a height 2 m above the trough requires $(3-y)98\,100y\,dy$ J of work. Hence the work required to empty the trough is

$$W = \int_0^1 98\,100y(3-y)\,dy$$
$$= 98\,100\left\{\frac{3y^2}{2} - \frac{y^3}{3}\right\}\Bigg|_0^1 = 114\,450 \text{ J}.$$

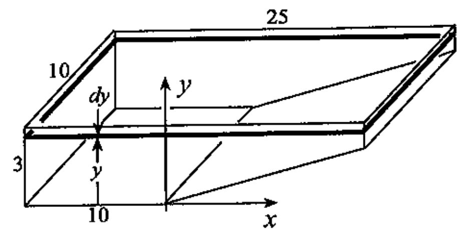


13. The force of gravity on slabs of water dy m thick for $2 \le y \le 3$ is

$$-9.81(1000)(10)(25)\,dy = -2\,452\,500\,dy \text{ N}.$$

The work to lower the level of the water by 1/2 m is

$$W = \int_{5/2}^3 (3-y)(2\,452\,500)\,dy$$
$$= 2\,452\,500\left\{-\frac{1}{2}(3-y)^2\right\}\Bigg|_{5/2}^3 = 3.07 \times 10^5 \text{ J}.$$

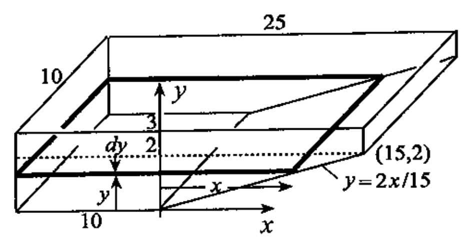


14. The force of gravity on slabs of water dy m thick for $0 \le y \le 2$ is

$$-9.81(1000)(10)(x+10)\,dy$$
$$= -98\,100\left(\frac{15y}{2} + 10\right)dy \text{ N}.$$

For slabs above $y = 2$, the force is

$$-9.81(1000)(10)(25)\,dy$$
$$= -2\,452\,500\,dy \text{ N}.$$

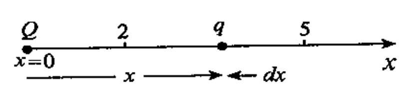


The work to empty the pool over its edge is therefore

$$W = \int_0^2 98\,100\left(\frac{15y}{2} + 10\right)(3-y)\,dy + \int_2^3 2\,452\,500(3-y)\,dy$$
$$= 245\,250\int_0^2 (12 + 5y - 3y^2)\,dy + 2\,452\,500\int_2^3 (3-y)\,dy$$
$$= 245\,250\left\{12y + \frac{5y^2}{2} - y^3\right\}\Bigg|_0^2 + 2\,452\,500\left\{3y - \frac{y^2}{2}\right\}\Bigg|_2^3 = 7.60 \times 10^6 \text{ J}.$$

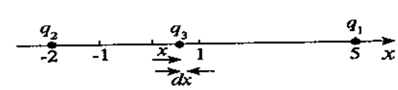
15. When q is at position x, the force on it is $qQ/(4\pi\epsilon_0 x^2)$. The work done by this force as q moves from $x = 2$ to $x = 5$ is

$$W = \int_2^5 \frac{qQ}{4\pi\epsilon_0 x^2}dx = \frac{qQ}{4\pi\epsilon_0}\left\{-\frac{1}{x}\right\}\Bigg|_2^5 = \frac{3qQ}{40\pi\epsilon_0}.$$



16. When q_3 is at position x, the total force on it due to q_1 and q_2 is

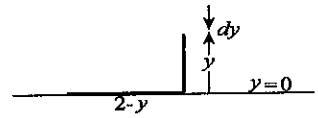
$$F(x) = \frac{-q_1q_3}{4\pi\epsilon_0(5-x)^2} + \frac{q_2q_3}{4\pi\epsilon_0(x+2)^2}.$$



The work done by this force as q_3 moves from $x = 1$ to $x = -1$ is

$$W = \int_1^{-1}\left[\frac{-q_1q_3}{4\pi\epsilon_0(5-x)^2} + \frac{q_2q_3}{4\pi\epsilon_0(x+2)^2}\right]dx = \frac{-q_1q_3}{4\pi\epsilon_0}\left\{\frac{1}{5-x}\right\}\Bigg|_1^{-1} + \frac{q_2q_3}{4\pi\epsilon_0}\left\{\frac{-1}{x+2}\right\}\Bigg|_1^{-1}$$
$$= \frac{q_1q_3}{48\pi\epsilon_0} - \frac{q_2q_3}{6\pi\epsilon_0} = \frac{q_3}{48\pi\epsilon_0}(q_1 - 8q_2).$$

17. When the end of the chain has been lifted a distance y, the force necessary to overcome gravity on the hanging part and friction on that part on the floor is $9.81(10)y + 0.01(9.81)(10)(2-y) = 1.962 + 97.119y$ N.

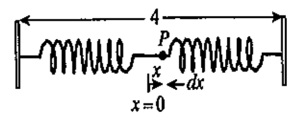


The work done by this force in lifting the end of chain 2 m is therefore

$$W = \int_0^2 (1.962 + 97.119y)\,dy = \left\{1.962y + \frac{97.119y^2}{2}\right\}_0^2 = 198.162 \text{ J.}$$

18. When P is moved x m to the right, the resultant force of the two springs on P is

$-k(1+x) + k(1-x) = -2kx$ N.



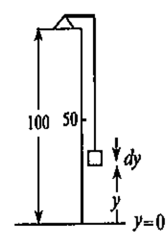
The work done by an equal and opposite force in moving P a distance b m to the right is

$$W = \int_0^b 2kx\,dx = \left\{kx^2\right\}_0^b = kb^2 \text{ J.}$$

19. The force necessary to maintain a draw of x m is $F = kx$. Since $F = 200$ when $x = 0.5$, it follows that $k = 400$ N/m. The work to fully draw the bow is $W = \int_0^{1/2} 400x\,dx = \left\{200x^2\right\}_0^{1/2} = 50$ J.

20. When the bucket has been raised y m ($y < 50$), the force of gravity on what remains in the bucket and the hanging cable is

$-9.81[(100 - y/5) + 5(100 - y)] = -9.81(600 - 26y/5)$ N.



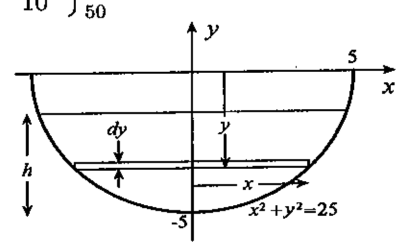
When $y \geq 50$, the force of gravity on bucket, cable and pigeon is

$$-9.81[(100 - y/5) + 5(100-y) + 2 - (y-50)]$$
$$= -9.81(652 - 31y/5) \text{ N.}$$

The work to overcome these forces is

$$W = \int_0^{50} 9.81\left(600 - \frac{26y}{5}\right)dy + \int_{50}^{100} 9.81\left(652 - \frac{31y}{5}\right)dy$$
$$= 9.81\left\{600y - \frac{13y^2}{5}\right\}_0^{50} + 9.81\left\{652y - \frac{31y^2}{10}\right\}_{50}^{100} = 3.22 \times 10^5 \text{ J.}$$

21. (a) When the depth of oil is h m, the volume of oil in the tank is



$$V = \int_{-5}^{h-5} \pi(25 - y^2)\,dy = \pi\left\{25y - \frac{y^3}{3}\right\}_{-5}^{h-5}$$
$$= \frac{\pi}{3}(15h^2 - h^3).$$

This volume will be $(1/3)\pi(125)$ when

$$\frac{125\pi}{3} = \frac{\pi}{3}(15h^2 - h^3),$$

and this equation reduces to $h^3 - 15h^2 + 125 = 0$. Newton's iterative procedure with $h_1 = 3$ and $h_{n+1} = h_n - \dfrac{h_n^3 - 15h_n^2 + 125}{3h_n^2 - 30h_n}$ leads to $h = 3.2635$ m.

(b) The work to empty the half-full tank is

$$W = \int_{-5}^{-1.7365} 9.81(750)(-y)(\pi x^2)\,dy = -9.81(750)\pi \int_{-5}^{-1.7365} y(25 - y^2)\,dy$$
$$= -9.81(750)\pi\left\{\frac{25y^2}{2} - \frac{y^4}{4}\right\}_{-5}^{-1.7365} = 2.79 \times 10^6 \text{ J.}$$

22. (a) Assuming that the earth is a sphere

$$F = \frac{6.67\times 10^{-11}(m)(4/3)\pi(6.37\times 10^6)^3(5.52\times 10^3)}{(6.37\times 10^6)^2} = 9.82m \text{ N}.$$

(b) The work required is $W = \int_{6.37\times 10^6}^{6.38\times 10^6} \frac{G(10)M}{r^2}\,dr = 10GM\left\{-\frac{1}{r}\right\}_{6.37\times 10^6}^{6.38\times 10^6}$

$$= 10(6.67\times 10^{-11})\frac{4}{3}\pi(6.37\times 10^6)^3(5.52\times 10^3)\left(\frac{-1}{6.38\times 10^6} + \frac{1}{6.37\times 10^6}\right)$$

$$= 9.8087\times 10^5 \text{ J}.$$

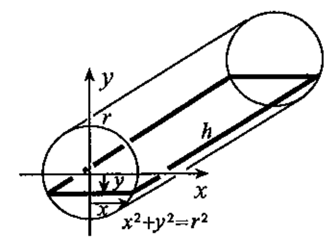
(c) With a constant $F = 9.82m$, $\quad W = \int_{6.37\times 10^6}^{6.38\times 10^6} 9.82(10)\,dy = 9.82\times 10^5$ J.

23. The force of gravity on a slab of oil dy m thick is $-9.81(\rho)(2x)(h)dy$ N. To lift this slab to the top of the tank requires $(r-y)(19.62\rho hx\,dy)$ J of work. Hence the work required to empty the tank is

$$W = \int_{-r}^{r}(r-y)(19.62\rho hx)\,dy = 19.62\rho h\int_{-r}^{r}(r-y)\sqrt{r^2-y^2}\,dy$$

$$= 19.62\rho h\left[\int_{-r}^{r} r\sqrt{r^2-y^2}\,dy - \int_{-r}^{r} y\sqrt{r^2-y^2}\,dy\right]$$

$$= 19.62\rho rh\int_{-r}^{r}\sqrt{r^2-y^2}\,dy - 19.62\rho h\left\{-\frac{1}{3}(r^2-y^2)^{3/2}\right\}_{-r}^{r}.$$



Since the remaing integral represents one-half the area of the end of the tank,

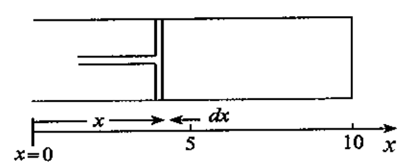
$$W = 19.62\rho hr\left(\frac{1}{2}\pi r^2\right) = 9.81\pi\rho r^3 h \text{ J}.$$

24. At position x, the force exerted by the piston is

$$F(x) = PA = \frac{C}{V}A = \frac{CA}{A(10-x)} = \frac{C}{10-x}.$$

The work done is therefore

$$W = \int_0^5 \frac{C}{10-x}\,dx = C\Big\{-\ln|10-x|\Big\}_0^5$$
$$= C(-\ln 5 + \ln 10) = C\ln 2.$$



25. We divide the work into four parts as shown,

$$W = \int_0^{70}\rho g(y)\pi\left(\frac{1}{20}\right)^2 dy + \int_{70}^{73}\rho g(y)\pi[9-(y-73)^2]\,dy$$

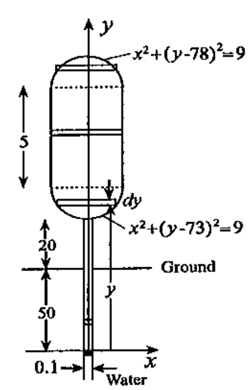
$$+ \int_{73}^{78}\rho g(y)\pi(3)^2\,dy + \int_{78}^{81}\rho g(y)\pi[9-(y-78)^2]\,dy$$

$$= \rho g\pi\left[\frac{1}{400}\int_0^{70} y\,dy + \int_{70}^{73}(-y^3+146y^2-5320y)\,dy\right.$$

$$\left. +9\int_{73}^{78} y\,dy + \int_{78}^{81}(-y^3+156y^2-6075y)\,dy\right]$$

$$= \rho g\pi\left[\frac{1}{400}\left\{\frac{y^2}{2}\right\}_0^{70} + \left\{-\frac{y^4}{4}+\frac{146y^3}{3}-2660y^2\right\}_{70}^{73}\right.$$

$$\left. +9\left\{\frac{y^2}{2}\right\}_{73}^{78} + \left\{-\frac{y^4}{4}+\frac{156y^3}{3}-\frac{6075y^2}{2}\right\}_{78}^{81}\right] = 1.89\times 10^8 \text{ J}.$$



26. Using points A and D we obtain $k_2 = 12\,000$ and $k_1 = 20\,000$. Since the work is the area bounded by the curves

$$W = \int_{1/5}^{3/5} \left(\frac{k_1}{V} - \frac{k_2}{V}\right) dV = (k_1 - k_2)\left\{\ln V\right\}_{1/5}^{3/5} = 8.8 \times 10^3 \text{ J}.$$

27. Using points A and D we obtain $k_2 = 1000$ and $k_1 = 20\,000$. Since the work is the area bounded by the curves

$$W = \int_{10\,000}^{100\,000} \left(\frac{k_1}{P} - \frac{k_2}{P}\right) dP = (k_1 - k_2)\left\{\ln P\right\}_{10\,000}^{100\,000} = 4.4 \times 10^4 \text{ J}.$$

28. Using points B and C we obtain $k_2 = 4.64$ and $k_1 = 6.89$. Since the work is the area bounded by the curves

$$W = \int_{2\times 10^{-4}}^{8\times 10^{-4}} \left(\frac{k_1}{V^{1/4}} - \frac{k_2}{V^{1.4}}\right) dV = (k_1 - k_2)\left\{\frac{-1}{0.4V^{0.4}}\right\}_{2\times 10^{-4}}^{8\times 10^{-4}} = 72 \text{ J}.$$

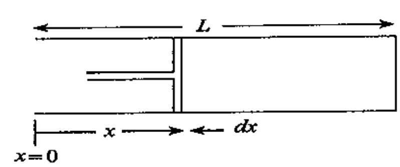
29. Using points A and D we obtain $k_2 = 15.0$ and $k_1 = 66.6$. Since the work is the area bounded by the curves

$$\begin{aligned} W &= \int_{2\times 10^{-4}}^{5.75\times 10^{-4}} \left(23 \times 10^5 - \frac{k_2}{V^{1.4}}\right) dV + \int_{5.75\times 10^{-4}}^{20\times 10^{-4}} \left(\frac{k_1}{V^{1.4}} - \frac{k_2}{V^{1.4}}\right) dV \\ &= \left\{23 \times 10^5 V + \frac{k_2}{0.4V^{0.4}}\right\}_{2\times 10^{-4}}^{5.75\times 10^{-4}} + (k_1 - k_2)\left\{\frac{-1}{0.4V^{0.4}}\right\}_{5.75\times 10^{-4}}^{20\times 10^{-4}} = 1.5 \times 10^3 \text{ J}. \end{aligned}$$

30. Using points A and D we obtain $k_2 = 2.62$ and $k_1 = 32.2$. Since the work is the area bounded by the curves

$$W = \int_{150\,000}^{1\,040\,000} \left[\left(\frac{k_1}{P}\right)^{5/7} - \left(\frac{k_2}{P}\right)^{5/7}\right] dP = \left(k_1^{5/7} - k_2^{5/7}\right)\left\{\frac{7}{2}P^{2/7}\right\}_{150\,000}^{1\,040\,000} = 7.8 \times 10^2 \text{ J}.$$

31. Let $x = 0$ represent the position of the piston face when $V = V_0$. At position x, the force exerted by the piston is



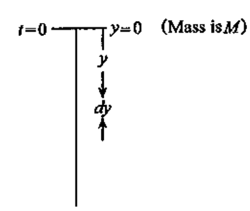
$$F(x) = PA = \frac{CA}{V^{7/5}} = \frac{CA}{[A(L-x)]^{7/5}} = \frac{C}{A^{2/5}(L-x)^{7/5}}.$$

The work done is therefore

$$W = \int_0^{L/2} \frac{C}{A^{2/5}(L-x)^{7/5}} dx = \frac{C}{A^{2/5}}\left\{\frac{5}{2(L-x)^{2/5}}\right\}_0^{L/2} = \frac{5C}{2A^{2/5}}\left[\frac{1}{(L/2)^{2/5}} - \frac{1}{L^{2/5}}\right] = \frac{5(2^{2/5}-1)C}{2(AL)^{2/5}}.$$

Since $V_0 = AL$, it follows that $W = \dfrac{5(2^{2/5}-1)C}{2V_0^{2/5}}$.

32. After time t, the mass of liquid remaining in the drop is $M - mt$, so that it completely disappears after time $t = M/m$. If we choose y as positive downward, then the work done by gravity in a small distance dy is $g(M - mt)\,dy$. Total work done is therefore



$$W = \int_0^{t=M/m} g(M - mt)\,dy = \int_0^{M/m} g(M - mt)\frac{dy}{dt}\,dt.$$

To find the velocity $v = dy/dt$ of the drop, we use Newton's second law in the form $F = d[(M - mt)v]/dt$. It requires

$$(M - mt)g = \frac{d}{dt}[(M - mt)v].$$

Integration gives

$$-\frac{g}{2m}(M-mt)^2 = (M-mt)v + C.$$

Since the initial velocity at time $t = 0$ is $v = 0$, we find that $C = -M^2g/(2m)$, and therefore

$$-\frac{g}{2m}(M-mt)^2 = (M-mt)v - \frac{M^2g}{2m} \qquad \Longrightarrow \qquad v = \frac{dy}{dt} = -\frac{g}{2m}(M-mt) + \frac{M^2g}{2m(M-mt)}.$$

The work done by gravity can now be calculated

$$\begin{aligned} W &= \int_0^{M/m} g(M-mt)\left[-\frac{g}{2m}(M-mt) + \frac{M^2g}{2m(M-mt)}\right] dt \\ &= \frac{g^2}{2m}\int_0^{M/m} [-(M-mt)^2 + M^2]\, dt = \frac{g^2}{2m}\left\{\frac{(M-mt)^3}{3m} + M^2t\right\}_0^{M/m} = \frac{g^2M^3}{3m^2} \text{ J.} \end{aligned}$$

33. During the time that the bell is completely submerged ($0 \le y \le 98$), the force exerted by the winch is

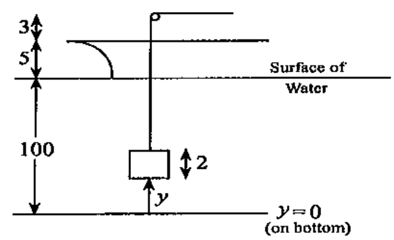
$$\begin{aligned} F(y) &= 9.81[(10\,000 - 8\,000) + 5(106-y) - (98-y)] \\ &= 9.81(2432 - 4y) \text{ N.} \end{aligned}$$

When the bell is partially submerged ($98 \le y \le 100$),

$$\begin{aligned} F(y) &= 9.81[10\,000 - 1000(2)(2)(100-y) + 5(106-y)] \\ &= 9.81(3995y - 389\,470) \text{ N.} \end{aligned}$$

When the bell is out of the water ($100 \le y \le 105$),

$$F(y) = 9.81[10\,000 + 5(106-y)] = 9.81(10\,530 - 5y) \text{ N.}$$



The work to lift the bell is therefore

$$\begin{aligned} W &= \int_0^{98} 9.81(2432-4y)\, dy + \int_{98}^{100} 9.81(3995y - 389\,470)\, dy + \int_{100}^{105} 9.81(10\,530 - 5y)\, dy \\ &= 9.81\left\{2432y - 2y^2\right\}_0^{98} + 9.81\left\{\frac{3995y^2}{2} - 389\,470y\right\}_{98}^{100} + 9.81\left\{10\,530y - \frac{5y^2}{2}\right\}_{100}^{105} \\ &= 2.76 \times 10^6 \text{ J.} \end{aligned}$$

EXERCISES 7.5

1. According to Example 7.20, the energy stored in a spring when it is stretched an amount x is $kx^2/2$. If the stretch is doubled the energy is $k(2x)^2/2 = 2kx^2$. The energy has therefore been quadrupled.

2. (a) If x represents the stretch in the spring, then the sum of the potential energy in the spring and the kinetic energy of the mass must always be constant, $C = kx^2/2 + mv^2/2$. Initially, $C = kx_0^2/2 + mv_0^2/2$. Consequently, x and v are related thereafter by the equation $kx_0^2/2 + mv_0^2/2 = kx^2/2 + mv^2/2 \Longrightarrow kx^2 + mv^2 = kx_0^2 + mv_0^2$.

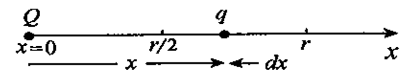
(b) Maximum stretch occurs when $v = 0$ in which case $kx^2 = kx_0^2 + mv_0^2$. This equation can be solved for $x = \sqrt{x_0^2 + mv_0^2/k}$.

(c) Maximum speed occurs when $x = 0$ in which case $mv^2 = kx_0^2 + mv_0^2$. This equation can be solved for $v = \sqrt{v_0^2 + kx_0^2/m}$.

3. From Exercise 19 in Section 7.4, the potential energy stored in the fully-drawn crossbow is 50 J. When we equate this to $mv^2/2$, and substitute for m, we obtain $50 = \frac{1}{2}\left(\frac{20}{1000}\right)v^2 \quad \Longrightarrow \quad v = 50\sqrt{2}$ m/s.

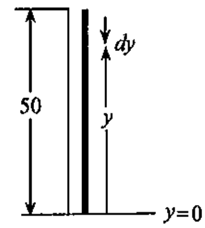
4. At position x, the force of repulsion on q is $qQ/(4\pi\epsilon_0 x^2)$. The gain in potential energy as q moves from $x = r$ to $x = r/2$ is the work done by an equal and opposite force in causing the motion,

$$W = \int_r^{r/2} \frac{-qQ}{4\pi\epsilon_0 x^2}\,dx = \left\{\frac{qQ}{4\pi\epsilon_0 x}\right\}_r^{r/2} = \frac{qQ}{4\pi\epsilon_0 r}.$$



5. (a) A length dy of the chain at height y has gravitational potential energy $9.81(2dy)y = 19.62y\,dy$. The total gravitational potential energy of the chain is therefore

$$\int_0^{50} 19.62y\,dy = \left\{9.81y^2\right\}_0^{50} = 24\,525 \text{ J}.$$



(b) The work to lift the chain to the top of the building is the gravitational potential energy that it has on the top of the building less its present potential energy, $9.81(100)(50) - 24\,525 = 24\,525$ J.

6. (a) The ultimate compression x occurs when the energy stored in the spring is equal to the original gravitational potential energy of the mass relative to this position,

$$\frac{1}{2}kx^2 = 9.81mx \quad \Longrightarrow \quad x = \frac{19.62m}{k}.$$

(b) During the oscillations of the mass, the sum of spring potential energy, gravitational potential energy, and kinetic energy will be constant. If we equate initial values of these energies (taking $x = 0$ at the uncompressed position of the spring, and x positive upward) and values at maximum compression, denoted by x,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx^2 + 9.81mx \quad \Longrightarrow \quad kx^2 + 19.62mx - mv_0^2 = 0.$$

Solutions of this quadratic equation are $x = \dfrac{-19.62m \pm \sqrt{(19.62m)^2 + 4kmv_0^2}}{2k}$. Maximum compression of the spring is therefore $\left(9.81m + \sqrt{(9.81m)^2 + kmv_0^2}\right)/k$.

7. The work done from $x = a$ to $x = b$ is $W = \displaystyle\int_a^b F(x)\,dx$. Newton's second law states that force F and acceleration a are related by $F = ma = m\dfrac{dv}{dt}$. Hence,

$$W = \int_a^b m\frac{dv}{dt}\,dx = \int_{x=a}^{x=b} m\frac{dv}{dt}\frac{dx}{dt}\,dt = \int_{x=a}^{x=b} mv\frac{dv}{dt}\,dt = \int_{x=a}^{x=b} \frac{d}{dt}\left(\frac{1}{2}mv^2\right)dt = \left\{\frac{1}{2}mv^2\right\}_{x=a}^{x=b}.$$

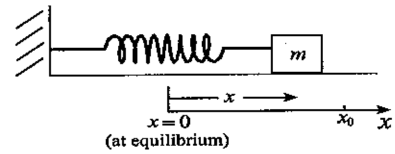
This is the difference in kinetic energies at $x = b$ and $x = a$.

8. (a) The magnitude of the force of attraction on the mass at distance x from the centre of the earth is GmM/x^2. The work done by a force equal and opposite to this in raising the mass from the earth's surface to height 10^5 m is $W = \displaystyle\int_{6.37\times 10^6}^{6.47\times 10^6} \frac{GmM}{x^2}\,dx = GmM\left\{-\frac{1}{x}\right\}_{6.37\times 10^6}^{6.47\times 10^6} = 9.67 \times 10^6$ J.

(b) If the mass is dropped from this height, this gravitational potential energy is converted into kinetic energy. If it strikes the earth with speed v, then

$$\frac{1}{2}mv^2 = 9.67 \times 10^6 \quad \Longrightarrow \quad v = \sqrt{\frac{2(9.67 \times 10^6)}{10}} = 1.4 \times 10^3 \text{ m/s}.$$

9. (a) When the mass is at position x, its kinetic energy plus spring potential energy plus the work done against friction is equal to its initial spring potential energy,



$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 + \mu mg(x_0 - x) = \frac{1}{2}kx_0^2$$
$$\Longrightarrow$$
$$kx_0^2 = mv^2 + kx^2 + 2\mu mg(x_0 - x).$$

(b) When the mass comes to a stop for the first time $v = 0$, in which case

$$kx_0^2 = kx^2 + 2\mu mg(x_0 - x) = 0 \quad \Longrightarrow \quad kx^2 - 2\mu mgx + (2\mu mgx_0 - kx_0^2) = 0.$$

Solutions of this quadratic equation are

$$x = \frac{2\mu mg \pm \sqrt{4\mu^2 m^2 g^2 - 4k(2\mu mgx_0 - kx_0^2)}}{2k} \quad \Longrightarrow \quad x = x_0,\ \frac{2\mu mg}{k} - x_0.$$

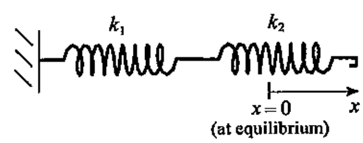
Thus, the mass stops at position $x = 2\mu mg/k - x_0$.

(c) As $\mu \to 0$, the stopping position approaches $-x_0$. This is to be expected, because without friction, the mass oscillates back and forth between $\pm x_0$.

(d) As $\mu \to 1$, the stopping position approaches $2mg/k - x_0$. This is to the left of its starting position.

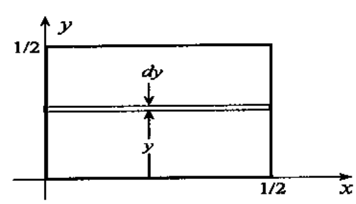
10. When the right end of the spring is moved a distance x, then $x_1 + x_2 = x$, where $x_2/x_1 = k_1/k_2$. These equations imply that $x_1 = k_2x/(k_1 + k_2)$, and $x_2 = k_1x/(k_1 + k_2)$. Since the force necessary to maintain a combined stretch x is $F(x) = k_1x_1 + k_2x_2 = \dfrac{2k_1k_2x}{k_1 + k_2}$, the work to produce total stretch L is

$$W = \int_0^L \frac{2k_1k_2x}{k_1 + k_2}\,dx = \frac{2k_1k_2}{k_1 + k_2}\left\{\frac{x^2}{2}\right\}_0^L = \frac{k_1k_2L^2}{k_1 + k_2}.$$



EXERCISES 7.6

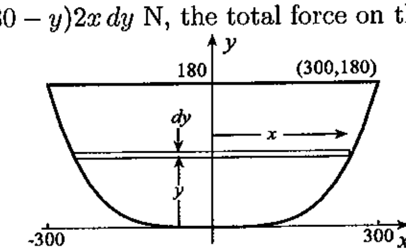
1. The force on the bottom is simply the weight of water in the tank, namely, $(1/4)(1000)(9.81) = 2452.5$ N. On each end, the force is



$$F = \int_0^{1/2} 9.81(1000)(1/2 - y)(1/2)\,dy$$
$$= \frac{4905}{2}\int_0^{1/2}(1 - 2y)\,dy = \frac{4905}{2}\left\{\frac{(1 - 2y)^2}{-4}\right\}_0^{1/2} = \frac{4905}{8}\text{ N.}$$

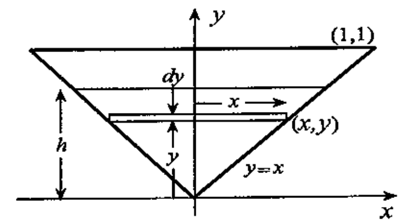
The force on the side is twice that on the ends, 4905/4 N.

2. Since the force on the representative rectangle is $9.81(1000)(180 - y)2x\,dy$ N, the total force on the dam is



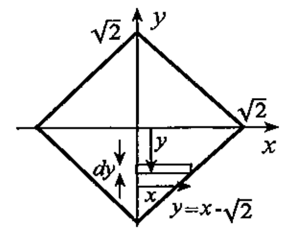
$$F = \int_0^{180} 19\,620(180 - y)(45 \times 10^6)^{1/4}y^{1/4}\,dy$$
$$= 196\,200(4500)^{1/4}\left\{144y^{5/4} - \frac{4}{9}y^{9/4}\right\}_0^{180}$$
$$= 6.78 \times 10^{10}\text{ N.}$$

3. If h is the depth of water in the trough when it is half-full, then $h^2 = 1/2 \Longrightarrow h = 1/\sqrt{2}$. Since the force on the representative rectangle is $9.81(1000)(1/\sqrt{2} - y)(2x)\,dy$, the total force on the end of the trough is



$$F = \int_0^{1/\sqrt{2}} 19\,620\left(\frac{1}{\sqrt{2}} - y\right) y\,dy$$
$$= 19\,620\left\{\frac{y^2}{2\sqrt{2}} - \frac{y^3}{3}\right\}_0^{1/\sqrt{2}} = \frac{1635}{\sqrt{2}}\ \text{N}.$$

4. The force on the rectangle is $9810(1000)(-y)x\,dy$ N. The total force on the plate is



$$F = 2\int_{-\sqrt{2}}^{0} 9810(-y)(y+\sqrt{2})\,dy$$
$$= -19\,620\left\{\frac{y^3}{3} + \frac{\sqrt{2}y^2}{2}\right\}_{-\sqrt{2}}^{0} = 9.25 \times 10^3\ \text{N}.$$

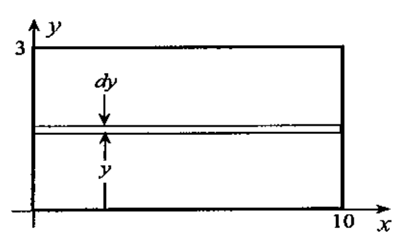
5. The force is the weight of oil in the tank, $\rho g(\pi r^2 h)$, where $g = 9.81$.

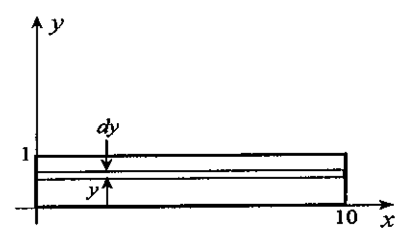
6. The force on the deep end of the pool (left figure below) is

$$F = \int_0^3 9810(3-y)(10)\,dy = 98\,100\left\{3y - \frac{y^2}{2}\right\}_0^3 = 4.41 \times 10^5\ \text{N}.$$

The force on the shallow end (right figure) is

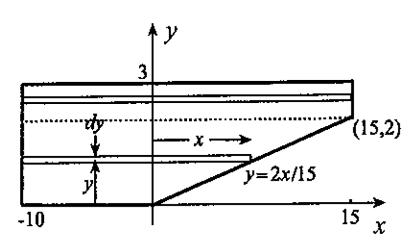
$$F = \int_0^1 9810(1-y)(10)\,dy = 98\,100\left\{y - \frac{y^2}{2}\right\}_0^1 = 4.91 \times 10^4\ \text{N}.$$





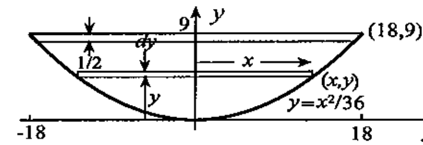
The force on each side of the pool is

$$F = \int_0^2 9810(3-y)(x+10)\,dy + \int_2^3 9810(3-y)(25)\,dy$$
$$= 9810\int_0^2 (3-y)\left(\frac{15y}{2} + 10\right) dy + 25(9810)\int_2^3 (3-y)\,dy$$
$$= \frac{5(9810)}{2}\int_0^2 (12 + 5y - 3y^2)\,dy + 25(9810)\int_2^3 (3-y)\,dy$$
$$= \frac{5(9810)}{2}\left\{12y + \frac{5y^2}{2} - y^3\right\}_0^2 + 25(9810)\left\{3y - \frac{y^2}{2}\right\}_2^3$$
$$= 7.60 \times 10^5\ \text{N}.$$



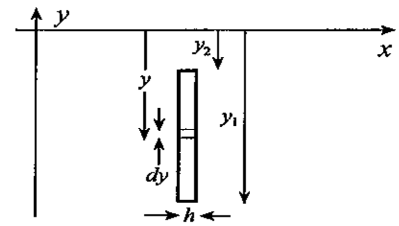
7. Since the force on the representative rectangle is $9.81(1000)(17/2 - y)(2x)\,dy$, the force on the dam is

$$F = 19\,620 \int_0^{17/2} \left(\frac{17}{2} - y\right) 6\sqrt{y}\,dy$$
$$= 58\,860 \int_0^{17/2} (17\sqrt{y} - 2y^{3/2})\,dy$$
$$= 58\,860 \left\{\frac{34y^{3/2}}{3} - \frac{4y^{5/2}}{5}\right\}_0^{17/2} = 6.61 \times 10^6 \text{ N}.$$



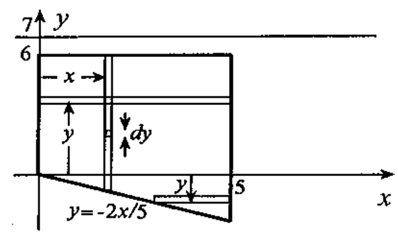
8. Since the force on the tiny rectangle shown is $9.81\rho(-y)(h dy)$ N, the force on the long rectangle is

$$F = \int_{y_1}^{y_2} -9.81\rho h y\,dy = -9.81\rho h \left\{\frac{y^2}{2}\right\}_{y_1}^{y_2}$$
$$= \frac{9.81\rho h}{2}(y_1^2 - y_2^2) \text{ N}.$$



9. With horizontal rectangles,

$$F = \int_{-2}^{0} 9.81\rho(7-y)(5-x)\,dy + \int_0^6 9.81\rho(7-y)(5)\,dy$$
$$= 9.81\rho \int_{-2}^{0} (7-y)\left(5 + \frac{5y}{2}\right) dy + 49.05\rho \left\{-\frac{1}{2}(7-y)^2\right\}_0^6$$
$$= 24.525\rho \int_{-2}^{0} (14 + 5y - y^2)\,dy + 1177.2\rho$$
$$= 24.525\rho \left\{14y + \frac{5y^2}{2} - \frac{y^3}{3}\right\}_{-2}^{0} + 1177.2\rho = 1553.25\rho \text{ N}.$$



There is a temptation to use Exercise 8 as a formula for vertical rectangles. This would be a mistake since the formula was based on a coordinate system different from the one in this problem. Instead, we divide the long vertical rectangle in the figure into smaller rectangles of width dy. Since the force on this rectangle is $9.81\rho(7-y)dx\,dy$, the force on the long vertical rectangle is

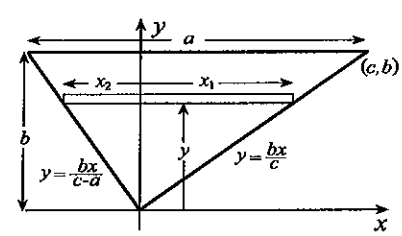
$$\int_{-2x/5}^{6} 9.81\rho(7-y)dx\,dy = 9.81\rho\,dx \left\{-\frac{(7-y)^2}{2}\right\}_{-2x/5}^{6} = 4.905\rho\,dx \left[\left(7 + \frac{2x}{5}\right)^2 - 1\right].$$

The force on the plate is therefore

$$F = \int_0^5 4.905\rho \left[\left(7 + \frac{2x}{5}\right)^2 - 1\right] dx = 4.905\rho \left\{\frac{5}{6}\left(7 + \frac{2x}{5}\right)^3 - x\right\}_0^5 = 1553.25\rho \text{ N}.$$

10. The force on the rectangle is

$$9.81\rho(b-y)(x_1 - x_2)\,dy$$
$$= 9.81\rho(b-y)\left[\frac{cy}{b} - \left(\frac{c-a}{b}\right)y\right] dy$$
$$= \frac{9.81\rho a}{b}(b-y)y\,dy \text{ N}.$$

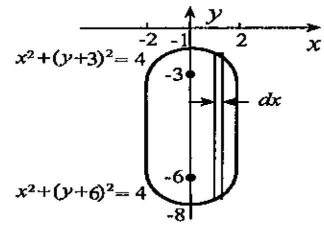


The total force on the plate is

$$F = \int_0^b \frac{9.81\rho a}{b}(by - y^2)\,dy = \frac{9.81\rho a}{b}\left\{\frac{by^2}{2} - \frac{y^3}{3}\right\}_0^b = \frac{9.81\rho a b^2}{6} \text{ N}.$$

11. Since the x-axis is in the surface of the water, we may use the formula of Exercise 8 with vertical rectangles,

$$F = 2\int_0^2 \frac{9.81(1000)}{2}\left[\left(-6-\sqrt{4-x^2}\right)^2 - \left(-3+\sqrt{4-x^2}\right)^2\right] dx$$
$$= 9810\int_0^2 \left(27+18\sqrt{4-x^2}\right) dx$$
$$= 88\,290\int_0^2 \left(3+2\sqrt{4-x^2}\right) dx \text{ N}.$$

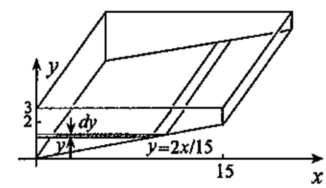


12. The force on the horizontal part of the bottom is the weight of the water directly above it:

$9810(10)(3)(10) = 2.943 \times 10^6$ N.

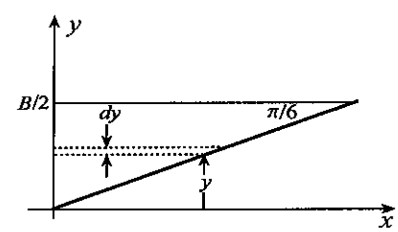
For the slanted part of the bottom we notice that differential dy gives rise to a rectangle of width $\sqrt{229}dy/2$ across the bottom. The force on this part is therefore

$$F = \int_0^2 9810(3-y)(10)\left(\frac{\sqrt{229}}{2}dy\right)$$
$$= 5\sqrt{229}(9810)\left\{3y - \frac{y^2}{2}\right\}_0^2 = 2.969 \times 10^6 \text{ N}.$$



13. Differential dy gives rise to a rectangle of width $2\,dy$. The force on the bow is

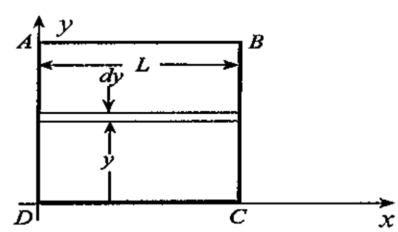
$$F = \int_0^{B/2} 9.81(1000)\left(\frac{B}{2}-y\right)A(2\,dy)$$
$$= 9810A\left\{-\frac{1}{4}(B-2y)^2\right\}_0^{B/2}$$
$$= \frac{4905AB^2}{2} \text{ N}.$$



14. If lengths of AB and BC are L m, the force on face $ABCD$ is

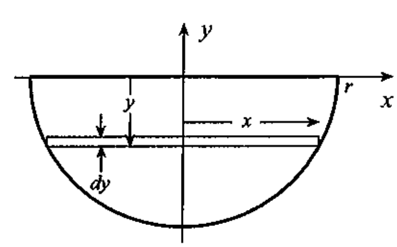
$$F = \int_0^L 9810(L-y)L\,dy$$
$$= 9810L\left\{Ly - \frac{y^2}{2}\right\}_0^L = 4905L^3 \text{ N}.$$

When we set this equal to 20 000 and solve, the result is $L = (20\,000/4905)^{1/3} = 1.60$ m.



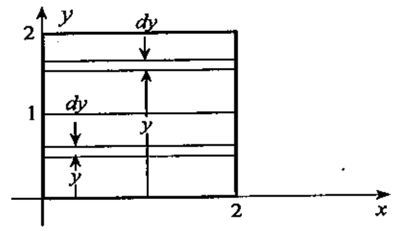
15. The force is

$$F = \int_{-r}^0 9.81\rho(-y)(2x)\,dy$$
$$= -19.62\rho\int_{-r}^0 y\sqrt{r^2-y^2}\,dy$$
$$= -19.62\rho\left\{-\frac{1}{3}(r^2-y^2)^{3/2}\right\}_{-r}^0 = 6.54\rho r^3.$$



16. If the top of the cylinder is at depth d, then the magnitude of the force on this face is $9.81\rho d(\pi r^2)$. Since the force on the bottom is $9.81\rho(d+h)\pi r^2$, the resultant vertical force on the cylinder is $9.81\rho(d+h)\pi r^2 - 9.82\rho d(\pi r^2) = 9.81\rho(\pi r^2 h)$. This is the weight of the fluid displaced by the cylinder.

17. (a) The force F_1 on the upper half is straightforward,



$$F_1 = \int_1^2 9.81(900)(2-y)(2)\,dy$$

$$= 17\,658\left\{-\frac{1}{2}(2-y)^2\right\}_1^2 = 8829 \text{ N}.$$

Pressure at a point y on the lower half of the surface is that due to the weight of fluid of unit cross-sectional area above the point,

$$9.81[900 + 1000(1-y)] = 981(19-10y).$$

The force on the lower half is therefore

$$F_2 = \int_0^1 981(19-10y)(2)\,dy = 1962\left\{-\frac{1}{20}(19-10y)^2\right\}_0^1 = 27\,468 \text{ N}.$$

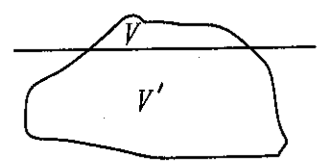
The total force is $F_1 + F_2 = 36\,297$ N.

(b) If the water and oil create a mixture with density 0.95 gm/cm^3, the force on each side is

$$F = \int_0^2 9.81(950)(2-y)(2)\,dy = 18\,639\left\{-\frac{1}{2}(2-y)^2\right\}_0^2 = 37\,278 \text{ N}.$$

Thus, the force increases by 9810 N.

18. (a) Suppose V and V' are the volumes of the object above and below the surface respectively. The buoyant force on the object is therefore $9.81\rho_w V'$. Because the object is floating at this position, this force is equal to that of gravity on the object;



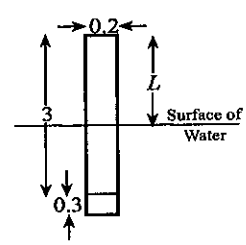
$$9.81\rho_w V' = 9.81\rho_o(V+V').$$

Thus, $\dfrac{V'}{V+V'} = \dfrac{\rho_o}{\rho_w}$. The percentage of the volume of the object above water is

$$100\frac{V}{V+V'} = 100\left(1 - \frac{V'}{V+V'}\right) = 100\left(1 - \frac{\rho_o}{\rho_w}\right) = 100\frac{\rho_w - \rho_o}{\rho_w}.$$

(b) For an iceberg, this percentage is $100\dfrac{1000-915}{1000} = 8.5$.

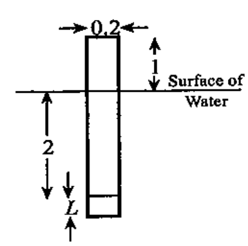
19. Suppose L represents the length of buoy above water. Archimedes' principle states that the weight of the buoy is equal to the weight of water displaced by the buoy,



$$\pi(0.1)^2(3)(500)g + \pi(0.1)^2(0.3)(3000)g =$$
$$\pi(0.1)^2(3.3-L)(1000)g.$$

The solution of this equation is $L = 0.9$ m.

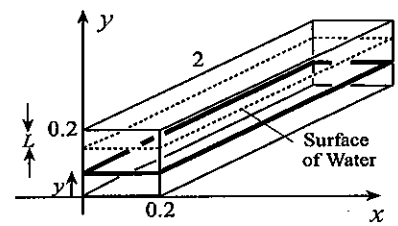
20. Suppose L is the length of the concrete attachment. Archimedes' principle states that the weight of the buoy must be equal to the weight of water displaced by the buoy,



$$\pi(0.1)^2(3)(500)g + \pi(0.1)^2(L)(3000)g =$$
$$\pi(0.1)^2(2+L)(1000)g.$$

The solution of this equation is $L = 1/4$ m.

21. Suppose L denotes the height of log above water.
The density of the log is $\rho(y) = 1000 - 2500y$ kg/m^3.
The weight of the log is



$$\begin{aligned} W_{\text{log}} &= \int_0^{1/5} (1000 - 2500y) g \left(\frac{1}{5}\right)(2)\,dy \\ &= 200g \int_0^{1/5} (2 - 5y)\,dy \\ &= 200g \left\{ 2y - \frac{5y^2}{2} \right\}_0^{1/5} = 60g \text{ N.} \end{aligned}$$

The weight of the water displaced by the log is

$$W_{\text{water}} = 1000g\left(\frac{1}{5} - L\right)\left(\frac{1}{5}\right)(2) = 80g(1 - 5L) \text{ N.}$$

Archimedes' principle requires $W_{\text{log}} = W_{\text{water}}$ so that $60g = 80g(1 - 5L) \Longrightarrow L = 1/20$ m.

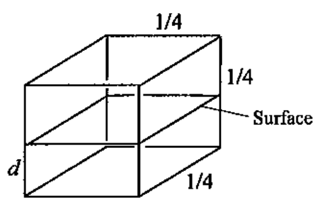
22. (a) The pressure at A due to the mercury in the tube is equal to the weight of a column of mercury of unit cross-sectional area and height h. With density of mercury equal to 13.6×10^3 kg/m^3, this weight is $(9.81)(13.6 \times 10^3)h = 1.33 \times 10^5 h$ N/m^2.
(b) When $h = .761$, atmospheric pressure is $(1.33 \times 10^5)(.761) = 1.01 \times 10^5$ N/m^2.

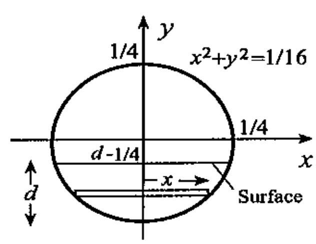
23. (a) If d is the depth of the lower face below the surface (left figure below), then the buoyant force due to water pressure is the weight of water displaced:

$$F = 9810\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)d = \frac{9810d}{16}.$$

Because the block is floating, this force is equal to the weight of the block; that is,

$$\frac{9810d}{16} = \left(\frac{1}{4}\right)^3 (400)(9.81) \quad \Longrightarrow \quad d = 10 \text{ cm.}$$





(b) When the lowest point of the sphere is at depth d (right figure above), the volume of water displaced is

$$V = \int_{-1/4}^{d-1/4} \pi x^2\,dy = \pi \int_{-1/4}^{d-1/4} \left(\frac{1}{16} - y^2\right) dy = \pi \left\{ \frac{y}{16} - \frac{y^3}{3} \right\}_{-1/4}^{d-1/4} = \frac{\pi}{12}(3d^2 - 4d^3).$$

At this position, the weight of water displaced is equal to the weight of the object:

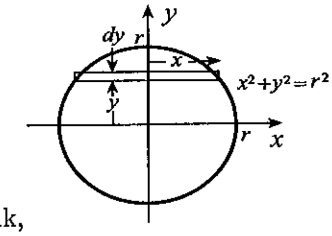
$$\frac{\pi}{12}(3d^2 - 4d^3)(9810) = \frac{4}{3}\pi\left(\frac{1}{4}\right)^3 (9.81)(400).$$

This equation simplifies to $40d^3 - 30d^2 + 1 = 0$. Newton's iterative procedure

$$d_{n+1} = d_n - \frac{40d_n^3 - 30d_n^2 + 1}{120d_n^2 - 60d_n}$$

with an initial approximation $d_1 = 1/8$ gives $d = 0.2165$. The lowest point is therefore 21.65 cm below the surface.

24. The force on the end of the tank when the radius is r is



$$F = \int_{-r}^{r} 9.81\rho(r-y)2\sqrt{r^2-y^2}\,dy$$
$$= 19.62\rho r \int_{-r}^{r} \sqrt{r^2-y^2}\,dy + 19.62\rho\left\{\frac{1}{3}(r^2-y^2)^{3/2}\right\}_{-r}^{r}.$$

Since the integral represents half the area of the end of the tank,

$$F = 19.62\rho r\left(\frac{1}{2}\pi r^2\right) = 9.81\pi\rho r^3.$$

The radius of the end of the tank must satisfy $40\,000 - 2\pi r(1000) = 9.81\pi\rho r^3$. We use Newton's iterative procedure with initial approximation $r = 1$ to solve this equation,

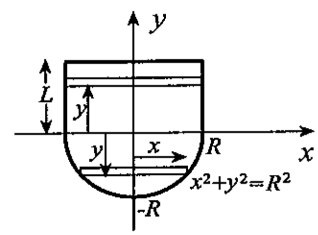
$$r_1 = 1, \qquad r_{n+1} = r_n - \frac{9.81(1019)\pi r_n^3 - 2000\pi r_n - 40\,000}{29.43(1019)\pi r_n^2 - 2000\pi}.$$

The result is $r = 1.02$ m.

25. The force on the rectangular part is

$$F_R = \int_0^L 9.81\rho(L-y)2R\,dy = 19.62\rho R\left\{Ly - \frac{y^2}{2}\right\}_0^L = 9.81\rho RL^2.$$

The force on the semicircular part is



$$F_S = \int_{-R}^{0} 9.81\rho(L-y)2x\,dy$$
$$= 19.62\rho\int_{-R}^{0}(L-y)\sqrt{R^2-y^2}\,dy$$
$$= 19.62\rho L\int_{-R}^{0}\sqrt{R^2-y^2}\,dy - 19.62\rho\int_{-R}^{0} y\sqrt{R^2-y^2}\,dy.$$

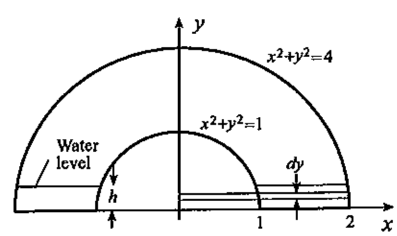
The first integral is the area of half the semicircular part, and therefore

$$F_S = 19.62\rho L\left(\frac{1}{4}\pi R^2\right) - 19.62\rho\left\{-\frac{1}{3}(R^2-y^2)^{3/2}\right\}_{-R}^{0} = 4.905\pi\rho LR^2 + 6.54\rho R^3.$$

These forces are equal when $9.81\rho RL^2 = 4.905\pi\rho LR^2 + 6.54\rho R^3$, and this equation simplifies to $6\left(\frac{L}{R}\right)^2 - 3\pi\left(\frac{L}{R}\right) - 4 = 0$. The positive solution of this quadratic equation in L/R is $L/R = 1.92$.

26. (a) If the shell is placed carefully on the water, no water will penetrate the hemispherical cavity. If the depth of the flat edge is h (figure to the right), then the volume of water displaced by the shell is

$$V = \int_0^h \pi x^2\,dy = \pi\int_0^h (4-y^2)\,dy$$
$$= \pi\left\{4y - \frac{y^3}{3}\right\}_0^h = \pi\left(4h - \frac{h^3}{3}\right)$$
$$= \frac{\pi h(12-h^2)}{3}.$$



According to Archimedes' principle, the weight of water displaced must be equal to the weight of the shell; that is,

$$9.81\left(\frac{2}{3}\pi\right)(2^3-1^3)(2) = 9.81(1000)\frac{\pi h(12-h^2)}{3}.$$

This equation reduces to $250h^3 - 3000h + 7 = 0$. With Newton's iterative procedure

$$h_1 = 0.0025, \qquad h_{n+1} = h_n - \frac{250h_n^3 - 3000h_n + 7}{750h_n^2 - 3000},$$

we obtain $h = 0.00233$. Hence, the shell sinks 2.33 mm.

(b) With a hole in the top of the shell, air will escape from the cavity and the water level inside the cavity will be the same as outside; only the water displaced by the shell itself is taken into account. The volume of water displaced is

$$V = \int_0^h \pi[(4 - y^2 - (1 - y^2)]\,dy = \pi \int_0^h 3\,dy = 3\pi h.$$

In this case, Archimedes' principle requires

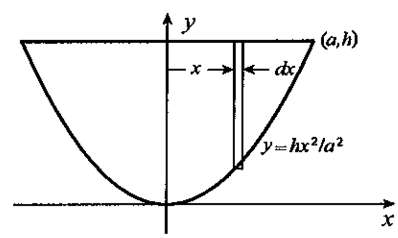
$$9.81\left(\frac{2}{3}\pi\right)(2^3 - 1^3)(2) = 9.81(1000)3\pi h,$$

and therefore $h = 0.00311$. The shell now sinks 3.11 mm.

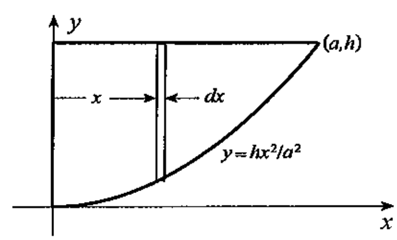
EXERCISES 7.7

1. By symmetry, $\overline{x} = 0$. $\quad M = 2\int_0^a \rho\left(h - \frac{hx^2}{a^2}\right)dx = 2\rho h\left\{x - \frac{x^3}{3a^2}\right\}_0^a = \frac{4\rho a h}{3}$

Since
$$M\overline{y} = 2\int_0^a \rho\left(\frac{h + hx^2/a^2}{2}\right)\left(h - \frac{hx^2}{a^2}\right)dx$$
$$= \rho h^2 \int_0^a \left(1 - \frac{x^4}{a^4}\right)dx$$
$$= \rho h^2\left\{x - \frac{x^5}{5a^4}\right\}_0^a = \frac{4\rho a h^2}{5},$$
we find $\overline{y} = \frac{4\rho a h^2}{5}\frac{3}{4\rho a h} = \frac{3h}{5}$.



2. $M = \int_0^a \rho\left(h - \frac{hx^2}{a^2}\right)dx$
$$= \rho h\left\{x - \frac{x^3}{3a^2}\right\}_0^a = \frac{2\rho a h}{3}$$
Since
$$M\overline{x} = \int_0^a x\rho\left(h - \frac{hx^2}{a^2}\right)dx$$
$$= \rho h\left\{\frac{x^2}{2} - \frac{x^4}{4a^2}\right\}_0^a = \frac{\rho h a^2}{4},$$

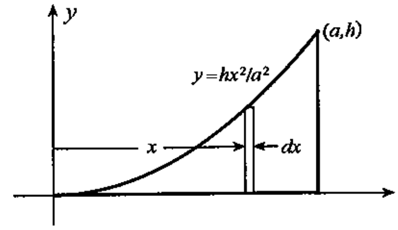


it follows that $\overline{x} = \frac{\rho h a^2}{4}\frac{3}{2\rho a h} = \frac{3a}{8}$. Since

$$M\overline{y} = \int_0^a \rho\left(\frac{h + hx^2/a^2}{2}\right)\left(h - \frac{hx^2}{a^2}\right)dx = \frac{\rho h^2}{2}\int_0^a\left(1 - \frac{x^4}{a^4}\right)dx = \frac{\rho h^2}{2}\left\{x - \frac{x^5}{5a^4}\right\}_0^a = \frac{2\rho a h^2}{5},$$

we find $\overline{y} = \frac{2\rho a h^2}{5}\frac{3}{2\rho a h} = \frac{3h}{5}$.

3. $M = \int_0^a \rho \left(\frac{hx^2}{a^2}\right) dx = \frac{\rho h}{a^2} \left\{\frac{x^3}{3}\right\}_0^a = \frac{\rho h a}{3}$

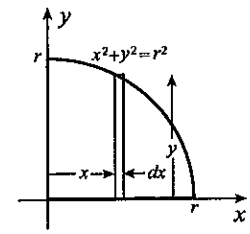


Since $M\overline{x} = \int_0^a \rho \left(\frac{hx^2}{a^2}\right) x\, dx = \frac{\rho h}{a^2} \left\{\frac{x^4}{4}\right\}_0^a = \frac{\rho h a^2}{4}$,

it follows that $\overline{x} = \frac{\rho h a^2}{4} \frac{3}{\rho h a} = \frac{3a}{4}$.

Since $M\overline{y} = \int_0^a \rho \left(\frac{hx^2}{a^2}\right)\left(\frac{hx^2}{2a^2}\right) dx = \frac{\rho h^2}{2a^4} \left\{\frac{x^5}{5}\right\}_0^a = \frac{\rho h^2 a}{10}$, we obtain $\overline{y} = \frac{\rho h^2 a}{10} \frac{3}{\rho h a} = \frac{3h}{10}$.

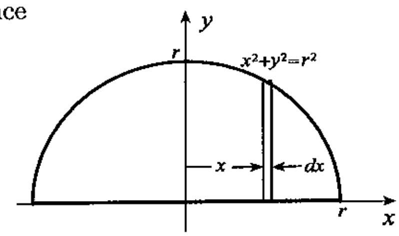
4. The mass of the plate is $M = \rho \pi r^2/4$. Since



$$M\overline{x} = \int_0^r x \rho y\, dx = \rho \int_0^r x\sqrt{r^2 - x^2}\, dx$$
$$= \rho \left\{-\frac{1}{3}(r^2 - x^2)^{3/2}\right\}_0^r = \frac{\rho r^3}{3},$$

it follows that $\overline{x} = \frac{\rho r^3}{3} \frac{4}{\pi \rho r^2} = \frac{4r}{3\pi}$. By symmetry, $\overline{y} = \overline{x} = 4r/(3\pi)$.

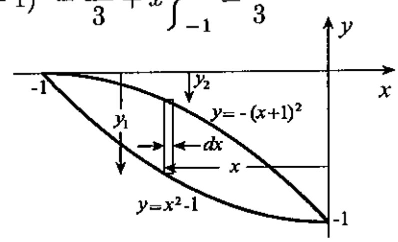
5. By symmetry, $\overline{x} = 0$. The mass of the plate is $M = \rho \pi r^2/2$. Since



$$M\overline{y} = 2\int_0^r \rho\sqrt{r^2 - x^2}\left(\frac{\sqrt{r^2 - x^2}}{2}\right) dx$$
$$= \rho \left\{r^2 x - \frac{x^3}{3}\right\}_0^r = \frac{2\rho r^3}{3},$$

it follows that $\overline{y} = \frac{2\rho r^3}{3} \frac{2}{\rho \pi r^2} = \frac{4r}{3\pi}$.

6. $A = \int_{-1}^0 (y_2 - y_1)\, dx = \int_{-1}^0 [-(x+1)^2 - (x^2 - 1)]\, dx = \left\{-\frac{1}{3}(x+1)^3 - \frac{x^3}{3} + x\right\}_{-1}^0 = \frac{1}{3}$



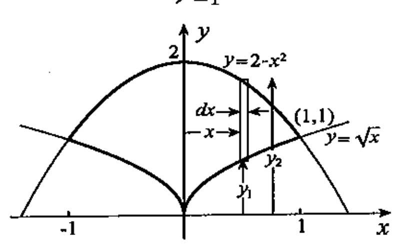
Since $A\overline{x} = \int_{-1}^0 x(y_2 - y_1)\, dx = \int_{-1}^0 x[-(x+1)^2 - (x^2 - 1)]\, dx$
$$= \int_{-1}^0 (-2x^3 - 2x^2)\, dx = \left\{-\frac{x^4}{2} - \frac{2x^3}{3}\right\}_{-1}^0 = -\frac{1}{6},$$

it follows that $\overline{x} = -\frac{1}{6}(3) = -\frac{1}{2}$. Since

$$A\overline{y} = \int_{-1}^0 \left(\frac{y_1 + y_2}{2}\right)(y_2 - y_1)\, dx = \frac{1}{2}\int_{-1}^0 (y_2^2 - y_1^2)\, dx$$
$$= \frac{1}{2}\int_{-1}^0 [(x+1)^4 - (x^4 - 2x^2 + 1)]\, dx = \frac{1}{2}\left\{\frac{1}{5}(x+1)^5 - \frac{x^5}{5} + \frac{2x^3}{3} - x\right\}_{-1}^0 = -\frac{1}{6},$$

we find $\overline{y} = -3/6 = -1/2$.

7. By symmetry, $\overline{x} = 0$. The area is

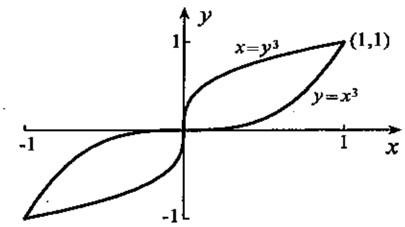


$$A = 2\int_0^1 (y_2 - y_1)\, dx = 2\int_0^1 (2 - x^2 - \sqrt{x})\, dx$$
$$= 2\left\{2x - \frac{x^3}{3} - \frac{2x^{3/2}}{3}\right\}_0^1 = 2.$$

Since $A\overline{y} = 2\int_0^1 \left(\frac{y_1 + y_2}{2}\right)(y_2 - y_1)\, dx = \int_0^1 [(2 - x^2)^2 - x]\, dx$
$$= \int_0^1 (4 - x - 4x^2 + x^4)\, dx = \left\{4x - \frac{x^2}{2} - \frac{4x^3}{3} + \frac{x^5}{5}\right\}_0^1 = \frac{71}{30},$$

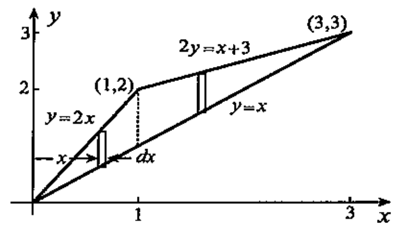
it follows that $\overline{y} = (71/30)(1/2) = 71/60$.

8. By symmetry, $\overline{x} = \overline{y} = 0$.



9. $A = \int_0^1 (2x - x)\, dx + \int_1^3 \left(\frac{x+3}{2} - x\right) dx$

$$= \left\{\frac{x^2}{2}\right\}_0^1 + \left\{\frac{3x}{2} - \frac{x^2}{4}\right\}_1^3 = \frac{3}{2}$$



Since $A\overline{x} = \int_0^1 x(2x - x)\, dx + \int_1^3 x\left(\frac{x+3}{2} - x\right) dx$

$$= \left\{\frac{x^3}{3}\right\}_0^1 + \left\{\frac{3x^2}{4} - \frac{x^3}{6}\right\}_1^3 = 2,$$

it follows that $\overline{x} = 2(2/3) = 4/3$. Since

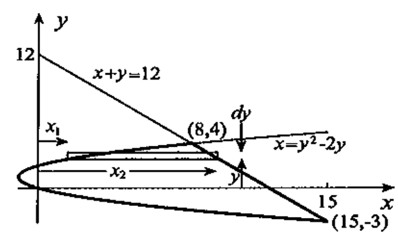
$$A\overline{y} = \int_0^1 \left(\frac{2x+x}{2}\right)(2x - x)\, dx + \int_1^3 \left[\frac{(x+3)/2 + x}{2}\right)\left(\frac{x+3}{2} - x\right) dx$$

$$= \frac{1}{2}\int_0^1 3x^2\, dx + \frac{1}{2}\int_1^3 \left[\left(\frac{x+3}{2}\right)^2 - x^2\right] dx = \frac{1}{2}\{x^3\}_0^1 + \frac{1}{2}\left\{\frac{2}{3}\left(\frac{x+3}{2}\right)^3 - \frac{x^3}{3}\right\}_1^3 = \frac{5}{2},$$

we find $\overline{y} = (5/2)(2/3) = 5/3$.

10. $A = \int_{-3}^4 (x_2 - x_1)\, dy = \int_{-3}^4 [(12 - y) - (y^2 - 2y)]\, dy$

$$= \left\{12y + \frac{y^2}{2} - \frac{y^3}{3}\right\}_{-3}^4 = \frac{343}{6}$$



Since $A\overline{x} = \int_{-3}^4 \left(\frac{x_2 + x_1}{2}\right)(x_2 - x_1)\, dy$

$$= \frac{1}{2}\int_{-3}^4 (x_2^2 - x_1^2)\, dy$$

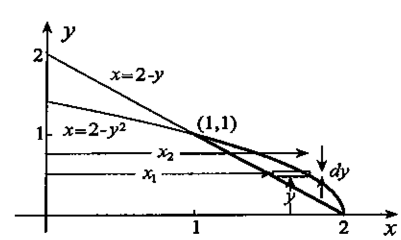
$$= \frac{1}{2}\int_{-3}^4 [(12 - y)^2 - (y^4 - 4y^3 + 4y^2)]\, dy = \frac{1}{2}\left\{-\frac{1}{3}(12 - y)^3 - \frac{y^5}{5} + y^4 - \frac{4y^3}{3}\right\}_{-3}^4 = \frac{3773}{10},$$

it follows that $\overline{x} = (3773/10)(6/343) = 33/5$. Since

$$A\overline{y} = \int_{-3}^4 y(x_2 - x_1)\, dy = \int_{-3}^4 y(12 + y - y^2)\, dy = \left\{6y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right\}_{-3}^4 = \frac{343}{12},$$

we find $\overline{y} = (343/12)(6/343) = 1/2$.

11. $A = \int_0^1 (x_2 - x_1)\, dy = \int_0^1 [(2 - y^2) - (2 - y)]\, dy = \left\{\frac{y^2}{2} - \frac{y^3}{3}\right\}_0^1 = \frac{1}{6}$



Since $A\overline{x} = \int_0^1 \left(\frac{x_1 + x_2}{2}\right)(x_2 - x_1)\, dy = \frac{1}{2}\int_0^1 (x_2^2 - x_1^2)\, dy$

$$= \frac{1}{2}\int_0^1 [(2 - y^2)^2 - (2 - y)^2]\, dy$$

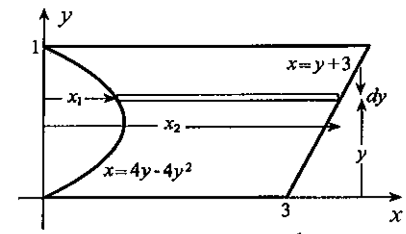
$$= \frac{1}{2}\int_0^1 (y^4 - 5y^2 + 4y)\, dy = \frac{1}{2}\left\{\frac{y^5}{5} - \frac{5y^3}{3} + 2y^2\right\}_0^1 = \frac{4}{15},$$

it follows that $\overline{x} = (4/15)(6) = 8/5$. Since

$$A\overline{y} = \int_0^1 y[(2-y^2)-(2-y)]\,dy = \int_0^1 (y^2-y^3)\,dy = \left\{\frac{y^3}{3}-\frac{y^4}{4}\right\}_0^1 = \frac{1}{12},$$

we find that $\overline{y} = (1/12)(6) = 1/2$.

12. $A = \int_0^1 (x_2 - x_1)\,dy = \int_0^1 [(y+3)-(4y-4y^2)]\,dy$



$$= \left\{3y - \frac{3y^2}{2} + \frac{4y^3}{3}\right\}_0^1 = \frac{17}{6}$$

Since $A\overline{x} = \int_0^1 \left(\frac{x_2+x_1}{2}\right)(x_2-x_1)\,dy = \frac{1}{2}\int_0^1 (x_2^2 - x_1^2)\,dy$

$$= \frac{1}{2}\int_0^1 [(y+3)^2 - (16y^2-32y^3+16y^4)]\,dy = \frac{1}{2}\left\{\frac{1}{3}(y+3)^3 - \frac{16y^3}{3} + 8y^4 - \frac{16y^5}{5}\right\}_0^1 = \frac{59}{10},$$

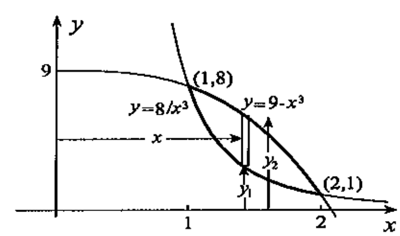
we find that $\overline{x} = (59/10)(6/17) = (177/85)$. Because

$$A\overline{y} = \int_0^1 y(x_2-x_1)\,dy = \int_0^1 y(3-3y+4y^2)\,dy = \left\{\frac{3y^2}{2} - y^3 + y^4\right\}_0^1 = \frac{3}{2},$$

it follows that $\overline{y} = (3/2)(6/17) = (9/17)$.

13. $A = \int_1^2 (y_2 - y_1)\,dx = \int_1^2 \left(9 - x^3 - \frac{8}{x^3}\right)dx$

$$= \left\{9x - \frac{x^4}{4} + \frac{4}{x^2}\right\}_1^2 = \frac{9}{4}$$



Since $A\overline{x} = \int_1^2 x\left(9 - x^3 - \frac{8}{x^3}\right)dx$

$$= \left\{\frac{9x^2}{2} - \frac{x^5}{5} + \frac{8}{x}\right\}_1^2 = \frac{33}{10},$$

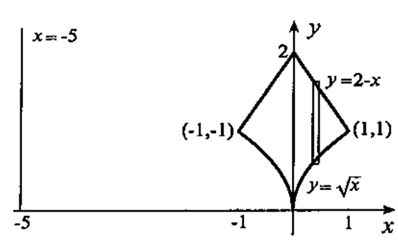
it follows that $\overline{x} = (33/10)(4/9) = 22/15$. Since

$$A\overline{y} = \int_1^2 \left(\frac{y_1+y_2}{2}\right)(y_2-y_1)\,dx = \frac{1}{2}\int_1^2 (y_2^2 - y_1^2)\,dx = \frac{1}{2}\int_1^2 \left[(9-x^3)^2 - \frac{64}{x^6}\right]dx$$

$$= \frac{1}{2}\int_1^2 \left(81 - 18x^3 + x^6 - \frac{64}{x^6}\right)dx = \frac{1}{2}\left\{81x - \frac{9x^4}{2} + \frac{x^7}{7} + \frac{64}{5x^5}\right\}_1^2 = \frac{4041}{420},$$

we find that $\overline{y} = (4041/420)(4/9) = 449/105$.

14. Since the centre of mass is on the y-axis, the first moment is $5M$ where M is the mass of the plate. Because the plate is symmetric about the y-axis, we find the mass of the right half and double the result:

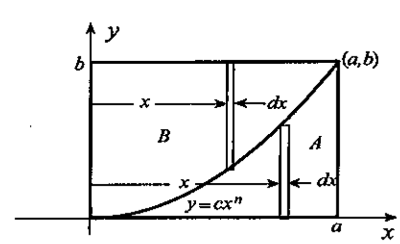


$$\text{Moment } = 5(2)\int_0^1 \rho[(2-x)-\sqrt{x}]\,dx$$

$$= 10\rho\left\{2x - \frac{x^2}{2} - \frac{2}{3}x^{3/2}\right\}_0^1 = \frac{25\rho}{3}.$$

15. First we notice that $b = ca^n \implies c = b/a^n$.
Areas of the regions are

$$A = \int_0^a cx^n\,dx = c\left\{\frac{x^{n+1}}{n+1}\right\}_0^a = \frac{ca^{n+1}}{n+1}, \quad B = ab - \frac{ca^{n+1}}{n+1}.$$



We now calculate moments of A and B about the axes,

$$\begin{aligned}
A\overline{x}_A &= \int_0^a x(cx^n)\,dx = c\left\{\frac{x^{n+2}}{n+2}\right\}_0^a = \frac{ca^{n+2}}{n+2},\\
B\overline{x}_B &= \int_0^a x(b-cx^n)\,dx = \left\{\frac{bx^2}{2}-\frac{cx^{n+2}}{n+2}\right\}_0^a = \frac{ba^2}{2}-\frac{ca^{n+2}}{n+2},\\
A\overline{y}_A &= \int_0^a \frac{1}{2}(cx^n)(cx^n)\,dx = \frac{c^2}{2}\left\{\frac{x^{2n+1}}{2n+1}\right\}_0^a = \frac{c^2a^{2n+1}}{2(2n+1)},\\
B\overline{y}_B &= \int_0^a \frac{1}{2}(b+cx^n)(b-cx^n)\,dx = \frac{1}{2}\int_0^a (b^2-c^2x^{2n})\,dx = \frac{1}{2}\left\{b^2x-\frac{c^2x^{2n+1}}{2n+1}\right\}_0^a = \frac{1}{2}\left(b^2a-\frac{c^2a^{2n+1}}{2n+1}\right).
\end{aligned}$$

We can now use $c = b/a^n$ to find

$$\overline{x}_A = \frac{ca^{n+2}}{n+2}\frac{n+1}{ca^{n+1}} = \left(\frac{n+1}{n+2}\right)a, \quad \overline{x}_B = \left(\frac{ba^2}{2}-\frac{ca^{n+2}}{n+2}\right)\left[\frac{n+1}{ab(n+1)-ca^{n+1}}\right] = \left(\frac{n+1}{2n+4}\right)a,$$

$$\overline{y}_A = \frac{c^2a^{2n+1}}{2(2n+1)}\frac{n+1}{ca^{n+1}} = \left(\frac{n+1}{4n+2}\right)b, \quad \overline{y}_B = \frac{1}{2}\left(ab^2-\frac{c^2a^{2n+1}}{2n+1}\right)\left[\frac{n+1}{ab(n+1)-ca^{n+1}}\right] = \left(\frac{n+1}{2n+1}\right)b.$$

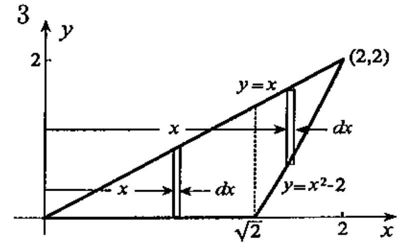
16. The mass of the seesaw itself is $2\rho L$, and this mass may be considered to act at its centre of mass, the midpoint of the seesaw. For balance to occur, the total first moment about the fulcrum must vanish:

$$0 = \sum_{i=1}^{6} m_i(x_i-\overline{x}) + 2\rho L(L-\overline{x}).$$

If we set $M = \sum_{i=1}^{6} m_i$, then $\overline{x}(M+2\rho L) = \sum_{i=1}^{6} m_ix_i + 2\rho L^2$, and therefore

$$\overline{x} = \frac{1}{M+2\rho L}\left(\sum_{i=1}^{6} m_ix_i + 2\rho L^2\right).$$

17. $A = \frac{1}{2}(2) + \int_{\sqrt{2}}^{2}(x-x^2+2)\,dx = 1 + \left\{\frac{x^2}{2}-\frac{x^3}{3}+2x\right\}_{\sqrt{2}}^{2} = \frac{10-4\sqrt{2}}{3}$



Since $A\overline{x} = \int_0^{\sqrt{2}} x(x)\,dx + \int_{\sqrt{2}}^{2} x(x-x^2+2)\,dx$

$$= \left\{\frac{x^3}{3}\right\}_0^{\sqrt{2}} + \left\{\frac{x^3}{3}-\frac{x^4}{4}+x^2\right\}_{\sqrt{2}}^{2} = \frac{5}{3},$$

we obtain $\overline{x} = \frac{5}{3}\frac{3}{10-4\sqrt{2}} = \frac{5}{10-4\sqrt{2}}$. Since

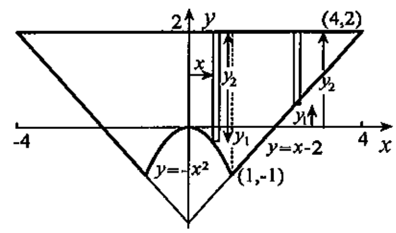
$$\begin{aligned}
A\overline{y} &= \int_0^{\sqrt{2}}\left(\frac{x}{2}\right)(x)\,dx + \int_{\sqrt{2}}^{2}\frac{1}{2}(x+x^2-2)(x-x^2+2)\,dx = \left\{\frac{x^3}{6}\right\}_0^{\sqrt{2}} + \frac{1}{2}\int_{\sqrt{2}}^{2}(-4+5x^2-x^4)\,dx\\
&= \frac{\sqrt{2}}{3} + \frac{1}{2}\left\{-4x+\frac{5x^3}{3}-\frac{x^5}{5}\right\}_{\sqrt{2}}^{2} = \frac{16\sqrt{2}-8}{15},
\end{aligned}$$

we find $\overline{y} = \frac{16\sqrt{2}-8}{15}\frac{3}{10-4\sqrt{2}} = \frac{8\sqrt{2}-4}{25-10\sqrt{2}}$.

18. $A = 2\int_0^1 (y_2 - y_1)\,dx + 2\int_1^4 (y_2 - y_1)\,dx$

$= 2\int_0^1 (2 + x^2)\,dx + 2\int_1^4 [2 - (x-2)]\,dx$

$= 2\left\{2x + \dfrac{x^3}{3}\right\}_0^1 + 2\left\{4x - \dfrac{x^2}{2}\right\}_1^4 = \dfrac{41}{3}$



By symmetry, $\overline{x} = 0$, and because

$$A\overline{y} = 2\int_0^1 \left(\frac{y_2 + y_1}{2}\right)(y_2 - y_1)\,dx + 2\int_1^4 \left(\frac{y_2 + y_1}{2}\right)(y_2 - y_1)\,dx$$

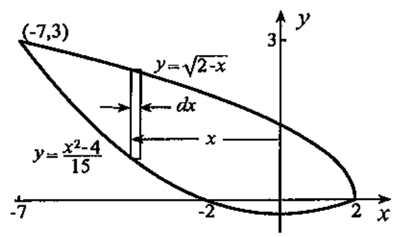
$$= \int_0^1 (y_2^2 - y_1^2)\,dx + \int_1^4 (y_2^2 - y_1^2)\,dx = \int_0^1 (4 - x^4)\,dx + \int_1^4 [4 - (x-2)^2]\,dx$$

$$= \left\{4x - \frac{x^5}{5}\right\}_0^1 + \left\{4x - \frac{1}{3}(x-2)^3\right\}_1^4 = \frac{64}{5},$$

it follows that $\overline{y} = \dfrac{64}{5}\dfrac{3}{41} = \dfrac{192}{205}$.

19. $A = \int_{-7}^2 \left(\sqrt{2-x} - \dfrac{x^2}{15} + \dfrac{4}{15}\right) dx$

$= \left\{-\dfrac{2}{3}(2-x)^{3/2} - \dfrac{x^3}{45} + \dfrac{4x}{15}\right\}_{-7}^2 = \dfrac{63}{5}$



If we set $u = 2 - x$ and $du = -dx$ in the first term of

$A\overline{x} = \int_{-7}^2 \left(x\sqrt{2-x} - \dfrac{x^3}{15} + \dfrac{4x}{15}\right) dx$, then

$$A\overline{x} = \int_9^0 (2-u)\sqrt{u}\,(-du) + \left\{\frac{-x^4}{60} + \frac{2x^2}{15}\right\}_{-7}^2 = \left\{-\frac{4u^{3/2}}{3} + \frac{2u^{5/2}}{5}\right\}_9^0 + \left\{\frac{-x^4}{60} + \frac{2x^2}{15}\right\}_{-7}^2 = -\frac{549}{20}.$$

Thus, $\overline{x} = -(549/20)(5/63) = -61/28$. Since

$$A\overline{y} = \int_{-7}^2 \left(\frac{\sqrt{2-x} + x^2/15 - 4/15}{2}\right)\left(\sqrt{2-x} - \frac{x^2}{15} + \frac{4}{15}\right) dx = \frac{1}{2}\int_{-7}^2 \left(2 - x - \frac{x^4}{225} + \frac{8x^2}{225} - \frac{16}{225}\right) dx$$

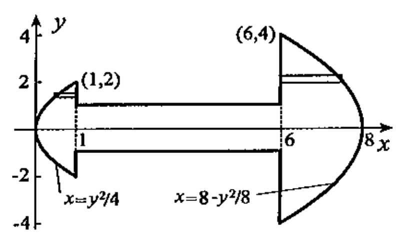
$$= \frac{1}{2}\left\{2x - \frac{x^2}{2} - \frac{x^5}{1125} + \frac{8x^3}{675} - \frac{16x}{225}\right\}_{-7}^2 = \frac{7263}{500},$$

it follows that $\overline{y} = (7263/500)(5/63) = 807/700$.

20. If ρ is the mass per unit area,

$$M = 2\int_0^2 \rho\left(1 - \frac{y^2}{4}\right) dy + 10\rho + 2\int_0^4 \rho\left(8 - \frac{y^2}{8} - 6\right) dy$$

$$= 2\rho\left\{y - \frac{y^3}{12}\right\}_0^2 + 10\rho + 2\rho\left\{2y - \frac{y^3}{24}\right\}_0^4 = \frac{70\rho}{3}.$$



Clearly $\overline{y} = 0$, and because

$$M\overline{x} = 2\int_0^2 \frac{1}{2}\left(1 + \frac{y^2}{4}\right)\rho\left(1 - \frac{y^2}{4}\right) dy + 10\rho\left(\frac{7}{2}\right) + 2\int_0^4 \frac{1}{2}\left(8 - \frac{y^2}{8} + 6\right)\rho\left(8 - \frac{y^2}{8} - 6\right) dy$$

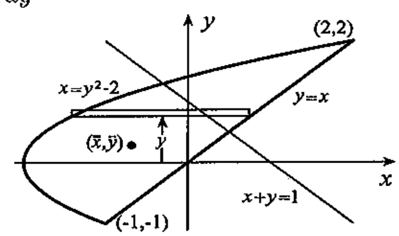
$$= \rho\int_0^2 \left(1 - \frac{y^4}{16}\right) dy + 35\rho + \rho\int_0^4 \left(28 - 2y^2 + \frac{y^4}{64}\right) dy = \rho\left\{y - \frac{y^5}{80}\right\}_0^2 + 35\rho + \rho\left\{28y - \frac{2y^3}{3} + \frac{y^5}{320}\right\}_0^4$$

$$= \frac{1637\rho}{15},$$

it follows that $\overline{x} = (1637\rho/15)[3/(70\rho)] = 1637/350$.

21. We find the centre of mass of the plate. Since $M = \int_{-1}^{2} \rho(y - y^2 + 2)\,dy = \rho\left\{\frac{y^2}{2} - \frac{y^3}{3} + 2y\right\}_{-1}^{2} = \frac{9\rho}{2}$, and

$$M\overline{x} = \int_{-1}^{2} \frac{\rho}{2}(y + y^2 - 2)(y - y^2 + 2)\,dy = \frac{\rho}{2}\int_{-1}^{2}(-y^4 + 5y^2 - 4)\,dy$$
$$= \frac{\rho}{2}\left\{-\frac{y^5}{5} + \frac{5y^3}{3} - 4y\right\}_{-1}^{2} = -\frac{9\rho}{5},$$



it follows that $\overline{x} = -\frac{9\rho}{5}\cdot\frac{2}{9\rho} = -\frac{2}{5}$. With

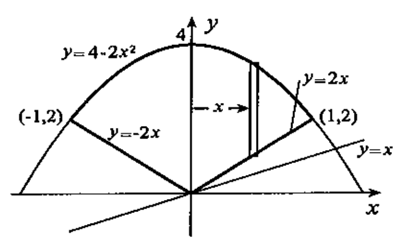
$$M\overline{y} = \int_{-1}^{2} \rho y(y - y^2 + 2)\,dy = \rho\left\{\frac{y^3}{3} - \frac{y^4}{4} + y^2\right\}_{-1}^{2} = \frac{9\rho}{4},$$

we find $\overline{y} = \frac{9\rho}{4}\cdot\frac{2}{9\rho} = \frac{1}{2}$. If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $x + y = 1$, we obtain for the required moment

$$\left(\frac{9\rho}{2}\right)\frac{|-2/5 + 1/2 - 1|}{\sqrt{2}} = \frac{81\sqrt{2}\rho}{40}.$$

22. We find the centre of mass of the plate. Since $M = 2\int_{0}^{1} \rho(4 - 2x^2 - 2x)\,dx = 2\rho\left\{4x - \frac{2x^3}{3} - x^2\right\}_{0}^{1} = \frac{14\rho}{3}$, and

$$M\overline{y} = 2\int_{0}^{1} \frac{\rho}{2}(4 - 2x^2 + 2x)(4 - 2x^2 - 2x)\,dx = 4\rho\int_{0}^{1}(4 - 5x^2 + x^4)\,dx$$
$$= 4\rho\left\{4x - \frac{5x^3}{3} + \frac{x^5}{5}\right\}_{0}^{1} = \frac{152\rho}{15},$$



it follows that $\overline{y} = \frac{152\rho}{15}\cdot\frac{3}{14\rho} = \frac{76}{35}$. Symmetry of the plate indicates that $\overline{x} = 0$. If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $y = x$, we obtain for the required moment

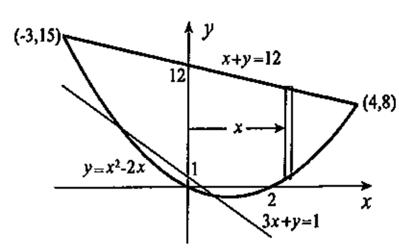
$$\left(\frac{14\rho}{3}\right)\frac{|76/35 - 0|}{\sqrt{2}} = \frac{76\sqrt{2}\rho}{15}.$$

23. We find the centre of mass of the plate. Since $M = \int_{-3}^{4} \rho(12 - x - x^2 + 2x)\,dx = \rho\left\{12x + \frac{x^2}{2} - \frac{x^3}{3}\right\}_{-3}^{4} = \frac{343\rho}{6}$, and

$$M\overline{x} = \int_{-3}^{4} \rho x(12 - x^2 + x)\,dx = \rho\left\{6x^2 - \frac{x^4}{4} + \frac{x^3}{3}\right\}_{-3}^{4} = \frac{343\rho}{12},$$

it follows that $\overline{x} = \frac{343\rho}{12}\cdot\frac{6}{343\rho} = \frac{1}{2}$. With

$$M\overline{y} = \int_{-3}^{4} \frac{\rho}{2}(12 - x + x^2 - 2x)(12 - x - x^2 + 2x)\,dx$$
$$= \frac{\rho}{2}\int_{-3}^{4}(144 - 24x - 3x^2 + 4x^3 - x^4)\,dx$$
$$= \frac{\rho}{2}\left\{144x - 12x^2 - x^3 + x^4 - \frac{x^5}{5}\right\}_{-3}^{4} = \frac{3773\rho}{10},$$

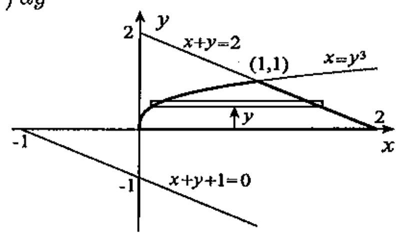


we find $\overline{y} = \dfrac{3773\rho}{10} \cdot \dfrac{6}{343\rho} = \dfrac{33}{5}$. If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $3x + y = 1$, we obtain for the required moment

$$\left(\frac{343\rho}{6}\right) \frac{|3(1/2) + 33/5 - 1|}{\sqrt{10}} = \frac{24353\sqrt{10}\rho}{600}.$$

24. We find the centre of mass of the plate. Since $M = \displaystyle\int_0^1 \rho(2 - y - y^3)\,dy = \rho\left\{2y - \frac{y^2}{2} - \frac{y^4}{4}\right\}_0^1 = \frac{5\rho}{4}$, and

$$M\overline{x} = \int_0^1 \frac{\rho}{2}(2 - y + y^3)(2 - y - y^3)\,dy = \frac{\rho}{2}\int_0^1 (4 - 4y + y^2 - y^6)\,dy$$
$$= \frac{\rho}{2}\left\{4y - 2y^2 + \frac{y^3}{3} - \frac{y^7}{7}\right\}_0^1 = \frac{23\rho}{21},$$



it follows that $\overline{x} = \dfrac{23\rho}{21} \cdot \dfrac{4}{5\rho} = \dfrac{92}{105}$. With

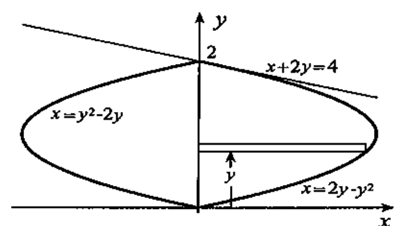
$$M\overline{y} = \int_0^1 \rho y(2 - y - y^3)\,dy = \rho\left\{y^2 - \frac{y^3}{3} - \frac{y^5}{5}\right\}_0^1 = \frac{7\rho}{15},$$

we find $\overline{y} = \dfrac{7\rho}{15} \cdot \dfrac{4}{5\rho} = \dfrac{28}{75}$. If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $x + y + 1 = 0$, we obtain for the required moment

$$\left(\frac{5\rho}{4}\right) \frac{|92/105 + 28/75 + 1|}{\sqrt{2}} = \frac{1181\sqrt{2}\rho}{840}.$$

25. Symmetry of the plate indicates that its centre of mass is $(0, 1)$. It mass is

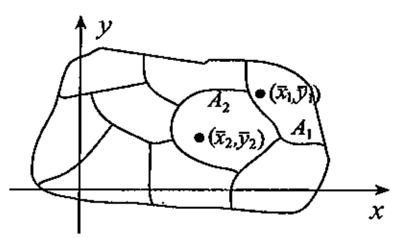
$$M = 2\int_0^2 \rho(2y - y^2)\,dy = 2\rho\left\{y^2 - \frac{y^3}{3}\right\}_0^2 = \frac{8\rho}{3}.$$



If we concentrate the mass at its centre of mass, and use formula 1.16 for the distance from the centre of mass to the line $x + 2y - 4 = 0$, we obtain for the required moment

$$\left(\frac{8\rho}{3}\right) \frac{|0 + 2(1) - 4|}{\sqrt{5}} = \frac{16\sqrt{5}\rho}{15}.$$

26. The first moment of A about the y-axis is $\quad A\overline{x} = \displaystyle\sum_{i=1}^n A_i\overline{x}_i,$

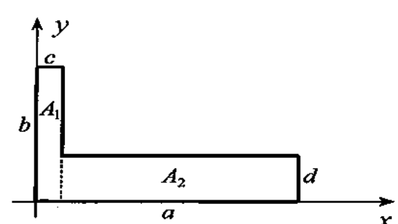


and therefore $\quad \overline{x} = \dfrac{1}{A}\displaystyle\sum_{i=1}^n A_i\overline{x}_i.$

Similarly, $\overline{y} = \dfrac{1}{A}\displaystyle\sum_{i=1}^n A_i\overline{y}_i.$

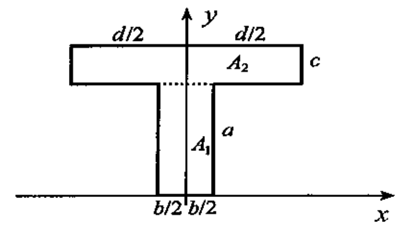
27. If we divide A into two parts as shown, then

$$\overline{x} = \frac{1}{A}(A_1\overline{x}_1 + A_2\overline{x}_2)$$
$$= \frac{1}{bc + d(a - c)}\left[\frac{c}{2}(bc) + \frac{a + c}{2}(ad - cd)\right]$$
$$= \frac{c^2(b - d) + a^2 d}{2(ad + bc - cd)};$$



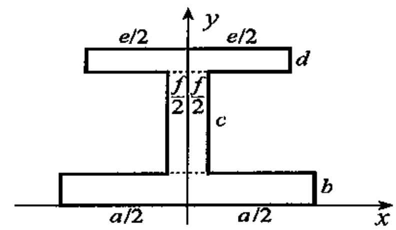
and

$$\overline{y} = \frac{1}{A}(A_1\overline{y}_1 + A_2\overline{y}_2) = \frac{1}{bc + d(a - c)}\left[\frac{b}{2}(bc) + \frac{d}{2}(ad - cd)\right] = \frac{c(b^2 - d^2) + ad^2}{2(ad + bc - cd)}.$$



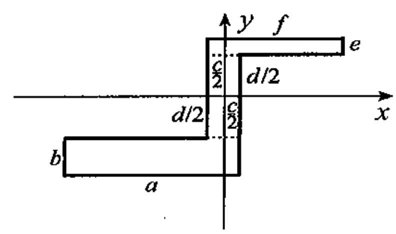
28. Symmetry gives $\bar{x} = 0$. If we divide the plate into two parts as shown, then

$$\bar{y} = \frac{1}{A}(A_1\bar{y}_1 + A_2\bar{y}_2) = \frac{1}{ab+cd}\left[\frac{a}{2}(ab) + \left(a + \frac{c}{2}\right)(cd)\right]$$
$$= \frac{a^2b + cd(2a+c)}{2(ab+cd)}.$$



29. Symmetry give $\bar{x} = 0$. If we divide the plate into three parts as shown, then

$$\bar{y} = \frac{1}{ab+cf+de}\left[\frac{b}{2}(ab) + \left(b + \frac{c}{2}\right)(cf) + \left(b + c + \frac{d}{2}\right)(de)\right]$$
$$= \frac{ab^2 + cf(2b+c) + de(2b+2c+d)}{2(ab+cf+de)}.$$

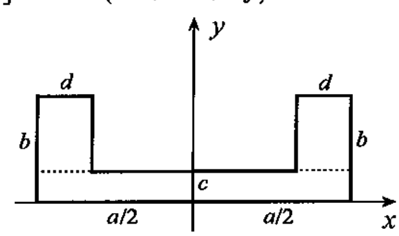


30. If we divide the plate into three parts as shown, then

$$\bar{x} = \frac{1}{ab+cd+ef}\left[\left(\frac{c/2 - a + c/2}{2}\right)(ab) + \left(\frac{f - c/2 - c/2}{2}\right)(ef)\right]$$
$$= \frac{ab(c-a) + ef(f-c)}{2(ab+cd+ef)};$$

and

$$\bar{y} = \frac{1}{ab+cd+ef}\left[\left(-\frac{d}{2} - \frac{b}{2}\right)(ab) + \left(\frac{d}{2} + \frac{e}{2}\right)(ef)\right] = \frac{ef(d+e) - ab(b+d)}{2(ab+cd+ef)}.$$

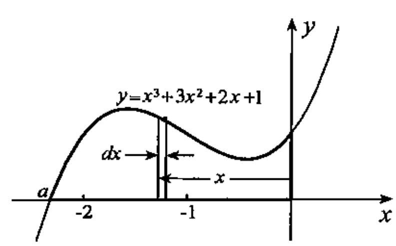


31. Symmetry gives $\bar{x} = 0$. If we divide the plate into three parts as shown, then

$$\bar{y} = \frac{1}{ac + 2d(b-c)}\left[\frac{c}{2}(ac) + \left(\frac{b+c}{2}\right)(2d)(b-c)\right]$$
$$= \frac{ac^2 + 2d(b^2 - c^2)}{2ac + 4d(b-c)}.$$

32. To six decimal places, the x-intercept of the curve is $a = -2.324\,718$. The area of the region is

$$A = \int_a^0 (x^3 + 3x^2 + 2x + 1)\,dx$$
$$= \left\{\frac{x^4}{4} + x^3 + x^2 + x\right\}\Bigg|_a^0 = 2.182\,258.$$



Since $A\bar{x} = \int_a^0 x(x^3 + 3x^2 + 2x + 1)\,dx$

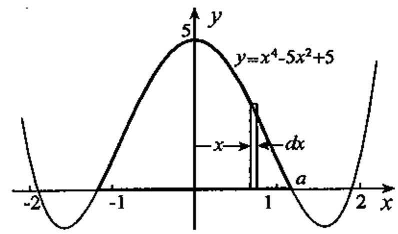
$$= \left\{\frac{x^5}{5} + \frac{3x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right\}\Bigg|_a^0 = -2.652\,017,$$

we get $\bar{x} = -2.652\,017/2.182\,258 = -1.215$. Since

$$A\bar{y} = \int_a^0 \frac{1}{2}(x^3 + 3x^2 + 2x + 1)^2\,dx = \frac{1}{2}\int_a^0 (x^6 + 6x^5 + 13x^4 + 14x^3 + 10x^2 + 4x + 1)\,dx$$
$$= \frac{1}{2}\left\{\frac{x^7}{7} + x^6 + \frac{13x^5}{5} + \frac{7x^4}{2} + \frac{10x^3}{3} + 2x^2 + x\right\}\Bigg|_a^0 = 1.140\,899,$$

it follows that $\bar{y} = 1.140\,899/2.182\,258 = 0.523$.

33. The positive x-intercept closest to the origin is $a = \sqrt{(5-\sqrt{5})/2}$. Symmetry makes it clear that $\overline{x} = 0$. The area of the plate is



$$A = 2\int_0^a (x^4 - 5x^2 + 5)\,dx$$
$$= 2\left\{\frac{x^5}{5} - \frac{5x^3}{3} + 5x\right\}_0^a = 7.238\,433.$$

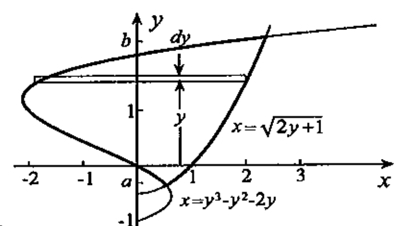
Since

$$A\overline{y} = 2\int_0^a \frac{1}{2}(x^4 - 5x^2 + 5)^2\,dx = \int_0^a (x^8 - 10x^6 + 35x^4 - 50x^2 + 25)\,dx$$
$$= \left\{\frac{x^9}{9} - \frac{10x^7}{7} + 7x^5 - \frac{50x^3}{3} + 25x\right\}_0^a = 14.072\,588,$$

it follows that $\overline{y} = \dfrac{14.072\,588}{7.238\,433} = 1.944.$

34. The y-coordinates of the points of intersection are $a = -0.354740$ and $b = 2.310040$. The area of the region is

$$A = \int_a^b (\sqrt{2y+1} - y^3 + y^2 + 2y)\,dy$$
$$= \left\{\frac{1}{3}(2y+1)^{3/2} - \frac{y^4}{4} + \frac{y^3}{3} + y^2\right\}_a^b = 6.608\,233.$$



Since

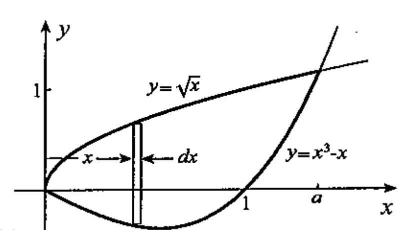
$$A\overline{x} = \int_a^b \left(\frac{\sqrt{2y+1} + y^3 - y^2 - 2y}{2}\right)(\sqrt{2y+1} - y^3 + y^2 + 2y)\,dy$$
$$= \frac{1}{2}\int_a^b (1 + 2y - y^6 + 2y^5 + 3y^4 - 4y^3 - 4y^2)\,dy = \frac{1}{2}\left\{y + y^2 - \frac{y^7}{7} + \frac{y^6}{3} + \frac{3y^5}{5} - y^4 - \frac{4y^3}{3}\right\}_a^b$$
$$= 1.448\,074,$$

it follows that $\overline{x} = 1.448\,074/6.608\,233 = 0.219$. If we set $u = 2y + 1$ and $du = 2\,dy$ in the first term of the following integral, then

$$A\overline{y} = \int_a^b y(\sqrt{2y+1} - y^3 + y^2 + 2y)\,dy = \int_{2a+1}^{2b+1} \left(\frac{u-1}{2}\right)\sqrt{u}\left(\frac{du}{2}\right) + \left\{-\frac{y^5}{5} + \frac{y^4}{4} + \frac{2y^3}{3}\right\}_a^b$$
$$= \frac{1}{4}\left\{\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right\}_{2a+1}^{2b+1} + \left\{-\frac{y^5}{5} + \frac{y^4}{4} + \frac{2y^3}{3}\right\}_a^b = 7.494\,397.$$

Hence, $\overline{y} = 7.494\,397/6.608\,233 = 1.134$.

35. The x-coordinate of the point of intersection of the curves is $a = 1.362\,599$. The area of the region is



$$A = \int_0^a (\sqrt{x} - x^3 + x)\,dx = \left\{\frac{2}{3}x^{3/2} - \frac{x^4}{4} + \frac{x^2}{2}\right\}_0^a$$
$$= 1.126\,905.$$

Since $A\overline{x} = \displaystyle\int_0^a x(\sqrt{x} - x^3 + x)\,dx = \left\{\frac{2}{5}x^{5/2} - \frac{x^5}{5} + \frac{x^3}{3}\right\}_0^a = 0.770\,781,$

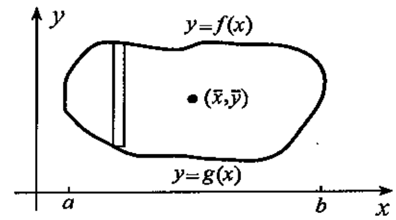
it follows that $\overline{x} = 0.770\,781/1.126\,905 = 0.684$. Since

$$A\overline{y} = \int_0^a \frac{1}{2}(\sqrt{x} + x^3 - x)(\sqrt{x} - x^3 + x)\,dx = \frac{1}{2}\int_0^a (x - x^2 + 2x^4 - x^6)\,dx$$
$$= \frac{1}{2}\left\{\frac{x^2}{2} - \frac{x^3}{3} + \frac{2x^5}{5} - \frac{x^7}{7}\right\}_0^a = 0.359\,018,$$

we obtain $\overline{y} = 0.359\,018/1.126\,905 = 0.319$.

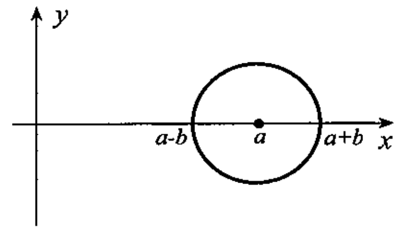
36. If we choose the y-axis along the axis of rotation, the volume generated is

$$V = \int_a^b 2\pi x[f(x) - g(x)]\,dx$$
$$= 2\pi \int_a^b x[f(x) - g(x)]\,dx$$
$$= 2\pi(A\overline{x}) = (2\pi\overline{x})A.$$



37. Since the centroid of the circle is at $(a, 0)$, the volume of the donut is

$$V = (\pi b^2)(2\pi a) = 2\pi^2 ab^2.$$



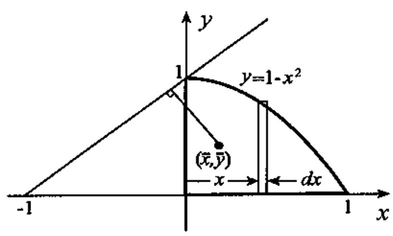
38. We find the centroid of the area. Since

$$A = \int_0^1 (1 - x^2)\,dx = \left\{x - \frac{x^3}{3}\right\}_0^1 = \frac{2}{3},$$

and

$$A\overline{x} = \int_0^1 x(1 - x^2)\,dx = \left\{\frac{x^2}{2} - \frac{x^4}{4}\right\}_0^1 = \frac{1}{4},$$

we find $\overline{x} = (1/4)(3/2) = 3/8$. Since



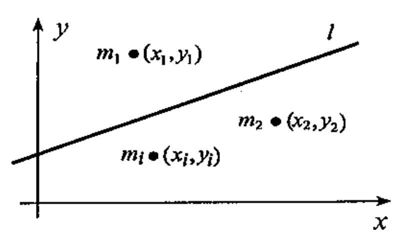
$$A\overline{y} = \int_0^1 \frac{1}{2}(1 - x^2)^2\,dx = \frac{1}{2}\int_0^1 (1 - 2x^2 + x^4)\,dx = \frac{1}{2}\left\{x - \frac{2x^3}{3} + \frac{x^5}{5}\right\}_0^1 = \frac{4}{15},$$

we get $\overline{y} = (4/15)(3/2) = 2/5$. Using the result of Exercise 36 and distance formula 1.16, the volume is

$$V = 2\pi\left(\frac{|3/8 - 2/5 + 1|}{\sqrt{2}}\right)\left(\frac{2}{3}\right) = \frac{13\sqrt{2}\pi}{20}.$$

39. (a) If $Ax + By + C = 0$ is the equation of the line l, then the first moment of the system about l is $\sum_{i=1}^{n} m_i d_i$ where d_i is the distance from the m_i to l.

Using formula 1.16, $d_i = \dfrac{Ax_i + By_i + C}{\sqrt{A^2 + B^2}}$, and therefore



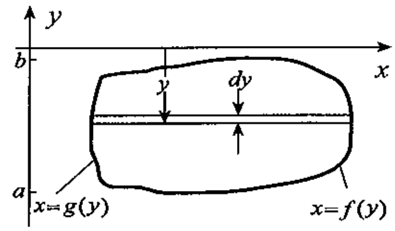
$$\sum_{i=1}^{n} m_i d_i = \sum_{i=1}^{n} \frac{m_i(Ax_i + By_i + C)}{\sqrt{A^2 + B^2}}$$
$$= \frac{1}{\sqrt{A^2 + B^2}}\left[A\sum_{i=1}^{n} m_i x_i + B\sum_{i=1}^{n} m_i y_i + C\sum_{i=1}^{n} m_i\right]$$
$$= \frac{1}{\sqrt{A^2 + B^2}}[A(M\overline{x}) + B(M\overline{y}) + CM] = M\left[\frac{A\overline{x} + B\overline{y} + C}{\sqrt{A^2 + B^2}}\right] = Md$$

where d is the distance from $(\overline{x}, \overline{y})$ to l.

(b) Yes, since for such a line $A\overline{x} + B\overline{y} + C = 0$.

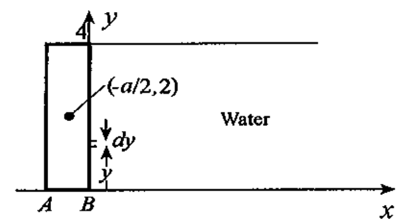
40. For the plate shown,

$$F = \int_a^b 9.81\rho(-y)[f(y) - g(y)]\,dy$$
$$= -9.81\rho \int_a^b y[f(y) - g(y)]\,dy$$
$$= -9.81\rho(A\overline{y}) = 9.81\rho A(-\overline{y}).$$



41. (a) For equilibrium, moments of concrete dam and water on the face of the dam about the line through A (perpendicular to the page) must sum to zero,

$$0 = -4(10)a(2400g)\left(\frac{a}{2}\right) + \int_0^4 (y)\rho g(4 - y)(10)\,dy$$
$$= -48\,000ga^2 + 10\rho g\left\{2y^2 - \frac{y^3}{3}\right\}_0^4$$
$$= -48\,000ga^2 + \frac{320\,000g}{3}.$$



This implies that $a = 2\sqrt{5}/3$ m.

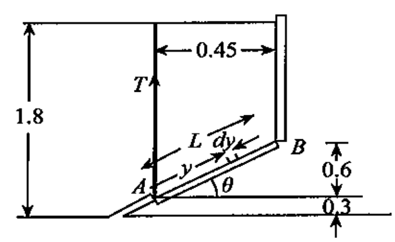
(b) In this case we include the moment due to water pressure on AB,

$$0 = -4(10)a(2400g)\left(\frac{a}{2}\right) + \int_0^4 (y)\rho g(4-y)(10)\,dy + \int_0^a (x)\rho g(4)(10)\,dx$$
$$= -48\,000ga^2 + \frac{320\,000g}{3} + 40\,000g\left\{\frac{x^2}{2}\right\}_0^a = -28\,000ga^2 + \frac{320\,000g}{3}.$$

This implies that $a = 4\sqrt{105}/21$ m.

42. The sum of the moments of the tension T in the cable and the force of the water on the gate about B must be zero,

$$0 = -T(0.45) + \int_0^L (L-y)(1.5 - y\sin\theta)\rho g(0.525)\,dy.$$

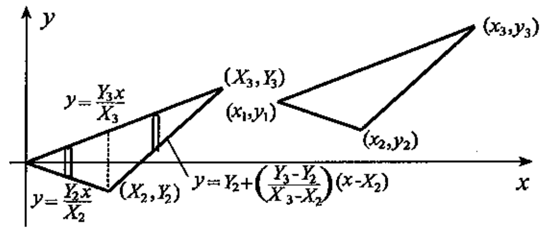


Since $\sin\theta = 600/750 = 4/5$, and
$L = \sqrt{(0.6)^2 + (0.45)^2} = 3/4$,

$$T = \frac{0.525\rho g}{0.45}\int_0^{3/4}\left(\frac{3}{4} - y\right)\left(\frac{3}{2} - \frac{4y}{5}\right)dy$$
$$= \frac{0.525\rho g}{0.45}\left\{\frac{9y}{8} - \frac{21y^2}{20} + \frac{4y^3}{15}\right\}_0^{3/4}$$
$$= 4185 \text{ N}.$$

43. If the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) in the figure is translated so that the vertex (x_1, y_1) coincides with the origin, then vertices of the new triangle corresponding to (x_2, y_2) and (x_3, y_3) are $(X_2, Y_2) = (x_2 - x_1, y_2 - y_1)$ and $(X_3, Y_3) = (x_3 - x_1, y_3 - y_1)$ respectively. We first find the centroid of the translated triangle. Its area is

$$A = \int_0^{X_2}\left(\frac{Y_3 x}{X_3} - \frac{Y_2 x}{X_2}\right)dx + \int_{X_2}^{X_3}\left[\frac{Y_3 x}{X_3} - Y_2 - \left(\frac{Y_3 - Y_2}{X_3 - X_2}\right)(x - X_2)\right]dx$$
$$= \left\{\left(\frac{Y_3}{X_3} - \frac{Y_2}{X_2}\right)\frac{x^2}{2}\right\}_0^{X_2} + \left\{\frac{Y_3}{X_3}\frac{x^2}{2} - Y_2 x - \frac{1}{2}\left(\frac{Y_3 - Y_2}{X_3 - X_2}\right)(x - X_2)^2\right\}_{X_2}^{X_3}$$
$$= \frac{1}{2}(X_2 Y_3 - X_3 Y_2).$$



The first moment of the triangle about the y-axis is

$$\begin{aligned}
A\overline{X} &= \int_0^{X_2}\left(\frac{Y_3}{X_3}-\frac{Y_2}{X_2}\right)x^2\,dx + \int_{X_2}^{X_3}\left\{\left(\frac{Y_3}{X_3}-\frac{Y_3-Y_2}{X_3-X_2}\right)x^2+\left[-Y_2+X_2\left(\frac{Y_3-Y_2}{X_3-X_2}\right)\right]x\right\}dx\\
&= \left(\frac{Y_3}{X_3}-\frac{Y_2}{X_2}\right)\left\{\frac{x^3}{3}\right\}_0^{X_2}+\left(\frac{Y_3}{X_3}-\frac{Y_3-Y_2}{X_3-X_2}\right)\left\{\frac{x^3}{3}\right\}_{X_2}^{X_3}+\left[-Y_2+X_2\left(\frac{Y_3-Y_2}{X_3-X_2}\right)\right]\left\{\frac{x^2}{2}\right\}_{X_2}^{X_3}\\
&= \frac{1}{6}(X_2Y_3-X_3Y_2)(X_2+X_3).
\end{aligned}$$

Thus, $\overline{X}=\dfrac{(X_2Y_3-X_3Y_2)(X_2+X_3)}{6}\dfrac{2}{X_2Y_3-X_3Y_2}=\dfrac{X_2+X_3}{3}$. The first moment of the triangle about the y-axis is

$$\begin{aligned}
A\overline{Y} &= \int_0^{X_2}\frac{1}{2}\left[\left(\frac{Y_3}{X_3}\right)^2-\left(\frac{Y_2}{X_2}\right)^2\right]x^2\,dx+\frac{1}{2}\int_{X_2}^{X_3}\left[\left(\frac{Y_3}{X_3}\right)^2x^2-Y_2^2-\left(\frac{Y_3-Y_2}{X_3-X_2}\right)^2(x-X_2)^2\right.\\
&\qquad\left.-2Y_2\left(\frac{Y_3-Y_2}{X_3-X_2}\right)(x-X_2)\right]dx\\
&= \frac{1}{2}\left[\left(\frac{Y_3}{X_3}\right)^2-\left(\frac{Y_2}{X_2}\right)^2\right]\left\{\frac{x^3}{3}\right\}_0^{X_2}+\frac{1}{2}\left(\frac{Y_3}{X_3}\right)^2\left\{\frac{x^3}{3}\right\}_{X_2}^{X_3}-\frac{1}{2}Y_2^2\Big\{x\Big\}_{X_2}^{X_3}\\
&\qquad-\frac{1}{6}\left(\frac{Y_3-Y_2}{X_3-X_2}\right)^2\Big\{(x-X_2)^3\Big\}_{X_2}^{X_3}-Y_2\left(\frac{Y_3-Y_2}{X_3-X_2}\right)\Big\{(x-X_2)^2\Big\}_{X_2}^{X_3}\\
&= \frac{1}{6}(X_2Y_3-X_3Y_2)(Y_2+Y_3).
\end{aligned}$$

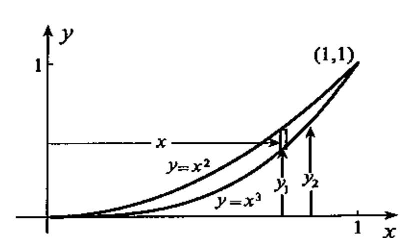
Thus, $\overline{Y}=\dfrac{(X_2Y_3-X_3Y_2)(Y_2+Y_3)}{6}\dfrac{2}{X_2Y_3-X_3Y_2}=\dfrac{Y_2+Y_3}{3}$. The centroid $(\overline{x},\overline{y})$ of the original triangle is therefore

$$\begin{aligned}
(\overline{x},\overline{y}) &= (\overline{X}+x_1,\overline{Y}+y_1)=\left(\frac{X_2+X_3}{3}+x_1,\frac{Y_2+Y_3}{3}+y_1\right)\\
&= \left(\frac{x_2-x_1+x_3-x_1+3x_1}{3},\frac{y_2-y_1+y_3-y_1+3y_1}{3}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right).
\end{aligned}$$

EXERCISES 7.8

1. $I=\displaystyle\int_0^1 x^2\rho(y_2-y_1)\,dx=\rho\int_0^1 x^2(x^2-x^3)\,dx$

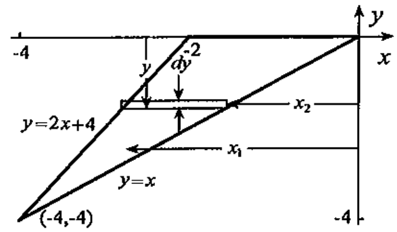
$\displaystyle=\rho\int_0^1(x^4-x^5)\,dx=\rho\left\{\frac{x^5}{5}-\frac{x^6}{6}\right\}_0^1=\frac{\rho}{30}$



2. $I = \int_{-4}^{0} (-y)^2 \rho (x_2 - x_1)\, dy$

$$= \rho \int_{-4}^{0} y^2 \left[y - \left(\frac{y-4}{2} \right) \right] dy = \frac{\rho}{2} \int_{-4}^{0} (y^3 + 4y^2)\, dy$$

$$= \frac{\rho}{2} \left\{ \frac{y^4}{4} + \frac{4y^3}{3} \right\}_{-4}^{0} = \frac{32\rho}{3}$$

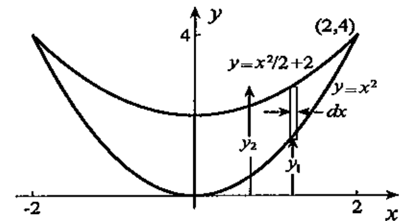


3. Using formula 7.42,

$$I = 2 \int_{0}^{2} \frac{\rho}{3} (y_2^3 - y_1^3)\, dx = \frac{2\rho}{3} \int_{0}^{2} [(2 + x^2/2)^3 - (x^2)^3]\, dx$$

$$= \frac{2\rho}{3} \int_{0}^{2} \left(8 + 6x^2 + \frac{3x^4}{2} - \frac{7x^6}{8} \right) dx$$

$$= \frac{2\rho}{3} \left\{ 8x + 2x^3 + \frac{3x^5}{10} - \frac{x^7}{8} \right\}_{0}^{2} = \frac{256\rho}{15}$$

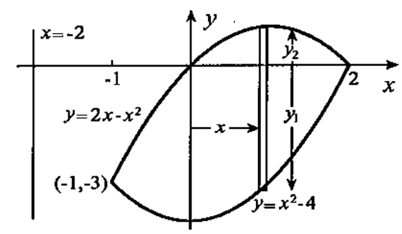


4. $I = \int_{-1}^{2} (x+2)^2 \rho (y_2 - y_1)\, dx$

$$= \rho \int_{-1}^{2} (x+2)^2 (2x - x^2 - x^2 + 4)\, dx$$

$$= \rho \int_{-1}^{2} (16 + 24x + 4x^2 - 6x^3 - 2x^4)\, dx$$

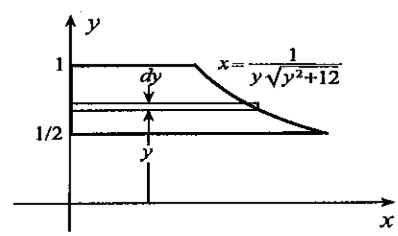
$$= \rho \left\{ 16x + 12x^2 + \frac{4x^3}{3} - \frac{3x^4}{2} - \frac{2x^5}{5} \right\}_{-1}^{2} = \frac{603\rho}{10}$$



5. $I = \int_{1/2}^{1} y^2 \rho \left(\frac{1}{y\sqrt{y^2+12}} \right) dy$

$$= \rho \int_{1/2}^{1} \frac{y}{\sqrt{y^2+12}} dy$$

$$= \rho \left\{ \sqrt{y^2+12} \right\}_{1/2}^{1} = \rho(\sqrt{13} - 7/2)$$

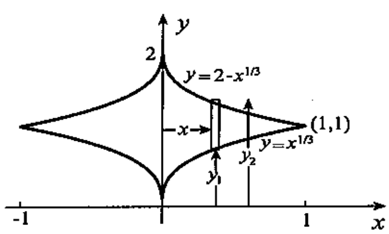


6. $I = 2 \int_{0}^{1} x^2 \rho (y_2 - y_1)\, dx$

$$= 2\rho \int_{0}^{1} x^2 (2 - x^{1/3} - x^{1/3})\, dx$$

$$= 4\rho \int_{0}^{1} (x^2 - x^{7/3})\, dx$$

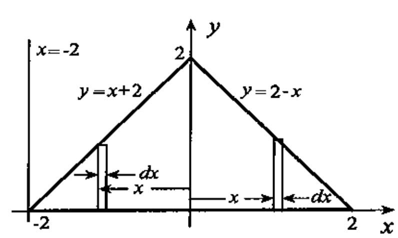
$$= 4\rho \left\{ \frac{x^3}{3} - \frac{3}{10} x^{10/3} \right\}_{0}^{1} = \frac{2\rho}{15}$$



7. $I = \int_{-2}^{0} (x+2)^2 \rho (x+2)\, dx + \int_{0}^{2} (x+2)^2 \rho (2-x)\, dx$

$$= \rho \int_{-2}^{0} (x+2)^3\, dx + \rho \int_{0}^{2} (8 + 4x - 2x^2 - x^3)\, dx$$

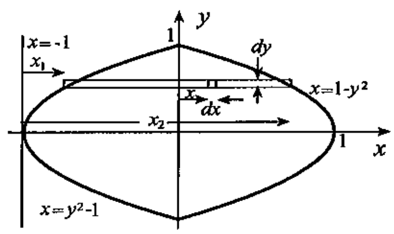
$$= \rho \left\{ \frac{(x+2)^4}{4} \right\}_{-2}^{0} + \rho \left\{ 8x + 2x^2 - \frac{2x^3}{3} - \frac{x^4}{4} \right\}_{0}^{2} = \frac{56\rho}{3}$$



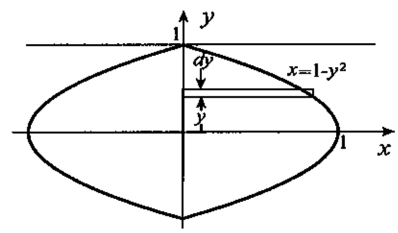
8. If we divide the long horizontal rectangle with width dy into tiny rectangles with length dx, then the moment of inertia of the long rectangle about the line $x = -1$ is

$$\int_{x_1}^{x_2} (x+1)^2 \rho dy\, dx = \rho\, dy \left\{ \frac{1}{3}(x+1)^3 \right\}_{x_1}^{x_2} = \frac{\rho}{3}[(x_2+1)^3 - (x_1+1)^3]\, dy.$$

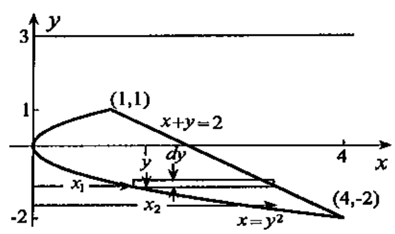
The moment of inertia of the plate is



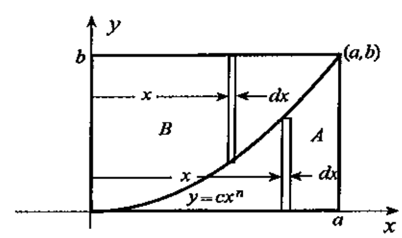
$$\begin{aligned} I &= \frac{2\rho}{3}\int_0^1 [(1-y^2+1)^3 - (y^2-1+1)^3]\, dy \\ &= \frac{2\rho}{3}\int_0^1 (8 - 12y^2 + 6y^4 - 2y^6)\, dy \\ &= \frac{2\rho}{3}\left\{ 8y - 4y^3 + \frac{6y^5}{5} - \frac{2y^7}{7} \right\}_0^1 = \frac{344\rho}{105} \end{aligned}$$



9. $$\begin{aligned} I &= 2\int_{-1}^1 (y-1)^2 \rho (1-y^2)\, dy \\ &= 2\rho \int_{-1}^1 (-y^4 + 2y^3 - 2y + 1)\, dy \\ &= 2\rho \left\{ -\frac{y^5}{5} + \frac{y^4}{2} - y^2 + y \right\}_{-1}^1 = \frac{16\rho}{5} \end{aligned}$$



10. $$\begin{aligned} I &= \int_{-2}^1 (y-3)^2 \rho (x_2 - x_1)\, dy = \rho \int_{-2}^1 (y-3)^2 [(2-y) - y^2]\, dy \\ &= \rho \int_{-2}^1 (18 - 21y - y^2 + 5y^3 - y^4)\, dy \\ &= \rho \left\{ 18y - \frac{21y^2}{2} - \frac{y^3}{3} + \frac{5y^4}{4} - \frac{y^5}{5} \right\}_{-2}^1 = \frac{1143\rho}{20} \end{aligned}$$



11. (a) Using formula 7.42,

$$\begin{aligned} I_{Bx} &= \int_0^a \frac{1}{3}[b^3 - (cx^n)^3]\, dx = \frac{1}{3}\int_0^a (b^3 - c^3x^{3n})\, dx \\ &= \frac{1}{3}\left\{ b^3x - \frac{c^3x^{3n+1}}{3n+1} \right\}_0^a = \frac{1}{3}\left(b^3a - \frac{c^3a^{3n+1}}{3n+1} \right) \\ &= \frac{nb^3a}{3n+1}, \text{ (using } b = ca^n) \end{aligned}$$

$$I_{Ax} = \int_0^a \frac{1}{3}(cx^n)^3\, dx = \frac{c^3}{3}\int_0^a x^{3n}\, dx = \frac{c^3}{3}\left\{ \frac{x^{3n+1}}{3n+1} \right\}_0^a = \frac{c^3a^{3n+1}}{3(3n+1)} = \frac{b^3a}{3(3n+1)}$$

This shows that $I_{Bx} = 3nI_{Ax}$.

(b) The following calculations show that $nI_{Ay} = 3I_{By}$.

$$I_{Ay} = \int_0^a x^2(cx^n)\, dx = c\int_0^a x^{n+2}\, dx = c\left\{ \frac{x^{n+3}}{n+3} \right\}_0^a = \frac{ca^{n+3}}{n+3} = \frac{ba^3}{n+3}$$

$$I_{By} = \int_0^a x^2(b - cx^n)\, dx = \int_0^a (bx^2 - cx^{n+2})\, dx = \left\{ \frac{bx^3}{3} - \frac{cx^{n+3}}{n+3} \right\}_0^a = \frac{ba^3}{3} - \frac{ca^{n+3}}{n+3} = \frac{ba^3n}{3(n+3)}$$

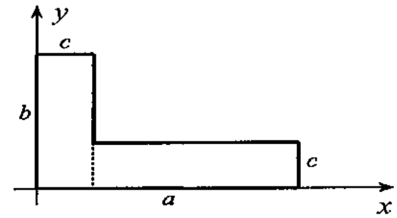
12. The product of the mass and the square of the distance from the x-axis to the centre of mass of the rectangle is $\rho(y_2 - y_1)h\left(\frac{y_1 + y_2}{2}\right)^2$. This is not the same as 7.42. For example, if $y_1 = 0$, so that the rectangle has its base on the x-axis, then 7.42 gives $\rho h y_2^3/3$ whereas the above expression gives $\rho h y_2^3/4$.

13. If we divide the section into two parts as shown, and use formula 7.42 with $\rho = 1$, we obtain

$$I_x = \frac{c}{3}(b^3) + \frac{a-c}{3}(c^3) = \frac{c}{3}(b^3 + ac^2 - c^3),$$

and

$$I_y = \frac{b}{3}(c^3) + \frac{c}{3}(a^3 - c^3) = \frac{c}{3}(a^3 + bc^2 - c^3).$$

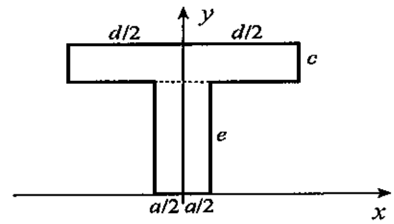


14. If we divide the right half of the section into two parts as shown, and use formula 7.42 with $\rho = 1$, we obtain

$$I_x = \frac{2}{3}\left(\frac{d}{2}\right)[(c+e)^3 - e^3] + \frac{2}{3}\left(\frac{a}{2}\right)e^3$$
$$= \frac{1}{3}[ae^3 + cd(c^2 + 3ce + 3e^2),$$

and

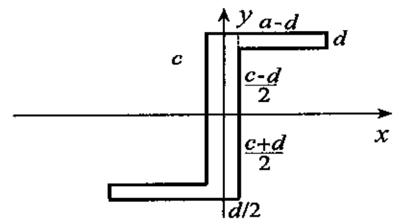
$$I_y = 2\left(\frac{c}{3}\right)\left(\frac{d}{2}\right)^3 + 2\left(\frac{e}{3}\right)\left(\frac{a}{2}\right)^3 = \frac{1}{12}(cd^3 + ea^3),$$



15. If we divide the first quadrant part of the section into subareas as shown, then formula 7.42 with $\rho = 1$ gives

$$I_x = \frac{2}{3}(a-d)\left[\left(\frac{c+d}{2}\right)^3 - \left(\frac{c-d}{2}\right)^3\right] + \frac{4}{3}\left(\frac{d}{2}\right)\left(\frac{c+d}{2}\right)^3,$$

and this simplifies to $I_x = \dfrac{ad}{6}(3c^2 + d^2) + \dfrac{d}{12}(c-d)^3$.



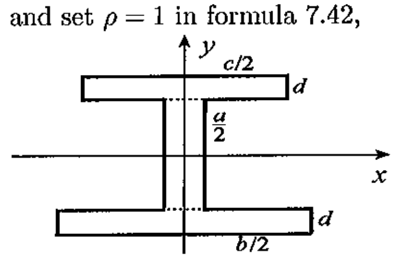
The second moment about the y-axis is $I_y = \dfrac{2}{3}(d)\left[\left(a - \dfrac{d}{2}\right)^3 - \left(\dfrac{d}{2}\right)^3\right] + \dfrac{4}{3}\left(\dfrac{c+d}{2}\right)\left(\dfrac{d}{2}\right)^3$, and this simplifies to $I_y = \dfrac{d}{12}(2a-d)^3 + \dfrac{cd^3}{12}$.

16. If we divide the right half of the section into subareas as shown, and set $\rho = 1$ in formula 7.42,

$$I_x = \frac{2}{3}\left(\frac{c}{2}\right)\left[\left(d + \frac{a}{2}\right)^3 - \left(\frac{a}{2}\right)^3\right] + \frac{4}{3}\left(\frac{d}{2}\right)\left(\frac{a}{2}\right)^3$$
$$+ \frac{2}{3}\left(\frac{b}{2}\right)\left[\left(d + \frac{a}{2}\right)^3 - \left(\frac{a}{2}\right)^3\right]$$
$$= \frac{1}{24}[2a^3 d + (b+c)(6a^2 d + 12ad^2 + 8d^3)],$$

and

$$I_y = \frac{2}{3}(d)\left(\frac{c}{2}\right)^3 + \frac{4}{3}\left(\frac{a}{2}\right)\left(\frac{d}{2}\right)^3 + \frac{2}{3}(d)\left(\frac{b}{2}\right)^3 = \frac{d}{12}(ad^2 + b^3 + c^3).$$

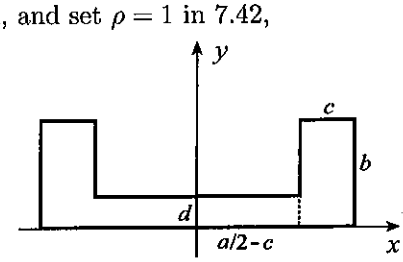


17. If we divide the right half of the section into subareas as shown, and set $\rho = 1$ in 7.42,

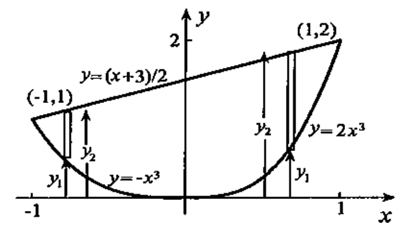
$$I_x = \frac{2}{3}(c)(b)^3 + \frac{2}{3}\left(\frac{a}{2} - c\right)(d)^3 = \frac{1}{3}(ad^3 + 2b^3 c - 2cd^3),$$

and

$$I_y = \frac{2}{3}(b)\left[\left(\frac{a}{2}\right)^3 - \left(\frac{a}{2} - c\right)^3\right] + \frac{2}{3}(d)\left(\frac{a}{2} - c\right)^3$$
$$= \frac{1}{12}[a^3 d + (b-d)(6a^2 c - 12ac^2 + 8c^3)].$$



18. The mass of the plate is



$$\begin{aligned} M &= \int_{-1}^{0} \rho(y_2 - y_1)\,dx + \int_{0}^{1} \rho(y_2 - y_1)\,dx \\ &= \rho\int_{-1}^{0}\left(\frac{x+3}{2} + x^3\right)dx + \rho\int_{0}^{1}\left(\frac{x+3}{2} - 2x^3\right)dx \\ &= \rho\left\{\frac{(x+3)^2}{4} + \frac{x^4}{4}\right\}_{-1}^{0} + \rho\left\{\frac{(x+3)^2}{4} - \frac{x^4}{2}\right\}_{0}^{1} = \frac{9\rho}{4}. \end{aligned}$$

The moment of inertia about the x-axis is

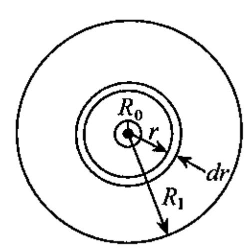
$$\begin{aligned} I_x &= \int_{-1}^{0} \frac{\rho}{3}(y_2^3 - y_1^3)\,dx + \int_{0}^{1} \frac{\rho}{3}(y_2^3 - y_1^3)\,dx \\ &= \frac{\rho}{3}\int_{-1}^{0}\left[\left(\frac{x+3}{2}\right)^3 - (-x^3)^3\right]dx + \frac{\rho}{3}\int_{0}^{1}\left[\left(\frac{x+3}{2}\right)^3 - (2x^3)^3\right]dx \\ &= \frac{\rho}{3}\left\{\frac{(x+3)^4}{32} + \frac{x^{10}}{10}\right\}_{-1}^{0} + \frac{\rho}{3}\left\{\frac{(x+3)^4}{32} - \frac{4x^{10}}{5}\right\}_{0}^{1} = \frac{11\rho}{5}. \end{aligned}$$

If r_x is the moment of gyration about the x-axis, then $\frac{11\rho}{5} = \frac{9\rho}{4}r_x^2 \implies r_x = \sqrt{44/45}$. The moment of inertia about the y-axis is

$$\begin{aligned} I_y &= \int_{-1}^{0} x^2\rho\left(\frac{x+3}{2} + x^3\right)dx + \int_{0}^{1} x^2\rho\left(\frac{x+3}{2} - 2x^3\right)dx \\ &= \frac{\rho}{2}\int_{-1}^{0}(x^3 + 3x^2 + 2x^5)\,dx + \frac{\rho}{2}\int_{0}^{1}(x^3 + 3x^2 - 4x^5)\,dx \\ &= \frac{\rho}{2}\left\{\frac{x^4}{4} + x^3 + \frac{x^6}{3}\right\}_{-1}^{0} + \frac{\rho}{2}\left\{\frac{x^4}{4} + x^3 - \frac{2x^6}{3}\right\}_{0}^{1} = \frac{\rho}{2}. \end{aligned}$$

If r_y is the radius of gyration about the y-axis, then $\frac{\rho}{2} = \frac{9\rho}{4}r_y^2 \implies r_y = \sqrt{2}/3$. The radius of gyration is the distance from a line at which a single particle of mass equal to that of the plate has the same moment of inertia as the plate itself.

19. Let inner and outer radii of the record be denoted by R_0 and R_1 respectively. Suppose the mass per unit area of the material in the record is a constant ρ, and the record rotates with angular speed ω (radians per unit time). We divide the record into thin rings of width dr. The ring with inner radius r has area approximately equal to $2\pi r\,dr$. Each point in this ring moves with speed $v = \omega r$, and therefore the kinetic energy of the ring is approximately



$$\frac{1}{2}(2\pi r\,dr)\rho(\omega r)^2 = \pi\rho\omega^2 r^3\,dr.$$

The kinetic energy of the record is therefore

$$K = \int_{R_0}^{R_1} \pi\rho\omega^2 r^3\,dr = \pi\rho\omega^2\left\{\frac{r^4}{4}\right\}_{R_0}^{R_1} = \frac{1}{4}\pi\rho\omega^2(R_1^4 - R_0^4).$$

The moment of inertia of the ring at radius r about a line through the centre of the record and perpendicular to its face is $(2\pi r\,dr)\rho(r^2) = 2\pi\rho r^3\,dr$. Consequently, the moment of inertia of the record about this line is

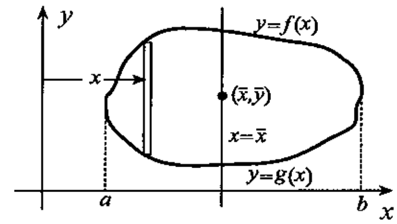
$$I = \int_{R_0}^{R_1} 2\pi\rho r^3\,dr = 2\pi\rho\left\{\frac{r^4}{4}\right\}_{R_0}^{R_1} = \frac{\pi\rho}{2}(R_1^4 - R_0^4).$$

Thus, $K = \frac{1}{4}\pi\rho\omega^2(R_1^4 - R_0^4) = \frac{1}{2}\left[\frac{\pi\rho}{2}(R_1^4 - R_0^4)\right]\omega^2 = \frac{1}{2}I\omega^2.$

20. Let us take the coplanar line to be the y-axis. The moment of inertia about the y-axis is

$$I_y = \int_a^b x^2 \rho[f(x) - g(x)]\,dx.$$

The moment of inertia about the line $x = \overline{x}$ is



$$\begin{aligned} I &= \int_a^b (x - \overline{x})^2 \rho[f(x) - g(x)]\,dx \\ &= \int_a^b x^2\rho[f(x) - g(x)]\,dx - 2\overline{x}\int_a^b x\rho[f(x) - g(x)]\,dx + \overline{x}^2\int_a^b \rho[f(x) - g(x)]\,dx \\ &= I_y - 2\overline{x}(M\overline{x}) + \overline{x}^2 M. \end{aligned}$$

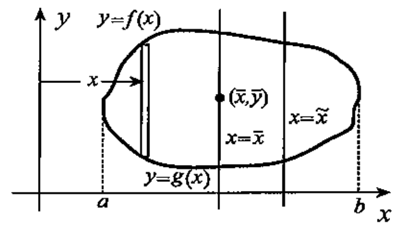
Thus, $I_y = I + M\overline{x}^2$.

21. $I_{\tilde{x}} = \displaystyle\int_a^b (x - \tilde{x})^2[f(x) - g(x)]\,dx$

According to the parallel axis theorem in Exercise 20, we may write

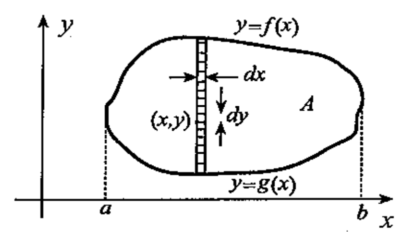
$$I_{\tilde{x}} = I_{\overline{x}} + (\tilde{x} - \overline{x})^2 A,$$

where A is the area of the plate. Since $I_{\overline{x}}$ is a fixed quantity, it follows that $I_{\tilde{x}}$ is a minimum when $\tilde{x} = \overline{x}$.



22. We divide the area A into vertical rectangles of width dx, and then divide this rectangle further into horizontal rectangles of width dy. Because the polar moment of inertia of the tiny rectangle is $(x^2 + y^2)\rho\,dy\,dx$, the polar moment of inertia of the vertical rectangle is

$$\begin{aligned} \int_{g(x)}^{f(x)} [(x^2 + y^2)\rho\,dx]\,dy &= \left\{\rho\,dx\left(x^2 y + \frac{y^3}{3}\right)\right\}_{g(x)}^{f(x)} \\ &= \left\{x^2[f(x) - g(x)] + \frac{1}{3}\{[f(x)]^3 - [g(x)]^3\}\right\}\rho\,dx. \end{aligned}$$



The polar moment of inertia of the plate is therefore

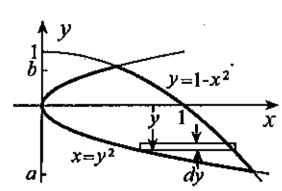
$$\begin{aligned} J_0 &= \int_a^b \left\{x^2[f(x) - g(x)] + \frac{1}{3}\{[f(x)]^3 - [g(x)]^3\}\right\}\rho\,dx \\ &= \int_a^b x^2\rho[f(x) - g(x)]\,dx + \int_a^b \frac{1}{3}\rho\{[f(x)]^3 - [g(x)]^3\}\,dx = I_x + I_y. \end{aligned}$$

23. The y-coordinates of the points of intersection of the curves are $a = -1.220\,744$ and $b = 0.724\,492$. The moment of inertia is

$$I = \int_a^b 2y^2(\sqrt{1-y} - y^2)\,dy.$$

If we set $u = 1 - y$ and $du = -dy$, then

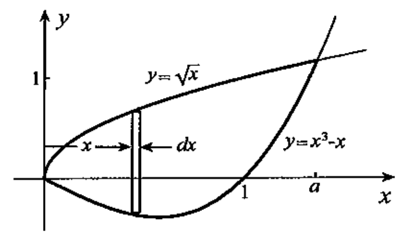
$$\begin{aligned} \int y^2\sqrt{1-y}\,dy &= \int (1-u)^2\sqrt{u}(-du) \\ &= -\int(\sqrt{u} - 2u^{3/2} + u^{5/2})\,du \\ &= -\frac{2}{3}u^{3/2} + \frac{4}{5}u^{5/2} - \frac{2}{7}u^{7/2} + C \\ &= -\frac{2}{3}(1-y)^{3/2} + \frac{4}{5}(1-y)^{5/2} - \frac{2}{7}(1-y)^{7/2} + C. \end{aligned}$$



Thus, $I = 2\left\{-\dfrac{2}{3}(1-y)^{3/2} + \dfrac{4}{5}(1-y)^{5/2} - \dfrac{2}{7}(1-y)^{7/2} - \dfrac{y^5}{5}\right\}_a^b = 0.680.$

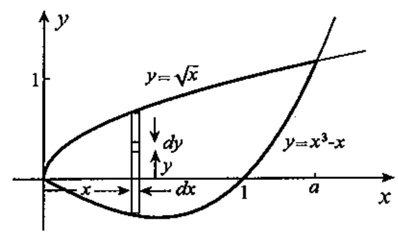
24. The x-coordinate of the point of intersection of the curves is $a = 1.362\,599$. The moment of inertia is

$$I = \int_0^a x^2(2)(\sqrt{x} - x^3 + x)\,dx$$
$$= 2\int_0^a (x^{5/2} - x^5 + x^3)\,dx$$
$$= 2\left\{\frac{2x^{7/2}}{7} - \frac{x^6}{6} + \frac{x^4}{4}\right\}_0^a = 1.278.$$



25. Since the moment of inertia of the tiny rectangle about the x-axis is $2y^2\,dx\,dy$, the moment of inertia of the long vertical rectangle is

$$\int_{x^3-x}^{\sqrt{x}} 2y^2\,dx\,dy = 2\left\{\frac{y^3}{3}\right\}_{x^3-x}^{\sqrt{x}} dx$$
$$= \frac{2}{3}[x^{3/2} - (x^3 - x)^3]\,dx.$$

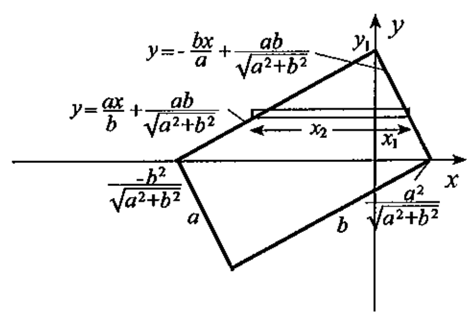


The x-coordinate of the point of intersection of the two curves is $a = 1.362\,599$. The moment of inertia of the plate about the x-axis is

$$I = \int_0^a \frac{2}{3}[x^{3/2} - (x^3 - x)^3]\,dx = \frac{2}{3}\int_0^a (x^{3/2} - x^9 + 3x^7 - 3x^5 + x^3)\,dx$$
$$= \frac{2}{3}\left\{\frac{2}{5}x^{5/2} - \frac{x^{10}}{10} + \frac{3x^8}{8} - \frac{x^6}{2} + \frac{x^4}{4}\right\}_0^a = 0.519.$$

26. If we choose the coordinate system shown, the second moment of area about the x-axis is

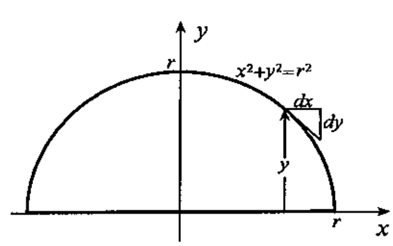
$$I = 2\int_0^{y_1} y^2(x_1 - x_2)\,dy \qquad (\text{where } y_1 = ab/\sqrt{a^2+b^2})$$
$$= 2\int_0^{y_1} y^2\left[\frac{a}{b}\left(\frac{ab}{\sqrt{a^2+b^2}} - y\right) - \frac{b}{a}\left(y - \frac{ab}{\sqrt{a^2+b^2}}\right)\right] dy$$
$$= 2\int_0^{y_1} y^2\left[\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}} - y\left(\frac{a}{b} + \frac{b}{a}\right)\right] dy$$
$$= 2\left\{\sqrt{a^2+b^2}\,\frac{y^3}{3} - \frac{a^2+b^2}{ab}\frac{y^4}{4}\right\}_0^{y_1}$$
$$= 2\left[\frac{1}{3}\sqrt{a^2+b^2}\,\frac{a^3b^3}{(a^2+b^2)^{3/2}} - \frac{a^2+b^2}{ab}\,\frac{a^4b^4}{4(a^2+b^2)^2}\right]$$
$$= \frac{a^3b^3}{6(a^2+b^2)}.$$



EXERCISES 7.9

1. The sphere can be formed by rotating the semicircle $y = \sqrt{r^2 - x^2}$ around the x-axis. Small lengths along the curve corresponding to lengths dx along the x-axis are given by

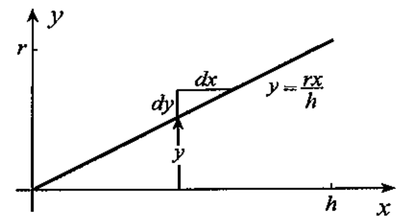
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$$
$$= \frac{r}{\sqrt{r^2-x^2}}dx.$$



The area of the sphere is therefore $A = 2\int_0^r 2\pi y\left(\frac{r}{\sqrt{r^2-x^2}}\right)dx = 4\pi r\int_0^r dx = 4\pi r\Big\{x\Big\}_0^r = 4\pi r^2.$

2. The cone can be formed by rotating the straight line segment shown about the x-axis. Small lengths along the curve corresponding to lengths dx along the x-axis are given by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+(r/h)^2}\,dx.$$

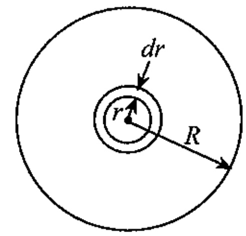


The area of the curved portion of the cone is therefore

$$A=\int_0^h 2\pi y\sqrt{1+(r/h)^2}\,dx = 2\pi\sqrt{1+(r/h)^2}\int_0^h \frac{rx}{h}dx = \frac{2\pi r}{h^2}\sqrt{r^2+h^2}\left\{\frac{x^2}{2}\right\}_0^h = \pi r\sqrt{r^2+h^2}.$$

3. (a) Since the amount of blood flowing through the ring per unit time is $v(r)(2\pi r\,dr)$, the total flow through the vessel is

$$F=\int_0^R cR\sqrt{R^2-r^2}(2\pi r)\,dr$$
$$=2\pi cR\left\{-\frac{1}{3}(R^2-r^2)^{3/2}\right\}_0^R = \frac{2\pi cR^4}{3}.$$



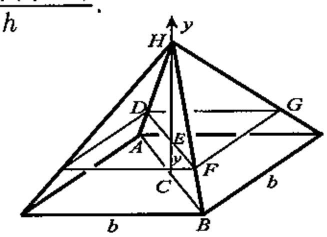
(b) With $v(r)=(c/R^2)(R^2-r^2)^2$,

$$F=\int_0^R \frac{c}{R^2}(R^2-r^2)^2(2\pi r)\,dr = \frac{2\pi c}{R^2}\left\{-\frac{1}{6}(R^2-r^2)^3\right\}_0^R = \frac{\pi cR^4}{3}.$$

4. Cross sections of the pyramid parallel to the base are always square. At height y above the base, similar triangles give $\|DE\|/\|EH\| = \|AC\|/\|HC\|$, or $\|DE\| = \dfrac{(h-y)(b/\sqrt{2})}{h}$.

Consequently,

$$\|FG\| = \frac{\|DF\|}{\sqrt{2}} = \frac{2\|DE\|}{\sqrt{2}}$$
$$=\sqrt{2}\left[\frac{b}{\sqrt{2}h}(h-y)\right] = \frac{b}{h}(h-y).$$



The area of the square at height y is therefore $b^2(h-y)^2/h^2$, and the volume of the pyramid is

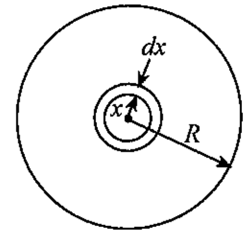
$$V=\int_0^h \frac{b^2}{h^2}(h-y)^2\,dy = \frac{b^2}{h^2}\left\{-\frac{1}{3}(h-y)^3\right\}_0^h = \frac{b^2h}{3}.$$

5. The total amount of water consumed is

$$\int_0^6 f(t)\,dt = \int_0^6 (5000+21.65t^2-249.7t^3+97.52t^4-9.680t^5)\,dt$$
$$=\left\{5000t+\frac{21.65t^3}{3}-\frac{249.7t^4}{4}+\frac{97.52t^5}{5}-\frac{9.680t^6}{6}\right\}_0^6 = 2.70\times 10^4\ \text{m}^3.$$

6. (a) The number of bees in a ring of width dx at distance x from the hive is $\rho(2\pi x\,dx)$. The total number of bees in the colony is therefore

$$N=\int_0^R \rho(2\pi x\,dx)$$
$$=2\pi\int_0^R x\left[\frac{600\,000}{31\pi R^5}(R^3+2R^2x-Rx^2-2x^3)\right]dx$$
$$=\frac{1\,200\,000}{31R^5}\left\{\frac{R^3x^2}{2}+\frac{2R^2x^3}{3}-\frac{Rx^4}{4}-\frac{2x^5}{5}\right\}_0^R = 20\,000.$$



(b) The number of bees within $R/2$ of the hive is

$$\overline{N} = \int_0^{R/2} \rho(2\pi x\,dx) = \frac{1\,200\,000}{31R^5}\left\{\frac{R^3x^2}{2} + \frac{2R^2x^3}{3} - \frac{Rx^4}{4} - \frac{2x^5}{5}\right\}_0^{R/2} = 6976,$$

or approximately 7000.

7. The amount of blood flowing through the hardened vessel is

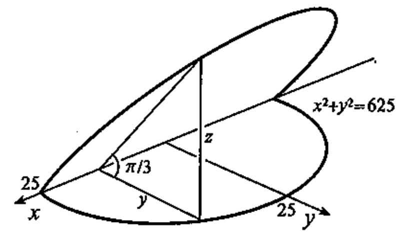
$$\int_0^{R/2} c(R^2 - 4r^2)(2\pi r)\,dr = 2\pi c\left\{\frac{R^2r^2}{2} - r^4\right\}_0^{R/2} = \frac{\pi c R^4}{8}.$$

Thus, only 25% of the flow gets through the hardened vessel.

8. Cross sections of the wedge parallel to the axis of the trunk and perpendicular to the diameter are triangles. At distance x from the centre of the wedge, $y = \sqrt{625 - x^2}$, and $z = y\tan(\pi/3) = \sqrt{3}\sqrt{625 - x^2}$. The area of the triangle at position x is therefore

$$\frac{1}{2}yz = \frac{\sqrt{3}}{2}(625 - x^2),$$

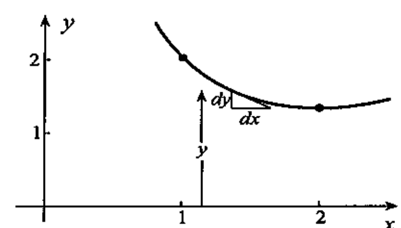
and the volume of the wedge is



$$V = 2\int_0^{25} \frac{\sqrt{3}}{2}(625 - x^2)\,dx = \sqrt{3}\left\{625x - \frac{x^3}{3}\right\}_0^{25} = \frac{31\,250}{\sqrt{3}}\ \text{cm}^3.$$

9. Small lengths along the curve corresponding to lengths dx along the x-axis are given by

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{x^2}{8} - \frac{2}{x^2}\right)^2}\,dx$$
$$= \sqrt{1 + \frac{x^4}{64} - \frac{1}{2} + \frac{4}{x^4}}\,dx$$
$$= \sqrt{\left(\frac{x^2}{8} + \frac{2}{x^2}\right)^2}\,dx = \left(\frac{x^2}{8} + \frac{2}{x^2}\right)dx.$$



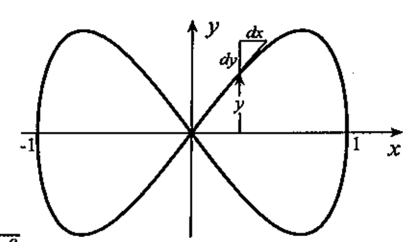
The area of the surface is

$$A = \int_1^2 2\pi y\left(\frac{x^2}{8} + \frac{2}{x^2}\right)dx = 2\pi\int_1^2\left(\frac{x^3}{24} + \frac{2}{x}\right)\left(\frac{x^2}{8} + \frac{2}{x^2}\right)dx = 2\pi\int_1^2\left(\frac{x^5}{192} + \frac{x}{3} + \frac{4}{x^3}\right)dx$$
$$= 2\pi\left\{\frac{x^6}{1152} + \frac{x^2}{6} - \frac{2}{x^2}\right\}_1^2 = \frac{263\pi}{64}.$$

10. If we differentiate $8y^2 = x^2(1 - x^2)$ with respect to x,

$$16y\frac{dy}{dx} = 2x - 4x^3 \implies \frac{dy}{dx} = \frac{x - 2x^3}{8y}.$$

Small lengths along the curve in the first quadrant corresponding to lengths dx along the x-axis are given by



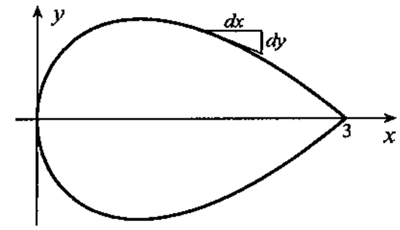
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{x - 2x^3}{8y}\right)^2}\,dx = \frac{\sqrt{64y^2 + x^2 - 4x^4 + 4x^6}}{8y}\,dx$$
$$= \frac{\sqrt{8x^2 - 8x^4 + x^2 - 4x^4 + 4x^6}}{8y}\,dx = \frac{3x - 2x^3}{2\sqrt{2}x\sqrt{1 - x^2}}\,dx = \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}}\,dx.$$

The area of the surface of revolution is therefore

$$A = 2\int_0^1 2\pi y\left(\frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\right)dx = \sqrt{2}\pi\int_0^1 \frac{x\sqrt{1-x^2}}{2\sqrt{2}}\left(\frac{3-2x^2}{\sqrt{1-x^2}}\right)dx$$
$$= \frac{\pi}{2}\int_0^1 (3x-2x^3)\,dx = \frac{\pi}{2}\left\{\frac{3x^2}{2} - \frac{x^4}{2}\right\}_0^1 = \frac{\pi}{2}.$$

11. If we differentiate $9y^2 = 9x - 6x^2 + x^3$ with respect to x,

$$18y\frac{dy}{dx} = 9 - 12x + 3x^2 \implies \frac{dy}{dx} = \frac{3-4x+x^2}{6y}.$$



Small lengths along the curve in the first quadrant corresponding to lengths dx along the x-axis are given by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+\left(\frac{3-4x+x^2}{6y}\right)^2}dx$$
$$= \sqrt{1+\frac{(3-x)^2(1-x)^2}{4x(3-x)^2}}dx = \sqrt{\frac{4x+(1-x)^2}{4x}}dx = \sqrt{\frac{(1+x)^2}{4x}}dx = \frac{1+x}{2\sqrt{x}}dx.$$

(a) The area of the surface of revolution for rotation around the y-axis is

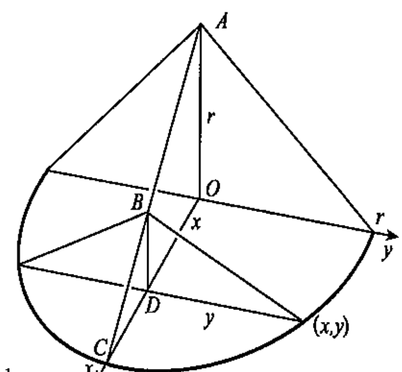
$$2\int_0^3 2\pi x\left(\frac{1+x}{2\sqrt{x}}\right)dx = 2\pi\int_0^3(\sqrt{x}+x^{3/2})\,dx = 2\pi\left\{\frac{2x^{3/2}}{3}+\frac{2x^{5/2}}{5}\right\}_0^3 = \frac{56\sqrt{3}\pi}{5}.$$

(b) For rotation around the x-axis, the area is

$$\int_0^3 2\pi y\left(\frac{1+x}{2\sqrt{x}}\right)dx = \pi\int_0^3\frac{1}{3}(3-x)\sqrt{x}\left(\frac{1+x}{\sqrt{x}}\right)dx = \frac{\pi}{3}\int_0^3(3+2x-x^2)\,dx = \frac{\pi}{3}\left\{3x+x^2-\frac{x^3}{3}\right\}_0^3 = 3\pi.$$

12. For the triangle at position x, we note that by similar triangles $\|BD\|/\|CD\| = \|AO\|/\|CO\|$, or,

$$\|BD\| = (r-x)\frac{r}{r} = r - x.$$



Because the base of the triangle has length $2y = 2\sqrt{r^2-x^2}$, the area of the triangle at x is

$$\frac{1}{2}(r-x)2\sqrt{r^2-x^2} = (r-x)\sqrt{r^2-x^2}.$$

The volume of the solid is therefore

$$V = 2\int_0^r (r-x)\sqrt{r^2-x^2}\,dx$$
$$= 2r\int_0^r\sqrt{r^2-x^2}\,dx - 2\int_0^r x\sqrt{r^2-x^2}\,dx.$$

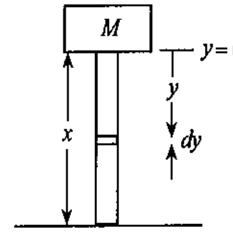
Since the first integral is one-quarter of the area of the base circle,

$$V = 2r\left(\frac{1}{4}\pi r^2\right) - 2\left\{-\frac{1}{3}(r^2-x^2)^{3/2}\right\}_0^r = \frac{\pi r^3}{2} - \frac{2r^3}{3} = \frac{(3\pi-4)r^3}{6}.$$

13. According to formula 7.43, the amount of stretch in the rod is $FL/(AE)$. Hence, the length of the rod is $L + FL/(AE)$.

14. If x is the original length of the rod, then according to equation 7.43, the compression when M is placed on top is $Mgx/(AE)$, where $g = 9.81$. It follows that $x - \dfrac{Mgx}{AE} = L \implies x = \dfrac{AEL}{AE - Mg}$.

15. Let x and ρ be the unconstrained length and density of the rod. The force on each cross section in an element dy is $Mg + \rho g A y$, and therefore its compression is $\dfrac{(M+\rho A y)g}{AE}\,dy$.



Total compression of the rod is

$$\int_0^x \frac{(M+\rho A y)g}{AE}\,dy = \frac{g}{AE}\left\{\frac{(M+\rho A y)^2}{2\rho A}\right\}_0^x = \frac{g[(M+\rho A x)^2 - M^2]}{2\rho E A^2}.$$

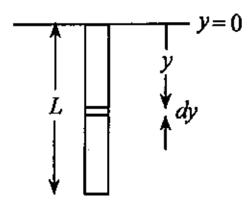
The length of the compressed rod is x minus this value and therefore

$$x - \frac{g[(M+\rho A x)^2 - M^2]}{2\rho E A^2} = L \quad \Longrightarrow \quad \frac{\rho g}{E}x^2 + 2\left(\frac{Mg}{EA} - 1\right)x + 2L = 0.$$

Of the two solutions to this quadratic equation, we choose

$$x = \frac{-2\left(\dfrac{Mg}{EA} - 1\right) + \sqrt{4\left(\dfrac{Mg}{EA} - 1\right)^2 - \dfrac{8\rho g L}{E}}}{2\rho g/E} = \frac{E}{\rho g}\left[1 - \frac{Mg}{EA} + \sqrt{\left(1 - \frac{Mg}{EA}\right)^2 - \frac{2\rho g L}{E}}\right].$$

16. If we consider a small length dy at position y, the force on each cross section in this element is approximately the same, and equal to the weight of that part of the rod below it plus F, $\rho g(L-y)A + F$, where ρ is the density of the material in the rod. According to equation 7.43, the element dy stretches by



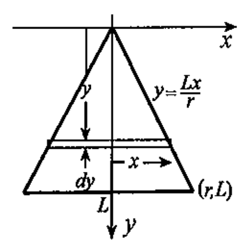
$$\frac{[\rho g(L-y)A + F]\,dy}{AE}.$$

Total stretch in the rod is therefore

$$\int_0^L \frac{\rho g(L-y)A + F}{AE}dy = \frac{1}{AE}\left\{-\frac{\rho g A}{2}(L-y)^2 + Fy\right\}_0^L = \frac{1}{AE}\left(FL + \frac{\rho g A L^2}{2}\right).$$

The length of the rod is therefore $L + \dfrac{FL}{AE} + \dfrac{\rho g L^2}{2E}$.

17. The force on the cross section of width dy at position y is the weight of the rod below it,



$$\begin{aligned} W &= \rho g\left(\frac{1}{3}\pi L r^2 - \frac{1}{3}\pi x^2 y\right) \\ &= \frac{\rho g \pi}{3}\left[L r^2 - y\left(\frac{ry}{L}\right)^2\right] \\ &= \frac{\rho g \pi r^2}{3L^2}(L^3 - y^3). \end{aligned}$$

The element dy therefore stretches by

$$\frac{\rho g \pi r^2}{3L^2 A E}(L^3 - y^3)\,dy = \frac{\rho g \pi r^2}{3L^2 E\pi(ry/L)^2}(L^3 - y^3)\,dy.$$

To find total stretch we should integrate this function from 0 to L. But this cannot be done because the function has an infinite discontinuity at $y = 0$.

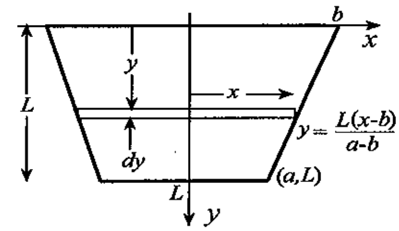
18. Since the force on each cross section is Mg, the stretch of the element of width dy at position y is $[Mg/(AE)]\,dy$. Since the cross-sectional area at y is $A = \pi x^2 = \pi r^2(1 - y/L)^2$, the stretch of element dy is

$$\frac{Mg}{E\pi r^2(1-y/L)^2}\,dy.$$

To find total stretch we should integrate this function from 0 to L. But this cannot be done because the function has an infinite discontinuity at $y = L$.

19. The answer to this problem is the same as that in Exercise 20 with a and b interchanged.

20. The force on the cross section of width dy at position y is the weight of the rod below it. To find this weight we calculate the volume by slicing,



$$\int_y^L \left[b + \frac{(a-b)}{L}y\right]^2 dy = \left\{\frac{L}{3(a-b)}\left[b + \frac{(a-b)}{L}y\right]^3\right\}_y^L$$
$$= \frac{1}{3(a-b)L^2}\left\{a^3L^3 - [bL + (a-b)y]^3\right\}.$$

The element dy therefore stretches by

$$\frac{\rho g\{a^3L^3 - [bL + (a-b)y]^3\}}{3(a-b)L^2[b + (a-b)y/L]^2 E}\,dy = \frac{\rho g\{a^3L^3 - [bL + (a-b)y]^3\}}{3E(a-b)[bL + (a-b)y]^2}\,dy.$$

Total stretch of the rod is

$$\int_0^L \frac{\rho g\{a^3L^3 - [bL + (a-b)y]^3\}}{3E(a-b)[bL + (a-b)y]^2}\,dy = \frac{\rho g}{3E(a-b)}\int_0^L \left\{\frac{a^3L^3}{[bL + (a-b)y]^2} - [bL + (a-b)y]\right\} dy$$
$$= \frac{\rho g}{3E(a-b)}\left\{\frac{-a^3L^3}{(a-b)[bL + (a-b)y]} - \frac{[bL + (a-b)y]^2}{2(a-b)}\right\}_0^L$$
$$= \frac{\rho g L^2(2a^3 - 3a^2b + b^3)}{6bE(a-b)^2}.$$

The total length of the rod is this quantity plus L.

21. The answer to this problem is the same as that in Exercise 22 with a and b interchanged.

22. Without the extra weight, the answer to the stretch of the rod in Exercise 19 is the same as that in Exercise 20 with a and b interchanged. With M added, we must add Mg to the weight of each cross section. Stretch is therefore given by the answer in Exercise 20 with a and b interchanged plus the following integral

$$\int_0^L \frac{Mg}{[a + (b-a)y/L]^2 E}\,dy = \frac{MgL^2}{E}\int_0^L \frac{1}{[aL + (b-a)y]^2}\,dy$$
$$= \frac{MgL^2}{E}\left\{\frac{-1}{(b-a)[aL + (b-a)y]}\right\}_0^L = \frac{MgL}{abE}.$$

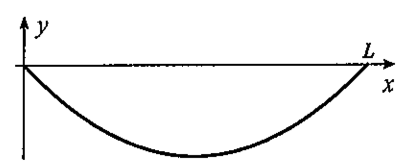
Thus, the length of the rod is $L + \dfrac{MgL}{abE} + \dfrac{\rho g L^2(2b^3 - 3ab^2 + a^3)}{6bE(a-b)^2}$.

23. The probability of an electron striking a ring of width dx at position x is $f(x)(2\pi x)\,dx$. The percentage of electrons striking within distance r from the centre of the target is

$$100\int_0^r 2\pi x\, f(x)\,dx = 200\pi\int_0^r \frac{5x}{3\pi R^5}(R^3 - x^3)\,dx = \frac{1000}{3R^5}\left\{\frac{R^3x^2}{2} - \frac{x^5}{5}\right\}_0^r = \frac{100r^2(5R^3 - 2r^3)}{3R^5}.$$

24. (a) According to equation 7.46, deflections are given by

$$y(x) = \int_0^L \frac{1}{L\tau}[x(L-X)h(X-x) + X(L-x)h(x-X)]k\,dX$$
$$= \frac{k}{L\tau}\int_0^x X(L-x)\,dX + \frac{k}{L\tau}\int_x^L x(L-X)\,dX$$
$$= \frac{k(L-x)}{L\tau}\left\{\frac{X^2}{2}\right\}_0^x + \frac{kx}{L\tau}\left\{\frac{-(L-X)^2}{2}\right\}_x^L$$
$$= \frac{kx(L-x)}{2\tau}.$$



(b) The graph of deflections is the parabola shown above. It is symmetric about $x = L/2$ with minimum at $x = L/2$. We would expect this for constant loading.

25. With $F(x) = k - F\delta(x - L/2)$, formla 7.46 gives

$$\begin{aligned}
y(x) &= \int_0^L \frac{1}{L\tau}[x(L-X)h(X-x) + X(L-x)h(x-X)][k - F\delta(X - L/2)]\,dX \\
&= \frac{k}{L\tau}\int_0^L [x(L-X)h(X-x) + X(L-x)h(x-X)]\,dX \\
&\qquad - \frac{F}{L\tau}\int_0^L [x(L-X)h(X-x) + X(L-x)h(x-X)]\delta(X - L/2)\,dX \\
&= \frac{k}{L\tau}\int_0^x X(L-x)\,dX + \frac{k}{L\tau}\int_x^L x(L-X)\,dX \\
&\qquad - \frac{F}{L\tau}\left[x\left(L - \frac{L}{2}\right)h\left(\frac{L}{2} - x\right) + \frac{L}{2}(L-x)\,h\left(x - \frac{L}{2}\right)\right] \\
&= \frac{k(L-x)}{L\tau}\left\{\frac{X^2}{2}\right\}_0^x + \frac{kx}{L\tau}\left\{\frac{-(L-X)^2}{2}\right\}_x^L - \frac{F}{2\tau}\left[x\,h\left(\frac{L}{2} - x\right) + (L-x)\,h\left(x - \frac{L}{2}\right)\right] \\
&= \frac{kx(L-x)}{2\tau} - \frac{F}{2\tau}\begin{cases} x & 0 < x < L/2 \\ L - x, & L/2 < x < L \end{cases} \\
&= \frac{1}{2\tau}\begin{cases} -kx^2 + (kL - F)x, & 0 < x \le L/2 \\ (L-x)(kx - F), & L/2 < x < L \end{cases},
\end{aligned}$$

where we have removed the discontinuity at $x = L/2$.

26. (a) According to equation 7.46,

$$\begin{aligned}
y(x) &= \int_0^L \frac{1}{6EI}[(x-X)^3 h(x-X) - x^3 + 3Xx^2]k\,dX \\
&= \frac{k}{6EI}\int_0^x [(x-X)^3 - x^3 + 3Xx^2]\,dX + \frac{k}{6EI}\int_x^L (3Xx^2 - x^3)\,dX \\
&= \frac{k}{6EI}\left\{\frac{-(x-X)^4}{4} - x^3X + \frac{3X^2x^2}{2}\right\}_0^x + \frac{k}{6EI}\left\{\frac{3X^2x^2}{2} - x^3X\right\}_x^L \\
&= \frac{kx^2(x^2 - 4Lx + 6L^2)}{24EI}.
\end{aligned}$$

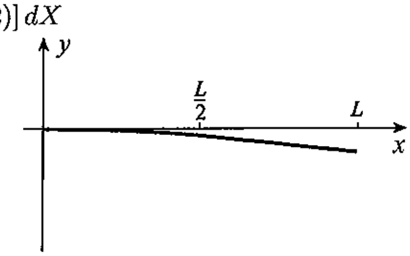
(b) With uniform loading and a free end at $x = L$, maximum deflection, meaning minimum y, should occur at $x = L$. We can verify this by finding critical points of $y(x)$,

$$0 = y'(x) = \frac{k}{24EI}(4x^3 - 12Lx^2 + 12L^2x) = \frac{4kx}{24EI}(x^2 - 3Lx + 3L^2).$$

Since $x = 0$ is the only solution (as expected), minimum $y(x)$ must occur at either $x = 0$ or $x = L$. Since $y(0) = 0$ and $y(L) < 0$, maximum deflection occurs at $x = L$.

27. According to equation 7.46,

$$\begin{aligned}
y(x) &= \int_0^L \left[\frac{1}{6EI}(x-X)^3 h(x-X) - \frac{x^3}{6EI} + \frac{Xx^2}{2EI}\right][-F\delta(X - L/2)]\,dX \\
&= -F\left[\frac{1}{6EI}(x - L/2)^3 h(x - L/2) - \frac{x^3}{6EI} + \frac{Lx^2}{4EI}\right] \\
&= \frac{F}{48EI}\begin{cases} 8x^3 - 12Lx^2, & 0 < x \le L/2 \\ L^3 - 6L^2x, & L/2 < x \le L \end{cases}.
\end{aligned}$$



We have removed the discontinuity at $x = L/2$.

(b) A graph is shown to the right. It is straight for $L/2 < x \le L$ since there is no loading on this part of the board.

28. (a) According to equation 7.46,

$$\begin{aligned}
y(x) &= \int_0^L \left[\frac{1}{6EI}(x-X)^3 h(x-X) + \frac{x^3}{6EIL^3}(-L^3+3LX^2-2X^3) + \frac{x^2}{2EIL^2}(X^3-2LX^2+L^2X)\right] k\,dX \\
&= \frac{k}{6EIL^3}\int_0^x \left[L^3(x-X)^3 + x^3(-L^3+3LX^2-2X^3) + 3x^2L(X^3-2LX^2+L^2X)\right] dX \\
&\quad + \frac{k}{6EIL^3}\int_x^L \left[x^3(-L^3+3LX^2-2X^3) + 3x^2L(X^3-2LX^2+L^2X)\right] dX \\
&= \frac{k}{6EIL^3}\left\{\frac{-L^3(x-X)^4}{4} + x^3\left(-L^3X + LX^3 - \frac{X^4}{2}\right) + 3x^2L\left(\frac{X^4}{4} - \frac{2LX^3}{3} + \frac{L^2X^2}{2}\right)\right\}_0^x \\
&\quad + \frac{k}{6EIL^3}\left\{x^3\left(-L^3X + LX^3 - \frac{X^4}{2}\right) + 3x^2L\left(\frac{X^4}{4} - \frac{2LX^3}{3} + \frac{L^2X^2}{2}\right)\right\}_x^L \\
&= \frac{x^2(L-x)^2}{24EI}.
\end{aligned}$$

(b) Phsically we expect maximum deflection, meaning minimum y, at $x = L/2$. To confirm this, we find critical points of $y(x)$,

$$0 = y'(x) = \frac{1}{24EI}[2x(L-x)^2 - 2x^2(L-x)] = \frac{1}{12EI}x(L-x)(L-2x).$$

This gives the expected $x = 0$ and $x = L$ (the beam is horizontal at its ends), and the hoped for $x = L/2$.

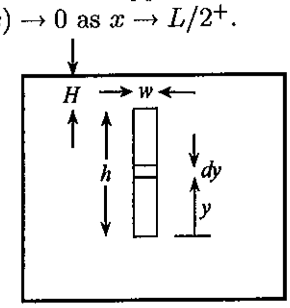
29. (a) According to equation 7.46,

$$\begin{aligned}
y(x) &= \int_0^L \left[\frac{1}{6EI}(x-X)^3 h(x-X) + \frac{x^3}{6EIL^3}(-L^3+3LX^2-2X^3)\right. \\
&\qquad \left. + \frac{x^2}{2EIL^2}(X^3-2LX^2+L^2X)\right](-F)\delta(x-L/2)\,dX \\
&= -F\left[\frac{1}{6EI}(x-L/2)^3 h(x-L/2) + \frac{x^3}{6EIL^3}\left(-L^3 + \frac{3L^3}{4} - \frac{L^3}{4}\right)\right. \\
&\qquad \left. + \frac{x^2}{2EIL^2}\left(\frac{L^3}{8} - \frac{L^3}{2} + \frac{L^3}{2}\right)\right] \\
&= \frac{F}{48EI}\left[-8(x-L/2)^3 h(x-L/2) - 4x^3 + 3Lx^2\right].
\end{aligned}$$

(b) Physically, we expect maximum deflection, meaning minimum y, at $x = L/2$. To confirm this we find critical points of $y(x)$ on the intervals $0 \le x < L/2$ and $L/2 < x \le L$. For $0 \le x < L/2$, the shape is $y(x) = F(3Lx^2 - 4x^3)/(48EI)$. For critical points, we solve $0 = y'(x) = F(6Lx - 12x^2)/(48EI) = Fx(L-2x)/(8EI)$, giving $x = 0$, as expected. In addition, the derivative approaches zero as $x \to L/2^-$. Similarly, for $L/2 < x \le L$, the critical point is $x = L$, and $y'(x) \to 0$ as $x \to L/2^+$.

30. According to the modified Torricelli law, the velocity of water through a vertical rectangle of height dy at a distance y above the bottom of the slit (see figure) is

$$v = c\sqrt{2g(H+h-y)}.$$

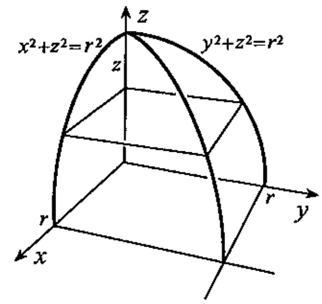


The rate at which water passes through this tiny rectangle is $v(w\,dy)$, and therefore the volume of water passing through the slit per unit time is

$$Q = \int_0^h c\sqrt{2g(H+h-y)}\,w\,dy = \sqrt{2g}cw\left\{-\frac{2}{3}(H+h-y)^{3/2}\right\}_0^h = \frac{2\sqrt{2g}cw}{3}[(H+h)^{3/2} - H^{3/2}].$$

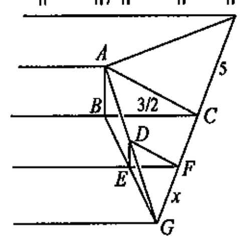
31. The volume is eight times that shown in the figure to the right. Horizontal cross sections of this volume are squares with length and width $\sqrt{r^2 - z^2}$. Consequently, the required volume is

$$V = 8\int_0^r (r^2 - z^2)\,dz = 8\left\{r^2 z - \frac{z^3}{3}\right\}_0^r = \frac{16r^3}{3}.$$



32. Vertical cross sections of the attic parallel to the length of the roof are trapezoids. Because BGC and EGF are similar triangles, ratios of corresponding sides are equal: $\|EF\|/\|FG\| = \|BC\|/\|GC\|$, or

$$\|EF\| = \frac{x(3/2)}{5} = \frac{3x}{10}.$$



Since ABC and DEF are similar triangles, we can say that $\|DE\|/\|EF\| = \|AB\|/\|BC\|$, or

$$\|DE\| = \frac{3x}{10}\left(\frac{2}{3/2}\right) = \frac{2x}{5}.$$

The area of the trapezoid at position x is therefore

$$\|DE\|\left(\frac{1}{2}\right)[15 + (15 - 2\|EF\|)] = \frac{2x}{5}\left(\frac{1}{2}\right)\left[30 - 2\left(\frac{3x}{10}\right)\right] = \frac{3x}{25}(50 - x).$$

The volume in the attic can now be calculated as

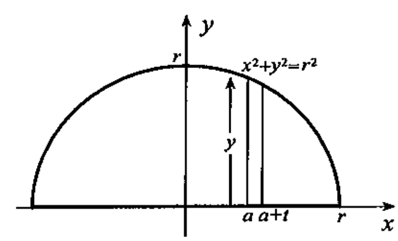
$$V = 2\int_0^5 \frac{3x}{25}(50 - x)\,dx = \frac{6}{25}\left\{25x^2 - \frac{x^3}{3}\right\}_0^5 = 140 \text{ m}^3.$$

33. In a small interval dt of time at time t before T, the number of births is $r(t)\,dt$. The number of these that survive until time T is $p(T-t)r(t)\,dt$. Integration of this gives the number of individuals born after $t = 0$ that survive to time T,

$$N(T) = \int_0^T p(T - t)r(t)\,dt.$$

34. (a) The area of the peel for a slice of thickness t at any value $x = a$ is

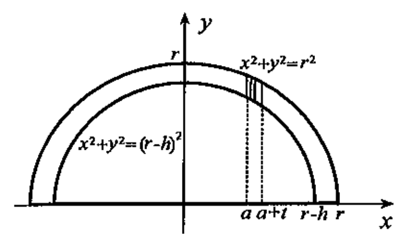
$$\begin{aligned}\text{Area } &= \int_a^{a+t} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi\int_a^{a+t} \sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx \\ &= 2\pi\int_a^{a+t} r\,dx = 2\pi rt.\end{aligned}$$



Since this is independent of a, any slice of width t has the same peel area. If there are n slices, each of width $2r/n$, the area of peel is $2\pi r(2r/n) = 4\pi r^2/n$. This is reasonable in that this is the area of the sphere divided by the number of slices.

(b) The volume in the peel for a slice of thickness t at $x = a$ is

$$\begin{aligned}\text{Volume } &= \int_a^{a+t} \{\pi(r^2 - x^2) - \pi[(r - h)^2 - x^2]\}\,dx \\ &= \pi\int_a^{a+t} (2rh - h^2)\,dx = \pi(2rh - h^2)t.\end{aligned}$$



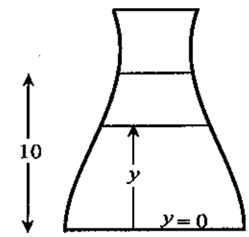
Since this is independent of a, the volume of peel is the same for each slice. This is true, however, only for slices which have holes. Clearly, slices between $r - h$ and r do not all have the same volume.

35. Suppose that $A(y)$ is the cross-sectional area of the container when the depth is y. Then, $dV/dt = k\,A(y)$. But the volume of water in the container when the depth is y is

$$V = \int_0^y A(y)\,dy.$$

The derivative of this equation with respect to t (using 6.19) is

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}.$$



We now equate the two expressions for dV/dt,

$$A(y)\frac{dy}{dt} = kA(y).$$

Hence $\dfrac{dy}{dt} = k$, and the solution of this equation is $y = kt + C$. The conditions $y(0) = 10$ and $y(5) = 9$ require $k = -1/5$ and $C = 10$. Consequently $y(t) = 10 - t/5$. The container empties when $y = 0$, and this occurs when $t = 50$ days.

36. (a) Yearly ordering costs are $F(N/x) + Nf$.
(b) The stocking cost at time t (in years) from the beginning of one stock period for a period dt is $p(x - Nt)\,dt$, where $x - Nt$ represents the number of refrigerators at time t. Total yearly stocking charges are

$$\frac{N}{x}\int_0^{x/N} p(x - Nt)\,dt = \frac{Np}{x}\left\{xt - \frac{Nt^2}{2}\right\}_0^{x/N} = \frac{px}{2}.$$

(c) Total yearly inventory costs are $I(x) = \dfrac{FN}{x} + Nf + \dfrac{px}{2}$. For critical points of this function, we solve

$$0 = I'(x) = -\frac{FN}{x^2} + \frac{p}{2} \quad\Longrightarrow\quad x = \sqrt{\frac{2FN}{p}}.$$

Since $\displaystyle\lim_{x\to 0^+} I(x) = \infty = \lim_{x\to\infty} I(x)$, it follows that $I(x)$ is minimized by $x = \sqrt{2FN/p}$.

EXERCISES 7.10

1. $$\int_3^\infty \frac{1}{(x+4)^2}dx = \lim_{b\to\infty}\int_3^b \frac{1}{(x+4)^2}\,dx = \lim_{b\to\infty}\left\{\frac{-1}{x+4}\right\}_3^b = \lim_{b\to\infty}\left(\frac{-1}{b+4} + \frac{1}{7}\right) = \frac{1}{7}$$

2. $$\int_3^\infty \frac{1}{(x+4)^{1/3}}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{(x+4)^{1/3}}\,dx = \lim_{b\to\infty}\left\{\frac{3}{2}(x+4)^{2/3}\right\}_3^b = \lim_{b\to\infty}\left[\frac{3}{2}(b+4)^{2/3} - \frac{3}{2}(7)^{2/3}\right] = \infty$$

3. $$\int_{-\infty}^{-4} \frac{x}{\sqrt{x^2-2}}\,dx = \lim_{a\to-\infty}\int_a^{-4} \frac{x}{\sqrt{x^2-2}}\,dx = \lim_{a\to-\infty}\left\{\sqrt{x^2-2}\right\}_a^{-4} = \lim_{a\to-\infty}(\sqrt{14} - \sqrt{a^2-2}) = -\infty$$

4. $$\int_{-\infty}^{-4} \frac{x}{(x^2-2)^4}\,dx = \lim_{a\to-\infty}\int_a^{-4} \frac{x}{(x^2-2)^4}\,dx = \lim_{a\to-\infty}\left\{\frac{-1}{6(x^2-2)^3}\right\}_a^{-4}$$
$$= \lim_{a\to-\infty}\left[-\frac{1}{6(14)^3} + \frac{1}{6(a^2-2)^3}\right] = \frac{-1}{16\,464}$$

5. $$\int_{-\infty}^\infty \frac{10^{10}x^3}{(x^4+5)^2}\,dx = 10^{10}\lim_{a\to-\infty}\int_a^0 \frac{x^3}{(x^4+5)^2}\,dx + 10^{10}\lim_{b\to\infty}\int_0^b \frac{x^3}{(x^4+5)^2}\,dx$$
$$= 10^{10}\lim_{a\to-\infty}\left\{\frac{-1}{4(x^4+5)}\right\}_a^0 + 10^{10}\lim_{b\to\infty}\left\{\frac{-1}{4(x^4+5)}\right\}_0^b$$
$$= 10^{10}\lim_{a\to-\infty}\left[-\frac{1}{20} + \frac{1}{4(a^4+5)}\right] + 10^{10}\lim_{b\to\infty}\left[\frac{-1}{4(b^4+5)} + \frac{1}{20}\right] = -\frac{10^{10}}{20} + \frac{10^{10}}{20} = 0$$

6. $$\int_{-\infty}^{\infty}\frac{x^3}{(x^4+5)^{1/4}}\,dx = \lim_{a\to-\infty}\int_a^0\frac{x^3}{(x^4+5)^{1/4}}\,dx + \lim_{b\to\infty}\int_0^b\frac{x^3}{(x^4+5)^{1/4}}\,dx$$
$$= \lim_{a\to-\infty}\left\{\frac{1}{3}(x^4+5)^{3/4}\right\}_a^0 + \lim_{b\to\infty}\left\{\frac{1}{3}(x^4+5)^{3/4}\right\}_0^b$$
$$= \lim_{a\to-\infty}\left[\frac{5^{3/4}-(a^4+5)^{3/4}}{3}\right] + \lim_{b\to\infty}\left[\frac{(b^4+5)^{3/4}-5^{3/4}}{3}\right]$$

Since neither of these limits exists, the integral diverges.

7. $$\int_0^1\frac{1}{(1-x)^{5/3}}\,dx = \lim_{c\to1^-}\int_0^c\frac{1}{(1-x)^{5/3}}\,dx = \lim_{c\to1^-}\left\{\frac{3}{2(1-x)^{2/3}}\right\}_0^c = \lim_{c\to1^-}\left[\frac{3}{2(1-c)^{2/3}}-\frac{3}{2}\right]=\infty$$

8. $$\int_0^1\frac{1}{\sqrt{1-x}}\,dx = \lim_{c\to1^-}\int_0^c\frac{1}{\sqrt{1-x}}\,dx = \lim_{c\to1^-}\left\{-2\sqrt{1-x}\right\}_0^c = \lim_{c\to1^-}(-2\sqrt{1-c}+2)=2$$

9. $$\int_1^{\infty}x\sqrt{x^2-1}\,dx = \lim_{b\to\infty}\int_1^b x\sqrt{x^2-1}\,dx = \lim_{b\to\infty}\left\{\frac{1}{3}(x^2-1)^{3/2}\right\}_1^b = \frac{1}{3}\lim_{b\to\infty}(b^2-1)^{3/2}=\infty$$

10. $$\int_2^5\frac{x}{\sqrt{x^2-4}}\,dx = \lim_{c\to2^+}\int_c^5\frac{x}{\sqrt{x^2-4}}\,dx = \lim_{c\to2^+}\left\{\sqrt{x^2-4}\right\}_c^5 = \lim_{c\to2^+}(\sqrt{21}-\sqrt{c^2-4})=\sqrt{21}$$

11. $$\int_{-1}^1\frac{x}{(1-x^2)^2}\,dx = \lim_{c\to-1^+}\int_c^0\frac{x}{(1-x^2)^2}\,dx + \lim_{d\to1^-}\int_0^d\frac{x}{(1-x^2)^2}\,dx$$
$$= \lim_{c\to-1^+}\left\{\frac{1}{2(1-x^2)}\right\}_c^0 + \lim_{d\to1^-}\left\{\frac{1}{2(1-x^2)}\right\}_0^d$$
$$= \lim_{c\to-1^+}\left[\frac{1}{2}-\frac{1}{2(1-c^2)}\right] + \lim_{d\to1^-}\left[\frac{1}{2(1-d^2)}-\frac{1}{2}\right]$$

Since neither of these limits exists, the integral diverges.

12. $$\int_{-\infty}^{\infty}\frac{1}{x^2}\,dx = \lim_{a\to-\infty}\int_a^{-1}\frac{1}{x^2}\,dx + \lim_{b\to0^-}\int_{-1}^b\frac{1}{x^2}\,dx + \lim_{c\to0^+}\int_c^1\frac{1}{x^2}\,dx + \lim_{d\to\infty}\int_1^d\frac{1}{x^2}\,dx$$

Since $\lim_{c\to0^+}\int_c^1\frac{1}{x^2}\,dx = \lim_{c\to0^+}\left\{-\frac{1}{x}\right\}_c^1 = \lim_{c\to0^+}\left(-1+\frac{1}{c}\right)=\infty$, the integral diverges.

13. $$\int_0^{\infty}\frac{1}{\sqrt{x}}\,dx = \lim_{c\to0^+}\int_c^1\frac{1}{\sqrt{x}}\,dx + \lim_{b\to\infty}\int_1^b\frac{1}{\sqrt{x}}\,dx = \lim_{c\to0^+}\left\{2\sqrt{x}\right\}_c^1 + \lim_{b\to\infty}\left\{2\sqrt{x}\right\}_1^b$$
$$= 2\lim_{c\to0^+}(1-\sqrt{c}) + 2\lim_{b\to\infty}(\sqrt{b}-1)$$

Since the second limit does not exist, the integral diverges.

14. $$\int_{-\infty}^{\pi/2}\frac{x}{(x^2-4)^2}\,dx = \lim_{a\to-\infty}\int_a^{-3}\frac{x}{(x^2-4)^2}\,dx + \lim_{b\to-2^-}\int_{-3}^b\frac{x}{(x^2-4)^2}\,dx + \lim_{c\to-2^+}\int_c^{\pi/2}\frac{x}{(x^2-4)^2}\,dx$$

Since $\lim_{b\to-2^-}\int_{-3}^b\frac{x}{(x^2-4)^2}\,dx = \lim_{b\to-2^-}\left\{\frac{-1}{2(x^2-4)}\right\}_{-3}^b = \lim_{b\to-2^-}\left[\frac{-1}{2(b^2-4)}+\frac{1}{10}\right]=-\infty$,
the integral diverges.

15. $\int_4^{\infty}\cos x\,dx = \lim_{b\to\infty}\int_4^b\cos x\,dx = \lim_{b\to\infty}\left\{\sin x\right\}_4^b = \lim_{b\to\infty}(\sin b-\sin 4)$, and this limit does not exist.

16. $$\int_{-\infty}^{\infty}\sin x\,dx = \lim_{a\to-\infty}\int_a^0\sin x\,dx + \lim_{b\to\infty}\int_0^b\sin x\,dx$$

Since $\lim_{b\to\infty}\int_0^b\sin x\,dx = \lim_{b\to\infty}\left\{-\cos x\right\}_0^b = \lim_{b\to\infty}(-\cos b+1)$, and this limit does not exist, neither does the integral.

17. If we set $u = x + 3$ and $du = dx$, then

$$\int \frac{x}{\sqrt{x+3}}\,dx = \int \frac{u-3}{\sqrt{u}}\,du = \frac{2u^{3/2}}{3} - 6\sqrt{u} + C = \frac{2}{3}(x+3)^{3/2} - 6\sqrt{x+3} + C.$$

Thus $\displaystyle\int_0^\infty \frac{x}{\sqrt{x+3}}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{\sqrt{x+3}}\,dx = \lim_{b\to\infty}\left\{\frac{2}{3}(x+3)^{3/2} - 6\sqrt{x+3}\right\}_0^b$

$$= \lim_{b\to\infty}\left[\frac{2}{3}(b+3)^{3/2} - 6\sqrt{b+3} - 2\sqrt{3} + 6\sqrt{3}\right] = \infty.$$

18. If we set $u = x^2 - 4$, then $du = 2x\,dx$, and

$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{(u+4)}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\left\{\frac{2}{3}u^{3/2} + 8\sqrt{u}\right\} + C = \frac{1}{3}(x^2-4)^{3/2} + 4\sqrt{x^2-4} + C.$$

Thus $\displaystyle\int_2^3 \frac{x^3}{\sqrt{x^2-4}}\,dx = \lim_{c\to 2^+}\int_c^3 \frac{x^3}{\sqrt{x^2-4}}\,dx = \lim_{c\to 2^+}\left\{\frac{1}{3}(x^2-4)^{3/2} + 4\sqrt{x^2-4}\right\}_c^3$

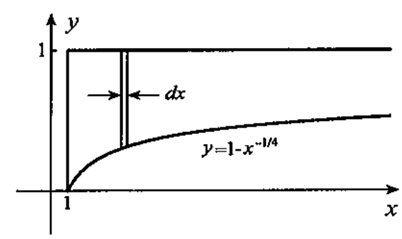
$$= \lim_{c\to 2^+}\left[\frac{5\sqrt{5}}{3} + 4\sqrt{5} - \frac{1}{3}(c^2-4)^{3/2} - 4\sqrt{c^2-4}\right] = \frac{17\sqrt{5}}{3}.$$

19. No. The sine function is odd in Exercise 16 but the integral diverges.

20. (a) If it is possible, the area must be defined by

$$\int_1^\infty \left[1 - (1 - x^{-1/4})\right] dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{1/4}}\,dx$$

$$= \lim_{b\to\infty}\left\{\frac{4}{3}x^{3/4}\right\}_1^b = \lim_{b\to\infty}\left(\frac{4}{3}b^{3/4} - \frac{4}{3}\right) = \infty.$$



Consequently, we cannot assign an area to the region.
(b) If it is possible, the volume must be defined by

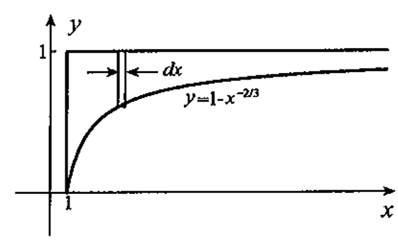
$$\int_1^\infty \pi\left[1 - (1 - x^{-1/4})\right]^2 dx = \lim_{b\to\infty}\pi\int_1^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\pi\left\{2\sqrt{x}\right\}_1^b = \lim_{b\to\infty}\pi(2\sqrt{b} - 2) = \infty.$$

Thus, no volume can be assigned to the solid of revolution.

21. (a) If it is possible, the area must be defined by

$$\int_1^\infty \left[1 - (1 - x^{-2/3})\right] dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{2/3}}\,dx$$

$$= \lim_{b\to\infty}\left\{3x^{1/3}\right\}_1^b = 3\lim_{b\to\infty}(b^{1/3} - 1) = \infty.$$



Consequently, we cannot assign an area to the region.
(b) If it is possible, the volume must be defined by

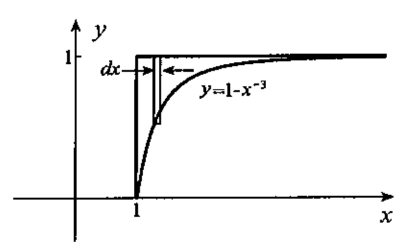
$$\int_1^\infty \pi\left[1 - (1 - x^{-2/3})\right]^2 dx = \pi\lim_{b\to\infty}\int_1^b \frac{1}{x^{4/3}}\,dx = \pi\lim_{b\to\infty}\left\{\frac{-3}{x^{1/3}}\right\}_1^b = 3\pi\lim_{b\to\infty}\left(1 - \frac{1}{b^{1/3}}\right) = 3\pi.$$

Thus, the volume is 3π.

22. (a) If it is possible, the area must be defined by

$$\int_1^\infty \left[1 - (1 - x^{-3})\right] dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty}\left\{\frac{-1}{2x^2}\right\}_1^b = \lim_{b\to\infty}\left(\frac{-1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2}.$$



(b) If it is possible, the volume must be defined by

$$\int_1^\infty \pi\left[1 - (1 - x^{-3})\right]^2 dx = \lim_{b\to\infty}\pi\int_1^b \frac{1}{x^6}\,dx = \lim_{b\to\infty}\pi\left\{\frac{-1}{5x^5}\right\}_1^b = \lim_{b\to\infty}\pi\left(\frac{-1}{5b^5} + \frac{1}{5}\right) = \frac{\pi}{5}.$$

23. (a) Both functions are nonnegative for $x \geq 0$. They are pdf's since

$$\int_0^\infty \frac{6x}{(1+3x^2)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{6x}{(1+3x^2)^2}\,dx = \lim_{b\to\infty}\left\{\frac{-1}{1+3x^2}\right\}_0^b = \lim_{b\to\infty}\left(1-\frac{1}{1+3b^2}\right) = 1,$$

$$\int_0^\infty \frac{2x}{(1+x)^3}\,dx = \lim_{b\to\infty}\int_0^b \frac{2(x+1)-2}{(1+x)^3}\,dx = \lim_{b\to\infty}\int_0^b\left[\frac{2}{(1+x)^2}-\frac{2}{(1+x)^3}\right]dx$$
$$= \lim_{b\to\infty}\left\{\frac{-2}{1+x}+\frac{1}{(1+x)^2}\right\}_0^b = \lim_{b\to\infty}\left[\frac{-2}{1+b}+\frac{1}{(1+b)^2}+2-1\right] = 1.$$

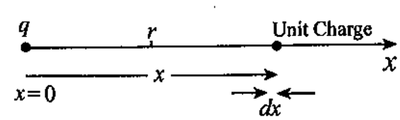
(b) $P(x \geq 3)$ for these pdf's are:

$$P(x\geq 3) = \int_3^\infty \frac{6x}{(1+3x^2)^2}dx = \lim_{b\to\infty}\int_3^b \frac{6x}{(1+3x^2)^2}dx = \lim_{b\to\infty}\left\{\frac{-1}{1+3x^2}\right\}_3^b$$
$$= \lim_{b\to\infty}\left(\frac{1}{28}-\frac{1}{1+3b^2}\right) = \frac{1}{28};$$
$$P(x\geq 3) = \int_3^\infty \frac{2x}{(1+x)^3}dx = \lim_{b\to\infty}\int_3^b \frac{2x}{(1+x)^3}dx = \lim_{b\to\infty}\int_3^b\left[\frac{2}{(1+x)^2}-\frac{2}{(1+x)^3}\right]dx$$
$$= \lim_{b\to\infty}\left\{\frac{-2}{1+x}+\frac{1}{(1+x)^2}\right\}_3^b = \lim_{b\to\infty}\left[\frac{-2}{1+b}+\frac{1}{(1+b)^2}+\frac{2}{4}-\frac{1}{16}\right] = \frac{7}{16}.$$

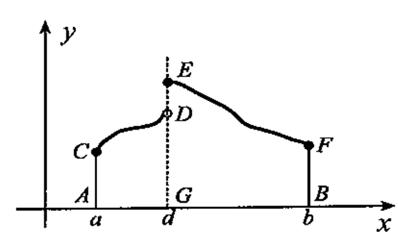
24. $$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left\{\frac{1}{(1-p)x^{p-1}}\right\}_1^b = \lim_{b\to\infty}\left[\frac{1}{(1-p)b^{p-1}}+\frac{1}{p-1}\right] = \begin{cases}\infty, & p<1\\ \dfrac{1}{p-1}, & p>1\end{cases}$$

25. The potential is

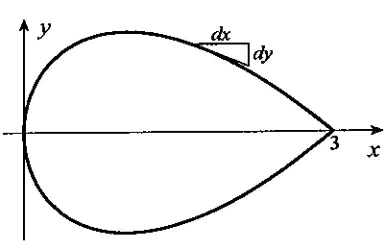
$$V = \int_\infty^r \frac{-q}{4\pi\epsilon_0 x^2}\,dx = \frac{q}{4\pi\epsilon_0}\lim_{b\to\infty}\int_r^b \frac{1}{x^2}\,dx$$
$$= \frac{q}{4\pi\epsilon_0}\lim_{b\to\infty}\left\{-\frac{1}{x}\right\}_r^b = \frac{q}{4\pi\epsilon_0}\lim_{b\to\infty}\left(\frac{1}{r}-\frac{1}{b}\right)$$
$$= \frac{q}{4\pi\epsilon_0 r}.$$



26. $$\int_a^b f(x)\,dx = \lim_{c\to d^-}\int_a^c f(x)\,dx + \lim_{c\to d^+}\int_c^b f(x)\,dx$$
The first limit can be interpreted as area $ACDG$ were the hole to be filled in at D. The second limit is area $GEFB$. Since both areas are clearly defined, the improper integral does indeed exist.



27. If we differentiate $9y^2 = 9x - 6x^2 + x^3$ with respect to x,
$$18y\frac{dy}{dx} = 9-12x+3x^2 \Longrightarrow \frac{dy}{dx} = \frac{3-4x+x^2}{6y}.$$
Small lengths along the curve in the first quadrant corresponding to lengths dx along the x-axis are given by
$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{3-4x+x^2}{6y}\right)^2}\,dx$$



$$= \sqrt{1 + \frac{(3-x)^2(1-x)^2}{4x(3-x)^2}}dx = \sqrt{\frac{4x+(1-x)^2}{4x}}dx = \sqrt{\frac{(1+x)^2}{4x}}dx = \frac{1+x}{2\sqrt{x}}dx.$$

The length of the loop is

$$2\int_0^3 \frac{1+x}{2\sqrt{x}}\,dx = \lim_{c\to 0^+}\int_c^3 \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right)dx = \lim_{c\to 0^+}\left\{2\sqrt{x} + \frac{2x^{3/2}}{3}\right\}_c^3$$
$$= \lim_{c\to 0^+}\left(2\sqrt{3} + 2\sqrt{3} - 2\sqrt{c} - \frac{2c^{3/2}}{3}\right) = 4\sqrt{3}.$$

28. No. For example, according to definition 7.49 the integral in Exercise 16 does not exist. But were we to use this definition:

$$\int_{-\infty}^{\infty} \sin x\,dx = \lim_{a\to\infty}\int_{-a}^{a} \sin x\,dx = \lim_{a\to\infty}\{-\cos x\}_{-a}^{a} = \lim_{a\to\infty}[-\cos a + \cos(-a)] = \lim_{a\to\infty}(0) = 0,$$

and the improper integral would exist.

29. Since $\dfrac{x^2}{\sqrt{x^2-1}} \geq \dfrac{x}{\sqrt{x^2-1}}$ for $x \geq 2$, we can say that

$$\int_2^{\infty} \frac{x^2}{\sqrt{x^2-1}}\,dx \geq \int_2^{\infty} \frac{x}{\sqrt{x^2-1}}\,dx = \lim_{b\to\infty}\int_2^b \frac{x}{\sqrt{x^2-1}}\,dx$$
$$= \lim_{b\to\infty}\left\{\sqrt{x^2-1}\right\}_2^b = \lim_{b\to\infty}(\sqrt{b^2-1} - \sqrt{3}) = \infty.$$

30. Since $\dfrac{x^3}{(27-x^3)^2} \geq \dfrac{x^2}{(27-x^3)^2} > 0$ for $1 \leq x \leq 3$, we can say that

$$\int_1^3 \frac{x^3}{(27-x^3)^2}\,dx \geq \int_1^3 \frac{x^2}{(27-x^3)^2}\,dx = \lim_{c\to 3^-}\int_1^c \frac{x^2}{(27-x^3)^2}\,dx$$
$$= \lim_{c\to 3^-}\left\{\frac{1}{3(27-x^3)}\right\}_1^c = \lim_{c\to 3^-}\left[\frac{1}{3(27-c^3)} - \frac{1}{78}\right] = \infty.$$

31. Since $\dfrac{x^2}{\sqrt{1-x^2}} \leq \dfrac{x}{\sqrt{1-x^2}}$ for $0 \leq x \leq 1$, we can say that

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx \leq \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{c\to 1^-}\int_0^c \frac{x}{\sqrt{1-x^2}}\,dx$$
$$= \lim_{c\to 1^-}\left\{-\sqrt{1-x^2}\right\}_0^c = \lim_{c\to 1^-}\left(1 - \sqrt{1-c^2}\right) = 1.$$

The given intergal therefore converges.

32. Since $\dfrac{\sqrt{-x}}{(x^2+5)^2} < \dfrac{-x}{(x^2+5)^2}$ for $x \leq -2$, we can say that

$$\int_{-\infty}^{-2} \frac{\sqrt{-x}}{(x^2+5)^2}\,dx < \int_{-\infty}^{-2} \frac{-x}{(x^2+5)^2}\,dx = \lim_{a\to -\infty}\int_a^{-2} \frac{-x}{(x^2+5)^2}\,dx$$
$$= \lim_{a\to -\infty}\left\{\frac{1}{2(x^2+5)}\right\}_a^{-2} = \lim_{a\to -\infty}\left[\frac{1}{18} - \frac{1}{2(a^2+5)}\right] = \frac{1}{18}.$$

The given integral therefore converges.

33. By changing the value of c, to d say, where $d > c$,we are simply moving the integral of a continuous function $f(x)$ from the second integral on the right of 7.49 to the first integral on the right. This does not affect whether either integral converges or diverges.

34. $\displaystyle\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \int_{-\infty}^{\infty} f(x)\left[\lim_{\epsilon\to 0} P_\epsilon(x-a)\right]dx = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(x)\,P_\epsilon(x-a)\,dx$

$$= \lim_{\epsilon\to 0}\int_a^{a+\epsilon} f(x)\frac{1}{\epsilon}dx = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_a^{a+\epsilon} f(x)\,dx.$$

If we let $F(x)$ be an antiderivative of $f(x)$, then

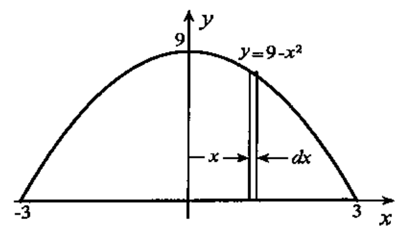
$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[F(a+\epsilon)-F(a)] = F'(a) = f(a).$$

REVIEW EXERICSES

1. (a) $A = 2\int_0^3 (9-x^2)\,dx = 2\left\{9x - \frac{x^3}{3}\right\}_0^3 = 36$

(b) $V_x = 2\int_0^3 \pi(9-x^2)^2\,dx = 2\pi\int_0^3 (81-18x^2+x^4)\,dx$

$$= 2\pi\left\{81x - 6x^3 + \frac{x^5}{5}\right\}_0^3 = \frac{1296\pi}{5}$$



$$V_y = \int_0^3 2\pi x(9-x^2)\,dx = 2\pi\left\{\frac{9x^2}{2} - \frac{x^4}{4}\right\}_0^3 = \frac{81\pi}{2}$$

(c) By symmetry, $\overline{x} = 0$. Since

$$A\overline{y} = 2\int_0^3 \frac{1}{2}(9-x^2)^2\,dx = \int_0^3 (81-18x^2+x^4)\,dx = \left\{81x - 6x^3 + \frac{x^5}{5}\right\}_0^3 = \frac{648}{5},$$

it follows that $\overline{y} = \frac{648}{5}\frac{1}{36} = \frac{18}{5}$.

(d) $I_x = 2\int_0^3 \frac{1}{3}(9-x^2)^3\,dx = \frac{2}{3}\int_0^3 (729 - 243x^2 + 27x^4 - x^6)\,dx$

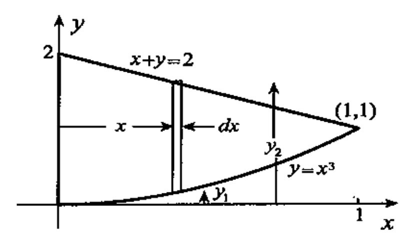
$$= \frac{2}{3}\left\{729x - 81x^3 + \frac{27x^5}{5} - \frac{x^7}{7}\right\}_0^3 = \frac{23\,328}{35}$$

$$I_y = 2\int_0^3 x^2(9-x^2)\,dx = 2\left\{3x^3 - \frac{x^5}{5}\right\}_0^3 = \frac{324}{5}$$

2. (a) $A = \int_0^1 (2-x-x^3)\,dx = \left\{2x - \frac{x^2}{2} - \frac{x^4}{4}\right\}_0^1 = \frac{5}{4}$

(b) $V_x = \int_0^1 \pi(y_2^2 - y_1^2)\,dx = \pi\int_0^1 [(2-x)^2 - (x^3)^2]\,dx$

$$= \pi\left\{-\frac{1}{3}(2-x)^3 - \frac{x^7}{7}\right\}_0^1 = \frac{46\pi}{21}$$



$$V_y = \int_0^1 2\pi x(y_2-y_1)\,dx = 2\pi\int_0^1 x(2-x-x^3)\,dx = 2\pi\left\{x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right\}_0^1 = \frac{14\pi}{15}$$

(c) Since $A\overline{x} = \int_0^1 x(y_2-y_1)\,dx = \frac{1}{2\pi}V_y = \frac{7}{15}$, we find $\overline{x} = \frac{7}{15}\frac{4}{5} = \frac{28}{75}$. Since

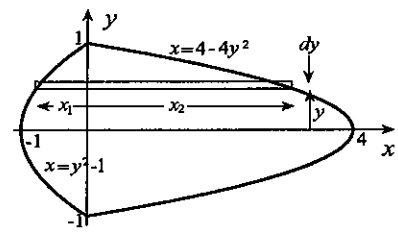
$A\overline{y} = \int_0^1 \frac{1}{2}(y_2+y_1)(y_2-y_1)\,dx = \frac{1}{2}\int_0^1 (y_2^2 - y_1^2)\,dx = \frac{1}{2\pi}V_x = \frac{23}{21}$, it follows that $\overline{y} = \frac{23}{21}\frac{4}{5} = \frac{92}{105}$.

(d) $I_x = \int_0^1 \frac{1}{3}(y_2^3 - y_1^3)\,dx = \frac{1}{3}\int_0^1 [(2-x)^3 - (x^3)^3]\,dx = \frac{1}{3}\left\{-\frac{1}{4}(2-x)^4 - \frac{x^{10}}{10}\right\}_0^1 = \frac{73}{60}$

$$I_y = \int_0^1 x^2(2-x-x^3)\,dx = \left\{\frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^6}{6}\right\}_0^1 = \frac{1}{4}$$

3. (a) $A = 2\int_0^1 [(4-4y^2)-(y^2-1)]\,dy$

$= 2\left\{5y - \frac{5y^3}{3}\right\}_0^1 = \frac{20}{3}$



(b) $V_x = \int_0^1 2\pi y(x_2-x_1)\,dy = 2\pi\int_0^1 y(5-5y^2)\,dy$

$= 2\pi\left\{\frac{5y^2}{2} - \frac{5y^4}{4}\right\}_0^1 = \frac{5\pi}{2}$

$V_y = 2\int_0^1 \pi(x_2)^2\,dy = 2\pi\int_0^1 (4-4y^2)^2\,dy = 32\pi\int_0^1 (1-2y^2+y^4)\,dy = 32\pi\left\{y - \frac{2y^3}{3} + \frac{y^5}{5}\right\}_0^1 = \frac{256\pi}{15}$

(c) By symmetry, $\overline{y} = 0$. Since

$$A\overline{x} = 2\int_0^1 \frac{1}{2}(x_1+x_2)(x_2-x_1)\,dy = \int_0^1 [(4-4y^2)^2-(y^2-1)^2]\,dy = 15\int_0^1 (1-2y^2+y^4)\,dy$$
$$= 15\left\{y - \frac{2y^3}{3} + \frac{y^5}{5}\right\}_0^1 = 8,$$

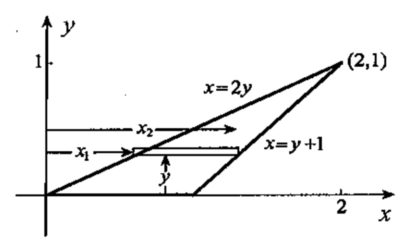
it follows that $\overline{x} = 8(3/20) = 6/5$.

(d) $I_x = 2\int_0^1 y^2(x_2-x_1)\,dy = 2\int_0^1 y^2(5-5y^2)\,dy = 2\left\{\frac{5y^3}{3} - y^5\right\}_0^1 = \frac{4}{3}$

$I_y = 2\int_0^1 \frac{1}{3}(x_2^3-x_1^3)\,dy = \frac{2}{3}\int_0^1 [(4-4y^2)^3-(y^2-1)^3]\,dy = \frac{130}{3}\int_0^1 (1-3y^2+3y^4-y^6)\,dy$

$= \frac{130}{3}\left\{y - y^3 + \frac{3y^5}{5} - \frac{y^7}{7}\right\}_0^1 = \frac{416}{21}$

4. (a) $A = \int_0^1 (y+1-2y)\,dy = \left\{y - \frac{y^2}{2}\right\}_0^1 = \frac{1}{2}$



(b) $V_x = \int_0^1 2\pi y(x_2-x_1)\,dy = 2\pi\int_0^1 y(1-y)\,dy$

$= 2\pi\left\{\frac{y^2}{2} - \frac{y^3}{3}\right\}_0^1 = \frac{\pi}{3}$

$V_y = \int_0^1 \pi(x_2^2-x_1^2)\,dy$

$= \pi\int_0^1 [(y+1)^2-(2y)^2]\,dy = \pi\left\{\frac{1}{3}(y+1)^3 - \frac{4y^3}{3}\right\}_0^1 = \pi$

(c) Since $A\overline{x} = \int_0^1 \frac{1}{2}(x_2+x_1)(x_2-x_1)\,dy = \frac{1}{2}\int_0^1 (x_2^2-x_1^2)\,dy = \frac{1}{2\pi}V_y = \frac{1}{2}$, we find $\overline{x} = \frac{1}{2}(2) = 1$.

Since $A\overline{y} = \int_0^1 y(x_2-x_1)\,dy = \frac{1}{2\pi}V_x = \frac{1}{6}$, it follows that $\overline{y} = \frac{1}{6}(2) = \frac{1}{3}$.

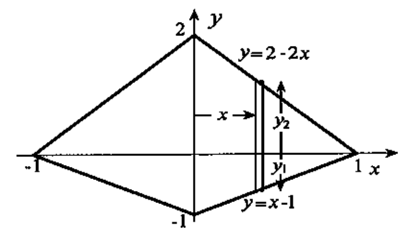
(d) $I_x = \int_0^1 y^2(1-y)\,dy = \left\{\frac{y^3}{3} - \frac{y^4}{4}\right\}_0^1 = \frac{1}{12}$

$I_y = \int_0^1 \frac{1}{3}(x_2^3-x_1^3)\,dy = \frac{1}{3}\int_0^1 [(y+1)^3-(2y)^3]\,dy = \frac{1}{3}\left\{\frac{1}{4}(y+1)^4 - 2y^4\right\}_0^1 = \frac{7}{12}$

5. (a) Area $= 2 + 1 = 3$

(b) $V_x = (2/3)\pi(2)^2(1) = 8\pi/3, \qquad V_y = (1/3)\pi(1)^2(2) + (1/3)\pi(1)^2(1) = \pi$

(c) By symmetry, $\overline{x} = 0$. Since



$$A\overline{y} = 2\int_0^1 \frac{1}{2}(y_2 + y_1)(y_2 - y_1)\,dx$$

$$= \int_0^1 [(2-2x)^2 - (x-1)^2]\,dx$$

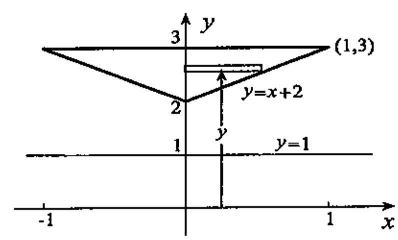
$$= \left\{\frac{(2-2x)^3}{-6} - \frac{(x-1)^3}{3}\right\}_0^1 = 1,$$

it follows that $\overline{y} = 1/3$.

(d) $I_x = 2\displaystyle\int_0^1 \frac{1}{3}(y_2^3 - y_1^3)\,dx = \frac{2}{3}\int_0^1 [(2-2x)^3 - (x-1)^3]\,dx = \frac{2}{3}\left\{\frac{(2-2x)^4}{-8} - \frac{(x-1)^4}{4}\right\}_0^1 = \frac{3}{2}$

$$I_y = 2\int_0^1 x^2(y_2 - y_1)\,dx = 2\int_0^1 x^2(3-3x)\,dx = 2\left\{x^3 - \frac{3x^4}{4}\right\}_0^1 = \frac{1}{2}$$

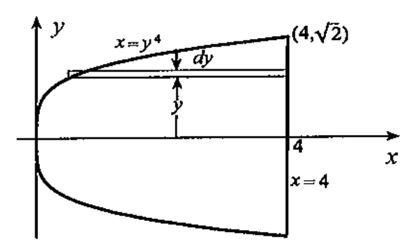
6. $V = 2\displaystyle\int_2^3 2\pi(y-1)(y-2)\,dy$



$$= 4\pi\int_2^3 (y^2 - 3y + 2)\,dy$$

$$= 4\pi\left\{\frac{y^3}{3} - \frac{3y^2}{2} + 2y\right\}_2^3 = \frac{10\pi}{3}$$

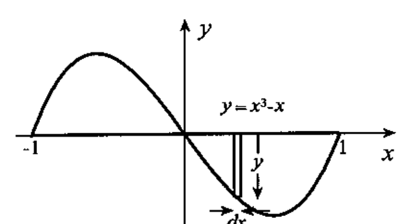
7. $V = 2\displaystyle\int_0^{\sqrt{2}} \pi(4 - y^4)^2\,dy$



$$= 2\pi\int_0^{\sqrt{2}} (16 - 8y^4 + y^8)\,dy$$

$$= 2\pi\left\{16y - \frac{8y^5}{5} + \frac{y^9}{9}\right\}_0^{\sqrt{2}} = \frac{1024\sqrt{2}\pi}{45}$$

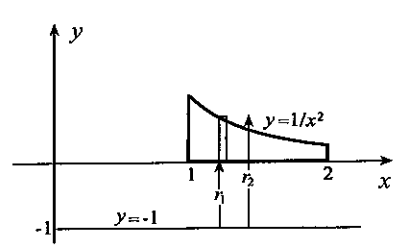
8. $V = 2\displaystyle\int_0^1 \pi(-y)^2\,dx = 2\pi\int_0^1 (x^3 - x)^2\,dx$



$$= 2\pi\int_0^1 (x^6 - 2x^4 + x^2)\,dx$$

$$= 2\pi\left\{\frac{x^7}{7} - \frac{2x^5}{5} + \frac{x^3}{3}\right\}_0^1 = \frac{16\pi}{105}$$

9. $V = \displaystyle\int_1^2 (\pi r_2^2 - \pi r_1^2)\,dx = \pi\int_1^2 [(1/x^2 + 1)^2 - (1)^2]\,dx$

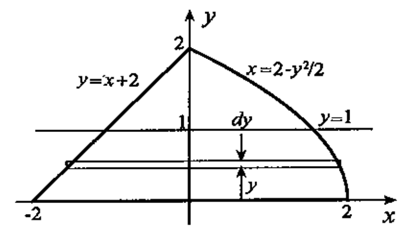


$$= \pi\int_1^2 \left(\frac{1}{x^4} + \frac{2}{x^2}\right)dx$$

$$= \pi\left\{-\frac{1}{3x^3} - \frac{2}{x}\right\}_1^2 = \frac{31\pi}{24}$$

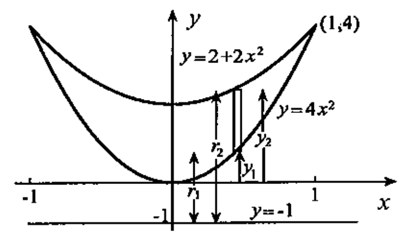
10. We reject the area above $y = 1$.

$$V = \int_0^1 2\pi(1-y)\left(2 - \frac{y^2}{2} - y + 2\right) dy$$
$$= \pi \int_0^1 (y^3 + y^2 - 10y + 8)\, dy$$
$$= \pi \left\{ \frac{y^4}{4} + \frac{y^3}{3} - 5y^2 + 8y \right\}_0^1 = \frac{43\pi}{12}$$



11. The first moment is

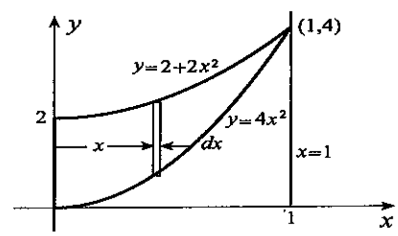
$$2\int_0^1 \left[\frac{y_1 + y_2}{2} + 1\right](y_2 - y_1)\, dx$$
$$= \int_0^1 [(y_2^2 - y_1^2 + 2(y_2 - y_1)]\, dx$$
$$= \int_0^1 [(2+2x^2)^2 - (4x^2)^2 + 2(2 - 2x^2)]\, dx$$
$$= 4\int_0^1 (2 + x^2 - 3x^4)\, dx = 4\left\{2x + \frac{x^3}{3} - \frac{3x^5}{5}\right\}_0^1 = \frac{104}{15}$$



$$I = 2\int_0^1 \frac{1}{3}(r_2^3 - r_1^3)\, dx = \frac{2}{3}\int_0^1 [(2+2x^2+1)^3 - (4x^2+1)^3]\, dx = \frac{4}{3}\int_0^1 (13 + 21x^2 - 6x^4 - 28x^6)\, dx$$
$$= \frac{4}{3}\left\{13x + 7x^3 - \frac{6x^5}{5} - 4x^7\right\}_0^1 = \frac{296}{15}$$

12. The first moment is

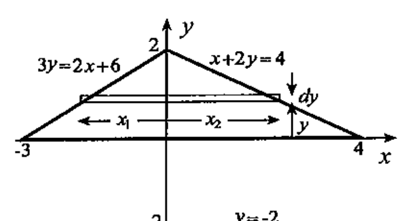
$$\int_0^1 (x-1)(2 + 2x^2 - 4x^2)\, dx$$
$$= 2\int_0^1 (-1 + x + x^2 - x^3)\, dx$$
$$= 2\left\{-x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right\}_0^1 = -\frac{5}{6}$$



$$I = \int_0^1 (x-1)^2(2 + 2x^2 - 4x^2)\, dx = 2\int_0^1 (1 - 2x + 2x^3 - x^4)\, dx = 2\left\{x - x^2 + \frac{x^4}{2} - \frac{x^5}{5}\right\}_0^1 = \frac{3}{5}$$

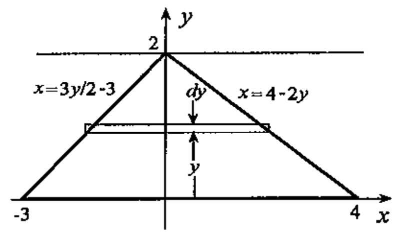
13. The first moment is

$$\int_0^2 (y+2)(x_2 - x_1)\, dy$$
$$= \int_0^2 (y+2)[(4 - 2y) - (3y/2 - 3)]\, dy$$
$$= \frac{7}{2}\int_0^2 (4 - y^2)\, dy = \frac{7}{2}\left\{4y - \frac{y^3}{3}\right\}_0^2 = \frac{56}{3}$$



$$I = \int_0^2 (y+2)^2(x_2 - x_1)\, dy = \int_0^2 (y+2)^2[(4 - 2y) - (3y/2 - 3)]\, dy$$
$$= \frac{7}{2}\int_0^2 (8 + 4y - 2y^2 - y^3)\, dy = \frac{7}{2}\left\{8y + 2y^2 - \frac{2y^3}{3} - \frac{y^4}{4}\right\}_0^2 = \frac{154}{3}$$

14. The first moment is

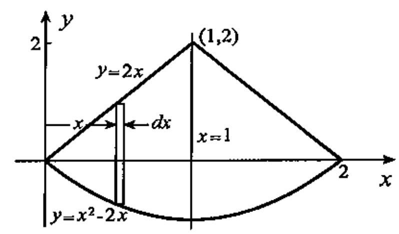


$$\int_0^2 (y-2)\left(4-2y-\frac{3y}{2}+3\right)dy = -\frac{7}{2}\int_0^2 (y-2)^2\,dy$$

$$= -\frac{7}{2}\left\{\frac{1}{3}(y-2)^3\right\}_0^2 = -\frac{28}{3}$$

$$I = \int_0^2 (y-2)^2\left(4-2y-\frac{3y}{2}+3\right)dy = -\frac{7}{2}\int_0^2 (y-2)^3\,dy = -\frac{7}{2}\left\{\frac{1}{4}(y-2)^4\right\}_0^2 = 14$$

15. Because the area is symmetric about $x = 1$, its first moment about this line is zero. Its moment of inertia is

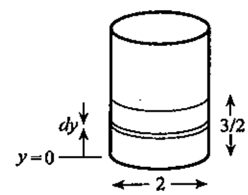


$$I = 2\int_0^1 (x-1)^2(2x-x^2+2x)\,dx$$

$$= 2\int_0^1 (4x-9x^2+6x^3-x^4)\,dx$$

$$= 2\left\{2x^2-3x^3+\frac{3x^4}{2}-\frac{x^5}{5}\right\}_0^1 = \frac{3}{5}$$

16. $W = \int_0^{3/2} (3-y)1000(9.81)\pi(1)^2\,dy$



$$= 9810\pi\left\{3y-\frac{y^2}{2}\right\}_0^{3/2} = 1.04\times 10^5 \text{ J.}$$

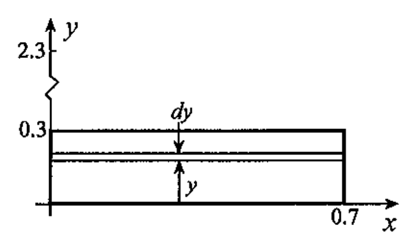
17. The force on the rear window (left figure below) is

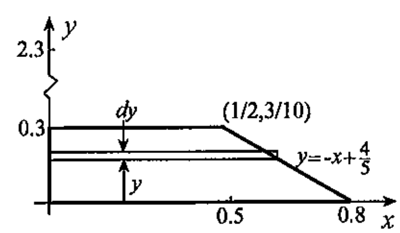
$$F = \int_0^{3/10} 9.81(1000)(23/10-y)(7/10)\,dy = 6867\left\{-\frac{1}{2}\left(\frac{23}{10}-y\right)^2\right\}_0^{3/10} = 4.43\times 10^3 \text{ N.}$$

The force on the front window (right figure below) is

$$F = \int_0^{3/10} 9.81(1000)(23/10-y)(4/5-y)\,dy = \frac{981}{5}\int_0^{3/10}(92-155y+50y^2)\,dy$$

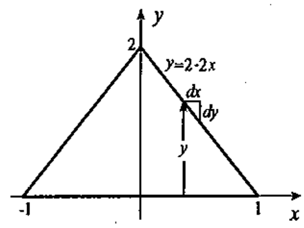
$$= \frac{981}{5}\left\{92y-\frac{155y^2}{2}+\frac{50y^3}{3}\right\}_0^{3/10} = 4.13\times 10^3 \text{ N.}$$





18. The surface area is twice the area of the surface of revolution obtained by rotating that part of $y = 2 - 2x$ in the first quadrant about the x-axis. Small lengths along the line are given by

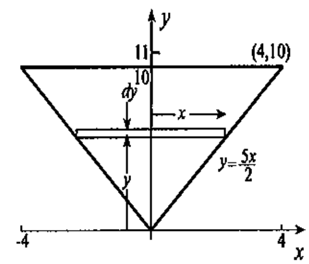
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + (-2)^2}\,dx = \sqrt{5}\,dx.$$



The required area is therefore

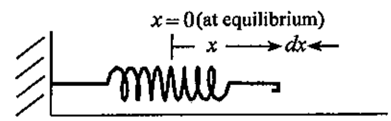
$$A = 2\int_0^1 2\pi y\sqrt{5}\,dx = 4\sqrt{5}\pi\int_0^1 (2-2x)\,dx = 8\sqrt{5}\pi\left\{x - \frac{x^2}{2}\right\}_0^1 = 4\sqrt{5}\pi.$$

19.
$$\begin{aligned} W &= \int_8^{10} (11-y)(9.81)(1000)(\pi x^2)\,dy \\ &= 9810\pi\int_8^{10}(11-y)\left(\frac{4y^2}{25}\right)dy \\ &= 1569.6\pi\left\{\frac{11y^3}{3} - \frac{y^4}{4}\right\}_8^{10} = 1.55\times 10^6 \text{ J} \end{aligned}$$



20. Suppose x_0 is the stretch that requires W units of work. When the spring is stretched x, the force required to maintain this stretch is kx. The work to stretch the spring x_0 is

$$W = \int_0^{x_0} kx\,dx = k\left\{\frac{x^2}{2}\right\}_0^{x_0} = \frac{1}{2}kx_0^2.$$



The work required to obtain a stretch of $2x_0$ is $\displaystyle\int_0^{2x_0} kx\,dx = k\left\{\frac{x^2}{2}\right\}_0^{2x_0} = 2kx_0^2 = 4W$. Thus, it requires $3W$ more unit of work to increase the stretch from x_0 to $2x_0$.

21. $$\int_1^\infty \frac{1}{x^{3/2}}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{3/2}}dx = \lim_{b\to\infty}\left\{\frac{-2}{x^{1/2}}\right\}_1^b = \lim_{b\to\infty}\left(2 - \frac{2}{\sqrt{b}}\right) = 2$$

22. $$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{c\to 3^-}\int_0^c \frac{1}{\sqrt{3-x}}\,dx = \lim_{c\to 3^-}\left\{-2\sqrt{3-x}\right\}_0^c = \lim_{c\to 3^-}(-2\sqrt{3-c} + 2\sqrt{3}) = 2\sqrt{3}$$

23. $$\int_{-1}^0 \frac{1}{(x+1)^2}dx = \lim_{c\to -1^+}\int_c^0 \frac{1}{(x+1)^2}dx = \lim_{c\to -1^+}\left\{\frac{-1}{x+1}\right\}_c^0 = \lim_{c\to -1^+}\left(-1 + \frac{1}{c+1}\right) = \infty$$

24.
$$\begin{aligned} \int_{-2}^2 \frac{x}{\sqrt{4-x^2}}\,dx &= \lim_{c\to -2^+}\int_c^0 \frac{x}{\sqrt{4-x^2}}\,dx + \lim_{d\to 2^-}\int_0^d \frac{x}{\sqrt{4-x^2}}\,dx \\ &= \lim_{c\to -2^+}\left\{-\sqrt{4-x^2}\right\}_c^0 + \lim_{d\to 2^-}\left\{-\sqrt{4-x^2}\right\}_0^d \\ &= \lim_{c\to -2^+}(\sqrt{4-c^2} - 2) + \lim_{d\to 2^-}(2 - \sqrt{4-d^2}) = -2 + 2 = 0 \end{aligned}$$

25.
$$\begin{aligned} \int_{-\infty}^\infty \frac{1}{(x+3)^3}dx = {} & \lim_{a\to -\infty}\int_a^{-4}\frac{1}{(x+3)^3}dx + \lim_{c\to -3^-}\int_{-4}^c \frac{1}{(x+3)^3}dx + \lim_{d\to -3^+}\int_d^0 \frac{1}{(x+3)^3}dx \\ & + \lim_{b\to\infty}\int_0^b \frac{1}{(x+3)^3}dx \end{aligned}$$

Since $\lim\limits_{d\to-3^+}\int_d^0 \frac{1}{(x+3)^3}dx = \lim\limits_{d\to-3^+}\left\{\frac{-1}{2(x+3)^2}\right\}_d^0 = \lim\limits_{d\to-3^+}\left[\frac{1}{2(d+3)^2}-\frac{1}{18}\right] = \infty$, the original improper integral diverges.

26. $\int_{-\infty}^{-3}\frac{1}{\sqrt{-x}}\,dx = \lim\limits_{a\to-\infty}\int_a^{-3}\frac{1}{\sqrt{-x}}\,dx = \lim\limits_{a\to-\infty}\left\{-2\sqrt{-x}\right\}_a^{-3} = \lim\limits_{a\to-\infty}(2\sqrt{-a}-2\sqrt{3}) = \infty$

27. $\int_{-6}^{\infty}x\sqrt{x^2+4}\,dx = \lim\limits_{b\to\infty}\int_{-6}^{b}x\sqrt{x^2+4}\,dx = \lim\limits_{b\to\infty}\left\{\frac{1}{3}(x^2+4)^{3/2}\right\}_{-6}^{b} = \frac{1}{3}\lim\limits_{b\to\infty}[(b^2+4)^{3/2}-40^{3/2}] = \infty$

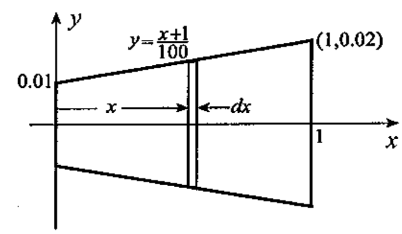
28. $\int_{-\infty}^{\infty}\frac{x}{(x^2-1)^2}\,dx = \lim\limits_{a\to-\infty}\int_a^{-2}\frac{x}{(x^2-1)^2}\,dx + \lim\limits_{b\to-1^-}\int_{-2}^{b}\frac{x}{(x^2-1)^2}\,dx + \lim\limits_{c\to-1^+}\int_c^0\frac{x}{(x^2-1)^2}\,dx$

$$+\lim_{d\to1^-}\int_0^d\frac{x}{(x^2-1)^2}\,dx + \lim_{e\to1^+}\int_e^2\frac{x}{(x^2-1)^2}\,dx + \lim_{f\to\infty}\int_2^f\frac{x}{(x^2-1)^2}\,dx.$$

Since $\lim\limits_{e\to1^+}\int_e^2\frac{x}{(x^2-1)^2}\,dx = \lim\limits_{e\to1^+}\left\{\frac{-1}{2(x^2-1)}\right\}_e^2 = \lim\limits_{e\to1^+}\left[\frac{1}{2(e^2-1)}-\frac{1}{6}\right] = \infty$, we conclude that the given improper integral diverges.

29. (a) Cross sections are circles with radius y where $y=(x+1)/100$. With the volume of a slab of width dx as $\pi y^2\,dx$, we obtain

$$V = \int_0^1 \pi\left[\frac{1}{100}(x+1)\right]^2 dx = \frac{\pi}{10\,000}\left\{\frac{(x+1)^3}{3}\right\}_0^1 = \frac{7\pi}{30\,000}\ \text{m}^3.$$



(b) Using the disc method, the volume integral is identical.

30. The force of gravity on the slab shown is

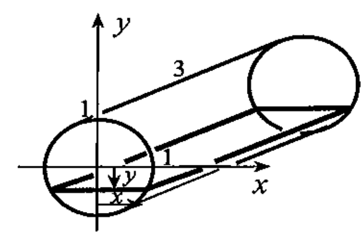
$$-9.81(1000)(2x)(3)\,dy\ \text{N}.$$

Since the work to move this slab from its present position to the top of the tank is

$$(1-y)(9810)(6x)\,dy\ \text{J},$$

the total work to empty the tank is

$$W = \int_{-1}^{0}6(9810)(1-y)\sqrt{1-y^2}\,dy = 58\,860\int_{-1}^{0}\sqrt{1-y^2}\,dy - 58\,860\int_{-1}^{0}y\sqrt{1-y^2}\,dy.$$



Since the first integral represents one-quarter of the area of the circle on the end of the tank,

$$W = 58\,860\left(\frac{1}{4}\right)\pi(1)^2 - 58\,860\left\{-\frac{1}{3}(1-y^2)^{3/2}\right\}_{-1}^{0} = 14\,715\pi + 19\,620 = 6.58\times10^4\ \text{J}.$$

CHAPTER 8

EXERCISES 8.1

1. $\displaystyle\int \frac{x^2}{5-3x^3}\,dx = -\frac{1}{9}\ln|5-3x^3| + C$

2. $\displaystyle\int xe^{-2x^2}\,dx = -\frac{1}{4}e^{-2x^2} + C$

3. $\displaystyle\int \frac{x}{(x^2+2)^{1/3}}\,dx = \frac{3}{4}(x^2+2)^{2/3} + C$

4. $\displaystyle\int \frac{e^x}{1+e^x}\,dx = \ln(1+e^x) + C$

5. $\displaystyle\int \frac{4t+8}{t^2+4t+5}\,dt = 2\ln|t^2+4t+5| + C$

6. $\displaystyle\int x^2\sqrt{1-3x^3}\,dx = -\frac{2}{27}(1-3x^3)^{3/2} + C$

7. $\displaystyle\int (x+1)(x^2+2x)^{1/3}\,dx = \frac{3}{8}(x^2+2x)^{4/3} + C$

8. $\displaystyle\int \frac{x^2}{(1+x^3)^3}\,dx = \frac{-1}{6(1+x^3)^2} + C$

9. If we set $u = 1-\sqrt{x}$, then $du = \dfrac{-1}{2\sqrt{x}}\,dx$, and

$$\begin{aligned}\int \frac{\sqrt{x}}{1-\sqrt{x}}\,dx &= \int \frac{1-u}{u}[-2(1-u)\,du] = 2\int\left(-\frac{1}{u}+2-u\right)du = 2\left(-\ln|u|+2u-\frac{u^2}{2}\right)+C\\ &= -2\ln|1-\sqrt{x}| + 4(1-\sqrt{x}) - (1-\sqrt{x})^2 + C = -2\ln|1-\sqrt{x}| - 2\sqrt{x} - x + D.\end{aligned}$$

10. $\displaystyle\int \frac{1-\sqrt{x}}{\sqrt{x}}\,dx = \int (x^{-1/2}-1)\,dx = 2\sqrt{x} - x + C$

11. $\displaystyle\int \frac{x+2}{x+1}\,dx = \int\left(1+\frac{1}{x+1}\right)dx = x + \ln|x+1| + C$

12. $\displaystyle\int \frac{x^2+2}{x^2+1}\,dx = \int\left(1+\frac{1}{x^2+1}\right)dx = x + \mathrm{Tan}^{-1}x + C$

13. If we set $u = \cos\theta - 1$, then $du = -\sin\theta\,d\theta$, and

$$\int \frac{\sin\theta}{\cos\theta-1}\,d\theta = \int \frac{1}{u}(-du) = -\ln|u| + C = -\ln|\cos\theta-1| + C = -\ln(1-\cos\theta) + C.$$

14. If we set $u = 2x+4$, then $du = 2\,dx$, and

$$\begin{aligned}\int \frac{x+3}{\sqrt{2x+4}}\,dx &= \int \frac{(u-4)/2+3}{\sqrt{u}}\frac{du}{2} = \frac{1}{4}\int\left(\sqrt{u}+\frac{2}{\sqrt{u}}\right)du\\ &= \frac{1}{4}\left(\frac{2}{3}u^{3/2}+4\sqrt{u}\right)+C = \frac{1}{6}(2x+4)^{3/2} + \sqrt{2x+4} + C.\end{aligned}$$

15. If we set $u = e^x$, then $du = e^x\,dx$, and $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \mathrm{Tan}^{-1}u + C = \mathrm{Tan}^{-1}(e^x) + C.$

16. $\displaystyle\int \sin^3 2x\cos 2x\,dx = \frac{1}{8}\sin^4 2x + C$

17. If we set $u = 2x^2-5$, then $du = 4x\,dx$, and

$$\begin{aligned}\int x^5(2x^2-5)^4\,dx &= \int (x^2)^2(2x^2-5)^4\,x\,dx = \int\left(\frac{u+5}{2}\right)^2 u^4\left(\frac{du}{4}\right) = \frac{1}{16}\int(u^6+10u^5+25u^4)\,du\\ &= \frac{1}{16}\left(\frac{u^7}{7}+\frac{5u^6}{3}+5u^5\right)+C = \frac{1}{112}(2x^2-5)^7 + \frac{5}{48}(2x^2-5)^6 + \frac{5}{16}(2x^2-5)^5 + C.\end{aligned}$$

18. If we set $u = x+5$, then $du = dx$, and

$$\begin{aligned}\int \frac{x^3}{(x+5)^2}\,dx &= \int \frac{(u-5)^3}{u^2}\,du = \int\left(u-15+\frac{75}{u}-\frac{125}{u^2}\right)du = \frac{u^2}{2} - 15u + 75\ln|u| + \frac{125}{u} + C\\ &= \frac{1}{2}(x+5)^2 - 15(x+5) + 75\ln|x+5| + \frac{125}{x+5} + C\\ &= \frac{x^2}{2} - 10x + 75\ln|x+5| + \frac{125}{x+5} + D.\end{aligned}$$

19. If we set $u = 3 - z$, then $du = -dz$, and

$$\int z^2\sqrt{3-z}\,dz = \int (3-u)^2\sqrt{u}(-du) = \int (-9\sqrt{u} + 6u^{3/2} - u^{5/2})\,du$$
$$= -6u^{3/2} + \frac{12u^{5/2}}{5} - \frac{2u^{7/2}}{7} + C = -6(3-z)^{3/2} + \frac{12}{5}(3-z)^{5/2} - \frac{2}{7}(3-z)^{7/2} + C.$$

20. If we set $u = \cos 3x$, then $du = -3\sin 3x\,dx$, and

$$\int \tan 3x\,dx = \int \frac{\sin 3x}{\cos 3x}dx = \int \frac{1}{u}\left(-\frac{du}{3}\right) = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|\cos 3x| + C = \frac{1}{3}\ln|\sec 3x| + C.$$

21. If we set $u = (x-3)^{1/3}$, then $x = 3 + u^3$, from which $dx = 3u^2\,du$, and

$$\int \frac{(x-3)^{2/3}}{(x-3)^{2/3}+1}dx = \int \frac{u^2}{u^2+1}(3u^2\,du) = 3\int \frac{u^4}{u^2+1}du = 3\int\left(u^2 - 1 + \frac{1}{u^2+1}\right)du$$
$$= 3\left(\frac{u^3}{3} - u + \mathrm{Tan}^{-1}u\right) + C = (x-3) - 3(x-3)^{1/3} + 3\,\mathrm{Tan}^{-1}(x-3)^{1/3} + C$$
$$= x - 3(x-3)^{1/3} + 3\,\mathrm{Tan}^{-1}(x-3)^{1/3} + D.$$

22. If we set $u = x^{1/4}$, or, $x = u^4$, then $dx = 4u^3\,du$, and

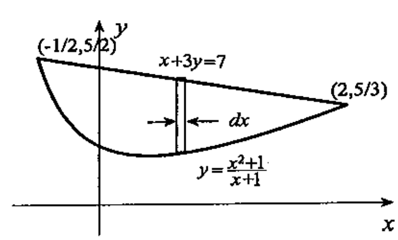
$$\int \frac{\sqrt{x}}{1+x^{1/4}}dx = \int \frac{u^2}{1+u}4u^3\,du = 4\int \frac{u^5}{u+1}du = 4\int\left(u^4 - u^3 + u^2 - u + 1 - \frac{1}{u+1}\right)du$$
$$= 4\left(\frac{u^5}{5} - \frac{u^4}{4} + \frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right) + C$$
$$= \frac{4}{5}x^{5/4} - x + \frac{4}{3}x^{3/4} - 2\sqrt{x} + 4x^{1/4} - 4\ln(x^{1/4}+1) + C.$$

23. Since small lengths along the curve can be approximated by

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2}dx = \sqrt{1+\tan^2 x}\,dx = |\sec x|\,dx,$$

the total length of the curve is $\displaystyle\int_0^{\pi/4}|\sec x|\,dx = \int_0^{\pi/4}\sec x\,dx = \Big\{\ln|\sec x + \tan x|\Big\}_0^{\pi/4} = \ln(\sqrt{2}+1)$.

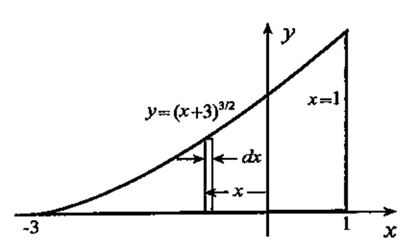
24.
$$A = \int_{-1/2}^{2}\left(\frac{7-x}{3} - \frac{x^2+1}{x+1}\right)dx$$
$$= \int_{-1/2}^{2}\left(\frac{7}{3} - \frac{x}{3} - x + 1 - \frac{2}{x+1}\right)dx$$
$$= \int_{-1/2}^{2}\left(\frac{10}{3} - \frac{4x}{3} - \frac{2}{x+1}\right)dx$$
$$= \left\{\frac{10x}{3} - \frac{2x^2}{3} - 2\ln|x+1|\right\}_{-1/2}^{2} = \frac{35 - 12\ln 6}{6}$$



25. (a) $A = \displaystyle\int_{-3}^{1}(3+x)^{3/2}\,dx = \left\{\frac{2(3+x)^{5/2}}{5}\right\}_{-3}^{1} = \frac{64}{5}$

If we set $u = 3 + x$ and $du = dx$, then

$$A\overline{x} = \int_{-3}^{1}x(3+x)^{3/2}\,dx = \int_0^4 (u-3)u^{3/2}\,du$$
$$= \left\{\frac{2u^{7/2}}{7} - \frac{6u^{5/2}}{5}\right\}_0^4 = \frac{-64}{35}.$$



Thus, $\overline{x} = -(64/35)(5/64) = -1/7$. Since

$$A\overline{y} = \int_{-3}^{1} \frac{1}{2}(3+x)^{3/2}(3+x)^{3/2}\,dx = \frac{1}{2}\int_{-3}^{1}(3+x)^3\,dx = \frac{1}{2}\left\{\frac{(3+x)^4}{4}\right\}_{-3}^{1} = 32,$$

we find $\overline{y} = 32(5/64) = 5/2$.

(b) If we again set $u = 3 + x$ and $du = dx$, then

$$\begin{aligned} I &= \int_{-3}^{1}(x-1)^2(3+x)^{3/2}\,dx = \int_0^4 (u-4)^2 u^{3/2}\,du \\ &= \int_0^4 (u^{7/2} - 8u^{5/2} + 16u^{3/2})\,du = \left\{\frac{2u^{9/2}}{9} - \frac{16u^{7/2}}{7} + \frac{32u^{5/2}}{5}\right\}_0^4 = \frac{2^{13}}{315}. \end{aligned}$$

26. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

If we set $u = -x$ and $du = -dx$ in the first integral on the right, then when $f(x)$ is an even function,

$$\int_{-a}^{a} f(x)\,dx = \int_a^0 f(-u)(-du) + \int_0^a f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx;$$

and when $f(x)$ is an odd function,

$$\int_{-a}^{a} f(x)\,dx = \int_a^0 f(-u)(-du) + \int_0^a f(x)\,dx = \int_0^a -f(u)\,du + \int_0^a f(x)\,dx = 0.$$

27. (a) Certainly, $f(x) > 0$ for $x \geq 0$. Furthermore,

$$\int_0^\infty \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty}\int_0^b \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty}\left\{-e^{-\lambda x}\right\}_0^b = \lim_{b\to\infty}(1 - e^{-\lambda b}) = 1.$$

Thus, the function qualifies as a pdf.

(b) The probability that $x \geq 3$ is

$$0.5 = \int_3^\infty \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty}\left\{-e^{-\lambda x}\right\}_3^b = \lim_{b\to\infty}(e^{-3\lambda} - e^{-b\lambda}) = e^{-3\lambda}.$$

The solution of this equation is $\lambda = -(1/3)\ln 0.5 = (1/3)\ln 2$.

28. (a) When we separate variables $\dfrac{dv}{1962 - v} = \dfrac{dt}{200}$, solutions are defined implicitly by

$$-\ln|1962 - v| = \frac{t}{200} + C \quad\Longrightarrow\quad \ln|1962 - v| = -\frac{t}{200} - C.$$

Exponentiation gives

$$|1962 - v| = e^{-C}e^{-t/200} \quad\Longrightarrow\quad 1962 - v = \pm e^{-C}e^{-t/200} = De^{-t/200},$$

where $D = \pm e^{-C}$. If we choose time $t = 0$ when descent begins, then $v(0) = 0$, and this requires $D = 1962$. Hence, $v = 1962 - 1962e^{-t/200} = 1962(1 - e^{-t/200})$ m/s.

(b) We set the velocity equal to dx/dt and integrate again,

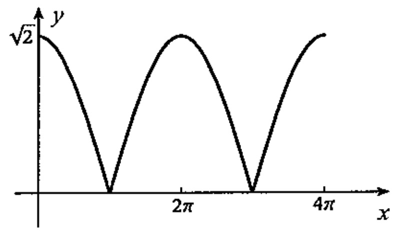
$$x = 1962(t + 200e^{-t/200}) + E.$$

Since $x(0) = 0$, we find $E = -1962(200)$, and therefore

$$x = 1962(t + 200e^{-t/200}) - 1962(200) = 1962t + 392\,400(e^{-t/200} - 1) \text{ m}.$$

29. Because of the symmetry of the function, we may integrate over $0 \le x \le \pi$ and quadruple the result. This saves later difficulties.

$$\begin{aligned}\int_0^{4\pi} \sqrt{1+\cos x}\,dx &= 4\int_0^{\pi} \sqrt{1+[2\cos^2(x/2)-1]}\,dx\\ &= 4\sqrt{2}\int_0^{\pi} |\cos(x/2)|\,dx\\ &= 4\sqrt{2}\int_0^{\pi} \cos(x/2)\,dx\\ &= 4\sqrt{2}\Big\{2\sin(x/2)\Big\}_0^{\pi} = 8\sqrt{2}.\end{aligned}$$



30. If we write $\dfrac{1}{x(3+2x^n)} = \dfrac{A}{x} + \dfrac{Bx^{n-1}}{3+2x^n} = \dfrac{3A+2Ax^n+Bx^n}{x(3+2x^n)}$, and equate numerators, then $3A+(2A+B)x^n = 1$. This equation is satisfied for all x if we choose $A = 1/3$ and $B = -2/3$. Then

$$\int \frac{1}{x(3+2x^n)}dx = \int \left[\frac{1}{3x} - \frac{2x^{n-1}}{3(3+2x^n)}\right]dx.$$

In the second integral on the right we set $u = 3+2x^n$ and $du = 2nx^{n-1}\,dx$,

$$\begin{aligned}\int \frac{1}{x(3+2x^n)}dx &= \frac{1}{3}\ln|x| - \frac{2}{3}\int \frac{1}{u}\left(\frac{du}{2n}\right) = \frac{1}{3}\ln|x| - \frac{1}{3n}\ln|u| + C\\ &= \frac{1}{3}\ln|x| - \frac{1}{3n}\ln|3+2x^n| + C = \frac{1}{3n}\ln\left|\frac{x^n}{3+2x^n}\right| + C.\end{aligned}$$

31. If $u - x = \sqrt{x^2+3x+4}$, then $(u-x)^2 = x^2+3x+4$, and when this equation is solved for x, the result is $x = (u^2-4)/(2u+3)$. Thus,

$$dx = \frac{(2u+3)(2u)-(u^2-4)(2)}{(2u+3)^2}du = \frac{2(u^2+3u+4)}{(2u+3)^2}du.$$

Since $\sqrt{x^2+3x+4} = u - x = u - \dfrac{u^2-4}{2u+3} = \dfrac{u^2+3u+4}{2u+3}$,

$$\begin{aligned}\int \frac{1}{(x^2+3x+4)^{3/2}}dx &= \int \frac{(2u+3)^3}{(u^2+3u+4)^3}\frac{2(u^2+3u+4)}{(2u+3)^2}du\\ &= 2\int \frac{2u+3}{(u^2+3u+4)^2}du = \frac{-2}{u^2+3u+4} + C\\ &= \frac{-2}{\left(x+\sqrt{x^2+3x+4}\right)^2 + 3\left(x+\sqrt{x^2+3x+4}\right) + 4} + C\\ &= \frac{-2}{2(x^2+3x+4)+(2x+3)\sqrt{x^2+3x+4}} + C\\ &= \frac{-2}{\sqrt{x^2+3x+4}\left(2x+3+2\sqrt{x^2+3x+4}\right)}\frac{2x+3-2\sqrt{x^2+3x+4}}{2x+3-2\sqrt{x^2+3x+4}} + C\\ &= \frac{2(2\sqrt{x^2+3x+4}-2x-3)}{\sqrt{x^2+3x+4}(-7)} + C = \frac{2(2x+3)}{7\sqrt{x^2+3x+4}} + D.\end{aligned}$$

32. If $u - x = \sqrt{x^2+bx+c}$, then $(u-x)^2 = x^2+bx+c$, and when this equation is solved for x, the result is $x = (u^2-c)/(2u+b)$. Thus,

$$dx = \frac{(2u+b)(2u)-(u^2-c)(2)}{(2u+b)^2}du = \frac{2(u^2+bu+c)}{(2u+b)^2}du.$$

Since $\sqrt{x^2+bx+c} = u - x = u - \dfrac{u^2-c}{2u+b} = \dfrac{u^2+bu+c}{2u+b}$,

$$\int \sqrt{x^2+bx+c}\,dx = \int \frac{u^2+bu+c}{2u+b}\frac{2(u^2+bu+c)}{(2u+b)^2}du = 2\int \frac{(u^2+bu+c)^2}{(2u+b)^3}du.$$

The integrand is a rational function of u.

33. If $(p+x)u = \sqrt{c+bx-x^2} = \sqrt{(p+x)(q-x)}$, then $u = \sqrt{(q-x)/(p+x)}$. If we square, $u^2 = (q-x)/(p+x)$, and when this is solved for x, the result is $x = (q-pu^2)/(u^2+1)$. Thus,

$$dx = \frac{(u^2+1)(-2pu)-(q-pu^2)(2u)}{(u^2+1)^2}du = \frac{-2(p+q)u}{(u^2+1)^2}du.$$

Since $\sqrt{c+bx-x^2} = (p+x)u = u\left(p+\dfrac{q-pu^2}{u^2+1}\right) = \dfrac{(p+q)u}{u^2+1}$,

$$\begin{aligned}\int \frac{1}{\sqrt{c+bx-x^2}}dx &= \int \frac{u^2+1}{(p+q)u}\left[\frac{-2(p+q)u}{(u^2+1)^2}du\right] = -2\int\frac{1}{u^2+1}du\\ &= -2\,\mathrm{Tan}^{-1}u + C = -2\,\mathrm{Tan}^{-1}\sqrt{\frac{q-x}{p+x}} + C.\end{aligned}$$

A similar derivation with the substitution $(q-x)u = \sqrt{c+bx-x^2}$ leads to

$$\int \frac{1}{\sqrt{c+bx-x^2}}dx = 2\,\mathrm{Tan}^{-1}\sqrt{\frac{p+x}{q-x}} + C.$$

34. If we set $u = a+bx$, then $du = b\,dx$, and

$$\begin{aligned}\int x(a+bx)^n\,dx &= \int\left(\frac{u-a}{b}\right)u^n\,\frac{du}{b} = \frac{1}{b^2}\int(u^{n+1}-au^n)\,du\\ &= \begin{cases}\dfrac{1}{b^2}\left(\dfrac{u^{n+2}}{n+2}-\dfrac{au^{n+1}}{n+1}\right)+C, & n\neq -2,\ -1\\ \dfrac{1}{b^2}\left(u-a\ln|u|\right)+C, & n=-1\\ \dfrac{1}{b^2}\left(\ln|u|+\dfrac{a}{u}\right)+C, & n=-2\end{cases}\\ &= \begin{cases}\dfrac{1}{b^2}\left[\dfrac{(a+bx)^{n+2}}{n+2}-\dfrac{a(a+bx)^{n+1}}{n+1}\right]+C, & n\neq -2,\ -1\\ \dfrac{1}{b^2}\left(bx-a\ln|a+bx|\right)+D, & n=-1\\ \dfrac{1}{b^2}\left(\ln|a+bx|+\dfrac{a}{a+bx}\right)+C, & n=-2.\end{cases}\end{aligned}$$

35. If we take the liberty of understanding the limiting procedure for the limits of $\pm\infty$, then

$$\begin{aligned}\int_{-\infty}^{\infty} x\,f(x)\,dx &= \int_{-\infty}^{\infty}\frac{x}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}[(x-\mu)+\mu]e^{-(x-\mu)^2/(2\sigma^2)}dx\\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}dx + \mu\int_{-\infty}^{\infty}f(x)\,dx\\ &= \frac{1}{\sqrt{2\pi}\sigma}\left\{-\sigma^2e^{-(x-\mu)^2/(2\sigma^2)}\right\}_{-\infty}^{\infty} + \mu = \mu.\end{aligned}$$

36. We take the liberty of understanding limiting procedures for the limits of $\pm\infty$. When $a \geq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-|a-x|}e^{-|x|}\,dx &= \int_{-\infty}^{0} e^{-(a-x)}e^{x}\,dx + \int_{0}^{a} e^{-(a-x)}e^{-x}\,dx + \int_{a}^{\infty} e^{a-x}e^{-x}\,dx\\ &= \int_{-\infty}^{0} e^{2x-a}\,dx + \int_{0}^{a} e^{-a}\,dx + \int_{a}^{\infty} e^{a-2x}\,dx\\ &= \left\{\frac{e^{2x-a}}{2}\right\}_{-\infty}^{0} + \left\{e^{-a}x\right\}_{0}^{a} + \left\{\frac{e^{a-2x}}{-2}\right\}_{a}^{\infty} = e^{-a}(a+1).\end{aligned}$$

When $a < 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-|a-x|}e^{-|x|}\,dx &= \int_{-\infty}^{a} e^{-(a-x)}e^{x}\,dx + \int_{a}^{0} e^{a-x}e^{x}\,dx + \int_{0}^{\infty} e^{a-x}e^{-x}\,dx\\ &= \int_{-\infty}^{a} e^{2x-a}\,dx + \int_{a}^{0} e^{a}\,dx + \int_{0}^{\infty} e^{a-2x}\,dx\\ &= \left\{\frac{e^{2x-a}}{2}\right\}_{-\infty}^{a} + \left\{e^{a}x\right\}_{a}^{0} + \left\{\frac{e^{a-2x}}{-2}\right\}_{0}^{\infty} = e^{a}(1-a).\end{aligned}$$

These may be combined into $\displaystyle\int_{-\infty}^{\infty} e^{-|a-x|}e^{-|x|}\,dx = e^{-|a|}(1+|a|)$.

EXERCISES 8.2

1. When we set $u = x$, $dv = \sin x\,dx$, then $du = dx$, $v = -\cos x$, and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C.$$

2. When we set $u = x^2$, $dv = e^{2x}\,dx$, then $du = 2x\,dx$, $v = e^{2x}/2$, and

$$\int x^2 e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \int 2x\frac{e^{2x}}{2}dx.$$

We now set $u = x$, $dv = e^{2x}\,dx$, in which case $du = dx$, $v = e^{2x}/2$, and

$$\int x^2 e^{2x}\,dx = \frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \int\frac{e^{2x}}{2}dx\right) = \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C.$$

3. When we set $u = \ln x$, $dv = x^4\,dx$, then $du = (1/x)dx$, $v = x^5/5$, and

$$\int x^4\ln x\,dx = \frac{x^5}{5}\ln x - \int\frac{x^4}{5}dx = \frac{x^5}{5}\ln x - \frac{x^5}{25} + C.$$

4. When we set $u = \ln(2x)$, $dv = \sqrt{x}\,dx$, then $du = (1/x)\,dx$, $v = (2/3)x^{3/2}$, and

$$\int\sqrt{x}\ln(2x)\,dx = \frac{2}{3}x^{3/2}\ln(2x) - \int\frac{2}{3}x^{3/2}\frac{1}{x}dx = \frac{2}{3}x^{3/2}\ln(2x) - \frac{4}{9}x^{3/2} + C.$$

5. When we set $u = z$, $dv = \sec^2(z/3)\,dz$, then $du = dz$, $v = 3\tan(z/3)$, and

$$\int z\sec^2(z/3)\,dz = 3z\tan(z/3) - \int 3\tan(z/3)\,dz = 3z\tan(z/3) + 9\ln|\cos(z/3)| + C.$$

6. When we set $u = x$, $dv = \sqrt{3-x}\,dx$, then $du = dx$, $v = -\dfrac{2}{3}(3-x)^{3/2}$, and

$$\int x\sqrt{3-x}\,dx = -\frac{2x}{3}(3-x)^{3/2} - \int -\frac{2}{3}(3-x)^{3/2}\,dx = -\frac{2x}{3}(3-x)^{3/2} - \frac{4}{15}(3-x)^{5/2} + C.$$

7. When we set $u = \mathrm{Sin}^{-1}x$, $dv = dx$, then $du = \dfrac{1}{\sqrt{1-x^2}}dx$, $v = x$, and

$$\int \mathrm{Sin}^{-1}x\,dx = x\,\mathrm{Sin}^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx = x\,\mathrm{Sin}^{-1}x + \sqrt{1-x^2} + C.$$

8. When we set $u = x^2$, $dv = \sqrt{x+5}\,dx$, then $du = 2x\,dx$, $v = \dfrac{2}{3}(x+5)^{3/2}$, and

$$\int x^2\sqrt{x+5}\,dx = \frac{2}{3}x^2(x+5)^{3/2} - \int \frac{4}{3}x(x+5)^{3/2}\,dx.$$

We now set $u = x$, $dv = (x+5)^{3/2}\,dx$, in which case $du = dx$, $v = \dfrac{2}{5}(x+5)^{5/2}$, and

$$\begin{aligned}\int x^2\sqrt{x+5}\,dx &= \frac{2}{3}x^2(x+5)^{3/2} - \frac{4}{3}\left[\frac{2}{5}x(x+5)^{5/2} - \int \frac{2}{5}(x+5)^{5/2}\,dx\right]\\ &= \frac{2}{3}x^2(x+5)^{3/2} - \frac{8}{15}x(x+5)^{5/2} + \frac{16}{105}(x+5)^{7/2} + C.\end{aligned}$$

9. When we set $u = x$, $dv = \dfrac{1}{\sqrt{2+x}}dx$, then $du = dx$ and $v = 2\sqrt{2+x}$, and

$$\int \frac{x}{\sqrt{2+x}}dx = 2x\sqrt{2+x} - \int 2\sqrt{2+x}\,dx = 2x\sqrt{2+x} - \frac{4}{3}(2+x)^{3/2} + C.$$

10. When we set $u = x^2$, $dv = \dfrac{1}{\sqrt{2+x}}dx$, then $du = 2x\,dx$, $v = 2\sqrt{2+x}$, and

$$\int \frac{x^2}{\sqrt{2+x}}dx = 2x^2\sqrt{2+x} - \int 4x\sqrt{2+x}\,dx.$$

We now set $u = x$, $dv = \sqrt{2+x}\,dx$, in which case $du = dx$, $v = \dfrac{2}{3}(2+x)^{3/2}$, and

$$\begin{aligned}\int \frac{x^2}{\sqrt{2+x}}dx &= 2x^2\sqrt{2+x} - 4\left[\frac{2x}{3}(2+x)^{3/2} - \int \frac{2}{3}(2+x)^{3/2}\,dx\right]\\ &= 2x^2\sqrt{2+x} - \frac{8}{3}x(2+x)^{3/2} + \frac{16}{15}(2+x)^{5/2} + C.\end{aligned}$$

11. $\displaystyle\int \frac{x}{\sqrt{2+x^2}}dx = \sqrt{2+x^2} + C$

12. When we set $u = \ln x$, $dv = (x-1)^2\,dx$, then $du = \dfrac{1}{x}dx$, $v = \dfrac{1}{3}(x-1)^3$, and

$$\begin{aligned}\int (x-1)^2\ln x\,dx &= \frac{(x-1)^3}{3}\ln x - \int \frac{(x-1)^3}{3}\frac{1}{x}dx = \frac{(x-1)^3}{3}\ln x - \frac{1}{3}\int\left(x^2 - 3x + 3 - \frac{1}{x}\right)dx\\ &= \frac{(x-1)^3}{3}\ln x - \frac{1}{3}\left(\frac{x^3}{3} - \frac{3x^2}{2} + 3x - \ln x\right) + C.\end{aligned}$$

13. When we set $u = e^x$, $dv = \cos x\,dx$, then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx.$$

We now set $u = e^x$, $dv = \sin x\,dx$, in which case $du = e^x\,dx$, $v = -\cos x$, and

$$\int e^x\cos x\,dx = e^x\sin x - \left(-e^x\cos x - \int -e^x\cos x\,dx\right).$$

If we bring both integrals to the left,

$$2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x \quad\Longrightarrow\quad \int e^x\cos x\,dx = \frac{e^x}{2}(\sin x + \cos x) + C.$$

14. When we set $u = \mathrm{Tan}^{-1}x$, $dv = dx$, then $du = \dfrac{1}{1+x^2}dx$, $v = x$, and

$$\int \mathrm{Tan}^{-1}x\,dx = x\,\mathrm{Tan}^{-1}x - \int \frac{x}{1+x^2}dx = x\,\mathrm{Tan}^{-1}x - \frac{1}{2}\ln(1+x^2) + C.$$

15. When we set $u = \cos(\ln x)$, $dv = dx$, then $du = -\dfrac{1}{x}\sin(\ln x)\,dx$, $v = x$, and

$$\int \cos(\ln x)\,dx = x\,\cos(\ln x) - \int -\sin(\ln x)\,dx.$$

We now set $u = \sin(\ln x)$, $dv = dx$, in which case $du = \dfrac{1}{x}\cos(\ln x)\,dx$, $v = x$, and

$$\int \cos(\ln x)\,dx = x\,\cos(\ln x) + x\,\sin(\ln x) - \int \cos(\ln x)\,dx.$$

If we bring both integrals to the left,

$$2\int \cos(\ln x)\,dx = x[\cos(\ln x) + \sin(\ln x)] \quad \Longrightarrow \quad \int \cos(\ln x)\,dx = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C.$$

16. When we set $u = e^{2x}$, $dv = \cos 3x\,dx$, then $du = 2e^{2x}\,dx$, $v = \dfrac{1}{3}\sin 3x$, and

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \int \frac{2}{3}e^{2x}\sin 3x\,dx.$$

We now set $u = e^{2x}$, $dv = \sin 3x\,dx$, in which case $du = 2e^{2x}\,dx$, $v = -\dfrac{1}{3}\cos 3x$, and

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left(-\frac{1}{3}e^{2x}\cos 3x - \int -\frac{2}{3}e^{2x}\cos 3x\,dx\right).$$

If we bring both integrals to the left,

$$\left(1 + \frac{4}{9}\right)\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x,$$

and therefore $\displaystyle\int e^{2x}\cos 3x\,dx = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C.$

17. When we set $u = x^2$, $dv = \dfrac{x}{\sqrt{5+3x^2}}dx$, then $du = 2x\,dx$, $v = (1/3)\sqrt{5+3x^2}$, and

$$\begin{aligned}\int \frac{x^3}{\sqrt{5+3x^2}}dx &= \frac{x^2}{3}\sqrt{5+3x^2} - \int \frac{2x}{3}\sqrt{5+3x^2}\,dx = \frac{x^2}{3}\sqrt{5+3x^2} - \frac{2}{3}\left[\frac{1}{9}(5+3x^2)^{3/2}\right] + C\\ &= \frac{x^2}{3}\sqrt{5+3x^2} - \frac{2}{27}(5+3x^2)^{3/2} + C.\end{aligned}$$

18. When we set $u = \ln(x^2+4)$, $dv = dx$, then $du = \dfrac{2x}{x^2+4}dx$, $v = x$, and

$$\int \ln(x^2+4)\,dx = x\ln(x^2+4) - \int \frac{2x^2}{x^2+4}dx = x\ln(x^2+4) - 2\int\left(1 - \frac{4}{x^2+4}\right)dx.$$

We now set $u = x/2$ and $du = dx/2$,

$$\begin{aligned}\int \ln(x^2+4)\,dx &= x\ln(x^2+4) - 2x + 8\int \frac{1}{4u^2+4}(2\,du) = x\ln(x^2+4) - 2x + 4\int \frac{1}{u^2+1}du\\ &= x\ln(x^2+4) - 2x + 4\,\mathrm{Tan}^{-1}u + C = x\ln(x^2+4) - 2x + 4\,\mathrm{Tan}^{-1}\left(\frac{x}{2}\right) + C.\end{aligned}$$

19. If we differentiate the equation,

$$\begin{aligned} x^5e^x &= (Ax^5e^x + 5Ax^4e^x) + (Bx^4e^x + 4Bx^3e^x) + (Cx^3e^x + 3Cx^2e^x) + (Dx^2e^x + 2Dxe^x) \\ &\quad + (Exe^x + Ee^x) + Fe^x \\ &= Ax^5e^x + (5A+B)x^4e^x + (4B+C)x^3e^x + (3C+D)x^2e^x + (2D+E)xe^x + (E+F)e^x. \end{aligned}$$

When we equate coefficients of like terms, left and right, we obtain

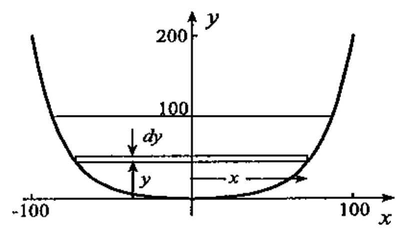
$$A = 1,\ 5A + B = 0,\ 4B + C = 0,\ 3C + D = 0,\ 2D + E = 0,\ E + F = 0.$$

These gives $B = -5$, $C = 20$, $D = -60$, $E = 120$, and $F = -120$. Thus,

$$\int x^5e^x\,dx = (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + G.$$

20. $F = \displaystyle\int_0^{100} 9.81(1000)(100-y)(2x)\,dy$

$= 19\,620 \displaystyle\int_0^{100} (100-y)\frac{1}{k}\ln(y+1)\,dy.$



If we set $u = \ln(y+1)$, $dv = (100-y)\,dy$, then $du = \dfrac{1}{y+1}dy$, $v = -\dfrac{1}{2}(100-y)^2$, and

$$\begin{aligned} F &= \frac{19\,620}{k}\left[-\frac{1}{2}(100-y)^2\ln(y+1)\Big|_0^{100} - \int_0^{100} -\frac{1}{2}(100-y)^2\frac{1}{y+1}dy\right] \\ &= \frac{9810}{k}\int_0^{100}\frac{10\,000 - 200y + y^2}{y+1}dy = \frac{9810}{k}\int_0^{100}\left(y - 201 + \frac{10\,201}{y+1}\right)dy \\ &= \frac{9810}{k}\left\{\frac{y^2}{2} - 201y + 10\,201\ln|y+1|\right\}_0^{100} = 5.92\times 10^9\text{ N}. \end{aligned}$$

21. When a constant of integration is included, there is no contradiction.

22. If we set $u = x$, $dv = \sin\dfrac{n\pi x}{L}dx$, $du = dx$, and $v = \dfrac{-L}{n\pi}\cos\dfrac{n\pi x}{L}$, then

$$\begin{aligned} \int_{-L}^{L} x\sin\frac{n\pi x}{L}\,dx &= \left\{\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}dx \\ &= -\frac{L^2}{n\pi}\cos n\pi - \frac{L^2}{n\pi}\cos(-n\pi) + \left\{\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} = \frac{2(-1)^{n+1}L^2}{n\pi}. \end{aligned}$$

If we set $u = x$, $dv = \cos\dfrac{n\pi x}{L}dx$, $du = dx$, and $v = \dfrac{L}{n\pi}\sin\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L} x\cos\frac{n\pi x}{L}\,dx = \left\{\frac{Lx}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}dx = -\left\{-\frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} = 0.$$

23. If we set $u = x^2$, $dv = \sin\dfrac{n\pi x}{L}dx$, $du = 2x\,dx$, and $v = \dfrac{-L}{n\pi}\cos\dfrac{n\pi x}{L}$, then

$$\int_{-L}^{L} x^2\sin\frac{n\pi x}{L}\,dx = \left\{\frac{-Lx^2}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{-2Lx}{n\pi}\cos\frac{n\pi x}{L}dx = \frac{2L}{n\pi}\int_{-L}^{L} x\cos\frac{n\pi x}{L}dx.$$

We now set $u = x$, $dv = \cos\dfrac{n\pi x}{L}dx$, $du = dx$, and $v = \dfrac{L}{n\pi}\sin\dfrac{n\pi x}{L}$, in which case

$$\int_{-L}^{L} x^2 \sin\frac{n\pi x}{L}\,dx = \frac{2L}{n\pi}\left\{\frac{Lx}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \frac{2L}{n\pi}\int_{-L}^{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}dx$$
$$= -\frac{2L^2}{n^2\pi^2}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} = 0.$$

If we set $u = x^2$, $dv = \cos\frac{n\pi x}{L}dx$, $du = 2x\,dx$, and $v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$, then

$$\int_{-L}^{L} x^2 \cos\frac{n\pi x}{L}\,dx = \left\{\frac{Lx^2}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{2Lx}{n\pi}\sin\frac{n\pi x}{L}dx = -\frac{2L}{n\pi}\int_{-L}^{L} x\sin\frac{n\pi x}{L}dx.$$

We now set $u = x$, $dv = \sin\frac{n\pi x}{L}dx$, $du = dx$, and $v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$, in which case

$$\int_{-L}^{L} x^2 \cos\frac{n\pi x}{L}\,dx = -\frac{2L}{n\pi}\left\{\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} + \frac{2L}{n\pi}\int_{-L}^{L}\frac{-L}{n\pi}\cos\frac{n\pi x}{L}dx$$
$$= \frac{2L^2}{n^2\pi^2}[L\cos n\pi + L\cos(-n\pi)] - \frac{2L^2}{n^2\pi^2}\left\{\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} = \frac{4(-1)^n L^3}{n^2\pi^2}.$$

24. If we set $u = 1-2x$, $dv = \sin\frac{n\pi x}{L}dx$, $du = -2\,dx$, and $v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$, then

$$\int_{-L}^{L}(1-2x)\sin\frac{n\pi x}{L}\,dx = \left\{\frac{-L(1-2x)}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{2L}{n\pi}\cos\frac{n\pi x}{L}dx$$
$$= \frac{-L}{n\pi}[(1-2L)\cos n\pi - (1+2L)\cos(-n\pi)] - \frac{2L}{n\pi}\left\{\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} = \frac{4(-1)^n L^2}{n\pi}.$$

If we set $u = 1-2x$, $dv = \cos\frac{n\pi x}{L}dx$, $du = -2\,dx$, and $v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$, then

$$\int_{-L}^{L}(1-2x)\cos\frac{n\pi x}{L}\,dx = \left\{\frac{L(1-2x)}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{-2L}{n\pi}\sin\frac{n\pi x}{L}dx = \frac{2L}{n\pi}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} = 0.$$

25. If we set $u = 2x^2-3x$, $dv = \sin\frac{n\pi x}{L}dx$, $du = (4x-3)\,dx$, and $v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$, then

$$\int_{-L}^{L}(2x^2-3x)\sin\frac{n\pi x}{L}\,dx = \left\{\frac{-L(2x^2-3x)}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}-\frac{(4x-3)L}{n\pi}\cos\frac{n\pi x}{L}dx$$
$$= -\frac{L}{n\pi}[(2L^2-3L)\cos n\pi - (2L^2+3L)\cos(-n\pi)] + \frac{L}{n\pi}\int_{-L}^{L}(4x-3)\cos\frac{n\pi x}{L}dx$$
$$= \frac{6(-1)^n L^2}{n\pi} + \frac{L}{n\pi}\int_{-L}^{L}(4x-3)\cos\frac{n\pi x}{L}dx.$$

If we now set $u = 4x-3$, $dv = \cos\frac{n\pi x}{L}dx$, $du = 4\,dx$, and $v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$, then

$$\int_{-L}^{L}(2x^2-3x)\sin\frac{n\pi x}{L}\,dx = \frac{6(-1)^n L^2}{n\pi} + \frac{L}{n\pi}\left\{\frac{(4x-3)L}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L}\frac{4L}{n\pi}\sin\frac{n\pi x}{L}dx$$
$$= \frac{6(-1)^n L^2}{n\pi} - \frac{4L^2}{n^2\pi^2}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} = \frac{6(-1)^n L^2}{n\pi}.$$

If we set $u = 2x^2-3x$, $dv = \cos\frac{n\pi x}{L}dx$, $du = (4x-3)\,dx$, and $v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$, then

$$\int_{-L}^{L}(2x^2-3x)\cos\frac{n\pi x}{L}\,dx = \left\{\frac{L(2x^2-3x)}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} - \int_{-L}^{L}\frac{(4x-3)L}{n\pi}\sin\frac{n\pi x}{L}dx$$
$$= -\frac{L}{n\pi}\int_{-L}^{L}(4x-3)\sin\frac{n\pi x}{L}dx.$$

If we now set $u = 4x-3$, $dv = \sin\frac{n\pi x}{L}dx$, $du = 4\,dx$, and $v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$, then

$$\int_{-L}^{L}(2x^2-3x)\cos\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}\left\{\frac{-(4x-3)L}{n\pi}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L}\frac{-4L}{n\pi}\cos\frac{n\pi x}{L}dx$$
$$= \frac{L^2}{n^2\pi^2}[(4L-3)\cos n\pi - (-4L-3)\cos(-n\pi)] - \frac{4L^2}{n^2\pi^2}\left\{\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right\}_{-L}^{L}$$
$$= \frac{8(-1)^n L^3}{n^2\pi^2}.$$

26. When we set $u = x^{n-1}$, $dv = e^{-x}\,dx$, then $du = (n-1)x^{n-2}\,dx$, $v = -e^{-x}$, and

$$\Gamma(n) = \lim_{b\to\infty}\int_0^b x^{n-1}e^{-x}\,dx = \lim_{b\to\infty}\left[\left\{-x^{n-1}e^{-x}\right\}_0^b - \int_0^b -(n-1)x^{n-2}e^{-x}\,dx\right]$$
$$= \lim_{b\to\infty}\left[-b^{n-1}e^{-b} + (n-1)\int_0^b x^{n-2}e^{-x}\,dx\right] = (n-1)\int_0^\infty x^{n-2}e^{-x}\,dx.$$

Further integrations by parts lead to

$$\Gamma(n) = (n-1)(n-2)(n-3)\cdots(2)(1)\int_0^\infty e^{-x}\,dx = (n-1)!\lim_{b\to\infty}\int_0^b e^{-x}\,dx$$
$$= (n-1)!\lim_{b\to\infty}\left\{-e^{-x}\right\}_0^b = (n-1)!\lim_{b\to\infty}(1-e^{-b}) = (n-1)!.$$

27. $F(s) = \int_0^\infty e^{-st}e^{3t}\,dt = \lim_{b\to\infty}\int_0^b e^{(3-s)t}\,dt = \lim_{b\to\infty}\left\{\frac{e^{(3-s)t}}{3-s}\right\}_0^b = \lim_{b\to\infty}\left[\frac{e^{(3-s)b}-1}{3-s}\right]$
$= \frac{1}{s-3}$ (provided $s > 3$)

28. $F(s) = \int_0^\infty t^2 e^{-st}\,dt = \lim_{b\to\infty}\int_0^b t^2 e^{-st}\,dt$ If we set $u = t^2$, $dv = e^{-st}\,dt$, $du = 2t\,dt$, and $v = -e^{-st}/s$, then

$$F(s) = \lim_{b\to\infty}\left[\left\{\frac{t^2e^{-st}}{-s}\right\}_0^b - \int_0^b\frac{2te^{-st}}{-s}dt\right] = \lim_{b\to\infty}\frac{b^2e^{-bs}}{-s} + \frac{2}{s}\lim_{b\to\infty}\int_0^b te^{-st}\,dt.$$

The first limit is zero provided $s > 0$. In the integral we set $u = t$, $dv = e^{-st}\,dt$, $du = dt$, and $v = -e^{-st}/s$, in which case

$$F(s) = \frac{2}{s}\lim_{b\to\infty}\left[\left\{\frac{te^{-st}}{-s}\right\}_0^b - \int_0^b -\frac{e^{-st}}{s}dt\right] = \frac{2}{s}\lim_{b\to\infty}\left(\frac{be^{-bs}}{-s}\right) + \frac{2}{s^2}\lim_{b\to\infty}\int_0^b e^{-st}\,dt$$
$$= \frac{2}{s^2}\lim_{b\to\infty}\left\{\frac{e^{-st}}{-s}\right\}_0^b = -\frac{2}{s^3}\lim_{b\to\infty}(e^{-bs}-1) = \frac{2}{s^3}.$$

29. If we set $u = e^{-st}$, $dv = \sin t\,dt$, $du = -se^{-st}\,dt$, and $v = -\cos t$, then

$$\int e^{-st}\sin t\,dt = -e^{-st}\cos t - \int se^{-st}\cos t\,dt.$$

If we now set $u = e^{-st}$, $dv = \cos t\,dt$, $du = -se^{-st}\,dt$, and $v = \sin t$, then

$$\begin{aligned}\int e^{-st}\sin t\,dt &= -e^{-st}\cos t - s\left(e^{-st}\sin t - \int -se^{-st}\sin t\,dt\right)\\ &= -e^{-st}(\cos t + s\sin t) - s^2\int e^{-st}\sin t\,dt.\end{aligned}$$

When we solve for the integral, we obtain $\displaystyle\int e^{-st}\sin t\,dt = \frac{-e^{-st}(\cos t + s\sin t)}{1+s^2}$. Thus,

$$\begin{aligned}F(s) &= \int_0^\infty e^{-st}\sin t\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\sin t\,dt = \lim_{b\to\infty}\left\{\frac{-e^{-st}(\cos t + s\sin t)}{1+s^2}\right\}_0^b\\ &= -\frac{1}{1+s^2}\lim_{b\to\infty}[e^{-bs}(\cos b + s\sin b) - 1] = \frac{1}{1+s^2}\quad\text{(provided } s > 0)\end{aligned}$$

30. If we set $u = t$, $dv = e^{-(s+1)t}\,dt$, $du = dt$, and $v = \dfrac{e^{-(s+1)t}}{-(s+1)}$, then

$$\int te^{-t}e^{-st}\,dt = \int te^{-(s+1)t}\,dt = -\frac{te^{-(s+1)t}}{s+1} - \int\frac{e^{-(s+1)t}}{-(s+1)}dt = -\frac{te^{-(s+1)t}}{s+1} - \frac{e^{-(s+1)t}}{(s+1)^2}.$$

Thus,

$$\begin{aligned}F(s) &= \int_0^\infty te^{-t}e^{-st}\,dt = \lim_{b\to\infty}\int_0^b te^{-(s+1)t}\,dt = \lim_{b\to\infty}\left\{-\frac{te^{-(s+1)t}}{s+1} - \frac{e^{-(s+1)t}}{(s+1)^2}\right\}_0^b\\ &= \lim_{b\to\infty}\left[-\frac{be^{-(s+1)b}}{s+1} - \frac{e^{-(s+1)b}}{(s+1)^2} + \frac{1}{(s+1)^2}\right] = \frac{1}{(s+1)^2}\quad\text{(provided } s > -1)\end{aligned}$$

31. Certainly $f(x) \geq 0$, and $\displaystyle\int_0^\infty f(x)\,dx = \int_0^\infty \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}dx$. If we set $u = x/\beta$, then $du = (1/\beta)\,dx$, and

$$\int_0^\infty f(x)\,dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty(\beta u)^{\alpha-1}e^{-u}(\beta\,du) = \frac{1}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}e^{-u}\,du = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1.$$

32.
$$\begin{aligned}F(\omega) &= \int_{-\infty}^\infty f(t)e^{-i\omega t}\,dt = \int_{-L/2}^{L/2}e^{-i\omega t}dt = \left\{\frac{e^{-i\omega t}}{-i\omega}\right\}_{-L/2}^{L/2}\\ &= \frac{-1}{i\omega}(e^{-i\omega L/2} - e^{i\omega L/2}) = \frac{1}{i\omega}\left(e^{i\omega L/2} - e^{-i\omega L/2}\right) = \frac{2}{i\omega}\sinh\left(\frac{i\omega L}{2}\right).\end{aligned}$$

33. $\displaystyle F(\omega) = \int_{-\infty}^\infty f(t)e^{-i\omega t}\,dt = \int_{-T}^0\left(1+\frac{t}{T}\right)e^{-i\omega t}\,dt + \int_0^T\left(1-\frac{t}{T}\right)e^{-i\omega t}\,dt$

If we set $u = 1 + t/T$, $dv = e^{-i\omega t}\,dt$, $du = dt/T$, $v = \dfrac{e^{-i\omega t}}{-i\omega}$ in the first integral, and $u = 1 - t/T$, $dv = e^{-i\omega t}\,dt$, $du = -dt/T$, $v = \dfrac{e^{-i\omega t}}{-i\omega}$ in the second,

$$\begin{aligned}F(\omega) &= \left\{\left(1+\frac{t}{T}\right)\left(\frac{e^{-i\omega t}}{-i\omega}\right)\right\}_{-T}^0 - \int_{-T}^0\frac{e^{-i\omega t}}{-i\omega T}dt + \left\{\left(1-\frac{t}{T}\right)\left(\frac{e^{-i\omega t}}{-i\omega}\right)\right\}_0^T - \int_0^T\frac{e^{-i\omega t}}{i\omega T}dt\\ &= -\frac{1}{i\omega} + \left\{\frac{e^{-i\omega t}}{-i^2\omega^2 T}\right\}_{-T}^0 + \frac{1}{i\omega} - \left\{\frac{e^{-i\omega t}}{-i^2\omega^2 T}\right\}_0^T = \frac{1}{i^2\omega^2 T}\left(-1 + e^{i\omega T} + e^{-i\omega T} - 1\right)\\ &= \frac{1}{\omega^2 T}\left[2 - 2\left(\frac{e^{i\omega T} + e^{-i\omega T}}{2}\right)\right] = \frac{2}{\omega^2 T}(1 - \cosh i\omega T).\end{aligned}$$

34. $F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt=\int_{-\infty}^{\infty} e^{-a|t|}e^{-i\omega t}\,dt=\int_{-\infty}^{0} e^{(a-i\omega)t}\,dt+\int_{0}^{\infty} e^{-(a+i\omega)t}\,dt$

$$=\lim_{b\to-\infty}\left\{\frac{e^{(a-i\omega)t}}{a-i\omega}\right\}_b^0+\lim_{b\to\infty}\left\{\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right\}_0^b$$

$$=\frac{1}{a-i\omega}\lim_{b\to-\infty}[1-e^{(a-i\omega)b}]-\frac{1}{a+i\omega}\lim_{b\to\infty}[e^{-(a+i\omega)b}-1]=\frac{1}{a-i\omega}+\frac{1}{a+i\omega}=\frac{2a}{\omega^2+a^2}.$$

35. $F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt=\int_a^b e^{-i\omega t}\,dt=\left\{\frac{e^{-i\omega t}}{-i\omega}\right\}_a^b=\frac{e^{-i\omega a}-e^{-i\omega b}}{i\omega}$

36. If we set $u=\operatorname{Tan}^{-1}\sqrt{x}$, $dv=dx$, $du=\dfrac{1}{2\sqrt{x}(1+x)}dx$, and $v=x$,

$$\int \operatorname{Tan}^{-1}\sqrt{x}\,dx=x\operatorname{Tan}^{-1}\sqrt{x}-\int\frac{x}{2\sqrt{x}(1+x)}dx=x\operatorname{Tan}^{-1}\sqrt{x}-\frac{1}{2}\int\frac{\sqrt{x}}{1+x}dx.$$

If we now set $u=\sqrt{x}$, from which $x=u^2$, and $dx=2u\,du$, then

$$\int \operatorname{Tan}^{-1}\sqrt{x}\,dx=x\operatorname{Tan}^{-1}\sqrt{x}-\frac{1}{2}\int\frac{u}{1+u^2}(2u\,du)=x\operatorname{Tan}^{-1}\sqrt{x}-\int\left(1-\frac{1}{1+u^2}\right)du$$

$$=x\operatorname{Tan}^{-1}\sqrt{x}-u+\operatorname{Tan}^{-1}u+C=x\operatorname{Tan}^{-1}\sqrt{x}-\sqrt{x}+\operatorname{Tan}^{-1}\sqrt{x}+C.$$

37. $\int x^2\cos^2 x\,dx=\int x^2\left(\frac{1+\cos 2x}{2}\right)dx=\frac{x^3}{6}+\frac{1}{2}\int x^2\cos 2x\,dx$

When we set $u=x^2$, $dv=\cos 2x\,dx$, then $du=2x\,dx$, $v=\dfrac{1}{2}\sin 2x$, and

$$\int x^2\cos^2 x\,dx=\frac{x^3}{6}+\frac{1}{2}\left(\frac{x^2}{2}\sin 2x-\int x\sin 2x\,dx\right).$$

We now set $u=x$, $dv=\sin 2x\,dx$, in which case $du=dx$, $v=-\dfrac{1}{2}\cos 2x$, and

$$\int x^2\cos^2 x\,dx=\frac{x^3}{6}+\frac{x^2}{4}\sin 2x-\frac{1}{2}\left(-\frac{x}{2}\cos 2x-\int -\frac{1}{2}\cos 2x\,dx\right)$$

$$=\frac{x^3}{6}+\frac{x^2}{4}\sin 2x+\frac{x}{4}\cos 2x-\frac{1}{8}\sin 2x+C.$$

38. First we evaluate the integral of $e^x\sin x$. When we set $u=e^x$, $dv=\sin x\,dx$, then $du=e^x\,dx$, $v=-\cos x$, and

$$\int e^x\sin x\,dx=-e^x\cos x-\int -e^x\cos x\,dx.$$

We now set $u=e^x$, $dv=\cos x\,dx$, in which case $du=e^x\,dx$, $v=\sin x$, and

$$\int e^x\sin x\,dx=-e^x\cos x+\left(e^x\sin x-\int e^x\sin x\,dx\right).$$

If we bring both integrals to the left,

$$2\int e^x\sin x\,dx=e^x\sin x-e^x\cos x\quad\Longrightarrow\quad\int e^x\sin x\,dx=\frac{e^x}{2}(\sin x-\cos x)+C.$$

In the given integral we now set $u=x$, $dv=e^x\sin x\,dx$, $du=dx$, and $v=e^x(\sin x-\cos x)/2$, in which case

$$\int x\,e^x\sin x\,dx=\frac{x\,e^x(\sin x-\cos x)}{2}-\int\frac{e^x(\sin x-\cos x)}{2}dx.$$

We now need the integral of $e^x\cos x$. When we set $u=e^x$, $dv=\cos x\,dx$, $du=e^x\,dx$, $v=\sin x$, then

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

We now set $u = e^x$, $dv = \sin x\, dx$, in which case $du = e^x\, dx$, $v = -\cos x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\, dx\right).$$

If we bring both integrals to the left,

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x \quad \Longrightarrow \quad \int e^x \cos x\, dx = \frac{e^x}{2}(\sin x + \cos x) + C.$$

We can now say that

$$\frac{1}{2}\int e^x(\sin x - \cos x)\, dx = \frac{1}{2}\left[\frac{e^x(\sin x - \cos x)}{2} - \frac{e^x(\sin x + \cos x)}{2}\right] = -\frac{e^x \cos x}{2}.$$

Finally,

$$\int x\, e^x \sin x\, dx = \frac{x\, e^x(\sin x - \cos x)}{2} + \frac{e^x \cos x}{2} + C.$$

39. If we set $x = \sin^2\theta$, then $dx = 2\sin\theta\cos\theta\, d\theta$, and

$$\int_0^1 x^n(1-x)^m\, dx = \int_0^{\pi/2} \sin^{2n}\theta\,(\cos^2\theta)^m\, 2\sin\theta\cos\theta\, d\theta = 2\int_0^{\pi/2} \sin^{2n+1}\theta\cos^{2m+1}\theta\, d\theta.$$

Consider now using integration by parts on the integral of $\sin^p\theta\cos^q\theta$ where $p \ge 2$ and $q \ge 1$ are integers. With $u = \sin^{p-1}\theta$, $dv = \cos^q\theta\sin\theta\, d\theta$, $du = (p-1)\sin^{p-2}\theta\cos\theta\, d\theta$, and $v = -\dfrac{\cos^{q+1}\theta}{q+1}$,

$$\begin{aligned}
\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta &= \left\{\frac{-1}{q+1}\sin^{p-1}\theta\cos^{q+1}\theta\right\}_0^{\pi/2} + \int_0^{\pi/2} \frac{p-1}{q+1}\sin^{p-2}\theta\cos^{q+2}\theta\, d\theta \\
&= \frac{p-1}{q+1}\int_0^{\pi/2} \sin^{p-2}\theta(1-\sin^2\theta)\cos^q\theta\, d\theta \\
&= \frac{p-1}{q+1}\int_0^{\pi/2} \sin^{p-2}\theta\cos^q\theta\, d\theta - \frac{p-1}{q+1}\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta.
\end{aligned}$$

If we combine the integrals,

$$\left(1 + \frac{p-1}{q+1}\right)\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{p-1}{q+1}\int_0^{\pi/2} \sin^{p-2}\theta\cos^q\theta\, d\theta,$$

from which

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{p-1}{p+q}\int_0^{\pi/2} \sin^{p-2}\theta\cos^q\theta\, d\theta.$$

We use this as a formula on the integral of $\sin^{2n+1}\theta\cos^{2m+1}\theta$ to eliminate the power on $\sin\theta$,

$$\begin{aligned}
\int_0^1 x^n(1-x)^m\, dx = 2\int_0^{\pi/2} \sin^{2n+1}\theta\cos^{2m+1}\theta\, d\theta &= \frac{2(2n)}{2n+2m+2}\int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2m+1}\theta\, d\theta \\
&= \frac{2n}{n+m+1}\frac{2n-2}{2n+2m}\int_0^{\pi/2} \sin^{2n-3}\theta\cos^{2m+1}\theta\, d\theta \\
&= \frac{2n(n-1)}{(n+m+1)(n+m)}\frac{2n-4}{2n+2m-2}\int_0^{\pi/2} \sin^{2n-5}\theta\cos^{2m+1}\theta\, d\theta \\
&\quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

$$= \frac{2n(n-1)(n-2)\cdots(1)}{(n+m+1)(n+m)(n+m-1)\cdots(m+2)} \int_0^{\pi/2} \sin\theta \cos^{2m+1}\theta\, d\theta$$
$$= \frac{2\,n!}{(n+m+1)(n+m)(n+m-1)\cdots(m+2)} \left\{-\frac{\cos^{2m+2}\theta}{2m+2}\right\}_0^{\pi/2}$$
$$= \frac{n!}{(n+m+1)(n+m)(n+m-1)\cdots(m+1)} = \frac{n!\,m!}{(n+m+1)!}.$$

40. We omit constants of integration in the following proof by mathematical induction. When $n=1$, the left side is the integral of $x\cos x$. If we set $u=x$, $dv=\cos x\,dx$, $du=dx$, and $v=\sin x$, then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x.$$

When $n=1$, the right side of the formula gives

$$\sin x \sum_{r=0}^{\lfloor 1/2\rfloor} \frac{(-1)^r}{(1-2r)!}x^{1-2r} + \cos x \sum_{r=0}^{\lfloor 0/2\rfloor} \frac{(-1)^r}{(1-2r-1)!}x^{-2r} = \sin x\left[\frac{(-1)^0 x}{1!}\right] + \cos x\left[\frac{(-1)^0}{0!}\right] = x\sin x + \cos x.$$

Thus, the formula is correct for $n=1$. When $n=2$, the left side is the integral of $x^2\cos x$. If we set $u=x^2$, $dv=\cos x\,dx$, $du=2x\,dx$, and $v=\sin x$, then

$$\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx.$$

If we set $u=x$, $dv=\sin x\,dx$, $du=dx$, and $v=-\cos x$, then

$$\int x^2\cos x\,dx = x^2\sin x - 2\left(-x\cos x - \int -\cos x\,dx\right) = x^2\sin x + 2x\cos x - 2\sin x.$$

When $n=2$, the right side of the formula gives

$$\sin x \sum_{r=0}^{\lfloor 1\rfloor} \frac{(-1)^r 2!}{(2-2r)!}x^{2-2r} + \cos x \sum_{r=0}^{\lfloor 1/2\rfloor} \frac{(-1)^r 2!}{(2-2r-1)!}x^{2-2r-1} = \sin x\left[\frac{2x^2}{2} + \frac{(-1)2}{0!}\right] + \cos x\left(\frac{2x}{1!}\right)$$
$$= (x^2-2)\sin x + 2x\cos x.$$

The formula is valid for $n=2$ also. Suppose the result is valid for some integer k; that is, assume that

$$\int x^k\cos x\,dx = \sin x \sum_{r=0}^{\lfloor k/2\rfloor} \frac{(-1)^r k!}{(k-2r)!}x^{k-2r} + \cos x \sum_{r=0}^{\lfloor (k-1)/2\rfloor} \frac{(-1)^r k!}{(k-2r-1)!}x^{k-2r-1}.$$

If we set $u=x^{k+2}$, $dv=\cos x\,dx$, $du=(k+2)x^{k+1}\,dx$, and $v=\sin x$, then

$$\int x^{k+2}\cos x\,dx = x^{k+2}\sin x - \int (k+2)x^{k+1}\sin x\,dx.$$

We now set $u=x^{k+1}$, $dv=\sin x\,dx$, $du=(k+1)x^k\,dx$, and $v=-\cos x$, in which case

$$\int x^{k+2}\cos x\,dx = x^{k+2}\sin x - (k+2)\left[-x^{k+1}\cos x - \int -(k+1)x^k\cos x\,dx\right]$$
$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x - (k+2)(k+1)\int x^k\cos x\,dx$$
$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x - (k+2)(k+1)\left[\sin x \sum_{r=0}^{\lfloor k/2\rfloor} \frac{(-1)^r k!}{(k-2r)!}x^{k-2r}\right.$$
$$\left. + \cos x \sum_{r=0}^{\lfloor (k-1)/2\rfloor} \frac{(-1)^r k!}{(k-2r-1)!}x^{k-2r-1}\right]$$

$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x - \sin x \sum_{r=0}^{\lfloor k/2\rfloor} \frac{(-1)^r(k+2)!}{(k-2r)!}x^{k-2r}$$

$$- \cos x \sum_{r=0}^{\lfloor (k-1)/2\rfloor} \frac{(-1)^r(k+2)!}{(k-2r-1)!}x^{k-2r-1}$$

$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x - \sin x \sum_{r=1}^{1+\lfloor k/2\rfloor} \frac{(-1)^{r-1}(k+2)!}{[k-2(r-1)]!}x^{k-2(r-1)}$$

$$- \cos x \sum_{r=1}^{1+\lfloor (k-1)/2\rfloor} \frac{(-1)^{r-1}(k+2)!}{[k-2(r-1)-1]!}x^{k-2(r-1)-1}$$

$$= x^{k+2}\sin x + (k+2)x^{k+1}\cos x + \sin x \sum_{r=1}^{\lfloor (k+2)/2\rfloor} \frac{(-1)^r(k+2)!}{[k+2-2r]!}x^{k+2-2r}$$

$$+ \cos x \sum_{r=1}^{\lfloor ((k+2)-1)/2\rfloor} \frac{(-1)^r(k+2)!}{[(k+2)-2r-1]!}x^{k+2-2r-1}$$

$$= \sin x \sum_{r=0}^{\lfloor (k+2)/2\rfloor} \frac{(-1)^r(k+2)!}{[k+2-2r]!}x^{k+2-2r} + \cos x \sum_{r=0}^{\lfloor ((k+2)-1)/2\rfloor} \frac{(-1)^r(k+2)!}{[(k+2)-2r-1]!}x^{k+2-2r-1}.$$

But this is the formula for $n = k+2$. By mathematical induction then, the result is valid for all $n \geq 1$.

EXERCISES 8.3

1. $\displaystyle\int \cos^3 x \sin x\, dx = -\frac{1}{4}\cos^4 x + C$

2. $\displaystyle\int \frac{\cos x}{\sin^3 x}dx = -\frac{1}{2\sin^2 x} + C$

3. $\displaystyle\int \tan^5 x \sec^2 x\, dx = \frac{1}{6}\tan^6 x + C$

4. $\displaystyle\int \csc^3 x \cot x\, dx = -\frac{1}{3}\csc^3 x + C$

5. $\displaystyle\int \cos^3(x+2)\, dx = \int [1-\sin^2(x+2)]\cos(x+2)\, dx = \sin(x+2) - \frac{1}{3}\sin^3(x+2) + C$

6. $\displaystyle\int \sqrt{\tan x}\sec^4 x\, dx = \int \sqrt{\tan x}(1+\tan^2 x)\sec^2 x\, dx$

$$= \int (\tan^{1/2} x + \tan^{5/2} x)\sec^2 x\, dx = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + C$$

7. $\displaystyle\int \frac{1}{\sin^4 t}dt = \int \csc^4 t\, dt = \int \csc^2 t(1+\cot^2 t)\, dt = -\cot t - \frac{1}{3}\cot^3 t + C$

8. $\displaystyle\int \sec^6 3x \tan 3x\, dx = \frac{1}{18}\sec^6 3x + C$

9. $\displaystyle\int \cos^2 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right) dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

10. $\displaystyle\int \frac{\tan^3 x \sec^2 x}{\sin^2 x}dx = \int \frac{\sin x}{\cos^5 x}dx = \frac{1}{4\cos^4 x} + C = \frac{1}{4}\sec^4 x + C$

11. $\displaystyle\int \sin^3 y \cos^2 y\, dy = \int \sin y(1-\cos^2 y)\cos^2 y\, dy = -\frac{1}{3}\cos^3 y + \frac{1}{5}\cos^5 y + C$

12. $\displaystyle\int \frac{\csc^2\theta}{\cot^2\theta}d\theta = \frac{1}{\cot\theta} + C = \tan\theta + C$

13. $\displaystyle\int \frac{\sin\theta}{1+\cos\theta}d\theta = -\ln|1+\cos\theta| + C = -\ln(1+\cos\theta) + C$

14. $\displaystyle\int \frac{\sec^2 x}{\sqrt{1+\tan x}}dx = 2\sqrt{1+\tan x} + C$

15. $\displaystyle\int \cos\theta \sin 2\theta\, d\theta = \int 2\sin\theta\cos^2\theta\, d\theta = -\frac{2}{3}\cos^3\theta + C$

16. $\displaystyle\int \frac{3+4\csc^2 x}{\cot^2 x}dx = \int (3\tan^2 x + 4\sec^2 x)\,dx = \int (3\sec^2 x - 3 + 4\sec^2 x)\,dx = 7\tan x - 3x + C$

17. $\displaystyle\int \sin^5 x\cos^5 x\,dx = \int \sin^5 x(1-\sin^2 x)^2\cos x\,dx = \int \sin^5 x(1 - 2\sin^2 x + \sin^4 x)\cos x\,dx$

$$= \frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C$$

18. $\displaystyle\int \sin^4 x\,dx = \int\left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$

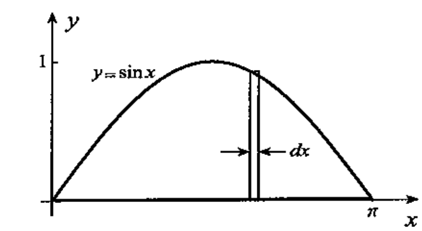
$$= \frac{1}{4}\int\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx = \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

19. $\displaystyle\int \frac{\tan^3 x}{\sec^4 x}dx = \int \sin^3 x\cos x\,dx = \frac{1}{4}\sin^4 x + C$

20. $\displaystyle\int \frac{\csc^4 x}{\cot^3 x}dx = \int \frac{(1+\cot^2 x)\csc^2 x}{\cot^3 x}dx = \int\left(\frac{1}{\cot^3 x} + \frac{1}{\cot x}\right)\csc^2 x\,dx$

$$= \frac{1}{2\cot^2 x} - \ln|\cot x| + C = \frac{1}{2}\tan^2 x + \ln|\tan x| + C$$

21. $\displaystyle A = \int_0^{\pi} \sin x\,dx = \Big\{-\cos x\Big\}_0^{\pi} = 2$

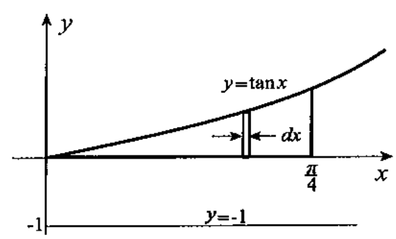


22. $\displaystyle V = \int_0^{\pi/4} [\pi(1+\tan x)^2 - \pi(1)^2]\,dx$

$$= \pi\int_0^{\pi/4}(\tan^2 x + 2\tan x)\,dx$$

$$= \pi\int_0^{\pi/4}(\sec^2 x - 1 + 2\tan x)\,dx$$

$$= \pi\Big\{\tan x - x + 2\ln|\sec x|\Big\}_0^{\pi/4} = \pi(1 - \pi/4 + \ln 2)$$



23. $\displaystyle\int_0^{\pi}\sqrt{1-\sin^2 x}\,dx = \int_0^{\pi}|\cos x|\,dx = \int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx = \Big\{\sin x\Big\}_0^{\pi/2} - \Big\{\sin x\Big\}_{\pi/2}^{\pi} = 2$

24. $\displaystyle\int \cot^4 z\,dz = \int \cot^2 z(\csc^2 z - 1)\,dz = \int(\cot^2 z\csc^2 z - \csc^2 z + 1)\,dz = -\frac{1}{3}\cot^3 z + \cot z + z + C$

25. If we set $u = 3 + \sin\theta$ and $du = \cos\theta\,d\theta$, then

$$\int \frac{\cos^3\theta}{3+\sin\theta}d\theta = \int\frac{(1-\sin^2\theta)\cos\theta}{3+\sin\theta}d\theta = \int\frac{1-(u-3)^2}{u}du = \int\left(-\frac{8}{u} + 6 - u\right)du$$

$$= -8\ln|u| + 6u - \frac{u^2}{2} + C = -8\ln|3+\sin\theta| + 6(3+\sin\theta) - \frac{1}{2}(3+\sin\theta)^2 + C$$

$$= 3\sin\theta - 8\ln(3+\sin\theta) - \frac{1}{2}\sin^2\theta + D.$$

26. $\displaystyle\int\frac{\cos^4\theta}{1+\sin\theta}d\theta = \int\frac{(1-\sin^2\theta)\cos^2\theta}{1+\sin\theta}d\theta = \int\frac{(1-\sin\theta)(1+\sin\theta)\cos^2\theta}{1+\sin\theta}d\theta = \int(1-\sin\theta)\cos^2\theta\,d\theta$

$$= \int\left(\frac{1+\cos 2\theta}{2} - \cos^2\theta\sin\theta\right)d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + \frac{1}{3}\cos^3\theta + C$$

27. $\displaystyle\int \sin^4 x \cos^2 x\, dx = \int \sin^2 x(\sin x \cos x)^2\, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{\sin 2x}{2}\right)^2 dx$

$$= \frac{1}{8}\int (\sin^2 2x - \sin^2 2x \cos 2x)\, dx = \frac{1}{8}\int \left(\frac{1-\cos 4x}{2} - \sin^2 2x \cos 2x\right) dx$$

$$= \frac{x}{16} - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$$

28. Using identity 1.48c, $\displaystyle\int \cos 6x \cos 2x\, dx = \int \frac{1}{2}(\cos 8x + \cos 4x)\, dx = \frac{1}{16}\sin 8x + \frac{1}{8}\sin 4x + C$

29. $\displaystyle\int \cos^2 2x \sin 3x\, dx = \int \left(\frac{1+\cos 4x}{2}\right)\sin 3x\, dx$ We use identity 1.48b on the second term,

$$\int \cos^2 2x \sin 3x\, dx = \frac{1}{2}\int \left(\sin 3x + \frac{1}{2}\sin 7x - \frac{1}{2}\sin x\right) dx = -\frac{1}{6}\cos 3x - \frac{1}{28}\cos 7x + \frac{1}{4}\cos x + C.$$

30. $\displaystyle\int \frac{1}{\sin x \cos^2 x} dx = \int \csc x \sec^2 x\, dx = \int \csc x(1+\tan^2 x)\, dx$

$$= \int (\csc x + \sec x \tan x)\, dx = \ln|\csc x - \cot x| + \sec x + C$$

31. If we set $u = \sec^3 x$, $dv = \sec^2 x\, dx$, $du = 3\sec^3 x \tan x\, dx$, and $v = \tan x$, then

$$\int \sec^5 x\, dx = \sec^3 x \tan x - \int 3\sec^3 x \tan^2 x\, dx = \sec^3 x \tan x - 3\int \sec^3 x(\sec^2 x - 1)\, dx$$

$$= \sec^3 x \tan x - 3\int \sec^5 x\, dx + 3\int \sec^3 x\, dx.$$

If we solve for the integral of $\sec^5 x$, we obtain

$$\int \sec^5 x\, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\, dx.$$

We now use the result of Example 8.9,

$$\int \sec^5 x\, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C.$$

32. The average power is the integral of Vi over one period $2\pi/\omega$ divided by the period,

$$\begin{aligned} P_{av} &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V_m \cos(\omega t + \phi_2)\, i_m \cos(\omega t + \phi_1)\, dt \\ &= \frac{\omega V_m i_m}{2\pi}\int_0^{2\pi/\omega} \frac{1}{2}[\cos(2\omega t + \phi_1 + \phi_2) + \cos(\phi_1 - \phi_2)]\, dt \\ &= \frac{\omega V_m i_m}{4\pi}\left\{\frac{1}{2\omega}\sin(2\omega t + \phi_1 + \phi_2) + t\cos(\phi_1 - \phi_2)\right\}_0^{2\pi/\omega} \\ &= \frac{\omega V_m i_m}{4\pi}\left[\frac{1}{2\omega}\sin(4\pi + \phi_1 + \phi_2) - \frac{1}{2\omega}\sin(\phi_1 + \phi_2) + \frac{2\pi}{\omega}\cos(\phi_1 - \phi_2)\right] \\ &= \frac{V_m i_m}{2}\cos(\phi_1 - \phi_2). \end{aligned}$$

33. The current can be expressed in the form $R\sin(\omega t + \phi)$ by setting

$$A\cos\omega t + B\sin\omega t = R\sin(\omega t + \phi) = R(\sin\omega t\cos\phi + \cos\omega t\sin\phi).$$

This equation is satisfied for all t if we choose R and ϕ such that $R\cos\phi = B$ and $R\sin\phi = A$. Squaring and adding these gives $R^2 = A^2 + B^2$, and therefore the amplitude of the current is $R = \sqrt{A^2 + B^2}$. If we choose $T = 2\pi/\omega$, the I_{rms} current is given by

$$\begin{aligned}
(I_{\text{rms}})^2 &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} (A\cos\omega t + B\sin\omega t)^2\,dt \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} (A^2\cos^2\omega t + 2AB\,\sin\omega t\,\cos\omega t + B^2\sin^2\omega t)\,dt \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[A^2\left(\frac{1+\cos 2\omega t}{2}\right) + AB\,\sin 2\omega t + B^2\left(\frac{1-\cos 2\omega t}{2}\right)\right]dt \\
&= \frac{\omega}{2\pi}\left\{\frac{A^2}{2}\left(t+\frac{1}{2\omega}\sin 2\omega t\right) - \frac{AB}{2\omega}\cos 2\omega t + \frac{B^2}{2}\left(t-\frac{1}{2\omega}\sin 2\omega t\right)\right\}\Bigg|_0^{2\pi/\omega} \\
&= \frac{1}{2}(A^2+B^2).
\end{aligned}$$

Thus, $I_{\text{rms}} = \sqrt{A^2+B^2}/\sqrt{2} = R/\sqrt{2}$.

34. $F_{dc} = \dfrac{\omega}{2\pi}\displaystyle\int_{-\pi/\omega}^{\pi/\omega}(A\,\cos\omega t + B\,\sin\omega t)\,dt = \dfrac{\omega}{2\pi}\left\{\dfrac{A}{\omega}\sin\omega t - \dfrac{B}{\omega}\cos\omega t\right\}\Bigg|_{-\pi/\omega}^{\pi/\omega} = 0$

35. $F_{dc} = \dfrac{\omega}{2\pi}\displaystyle\int_{-\pi/\omega}^{\pi/\omega}(A + B\,\cos\omega t)\,dt = \dfrac{\omega}{2\pi}\left\{At + \dfrac{B}{\omega}\sin\omega t\right\}\Bigg|_{-\pi/\omega}^{\pi/\omega} = A$

36. $F_{dc} = \dfrac{\omega}{\pi}\displaystyle\int_{-\pi/(2\omega)}^{\pi/(2\omega)}\sin^2\omega t\,dt = \dfrac{\omega}{\pi}\displaystyle\int_{-\pi/(2\omega)}^{\pi/(2\omega)}\left(\dfrac{1-\cos 2\omega t}{2}\right)dt = \dfrac{\omega}{2\pi}\left\{t - \dfrac{\sin 2\omega t}{2\omega}\right\}\Bigg|_{-\pi/(2\omega)}^{\pi/(2\omega)} = \dfrac{1}{2}$

37. Since we must integrate the function over one full period, we choose to integrate over $0 \le t \le 1$,

$$F_{dc} = \frac{1}{1}\int_0^1 t\,dt = \left\{\frac{t^2}{2}\right\}_0^1 = \frac{1}{2}.$$

38. First we consider summing $1+\sin\theta+\sin^2\theta+\cdots$. If we denote the sum of the first n terms by $S_n = 1+\sin\theta+\cdots+\sin^{n-1}\theta$, then multiplication of this by $\sin\theta$ gives $(\sin\theta)S_n = \sin\theta+\sin^2\theta+\cdots+\sin^n\theta$. If we subtract, many cancellations occur, leaving

$$S_n - (\sin\theta)S_n = 1-\sin^n\theta \quad\Longrightarrow\quad S_n = \frac{1-\sin^n\theta}{1-\sin\theta},\ \text{provided } \sin\theta \ne 1.$$

If we take limits as $n\to\infty$, we obtain $1+\sin\theta+\cdots = \dfrac{1}{1-\sin\theta}$. Hence,

$$\begin{aligned}
\int(1+\sin\theta+\sin^2\theta+\cdots)\,d\theta &= \int\frac{1}{1-\sin\theta}d\theta = \int\frac{1+\sin\theta}{1-\sin^2\theta}d\theta = \int\frac{1+\sin\theta}{\cos^2\theta}d\theta \\
&= \int(\sec^2\theta + \sec\theta\,\tan\theta)\,d\theta = \tan\theta + \sec\theta + C.
\end{aligned}$$

39.
$$\begin{aligned}
\int\sec^n x\,dx &= \int\sec^{n-2}x\,\sec^2 x\,dx = \int(1+\tan^2 x)^{n/2-1}\sec^2 x\,dx \\
&= \int\sec^2 x\left[\sum_{r=0}^{n/2-1}\binom{n/2-1}{r}\tan^{2r}x\right]dx = \sum_{r=0}^{n/2-1}\binom{n/2-1}{r}\int\tan^{2r}x\,\sec^2 x\,dx \\
&= \sum_{r=0}^{n/2-1}\binom{n/2-1}{r}\frac{\tan^{2r+1}x}{2r+1} + C
\end{aligned}$$

40. We integrate the functions in pairs:

$$\int_0^{2\pi}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}dx = \frac{1}{2\pi}\Big\{x\Big\}_0^{2\pi} = 1;$$
$$\int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{\pi}}\sin nx\right)dx = \frac{1}{\sqrt{2\pi}}\left\{-\frac{1}{n}\cos nx\right\}_0^{2\pi} = 0;$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{\pi}}\cos nx\right)dx = \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{n}\sin nx\right\}_0^{2\pi} = 0;$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{\pi}}\sin nx\right)\left(\frac{1}{\sqrt{\pi}}\cos mx\right)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}[\sin(n+m)x + \sin(n-m)x]\,dx$$

$$= \frac{1}{2\pi}\begin{cases}\left\{-\frac{1}{2n}\cos 2nx\right\}_0^{2\pi}, & m = n\\ \left\{\dfrac{-1}{n+m}\cos(n+m)x - \dfrac{1}{n-m}\cos(n-m)x\right\}_0^{2\pi}, & m \neq n\end{cases}$$

$$= 0;$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{\pi}}\sin nx\right)\left(\frac{1}{\sqrt{\pi}}\sin mx\right)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}[-\cos(n+m)x + \cos(n-m)x]\,dx$$

$$= \frac{1}{2\pi}\begin{cases}\left\{x - \dfrac{1}{2n}\sin 2nx\right\}_0^{2\pi}, & m = n\\ \left\{\dfrac{-1}{n+m}\sin(n+m)x + \dfrac{1}{n-m}\sin(n-m)x\right\}_0^{2\pi}, & m \neq n\end{cases}$$

$$= \begin{cases}1, & m = n\\ 0, & m \neq n\end{cases};$$

$$\int_0^{2\pi}\left(\frac{1}{\sqrt{\pi}}\cos nx\right)\left(\frac{1}{\sqrt{\pi}}\cos mx\right)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}[\cos(n+m)x + \cos(n-m)x]\,dx$$

$$= \frac{1}{2\pi}\begin{cases}\left\{x + \dfrac{1}{2n}\sin 2nx\right\}_0^{2\pi}, & m = n\\ \left\{\dfrac{1}{n+m}\sin(n+m)x + \dfrac{1}{n-m}\sin(n-m)x\right\}_0^{2\pi}, & m \neq n\end{cases}$$

$$= \begin{cases}1, & m = n\\ 0, & m \neq n\end{cases}.$$

EXERCISES 8.4

1. If we set $x = \sqrt{2}\sec\theta$, then $dx = \sqrt{2}\sec\theta\,\tan\theta\,d\theta$, and

$$\int\frac{1}{x\sqrt{2x^2-4}}dx = \int\frac{1}{\sqrt{2}\sec\theta\,2\tan\theta}\sqrt{2}\sec\theta\,\tan\theta\,d\theta = \frac{\theta}{2} + C = \frac{1}{2}\,\mathrm{Sec}^{-1}\left(\frac{x}{\sqrt{2}}\right) + C.$$

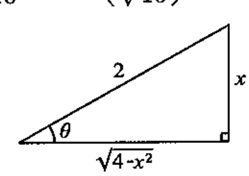
2. If we set $x = \dfrac{3}{\sqrt{5}}\sin\theta$, then $dx = \dfrac{3}{\sqrt{5}}\cos\theta\,d\theta$, and

$$\int\frac{1}{\sqrt{9-5x^2}}dx = \int\frac{1}{3\cos\theta}\left(\frac{3}{\sqrt{5}}\right)\cos\theta\,d\theta = \frac{\theta}{\sqrt{5}} + C = \frac{1}{\sqrt{5}}\,\mathrm{Sin}^{-1}\left(\frac{\sqrt{5}x}{3}\right) + C.$$

3. If we set $x = \sqrt{10}\tan\theta$, then $dx = \sqrt{10}\sec^2\theta\,d\theta$, and

$$\int\frac{1}{10+x^2}dx = \int\frac{1}{10\sec^2\theta}\sqrt{10}\sec^2\theta\,d\theta = \frac{\theta}{\sqrt{10}} + C = \frac{1}{\sqrt{10}}\,\mathrm{Tan}^{-1}\left(\frac{x}{\sqrt{10}}\right) + C.$$

4. If we set $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$, and

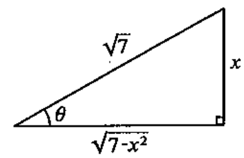


$$\int\frac{1}{x^2\sqrt{4-x^2}}dx = \int\frac{1}{4\sin^2\theta\,2\cos\theta}2\cos\theta\,d\theta = \frac{1}{4}\int\csc^2\theta\,d\theta$$

$$= -\frac{1}{4}\cot\theta + C = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C.$$

5. If we set $x = \sqrt{7}\sin\theta$, then $dx = \sqrt{7}\cos\theta\, d\theta$, and

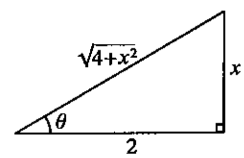
$$\begin{aligned}\int \sqrt{7 - x^2}\, dx &= \int \sqrt{7}\cos\theta\,\sqrt{7}\cos\theta\, d\theta = 7\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta \\ &= \frac{7}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{7\theta}{2} + \frac{7}{2}\sin\theta\,\cos\theta + C \\ &= \frac{7}{2}\,\mathrm{Sin}^{-1}\!\left(\frac{x}{\sqrt{7}}\right) + \frac{7}{2}\left(\frac{x}{\sqrt{7}}\right)\left(\frac{\sqrt{7-x^2}}{\sqrt{7}}\right) + C \\ &= \frac{7}{2}\,\mathrm{Sin}^{-1}\!\left(\frac{x}{\sqrt{7}}\right) + \frac{x}{2}\sqrt{7-x^2} + C.\end{aligned}$$



6. $\displaystyle\int x\sqrt{5x^2+3}\, dx = \frac{1}{15}(5x^2+3)^{3/2} + C$

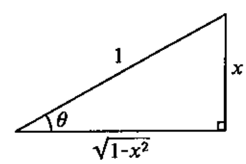
7. If we set $x = 2\tan\theta$, then $dx = 2\sec^2\theta\, d\theta$, and

$$\begin{aligned}\int x^3\sqrt{4+x^2}\, dx &= \int 8\tan^3\theta\, 2\sec\theta\, 2\sec^2\theta\, d\theta = 32\int \sec^3\theta(\sec^2\theta - 1)\tan\theta\, d\theta \\ &= 32\left(\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta\right) + C \\ &= \frac{32}{5}\left(\frac{\sqrt{4+x^2}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 + C \\ &= \frac{1}{5}(4+x^2)^{5/2} - \frac{4}{3}(4+x^2)^{3/2} + C.\end{aligned}$$



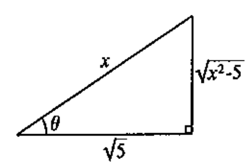
8. If we set $x = \sin\theta$, then $dx = \cos\theta\, d\theta$, and

$$\begin{aligned}\int \frac{1}{1-x^2} dx &= \int \frac{1}{\cos^2\theta}\cos\theta\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C = \ln\left|\frac{1+x}{\sqrt{(1-x)(1+x)}}\right| + C \\ &= \ln\left|\frac{\sqrt{1+x}}{\sqrt{1-x}}\right| + C = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C.\end{aligned}$$



9. If we set $x = \sqrt{5}\sec\theta$, then $dx = \sqrt{5}\sec\theta\,\tan\theta\, d\theta$, and

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-5}} dx &= \int \frac{1}{\sqrt{5}\tan\theta}\sqrt{5}\sec\theta\,\tan\theta\, d\theta = \int \sec\theta\, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{\sqrt{5}} + \frac{\sqrt{x^2-5}}{\sqrt{5}}\right| + C \\ &= \ln\left|x + \sqrt{x^2-5}\right| + D.\end{aligned}$$



10. $\displaystyle\int \frac{x+5}{10x^2+2} dx = \int \frac{x}{10x^2+2} dx + \frac{5}{2}\int \frac{1}{5x^2+1} dx$

In the second term we set $x = \dfrac{1}{\sqrt{5}}\tan\theta$ and $dx = \dfrac{1}{\sqrt{5}}\sec^2\theta\, d\theta$,

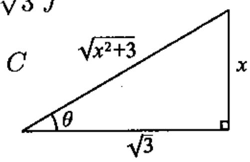
$$\begin{aligned}\int \frac{x+5}{10x^2+2} dx &= \frac{1}{20}\ln(10x^2+2) + \frac{5}{2}\int \frac{1}{\sec^2\theta}\left(\frac{1}{\sqrt{5}}\right)\sec^2\theta\, d\theta \\ &= \frac{1}{20}\ln(10x^2+2) + \frac{\sqrt{5}}{2}\theta + C = \frac{1}{20}\ln(5x^2+1) + \frac{\sqrt{5}}{2}\,\mathrm{Tan}^{-1}(\sqrt{5}x) + D.\end{aligned}$$

11. If we set $x = \sqrt{3}\tan\theta$, then $dx = \sqrt{3}\sec^2\theta\,d\theta$, and

$$\int \frac{1}{x\sqrt{x^2+3}}dx = \int \frac{1}{\sqrt{3}\tan\theta\,\sqrt{3}\sec\theta}\sqrt{3}\sec^2\theta\,d\theta = \frac{1}{\sqrt{3}}\int\frac{\sec\theta}{\tan\theta}d\theta = \frac{1}{\sqrt{3}}\int\csc\theta\,d\theta$$

$$= \frac{1}{\sqrt{3}}\ln|\csc\theta - \cot\theta| + C = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+3}}{x} - \frac{\sqrt{3}}{x}\right| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+3}-\sqrt{3}}{x}\right| + C.$$

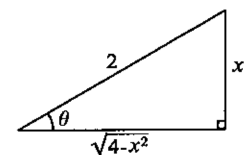


12. If we set $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$, and

$$\int\frac{\sqrt{4-x^2}}{x}dx = \int\frac{2\cos\theta}{2\sin\theta}2\cos\theta\,d\theta = 2\int\frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$= 2\int(\csc\theta - \sin\theta)\,d\theta = 2(\ln|\csc\theta - \cot\theta| + \cos\theta) + C$$

$$= 2\left(\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \frac{\sqrt{4-x^2}}{2}\right) + C = 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C.$$



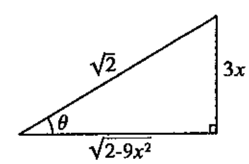
13. If we set $x = (\sqrt{2}/3)\sin\theta$, then $dx = (\sqrt{2}/3)\cos\theta\,d\theta$, and

$$\int\frac{x^2}{(2-9x^2)^{3/2}}dx = \int\frac{(2/9)\sin^2\theta}{2\sqrt{2}\cos^3\theta}\frac{\sqrt{2}}{3}\cos\theta\,d\theta = \frac{1}{27}\int\frac{\sin^2\theta}{\cos^2\theta}d\theta = \frac{1}{27}\int\tan^2\theta\,d\theta$$

$$= \frac{1}{27}\int(\sec^2\theta - 1)\,d\theta = \frac{1}{27}(\tan\theta - \theta) + C$$

$$= \frac{1}{27}\left(\frac{3x}{\sqrt{2-9x^2}}\right) - \frac{1}{27}\operatorname{Sin}^{-1}\left(\frac{3x}{\sqrt{2}}\right) + C$$

$$= \frac{x}{9\sqrt{2-9x^2}} - \frac{1}{27}\operatorname{Sin}^{-1}\left(\frac{3x}{\sqrt{2}}\right) + C.$$

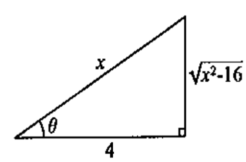


14. If we set $x = 4\sec\theta$, then $dx = 4\sec\theta\tan\theta\,d\theta$, and

$$\int\frac{\sqrt{x^2-16}}{x^2}dx = \int\frac{4\tan\theta}{16\sec^2\theta}4\sec\theta\tan\theta\,d\theta = \int\frac{\sec^2\theta - 1}{\sec\theta}d\theta$$

$$= \int(\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$= \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| - \frac{\sqrt{x^2-16}}{x} + C = \ln\left|x + \sqrt{x^2-16}\right| - \frac{\sqrt{x^2-16}}{x} + D.$$

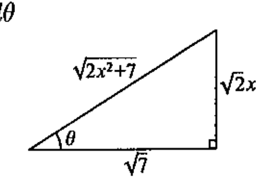


15. If we set $x = \sqrt{7/2}\tan\theta$, then $dx = \sqrt{7/2}\sec^2\theta\,d\theta$, and

$$\int\frac{1}{x^2\sqrt{2x^2+7}}dx = \int\frac{1}{(7/2)\tan^2\theta\sqrt{7}\sec\theta}\sqrt{\frac{7}{2}}\sec^2\theta\,d\theta = \frac{\sqrt{2}}{7}\int\frac{\sec\theta}{\tan^2\theta}d\theta$$

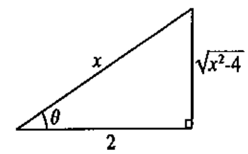
$$= \frac{\sqrt{2}}{7}\int\csc\theta\cot\theta\,d\theta = \frac{\sqrt{2}}{7}(-\csc\theta) + C$$

$$= -\frac{\sqrt{2}}{7}\left(\frac{\sqrt{2x^2+7}}{\sqrt{2}x}\right) + C = -\frac{\sqrt{2x^2+7}}{7x} + C.$$



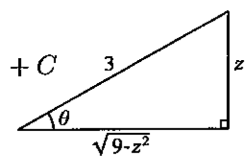
16. If we set $x = 2\sec\theta$, then $dx = 2\sec\theta\,\tan\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{x^3\sqrt{x^2-4}}dx &= \int \frac{1}{8\sec^3\theta\,2\tan\theta}2\sec\theta\,\tan\theta\,d\theta = \frac{1}{8}\int \cos^2\theta\,d\theta\\ &= \frac{1}{8}\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C\\ &= \frac{\theta}{16} + \frac{1}{16}\sin\theta\,\cos\theta + C\\ &= \frac{1}{16}\operatorname{Sec}^{-1}\left(\frac{x}{2}\right) + \frac{1}{16}\frac{\sqrt{x^2-4}}{x}\frac{2}{x} + C = \frac{1}{16}\operatorname{Sec}^{-1}\left(\frac{x}{2}\right) + \frac{\sqrt{x^2-4}}{8x^2} + C.\end{aligned}$$



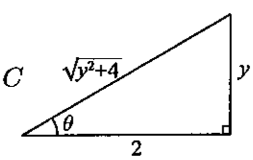
17. If we set $z = 3\sin\theta$, then $dz = 3\cos\theta\,d\theta$, and

$$\begin{aligned}\int \frac{\sqrt{9-z^2}}{z^4}dz &= \int \frac{3\cos\theta}{81\sin^4\theta}3\cos\theta\,d\theta = \frac{1}{9}\int\cot^2\theta\,\csc^2\theta\,d\theta = \frac{1}{9}\left(-\frac{1}{3}\cot^3\theta\right) + C\\ &= -\frac{1}{27}\left(\frac{\sqrt{9-z^2}}{z}\right)^3 + C = -\frac{(9-z^2)^{3/2}}{27z^3} + C.\end{aligned}$$



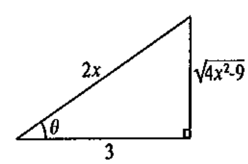
18. If we set $y = 2\tan\theta$, then $dy = 2\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{y^3}{\sqrt{y^2+4}}dy &= \int \frac{8\tan^3\theta}{2\sec\theta}2\sec^2\theta\,d\theta = 8\int \tan\theta\,(\sec^2\theta - 1)\,\sec\theta\,d\theta\\ &= 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C = \frac{8}{3}\left(\frac{\sqrt{y^2+4}}{2}\right)^3 - \frac{8\sqrt{y^2+4}}{2} + C\\ &= \frac{1}{3}(y^2+4)^{3/2} - 4\sqrt{y^2+4} + C.\end{aligned}$$



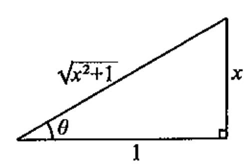
19. If we set $x = (3/2)\sec\theta$, then $dx = (3/2)\sec\theta\,\tan\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{(4x^2-9)^{3/2}}dx &= \int \frac{1}{27\tan^3\theta}\frac{3}{2}\sec\theta\,\tan\theta\,d\theta = \frac{1}{18}\int\frac{\sec\theta}{\tan^2\theta}d\theta\\ &= \frac{1}{18}\int\csc\theta\,\cot\theta\,d\theta = \frac{1}{18}(-\csc\theta) + C\\ &= -\frac{1}{18}\left(\frac{2x}{\sqrt{4x^2-9}}\right) + C = \frac{-x}{9\sqrt{4x^2-9}} + C.\end{aligned}$$



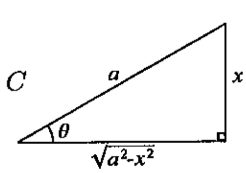
20. If we set $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{x^2+2}{x^3+x}dx &= \int \frac{x^2+2}{x(x^2+1)}dx = \int \frac{\tan^2\theta + 2}{\tan\theta\,\sec^2\theta}\sec^2\theta\,d\theta\\ &= \int(\tan\theta + 2\cot\theta)\,d\theta = \ln|\sec\theta| + 2\ln|\sin\theta| + C\\ &= \ln|\sqrt{x^2+1}| + 2\ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C = 2\ln|x| - \frac{1}{2}\ln(x^2+1) + C.\end{aligned}$$



21. If we set $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{a^2-x^2}dx &= \int \frac{1}{a^2\cos^2\theta}a\cos\theta\,d\theta = \frac{1}{a}\int\sec\theta\,d\theta = \frac{1}{a}\ln|\sec\theta + \tan\theta| + C\\ &= \frac{1}{a}\ln\left|\frac{a}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}}\right| + C = \frac{1}{a}\ln\left|\frac{a+x}{\sqrt{(a-x)(a+x)}}\right| + C\\ &= \frac{1}{a}\ln\left|\frac{\sqrt{a+x}}{\sqrt{a-x}}\right| + C = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C\end{aligned}$$

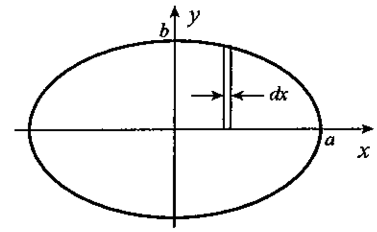


22. $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$

If we set $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$, and

$$A = \frac{4b}{a}\int_0^{\pi/2} a\cos\theta\, a\cos\theta\,d\theta = 4ab\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 2ab\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \pi ab.$$

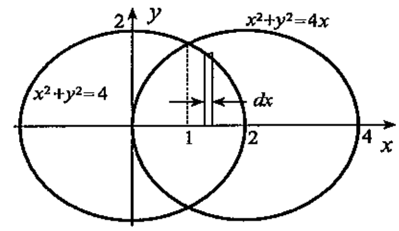


23. $A = 4\int_1^2 \sqrt{4 - x^2}\,dx$

If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$, then

$$A = 4\int_{\pi/6}^{\pi/2} 2\cos\theta\, 2\cos\theta\,d\theta = 16\int_{\pi/6}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 8\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\pi/6}^{\pi/2} = \frac{8\pi}{3} - 2\sqrt{3}.$$



24. A plot of the ellipse is shown to the right. If we solve

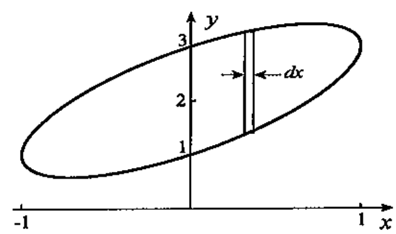
$$y^2 - (2x+4)y + (2x^2 + 4x + 3) = 0$$

for y in terms of x, we obtain

$$y = \frac{2x + 4 \pm \sqrt{(2x+4)^2 - 4(2x^2 + 4x + 3)}}{2}$$

$$= \frac{2x + 4 \pm \sqrt{4 - 4x^2}}{2}$$

$$= x + 2 \pm \sqrt{1 - x^2}.$$



It follows that a rectangle of width dx at position x between the top and bottom of the ellipse has area

$$[(x + 2 + \sqrt{1-x^2}) - (x + 2 - \sqrt{1-x^2})]dx = 2\sqrt{1-x^2}\,dx.$$

Since the ellipse extends from $x = -1$ to $x = 1$, its area must be $A = \int_{-1}^{1} 2\sqrt{1-x^2}\,dx.$

Setting $x = \sin\theta$ and $dx = \cos\theta\,d\theta$,

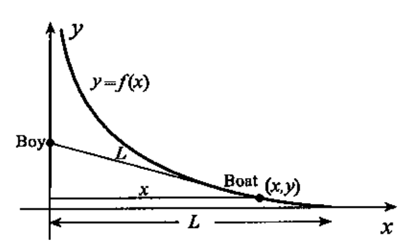
$$A = 2\int_{-\pi/2}^{\pi/2}\cos\theta\cos\theta\,d\theta = 2\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/2}^{\pi/2} = \pi.$$

25. (a) Since the slope of the curve is also the slope of the string,

$$\frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}.$$

(b) If we integrate with respect to x,

$$y = -\int \frac{\sqrt{L^2 - x^2}}{x}\,dx.$$



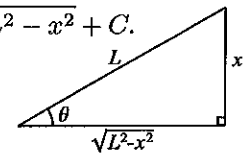
We now set $x = L\sin\theta$ and $dx = L\cos\theta\,d\theta$,

$$y = -\int \frac{L\cos\theta}{L\sin\theta}L\cos\theta\,d\theta = -L\int\frac{1-\sin^2\theta}{\sin\theta}d\theta = L\int(\sin\theta - \csc\theta)\,d\theta$$

$$= L[-\cos\theta - \ln|\csc\theta - \cot\theta|] + C = -L\left(\frac{\sqrt{L^2-x^2}}{L} + \ln\left|\frac{L}{x} - \frac{\sqrt{L^2-x^2}}{x}\right|\right) + C$$

$$= -\sqrt{L^2-x^2} - L\ln\left|\frac{L - \sqrt{L^2-x^2}}{x}\right| + C = -\sqrt{L^2-x^2} - L\ln\left|\frac{L-\sqrt{L^2-x^2}}{x}\frac{L+\sqrt{L^2-x^2}}{L+\sqrt{L^2-x^2}}\right| + C$$

$$= -\sqrt{L^2 - x^2} - L \ln\left|\frac{x}{L + \sqrt{L^2 - x^2}}\right| + C = L \ln\left|\frac{L + \sqrt{L^2 - x^2}}{x}\right| - \sqrt{L^2 - x^2} + C.$$



Since $y = 0$ when $x = L$, it follows that $C = 0$, and

$$y = L \ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right) - \sqrt{L^2 - x^2}.$$

26. $I = 2\int_0^{\tilde{y}} y^2(\sqrt{4 - y^2} - y^2)\, dy$

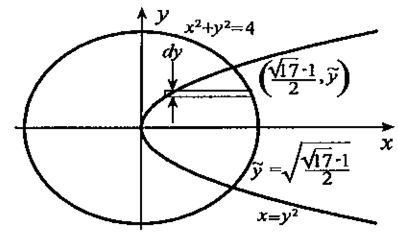
In the first integral we set $y = 2\sin\theta$ and $dy = 2\cos\theta\, d\theta$. If $\tilde{\theta} = \text{Sin}^{-1}(\tilde{y}/2)$, then

$$I = 2\int_0^{\tilde{\theta}} 4\sin^2\theta\, 2\cos\theta\, 2\cos\theta\, d\theta - 2\left\{\frac{y^5}{5}\right\}_0^{\tilde{y}}$$

$$= 32\int_0^{\tilde{\theta}} \sin^2\theta\, \cos^2\theta\, d\theta - \frac{2\tilde{y}^5}{5}$$

$$= 32\int_0^{\tilde{\theta}} \frac{1}{4}\sin^2 2\theta\, d\theta - \frac{2\tilde{y}^5}{5} = 8\int_0^{\tilde{\theta}} \left(\frac{1 - \cos 4\theta}{2}\right) d\theta - \frac{2\tilde{y}^5}{5}$$

$$= 4\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\tilde{\theta}} - \frac{2\tilde{y}^5}{5} = 4\tilde{\theta} - \sin 4\tilde{\theta} - \frac{2\tilde{y}^5}{5} = 1.053.$$



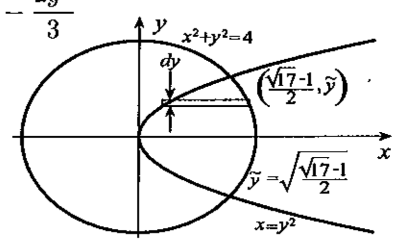
27. By symmetry, $\overline{y} = 0$. The area is $A = 2\int_0^{\tilde{y}} (\sqrt{4 - y^2} - y^2)\, dy$.

We set $y = 2\sin\theta$ and $dy = 2\cos\theta\, d\theta$ in the first term. If $\tilde{\theta} = \text{Sin}^{-1}(\tilde{y}/2)$, then

$$A = 2\int_0^{\tilde{\theta}} 2\cos\theta\, 2\cos\theta\, d\theta - 2\left\{\frac{y^3}{3}\right\}_0^{\tilde{y}} = 8\int_0^{\tilde{\theta}} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta - \frac{2\tilde{y}^3}{3}$$

$$= 4\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\tilde{\theta}} - \frac{2\tilde{y}^3}{3}$$

$$= 4\tilde{\theta} + 2\sin 2\tilde{\theta} - \frac{2\tilde{y}^3}{3} = 3.3500.$$



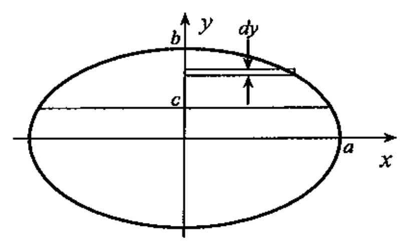
Since

$$A\overline{x} = 2\int_0^{\tilde{y}} \frac{1}{2}(\sqrt{4 - y^2} + y^2)(\sqrt{4 - y^2} - y^2)\, dy = \int_0^{\tilde{y}} (4 - y^2 - y^4)\, dy = \left\{4y - \frac{y^3}{3} - \frac{y^5}{5}\right\}_0^{\tilde{y}} = 3.7386,$$

it follows that $\overline{x} = 3.7386/3.3500 = 1.116$.

28. According to Exercise 22, the area of the ellipse is πab. If we let the required line be $y = c$, then c must satisfy the equation

$$\frac{\pi ab}{3} = 2\int_c^b \frac{a}{b}\sqrt{b^2 - y^2}\, dy.$$



We let $y = b\sin\theta$ and $dy = b\cos\theta\, d\theta$. If $\tilde{\theta} = \text{Sin}^{-1}(c/b)$, then

$$\frac{\pi ab}{3} = \frac{2a}{b}\int_{\tilde{\theta}}^{\pi/2} b\cos\theta\, b\cos\theta\, d\theta$$

$$= 2ab\int_{\tilde{\theta}}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta = ab\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\tilde{\theta}}^{\pi/2} = ab\left(\frac{\pi}{2} - \tilde{\theta} - \frac{1}{2}\sin 2\tilde{\theta}\right).$$

Thus, $\tilde{\theta}$ must satisfy the equation $\dfrac{\pi}{3} = \dfrac{\pi}{2} - \tilde{\theta} - \dfrac{1}{2}\sin 2\tilde{\theta}$, or, $6\tilde{\theta} + 3\sin 2\tilde{\theta} - \pi = 0$. Newton's iterative procedure with $\tilde{\theta}_1 = 0.25$, $\tilde{\theta}_{n+1} = \tilde{\theta}_n - \dfrac{6\tilde{\theta}_n + 3\sin 2\tilde{\theta}_n - \pi}{6 + 6\cos 2\tilde{\theta}_n}$ gives the iterations $\tilde{\theta}_2 = 0.268$, $\tilde{\theta}_3 = 0.268\,133$, $\tilde{\theta}_4 = 0.268\,133$. Hence, the required line is $y = b\sin\tilde{\theta} = 0.265b$.

29. We find the moment of inertia about the line $x = 2$, $\quad I = 2\int_{-2}^{2}(x-2)^2\rho\sqrt{4-x^2}\,dx.$

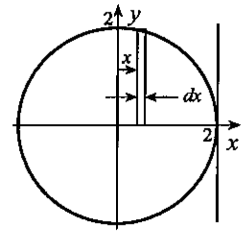
If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

$$I = 2\rho\int_{-\pi/2}^{\pi/2}(2\sin\theta - 2)^2\, 2\cos\theta\, 2\cos\theta\,d\theta = 32\rho\int_{-\pi/2}^{\pi/2}(\sin^2\theta - 2\sin\theta + 1)\cos^2\theta\,d\theta$$

$$= 32\rho\int_{-\pi/2}^{\pi/2}\left[\left(\frac{\sin 2\theta}{2}\right)^2 - 2\cos^2\theta\sin\theta + \cos^2\theta\right]d\theta$$

$$= 8\rho\int_{-\pi/2}^{\pi/2}\left[\frac{1-\cos 4\theta}{2} - 8\cos^2\theta\sin\theta + 2(1+\cos 2\theta)\right]d\theta$$

$$= 8\rho\left\{\frac{5\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{8}{3}\cos^3\theta + \sin 2\theta\right\}_{-\pi/2}^{\pi/2} = 20\rho\pi.$$



30. $F = \int_{-r}^{r} 9.81\rho(r-y)2x\,dy = 19.62\rho\int_{-r}^{r}(r-y)\sqrt{r^2-y^2}\,dy$

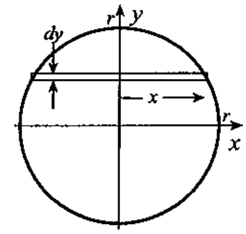
If we set $y = r\sin\theta$, then $dy = r\cos\theta\,d\theta$, and

$$F = 19.62\rho\int_{-\pi/2}^{\pi/2}(r - r\sin\theta)r\cos\theta\, r\cos\theta\,d\theta$$

$$= 19.62\rho r^3\int_{-\pi/2}^{\pi/2}(1-\sin\theta)\cos^2\theta\,d\theta$$

$$= 19.62\rho r^3\int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}(1+\cos 2\theta) - \cos^2\theta\sin\theta\right]d\theta$$

$$= 19.62\rho r^3\left\{\frac{\theta}{2} + \frac{1}{4}\sin 2\theta + \frac{1}{3}\cos^3\theta\right\}_{-\pi/2}^{\pi/2} = 9.81\pi\rho r^3.$$



31. It is advantageous first to divide, $\int\frac{2x^4-x^2}{2x^2+1}dx = \int\left(x^2 - 1 + \frac{1}{2x^2+1}\right)dx$. If we set $x = (1/\sqrt{2})\tan\theta$ and $dx = (1/\sqrt{2})\sec^2\theta\,d\theta$ in the last term,

$$\int\frac{2x^4-x^2}{2x^2+1}dx = \frac{x^3}{3} - x + \int\frac{1}{\sec^2\theta}\frac{1}{\sqrt{2}}\sec^2\theta\,d\theta = \frac{x^3}{3} - x + \frac{\theta}{\sqrt{2}} + C = \frac{x^3}{3} - x + \frac{1}{\sqrt{2}}\mathrm{Tan}^{-1}(\sqrt{2}x) + C.$$

32. If we set $x = \sqrt{7}\sin\theta$, then $dx = \sqrt{7}\cos\theta\,d\theta$, and

$$\int(7-x^2)^{3/2}\,dx = \int 7\sqrt{7}\cos^3\theta\,\sqrt{7}\cos\theta\,d\theta = 49\int\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{49}{4}\int\left[1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right]d\theta$$

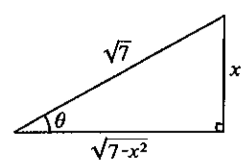
$$= \frac{49}{4}\left(\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right) + C$$

$$= \frac{147}{8}\theta + \frac{49}{2}\sin\theta\cos\theta + \frac{49}{16}\sin 2\theta\cos 2\theta + C$$

$$= \frac{147}{8}\theta + \frac{49}{2}\sin\theta\cos\theta + \frac{49}{8}\sin\theta\cos\theta(1-2\sin^2\theta) + C$$

$$= \frac{147}{8}\mathrm{Sin}^{-1}\left(\frac{x}{\sqrt{7}}\right) + \frac{245}{8}\frac{x}{\sqrt{7}}\frac{\sqrt{7-x^2}}{\sqrt{7}} - \frac{49}{4}\left(\frac{x}{\sqrt{7}}\right)^3\frac{\sqrt{7-x^2}}{\sqrt{7}} + C$$

$$= \frac{147}{8}\mathrm{Sin}^{-1}\left(\frac{x}{\sqrt{7}}\right) + \frac{x}{8}(35-2x^2)\sqrt{7-x^2} + C.$$



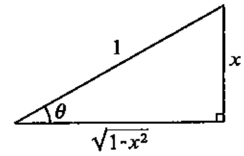
33. If we set $x = \sin\theta$ and $dx = \cos\theta\, d\theta$, then

$$\int \frac{1}{x - x^3}dx = \int \frac{1}{x(1-x^2)}dx = \int \frac{1}{\sin\theta\,\cos^2\theta}\cos\theta\, d\theta = \int \frac{1}{\sin\theta\,\cos\theta}d\theta$$

$$= \int 2\,\csc 2\theta\, d\theta = \ln|\csc 2\theta - \cot 2\theta| + C$$

$$= \ln\left|\frac{1-\cos 2\theta}{\sin 2\theta}\right| + C = \ln\left|\frac{1-(1-2\,\sin^2\theta)}{2\,\sin\theta\,\cos\theta}\right| + C$$

$$= \ln|\tan\theta| + C = \ln\left|\frac{x}{\sqrt{1-x^2}}\right| + C.$$



34. If we set $x = (1/2)\sec\theta$, then $dx = (1/2)\sec\theta\,\tan\theta\, d\theta$, and

$$\int \frac{1}{x^3(4x^2-1)^{3/2}}dx = \int \frac{1}{(1/8)\sec^3\theta\,\tan^3\theta}(1/2)\sec\theta\,\tan\theta\, d\theta = 4\int \frac{\cos^4\theta}{\sin^2\theta}d\theta$$

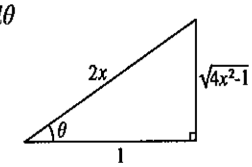
$$= 4\int \frac{\cos^2\theta(1-\sin^2\theta)}{\sin^2\theta}d\theta = 4\int(\cot^2\theta - \cos^2\theta)\, d\theta$$

$$= 4\int\left(\csc^2\theta - 1 - \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 4\left(-\cot\theta - \frac{3\theta}{2} - \frac{1}{4}\sin 2\theta\right) + C$$

$$= 4\left[-\frac{1}{\sqrt{4x^2-1}} - \frac{3}{2}\,\mathrm{Sec}^{-1}(2x) - \frac{1}{2}\frac{\sqrt{4x^2-1}}{2x}\frac{1}{2x}\right] + C$$

$$= -6\,\mathrm{Sec}^{-1}(2x) + \frac{1-12x^2}{2x^2\sqrt{4x^2-1}} + C.$$



35. If we set $x = 2\,\sec\theta$ and $dx = 2\,\sec\theta\,\tan\theta\, d\theta$, then

$$\int\sqrt{x^2-4}\,dx = \int 2\,\tan\theta\, 2\,\sec\theta\,\tan\theta\, d\theta = 4\int\tan^2\theta\,\sec\theta\, d\theta$$

$$= 4\int(\sec^2\theta - 1)\sec\theta\, d\theta = 4\int(\sec^3\theta - \sec\theta)\, d\theta.$$

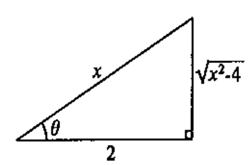
We use Example 8.9 to write

$$\int\sqrt{x^2-4}\,dx = 2\,\sec\theta\,\tan\theta + 2\ln|\sec\theta + \tan\theta| - 4\ln|\sec\theta + \tan\theta| + C$$

$$= 2\,\sec\theta\,\tan\theta - 2\ln|\sec\theta + \tan\theta| + C$$

$$= 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{x^2-4}}{2}\right) - 2\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

$$= \frac{x\sqrt{x^2-4}}{2} - 2\ln|x + \sqrt{x^2-4}| + D.$$



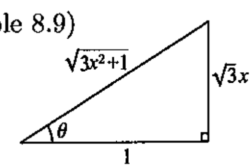
36. If we set $x = (1/\sqrt{3})\tan\theta$, then $dx = (1/\sqrt{3})\sec^2\theta\, d\theta$, and

$$\int\sqrt{1+3x^2}\,dx = \int\sec\theta\left(\frac{1}{\sqrt{3}}\right)\sec^2\theta\, d\theta = \frac{1}{\sqrt{3}}\int\sec^3\theta\, d\theta$$

$$= \frac{1}{2\sqrt{3}}(\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|) + C \quad \text{(using Example 8.9)}$$

$$= \frac{1}{2\sqrt{3}}(\sqrt{1+3x^2}\,\sqrt{3}x + \ln|\sqrt{1+3x^2} + \sqrt{3}x|) + C$$

$$= \frac{x}{2}\sqrt{1+3x^2} + \frac{1}{2\sqrt{3}}\ln|\sqrt{1+3x^2} + \sqrt{3}x| + C.$$

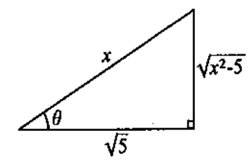


37. If we set $x = \sqrt{5}\sec\theta$ and $dx = \sqrt{5}\sec\theta\,\tan\theta\,d\theta$, then

$$\int \frac{x^2}{\sqrt{x^2-5}}\,dx = \int \frac{5\sec^2\theta}{\sqrt{5}\,\tan\theta}\sqrt{5}\sec\theta\,\tan\theta\,d\theta = 5\int \sec^3\theta\,d\theta.$$

We use Example 8.9 to write

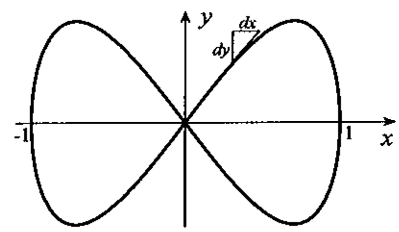
$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-5}}\,dx &= \frac{5}{2}\sec\theta\,\tan\theta + \frac{5}{2}\ln|\sec\theta+\tan\theta| + C\\ &= \frac{5}{2}\left(\frac{x}{\sqrt{5}}\right)\left(\frac{\sqrt{x^2-5}}{\sqrt{5}}\right) + \frac{5}{2}\ln\left|\frac{x}{\sqrt{5}} + \frac{\sqrt{x^2-5}}{\sqrt{5}}\right| + C\\ &= \frac{x\sqrt{x^2-5}}{2} + \frac{5}{2}\ln|x+\sqrt{x^2-5}| + D.\end{aligned}$$



38. Differentiation of $8y^2 = x^2 - x^4$ gives

$$16y\frac{dy}{dx} = 2x - 4x^3.$$

Therefore, $\dfrac{dy}{dx} = \dfrac{x-2x^3}{8y}$. Small lengths along that part of the curve in the first quadrant are approximated by



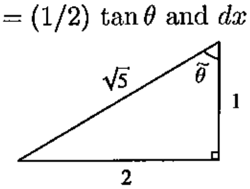
$$\begin{aligned}\sqrt{1+\left(\frac{x-2x^3}{8y}\right)^2}\,dx &= \sqrt{1+\frac{(x-2x^3)^2}{64y^2}}\,dx = \sqrt{1+\frac{x^2(1-2x^2)^2}{8x^2(1-x^2)}}\,dx\\ &= \sqrt{\frac{9-12x^2+4x^4}{8(1-x^2)}}\,dx = \frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\,dx.\end{aligned}$$

The length of the curve is therefore $L = 4\displaystyle\int_0^1 \frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\,dx$. When we set $x = \sin\theta$, and $dx = \cos\theta\,d\theta$,

$$L = \sqrt{2}\int_0^{\pi/2}\frac{3-2\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \sqrt{2}\int_0^{\pi/2}[3-(1-\cos 2\theta)]\,d\theta = \sqrt{2}\left\{2\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \sqrt{2}\pi.$$

39. The length of the parabola is $L = \displaystyle\int_0^1 \sqrt{1+4x^2}\,dx$. We set $x = (1/2)\tan\theta$ and $dx = (1/2)\sec^2\theta\,d\theta$. If $\tilde{\theta} = \mathrm{Tan}^{-1}2$, then

$$L = \int_0^{\tilde{\theta}} \sec\theta\frac{1}{2}\sec^2\theta\,d\theta = \frac{1}{2}\int_0^{\tilde{\theta}}\sec^3\theta\,d\theta.$$

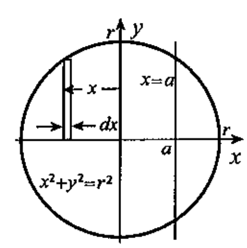


We now use Example 8.9 to write

$$L = \frac{1}{4}\left\{\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\right\}_0^{\tilde{\theta}} = \frac{1}{4}(\sec\tilde{\theta}\,\tan\tilde{\theta} + \ln|\sec\tilde{\theta}+\tan\tilde{\theta}|) = \frac{1}{4}[2\sqrt{5}+\ln(2+\sqrt{5})].$$

40. Let the radius of the circle be r, and let the position of the line be denoted by $x = a$. Then the requirement that the second moment of area about $x = a$ be twice that about $x = 0$ can be expressed as

$$\begin{aligned}2\int_{-r}^{r}(x-a)^2\sqrt{r^2-x^2}\,dx\\ = 2(2)\int_{-r}^{r}x^2\sqrt{r^2-x^2}\,dx.\end{aligned}$$



If we set $x = r\sin\theta$ and $dx = r\cos\theta\,d\theta$ in these integrals, then

$$\int_{-\pi/2}^{\pi/2}(r\sin\theta-a)^2 r\cos\theta\, r\cos\theta\, d\theta = 2\int_{-\pi/2}^{\pi/2} r^2\sin^2\theta\, r\cos\theta\, r\cos\theta\, d\theta,$$

or,

$$\begin{aligned}
0 &= r^2\int_{-\pi/2}^{\pi/2}(2r^2\sin^2\theta\,\cos^2\theta - r^2\sin^2\theta\,\cos^2\theta + 2ar\cos^2\theta\,\sin\theta - a^2\cos^2\theta)\,d\theta\\
&= r^2\int_{-\pi/2}^{\pi/2}\left(\frac{r^2}{4}\sin^2 2\theta + 2ar\cos^2\theta\,\sin\theta - a^2\cos^2\theta\right)d\theta\\
&= r^2\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{4}\left(\frac{1-\cos 4\theta}{2}\right) + 2ar\cos^2\theta\,\sin\theta - a^2\left(\frac{1+\cos 2\theta}{2}\right)\right]d\theta\\
&= r^2\left\{\frac{r^2}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right) - \frac{2ar}{3}\cos^3\theta - \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right\}_{-\pi/2}^{\pi/2}\\
&= r^2\left(\frac{\pi r^2}{8} - \frac{\pi a^2}{2}\right).
\end{aligned}$$

Thus, $a = r/2$.

41. If we substitute $K(x) = k/(cx^2+1)$,

$$z(x) = -\int \frac{1}{1+\dfrac{V(cx^2+1)}{k}}dx = -k\int\frac{1}{(k+V)+Vcx^2}dx.$$

If we set $x = \sqrt{\dfrac{k+V}{cV}}\tan\theta$ and $dx = \sqrt{\dfrac{k+V}{cV}}\sec^2\theta\,d\theta$, then

$$z(x) = -k\int\frac{\sqrt{\dfrac{k+V}{cV}}\sec^2\theta}{(k+V)\sec^2\theta}d\theta = -\frac{k}{\sqrt{Vc(k+V)}}\theta + C = \frac{-k}{\sqrt{Vc(k+V)}}\mathrm{Tan}^{-1}\sqrt{\frac{Vc}{k+V}}x + C.$$

When $z(0) = H$, we find that $H = C$, and $z(x) = H - \dfrac{k}{\sqrt{Vc(k+V)}}\mathrm{Tan}^{-1}\sqrt{\dfrac{Vc}{k+V}}x$. On other hand, when $z(H_w - L) = L$,

$$L = \frac{-k}{\sqrt{Vc(k+L)}}\mathrm{Tan}^{-1}\sqrt{\frac{Vc}{k+V}}(H_w - L) + C \quad\Longrightarrow\quad C = L + \frac{k}{\sqrt{Vc(k+L)}}\mathrm{Tan}^{-1}\sqrt{\frac{Vc}{k+V}}(H_w - L).$$

Thus, $z(x) = L + \dfrac{k}{\sqrt{Vc(k+L)}}\mathrm{Tan}^{-1}\sqrt{\dfrac{Vc}{k+V}}(H_w - L) - \dfrac{k}{\sqrt{Vc(k+V)}}\mathrm{Tan}^{-1}\sqrt{\dfrac{Vc}{k+V}}x.$

42. (a) If we set $p = \tan\theta$ and $dp = \sec^2\theta\,d\theta$, then

$$kx + C = \int\frac{1}{\sec\theta}\sec^2\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1+p^2}+p|.$$

Exponentiation gives $\sqrt{1+p^2}+p = De^{kx}$, where $D = e^C$. Since $p(0) = f'(0) = 0$, we obtain $D = 1$. Hence,

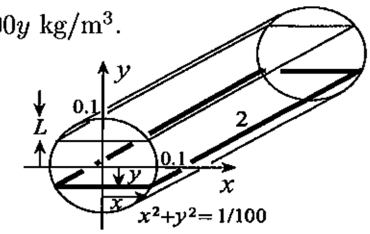
$$\sqrt{1+p^2} = e^{kx} - p \quad\Longrightarrow\quad 1 + p^2 = e^{2kx} - 2pe^{kx} + p^2.$$

This can be solved for $\quad p = \dfrac{dy}{dx} = \dfrac{e^{2kx}-1}{2e^{kx}} = \dfrac{1}{2}(e^{kx} - e^{-kx}).$

(b) Integration now yields $y = \dfrac{1}{2k}(e^{kx} + e^{-kx}) + C.$

43. Suppose L denotes the height of log above water. The density of the log is

$$\rho(y) = 1000 - \frac{500}{0.2}(y+0.1) = 750 - 2500y \text{ kg/m}^3.$$



The weight of the log is

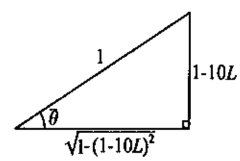
$$\begin{aligned} W_{\text{log}} &= \int_{-0.1}^{0.1} (750-2500y)g(2x)(2)\,dy \\ &= 1000g\int_{-0.1}^{0.1}(3-10y)\sqrt{1/100-y^2}\,dy \\ &= 3000g\int_{-0.1}^{0.1}\sqrt{1/100-y^2}\,dy + 10\,000g\left\{\frac{1}{3}\left(\frac{1}{100}-y^2\right)^{3/2}\right\}_{-0.1}^{0.1}. \end{aligned}$$

Since the integral represents half the area of the end of the log, $W = 3000g(1/2)\pi(1/100) = 15\pi g$ N. The weight of the water displaced by the log is

$$W_{\text{water}} = \int_{-1/10}^{1/10-L} 1000g(2x)(2)\,dy = 4000g\int_{-1/10}^{1/10-L}\sqrt{1/100-y^2}\,dy.$$

If we set $y = (1/10)\sin\theta$ and $dy = (1/10)\cos\theta d\theta$, then

$$\begin{aligned} W_{\text{water}} &= 4000g\int_{-\pi/2}^{\overline{\theta}}\frac{1}{10}\cos\theta\frac{1}{10}\cos\theta\,d\theta \qquad \text{where } \overline{\theta} = \operatorname{Sin}^{-1}(1-10L) \\ &= 40g\int_{-\pi/2}^{\overline{\theta}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 20g\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/2}^{\overline{\theta}} \\ &= 20g\left\{\theta + \sin\theta\cos\theta\right\}_{-\pi/2}^{\overline{\theta}} = 20g\left(\overline{\theta} + \sin\overline{\theta}\cos\overline{\theta} + \frac{\pi}{2}\right) \\ &= 20g[\operatorname{Sin}^{-1}(1-10L) + (1-10L)\sqrt{20L-100L^2}] \text{ N}. \end{aligned}$$



Archimedes' principle requires $W_{\text{log}} = W_{\text{water}}$ so that

$$15\pi g = 20g[\operatorname{Sin}^{-1}(1-10L) + (1-10L)\sqrt{20L-100L^2}].$$

Instead of solving this equation for L, we return to the expression for W_{water} in terms of $\overline{\theta}$, drop the overbars, and equate

$$15\pi g = 20g\left(\theta + \frac{1}{2}\sin 2\theta + \frac{\pi}{2}\right) \quad\Longrightarrow\quad 4\theta + 2\sin 2\theta - \pi = 0.$$

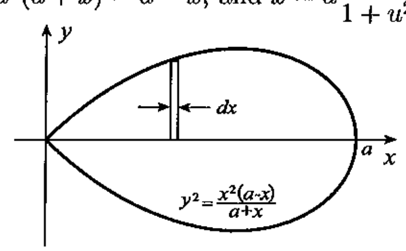
We use Newton's iterative procedure to solve this equation numerically,

$$\theta_1 = 0.5, \qquad \theta_{n+1} = \theta_n - \frac{4\theta_n + 2\sin 2\theta_n - \pi}{4 + 4\cos 2\theta_n}.$$

Iteration gives $\theta_2 = 0.412$, $\theta_3 = 0.415\,849$, $\theta_4 = 0.415\,856$, $\theta_5 = 0.415\,856$. Using $\theta = \overline{\theta} = 0.415\,856$ in $\overline{\theta} = \operatorname{Sin}^{-1}(1-10L)$, we obtain $L = (1-\sin\overline{\theta})/10 = 0.06$; that is, only 6 cm of the log is above water.

44. $A = 2\int_0^a x\sqrt{\dfrac{a-x}{a+x}}\,dx$ If we set $u = \sqrt{(a-x)/(a+x)}$, then $u^2(a+x) = a-x$, and $x = a\dfrac{1-u^2}{1+u^2}$.

Thus, $$\begin{aligned} dx &= a\frac{(1+u^2)(-2u)-(1-u^2)(2u)}{(1+u^2)^2}du \\ &= \frac{-4au}{(1+u^2)^2}du, \end{aligned}$$



and

$$A = 2\int_1^0 \frac{a(1-u^2)}{1+u^2}u\frac{-4au}{(1+u^2)^2}du = 8a^2\int_0^1\frac{u^2(1-u^2)}{(1+u^2)^3}du.$$

We now set $u = \tan\theta$, and $du = \sec^2\theta\, d\theta$,

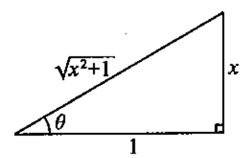
$$\begin{aligned}
A &= 8a^2 \int_0^{\pi/4} \frac{\tan^2\theta\,(1-\tan^2\theta)}{\sec^6\theta} \sec^2\theta\, d\theta = 8a^2 \int_0^{\pi/4} (\sin^2\theta\, \cos^2\theta - \sin^4\theta)\, d\theta \\
&= 8a^2 \int_0^{\pi/4} \left[\frac{\sin^2 2\theta}{4} - \left(\frac{1-\cos 2\theta}{2}\right)^2\right] d\theta \\
&= 2a^2 \int_0^{\pi/4} \left[\frac{1}{2}(1-\cos 4\theta) - 1 + 2\cos 2\theta - \frac{1}{2}(1+\cos 4\theta)\right] d\theta \\
&= 2a^2 \left\{-\theta + \sin 2\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/4} = \frac{a^2(4-\pi)}{2}.
\end{aligned}$$

45. (a) $\dfrac{1}{x+1+\sqrt{x^2+1}} = \dfrac{1}{x+1+\sqrt{x^2+1}} \dfrac{x+1-\sqrt{x^2+1}}{x+1-\sqrt{x^2+1}} = \dfrac{x+1-\sqrt{x^2+1}}{2x}$

(b) $\displaystyle\int_0^1 \frac{1}{x+1+\sqrt{x^2+1}} dx = \int_0^1 \frac{x+1-\sqrt{x^2+1}}{2x} dx = \frac{1}{2}\int_0^1 \left(1 + \frac{1}{x} - \frac{\sqrt{x^2+1}}{x}\right) dx$

In the last term we set $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$,

$$\begin{aligned}
\int \frac{\sqrt{x^2+1}}{x} dx &= \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\, d\theta = \int \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta = \int (\csc\theta + \tan\theta\, \sec\theta)\, d\theta \\
&= \ln|\csc\theta - \cot\theta| + \sec\theta + C \\
&= \ln\left|\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}\right| + \sqrt{x^2+1} + C.
\end{aligned}$$



Thus,

$$\begin{aligned}
\int_0^1 \frac{1}{x+1+\sqrt{x^2+1}} dx &= \frac{1}{2}\left\{x + \ln|x| - \ln|\sqrt{x^2+1} - 1| + \ln|x| - \sqrt{x^2+1}\right\}_0^1 \\
&= \frac{1}{2}\left\{x - \sqrt{x^2+1} + \ln\left(\frac{x^2}{\sqrt{x^2+1}-1}\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}\right)\right\}_0^1 \\
&= \frac{1}{2}\left\{x - \sqrt{x^2+1} + \ln(\sqrt{x^2+1}+1)\right\}_0^1 = 1 - \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\frac{1+\sqrt{2}}{2}\right).
\end{aligned}$$

46. If we set $u = \sqrt{(1+x)/(1-x)}$, then $(1-x)u^2 = 1+x$, and $x = (u^2-1)/(u^2+1)$. Thus,

$$dx = \frac{(u^2+1)(2u) - (u^2-1)(2u)}{(u^2+1)^2} du = \frac{4u}{(u^2+1)^2} du,$$

and

$$\int_{-1}^1 \sqrt{\frac{1+x}{1-x}} dx = \int_0^\infty \frac{u(4u)}{(u^2+1)^2} du.$$

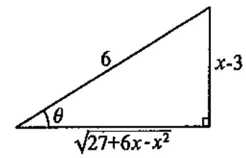
We now set $u = \tan\theta$ and $du = \sec^2\theta\, d\theta$,

$$\begin{aligned}
\int_{-1}^1 \sqrt{\frac{1+x}{1-x}} dx &= 4\int_0^{\pi/2} \frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\, d\theta = 4\int_0^{\pi/2} \sin^2\theta\, d\theta = 4\int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)\, d\theta \\
&= 2\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \pi.
\end{aligned}$$

EXERCISES 8.5

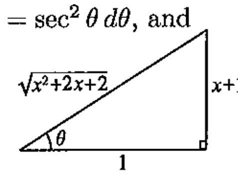
1. Since $27+6x-x^2 = 36-(x-3)^2$, we set $x-3 = 6\sin\theta$, in which case $dx = 6\cos\theta\,d\theta$, and

$$\begin{aligned}\int \frac{x}{\sqrt{27+6x-x^2}}dx &= \int \frac{x}{\sqrt{36-(x-3)^2}}dx = \int \frac{3+6\sin\theta}{6\cos\theta}6\cos\theta\,d\theta \\ &= 3\theta - 6\cos\theta + C \\ &= 3\,\mathrm{Sin}^{-1}\left(\frac{x-3}{6}\right) - \sqrt{27+6x-x^2} + C.\end{aligned}$$



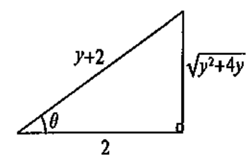
2. Since $x^2+2x+2 = (x+1)^2+1$, we set $x+1 = \tan\theta$, in which case $dx = \sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2x+2}}dx &= \int \frac{1}{\sec\theta}\sec^2\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|\sqrt{x^2+2x+2} + x + 1| + C.\end{aligned}$$



3. Since $y^2+4y = (y+2)^2-4$, we set $y+2 = 2\sec\theta$, in which case $dy = 2\sec\theta\tan\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{(y^2+4y)^{3/2}}dy &= \int \frac{1}{[(y+2)-4]^{3/2}}dy = \int \frac{1}{8\tan^3\theta}2\sec\theta\tan\theta\,d\theta \\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{4}\int \csc\theta\cot\theta\,d\theta \\ &= \frac{1}{4}(-\csc\theta) + C = -\frac{y+2}{4\sqrt{y^2+4y}} + C.\end{aligned}$$

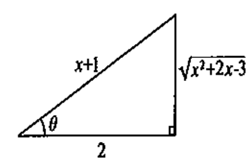


4. Since $-x^2+3x-4 = -(x-3/2)^2-7/4$, we set $x-3/2 = (\sqrt{7}/2)\tan\theta$, in which case $dx = (\sqrt{7}/2)\sec^2\theta\,d\theta$, and

$$\int \frac{1}{3x-x^2-4}dx = \int \frac{1}{-(7/4)\sec^2\theta}(\sqrt{7}/2)\sec^2\theta\,d\theta = -\frac{2}{\sqrt{7}}\theta + C = -\frac{2}{\sqrt{7}}\,\mathrm{Tan}^{-1}\left(\frac{2x-3}{\sqrt{7}}\right) + C.$$

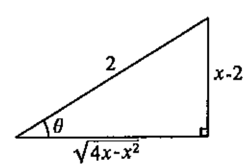
5. Since $x^2+2x-3 = (x+1)^2-4$, we set $x+1 = 2\sec\theta$, in which case $dx = 2\sec\theta\tan\theta\,d\theta$, and

$$\begin{aligned}\int \frac{\sqrt{x^2+2x-3}}{x+1}dx &= \int \frac{\sqrt{(x+1)^2-4}}{x+1}dx = \int \frac{2\tan\theta}{2\sec\theta}2\sec\theta\tan\theta\,d\theta \\ &= 2\int \tan^2\theta\,d\theta = 2\int(\sec^2\theta - 1)\,d\theta = 2(\tan\theta - \theta) + C \\ &= \sqrt{x^2+2x-3} - 2\,\mathrm{Sec}^{-1}\left(\frac{x+1}{2}\right) + C.\end{aligned}$$



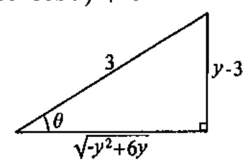
6. Since $4x-x^2 = -(x-2)^2+4$, we set $x-2 = 2\sin\theta$, in which case $dx = 2\cos\theta\,d\theta$, and

$$\begin{aligned}\int \frac{x}{(4x-x^2)^{3/2}}dx &= \int \frac{2+2\sin\theta}{8\cos^3\theta}2\cos\theta\,d\theta = \frac{1}{2}\int \frac{1+\sin\theta}{\cos^2\theta}d\theta \\ &= \frac{1}{2}\int\left(\sec^2\theta + \frac{\sin\theta}{\cos^2\theta}\right)d\theta = \frac{1}{2}\left(\tan\theta + \frac{1}{\cos\theta}\right) + C \\ &= \frac{1}{2}\left(\frac{x-2}{\sqrt{4x-x^2}} + \frac{2}{\sqrt{4x-x^2}}\right) + C \\ &= \frac{x}{2\sqrt{4x-x^2}} + C.\end{aligned}$$



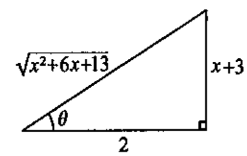
7. Since $-y^2+6y = 9-(y-3)^2$, we set $y-3 = 3\sin\theta$, in which case $dy = 3\cos\theta\,d\theta$, and

$$\begin{aligned}\int\sqrt{-y^2+6y}\,dy &= \int\sqrt{9-(y-3)^2}\,dy = \int 3\cos\theta\,3\cos\theta\,d\theta\\ &= 9\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{9}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C = \frac{9}{2}(\theta+\sin\theta\cos\theta)+C\\ &= \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{y-3}{3}\right)+\frac{9}{2}\left(\frac{y-3}{3}\right)\left(\frac{\sqrt{-y^2+6y}}{3}\right)+C\\ &= \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{y-3}{3}\right)+\frac{1}{2}(y-3)\sqrt{-y^2+6y}+C.\end{aligned}$$



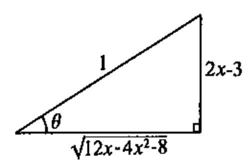
8. Since $x^2+6x+13 = (x+3)^2+4$, we set $x+3 = 2\tan\theta$, in which case $dx = 2\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int\frac{2x-3}{x^2+6x+13}dx &= \int\frac{2(2\tan\theta-3)-3}{4\sec^2\theta}2\sec^2\theta\,d\theta = \frac{1}{2}(4\ln|\sec\theta|-9\theta)+C\\ &= 2\ln\left|\frac{\sqrt{x^2+6x+13}}{2}\right|-\frac{9}{2}\,\mathrm{Tan}^{-1}\left(\frac{x+3}{2}\right)+C\\ &= \ln(x^2+6x+13)-\frac{9}{2}\,\mathrm{Tan}^{-1}\left(\frac{x+3}{2}\right)+D.\end{aligned}$$



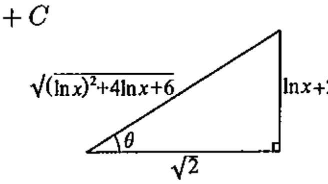
9. Since $12x-4x^2-8 = 4(-x^2+3x-2) = 4[-(x-3/2)^2+1/4]$, we set $x-3/2 = (1/2)\sin\theta$, in which case $dx = (1/2)\cos\theta\,d\theta$, and

$$\begin{aligned}\int\frac{5-4x}{\sqrt{12x-4x^2-8}}dx &= \int\frac{5-4x}{2\sqrt{1/4-(x-3/2)^2}}dx = \int\frac{5-2(3+\sin\theta)}{\cos\theta}\frac{1}{2}\cos\theta\,d\theta\\ &= -\frac{1}{2}\int(1+2\sin\theta)\,d\theta\\ &= -\frac{1}{2}(\theta-2\cos\theta)+C\\ &= -\frac{1}{2}\,\mathrm{Sin}^{-1}(2x-3)+\sqrt{12x-4x^2-8}+C.\end{aligned}$$



10. Since $6+4\ln x+(\ln x)^2 = (\ln x+2)^2+2$, we set $\ln x+2 = \sqrt{2}\tan\theta$. Then $(1/x)dx = \sqrt{2}\sec^2\theta\,d\theta$ and

$$\begin{aligned}\int\frac{1}{x\sqrt{6+4\ln x+(\ln x)^2}}dx &= \int\frac{1}{\sqrt{2}\sec\theta}\sqrt{2}\sec^2\theta\,d\theta = \ln|\sec\theta+\tan\theta|+C\\ &= \ln\left|\frac{\sqrt{(\ln x)^2+4\ln x+6}}{\sqrt{2}}+\frac{\ln x+2}{\sqrt{2}}\right|+C\\ &= \ln\left|\sqrt{(\ln x)^2+4\ln x+6}+\ln x+2\right|+D.\end{aligned}$$

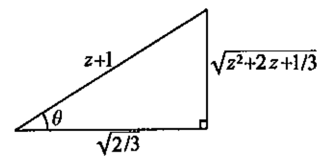


11. If we set $z = 1/x$ and $dx = -(1/z^2)dz$, then

$$\begin{aligned}\int\frac{1}{x\sqrt{x^2+6x+3}}dx &= \int\frac{1}{\frac{1}{z}\sqrt{\frac{1}{z^2}+\frac{6}{z}+3}}\left(\frac{dz}{-z^2}\right) = -\int\frac{|z|}{z\sqrt{3z^2+6z+1}}dz\\ &= \frac{-1}{\sqrt{3}}\int\frac{|z|}{z\sqrt{z^2+2z+1/3}}dz = \frac{-1}{\sqrt{3}}\int\frac{|z|}{z\sqrt{(z+1)^2-2/3}}dz.\end{aligned}$$

When $z>0$, $|z|/z = 1$, and we set $z+1 = \sqrt{2/3}\sec\theta$ and $dz = \sqrt{2/3}\sec\theta\tan\theta\,d\theta$, in which case

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2+6x+3}}dx &= \frac{-1}{\sqrt{3}}\int \frac{1}{\sqrt{2/3}\tan\theta}\sqrt{\frac{2}{3}}\sec\theta\,\tan\theta\,d\theta = \frac{-1}{\sqrt{3}}\int \sec\theta\,d\theta\\
&= \frac{-1}{\sqrt{3}}\ln|\sec\theta+\tan\theta| + C = \frac{-1}{\sqrt{3}}\ln\left|\frac{z+1}{\sqrt{2/3}} + \frac{\sqrt{z^2+2z+1/3}}{\sqrt{2/3}}\right| + C\\
&= \frac{-1}{\sqrt{3}}\ln\left|\frac{1}{x}+1+\sqrt{\frac{1}{x^2}+\frac{2}{x}+\frac{1}{3}}\right| + D = \frac{-1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}{\sqrt{3}x}\right| + D\\
&= \frac{-1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}{x}\right| + E
\end{aligned}$$



When $z<0$, $|z|/z=-1$. We make the same substitution as above, in which case

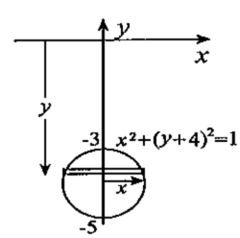
$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2+6x+3}}dx &= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{2/3}\tan\theta}\sqrt{\frac{2}{3}}\sec\theta\,\tan\theta\,d\theta = \frac{1}{\sqrt{3}}\int \sec\theta\,d\theta\\
&= \frac{1}{\sqrt{3}}\ln|\sec\theta+\tan\theta| + C = \frac{1}{\sqrt{3}}\ln\left|\frac{z+1}{\sqrt{2/3}} + \frac{\sqrt{z^2+2z+1/3}}{\sqrt{2/3}}\right| + C\\
&= \frac{1}{\sqrt{3}}\ln\left|\frac{1}{x}+1+\sqrt{\frac{1}{x^2}+\frac{2}{x}+\frac{1}{3}}\right| + D = \frac{1}{\sqrt{3}}\ln\left|\frac{1}{x}+1+\frac{\sqrt{x^2+6x+3}}{-\sqrt{3}x}\right| + D\\
&= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)-\sqrt{x^2+6x+3}}{x}\right| + E\\
&= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)-\sqrt{x^2+6x+3}}{x}\,\frac{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}\right| + E\\
&= \frac{1}{\sqrt{3}}\ln\left|\frac{3(x+1)^2-(x^2+6x+3)}{x[\sqrt{3}(x+1)+\sqrt{x^2+6x+3}]}\right| + E\\
&= \frac{1}{\sqrt{3}}\ln\left|\frac{2x}{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}\right| + E\\
&= \frac{-1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}(x+1)+\sqrt{x^2+6x+3}}{x}\right| + F,
\end{aligned}$$

the same antiderivative as when $x>0$.

12. $F = \displaystyle\int_{-5}^{-3} 9810(-y)2x\,dy = -19\,620\int_{-5}^{-3} y\sqrt{1-(y+4)^2}\,dy$

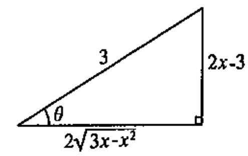
If we set $y+4=\sin\theta$, then $dy=\cos\theta\,d\theta$, and

$$\begin{aligned}
F &= -19\,620\int_{-\pi/2}^{\pi/2}(\sin\theta-4)\cos\theta\,\cos\theta\,d\theta\\
&= -19\,620\int_{-\pi/2}^{\pi/2}[\cos^2\theta\,\sin\theta - 2(1+\cos 2\theta)]\,d\theta\\
&= -19\,620\left\{-\frac{1}{3}\cos^3\theta - 2\theta - \sin 2\theta\right\}\Bigg|_{-\pi/2}^{\pi/2} = 39\,240\pi \text{ N.}
\end{aligned}$$



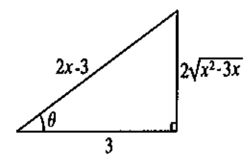
13. (a) With $3x - x^2 = -(x-3/2)^2 + 9/4$, we set $x - 3/2 = (3/2)\sin\theta$ and $dx = (3/2)\cos\theta\, d\theta$. Then

$$\int \frac{1}{3x - x^2}dx = \int \frac{1}{-(x-3/2)^2 + 9/4}dx = \int \frac{1}{(9/4)\cos^2\theta}\frac{3}{2}\cos\theta\, d\theta$$
$$= \frac{2}{3}\int \sec\theta\, d\theta = \frac{2}{3}\ln|\sec\theta + \tan\theta| + C$$
$$= \frac{2}{3}\ln\left|\frac{3}{2\sqrt{3x-x^2}} + \frac{2x-3}{2\sqrt{3x-x^2}}\right| + C = \frac{2}{3}\ln\left|\frac{x}{\sqrt{3x-x^2}}\right| + C.$$



(b) With $x^2 - 3x = (x-3/2)^2 - 9/4$, we set $x - 3/2 = (3/2)\sec\theta$ and $dx = (3/2)\sec\theta\tan\theta\, d\theta$. Then

$$\int \frac{1}{3x - x^2}dx = \int \frac{-1}{x^2 - 3x}dx = \int \frac{-1}{(x-3/2)^2 - 9/4}dx$$
$$= \int \frac{-1}{(9/4)\tan^2\theta}\frac{3}{2}\sec\theta\tan\theta\, d\theta = -\frac{2}{3}\int \frac{\sec\theta}{\tan\theta}d\theta = -\frac{2}{3}\int \csc\theta\, d\theta$$
$$= -\frac{2}{3}\ln|\csc\theta - \cot\theta| + C$$
$$= -\frac{2}{3}\ln\left|\frac{2x-3}{2\sqrt{x^2-3x}} - \frac{3}{2\sqrt{x^2-3x}}\right| + C$$
$$= -\frac{2}{3}\ln\left|\frac{x-3}{\sqrt{x^2-3x}}\right| + C.$$



(c) The first answer should only be used when $3x - x^2 > 0$; that is, when $0 < x < 3$. The second answer should be used when $x < 0$ or $x > 3$. We can find a single expression combining both answers, valid for all x except $x = 0$ and $x = 3$. The solution in part (a) can be rewritten

$$\frac{2}{3}\ln\left|\frac{x}{\sqrt{x(3-x)}}\right| + C = \frac{2}{3}\ln\left|\sqrt{\frac{x}{3-x}}\right| + C = \frac{1}{3}\ln\left(\frac{x}{3-x}\right) + C,$$

valid for $0 < x < 3$. For the solution in part (b), we write

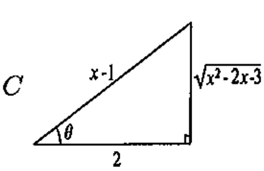
$$-\frac{2}{3}\ln\left|\frac{x-3}{\sqrt{x(x-3)}}\right| + C = -\frac{2}{3}\ln\left|\sqrt{\frac{x-3}{x}}\right| + C = -\frac{1}{3}\ln\left(\frac{x-3}{x}\right) + C = \frac{1}{3}\ln\left(\frac{x}{x-3}\right) + C,$$

valid for $x < 0$ and $x > 3$. Both of these can be combined into

$$\frac{1}{3}\ln\left|\frac{x}{x-3}\right| + C.$$

14. Since $x^2 - 2x - 3 = (x-1)^2 - 4$, we set $x - 1 = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta\, d\theta$,

$$\int \sqrt{x^2 - 2x - 3}\, dx = \int 2\tan\theta\, 2\sec\theta\tan\theta\, d\theta = 4\int \tan^2\theta\sec\theta\, d\theta$$
$$= 4\int (\sec^2\theta - 1)\sec\theta\, d\theta = 4\int (\sec^3\theta - \sec\theta)\, d\theta$$
$$= 4\left[\frac{1}{2}\ln|\sec\theta + \tan\theta| + \frac{1}{2}\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right] + C \text{ (see Example 8.9)}$$
$$= 2[\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|] + C$$
$$= 2\left(\frac{x-1}{2}\right)\frac{\sqrt{x^2-2x-3}}{2} - 2\ln\left|\frac{x-1}{2} + \frac{\sqrt{x^2-2x-3}}{2}\right| + C$$
$$= \frac{1}{2}(x-1)\sqrt{x^2-2x-3} - 2\ln|x - 1 + \sqrt{x^2-2x-3}| + D.$$

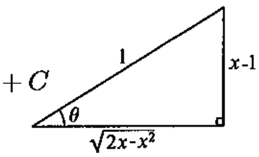


15. Since $2x - x^2 = 1 - (x-1)^2$, we set $x - 1 = \sin\theta$, in which case $dx = \cos\theta\, d\theta$, and

$$\int \frac{1}{x\sqrt{2x-x^2}}dx = \int \frac{1}{(1+\sin\theta)\cos\theta}\cos\theta\, d\theta = \int \frac{1}{1+\sin\theta}\frac{1-\sin\theta}{1-\sin\theta}d\theta$$

$$= \int \frac{1-\sin\theta}{\cos^2\theta}d\theta = \int (\sec^2\theta - \sec\theta\,\tan\theta)\,d\theta = \tan\theta - \sec\theta + C$$

$$= \frac{x-1}{\sqrt{2x-x^2}} - \frac{1}{\sqrt{2x-x^2}} + C = \frac{x-2}{\sqrt{2x-x^2}} + C.$$



16. If we set $u = \sqrt{2x-3}$, then $du = \dfrac{1}{\sqrt{2x-3}}dx$, and

$$\int \frac{1}{(2x+5)\sqrt{2x-3}+8x-12}dx = \int \frac{1}{(u^2+8)u+4u^2}u\,du = \int \frac{1}{u^2+4u+8}du = \int \frac{1}{(u+2)^2+4}du.$$

If we now set $u + 2 = 2\tan\theta$, then $du = 2\sec^2\theta\, d\theta$, and

$$\int \frac{1}{(2x+5)\sqrt{2x-3}+8x-12}dx = \int \frac{1}{4\sec^2\theta}2\sec^2\theta\, d\theta = \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\,\mathrm{Tan}^{-1}\left(\frac{u+2}{2}\right) + C = \frac{1}{2}\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{2x-3}+2}{2}\right) + C.$$

EXERCISES 8.6

1. If we set $\dfrac{x+2}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$, then $A = 1$ and $B = 3$, and

$$\int \frac{x+2}{x^2-2x+1}dx = \int \left[\frac{1}{x-1} + \frac{3}{(x-1)^2}\right]dx = \ln|x-1| - \frac{3}{x-1} + C.$$

2. $\displaystyle\int \frac{1}{y^3+3y^2+3y+1}dy = \int \frac{1}{(y+1)^3}dy = \frac{-1}{2(y+1)^2} + C$

3. If we set $\dfrac{1}{z(z^2+1)} = \dfrac{A}{z} + \dfrac{Bz+C}{z^2+1}$, then $A = 1$, $B = -1$, and $C = 0$, and

$$\int \frac{1}{z^3+z}dz = \int \left(\frac{1}{z} - \frac{z}{z^2+1}\right)dz = \ln|z| - \frac{1}{2}\ln(z^2+1) + C.$$

4. $\displaystyle\int \frac{x^2+2x-4}{x^2-2x-8}dx = \int \left(1 + \frac{4x+4}{x^2-2x-8}\right)dx = x + 4\int \frac{x+1}{(x-4)(x+2)}dx$

If we set $\dfrac{x+1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$, then $A = 5/6$ and $B = 1/6$, and

$$\int \frac{x^2+2x-4}{x^2-2x-8}dx = x + 4\int \left(\frac{5/6}{x-4} + \frac{1/6}{x+2}\right)dx = x + \frac{10}{3}\ln|x-4| + \frac{2}{3}\ln|x+2| + C.$$

5. $\displaystyle\int \frac{x}{(x-4)^2}dx = \int \frac{(x-4)+4}{(x-4)^2}dx = \int \left[\frac{1}{x-4} + \frac{4}{(x-4)^2}\right]dx = \ln|x-4| - \frac{4}{x-4} + C.$

6. If we set $\dfrac{y+1}{y(y+3)(y-2)} = \dfrac{A}{y} + \dfrac{B}{y+3} + \dfrac{C}{y-2}$, then $A = -1/6$, $B = -2/15$, $C = 3/10$, and

$$\int \frac{y+1}{y^3+y^2-6y}dy = \int \left(\frac{-1/6}{y} - \frac{2/15}{y+3} + \frac{3/10}{y-2}\right)dy = -\frac{1}{6}\ln|y| - \frac{2}{15}\ln|y+3| + \frac{3}{10}\ln|y-2| + C.$$

7. If we set $\dfrac{3x+5}{(x-1)^2(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$, then $A = -1/2$, $B = 4$, and $C = 1/2$, and

$$\int \frac{3x+5}{x^3-x^2-x+1}dx = \int \left(\frac{-1/2}{x-1} + \frac{4}{(x-1)^2} + \frac{1/2}{x+1}\right) dx = -\frac{1}{2}\ln|x-1| - \frac{4}{x-1} + \frac{1}{2}\ln|x+1| + C.$$

8. If we set $\dfrac{x^3}{(x^2+2)^2} = \dfrac{Ax+B}{x^2+2} + \dfrac{Cx+D}{(x^2+2)^2}$, then $A = 1$, $B = 0$, $C = -2$, $D = 0$, and

$$\int \frac{x^3}{(x^2+2)^2}dx = \int \left[\frac{x}{x^2+2} - \frac{2x}{(x^2+2)^2}\right] dx = \frac{1}{2}\ln(x^2+2) + \frac{1}{x^2+2} + C.$$

9. If we set $\dfrac{1}{x^2-3} = \dfrac{A}{x+\sqrt{3}} + \dfrac{B}{x-\sqrt{3}}$, then $A = -1/(2\sqrt{3})$ and $b = 1/(2\sqrt{3})$, and

$$\int \frac{1}{x^2-3}dx = \int \left(\frac{-1/(2\sqrt{3})}{x+\sqrt{3}} + \frac{1/(2\sqrt{3})}{x-\sqrt{3}}\right) dx = \frac{1}{2\sqrt{3}}(\ln|x-\sqrt{3}| - \ln|x+\sqrt{3}|) + C.$$

10. $\displaystyle\int \frac{y^2}{y^2+3y+2}dy = \int \left(1 + \frac{-3y-2}{y^2+3y+2}\right) dy$ If we set $\dfrac{3y+2}{y^2+3y+2} = \dfrac{A}{y+2} + \dfrac{B}{y+1}$, then $A = 4$, $B = -1$, and

$$\int \frac{y^2}{y^2+3y+2}dy = y + \int \left(\frac{-4}{y+2} + \frac{1}{y+1}\right) dy = y - 4\ln|y+2| + \ln|y+1| + C.$$

11. If we set $\dfrac{z^2+3z-2}{z^3+5z} = \dfrac{A}{z} + \dfrac{Bz+C}{z^2+5}$, then $A = -2/5$, $B = 7/5$ and $C = 3$, and

$$\int \frac{z^2+3z-2}{z^3+5z}dz = \int \left(\frac{-2/5}{z} + \frac{7z/5+3}{z^2+5}\right) dz.$$

In the term $3/(z^2+5)$, we set $z = \sqrt{5}\tan\theta$ and $dz = \sqrt{5}\sec^2\theta\, d\theta$,

$$\begin{aligned}\int \frac{z^2+3z-2}{z^3+5z}dz &= -\frac{2}{5}\ln|z| + \frac{7}{10}\ln(z^2+5) + 3\int \frac{1}{5\sec^2\theta}\sqrt{5}\sec^2\theta\, d\theta \\ &= -\frac{2}{5}\ln|z| + \frac{7}{10}\ln(z^2+5) + \frac{3}{\sqrt{5}}\theta + C \\ &= -\frac{2}{5}\ln|z| + \frac{7}{10}\ln(z^2+5) + \frac{3}{\sqrt{5}}\text{Tan}^{-1}\left(\frac{z}{\sqrt{5}}\right) + C.\end{aligned}$$

12. If we set $\dfrac{y^2+6y+4}{(y^2+4)(y^2+1)} = \dfrac{Ay+B}{y^2+4} + \dfrac{Cy+D}{y^2+1}$, then $A = -2$, $B = 0$, $C = 2$, $D = 1$, and

$$\int \frac{y^2+6y+4}{y^4+5y^2+4}dy = \int \left(\frac{-2y}{y^2+4} + \frac{2y+1}{y^2+1}\right) dy = -\ln(y^2+4) + \ln(y^2+1) + \text{Tan}^{-1}y + C.$$

13. If we set $\dfrac{x}{(x^2+6)(x^2+1)} = \dfrac{Ax+B}{x^2+6} + \dfrac{Cx+D}{x^2+1}$, then $A = -1/5$, $B = 0$, $C = 1/5$, and $D = 0$, and

$$\int \frac{x}{x^4+7x^2+6}dx = \int \left(\frac{x/5}{x^2+1} - \frac{x/5}{x^2+6}\right) dx = \frac{1}{10}\ln(x^2+1) - \frac{1}{10}\ln(x^2+6) + C.$$

14. If we set $\dfrac{x^2+3}{(x^2+2)(x-1)(x+1)} = \dfrac{Ax+B}{x^2+2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$, then $A = 0$, $B = -1/3$, $C = 2/3$, $D = -2/3$, and

$$\int \frac{x^2+3}{x^4+x^2-2}dx = \int \left(\frac{-1/3}{x^2+2} + \frac{2/3}{x-1} - \frac{2/3}{x+1}\right) dx.$$

In the first term we set $x = \sqrt{2}\tan\theta$ and $dx = \sqrt{2}\sec^2\theta\,d\theta$,

$$\int \frac{x^2+3}{x^4+x^2-2}dx = -\frac{1}{3}\int \frac{1}{2\sec^2\theta}\sqrt{2}\sec^2\theta\,d\theta + \frac{2}{3}\ln|x-1| - \frac{2}{3}\ln|x+1|$$
$$= -\frac{1}{3\sqrt{2}}\theta + \frac{2}{3}\ln\left|\frac{x-1}{x+1}\right| + C = -\frac{1}{3\sqrt{2}}\mathrm{Tan}^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{2}{3}\ln\left|\frac{x-1}{x+1}\right| + C.$$

15. If we set $\dfrac{3t+4}{t(t-1)^3} = \dfrac{A}{t} + \dfrac{B}{t-1} + \dfrac{C}{(t-1)^2} + \dfrac{D}{(t-1)^3}$, then $A=-4$, $B=4$, $C=-4$, and $D=7$, and

$$\int \frac{3t+4}{t^4-3t^3+3t^2-t}dt = \int\left[-\frac{4}{t} + \frac{4}{t-1} - \frac{4}{(t-1)^2} + \frac{7}{(t-1)^3}\right]dt$$
$$= -4\ln|t| + 4\ln|t-1| + \frac{4}{t-1} - \frac{7}{2(t-1)^2} + C.$$

16. If we set $\dfrac{x^3+6}{(x-1)^2(x+2)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2}$, then $A = -5/27$, $B = 7/9$, $C = 32/27$, $D = -2/9$, and

$$\int \frac{x^3+6}{x^4+2x^3-3x^2-4x+4}dx = \int\left[\frac{-5/27}{x-1} + \frac{7/9}{(x-1)^2} + \frac{32/27}{x+2} - \frac{2/9}{(x+2)^2}\right]dx$$
$$= -\frac{5}{27}\ln|x-1| - \frac{7}{9(x-1)} + \frac{32}{27}\ln|x+2| + \frac{2}{9(x+2)} + C.$$

17. The length of the curve is

$$L = \int_0^{1/2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_0^{1/2}\sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}dx = \int_0^{1/2}\sqrt{\frac{1-2x^2+x^4+4x^2}{(1-x^2)^2}}dx$$
$$= \int_0^{1/2}\sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}dx = \int_0^{1/2}\frac{1+x^2}{1-x^2}dx = \int_0^{1/2}\left(-1+\frac{2}{1-x^2}\right)dx$$
$$= \int_0^{1/2}\left(-1+\frac{1}{1-x}+\frac{1}{1+x}\right)dx = \left\{-x-\ln|1-x|+\ln|1+x|\right\}_0^{1/2} = \ln 3 - \frac{1}{2}.$$

18. Separation of variables and partial fractions give

$$\int -\frac{dt}{1500} = \int \frac{dv}{v^2-2500} = \int\left(\frac{1/100}{v-50} - \frac{1/100}{v+50}\right)dv = \frac{1}{100}\int\left(\frac{1}{v-50} - \frac{1}{v+50}\right)dv.$$

Thus,

$$\frac{-t}{1500} + C = \frac{1}{100}\left(\ln|v-50| - \ln|v+50|\right) \quad\Longrightarrow\quad -\frac{t}{15} + 100C = \ln\left|\frac{v-50}{v+50}\right|.$$

When we exponentiate,

$$\left|\frac{v-50}{v+50}\right| = e^{100C-t/15} \quad\Longrightarrow\quad v-50 = (v+50)De^{-t/15},$$

where $D = \pm e^{100C}$. When we solve this for v, the result is $v(t) = \dfrac{50(1+De^{-t/15})}{1-De^{-t/15}}$. If we choose time $t=0$ when the car begins motion, then $v(0)=0$, and this requires $0 = 50(1+D)/(1-D) \Longrightarrow D = -1$. Hence, $v(t) = 50(1-e^{-t/15})/(1+e^{-t/15})$.
(b) If we set $u = 1 + e^{-t/15}$ and $du = -(1/15)e^{-t/15}dt$,

$$x(t) = \int \frac{50(1-e^{-t/15})}{1+e^{-t/15}}dt = 50\int \frac{1-(u-1)}{u}\left(\frac{-15\,du}{u-1}\right) = 750\int \frac{u-2}{u(u-1)}du$$
$$= 750\int \left(\frac{2}{u} - \frac{1}{u-1}\right)du = 750(2\ln|u| - \ln|u-1|) + E$$
$$= 750[2\ln(1+e^{-t/15}) - \ln(e^{-t/15})] + E = 750\left[\frac{t}{15} + 2\ln(1+e^{-t/15})\right] + E.$$

If we choose $x(0) = 0$, then $0 = 750(2\ln 2) + E \Longrightarrow E = -1500\ln 2$, and

$$x(t) = 750\left[\frac{t}{15} + 2\ln(1+e^{-t/15})\right] - 1500\ln 2 = 750\left[\frac{t}{15} + 2\ln\left(\frac{1+e^{-t/15}}{2}\right)\right].$$

19. If we set $V = \sqrt{mg/k}$, the differential equation can be expressed in the form

$$\frac{m}{k}\frac{dv}{dt} = \frac{mg}{k} - v^2 = V^2 - v^2 \quad \Longrightarrow \quad \int \frac{dv}{v^2 - V^2} = \int -\frac{k}{m}dt.$$

Partial fractions gives

$$-\frac{kt}{m} + C = \int \left[\frac{1/(2V)}{v-V} - \frac{1/(2V)}{v+V}\right]dv = \frac{1}{2V}(\ln|v-V| - \ln|v+V|) = \frac{1}{2V}\ln\left|\frac{v-V}{v+V}\right|.$$

When we exponentiate,

$$\left|\frac{v-V}{v+V}\right| = e^{2VC-2kVt/m} \quad \Longrightarrow \quad v - V = (v+V)De^{-2kVt/m},$$

where $D = \pm e^{2VC}$. When we solve this for v, the result is $v(t) = \dfrac{V(1+De^{-2kVt/m})}{1-De^{-2kVt/m}}$. If we choose time $t = 0$ when the raindrop exits the cloud, then $v(0) = v_0$, and this requires

$$v_0 = \frac{V(1+D)}{1-D} \quad \Longrightarrow \quad v_0(1-D) = V(1+D) \quad \Longrightarrow \quad D = \frac{v_0 - V}{v_0 + V}.$$

Hence, $v(t) = \dfrac{V\left[1 - \left(\dfrac{V-v_0}{V+v_0}\right)e^{-2kVt/m}\right]}{1 + \left(\dfrac{V-v_0}{V+v_0}\right)e^{-2kVt/m}}$. Since $V = \lim_{t\to\infty} v(t)$, it follows that V is the limiting velocity of the raindrop.

20. With $mv\dfrac{dv}{dy} = mg - kv^2$ expressed in the form $\dfrac{v\,dv}{mg-kv^2} = \dfrac{dy}{m}$, solutions are defined implicitly by

$$\frac{y}{m} + C = \int \frac{v\,dv}{mg - kv^2} = -\frac{1}{2k}\ln|mg - kv^2|.$$

When we multiply by $-2k$ and exponentiate,

$$|mg - kv^2| = e^{-2kC-2ky/m} \quad \Longrightarrow \quad mg - kv^2 = De^{-2ky/m} \quad \Longrightarrow \quad v = \sqrt{\frac{mg}{k} - \frac{D}{k}e^{-2ky/m}},$$

where $D = \pm e^{-2kC}$. Since $v(0) = v_0$, we have $v_0 = \sqrt{\dfrac{mg}{k} - \dfrac{D}{k}} \Longrightarrow \dfrac{D}{k} = \dfrac{mg}{k} - v_0^2$. Hence,

$$v(y) = \sqrt{\frac{mg}{k} - \left(\frac{mg}{k} - v_0^2\right)e^{-2ky/m}}.$$

The velocity of the raindrop when it strikes the earth is $\sqrt{\dfrac{mg}{k} - \left(\dfrac{mg}{k} - v_0^2\right)e^{-2kh/m}}$.

21. With $k = 1$ and $C = 10^6$, the differential equation can be expressed in the form

$$\frac{dN}{dt} = N\left(1 - \frac{N}{10^6}\right) = 10^{-6}N(10^6 - N) \quad \Longrightarrow \quad \frac{dN}{N(10^6 - N)} = 10^{-6}dt.$$

Partial fractions gives

$$\int \left(\frac{10^{-6}}{N} + \frac{10^{-6}}{10^6 - N}\right) dN = \int 10^{-6}\, dt.$$

If we divide by 10^{-6}, solutions are defined implicitly by

$$t + D = \ln|N| - \ln|10^6 - N| = \ln\left|\frac{N}{10^6 - N}\right|.$$

Exponentiation gives $\left|\dfrac{N}{10^6 - N}\right| = e^{t+D} \quad \Longrightarrow \quad N = (10^6 - N)Ee^t \quad \Longrightarrow \quad N = \dfrac{10^6 Ee^t}{1 + Ee^t}$, where $E = \pm e^D$. For $N(0) = 100$, we must have $100 = \dfrac{10^6 E}{1 + E} \Longrightarrow E = \dfrac{1}{9999}$. Hence, $N(t) = 10^6/(1 + 9999e^{-t})$.

22. The differential equation can be expressed in the form $\dfrac{dN}{N(C - N)} = \dfrac{k}{C}\, dt$. Partial fractions gives

$$\int \frac{k}{C}dt = \int \left(\frac{1/C}{N} + \frac{1/C}{C - N}\right) dN = \frac{1}{C}\int \left(\frac{1}{N} + \frac{1}{C - N}\right) dN.$$

When we multiply by C, solutions are defined implicitly by

$$kt + D = \ln|N| - \ln|C - N| = \ln\left|\frac{N}{C - N}\right|.$$

Exponentiation gives $\left|\dfrac{N}{C - N}\right| = e^{kt+D} \Longrightarrow N = (C - N)Ee^{kt} \Longrightarrow N = \dfrac{CEe^{kt}}{1 + Ee^{kt}}$, where $E = \pm e^D$. For $N(0) = N_0$, we must have $N_0 = \dfrac{CE}{1 + E} \Longrightarrow N_0(1 + E) = CE \Longrightarrow E = \dfrac{N_0}{C - N_0}$. Hence,

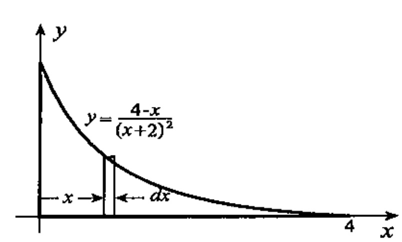
$$N(t) = \frac{C\left(\dfrac{N_0}{C - N_0}\right)e^{kt}}{1 + \left(\dfrac{N_0}{C - N_0}\right)e^{kt}} = \frac{C}{1 + \left(\dfrac{C - N_0}{N_0}\right)e^{-kt}}.$$

23. $A = \displaystyle\int_0^4 \frac{4 - x}{(x + 2)^2}dx = \int_0^4 \left[\frac{6}{(x + 2)^2} - \frac{1}{x + 2}\right] dx = \left\{\frac{-6}{x + 2} - \ln|x + 2|\right\}_0^4 = 2 - \ln 3$

Since $A\overline{x} = \displaystyle\int_0^4 \frac{x(4 - x)}{(x + 2)^2}dx = \int_0^4 \left[-1 + \frac{8x + 4}{(x + 2)^2}\right] dx$

$$= \int_0^4 \left[-1 + \frac{8}{x + 2} - \frac{12}{(x + 2)^2}\right] dx$$

$$= \left\{-x + 8\ln|x + 2| + \frac{12}{x + 2}\right\}_0^4 = 8(-1 + \ln 3),$$



it follows that $\overline{x} = 8(-1 + \ln 3)/(2 - \ln 3) = 0.875$. Since

$$A\overline{y} = \int_0^4 \frac{1}{2}\left[\frac{4 - x}{(x + 2)^2}\right]^2 dx = \frac{1}{2}\int_0^4 \frac{(4 - x)^2}{(x + 2)^4}dx = \frac{1}{2}\int_0^4 \left[\frac{1}{(x + 2)^2} - \frac{12}{(x + 2)^3} + \frac{36}{(x + 2)^4}\right] dx$$

$$= \frac{1}{2}\left\{-\frac{1}{x + 2} + \frac{6}{(x + 2)^2} - \frac{12}{(x + 2)^3}\right\}_0^4 = \frac{2}{9},$$

we obtain $\overline{y} = (2/9)/(2 - \ln 3) = 0.247$.

24. We can separate the differential equation,

$$k\,dt = \frac{1}{x(N-x)}\,dx \qquad \Longrightarrow \qquad k\,dt = \left(\frac{1/N}{x} + \frac{1/N}{N-x}\right)dx.$$

Solutions are defined implicitly by

$$\frac{1}{N}\left(\ln|x| - \ln|N-x|\right) = kt + C.$$

Since x and $N-x$ are both positive, we may drop the absolute values and write

$$\ln\left(\frac{x}{N-x}\right) = N(kt+C) \qquad \Longrightarrow \qquad \frac{x}{N-x} = De^{Ft},$$

where we have substituted $D = e^{NC}$ and $F = kN$. Multiplication by $N-x$ gives

$$x = De^{Ft}(N-x) \quad \Longrightarrow \quad x\left(1 + De^{Ft}\right) = NDe^{Ft}.$$

Thus,

$$x(t) = \frac{NDe^{Ft}}{1+De^{Ft}} = \frac{ND}{D+e^{-Ft}}.$$

Since $x(0) = 1$, it follows that $1 = \dfrac{ND}{1+D} \Longrightarrow D = 1/(N-1)$, and

$$x(t) = \frac{\dfrac{N}{N-1}}{\dfrac{1}{N-1} + e^{-Ft}} = \frac{N}{1+(N-1)e^{-Ft}}.$$

25. (a) When $a = b$, the differential equation becomes

$$\frac{dx}{dt} = k(a-x)^2 \quad \Longrightarrow \quad \frac{1}{(a-x)^2}\,dx = k\,dt,$$

a separated equation. Solutions are defined implicitly by

$$\frac{1}{a-x} = kt + C \quad \Longrightarrow \quad x - a = -\frac{1}{kt+C} \quad \Longrightarrow \quad x(t) = a - \frac{1}{kt+C}.$$

(b) When $a \neq b$, we again separate the differential equation, but use partial fractions to write,

$$\frac{1}{(a-x)(b-x)}\,dx = k\,dt \quad \Longrightarrow \quad \left[\frac{-1/(a-b)}{a-x} + \frac{1/(a-b)}{b-x}\right]dx = k\,dt.$$

Solutions are defined implicitly by

$$\frac{1}{a-b}\left[\ln|a-x| - \ln|b-x|\right] = kt + C.$$

To find explicit solutions we write

$$\ln\left|\frac{a-x}{b-x}\right| = (a-b)kt + C(a-b) \quad \Longrightarrow \quad \left|\frac{a-x}{b-x}\right| = e^{(a-b)kt+C(a-b)} \quad \Longrightarrow \quad \frac{a-x}{b-x} = De^{(a-b)kt},$$

where $D = \pm e^{C(a-b)}$. Multiplication by $b-x$ gives

$$a - x = (b-x)De^{(a-b)kt} \quad \Longrightarrow \quad x(t) = \frac{a - bDe^{(a-b)kt}}{1 - De^{(a-b)kt}}.$$

26. We separate the differential equation and use partial fractions to write

$$\frac{1}{v_0^2 - v^2}\, dv = \frac{1}{a}\, dt \quad \Longrightarrow \quad \left[\frac{1/(2v_0)}{v_0 - v} + \frac{1/(2v_0)}{v_0 + v}\right] dv = \frac{1}{a}\, dt.$$

Solutions are defined implicitly by

$$\frac{1}{2v_0}\left[-\ln(v_0 - v) + \ln(v_0 + v)\right] = \frac{t}{a} + C \quad \Longrightarrow \quad \ln\left(\frac{v_0 + v}{v_0 - v}\right) = \frac{2v_0 t}{a} + 2v_0 C.$$

Exponentiation gives

$$\frac{v_0 + v}{v_0 - v} = e^{2v_0 t/a + 2v_0 C} = De^{2v_0 t/a},$$

where $D = e^{2v_0 C}$. We can now solve for $v(t)$,

$$v_0 + v = (v_0 - v)De^{2v_0 t/a} \quad \Longrightarrow \quad v = \frac{v_0(De^{2v_0 t/a} - 1)}{De^{2v_0 t/a} + 1}.$$

The initial condition $v(0) = 0$ requires $D = 1$, and therefore

$$v(t) = \frac{v_0(e^{2v_0 t/a} - 1)}{e^{2v_0 t/a} + 1}.$$

27. If we set $\dfrac{x^3 + x + 2}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1} + \dfrac{Dx + E}{(x^2+1)^2}$, then $A = 2$, $B = -2$, $C = 1$, $D = -2$, and $E = 0$, so that

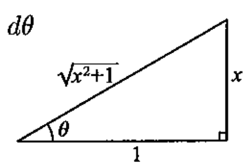
$$\int \frac{x^3 + x + 2}{x^5 + 2x^3 + x} dx = \int \left[\frac{2}{x} + \frac{-2x + 1}{x^2 + 1} - \frac{2x}{(1 + x^2)^2}\right] dx = 2\ln|x| - \ln(x^2 + 1) + \mathrm{Tan}^{-1} x + \frac{1}{x^2 + 1} + C.$$

28. If we set $\dfrac{1}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1} = \dfrac{1}{(x+1)(x^2+1)^2} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2 + 1} + \dfrac{Dx + E}{(x^2+1)^2}$, then $A = 1/4$, $B = -1/4$, $C = 1/4$, $D = -1/2$, $E = 1/2$, and

$$\int \frac{1}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1} dx = \int \left[\frac{1/4}{x+1} + \frac{-x/4 + 1/4}{x^2 + 1} + \frac{-x/2 + 1/2}{(x^2+1)^2}\right] dx.$$

In the very last term we set $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$, in which case

$$\begin{aligned}\int \frac{1}{(x^2+1)^2} dx &= \int \frac{1}{\sec^4\theta} \sec^2\theta\, d\theta = \int \cos^2\theta\, d\theta = \int \left(\frac{1 + \cos 2\theta}{2}\right) d\theta \\ &= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C = \frac{\theta}{2} + \frac{1}{2}\sin\theta\, \cos\theta + C \\ &= \frac{1}{2}\mathrm{Tan}^{-1} x + \frac{1}{2}\frac{x}{x^2 + 1} + C.\end{aligned}$$



Consequently,

$$\begin{aligned}\int \frac{1}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1} dx &= \frac{1}{4}\ln|x + 1| - \frac{1}{8}\ln(x^2 + 1) + \frac{1}{4}\mathrm{Tan}^{-1} x \\ &\quad + \frac{1}{4(x^2+1)} + \frac{1}{4}\mathrm{Tan}^{-1} x + \frac{x}{4(x^2+1)} + C \\ &= \frac{1}{4}\ln|x + 1| - \frac{1}{8}\ln(x^2 + 1) + \frac{1}{2}\mathrm{Tan}^{-1} x + \frac{x + 1}{4(x^2+1)} + C.\end{aligned}$$

29. If we set $\dfrac{1}{(x^2+5)(x^2+2x+3)} = \dfrac{Ax + B}{x^2 + 5} + \dfrac{Cx + D}{x^2 + 2x + 3}$, then $A = -1/12$, $B = -1/12$, $C = 1/12$, and $D = 1/4$, so that

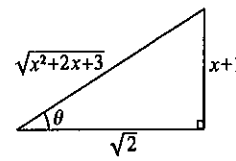
$$\int \frac{1}{(x^2+5)(x^2+2x+3)} dx = \frac{1}{12}\int \left(\frac{-x - 1}{x^2 + 5} + \frac{x + 3}{x^2 + 2x + 3}\right) dx.$$

If we set $x = \sqrt{5}\tan\theta$ and $dx = \sqrt{5}\sec^2\theta\,d\theta$, then

$$\int \frac{1}{x^2+5}dx = \int \frac{1}{5\sec^2\theta}\sqrt{5}\sec^2\theta\,d\theta = \frac{\theta}{\sqrt{5}} + C = \frac{1}{\sqrt{5}}\operatorname{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right) + C.$$

Since $x^2+2x+3 = (x+1)^2+2$, we set $x+1 = \sqrt{2}\tan\theta$ and $dx = \sqrt{2}\sec^2\theta\,d\theta$ in

$$\begin{aligned}\int \frac{x+3}{(x+1)^2+2}dx &= \int \frac{2+\sqrt{2}\tan\theta}{2\sec^2\theta}\sqrt{2}\sec^2\theta\,d\theta = \int(\sqrt{2}+\tan\theta)\,d\theta\\ &= \sqrt{2}\,\theta + \ln|\sec\theta| + C\\ &= \sqrt{2}\operatorname{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \ln\left|\frac{\sqrt{x^2+2x+3}}{\sqrt{2}}\right| + C\\ &= \sqrt{2}\operatorname{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{2}\ln(x^2+2x+3) + D.\end{aligned}$$



Thus,

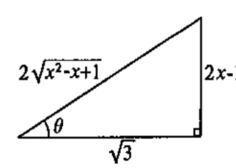
$$\begin{aligned}\int \frac{1}{(x^2+5)(x^2+2x+3)}dx &= \frac{1}{12}\left[-\frac{1}{2}\ln(x^2+5) - \frac{1}{\sqrt{5}}\operatorname{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right) + \sqrt{2}\operatorname{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right.\\ &\qquad \left. + \frac{1}{2}\ln(x^2+2x+3)\right] + C\\ &= -\frac{1}{24}\ln(x^2+5) - \frac{1}{12\sqrt{5}}\operatorname{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right) + \frac{\sqrt{2}}{12}\operatorname{Tan}^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\\ &\qquad + \frac{1}{24}\ln(x^2+2x+3) + C.\end{aligned}$$

30. If we set $\dfrac{1}{x^3+1} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$, then $A = 1/3$, $B = -1/3$, $C = 2/3$, and

$$\int \frac{1}{x^3+1}dx = \int\left(\frac{1/3}{x+1} + \frac{-x/3+2/3}{x^2-x+1}\right)dx = \frac{1}{3}\ln|x+1| + \frac{1}{3}\int \frac{-x+2}{(x-1/2)^2+3/4}dx.$$

In the remaining integral we set $x - 1/2 = (\sqrt{3}/2)\tan\theta$, and $dx = (\sqrt{3}/2)\sec^2\theta\,d\theta$,

$$\begin{aligned}\int \frac{1}{x^3+1}dx &= \frac{1}{3}\ln|x+1| + \frac{1}{3}\int \frac{-1/2-(\sqrt{3}/2)\tan\theta+2}{(3/4)\sec^2\theta}\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta\\ &= \frac{1}{3}\ln|x+1| + \frac{1}{3}(\sqrt{3}\theta + \ln|\cos\theta|) + C\\ &= \frac{1}{3}\ln|x+1| + \frac{1}{\sqrt{3}}\operatorname{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{3}\ln\left|\frac{\sqrt{3}}{2\sqrt{x^2-x+1}}\right| + C\\ &= \frac{1}{3}\ln|x+1| + \frac{1}{\sqrt{3}}\operatorname{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{6}\ln(x^2-x+1) + D.\end{aligned}$$



31. If we set $u = \cos x$ and $du = -\sin x\,dx$, then

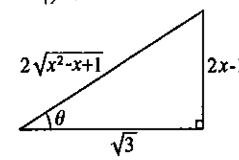
$$\begin{aligned}\int \frac{\sin x}{\cos x(1+\cos^2 x)}dx &= \int \frac{1}{u(1+u^2)}(-du) = -\int\left(\frac{1}{u} - \frac{u}{1+u^2}\right)du\\ &= -\ln|u| + \frac{1}{2}\ln(1+u^2) + C = \frac{1}{2}\ln(1+\cos^2 x) - \ln|\cos x| + C.\end{aligned}$$

32. If we set $\dfrac{x^4+8x^3-x^2+2x+1}{x^5+x^4+x^2+x} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} + \dfrac{Dx+E}{x^2-x+1}$, then $A = 1$, $B = -2$, $C = 3$, $D = 2$, $E = 0$, and

$$\int \frac{x^4+8x^3-x^2+2x+1}{x^5+x^4+x^2+x}dx = \int\left[\frac{1}{x} - \frac{2}{x+1} + \frac{3}{(x+1)^2} + \frac{2x}{(x-1/2)^2+3/4}\right]dx.$$

In the last term we set $x - 1/2 = (\sqrt{3}/2)\tan\theta$ and $dx = (\sqrt{3}/2)\sec^2\theta\, d\theta$, in which case

$$\int \frac{2x}{(x-1/2)^2+3/4}dx = 2\int \frac{1/2+(\sqrt{3}/2)\tan\theta}{(3/4)\sec^2\theta}\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta = \frac{2}{\sqrt{3}}(\theta+\sqrt{3}\ln|\sec\theta|)+C$$
$$= \frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + 2\ln\left|\frac{2\sqrt{x^2-x+1}}{\sqrt{3}}\right| + C$$
$$= \frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \ln(x^2-x+1) + D.$$



Thus,
$$\int \frac{x^4+8x^3-x^2+2x+1}{x^5+x^4+x^2+x}dx = \ln|x| - 2\ln|x+1| - \frac{3}{x+1} + \frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$
$$+\ln(x^2-x+1)+D.$$

33. (a) $\dfrac{dv}{dt} = \dfrac{gH}{L}\left(1-\dfrac{v^2}{v_f^2}\right) = \dfrac{gH}{Lv_f^2}(v_f^2-v^2)$, a separable differential equation $\dfrac{dv}{v_f^2-v^2} = \dfrac{gH}{Lv_f^2}dt$. Partial fractions gives

$$\frac{gHt}{Lv_f^2}+C = \int \frac{1}{v_f^2-v^2}dv = \frac{1}{2v_f}\int\left(\frac{1}{v_f+v}+\frac{1}{v_f-v}\right)dv$$
$$= \frac{1}{2v_f}[\ln(v_f+v)-\ln(v_f-v)] = \frac{1}{2v_f}\ln\left(\frac{v_f+v}{v_f-v}\right).$$

The initial condition $v(0)=0$ implies that $C=0$, and therefore

$$\frac{gHt}{Lv_f^2} = \frac{1}{2v_f}\ln\left(\frac{v_f+v}{v_f-v}\right) \quad\Longrightarrow\quad t = \frac{Lv_f}{2gH}\ln\left(\frac{v_f+v}{v_f-v}\right).$$

(b) If we exponentiate $\ln\left(\dfrac{v_f+v}{v_f-v}\right) = \dfrac{2gHt}{Lv_f}$, we obtain

$$\frac{v_f+v}{v_f-v} = e^{2gHt/(Lv_f)} \quad\Longrightarrow\quad v_f+v = (v_f-v)e^{2gHt/(Lv_f)}.$$

Hence, $v = v_f\left[\dfrac{e^{2gHt/(Lv_f)}-1}{e^{2gHt/(Lv_f)}+1}\right] = v_f\tanh\left(\dfrac{gHt}{Lv_f}\right)$.

34. We could use partial fractions on the integrand as it now stands, but it is easier if we first substitute $x = M^2$ and $dx = 2M\,dM$. At the same time, let us set $a = (k-1)/2$ and denote the integral by I:

$$I = \int \frac{M(1-M^2)}{M^4\left(1+\dfrac{k-1}{2}M^2\right)}dM = \frac{1}{2}\int\frac{1-x}{x^2(1+ax)}dx = \frac{1}{2}\int\left(\frac{-a-1}{x}+\frac{1}{x^2}+\frac{a+a^2}{1+ax}\right)dx$$
$$= \frac{1}{2}\left[-(a+1)\ln x - \frac{1}{x} + (1+a)\ln(1+ax)\right] + C = -\frac{1}{2x} + \left(\frac{a+1}{2}\right)\ln\left(\frac{1+ax}{x}\right)+C$$
$$= -\frac{1}{2M^2} + \left(\frac{k+1}{4}\right)\ln\left[\frac{1+\left(\dfrac{k-1}{2}\right)M^2}{M^2}\right]+C.$$

35. If $t = \tan(x/2)$, then $x = 2\,\mathrm{Tan}^{-1}t$, from which $dx = \dfrac{2}{1+t^2}dt$. Since $t = \sin(x/2)/\cos(x/2)$, it follows that $\sin(x/2) = t\cos(x/2)$. Using the fact that $\sin^2(x/2)+\cos^2(x/2) = 1$, we obtain

$$1 = t^2\cos^2(x/2)+\cos^2(x/2) \quad\Longrightarrow\quad \cos^2(x/2) = \frac{1}{1+t^2}.$$

Thus, $\cos x = 2\cos^2(x/2) - 1 = \dfrac{2}{1+t^2} - 1 = \dfrac{1-t^2}{1+t^2}$. Furthermore, $\sin x = 2\sin(x/2)\cos(x/2)$ $= 2t\cos^2(x/2) = \dfrac{2t}{1+t^2}$.

36. With the substitution from Exercise 35,

$$\int \sec x\,dx = \int \frac{1}{\cos x}dx = \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}dt = 2\int \frac{1}{1-t^2}dt = 2\int\left(\frac{1/2}{1-t} + \frac{1/2}{1+t}\right)dt$$

$$= -\ln|1-t| + \ln|1+t| + C = \ln\left|\frac{1+t}{1-t}\right| + C = \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C.$$

37. With the substitution from Exercise 35,

$$\int \frac{1}{3+5\sin x}dx = \int \frac{1}{3+\dfrac{10t}{1+t^2}}\frac{2}{1+t^2}dt = 2\int \frac{1}{3t^2+10t+3}dt$$

$$= 2\int \frac{1}{(3t+1)(t+3)}dt = 2\int\left(\frac{3/8}{3t+1} - \frac{1/8}{t+3}\right)dt$$

$$= \frac{1}{4}\ln|3t+1| - \frac{1}{4}\ln|t+3| + C = \frac{1}{4}\ln|3\tan(x/2)+1| - \frac{1}{4}\ln|\tan(x/2)+3| + C.$$

38. With the substitution from Exercise 35,

$$\int \frac{1}{1-2\cos x}dx = \int \frac{1}{1-2\left(\dfrac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}dt = 2\int \frac{1}{3t^2-1}dt$$

$$= 2\int\left(\frac{-1/2}{\sqrt{3}t+1} + \frac{1/2}{\sqrt{3}t-1}\right)dt = -\frac{1}{\sqrt{3}}\ln|\sqrt{3}t+1| + \frac{1}{\sqrt{3}}\ln|\sqrt{3}t-1| + C$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}t-1}{\sqrt{3}t+1}\right| + C = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}\tan(x/2)-1}{\sqrt{3}\tan(x/2)+1}\right| + C.$$

39. With the substitution from Exercise 35,

$$\int \frac{1}{\sin x+\cos x}dx = \int \frac{1}{\dfrac{2t}{1+t^2} + \dfrac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt = 2\int \frac{1}{1+2t-t^2}dt = 2\int \frac{1}{2-(t-1)^2}dt.$$

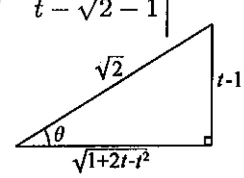
We now set $t-1 = \sqrt{2}\sin\theta$ and $dt = \sqrt{2}\cos\theta\,d\theta$,

$$\int \frac{1}{\sin x+\cos x}dx = 2\int \frac{1}{2\cos^2\theta}\sqrt{2}\cos\theta\,d\theta = \sqrt{2}\int \sec\theta\,d\theta$$

$$= \sqrt{2}\ln|\sec\theta + \tan\theta| + C = \sqrt{2}\ln\left|\frac{\sqrt{2}}{\sqrt{1+2t-t^2}} + \frac{t-1}{\sqrt{1+2t-t^2}}\right| + C$$

$$= \sqrt{2}\ln\left|\frac{t+\sqrt{2}-1}{\sqrt{-(t+\sqrt{2}-1)(t-\sqrt{2}-1)}}\right| + C = \sqrt{2}\ln\left|\sqrt{-\frac{t+\sqrt{2}-1}{t-\sqrt{2}-1}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan(x/2)+\sqrt{2}-1}{\tan(x/2)-\sqrt{2}-1}\right| + C.$$



40. We must show that $\dfrac{1+\tan(x/2)}{1-\tan(x/2)} = \sec x + \tan x$.

$$\frac{1+\tan(x/2)}{1-\tan(x/2)} = \frac{1+\dfrac{\sin(x/2)}{\cos(x/2)}}{1-\dfrac{\sin(x/2)}{\cos(x/2)}} = \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)} = \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\,\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)+\sin(x/2)}$$

$$= \frac{\cos^2(x/2)+2\sin(x/2)\cos(x/2)+\sin^2(x/2)}{\cos^2(x/2)-\sin^2(x/2)} = \frac{1+\sin x}{\cos x} = \sec x + \tan x.$$

41. (a) With the substitution from Exercise 35,

$$\int \frac{1}{5-4\cos x}dx = \int \frac{1}{5-\dfrac{4(1-t^2)}{1+t^2}}\frac{2}{1+t^2}dt = 2\int \frac{1}{1+9t^2}dt.$$

If we now set $t = (1/3)\tan\theta$ and $dt = (1/3)\sec^2\theta\,d\theta$, then

$$\int \frac{1}{5-4\cos x}dx = 2\int \frac{1}{\sec^2\theta}\frac{1}{3}\sec^2\theta\,d\theta = \frac{2\theta}{3}+C = \frac{2}{3}\operatorname{Tan}^{-1}3t + C = \frac{2}{3}\operatorname{Tan}^{-1}\left[3\tan\left(\frac{x}{2}\right)\right] + C.$$

(b) $\displaystyle\int_0^{2\pi} \frac{1}{5-4\cos x}dx = \left\{\frac{2}{3}\operatorname{Tan}^{-1}\left[3\tan\left(\frac{x}{2}\right)\right]\right\}_0^{2\pi} = 0$

This cannot be correct because the integrand is always positive.

(c) To verify that the function is an antiderivative, we differentiate it, obtaining

$$\frac{1}{3} + \frac{2/3}{1+\dfrac{\sin^2 x}{(2-\cos x)^2}}\left[\frac{(2-\cos x)(\cos x) - \sin x(\sin x)}{(2-\cos x)^2}\right],$$

and this simplifies to $1/(5-4\cos x)$. When we use this antiderivative,

$$\int_0^{2\pi} \frac{1}{5-4\cos x}dx = \left\{\frac{x}{3} + \frac{2}{3}\operatorname{Tan}^{-1}\left(\frac{\sin x}{2-\cos x}\right)\right\}_0^{2\pi} = \frac{2\pi}{3}.$$

42. If we set $x^4+x^3+2x^2+11x-5 = (x^2+bx+c)(x^2+dx+e)$, multiply the right side out, and equate coefficients, we obtain the equations

$$b+d=1, \quad c+bd+e=2, \quad be+cd=11, \quad ce=-5.$$

Solutions of these are $b=-1$, $c=5$, $d=2$, and $e=-1$. The partial fraction decomposition of the integrand therefore takes the form

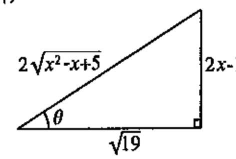
$$\frac{x^2+x+3}{(x^2-x+5)(x^2+2x-1)} = \frac{Ax+B}{x^2-x+5} + \frac{Cx+D}{x^2+2x-1}.$$

We find that $A=-2/21$, $B=4/7$, $C=2/21$, and $D=5/7$. Hence

$$\int \frac{x^2+x+3}{x^4+x^3+2x^2+11x-5}dx = \frac{1}{21}\int\left(\frac{-2x+12}{x^2-x+5} + \frac{2x+15}{x^2+2x-1}\right)dx$$
$$= \frac{2}{21}\int \frac{-x+6}{(x-1/2)^2+19/4}dx + \frac{1}{21}\int \frac{2x+15}{(x+1)^2-2}dx.$$

In the first integral we set $x-1/2 = (\sqrt{19}/2)\tan\theta$ and $dx = (\sqrt{19}/2)\sec^2\theta\,d\theta$,

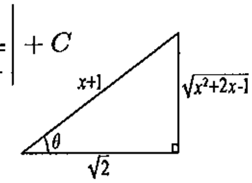
$$\int \frac{-x+6}{(x-1/2)^2+19/4}dx = \int \frac{-1/2-(\sqrt{19}/2)\tan\theta+6}{(19/4)\sec^2\theta}\frac{\sqrt{19}}{2}\sec^2\theta\, d\theta$$
$$= \frac{1}{\sqrt{19}}\int(11-\sqrt{19}\tan\theta)\,d\theta = \frac{1}{\sqrt{19}}(11\theta+\sqrt{19}\ln|\cos\theta|)+C$$
$$= \frac{11}{\sqrt{19}}\operatorname{Tan}^{-1}\left(\frac{2x-1}{\sqrt{19}}\right)+\ln\left|\frac{\sqrt{19}}{2\sqrt{x^2-x+5}}\right|+C$$



$$= \frac{11}{\sqrt{19}}\operatorname{Tan}^{-1}\left(\frac{2x-1}{\sqrt{19}}\right)-\frac{1}{2}\ln(x^2-x+5)+D.$$

In the second integral we set $x+1=\sqrt{2}\sec\theta$ and $dx=\sqrt{2}\sec\theta\tan\theta\, d\theta$,

$$\int\frac{2x+15}{(x+1)^2-2}dx = \int\frac{2(\sqrt{2}\sec\theta-1)+15}{2\tan^2\theta}\sqrt{2}\sec\theta\tan\theta\,d\theta = \frac{1}{\sqrt{2}}\int\frac{\sec\theta}{\tan\theta}(2\sqrt{2}\sec\theta+13)\,d\theta$$
$$= \frac{1}{\sqrt{2}}\int\left(\frac{2\sqrt{2}\sec^2\theta}{\tan\theta}+13\csc\theta\right)d\theta = 2\ln|\tan\theta|+\frac{13}{\sqrt{2}}\ln|\csc\theta-\cot\theta|+C$$
$$= 2\ln\left|\frac{\sqrt{x^2+2x-1}}{\sqrt{2}}\right|+\frac{13}{\sqrt{2}}\ln\left|\frac{x+1}{\sqrt{x^2+2x-1}}-\frac{\sqrt{2}}{\sqrt{x^2+2x-1}}\right|+C$$



$$= \frac{13}{\sqrt{2}}\ln|x+1-\sqrt{2}|+\frac{2\sqrt{2}-13}{2\sqrt{2}}\ln(x^2+2x-1)+D.$$

Thus,
$$\int\frac{x^2+x+3}{x^4+x^3+2x^2+11x-5}dx = \frac{22}{21\sqrt{19}}\operatorname{Tan}^{-1}\left(\frac{2x-1}{\sqrt{19}}\right)-\frac{1}{21}\ln(x^2-x+5)$$
$$+\frac{13}{21\sqrt{2}}\ln|x+1-\sqrt{2}|+\frac{2\sqrt{2}-13}{42\sqrt{2}}\ln(x^2+2x-1)+G.$$

43. If we set $x^4+3x^3+x^2+2x-12=(x^2+bx+c)(x^2+dx+e)$, multiply the right side out, and equate coefficients, we obtain the equations

$$b+d=3,\quad c+bd+e=1,\quad be+cd=2,\quad ce=-12.$$

Solutions of these are $b=1$, $c=3$, $d=2$, and $e=-4$. The partial fraction decomposition of the integrand therefore takes the form

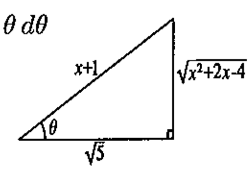
$$\frac{2x^3+8x^2-3x+5}{x^4+3x^3+x^2+2x-12}=\frac{Ax+B}{x^2+x+3}+\frac{Cx+D}{x^2+2x-4}.$$

We find that $A=2$, $B=1$, $C=0$, and $D=3$, so that

$$\int\frac{2x^3+8x^2-3x+5}{x^4+3x^3+x^2+2x-12}dx=\int\left[\frac{2x+1}{x^2+x+3}+\frac{3}{(x+1)^2-5}\right]dx.$$

In the second integral we set $x+1=\sqrt{5}\sec\theta$ and $dx=\sqrt{5}\sec\theta\tan\theta\,d\theta$,

$$\int\frac{2x^3+8x^2-3x+5}{x^4+3x^3+x^2+2x-12}dx=\ln(x^2+x+3)+3\int\frac{1}{5\tan^2\theta}\sqrt{5}\sec\theta\tan\theta\,d\theta$$
$$=\ln(x^2+x+3)+\frac{3}{\sqrt{5}}\int\csc\theta\,d\theta$$



$$=\ln(x^2+x+3)+\frac{3}{\sqrt{5}}\ln|\csc\theta-\cot\theta|+C$$
$$=\ln(x^2+x+3)+\frac{3}{\sqrt{5}}\ln\left|\frac{x+1}{\sqrt{x^2+2x-4}}-\frac{\sqrt{5}}{\sqrt{x^2+2x-4}}\right|+C$$
$$=\ln(x^2+x+3)+\frac{3}{\sqrt{5}}\ln|x+1-\sqrt{5}|-\frac{3}{2\sqrt{5}}\ln(x^2+2x-4)+C.$$

EXERCISES 8.7

1. With the trapezoidal rule,

$$\int_1^2 \frac{1}{x}dx \approx \frac{1/10}{2}\left(\frac{1}{1}+2\sum_{i=1}^{9}\frac{1}{1+i/10}+\frac{1}{2}\right)=\frac{1}{20}\left(1+20\sum_{i=1}^{9}\frac{1}{10+i}+\frac{1}{2}\right)=0.693\,77.$$

With Simpson's rule, $\int_1^2 \frac{1}{x}dx \approx \frac{1/10}{3}\left(1+\frac{4}{1.1}+\frac{2}{1.2}+\cdots+\frac{2}{1.8}+\frac{4}{1.9}+\frac{1}{2}\right)=0.693\,15.$

Analytically, $\int_1^2 \frac{1}{x}dx = \left\{\ln x\right\}_1^2 = \ln 2 \approx 0.693\,15.$

2. With the trapezoidal rule,

$$\int_2^3 \frac{1}{\sqrt{x+2}}dx \approx \frac{1/10}{2}\left(\frac{1}{\sqrt{4}}+2\sum_{i=1}^{9}\frac{1}{\sqrt{\frac{i}{10}+4}}+\frac{1}{\sqrt{5}}\right)=\frac{1}{20}\left(\frac{1}{2}+2\sqrt{10}\sum_{i=1}^{9}\frac{1}{\sqrt{40+i}}+\frac{1}{\sqrt{5}}\right)=0.472\,15.$$

With Simpson's rule,

$$\int_2^3 \frac{1}{\sqrt{x+2}}dx \approx \frac{1/10}{3}\left(\frac{1}{2}+\frac{4}{\sqrt{4.1}}+\frac{2}{\sqrt{4.2}}+\cdots+\frac{2}{\sqrt{4.8}}+\frac{4}{\sqrt{4.9}}+\frac{1}{\sqrt{5}}\right)=0.472\,14.$$

Analytically, $\int_2^3 \frac{1}{\sqrt{x+2}}dx = \left\{2\sqrt{x+2}\right\}_2^3 = 2\sqrt{5}-4 \approx 0.472\,14.$

3. With the trapezoidal rule, $\int_0^1 \tan x\,dx \approx \frac{1/10}{2}\left(\tan 0 + 2\sum_{i=1}^{9}\tan(i/10)+\tan 1\right)=0.617\,64.$

With Simpson's rule,

$$\int_0^1 \tan x\,dx \approx \frac{1/10}{3}\left[\tan 0 + 4\,\tan(0.1)+2\,\tan(0.2)+\cdots+2\,\tan(0.8)+4\,\tan(0.9)+\tan 1\right]=0.615\,65.$$

Analytically, $\int_0^1 \tan x\,dx = \left\{\ln(\sec x)\right\}_0^1 = \ln(\sec 1) \approx 0.615\,63.$

4. With the trapezoidal rule,

$$\int_0^{1/2} e^x\,dx \approx \frac{1/20}{2}\left(e^0+2\sum_{i=1}^{9}e^{i/20}+e^{1/2}\right)=\frac{1}{40}\left(1+2\sum_{i=1}^{9}e^{i/20}+\sqrt{e}\right)=0.648\,86.$$

With Simpson's rule, $\int_0^{1/2} e^x\,dx \approx \frac{1/20}{3}\left(e^0+4e^{1/20}+2e^{1/10}+\cdots+2e^{2/5}+4e^{9/20}+e^{1/2}\right)=0.648\,72.$

Analytically, $\int_0^{1/2} e^x\,dx = \left\{e^x\right\}_0^{1/2} = \sqrt{e}-1 \approx 0.648\,72.$

5. With the trapezoidal rule,

$$\int_{-1}^1 \sqrt{x+1}\,dx \approx \frac{1/5}{2}\left(0+2\sum_{i=1}^{9}\sqrt{(-1+i/5)+1}+\sqrt{2}\right)=\frac{1}{10}\left(\frac{2}{\sqrt{5}}\sum_{i=1}^{9}\sqrt{i}+\sqrt{2}\right)=1.868\,2.$$

With Simpson's rule,

$$\int_{-1}^1 \sqrt{x+1}\,dx \approx \frac{1/5}{3}\left(0+4\sqrt{1-4/5}+2\sqrt{1-3/5}+\cdots+2\sqrt{1+3/5}+4\sqrt{1+4/5}+\sqrt{2}\right)=1.878\,4.$$

Analytically, $\int_{-1}^1 \sqrt{x+1}\,dx = \left\{\frac{2(x+1)^{3/2}}{3}\right\}_{-1}^1 = \frac{4\sqrt{2}}{3} \approx 1.885\,6.$

6. With the trapezoidal rule,

$$\int_{-3}^{-2} \frac{1}{x^3}dx \approx \frac{1/10}{2}\left[-\frac{1}{27}+2\sum_{i=1}^{9}\frac{1}{(-3+i/10)^3}-\frac{1}{8}\right]=\frac{1}{20}\left[-\frac{1}{27}-2000\sum_{i=1}^{9}\frac{1}{(30-i)^3}-\frac{1}{8}\right]=-0.069\,570.$$

With Simpson's rule,

$$\int_{-3}^{-2} \frac{1}{x^3}dx \approx \frac{1/10}{3}\left[-\frac{1}{27}+\frac{4}{(-2.9)^3}+\frac{2}{(-2.8)^3}+\cdots+\frac{2}{(-2.2)^3}+\frac{4}{(-2.1)^3}-\frac{1}{8}\right]=-0.069\,445.$$

Analytically, $\displaystyle\int_{-3}^{-2}\frac{1}{x^3}dx=\left\{-\frac{1}{2x^2}\right\}_{-3}^{-2}=-\frac{5}{72}\approx -0.069\,444.$

7. With the trapezoidal rule, $\displaystyle\int_{1/2}^{1}\cos x\,dx \approx \frac{1/20}{2}\left[\cos(1/2)+2\sum_{i=1}^{9}\cos(1/2+i/20)+\cos 1\right]=0.361\,97.$

With Simpson's rule,

$$\int_{1/2}^{1}\cos x\,dx \approx \frac{1/20}{3}[\cos(0.5)+4\cos(0.55)+2\cos(0.6)+\cdots+2\cos(0.9)+4\cos(0.95)+\cos 1]=0.362\,05.$$

Analytically, $\displaystyle\int_{1/2}^{1}\cos x\,dx=\left\{\sin x\right\}_{1/2}^{1}=\sin 1-\sin(1/2)\approx 0.362\,05.$

8. With the trapezoidal rule,

$$\int_0^1 \frac{1}{3+x^2}dx \approx \frac{1/10}{2}\left[\frac{1}{3}+2\sum_{i=1}^{9}\frac{1}{3+(i/10)^2}+\frac{1}{4}\right]=\frac{1}{20}\left(\frac{7}{12}+200\sum_{i=1}^{9}\frac{1}{300+i^2}\right)=0.302\,20.$$

With Simpson's rule,

$$\int_0^1 \frac{1}{3+x^2}dx \approx \frac{1/10}{3}\left[\frac{1}{3}+\frac{4}{3+(1/10)^2}+\frac{2}{3+(2/10)^2}+\cdots+\frac{2}{3+(8/10)^2}+\frac{4}{3+(9/10)^2}+\frac{1}{4}\right]=0.302\,30.$$

Analytically, we set $x=\sqrt{3}\tan\theta$ and $dx=\sqrt{3}\sec^2\theta\,d\theta$,

$$\int_0^1\frac{1}{3+x^2}dx=\int_0^{\pi/6}\frac{1}{3\sec^2\theta}\sqrt{3}\sec^2\theta\,d\theta=\frac{1}{\sqrt{3}}\left\{\theta\right\}_0^{\pi/6}=\frac{\pi}{6\sqrt{3}}\approx 0.302\,30.$$

9. With the trapezoidal rule, $\displaystyle\int_1^3\frac{1}{x^2+x}dx\approx\frac{1/5}{2}\left[\frac{1}{2}+2\sum_{i=1}^{9}\frac{1}{(1+i/5)^2+(1+i/5)}+\frac{1}{12}\right]=0.407\,79.$

With Simpson's rule,

$$\int_1^3\frac{1}{x^2+x}dx\approx\frac{1/5}{3}\left[\frac{1}{2}+\frac{4}{(1.2)^2+1.2}+\frac{2}{(1.4)^2+1.4}+\cdots+\frac{2}{(2.6)^2+2.6}+\frac{4}{(2.8)^2+2.8}+\frac{1}{12}\right]=0.405\,51.$$

Analytically, $\displaystyle\int_1^3\frac{1}{x^2+x}dx=\int_1^3\left(\frac{1}{x}-\frac{1}{x+1}\right)dx=\left\{\ln x-\ln(x+1)\right\}_1^3=\ln(3/2)\approx 0.405\,47.$

10. With the trapezoidal rule,

$$\int_0^{1/2}xe^{x^2}\,dx\approx\frac{1/20}{2}\left[0+2\sum_{i=1}^{9}(i/20)e^{(i/20)^2}+\frac{1}{2}e^{1/4}\right]=\frac{1}{40}\left(\frac{1}{10}\sum_{i=1}^{9}ie^{i^2/400}+\frac{1}{2}e^{1/4}\right)=0.142\,21.$$

With Simpson's rule,

$$\int_0^{1/2}xe^{x^2}\,dx\approx\frac{1/20}{3}\Big[0+4(1/20)e^{1/400}+2(1/10)e^{1/100}+\cdots$$
$$+2(2/5)e^{4/25}+4(9/20)e^{81/400}+(1/2)e^{1/4}\Big]=0.142\,01.$$

Analytically, $\displaystyle\int_0^{1/2} xe^{x^2}\,dx = \left\{\frac{e^{x^2}}{2}\right\}_0^{1/2} = \frac{e^{1/4}-1}{2} \approx 0.142\,01.$

11. With the trapezoidal rule, $\displaystyle\int_0^2 \frac{1}{1+x^3}dx \approx \frac{1/5}{2}\left[1+2\sum_{i=1}^{9}\frac{1}{1+(i/5)^3}+\frac{1}{9}\right] = 1.0895.$

With Simpson's rule,

$$\int_0^2 \frac{1}{1+x^3}dx \approx \frac{1/5}{3}\left[1+\frac{4}{1+(0.2)^3}+\frac{2}{1+(0.4)^3}+\cdots+\frac{2}{1+(1.6)^3}+\frac{4}{1+(1.8)^3}+\frac{1}{9}\right] = 1.0900.$$

12. With the trapezoidal rule, $\displaystyle\int_0^1 e^{x^2}\,dx \approx \frac{1/10}{2}\left[1+2\sum_{i=1}^{9}e^{(i/10)^2}+e\right] = 1.4672.$

With Simpson's rule, $\displaystyle\int_0^1 e^{x^2}\,dx \approx \frac{1/10}{3}\left(1+4e^{0.01}+2e^{0.04}+\cdots+2e^{0.64}+4e^{0.81}+e\right) = 1.4627.$

13. With the trapezoidal rule, $\displaystyle\int_1^2 \sqrt{1+x^4}\,dx \approx \frac{1/10}{2}\left[\sqrt{2}+2\sum_{i=1}^{9}\sqrt{1+(1+i/10)^4}+\sqrt{17}\right] = 2.5661.$

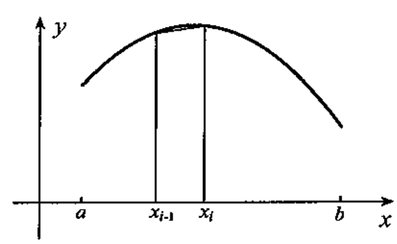
With Simpson's rule,
$$\int_1^2 \sqrt{1+x^4}\,dx \approx \frac{1/10}{3}\Big[\sqrt{2}+4\sqrt{1+(1.1)^4}+2\sqrt{1+(1.2)^4}+\cdots$$
$$+2\sqrt{1+(1.8)^4}+4\sqrt{1+(1.9)^4}+\sqrt{17}\Big] = 2.5641.$$

14. With the trapezoidal rule, and noting that $\sin(x^2)$ is an even function,

$$\int_{-1}^0 \sin(x^2)\,dx = \int_0^1 \sin(x^2)\,dx \approx \frac{1/10}{2}\left[0+2\sum_{i=1}^{9}\sin(i/10)^2+\sin 1\right] = 0.311\,17.$$

With Simpson's rule,
$$\int_{-1}^0 \sin(x^2)\,dx = \int_0^1 \sin(x^2)\,dx \approx \frac{1/10}{3}\Big[0+4\sin(0.01)+2\sin(0.04)+\cdots$$
$$+2\sin(0.64)+4\sin(0.81)+\sin 1\Big] = 0.310\,26.$$

15. The trapezoid on the i^{th} interval between x_{i-1} and x_i underestimates the area under the graph on that subinterval.



16. In equation 8.15 the error is reduced by a factor of 1/4; in equation 8.16 it is reduced by 1/16.

17. With 16 subdivisions, Simpson's rule gives

$$\int_0^1 e^{-x^2}\,dx \approx \frac{1/16}{3}\left[e^0+4e^{-(1/16)^2}+2e^{-(1/8)^2}+\cdots+2e^{-(7/8)^2}+4e^{-(15/16)^2}+e^{-1}\right] = 0.746\,82.$$

18. The length of the parabola is
$$\int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx \approx \frac{1/10}{3}\Big[1+4\sqrt{1+4(1/10)^2}+2\sqrt{1+4(1/5)^2}+\cdots$$
$$+2\sqrt{1+4(4/5)^2}+4\sqrt{1+4(9/10)^2}+\sqrt{5}\Big] = 1.4789.$$

According to Exercise 8.4–39, the length of the parabola is $[2\sqrt{5}+\ln(2+\sqrt{5})]/4$, which to four decimals is also 1.4789.

19. The length of the curve is given by $L = \int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx$. When we use the trapezoidal rule with 10 subdivisions, $L \approx \frac{\pi/20}{2}\left[\sqrt{2} + 2\sum_{i=1}^{9}\sqrt{1+\cos^2(\pi i/20)} + 1\right] = 1.910.$

With Simpson's rule, $L \approx \frac{\pi/20}{3}\left[\sqrt{2} + 4\sqrt{1+\cos^2(\pi/20)} + 2\sqrt{1+\cos^2(\pi/10)} + \cdots\right.$

$\left.+2\sqrt{1+\cos^2(2\pi/5)} + 4\sqrt{1+\cos^2(9\pi/20)} + 1\right] = 1.910.$

20. Using Simpson's rule, the volume in cubic metres is approximately

$$(1.8)\left(\frac{1}{3}\right)[0 + 4(6.0) + 2(7.0) + 4(6.8) + 2(5.8) + 4(4.6) + 2(3.8) + 4(3.6) + 2(3.6) + 4(3.8) + 0] = 83.76.$$

21. Since there is an odd number of subdivisions, we use the trapezoidal rule to approximate the area of the spill

$$\frac{50}{2}[0 + 2(180) + 2(190) + 2(200) + 2(440) + 2(210) + 2(180) + 0] = 70\,000.$$

The volume of oil is approximately 700 m^3.

22. (a) Both rules require the value of the integrand at the lower limit of integration, but $e^x/\sqrt{x}$ is undefined at $x = 0$.

(b) If we set $u = \sqrt{x}$ and $du = 1/(2\sqrt{x})\,dx$, then

$$\int_0^4 \frac{e^x}{\sqrt{x}}dx = \int_0^2 e^{u^2}2\,du = 2\int_0^2 e^{u^2}du,$$

and this integral is no longer improper. With Simpson's rule and 20 equal subdivisions,

$$2\int_0^2 e^{u^2}du \approx 2\left(\frac{1/10}{3}\right)\left[e^0 + 4e^{(0.1)^2} + 2e^{(0.2)^2} + \cdots + 2e^{(1.8)^2} + 4e^{(1.9)^2} + e^4\right] = 32.91.$$

(c) Rectangular rule 8.11 can be used since it does not require the value of $e^x/\sqrt{x}$ at $x = 0$.

23. (a) Since $y = (2/3)\sqrt{9-x^2}$ on the first quadrant part of the ellipse, small lengths thereon can be approximated by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+\left[\frac{(2/3)(-x)}{\sqrt{9-x^2}}\right]^2}dx = \sqrt{\frac{81-9x^2+4x^2}{9(9-x^2)}}dx = \frac{1}{3}\sqrt{\frac{81-5x^2}{9-x^2}}dx.$$

The total length of the ellipse is therefore $L = \frac{4}{3}\int_0^3 \sqrt{\frac{81-5x^2}{9-x^2}}dx$.

(b) If we set $x = 3\sin\theta$ and $dx = 3\cos\theta\,d\theta$, then

$$L = \frac{4}{3}\int_0^{\pi/2}\frac{\sqrt{81-45\sin^2\theta}}{3\cos\theta}3\cos\theta\,d\theta = 4\int_0^{\pi/2}\sqrt{9-5\sin^2\theta}\,d\theta.$$

(c) If we use the trapezoidal rule with 8 subdivisions to approximate the integral,

$$L \approx \frac{4(\pi/16)}{2}\left[3 + 2\sum_{i=1}^{7}\sqrt{9-5\sin^2(\pi i/16)} + 2\right] = 15.865.$$

With Simpson's rule, $L \approx \frac{4(\pi/16)}{3}\left[3 + 4\sqrt{9-5\sin^2(\pi/16)} + 2\sqrt{9-5\sin^2(\pi/8)} + \cdots\right.$

$\left.+2\sqrt{9-5\sin^2(3\pi/8)} + 4\sqrt{9-5\sin^2(7\pi/16)} + 2\right] = 15.865.$

24. If we set $x = 1/t$, then $dx = -(1/t^2)\,dt$, and

$$\int_1^\infty \frac{1}{1+x^4}dx = \int_1^0 \frac{1}{1+t^{-4}}\left(-\frac{1}{t^2}dt\right) = \int_0^1 \frac{t^2}{1+t^4}dt.$$

The trapezoidal rule with 10 subdivisions gives

$$\int_0^1 \frac{t^2}{1+t^4}dt \approx \frac{1/10}{2}\left[0 + 2\sum_{i=1}^{9}\frac{(i/10)^2}{1+(i/10)^4} + \frac{1}{2}\right] = 0.2437.$$

Simpson's rule with the same subdivision yields

$$\int_0^1 \frac{t^2}{1+t^4}dt \approx \frac{1/10}{3}\left[0 + \frac{4(0.1)^2}{1+(0.1)^4} + \frac{2(0.2)^2}{1+(0.2)^4} + \cdots + \frac{4(0.9)^2}{1+(0.9)^4} + \frac{1}{2}\right] = 0.2438.$$

25. If we set $x = 1/t$ and $dx = -dt/t^2$, then

$$\int_1^\infty \frac{x^2}{x^4+x^2+1}dx = \int_1^0 \frac{1/t^2}{1/t^4+1/t^2+1}\left(\frac{dt}{-t^2}\right) = \int_0^1 \frac{1}{t^4+t^2+1}dt.$$

If we use Simpson's rule with 10 subdivisions to approximate this integral,

$$\int_0^1 \frac{1}{t^4+t^2+1}dt \approx \frac{1/10}{3}\left[1 + \frac{4}{(1/10)^4+(1/10)^2+1} + \frac{2}{(1/5)^4+(1/5)^2+1} + \cdots \right.$$
$$\left. + \frac{2}{(4/5)^4+(4/5)^2+1} + \frac{4}{(9/10)^4+(9/10)^2+1} + \frac{1}{3}\right] = 0.728.$$

26. If $f(x)$ is the function, then the trapezoidal rule gives

$$\int_{-1}^4 f(x)\,dx \approx \frac{1/2}{2}\{f(-1) + 2[f(-0.5) + f(0) + \cdots + f(3.5)] + f(4)\} = 2.113.$$

Simpson's rule gives

$$\int_{-1}^4 f(x)\,dx \approx \frac{1/2}{3}[f(-1) + 4f(-0.5) + 2f(0) + \cdots + 2f(3) + 4f(3.5) + f(4)] = 1.729.$$

27. If $f(x)$ is the function, then the trapezoidal rule gives

$$\int_1^3 f(x)\,dx \approx \frac{1/5}{2}\{f(1.0) + 2[f(1.2) + f(1.4) + \cdots + f(2.6) + f(2.8)] + f(3.0)\} = 2.80.$$

Simpson's rule gives

$$\int_1^3 f(x)\,dx \approx \frac{1/5}{3}[f(1.0) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3.0)] = 2.81.$$

28. According to equation 8.16, the maximum error in approximating the definite integral of $f(x)$ from $x = a$ to $x = b$ with n equal subdivisions is given by $|S_n| = M(b-a)^5/(180n^4)$ where M is the maximum value of $|f''''(x)|$ on $a \le x \le b$. But if $f(x)$ is a cubic polynomial, $f''''(x) = 0$ for all x. Hence, $S_n = 0$. For example,

$$\int_1^2 (x^3+1)\,dx = \left\{\frac{x^4}{4} + x\right\}_1^2 = \frac{19}{4}.$$

Simpson's rule with 10 equal subdivisions, and $f(x) = x^3 + 1$, gives

$$\int_1^2 (x^3+1)\,dx \approx \frac{1/10}{3}[f(1) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2)] = 4.75.$$

29. (a) According to equation 8.15, the error in using the trapezoidal rule with n equal partitions is $|T_n| \leq M(3)^3/(12n^2)$ where M is the maximum of the absolute value of the second derivative of $1/x$ on $1 \leq x \leq 4$. Since $d^2(1/x)/dx^2 = 2/x^3$, it follows that $M = 2$, and $|T_n| \leq 2(3)^3/(12n^2) = 9/(2n^2)$. For $|T_n|$ to be less than 10^{-4}, we require

$$\frac{9}{2n^2} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt{\frac{9(10^4)}{2}} = 212.1.$$

At least 213 subdivisions should be used.

(b) According to equation 8.16, the error in using Simpson's rule with n equal partitions is $|S_n| \leq M(3)^5/(180n^4)$ where M is the maximum of the absolute value of the fourth derivative of $1/x$ on $1 \leq x \leq 4$. Since $d^4(1/x)/dx^4 = 24/x^5$, it follows that $M = 24$, and $|S_n| \leq 24(3)^5/(180n^4) = 162/(5n^4)$. For $|S_n|$ to be less than 10^{-4}, we require

$$\frac{162}{5n^4} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt[4]{\frac{162(10^4)}{5}} = 23.9.$$

We should use at least 24 subdivisions.

30. (a) According to equation 8.15, the error in using the trapezoidal rule with n equal partitions is $|T_n| \leq M(\pi/4)^3/(12n^2)$ where M is the maximum of the absolute value of the second derivative of $\cos x$ on $0 \leq x \leq \pi/4$. Since $d^2(\cos x)/dx^2 = -\cos x$, it follows that $M = 1$, and $|T_n| \leq \pi^3/(768n^2)$. For $|T_n|$ to be less than 10^{-4}, we require

$$\frac{\pi^3}{768n^2} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt{\frac{10^4\pi^3}{768}} = 20.09.$$

Thus, at least 21 subdivisions should be used.

(b) According to equation 8.16, the error in using Simpson's rule with n equal partitions is $|S_n| \leq M(\pi/4)^5/(180n^4)$ where M is the maximum of the absolute value of the fourth derivative of $\cos x$ on $0 \leq x \leq \pi/4$. Since $d^4(\cos x)/dx^4 = \cos x$, it follows that $M = 1$, and $|S_n| \leq (\pi/4)^5/(180n^4)$. For $|S_n|$ to be less than 10^{-4}, we require

$$\frac{\pi^5}{180(4)^5n^4} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt[4]{\frac{10^4\pi^5}{180(4)^5}} = 2.02.$$

Since n must be even, we should use at least 4 subdivisions.

31. (a) According to equation 8.15, the error in using the trapezoidal rule with n equal partitions is $|T_n| \leq M(1/3)^3/(12n^2)$ where M is the maximum of the absolute value of the second derivative of e^{2x} on $0 \leq x \leq 1/3$. Since $d^2(e^{2x})/dx^2 = 4e^{2x}$, it follows that $M = 4e^{2/3}$, and $|T_n| \leq 4e^{2/3}(1/3)^3/(12n^2) = e^{2/3}/(81n^2)$. For $|T_n|$ to be less than 10^{-4}, we require

$$\frac{e^{2/3}}{81n^2} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt{\frac{10^4(e^{2/3})}{81}} = 15.5.$$

At least 16 subdivisions should be used.

(b) According to equation 8.16, the error in using Simpson's rule with n equal partitions is $|S_n| \leq M(1/3)^5/(180n^4)$ where M is the maximum of the absolute value of the fourth derivative of e^{2x} on $0 \leq x \leq 1/3$. Since $d^4(e^{2x})/dx^4 = 16e^{2x}$, it follows that $M = 16e^{2/3}$, and $|S_n| \leq 16e^{2/3}(1/3)^5/(180n^4) = 4e^{2/3}/[45(3^5)n^4]$. For $|S_n|$ to be less than 10^{-4}, we require

$$\frac{4e^{2/3}}{45(3)^5n^4} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt[4]{\frac{4(10^4)e^{2/3}}{45(3)^5}} = 1.6.$$

We need only use 2 subdivisions.

32. (a) According to equation 8.15, the error in using the trapezoidal rule with n equal partitions is $|T_n| \le M(5-4)^3/(12n^2)$ where M is the maximum of the absolute value of the second derivative of $1/\sqrt{x+2}$ on $4 \le x \le 5$. Since $d^2(1/\sqrt{x+2})/dx^2 = (3/4)(x+2)^{-5/2}$, it follows that $M = (3/4)6^{-5/2}$, and $|T_n| \le (3/4)6^{-5/2}/(12n^2)$. For $|T_n|$ to be less than 10^{-4}, we require

$$\frac{1}{16(6^{5/2})n^2} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt{\frac{10^4}{16(6^{5/2})}} = 2.66.$$

Thus, at least 3 subdivisions should be used.

(b) According to equation 8.16, the error in using Simpson's rule with n equal partitions is $|S_n| \le M(5-4)^5/(180n^4)$ where M is the maximum of the absolute value of the fourth derivative of $1/\sqrt{x+2}$ on $4 \le x \le 5$. Since $d^4(1/\sqrt{x+2})/dx^4 = (105/16)(x+2)^{-9/2}$, it follows that $M = (105/16)6^{-9/2}$, and $|S_n| \le (105/16)6^{-9/2}/(180n^4)$. For $|S_n|$ to be less than 10^{-4}, we require

$$\frac{105}{16(180)(6^{9/2})n^4} < 10^{-4} \qquad \Longrightarrow \qquad n > \sqrt[4]{\frac{105(10^4)}{16(180)(6^{9/2})}} = 0.58.$$

Since n must be even, only 2 subdivisions are needed.

REVIEW EXERCISES

1. $\displaystyle\int \sqrt{2-x}\,dx = -\frac{2}{3}(2-x)^{3/2} + C$

2. $\displaystyle\int \frac{1}{(x+3)^2}dx = -\frac{1}{x+3} + C$

3. $\displaystyle\int \frac{x^2+3}{x}dx = \int \left(x + \frac{3}{x}\right)dx = \frac{x^2}{2} + 3\ln|x| + C$

4. $\displaystyle\int \frac{x^2+3}{x+1}dx = \int \left(x - 1 + \frac{4}{x+1}\right)dx = \frac{x^2}{2} - x + 4\ln|x+1| + C$

5. $\displaystyle\int \frac{x^2+3}{x^2+1}dx = \int \left(1 + \frac{2}{1+x^2}\right)dx = x + 2\,\mathrm{Tan}^{-1}x + C$

6. If we set $u = x+3$ and $du = dx$, then

$$\int \frac{x}{\sqrt{x+3}}dx = \int \frac{u-3}{\sqrt{u}}du = \int (\sqrt{u} - 3u^{-1/2})\,du = \frac{2}{3}u^{3/2} - 6u^{1/2} + C = \frac{2}{3}(x+3)^{3/2} - 6\sqrt{x+3} + C.$$

7. $\displaystyle\int \sin^2 x\,\cos^3 x\,dx = \int \sin^2 x(1-\sin^2 x)\cos x\,dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

8. If we set $u = x$, $dv = \sin x\,dx$, then $du = dx$, $v = -\cos x$, and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C.$$

9. $\displaystyle\int \tan^2(2x)\,dx = \int [\sec^2(2x) - 1]\,dx = \frac{1}{2}\tan(2x) - x + C$

10. $\displaystyle\int \frac{x}{x^2+2x-3}dx = \int \left(\frac{3/4}{x+3} + \frac{1/4}{x-1}\right)dx = \frac{3}{4}\ln|x+3| + \frac{1}{4}\ln|x-1| + C$

11. If we set $x = (2/\sqrt{3})\sin\theta$ and $dx = (2/\sqrt{3})\cos\theta\,d\theta$, then

$$\int \frac{1}{\sqrt{4-3x^2}}dx = \int \frac{1}{2\cos\theta}\frac{2}{\sqrt{3}}\cos\theta\,d\theta = \frac{\theta}{\sqrt{3}} + C = \frac{1}{\sqrt{3}}\mathrm{Sin}^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C.$$

12. If we set $u = \sqrt{x} + 5$, then $du = 1/(2\sqrt{x})dx$, and

$$\int \frac{2-\sqrt{x}}{\sqrt{x}+5}dx = \int \frac{2-(u-5)}{u}(2)(u-5)\,du = 2\int \frac{(7-u)(u-5)}{u}du$$

$$= 2\int\left(-\frac{35}{u} + 12 - u\right)du = 2\left(-35\ln|u| + 12u - \frac{u^2}{2}\right) + C$$
$$= -70\ln|\sqrt{x}+5| + 24(\sqrt{x}+5) - (\sqrt{x}+5)^2 + C$$
$$= -70\ln(\sqrt{x}+5) + 14\sqrt{x} - x + D.$$

13. $\displaystyle\int \frac{x}{3x^2+4}dx = \frac{1}{6}\ln(3x^2+4) + C$

14. If we set $u = e^x$, then $du = e^x\,dx$, and $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{1}{\sqrt{1-u^2}}du = \mathrm{Sin}^{-1}u + C = \mathrm{Sin}^{-1}(e^x) + C.$

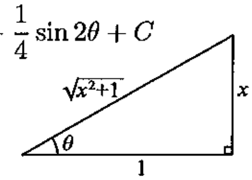
15. If we set $u = \ln x$, $dv = x^2\,dx$, $du = (1/x)\,dx$, and $v = x^3/3$, then

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

16. $\displaystyle\int \frac{x}{(x^2+1)^2}dx = -\frac{1}{2(x^2+1)} + C$

17. If we set $x = \tan\theta$ and $dx = \sec^2\theta\,d\theta$, then

$$\int \frac{x^2}{(1+x^2)^2}dx = \int \frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\,d\theta = \int \frac{\tan^2\theta}{\sec^2\theta}d\theta = \int \frac{\sec^2\theta - 1}{\sec^2\theta}d\theta$$
$$= \int (1-\cos^2\theta)\,d\theta = \int \sin^2\theta\,d\theta = \int\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C$$
$$= \frac{1}{2}\mathrm{Tan}^{-1}x - \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + C$$
$$= \frac{1}{2}\mathrm{Tan}^{-1}x - \frac{x}{2(x^2+1)} + C.$$



18. If we set $u = x^2+1$, then $du = 2x\,dx$, and

$$\int \frac{x^3}{(x^2+1)^2}dx = \int \frac{u-1}{u^2}\left(\frac{du}{2}\right) = \frac{1}{2}\int\left(\frac{1}{u} - \frac{1}{u^2}\right)du = \frac{1}{2}\left(\ln|u| + \frac{1}{u}\right) + C$$
$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.$$

19. If we set $\dfrac{x+1}{x(x-2)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$, then $A = -1/4$, $B = 3/8$, and $C = -1/8$. Hence,

$$\int \frac{x+1}{x^3-4x}dx = \int\left(\frac{-1/4}{x} + \frac{3/8}{x-2} - \frac{1/8}{x+2}\right)dx = -\frac{1}{4}\ln|x| + \frac{3}{8}\ln|x-2| - \frac{1}{8}\ln|x+2| + C.$$

20. $\displaystyle\int\left(\frac{x+1}{x-1}\right)^2 dx = \int\left(1 + \frac{2}{x-1}\right)^2 dx = \int\left[1 + \frac{4}{x-1} + \frac{4}{(x-1)^2}\right]dx = x + 4\ln|x-1| - \frac{4}{x-1} + C$

21. $\displaystyle\int \frac{x^2}{(1+3x^3)^4}dx = \frac{-1}{27(1+3x^3)^3} + C$

22. If we set $u = \mathrm{Cos}^{-1}x$, $dv = dx$, then $du = (-1/\sqrt{1-x^2})dx$, $v = x$, and

$$\int \mathrm{Cos}^{-1}x\,dx = x\,\mathrm{Cos}^{-1}x - \int \frac{-x}{\sqrt{1-x^2}}dx = x\,\mathrm{Cos}^{-1}x - \sqrt{1-x^2} + C.$$

23. $\displaystyle\int \sin x\,\cos 2x\,dx = \int \sin x(2\cos^2 x - 1)\,dx = -\frac{2}{3}\cos^3 x + \cos x + C$

24. Using identity 1.48b, $\displaystyle\int \sin x \cos 5x\, dx = \frac{1}{2}\int (\sin 6x - \sin 4x)\, dx = -\frac{1}{12}\cos 6x + \frac{1}{8}\cos 4x + C.$

25. If we set $u = e^{3x}$, $dv = \cos 2x\, dx$, $du = 3e^{3x}\, dx$, and $v = (1/2)\sin 2x$,

$$\int e^{3x}\cos 2x\, dx = \frac{1}{2}e^{3x}\sin 2x - \int \frac{3}{2}e^{3x}\sin 2x\, dx.$$

We now set $u = e^{3x}$, $dv = \sin 2x\, dx$, $du = 3e^{3x}\, dx$, and $v = -(1/2)\cos 2x$,

$$\int e^{3x}\cos 2x\, dx = \frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\left(-\frac{1}{2}e^{3x}\cos 2x - \int -\frac{3}{2}e^{3x}\cos 2x\, dx\right).$$

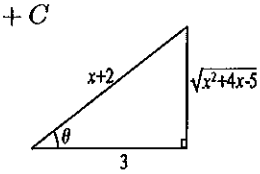
When we combine both integrals of $e^{3x}\cos 2x$, we obtain

$$\left(1 + \frac{9}{4}\right)\int e^{3x}\cos 2x\, dx = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x,$$

from which $\displaystyle\int e^{3x}\cos 2x\, dx = \frac{e^{3x}}{13}(2\sin 2x + 3\cos 2x) + C.$

26. Since $x^2 + 4x - 5 = (x+2)^2 - 9$, we set $x + 2 = 3\sec\theta$, in which case $dx = 3\sec\theta\,\tan\theta\, d\theta$, and

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+4x-5}}dx &= \int \frac{1}{3\tan\theta}3\sec\theta\,\tan\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C\\ &= \ln\left|\frac{x+2}{3} + \frac{\sqrt{x^2+4x-5}}{3}\right| + C\\ &= \ln|x + 2 + \sqrt{x^2+4x-5}| + D.\end{aligned}$$



27. If we set $\dfrac{1}{(x+5)(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{x-1}$, then $A = -1/6$ and $B = 1/6$. Hence,

$$\int \frac{1}{x^2+4x-5}dx = \int\left(\frac{-1/6}{x+5} + \frac{1/6}{x-1}\right)dx = -\frac{1}{6}\ln|x+5| + \frac{1}{6}\ln|x-1| + C.$$

28. If we set $u = 4 - x^2$, then $du = -2x\, dx$, and

$$\begin{aligned}\int x^3\sqrt{4-x^2}\, dx &= \int (4-u)\sqrt{u}\left(-\frac{du}{2}\right) = \frac{1}{2}\int (u^{3/2} - 4\sqrt{u})\, du\\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right) + C = \frac{1}{5}(4-x^2)^{5/2} - \frac{4}{3}(4-x^2)^{3/2} + C.\end{aligned}$$

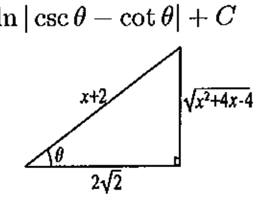
29. $\displaystyle\int \frac{\cos 2x}{1 - \sin 2x}dx = -\frac{1}{2}\ln(1 - \sin 2x) + C$

30. $\displaystyle\int \frac{6x}{4-x^2}dx = -3\ln|4 - x^2| + C$

31. If we set $u = \ln x$, then $du = (1/x)\, dx$, and $\displaystyle\int \frac{1}{x\sqrt{\ln x}}dx = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} + C = 2\sqrt{\ln x} + C.$

32. Since $x^2 + 4x - 4 = (x+2)^2 - 8$, we set $x + 2 = 2\sqrt{2}\sec\theta$, in which case $dx = 2\sqrt{2}\sec\theta\,\tan\theta\, d\theta$, and

$$\begin{aligned}\int \frac{1}{x^2+4x-4}dx &= \int \frac{1}{8\tan^2\theta}2\sqrt{2}\sec\theta\,\tan\theta\, d\theta = \frac{1}{2\sqrt{2}}\int \csc\theta\, d\theta = \frac{1}{2\sqrt{2}}\ln|\csc\theta - \cot\theta| + C\\ &= \frac{1}{2\sqrt{2}}\ln\left|\frac{x+2}{\sqrt{x^2+4x-4}} - \frac{2\sqrt{2}}{\sqrt{x^2+4x-4}}\right| + C\\ &= \frac{1}{2\sqrt{2}}\ln\left|\frac{x+2-2\sqrt{2}}{\sqrt{x^2+4x-4}}\right| + C.\end{aligned}$$



33. If we set $u = \cos x$, then $du = -\sin x\, dx$, and

$$\int \frac{\sin x}{1+\cos^2 x}dx = \int \frac{1}{1+u^2}(-du) = -\mathrm{Tan}^{-1}u + C = -\mathrm{Tan}^{-1}(\cos x) + C.$$

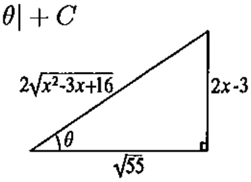
34. When we set $\dfrac{1}{x^4+x^3} = \dfrac{1}{x^3(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x+1}$, we find that $A = 1$, $B = -1$, $C = 1$, and $D = -1$. Thus, $\displaystyle\int \frac{1}{x^4+x^3}dx = \int \left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x+1}\right) dx = \ln|x| + \frac{1}{x} - \frac{1}{2x^2} - \ln|x+1| + C.$

35. If we set $u = x$, $dv = \sec^2(3x)\,dx$, $du = dx$, and $v = (1/3)\tan(3x)$, then

$$\int x\sec^2(3x)\,dx = \frac{x}{3}\tan(3x) - \int \frac{1}{3}\tan(3x)\,dx = \frac{x}{3}\tan(3x) - \frac{1}{9}\ln|\sec(3x)| + C.$$

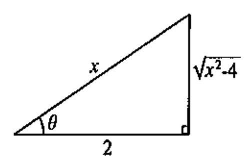
36. Since $16 - 3x + x^2 = (x - 3/2)^2 + 55/4$, we set $x - 3/2 = (\sqrt{55}/2)\tan\theta$. Then $dx = (\sqrt{55}/2)\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{\sqrt{16-3x+x^2}}dx &= \int \frac{1}{(\sqrt{55}/2)\sec\theta}(\sqrt{55}/2)\sec^2\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{2\sqrt{x^2-3x+16}}{\sqrt{55}} + \frac{2x-3}{\sqrt{55}}\right| + C \\ &= \ln|2\sqrt{x^2-3x+16} + 2x - 3| + D.\end{aligned}$$



37. If we set $x = 2\sec\theta$, then $dx = 2\sec\theta\tan\theta\,d\theta$, and

$$\begin{aligned}\int \frac{\sqrt{x^2-4}}{x^2}dx &= \int \frac{2\tan\theta}{4\sec^2\theta}2\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}d\theta \\ &= \int (\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C \\ &= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + C \\ &= \ln|x + \sqrt{x^2-4}| - \frac{\sqrt{x^2-4}}{x} + D.\end{aligned}$$



38. When we set $u = \operatorname{Tan}^{-1}x$, $dv = x^2\,dx$, then $du = \dfrac{1}{1+x^2}dx$, $v = x^3/3$, and

$$\begin{aligned}\int x^2\operatorname{Tan}^{-1}x\,dx &= \frac{x^3}{3}\operatorname{Tan}^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx = \frac{x^3}{3}\operatorname{Tan}^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx \\ &= \frac{x^3}{3}\operatorname{Tan}^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C.\end{aligned}$$

39. If we set $u = x + 1$ and $du = dx$, then

$$\begin{aligned}\int \frac{x^2}{x^3+3x^2+3x+1}dx &= \int \frac{x^2}{(x+1)^3}dx = \int \frac{(u-1)^2}{u^3}du = \int\left(\frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3}\right)du \\ &= \ln|u| + \frac{2}{u} - \frac{1}{2u^2} + C = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.\end{aligned}$$

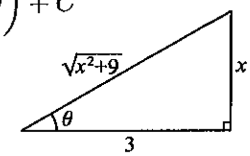
40. If we set $u = \ln x$, then $du = (1/x)\,dx$, and $\displaystyle\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$

41. If we set $2x^3 = \tan\theta$, then $6x^2\,dx = \sec^2\theta\,d\theta$, and

$$\int \frac{x^2}{1+4x^6}dx = \int \frac{1}{\sec^2\theta}\frac{1}{6}\sec^2\theta\,d\theta = \frac{\theta}{6} + C = \frac{1}{6}\operatorname{Tan}^{-1}(2x^3) + C.$$

42. If we set $x = 3\tan\theta$, then $dx = 3\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{1}{x(9+x^2)^2}dx &= \int \frac{1}{3\tan\theta(81\sec^4\theta)}3\sec^2\theta\,d\theta = \frac{1}{81}\int \frac{\cos^3\theta}{\sin\theta}d\theta \\ &= \frac{1}{81}\int \frac{\cos\theta(1-\sin^2\theta)}{\sin\theta}d\theta = \frac{1}{81}\left(\ln|\sin\theta| + \frac{1}{2}\cos^2\theta\right) + C \\ &= \frac{1}{81}\ln\left|\frac{x}{\sqrt{x^2+9}}\right| + \frac{1}{162}\left(\frac{3}{\sqrt{x^2+9}}\right)^2 + C \\ &= \frac{1}{81}\ln|x| - \frac{1}{162}\ln(x^2+9) + \frac{1}{18(x^2+9)} + C.\end{aligned}$$



43. If we set $\dfrac{x^2+2}{x(x+1)(x+4)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x+4}$, then $A = 1/2$, $B = -1$, and $C = 3/2$. Hence,

$$\int \frac{x^2+2}{x^3+5x^2+4x}dx = \int\left(\frac{1/2}{x} - \frac{1}{x+1} + \frac{3/2}{x+4}\right)dx = \frac{1}{2}\ln|x| - \ln|x+1| + \frac{3}{2}\ln|x+4| + C.$$

44. If we set $\dfrac{x^2+2}{x^3+4x^2+4x} = \dfrac{x^2+2}{x(x+2)^2} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$, we find that $A = 1/2$, $B = 1/2$, and $C = -3$. Thus,

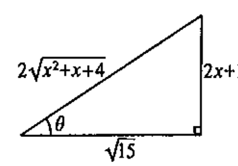
$$\int \frac{x^2+2}{x^3+4x^2+4x}dx = \int\left[\frac{1/2}{x} + \frac{1/2}{x+2} - \frac{3}{(x+2)^2}\right]dx = \frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+2| + \frac{3}{x+2} + C.$$

45. If we set $\dfrac{x^2+2}{x(x^2+x+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+4}$, then $A = 1/2$, $B = 1/2$, and $C = -1/2$. Hence

$$\int \frac{x^2+2}{x^3+x^2+4x}dx = \frac{1}{2}\int\left(\frac{1}{x} + \frac{x-1}{x^2+x+4}\right)dx = \frac{1}{2}\ln|x| + \frac{1}{2}\int\frac{x-1}{(x+1/2)^2+15/4}dx.$$

We set $x + 1/2 = (\sqrt{15}/2)\tan\theta$ and $dx = (\sqrt{15}/2)\sec^2\theta\,d\theta$, in which case

$$\begin{aligned}\int \frac{x^2+2}{x^3+x^2+4x}dx &= \frac{1}{2}\ln|x| + \frac{1}{2}\int\frac{-3/2 + (\sqrt{15}/2)\tan\theta}{(15/4)\sec^2\theta}\frac{\sqrt{15}}{2}\sec^2\theta\,d\theta \\ &= \frac{1}{2}\ln|x| + \frac{1}{2\sqrt{15}}\int(\sqrt{15}\tan\theta - 3)\,d\theta \\ &= \frac{1}{2}\ln|x| + \frac{1}{2\sqrt{15}}(\sqrt{15}\ln|\sec\theta| - 3\theta) + C \\ &= \frac{1}{2}\ln|x| + \frac{1}{2}\ln\left|\frac{2\sqrt{x^2+x+4}}{\sqrt{15}}\right| - \frac{3}{2\sqrt{15}}\mathrm{Tan}^{-1}\left(\frac{2x+1}{\sqrt{15}}\right) + C \\ &= \frac{1}{2}\ln|x| + \frac{1}{4}\ln(x^2+x+4) - \frac{3}{2\sqrt{15}}\mathrm{Tan}^{-1}\left(\frac{2x+1}{\sqrt{15}}\right) + D.\end{aligned}$$



46. If we set $\dfrac{3x^2+2x+4}{x^3+x^2+4x} = \dfrac{3x^2+2x+4}{x(x^2+x+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+4}$, we find that $A = 1$, $B = 2$, and $C = 1$. Thus,

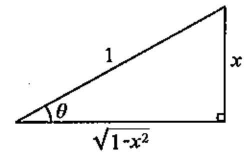
$$\int \frac{3x^2+2x+4}{x^3+x^2+4x}dx = \int\left(\frac{1}{x} + \frac{2x+1}{x^2+x+4}\right)dx = \ln|x| + \ln(x^2+x+4) + C.$$

47. If we set $u = \mathrm{Sin}^{-1}x$, $dv = x\,dx$, $du = \dfrac{1}{\sqrt{1-x^2}}dx$, and $v = \dfrac{x^2}{2}$, then

$$\int x\,\mathrm{Sin}^{-1}x\,dx = \frac{x^2}{2}\mathrm{Sin}^{-1}x - \int\frac{x^2}{2\sqrt{1-x^2}}dx.$$

We now set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$,

$$\int x\,\mathrm{Sin}^{-1}x\,dx = \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{1}{2}\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$
$$= \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$
$$= \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{\theta}{4} + \frac{1}{4}\sin\theta\,\cos\theta + C$$
$$= \frac{x^2}{2}\,\mathrm{Sin}^{-1}x - \frac{1}{4}\,\mathrm{Sin}^{-1}x + \frac{1}{4}(x)\sqrt{1-x^2} + C.$$



48. $\displaystyle\int \sqrt{\cot x}\,\csc^4 x\,dx = \int \sqrt{\cot x}(1+\cot^2 x)\csc^2 x\,dx = -\frac{2}{3}\cot^{3/2}x - \frac{2}{7}\cot^{7/2}x + C$

49. If we set $u = \ln(\sqrt{x}+1)$, $dv = dx$, $du = \dfrac{1}{2\sqrt{x}(\sqrt{x}+1)}dx$, and $v = x$, then

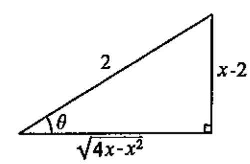
$$\int \ln(\sqrt{x}+1)\,dx = x\,\ln(\sqrt{x}+1) - \int \frac{x}{2\sqrt{x}(\sqrt{x}+1)}dx = x\,\ln(\sqrt{x}+1) - \frac{1}{2}\int \frac{\sqrt{x}}{\sqrt{x}+1}dx.$$

We now set $u = \sqrt{x} \implies x = u^2$, and $dx = 2u\,du$,

$$\int \ln(\sqrt{x}+1)\,dx = x\,\ln(\sqrt{x}+1) - \frac{1}{2}\int \frac{u}{u+1}(2u\,du) = x\,\ln(\sqrt{x}+1) - \int\left(u - 1 + \frac{1}{u+1}\right)du$$
$$= x\,\ln(\sqrt{x}+1) - \frac{u^2}{2} + u - \ln|u+1| + C = x\,\ln(\sqrt{x}+1) - \frac{x}{2} + \sqrt{x} - \ln(\sqrt{x}+1) + C.$$

50. Since $4x - x^2 = -(x-2)^2 + 4$, we set $x - 2 = 2\sin\theta$. Then $dx = 2\cos\theta\,d\theta$, and

$$\int \frac{1}{(4x-x^2)^{3/2}}dx = \int \frac{1}{8\cos^3\theta}2\cos\theta\,d\theta = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C$$
$$= \frac{1}{4}\frac{x-2}{\sqrt{4x-x^2}} + C.$$



51. With the trapezoidal rule and 10 equal partitions,

$$\int_1^2 \frac{\sin x}{x}dx \approx \frac{1/10}{2}\left[\sin 1 + 2\sum_{i=1}^{9}\frac{\sin(1+i/10)}{1+i/10} + \frac{\sin 2}{2}\right] = 0.659\,22.$$

With Simpson's rule, $\displaystyle\int_1^2 \frac{\sin x}{x}dx \approx \frac{1/10}{3}\left[\sin 1 + \frac{4\,\sin(1.1)}{1.1} + \frac{2\,\sin(1.2)}{1.2} + \cdots\right.$

$$\left. + \frac{2\,\sin(1.8)}{1.8} + \frac{4\,\sin(1.9)}{1.9} + \frac{\sin 2}{2}\right] = 0.659\,33.$$

52. With the trapezoidal rule and 10 equal partitions,

$$\int_0^1 \sqrt{\sin x}\,dx \approx \frac{1/10}{2}\left[\sqrt{\sin 0} + 2\sum_{i=1}^{9}\sqrt{\sin(i/10)} + \sqrt{\sin 1}\right] = 0.636\,65.$$

With Simpson's rule, $\displaystyle\int_0^1 \sqrt{\sin x}\,dx \approx \frac{1/10}{3}[\sqrt{\sin 0} + 4\sqrt{\sin(1/10)} + 2\sqrt{\sin(1/5)} + \cdots$

$$+2\sqrt{\sin(4/5)} + 4\sqrt{\sin(9/10)} + \sqrt{\sin 1}] = 0.640\,41.$$

53. With the trapezoidal rule and 10 equal partitions,

$$\int_2^4 \frac{1}{\ln x}dx \approx \frac{1/5}{2}\left[\frac{1}{\ln 2} + 2\sum_{i=1}^{9}\frac{1}{\ln(2+i/5)} + \frac{1}{\ln 4}\right] = 1.9254.$$

With Simpson's rule, $\displaystyle\int_2^4 \frac{1}{\ln x}dx \approx \frac{1/5}{3}\left[\frac{1}{\ln 2} + \frac{4}{\ln 2.2} + \frac{2}{\ln 2.4} + \cdots + \frac{2}{\ln 3.6} + \frac{4}{\ln 3.8} + \frac{1}{\ln 4}\right] = 1.9225.$

54. With the trapezoidal rule and 10 equal subdivisions,

$$\int_{-1}^{3} \frac{1}{1+e^x}dx \approx \frac{2/5}{2}\left[\frac{1}{1+e^{-1}} + 2\sum_{i=1}^{9}\frac{1}{1+e^{-1+2i/5}} + \frac{1}{1+e^3}\right] = 1.2667.$$

With Simpson's rule, $\displaystyle\int_{-1}^{3}\frac{1}{1+e^x}dx \approx \frac{2/5}{3}\left(\frac{1}{1+e^{-1}} + \frac{4}{1+e^{-3/5}} + \frac{2}{1+e^{-1/5}} + \cdots\right.$

$$\left. + \frac{2}{1+e^{11/5}} + \frac{4}{1+e^{13/5}} + \frac{1}{1+e^3}\right) = 1.2647.$$

55. With the trapezoidal rule and 10 equal partitions,

$$\int_0^1 \frac{1}{(1+x^4)^2}\,dx \approx \frac{1/10}{2}\left[1 + 2\sum_{i=1}^{9}\frac{1}{[1+(i/10)^4]^2} + \frac{1}{(1+1^4)^2}\right] = 0.77440.$$

With Simpson's rule, $\displaystyle\int_0^1 \frac{1}{(1+x^4)^2}\,dx \approx \frac{1/10}{3}\left\{1 + 4\left[\frac{1}{(1+0.1^4)^2}\right] + 2\left[\frac{1}{(1+0.2^4)^2}\right] + \cdots\right.$

$$\left. + 2\left[\frac{1}{(1+0.8^4)^2}\right] + 4\left[\frac{1}{(1+0.9^4)^2}\right] + \frac{1}{(1+1^4)^2}\right\}$$

$$= 0.77523.$$

56. If we set $u = x^{1/6}$, or, $x = u^6$, then $dx = 6u^5\,du$, and

$$\int \frac{1}{x^{1/3}-\sqrt{x}}dx = \int \frac{1}{u^2-u^3}6u^5\,du = 6\int\frac{u^3}{1-u}du = 6\int\left(-u^2-u-1+\frac{1}{1-u}\right)du$$

$$= 6\left(-\frac{u^3}{3} - \frac{u^2}{2} - u - \ln|1-u|\right) + C = -2\sqrt{x} - 3x^{1/3} - 6x^{1/6} - 6\ln|1-x^{1/6}| + C.$$

57. If we set $u = \ln(1+x^2)$, $dv = dx$, $du = \dfrac{2x}{1+x^2}dx$, and $v = x$, then

$$\int \ln(1+x^2)\,dx = x\,\ln(1+x^2) - \int\frac{2x^2}{1+x^2}dx = x\,\ln(1+x^2) - 2\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$= x\,\ln(1+x^2) - 2x + 2\,\mathrm{Tan}^{-1}x + C.$$

58. If we set $x^2 = 4\tan\theta$, then $2x\,dx = 4\sec^2\theta\,d\theta$, and

$$\int\frac{x}{x^4+16}dx = \int\frac{1}{16\sec^2\theta}2\sec^2\theta\,d\theta = \frac{\theta}{8} + C = \frac{1}{8}\,\mathrm{Tan}^{-1}\left(\frac{x^2}{4}\right) + C.$$

59. If we set $u = \csc x$, $dv = \csc^2 x\,dx$, $du = -\csc x\,\cot x\,dx$, $v = -\cot x$, then

$$\int\csc^3 x\,dx = -\csc x\,\cot x - \int\csc x\,\cot^2 x\,dx = -\csc x\,\cot x - \int\csc x(\csc^2 x - 1)\,dx$$

$$= -\csc x\,\cot x - \int\csc^3 x\,dx + \ln|\csc x - \cot x|.$$

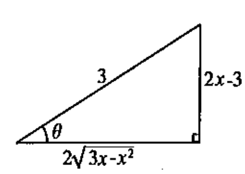
We now solve for $\displaystyle\int\csc^3 x\,dx = \frac{1}{2}\ln|\csc x - \cot x| - \frac{1}{2}\csc x\,\cot x + C.$

60. Since $3x - x^2 = -(x-3/2)^2 + 9/4$, we set $x - 3/2 = (3/2)\sin\theta$, in which case $dx = (3/2)\cos\theta\,d\theta$, and

$$\int\frac{1}{(3x-x^2)^{3/2}}dx = \int\frac{1}{(27/8)\cos^3\theta}(3/2)\cos\theta\,d\theta = \frac{4}{9}\int\sec^2\theta\,d\theta$$

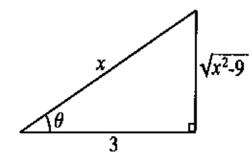
$$= \frac{4}{9}\tan\theta + C = \frac{4}{9}\left(\frac{2x-3}{2\sqrt{3x-x^2}}\right) + C$$

$$= \frac{4x-6}{9\sqrt{3x-x^2}} + C.$$



61. If we set $x = 3\sec\theta$ and $dx = 3\sec\theta\,\tan\theta\,d\theta$, then

$$\int \frac{1}{x^3\sqrt{x^2-9}}dx = \int \frac{1}{27\sec^3\theta(3\tan\theta)}3\sec\theta\,\tan\theta\,d\theta = \frac{1}{27}\int \cos^2\theta\,d\theta = \frac{1}{27}\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{54}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{54}(\theta + \sin\theta\,\cos\theta) + C$$
$$= \frac{1}{54}\operatorname{Sec}^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\left(\frac{\sqrt{x^2-9}}{x}\right)\left(\frac{3}{x}\right) + C$$
$$= \frac{1}{54}\operatorname{Sec}^{-1}\left(\frac{x}{3}\right) + \frac{\sqrt{x^2-9}}{18x^2} + C.$$



62. If we set $y = \sqrt{x}$ and $dy = 1/(2\sqrt{x})\,dx$, then $\int \sin\sqrt{x}\,dx = \int \sin y\,(2y\,dy)$. Now we set $u = y$, $dv = \sin y\,dy$, $du = dy$, $v = -\cos y$, and use integration by parts,

$$\int \sin\sqrt{x}\,dx = 2\left(-y\cos y - \int -\cos y\,dy\right) = -2y\cos y + 2\sin y + C = -2\sqrt{x}\,\cos\sqrt{x} + 2\sin\sqrt{x} + C.$$

63. If we set $u = \sin(\ln x)$, $dv = dx$, $du = \frac{1}{x}\cos(\ln x)$, and $v = x$, then

$$\int \sin(\ln x)\,dx = x\,\sin(\ln x) - \int \cos(\ln x)\,dx.$$

We now set $u = \cos(\ln x)$, $dv = dx$, $du = -\frac{1}{x}\sin(\ln x)$, and $v = x$,

$$\int \sin(\ln x)\,dx = x\,\sin(\ln x) - \left[x\,\cos(\ln x) - \int -\sin(\ln x)\,dx\right].$$

We can now solve for $\int \sin(\ln x)\,dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C$.

64. Using identity 1.48b, $\int x\,\cos x\,\sin 3x\,dx = \int \frac{x}{2}(\sin 2x + \sin 4x)\,dx$. We now set $u = x$, $dv = (\sin 2x + \sin 4x)dx$, in which case $du = dx$, $v = -(1/2)\cos 2x - (1/4)\cos 4x$, and

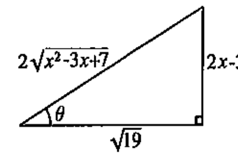
$$\int x\,\cos x\,\sin 3x\,dx = \frac{1}{2}\left[x\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) - \int\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right)dx\right]$$
$$= -\frac{x}{8}(2\cos 2x + \cos 4x) + \frac{1}{2}\left(\frac{1}{4}\sin 2x + \frac{1}{16}\sin 4x\right) + C.$$

65. $\displaystyle\int \frac{x^4+3x^2+1}{x(x^2+1)^2}dx = \int \frac{(x^4+2x^2+1)+x^2}{x(x^2+1)^2}dx = \int\left[\frac{1}{x} + \frac{x}{(x^2+1)^2}\right]dx = \ln|x| - \frac{1}{2(x^2+1)} + C$

66. $\displaystyle\int \frac{1}{1+\cos 2x}dx = \int \frac{1}{1+(2\cos^2 x - 1)}dx = \frac{1}{2}\int \sec^2 x\,dx = \frac{1}{2}\tan x + C$

67. Long division gives $\dfrac{x^4+3x^2-2x+5}{x^2-3x+7} = x^2 + 3x + 5 - \dfrac{8x+30}{x^2-3x+7}$. Consider the integral of $\dfrac{4x+15}{x^2-3x+7} = \dfrac{4x+15}{(x-3/2)^2+19/4}$. If we set $x - 3/2 = (\sqrt{19}/2)\tan\theta$ and $dx = (\sqrt{19}/2)\sec^2\theta\,d\theta$, then

$$\int \frac{4x+15}{(x-3/2)^2+19/4}dx = \int \frac{21+2\sqrt{19}\,\tan\theta}{(19/4)\sec^2\theta}\frac{\sqrt{19}}{2}\sec^2\theta\,d\theta = \frac{2}{\sqrt{19}}\int(21+2\sqrt{19}\,\tan\theta)\,d\theta$$
$$= \frac{2}{\sqrt{19}}(21\theta + 2\sqrt{19}\,\ln|\sec\theta|)+C$$
$$= \frac{42}{\sqrt{19}}\operatorname{Tan}^{-1}\left(\frac{2x-3}{\sqrt{19}}\right)+4\ln\left|\frac{2\sqrt{x^2-3x+7}}{\sqrt{19}}\right|+C$$
$$= \frac{42}{\sqrt{19}}\operatorname{Tan}^{-1}\left(\frac{2x-3}{\sqrt{19}}\right)+2\ln(x^2-3x+7)+D.$$



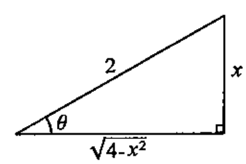
Finally then,

$$\int \frac{x^4+3x^2-2x+5}{x^2-3x+7}dx = \frac{x^3}{3}+\frac{3x^2}{2}+5x-\frac{84}{\sqrt{19}}\operatorname{Tan}^{-1}\left(\frac{2x-3}{\sqrt{19}}\right)-4\ln(x^2-3x+7)+C.$$

68. $$\int \sin^2 x\,\cos 3x\,dx = \int\left(\frac{1-\cos 2x}{2}\right)\cos 3x\,dx = \frac{1}{2}\int\left[\cos 3x - \frac{1}{2}(\cos 5x+\cos x)\right]dx$$
$$= \frac{1}{6}\sin 3x - \frac{1}{20}\sin 5x - \frac{1}{4}\sin x + C.$$

69. If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$, then

$$\int \frac{1}{x^3(4-x^2)^{3/2}}dx = \int \frac{1}{8\sin^3\theta(8\cos^3\theta)}2\cos\theta\,d\theta = \frac{1}{32}\int\frac{\sec^5\theta}{\tan^3\theta}d\theta$$
$$= \frac{1}{32}\int\frac{(\tan^2\theta+1)^2\sec\theta}{\tan^3\theta}d\theta = \frac{1}{32}\int\left(\tan\theta+\frac{2}{\tan\theta}+\frac{1}{\tan^3\theta}\right)\sec\theta\,d\theta$$
$$= \frac{1}{32}\int(\sec\theta\,\tan\theta + 2\csc\theta + \cot^2\theta\,\csc\theta)\,d\theta$$
$$= \frac{1}{32}\int[\sec\theta\,\tan\theta + (\csc^2\theta+1)\csc\theta]\,d\theta$$
$$= \frac{1}{32}\int(\sec\theta\,\tan\theta + \csc\theta + \csc^3\theta)\,d\theta.$$



For the integral of $\csc^3\theta$, we use Exercise 59,

$$\int\frac{1}{x^3(4-x^2)^{3/2}}dx = \frac{1}{32}\left(\sec\theta + \ln|\csc\theta-\cot\theta| + \frac{1}{2}\ln|\csc\theta-\cot\theta| - \frac{1}{2}\csc\theta\,\cot\theta\right)+C$$
$$= \frac{1}{32}\left[\frac{2}{\sqrt{4-x^2}}+\frac{3}{2}\ln\left|\frac{2}{x}-\frac{\sqrt{4-x^2}}{x}\right| - \frac{1}{2}\left(\frac{2}{x}\right)\left(\frac{\sqrt{4-x^2}}{x}\right)\right]+C$$
$$= \frac{1}{16\sqrt{4-x^2}}+\frac{3}{64}\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right| - \frac{\sqrt{4-x^2}}{32x^2}+C.$$

70. If we set $y = \operatorname{Sin}^{-1}x$, then $x = \sin y$ and $dx = \cos y\,dy$. With these,

$$\int\sqrt{1-x^2}\,\operatorname{Sin}^{-1}x\,dx = \int\cos y\,(y)\,\cos y\,dy = \int y\left(\frac{1+\cos 2y}{2}\right)dy = \frac{y^2}{4}+\frac{1}{2}\int y\,\cos 2y\,dy.$$

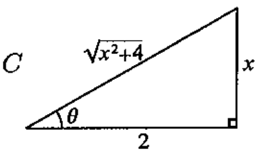
We now set $u = y$, $dv = \cos 2y\,dy$, $du = dy$, $v = (1/2)\sin 2y$, and use integration by parts,

$$\int\sqrt{1-x^2}\,\operatorname{Sin}^{-1}x\,dx = \frac{y^2}{4}+\frac{1}{2}\left(\frac{y}{2}\sin 2y - \int\frac{1}{2}\sin 2y\,dy\right)$$
$$= \frac{y^2}{4}+\frac{y}{4}\sin 2y + \frac{1}{8}\cos 2y + C$$
$$= \frac{y^2}{4}+\frac{y}{2}\sin y\,\cos y + \frac{1}{8}(1-2\sin^2 y)+C$$
$$= \frac{1}{4}(\operatorname{Sin}^{-1}x)^2 + \frac{1}{2}(\operatorname{Sin}^{-1}x)x\sqrt{1-x^2} - \frac{1}{4}x^2 + D.$$

71. $\displaystyle\int \frac{1}{x+\sqrt{x^2+4}}dx = \int \frac{1}{x+\sqrt{x^2+4}}\frac{x-\sqrt{x^2+4}}{x-\sqrt{x^2+4}}dx = \int \frac{x-\sqrt{x^2+4}}{x^2-(x^2+4)}dx = \frac{1}{4}\int(-x+\sqrt{x^2+4})\,dx$

If we set $x = 2\tan\theta$ and $dx = 2\sec^2\theta\, d\theta$, then

$$\begin{aligned}\int \frac{1}{x+\sqrt{x^2+4}}dx &= -\frac{x^2}{8} + \frac{1}{4}\int 2\sec\theta\, 2\sec^2\theta\, d\theta \quad \text{(and using Example 8.9)}\\ &= -\frac{x^2}{8} + \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C\\ &= -\frac{x^2}{8} + \frac{1}{2}\left(\frac{\sqrt{x^2+4}}{2}\right)\left(\frac{x}{2}\right) + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right| + C\\ &= -\frac{x^2}{8} + \frac{x\sqrt{x^2+4}}{8} + \frac{1}{2}\ln|\sqrt{x^2+4}+x| + D.\end{aligned}$$



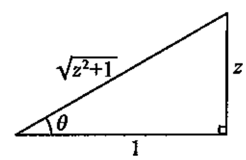
72. (a) If $z^2 = (1+x)/(1-x)$, then $z^2(1-x) = 1+x \Longrightarrow x = (z^2-1)/(z^2+1)$, and

$$dx = \frac{(z^2+1)(2z)-(z^2-1)(2z)}{(z^2+1)^2}dz = \frac{4z}{(z^2+1)^2}dz.$$

Thus, $\displaystyle\int\sqrt{\frac{1+x}{1-x}}dx = \int z\frac{4z}{(z^2+1)^2}dz = 4\int\frac{z^2}{(z^2+1)^2}dz.$

We now set $z = \tan\theta$ and $dz = \sec^2\theta\, d\theta$,

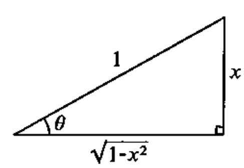
$$\begin{aligned}\int\sqrt{\frac{1+x}{1-x}}dx &= 4\int\frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\, d\theta = 4\int\sin^2\theta\, d\theta = 2\int(1-\cos 2\theta)\, d\theta\\ &= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = 2\theta - 2\sin\theta\cos\theta + C\\ &= 2\,\mathrm{Tan}^{-1}z - 2\frac{z}{\sqrt{z^2+1}}\frac{1}{\sqrt{z^2+1}} + C\\ &= 2\,\mathrm{Tan}^{-1}z - \frac{2z}{z^2+1} + C\\ &= 2\,\mathrm{Tan}^{-1}\sqrt{\frac{1+x}{1-x}} - \frac{2\sqrt{\dfrac{1+x}{1-x}}}{\dfrac{1+x}{1-x}+1} + C = 2\,\mathrm{Tan}^{-1}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + C.\end{aligned}$$



(b) $\displaystyle\int\sqrt{\frac{1+x}{1-x}}dx = \int\sqrt{\frac{1+x}{1-x}}\frac{\sqrt{1+x}}{\sqrt{1+x}}dx = \int\frac{1+x}{\sqrt{1-x^2}}dx$

We now set $x = \sin\theta$ and $dx = \cos\theta\, d\theta$,

$$\begin{aligned}\int\sqrt{\frac{1+x}{1-x}}dx &= \int\frac{1+\sin\theta}{\cos\theta}\cos\theta\, d\theta\\ &= \theta - \cos\theta + C\\ &= \mathrm{Sin}^{-1}x - \sqrt{1-x^2} + C.\end{aligned}$$



If we set $\phi = 2\,\mathrm{Tan}^{-1}\sqrt{\dfrac{1+x}{1-x}}$, then $\dfrac{1+x}{1-x} = \tan^2(\phi/2)$. When we solve this equation for x,

$$x = \frac{\tan^2(\phi/2)-1}{\tan^2(\phi/2)+1} = \frac{\sin^2(\phi/2)-\cos^2(\phi/2)}{\sin^2(\phi/2)+\cos^2(\phi/2)} = -\cos\phi = -\sin\left(\frac{\pi}{2}-\phi\right).$$

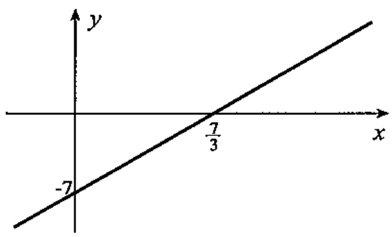
Thus, $\dfrac{\pi}{2} - \phi = \mathrm{Sin}^{-1}(-x) = -\mathrm{Sin}^{-1}x$, or $\phi = \mathrm{Sin}^{-1}x + \pi/2$, and it follows that

$$2\,\mathrm{Tan}^{-1}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + C = \mathrm{Sin}^{-1}x + \frac{\pi}{2} - \sqrt{1-x^2} + C = \mathrm{Sin}^{-1}x - \sqrt{1-x^2} + D.$$

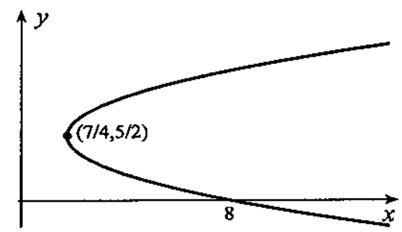
CHAPTER 9

EXERCISES 9.1

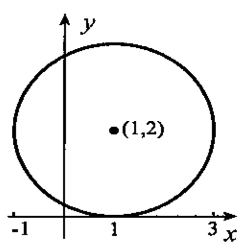
1. This is the straight line
$y = 3(x-2) - 1 = 3x - 7$.



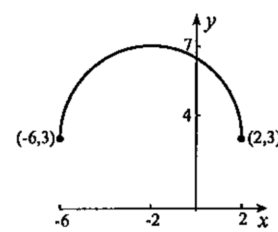
2. This is the parabola
$x = (1-y)^2 + 3(1-y) + 4 = y^2 - 5y + 8$.



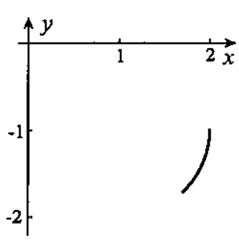
3. Since $(x-1)^2 + (y-2)^2 = 4\cos^2 t + 4\sin^2 t = 4$, this is a circle with centre $(1,2)$ and radius 2.



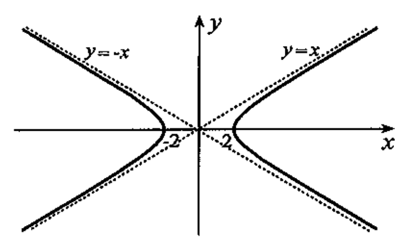
4. Since $(x+2)^2 + (y-3)^2 = 16\cos^2 t + 16\sin^2 t$ $= 16$, points lie on a circle with centre $(-2,3)$ and radius 4. Values of t in the interval $0 \le t \le \pi$ yield only the upper semicircle.



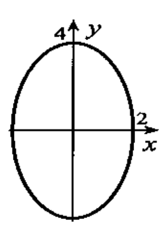
5. Since $(x-1)^2 + (y+1)^2 = \cos^2 t + \sin^2 t = 1$, points lie on a circle with centre $(1,-1)$ and radius 1. Values $0 \le t \le \pi/4$ give only one eighth of the circle.



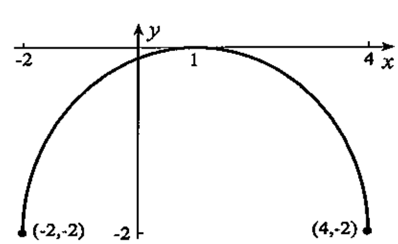
6. Since $x + y = 2t$ and $x - y = 2/t$, multiplication gives $x^2 - y^2 = 4$, a hyperbola.



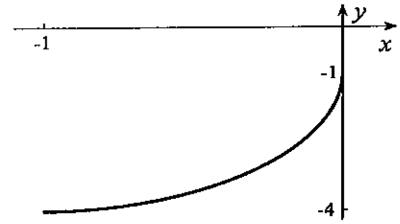
7. Since $\dfrac{x^2}{4} + \dfrac{y^2}{16} = \cos^2 t + \sin^2 t = 1$, this is an ellipse.



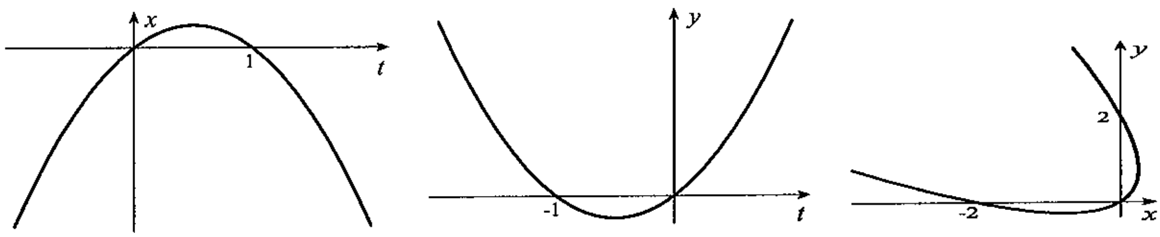
8. Since $\dfrac{(x-1)^2}{9} + \dfrac{(y+2)^2}{4} = \cos^2 t + \sin^2 t = 1$, points lie on an ellipse with centre $(1,-2)$. Values $0 \le t \le \pi$ yield only the top half of the ellipse.



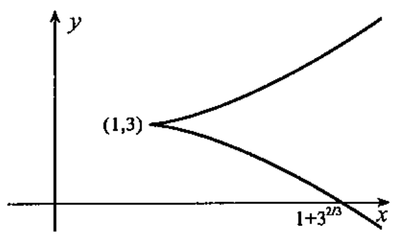
9. Since $(x+1)^2 + \dfrac{(y+1)^2}{9} = \sin^2 t + \cos^2 t = 1$, points lie on an ellipse with centre $(-1,-1)$. Values $0 \le t \le \pi/2$ yield only one quarter of the ellipse.



10. Sketches of x and y against t in the left and middle figures below yield the curve to the right.

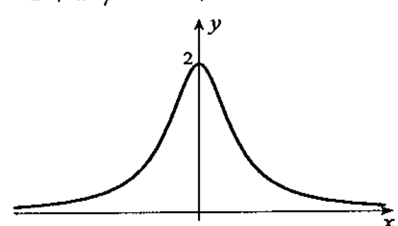


11. When we eliminate the parameter, $x = (y-3)^{2/3} + 1$.



12. When we eliminate the parameter,

$$y = 2\sin^2\theta = \frac{2}{\csc^2\theta} = \frac{2}{1+\cot^2\theta} = \frac{2}{1+x^2/4} = \frac{8}{4+x^2}.$$



13. $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2+3}$$

14. $$\frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{\dfrac{(u^2-1)(2u)-u^2(2u)}{(u^2-1)^2}}{\dfrac{(u-1)(1)-u(1)}{(u-1)^2}} = \frac{-2u}{(u^2-1)^2}\,\frac{(u-1)^2}{-1} = \frac{2u}{(u+1)^2}$$

15. $$\frac{dy}{dx} = \frac{dy/dv}{dx/dv} = \frac{6v^2}{3v^2\sqrt{v-1} + \dfrac{v^3+2}{2\sqrt{v-1}}} = \frac{12v^2\sqrt{v-1}}{7v^3-6v^2+2}$$

16. Since $y = 1/x$, it follows that $dy/dx = -1/x^2$.

17. $$\frac{dy}{dx} = \frac{dy/ds}{dx/ds} = \frac{2s+2}{(3/2)\sqrt{s}-(2/3)s^{-1/3}} = \frac{12s^{1/3}(s+1)}{9s^{5/6}-4}$$

18. $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dfrac{(t+6)(1)-t(1)}{(t+6)^2}}{4(2t+3)^3(2)} = \frac{6}{(t+6)^2 8(2t+3)^3} = \frac{3}{4(t+6)^2(2t+3)^3}$$

19. Since $\dfrac{1+u}{1-u} = x^3$, it follows that $y = 1/x^{12}$. Hence $dy/dx = -12/x^{13}$.

20. $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dfrac{-(2t+2)}{(t^2+2t-5)^2}}{\dfrac{-2t+3}{2\sqrt{-t^2+3t+5}}} = \frac{4(t+1)\sqrt{-t^2+3t+5}}{(2t-3)(t^2+2t-5)^2}$$

21. Since $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + 1/t^2}{1 - 1/t^2} = \dfrac{t^2+1}{t^2-1}$, the slope of the tangent line at the point $(17/4, 15/4)$, corresponding to $t = 4$ is $17/15$. Equations for the tangent and normal lines are $y - 15/4 = (17/15)(x - 17/4)$ and $y - 15/4 = -(15/17)(x - 17/4)$. These simplify to $17x - 15y = 16$ and $30x + 34y = 255$.

22. The slope will be one when $1 = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t+1}{t^2-3}$, or, $t^2 - 3 = 3t + 1$. Thus, $0 = t^2 - 3t - 4 = (t-4)(t+1) \Longrightarrow t = -1, 4$. The required points are $(8/3, 1/2)$ and $(28/3, 28)$.

23. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 1/t^2}{2t - 1/t^2} = \dfrac{2t^3+1}{2t^3-1}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{2t^3+1}{2t^3-1}\right) = \frac{d}{dt}\left(\frac{2t^3+1}{2t^3-1}\right)\frac{dt}{dx} = \frac{\dfrac{d}{dt}\left(\dfrac{2t^3+1}{2t^3-1}\right)}{dx/dt} = \frac{\dfrac{(2t^3-1)(6t^2)-(2t^3+1)(6t^2)}{(2t^3-1)^2}}{2t - 1/t^2} = \frac{-12t^4}{(2t^3-1)^3}.$$

24. Since $x^2 = t - 1$ and $y^2 = t + 1$, it follows that $y^2 - x^2 = 2$. Thus, $2y\dfrac{dy}{dx} - 2x = 0 \implies \dfrac{dy}{dx} = \dfrac{x}{y}$.
Consequently, $\dfrac{d^2y}{dx^2} = \dfrac{y(1) - x(dy/dx)}{y^2} = \dfrac{y - x(x/y)}{y^2} = \dfrac{y^2 - x^2}{y^3} = \dfrac{2}{y^3}$.

25. Since $y = 7 - 14(x-5)/2 = 42 - 7x$, it follows that $dy/dx = -7$ and $d^2y/dx^2 = 0$.

26. Since $\dfrac{dy}{dx} = \dfrac{dy/dv}{dx/dv} = \dfrac{2}{2v+2} = \dfrac{1}{v+1}$, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{v+1}\right) = \frac{d}{dv}\left(\frac{1}{v+1}\right)\frac{dv}{dx} = \frac{\dfrac{d}{dv}\left(\dfrac{1}{v+1}\right)}{dx/dv} = \frac{\dfrac{-1}{(v+1)^2}}{2v+2} = \frac{-1}{2(v+1)^3}.$$

27. Yes The parabola $y = 2x^2 - 1$ is defined for all x, whereas the parametric equations define only those points on the parabola with x-coordinates in the interval $-1 \le x \le 1$.

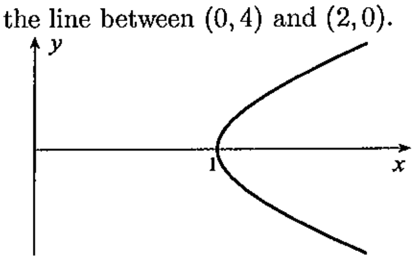
28. Since $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = \cos^2\theta + \sin^2\theta = 1$, the equations describe an ellipse. Values of the parameter yield all points on the ellipse exactly once.

29. Based on the equation $(x-h)^2 + (y-k)^2 = r^2$, parametric equations are $x = h + r\cos\theta$, $y = k + r\sin\theta$. Values $0 \le \theta < 2\pi$ traverse the circle once.

30. By solving each equation for t and equating results $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} \Longrightarrow y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$. This is the equation of the line through P_1 and P_2. By permitting t to take on all possible values, all points on the line are obtained.

31. When we add the second equation to twice the first, $2x + y = 4\sin^2 t + 4\cos^2 t = 4$, thus indicating that points lie on the straight line. Values of t only yield points on the line between $(0, 4)$ and $(2, 0)$.

32. (a) Since $x^2 - y^2 = \sec^2\theta - \tan^2\theta = 1$, points lie on the hyperbola. Values $-\pi/2 < \theta < \pi/2$ yield only the right half of the hyperbola.
(b) Since $x^2 - y^2 = \cosh^2\phi - \sinh^2\phi = 1$, points are again on the hyperbola. Because $\cosh\phi$ is always positive, only the right half of the hyperbola is defined by the parametric equations.



(c) Since $x^2 - y^2 = \dfrac{1}{4}\left(t + \dfrac{1}{t}\right)^2 - \dfrac{1}{4}\left(t - \dfrac{1}{t}\right)^2 = 1$, points are once again on the hyperbola. With $t > 0$, so also is x, and therefore only the right half of the hyperbola is defined by the parametric equations.

33. If we set $x = t$, then $y = \dfrac{t+1}{t-2}$.

34. If we set $y = t$, then $x = \dfrac{5y^2 - y^3}{1+y} = \dfrac{5t^2 - t^3}{1+t}$.

35. Written in the form $(x+1)^2+(y-2)^2=5$, the curve is a circle with centre $(-1,2)$ and radius $\sqrt{5}$. According to Exercise 29, parametric equations are $x=-1+\sqrt{5}\cos t$, $y=2+\sqrt{5}\sin t$, $0\le t<2\pi$.

36. Using Exercise 32, parametric equation for the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{2}=1$ are $x=2\sec\theta$, $y=\sqrt{2}\tan\theta$. To get both halves of the hyperbola, we use $-\pi\le\theta\le\pi$, but do not consider $\theta=\pm\pi/2$.

37. The distance D between the particles at any time is given by

$$D^2=(1-t-4t+5)^2+(t-2t+1)^2=(6-5t)^2+(1-t)^2=26t^2-62t+37,\quad t\ge 0.$$

For critical points of D^2, we solve $0=\dfrac{dD^2}{dt}=52t-62$, which implies $t=31/26$. Since

$$D^2(0)=37,\qquad D^2(31/26)=0.04,\qquad \lim_{t\to\infty}D^2=\infty,$$

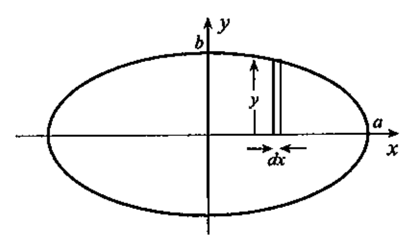
the particles are closest together at $t=31/26$.

38. If we differentiate $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}$ with respect to x, we obtain

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)=\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\frac{dt}{dx}=\frac{\dfrac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{(dx/dt)^2}}{dx/dt}=\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{(dx/dt)^3}.$$

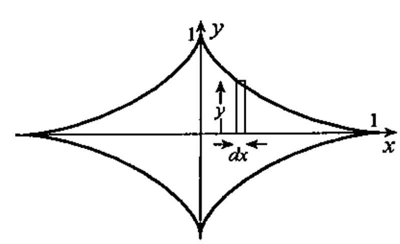
39. The area is four times that in the first quadrant (figure to the right),

$$\begin{aligned}A&=4\int_0^a y\,dx=4\int_{\pi/2}^0 b\sin t(-a\sin t\,dt)\\&=4ab\int_0^{\pi/2}\sin^2 t\,dt=4ab\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)dt\\&=2ab\left\{t-\frac{1}{2}\sin 2t\right\}_0^{\pi/2}=\pi ab.\end{aligned}$$

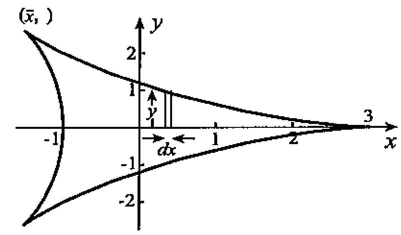


40. The area is four times that in the first quadrant (figure to the right),

$$\begin{aligned}A&=4\int_0^1 y\,dx=4\int_{\pi/2}^0\sin^3 t(-3\cos^2 t\sin t\,dt)\\&=12\int_0^{\pi/2}\sin^2 t(\sin t\cos t)^2\,dt\\&=12\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)\left(\frac{\sin 2t}{2}\right)^2dt\\&=\frac{3}{2}\int_0^{\pi/2}(\sin^2 2t-\sin^2 2t\cos 2t)\,dt=\frac{3}{2}\int_0^{\pi/2}\left(\frac{1-\cos 4t}{2}-\sin^2 2t\cos 2t\right)dt\\&=\frac{3}{2}\left\{\frac{t}{2}-\frac{1}{8}\sin 4t-\frac{1}{6}\sin^3 2t\right\}_0^{\pi/2}=\frac{3\pi}{8}.\end{aligned}$$

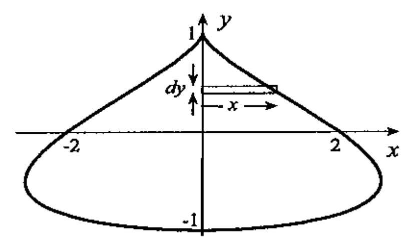


41. The area is twice that above the x-axis (figure to the right). We set up two integrals for this area. If $\overline{t}$ is the value of t giving the point with x-coordinate $\overline{x}$ shown, then



$$
\begin{aligned}
A &= 2\int_{\overline{x}}^{3} y\,dx - 2\int_{\overline{x}}^{-1} y\,dx \\
&= 2\int_{\overline{x}}^{3} y\,dx + 2\int_{-1}^{\overline{x}} y\,dx \\
&= 2\int_{\overline{t}}^{0} (2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\,dt + 2\int_{\pi}^{\overline{t}} (2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\,dt \\
&= 2\int_{\pi}^{0} (2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\,dt \\
&= 4\int_{0}^{\pi} (2\sin^2 t - \sin^2 2t + \sin t \sin 2t)\,dt = 4\int_{0}^{\pi}\left(1 - \cos 2t - \frac{1-\cos 4t}{2} + 2\sin^2 t\cos t\right)dt \\
&= 4\left\{\frac{t}{2} - \frac{1}{2}\sin 2t + \frac{1}{8}\sin 4t + \frac{2}{3}\sin^3 t\right\}_{0}^{\pi} = 2\pi.
\end{aligned}
$$

42. The area is twice that to the right of the y-axis (figure to the right),



$$
\begin{aligned}
A &= 2\int_{-1}^{1} x\,dy = 2\int_{-\pi/2}^{\pi/2} (2\cos t - \sin 2t)(\cos t\,dt) \\
&= 2\int_{-\pi/2}^{\pi/2} (1 + \cos 2t - 2\cos^2 t \sin t)\,dt \\
&= 2\left\{t + \frac{1}{2}\sin 2t + \frac{2}{3}\cos^3 t\right\}_{-\pi/2}^{\pi/2} = 2\pi.
\end{aligned}
$$

43. We revolve that part of the area in the first quadrant and double the result,

$$
\begin{aligned}
V &= 2\int_{0}^{a} \pi y^2\,dx = 2\pi\int_{\pi/2}^{0} b^2 \sin^2 2t(-a\sin t\,dt) = 2\pi ab^2\int_{0}^{\pi/2} (2\sin t\cos t)^2 \sin t\,dt \\
&= 8\pi ab^2\int_{0}^{\pi/2} \sin^3 t\cos^2 t\,dt = 8\pi ab^2\int_{0}^{\pi/2} (1-\cos^2 t)\cos^2 t\sin t\,dt \\
&= 8\pi ab^2\left\{-\frac{1}{3}\cos^3 t + \frac{1}{5}\cos^5 t\right\}_{0}^{\pi/2} = \frac{16\pi ab^2}{15}.
\end{aligned}
$$

44. The volume is

$$
\begin{aligned}
V &= \int_{0}^{2\pi R} \pi y^2\,dx = \pi\int_{0}^{2\pi} R^2(1-\cos\theta)^2(R)(1-\cos\theta)\,d\theta \\
&= \pi R^3\int_{0}^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)\,d\theta \\
&= \pi R^3\int_{0}^{2\pi}\left[1 - 3\cos\theta + \frac{3}{2}(1+\cos 2\theta) - \cos\theta(1-\sin^2\theta)\right]d\theta \\
&= \pi R^3\left\{\frac{5\theta}{2} - 4\sin\theta + \frac{3}{4}\sin 2\theta + \frac{1}{3}\sin^3\theta\right\}_{0}^{2\pi} = 5\pi^2 R^3.
\end{aligned}
$$

45. Since these equations define the circle $(x-3)^2 + (y+2)^2 = 16$, its length is $2\pi(4) = 8\pi$.

46. Since small lengths along the curve are given by

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{(e^{-t}\cos t-e^{-t}\sin t)^2+(-e^{-t}\sin t-e^{-t}\cos t)^2}=\sqrt{2}\,e^{-t}\,dt,$$

the length of the curve is $\displaystyle\int_0^1\sqrt{2}\,e^{-t}\,dt=\sqrt{2}\left\{-e^{-t}\right\}_0^1=\sqrt{2}(1-e^{-1})$.

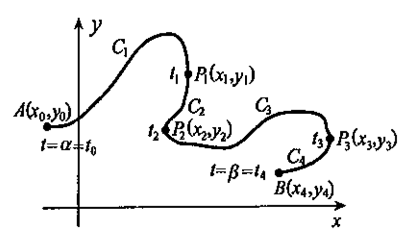
47. By equation 9.3, $L=\displaystyle\int_1^2\sqrt{(1+1/t)^2+(1-1/t)^2}\,dt=\sqrt{2}\int_1^2\frac{\sqrt{1+t^2}}{t}dt$. Suppose we set $t=\tan\theta$ and $dt=\sec^2\theta\,d\theta$. With $\tilde{\theta}=\mathrm{Tan}^{-1}2$,

$$\begin{aligned}L&=\sqrt{2}\int_{\pi/4}^{\tilde{\theta}}\frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta=\sqrt{2}\int_{\pi/4}^{\tilde{\theta}}\frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta=\sqrt{2}\int_{\pi/4}^{\tilde{\theta}}(\csc\theta+\sec\theta\,\tan\theta)\,d\theta\\&=\sqrt{2}\left\{\ln|\csc\theta-\cot\theta|+\sec\theta\right\}_{\pi/4}^{\tilde{\theta}}=1.73.\end{aligned}$$

48. Quadrupling the first quadrant length, we get

$$4\int_0^{\pi/2}\sqrt{(-a\,\sin\theta)^2+(b\,\cos\theta)^2}\,d\theta=4\int_0^{\pi/2}\sqrt{a^2\,\sin^2\theta+b^2\,\cos^2\theta}\,d\theta.$$

49. To verify 9.3 we first note that any curve defined parametrically by 9.1 can be divided into subcurves in such a way that each subcurve represents a function $y=f(x)$. Suppose we denote these subcurves by $C_1, C_2, \cdots, C_n$ and let P_i be the point joining C_i and C_{i+1}. According to equation 7.15, the length of C_i is given by



$$L_i=\int_{x_{i-1}}^{x_i}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

when $x_i>x_{i-1}$, and by

$$L_i=\int_{x_i}^{x_{i-1}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

when $x_{i-1}>x_i$. Let t_i be the value of t yielding P_i. If we set $x=x(t)$, $y=y(t)$ in these integrals and use 10.2, the first integral becomes

$$L_i=\int_{t_{i-1}}^{t_i}\sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\,dx=\int_{t_{i-1}}^{t_i}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,\frac{dt}{dx}dx=\int_{t_{i-1}}^{t_i}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

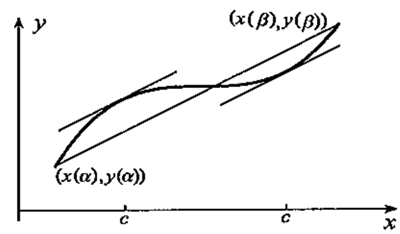
The second integral for L_i leads to the same integral in t when $x_{i-1}>x_i$. The total length of the curve is therefore

$$L=\sum_{i=1}^n L_i=\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

50. (a) If we set $u=t-1$, then $x=(u+1)^2+2(u+1)-1=u^2+4u+2$, and $y=(u+1)+5=u+6$, define the same curve where $0\le u\le 3$.
(b) If we set $v=u/3$, then $x=(3v)^2+4(3v)+2=9v^2+12v+2$, and $y=3v+6$ define the same curve where $0\le v\le 1$.

51. Theorem 3.18 with $f(x)$ replaced by $y(t)$ and $g(x)$ by $x(t)$ on $\alpha \le t \le \beta$ states that

$$\frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)} = \frac{y'(c)}{x'(c)}.$$

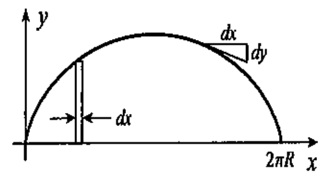


If we interpret $x = x(t)$, $y = y(t)$, $\alpha \le t \le \beta$ as parametric equations for a curve, then $[y(\beta) - y(\alpha)]/[x(\beta) - x(\alpha)]$ is the slope of the line joining the end points of the curve. The ratio $y'(c)/x'(c)$ is the slope of the tangent line to the curve at the point corresponding to $t = c$. Thus, the theorem states that there is at least one point on the curve at which the tangent line is parallel to the line joining the end points of the curve.

52. Parametric equations for the circle are $x = 2\cos\theta$, $y = 2\sin\theta$. If the particle makes two revolutions each sescond, then $\theta = 4\pi t$, where $t \ge 0$. Consequently, parametric equations for the position are $x = 2\cos 4\pi t$, $y = 2\sin 4\pi t$, $t \ge 0$.

53. (a) If we use vertical rectangles, the area is

$$\begin{aligned}A &= \int_0^{2\pi R} y\,dx = \int_0^{2\pi} R(1-\cos\theta)R(1-\cos\theta)\,d\theta \\ &= R^2\int_0^{2\pi}\left[1 - 2\cos\theta + \left(\frac{1+\cos 2\theta}{2}\right)\right]d\theta \\ &= R^2\left\{\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{2\pi} = 3\pi R^2.\end{aligned}$$



(b) Small lengths along the cycloid are approximated by

$$\sqrt{(dx)^2+(dy)^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \sqrt{(R - R\cos\theta)^2 + (R\sin\theta)^2}\,d\theta = R\sqrt{2-2\cos\theta}\,d\theta.$$

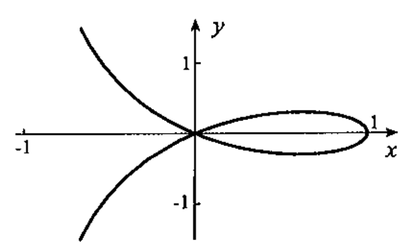
The length of one arch is therefore

$$\begin{aligned}L &= \int_0^{2\pi} R\sqrt{2-2\cos\theta}\,d\theta = \sqrt{2}R\int_0^{2\pi}\sqrt{1-[1-2\sin^2(\theta/2)]}\,d\theta \\ &= \sqrt{2}R\int_0^{2\pi}\sqrt{2}\sin\left(\frac{\theta}{2}\right)d\theta = 2R\left\{-2\cos\left(\frac{\theta}{2}\right)\right\}_0^{2\pi} = 8R.\end{aligned}$$

It represents the distance travelled by the stone as the tire makes one revolution.

54. The curve is shown to the right. Points at which the tangent line is horizontal can be found by solving

$$\begin{aligned}0 = \frac{dy}{dx} &= \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \\ &= \frac{\dfrac{(1+t^2)(1-3t^2)-(t-t^3)(2t)}{(1+t^2)^2}}{\dfrac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}} \\ &= \frac{1-4t^2-t^4}{-4t}.\end{aligned}$$



But this implies that $t^4 + 4t^2 - 1 = 0$, a quadratic equation in t^2,

$$t^2 = \frac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}.$$

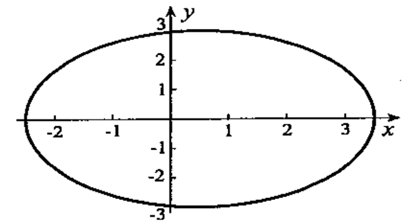
Since t^2 must be nonnegative, it follows that $t^2 = \sqrt{5}-2$, and therefore $t = \pm\sqrt{\sqrt{5}-2}$. For these values of t, $x = 0.62$ and $y = \pm 0.30$. Points with a horizontal tangent line are therefore $(0.62, \pm 0.30)$.

55. When $l_1 = 1$, $l_2 = 3$, and $d = 1/2$, parametric equations are

$$x_c = \frac{1}{2} - \frac{3(1/2 - \cos\theta)}{\sqrt{1 + 1/4 - 2(1/2)\cos\theta}} = \frac{1}{2} - \frac{3(1 - 2\cos\theta)}{\sqrt{5 - 4\cos\theta}},$$

$$y_c = \frac{1(3)\sin\theta}{\sqrt{1 + 1/4 - 2(1/2)\cos\theta}} = \frac{6\sin\theta}{\sqrt{5 - 4\cos\theta}}.$$

The plot is shown to the right.



56. (a) Since the length of CE is l_3,

$$(x_E - x_c)^2 + (y_E - y_c)^2 = l_3^2.$$

Substituting for x_c, and y_c gives

$$\left[x_E - d + \frac{l_2(d - l_1\cos\theta)}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}}\right]^2 + \left[y_E - \frac{l_1 l_2 \sin\theta}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}}\right]^2 = l_3^2.$$

Hence,

$$x_E = d - \frac{l_2(d - l_1\cos\theta)}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}} \pm \sqrt{l_3^2 - \left[y_E - \frac{l_1 l_2 \sin\theta}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}}\right]^2}.$$

Since x_E must be greater than x_c, we choose

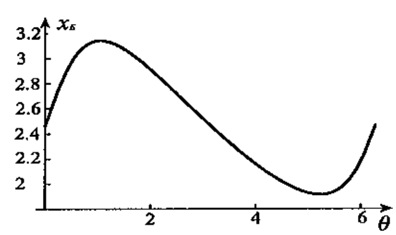
$$x_E = d - \frac{l_2(d - l_1\cos\theta)}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}} + \sqrt{l_3^2 - \left[y_E - \frac{l_1 l_2 \sin\theta}{\sqrt{d^2 + l_1^2 - 2dl_1\cos\theta}}\right]^2}.$$

(b) When $l_1 = 1/2$, $l_2 = 2$, $l_3 = 4$, $d = 1$, and $y_E = 2$,

$$x_E = 1 - \frac{2[1 - (1/2)\cos\theta]}{\sqrt{1 + 1/4 - 2(1/2)\cos\theta}} + \sqrt{16 - \left[2 - \frac{(1/2)(2)\sin\theta}{\sqrt{1 + 1/4 - 2(1/2)\cos\theta}}\right]^2}$$

$$= 1 - \frac{2(2 - \cos\theta)}{\sqrt{5 - 4\cos\theta}} + \sqrt{16 - \left[2 - \frac{2\sin\theta}{\sqrt{5 - 4\cos\theta}}\right]^2}.$$

A plot is shown below. The estimated stroke is 1.2 m.



57. (a) Since A rotates at 60 rpm, or $\dfrac{60(2\pi)}{60} = 2\pi$ radians per second, coordinates of A are $(\cos 2\pi t, \sin 2\pi t)$. The x-coordinate of B when A starts in the first quadrant (left figure below) is

$$x = \|DC\| + \|CB\| = \cos 2\pi t + \sqrt{16 - (3 + \sin 2\pi t)^2}, \quad 0 \le t \le 1/4.$$

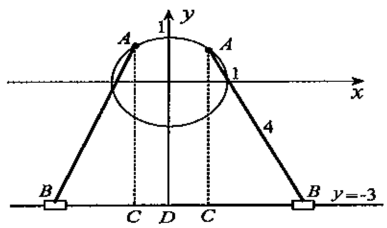
When A now moves through a complete revolution, B will be in the third quadrant, and its x-coordinate is given by

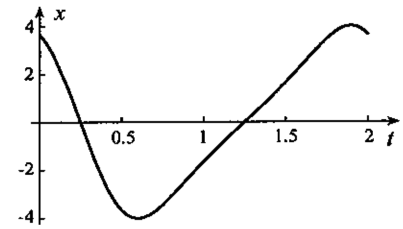
$$x = -\|DC\| - \|CB\| = \cos 2\pi t - \sqrt{16 - (3 + \sin 2\pi t)^2}, \quad 1/4 < t \le 5/4.$$

During the next three-quarters of a revolution of A, B passes into and remains in the fourth quadrant. Its x-coordinate is once again given by

$$x = \cos 2\pi t + \sqrt{16 - (3 + \sin 2\pi t)^2}, \quad 5/4 < t \le 2.$$

A plot is shown in the right figure below.





(b) The graph suggest that maximum and minimum values for x are $x = \pm 4$. To confirm the positive value, we determine when velocity is zero for that part of the function in the interval $5/4 < t < 2$,

$$0 = \frac{dx}{dt} = -2\pi \sin 2\pi t - \frac{2(2\pi)\cos 2\pi t(3 + \sin 2\pi t)}{2\sqrt{16 - (3 + \sin 2\pi t)^2}}.$$

This implies that

$$\sqrt{16 - (3 + \sin 2\pi t)^2} = -\cot 2\pi t\,(3 + \sin 2\pi t).$$

Squaring gives

$$16 - (3 + \sin 2\pi t)^2 = \cot^2 2\pi t\,(3 + \sin 2\pi t)^2 \quad \Longrightarrow \quad 16 = (3 + \sin 2\pi t)^2 \csc^2 2\pi t.$$

Square roots now yield

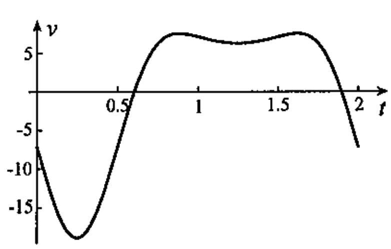
$$\pm 4 = \csc 2\pi t\,(3 + \sin 2\pi t) = 3\csc 2\pi t + 1.$$

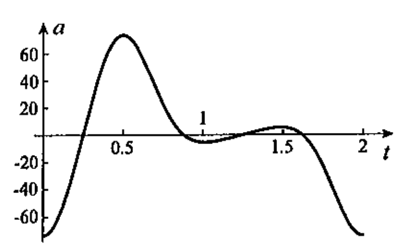
Thus,

$$3\csc 2\pi t = 3,\ -5 \quad \Longrightarrow \quad \sin 2\pi t = 1,\ -\frac{3}{5}.$$

Since dx/dt is undefined when $\sin 2\pi t = 1$, the required critical point satisfies $\sin 2\pi t = -3/5$. Instead of solving this for t, notice that the graph makes it clear that $1.8 < t < 2 \Longrightarrow 1.8(2\pi) < 2\pi t < 2(2\pi) \Longrightarrow 7\pi/2 < 2\pi t < 4\pi$. It follows that $2\pi t$ is an angle in the fourth quadrant, and therefore $\cos 2\pi t = 4/5$. Consequently, the value of x, when the velocity is zero is $x = 4/5 + \sqrt{16 - (3 - 3/5)^2} = 4$.

(c) We analyze the velocity function at $t = 1/4$ as typical. Because $v(t)$ is undefined at this value of t, velocity is not continuous at $t = 1/4$. The plot of $v(t)$ in the left figure below does not show the discontinuity, but it does indicate that the limit of $v(t)$ exists as $t \to 1/4$, and therefore the discontinuity is removable.



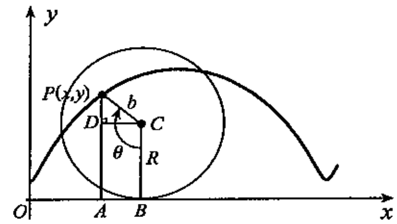


(d) The graph of the velocity function suggests that maximum speed is approximately 19 cm/s.

(e) The graph of the acceleration function in the right figure above suggests that the maximum value of $|a(t)|$ is approximately 73 cm/s^2.

58. From the figure to the right,

$$x = \|OB\| - \|AB\| = R\theta - \|CD\|$$
$$= R\theta - b\cos(\theta - \pi/2) = R\theta - b\sin\theta,$$
$$y = \|AD\| + \|DP\| = R + b\sin(\theta - \pi/2)$$
$$= R - b\cos\theta.$$

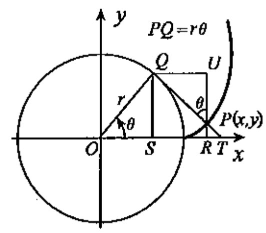


59. From the figure,

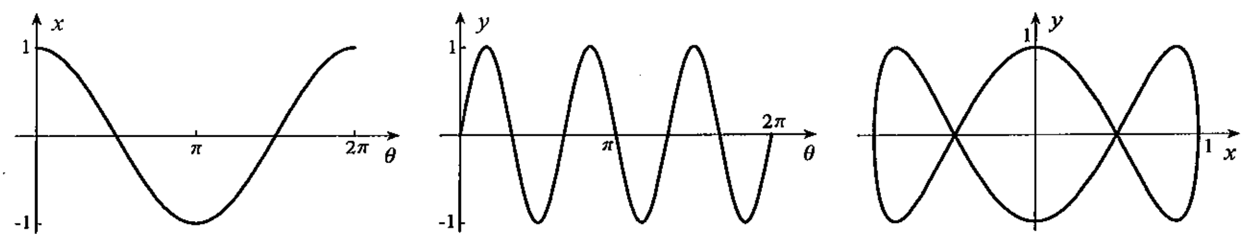
$$x = \|OS\| + \|SR\|$$
$$= r\cos\theta + \|QU\|$$
$$= r\cos\theta + r\theta\sin\theta,$$

and

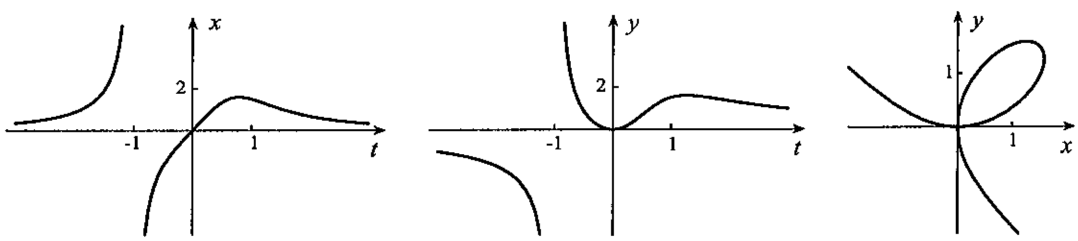
$$y = \|QS\| - \|PU\|$$
$$= r\sin\theta - r\theta\cos\theta.$$



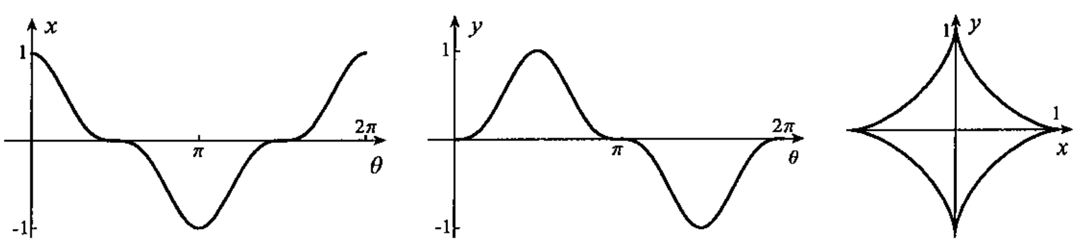
60. The graphs of x and y as functions of θ in the left and middle figures lead to the curve in the right figure.



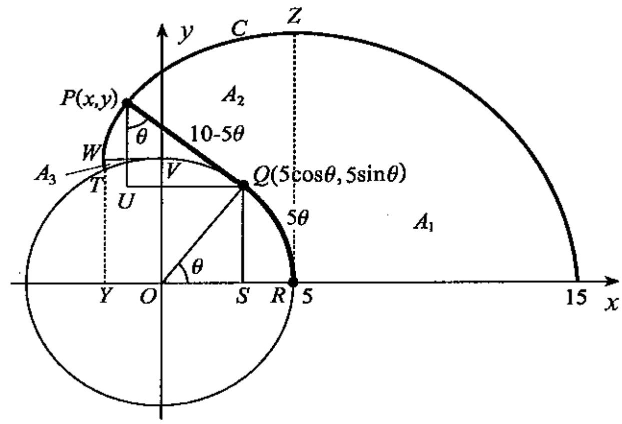
61. The graphs of x and y as functions of t in the left and middle figures lead to the curve in the right figure.



62. The graphs of x and y as functions of θ in the left and middle figures lead to the curve in the right figure.



63. We double the grazing area above the x-axis. We divide this area into three parts: A_1 to the right of the line $x = 5$; A_2 under the curve C, to the left of $x = 5$, and above the horizontal straight line segment WV and also above the quarter circle VR ; and A_3 to the right of the curve C, under the straight line segment WV and above that part $TV$of the circle (figure below).



To find A_2 and A_3, we require the equation for the curve C followed by the end of the rope; we will find parametric equations for C in terms of the angle θ between the x-axis and the final point of contact of the rope when the end of the rope is at position $P(x, y)$. The diagram shows that $x = \|OS\| - \|UQ\| = 5\cos\theta - (10 - 5\theta)\sin\theta$, and $y = \|SQ\| + \|UP\| = 5\sin\theta + (10 - 5\theta)\cos\theta$. These are parametric equations for the end of the rope. The initial value of θ is 0 and the final value occurs when $5\theta - 10 = 0 \Longrightarrow \theta = 2$. Clearly, A_1 is the area of one-quarter of a circle with radius 10, $A_1 = (1/4)\pi(10)^2 = 25\pi$ m^2. We calculate area A_2 by finding the area under C and above the x-axis from W to R and subtracting from this the sum of the areas of quarter circle ORV and rectangle $OVWY$.

$$\begin{aligned}
A_2 &= \int_{\theta=\pi/2}^{\theta=0} y\,dx - \frac{1}{4}\pi(5)^2 - 5\left(10 - \frac{5\pi}{2}\right)\\
&= \int_{\pi/2}^{0} [5\sin\theta + (10-5\theta)\cos\theta][-5\sin\theta + 5\sin\theta - (10-5\theta)\cos\theta]\,d\theta + \frac{25\pi}{4} - 50\\
&= 25\int_0^{\pi/2} [(2-\theta)\sin\theta\cos\theta + (2-\theta)^2\cos^2\theta]\,d\theta + \frac{25\pi}{4} - 50\\
&= \frac{25}{2}\int_0^{\pi/2} [(2-\theta)\sin 2\theta + (2-\theta)^2(1+\cos 2\theta)]\,d\theta + \frac{25\pi}{4} - 50.
\end{aligned}$$

Because we need the antiderivative of this integrand again later, we digress to develop it. Integration by parts on the second term with $u = (2-\theta)^2$, $dv = \cos 2\theta\,d\theta$, $du = -2(2-\theta)\,d\theta$, and $v = (1/2)\sin 2\theta$, gives

$$\begin{aligned}
I &= \int [(2-\theta)\sin 2\theta + (2-\theta)^2(1+\cos 2\theta)]\,d\theta\\
&= -\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta + \int [(2-\theta)\sin 2\theta + (2-\theta)\sin 2\theta]\,d\theta\\
&= -\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta + 2\int (2-\theta)\sin 2\theta\,d\theta.
\end{aligned}$$

Integration by parts with $u = 2 - \theta$, $dv = \sin 2\theta\,d\theta$, $du = -d\theta$, and $v = -(1/2)\cos 2\theta$, now gives

$$\begin{aligned}
I &= -\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta - (2-\theta)\cos 2\theta - \int \cos 2\theta\,d\theta\\
&= -\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta - (2-\theta)\cos 2\theta - \frac{1}{2}\sin 2\theta + C.
\end{aligned}$$

Using this antiderivative,

$$A_2 = \frac{25}{2}\left\{-\frac{1}{3}(2-\theta)^3 + \frac{1}{2}(2-\theta)^2\sin 2\theta - (2-\theta)\cos 2\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} + \frac{25\pi}{4} - 50 = 33.0350 \text{ m}^2.$$

Area A_3 must be calculated with two integrals:

$$\begin{aligned}
A_3 &= \int_{\theta=\pi/2}^{\theta=2}(5-y)\,dx + \int_{5\cos 2}^{0}(5-y)\,dx \\
&= \int_{\pi/2}^{2}[5-5\sin\theta-(10-5\theta)\cos\theta][-5\sin\theta+5\sin\theta-(10-5\theta)\cos\theta]\,d\theta + \int_{5\cos 2}^{0}(5-\sqrt{25-x^2})\,dx \\
&= 25\int_{\pi/2}^{2}[-(2-\theta)\sin\theta\cos\theta-(2-\theta)^2\cos^2\theta+(2-\theta)\cos\theta]\,d\theta + \int_{5\cos 2}^{0}(5-\sqrt{25-x^2})\,dx \\
&= \frac{25}{2}\int_{\pi/2}^{2}[-(2-\theta)\sin 2\theta-(2-\theta)^2(1+\cos 2\theta)+2(2-\theta)\cos\theta]\,d\theta + \int_{5\cos 2}^{0}(5-\sqrt{25-x^2})\,dx.
\end{aligned}$$

The first two terms in the first integral were integrated above. For the integral of $(2-\theta)\cos\theta$, we use integration by parts with $u = 2-\theta$, $dv = \cos\theta\,d\theta$, $du = -d\theta$, and$v = \sin\theta$,

$$\int(2-\theta)\cos\theta\,d\theta = (2-\theta)\sin\theta - \int -\sin\theta\,d\theta = (2-\theta)\sin\theta - \cos\theta + C.$$

For the antiderivative of $\sqrt{25-x^2}$, we use the trigonometric substitution $x = 5\sin\phi$ and $dx = 5\cos\phi\,d\phi$,

$$\begin{aligned}
\int\sqrt{25-x^2}\,dx &= \int 5\cos\phi\,5\cos\phi\,d\phi = \frac{25}{2}\int(1+\cos 2\phi)\,d\phi \\
&= \frac{25}{2}\left(\phi+\frac{1}{2}\sin 2\phi\right)+C = \frac{25}{2}(\phi+\sin\phi\cos\phi)+C \\
&= \frac{25}{2}\text{Sin}^{-1}(x/5)+\frac{25}{2}\left(\frac{x}{5}\right)\sqrt{1-x^2/25}+C = \frac{25}{2}\text{Sin}^{-1}(x/5)+\frac{x}{2}\sqrt{25-x^2}.
\end{aligned}$$

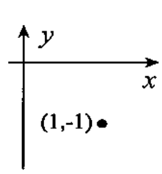
When we put these all together,

$$\begin{aligned}
A_3 = &= \frac{25}{2}\left\{\frac{1}{3}(2-\theta)^3-\frac{1}{2}(2-\theta)^2\sin 2\theta+(2-\theta)\cos 2\theta+\frac{1}{2}\sin 2\theta+2(2-\theta)\sin\theta-2\cos\theta\right\}_{\pi/2}^{2} \\
&\quad + \left\{5x-\frac{25}{2}\text{Sin}^{-1}(x/5)-\frac{x}{2}\sqrt{25-x^2}\right\}_{5\cos 2}^{0} \\
&= 0.2878 \text{ m}^2.
\end{aligned}$$

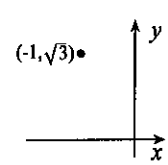
Finally then the grazing area is $2(A_1+A_2+A_3) = 2(25\pi+33.0350+0.2878) = 223.7 \text{ m}^2$.

EXERCISES 9.2

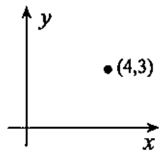
1. $r = \sqrt{1^2+(-1)^2} = \sqrt{2}$. Angles satisfying $\tan\theta = -1$ are $-\pi/4+n\pi$. Since the point is in the fourth quadrant, polar coordinates are $(\sqrt{2}, 2n\pi-\pi/4)$.



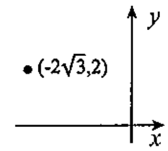
2. $r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$. Angles satisfying $\tan\theta = -\sqrt{3}$ are $-\pi/3+n\pi$. Since the point is in the second quadrant, polar coordinates are $(2, 2\pi/3+2n\pi)$.



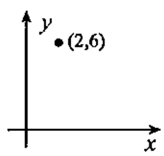
3. $r = \sqrt{4^2 + 3^2} = 5$. Angles satisfying $\tan\theta = 3/4$ are $0.644 + n\pi$. Since the point is in the first quadrant, polar coordinates are $(5, 0.644 + 2n\pi)$.



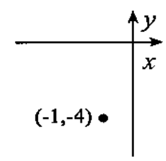
4. $r = \sqrt{(-2\sqrt{3})^2 + 2^2} = 4$. Angles satisfying $\tan\theta = -1/\sqrt{3}$ are $-\pi/6 + n\pi$. Since the point is in the second quadrant, polar coordinates are $(4, 5\pi/6 + 2n\pi)$.



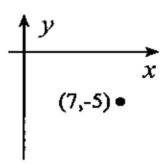
5. $r = \sqrt{2^2 + 6^2} = 2\sqrt{10}$. Angles satisfying $\tan\theta = 3$ are $1.25 + n\pi$. Since the point is in the first quadrant, polar coordinates are $(2\sqrt{10}, 1.25 + 2n\pi)$.



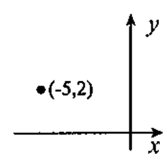
6. $r = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$. Angles satisfying $\tan\theta = 4$ are $1.33 + n\pi$. Since the point is in the third quadrant, polar coordinates are $(\sqrt{17}, -1.82 + 2n\pi)$.



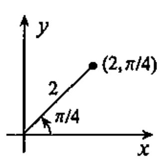
7. $r = \sqrt{7^2 + (-5)^2} = \sqrt{74}$. Angles satisfying $\tan\theta = -5/7$ are $-0.620 + n\pi$. Since the point is in the fourth quadrant, polar polar coordinates are $(\sqrt{74}, -0.620 + 2n\pi)$.



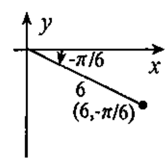
8. $r = \sqrt{(-5)^2 + 2^2} = \sqrt{29}$. Angles satisfying $\tan\theta = -2/5$ are $-0.38 + n\pi$. Since the point is in the second quadrant, polar coordinates are $(\sqrt{29}, 2.76 + 2n\pi)$.



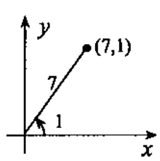
9. $(x, y) = (2\cos(\pi/4), 2\sin(\pi/4)) = (\sqrt{2}, \sqrt{2})$



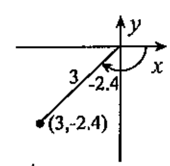
10. $(x, y) = (6\cos(-\pi/6), 6\sin(-\pi/6))$
$= (3\sqrt{3}, -3)$



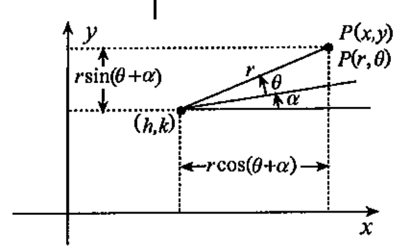
11. $(x, y) = (7\cos 1, 7\sin 1) = (3.78, 5.89)$



12. $(x, y) = (3\cos(-2.4), 3\sin(-2.4))$
$= (-2.21, -2.03)$

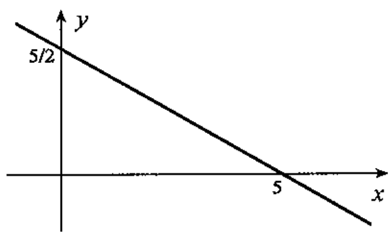


13. The diagram indicates that equations 9.9 are valid.

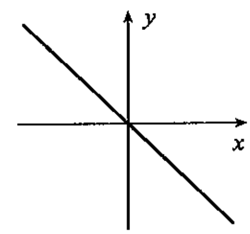


EXERCISES 9.3

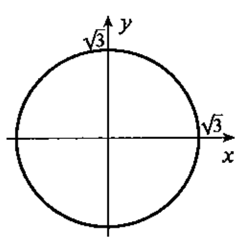
1. If we set $x = r\cos\theta$ and $y = r\sin\theta$ in $x + 2y = 5$, we obtain $r\cos\theta + 2r\sin\theta = 5$.



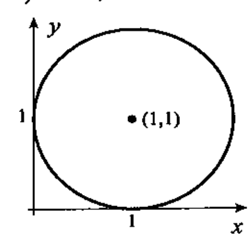
2. If we set $x = r\cos\theta$ and $y = r\sin\theta$ in $y = -x$, we obtain $r\sin\theta = -r\cos\theta$. Either $r = 0$ or $\sin\theta = -\cos\theta$. The first describes the origin (or pole), and the second can be expressed in the form $\tan\theta = -1$. This is equivalent to the two half lines $\theta = 3\pi/4$ and $\theta = -\pi/4$, both of which contain $r = 0$.



3. If we set $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + y^2 = 3$, we obtain $r = \sqrt{3}$.

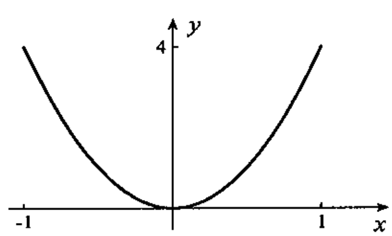


4. In polar coordinates, $r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 0$, or, $r^2 - 2r(\cos\theta + \sin\theta) + 1 = 0$. The curve is more easily sketched in Cartesian coordinates by writing its equation in the form $(x-1)^2 + (y-1)^2 = 1$, a circle.



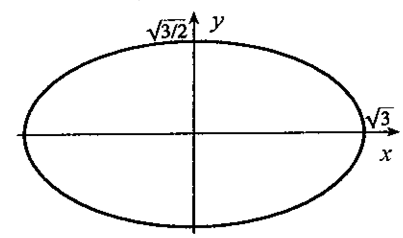
5. In polar coordinates, the equation of the parabola is $r\sin\theta = 4r^2\cos^2\theta$,

or, $4r = \sec\theta\tan\theta$.

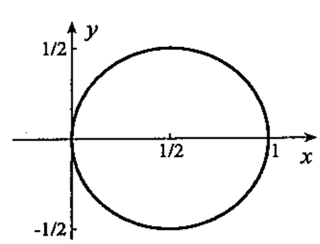


6. The equation of this ellipse in polar coordinates is $r^2\cos^2\theta + 2r^2\sin^2\theta = 3$, or,

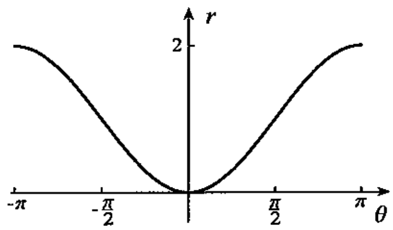
$$r^2 = \frac{3}{\cos^2\theta + 2\sin^2\theta} = \frac{3}{1 + \sin^2\theta}.$$

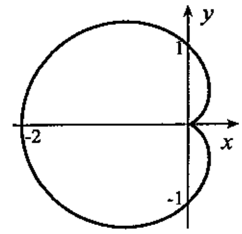


7. In polar coordinates, $r^2 = r\cos\theta$. Either $r = 0$ or $r = \cos\theta$. Since $r = 0$ also satisfies $r = \cos\theta$ (for $\theta = \pi/2$), we need only write $r = \cos\theta$. It is most esily drawn by completing the square, $(x - 1/2)^2 + y^2 = 1/4$, a circle.

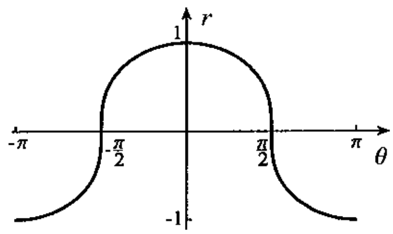


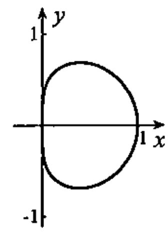
8. In polar coordinates, $r^2 = r - r\cos\theta$. Either $r = 0$ or $r = 1 - \cos\theta$. Since $r = 0$ also satisfies $r = 1 - \cos\theta$ (for $\theta = 0$), we need only write $r = 1 - \cos\theta$. The graph on the left leads to the curve on the right.



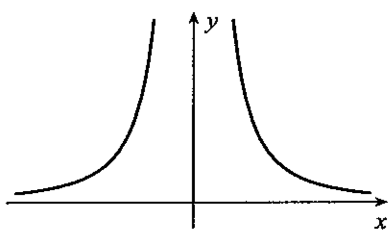


9. In polar coordinates, $r^4 = r\,\cos\theta$. Either $r = 0$ or $r^3 = \cos\theta$. Since $r = 0$ also satisfies $r^3 = \cos\theta$ (for $\theta = \pi/2$), we need only write $r^3 = \cos\theta$. The graph on the left leads to the curve on the right.

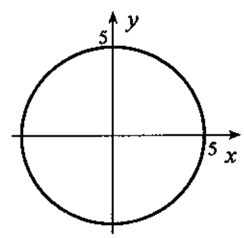




10. In polar coordinates, $r\sin\theta = \dfrac{1}{r^2\cos^2\theta}$, or, $r^3 = \sec^2\theta\,\csc\theta$. It is easily drawn in Cartesian coordinates.

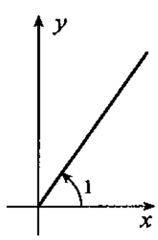


11. In Cartesian coordinates, $\sqrt{x^2 + y^2} = 5$, or, $x^2 + y^2 = 25$, a circle.

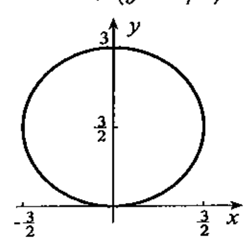


12. This equation describes a half line. Its equation in Cartesian coordinates is

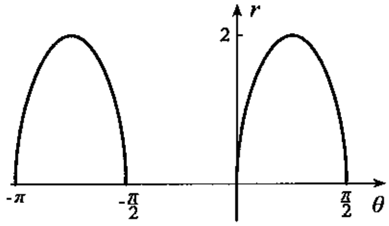
$y = (\tan 1)x$, $x \geq 0$.

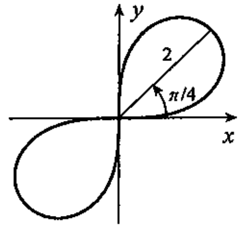


13. In Cartesian coordinates, $\sqrt{x^2 + y^2} = \dfrac{3y}{\sqrt{x^2 + y^2}}$, or, $x^2 + y^2 = 3y$. This is the circle $x^2 + (y - 3/2)^2 = 9/4$.

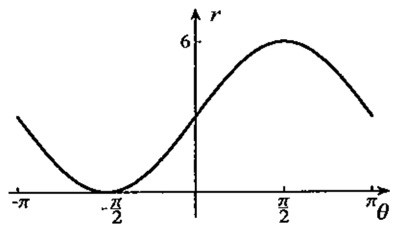


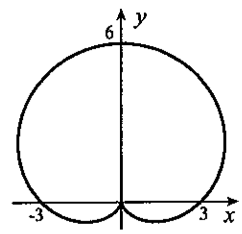
14. In Cartesian coordinates, $x^2 + y^2 = 8\sin\theta\,\cos\theta = 8\dfrac{y}{\sqrt{x^2 + y^2}}\dfrac{x}{\sqrt{x^2 + y^2}} \implies (x^2 + y^2)^2 = 8xy$. The graph on the left leads to the curve on the right.



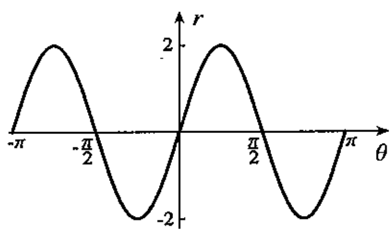


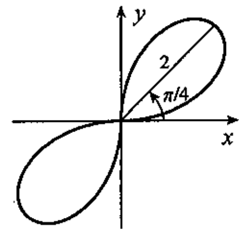
15. In Cartesian coordinates, $\sqrt{x^2+y^2} = 3 + \dfrac{3y}{\sqrt{x^2+y^2}} \Longrightarrow x^2+y^2 = 3(\sqrt{x^2+y^2}+y)$. The graph on the left leads to the curve on the right.



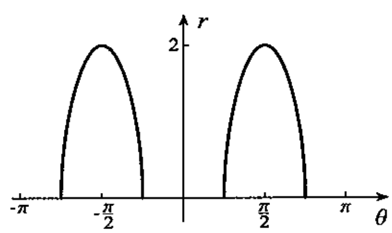


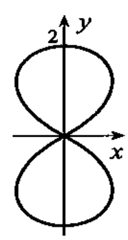
16. In Cartesian coordinates, $\sqrt{x^2+y^2} = 4\sin\theta\,\cos\theta = 4\dfrac{y}{\sqrt{x^2+y^2}}\dfrac{x}{\sqrt{x^2+y^2}} \Longrightarrow (x^2+y^2)^{3/2} = 4xy$. The graph on the left leads to the curve on the right.



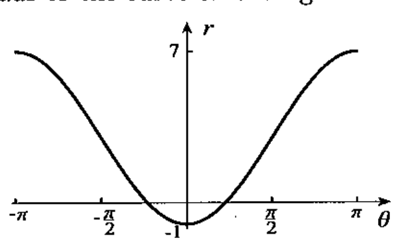


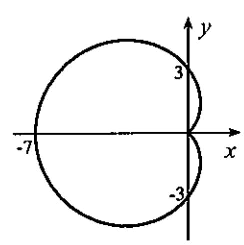
17. Since $r^2 = -4(\cos^2\theta - \sin^2\theta)$, we find that in Cartesian coordinates, $x^2+y^2 = -4\left(\dfrac{x^2}{x^2+y^2} - \dfrac{y^2}{x^2+y^2}\right)$, or, $(x^2+y^2)^2 = 4(y^2-x^2)$. The graph on the left leads to the curve on the right.



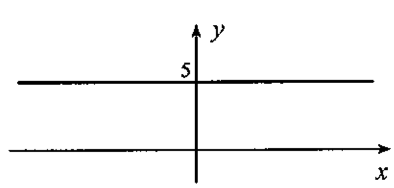


18. In Cartesian coordinates, $\sqrt{x^2+y^2} = 3 - \dfrac{4x}{\sqrt{x^2+y^2}} \Longrightarrow x^2+y^2 = 3\sqrt{x^2+y^2} - 4x$. The graph on the left leads to the curve on the right.

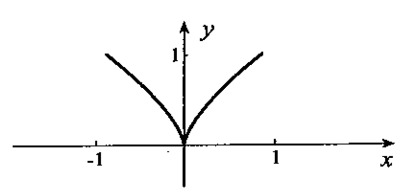




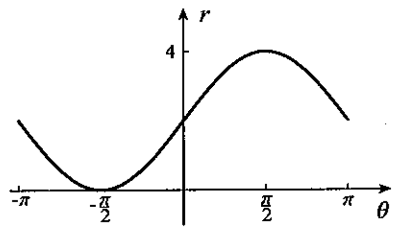
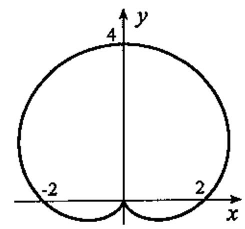
19. In Cartesian coordinates, $\sqrt{x^2+y^2} = \dfrac{5\sqrt{x^2+y^2}}{y}$, or $y = 5$.



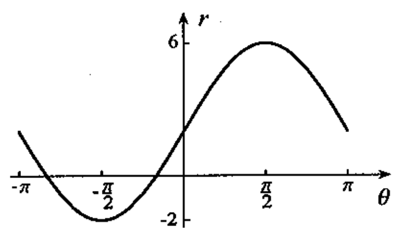
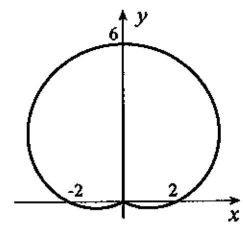
20. In Cartesian coordinates,
$\sqrt{x^2+y^2} = \dfrac{\cos^2\theta}{\sin^3\theta} = \dfrac{x^2}{x^2+y^2}\dfrac{(x^2+y^2)^{3/2}}{y^3}$,
or, $y^3 = x^2$.



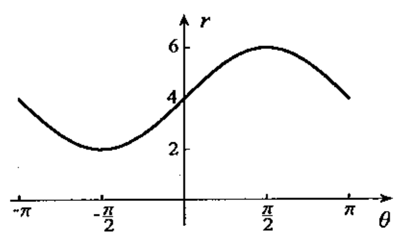
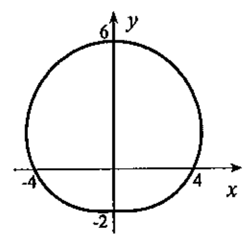
21. (a) The graph on the left leads to the curve on the right.

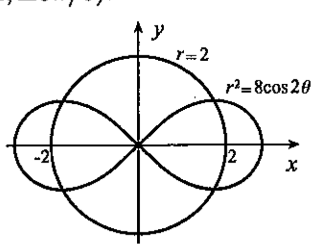
(b) The graph on the left leads to the curve on the right.

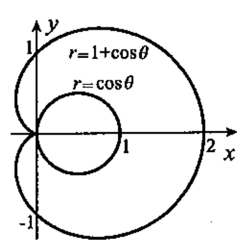
(c) The graph on the left below leads to the curve on the right.

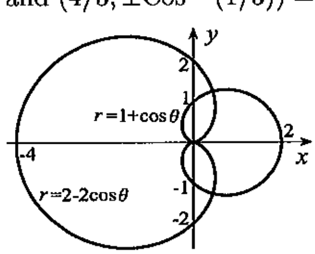
22. When we set $4 = 8\cos 2\theta$, we obtain $\cos 2\theta = 1/2 \Longrightarrow 2\theta = \pm\pi/3 + 2n\pi$. Thus, $\theta = \pm\pi/6 + n\pi$. The figure indicates four points of intersection, $(2, \pm\pi/6)$ and $(2, \pm 5\pi/6)$.



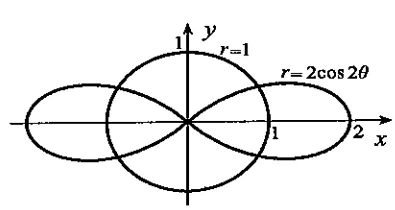
23. When we set $\cos\theta = 1 + \cos\theta$, we obtain no solution. The figure indicates that the curves intersect at the pole.



24. When we set $1 + \cos\theta = 2 - 2\cos\theta$, we obtain $\cos\theta = 1/3$. Thus, $\theta = \pm\mathrm{Cos}^{-1}(1/3) + 2n\pi$. The figure indicates three points of intersection, the pole $(0, \theta)$, and $(4/3, \pm\mathrm{Cos}^{-1}(1/3)) = (4/3, \pm 1.23)$.



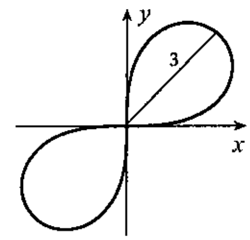
25. When we set $1 = 2\cos 2\theta$, we obtain $2\theta = \pm\pi/3 + 2n\pi$, from which $\theta = \pm\pi/6 + n\pi$. The figure indicates four points of intersection $(1, \pm\pi/6)$ and $(1, \pm 5\pi/6)$.



26. According to equation 9.13, the slope of the tangent line to $r = 9\cos 2\theta$ at $\theta = \pi/6$ is

$$\frac{dy}{dx}_{|\theta=\pi/6} = \frac{-18\sin 2\theta\sin\theta + 9\cos 2\theta\cos\theta}{-18\sin 2\theta\cos\theta - 9\cos 2\theta\sin\theta}_{|\theta=\pi/6} = \frac{\sqrt{3}}{7}.$$

27. According to equation 9.13, the slope of the tangent line to $r = 3\sqrt{\sin 2\theta}$ at $\theta = -5\pi/6$ is



$$\frac{dy}{dx}_{|\theta=-5\pi/6} = \frac{\dfrac{6\cos 2\theta}{2\sqrt{\sin 2\theta}}\sin\theta + 3\sqrt{\sin 2\theta}\cos\theta}{\dfrac{6\cos 2\theta}{2\sqrt{\sin 2\theta}}\cos\theta - 3\sqrt{\sin 2\theta}\sin\theta}_{|\theta=-5\pi/6}$$

$$= \frac{\dfrac{3\cos(-5\pi/3)}{\sqrt{\sin(-5\pi/3)}}\sin(-5\pi/6) + 3\sqrt{\sin(-5\pi/3)}\cos(-5\pi/6)}{\dfrac{3\cos(-5\pi/3)}{\sqrt{\sin(-5\pi/3)}}\cos(-5\pi/6) - 3\sqrt{\sin(-5\pi/3)}\sin(-5\pi/6)}.$$

Since the denominator is equal to 0, either the curve does not have a tangent line at the point corresponding to $\theta = -5\pi/6$, or the tangent line is vertical. The graph of the curve to the right shows that the tangent line is vertical.

28. According to equation 9.13, the slope of the tangent line to $r = 3 - 5\cos\theta$ at $\theta = 3\pi/4$ is

$$\frac{dy}{dx}_{|\theta=3\pi/4} = \frac{5\sin\theta\sin\theta + (3 - 5\cos\theta)\cos\theta}{5\sin\theta\cos\theta - (3 - 5\cos\theta)\sin\theta}_{|\theta=3\pi/4} = \frac{3}{5\sqrt{2}+3}.$$

29. According to equation 9.13, the slope of the tangent line to $r = 2\cos(\theta/2)$ at $\theta = \pi/2$ is

$$\frac{dy}{dx}_{|\theta=\pi/2} = \frac{-\sin(\theta/2)\sin\theta + 2\cos(\theta/2)\cos\theta}{-\sin(\theta/2)\cos\theta - 2\cos(\theta/2)\sin\theta}_{|\theta=\pi/2} = \frac{-1/\sqrt{2}}{-2(1/\sqrt{2})} = \frac{1}{2}.$$

30. (a) According to 9.13,

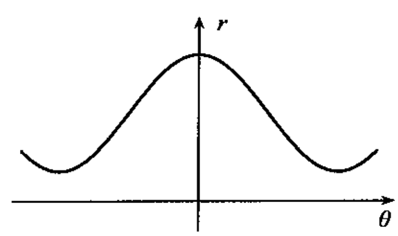
$$\frac{dy}{dx}_{|\theta=\pi/6} = \frac{\dfrac{3\cos\theta\sin\theta}{(1-\sin\theta)^2} + \dfrac{3\cos\theta}{1-\sin\theta}}{\dfrac{3\cos\theta\cos\theta}{(1-\sin\theta)^2} - \dfrac{3\sin\theta}{1-\sin\theta}}_{|\theta=\pi/6} = \sqrt{3}.$$

(b) The equation of the curve in Cartesian coordinates is

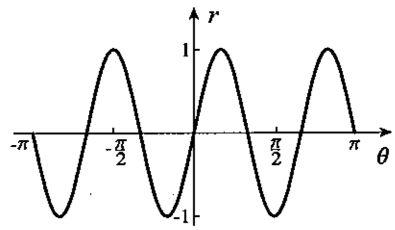
$$\sqrt{x^2+y^2} = \frac{3}{1 - \dfrac{y}{\sqrt{x^2+y^2}}} = \frac{3\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} - y} \quad \Longrightarrow \quad \sqrt{x^2+y^2} - y = 3.$$

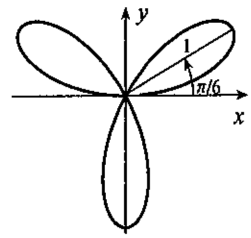
When we transpose the the y, square and simplify, the result is $y = (x^2 - 9)/6$. Hence, $dy/dx = x/3$. Since the x-coordinate of the point is $x = 6(\sqrt{3}/2) = 3\sqrt{3}$, the slope of the tangent line at the point is $(3\sqrt{3})/3 = \sqrt{3}$.

31. If $f(\theta)$ is even, then a sketch of the function is symmetric about the r-axis (figure to the right). This sketch indicates that the radial distance r is the same for a positive rotation θ and for a negative rotation $-\theta$. In other words the curve $r = f(\theta)$ is symmetric about the lines $\theta = 0$ and $\theta = \pi$. The curves in Exercise 18 and Example 9.16 are represented by even functions $f(\theta)$.

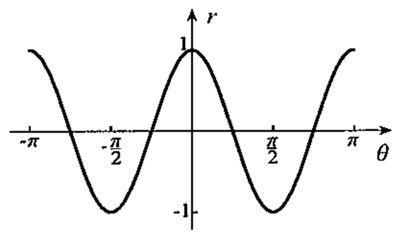


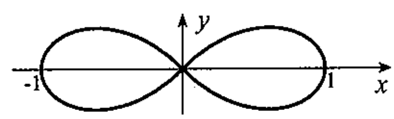
32. The graph on the left leads to the curve on the right.



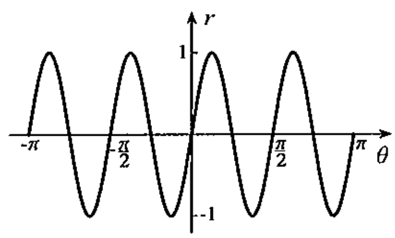


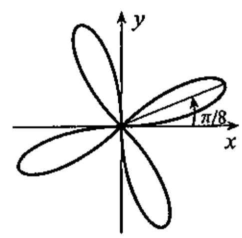
33. The graph on the left leads to the curve on the right.



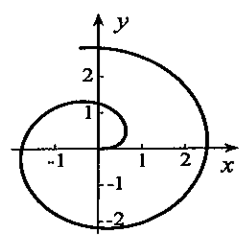


34. The graph on the left leads to the curve on the right.

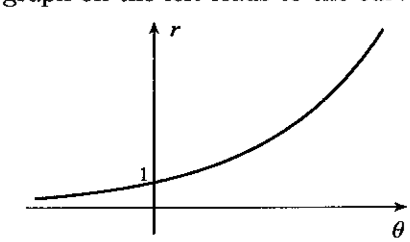


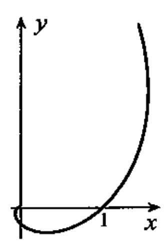


35. The graph on the left leads to the curve on the right.

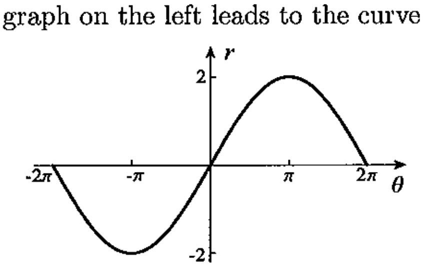


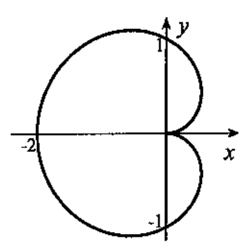
36. The graph on the left leads to the curve on the right.



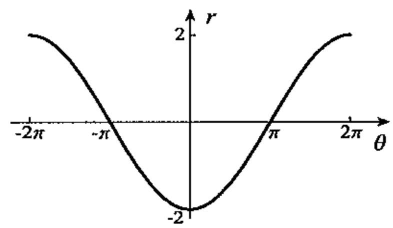


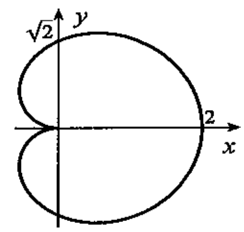
37. The graph on the left leads to the curve on the right.



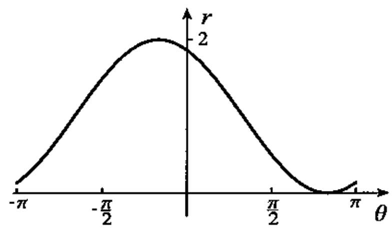


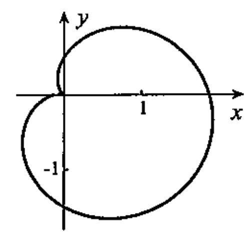
38. The graph on the left leads to the curve on the right.





39. The graph on the left leads to the curve on the right.

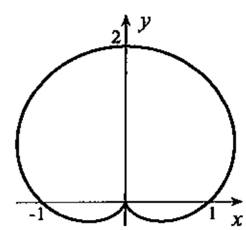




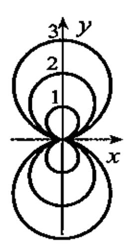
40. Maximum speed occurs when $2\pi\omega t = \pm\dfrac{\pi}{2} + n\pi \Longrightarrow t = \pm\dfrac{1}{4\omega} + \dfrac{n}{2\omega}$, where $n \geq 0$ is an integer. At these times $x = a$. Minimum speed is zero when $2\pi\omega t = n\pi \Longrightarrow t = \dfrac{n}{2\omega}$, where $n \geq 0$ is an integer. At these times, $x = a \pm b$, positions where the follower is furthest from and closest to the axis of rotation.

41. With formula 9.14,

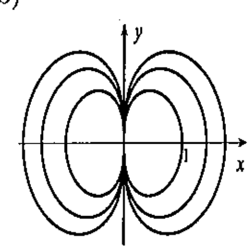
$$\begin{aligned}
L &= 2\int_{-\pi/2}^{\pi/2} \sqrt{a^2(1+\sin\theta)^2 + a^2\cos^2\theta}\,d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} \sqrt{2+2\sin\theta}\,d\theta \\
&= 2\sqrt{2}a\int_{-\pi/2}^{\pi/2} \sqrt{1+\sin\theta}\,\frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}d\theta \\
&= 2\sqrt{2}a\int_{-\pi/2}^{\pi/2} \frac{\sqrt{1-\sin^2\theta}}{\sqrt{1-\sin\theta}}d\theta = 2\sqrt{2}a\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}}d\theta = 2\sqrt{2}a\left\{-2\sqrt{1-\sin\theta}\right\}_{-\pi/2}^{\pi/2} = 8a.
\end{aligned}$$



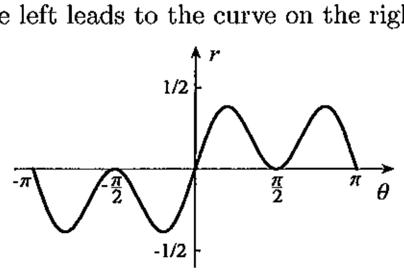
42. (a)

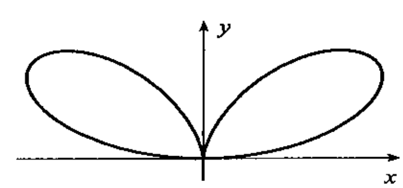


(b)

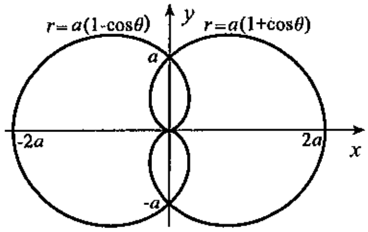


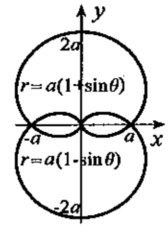
43. The equation of the bifolium in polar coordinates is $r^4 = r^3\cos^2\theta\sin\theta$, or, $r = \cos^2\theta\sin\theta$. The graph on the left leads to the curve on the right.





44. (a) Graphs of the cardioids are shown below.



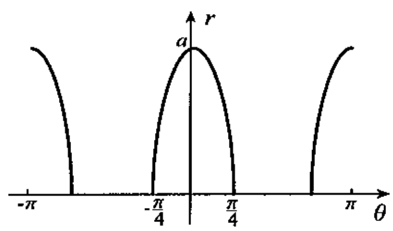


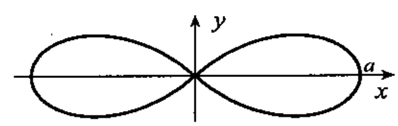
(b) Equations for $r = a(1 \pm \cos\theta)$ in Caresian coordinates are

$$\sqrt{x^2+y^2} = a\left(1 \pm \frac{x}{\sqrt{x^2+y^2}}\right) \implies x^2+y^2 = a(\sqrt{x^2+y^2} \pm x).$$

Similarly, equations for the other cardioids are $x^2+y^2 = a(\sqrt{x^2+y^2} \pm y)$.

45. (a) For $r = a\sqrt{\cos 2\theta}$, the graph on the left leads to the curve on the right.

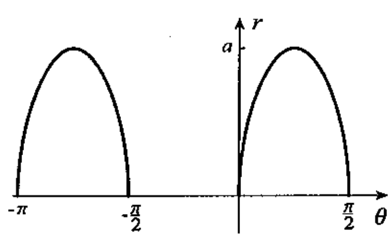


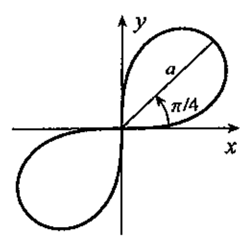


(b) From $r^2 = a^2(2\cos^2\theta - 1)$, we obtain the equation of the curve in Cartesian coordinates as

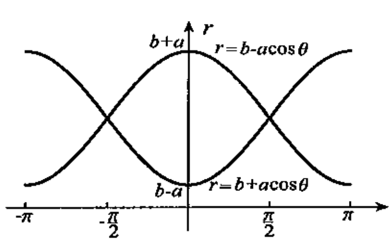
$$x^2+y^2 = a^2\left(\frac{2x^2}{x^2+y^2} - 1\right) \implies (x^2+y^2)^2 = a^2(x^2-y^2).$$

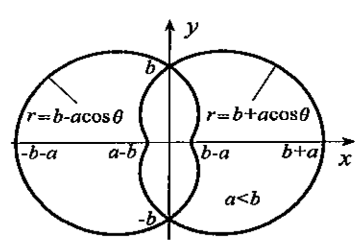
A similar analysis gives the equation $(x^2+y^2)^2 = 2a^2xy$ for $r^2 = a^2\sin 2\theta$ and the curve below.



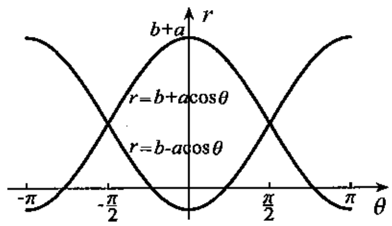


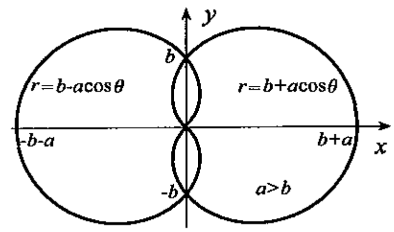
46. (a) We draw the curves $r = b \pm a\cos\theta$; curves $r = b \pm a\sin\theta$ are rotated $\pi/2$ radians. For the case $b = a$, see the cardioids in Exercise 44. When $a < b$, the graph on the left leads to the curve on the right.





When $a > b$, the graph on the left leads to the curve on the right.





(b) Equations for $r = b \pm a\cos\theta$ in Caresian coordinates are

$$\sqrt{x^2+y^2} = b \pm \frac{ax}{\sqrt{x^2+y^2}} \quad \Longrightarrow \quad x^2+y^2 = b\sqrt{x^2+y^2} \pm ax.$$

Similarly, equations for the other curves are $x^2+y^2 = b\sqrt{x^2+y^2} \pm ay$.

47. The function $r = f(\theta) = a\sin n\theta$ (or $a\cos n\theta$) is $2\pi/n$ periodic. This means that the graph of the function has n distinct parts above the θ-axis in the interval $-\pi \le \theta \le \pi$. These yield n loops (or petals) for the curve $r = a\sin n\theta$.

48. The function $r = f(\theta) = a\sin n\theta$ (or $a\cos n\theta$) is $2\pi/n$ periodic. This means that the graph of $r = |a\sin n\theta|$ has $2n$ distinct parts above the θ-axis in the interval $-\pi \le \theta \le \pi$. These yield $2n$ loops (or petals) for the curve $r = |a\sin n\theta|$.

49. (a) When we substitute $x = r\cos\theta$ and $y = r\sin\theta$,

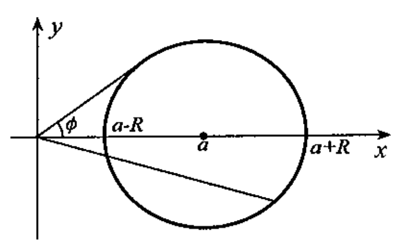
$$(r\cos\theta - a)^2 + r^2\sin^2\theta = R^2 \quad \Longrightarrow \quad r^2 - 2ar\cos\theta + (a^2 - R^2) = 0.$$

Solutions of this quadratic equation in r are

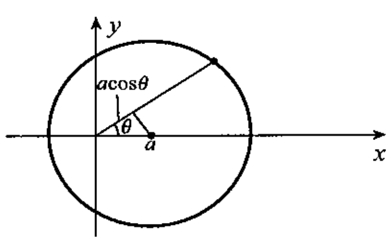
$$r = \frac{2a\cos\theta \pm \sqrt{4a^2\cos^2\theta - 4(a^2-R^2)}}{2} = a\cos\theta \pm \sqrt{R^2 - a^2\sin^2\theta}.$$

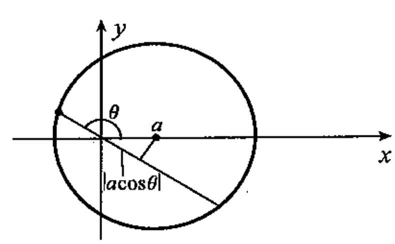
(b) We could use the final result in part (a) to reduce the equation when $a = R$. It is easier however to return to $r^2 - 2ar\cos\theta + (a^2 - R^2) = 0$. When $a = R$, this immediately gives $r = 2a\cos\theta$. It represents the entire circle.

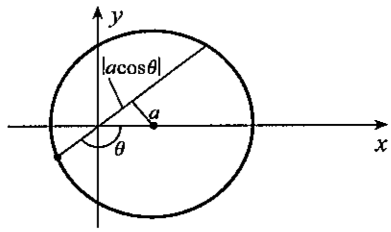
(c) When $a > R$, we require both parts since for each angle in the interval $-\phi < \theta < \phi$, there are two values of r. The positive radical gives point on the right portion of the circle, and the negative radical gives points on the left portion. We can find angle ϕ by setting $a\cos\phi + \sqrt{R^2 - a^2\sin^2\phi} = a\cos\phi - \sqrt{R^2 - a^2\sin^2\phi}$. This gives $R^2 = a^2\sin^2\phi$, from which $\phi = \mathrm{Sin}^{-1}(R/a)$.

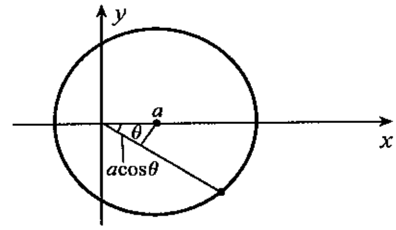


(d) When $a < R$, we have shown $a\cos\theta$ in the following figures for θ in each of the four quadrants. In all cases, r is greater than $a\cos\theta$, with the result that we must choose $r = a\cos\theta + \sqrt{R^2 - a^2\sin^2\theta}$ for the entire circle.



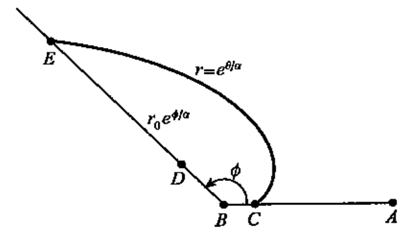






50. (a) If D is the position of the submarine after time $k/(V+v)$, then $\|BD\| = kv/(V+v)$. On the other hand,

$$\|BC\| = k - \|AC\| = k - \frac{kV}{V+v} = \frac{kv}{V+v}.$$



(b) The point of intersection, E, of the paths followed by the two boats is a distance $r_0\, e^{\phi/\alpha}$ from the pole. The submarine reaches E at time $t = (r_0/v)\, e^{\phi/\alpha}$.
To find the time at which the patrol boat reaches E, we first calculate the length of the spiral from C to this point,

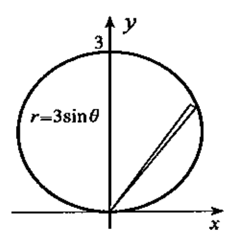
$$\int_0^{\phi} \sqrt{(r_0\, e^{\theta/\alpha})^2 + \left(\frac{r_0}{\alpha} e^{\theta/\alpha}\right)^2}\, d\theta = \frac{r_0}{\alpha}\sqrt{\alpha^2+1}\int_0^{\phi} e^{\theta/\alpha}\, d\theta = \frac{r_0}{\alpha}\sqrt{\alpha^2+1}\left\{\alpha\, e^{\theta/\alpha}\right\}_0^{\phi} = r_0\sqrt{\alpha^2+1}(e^{\phi/\alpha}-1).$$

The time at which the patrol boat reaches E is

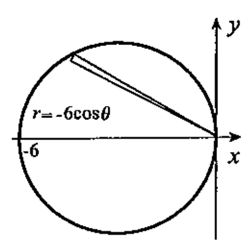
$$\frac{k}{V+v} + \frac{r_0\sqrt{\alpha^2+1}(e^{\phi/\alpha}-1)}{V} = \frac{k}{V+v} + \frac{r_0}{V}\sqrt{\frac{V^2}{v^2} - 1 + 1}(e^{\phi/\alpha}-1) = \frac{k}{V+v} + \frac{r_0}{v}e^{\phi/\alpha} - \frac{r_0}{v}$$
$$= \frac{k}{V+v} + \frac{r_0}{v}e^{\phi/\alpha} - \frac{kv}{v(V+v)} = \frac{r_0}{v}e^{\phi/\alpha}.$$

EXERCISES 9.4

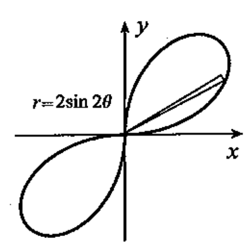
1.
$$A = 2\int_0^{\pi/2} \frac{1}{2}(3\sin\theta)^2\, d\theta = 9\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right) d\theta = \frac{9}{2}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{9\pi}{4}$$



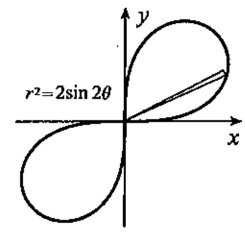
2.
$$A = 2\int_{\pi/2}^{\pi} \frac{1}{2}(-6\cos\theta)^2\, d\theta = 36\int_{\pi/2}^{\pi}\left(\frac{1+\cos 2\theta}{2}\right) d\theta = 18\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\pi/2}^{\pi} = 9\pi$$



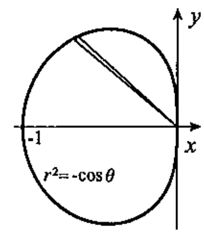
3.
$$A = 2\int_0^{\pi/2} \frac{1}{2}(2\sin 2\theta)^2\, d\theta = 4\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = 2\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \pi$$



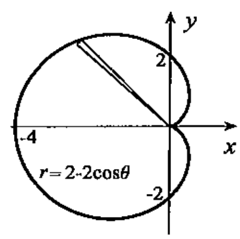
4. $$A = 2\int_0^{\pi/2} \frac{1}{2}(2\sin 2\theta)\,d\theta = \Big\{-\cos 2\theta\Big\}_0^{\pi/2} = 2$$



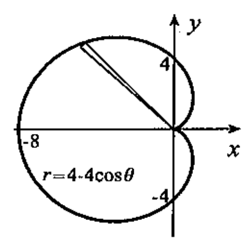
5. $$A = 2\int_{\pi/2}^{\pi} \frac{1}{2}(-\cos\theta)\,d\theta = \Big\{-\sin\theta\Big\}_{\pi/2}^{\pi} = 1$$



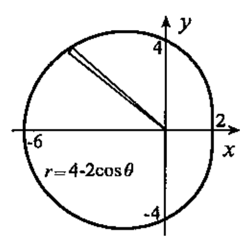
6. $$\begin{aligned} A &= 2\int_0^{\pi} \frac{1}{2}(2-2\cos\theta)^2\,d\theta \\ &= 4\int_0^{\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta \\ &= 4\int_0^{\pi}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 4\left\{\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right\}_0^{\pi} = 6\pi \end{aligned}$$



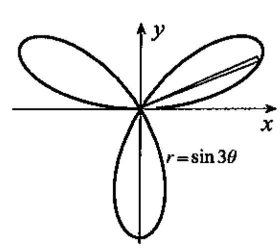
7. $$\begin{aligned} A &= 2\int_0^{\pi} \frac{1}{2}(4-4\cos\theta)^2\,d\theta \\ &= 16\int_0^{\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta \\ &= 16\int_0^{\pi}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 16\left\{\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right\}_0^{\pi} = 24\pi \end{aligned}$$



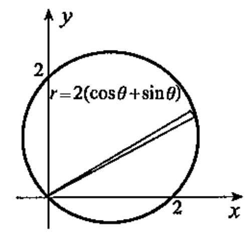
8. $$\begin{aligned} A &= 2\int_0^{\pi} \frac{1}{2}(4-2\cos\theta)^2\,d\theta = 4\int_0^{\pi}(4-4\cos\theta+\cos^2\theta)\,d\theta \\ &= 4\int_0^{\pi}\left(4-4\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 4\left\{\frac{9\theta}{2}-4\sin\theta+\frac{1}{4}\sin 2\theta\right\}_0^{\pi} = 18\pi \end{aligned}$$



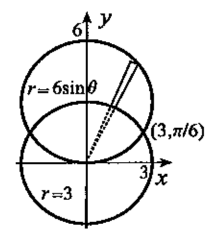
9. $$\begin{aligned} A &= 3\int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2\,d\theta \\ &= \frac{3}{2}\int_0^{\pi/3}\left(\frac{1-\cos 6\theta}{2}\right)d\theta \\ &= \frac{3}{4}\left\{\theta-\frac{1}{6}\sin 6\theta\right\}_0^{\pi/3} = \frac{\pi}{4} \end{aligned}$$



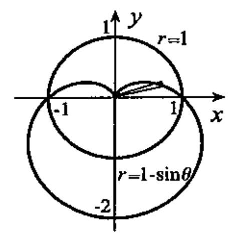
10. $$A = \int_{-\pi/4}^{3\pi/4} \frac{1}{2}[2(\cos\theta + \sin\theta)]^2\, d\theta$$
$$= 2\int_{-\pi/4}^{3\pi/4} (1 + 2\cos\theta\sin\theta)\, d\theta$$
$$= 2\Big\{\theta + \sin^2\theta\Big\}_{-\pi/4}^{3\pi/4} = 2\pi$$



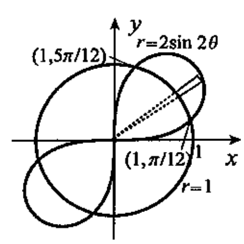
11. $$A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}[(6\sin\theta)^2 - 9]\, d\theta = 9\int_{\pi/6}^{\pi/2} (4\sin^2\theta - 1)\, d\theta$$
$$= 9\int_{\pi/6}^{\pi/2} [2(1 - \cos 2\theta) - 1]\, d\theta$$
$$= 9\Big\{\theta - \sin 2\theta\Big\}_{\pi/6}^{\pi/2} = 3\pi + \frac{9\sqrt{3}}{2}$$



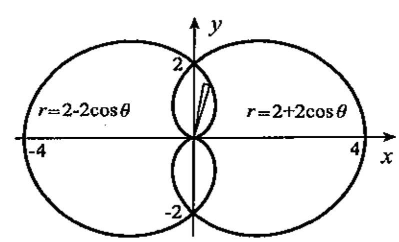
12. $$A = \frac{\pi}{2} + 2\int_0^{\pi/2} \frac{1}{2}(1 - \sin\theta)^2\, d\theta$$
$$= \frac{\pi}{2} + \int_0^{\pi/2} \left(1 - 2\sin\theta + \frac{1 - \cos 2\theta}{2}\right) d\theta$$
$$= \frac{\pi}{2} + \left\{\frac{3\theta}{2} + 2\cos\theta - \frac{1}{4}\sin 2\theta\right\}_0^{\pi/2} = \frac{1}{4}(5\pi - 8)$$



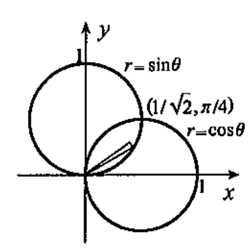
13. $$A = 2\int_{\pi/12}^{5\pi/12} \frac{1}{2}[(2\sin 2\theta)^2 - 1]\, d\theta$$
$$= \int_{\pi/12}^{5\pi/12} [2(1 - \cos 4\theta) - 1]\, d\theta$$
$$= \left\{\theta - \frac{1}{2}\sin 4\theta\right\}_{\pi/12}^{5\pi/12} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$



14. $$A = 4\int_0^{\pi/2} \frac{1}{2}(2 - 2\cos\theta)^2\, d\theta$$
$$= 8\int_0^{\pi/2} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= 8\left\{\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi/2} = 6\pi - 16$$



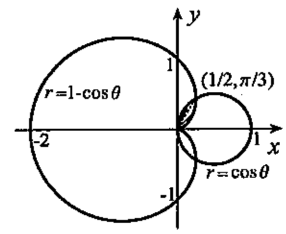
15. $$A = 2\int_0^{\pi/4} \frac{1}{2}(\sin\theta)^2\, d\theta$$
$$= \int_0^{\pi/4} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta$$
$$= \frac{1}{2}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}$$



16. $A = 2\int_0^{\pi/3} \frac{1}{2}(1-\cos\theta)^2\,d\theta + 2\int_{\pi/3}^{\pi/2} \frac{1}{2}\cos^2\theta\,d\theta$

$$= \int_0^{\pi/3}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta + \int_{\pi/3}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \left\{\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi/3} + \frac{1}{2}\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\pi/3}^{\pi/2} = \frac{1}{12}(7\pi - 12\sqrt{3})$$



17. $A = 2\int_{\tilde\theta}^{\pi} \frac{1}{2}(1-4\cos\theta)^2\,d\theta$

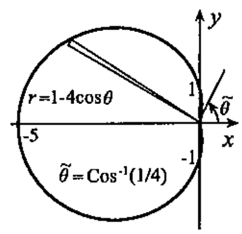
$$= \int_{\tilde\theta}^{\pi}(1 - 8\cos\theta + 16\cos^2\theta)\,d\theta$$

$$= \int_{\tilde\theta}^{\pi}[1 - 8\cos\theta + 8(1+\cos 2\theta)]\,d\theta$$

$$= \left\{9\theta - 8\sin\theta + 4\sin 2\theta\right\}_{\tilde\theta}^{\pi}$$

$$= 9\pi - 9\,\mathrm{Cos}^{-1}(1/4) + 8\sin\tilde\theta - 4\sin 2\tilde\theta = 9\pi - 9\,\mathrm{Cos}^{-1}(1/4) + 8\sin\tilde\theta - 8\sin\tilde\theta\cos\tilde\theta$$

$$= 9\pi - 9\,\mathrm{Cos}^{-1}(1/4) + 8\left(\frac{\sqrt{15}}{4}\right) - 8\left(\frac{\sqrt{15}}{4}\right)\left(\frac{1}{4}\right) = 9\pi - 9\,\mathrm{Cos}^{-1}(1/4) + \frac{3\sqrt{15}}{2}$$



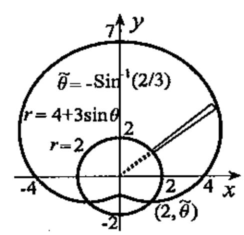
18. $A = 2\int_{\tilde\theta}^{\pi/2} \frac{1}{2}[(4+3\sin\theta)^2 - 4]\,d\theta = \int_{\tilde\theta}^{\pi/2}\left[12 + 24\sin\theta + \frac{9}{2}(1-\cos 2\theta)\right]d\theta$

$$= \left\{\frac{33\theta}{2} - 24\cos\theta - \frac{9}{4}\sin 2\theta\right\}_{\tilde\theta}^{\pi/2}$$

$$= \frac{33\pi}{4} - \frac{33\tilde\theta}{2} + 24\cos\tilde\theta + \frac{9}{4}\sin 2\tilde\theta$$

$$= \frac{33\pi}{4} - \frac{33\tilde\theta}{2} + 24\cos\tilde\theta + \frac{9}{2}\sin\tilde\theta\cos\tilde\theta$$

$$= \frac{33\pi}{4} + \frac{33}{2}\mathrm{Sin}^{-1}\left(\frac{2}{3}\right) + 24\left(\frac{\sqrt{5}}{3}\right) + \frac{9}{2}\left(-\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) = \frac{33\pi}{4} + \frac{33}{2}\mathrm{Sin}^{-1}\left(\frac{2}{3}\right) + 7\sqrt{5}$$

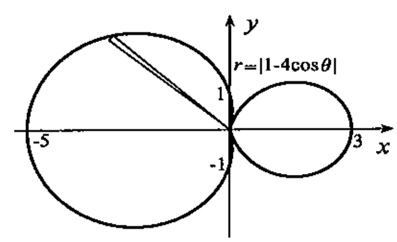


19. $A = 2\int_0^{\pi} \frac{1}{2}(1-4\cos\theta)^2\,d\theta$

$$= \int_0^{\pi}(1 - 8\cos\theta + 16\cos^2\theta)\,d\theta$$

$$= \int_0^{\pi}[1 - 8\cos\theta + 8(1+\cos 2\theta)]\,d\theta$$

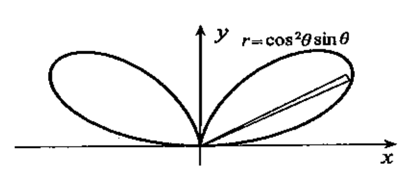
$$= \left\{9\theta - 8\sin\theta + 4\sin 2\theta\right\}_0^{\pi} = 9\pi$$



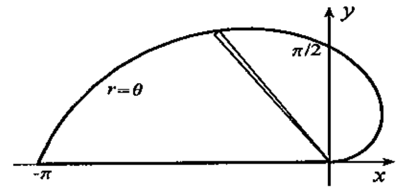
20. $A = 2\int_0^{\pi/2} \frac{1}{2}\cos^4\theta\sin^2\theta\,d\theta = \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\sin^2 2\theta}{4}\right)d\theta$

$$= \frac{1}{8}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2} + \sin^2 2\theta\cos 2\theta\right)d\theta$$

$$= \frac{1}{8}\left\{\frac{\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{\pi}{32}$$



21. $A = \int_0^{\pi} \frac{1}{2}\theta^2 \, d\theta$
$= \frac{1}{2}\left\{\frac{\theta^3}{3}\right\}_0^{\pi} = \frac{\pi^3}{6}$



22. (a) In polar coordinates,

$$r^2 \sin^2\theta = r^2\cos^2\theta \left(\frac{a - r\cos\theta}{a + r\cos\theta}\right), \quad \Longrightarrow \quad a\sin^2\theta + r\sin^2\theta\cos\theta = a\cos^2\theta - r\cos^3\theta.$$

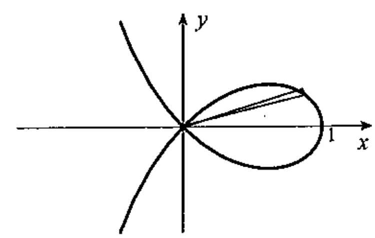
Thus, $r = \dfrac{a(\cos^2\theta - \sin^2\theta)}{\sin^2\theta\cos\theta + \cos^3\theta} = \dfrac{a\cos 2\theta}{\cos\theta} = a\cos 2\theta \sec\theta.$

(b) $A = 2\int_0^{\pi/4} \frac{1}{2}a^2\cos^2 2\theta \sec^2\theta \, d\theta = a^2 \int_0^{\pi/4} (2\cos^2\theta - 1)^2 \sec^2\theta \, d\theta$

$= a^2 \int_0^{\pi/4} (4\cos^2\theta - 4 + \sec^2\theta) \, d\theta$

$= a^2 \int_0^{\pi/4} [2(1 + \cos 2\theta) - 4 + \sec^2\theta] \, d\theta$

$= a^2\left\{-2\theta + \sin 2\theta + \tan\theta\right\}_0^{\pi/4} = \frac{a^2}{2}(4 - \pi)$



EXERCISES 9.5

1. This is a parabola.

2. Completion of squares leads to $(x - 3/2)^2 + (y + 1)^2 = 25 + 9/4 + 1 = 113/4$, a circle.

3. This is a straight line.

4. The presence of y^3 means that the equation describes none of the given curves.

5. This is an ellipse.

6. The equation can be expressed in form 9.26 for a hyperbola, $\dfrac{y^2}{5/3} - \dfrac{x^2}{5/2} = 1$.

7. Completion of squares leads to $(x - 1/2)^2 + (y + 3/2)^2 = 33/2$, a circle.

8. In the form $y = x^2/3 + 2x/3 - 4/3$, we have a parabola.

9. Completion of squares leads to $(x - 1)^2 + (y + 3)^2 = -5$. No point can satisfy this equation.

10. No x and y can make $x^2 + 2y^2 + 24$ equal to zero.

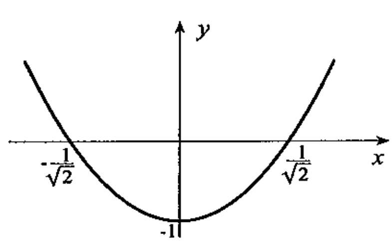
11. This is a hyperbola.

12. This is an ellipse.

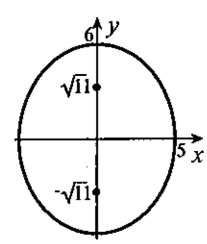
13. The presence of y^3 means that the equation describes none of the given curves.

14. Equation $x + 4y = 3$ represents a straight line.

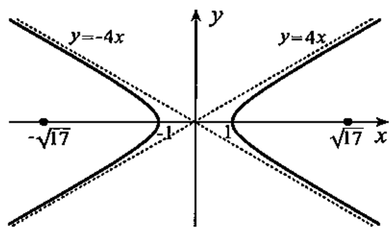
15. This is a parabola.



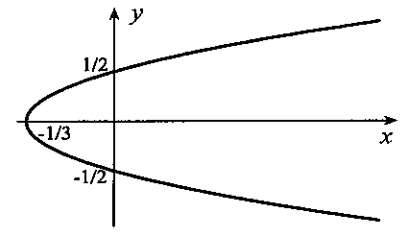
16. Foci for the ellipse are on the y-axis at distances $\pm\sqrt{36 - 25} = \pm\sqrt{11}$ from the origin.



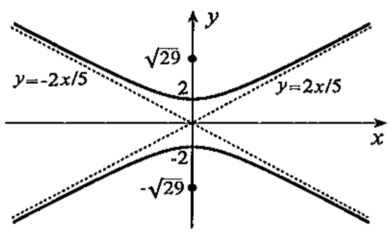
17. Foci for the hyperbola are on the x-axis at distances $\pm\sqrt{1+16} = \pm\sqrt{17}$ from the origin.



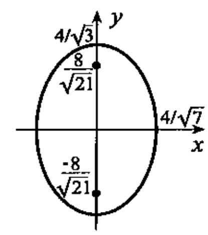
18. This is a parabola.



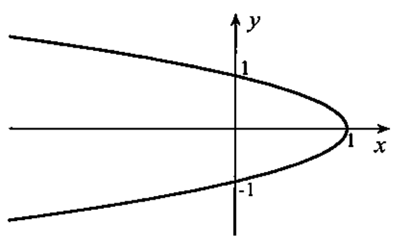
19. Foci for the hyperbola are on the y-axis at distances $\pm\sqrt{4+25} = \pm\sqrt{29}$ from the origin.



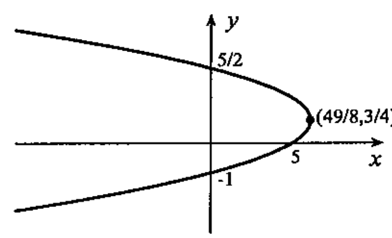
20. Foci for the ellipse $\dfrac{x^2}{16/7} + \dfrac{y^2}{16/3} = 1$ are on the y-axis at distances $\pm\sqrt{16/3 - 16/7} = \pm 8/\sqrt{21}$ from the origin.



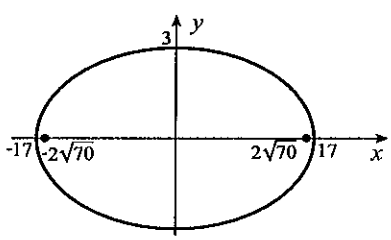
21. This is a parabola.



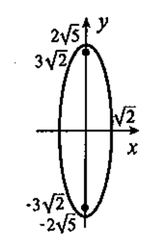
22. This is the parabola $x = -2y^2 + 3y + 5$ $= -(2y-5)(y+1)$.



23. Foci for the ellipse $\dfrac{x^2}{289} + \dfrac{y^2}{9} = 1$ are on the x-axis at distances $\pm\sqrt{289-9} = \pm 2\sqrt{70}$ from the origin.

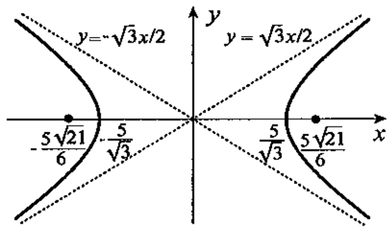


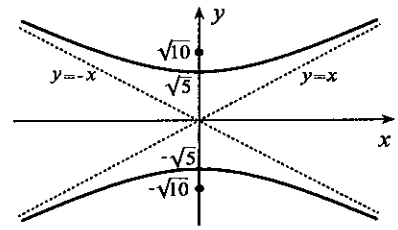
24. Foci for the ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{20} = 1$ are on the y-axis at distances $\pm\sqrt{20-2} = \pm 3\sqrt{2}$ from the origin.



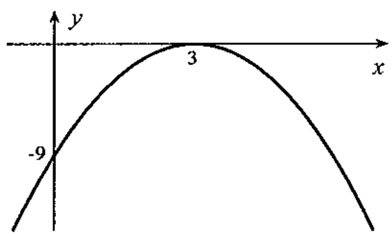
25. Foci for the hyperbola $\dfrac{x^2}{25/3} - \dfrac{y^2}{25/4} = 1$ are on the x-axis at distances $\pm\sqrt{25/3 + 25/4} = \pm 5\sqrt{21}/6$ from the origin.

26. Foci for the hyperbola $\dfrac{y^2}{5} - \dfrac{x^2}{5} = 1$ are on the y-axis at distances $\pm\sqrt{5+5} = \pm\sqrt{10}$ from the origin.

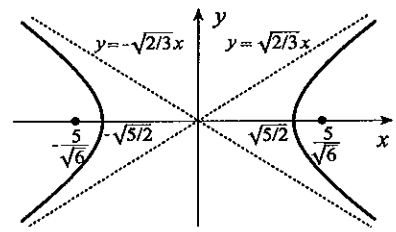




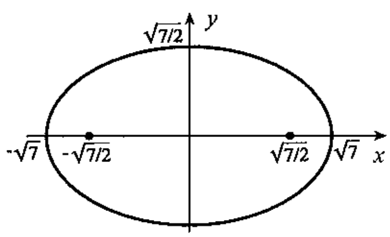
27. This is the parabola $y = -(x-3)^2$.



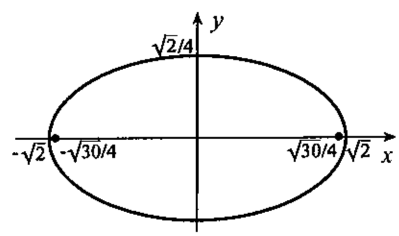
28. Foci for the hyperbola $\frac{x^2}{5/2} - \frac{y^2}{5/3} = 1$ are on the x-axis at distances $\pm\sqrt{5/2 + 5/3} = \pm 5/\sqrt{6}$ from the origin.



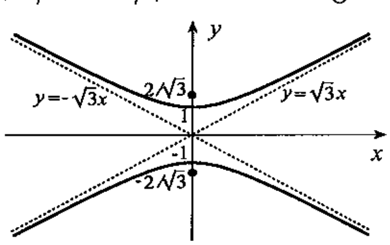
29. Foci for the ellipse $\frac{x^2}{7} + \frac{y^2}{7/2} = 1$ are on the x-axis at distances $\pm\sqrt{7 - 7/2} = \pm\sqrt{7/2}$ from the origin.



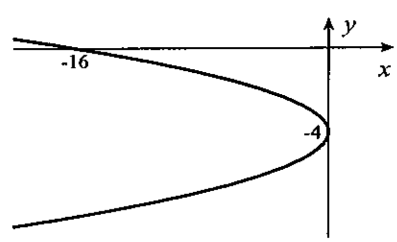
30. Foci for the ellipse $\frac{x^2}{2} + \frac{y^2}{1/8} = 1$ are on the x-axis at distances $\pm\sqrt{2 - 1/8} = \pm\sqrt{30}/4$ from the origin.



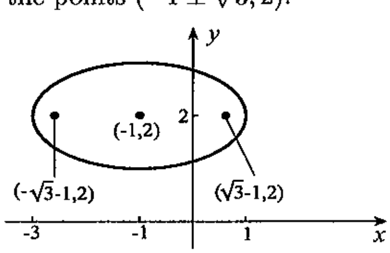
31. Foci for the hyperbola $y^2 - \frac{x^2}{1/3} = 1$ are on the y-axis at distances $\pm\sqrt{1 + 1/3} = \pm 2/\sqrt{3}$ from the origin.



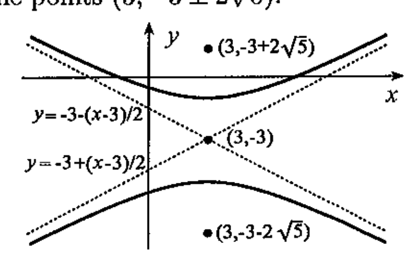
32. This is a parabola.



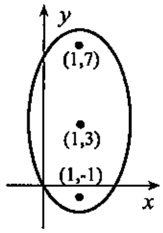
33. Foci for the ellipse $\frac{(x+1)^2}{4} + (y-2)^2 = 1$ are at the points $(-1 \pm \sqrt{3}, 2)$.



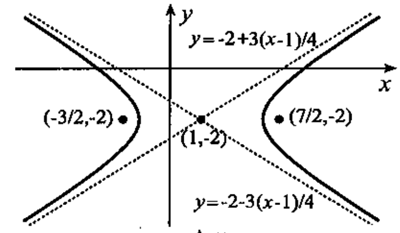
34. Foci for the hyperbola $\frac{(y+3)^2}{4} - \frac{(x-3)^2}{16} = 1$ are at the points $(3, -3 \pm 2\sqrt{5})$.



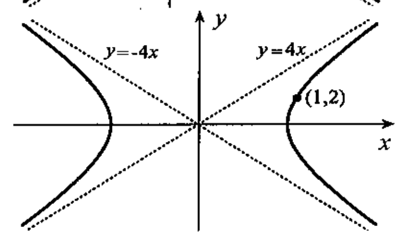
35. Foci for the ellipse $\dfrac{(x-1)^2}{2}+\dfrac{(y-3)^2}{18}=1$ are at the points $(1,-1)$ and $(1,7)$.



36. Foci for the hyperbola $\dfrac{(x-1)^2}{4}-\dfrac{(y+2)^2}{9/4}=1$ are at the points $(7/2,-2)$ and $(-3/2,-2)$.



37. With asymptotes $y=\pm 4x$, and the point $(1,2)$, we take the hyperbola in the form $x^2/a^2-y^2/b^2=1$ with $b/a=4$. Since $(1,2)$ is on the ellipse, $1/a^2-4/b^2=1$. These two equations give $a^2=3/4$ and $b^2=12$. The equation of the hyperbola is therefore $x^2/(3/4)-y^2/12=1$, or $16x^2-y^2=12$.



38. For an ellipse of the form $x^2/a^2+y^2/b^2=1$, a and b must satisfy

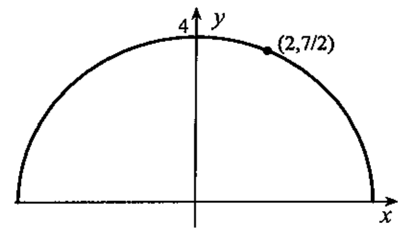
$$\frac{4}{a^2}+\frac{16}{b^2}=1 \qquad \text{and} \qquad \frac{9}{a^2}+\frac{1}{b^2}=1.$$

These imply that $a^2=28/3$ and $b^2=28$. Hence, $3x^2/28+y^2/28=1 \Longrightarrow 3x^2+y^2=28$.

39. With the coordinate system shown, the equation of the ellipse must be of the form $x^2/a^2+y^2/b^2=1$. Since $(0,4)$ and $(2,7/2)$ are points on the ellipse,

$$\frac{16}{b^2}=1, \qquad \frac{4}{a^2}+\frac{49/4}{b^2}=1.$$

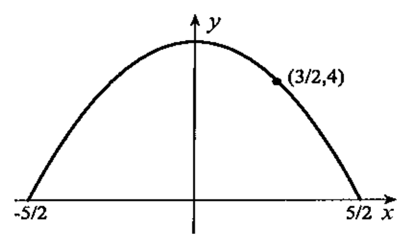
These imply that $a^2=256/15$ and $b^2=16$. The width of the arch is therefore $2a=32/\sqrt{15}$.



40. With the coordinate system shown to the right, the equation of the parabola is $y=ax^2+c$. Since $(5/2,0)$ and $(3/2,4)$ are points thereon,

$$0=\frac{25a}{4}+c, \quad 4=\frac{9a}{4}+c.$$

These imply that $a=-1$ and $c=25/4$. Thus the height of the arch is $25/4$.



41. Suppose the ends of the string are attached to the tacks at fixed positions. If the pencil is placed against the string and moved so that the string is always taut, the curve traced out is an ellipse with the tacks as foci.

42. (a) The foci of the ellipse are $(\pm 4,0)$, so that in equation 9.22, $a^2-b^2=16$. Since the sum of the distances from a point on the ellipse to the foci is $2a$, it follows that $a=5$, and hence $b=3$. The required equation is therefore $x^2/25+y^2/9=1 \Longrightarrow 9x^2+25y^2=225$.
(b) With foci $(\pm 4,0)$ and $2a=10$, equation 9.20 for the ellipse is

$$\sqrt{(x+4)^2+y^2}+\sqrt{(x-4)^2+y^2}=10.$$

If we transpose the second term and square both sides,

$$x^2+8x+16+y^2=100-20\sqrt{(x-4)^2+y^2}+x^2-8x+16+y^2 \implies 100-16x=20\sqrt{(x-4)^2+y^2}.$$

Dividing by 4, and squaring, $625-200x+16x^2=25(x^2-8x+16+y^2) \Longrightarrow 9x^2+25y^2=225$.

43. (a) The foci of the hyperbola are $(0, \pm 3)$, so that in equation 9.26, $a^2 + b^2 = 9$. Since the differences of the distances from a point on the hyperbola to the foci is $2b$, it follows that $b = 1/2$, and hence $a = \sqrt{35}/2$. The required equation is therefore $y^2/(1/4) - x^2/(35/4) = 1 \Longrightarrow 140y^2 - 4x^2 = 35$.
(b) With foci $(0, \pm 3)$ and $2b = 1$, the equation similar to 9.24 for the hyperbola is

$$|\sqrt{x^2 + (y+3)^2} - \sqrt{x^2 + (y-3)^2}| = 1.$$

If we write $\sqrt{x^2 + (y+3)^2} = \pm 1 + \sqrt{x^2 + (y-3)^2}$, and square both sides,

$$x^2 + (y+3)^2 = 1 \pm 2\sqrt{x^2 + (y-3)^2} + x^2 + (y-3)^2 \quad \Longrightarrow \quad 12y - 1 = \pm 2\sqrt{x^2 + (y-3)^2}.$$

Squaring again gives

$$144y^2 - 24y + 1 = 4[x^2 + (y-3)^2] \quad \Longrightarrow \quad 140y^2 - 4x^2 = 35.$$

44. For straight lines we set $A = C = 0$. For circles we set $A = C$. For parabolas of the form $y = ax^2 + bx + c$, we set $C = 0$. For parabolas of the the form $x = ay^2 + by + c$, we set $A = 0$. For ellipse 9.23 we demand that $AC > 0$. For hyperbolas 9.27 we demand that $AC < 0$.

45. If we differentiate $b^2x^2 + a^2y^2 = a^2b^2$ with respect to x, $2b^2x + 2a^2y\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$. At a point (x_0, y_0), the equation of the tangent line is

$$y - y_0 = -\frac{b^2x_0}{a^2y_0}(x - x_0) \qquad \Longrightarrow \qquad a^2yy_0 - a^2y_0^2 = -b^2xx_0 + b^2x_0^2.$$

Since $b^2x_0^2 + a^2y_0^2 = a^2b^2$, we may also write for the line, $b^2xx_0 + a^2yy_0 = a^2b^2$.

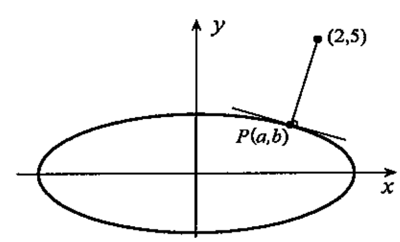
46. If we differentiate $b^2x^2 - a^2y^2 = a^2b^2$ with respect to x, $2b^2x - 2a^2y\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{b^2x}{a^2y}$. At a point (x_0, y_0), the equation of the tangent line is

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0) \qquad \Longrightarrow \qquad a^2yy_0 - a^2y_0^2 = b^2xx_0 - b^2x_0^2.$$

Since $b^2x_0^2 - a^2y_0^2 = a^2b^2$, we may also write for the line $b^2xx_0 - a^2yy_0 = a^2b^2$.

47. If we differentiate $2x^2 + 3y^2 = 14$ with respect to x

$$4x + 6y\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{2x}{3y}.$$



The slope of the tangent line to the ellipse at $P(a, b)$ is therefore $-2a/(3b)$. Since the slope of the line joining P and $(2, 5)$ is $(b-5)/(a-2)$, it follows that

$$-\frac{2a}{3b} = -\frac{a-2}{b-5} \quad \Longrightarrow \quad b = \frac{10a}{6-a}.$$

Since P is on the ellipse, we must also have $2a^2 + 3b^2 = 14$. When we substitute for b,

$$2a^2 + 3\left(\frac{10a}{6-a}\right)^2 = 14,$$

and this equation simplifies to

$$P(a) = a^4 - 12a^3 + 179a^2 + 84a - 252 = 0.$$

We find that $a = 1$ is a solution of this equation. Geometrically, we can see that this is the only solution. To show this algebraically, we factor $a - 1$ from $P(a)$,

$$P(a) = (a-1)(a^3 - 11a^2 + 168a + 252) = 0.$$

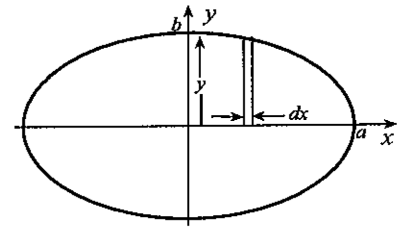
We now examine the cubic polynomial by expressing it in the form

$$Q(a) = a^3 - 11a^2 + 168a + 252 = a(a^2 - 11a + 168) + 252.$$

Since the discriminant of $a^2 - 11a + 168 < 0$, this quantity is always positive. It follows that for $0 < a < \sqrt{7}$, the polynomial $Q(a)$ cannot vanish; that is, $a = 1$ is the only solution of $P(a) = 0$. The required point is therefore $(1, 2)$.

48. $A = 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$ If we set $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$, and

$$\begin{aligned} A &= \frac{4b}{a}\int_0^{\pi/2} a\cos\theta\, a\cos\theta\,d\theta \\ &= 4ab\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 2ab\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \pi ab. \end{aligned}$$



49. The area of the rectangle is $A = 4xy$. When we solve the equation of the ellipse for the positive value of y, the result is $y = (b/a)\sqrt{a^2 - x^2}$. The area of the rectangle can therefore be expressed in the form

$$A = A(x) = \frac{4bx}{a}\sqrt{a^2 - x^2}, \quad 0 \le x \le a.$$

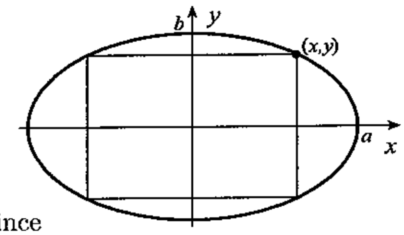
For critical points of $A(x)$ we solve

$$0 = A'(x) = \frac{4b}{a}\left(\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right).$$

This equation can be expressed in the form

$$\sqrt{a^2 - x^2} = \frac{x^2}{\sqrt{a^2 - x^2}},$$

from which $a^2 - x^2 = x^2$. The positive solution is $x = a/\sqrt{2}$. Since



$$A(0) = 0, \qquad A\left(\frac{a}{\sqrt{2}}\right) > 0, \qquad A(a) = 0,$$

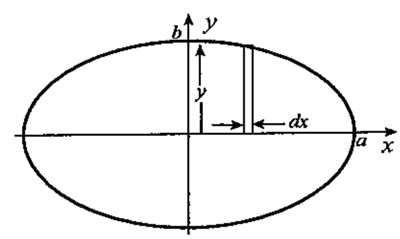
area is maximized when the length of the rectangle in the x-direction is $\sqrt{2}a$ and that in the y-direction is $\sqrt{2}b$.

50. For the prolate spheroid,

$$\begin{aligned} V &= 2\int_0^a \pi y^2\,dx = 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx \\ &= \frac{2\pi b^2}{a^2}\left\{a^2x - \frac{x^3}{3}\right\}_0^a = \frac{4}{3}\pi ab^2. \end{aligned}$$

For the oblate spheroid,

$$\begin{aligned} V &= 2\int_0^a 2\pi xy\,dx = 4\pi\int_0^a x\frac{b}{a}\sqrt{a^2 - x^2}\,dx \\ &= \frac{4\pi b}{a}\left\{-\frac{1}{3}(a^2 - x^2)^{3/2}\right\}_0^a = \frac{4}{3}\pi a^2 b. \end{aligned}$$



51. This is due to the fact that the sum of the distances from the foci to any point on the ellipse is always the same.

52. If we equate coefficients in $y = ax^2 + bx + c$ and those in equation 9.18,

$$a = \frac{1}{2(q - r)}, \qquad b = \frac{-p}{q - r}, \qquad c = \frac{p^2 + q^2 - r^2}{2(q - r)}.$$

The second divided by the first gives $b/a = -2p$, or, $p = -b/(2a)$. When the third is divided by the first and p set equal to $-b/(2a)$, the result is $q^2 - r^2 = (4ac - b^2)/(4a^2)$, or,

$$\frac{4ac - b^2}{4a^2} = (q + r)(q - r) = (q + r)\frac{1}{2a}.$$

Thus,

$$q + r = \frac{4ac - b^2}{2a} \qquad \text{and} \qquad q - r = \frac{1}{2a}.$$

Addition and subtraction of these results give the expressions for q and r.

53. Simply interchange the formulas for p and q.

54. Exercise 15: $p = 0$; $q = (1/8)(1-8) = -7/8$; $r = (1/8)(-1-8) = -9/8$. Thus, the focus is $(0, -7/8)$, and the directrix is $y = -9/8$.

Exercise 18: $p = 1/(16/3)[1 + 4(4/3)(-1/3)] = -7/48$; $q = 0$; $r = 1/(16/3)[-1 + 4(4/3)(-1/3)] = -25/48$. Thus, the focus is $(-7/48, 0)$, and the directrix is $x = -25/48$.

Exercise 21: $p = 1/(-4)[1 + 4(-1)(1)] = 3/4$; $q = 0$; $r = 1/(-4)[-1 + 4(-1)(1)] = 5/4$. Thus, the focus is $(3/4, 0)$, and the directrix is $x = 5/4$.

Exercise 22: $p = 1/(-8)[1 + 4(-2)(5) - 9] = 6$; $q = -3/(-4)$; $r = 1/(-8)[-1 + 4(-2)(5) - 9] = 25/4$. Thus, the focus is $(6, 3/4)$, and the directrix is $x = 25/4$.

Exercise 27: $p = -6/(-2) = 3$; $q = 1/(-4)[1 + 4(-1)(-9) - 36] = -1/4$; $r = 1/(-4)[-1 + 4(-1)(-9) - 36] = 1/4$. Thus, the focus is $(3, -1/4)$, and the directrix is $y = 1/4$.

Exercise 32: With $x = -y^2 - 8y - 16$, $p = [1/(-4)][1 + 4(-1)(-16) - 64] = -1/4$; $q = 8/(-2) = -4$; $r = [1/(-4)][-1 + 4(-1)(-16) - 64] = 1/4$. Thus, the focus is $(-1/4, -4)$ and the directrix $x = 1/4$.

55. When the centre of the ellipse is (h, k), and foci lie on $y = k$, the foci are $(h \pm c, k)$. If the sum of the distances from a point(x, y) on the ellipse to the foci is $2a$, then the equation of the ellipse is

$$\sqrt{(x-h-c)^2 + (y-k)^2} + \sqrt{(x-h+c)^2 + (y-k)^2} = 2a.$$

When we transpose the second square root and square both sides

$$(x-h-c)^2 + (y-k)^2 = (x-h+c)^2 + (y-k)^2 - 4a\sqrt{(x-h+c)^2 + (y-k)^2} + 4a^2,$$

and this equation simplifies to $a\sqrt{(x-h+c)^2 + (y-k)^2} = a^2 + c(x-h)$. Squaring again leads to

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2 - c^2} = 1.$$

If we set $b^2 = a^2 - c^2$, then 9.23 is obtained. A similar derivation can be given when foci are on the line $x = h$.

56. When the centre of the hyperbola is (h, k), and foci lie on $y = k$, the foci are $(h \pm c, k)$. If the difference of the distances from a point(x, y) on the hyperbola to the foci is $2a$, then the equation of the hyperbola is

$$\left| \sqrt{(x-h-c)^2 + (y-k)^2} - \sqrt{(x-h+c)^2 + (y-k)^2} \right| = 2a.$$

When we write $\sqrt{(x-h-c)^2 + (y-k)^2} = \sqrt{(x-h+c)^2 + (y-k)^2} \pm 2a$, and square both sides

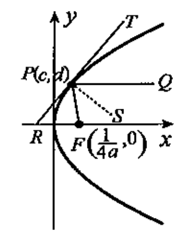
$$(x-h-c)^2 + (y-k)^2 = (x-h+c)^2 + (y-k)^2 \pm 4a\sqrt{(x-h+c)^2 + (y-k)^2} + 4a^2,$$

and this equation simplifies to $\pm a\sqrt{(x-h+c)^2 + (y-k)^2} = a^2 + c(x-h)$. Squaring again leads to

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{c^2 - a^2} = 1.$$

If we set $b^2 = c^2 - a^2$, then the first of equations 9.27 is obtained. The second equation in 9.27 is obtained in a similar fashion when foci are on the line $x = h$.

57. According to Exercise 53, the x-coordinate of the focus of the parabola is $1/(4a)$. If we differentiate the equation $x = ay^2$ with respect to x, then $1 = 2ay(dy/dx)$, from which $dy/dx = 1/(2ay)$. The slope of the tangent line at P is therefore $1/(2ad)$, and the equation of the tangent line is $y - d = [1/(2ad)](x - c)$. The x-intercept of this line is given by



$$-d = \frac{1}{2ad}(x - c) \quad \Longrightarrow \quad x = c - 2ad^2.$$

Taking into account the fact that $c = ad^2$, squares of the lengths of PF and RF are

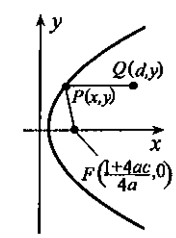
$$\|PF\|^2 = \left(c - \frac{1}{4a}\right)^2 + d^2 = \left(ad^2 - \frac{1}{4a}\right)^2 + d^2 = a^2d^4 + \frac{d^2}{2} + \frac{1}{16a^2},$$

$$\|RF\|^2 = \left(\frac{1}{4a} - c + 2ad^2\right)^2 = \left(\frac{1}{4a} + ad^2\right)^2 = \frac{1}{16a^2} + \frac{d^2}{2} + a^2d^4.$$

Thus, $\|PF\| = \|RF\|$. It now follows that angles FPR and FRP are equal, and both are equal to angle TPQ. Since angles FPR and SPF add to $\pi/2$ as do angles TPQ and QPS, it follows that angles SPF and QPS are equal.

58. What we must verify is that the sum of distances $\|PQ\| + \|PF\|$ is independent of the coordinates of P.

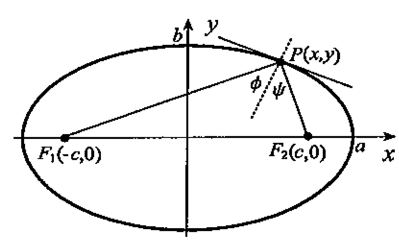
$$\begin{aligned}\|PQ\| + \|PF\| &= (d - x) + \sqrt{\left(x - \frac{1+4ac}{4a}\right)^2 + y^2} \\ &= d - (ay^2 + c) + \sqrt{\left(ay^2 + c - \frac{1+4ac}{4a}\right)^2 + y^2} \\ &= d - ay^2 - c + \sqrt{\left(\frac{4a^2y^2 - 1}{4a}\right)^2 + y^2} \\ &= d - ay^2 - c + \sqrt{\frac{16a^4y^4 - 8a^2y^2 + 1 + 16a^2y^2}{16a^2}} \\ &= d - ay^2 - c + \sqrt{\frac{(4a^2y^2+1)^2}{16a^2}} = d - ay^2 - c + \frac{4a^2y^2+1}{4a} = d - c + \frac{1}{4a},\end{aligned}$$



and this is indeed independent of x and y.

59. The slope of the tangent line to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at P is given by

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$



The slope of the normal line to the ellipse at P is $a^2y/(b^2x)$. Slopes of PF_2 and PF_1 are $y/(x-c)$ and $y/(x+c)$, respectively. Using formula 1.60, angles ψ and ϕ are given by

$$\tan\psi = \left|\frac{\dfrac{y}{x-c} - \dfrac{a^2y}{b^2x}}{1 + \dfrac{y}{x-c}\,\dfrac{a^2y}{b^2x}}\right| = \left|\frac{b^2xy - a^2y(x-c)}{b^2x(x-c)}\,\frac{b^2x(x-c)}{b^2x(x-c) + a^2y^2}\right| = \left|\frac{b^2xy - a^2y(x-c)}{a^2y^2 + b^2x^2 - b^2cx}\right|$$

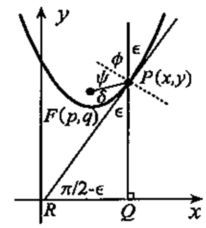
$$= \left|\frac{y[-x(a^2-b^2) + a^2c]}{a^2b^2 - b^2cx}\right| = \left|\frac{y(-xc^2 + a^2c)}{b^2(a^2 - cx)}\right| = \left|\frac{yc(-cx + a^2)}{b^2(a^2 - cx)}\right| = \left|\frac{cy}{b^2}\right|,$$

$$\tan\phi = \left|\frac{\dfrac{a^2y}{b^2x} - \dfrac{y}{x+c}}{1 + \dfrac{a^2y}{b^2x}\,\dfrac{y}{x+c}}\right| = \left|\frac{a^2y(x+c) - b^2xy}{b^2x(x+c)}\,\frac{b^2x(x+c)}{b^2x(x+c) + a^2y^2}\right| = \left|\frac{a^2y(x+c) - b^2xy}{b^2x^2 + a^2y^2 + b^2cx}\right|$$

$$= \left|\frac{y[x(-b^2 + a^2) + a^2c]}{a^2b^2 + b^2cx}\right| = \left|\frac{y(xc^2 + a^2c)}{b^2(a^2 + cx)}\right| = \left|\frac{yc(xc + a^2)}{b^2(a^2 + cx)}\right| = \left|\frac{cy}{b^2}\right|.$$

Since $\tan\psi = \tan\phi$ and both angles are betwen 0 and π, we conclude that $\psi = \phi$. In other words, the normal bisects the angle between the focal radii. A similar proof can be constructed for the hyperbola.

60. The figure indicates that $\psi = \phi$ if $\delta = \epsilon$. When we differentiate $y = ax^2 + bx + c$, we obtain $y' = 2ax + b$. The slope of PF is $(y-q)/(x-p)$ and hence, using equation 1.60,



$$\tan\delta = \frac{2ax + b - \dfrac{y-q}{x-p}}{1 + (2ax+b)\left(\dfrac{y-q}{x-p}\right)}$$

or,

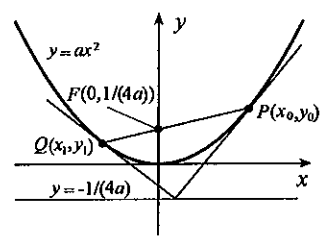
$$\cot\delta = \frac{x - p + (y-q)(2ax+b)}{(2ax+b)(x-p) - y + q}.$$

If we substitute $y = ax^2 + bx + c$, $p = -b/(2a)$, $q = (1 + 4ac - b^2)/(4a)$ (see Exercise 52),

$$\cot\delta = \frac{\left(x + \dfrac{b}{2a}\right) + (2ax+b)\left[ax^2 + bx + c - \dfrac{1}{4a}(1 + 4ac - b^2)\right]}{(2ax+b)\left(x + \dfrac{b}{2a}\right) - \left[ax^2 + bx + c - \dfrac{1}{4a}(1 + 4ac - b^2)\right]}$$

and this quantity simplifies to $\cot\delta = 2ax + b$. But $\cot\epsilon = \tan(\pi/2 - \epsilon) = f'(x) = 2ax + b$. Hence, $\cot\delta = \cot\epsilon$, and $\delta = \epsilon$.

61. Since $dy/dx = 2ax$, slopes of the tangent lines at P and Q are $2ax_0$ and $2ax_1$. The equation of PQ is $y - y_0 = \dfrac{y_1 - y_0}{x_1 - x_0}(x - x_0)$, and since F is on this line



$$\frac{1}{4a} - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(-x_0).$$

Furthermore, $y_0 = ax_0^2$ and $y_1 = ax_1^2$, so that

$$\frac{1}{4a} - ax_0^2 = \frac{ax_1^2 - ax_0^2}{x_1 - x_0}(-x_0).$$

When this equation is multiplied by $4a$,

$$1 - 4a^2x_0^2 = -4a^2x_0(x_1 + x_0) \quad\Longrightarrow\quad 1 = -4a^2x_0x_1 = -(2ax_0)(2ax_1).$$

In other words, the tangent lines at P and Q are perpendicular. Equations of the tangent lines at P and Q are $y - y_0 = 2ax_0(x - x_0)$ and $y - y_1 = 2ax_1(x - x_1)$. The point of intersection of these lines has a y-coordinate given by

$$\frac{y - y_0}{2ax_0} + x_0 = \frac{y - y_1}{2ax_1} + x_1 \quad\Longrightarrow\quad x_1(y - y_0) + 2ax_0^2x_1 = x_0(y - y_1) + 2ax_0x_1^2.$$

Thus, $y = \dfrac{2ax_0x_1^2 - 2ax_0^2x_1 - x_0y_1 + x_1y_0}{x_1 - x_0}$. When we substitute $y_0 = ax_0^2$ and $y_1 = ax_1^2$,

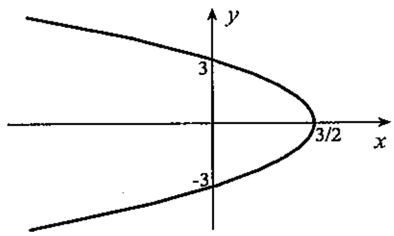
$$y = \frac{2ax_0x_1(x_1 - x_0) - x_0(ax_1^2) + x_1(ax_0^2)}{x_1 - x_0} = \frac{2ax_0x_1(x_1 - x_0) - ax_0x_1(x_1 - x_0)}{x_1 - x_0} = ax_0x_1.$$

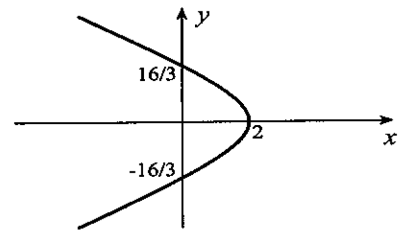
But $4a^2x_0x_1 = -1$, and hence $y = -1/(4a)$; that is, the point of intersection lies on the directrix.

EXERCISES 9.6

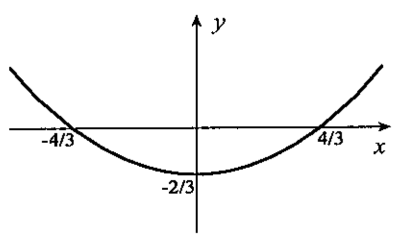
1. We have a parabola with focus at the origin that opens to the left. It crosses the x-axis at $r = 3/2$ when $\theta = 0$, and the y-axis at $r = 3$ when $\theta = \pm\pi/2$.

2. Since $r = \dfrac{16/3}{1 + (5/3)\cos\theta}$, we have a hyperbola with foci on the x-axis. It crosses the x-axis at $r = 2$ when $\theta = 0$, and the y-axis at $r = 16/3$ when $\theta = \pm\pi/2$. Only the left half of the hyperbola is obtained.

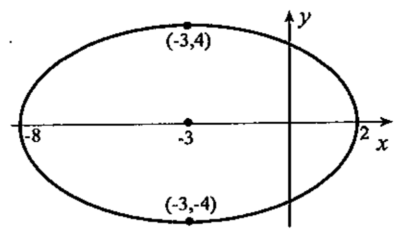




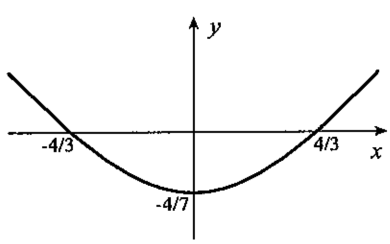
3. Since $r = \dfrac{4/3}{1 - \sin\theta}$, we have a parabola with focus at the origin that opens upward. It crosses the y-axis at $r = 2/3$ when $\theta = -\pi/2$, and the x-axis at $r = 4/3$ when $\theta = 0$ and $\theta = \pi$.



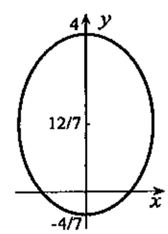
4. Since $r = \dfrac{16/5}{1 + (3/5)\cos\theta}$, we have an ellipse with foci on the x-axis. It crosses the x-axis at $r = 2$ when $\theta = 0$, and at $r = 8$ when $\theta = \pi$. The centre of the ellipse is $(-3, 0)$, and its maximum y-value is $b = \sqrt{25 - 9} = 4$.



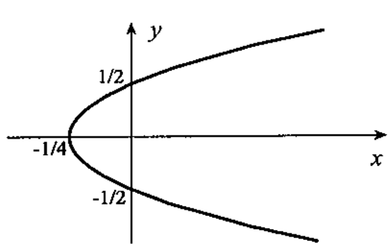
5. Since $r = \dfrac{4/3}{1 - (4/3)\sin\theta}$, we have a hyperbola with foci on the y-axis. It crosses the y-axis at $r = 4/7$ when $\theta = -\pi/2$, and the x-axis at $r = 4/3$ when $\theta = 0$ and $\theta = \pi$. Only the top half of the hyperbola is obtained.



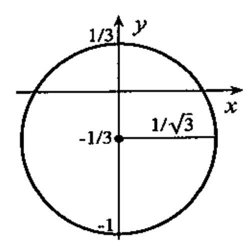
6. Since $r = \dfrac{1}{1 - (3/4)\sin\theta}$, we have an ellipse with foci on the y-axis. It crosses the y-axis at $r = 4$ when $\theta = \pi/2$, and at $r = 4/7$ when $\theta = -\pi/2$. The centre of the ellipse is $(0, 12/7)$ and its maximum x-value is $a = \sqrt{(16/7)^2 - (12/7)^2} = 4/\sqrt{7}$.



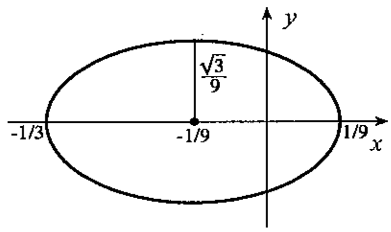
7. Since $r = \dfrac{1/2}{1 - \cos\theta}$, we have a parabola that opens to the right. It crosses the x-axis at $r = 1/4$ when $\theta = \pi$, and the y-axis at $r = 1/2$ when $\theta = \pm\pi/2$.



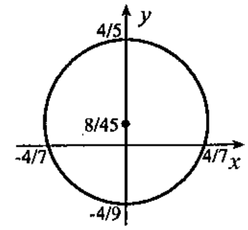
8. Since $r = \dfrac{1/2}{1 + (1/2)\sin\theta}$, we have an ellipse with foci on the y-axis. It crosses the y-axis at $r = 1/3$ when $\theta = \pi/2$, and at $r = 1$ when $\theta = -\pi/2$. The centre of the ellipse is $(0, -1/3)$ and its maximum x-value is $a = \sqrt{(2/3)^2 - (1/3)^2} = 1/\sqrt{3}$.



9. Since $r = \dfrac{1}{3\cos\theta + 6} = \dfrac{1/6}{1 + (1/2)\cos\theta}$, we have an ellipse with foci on the x-axis. It crosses the x-axis at $r = 1/9$ when $\theta = 0$, and at $r = 1/3$ when $\theta = \pi$. Its centre is $(-1/9, 0)$ and its maximum y-value is $\sqrt{(2/9)^2 - (1/9)^2} = \sqrt{3}/9$.



10. Since $r = \dfrac{4}{7 - 2\sin\theta} = \dfrac{4/7}{1 - (2/7)\sin\theta}$, we have an ellipse with foci on the y-axis. It crosses the y-axis at $r = 4/5$ when $\theta = \pi/2$, and at $r = 4/9$ when $\theta = -\pi/2$. The centre of the ellipse is $(0, 8/45)$ and its maximum x-value is $a = \sqrt{(28/45)^2 - (8/45)^2} = 4\sqrt{5}/15$.



11. We set $r = \sqrt{x^2+y^2}$ and $\sin\theta = y/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{3}{1 - \dfrac{y}{\sqrt{x^2+y^2}}} = \dfrac{3\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} - y}$. Thus, $\sqrt{x^2+y^2} - y = 3 \Longrightarrow x^2 + y^2 = (y+3)^2$. This simplifies to $x^2 = 6y + 9$.

12. We set $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{1}{3 + \dfrac{x}{\sqrt{x^2+y^2}}} = \dfrac{\sqrt{x^2+y^2}}{3\sqrt{x^2+y^2} + x}$. Thus, $3\sqrt{x^2+y^2} + x = 1 \Longrightarrow 9(x^2+y^2) = (1-x)^2$. This simplifies to $8x^2 + 9y^2 + 2x = 1$.

13. We set $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{1}{1 + \dfrac{2x}{\sqrt{x^2+y^2}}} = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} + 2x}$. Thus, $\sqrt{x^2+y^2} + 2x = 1 \Longrightarrow x^2 + y^2 = (1-2x)^2$. This simplifies to $3x^2 - y^2 - 4x + 1 = 0$.

14. We set $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{2}{1 - \dfrac{3x}{\sqrt{x^2+y^2}}} = \dfrac{2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} - 3x}$. Thus, $\sqrt{x^2+y^2} - 3x = 2 \Longrightarrow x^2 + y^2 = (3x+2)^2$. This simplifies to $8x^2 - y^2 + 12x + 4 = 0$.

15. We set $r = \sqrt{x^2+y^2}$ and $\sin\theta = y/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{4}{6 - \dfrac{3y}{\sqrt{x^2+y^2}}} = \dfrac{4\sqrt{x^2+y^2}}{6\sqrt{x^2+y^2} - 3y}$. Thus, $6\sqrt{x^2+y^2} - 3y = 4 \Longrightarrow 36(x^2+y^2) = (3y+4)^2$. This simplifies to $36x^2 + 27y^2 - 24y = 16$.

16. We set $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/\sqrt{x^2+y^2}$, $\quad \sqrt{x^2+y^2} = \dfrac{4}{5 + \dfrac{5x}{\sqrt{x^2+y^2}}} = \dfrac{4\sqrt{x^2+y^2}}{5\sqrt{x^2+y^2} + 5x}$. Thus, $5\sqrt{x^2+y^2} + 5x = 4 \Longrightarrow 25(x^2+y^2) = (4-5x)^2$. This simplifies to $25y^2 = 16 - 40x$.

17. For this hyperbola, $c = \sqrt{16+1} = \sqrt{17}$. If we choose the pole at the focus $(\sqrt{17}, 0)$ and the polar axis parallel to the positive x-axis, then

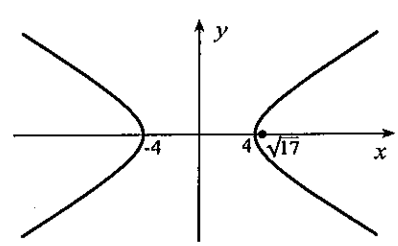
$$x = \sqrt{17} + r\cos\theta, \quad y = r\sin\theta.$$

Substitution of these into the equation of the hyperbola gives

$$17 + 2\sqrt{17}r\cos\theta + r^2\cos^2\theta - 16r^2\sin^2\theta = 16,$$

or,

$$r^2(\cos^2\theta - 16\sin^2\theta) + r(2\sqrt{17}\cos\theta) + 1 = 0.$$



Solutions for r are

$$r = \frac{-2\sqrt{17}\cos\theta \pm \sqrt{68\cos^2\theta - 4(\cos^2\theta - 16\sin^2\theta)}}{2(\cos^2\theta - 16\sin^2\theta)} = \frac{-2\sqrt{17}\cos\theta \pm 8}{2(17\cos^2\theta - 16)} = \frac{-\sqrt{17}\cos\theta \pm 4}{17\cos^2\theta - 16}.$$

If we choose -4, then

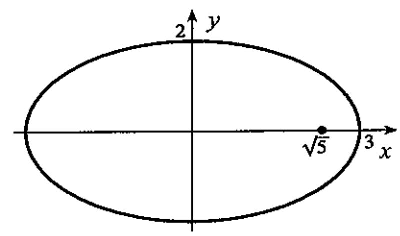
$$r = \frac{-\sqrt{17}\cos\theta - 4}{(\sqrt{17}\cos\theta + 4)(\sqrt{17}\cos\theta - 4)} = \frac{1}{4 - \sqrt{17}\cos\theta}.$$

This gives the right half of the hyperbola. The other half is obtained when we set

$$r = \frac{-\sqrt{17}\cos\theta + 4}{17\cos^2\theta - 16} = \frac{-1}{4 + \sqrt{17}\cos\theta}.$$

18. For this ellipse, $c = \sqrt{9-4} = \sqrt{5}$. If we choose the pole at the focus $(\sqrt{5}, 0)$ and the polar axis along the positive x-axis, then

$$x = \sqrt{5} + r\cos\theta, \quad y = r\sin\theta.$$



Substitution of these into the equation of the ellipse gives

$$36 = 4(5 + 2\sqrt{5}r\cos\theta + r^2\cos^2\theta) + 9r^2\sin^2\theta,$$

or,

$$r^2(4\cos^2\theta + 9\sin^2\theta) + r(8\sqrt{5}\cos\theta) - 16 = 0.$$

Solutions for r are

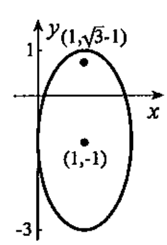
$$r = \frac{-8\sqrt{5}\cos\theta \pm \sqrt{320\cos^2\theta - 4(-16)(4\cos^2\theta + 9\sin^2\theta)}}{2(4\cos^2\theta + 9\sin^2\theta)} = \frac{-8\sqrt{5}\cos\theta \pm 24}{2(9 - 5\cos^2\theta)} = \frac{4(\pm 3 - \sqrt{5}\cos\theta)}{9 - 5\cos^2\theta}.$$

If we choose $+3$ (to correspond to positive r-values),

$$r = \frac{4(3 - \sqrt{5}\cos\theta)}{(3 - \sqrt{5}\cos\theta)(3 + \sqrt{5}\cos\theta)} = \frac{4}{3 + \sqrt{5}\cos\theta}.$$

19. For this ellipse, $c = \sqrt{4-1} = \sqrt{3}$. If we choose the pole at the focus $(1, \sqrt{3} - 1)$ and the polar axis parallel to the positive x-axis, then

$$x = 1 + r\cos\theta, \quad y = (\sqrt{3} - 1) + r\sin\theta.$$



Substitution of these into the equation of the ellipse gives

$$\begin{aligned} 4 &= 4r^2\cos^2\theta + (\sqrt{3} + r\sin\theta)^2 \\ &= 4r^2\cos^2\theta + r^2\sin^2\theta + 2\sqrt{3}r\sin\theta + 3, \end{aligned}$$

or,

$$r^2(4\cos^2\theta + \sin^2\theta) + r(2\sqrt{3}\sin\theta) - 1 = 0.$$

Solutions for r are

$$r = \frac{-2\sqrt{3}\sin\theta \pm \sqrt{12\sin^2\theta + 4(4\cos^2\theta + \sin^2\theta)}}{2(4\cos^2\theta + \sin^2\theta)} = \frac{-2\sqrt{3}\sin\theta \pm 4}{2(4 - 3\sin^2\theta)} = \frac{\pm 2 - \sqrt{3}\sin\theta}{4 - 3\sin^2\theta}.$$

If we choose $+2$ (to correspond to positive r-values),

$$r = \frac{2 - \sqrt{3}\sin\theta}{(2 - \sqrt{3}\sin\theta)(2 + \sqrt{3}\sin\theta)} = \frac{1}{2 + \sqrt{3}\sin\theta}.$$

20. For this hyperbola, $c = \sqrt{9+1} = \sqrt{10}$. If we choose the pole at the focus $(\sqrt{10}, 2)$ and the polar axis parallel to the positive x-axis, then

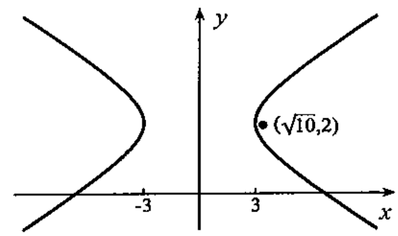
$$x = \sqrt{10} + r\cos\theta, \quad y = 2 + r\sin\theta.$$

Substitution of these into the equation of the hyperbola gives

$$10 + 2\sqrt{10}r\cos\theta + r^2\cos^2\theta - 9r^2\sin^2\theta = 9,$$

or,

$$r^2(\cos^2\theta - 9\sin^2\theta) + r(2\sqrt{10}\cos\theta) + 1 = 0.$$



Solutions for r are

$$r = \frac{-2\sqrt{10}\cos\theta \pm \sqrt{40\cos^2\theta - 4(\cos^2\theta - 9\sin^2\theta)}}{2(\cos^2\theta - 9\sin^2\theta)} = \frac{-2\sqrt{10}\cos\theta \pm 6}{2(10\cos^2\theta - 9)} = \frac{-\sqrt{10}\cos\theta \pm 3}{10\cos^2\theta - 9}.$$

If we choose -3, then

$$r = \frac{-\sqrt{10}\cos\theta - 3}{(\sqrt{10}\cos\theta + 3)(\sqrt{10}\cos\theta - 3)} = \frac{1}{3 - \sqrt{10}\cos\theta}.$$

This gives the right half of the hyperbola. The other half is obtained when we set

$$r = \frac{-\sqrt{10}\cos\theta + 3}{10\cos^2\theta - 9} = \frac{-1}{3 + \sqrt{10}\cos\theta}.$$

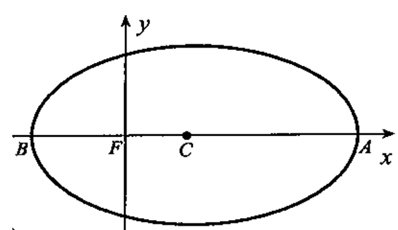
21. (a) We prove the case for the ellipse. Equation 9.29 gives the polar coordinate r of A as $\epsilon d/(1-\epsilon)$ (from $\theta = 0$) and that for B as $r = \epsilon d/(1+\epsilon)$ (from $\theta = \pi$). The r-coordinate of the centre of the ellipse is therefore

$$\frac{1}{2}\left(\frac{\epsilon d}{1-\epsilon} - \frac{\epsilon d}{1+\epsilon}\right),$$

and this is the length of CF. The length of AB is

$$\frac{\epsilon d}{1-\epsilon} + \frac{\epsilon d}{1+\epsilon}.$$

Consequently,



$$\frac{\|CF\|}{\|AB\|/2} = \frac{\frac{1}{2}\left(\frac{\epsilon d}{1-\epsilon} - \frac{\epsilon d}{1+\epsilon}\right)}{\frac{1}{2}\left(\frac{\epsilon d}{1-\epsilon} + \frac{\epsilon d}{1+\epsilon}\right)},$$

and this simplifies to ϵ.

(b) If we express equation 9.32 in the form $(x-h)^2/a^2 + y^2/b^2 = 1$, then

$$a^2 = \left(\frac{\epsilon d}{1-\epsilon^2}\right)^2, \qquad b^2 = (1-\epsilon^2)\left(\frac{\epsilon d}{1-\epsilon^2}\right)^2.$$

Consequently, $b^2/a^2 = 1 - \epsilon^2$, and this ratio determines the elongation of the ellipse. For small ϵ, $b \approx a$ and the ellipse is almost circular as in Figure 9.45a. For ϵ close to 1, b is very much less than a and the ellipse is very elongated as in Figure 9.45b.

22. According to equation 9.32, the distance between the foci of the ellipse determined by equation 9.29 is $2d\epsilon^2/(1-\epsilon^2)$. For this distance to approach zero, ϵ must approach zero.

23. (a) When $\epsilon = 0$, corresponding to a circular orbit, $I = \theta + C$.

(b) When $0 < \epsilon < 1$, the orbit is elliptic. We use the Weierstrass substitution $x = \tan(\theta/2)$ of Exercise 35 in Section 8.6. The integral becomes

$$I = \int \frac{1}{\left[1 + \dfrac{\epsilon(1-x^2)}{1+x^2}\right]^2} \frac{2}{1+x^2}dx = 2\int \frac{1+x^2}{[1+x^2+\epsilon-\epsilon x^2]^2}dx = 2\int \frac{1+x^2}{[(1+\epsilon)+(1-\epsilon)x^2]^2}dx.$$

We now set $x = \sqrt{\dfrac{1+\epsilon}{1-\epsilon}}\tan\phi$ and $dx = \sqrt{\dfrac{1+\epsilon}{1-\epsilon}}\sec^2\phi\, d\phi$:

$$\begin{aligned}
I &= 2\int \frac{1 + \dfrac{1+\epsilon}{1-\epsilon}\tan^2\phi}{[1+\epsilon+(1+\epsilon)\tan^2\phi]^2}\sqrt{\frac{1+\epsilon}{1-\epsilon}}\sec^2\phi\, d\phi = \frac{2}{(1-\epsilon^2)^{3/2}}\int \frac{(1-\epsilon)+(1+\epsilon)\tan^2\phi}{\sec^2\phi}d\phi \\
&= \frac{2}{(1-\epsilon^2)^{3/2}}\int [(1-\epsilon)\cos^2\phi + (1+\epsilon)\sin^2\phi]\, d\phi \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\int [(1-\epsilon)(1+\cos 2\phi) + (1+\epsilon)(1-\cos 2\phi)]\, d\phi \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}[2\phi - \epsilon\sin 2\phi] + C = \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\sqrt{\frac{1-\epsilon}{1+\epsilon}}x - \frac{2\epsilon\sqrt{\dfrac{1+\epsilon}{1-\epsilon}}x}{x^2 + \dfrac{1+\epsilon}{1-\epsilon}}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\sqrt{\frac{1-\epsilon}{1+\epsilon}}x - \frac{2\epsilon\sqrt{1-\epsilon^2}x}{(1-\epsilon)x^2+(1+\epsilon)}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{2\epsilon\sqrt{1-\epsilon^2}\tan(\theta/2)}{(1-\epsilon)\tan^2(\theta/2)+(1+\epsilon)}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{2\epsilon\sqrt{1-\epsilon^2}\sin(\theta/2)}{\cos(\theta/2)\left[(1-\epsilon)\dfrac{\sin^2(\theta/2)}{\cos^2(\theta/2)}+(1+\epsilon)\right]}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{2\epsilon\sqrt{1-\epsilon^2}\sin(\theta/2)\cos(\theta/2)}{(1-\epsilon)\sin^2(\theta/2)+(1+\epsilon)\cos^2(\theta/2)}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{\epsilon\sqrt{1-\epsilon^2}\sin\theta}{(1-\epsilon)(1-\cos\theta)/2+(1+\epsilon)(1+\cos\theta)/2}\right] + C \\
&= \frac{1}{(1-\epsilon^2)^{3/2}}\left[2\mathrm{Tan}^{-1}\left(\sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan(\theta/2)\right) - \frac{\epsilon\sqrt{1-\epsilon^2}\sin\theta}{1+\epsilon\cos\theta}\right] + C.
\end{aligned}$$

(c) When $\epsilon = 1$, the orbit is parabolic. Once again we use the Weierstrass substitution,

$$\begin{aligned}
I &= \int \frac{1}{(1+\cos\theta)^2}d\theta = \int \frac{1}{\left(1+\dfrac{1-x^2}{1+x^2}\right)^2}\frac{2}{1+x^2}dx \\
&= 2\int \frac{1+x^2}{4}dx = \frac{1}{2}\left(x + \frac{x^3}{3}\right) + C = \frac{1}{2}\tan(\theta/2) + \frac{1}{6}\tan^3(\theta/2) + C.
\end{aligned}$$

(d) When $\epsilon > 1$, the orbit is a hyperbola. With the Weierstrass substitution we arrive, as in the elliptic case, at

$$I = 2\int \frac{1+x^2}{[(1+\epsilon)+(1-\epsilon)x^2]^2}dx.$$

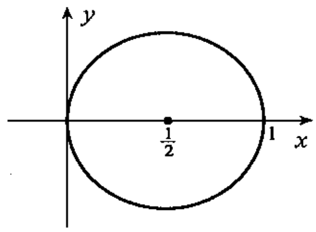
This time we set $x = \sqrt{\dfrac{1+\epsilon}{\epsilon-1}}\sin\phi$ and $dx = \sqrt{\dfrac{1+\epsilon}{\epsilon-1}}\cos\phi\, d\phi$,

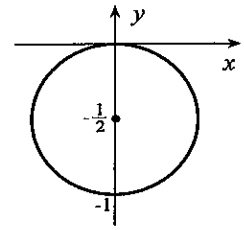
$$
\begin{aligned}
I &= 2\int \frac{1+\dfrac{1+\epsilon}{\epsilon-1}\sin^2\phi}{[(1+\epsilon)-(1+\epsilon)\sin^2\phi]^2}\sqrt{\frac{1+\epsilon}{\epsilon-1}}\cos\phi\,d\phi \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\int \frac{(\epsilon-1)+(1+\epsilon)\sin^2\phi}{\cos^4\phi}\cos\phi\,d\phi \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\int [(\epsilon-1)\sec^3\phi+(1+\epsilon)(\sec^3\phi-\sec\phi)]\,d\phi \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\int [2\epsilon\sec^3\phi-(1+\epsilon)\sec\phi]\,d\phi \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}[\epsilon(\sec\phi\tan\phi+\ln|\sec\phi+\tan\phi|)-(1+\epsilon)\ln|\sec\phi+\tan\phi|]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}[\epsilon\sec\phi\tan\phi-\ln|\sec\phi+\tan\phi|]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\left[\epsilon\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}x}{\dfrac{1+\epsilon}{\epsilon-1}-x^2}-\ln\left|\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}}{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}-x^2}}+\frac{x}{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}-x^2}}\right|\right]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\left[\frac{\epsilon\sqrt{\epsilon^2-1}x}{(1+\epsilon)-(\epsilon-1)x^2}-\ln\left|\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}+x}{\sqrt{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}-x}\sqrt{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}+x}}\right|\right]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\left[\frac{\epsilon\sqrt{\epsilon^2-1}x}{(1+\epsilon)-(\epsilon-1)x^2}-\frac{1}{2}\ln\left|\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}+x}{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}-x}\right|\right]+C \\
&= \frac{2}{(\epsilon^2-1)^{3/2}}\left[\frac{\epsilon\sqrt{\epsilon^2-1}\tan(\theta/2)}{(1+\epsilon)-(\epsilon-1)\tan^2(\theta/2)}-\frac{1}{2}\ln\left|\frac{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}+\tan(\theta/2)}{\sqrt{\dfrac{1+\epsilon}{\epsilon-1}}-\tan(\theta/2)}\right|\right]+C \\
&= \frac{1}{(\epsilon^2-1)^{3/2}}\left[\frac{2\epsilon\sqrt{\epsilon^2-1}\dfrac{\sin(\theta/2)}{\cos(\theta/2)}}{(1+\epsilon)-(\epsilon-1)\dfrac{\sin^2(\theta/2)}{\cos^2(\theta/2)}}-\ln\left|\frac{\sqrt{1+\epsilon}+\sqrt{\epsilon-1}\tan(\theta/2)}{\sqrt{1+\epsilon}-\sqrt{\epsilon-1}\tan(\theta/2)}\right|\right]+C \\
&= \frac{1}{(\epsilon^2-1)^{3/2}}\left[\frac{2\epsilon\sqrt{\epsilon^2-1}\sin(\theta/2)\cos(\theta/2)}{(1+\epsilon)\cos^2(\theta/2)-(\epsilon-1)\sin^2(\theta/2)}-\ln\left|\frac{\sqrt{1+\epsilon}+\sqrt{\epsilon-1}\tan(\theta/2)}{\sqrt{1+\epsilon}-\sqrt{\epsilon-1}\tan(\theta/2)}\right|\right]+C \\
&= \frac{1}{(\epsilon^2-1)^{3/2}}\left[\frac{\epsilon\sqrt{\epsilon^2-1}\sin\theta}{1+\epsilon\cos\theta}-\ln\left|\frac{\sqrt{1+\epsilon}+\sqrt{\epsilon-1}\tan(\theta/2)}{\sqrt{1+\epsilon}-\sqrt{\epsilon-1}\tan(\theta/2)}\right|\right]+C
\end{aligned}
$$

REVIEW EXERCISES

1. In Cartesian coordinates, $\sqrt{x^2+y^2} = \dfrac{x}{\sqrt{x^2+y^2}} \Longrightarrow (x-1/2)^2+y^2=1/4$.
This is a circle.

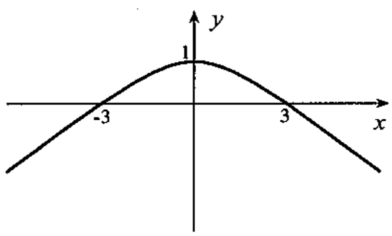
2. In Cartesian coordinates, $\sqrt{x^2+y^2} = \dfrac{-y}{\sqrt{x^2+y^2}} \Longrightarrow x^2+(y+1/2)^2=1/4$.
This is a circle.



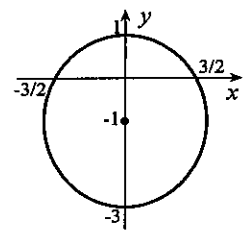


3. This is a hyperbola with foci on the y-axis.

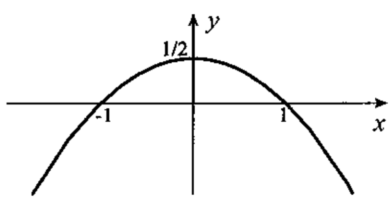
It crosses the y-axis at $r = 1$ when $\theta = \pi/2$ and the x-axis at $r = 3$ when $\theta = 0$ and $\theta = \pi$. Only the bottom half of the hyperbola is obtained.



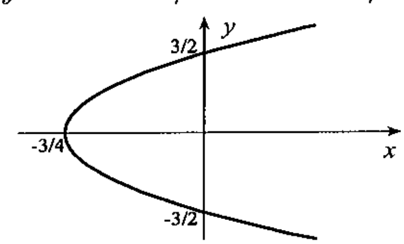
4. Since $r = \dfrac{3/2}{1 + (1/2)\sin\theta}$, the curve is an ellipse with foci on the y-axis. It crosses the y-axis at $r = 1$ when $\theta = \pi/2$, and at $r = 3$ when $\theta = -\pi/2$. The centre of the ellipse is $(0, -1)$, and its maximum x-value is $a = \sqrt{4-1} = \sqrt{3}$.



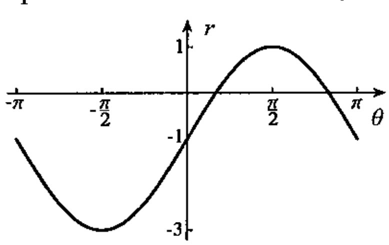
5. This is a parabola that opens downward. It crosses the y-axis at $r = 1/2$ when $\theta = \pi/2$ and the x-axis at $r = 1$ when $\theta = 0$ and $\theta = \pi$.

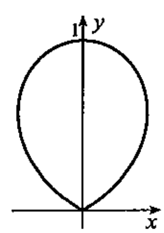


6. This is a parabola that opens to the right. It crosses the x-axis at $r = 3/4$ when $\theta = \pi$ and the y-axis at $r = 3/2$ when $\theta = \pm\pi/2$.

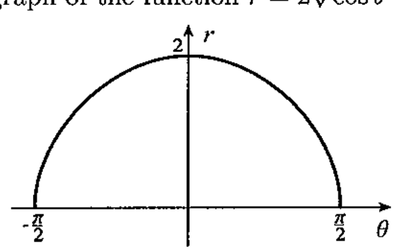


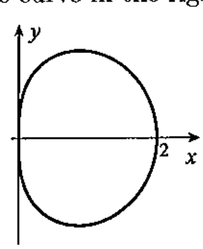
7. The graph of the function $r = -1 + 2\sin\theta$ in the left figure gives the curve in the right figure.



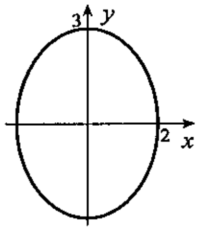


8. The graph of the function $r = 2\sqrt{\cos\theta}$ in the left figure gives the curve in the right figure.

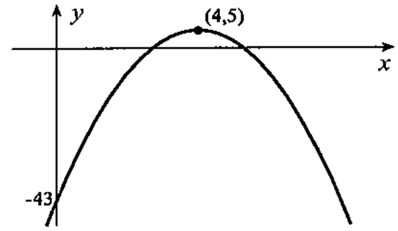




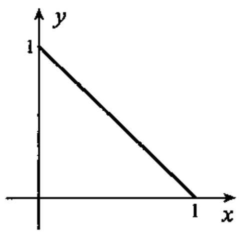
9. This is the ellipse $x^2/4 + y^2/9 = 1$.



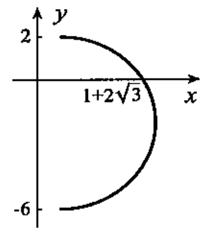
10. This is the parabola $y = 5 - 3(x-4)^2$.



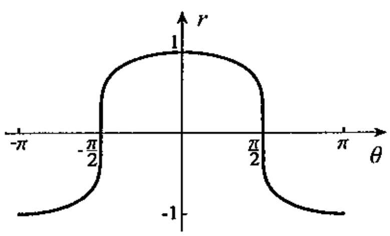
11. This is the line segment $x + y = 1$, $0 \le x \le 1$, traced four times.

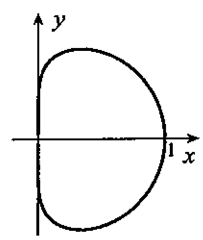


12. These equations describe the right half of of the circle $(x-1)^2 + (y+2)^2 = 16$.

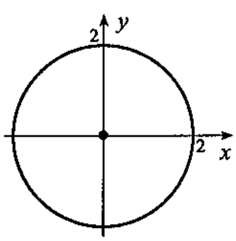


13. In polar coordinates $r^6 = r\cos\theta \implies r^5 = \cos\theta$. The graph of the function $r = (\cos\theta)^{1/5}$ in the left figure gives the curve in the right figure.

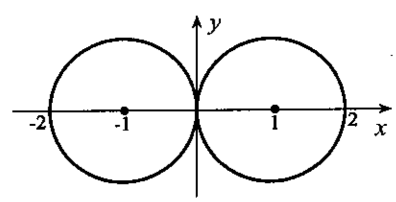




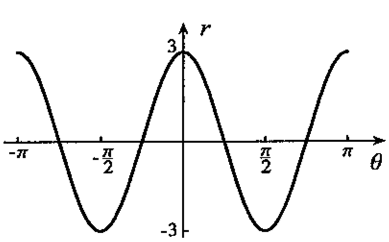
14. In polar coordinates, $r^2 = 2r$, and this describes the circle $r = 2$ and the origin $r = 0$.

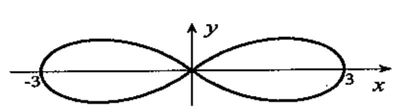


15. In Cartesian coordinates, $x^2 + y^2 = \dfrac{4x^2}{x^2+y^2} \implies x^2 + y^2 = \pm 2x$. From these $(x \pm 1)^2 + y^2 = 1$, two circles.



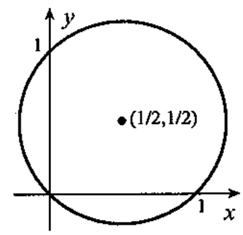
16. The graph of the function $r = 3\cos 2\theta$ in the left figure gives the curve in the right figure.



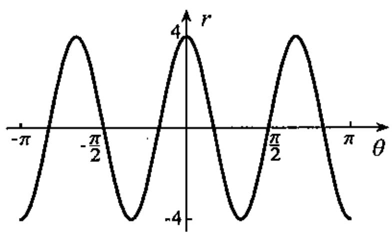


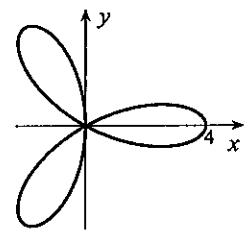
17. If we square the equation, $x^2 + 2xy + x^2 = x^2 + y^2$ and this implies that $x = 0$ or $y = 0$. Since x and y must be nonnegative, the equation defines the nonnegative x- and y-axes.

18. When squares are completed on x and y terms, $(x-1/2)^2+(y-1/2)^2=1/2$, a circle.

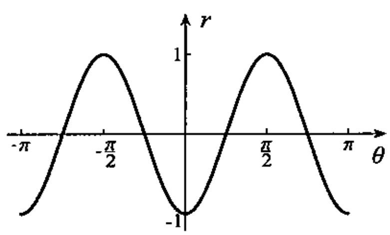


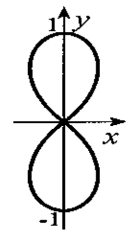
19. The graph of the function $r=4\cos 3\theta$ in the left figure gives the curve in the right figure.



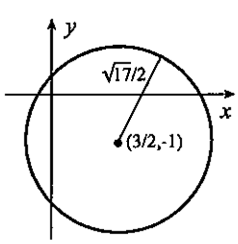


20. The graph of the function $r=\sin^2\theta-\cos^2\theta=-\cos 2\theta$ in the left figure gives the curve in the right figure.

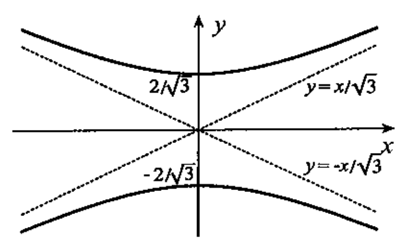




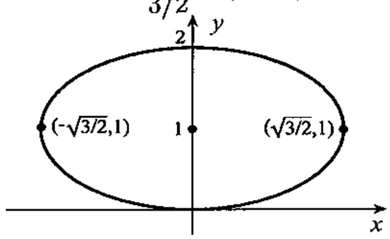
21. When we complete squares on x- and y-terms, $(x-3/2)^2+(y+1)^2=17/4$, a circle.



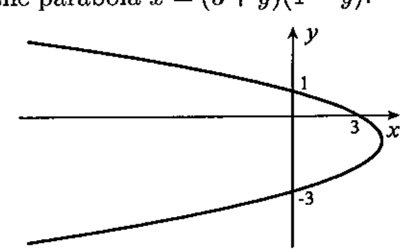
22. This is a hyperbola.



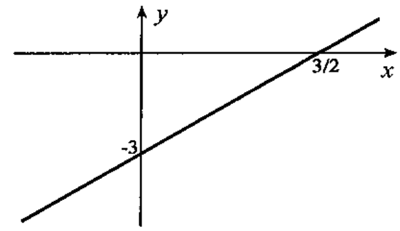
23. This is the ellipse $\dfrac{x^2}{3/2}+(y-1)^2=1$.



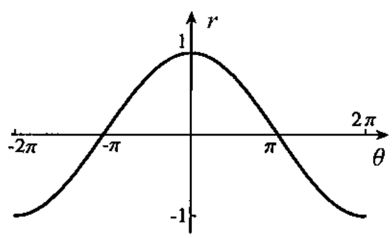
24. This is the parabola $x=(3+y)(1-y)$.

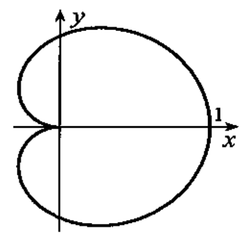


25. This is the line $2x - y = 3$.

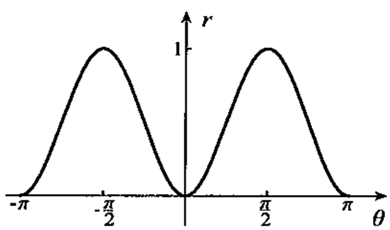


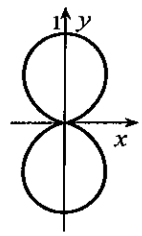
26. The graph of the function $r = \cos(\theta/2)$ in the left figure gives the curve in the right figure.





27. The graph of the function $r = \sin^2\theta$ in the left figure gives the curve in the right figure.

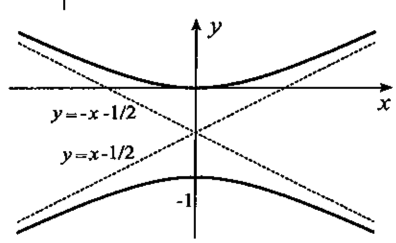




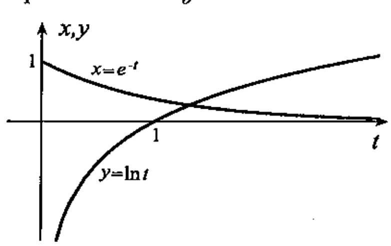
28. Completion of the square on the y term leads to

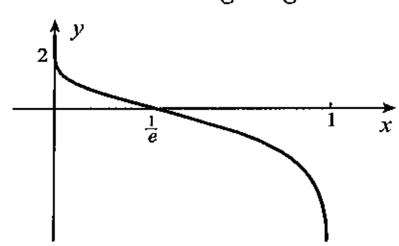
$$\frac{(y+1/2)^2}{1/4} - \frac{x^2}{1/4} = 1.$$

This is a hyperbola.

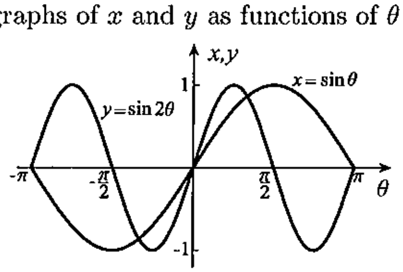


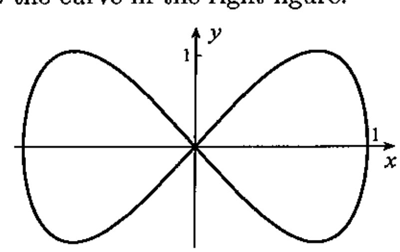
29. The graphs of x and y as functions of t in the left figure lead to the curve in the right figure.



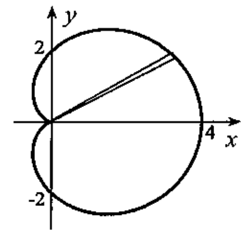


30. The graphs of x and y as functions of θ in the left figure lead to the curve in the right figure.

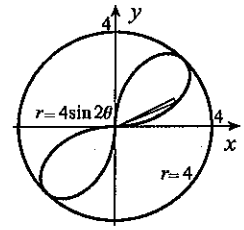




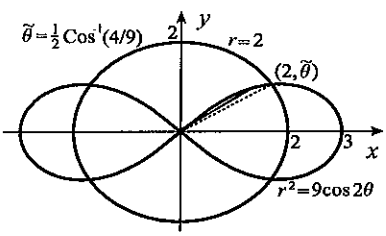
31. $$\begin{aligned} A &= 2\int_0^{\pi} \frac{1}{2}(2+2\cos\theta)^2\,d\theta \\ &= 4\int_0^{\pi} (1+2\cos\theta+\cos^2\theta)\,d\theta \\ &= 4\int_0^{\pi} \left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta \\ &= 4\left\{\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right\}_0^{\pi} = 6\pi \end{aligned}$$



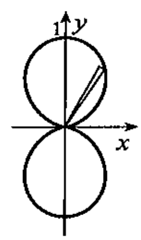
32. $$\begin{aligned} A &= 16\pi - 2\int_0^{\pi/2} \frac{1}{2}(4\sin 2\theta)^2\,d\theta \\ &= 16\pi - 16\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta \\ &= 16\pi - 8\left\{\theta-\frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = 12\pi \end{aligned}$$



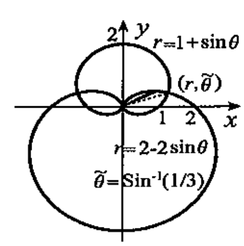
33. $$\begin{aligned} A &= 4\left[4\pi\left(\frac{\tilde\theta}{2\pi}\right)+\int_{\tilde\theta}^{\pi/4}\frac{1}{2}(9\cos 2\theta)\,d\theta\right] \\ &= 8\tilde\theta+18\left\{\frac{1}{2}\sin 2\theta\right\}_{\tilde\theta}^{\pi/4} = 8\tilde\theta+9(1-\sin 2\tilde\theta) \\ &= 8\tilde\theta+9-9\sin 2\tilde\theta = 9+4\,\mathrm{Cos}^{-1}(4/9)-9\sqrt{1-\frac{16}{81}} \\ &= 9+4\,\mathrm{Cos}^{-1}(4/9)-\sqrt{65} \end{aligned}$$



34. $$\begin{aligned} A &= 4\int_0^{\pi/2}\frac{1}{2}(\sin^2\theta)^2\,d\theta \\ &= 2\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta \\ &= \frac{1}{2}\int_0^{\pi/2}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta \\ &= \frac{1}{2}\left\{\frac{3\theta}{2}-\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{3\pi}{8} \end{aligned}$$



35. $$\begin{aligned} A &= 2\int_{-\pi/2}^{\tilde\theta}\frac{1}{2}(1+\sin\theta)^2\,d\theta + 2\int_{\tilde\theta}^{\pi/2}\frac{1}{2}(2-2\sin\theta)^2\,d\theta \\ &= \int_{-\pi/2}^{\tilde\theta}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta \\ &\quad +4\int_{\tilde\theta}^{\pi/2}\left(1-2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta \\ &= \left\{\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right\}_{-\pi/2}^{\tilde\theta} + 4\left\{\frac{3\theta}{2}+2\cos\theta-\frac{1}{4}\sin 2\theta\right\}_{\tilde\theta}^{\pi/2} \\ &= \frac{3\tilde\theta}{2}-2\cos\tilde\theta-\frac{1}{4}\sin 2\tilde\theta+\frac{3\pi}{4}+3\pi-6\tilde\theta-8\cos\tilde\theta+\sin 2\tilde\theta = \frac{15\pi}{4}-\frac{9\tilde\theta}{2}-10\cos\tilde\theta+\frac{3}{2}\sin\tilde\theta\cos\tilde\theta \\ &= \frac{15\pi}{4}-\frac{9}{2}\,\mathrm{Sin}^{-1}(1/3)-10\left(\frac{2\sqrt{2}}{3}\right)+\frac{3}{2}\left(\frac{1}{3}\right)\left(\frac{2\sqrt{2}}{3}\right) = \frac{15\pi}{4}-\frac{9}{2}\,\mathrm{Sin}^{-1}(1/3)-\frac{19\sqrt{2}}{3} \end{aligned}$$



36. Since $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3-3t^2}{3t^2+2}$,

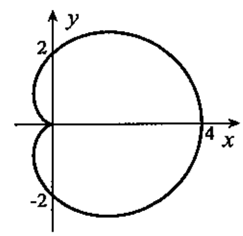
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3-3t^2}{3t^2+2}\right) = \frac{d}{dt}\left(\frac{3-3t^2}{3t^2+2}\right)\frac{dt}{dx} = \frac{\dfrac{d}{dt}\left(\dfrac{3-3t^2}{3t^2+2}\right)}{dx/dt} = \frac{\dfrac{(3t^2+2)(-6t)-(3-3t^2)(6t)}{(3t^2+2)^2}}{3t^2+2} = \frac{-30t}{(3t^2+2)^3}.$$

37. Since $\dfrac{dy}{dx} = \dfrac{dy/du}{dx/du} = \dfrac{-3\sin u}{2\cos u} = -\dfrac{3}{2}\tan u$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{3}{2}\tan u\right) = -\frac{3}{2}\frac{d}{du}(\tan u)\frac{du}{dx} = -\frac{3}{2}\frac{\dfrac{d}{du}(\tan u)}{dx/du} = -\frac{3}{2}\left(\frac{\sec^2 u}{2\cos u}\right) = -\frac{3}{4}\sec^3 u.$$

38. By equation 9.14,

$$\begin{aligned} L &= 2\int_0^{\pi}\sqrt{(2+2\cos\theta)^2+(-2\sin\theta)^2}\,d\theta \\ &= 4\sqrt{2}\int_0^{\pi}\sqrt{1+\cos\theta}\,d\theta \\ &= 4\sqrt{2}\int_0^{\pi}\sqrt{1+[2\cos^2(\theta/2)-1]}\,d\theta \\ &= 8\int_0^{\pi}\cos(\theta/2)\,d\theta = 8\Big\{2\sin(\theta/2)\Big\}_0^{\pi} = 16. \end{aligned}$$



39. According to equation 9.3, the length is

$$\int_0^1\sqrt{(2t)^2+(3t^2)^2}\,dt = \int_0^1 t\sqrt{4+9t^2}\,dt = \left\{\frac{1}{27}(4+9t^2)^{3/2}\right\}_0^1 = \frac{1}{27}(13\sqrt{13}-8).$$

40. By equation 9.3, the length is

$$\int_0^{\pi/2}\sqrt{(e^t\cos t - e^t\sin t)^2+(e^t\sin t+e^t\cos t)^2}\,dt = \int_0^{\pi/2}\sqrt{2}e^t\,dt = \sqrt{2}\Big\{e^t\Big\}_0^{\pi/2} = \sqrt{2}(e^{\pi/2}-1).$$

41. The slope of the tangent line is $\dfrac{dy}{dx}_{|\theta=\pi/6} = \dfrac{(-2\cos\theta)\sin\theta+(2-2\sin\theta)\cos\theta}{(-2\cos\theta)\cos\theta-(2-2\sin\theta)\sin\theta}_{|\theta=\pi/6} = 0$. Since Cartesian coordinates of the point are $(\sqrt{3}/2, 1/2)$, the equation of the tangent line is $y = 1/2$.

CHAPTER 10

EXERCISES 10.1

1. This sequence has limit 0.

2. This sequence diverges.

3. This sequence has limit 3.

4. This sequence has limit 0.

5. This sequence diverges.

6. This sequence has limit 0.

7. This sequence diverges.

8. This sequence has limit $\lim_{n\to\infty}\frac{n}{n^2+n+2}=\lim_{n\to\infty}\frac{1}{n+1+2/n}=0.$

9. This sequence has limit 0.

10. This sequence has limit $\pi/2$.

11. This sequence has limit 0.

12. This sequence diverges.

13. This sequence has limit 2 (since all terms are equal to 2).

14. This sequence has limit 0.

15. This sequence has limit $\lim_{n\to\infty}\frac{n+1}{2n+3}=\lim_{n\to\infty}\frac{1+1/n}{2+3/n}=\frac{1}{2}.$

16. This sequence has limit $\lim_{n\to\infty}\frac{2n+3}{n^2-5}=\lim_{n\to\infty}\frac{2+3/n}{n-5/n}=0.$

17. This sequence has limit $\lim_{n\to\infty}\frac{n^2+5n-4}{n^2+2n-2}=\lim_{n\to\infty}\frac{1+5/n-4/n^2}{1+2/n-2/n^2}=1.$

18. This sequence has limit 0.

19. This sequence has limit 0.

20. This sequence has limit $\lim_{n\to\infty}\frac{1}{1+1/n}\mathrm{Tan}^{-1}n=\frac{\pi}{2}.$

21. The general term is $\frac{2^n-1}{2^n}$.

22. The general term is $\frac{3n+1}{n^2}$.

23. The general term is $(-1)^{n+1}\frac{\ln(n+1)}{\sqrt{n+1}}$.

24. The general term is $\frac{1+(-1)^{n+1}}{2}$.

25. The general term is $\sqrt{2}\sin\frac{(2n-1)\pi}{4}$.

26. The limit of the sequence $\{\ln n/\sqrt{n}\}$ as $n\to\infty$ is equal to the limit of the function $\ln x/\sqrt{x}$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule on the limit of the function,

$$\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0.$$

27. The limit of the sequence $\{(n^3+1)/e^n\}$ as $n\to\infty$ is equal to the limit of the function $(x^3+1)/e^x$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule on the limit of the function,

$$\lim_{n\to\infty}\frac{n^3+1}{e^n}=\lim_{x\to\infty}\frac{x^3+1}{e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}=\lim_{x\to\infty}\frac{6}{e^x}=0.$$

28. The limit of the sequence $\{n\sin(4/n)\}$ as $n\to\infty$ is equal to the limit of the function $x\sin(4/x)$ as $x\to\infty$, provided the limit of the function exists. When we use L'Hôpital's rule,

$$\lim_{n\to\infty}n\sin\left(\frac{4}{n}\right)=\lim_{x\to\infty}x\sin\left(\frac{4}{x}\right)=\lim_{x\to\infty}\frac{\sin(4/x)}{1/x}=\lim_{x\to\infty}\frac{-(4/x^2)\cos(4/x)}{-1/x^2}=4.$$

29. The limit of the sequence $\{[(n+5)/(n+3)]^n\}$ as $n \to \infty$ is equal to the limit of the function $[(x+5)/(x+3)]^x$ as $x \to \infty$, provided the limit of the function exists. We set L equal to the limit of the function, take logarithms, and use L'Hôpital's rule,

$$\ln L = \ln\left[\lim_{x\to\infty}\left(\frac{x+5}{x+3}\right)^x\right] = \lim_{x\to\infty} x\ln\left(\frac{x+5}{x+3}\right) = \lim_{x\to\infty}\frac{\ln\left(\dfrac{x+5}{x+3}\right)}{1/x}$$

$$= \lim_{x\to\infty}\frac{\dfrac{x+3}{x+5}\left[\dfrac{(x+3)-(x+5)}{(x+3)^2}\right]}{-1/x^2} = \lim_{x\to\infty}\frac{2x^2}{(x+3)(x+5)} = 2.$$

Thus, $L = e^2$, and this is also the limit of the sequence.

30. Certainly the sequence diverges; terms get arbitrarily large for large n. On the other hand, as n increases, the difference between terms approaches $\lim_{n\to\infty}[\ln n - \ln(n+1)] = \lim_{n\to\infty}\ln\left(\dfrac{n}{n+1}\right) = 0.$

31. (a) The first ten terms are 2,3,5,7,11,13,17,19,23,29. (b) No one has developed a formula for all primes.

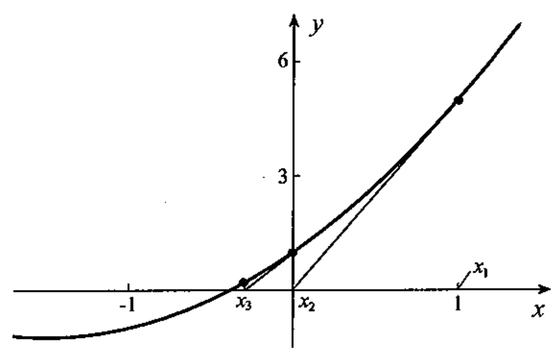
32. The figure indicates that with initial approximation $x_1 = 1$, the sequence defined by Newton's iterative procedure has a limit near $-1/2$. Iteration of

$$x_1 = 1, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

leads to

$$x_2 = 0, \qquad x_3 = -1/3,$$
$$x_4 = -0.381, \qquad x_5 = -0.381\,966,$$
$$x_6 = -0.381\,966\,01, \qquad x_7 = -0.381\,966\,01.$$

Since $f(-0.381\,965\,95) = 1.4 \times 10^{-7}$ and $f(-0.381\,966\,05) = -8.7 \times 10^{-8}$, we can say that to seven decimals $x = -0.381\,966\,0$.



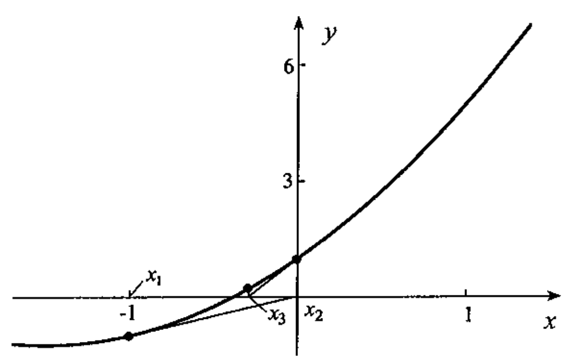
33. The figure indicates that with initial approximation $x_1 = -1$, the sequence defined by Newton's iterative procedure has a limit near $-1/2$. Iteration of

$$x_1 = -1, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

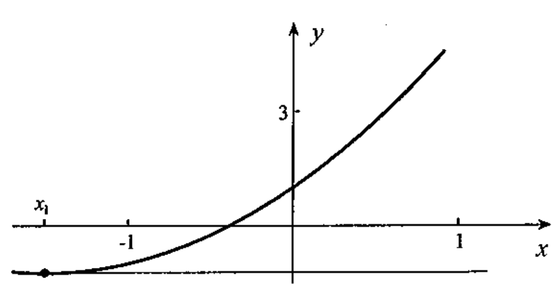
leads to

$$x_2 = 0, \qquad x_3 = -1/3,$$
$$x_4 = -0.381, \qquad x_5 = -0.381\,966,$$
$$x_6 = -0.381\,966\,01, \qquad x_7 = -0.381\,966\,01.$$

Since $f(-0.381\,965\,95) = 1.4 \times 10^{-7}$ and $f(-0.381\,966\,05) = -8.7 \times 10^{-8}$, we can say that to seven decimals $x = -0.381\,966\,0$.



34. The figure indicates that with initial approximation $x_1 = -1.5$, the sequence defined by Newton's iterative procedure does not have a limit. This is because $x_1 = -1.5$ is a critical point of the function.



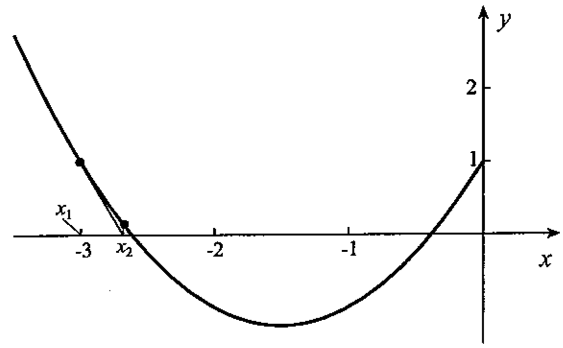
35. The figure indicates that with initial approximation $x_1 = -3$, the sequence defined by Newton's iterative procedure has a limit near -3. Iteration of

$$x_1 = -3, \quad x_{n+1} = x_n - \frac{x_n^2 + 3x_n + 1}{2x_n + 3}$$

leads to

$x_2 = -2.667$, $x_3 = -2.6191$,
$x_4 = -2.618\,034\,5$, $x_5 = -2.618\,033\,99$,
$x_6 = -2.618\,033\,99$.

Since $f(-2.618\,034\,05) = 1.4 \times 10^{-7}$ and $f(-2.618\,033\,95) = -8.7 \times 10^{-8}$, we can say that to seven decimals $x = -2.618\,034\,0$.



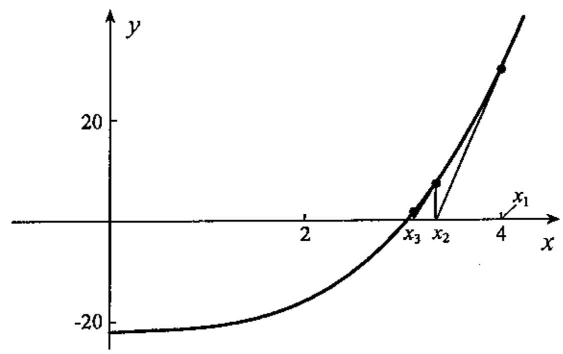
36. The figure indicates that with initial approximation $x_1 = 4$, the sequence defined by Newton's iterative procedure has a limit near 3. Iteration of

$$x_1 = 4, \quad x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n - 22}{3x_n^2 - 2x_n + 1}$$

leads to

$x_2 = 3.268$, $x_3 = 3.060\,9$,
$x_4 = 3.044\,8$, $x_5 = 3.044\,723\,15$,
$x_6 = 3.044\,723\,15$.

Since $f(3.044\,723\,05) = -2.2 \times 10^{-6}$ and $f(3.044\,723\,15) = 3.5 \times 10^{-8}$, we can say that to seven decimals $x = 3.044\,723\,1$.



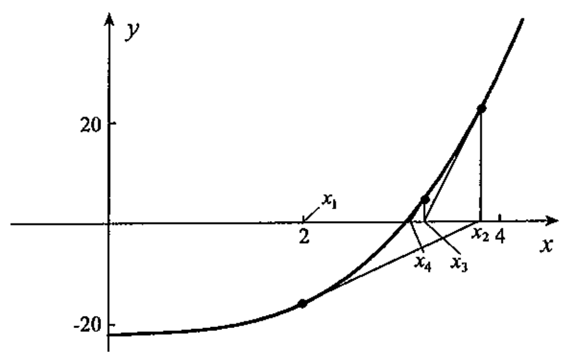
37. The figure indicates that with initial approximation $x_1 = 2$, the sequence defined by Newton's iterative procedure has a limit near 3. Iteration of

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^3 - x_n^2 + x_n - 22}{3x_n^2 - 2x_n + 1}$$

leads to

$x_2 = 3.778$, $x_3 = 3.187$,
$x_4 = 3.0515$, $x_5 = 3.044\,740$,
$x_6 = 3.044\,723\,15$, $x_7 = 3.044\,723\,15$.

Since $f(3.044\,723\,05) = -2.2 \times 10^{-6}$ and $f(3.044\,723\,15) = 3.5 \times 10^{-8}$, we can say that to seven decimals $x = 3.044\,723\,1$.



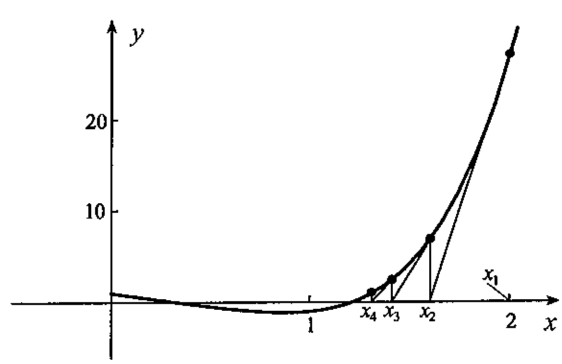
38. The figure indicates that with initial aproximation $x_1 = 2$, the sequence defined by Newton's iterative procedure has a limit near 1. Iteration of

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = 1.649$, $x_3 = 1.406$,
$x_4 = 1.268$, $x_5 = 1.220$,
$x_6 = 1.215$, $x_7 = 1.214\,65$,
$x_8 = 1.214\,648\,04$, $x_9 = 1.214\,648\,04$.

Since $f(1.214\,647\,95) = -7.3 \times 10^{-7}$ and $f(1.214\,648\,05) = 5.8 \times 10^{-8}$, we can say that to seven decimals $x = 1.214\,648\,0$.



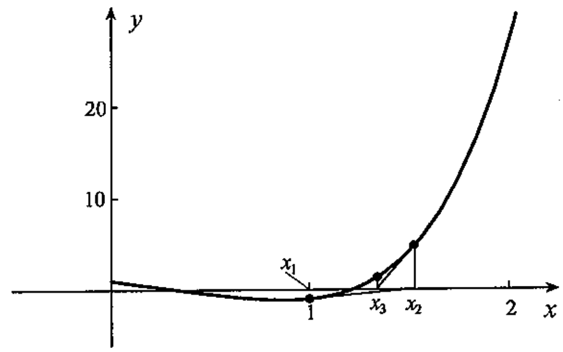
39. The figure indicates that the sequence defined by Newton's iterative procedure has a limit. Iteration of

$$x_1 = 1, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = 1.5,$ $\quad x_3 = 1.317,$
$x_4 = 1.233,$ $\quad x_5 = 1.2154,$
$x_6 = 1.214\,649,$ $\quad x_7 = 1.214\,648\,04,$
$x_8 = 1.214\,648\,04.$

Since $f(1.214\,647\,95) = -7.3 \times 10^{-7}$ and $f(1.214\,648\,05) = 5.8 \times 10^{-8}$, we can say that to seven decimals $x = 1.214\,648\,0$.



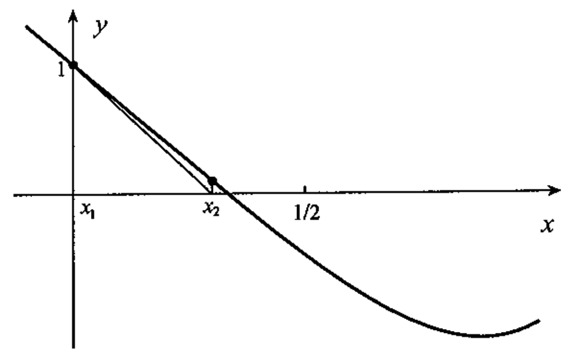
40. The figure indicates that with initial approximation $x_1 = 0$, the sequence defined by Newton's iterative procedure has a limit near 0.3. Iteration of

$$x_1 = 0, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = 1/3,$ $\quad x_3 = 0.334\,7,$
$x_4 = 0.334\,734\,14,$ $\quad x_5 = 0.334\,734\,14.$

Since $f(0.334\,734\,05) = 2.7 \times 10^{-7}$ and $f(0.334\,734\,15) = -2.4 \times 10^{-8}$, we can say that to seven decimals $x = 0.334\,734\,1$.



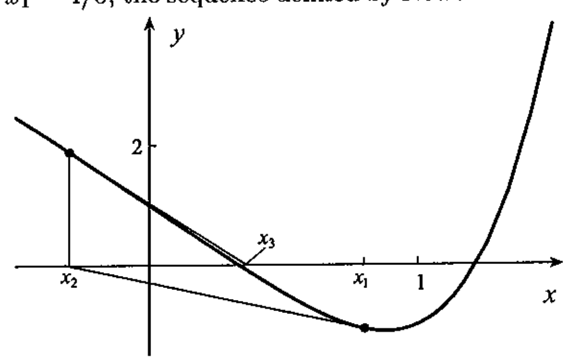
41. The figure indicates that with initial approximation $x_1 = 4/5$, the sequence defined by Newton's iterative procedure has a limit near 0.3. Iteration of

$$x_1 = 4/5, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = -0.326,$ $\quad x_3 = 0.345,$
$x_4 = 0.334\,72,$ $\quad x_5 = 0.334\,734\,14,$
$x_6 = 0.334\,734\,14.$

Since $f(0.334\,734\,05) = 2.7 \times 10^{-7}$ and $f(0.334\,734\,15) = -2.4 \times 10^{-8}$, we can say that to seven decimals $x = 0.334\,734\,1$.



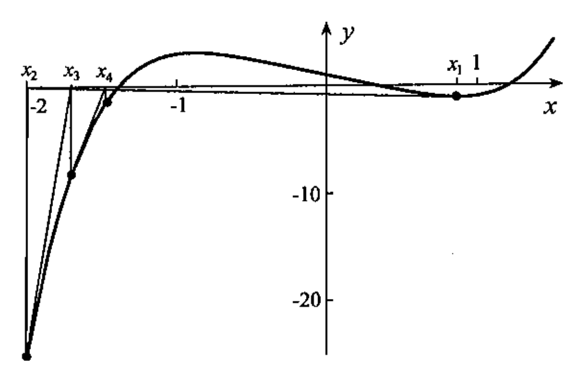
42. The figure indicates that with initial approximation $x_1 = 0.85$, the sequence defined by Newton's iterative procedure has a limit near -1.5. Iteration of

$$x_1 = 0.85, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

$x_2 = -1.987,$ $\quad x_3 = -1.667,$
$x_4 = -1.474,$ $\quad x_5 = -1.399,$
$x_6 = -1.389,$ $\quad x_7 = -1.388\,792\,06,$
$x_8 = -1.388\,791\,98,$ $\quad x_9 = -1.388\,791\,98.$

Since $f(-1.388\,791\,95) = 5.4 \times 10^{-7}$ and $f(-1.388\,792\,05) = -1.0 \times 10^{-6}$, we can say that to seven decimals $x = -1.388\,792\,0$.



43. The figure indicates that with initial approximation $x_1 = -2$, the sequence defined by Newton's iterative procedure has a limit near -1.5. Iteration of

$$x_1 = -2, \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 1}{5x_n^4 - 3}$$

gives

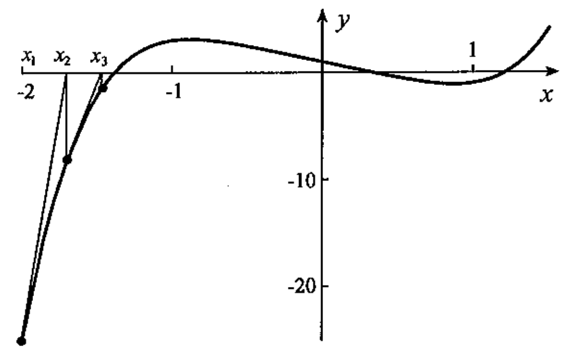
$x_2 = -1.675,\qquad x_3 = -1.478,$

$x_4 = -1.4004,\qquad x_5 = -1.3890,$

$x_6 = -1.388\,792,\qquad x_7 = -1.388\,791\,98,$

$x_8 = -1.388\,791\,98.$

Since $f(-1.388\,791\,95) = 5.4 \times 10^{-7}$ and $f(-1.388\,792\,05) = -1.0 \times 10^{-6}$, we can say that to seven decimals $x = -1.388\,792\,0$.



44. Iteration of $x_1 = 2, \quad x_{n+1} = 2 + \dfrac{1}{x_n}$ gives

$x_2 = 2.5,\quad x_3 = 2.4,\quad x_4 = 2.416\,67,\quad x_5 = 2.413\,79,\quad x_6 = 2.414\,29,$

$x_7 = 2.414\,20,\quad x_8 = 2.414\,22,\quad x_9 = 2.414\,21,\quad x_{10} = 2.414\,21.$

Since $f(2.414\,15) = -1.8 \times 10^{-4}$ and $f(2.414\,25) = 1.0 \times 10^{-4}$, it follows that to 4 decimals, $x = 2.4142$.

45. Iteration of $x_1 = -1, \quad x_{n+1} = -\dfrac{1}{6}(x_n^3 + 3)$ gives

$$x_2 = -1/3, \quad x_3 = -0.4938, \quad x_4 = -0.4799, \quad x_5 = -0.4816, \quad x_6 = -0.481\,38, \quad x_7 = -0.481\,41.$$

Since $f(-0.481\,35) = 3.7 \times 10^{-4}$ and $f(-0.481\,45) = -3.0 \times 10^{-4}$, it follows that to 4 decimals, $x = -0.4814$.

46. Iteration of $x_1 = 0, \quad x_{n+1} = \dfrac{1}{120}(x_n^4 + 20)$ gives

$$x_2 = 1/6, \quad x_3 = 0.166\,67, \quad x_4 = 0.166\,67.$$

Since $f(0.166\,65) = 2.8 \times 10^{-3}$ and $f(0.166\,75) = -9.2 \times 10^{-3}$, it follows that to 4 decimals, $x = 0.1667$.

47. Iteration of $x_1 = 3, \quad x_{n+1} = \dfrac{2x_n^2 + 3x_n - 1}{x_n^2}$ gives

$x_2 = 2.889,\quad x_3 = 2.9186,\quad x_4 = 2.910\,49,\quad x_5 = 2.912\,70,$

$x_6 = 2.912\,10,\quad x_7 = 2.912\,26,\quad x_8 = 2.912\,22.$

Since $f(2.912\,15) = -8.5 \times 10^{-4}$ and $f(2.912\,25) = 2.2 \times 10^{-4}$, the root is $x = 2.9122$ to 4 decimal places.

48. Iteration of $x_1 = 0, \quad x_{n+1} = \dfrac{1}{2}(1 + x_n^2)^{1/3}$ gives

$x_2 = 1/2,\quad x_3 = 0.5386,\quad x_4 = 0.5443,\quad x_5 = 0.545\,17,$

$x_6 = 0.545\,31,\quad x_7 = 0.545\,33,\quad x_8 = 0.545\,33.$

Since $f(0.545\,25) = -4.9 \times 10^{-4}$ and $f(0.545\,35) = 1.2 \times 10^{-4}$, it follows that to 4 decimals, $x = 0.5453$.

49. With $x_1 = 3.5$, and $x_{n+1} = \dfrac{6x_n^2 - 11x_n + 7}{x_n^2}$, iteration gives

$x_2 = 3.4286,\quad x_3 = 3.3872,\quad x_4 = 3.3626,\quad x_5 = 3.3478,\quad x_6 = 3.3388,$

$x_7 = 3.333\,34,\quad x_8 = 3.330\,00,\quad x_9 = 3.327\,96,\quad x_{10} = 3.326\,71,\quad x_{11} = 3.325\,94,$

$x_{12} = 3.325\,47,\quad x_{13} = 3.325\,18,\quad x_{14} = 3.325\,00,\quad x_{15} = 3.324\,89,\quad x_{16} = 3.324\,82,$

$x_{17} = 3.324\,78,\quad x_{18} = 3.324\,76,\quad x_{19} = 3.324\,74.$

With $f(x) = x^3 - 6x^2 + 11x - 7$, we calculate that $f(3.324\,65) = -2.9 \times 10^{-4}$ and $f(3.324\,75) = 1.4 \times 10^{-4}$. The root is therefore $x = 3.3247$ to 4 decimals.

50. With $x_1 = 0$, and $x_{n+1} = \dfrac{x_n^4 - 3x_n^2 + 1}{3}$, iteration gives

$$\begin{array}{lllll} x_2 = 1/3, & x_3 = 0.226, & x_4 = 0.283, & x_5 = 0.255, & x_6 = 0.270, \\ x_7 = 0.262, & x_8 = 0.266, & x_9 = 0.2642, & x_{10} = 0.2652, & x_{11} = 0.2647, \\ x_{12} = 0.2649, & x_{13} = 0.264\,80, & x_{14} = 0.264\,85, & x_{15} = 0.264\,83, & x_{16} = 0.264\,84. \end{array}$$

With $f(x) = x^4 - 3x^2 - 3x + 1$, we calculate that $f(0.264\,75) = 3.9 \times 10^{-4}$ and $f(0.264\,85) = -6.6 \times 10^{-5}$. The root is therefore $x = 0.2648$ to 4 decimals.

51. With $x_1 = 0.5$ and $x_{n+1} = \dfrac{50 + 50x_n^2 - 4x_n^3 - x_n^4}{100}$, iteration gives

$$\begin{array}{lllll} x_2 = 0.6194, & x_3 = 0.6809, & x_4 = 0.7170, & x_5 = 0.7397, & x_6 = 0.7544, \\ x_7 = 0.7641, & x_8 = 0.7707, & x_9 = 0.7751, & x_{10} = 0.7782, & x_{11} = 0.7803, \\ x_{12} = 0.7817, & x_{13} = 0.7827, & x_{14} = 0.7834, & x_{15} = 0.7839, & x_{16} = 0.7842, \\ x_{17} = 0.7844, & x_{18} = 0.7846, & x_{19} = 0.7847, & x_{20} = 0.7848, & x_{21} = 0.784\,83, \\ x_{22} = 0.784\,85, & x_{23} = 0.784\,86. & & & \end{array}$$

With $f(x) = x^4 + 4x^3 - 50x^2 + 100x - 50$, we calculate that $f(0.784\,85) = -1.2 \times 10^{-3}$ and $f(0.784\,95) = 1.9 \times 10^{-3}$. Thus to 4 decimals, $x = 0.7849$.

52. With $x_1 = 0.75$, and $x_{n+1} = \sqrt{1 - \sin^2 x_n} = \sqrt{\cos^2 x_n} = \cos x_n$, iteration gives

$$\begin{array}{lllll} x_2 = 0.732, & x_3 = 0.744, & x_4 = 0.736, & x_5 = 0.741, & x_6 = 0.738, \\ x_7 = 0.740, & x_8 = 0.7385, & x_9 = 0.7395. & & \end{array}$$

With $f(x) = \sin^2 x - 1 + x^2$, we calculate that $f(0.739\,05) = -8.7 \times 10^{-5}$ and $f(0.739\,15) = 1.6 \times 10^{-4}$. The root is therefore $x = 0.7391$ to 4 decimals.

53. By cross-multiplying, $(1 + x^4)\sec x = 2$, and therefore the equation can be rearranged into the form $x = (2\cos x - 1)^{1/4}$. With $x_1 = 0.5$ and $x_{n+1} = (2\cos x_n - 1)^{1/4}$, iteration gives

$$\begin{array}{lllll} x_2 = 0.932, & x_3 = 0.662, & x_4 = 0.872, & x_5 = 0.732, & x_6 = 0.836, \\ x_7 = 0.764, & x_8 = 0.816, & x_9 = 0.780, & x_{10} = 0.806, & x_{11} = 0.788, \\ x_{12} = 0.800, & x_{13} = 0.792, & x_{14} = 0.798, & x_{15} = 0.793, & x_{16} = 0.797, \\ x_{17} = 0.7941, & x_{18} = 0.7962, & x_{19} = 0.7947, & x_{20} = 0.7958, & x_{21} = 0.7950, \\ x_{22} = 0.7956, & x_{23} = 0.7951, & x_{24} = 0.7955, & x_{25} = 0.7952, & x_{26} = 0.7954, \\ x_{27} = 0.7953. & & & & \end{array}$$

With $f(x) = \sec x - 2(1 + x^4)^{-1}$, we calculate that $f(0.795\,25) = -2.6 \times 10^{-4}$ and $f(0.795\,35) = 9.2 \times 10^{-5}$. To 4 decimals then, $x = 0.7953$.

54. With $x_1 = 0.5$, and $x_{n+1} = \dfrac{e^{x_n} + e^{-x_n}}{10}$, iteration gives

$$x_2 = 0.226, \quad x_3 = 0.205, \quad x_4 = 0.2042, \quad x_5 = 0.204\,18, \quad x_6 = 0.204\,18.$$

With $f(x) = e^x + e^{-x} - 10x$, we calculate that $f(0.204\,15) = 3.2 \times 10^{-4}$ and $f(0.204\,25) = -6.4 \times 10^{-4}$. The root is therefore $x = 0.2042$ to 4 decimals.

55. (a) Iteration of $x_1 = 1$, $x_{n+1} = x_n - \dfrac{x_n^4 - 15x_n + 2}{4x_n^3 - 15}$ gives

$$x_2 = -0.09, \quad x_3 = 0.1333, \quad x_4 = 0.133\,354\,4, \quad x_5 = 0.133\,354\,4.$$

With $f(x) = x^4 - 15x + 2$, we calculate that $f(0.133\,353\,5) = 1.4 \times 10^{-5}$ and $f(0.133\,354\,5) = -1.2 \times 10^{-6}$. The root is therefore $x = 0.133\,354$ to 6 decimals.

(b) Iteration gives

$$x_2 = 0.2, \quad x_3 = 0.133\,44, \quad x_4 = 0.133\,354\,47, \quad x_5 = 0.133\,354\,42.$$

This leads to the same root as in part (a).

(c) Iteration of the sequence in part (a) beginning with $x_1 = 2.5$ gives

$$x_2 = 2.425, \quad x_3 = 2.4201, \quad x_3 = 2.420\,061\,9, \quad x_4 = 2.420\,061\,9.$$

Since $f(2.420\,061\,5) = -1.5 \times 10^{-5}$ and $f(2.420\,062\,5) = 2.7 \times 10^{-5}$, the root is $x = 2.420\,062$.
(d) Iteration beginning with $x_1 = 2$ gives

$$x_2 = 1.2, \quad x_3 = 0.271\,6, \quad x_4 = 0.133\,696, \quad x_5 = 0.133\,355.$$

The sequence is converging to the root in part (a). Beginning with $x_1 = 3$, we obtain $x_2 = 5.5$ and $x_3 = 61.1$. The sequence is diverging.

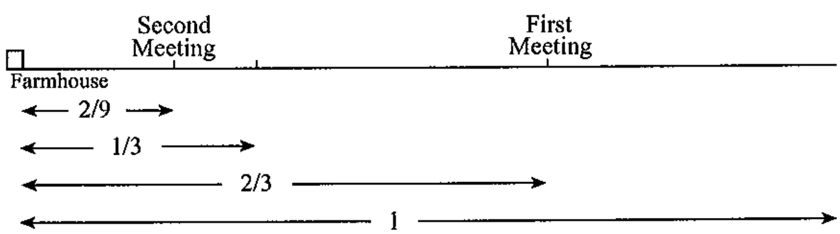
56. (a) $d_1 = 2(0.99)(20) = 40(0.99)$ m
$d_2 = 2(0.99)[(0.99)(20)] = 40(0.99)^2$ m
$d_3 = 2(0.99)[(20)(0.99)^2] = 40(0.99)^3$ m

The pattern emerging is $d_n = 40(0.99)^n$ m.

(b) When an object falls from rest under gravity, the distance that it falls as a function of time t is given by $d = 4.905t^2$. Consequently, the time to fall from peak height between n^{th} and $(n+1)^{\text{th}}$ bounces is given by $d_n/2 = 4.905t^2$. When this equation is solved for t, the result is $t = \sqrt{d_n/9.81}$, and therefore

$$t_n = 2\sqrt{d_n/9.81} = 2\sqrt{40(0.99)^n/9.81} = \frac{4}{\sqrt{0.981}}(0.99)^{n/2} \text{ s.}$$

57. The dog reaches the farmer for the first time 2/3 km from the farmhouse. When the dog returns to the farmhouse (travelling 2/3 km), the farmer moves to a distance 1/3 km from the farmhouse. The dog then runs $(2/3)(1/3) = 2/9$ km in reaching the farmer for the second time. Thus, $d_1 = 2/3 + 2/9 = 8/9$ km. When the dog returns to the farmhouse for the second time, the farmer moves to a distance 1/9 km from the farmhouse. The dog then runs $(2/3)(1/9) = 2/27$ km in reaching the farmer for the third time. Thus, $d_2 = 2/9 + 2/27 = 8/27$ km. The pattern emerging is $d_n = 8/3^{n+1}$ km.



58. Since each of the 12 straight line segments in the middle figure has length $P/9$,

$$P_1 = \frac{12P}{9} = \frac{4P}{3}.$$

Since each of the 48 straight line segments in the right figure has length $P/27$,

$$P_2 = \frac{48P}{27} = \frac{4^2P}{3^2}.$$

The next perimeter is $P_3 = 4(48)\dfrac{P}{81} = \dfrac{4^3P}{3^3}$. The pattern emerging is $P_n = \dfrac{4^nP}{3^n}$. The limit of P_n as $n \to \infty$ does not exist.

59. (a) Since $y(3) = 11.8$ and $y(4) = -3.0$, the solution is between 3 and 4. To find it more accurately we use

$$t_1 = 3.8, \qquad t_{n+1} = t_n - \frac{1181(1 - e^{-t_n/10}) - 98.1t_n}{118.1e^{-t_n/10} - 98.1}.$$

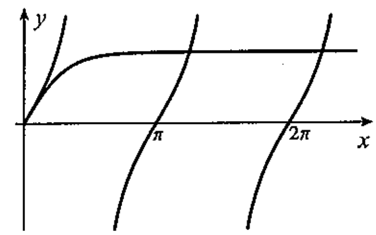
Iteration gives $t_2 = 3.833\,4$ and $t_3 = 3.833\,2$. Since $y(3.825) = 0.14$ and $y(3.835) = -0.03$, it follows that to 2 decimals $t = 3.83$ s.

(b) If air resistance is ignored, the acceleration of the stone is $a = dv/dt = -9.81$. Antidifferentiation gives $v(t) = -9.81t + C$. Since $v(0) = 20$, it follows that $C = 20$, and $v(t) = dy/dt = -9.81t + 20$. Antidifferentiation now gives $y(t) = -4.905t^2 + 20t + D$. Since $y(0) = 0$, we find that $D = 0$, and the height of the stone is $y(t) = -4.905t^2 + 20t$. When we set $0 = y = -4.905t^2 + 20t$, the positive solution is 4.08 s.

60. The figure shows graphs of $y = \tan x$ and $y = (e^x - e^{-x})/(e^x + e^{-x}) = \tanh x$ for $x \geq 0$. They intersect at $x = 0$ and values near 4 and 7. We use Newton's iterative procedure

$$x_{n+1} = x_n - \frac{\tan x_n - \tanh x_n}{\sec^2 x_n - \operatorname{sech}^2 x_n}$$

with $x_1 = 4$ to locate the smaller root.



Iteration gives $x_2 = 3.932\,25$, $x_3 = 3.926\,63$, $x_4 = 3.926\,60$, $x_5 = 3.926\,60$. When we divide this by 20π, the result is 0.0625. A similar procedure gives the next natural frequency 0.1125.

61. Since the area of an equilateral triangle with sides of length l is $\sqrt{3}l^2/4$, the area of the first triangle in Exercise 58 is $\dfrac{\sqrt{3}}{4}\left(\dfrac{P}{3}\right)^2 = \dfrac{\sqrt{3}P^2}{36}$. The middle figure adds three triangles each of area $\sqrt{3}(P/9)^2/4$ to the area in the first figure, and therefore

$$A_1 = \frac{\sqrt{3}P^2}{36} + \frac{3\sqrt{3}}{4}\left(\frac{P^2}{81}\right) = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36}.$$

The right figure adds twelve triangles each of area $\sqrt{3}(P/27)^2/4$ to the middle figure, and therefore

$$A_2 = A_1 + \frac{12\sqrt{3}}{4}\left(\frac{P}{27}\right)^2 = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36} + \frac{4\sqrt{3}P^2}{3^3 \cdot 36}.$$

The next figure in the sequence would add 48 triangles each of area $\sqrt{3}(P/81)^2/4$, and therefore

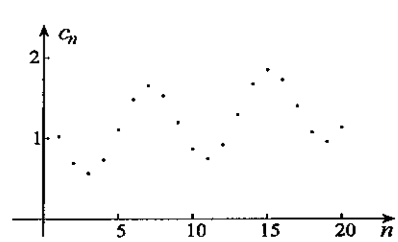
$$A_3 = A_2 + \frac{48\sqrt{3}}{4}\left(\frac{P}{81}\right)^2 = \frac{\sqrt{3}P^2}{36} + \frac{\sqrt{3}P^2}{3 \cdot 36} + \frac{4\sqrt{3}P^2}{3^3 \cdot 36} + \frac{4^2\sqrt{3}P^2}{3^5 \cdot 36}.$$

The pattern emerging is $\quad A_n = \dfrac{\sqrt{3}P^2}{36}\left(1 + \dfrac{1}{3} + \dfrac{4}{3^3} + \dfrac{4^2}{3^5} + \cdots + \dfrac{4^{n-1}}{3^{2n-1}}\right).$

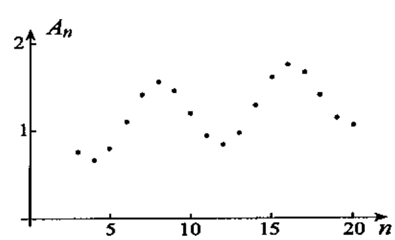
62. The next two terms are 1113213211, 31131211131221. Reason as follows: The second term is 11 because there is one 1 in the first term; the third term is 21 because the second term has two 1's; the fourth term is 1211 because the third term has one 2 followed by one 1; the fifth term is 111221 because the fourth term is one 1, followed by one 2, followed by two 1's; etc.

63. Plots of the sequences are shown below.

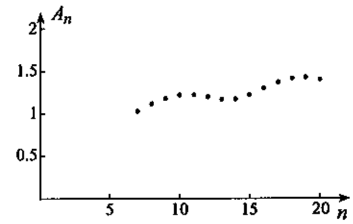
(a)



(b)



64. The plot of the seven-point averager is shown to the right.



65. An explicit formula for this FIR is $F_n = \dfrac{n}{n+1} + 2\left(\dfrac{n-1}{n}\right) - \left(\dfrac{n-2}{n-1}\right)$. When we substitute $n = 3, \ldots, 12$, we obtain the first 10 terms,

$$\frac{19}{12}, \frac{49}{30}, \frac{101}{60}, \frac{181}{105}, \frac{295}{168}, \frac{449}{252}, \frac{649}{360}, \frac{901}{495}, \frac{1211}{660}, \frac{1585}{858}.$$

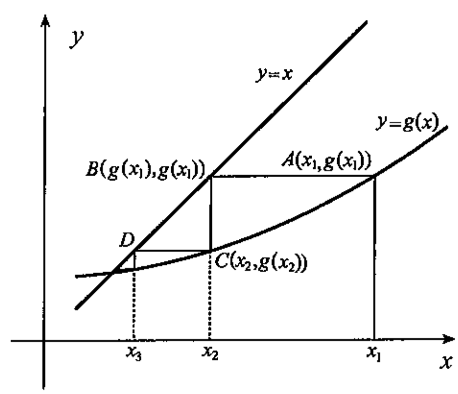
66. An explicit formula for this FIR is

$$F_n = \frac{1}{n^2}\sin\left(\frac{n}{3}\right) - \frac{2}{(n-1)^2}\sin\left(\frac{n-1}{3}\right) + \frac{3}{(n-2)^2}\sin\left(\frac{n-2}{3}\right) - \frac{4}{(n-3)^2}\sin\left(\frac{n-3}{3}\right).$$

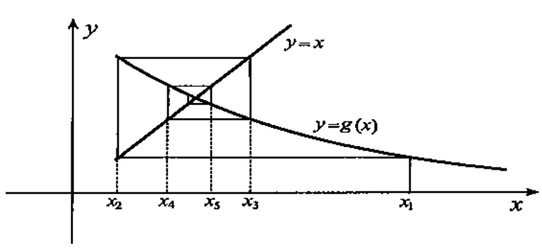
When we substitute $n = 4, \ldots, 13$, we obtain the first 10 terms,

$$-0.9712, -0.4196, -0.2461, -0.1593, -0.1059, -0.0693, -0.0430, -0.0237, -0.0096, 0.0002.$$

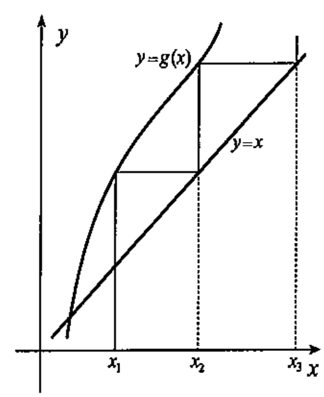
67. (a) The height of the curve $y = g(x)$ at the point A with x-coordinate x_1 is $y = g(x_1)$. If we proceed horizontally to the line $y = x$, the coordinates of the point B on the line are $\big(g(x_1), g(x_1)\big)$. But the second term in the sequence established by the method of successive substitutions is $x_2 = g(x_1)$. Hence the x-coordinate of B is x_2. The height of the curve $y = g(x)$ at C is $y = g(x_2)$. The point D has coordinates $\big(g(x_2), g(x_2)\big)$, and hence, the x-coordinate of D is $x_3 = g(x_2)$. Continuation leads to the interpretation of the $\{x_n\}$ as shown in the figure.



(b)

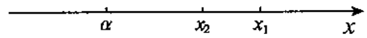


(c)



(d) It appears that the slope of $y = g(x)$ near the required root dictates whether the sequence converges. For slopes near zero (figures in (a) and (b)), the sequence converges, but for large slopes (figure in (c)), the sequence diverges.

(e)



If we apply the mean value theorem (Theorem 3.19) to $g(x)$ on the interval between α and x_1,

$$g(x_1) = g(\alpha) + g'(c)(x_1 - \alpha)$$

where c is between α and x_1. Since $\alpha = g(\alpha)$, $x_2 = g(x_1)$, and $|g'(c)| \le a$, we may write that

$$x_2 = \alpha + g'(c)(x_1 - \alpha) \quad \Longrightarrow \quad |x_2 - \alpha| = |g'(c)||x_1 - \alpha| \le a|x_1 - \alpha|.$$

What this means is that x_2 is closer to α than x_1. If we repeat this procedure for $x_3 = g(x_2)$ on the interval between α and x_2, we obtain

$$|x_3 - \alpha| \le a|x_2 - \alpha| \le a^2|x_1 - \alpha|.$$

Continuation of this process gives $|x_n - \alpha| \le a^{n-1}|x_1 - \alpha|$. It now follows that

$$\lim_{n\to\infty} |x_n - \alpha| \le \lim_{n\to\infty} a^{n-1}|x_1 - \alpha| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} x_n = \alpha.$$

68. Newton's iterative procedure defines the sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. If we define $F(x) = x - f(x)/f'(x)$, then $x_{n+1} = F(x_n)$. According to part (e) of Exercise 67, a sequence of this type converges to a root $x = \alpha$ of $x = F(x)$ if on the interval $|x - \alpha| \le |x_1 - \alpha|$ we have $|F'(x)| \le a < 1$. Since $F'(x) = 1 - [(f')^2 - ff'']/(f')^2$, we will have convergence if $1 > a \ge \left|1 - \dfrac{(f')^2 - ff''}{(f')^2}\right| = \left|\dfrac{ff''}{(f')^2}\right|$. Thus, Newton's sequence converges to α if on $|x-\alpha| \le |x_1-\alpha|$, $|ff''/(f')^2| \le a < 1$. In other words, if it is possible to choose x_1 close enough to α to guarantee $|ff''/(f')^2| \le a < 1$, on the interval $|x - \alpha| \le |x_1 - \alpha|$, then Newton's sequence converges to α. To show that this is always possible, we let M be the maximum value of $|f''|$ on the open interval containing α in which $f''(x)$ is known to exist. Because $f'(\alpha) \ne 0$, there exists an open interval I containing α in which $f'(x) \ne 0$ (by continuity of $f'(x)$). Let m be the minimum value of $|f'(x)|$ on I. Since $f(x)$ is continuous at $x = \alpha$, where $f(\alpha) = 0$, there exists an open interval $|x - \alpha| < \delta$ contained in I which $|f(x)| < am^2/M$ for any a such that $0 < a < 1$. Consequently, for $|x - \alpha| < \delta$,

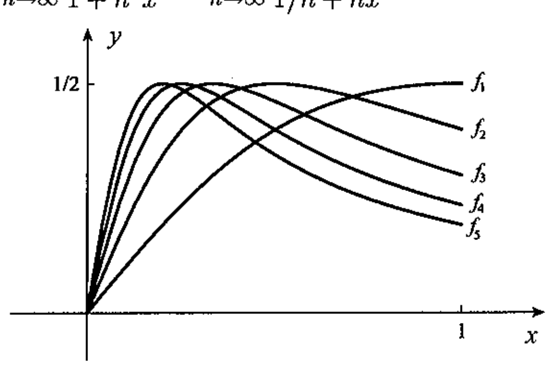
$$\left|\frac{ff''}{(f')^2}\right| < \frac{am^2}{M}\frac{M}{m^2} = a < 1.$$

Thus, if $|x_1 - \alpha| = \delta$, we may say that for all x in $|x - a| < |x_1 - \alpha|$, $|ff''/(f')^2| < a < 1$, and Newton's iterative sequence converges to α.

EXERCISES 10.2

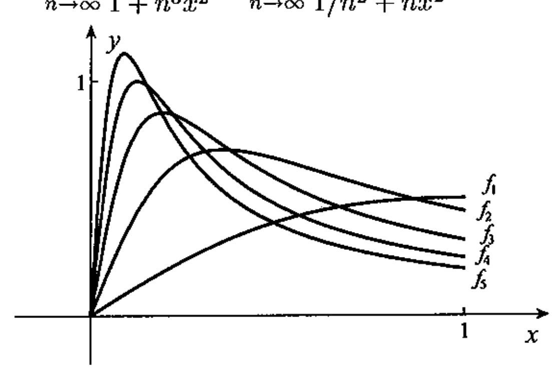
1. The limit function is $f(x) = 0$, since for each x in $0 \le x \le 1$,

$$\lim_{n\to\infty} \frac{nx}{1 + n^2x^2} = \lim_{n\to\infty} \frac{x}{1/n + nx^2} = 0.$$

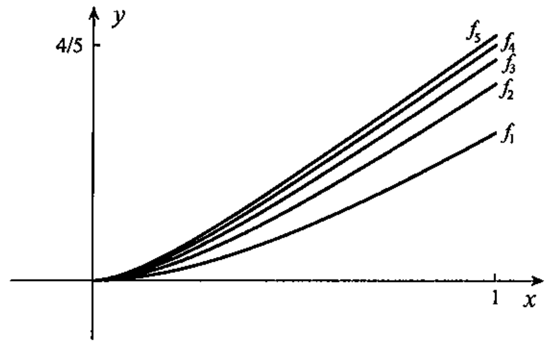


2. The limit function is $f(x) = 0$, since for each x in $0 \le x \le 1$,

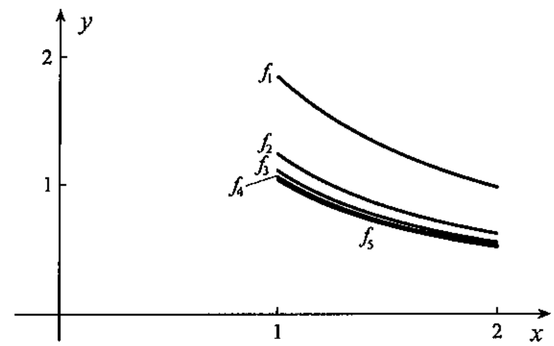
$$\lim_{n\to\infty} \frac{n^2x}{1 + n^3x^2} = \lim_{n\to\infty} \frac{x}{1/n^2 + nx^2} = 0.$$



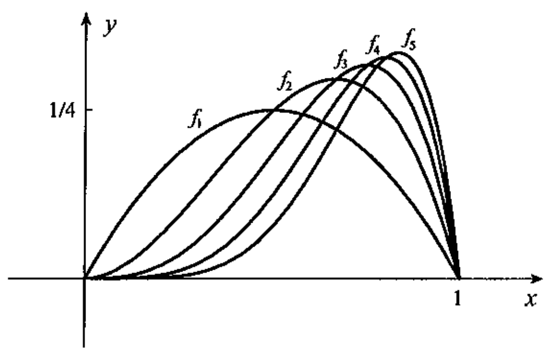
3. The limit function is $f(x) = x$, since for each x in $0 \le x \le 1$,
$$\lim_{n\to\infty} \frac{nx^2}{1+nx} = \lim_{n\to\infty} \frac{x^2}{1/n + x} = x.$$



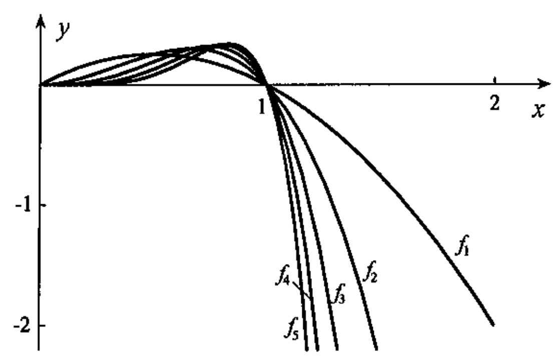
4. The limit function is $f(x) = 1/x$.



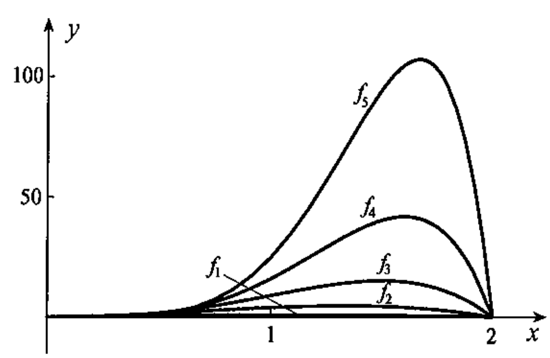
5. Since $f_n(0) = f_n(1) = 0$, the limit function $f(x)$ has values $f(0) = f(1) = 0$. For fixed x in $0 < x < 1$,
$$f(x) = \lim_{n\to\infty} nx^n(1-x) = \lim_{n\to\infty} \frac{n(1-x)}{x^{-n}}$$
$$= \lim_{n\to\infty} \frac{1-x}{-x^{-n}\ln x} = \lim_{n\to\infty} \frac{x^n(x-1)}{\ln x} = 0.$$



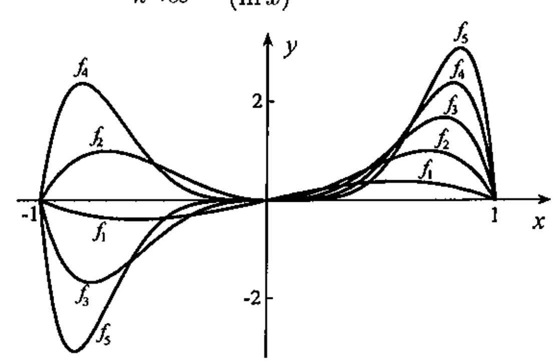
6. There is no limit function.



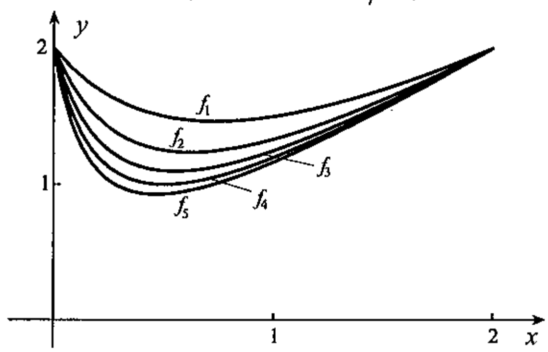
7. There is no limit function.



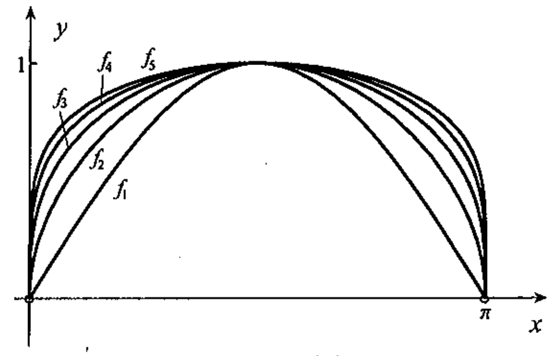
8. Since $f_n(0) = f_n(1) = 0$, the limit function $f(x)$ has values $f(0) = f(1) = 0$. For fixed x in $0 < x < 1$,
$$f(x) = \lim_{n\to\infty} n^2x^n(1-x^2) = \lim_{n\to\infty} \frac{n^2(1-x^2)}{x^{-n}}$$
$$= \lim_{n\to\infty} \frac{2n(1-x^2)}{-x^{-n}\ln x} = \lim_{n\to\infty} \frac{2(1-x^2)}{x^{-n}(\ln x)^2}$$
$$= \lim_{n\to\infty} \frac{2(1-x^2)x^n}{(\ln x)^2} = 0.$$



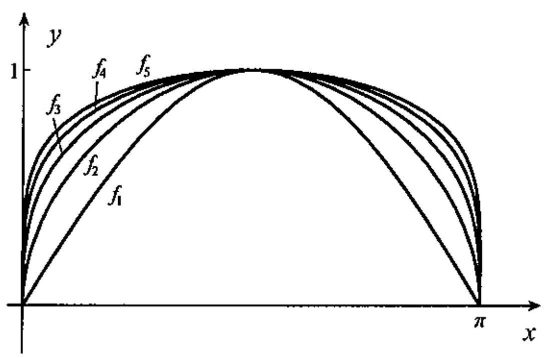
9. The limit function $f(x)$ has value 2 at $x = 0$, and for all other vaues of x,
$$f(x) = \lim_{n\to\infty} \frac{2 + nx^2}{1 + nx} = \lim_{n\to\infty} \frac{2/n + x^2}{1/n + x} = x.$$



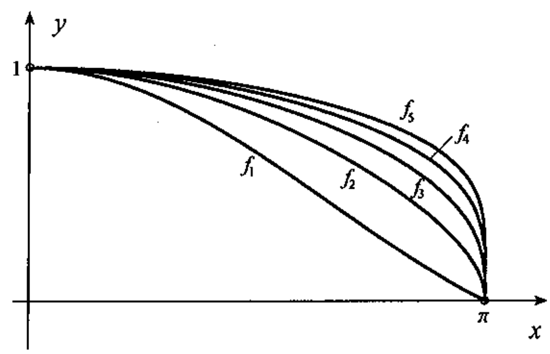
10. The limit function is $f(x) = 1$.



11. The limit function $f(x)$ has value 1 for all x except $x = 0, \pi$, where its value is 0.

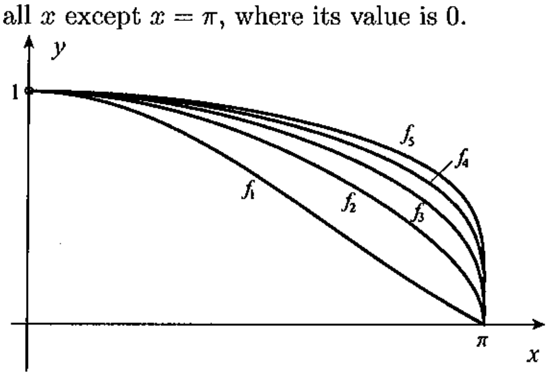


12. The limit function is $f(x) = 1$.

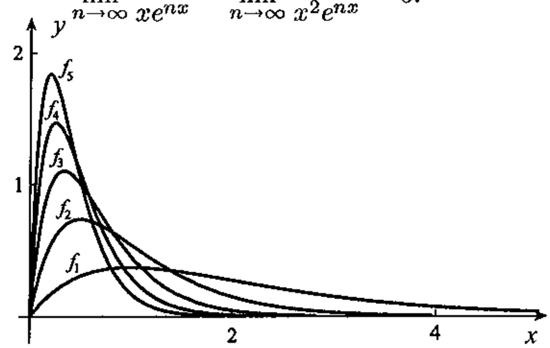


13. The limit function $f(x)$ has value 1 for

all x except $x = \pi$, where its value is 0.



14. The limit function is $f(x) = \lim_{n\to\infty} \frac{n^2x}{e^{nx}}$
$$= \lim_{n\to\infty} \frac{2nx}{xe^{nx}} = \lim_{n\to\infty} \frac{2x}{x^2e^{nx}} = 0.$$



15. The sequence $\{x^n\}$ converges to 0 for $-1 < x < 1$, to 1 for $x = 1$, and diverges for all other values of x. Hence, the sequence $\{(1 - x^n)/(1 - x)\}$ converges to $1/(1 - x)$ for $-1 < x < 1$ and diverges for all other values of x.

EXERCISES 10.3

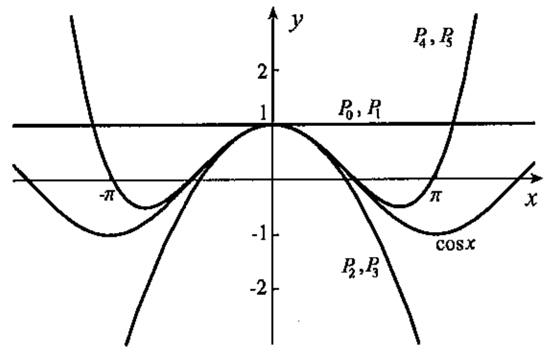
1. Since $f(0) = 1$, $f'(0) = -\sin 0 = 0$, $f''(0) = -\cos 0 = -1$, $f'''(0) = \sin 0 = 0$, $f''''(0) = \cos 0 = 1$, etc., Taylor's remainder formula gives

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \text{term in } x^n + R_n,$$

where $R_n = \dfrac{d^{n+1}}{dx^{n+1}}(\cos x)_{|x=z_n} \dfrac{x^{n+1}}{(n+1)!}$. The n^{th} derivative of $\cos x$ is $\pm \sin x$ or $\pm \cos x$, so that

$$\left| \frac{d^{n+1}}{dx^{n+1}} \cos x_{|x=z_n} \right| \le 1.$$

Hence, $|R_n| \le |x|^{n+1}/(n+1)!$. But according to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any



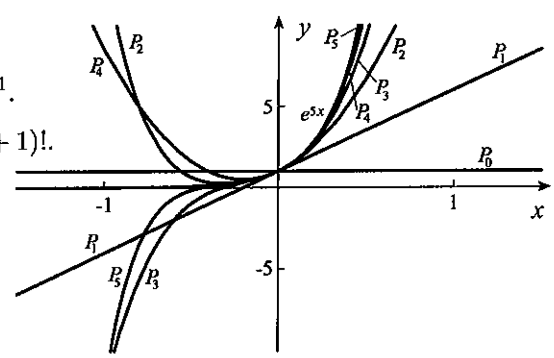
x whatsoever. It follows that $\lim_{n\to\infty} R_n = 0$, and the Maclaurin series for $\cos x$ therefore converges to $\cos x$ for all x. We may write

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \quad -\infty < x < \infty.$$

2. Since $f^{(n)}(x) = 5^n e^{5x}$, Taylor's remainder formula for e^{5x} and $c = 0$ gives

$$e^{5x} = 1 + 5x + \frac{5^2}{2!}x^2 + \frac{5^3}{3!}x^3 + \cdots + \frac{5^n}{n!}x^n + R_n,$$

where $R_n = \dfrac{d^{n+1}}{dx^{n+1}}(e^{5x})_{|x=z_n} \dfrac{x^{n+1}}{(n+1)!} = \dfrac{5^{n+1}e^{5z_n}}{(n+1)!}x^{n+1}$. If $x < 0$, then $x < z_n < 0$, and $|R_n| < 5^{n+1}|x|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever, and therefore $\lim_{n\to\infty} 5^{n+1}|x|^{n+1}/(n+1)! = 0$ also. Thus, if $x < 0$, $\lim_{n\to\infty} R_n = 0$. If $x > 0$,



then $0 < z_n < x$, and

$$|R_n| < \frac{5^{n+1}e^{5x}}{(n+1)!}|x|^{n+1} = e^{5x}\left(\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\right).$$

But we have just indicated that $\lim_{n\to\infty} 5^{n+1}|x|^{n+1}/(n+1)! = 0$, and therefore $\lim_{n\to\infty} R_n = 0$ for $x > 0$ also. Thus, for any x whatsoever, the sequence $\{R_n\}$ has limit 0, and the Maclaurin series for e^{5x} converges to e^{5x},

$$e^{5x} = \sum_{n=0}^{\infty} \frac{5^n}{n!}x^n, \quad -\infty < x < \infty.$$

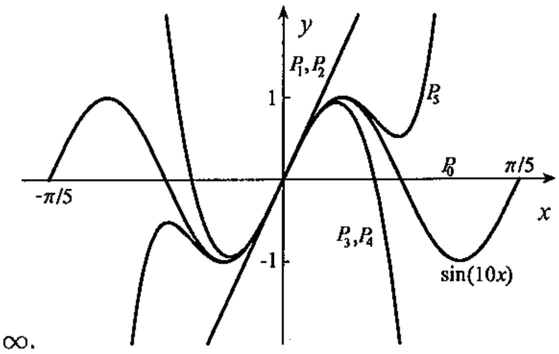
3. Since $f(0) = \sin(0) = 0$, $f'(0) = 10\cos 0 = 10$, $f''(0) = -10^2 \sin 0 = 0$, $f'''(0) = -10^3 \cos 0 = -10^3$, $f''''(0) = 10^4 \sin 0 = 0$, etc., Taylor's remainder formula gives

$$\sin(10x) = 10x - \frac{10^3x^3}{3!} + \frac{10^5x^5}{5!} + \cdots + \text{term in } x^n + R_n,$$

where $R_n = \frac{d^{n+1}}{dx^{n+1}}[\sin(10x)]_{|x=z_n} \frac{x^{n+1}}{(n+1)!}$. The n^{th} derivative of $\sin(10x)$ is $\pm 10^n \sin(10x)$ or $\pm 10^n \cos(10x)$, so that

$$\left| \frac{d^{n+1}}{dx^{n+1}}[\sin(10x)]_{|x=z_n} \right| \le 10^{n+1}.$$

Hence, $|R_n| \le 10^{n+1}|x|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever, and therefore $\lim_{n\to\infty} 10^{n+1}|x|^{n+1}/(n+1)! = 0$ also. It follows that $\lim_{n\to\infty} R_n = 0$, and the Maclaurin series for $\sin(10x)$ therefore converges to $\sin(10x)$ for all x.



We may write

$$\sin(10x) = 10x - \frac{10^3x^3}{3!} + \frac{10^5x^5}{5!} + \cdots, \quad -\infty < x < \infty.$$

4. Since $f(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, $f'(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, $f''(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}$, $f'''(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}$, $f''''(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, etc., Taylor's remainder formula gives

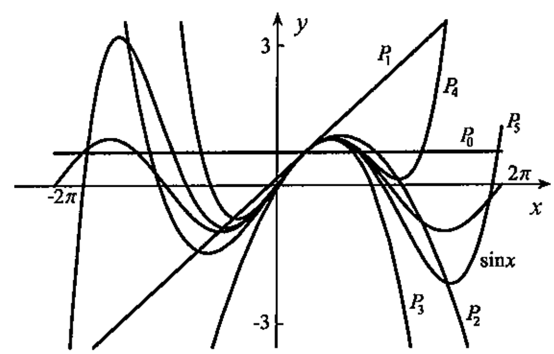
$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x-\pi/4) - \frac{1}{2!\sqrt{2}}(x-\pi/4)^2 - \frac{1}{3!\sqrt{2}}(x-\pi/4)^3 + \ldots + \text{term in } (x-\pi/4)^n + R_n,$$

where $R_n = \frac{d^{n+1}}{dx^{n+1}}(\sin x)_{|x=z_n} \frac{(x-\pi/4)^{n+1}}{(n+1)!}$.

The n^{th} derivative of $\sin x$ is $\pm \sin x$ or $\pm \cos x$, so that

$$\left| \frac{d^{n+1}}{dx^{n+1}}(\sin x)_{|x=z_n} \right| \le 1.$$

Hence, $|R_n| \le |x-\pi/4|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever, and therefore $\lim_{n\to\infty} |x-\pi/4|^{n+1}/(n+1)! = 0$ also. It follows that $\lim_{n\to\infty} R_n = 0$, and the Taylor series for $\sin x$ about $\pi/4$ therefore converges to $\sin x$ for all x.



We may write

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + (x-\pi/4) - \frac{1}{2!}(x-\pi/4)^2 - \frac{1}{3!}(x-\pi/4)^3 + \cdots\right], \quad -\infty < x < \infty.$$

5. Since $f^{(n)}(x) = 2^n e^{2x}$, Taylor's remainder formula for e^{2x} and $c = 1$ gives

$$e^{2x} = e^2 + 2e^2(x-1) + \frac{2^2e^2}{2!}(x-1)^2 + \frac{2^3e^2}{3!}(x-1)^3 + \cdots + \frac{2^ne^2}{n!}(x-1)^n + R_n,$$

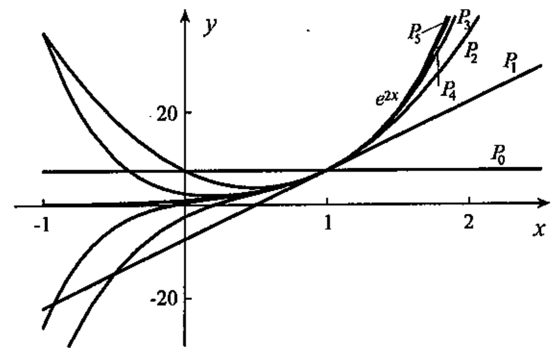
where $R_n = \frac{d^{n+1}}{dx^{n+1}}(e^{2x})_{|x=z_n} \frac{(x-1)^{n+1}}{(n+1)!} = \frac{2^{n+1}e^{2z_n}}{(n+1)!}(x-1)^{n+1}$. If $x < 1$, then $x < z_n < 1$, and $|R_n| < 2^{n+1}e^2|x-1|^{n+1}/(n+1)!$. According to Example 10.5, $\lim_{n\to\infty} |x|^n/n! = 0$ for any x whatsoever, and therefore $\lim_{n\to\infty} 2^{n+1}e^2|x-1|^{n+1}/(n+1)! = 0$ also.

Thus, if $x < 1$, $\lim_{n\to\infty} R_n = 0$. If $x > 1$, then $1 < z_n < x$, and

$$|R_n| < \frac{2^{n+1}e^{2x}}{(n+1)!}|x-1|^{n+1} = e^{2x}\left[\frac{2^{n+1}|x-1|^{n+1}}{(n+1)!}\right].$$

But we have just indicated that $\lim_{n\to\infty} 2^{n+1}|x-1|^{n+1}/(n+1)! = 0$, and therefore $\lim_{n\to\infty} R_n = 0$ for $x > 1$ also. Thus, for any x whatsoever, the sequence $\{R_n\}$ has limit 0, and the Taylor series for e^{2x} converges to e^{2x},

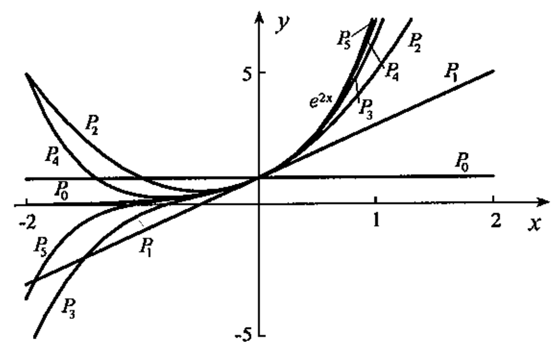
$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n, \quad -\infty < x < \infty.$$



6. Since $f^{(n)}(0) = 2^n$, the Maclaurin series for e^{2x} is

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}x^n = 1 + 2x + \frac{2^2x^2}{2!} + \cdots.$$

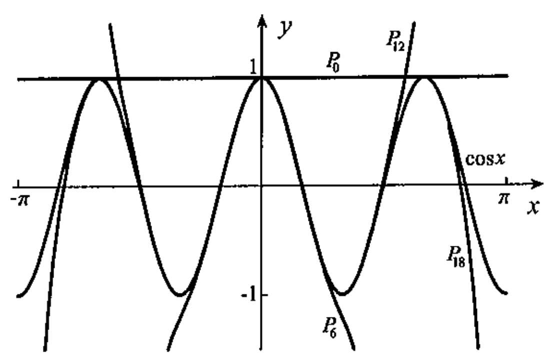
Plots of the polynomials suggest that the series converges to e^{2x} for all x.



7. Since $f(0) = 1$, $f'(0) = 0$, $f''(0) = -3^2$, $f'''(0) = 0$, $f''''(0) = 3^4$, etc., the Maclaurin series for $\cos 3x$ is

$$1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!}x^{2n}.$$

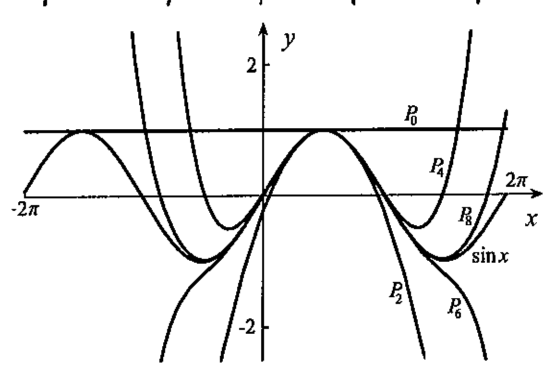
Plots of the polynomials suggest that the series converges to $\cos 3x$ for all x.



8. Since $f(\pi/2) = 1$, $f'(\pi/2) = 0$, $f''(\pi/2) = -1$, $f'''(\pi/2) = 0$, and $f''''(\pi/2) = 1$, the Taylor series for $\sin x$ about $x = \pi/2$ is

$$1 - \frac{(x-\pi/2)^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\pi/2)^{2n}.$$

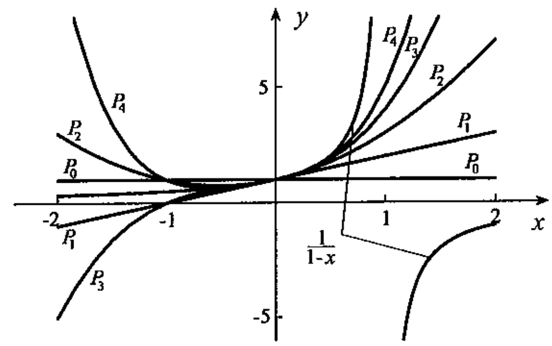
Plots of the polynomials suggest that the series converges to $\sin x$ for all x.



9. Since $f^{(n)}(0) = n!$, the Maclaurin series for $1/(1-x)$ is

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .$$

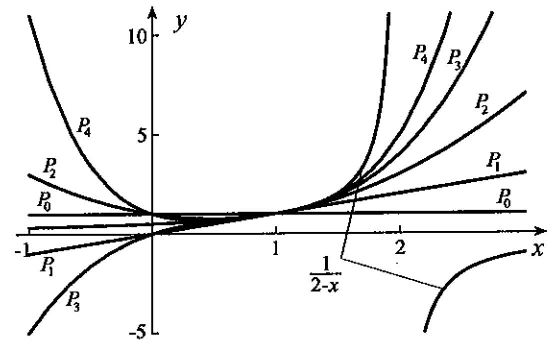
Plots of the polynomials suggest that the series converges to $1/(1-x)$ for $-1 < x < 1$.



10. Since $f^{(n)}(1) = n!$, the Taylor series for $1/(2-x)$ about $x = 1$ is

$$\sum_{n=0}^{\infty} (x-1)^n = 1 + (x-1) + (x-1)^2 + \cdots .$$

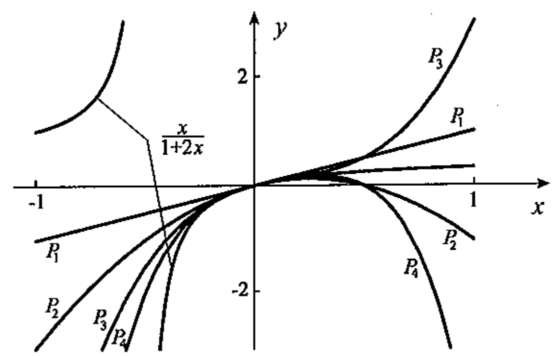
Plots of the polynomials suggest that the series converges to $1/(2-x)$ only for $0 < x < 2$.



11. By writing $f(x)$ in the form $1/2 - (1/2)/(1+2x)$ and taking derivatives, we quickly discover that $f^{(n)}(0) = (-1)^{n+1}2^{n-1}n!$ for $n \geq 1$. The Maclaurin series for $f(x)$ is therefore

$$\sum_{n=1}^{\infty} (-1)^{n+1}2^{n-1}x^n = x - 2x^2 + 4x^3 - 8x^4 + \cdots .$$

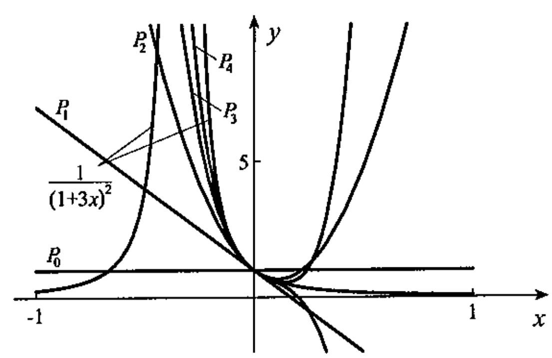
Plots of the polynomials suggest that the series converges to $x/(1+2x)$ only for $-1/2 < x < 1/2$.



12. Since $f^{(n)}(0) = (-1)^n 3^n (n+1)!$, the Maclaurin series for $1/(1+3x)^2$ is

$$\sum_{n=0}^{\infty} (-1)^n 3^n (n+1)x^n = 1 - 6x + 3^2(3)x^2 + \cdots .$$

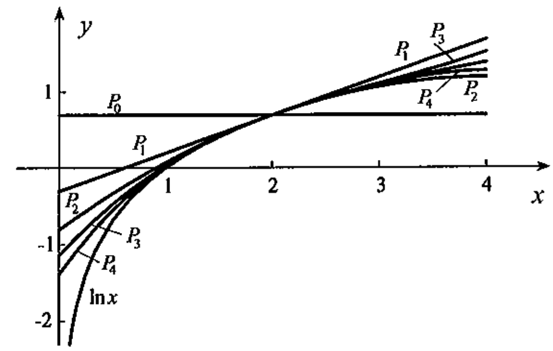
Plots of the polynomials suggest that the series converges to $1/(1+3x)^2$ only for $-1/3 < x < 1/3$.



13. Since $f^{(n)}(2) = (-1)^{n+1}(n-1)!/2^n$ for $n \geq 1$, the Taylor series for $\ln x$ about $x = 2$ is

$$\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,2^n}(x-2)^n$$
$$= \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{2 \cdot 2^2} + \cdots .$$

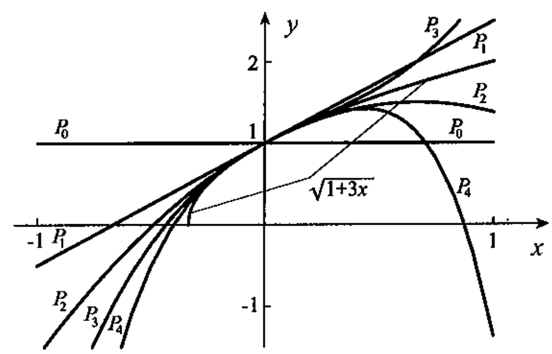
Plots of the polynomials suggest that the series converges to $\ln x$ only for $0 < x < 4$.



14. Calculating derivatives of the function leads to the formula $f^{(n)}(0) = \dfrac{(-1)^{n+1}3^n[1 \cdot 3 \cdot 5 \cdots (2n-3)]}{2^n}$ for $n \geq 2$, together with $f(0) = 1$ and $f'(0) = 3/2$. The Maclaurin series for $\sqrt{1+3x}$ is therefore

$$1 + \frac{3x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}3^n[1 \cdot 3 \cdot 5 \cdots (2n-3)]}{2^n\,n!}x^n.$$

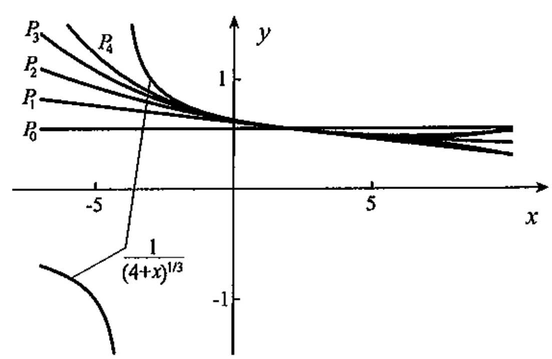
Plots of the polynomials suggest that the series converges to $\sqrt{1+3x}$ only for $-1/3 \leq x \leq 1/3$.



15. Calculating derivatives of the function leads to the formula $f^{(n)}(2) = \dfrac{(-1)^n[1 \cdot 4 \cdot 7 \cdots (3n-2)]}{2^n 3^{2n} 6^{1/3}}$ for $n \geq 1$. The Taylor series for $1/(4+x)^{1/3}$ about $x = 2$ is therefore

$$\frac{1}{6^{1/3}} + \sum_{n=1}^{\infty} \frac{(-1)^n[1 \cdot 4 \cdot 7 \cdots (3n-2)]}{2^n 3^{2n} 6^{1/3} n!}(x-2)^n.$$

Plots of the polynomials suggest that the series converges to $1/(4+x)^{1/3}$ only for $-4 < x < 8$.



16. If I' is the open interval in which $f'(x)$ and $f''(x)$ are continuous, and we apply Taylor's remainder formula to $f(x)$ at x_0 in I', we obtain

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(z_1)}{2!}(x-x_0)^2 = f(x_0) + \frac{f''(z_1)}{2}(x-x_0)^2,$$

where z_1 is between x_0 and x. Suppose that $f''(x_0) > 0$. Because $f''(x)$ is continuous at x_0, there exists an open interval I containing x_0 in which $f''(x) > 0$. For any x in this interval, it follows that $f''(z_1) > 0$ also. As a result, for any x in I, $f(x) > f(x_0)$, and $f(x)$ must have a relative minimum at x_0. A similar discussion shows that when $f''(x_0) < 0$, the function has a relative maximum at x_0. If $f''(x_0) = 0$, no conclusion can be reached.

17. If I' is the open interval in which $f(x)$ has derivatives of all orders, and we apply Taylor's remainder formula to $f(x)$ at x_0 in I', we obtain

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-x_0)^{n+1}$$
$$= f(x_0) + \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-x_0)^{n+1}$$

where z_n is between x_0 and x.

(i) Consider first the case that n is even, and suppose that $f^{(n+1)}(x_0) > 0$. (A similar proof follows in the case that $f^{(n+1)}(x_0) < 0$.) Because $f^{(n+1)}(x)$ is continuous at x_0, there exists an open interval I containing x_0 in which $f^{(n+1)}(x) > 0$. For any x in this interval, it follows that $f^{(n+1)}(z_n) > 0$ also. As a result, when $x < x_0$, $f(x) < f(x_0)$, and when $x > x_0$, $f(x) > f(x_0)$. This implies that x_0 must yield a horizontal point of inflection.

(ii) Consider now when n is odd and $f^{(n+1)}(x_0) > 0$. In this case, for any x in I, $f(x) > f(x_0)$ and $f(x)$ must have a relative minimum at x_0.

(iii) When n is odd and $f^{(n+1)}(x_0) < 0$, $f(x) < f(x_0)$ in I, and $f(x)$ has a relative maximum at x_0.

18. (a) This follows from $\displaystyle\int_c^x f'(t)\,dt = \Big\{f(t)\Big\}_c^x = f(x) - f(c)$.

(b) If we set $u = f'(t)$, $du = f''(t)\,dt$, $dv = dt$, and $v = t - x$, then

$$f(x) = f(c) + \Big\{(t-x)f'(t)\Big\}_c^x - \int_c^x (t-x)f''(t)\,dt = f(c) + f'(c)(x-c) + \int_c^x (x-t)f''(t)\,dt.$$

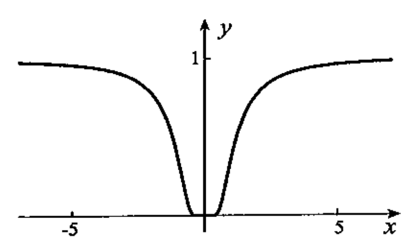
(c) If we now set $u = f''(t)$, $du = f'''(t)\,dt$, $dv = (x-t)\,dt$, and $v = -(1/2)(x-t)^2$,

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \left\{-\frac{(x-t)^2 f''(t)}{2}\right\}_c^x - \int_c^x -\frac{1}{2}(x-t)^2 f'''(t)\,dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{1}{2}\int_c^x (x-t)^2 f'''(t)\,dt. \end{aligned}$$

(d) One more integration by parts should convince us that the formula is correct. If we set $u = f'''(t)$, $du = f''''(t)\,dt$, $dv = (x-t)^2\,dt$, and $v = -(1/3)(x-t)^3$,

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{1}{2}\left\{-\frac{(x-t)^3 f'''(t)}{3}\right\}_c^x - \frac{1}{2}\int_c^x -\frac{1}{3}(x-t)^3 f''''(t)\,dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{1}{3!}\int_c^x (x-t)^3 f''''(t)\,dt. \end{aligned}$$

19. (a) Limits as $x \to 0^+$ and $x \to \infty$ together with symmetry about the y-axis give the graph to the right.



(b) If we can show that

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = 0,$$

then the limit from the left must also be zero.

Suppose we set $L = \displaystyle\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n}$, and take logarithms,

$$\ln L = -\lim_{x\to 0^+}\left(\frac{1}{x^2} + n\ln x\right) = -\lim_{x\to 0^+}\left(\frac{1 + nx^2\ln x}{x^2}\right).$$

Since $\displaystyle\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x^2} = \lim_{x\to 0^+}\frac{1/x}{-2/x^3} = \lim_{x\to 0^+}(-x^2/2) = 0$, it follows that $\ln L \to -\infty$ as $x \to 0^+$. Therefore, $L = 0$.

(c) $f'(0) = \displaystyle\lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{e^{-1/h^2}}{h} = 0$, by part (b). Suppose that k is some integer for which $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{h\to 0}\frac{f^{(k)}(h) - f^k(0)}{h} = \lim_{h\to 0}\frac{f^{(k)}(h)}{h}.$$

Now, any number of differentiations of $f(x) = e^{-1/x^2}$ gives rise to terms of the form $Ae^{-1/x^2}/x^n$, where n is a positive integer, and A is a constant. It follows that $f^{(k)}(h)/h$ must consist of terms of the form $Ae^{-1/h^2}/h^n$ which have limit zero as $h \to 0$. Thus, $f^{(k+1)}(0) = 0$, and by mathematical induction, $f^{(n)}(0) = 0$ for all $n \geq 1$.

(d) The Maclaurin series for $f(x)$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = 0 + 0 + 0 + \cdots .$$

(e) This series converges to $f(x)$ only at $x = 0$.

EXERCISES 10.4

1. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{1/n}{1/(n+1)}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

2. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

3. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{1/(n+1)^3}{1/(n+2)^3}\right| = 1$, the open interval of convergence is $-1 < x < 1$.

4. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{n^2 3^n}{(n+1)^2 3^{n+1}}\right| = \frac{1}{3}$, the open interval of convergence is $-1/3 < x < 1/3$.

5. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{1/2^n}{1/2^{n+1}}\right| = 2$, the open interval of convergence is $-1 < x < 3$.

6. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{(-1)^n n^3}{(-1)^{n+1}(n+1)^3}\right| = 1$, the open interval of convergence is $-4 < x < -2$.

7. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{1/\sqrt{n}}{1/\sqrt{n+1}}\right| = 1$, the open interval of convergence is $-3 < x < -1$.

8. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{2^n\left(\frac{n-1}{n+2}\right)^2}{2^{n+1}\left(\frac{n}{n+3}\right)^2}\right| = \frac{1}{2}$, the open interval of convergence is $7/2 < x < 9/2$.

9. If we set $y = x^2$, then $\sum_{n=1}^{\infty} \frac{1}{n^2}x^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}y^n$. Since $R_y = \lim_{n\to\infty} \left|\frac{1/n^2}{1/(n+1)^2}\right| = 1$, it follows that $R_x = \sqrt{R_y} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

10. If we set $y = x^3$, then $\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n y^n$. Since $R_y = \lim_{n\to\infty} \left|\frac{(-1)^n}{(-1)^{n+1}}\right| = 1$, it follows that $R_x = {R_y}^{1/3} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

11. Since the radius of convergence is $R = \lim_{n\to\infty} \left|\frac{2^n(n-1)/(n+1)}{2^{n+1}n/(n+2)}\right| = 1/2$, the open interval of convergence is $-1/2 < x < 1/2$.

12. If we set $y = x^3$, then $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}x^{3n+1} = y^{1/3}\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}y^n$. Since $R_y = \lim_{n\to\infty} \left|\frac{1/\sqrt{n+1}}{1/\sqrt{n+2}}\right| = 1$, it follows that $R_x = R_y^{1/3} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

13. If we set $y = x^2$, then $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^{2n+1} = \pm\sqrt{y} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} y^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{(-1)^n/3^n}{(-1)^{n+1}/3^{n+1}} \right| = 3$, it follows that $R_x = \sqrt{R_y} = \sqrt{3}$. The open interval of convergence is therefore $-\sqrt{3} < x < \sqrt{3}$.

14. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{(-e)^n/n^2}{(-e)^{n+1}/(n+1)^2} \right| = \frac{1}{e}$, the open interval of convergence is $-1/e < x < 1/e$.

15. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^2/3^{2n}}{(n+1)^2/3^{2n+2}} \right| = 9$, the open interval of convergence is $-9 < x < 9$.

16. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n \left(\frac{1}{n+1} \right) = \frac{1}{e}(0) = 0$, the series converges only for $x = 0$.

17. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{1/n^2}{1/(n+1)^2} \right| = 1$, the open interval of convergence is $-11 < x < -9$.

18. Since the radius of convergence is $R = \lim_{n\to\infty} \left| \frac{n^3 3^n}{(n+1)^3 3^{n+1}} \right| = \frac{1}{3}$, the open interval of convergence is $-1/3 < x < 1/3$.

19. If we set $y = x^2$, then $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^2} x^{2n} = \sum_{n=1}^{\infty} \frac{3^n}{(n+1)^2} y^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{3^n/(n+1)^2}{3^{n+1}/(n+2)^2} \right| = 1/3$, it follows that $R_x = \sqrt{R_y} = 1/\sqrt{3}$. The open interval of convergence is therefore $-1/\sqrt{3} < x < 1/\sqrt{3}$.

20. If we set $y = x^3$, the series becomes $\sum_{n=0}^{\infty} y^n/5^n$. Since $R_y = \lim_{n\to\infty} \left| \frac{1/5^n}{1/5^{n+1}} \right| = 5$, it follows that $R_x = {R_y}^{1/3} = 5^{1/3}$. The open interval of convergence is therefore $-5^{1/3} < x < 5^{1/3}$.

21. Using L'Hôpital's rule, $R = \lim_{n\to\infty} \left| \frac{1/\ln n}{1/\ln(n+1)} \right| = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

22. Using L'Hôpital's rule, $R = \lim_{n\to\infty} \left| \frac{\frac{1}{n^2 \ln n}}{\frac{1}{(n+1)^2 \ln(n+1)}} \right| = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$. The open interval of convergence is therefore $-1 < x < 1$.

23. Since $R = \lim_{n\to\infty} \left| \frac{(n!)^3/(3n)!}{[(n+1)!]^3/(3n+3)!} \right| = \lim_{n\to\infty} \frac{(n!)^3(3n+3)(3n+2)(3n+1)(3n)!}{(3n)!(n+1)^3(n!)^3} = 27$, the open interval of convergence is $-27 < x < 27$.

24. Since $R = \lim_{n\to\infty} \left| \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \frac{3 \cdot 5 \cdots (2n+3)}{2 \cdot 4 \cdots (2n+2)} \right| = \lim_{n\to\infty} \frac{2n+3}{2n+2} = 1$, the open interval of convergence is $-1 < x < 1$.

25. Since $R = \lim_{n\to\infty} \left| \frac{\frac{[1 \cdot 3 \cdots (2n+1)]^2}{2^{2n}(2n)!}}{\frac{[1 \cdot 3 \cdots (2n+3)]^2}{2^{2n+2}(2n+2)!}} \right| = \lim_{n\to\infty} \frac{4(2n+2)(2n+1)}{(2n+3)^2} = 4$, the open interval of convergence is $-4 < x < 4$.

26. $$\sum_{n=0}^{\infty} \frac{1}{4^n} x^{3n} = \sum_{n=0}^{\infty} \left(\frac{x^3}{4} \right)^n = \frac{1}{1 - x^3/4} = \frac{4}{4 - x^3} \quad \text{provided } \left| \frac{x^3}{4} \right| < 1 \implies |x| < 4^{1/3}$$

27. $$\sum_{n=1}^{\infty} (-e)^n x^n = \sum_{n=1}^{\infty} (-ex)^n = \frac{-ex}{1 + ex} \quad \text{provided } |-ex| < 1 \implies |x| < 1/e$$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{2n}}(x-1)^n = \sum_{n=1}^{\infty}\left(\frac{x-1}{9}\right)^n = \frac{\frac{x-1}{9}}{1-\frac{x-1}{9}} = \frac{x-1}{10-x}$ provided $\left|\frac{x-1}{9}\right| < 1 \Longrightarrow |x-1| < 9$

29. $\displaystyle\sum_{n=2}^{\infty}(x+5)^{2n} = \frac{(x+5)^4}{1-(x+5)^2}$ provided $|(x+5)^2| < 1 \Longrightarrow |x+5| < 1$

30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^2)^{2n} = \cos(x^2)$ valid for all x

31. $\displaystyle\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{1}{n!}(5x)^n = e^{5x}$ valid for all x

32. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n+1}(2n+1)!}x^{2n+2} = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{x}{3}\right)^{2n+1} = x\sin(x/3)$ valid for all x

33. $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n}{n!}(x+1)^n = \sum_{n=0}^{\infty}\frac{1}{n!}[-3(x+1)]^n = e^{-3(x+1)}$ valid for all x

34. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n = -1+\sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n = -1+e^{-x}$ valid for all x

35. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}(x+1)^{2n+3} = -(x+1)^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x+1)^{2n+1} = -(x+1)^2\sin(x+1)$ valid for all x

36. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}(x-1/2)^n = \sum_{n=0}^{\infty}\frac{1}{n!}(2x-1)^n = e^{2x-1}$ valid for all x

37. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(2n)!}x^{4n+4} = x^4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{x^2}{2}\right)^{2n} = x^4\cos(x^2/2)$ valid for all x

38. (a)
$$J_0(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n} = 1-\frac{x^2}{2^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+\frac{x^8}{2^8(4!)^2}-\cdots$$
$$J_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}(n!)(n+1)!}x^{2n+1} = \frac{x}{2}-\frac{x^3}{2^3 2!}+\frac{x^5}{2^5 2!3!}-\frac{x^7}{2^7 3!4!}+\frac{x^9}{2^9 4!5!}-\cdots$$
$$J_m(x) = \frac{x^m}{2^m m!}-\frac{x^{m+2}}{2^{m+2}(m+1)!}+\frac{x^{m+4}}{2^{m+4}2!(m+2)!}-\frac{x^{m+6}}{2^{m+6}3!(m+3)!}+\frac{x^{m+8}}{2^{m+8}4!(m+4)!}-\cdots\cdots$$

(b) $\displaystyle R = \lim_{n\to\infty}\left|\frac{(-1)^n}{2^{2n+m}n!(n+m)!}\frac{2^{2n+m+2}(n+1)!(n+m+1)!}{(-1)^{n+1}}\right| = \lim_{n\to\infty}2^2(n+1)(n+m+1) = \infty$

The interval of convergence is therefore $-\infty < x < \infty$.

39. (a) $\displaystyle 1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}x^n$

(b) $$R = \lim_{n\to\infty}\left|\frac{\dfrac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}}{\dfrac{\alpha(\alpha+1)\cdots(\alpha+n)\beta(\beta+1)\cdots(\beta+n)}{(n+1)!\,\gamma(\gamma+1)\cdots(\gamma+n)}}\right| = \lim_{n\to\infty}\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = 1$$

EXERCISES 10.5

1. $\displaystyle\frac{1}{3x+2} = \frac{1}{2(1+3x/2)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{3x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{2^{n+1}}x^n,\quad |-3x/2| < 1 \Longrightarrow |x| < 2/3$

2. $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\left(\dfrac{1}{1+x^2/4}\right) = \dfrac{1}{4}\sum_{n=0}^{\infty}\left(-\dfrac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{4^{n+1}}x^{2n}, \quad \left|-\dfrac{x^2}{4}\right| < 1 \Longrightarrow |x| < 2$

3. Since $\cos x = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!}x^{2n}, \quad -\infty < x < \infty,$ it follows that

$$\cos(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^2)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}, \quad -\infty < x < \infty.$$

4. Since $e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n, \quad -\infty < x < \infty,$ it follows that

$$e^{5x} = \sum_{n=0}^{\infty}\frac{1}{n!}(5x)^n = \sum_{n=0}^{\infty}\frac{5^n}{n!}x^n, \quad -\infty < x < \infty.$$

5. Since $f(x) = e^x = e^3\,e^{x-3}$, and the Maclaurin series $e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n$ converges for all x, it follows that

$$e^x = e^3\sum_{n=0}^{\infty}\frac{1}{n!}(x-3)^n = \sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n, \quad -\infty < x < \infty.$$

6. Since $f(x) = e^{1-2x} = e\,e^{-2x}$, and the Maclaurin series $e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n$ converges for all x, it follows that

$$e^{1-2x} = e\sum_{n=0}^{\infty}\frac{1}{n!}(-2x)^n = \sum_{n=0}^{\infty}\frac{e(-1)^n2^n}{n!}x^n, \quad -\infty < x < \infty.$$

7. Since $f(x) = e^{1-2x} = e^{3-2(x+1)} = e^3\,e^{-2(x+1)}$, and the Maclaurin series $e^x = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^n$ converges for all x, it follows that

$$e^{1-2x} = e^3\sum_{n=0}^{\infty}\frac{1}{n!}[-2(x+1)]^n = \sum_{n=0}^{\infty}\frac{e^3(-1)^n2^n}{n!}(x+1)^n, \quad -\infty < x < \infty.$$

8.
$$\begin{aligned}\cosh x &= \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n!}x^n + \sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[1+(-1)^n]}{n!}x^n\\ &= \frac{1}{2}\left(2 + \frac{2}{2!}x^2 + \frac{2}{4!}x^4 + \cdots\right) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}, \quad -\infty < x < \infty\end{aligned}$$

9.
$$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{1}{n!}x^n - \sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[1-(-1)^n]}{n!}x^n\\ &= \frac{1}{2}\left(2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots\right) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}, \quad -\infty < x < \infty\end{aligned}$$

10. This function is its own Maclaurin series.

11. Since $f(-2) = 33$, $f'(-2) = -46$, $f''(-2) = 54$, $f'''(-2) = -48$, $f''''(-2) = 24$, and $f^{(n)}(-2) = 0$ for $n \geq 5$, formula 10.17 gives

$$\begin{aligned}f(x) &= 33 - 46(x+2) + \frac{54}{2!}(x+2)^2 - \frac{48}{3!}(x+2)^3 + \frac{24}{4!}(x+2)^4\\ &= 33 - 46(x+2) + 27(x+2)^2 - 8(x+2)^3 + (x+2)^4.\end{aligned}$$

12. $\dfrac{1}{x+3} = \dfrac{1}{5+(x-2)} = \dfrac{1}{5\left(1+\dfrac{x-2}{5}\right)} = \dfrac{1}{5}\sum_{n=0}^{\infty}\left(-\dfrac{x-2}{5}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{5^{n+1}}(x-2)^n, \quad \left|-\dfrac{x-2}{5}\right| < 1 \Longrightarrow$
$-3 < x < 7$

13. Long division gives

$$\frac{x}{2x+5} = \frac{1}{2} - \frac{5/2}{2x+5} = \frac{1}{2} - \frac{5/2}{2(x-1)+7} = \frac{1}{2} - \frac{5}{14\left[1+\dfrac{2(x-1)}{7}\right]} = \frac{1}{2} - \frac{5}{14}\sum_{n=0}^{\infty}\left[-\frac{2}{7}(x-1)\right]^n$$

$$= \frac{1}{7} + \sum_{n=1}^{\infty}\frac{5(-1)^{n+1}2^{n-1}}{7^{n+1}}(x-1)^n, \quad \left|-\frac{2(x-1)}{7}\right| < 1 \Longrightarrow -\frac{5}{2} < x < \frac{9}{2}$$

14. Long division gives

$$\frac{x^2}{3-4x} = -\frac{x}{4} - \frac{3}{16} + \frac{9/16}{3-4x} = -\frac{1}{4}(x-2) - \frac{11}{16} + \frac{9/16}{-5-4(x-2)} = -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9/80}{1+\dfrac{4(x-2)}{5}}$$

$$= -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9}{80}\sum_{n=0}^{\infty}\left[-\frac{4}{5}(x-2)\right]^n$$

$$= -\frac{11}{16} - \frac{1}{4}(x-2) - \frac{9}{80}\left[1 - \frac{4}{5}(x-2) + \sum_{n=2}^{\infty}\frac{(-1)^n 4^n}{5^n}(x-2)^n\right]$$

$$= -\frac{4}{5} - \frac{4}{25}(x-2) + \sum_{n=2}^{\infty}\frac{9(-1)^{n+1}4^{n-2}}{5^{n+1}}(x-2)^n, \quad \left|-\frac{4(x-2)}{5}\right| < 1 \Longrightarrow \frac{3}{4} < x < \frac{13}{4}$$

15. With the binomial expansion 10.33b,

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{(-1/2)(-3/2)}{2!}x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^3 + \cdots, \quad -1 < x \le 1$$

$$= 1 - \frac{x}{2} + \frac{3}{2^2 2!}x^2 - \frac{3\cdot 5}{2^3 3!}x^3 + \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n n!}x^n$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 2\cdot 3\cdot 4\cdots(2n)]}{2^n n![2\cdot 4\cdot 6\cdots(2n)]}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}x^n$$

16. Term-by-term integration of $\dfrac{1}{1+2x} = \sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^n$ gives

$$\frac{1}{2}\ln|1+2x| = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{n+1}x^{n+1} + C.$$

Setting $x = 0$ gives $C = 0$, and therefore $\ln|1+2x| = \sum_{n=0}^{\infty}\dfrac{(-1)^n 2^{n+1}}{n+1}x^{n+1}$. Since the radius of convergence of the geometric series is $1/2$, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $-1/2 < x < 1/2$, and the absolute values may be dropped,

$$\ln(1+2x) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{n+1}}{n+1}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n}{n}x^n.$$

17. With the binomial expansion 10.33b,

$$\begin{aligned}(1+3x)^{3/2} &= 1+\left(\frac{3}{2}\right)(3x)+\frac{(3/2)(1/2)}{2!}(3x)^2+\frac{(3/2)(1/2)(-1/2)}{3!}(3x)^3+\cdots,\quad -1\le 3x\le 1\\ &= 1+\frac{9}{2}x+\frac{3^3}{2^2 2!}x^2-\frac{3^4}{2^3 3!}x^3+\frac{3^5(1)(3)}{2^4 4!}x^4-\frac{3^6(1)(3)(5)}{2^5 5!}x^5+\cdots\\ &= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-5)]3^{n+1}}{2^n n!}x^n\\ &= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n[1\cdot 2\cdot 3\cdot 4\cdots(2n-5)(2n-4)]3^{n+1}}{[2\cdot 4\cdots(2n-4)]2^n n!}x^n\\ &= 1+\frac{9}{2}x+\frac{27}{8}x^2-\frac{27}{16}x^3+\sum_{n=4}^{\infty}\frac{(-1)^n(2n-4)!\,3^{n+1}}{2^{2n-2}n!\,(n-2)!}x^n\\ &= 1+\frac{9}{2}x+\sum_{n=2}^{\infty}\frac{(-1)^n(2n-4)!\,3^{n+1}}{2^{2n-2}n!\,(n-2)!}x^n,\quad -\frac{1}{3}\le x\le\frac{1}{3}.\end{aligned}$$

18. Termwise integration of

$$\frac{1}{x}=\frac{1}{2+(x-2)}=\frac{1}{2[1+(x-2)/2]}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-2}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

gives $\ln|x|=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}+C$. Setting $x=2$ gives $C=\ln 2$, and therefore $\ln|x|=\ln 2+\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}$. Since the radius of convergence of the geometric series is 2, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $0<x<4$, and the absolute values may be dropped,

$$\ln x=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x-2)^{n+1}=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}(x-2)^n.$$

19. Termwise integration of

$$\frac{1}{x+3}=\frac{1}{2+(x+1)}=\frac{1}{2[1+(x+1)/2]}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x+1}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x+1)^n$$

gives $\ln|x+3|=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}+C$. Setting $x=-1$ gives $C=\ln 2$, and therefore $\ln|x+3|=\ln 2+\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}$. Since the radius of convergence of the geometric series is 2, this is also the radius of convergence for the series of the logarithm function. The open interval of convergence is therefore $-3<x<1$, and the absolute values may be dropped,

$$\ln(x+3)=\ln 2+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(x+1)^{n+1}=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}(x+1)^n.$$

20. $\displaystyle\frac{1}{x}=\frac{1}{4+(x-4)}=\frac{1}{4[1+(x-4)/4]}=\frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x-4}{4}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x-4)^n$, provided $\left|-\dfrac{x-4}{4}\right|<1 \implies 0<x<8$

21. With the binomial expansion 10.33b,

$$\frac{1}{(x+2)^3} = \frac{1}{8(1+x/2)^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\left[1+(-3)\left(\frac{x}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$= \frac{1}{8}\left[1-\frac{3x}{2}+\frac{3\cdot 4}{2^3}x^2-\frac{4\cdot 5}{2^4}x^3+\frac{5\cdot 6}{2^5}x^4+\cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2^{n+4}}x^n, \quad \text{valid for } -1<\frac{x}{2}<1 \Longrightarrow -2<x<2.$$

22. With the binomial expansion 10.33b,

$$\frac{1}{(2-x)^2} = \frac{1}{[-1-(x-3)]^2} = \frac{1}{[1+(x-3)]^2} = [1+(x-3)]^{-2}$$

$$= 1-2(x-3)+\frac{(-2)(-3)}{2!}(x-3)^2+\frac{(-2)(-3)(-4)}{3!}(x-3)^3+\cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n(n+1)(x-3)^n, \quad \text{provided } -1<x-3<1 \Longrightarrow 2<x<4.$$

23. With the binomial expansion 10.33b,

$$\frac{1}{(x+3)^2} = \frac{1}{[4+(x-1)]^2} = \frac{1}{16[1+(x-1)/4]^2} = \frac{1}{16}\left(1+\frac{x-1}{4}\right)^{-2}$$

$$= \frac{1}{16}\left[1-2\left(\frac{x-1}{4}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x-1}{4}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x-1}{4}\right)^3+\cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{4^{n+2}}(x-1)^n, \quad \text{provided } -1<\frac{x-1}{4}<1 \Longrightarrow -3<x<5.$$

24. $$\frac{1}{x^2+8x+15} = \frac{1}{(x+3)(x+5)} = \frac{1/2}{x+3}+\frac{-1/2}{x+5} = \frac{1/6}{1+x/3}-\frac{1/10}{1+x/5}$$

$$= \frac{1}{6}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n-\frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2(3^{n+1})}x^n+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2(5^{n+1})}x^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\left(\frac{1}{3^{n+1}}-\frac{1}{5^{n+1}}\right)x^n, \quad \text{valid for } -3<x<3.$$

25. Term-by-term integration of $\dfrac{1}{1+x^2} = 1-x^2+x^4-x^6+\cdots$ gives

$$\mathrm{Tan}^{-1}x = \left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)+C.$$

Substitution of $x=0$ gives $C=0$, and therefore $\mathrm{Tan}^{-1}x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$. The open interval of convergence is $-1<x<1$.

26. With the binomial expansion 10.33b,

$$\sqrt{x+3} = \sqrt{3}\sqrt{1+x/3} = \sqrt{3}\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{3}\right)+\frac{(1/2)(-1/2)}{2!}\left(\frac{x}{3}\right)^2+\frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x}{3}\right)^3+\cdots\right]$$

$$= \sqrt{3}\left[1+\frac{x}{6}-\frac{1}{2^2 3^2 2!}x^2+\frac{(1)(3)}{2^3 3^3 3!}x^3-\frac{(1)(3)(5)}{2^4 3^4 4!}x^4+\cdots\right]$$

$$= \sqrt{3}\left[1+\frac{x}{6}-\frac{x^2}{72}+\sum_{n=3}^{\infty}\frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{2^n 3^n n!}x^n\right]$$

$$= \sqrt{3}\left[1 + \frac{x}{6} - \frac{x^2}{72} + \sum_{n=3}^{\infty} \frac{(-1)^{n+1}[1\cdot 2\cdot 3\cdot 4\cdots(2n-3)(2n-2)]}{[2\cdot 4\cdot 6\cdots(2n-2)]6^n n!}x^n\right]$$

$$= \sqrt{3}\left[1 + \frac{x}{6} - \frac{x^2}{72} + \sum_{n=3}^{\infty} \frac{2(-1)^{n+1}(2n-2)!}{12^n n!\,(n-1)!}x^n\right]$$

$$= \sqrt{3} + \sum_{n=1}^{\infty} \frac{2\sqrt{3}(-1)^{n+1}(2n-2)!}{12^n n!\,(n-1)!}x^n, \quad \text{valid for } -1 \le \frac{x}{3} \le 1 \Longrightarrow |x| \le 3.$$

27. With the binomial expansion 10.33b,

$$\sqrt{x+3} = \sqrt{5+(x-2)} = \sqrt{5}\sqrt{1+(x-2)/5}$$

$$= \sqrt{5}\left[1 + \frac{1}{2}\left(\frac{x-2}{5}\right) + \frac{(1/2)(-1/2)}{2!}\left(\frac{x-2}{5}\right)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x-2}{5}\right)^3 + \cdots\right]$$

$$= \sqrt{5}\left[1 + \frac{1}{10}(x-2) - \frac{1}{10^2 2!}(x-2)^2 + \frac{1\cdot 3}{10^3 3!}(x-2)^3 + \cdots\right]$$

$$= \sqrt{5} + \frac{\sqrt{5}}{10}(x-2) + \sum_{n=2}^{\infty} \frac{\sqrt{5}(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{10^n n!}(x-2)^n$$

$$= \sqrt{5} + \frac{\sqrt{5}}{10}(x-2) + \sum_{n=2}^{\infty} \frac{\sqrt{5}(-1)^{n+1}[1\cdot 2\cdot 3\cdots(2n-2)]}{[2\cdot 4\cdot 6\cdots(2n-2)]10^n n!}(x-2)^n$$

$$= \sqrt{5} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-2)!}{5^{n-1/2}2^{2n-1}n!\,(n-1)!}(x-2)^n, \quad \text{valid for } -1 \le \frac{x-2}{5} \le 1 \Longrightarrow -3 \le x \le 7$$

28. With the binomial expansion 10.33b,

$$(1-2x)^{1/3} = [-1-2(x-1)]^{1/3} = -[1+2(x-1)]^{1/3}$$

$$= -\left\{1 + \frac{2(x-1)}{3} + \frac{(1/3)(-2/3)}{2!}[2(x-1)]^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}[2(x-1)])^3 + \cdots\right\}$$

$$= -1 - \frac{2}{3}(x-1) + \frac{2^2 2}{3^2 2!}(x-1)^2 - \frac{2^3(2\cdot 5)}{3^3 3!}(x-1)^3 + \cdots$$

$$= -1 - \frac{2}{3}(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n 2^n[2\cdot 5\cdot 8\cdots(3n-4)]}{3^n n!}(x-1)^n,$$

$$= -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n[(-1)\cdot 2\cdot 5\cdot 8\cdots(3n-4)]}{3^n n!}(x-1)^n,$$

valid for $-1 \le 2(x-1) \le 1 \Longrightarrow 1/2 \le x \le 3/2$.

29. With the binomial expansion 10.33b,

$$\frac{x^2}{(1+x^2)^2} = x^2(1+x^2)^{-2} = x^2\left[1 + (-2)(x^2) + \frac{(-2)(-3)}{2!}(x^2)^2 + \frac{(-2)(-3)(-4)}{3!}(x^2)^3 + \cdots\right]$$

$$= x^2 - 2x^4 + 3x^6 - 4x^8 + \cdots$$

$$= \sum_{n=1}^{\infty} n(-1)^{n+1}x^{2n}, \quad \text{valid for } -1 < x^2 < 1 \Longrightarrow -1 < x < 1$$

30. With the binomial expansion 10.33b,

$$x(1-x)^{1/3} = x\left[1 + \left(\frac{1}{3}\right)(-x) + \frac{(1/3)(-2/3)}{2!}(-x)^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}(-x)^3 + \cdots\right]$$

$$= x - \frac{x^2}{3} - \frac{2}{3^2 2!}x^3 - \frac{(2)(5)}{3^3 3!}x^4 - \frac{(2)(5)(8)}{3^4 4!}x^5 + \cdots$$

$$= x - \frac{x^2}{3} - \sum_{n=3}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n-7)}{3^{n-1}(n-1)!} x^n$$

$$= x + \sum_{n=2}^{\infty} \frac{(-1) \cdot 2 \cdot 5 \cdot 8 \cdots (3n-7)}{3^{n-1}(n-1)!} x^n, \quad \text{valid for } -1 \le x \le 1.$$

31. We extend the calculations in Example 10.24 to obtain another nonzero term. When we equate coefficients of x^6, we obtain $0 = a_6 - a_4/2! + a_2/4! - a_0/6!$, and this implies that $a_6 = 0$. Coefficients of x^7 give $-1/7! = a_7 - a_5/2! + a_3/4! - a_1/6! \Longrightarrow a_7 = 17/315$. Consequently,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots,$$

and if we replace x by $2x$, $\tan 2x = 2x + \dfrac{8x^3}{3} + \dfrac{64x^5}{15} + \dfrac{2176x^7}{315} + \cdots$.

32. If we set $\sec x = \dfrac{1}{\cos x} = a_0 + a_1 x + a_2 x^2 + \cdots$, then

$$1 = \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right).$$

We now multiply the power series on the right and equate coefficients:

$$\begin{array}{ll}
1: & 1 = a_0 \\
x: & 0 = a_1 \\
x^2: & 0 = -a_0/2! + a_2 \quad \Longrightarrow \quad a_2 = 1/2 \\
x^3: & 0 = -a_1/2! + a_3 \quad \Longrightarrow \quad a_3 = 0 \\
x^4: & 0 = a_0/4! - a_2/2! + a_4 \quad \Longrightarrow \quad a_4 = 5/24 \\
x^5: & 0 = a_1/4! - a_3/2! + a_5 \quad \Longrightarrow \quad a_5 = 0 \\
x^6: & 0 = -a_0/6! + a_2/4! - a_4/2! + a_6 \quad \Longrightarrow \quad a_6 = 61/720
\end{array}$$

Thus, $\sec x = 1 + \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 + \dfrac{61}{720}x^6 + \cdots$. Long division could also be used.

33.
$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$
$$= x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \left(\frac{1}{4!} - \frac{1}{2!3!} + \frac{1}{5!}\right)x^5 + \cdots$$
$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

34.
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}\right] = \frac{1}{2}\left[1 + 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}\right]$$
$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{2n}, \quad -\infty < x < \infty$$

35.
$$\frac{1}{x^6 - 3x^3 - 4} = \frac{1}{(x^3 - 4)(x^3 + 1)} = \frac{-1/5}{1 + x^3} + \frac{1/5}{x^3 - 4} = \frac{-1/5}{1 + x^3} - \frac{1/20}{1 - x^3/4}$$
$$= -\frac{1}{5}\sum_{n=0}^{\infty}(-x^3)^n - \frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{x^3}{4}\right)^n$$
$$= \sum_{n=0}^{\infty} -\frac{1}{5}\left[(-1)^n + \frac{1}{4^{n+1}}\right]x^{3n}, \quad \text{valid for } -1 < x < 1.$$

36. The Maclaurin series for $\mathrm{Sin}^{-1}(x^2)$ can be obtained by replacing x by x^2 in the series for $\mathrm{Sin}^{-1}x$ in Example 10.26:

$$\mathrm{Sin}^{-1}(x^2) = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)2^{2n}(n!)^2}(x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)2^{2n}(n!)^2}x^{4n+2}, \quad |x| < 1.$$

37.
$$\frac{2x^2+4}{x^2+4x+3} = 2 - \frac{8x+2}{(x+3)(x+1)} = 2 - \frac{11}{x+3} + \frac{3}{x+1} = 2 - \frac{11/3}{1+x/3} + \frac{3}{1+x}$$
$$= 2 - \frac{11}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n + 3\sum_{n=0}^{\infty}(-x)^n = \left(2 - \frac{11}{3} + 3\right) + \sum_{n=1}^{\infty}\left[-\frac{11}{3}\left(-\frac{1}{3}\right)^n + 3(-1)^n\right]x^n$$
$$= \frac{4}{3} + \sum_{n=1}^{\infty}(-1)^n\left(3 - \frac{11}{3^{n+1}}\right)x^n, \quad \text{valid for } -1 < x < 1.$$

38. If we integrate the series $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$, we obtain $\displaystyle -\ln|1-x| = \sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1} + C$. Substitution of $x = 0$ gives $C = 0$, and therefore $\displaystyle \ln|1-x| = \sum_{n=0}^{\infty}\frac{-1}{n+1}x^{n+1} = \sum_{n=1}^{\infty} -\frac{1}{n}x^n$. The open interval of convergence is $-1 < x < 1$ so that absolute values may be dropped. If we replace x by $x/\sqrt{2}$ and $-x/\sqrt{2}$, we find

$$f(x) = \ln(1 + x/\sqrt{2}) - \ln(1 - x/\sqrt{2}) = \sum_{n=1}^{\infty} -\frac{1}{n}\left(-\frac{x}{\sqrt{2}}\right)^n - \sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{x}{\sqrt{2}}\right)^n$$
$$= \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n2^{n/2}} + \frac{1}{n2^{n/2}}\right]x^n = \sum_{n=1}^{\infty}\left[\frac{1+(-1)^{n+1}}{n2^{n/2}}\right]x^n.$$

When n is even the coefficient of x^n is zero, and therefore

$$f(x) = \sum_{n=0}^{\infty}\frac{2}{(2n+1)2^{(2n+1)/2}}x^{2n+1} = \sum_{n=0}^{\infty}\frac{\sqrt{2}}{(2n+1)2^n}x^{2n+1}.$$

Since the added series both have open interval of convergence $-\sqrt{2} < x < \sqrt{2}$, this is the open interval of convergence for the combined series.

39. If $\sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} b_n(x-c)^n$, then $\sum_{n=0}^{\infty}(a_n - b_n)(x-c)^n = 0$. The right side of this equation is the Maclaurin series for the function identically equal to zero, and as such, its coefficients must all be zero; that is, $a_n - b_n = 0$ for all n.

40. The right side of this equation is the Maclaurin series for the function identically equal to zero, and as such, its coefficients must all be zero; that is, $a_n = 0$ for all n.

41. $\displaystyle\sum_{n=0}^{\infty} P_n(t) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{30}\right)^n e^{-t/30} = e^{-t/30}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{30}\right)^n = e^{-t/30}(e^{t/30}) = 1$ The sum represents the probability that either nobody, or just one person, or two people, or three people, etc., drink from the fountain. Since one of these situations must occur, the probability is one.

42. (a) $\displaystyle\sum_{n=1}^{\infty} np(1-p)^{n-1} = p\sum_{n=1}^{\infty} n(1-p)^{n-1}$ If we differentiate the series $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$, term-by-term, we obtain $\displaystyle\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$, $|x| < 1$. We now substitute $x = 1 - p$ into this result, $\displaystyle\frac{1}{[1-(1-p)]^2} = \sum_{n=1}^{\infty} n(1-p)^{n-1}$. Multiplication by p gives $\displaystyle\frac{1}{p} = \sum_{n=1}^{\infty} np(1-p)^{n-1}$.

(b) The probability of throwing a six is $p = 1/6$, and therefore $\displaystyle\sum_{n=1}^{\infty} np(1-p)^{n-1} = \frac{1}{1/6} = 6$.

43. $$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\int_0^x\left[\sum_{n=0}^{\infty}\frac{1}{n!}(-t^2)^n\right]dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x t^{2n}\,dt$$
$$= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left\{\frac{t^{2n+1}}{2n+1}\right\}_0^x = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1}$$

44. Integrating the Maclaurin series for $\cos(\pi t^2/2)$ (see Example 10.21) term-by-term gives
$$C(x) = \int_0^x\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi t^2}{2}\right)^{2n}\right]dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{\pi^{2n}}{2^{2n}}\int_0^x t^{4n}\,dt$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{\pi^{2n}}{2^{2n}}\left\{\frac{t^{4n+1}}{4n+1}\right\}_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(4n+1)2^{2n}(2n)!}x^{4n+1},$$
valid for $-\infty < x < \infty$. A similar procedure leads to the Maclaurin series for $S(x)$.

45. With the binomial expansion 10.33b,
$$\frac{x}{(4+3x)^2} = \frac{x}{16}\left(1+\frac{3x}{4}\right)^{-2} = \frac{x}{16}\left[1-2\left(\frac{3x}{4}\right)+\frac{(-2)(-3)}{2!}\left(\frac{3x}{4}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{3x}{4}\right)^3+\cdots\right]$$
$$= \frac{x}{16}\sum_{n=0}^{\infty}\frac{(-1)^n3^n(n+1)}{4^n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n3^n(n+1)}{4^{n+2}}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^{n-1}n}{4^{n+1}}x^n.$$
But the coefficient of x^n in the Maclaurin series is $f^{(n)}(0)/n!$, and therefore
$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}3^{n-1}n}{4^{n+1}} \implies f^{(n)}(0) = \frac{(-1)^{n+1}3^{n-1}n\,n!}{4^{n+1}}.$$

46. The Maclaurin series for $f(x) = xe^{-2x}$ is
$$xe^{-2x} = x\sum_{n=0}^{\infty}\frac{1}{n!}(-2x)^n = \sum_{n=0}^{\infty}\frac{(-1)^n2^n}{n!}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{n-1}}{(n-1)!}x^n.$$
But the coefficient of x^n in the Maclaurin series is $f^{(n)}(0)/n!$, and therefore
$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}2^{n-1}}{(n-1)!} \implies f^{(n)}(0) = \frac{(-1)^{n+1}2^{n-1}n!}{(n-1)!} = n(-1)^{n+1}2^{n-1}.$$

47. The Taylor series for $f(x) = 1/(3+x)$ about $x = 2$ is
$$\frac{1}{3+x} = \frac{1}{5+(x-2)} = \frac{1}{5[1+(x-2)/5]} = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x-2}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-2)^n.$$
But the coefficient of $(x-2)^n$ in the Taylor series is $f^{(n)}(2)/n!$, and therefore
$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{5^{n+1}} \implies f^{(n)}(2) = \frac{(-1)^n n!}{5^{n+1}}.$$

48. The Taylor series for $f(x) = xe^{-x}$ about $x = 2$ is
$$xe^{-x} = [(x-2)+2]e^{-(x-2)-2} = e^{-2}[2+(x-2)]\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-2)^n$$
$$= e^{-2}\left[\sum_{n=0}^{\infty}\frac{2(-1)^n}{n!}(x-2)^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-2)^{n+1}\right]$$
$$= e^{-2}\left[\sum_{n=0}^{\infty}\frac{2(-1)^n}{n!}(x-2)^n + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n-1)!}(x-2)^n\right]$$
$$= e^{-2}\left[2+\sum_{n=1}^{\infty}\frac{(-1)^n(2-n)}{n!}(x-2)^n\right].$$

But the coefficient of $(x-2)^n$ in the Taylor series is $f^{(n)}(2)/n!$, and therefore

$$\frac{f^{(n)}(2)}{n!}=\frac{(-1)^n(2-n)e^{-2}}{n!} \quad \Longrightarrow \quad f^{(n)}(2)=\frac{(-1)^n(2-n)n!}{e^2n!}=\frac{(n-2)(-1)^{n+1}}{e^2}.$$

49. Since the Maclaurin series for $x^2\sin 2x$

$$x^2\sin 2x=x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n2^{2n+1}}{(2n+1)!}x^{2n+3}$$

contains only odd powers of x, the even derivatives of $x^2\sin 2x$ must all be zero.

50. Since the Maclaurin series for e^{-x^2}, namely, $\quad e^{-x^2}=\sum_{n=0}^{\infty}\frac{1}{n!}(-x^2)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}\quad$ contains only even powers of x, the odd derivatives of e^{-x^2} must all be zero.

51. Using the definition of $J_m(x)$ as the Maclaurin series in Exercise 38 of Section 10.4, we may write

$$\begin{aligned}
2m\,J_m(x)-x\,J_{m-1}(x)&=2m\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m}n!(n+m)!}x^{2n+m}-x\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m-1}n!(n+m-1)!}x^{2n+m-1}\\
&=\sum_{n=0}^{\infty}\frac{m(-1)^n}{2^{2n+m-1}n!(n+m)!}x^{2n+m}-\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m-1}n!(n+m-1)!}x^{2n+m}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n(m-n-m)}{2^{2n+m-1}n!\,(n+m)!}x^{2n+m}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{2n+m-1}(n-1)!\,(n+m)!}x^{2n+m}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m+1}n!\,(n+m+1)!}x^{2n+m+2}=x\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m+1}n!\,(n+m+1)!}x^{2n+m+1}\\
&=x\,J_{m+1}(x).
\end{aligned}$$

52. Using the definition of $J_m(x)$ as the Maclaurin series in Exercise 38 of Section 10.4, we may write

$$J_{m-1}(x)-J_{m+1}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m-1}n!(n+m-1)!}x^{2n+m-1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+m+1}n!(n+m+1)!}x^{2n+m+1}.$$

We lower n by 1 in the second summation, and separate out the first term in the first summation,

$$\begin{aligned}
J_{m-1}(x)-J_{m+1}(x)&=\frac{1}{2^{m-1}(m-1)!}x^{m-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+m-1}n!(n+m-1)!}x^{2n+m-1}\\
&\quad+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m)!}x^{2n+m-1}\\
&=\frac{1}{2^{m-1}(m-1)!}x^{m-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m-1)!}\left(\frac{1}{n}+\frac{1}{n+m}\right)x^{2n+m-1}\\
&=\frac{1}{2^{m-1}(m-1)!}x^{m-1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n+m-1}(n-1)!(n+m-1)!}\left[\frac{2n+m}{n(n+m)}\right]x^{2n+m-1}\\
&=\frac{m}{2^{m-1}m!}x^{m-1}+\sum_{n=1}^{\infty}\frac{(2n+m)(-1)^n}{2^{2n+m-1}n!(n+m)!}x^{2n+m-1}\\
&=\sum_{n=0}^{\infty}\frac{(2n+m)(-1)^n}{2^{2n+m-1}n!(n+m)!}x^{2n+m-1}.
\end{aligned}$$

Term-by-term differentiation of the series for $J_m(x)$ gives $J_m'(x)=\sum_{n=0}^{\infty}\frac{(2n+m)(-1)^n}{2^{2n+m}n!(n+m)!}x^{2n+m-1}$.
Hence, $J_{m-1}(x)-J_{m+1}(x)=2J_m'(x)$.

53. With the binomial expansion 10.33b,

$$\begin{aligned}\frac{1}{\sqrt{1-2\mu x+x^2}} &= 1-\frac{1}{2}(x^2-2\mu x)+\frac{(-1/2)(-3/2)}{2!}(x^2-2\mu x)^2+\frac{(-1/2)(-3/2)(-5/2)}{3!}(x^2-2\mu x)^3+\cdots\\ &= 1+\frac{1}{2}(2\mu x-x^2)+\frac{3}{8}(4\mu^2x^2-4\mu x^3+x^4)+\frac{5}{16}(8\mu^3x^3-12\mu^2x^4+6\mu x^5-x^6)+\cdots\\ &= 1+(\mu)x+\left(-\frac{1}{2}+\frac{3\mu^2}{2}\right)x^2+\left(-\frac{3\mu}{2}+\frac{5\mu^3}{2}\right)x^3+\cdots.\end{aligned}$$

Thus, $P_0(\mu)=1$, $P_1(\mu)=\mu$, $P_2(\mu)=(3\mu^2-1)/2$, and $P_3(\mu)=(5\mu^3-3\mu)/2$.

54. (a) If we substitute the Maclaurin series for e^x into $x=(e^x-1)\left(1+B_1x+\frac{B_2}{2!}x^2+\cdots\right)$,

$$\begin{aligned}x &= \left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-1\right]\left(1+B_1x+\frac{B_2x^2}{2!}+\cdots\right)\\ &= \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots\right)\left(1+B_1x+\frac{B_2x^2}{2!}+\frac{B_3x^3}{3!}+\frac{B_4x^4}{4!}+\frac{B_5x^5}{5!}+\cdots\right).\end{aligned}$$

When we multiply the series on the right and equate coefficients of powers of x left and right:

x: $1=1$

x^2: $0=\frac{1}{2!}+B_1 \implies B_1=-\frac{1}{2}$

x^3: $0=\frac{1}{3!}+\frac{B_1}{2!}+\frac{B_2}{2!} \implies B_2=\frac{1}{6}$

x^4: $0=\frac{1}{4!}+\frac{B_1}{3!}+\frac{B_2}{(2!)^2}+\frac{B_3}{3!} \implies B_3=0$

x^5: $0=\frac{1}{5!}+\frac{B_1}{4!}+\frac{B_2}{2!3!}+\frac{B_3}{2!3!}+\frac{B_4}{4!} \implies B_4=-\frac{1}{30}$

x^6: $0=\frac{1}{6!}+\frac{B_1}{5!}+\frac{B_2}{2!4!}+\frac{B_3}{(3!)^2}+\frac{B_4}{2!4!}+\frac{B_5}{5!} \implies B_5=0$

(b) Suppose we set
$$\begin{aligned}f(x)&=\frac{x}{e^x-1}-1-B_1x=\frac{x}{e^x-1}-1+\frac{x}{2}=\frac{2x-2(e^x-1)+x(e^x-1)}{2(e^x-1)}\\ &=\frac{xe^x-2e^x+x+2}{2(e^x-1)}=\frac{B_2}{2!}x^2+\frac{B_3}{3!}x^3+\cdots.\end{aligned}$$

Since

$$\begin{aligned}f(-x)&=\frac{-x}{e^{-x}-1}-1-\frac{x}{2}=\frac{xe^x}{e^x-1}-1-\frac{x}{2}\\ &=\frac{2xe^x-2(e^x-1)-x(e^x-1)}{2(e^x-1)}=\frac{xe^x-2e^x+x+2}{2(e^x-1)}=f(x),\end{aligned}$$

$f(x)$ is an even function. But the Maclaurin series for $f(x)$ can represent an even function only if all odd powers are absent. In other words, $0=B_3=B_5=\cdots$.

55. $e^{x(t-1/t)/2}=e^{xt/2}\,e^{-x/(2t)}=\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{xt}{2}\right)^n\right]\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x}{2t}\right)^n\right]=\left[\sum_{n=0}^{\infty}\frac{(x/2)^n}{n!}t^n\right]\left[\sum_{n=0}^{\infty}\frac{(-x/2)^n}{n!}\left(\frac{1}{t}\right)^n\right].$

When these series are multiplied together, the coefficient of t^n is

$$\begin{aligned}\frac{(x/2)^n}{n!}+\frac{(x/2)^{n+1}}{(n+1)!}\frac{(-x/2)}{1!}+\frac{(x/2)^{n+2}}{(n+2)!}\frac{(-x/2)^2}{2!}+\cdots &= \sum_{m=0}^{\infty}\frac{(x/2)^{n+m}}{(n+m)!}\frac{(-x/2)^m}{m!}\\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m+n}m!\,(n+m)!}x^{2m+n}=J_n(x).\end{aligned}$$

EXERCISES 10.6

1. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty} nx^{n-1}$, then term-by-term integration gives

$$\int S(x)\,dx + C = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x},$$

since the series is geometric. Differentiation now gives $S(x) = \frac{(1-x)(1)-x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$.

2. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\frac{n(n-1)}{(n+1)n}\right| = 1$. If we set $S(x) = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$, then term-by-term integration gives $\int S(x)\,dx + C = \sum_{n=2}^{\infty} nx^{n-1}$. A second integration leads to

$$\int\left[\int S(x)\,dx + C\right]dx + D = \sum_{n=2}^{\infty} x^n = \frac{x^2}{1-x},$$

since the series is geometric. Differentiation now gives

$$\int S(x)\,dx + C = \frac{(1-x)(2x)-x^2(-1)}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2}.$$

A second differentiation provides $S(x)$,

$$S(x) = \frac{(1-x)^2(2-2x)-(2x-x^2)2(1-x)(-1)}{(1-x)^4} = \frac{2}{(1-x)^3}.$$

3. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty}(n+1)x^{n-1}$, then $x\,S(x) = \sum_{n=1}^{\infty}(n+1)x^n$. Term-by-term integration gives

$$\int x\,S(x)\,dx + C = \sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x},$$

since the series is geometric. Differentiation now gives

$$x\,S(x) = \frac{(1-x)(2x)-x^2(-1)}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2} \quad\Longrightarrow\quad S(x) = \frac{2-x}{(1-x)^2}.$$

4. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right| = 1$. If we set $S(x) = \sum_{n=1}^{\infty} n^2x^{n-1}$, then term-by-term integration gives $\int S(x)\,dx + C = \sum_{n=1}^{\infty} nx^n$. When $x \neq 0$, we can divide by x, $\frac{1}{x}\int S(x)\,dx + \frac{C}{x} = \sum_{n=1}^{\infty} nx^{n-1}$. Integration now gives,

$$\int\left[\frac{1}{x}\int S(x)\,dx\right]dx + C\ln|x| + D = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}.$$

If we now differentiate, $\dfrac{1}{x}\displaystyle\int S(x)\,dx + \dfrac{C}{x} = \dfrac{(1-x)(1)-x(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$. Multiplication by x and a further differentiation gives

$$S(x) = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{(1-x)^2(1)-x(2)(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3}.$$

Since the sum of the series at $x=0$ is 1, and this is $S(0)$, the formula $S(x)=(x+1)/(1-x)^3$ can be used for all x in $|x|<1$.

5. If we divide the series into two parts, $\displaystyle\sum_{n=1}^{\infty}(n^2+2n)x^n = \sum_{n=1}^{\infty}n^2x^n + 2\sum_{n=1}^{\infty}nx^n$, the first series is x times that in Exercise 4, and the second is x times that in Exercise 1. Hence,

$$\sum_{n=1}^{\infty}(n^2+2n)x^n = \frac{x(x+1)}{(1-x)^3} + \frac{2x}{(1-x)^2} = \frac{3x-x^2}{(1-x)^3}.$$

6. The radius of convergence of the series is $R = \displaystyle\lim_{n\to\infty}\left|\frac{1/(n+1)}{1/(n+2)}\right| = 1$. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}x^n$, then $x\,S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$. Term-by-term differentiation gives $\dfrac{d}{dx}[x\,S(x)] = \displaystyle\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$, since the series is geometric. We now integrate,

$$x\,S(x) = \int\frac{1}{1-x}dx = -\ln(1-x) + C.$$

Substitution of $x=0$ gives $C=0$, and therefore $S(x) = -\dfrac{1}{x}\ln(1-x)$. This is valid for $-1<x<1$, but not at $x=0$. It is interesting to note, however, that the limit of $S(x)$ as x approaches zero is 1 and this is the sum of the series at $x=0$.

7. If we set $y=x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = \pm\sqrt{y}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^n/(2n+1)}{(-1)^{n+1}/(2n+3)}\right| = 1$. The radius of convergence of the original series is therefore $R_x=1$. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$, then term-by-term differentiation gives $S'(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^nx^{2n} = \frac{1}{1+x^2}$, since the series is geometric. Integration now gives $S(x) = \mathrm{Tan}^{-1}x + C$. Since $S(0)=0$, it follows that $C=0$, and $S(x)=\mathrm{Tan}^{-1}x$.

8. If we set $y=x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^n/n}{(-1)^{n+1}/(n+1)}\right| = 1$. The radius of convergence of the original series is therefore $R_x=1$. If we set $S(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x^{2n}$, then term-by-term differentiation gives

$$S'(x) = \sum_{n=1}^{\infty}2(-1)^nx^{2n-1} = \frac{-2x}{1+x^2},$$

since the series is geometric. Integration now leads to $S(x) = -\ln(1+x^2)+C$. Since $S(0)=0$, it follows that $C=0$, and $S(x) = -\ln(1+x^2)$.

9. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=2}^{\infty} n3^n x^{2n} = \sum_{n=2}^{\infty} n3^n y^n$. The radius of convergence of this series is $R_y = \lim_{n\to\infty}\left|\dfrac{n3^n}{(n+1)3^{n+1}}\right| = 1/3$. The radius of convergence of the original series is therefore $R_x = 1/\sqrt{3}$. If we set $S(x) = \displaystyle\sum_{n=2}^{\infty} n3^n x^{2n}$, then $\dfrac{S(x)}{x} = \displaystyle\sum_{n=2}^{\infty} n3^n x^{2n-1}$, provided $x \neq 0$. Term-by-term integration of this equation gives

$$\int \frac{S(x)}{x}\,dx = \sum_{n=2}^{\infty} \frac{3^n}{2} x^{2n} = \frac{9x^4/2}{1-3x^2},$$

since the series is geometric. Differentiation now gives

$$\frac{S(x)}{x} = \frac{9}{2}\left[\frac{(1-3x^2)(4x^3) - x^4(-6x)}{(1-3x^2)^2}\right] = \frac{9(4x^3-6x^5)}{2(1-3x^2)^2} \quad\Longrightarrow\quad S(x) = \frac{9x^4(2-3x^2)}{(1-3x^2)^2}.$$

Since the sum of the series at $x = 0$ is 0, and this is $S(0)$, the formula for $S(x)$ can be used for all x in $|x| < 1/\sqrt{3}$.

10. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{(n+1)/(n+2)}{(n+2)/(n+3)}\right| = 1$. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)x^n$, and integrate, $\displaystyle\int S(x)\,dx = \sum_{n=0}^{\infty}\frac{1}{n+2}x^{n+1} + C$. Multiplication by x gives $x\displaystyle\int S(x)\,dx = \sum_{n=0}^{\infty}\frac{1}{n+2}x^{n+2} + Cx$. Differentiation now gives

$$\frac{d}{dx}\left[x\int S(x)\,dx\right] = \sum_{n=0}^{\infty} x^{n+1} + C = \frac{x}{1-x} + C,$$

since the series is geometric. Integration now yields

$$x\int S(x)\,dx = \int \frac{x}{1-x}dx + Cx + D = -x - \ln|1-x| + Cx + D.$$

If we set $x = 0$ in this equation we find that $D = 0$. When we drop absolute values and divide by x,

$$\int S(x)\,dx = -1 - \frac{1}{x}\ln(1-x) + C, \quad x \neq 0.$$

When we differentiate this equation, we obtain $S(x) = \dfrac{1}{x^2}\ln(1-x) + \dfrac{1}{x(1-x)}$. This formula can only be used for values of x in the interval $-1 < x < 1$, but not $x = 0$. The sum at $x = 0$ is $1/2$.

11. The radius of convergence of the series is $R = \lim_{n\to\infty}\left|\dfrac{(n+1)/n!}{(n+2)/(n+1)!}\right| = \infty$. If we set $S(x) = \displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{n!}\right)x^n$, and integrate,

$$\int S(x)\,dx = \sum_{n=1}^{\infty}\frac{x^{n+1}}{n!} + C = x\sum_{n=1}^{\infty}\frac{x^n}{n!} + C = x(e^x - 1) + C.$$

Differentiation now gives

$$S(x) = (e^x - 1) + x(e^x) = (x+1)e^x - 1.$$

12. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)!}x^{2n+1} = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = x\cos x$

13. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!} x^{2n} = \pm\sqrt{y} \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!} y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty} \left| \frac{(-1)^n(n+2)/(2n)!}{(-1)^{n+1}(n+3)/(2n+2)!} \right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!} x^{2n}$, and multiply by x^3, $x^3 S(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!} x^{2n+3}$. Integration now gives

$$\int x^3 S(x)\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{(2n)!(2n+4)} x^{2n+4} + C = \frac{x^4}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + C = \frac{x^4}{2} \cos x + C.$$

We now differentiate to get

$$x^3 S(x) = 2x^3 \cos x - \frac{x^4}{2} \sin x \qquad \Longrightarrow \qquad S(x) = 2\cos x - \frac{x}{2} \sin x.$$

14. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} x^{2n} = \sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty} \left| \frac{(2n+3)2^n/n!}{(2n+5)2^{n+1}/(n+1)!} \right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} x^{2n}$, and multiply by x^2, $x^2 S(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(2n+3)2^n}{n!} x^{2n+2}$. Integration now gives

$$\int x^2 S(x)\, dx = \sum_{n=1}^{\infty} \frac{2^n}{n!} x^{2n+3} + C = x^3 \sum_{n=1}^{\infty} \frac{1}{n!} (2x^2)^n + C = x^3 (e^{2x^2} - 1) + C.$$

We now differentiate to get

$$x^2 S(x) = 3x^2(e^{2x^2} - 1) + x^3(4xe^{2x^2}) \qquad \Longrightarrow \qquad S(x) = (4x^2 + 3)e^{2x^2} - 3.$$

15. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!} x^{2n+1} = \pm\sqrt{y} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!} y^n$. The radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(2n-1)/(2n)!}{(-1)^{n+2}(2n+1)/(2n+2)!} \right| = \infty$. The radius of convergence of the original series is therefore $R_x = \infty$ also. If we set $S(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!} x^{2n+1}$, and divide by x^3, $\displaystyle\frac{S(x)}{x^3} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n)!} x^{2n-2}$, $x \neq 0$. Integration now gives

$$\int \frac{S(x)}{x^3}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n-1} + C = -\frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + C = -\frac{1}{x} \cos x + C.$$

We now differentiate to get

$$\frac{S(x)}{x^3} = \frac{1}{x^2} \cos x + \frac{1}{x} \sin x \qquad \Longrightarrow \qquad S(x) = x\cos x + x^2 \sin x.$$

This gives the sum of the series at $x = 0$ also.

EXERCISES 10.7

1. Taylor's remainder formula for e^x and $c=0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $0 < z_3 < x$. Since $x \le 0.01$, we can say that

$$R_3 < e^x\frac{x^4}{24} \le e^{0.01}\frac{(0.01)^4}{24} = 4.2 \times 10^{-10}.$$

2. Taylor's remainder formula for e^x and $c=0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $0 < z_3 < x$. Since $x < 0.01$, we can say that

$$R_3 < e^x\frac{x^4}{24} < e^{0.01}\frac{(0.01)^4}{24} = 4.2 \times 10^{-10}.$$

3. Taylor's remainder formula for e^x and $c=0$ gives $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + R_3$, where $R_3 = \dfrac{d^4}{dx^4}e^x{}_{|x=z_3}\dfrac{x^4}{4!} = e^{z_3}\dfrac{x^4}{24}$, and $x < z_3 < 0$. Since $-0.01 \le x < 0$, we can say that

$$|R_3| < e^0\frac{|x|^4}{24} \le \frac{|-0.01|^4}{24} = 4.2 \times 10^{-10}.$$

4. According to Exercise 2, a maximum error on $0 \le x \le 0.01$ is 4.2×10^{-10}. For $-0.01 \le x < 0$, $R_3 = e^{z_3}\dfrac{x^4}{24}$ where $x < z_3 < 0$. Since $x \ge -0.01$, it follows that

$$|R_3| < e^0\frac{|x|^4}{24} \le \frac{|-0.01|^4}{24} < 4.2 \times 10^{-10}.$$

5. Taylor's remainder formula for $\sin x$ and $c=0$ gives $\sin x = x - \dfrac{x^3}{3!} + R_4$, where $R_4 = \dfrac{d^5}{dx^5}\sin x_{|x=z_4}\dfrac{x^5}{5!} = (\cos z_4)\dfrac{x^5}{120}$, and $0 < z_4 < x$. Since $0 \le x \le 1$, we can say that

$$R_4 < (1)\frac{x^5}{120} \le \frac{(1)^5}{120} = \frac{1}{120}.$$

6. Taylor's remainder formula for $\cos x$ and $c=0$ gives $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + R_5$, where $R_5 = \dfrac{d^6}{dx^6}\cos x_{|x=z_5}\dfrac{x^6}{6!} = -(\cos z_5)\dfrac{x^6}{6!}$, and z_5 is between 0 and x. Since $|x| \le 0.1$, we can say that

$$|R_5| < (1)\frac{|x|^6}{6!} \le \frac{(0.1)^6}{6!} < 1.4 \times 10^{-9}.$$

7. The first four derivatives of $f(x) = \ln(1-x)$ are $f'(x) = -1/(1-x)$, $f''(x) = -1/(1-x)^2$, $f'''(x) = -2/(1-x)^3$, and $f''''(x) = -6/(1-x)^4$. Taylor's remainder formula for $\ln(1-x)$ and $c=0$ gives $\ln(1-x) = -x - x^2/2 - x^3/3 + R_3(x)$, where $R_3(x) = f''''(z_3)\dfrac{x^4}{4!} = \dfrac{-x^4}{4(1-z_3)^4}$, and $0 < z_3 < x$. Since $0 \le x \le 0.01$, we can say that

$$|R_3| < \frac{x^4}{4(1-x)^4} \le \frac{(0.01)^4}{4(1-0.01)^4} < 2.7 \times 10^{-9}.$$

8. The first four derivatives of $f(x) = 1/(1-x)^3$ are $f'(x) = 3/(1-x)^4$, $f''(x) = 12/(1-x)^5$, $f'''(x) = 60/(1-x)^6$, and $f''''(x) = 360/(1-x)^7$. Taylor's remainder formula for $1/(1-x)^3$ and $c = 0$ gives $\dfrac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + R_3(x)$, where $R_3(x) = f''''(z_3)\dfrac{x^4}{4!} = \dfrac{15x^4}{(1-z_3)^7}$, and z_3 is between 0 and x. Since $|x| < 0.2$, we can say that

$$|R_3| < \frac{15|x|^4}{(1-0.2)^7} < \frac{15(0.2)^4}{(1-0.2)^7} < 0.115.$$

9. Taylor's remainder formula for $\sin 3x$ and $c = 0$ gives $\sin 3x = 3x - \dfrac{9x^3}{2} + \dfrac{81x^5}{40} + R_6$, where $R_6 = \dfrac{d^7}{dx^7}\sin 3x_{|x=z_6}\dfrac{x^7}{7!} = -3^7(\cos 3z_6)\dfrac{x^7}{7!}$, and z_6 is between 0 and x. Since $|x| < \pi/100$, we can say that

$$|R_6| < 3^7(1)\frac{|x|^7}{7!} < 3^7\frac{(\pi/100)^7}{7!} < 1.4 \times 10^{-11}.$$

10. The first five derivatives of $f(x) = \ln x$ are $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 2/x^3$, $f''''(x) = -6/x^4$, and $f'''''(x) = 24/x^5$. Taylor's remainder formula with $c = 1$ gives

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + R_4,$$

where $R_4 = f^{(5)}(z_4)\dfrac{(x-1)^5}{5!} = \dfrac{24}{z_4^5}\dfrac{(x-1)^5}{5!} = \dfrac{(x-1)^5}{5z_4^5}$ and z_4 is between 1 and x. Since $1/2 \le x \le 3/2$, we can say that

$$|R_4| < \frac{|x-1|^5}{5(1/2)^5} \le \frac{(1/2)^5}{5(1/2)^5} = 0.2.$$

11. Taylor's remainder formula for $\sin x$ gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{d^n}{dx^n}(\sin x)_{|x=0}\frac{x^n}{n!} + R_n(0,x)$$

where $R_n(0,x) = \dfrac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\dfrac{x^{n+1}}{(n+1)!}$ and z_n is between 0 and x. Therefore

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + \frac{1}{x}R_n(0,x).$$

When we take definite integrals,

$$\begin{aligned}\int_0^1 \frac{\sin x}{x}dx &= \int_0^1\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + \frac{1}{x}R_n(0,x)\right]dx\\ &= \left\{x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \cdots\right\}_0^1 + \int_0^1\frac{1}{x}R_n(0,x)\,dx,\\ &= 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \cdots + \int_0^1\frac{1}{x}R_n(0,x)\,dx.\end{aligned}$$

Now, $\left|\displaystyle\int_0^1\frac{1}{x}R_n(0,x)\,dx\right| \le \displaystyle\int_0^1\left|\frac{1}{x}\frac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\frac{x^{n+1}}{(n+1)!}\right|dx$. Since $\left|\dfrac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\right| \le 1$, it follows that

$$\left|\int_0^1\frac{1}{x}R_n(0,x)\,dx\right| \le \int_0^1\frac{x^n}{(n+1)!}dx = \left\{\frac{x^{n+1}}{(n+1)(n+1)!}\right\}_0^1 = \frac{1}{(n+1)(n+1)!}.$$

When $n = 6$, this is less than 0.000 029. Hence, if we approximate the integral with the first three terms, namely, $1 - \dfrac{1}{3\cdot 3!} + \dfrac{1}{5\cdot 5!} = 0.946\,111$, then we can say that

$$0.946\,111 - 0.000\,029 < \int_0^1 \frac{\sin x}{x} dx < 0.946\,111 + 0.000\,029;$$

that is, $0.946\,082 < \int_0^1 \frac{\sin x}{x} dx < 0.946\,140$. To three decimals, then, the value of the integral is 0.946.

12. If we set $u = x^2$ and $du = 2x\,dx$, then $\int_0^{1/2} \cos(x^2)\,dx = \frac{1}{2}\int_0^{1/4} \frac{\cos u}{\sqrt{u}} du$. Taylor's remainder formula for $\cos u$ gives

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + \frac{d^n(\cos u)}{du^n}_{|u=0} \frac{u^n}{n!} + R_n(0, u),$$

where $R_n(0, u) = \frac{d^{n+1}(\cos u)}{du^{n+1}}_{|u=z_n} \frac{u^{n+1}}{(n+1)!}$. Consequently,

$$\begin{aligned}
\int_0^{1/2} \cos(x^2)\,dx &= \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}}\left[1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + R_n(0, u)\right] du \\
&= \frac{1}{2}\int_0^{1/4} \left[\frac{1}{\sqrt{u}} - \frac{u^{3/2}}{2!} + \frac{u^{7/2}}{4!} - \cdots + \frac{1}{\sqrt{u}} R_n(0, u)\right] du \\
&= \frac{1}{2}\left\{2\sqrt{u} - \frac{2u^{5/2}}{5 \cdot 2!} + \frac{2u^{9/2}}{9 \cdot 4!} - \cdots\right\}_0^{1/4} + \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} R_n(0, u)\,du \\
&= \frac{1}{2} - \frac{1}{5 \cdot 2^5 \cdot 2!} + \frac{1}{9 \cdot 2^9 \cdot 4!} - \cdots + \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} R_n(0, u)\,du.
\end{aligned}$$

Now,

$$\begin{aligned}
\left|\frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} R_n(0, u)\,du\right| &\le \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} |R_n(0, u)|\,du \le \frac{1}{2}\int_0^{1/4} \frac{1}{\sqrt{u}} \frac{u^{n+1}}{(n+1)!} du \\
&= \frac{1}{2}\int_0^{1/4} \frac{u^{n+1/2}}{(n+1)!} du = \frac{1}{2(n+1)!}\left\{\frac{u^{n+3/2}}{n+3/2}\right\}_0^{1/4} = \frac{1}{(2n+3)(n+1)!4^{n+3/2}}.
\end{aligned}$$

When $n = 2$, this is less than 1.9×10^{-4}. Hence, if we approximate the integral with the first two terms, namely, $\frac{1}{2} - \frac{1}{5 \cdot 2^5 \cdot 2!} = \frac{159}{320}$, then we can say that

$$\frac{159}{320} - 0.000\,19 < \int_0^{1/2} \cos(x^2)\,dx < \frac{159}{320} + 0.000\,19,$$

that is, $0.496\,685 < \int_0^{1/2} \cos(x^2)\,dx < 0.497\,065$. To three decimals, the value of the integral is 0.497.

13. Taylor's remainder formula for $\sin x$ gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{d^n}{dx^n}(\sin x)_{|x=0} \frac{x^n}{n!} + R_n(0, x)$$

where $R_n(0, x) = \frac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n} \frac{x^{n+1}}{(n+1)!}$ and z_n is between 0 and x. Therefore

$$x^{11} \sin x = x^{12} - \frac{x^{14}}{3!} + \frac{x^{16}}{5!} - \cdots + x^{11} R_n(0, x).$$

When we take definite integrals,

$$\begin{aligned}\int_{-1}^{1} x^{11}\sin x\,dx &= \int_{-1}^{1}\left[x^{12} - \frac{x^{14}}{3!} + \frac{x^{16}}{5!} - \cdots + x^{11}R_n(0,x)\right]dx\\ &= \left\{\frac{x^{13}}{13} - \frac{x^{15}}{15\cdot 3!} + \frac{x^{17}}{17\cdot 5!} - \cdots\right\}_{-1}^{1} + \int_{-1}^{1} x^{11}R_n(0,x)\,dx,\\ &= \frac{2}{13} - \frac{2}{15\cdot 3!} + \frac{2}{17\cdot 5!} - \cdots + \int_{-1}^{1} x^{11}R_n(0,x)\,dx.\end{aligned}$$

Now,

$$\begin{aligned}\left|\int_{-1}^{1} x^{11}R_n(0,x)\,dx\right| &\le \int_{-1}^{1}\left|x^{11}\frac{d^{n+1}(\sin x)}{dx^{n+1}}_{|x=z_n}\frac{x^{n+1}}{(n+1)!}\right|dx \le \int_{-1}^{1}\frac{|x^{n+12}|}{(n+1)!}dx\\ &= \frac{2}{(n+1)!}\int_{0}^{1} x^{n+12}\,dx = \frac{2}{(n+1)!}\left\{\frac{x^{n+13}}{n+13}\right\}_{0}^{1} = \frac{2}{(n+13)(n+1)!}.\end{aligned}$$

When $n = 6$, this is less than 2.1×10^{-5}. Hence, if we approximate the integral with the first three terms, namely, $\dfrac{2}{13} - \dfrac{2}{15\cdot 3!} + \dfrac{2}{17\cdot 5!} = 0.132\,604$, then we can say that

$$0.132\,604 - 0.000\,021 < \int_{-1}^{1} x^{11}\sin x\,dx < 0.132\,604 + 0.000\,021,$$

that is, $0.132\,583 < \displaystyle\int_{-1}^{1} x^{11}\sin x\,dx < 0.132\,625$. To three decimals, the value of the integral is 0.133.

14. If we set $w = x^2$ and $dw = 2x\,dx$, then $\displaystyle\int_0^{0.3} e^{-x^2}\,dx = \frac{1}{2}\int_0^{0.09}\frac{e^{-w}}{\sqrt{w}}dw$. Taylor's remainder formula applied to e^{-w} gives

$$e^{-w} = 1 - w + \frac{w^2}{2!} - \frac{w^3}{3!} + \cdots + \frac{(-1)^n w^n}{n!} + R_n(0,w)$$

where $R_n(0,w) = \dfrac{d^{n+1}}{dw^{n+1}}(e^{-w})_{|w=w_n}\dfrac{w^{n+1}}{(n+1)!} = \dfrac{(-1)^{n+1}e^{-w_n}w^{n+1}}{(n+1)!}$. Consequently,

$$\begin{aligned}\int_0^{0.3} e^{-x^2}dx &= \frac{1}{2}\int_0^{0.09}\frac{1}{\sqrt{w}}\left[1 - w + \frac{w^2}{2!} - \frac{w^3}{3!} + \cdots + \frac{(-1)^n w^n}{n!} + R_n(0,w)\right]dw\\ &= \frac{1}{2}\int_0^{0.09}\left[\frac{1}{\sqrt{w}} - \sqrt{w} + \frac{w^{3/2}}{2!} - \frac{w^{5/2}}{3!} + \cdots + \frac{(-1)^n w^{n-1/2}}{n!} + \frac{1}{\sqrt{w}}R_n(0,w)\right]dw\\ &= \frac{1}{2}\left\{2\sqrt{w} - \frac{2w^{3/2}}{3} + \frac{2w^{5/2}}{5\cdot 2!} - \frac{2w^{7/2}}{7\cdot 3!} + \cdots + \frac{2(-1)^n w^{n+1/2}}{(2n+1)n!}\right\}_0^{0.09} + \frac{1}{2}\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\\ &= \sqrt{0.09} - \frac{(0.09)^{3/2}}{3} + \frac{(0.09)^{5/2}}{5\cdot 2!} - \frac{(0.09)^{7/2}}{7\cdot 3!} + \cdots + \frac{(-1)^n(0.09)^{n+1/2}}{(2n+1)n!} + \frac{1}{2}\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw.\end{aligned}$$

Now,

$$\frac{1}{2}\left|\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\right| \le \frac{1}{2}\int_0^{0.09}\frac{1}{\sqrt{w}}\left|\frac{(-1)^{n+1}e^{-w_n}w^{n+1}}{(n+1)!}\right|dw = \frac{1}{2(n+1)!}\int_0^{0.09} e^{-w_n}w^{n+1/2}dw.$$

Since $0 < w_n < w < 0.09$, we can say $e^{-w_n} \le 1$. Thus,

$$\frac{1}{2}\left|\int_0^{0.09}\frac{R_n(0,w)}{\sqrt{w}}dw\right| \le \frac{1}{2(n+1)!}\left\{\frac{2w^{n+3/2}}{2n+3}\right\}_0^{0.09} = \frac{(0.09)^{n+3/2}}{(2n+3)(n+1)!}.$$

When $n=2$, this is less than 3.0×10^{-6}. Hence, if we approximate the integral with the first three terms, namely, $\sqrt{0.09} - \dfrac{(0.09)^{3/2}}{3} + \dfrac{(0.09)^{5/2}}{5 \cdot 2!} = 0.291\,243$, then we can say that

$$0.291\,243 - 0.000\,003 < \int_0^{0.3} e^{-x^2}\,dx < 0.291\,243 + 0.000\,003,$$

that is, $0.291\,240 < \int_0^{0.3} e^{-x^2}\,dx < 0.291\,246$. To three decimals, the value of the integral is 0.291.

15. Using the result of Example 10.24, $\lim_{x\to 0} \dfrac{\tan x}{x} = \lim_{x\to 0}\left(1 + \dfrac{x^2}{3} + \dfrac{2x^4}{15} + \cdots\right) = 1.$

16. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2} = \lim_{x\to 0} \dfrac{1}{x^2}\left[1 - \left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots\right)\right] = \lim_{x\to 0}\left(\dfrac{1}{2!} - \dfrac{x^2}{4!} + \cdots\right) = \dfrac{1}{2}$

17.
$$\lim_{x\to 0} \frac{(1-\cos x)^2}{3x^4} = \lim_{x\to 0} \frac{1}{3x^4}\left[1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\right]^2 = \lim_{x\to 0} \frac{1}{3x^4}\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)^2$$
$$= \lim_{x\to 0} \frac{1}{3x^4}\left(\frac{x^4}{4} - \frac{x^6}{24} + \cdots\right) = \lim_{x\to 0}\left(\frac{1}{12} - \frac{x^2}{72} + \cdots\right) = \frac{1}{12}.$$

18. $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \dfrac{1}{x}\left[\left(1 + \dfrac{x}{2} - \dfrac{(1/2)(-1/2)}{2!}x^2 + \cdots\right) - 1\right] = \lim_{x\to 0}\left[\dfrac{1}{2} + \dfrac{x}{8} + \cdots\right] = \dfrac{1}{2}$

19. $\lim_{x\to\infty} x\sin\left(\dfrac{1}{x}\right) = \lim_{x\to\infty} x\left(\dfrac{1}{x} - \dfrac{1}{3!x^3} + \dfrac{1}{5!x^5} - \cdots\right) = \lim_{x\to\infty}\left(1 - \dfrac{1}{3!x^2} + \dfrac{1}{5!x^4} - \cdots\right) = 1$

20.
$$\frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1 + \left[1 - 2x + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \cdots\right]}{1 - \left[1 - 2x + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \cdots\right]} = \frac{2 - 2x + 2x^2 - \dfrac{4x^3}{3} + \cdots}{2x - 2x^2 + \dfrac{4x^3}{3} + \cdots}$$

Long division gives $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{1}{x} + \dfrac{x}{3} + \cdots$.

Thus, $\lim_{x\to 0}\left(\dfrac{e^x + e^{-x}}{e^x - e^{-x}} - \dfrac{1}{x}\right) = \lim_{x\to 0}\left[\left(\dfrac{1}{x} + \dfrac{x}{3} + \cdots\right) - \dfrac{1}{x}\right] = 0.$

21. Taylor's remainder formula for $\sin(x/3)$ gives

$$\sin(x/3) = \frac{x}{3} - \frac{x^3}{3^3 \cdot 3!} + \frac{x^5}{3^5 \cdot 5!} - \frac{x^7}{3^7 \cdot 7!} + \cdots + \frac{d^n}{dx^n}[\sin(x/3)]_{|x=0}\frac{x^n}{n!} + R_n(0,x)$$

where $R_n(0,x) = \dfrac{d^{n+1}[\sin(x/3)]}{dx^{n+1}}_{|x=z_n} \dfrac{x^{n+1}}{(n+1)!}$ and z_n is between 0 and x. Since the $(n+1)^{\text{th}}$ derivative of $\sin(x/3)$ is $\pm 3^{-n-1}\sin(x/3)$ or $\pm 3^{-n-1}\cos(x/3)$, and $|x| \le 4$, it follows that

$$|R_n(0,x)| \le \frac{|x|^{n+1}}{3^{n+1}(n+1)!} \le \frac{4^{n+1}}{3^{n+1}(n+1)!}.$$

The smallest integer for which this is less than 10^{-3} is $n=7$. Thus, the series should be truncated after $x^7/(3^7 \cdot 7!)$.

22. We set $u = x^3$ and consider the function $f(u) = 1/\sqrt{1+u}$ on the interval $0 < u < 1/8$. Since the n^{th} derivative of $f(u)$ is $f^{(n)}(u) = \dfrac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n(1+u)^{n+1/2}}$, Taylor's remainder formula gives

$$f(u) = 1 - \frac{u}{2} + \frac{3u^2}{8} - \frac{5u^3}{16} + \cdots + \frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n\,n!}u^n + R_n(0,u),$$

where $R_n(0,u) = \dfrac{f^{(n+1)}(z_n)}{(n+1)!}u^{n+1}$, and $0 < z_n < u$. Since $0 < u < 1/8$, we can say that

$$|R_n(0,u)| = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}|1+z_n|^{n+3/2}(n+1)!}|u|^{n+1} < \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}|1+0|^{n+3/2}(n+1)!}|u|^{n+1}$$
$$< \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}(n+1)!}\left(\frac{1}{8}\right)^{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{4n+4}(n+1)!}.$$

The smallest integer for which this is less than 10^{-4} is $n=3$. Thus, we should approximate $1/\sqrt{1+u}$ with $1-u/2+3u^2/8-5u^3/16$, or approximate $1/\sqrt{1+x^3}$ with

$$1-\frac{x^3}{2}+\frac{3x^6}{8}-\frac{5x^9}{16}.$$

23. Since the n^{th} derivative of $f(x)=\ln(1-x)$ is $f^{(n)}(x)=-(n-1)!/(1-x)^n$, Taylor's remainder formula gives

$$f(x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots-\frac{x^n}{n}+R_n(0,x),$$

where $R_n(0,x)=\dfrac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1}=\dfrac{-n!\,x^{n+1}}{(n+1)!(1-z_n)^{n+1}}=\dfrac{-x^{n+1}}{(n+1)(1-z_n)^{n+1}}$, and z_n is between 0 and x. Since $|x|<1/3$, we can say that

$$|R_n(0,x)| = \frac{|x|^{n+1}}{(n+1)|1-z_n|^{n+1}} < \frac{|x|^{n+1}}{(n+1)|1-1/3|^{n+1}} < \frac{(1/3)^{n+1}}{(n+1)(2/3)^{n+1}} = \frac{1}{(n+1)2^{n+1}}.$$

The smallest integer for which this is less than 10^{-2} is $n=4$. Thus, we should approximate $\ln(1-x)$ with $-x-x^2/2-x^3/3-x^4/4$.

24. Taylor's remainder formula for $\cos^2 x=(1+\cos 2x)/2$ gives

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\left[1+\left(1-\frac{2^2x^2}{2!}+\frac{2^4x^4}{4!}-\cdots+\frac{f^{(n)}(0)}{n!}x^n+R_n(0,x)\right)\right]$$
$$= 1-x^2+\frac{x^4}{3}-\cdots+\frac{f^{(n)}(0)}{2n!}x^n+\frac{1}{2}R_n(0,x),$$

where $R_n(0,x)=\dfrac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1}$ and z_n is between 0 and x. Since the $(n+1)^{\text{th}}$ derivative of $f(x)$ is $\pm 2^{n+1}\sin 2x$ or $\pm 2^{n+1}\cos 2x$, and $|x|\le 0.1$, it follows that

$$\frac{1}{2}|R_n(0,x)| \le \frac{2^{n+1}|x|^{n+1}}{2(n+1)!} < \frac{2^n}{(n+1)!10^{n+1}}.$$

The smallest integer for which this is less than 10^{-3} is $n=2$. Thus, the function should be approximated by $1-x^2$.

25. If we substitute $y=f(x)=\sum_{n=0}^{\infty}a_nx^n$ into the differential equation,

$$0 = -4+\sum_{n=0}^{\infty}3a_nx^n+\sum_{n=0}^{\infty}na_nx^{n-1} = -4+\sum_{n=0}^{\infty}3a_nx^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$
$$= (-4+3a_0+a_1)+\sum_{n=1}^{\infty}[3a_n+(n+1)a_{n+1}]x^n.$$

When we equate coefficients to zero:

$$-4+3a_0+a_1=0 \quad \text{and} \quad 3a_n+(n+1)a_{n+1}=0, \quad n\ge 1.$$

The first implies that $a_1=4-3a_0$ and the second gives the recursive formula $a_{n+1}=\dfrac{-3a_n}{(n+1)}$, $n\ge 1$. Iteration leads to

$$a_2 = -\frac{3a_1}{2} = \frac{-3(4-3a_0)}{2}, \quad a_3 = -\frac{3a_2}{3} = \frac{3^2(4-3a_0)}{3!}, \quad a_4 = -\frac{3a_3}{4} = -\frac{3^3(4-3a_0)}{4!}, \quad \cdots .$$

Thus,

$$\begin{aligned} y = f(x) &= a_0 + (4-3a_0)x - \frac{3(4-3a_0)}{2!}x^2 + \frac{3^2(4-3a_0)}{3!}x^3 + \cdots \\ &= a_0 + \frac{(4-3a_0)}{3}\left[3x - \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots\right] \\ &= a_0 + \frac{(4-3a_0)}{3}(1-e^{-3x}) = \frac{4}{3} + \frac{(3a_0-4)}{3}e^{-3x}. \end{aligned}$$

26. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty}(n+1)n a_{n+1}x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty}[(n+1)n a_{n+1} + n a_n]x^{n-1}. \end{aligned}$$

When we equate coefficients to zero:

$$(n+1)na_{n+1} + na_n = 0 \quad \Longrightarrow \quad a_{n+1} = -\frac{a_n}{n+1}, \quad n \geq 1.$$

This recursive definition implies that

$$a_2 = -\frac{a_1}{2}, \quad a_3 = -\frac{a_2}{3} = \frac{a_1}{3!}, \quad a_4 = -\frac{a_3}{4} = -\frac{a_1}{4!}, \quad \cdots .$$

Thus, $$y = f(x) = a_0 + a_1\left(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots\right) = a_0 + a_1\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}x^n$$
$$= a_0 - a_1(e^{-x} - 1) = (a_0 + a_1) - a_1 e^{-x}.$$

27. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$0 = -3x - \sum_{n=0}^{\infty} 4a_n x^n + \sum_{n=0}^{\infty} n a_n x^n = -4a_0 + (-3 - 4a_1 + a_1)x + \sum_{n=2}^{\infty}(n-4)a_n x^n.$$

When we equate coefficients to zero:

$$a_0 = 0, \qquad -3 - 3a_1 = 0, \qquad (n-4)a_n = 0, \quad n \geq 2.$$

These imply that $a_1 = -1$, a_4 is undetermined, and all other coefficients vanish. Thus, $y = f(x) = -x + a_4x^4$.

28. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= 4x\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} 4n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} 4n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=1}^{\infty} a_{n-1}x^{n-1} \\ &= (2a_1 + a_0) + \sum_{n=2}^{\infty}[(4n^2 - 4n + 2n)a_n + a_{n-1}]x^{n-1}. \end{aligned}$$

We now equate coefficients of powers of x to zero. From the coefficient of x^0 we obtain $2a_1 + a_0 = 0$ which implies that $a_1 = -a_0/2$. From the remaining coefficients, we obtain

$$(4n^2 - 2n)a_n + a_{n-1} = 0 \implies a_n = \frac{-a_{n-1}}{2n(2n-1)}, \quad n \geq 2.$$

When we iterate this recursive definition:

$$a_2 = \frac{-a_1}{4 \cdot 3} = \frac{a_0}{4!}, \quad a_3 = \frac{-a_2}{6 \cdot 5} = -\frac{a_0}{6!}, \quad \cdots$$

The solution is therefore $\quad y = f(x) = a_0\left(1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n.$

29. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n]x^n. \end{aligned}$$

When we equate coefficients of powers of x to zero, we obtain the recursive formula

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0.$$

Iteration gives

$$a_2 = -\frac{a_0}{2!}, \quad a_4 = \frac{a_0}{4!}, \quad a_6 = -\frac{a_0}{6!}, \quad \ldots, \quad \text{and} \quad a_3 = -\frac{a_1}{3!}, \quad a_5 = \frac{a_1}{5!}, \quad a_7 = -\frac{a_1}{7!}, \quad \ldots.$$

The solution is therefore

$$y = f(x) = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = a_0 \cos x + a_1 \sin x.$$

30. If we substitute $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$\begin{aligned} 0 &= x\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} a_{n-1}x^{n-1} = a_0 + \sum_{n=2}^{\infty} [n(n-1)a_n + a_{n-1}]x^{n-1}. \end{aligned}$$

We now equate coefficients of powers of x to zero. From the coefficient of x^0 we obtain $a_0 = 0$. From the remaining coefficients, we obtain

$$n(n-1)a_n + a_{n-1} = 0 \implies a_n = \frac{-a_{n-1}}{n(n-1)}, \quad n \geq 2.$$

When we iterate this recursive definition:

$$a_2 = \frac{-a_1}{2 \cdot 1}, \quad a_3 = \frac{-a_2}{3 \cdot 2} = \frac{a_1}{3!\,2!}, \quad a_4 = \frac{-a_3}{4 \cdot 3} = \frac{-a_1}{4!\,3!}, \quad \ldots.$$

The solution is therefore

$$y = f(x) = a_1\left(x - \frac{x^2}{2 \cdot 1} + \frac{x^3}{3! \cdot 2!} - \frac{x^4}{4! \cdot 3!} + \cdots\right) = a_1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!(n-1)!} x^n.$$

31. According to Exercise 23, Taylor's remainder formula for $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \cdots + R_n(0,x), \quad \text{where } R_n(0,x) = \frac{-x^{n+1}}{(n+1)(1-z_n)^{n+1}},$$

and z_n is between 0 and x. The maximum error when only the first term is used is $R_1(0,x) = \dfrac{-x^2}{2(1-z_1)^2}$. If we set $x = 0.000\,000\,000\,1$, then $z_1 < 0.000\,000\,000\,1$, and we can say that

$$|R_1(0, 0.000\,000\,000\,1)| < \frac{(0.000\,000\,000\,1)^2}{2(0.999\,999\,999\,9)^2} < 3.4 \times 10^{-21}.$$

Hence, $\ln(0.999\,999\,999\,9) = -10^{-10}$, and this is definitely accurate to more than 15 decimal places.

32. $$K = c^2(m - m_0) = c^2 m_0 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)$$
$$= c^2 m_0 \left\{\left[1 - \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{(-1/2)(-3/2)}{2!}\left(-\frac{v^2}{c^2}\right)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left(-\frac{v^2}{c^2}\right)^3 + \cdots\right] - 1\right\}$$
$$= c^2 m_0 \left\{\frac{v^2}{2c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right\} = \frac{1}{2}m_0 v^2 + m_0 c^2 \left(\frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)$$

33. Using the binomial expansion,

$$\frac{P_0}{P} = 1 + \left(\frac{k}{k-1}\right)\left(\frac{k-1}{2}\right)M^2 + \cdots = 1 + \frac{kM^2}{2} + \cdots .$$

34. When we expand P_s/P_0 with the binomial expansion,

$$\frac{P_s}{P_0} = 1 + \left(\frac{k}{k-1}\right)\left(\frac{k-1}{2}\right)M_0^2 + \frac{1}{2}\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1} - 1\right)\left(\frac{k-1}{2}\right)^2 M_0^4$$
$$+ \frac{1}{3!}\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1} - 1\right)\left(\frac{k}{k-1} - 2\right)\left(\frac{k-1}{2}\right)^3 M_0^6 + \cdots$$
$$= 1 + \frac{k}{2}M_0^2 + \frac{k}{8}M_0^4 + \frac{k(2-k)}{48}M_0^6 + \cdots$$
$$= 1 + \frac{1}{2}M_0^2\left(\frac{\rho_0 c_0^2}{P_0}\right) + \frac{1}{8}M_0^4\left(\frac{\rho_0 c_0^2}{P_0}\right) + \frac{1}{48}M_0^6(2-k)\left(\frac{\rho_0 c_0^2}{P_0}\right) + \cdots .$$

Multiplication by P_0, and replacement of M_0^2 by V_0^2/c_0^2 in the last three terms gives

$$P_s = P_0 + \frac{1}{2}\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right) + \frac{1}{8}\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right)M_0^2 + \frac{1}{48}(2-k)\rho_0 c_0^2\left(\frac{V_0^2}{c_0^2}\right)M_0^4 + \cdots$$
$$= P_0 + \frac{1}{2}\rho_0 V_0^2 + \frac{1}{8}\rho_0 V_0^2 M_0^2 + \frac{1}{48}(2-k)\rho_0 V_0^2 M_0^4 + \cdots$$
$$= P_0 + \frac{1}{2}\rho_0 V_0^2\left[1 + \frac{M_0^2}{4} + \left(\frac{2-k}{24}\right)M_0^4 + \cdots\right].$$

35. (a) Using formula 9.3, the length of the ellipse is four times that in the first quadrant,

$$L = 4\int_0^{\pi/2}\sqrt{(-a\sin t)^2 + (b\cos t)^2}\,dt = 4b\int_0^{\pi/2}\sqrt{\frac{a^2}{b^2}\sin^2 t + (1 - \sin^2 t)}\,dt = 4b\int_0^{\pi/2}\sqrt{1 - k^2\sin^2 t}\,dt,$$

where $k^2 = 1 - a^2/b^2$.
(b) If we expand the integrand with the binomial expansion 10.33b, and integrate term-by-term,

$$\begin{aligned}L &= 4b\int_0^{\pi/2}\left[1+\frac{1}{2}(-k^2\sin^2 t)+\frac{(1/2)(-1/2)}{2}(-k^2\sin^2 t)^2+\cdots\right]dt\\ &= 4b\int_0^{\pi/2}\left[1-\frac{k^2}{2}\left(\frac{1-\cos 2t}{2}\right)-\frac{k^4}{8}\left(\frac{1-\cos 2t}{2}\right)^2+\cdots\right]dt\\ &= 4b\int_0^{\pi/2}\left[1-\frac{k^2}{4}(1-\cos 2t)-\frac{k^4}{32}\left(1-2\cos 2t+\frac{1+\cos 4t}{2}\right)+\cdots\right]dt\\ &= 4b\left\{t-\frac{k^2}{4}\left(t-\frac{\sin 2t}{2}\right)-\frac{k^4}{32}\left(\frac{3t}{2}-\sin 2t+\frac{\sin 4t}{8}\right)+\cdots\right\}_0^{\pi/2}\\ &= 4b\left[\frac{\pi}{2}-\frac{k^2}{4}\left(\frac{\pi}{2}\right)-\frac{k^4}{32}\left(\frac{3\pi}{4}\right)+\cdots\right]\\ &= 2\pi b\left(1-\frac{k^2}{4}-\frac{3k^4}{64}+\cdots\right).\end{aligned}$$

36. (a) If we substitute $e^{-\beta^2/(4\alpha x)} = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha x}\right)^n$, we obtain

$$W(\alpha,\beta)=\int_1^\infty \frac{1}{x}e^{-\alpha x}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha x}\right)^n dx=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\beta^2}{4\alpha}\right)^n\int_1^\infty\frac{e^{-\alpha x}}{x^{n+1}}dx=\sum_{n=0}^{\infty}\frac{(-1)^n\beta^{2n}}{4^n\alpha^n n!}E_{n+1}(\alpha).$$

(b) We use integration by parts with $u = e^{-\alpha x}$ and $dv = \dfrac{1}{x^{n+1}}dx$,

$$\begin{aligned}E_{n+1}(\alpha) &= \int_1^\infty \frac{e^{-\alpha x}}{x^{n+1}}dx = \left\{-\frac{e^{-\alpha x}}{nx^n}\right\}_1^\infty - \int_1^\infty -\frac{1}{nx^n}(-\alpha)e^{-\alpha x}\,dx = \frac{e^{-\alpha}}{n}-\frac{\alpha}{n}\int_1^\infty\frac{e^{-\alpha x}}{x^n}dx\\ &= \frac{1}{n}[e^{-\alpha}-\alpha E_n(\alpha)].\end{aligned}$$

37. If we substitute the Maclaurin series for $e^{ch/(\lambda kT)}$,

$$\Psi(\lambda)=\frac{8\pi ch\lambda^{-5}}{\left(1+\dfrac{ch}{\lambda kT}+\dfrac{c^2h^2}{2\lambda^2k^2T^2}+\cdots\right)-1}=\frac{8\pi ch}{\lambda^5\left(\dfrac{ch}{\lambda kT}+\dfrac{c^2h^2}{2\lambda^2k^2T^2}+\cdots\right)}=\frac{8\pi ch}{\dfrac{ch}{kT}\lambda^4+\dfrac{c^2h^2}{2k^2T^2}\lambda^3+\cdots}.$$

If we long divide the denominator into the numerator, the result is

$$\Psi(\lambda)=\frac{8\pi kT}{\lambda^4}+\text{ terms in }\lambda^{-5},\ \lambda^{-6},\text{ etc..}$$

Thus, for large λ, $\Psi(\lambda)$ can be approximated by $8\pi kT/\lambda^4$.

38. (a) We write $E = \dfrac{q}{4\pi\epsilon_0 x^2\left(1-\dfrac{d}{2x}\right)^2}-\dfrac{q}{4\pi\epsilon_0 x^2\left(1+\dfrac{d}{2x}\right)^2}=\dfrac{q}{4\pi\epsilon_0 x^2}\left[\left(1-\dfrac{d}{2x}\right)^{-2}-\left(1+\dfrac{d}{2x}\right)^{-2}\right].$

(b) If we expand each term with the binomial expansion 10.33b,

$$E=\frac{q}{4\pi\epsilon_0 x^2}\left\{\left[1-2\left(-\frac{d}{2x}\right)+\cdots\right]-\left[1-2\left(\frac{d}{2x}\right)+\cdots\right]\right\}.$$

When d is very much less than x, we omit higher order terms in d/x, and write

$$E\approx\frac{q}{4\pi\epsilon_0 x^2}\left(1+\frac{d}{x}-1+\frac{d}{x}\right)=\frac{qd}{2\pi\epsilon_0 x^3}.$$

39. The cross-sectional area of the liquid is the area of the sector less the area of the triangle above it,

$$A = \frac{1}{2}R^2\theta - 2\left(\frac{1}{2}\right)\left(R\sin\frac{\theta}{2}\right)\left(R\cos\frac{\theta}{2}\right) = \frac{R^2}{2}(\theta - \sin\theta).$$

Since $d = 2R\sin\dfrac{\theta}{2}$ and $h = R - R\cos\dfrac{\theta}{2}$,

$$hd = 2R\sin\frac{\theta}{2}\left(R - R\cos\frac{\theta}{2}\right) = R^2\left(2\sin\frac{\theta}{2} - \sin\theta\right).$$

The required ratio is

$$\frac{A}{hd} = \frac{\dfrac{R^2}{2}(\theta - \sin\theta)}{R^2\left(2\sin\dfrac{\theta}{2} - \sin\theta\right)} = \frac{\theta - \sin\theta}{2\left(2\sin\dfrac{\theta}{2} - \sin\theta\right)}.$$

If we expand the sine functions in their Maclaurin series

$$\begin{aligned}\frac{A}{hd} &= \frac{\theta - \left(\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots\right)}{2\left[2\left(\dfrac{\theta}{2} - \dfrac{(\theta/2)^3}{3!} + \dfrac{(\theta/2)^5}{5!} - \cdots\right) - \left(\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots\right)\right]}\\ &= \frac{\dfrac{\theta^3}{3!} - \dfrac{\theta^5}{5!} + \cdots}{\dfrac{\theta^3}{4} - \dfrac{\theta^5}{64} + \cdots} \qquad \text{(and by long division)}\\ &= \frac{2}{3} + \frac{\theta^2}{120} + \cdots .\end{aligned}$$

For small θ, we can use the approximation $\dfrac{A}{hd} \approx \dfrac{2}{3} + \dfrac{\theta^2}{120}$.

EXERCISES 10.8

1. True If a sequence satisfies 10.35a, then it satisfies 10.35b; that is, every increasing sequence is nondecreasing.

2. False The sequence $\{n\}$ is increasing but has no upper bound.

3. True The first term of an increasing sequence is a lower bound.

4. False The sequence $\{-n\}$ is decreasing but has no lower bound.

5. False The sequence $\{n\}$ is increasing with lower bound 1, but it does not have a limit.

6. True An increasing sequence has a lower bound. If it also has an upper bound, then it has a limit according to Theorem 10.7.

7. False The sequence $\{(-1)^n\}$ does not converge, but its terms are all ± 1.

8. True For a sequence to be increasing and decreasing, its terms would have to satisfy $c_{n+1} > c_n$ and $c_{n+1} < c_n$ for all n. This is impossible.

9. True The sequence $\{1\}$ is an example.

10. True This is part of the corollary to Theorem 10.7.

11. False The sequence $\{(-1)^n/n\}$ is bounded and has limit 0, but it is not monotonic.

12. False The sequence $\{(-1)^n/n\}$ is bounded, not monotonic, and it has limit 0.

13. True Suppose that $\lim_{n\to\infty} c_n = L$. Then there exists an integer N such that for all $n > N$, all terms of the sequence satisfy $|c_n - L| < 1$. This implies that for $n > N$, terms satisfy $c_n < L + 1$. If we set U equal to the biggest of $L+1$ and the terms $c_1, c_2, \ldots, c_N$, then we can say that all terms of the sequence are less than U. Thus, U is an upper bound for the sequence. A similar proof shows that the sequence has a lower bound.

14. False The sequence $\{(-1)^n/n\}$ is not montonic, but it has limit 0.

15. False L could be equal to U. For example all terms of the sequence $\{(n-1)/n\}$ are less than 1, and the limit of the sequence is 1.

16. False The sequence $\{(-1)^n\}$ has no limit, but the sequence $\{[(-1)^n]^2\} = \{1\}$ has limit 1.

17. False The oscillating sequence $\{(-1)^n\}$ does not converge.

18. True Such a sequence displays the up-down-up-down nature required of an oscillating sequence.

19. True Terms of such a sequence must approach 0.

20. False Each term of the sequence $\{-3^n\}$ is less than half the previous term, but the sequence does not have a limit.

21. True If L is the limit of the sequence and U is the smallest of the upper bounds, then L cannot be smaller than U, otherwise there would be smaller upper bounds than U. On the other hand, L cannot be larger than U because U would not then be an upper bound. Hence, L must be equal to U.

22. True. The only other possibility is that the sequence $\{c_n\}$ has a limit L which is not equal to a. Suppose $L > a$ and we set $\epsilon = L - a$. Then there exists an integer N such that for $n > N$, $|c_n - L| < L - a$; that is $-(L-a) < c_n - L < L - a$. Thus, for $n > N$,

$$L - (L-a) < c_n < L + (L-a) \quad \Longrightarrow \quad a < c_n < 2L - a.$$

But this contradicts the fact that $c_n = a$ for an infinity of values of n. Hence L cannot be greater than a. A similar proof shows that L cannot be less than a.

23. False According to Theorem 10.8, absolute values of differences must also approach 0. For example, absolute values of the differences of the terms of the oscillating sequence $\{[1 + (-1)^n]/2 + (-1/2)^n\}$ decrease, but the sequence does not have a limit. The easiest way to see this is to plot about ten terms of the sequence.

24. False The oscillating sequence $\{[1 + (-1)^{n+1}]/(2n)\}$ has limit zero, but terms are all nonnegative.

25. The first four terms of the sequence are $c_1 = 1$, $c_2 = 13/10 = 1.3$, $c_3 = 1.4197$, $c_4 = 1.486\,147$. The sequence appears to be increasing; that is $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, ${c_{k+1}}^3 > {c_k}^3$, and ${c_{k+1}}^3 + 12 > {c_k}^3 + 12$. Thus, $({c_{k+1}}^3 + 12)/10 > ({c_k}^3 + 12)/10$, and this means that $c_{k+2} > c_{k+1}$. By mathematical induction then, $c_{n+1} > c_n$ for all $n \geq 1$. The first term $c_1 = 1$ must be a lower bound. We suspect that $U = 1.8$ is an upper bound; that is, $c_n \leq 1.8$. This is true for $n = 1$. Suppose $c_k \leq 1.8$ for some integer k. Then $c_{k+1} = ({c_k}^3 + 12)/10 \leq (1.8^3 + 12)/10 < 1.8$. Hence, by mathematical induction, $c_n \leq 1.8$ for all n.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{10}({c_n}^3 + 12).$$

It follows that L must satisfy $L = (L^3 + 12)/10 \Longrightarrow L^3 - 10L + 12 = 0$. Roots of this equation are 2 and $-1 \pm \sqrt{7}$. Since 1.8 is an upper bound for the sequence, it follows that $L = -1 + \sqrt{7}$.

26. The first four terms of the sequence are $c_1 = 0$, $c_2 = 5/12$, $c_3 = 0.419\,178$, $c_4 = 0.419\,240$. The sequence appears to be increasing; that is $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, ${c_{k+1}}^4 > {c_k}^4$, and ${c_{k+1}}^4 + 5 > {c_k}^4 + 5$. Thus, $({c_{k+1}}^4 + 5)/12 > ({c_k}^4 + 5)/12$, and this means that $c_{k+2} > c_{k+1}$. By mathematical induction then, $c_{n+1} > c_n$ for all $n \geq 1$. The first term $c_1 = 0$ must be a lower bound. We suspect that $U = 1$ is an upper bound; that is, $c_n \leq 1$. This is true for $n = 1$. Suppose $c_k \leq 1$ for some integer k. Then $c_{k+1} = ({c_k}^4 + 5)/12 \leq (1 + 5)/12 = 1/2 < 1$. Hence, by mathematical induction, $c_n \leq 1$ for all n.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{12}({c_n}^4 + 5).$$

It follows that L must satisfy $L = (L^4+5)/12 \Longrightarrow 0 = L^4 - 12L + 5 = f(L)$. Since $f(0.419\,240\,5) = 6.5 \times 10^{-6}$ and $f(0.419\,241\,5) = -5.2 \times 10^{-6}$, the limit of the sequence (to six decimals) is 0.419 241.

27. It is clear that all terms of the sequence are positive and therefore there is no difficulty with taking square roots. The first four terms of the sequence are $c_1 = 3$, $c_2 = \sqrt{8} = 2.828$, $c_3 = 2.7979$, $c_4 = 2.7925$. The sequence appears to be decreasing; that is $c_{n+1} < c_n$. This is certainly true for $n = 1$ as $c_2 < c_1$. Suppose $c_{k+1} < c_k$ for some integer k. Then, $5 + c_{k+1} < 5 + c_k$, and $\sqrt{5+c_{k+1}} < \sqrt{5+c_k}$. Thus, $c_{k+2} < c_{k+1}$, and by mathematical induction, $c_{n+1} < c_n$ for all $n \ge 1$. The first term $c_1 = 3$ must be an upper bound. Clearly $V = 0$ is a lower bound.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt{5+c_n}.$$

It follows that L must satisfy $L = \sqrt{5+L} \Longrightarrow L^2 - L - 5 = 0$. Of the two solutions $(1 \pm \sqrt{21})/2$ of this equation, only $(1+\sqrt{21})/2$ lies between the bounds. Hence, $L = (1+\sqrt{21})/2$.

28. It is clear that all terms of the sequence are positive and therefore there is no difficulty with taking square roots. The first four terms of the sequence are $c_1 = 1$, $c_2 = \sqrt{6} = 2.449$, $c_3 = 2.7294$, $c_4 = 2.7802$. The sequence appears to be increasing; that is $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, $5 + c_{k+1} > 5 + c_k$, and $\sqrt{5+c_{k+1}} > \sqrt{5+c_k}$. Thus, $c_{k+2} > c_{k+1}$, and by mathematical induction, $c_{n+1} > c_n$ for all $n \ge 1$. The first term $c_1 = 1$ must be a lower bound. We suspect that $U = 5$ is an upper bound; that is, $c_n \le 5$. This is true for $n = 1$. Suppose $c_k \le 5$ for some integer k. Then $c_{k+1} = \sqrt{5+c_k} \le \sqrt{5+5} = \sqrt{10} < 5$. Hence, by mathematical induction, $c_n \le 5$ for all n.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt{5+c_n}.$$

It follows that L must satisfy $L = \sqrt{5+L} \Longrightarrow L^2 - L - 5 = 0$. Of the two solutions $(1 \pm \sqrt{21})/2$ of this equation, only $(1+\sqrt{21})/2$ lies between the bounds. Hence, $L = (1+\sqrt{21})/2$.

29. It is clear that all terms of the sequence are positive and therefore there is no difficulty with taking square roots. The first four terms of the sequence are $c_1 = 5$, $c_2 = 1 + \sqrt{11} = 4.317$, $c_3 = 4.212$, $c_4 = 4.196$. The sequence appears to be decreasing; that is $c_{n+1} < c_n$. This is certainly true for $n = 1$ as $c_2 < c_1$. Suppose $c_{k+1} < c_k$ for some integer k. Then, $6 + c_{k+1} < 6 + c_k$, and $\sqrt{6+c_{k+1}} < \sqrt{6+c_k}$. Thus, $1 + \sqrt{6+c_{k+1}} < 1 + \sqrt{6+c_k}$, and this means that $c_{k+2} < c_{k+1}$. By mathematical induction then, $c_{n+1} < c_n$ for all $n \ge 1$. The first term $c_1 = 5$ must be an upper bound, and $V = 0$ is a lower bound.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \left[1 + \sqrt{6+c_n}\right].$$

It follows that L must satisfy $L = 1 + \sqrt{6+L} \Longrightarrow (L-1)^2 = 6 + L$. This equation simplifies to $L^2 - 3L - 5 = 0$, and of the two solutions $(3 \pm \sqrt{29})/2$ of this equation, only $(3+\sqrt{29})/2$ lies between the bounds. Hence, $L = (3+\sqrt{29})/2$.

30. It is clear that all terms of the sequence are positive and therefore there is no difficulty with taking square roots. The first four terms of the sequence are $c_1 = 3$, $c_2 = 4$, $c_3 = 4.1623$, $c_4 = 4.1878$. The sequence appears to be increasing; that is $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, $6 + c_{k+1} > 6 + c_k$, and $\sqrt{6+c_{k+1}} > \sqrt{6+c_k}$. Thus, $1 + \sqrt{6+c_{k+1}} > 1 + \sqrt{6+c_k}$, and this means that $c_{k+2} > c_{k+1}$. By mathematical induction then,

$c_{n+1} > c_n$ for all $n \geq 1$. The first term $c_1 = 3$ must be a lower bound. We suspect that $U = 10$ is an upper bound; that is, $c_n \leq 10$. This is true for $n = 1$. Suppose $c_k \leq 10$ for some integer k. Then $c_{k+1} = 1 + \sqrt{6 + c_k} \leq 1 + \sqrt{6 + 10} = 5 < 10$. Hence, by mathematical induction, $c_n \leq 10$ for all n.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \left[1 + \sqrt{6 + c_n}\right].$$

It follows that L must satisfy $L = 1 + \sqrt{6 + L} \Longrightarrow (L - 1)^2 = 6 + L$. This equation simplifies to $L^2 - 3L - 5 = 0$, and of the two solutions $(3 \pm \sqrt{29})/2$ of this equation, only $(3 + \sqrt{29})/2$ lies between the bounds. Hence, $L = (3 + \sqrt{29})/2$.

31. The first four terms of the sequence are $c_1 = 1$, $c_2 = 2$, $c_3 = 2.268$, $c_4 = 2.347$. Previous exercises have indicated that when it is anticipated that a sequence is monotonic and bounded, it is advantageous to first verify monotony. If a sequence is known to be increasing (or nondecreasing), then a lower bound must be the first term of the sequence. On the other hand, if a sequence is known to be decreasing (or nonincreasing), its first term is an upper bound. Unfortunately, it is not always possible to verify monotony without knowledge of bounds. This is such an example. Try to prove that this sequence is decreasing before reading the rest of this solution, and discover why the proof fails. In addition, to guarantee that all terms of the sequence are well-defined, we must know that no one of them can be greater than 5.

We begin by proving that upper and lower bounds are 4 and 0; that is, $0 \leq c_n \leq 4$. This is certainly true for $n = 1$. Suppose $0 \leq c_k \leq 4$ for some integer k. Then $0 \geq -c_k \geq -4$, from which $5 \geq 5 - c_k \geq 1$. It follows that $\sqrt{5} \geq \sqrt{5 - c_k} \geq 1$, and $-\sqrt{5} \leq -\sqrt{5 - c_k} \leq -1$. Thus, $4 - \sqrt{5} \leq 4 - \sqrt{5 - c_k} \leq 3$. This means that $0 < 4 - \sqrt{5} \leq c_{k+1} \leq 3 < 4$, and therefore by mathematical induction, $0 \leq c_n \leq 4$ for all n.

Now we verify that the sequence is increasing, $c_{n+1} > c_n$. This is certainly true for $n = 1$ as $c_2 > c_1$. Suppose $c_{k+1} > c_k$ for some integer k. Then, $5 - c_{k+1} < 5 - c_k$. Because all terms of the sequence are between 0 and 4, both sides of this inequality are positive, and we can take square roots, $\sqrt{5 - c_{k+1}} < \sqrt{5 - c_k}$. Thus, $4 - \sqrt{5 - c_{k+1}} > 4 - \sqrt{5 - c_k}$, and this means that $c_{k+2} > c_{k+1}$. By mathematical induction, $c_{n+1} > c_n$ for all $n \geq 1$.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \left[4 - \sqrt{5 - c_n}\right].$$

It follows that L must satisfy $L = 4 - \sqrt{5 - L} \Longrightarrow (L - 4)^2 = 5 - L$. This equation simplifies to $L^2 - 7L + 11 = 0$, and of the two solutions $(7 \pm \sqrt{5})/2$, only $(7 - \sqrt{5})/2$ lies between the bounds. Hence, $L = (7 - \sqrt{5})/2$.

32. The first four terms of the sequence are $c_1 = 4$, $c_2 = 3$, $c_3 = 2.5858$, $c_4 = 2.4462$. Previous exercises have indicated that when it is anticipated that a sequence is monotonic and bounded, it is advantageous to first verify monotony. If a sequence is known to be increasing (or nondecreasing), then a lower bound must be the first term of the sequence. On the other hand, if a sequence is known to be decreasing (or nonincreasing), its first term is an upper bound. Unfortunately, it is not always possible to verify monotony without knowledge of bounds. This is such an example. Try to prove that this sequence is decreasing before reading the rest of this solution, and discover why the proof fails. In addition, to guarantee that all terms of the sequence are well-defined, we must know that no one of them can be greater than 5.

We begin by proving that upper and lower bounds are 4 and 0; that is, $0 \leq c_n \leq 4$. This is certainly true for $n = 1$. Suppose $0 \leq c_k \leq 4$ for some integer k. Then $0 \geq -c_k \geq -4$, from which $5 \geq 5 - c_k \geq 1$. It follows that $\sqrt{5} \geq \sqrt{5 - c_k} \geq 1$, and $-\sqrt{5} \leq -\sqrt{5 - c_k} \leq -1$. Thus, $4 - \sqrt{5} \leq 4 - \sqrt{5 - c_k} \leq 3$. This means that $0 < 4 - \sqrt{5} \leq c_{k+1} \leq 3 < 4$, and therefore by mathematical induction, $0 \leq c_n \leq 4$ for all n.

Now we verify that the sequence is decreasing, $c_{n+1} < c_n$. This is certainly true for $n = 1$ as $c_2 < c_1$. Suppose $c_{k+1} < c_k$ for some integer k. Then, $5 - c_{k+1} > 5 - c_k$. Because all terms of the sequence are between 0 and 4, both sides of this inequality are positive, and we can take square roots,

$\sqrt{5-c_{k+1}} > \sqrt{5-c_k}$. Thus, $4-\sqrt{5-c_{k+1}} < 4-\sqrt{5-c_k}$, and this means that $c_{k+2} < c_{k+1}$. By mathematical induction, $c_{n+1} < c_n$ for all $n \geq 1$.

Theorem 10.7 now implies that the sequence has a limit, call it L. By taking limits on both sides of the recursive definition of the sequence we obtain

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty}\left[4-\sqrt{5-c_n}\right].$$

It follows that L must satisfy $L = 4-\sqrt{5-L} \Longrightarrow (L-4)^2 = 5-L$. This equation simplifies to $L^2-7L+11=0$, and of the two solutions $(7\pm\sqrt{5})/2$, only $(7-\sqrt{5})/2$ lies between the bounds. Hence, $L=(7-\sqrt{5})/2$.

33. The first four terms of the sequence are $c_1=2$, $c_2=1$, $c_3=1/2$, $c_4=2/5$. An initial attempt at proving that this sequence is decreasing fails. We require information about its bounds. Consider proving that upper and lower bounds are 2 and 0; that is, $0\leq c_n\leq 2$. This is certainly true for $n=1$. Suppose $0\leq c_k\leq 2$ for some integer k. Then, $0\geq -c_k\geq -2$, from which $3\geq 3-c_k\geq 1$. Inversion gives $1/3\leq 1/(3-c_k)\leq 1$, but this implies that $0<1/3\leq c_{k+1}\leq 1<2$. By mathematical induction then, $0\leq c_n\leq 2$ for all n. Now we verify that the sequence is decreasing, that its terms satisfy $c_{n+1}<c_n$. This is true for $n=1$. Suppose $c_{k+1}<c_k$ for some integer k. Then $-c_{k+1}>-c_k$, and, $3-c_{k+1}>3-c_k$. Because all terms of the sequence are between 0 and 2, both expressions in this inequality are positive. We may therefore invert and write $1/(3-c_{k+1})<1/(3-c_k)$; i.e., $c_{k+2}<c_{k+1}$. By mathematical induction then, $c_{n+1}<c_n$ for all n.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L. By writing

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty}\frac{1}{3-c_n},$$

we obtain the equation $L=1/(3-L)$. Of the two solutions $(3\pm\sqrt{5})/2$ of this equation, only $(3-\sqrt{5})/2$ is between the bounds. Hence, $L=(3-\sqrt{5})/2$.

34. The first four terms of the sequence are $c_1=1$, $c_2=1/2$, $c_3=1/3$, $c_4=3/10$. An initial attempt at proving that this sequence is decreasing fails. We require information about its bounds. Consider proving that upper and lower bounds are 1 and 0; that is, $0\leq c_n\leq 1$. This is certainly true for $n=1$. Suppose $0\leq c_k\leq 1$ for some integer k. Then, $0\geq -2c_k\geq -2$, from which $4\geq 4-2c_k\geq 2$. Inversion gives $1/4\leq 1/(4-2c_k)\leq 1/2$, but this implies that $0<1/4\leq c_{k+1}\leq 1/2<1$. By mathematical induction then, $0\leq c_n\leq 1$ for all n. Now we verify that the sequence is decreasing, that its terms satisfy $c_{n+1}<c_n$. This is true for $n=1$. Suppose $c_{k+1}<c_k$ for some integer k. Then $-2c_{k+1}>-2c_k$, and, $4-2c_{k+1}>4-2c_k$. Because all terms of the sequence are between 0 and 1, both expressions in this inequality are positive. We may therefore invert and write $1/(4-2c_{k+1})<1/(4-2c_k)$; i.e., $c_{k+2}<c_{k+1}$. By mathematical induction then, $c_{n+1}<c_n$ for all n.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L. By writing

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty}\frac{1}{4-2c_n},$$

we obtain the equation $L=1/(4-2L)$. Of the two solutions $1\pm 1/\sqrt{2}$ of this equation, only $1-1/\sqrt{2}$ is between the bounds. Hence, $L=1-1/\sqrt{2}$.

35. The first four terms of the sequence are $c_1=1$, $c_2=7/8$, $c_3=0.7089$, $c_4=0.5843$. An initial attempt at proving that this sequence is decreasing fails. We require information about its bounds. We prove first therefore that $0\leq c_n\leq 1$. This is true for $n=1$. Suppose $0\leq c_k\leq 1$ for some integer k. Then, $0\geq -8{c_k}^2\geq -8$, and $16\geq 16-8{c_k}^2\geq 8$. Inversion gives $1/16\leq 1/(16-8{c_k}^2)\leq 1/8$. But then $7/16\leq 7/(16-8{c_k}^2)\leq 7/8$, or, $0<7/16\leq c_{k+1}\leq 7/8<1$. By mathematical induction then, $0\leq c_n\leq 1$, and we have upper and lower bounds. Now we verify that the sequence is decreasing, that its terms satisfy $c_{n+1}<c_n$. This is true for $n=1$. Suppose $c_{k+1}<c_k$ for some integer k. Then $-8{c_{k+1}}^2>-8{c_k}^2$, and, $16-8{c_{k+1}}^2>16-8{c_k}^2$. Because all terms of the sequence are between 0 and 1, both expressions in this inequality are positive. We may therefore invert and write $1/(16-8{c_{k+1}}^2)<1/(16-8{c_k}^2)$. In other words, $7/(16-8c_{k+1}{}^2) < 7/(16-8c_k{}^2)$, or, $c_{k+2} < c_{k+1}$. By mathematical induction then, $c_{n+1} < c_n$ for all n.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L. By writing

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{7}{16-8c_n{}^2},$$

we obtain the equation $L = 7/(16-8L^2) \Longrightarrow 0 = 8L^3 - 16L + 7 = (2L-1)(4L^2+2L-7)$. Of the three solutions to this equation, only $1/2$ is between the bounds. Hence, $L = 1/2$.

36. The first four terms of the sequence are $c_1 = 0$, $c_2 = 7/16$, $c_3 = 0.443$, $c_4 = 0.485$. An initial attempt at proving that this sequence is increasing fails. We require information about its bounds. We prove first therefore that $0 \le c_n \le 1$. This is true for $n = 1$. Suppose $0 \le c_k \le 1$ for some integer k. Then, $0 \ge -8c_k{}^2 \ge -8$, and $16 \ge 16 - 8c_k{}^2 \ge 8$. Inversion gives $1/16 \le 1/(16-8c_k{}^2) \le 1/8$. But then $7/16 \le 7/(16-8c_k{}^2) \le 7/8$, or, $0 < 7/16 \le c_{k+1} \le 7/8 < 1$. By mathematical induction then, $0 \le c_n \le 1$, and we have upper and lower bounds. Now we verify that the sequence is increasing, that its terms satisfy $c_{n+1} > c_n$. This is true for $n = 1$. Suppose $c_{k+1} > c_k$ for some integer k. Then $-8c_{k+1}{}^2 < -8c_k{}^2$, and, $16 - 8c_{k+1}{}^2 < 16 - 8c_k{}^2$. Because all terms of the sequence are between 0 and 1, both expressions in this inequality are positive. We may therefore invert and write $1/(16-8c_{k+1}{}^2) > 1/(16-8c_k{}^2)$. In other words, $7/(16-8c_{k+1}{}^2) > 7/(16-8c_k{}^2)$, or, $c_{k+2} > c_{k+1}$. By mathematical induction then, $c_{n+1} > c_n$ for all n.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L. By writing

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{7}{16-8c_n{}^2},$$

we obtain the equation $L = 7/(16-8L^2) \Longrightarrow 0 = 8L^3 - 16L + 7 = (2L-1)(4L^2+2L-7)$. Of the three solutions to this equation, only $1/2$ is between the bounds. Hence, $L = 1/2$.

37. It is advantageous to express the recursive definition in the form $c_{n+1} = \dfrac{4}{1+\dfrac{4}{c_n}}$.

The first four terms of the sequence are $c_1 = 1$, $c_2 = 4/5$, $c_3 = 2/3$, $c_4 = 4/7$. The sequence appears to be decreasing, $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then,

$$\frac{1}{c_{k+1}} > \frac{1}{c_k} \quad \Longrightarrow \quad \frac{4}{c_{k+1}} > \frac{4}{c_k} \quad \Longrightarrow \quad 1 + \frac{4}{c_{k+1}} > 1 + \frac{4}{c_k}.$$

Thus,

$$\frac{1}{1+\dfrac{4}{c_{k+1}}} < \frac{1}{1+\dfrac{4}{c_k}}; \qquad \text{that is,} \qquad c_{k+2} < c_{k+1}.$$

By mathematical induction , $c_{n+1} < c_n$ for all n. It follows that $U = 1$ is an upper bound, and because no term can be negative, $V = 0$. By Theorem 10.7, the sequence has a limit L that we can obtain by solving the equation $L = 4L/(4+L)$. The only solution of this equation is $L = 0$.

38. It is advantageous to express the recursive definition in the form $c_{n+1} = \dfrac{3}{1+\dfrac{2}{c_n}}$.

The first four terms of the sequence are $c_1 = 4$, $c_2 = 2$, $c_3 = 3/2$, $c_4 = 9/7$. The sequence appears to be decreasing, $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then,

$$\frac{2}{c_{k+1}} > \frac{2}{c_k} \quad \Longrightarrow \quad 1 + \frac{2}{c_{k+1}} > 1 + \frac{2}{c_k}.$$

Thus,

$$\frac{1}{1+\dfrac{2}{c_{k+1}}} < \frac{1}{1+\dfrac{2}{c_k}} \quad \Longrightarrow \quad \frac{3}{1+\dfrac{2}{c_{k+1}}} < \frac{3}{1+\dfrac{2}{c_k}}; \qquad \text{that is,} \qquad c_{k+2} < c_{k+1}.$$

By mathematical induction , $c_{n+1} < c_n$ for all n. It follows that $U = 1$ is an upper bound, and because no term can be negative, $V = 0$. By Theorem 10.7, the sequence has a limit L that we can obtain by solving the equation $L = 3L/(2+L)$. Of the two solutions 0 and 1 of this equation, we must choose $L = 0$.

39. It is advantageous to express the recursive definition in the form $c_{n+1} = \dfrac{2}{\dfrac{1}{c_n} + \dfrac{3}{c_n^2}}$.

The first four terms of the sequence are $c_1 = 2$, $c_2 = 8/5$, $c_3 = 1.1130$, $c_4 = 0.6024$. The sequence appears to be decreasing, $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then, $\dfrac{1}{c_{k+1}} > \dfrac{1}{c_k}$ and $\dfrac{3}{{c_{k+1}}^2} > \dfrac{3}{{c_k}^2}$. Addition of these gives $\dfrac{1}{c_{k+1}} + \dfrac{3}{{c_{k+1}}^2} > \dfrac{1}{c_k} + \dfrac{3}{{c_k}^2}$. Hence,

$$\frac{1}{\dfrac{1}{c_{k+1}} + \dfrac{3}{{c_{k+1}}^2}} < \frac{1}{\dfrac{1}{c_k} + \dfrac{3}{{c_k}^2}} \quad \Longrightarrow \quad \frac{2}{\dfrac{1}{c_{k+1}} + \dfrac{3}{{c_{k+1}}^2}} < \frac{2}{\dfrac{1}{c_k} + \dfrac{3}{{c_k}^2}}.$$

But this states that $c_{k+2} < c_{k+1}$, and therefore by mathematical induction , $c_{n+1} < c_n$ for all n. It follows that $c_1 = 2$ is an upper bound, and because no term can be negative, $V = 0$ is a lower bound. By Theorem 10.7, the sequence has a limit L that we can obtain by solving the equation $L = 2L^2/(3+L)$. This equation can be expressed in the form $L^2 - 3L = 0$, and since $L = 3$ is greater that $U = 2$, the limit must be $L = 0$.

40. It is advantageous to express the recursive definition in the form $c_{n+1} = -1 + \dfrac{6}{4 - c_n}$.

The first four terms of the sequence are $c_1 = 3/2$, $c_2 = 7/5$, $c_3 = 17/13$, $c_4 = 43/35$. It would be prudent to first verify that the sequence is decreasing in which case an upper bound would be immediate. Unfortunately, information on bounds is required to complete the proof. We therefore begin by proving that $1 \le c_n \le 2$. This is certainly true for $n = 1$. Suppose $1 \le c_k \le 2$ for some integer k. Then, $-1 \ge -c_k \ge -2$, from which $3 \ge 4 - c_k \ge 2$. Thus, $1/3 \le 1/(4 - c_k) \le 1/2$, and $2 \le 6/(4 - c_k) \le 3$. Hence, $1 \le -1 + 6/(4 - c_k) \le 2$; that is, $1 \le c_{k+1} \le 2$. By mathematical induction then, $1 \le c_n \le 2$ for all n. Now we verify that $c_{n+1} < c_n$. This is true for $n = 1$. Suppose $c_{k+1} < c_k$ for some integer k. Then $-c_{k+1} > -c_k$, and, $4 - c_{k+1} > 4 - c_k$. Since both sides are positive, we may invert,

$$\frac{1}{4 - c_{k+1}} < \frac{1}{4 - c_k} \quad \Longrightarrow \quad -1 + \frac{6}{4 - c_{k+1}} < -1 + \frac{6}{4 - c_k}.$$

Thus, $c_{k+2} < c_{k+1}$, and by mathematical induction, $c_{n+1} < c_n$ for all n. Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L that must satisfy the equation $L = (L+2)/(4-L)$. This equation reduces to $L^2 - 3L + 2 = 0$, and of the two solutions $L = 1$ and $L = 2$ only $L = 1$ could be the limit.

41. It is advantageous to express the recursive definition in the form $c_{n+1} = \dfrac{1}{2} + \dfrac{1}{2(5 - 2c_n)}$.

The first four terms of the sequence are $c_1 = 0$, $c_2 = 3/5$, $c_3 = 12/19 = 0.6316$, $c_4 = 0.6338$. It would be prudent to first verify that the sequence is increasing in which case a lower bound would be immediate. Unfortunately, information on bounds is required to complete the proof. We therefore begin by proving that $0 \le c_n \le 1$. This is certainly true for $n = 1$. Suppose $0 \le c_k \le 1$ for some integer k. Then, $0 \ge -2c_k \ge -2$, from which $5 \ge 5 - 2c_k \ge 3$. Thus, $2(5) \ge 2(5 - 2c_k) \ge 2(3)$, and we can say that $\dfrac{1}{10} \le \dfrac{1}{2(5 - 2c_k)} \le \dfrac{1}{6}$. It follows that $\dfrac{1}{2} + \dfrac{1}{10} \le \dfrac{1}{2} + \dfrac{1}{2(5 - 2c_k)} \le \dfrac{1}{2} + \dfrac{1}{6}$; that is, $0 < \dfrac{3}{5} \le c_{k+1} \le \dfrac{2}{3} < 1$. By mathematical induction then, $0 \le c_n \le 1$ for all n. Now we verify that $c_{n+1} > c_n$. This is true for $n = 1$. Suppose $c_{k+1} > c_k$ for some integer k. Then $-2c_{k+1} < -2c_k$, and, $5 - 2c_{k+1} < 5 - 2c_k$. Thus, $2(5 - 2c_{k+1}) < 2(5 - 2c_k)$. Since both sides are positive, we may invert,

$$\frac{1}{2(5-2c_{k+1})} > \frac{1}{2(5-2c_k)} \quad \Longrightarrow \quad \frac{1}{2} + \frac{1}{2(5-2c_{k+1})} > \frac{1}{2} + \frac{1}{2(5-2c_k)}.$$

Thus, $c_{k+2} > c_{k+1}$, and by mathematical induction, $c_{n+1} > c_n$ for all n. Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit L that must satisfy the equation $L = (3-L)/(5-2L)$. This equation reduces to $2L^2 - 6L + 3 = 0$, and of the two solutions $(3 \pm \sqrt{3})/2$, only $(3-\sqrt{3})/2$ lies between the bounds of the sequence. Hence, $L = (3-\sqrt{3})/2$.

42. First we verify that 0 and 1 are bounds for the sequence; that is, $0 \le c_n \le 1$. This is true for $n = 1$. Suppose that $0 \le c_k \le 1$. Then, $0 \ge -c_k \ge -1$ and $0 \ge -{c_k}^2 \ge -1$. When we add these, $0 \ge -c_k - {c_k}^2 \ge -2$, and if we add 4, $4 \ge 4 - c_k - {c_k}^2 \ge 2$. Inverting this gives $1/4 \le 1/(4 - c_k - {c_k}^2) \le 1/2$. Hence, $0 < 1/4 \le c_{k+1} \le 1/2 < 1$. Consequently, by mathematical induction, $0 \le c_n \le 1$ for all $n \ge 1$.

We now verify that the sequence is decreasing by showing that $c_{n+1} < c_n$. This is true for $n = 1$ since $c_2 = 1/2 < c_1$. Suppose that $c_{k+1} < c_k$. Then $-c_{k+1} > -c_k$ and $-{c_{k+1}}^2 > -{c_k}^2$. When we add these, $-c_{k+1} - {c_{k+1}}^2 > -c_k - {c_k}^2$, and if we add 4, $4 - c_{k+1} - {c_{k+1}}^2 > 4 - c_k - {c_k}^2$. Because both sides of this inequality are positive, we may invert and reverse the sign, $1/(4 - c_{k+1} - {c_{k+1}}^2) < 1/(4 - c_k - {c_k}^2)$; that is, $c_{k+2} < c_{k+1}$. By mathematical induction, then, $c_{n+1} < c_n$ for all $n \ge 1$.

Because the sequence is monotonic and bounded, Theorem 10.7 guarantees that it has a limit that must satisfy the equation $L = 1/(4 - L - L^2)$. This equation reduces to $f(L) = L^3 + L^2 - 4L + 1 = 0$. The first nine terms of the sequence are

$$\begin{array}{lllll} x_1 = 1 & x_2 = 1/2 & x_3 = 4/13 & x_4 = 0.278 & x_5 = 0.274 \\ x_6 = 0.273\,903 & x_7 = 0.273\,892 & x_8 = 0.273\,891 & x_9 = 0.273\,891. & \end{array}$$

Since $f(0.273\,885) = 1.8 \times 10^{-5}$ and $f(0.273\,895) = -1.4 \times 10^{-5}$, the limit is 0.273 89 (to 5 decimals).

43. (a) If L is the limit of the sequence $\{c_n\}_1^\infty$, then L is also the limit of $\{c_{n-1}\}_2^\infty$. By Theorem 10.10, the sequence $\{c_n - c_{n-1}\}_2^\infty$ has limit $L - L = 0$.
(b) The sequence diverges because $\lim_{n\to\infty} \ln n = \infty$, yet

$$\lim_{n\to\infty} (c_{n+1} - c_n) = \lim_{n\to\infty} [\ln(n+1) - \ln n] = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = 0.$$

44. The first four terms of the sequence are $c_1 = 2$, $c_2 = 6/5 = 1.2$, $c_3 = 1.36$, and $c_4 = 1.328$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{8 - c_n}{5} - \frac{8 - c_{n-1}}{5} = -\frac{1}{5}(c_n - c_{n-1}).$$

This shows that the differences $c_{n+1} - c_n$ alternate in sign, and therefore the sequence oscillates. Because $|c_{n+1} - c_n| = |c_n - c_{n-1}|/5$, absolute values $|c_{n+1} - c_n|$ decrease and approach 0. According to Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{8 - c_n}{5} \quad \Longrightarrow \quad L = \frac{8 - L}{5} \quad \Longrightarrow \quad L = \frac{4}{3}.$$

45. The first four terms of the sequence are $c_1 = 20$, $c_2 = 243/20 = 12.15$, $c_3 = 12.247$, and $c_4 = 12.245$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \left(12 + \frac{3}{c_n}\right) - \left(12 + \frac{3}{c_{n-1}}\right) = -\frac{3(c_n - c_{n-1})}{c_n c_{n-1}}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{3|c_n - c_{n-1}|}{c_n c_{n-1}}.$$

Since all terms of the sequence are greater than 12 (the recursive definition makes this clear), it follows that

$$|c_{n+1} - c_n| < \frac{3|c_n - c_{n-1}|}{(12)(12)} = \frac{|c_n - c_{n-1}|}{48}.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty}\left(12 + \frac{3}{c_n}\right) \quad \Longrightarrow \quad L = 12 + \frac{3}{L}.$$

Of the two solutions $6 \pm \sqrt{39}$ of this equation, only $L = 6 + \sqrt{39}$ is positive.

46. The first four terms of the sequence are $c_1 = 1$, $c_2 = 1/3$, $c_3 = 3/7$, and $c_4 = 7/17$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{1}{2 + c_n} - \frac{1}{2 + c_{n-1}} = -\frac{(c_n - c_{n-1})}{(2 + c_n)(2 + c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{|c_n - c_{n-1}|}{(2 + c_n)(2 + c_{n-1})}.$$

Since all terms of the sequence are positive, it follows that

$$|c_{n+1} - c_n| < \frac{|c_n - c_{n-1}|}{(2)(2)} = \frac{1}{4}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{2 + c_n} \quad \Longrightarrow \quad L = \frac{1}{2 + L}.$$

Of the two solutions $-1 \pm \sqrt{2}$ of this equation, only $L = -1 + \sqrt{2}$ is positive.

47. The first four terms of the sequence are $c_1 = 10$, $c_2 = 1/23 = 0.044$, $c_3 = 23/71 = 0.324$, and $c_4 = 71/259 = 0.274$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{1}{3 + 2c_n} - \frac{1}{3 + 2c_{n-1}} = -\frac{2(c_n - c_{n-1})}{(3 + 2c_n)(3 + 2c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{2|c_n - c_{n-1}|}{(3 + 2c_n)(3 + 2c_{n-1})} < \frac{2|c_n - c_{n-1}|}{(3)(3)} = \frac{2}{9}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{3 + 2c_n} \quad \Longrightarrow \quad L = \frac{1}{3 + 2L}.$$

Of the two solutions $(-3 \pm \sqrt{17})/4$ of this equation, only $L = (-3 + \sqrt{17})/4$ is positive.

48. The inequality $1 \le c_n \le 2$ is valid for $n = 1$. Suppose $1 \le c_k \le 2$. Then, $2 \le 1 + c_k \le 3$, and inverting, $1/2 \ge 1/(1 + c_k) \ge 1/3$. Multiplication by 3 gives $3/2 \ge 3/(1 + c_k) \ge 1$, and this states that $1 \le c_{k+1} \le 3/2 < 2$. Hence, by mathematical induction, $1 \le c_n \le 2$ for all $n \ge 1$.

The first four terms of the sequence are $c_1 = 1$, $c_2 = 3/2$, $c_3 = 6/5$, and $c_4 = 15/11$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{3}{1+c_n} - \frac{3}{1+c_{n-1}} = -\frac{3(c_n - c_{n-1})}{(1+c_n)(1+c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{3|c_n - c_{n-1}|}{(1+c_n)(1+c_{n-1})} < \frac{3|c_n - c_{n-1}|}{(1+1)(1+1)} = \frac{3}{4}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{3}{1+c_n} \quad \Longrightarrow \quad L = \frac{3}{1+L}.$$

Of the two solutions $(-1 \pm \sqrt{13})/2$ of this equation, only $L = (-1+\sqrt{13})/2$ lies between 1 and 2.

49. The first six terms of the sequence are $c_1 = 0$, $c_2 = 3$, $c_3 = 3/4$, $c_4 = 12/7$, $c_5 = 21/19$, and $c_6 = 57/40$. They are oscillating. The inequality $1 \le c_n \le 2$ is valid for $n = 4$. Suppose $1 \le c_k \le 2$. Then, $2 \le 1 + c_k \le 3$, and inverting, $1/2 \ge 1/(1+c_k) \ge 1/3$. Multiplication by 3 gives $3/2 \ge 3/(1+c_k) \ge 1$, and this states that $1 \le c_{k+1} \le 3/2 < 2$. Hence, by mathematical induction, $1 \le c_n \le 2$ for all $n \ge 4$.

To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{3}{1+c_n} - \frac{3}{1+c_{n-1}} = -\frac{3(c_n - c_{n-1})}{(1+c_n)(1+c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{3|c_n - c_{n-1}|}{(1+c_n)(1+c_{n-1})} < \frac{3|c_n - c_{n-1}|}{(1+1)(1+1)} = \frac{3}{4}|c_n - c_{n-1}|, \quad n \ge 5.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{3}{1+c_n} \quad \Longrightarrow \quad L = \frac{3}{1+L}.$$

Of the two solutions $(-1 \pm \sqrt{13})/2$ of this equation, only $L = (-1+\sqrt{13})/2$ lies between 1 and 2.

50. The inequality $1/2 \le c_n \le 1$ is valid for $n = 1$. Suppose $1/2 \le c_k \le 1$. Then, $5/2 \le 5c_k \le 5 \Longrightarrow 9/2 \le 2 + 5c_k \le 7$, and inverting, $2/9 \ge 1/(2+5c_k) \ge 1/7$. Multiplication by 4 gives $8/9 \ge 4/(2+5c_k) \ge 4/7$, and this states that $1/2 < 4/7 \le c_{k+1} \le 8/9 < 1$. Hence, by mathematical induction, $1/2 \le c_n \le 1$ for all $n \ge 1$.

The first four terms of the sequence are $c_1 = 1$, $c_2 = 4/7$, $c_3 = 14/17$, and $c_4 = 17/26$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{4}{2+5c_n} - \frac{4}{2+5c_{n-1}} = -\frac{20(c_n - c_{n-1})}{(2+5c_n)(2+5c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{20|c_n - c_{n-1}|}{(2+5c_n)(2+5c_{n-1})} < \frac{20|c_n - c_{n-1}|}{(2+5/2)(2+5/2)} = \frac{80}{81}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{4}{2+5c_n} \implies L = \frac{4}{2+5L}.$$

Of the two solutions $(-1 \pm \sqrt{21})/5$ of this equation, only $L = (-1+\sqrt{21})/5$ is positive.

51. The first four terms of the sequence are $c_1 = 1$, $c_2 = 5$, $c_3 = 4.58$, and $c_4 = 4.63$. They are oscillating.

The inequality $4 \le c_n \le 5$ is valid for $n = 2$. Suppose $4 \le c_k \le 5$. Then, $-4 \ge -c_k \ge -5$, and adding 26, $22 \ge 26 - c_k \ge 21$. When we take square roots, $\sqrt{22} \ge \sqrt{26 - c_k} \ge \sqrt{21}$, and this states that $4 < \sqrt{21} \le c_{k+1} \le \sqrt{22} < 5$. Hence, by mathematical induction, $4 \le c_n \le 5$ for all $n \ge 2$. This shows that all terms of the sequence are defined. Consider now

$$(c_{n+1})^2 - (c_n)^2 = (26 - c_n) - (26 - c_{n-1}) = -(c_n - c_{n-1}).$$

When we factor the left side into $(c_{n+1} + c_n)(c_{n+1} - c_n)$, the above equation can be rewritten in the form

$$c_{n+1} - c_n = \frac{-(c_n - c_{n-1})}{c_{n+1} + c_n}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{|c_n - c_{n-1}|}{c_{n+1} + c_n} \le \frac{|c_n - c_{n-1}|}{4+4} = \frac{1}{8}|c_n - c_{n-1}|, \quad n \ge 2.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt{26 - c_n} \implies L = \sqrt{26 - L}.$$

Of the two solutions $(-1 \pm \sqrt{105})/2$ of this equation, only $L = (-1+\sqrt{105})/2$ lies between 4 and 5.

52. The first four terms of the sequence are $c_1 = 4$, $c_2 = 2.83$, $c_3 = 3.39$, and $c_4 = 3.13$. They are oscillating.

First we prove that $2 \le c_n \le 4$ which is valid for $n = 1$. Suppose $2 \le c_k \le 4$. Then, $-2 \ge -c_k \ge -4$, and multiplying by 3, $-6 \ge -3c_k \ge -12$. Adding 20 gives $14 \ge 20 - 3c_k \ge 8$. When we take square roots, $\sqrt{14} \ge \sqrt{20 - 3c_k} \ge \sqrt{8}$, and this states that $2 < \sqrt{8} \le c_{k+1} \le \sqrt{14} < 4$. Hence, by mathematical induction, $2 \le c_n \le 4$ for all $n \ge 1$. This shows that all terms of the sequence are defined. Consider now

$$(c_{n+1})^2 - (c_n)^2 = (20 - 3c_n) - (20 - 3c_{n-1}) = -3(c_n - c_{n-1}).$$

When we factor the left side into $(c_{n+1} + c_n)(c_{n+1} - c_n)$, the above equation can be rewritten in the form

$$c_{n+1} - c_n = \frac{-3(c_n - c_{n-1})}{c_{n+1} + c_n}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{3|c_n - c_{n-1}|}{c_{n+1} + c_n} \le \frac{3|c_n - c_{n-1}|}{2+2} = \frac{3}{4}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt{20 - 3c_n} \implies L = \sqrt{20 - 3L}.$$

Of the two solutions $(-3 \pm \sqrt{89})/2$ of this equation, only $L = (-3 + \sqrt{89})/2$ is positive.

53. The first four terms of the sequence are $c_1 = 2$, $c_2 = 2/7$, $c_3 = 2$, and $c_4 = 2/7$. In other words, the sequence oscillates back and forth between 2 and 2/7, never converging.

54. The third term is $c_3 = \dfrac{c_2}{4c_2 - 1} = \dfrac{\dfrac{c_1}{4c_1 - 1}}{4\left(\dfrac{c_1}{4c_1 - 1}\right) - 1} = \left(\dfrac{c_1}{4c_1 - 1}\right)\left(\dfrac{4c_1 - 1}{4c_1 - 4c_1 + 1}\right) = c_1.$ In other words, terms of the sequence oscillate back and forth between c_1 and c_2, never converging unless $c_1 = c_2$. This occurs only when $c_1 = 0$ or $c_1 = 1/2$.

55. The first four terms of the sequence are $c_1 = 1$, $c_2 = 5$, $c_3 = 25/9$, and $c_4 = 125/41$. They are oscillating. The inequality $2 \le c_n \le 4$ is valid for $n = 3$. Suppose $2 \le c_k \le 4$. Then, $1/2 \ge 1/c_k \ge 1/4$, and $-1/2 \le -1/c_k \le -1/4$. Adding 2 gives

$$\frac{3}{2} \le 2 - \frac{1}{c_k} \le \frac{7}{4} \quad \Longrightarrow \quad \frac{2}{3} \ge \frac{1}{2 - \dfrac{1}{c_k}} \ge \frac{4}{7}.$$

Multiplication be 5 yields

$$\frac{10}{3} \ge \frac{5}{2 - \dfrac{1}{c_k}} \ge \frac{20}{7} \quad \Longrightarrow \quad 2 < \frac{20}{7} \le c_{k+1} \le \frac{10}{3} < 4.$$

Hence, by mathematical induction, $2 \le c_n \le 4$ for all $n \ge 3$. To show that the entire sequence oscillates, we consider

$$c_{n+1} - c_n = \frac{5c_n}{2c_n - 1} - \frac{5c_{n-1}}{2c_{n-1} - 1} = \frac{-5(c_n - c_{n-1})}{(2c_n - 1)(2c_{n-1} - 1)}.$$

Since $c_1 = 1$, $c_2 = 5$, and $2 \le c_n \le 4$ for $n \ge 3$, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{5|c_n - c_{n-1}|}{(2c_n - 1)(2c_{n-1} - 1)} \le \frac{5|c_n - c_{n-1}|}{(3)(3)} = \frac{5}{9}|c_n - c_{n-1}|, \quad n \ge 4.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{5c_n}{2c_n - 1} \quad \Longrightarrow \quad L = \frac{5L}{2L - 1}.$$

Of the two solutions 0 and 3 of this equation, only $L = 3$ could be the limit.

56. The first four terms of the sequence are $c_1 = 3$, $c_2 = 1.913$, $c_3 = 2.007$, and $c_4 = 1.999$. They are oscillating. First we prove that $1 \le c_n \le 3$ which is valid for $n = 1$. Suppose $1 \le c_k \le 3$. Then, $-1 \ge -c_k \ge -3$, and adding 10 gives $9 \ge 10 - c_k \ge 7$. When we take cube roots, $\sqrt[3]{9} \ge \sqrt[3]{10 - c_k} \ge \sqrt[3]{7}$, and this states that $1 < \sqrt[3]{7} \le c_{k+1} \le \sqrt[3]{9} < 3$. Hence, by mathematical induction, $1 \le c_n \le 3$ for all $n \ge 1$. Consider now

$$(c_{n+1})^3 - (c_n)^3 = (10 - c_n) - (10 - c_{n-1}) = -(c_n - c_{n-1}).$$

When we factor the left side into $(c_{n+1} - c_n)({c_{n+1}}^2 + c_{n+1}c_n + {c_n}^2)$, the above equation can be rewritten in the form

$$c_{n+1} - c_n = \frac{-(c_n - c_{n-1})}{{c_{n+1}}^2 + c_{n+1}c_n + {c_n}^2}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{|c_n - c_{n-1}|}{{c_{n+1}}^2 + c_{n+1}c_n + {c_n}^2} \le \frac{|c_n - c_{n-1}|}{1+1+1} = \frac{1}{3}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \sqrt[3]{10 - c_n} \quad \Longrightarrow \quad L = \sqrt[3]{10 - L} \quad \Longrightarrow \quad L = 2.$$

57. The second and third terms of the sequence are

$$c_2 = \frac{a}{ab-1}, \qquad c_3 = \frac{c_2}{bc_2 - 1} = \frac{\dfrac{a}{ab-1}}{\dfrac{ab}{ab-1} - 1} = \left(\frac{a}{ab-1}\right)\left(\frac{ab-1}{ab-ab+1}\right) = a.$$

In other words, terms of the sequence oscillate back and forth between c_1 and c_2, never converging unless $c_1 = c_2$. This occurs for $a = 0$ which is not permitted, and for $ab = 2$.

58. (i) Suppose $\epsilon > 0$ is any given number. Since $\lim_{n\to\infty} c_n = C$, there exists an N such that $|c_n - C| < \epsilon/|k|$ for all $n > N$. For such n,

$$|kc_n - kC| = |k||c_n - C| < |k|\left(\frac{\epsilon}{|k|}\right) = \epsilon.$$

This proves that $\lim_{n\to\infty} kc_n = kC$.
(ii) Suppose $\epsilon > 0$ is any given number. Since $\lim_{n\to\infty} c_n = C$, there exists an N_1 such that $|c_n - C| < \epsilon/2$ for all $n > N_1$. Similarly, since $\lim_{n\to\infty} d_n = D$, there exists an N_2 such that $|d_n - D| < \epsilon/2$ for all $n > N_2$. For all n greater than the larger of N_1 and N_2,

$$|(c_n + d_n) - (C + D)| = |(c_n - C) + (d_n - D)| \le |c_n - C| + |d_n - D| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves that $\lim_{n\to\infty} (c_n + d_n) = C + D$. The proof that $\lim_{n\to\infty} (c_n - d_n) = C - D$ is similar.

59. The first four terms of the sequence are $c_1 = 1$, $c_2 = 2/3$, $c_3 = 5/7$, $c_4 = 12/17$. All terms of the sequence are clearly positive, and therefore a lower bound is $V = 0$. Since

$$1 - c_n = 1 - \frac{1 + c_{n-1}}{1 + 2c_{n-1}} = \frac{c_{n-1}}{1 + 2c_{n-1}} > 0,$$

it follows that $c_n < 1$ for all n, and therefore 1 is an upper bound.

60. The first four terms of the sequence are $c_1 = -30$, $c_2 = -20$, $c_3 = -15$, $c_4 = -55/6$. The sequence appears to be increasing, $c_{n+1} > c_n$. This is true for $n = 1, 2$. Suppose that $c_k > c_{k-1}$ and $c_{k+1} > c_k$. Then, $c_k/3 > c_{k-1}/3$ and $c_{k+1}/2 > c_k/2$. Addition of these gives

$$\frac{c_{k+1}}{2} + \frac{c_k}{3} > \frac{c_k}{2} + \frac{c_{k-1}}{3} \quad \Longrightarrow \quad 5 + \frac{c_{k+1}}{2} + \frac{c_k}{3} > 5 + \frac{c_k}{2} + \frac{c_{k-1}}{3}.$$

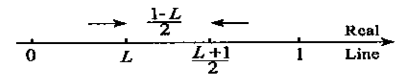
This states that $c_{k+2} > c_{k+1}$, and therefore, by mathematical induction, the sequence is increasing. The first term is therefore a lower bound. We now prove that 30 is an upper bound, $c_n \le 30$. Certainly c_1 and c_2 are both less than 30. Suppose that $c_{k-1} \le 30$ and $c_k \le 30$. Then

$$c_{k+1} = 5 + \frac{c_k}{2} + \frac{c_{k-1}}{3} \le 5 + \frac{30}{2} + \frac{30}{3} = 30.$$

Hence, by mathematical induction, $c_n \le 30$ for all $n \ge 1$. According to Theorem 10.7, the sequence has a limit that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty}\left(5 + \frac{c_n}{2} + \frac{c_{n-1}}{3}\right) \quad \Longrightarrow \quad L = 5 + \frac{L}{2} + \frac{L}{3} \quad \Longrightarrow \quad L = 30.$$

61. (a) If $\lim_{n\to\infty} c_n = L$, then according to Definition 10.4, given $\epsilon = (1-L)/2$, there exists an integer N such that for all $n > N$,

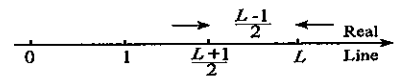


$$|c_n - L| < \epsilon = \frac{1-L}{2}.$$

This is equivalent to $\quad -\dfrac{1-L}{2} < c_n - L < \dfrac{1-L}{2} \quad \Longrightarrow \quad L - \dfrac{1-L}{2} < c_n < \dfrac{1-L}{2} + L.$

The right inequality implies that $c_n < (L+1)/2$.

(b) If $\lim_{n\to\infty} c_n = L$, then given $\epsilon = (L-1)/2$, there exists an integer N such that for all $n > N$,



$$|c_n - L| < \epsilon = \frac{L-1}{2}.$$

This is equivalent to

$$-\frac{L-1}{2} < c_n - L < \frac{L-1}{2} \quad \Longrightarrow \quad L - \frac{L-1}{2} < c_n < \frac{L-1}{2} + L.$$

The left inequality implies that $c_n > (L+1)/2$.

62. If $\lim_{n\to\infty} c_n = L$, then given $\epsilon = 1$, there exists an integer N such that for all $n > N$, $|c_n - L| < \epsilon = 1$. This is equivalent to $-1 < c_n - L < 1 \Longrightarrow L - 1 < c_n < L + 1$.

63. (a) The first ten terms of the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

(b) The sequence is clearly increasing. It is bounded below by $V = c_1 = 1$, but it has no upper bound or limit.

(c) This is true for $n = 2$ since ${c_2}^2 - c_1c_3 = (1)^2 - (1)(2) = -1 = (-1)^{2+1}$. Suppose ${c_k}^2 - c_{k-1}c_{k+1} = (-1)^{k+1}$ for some integer k. Then,

$$\begin{aligned}
{c_{k+1}}^2 - c_k c_{k+2} &= {c_{k+1}}^2 - c_k(c_k + c_{k+1}) \qquad (\text{since } c_{k+2} = c_k + c_{k+1})\\
&= {c_{k+1}}^2 - {c_k}^2 - c_k c_{k+1}\\
&= -{c_k}^2 + c_{k+1}(c_{k+1} - c_k)\\
&= -{c_k}^2 + c_{k+1}(c_{k-1}) \qquad (\text{since } c_{k+1} = c_k + c_{k-1})\\
&= -[{c_k}^2 - c_{k-1}c_{k+1}]\\
&= -(-1)^{k+1} \qquad (\text{by assumption})\\
&= (-1)^{(k+1)+1}.
\end{aligned}$$

The result is therefore valid for $k+1$, and by mathematical induction it is true for all $n \geq 2$.

(d) When $n = 1$, the formula gives $\dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right) - \left(\dfrac{1-\sqrt{5}}{2}\right)\right] = 1$, and this is c_1. Thus, the formula is correct for $n = 1$. When $n = 2$, the formula gives

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{4\sqrt{5}}[(1 + 2\sqrt{5} + 5) - (1 - 2\sqrt{5} + 5)] = 1.$$

Thus, the formula is also true for $n = 2$. Suppose it is valid for integers $k-1$ and k; that is,

$$c_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right] \quad \text{and} \quad c_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right].$$

Then,

$$c_{k+1} = c_{k-1} + c_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(1+\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(1+\frac{1-\sqrt{5}}{2}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)\right].$$

Since $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1}{4}(1+2\sqrt{5}+5) = \frac{3+\sqrt{5}}{2}$, and similarly, $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$, we may write that

$$c_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right].$$

Consequently the formula is valid for $k+1$, and by mathematical induction it is valid for all $n \ge 1$.

(e) The first four terms of the sequence are

$$b_1 = \frac{c_2}{c_1} = 1, \quad b_2 = \frac{c_3}{c_2} = 2, \quad b_3 = \frac{c_4}{c_3} = \frac{3}{2}, \quad b_4 = \frac{c_5}{c_4} = \frac{5}{3}.$$

The sequence is not monotonic. To investigate whether $\{b_n\}$ has a limit we use the explicit formula for c_n in part (d),

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}.$$

We now divide both numerator and denominator by $[(1+\sqrt{5})/2]^n$,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n} = \frac{1+\sqrt{5}}{2}$$

since $|(1-\sqrt{5})/(1+\sqrt{5})| < 1$.

64. We concentrate only on female rabbits. After the first month, we have the original adult female and a newborn female. Hence $R_1 = 1$. After the second month,we have the original adult female, a one-month old female, and a newborn female. Hence, $R_2 = 1$. If R_n is the number of adult females after n months, then R_{n+1} is the sum of R_n and the number of one-month old females after n months. But the number of one-month old females after n months is the number of adult females R_{n-1} after $n-1$ months. In other words, a recursive definition for the sequence is $R_1 = 1$, $R_2 = 1$, $R_{n+1} = R_n + R_{n-1}$. This is the Fibonacci sequence.

65. If we assume that $c_{k+1} > c_k$ for some integer k, then $2bc_{k+1} > 2bc_k$, and $a + 2bc_{k+1} > a + 2bc_k$. Taking square roots gives $\sqrt{a + 2bc_{k+1}} > \sqrt{a + 2bc_k}$; that is, $c_{k+2} > c_{k+1}$. Thus, the sequence is increasing if and only if

$$c_2 > c_1 \iff \sqrt{a + 2bc_1} > d \iff a + 2bd > d^2 \iff d^2 - 2bd - a < 0.$$

Since $d^2 - 2bd - a = 0$ when $d = (2b \pm \sqrt{4b^2 + 4a})/2 = b \pm \sqrt{a + b^2}$, it follows that the sequence is increasing if and only if $d < b + \sqrt{a + b^2}$. When $d = b + \sqrt{a + b^2}$, all terms of the sequence are equal to d.

66. (a) $|c_n d_n - CD| = |(c_n d_n - Dc_n) + (Dc_n - CD)| \le |c_n||d_n - D| + |D||c_n - C|$

(b) If $\lim_{n\to\infty} c_n = C$, then $\lim_{n\to\infty} |c_n| = |C|$. According to Exercise 62, there exists an N_1 such that for $n > N_1$, $|c_n| < |C| + 1$.

Because $\lim_{n\to\infty} d_n = D$, given any $\epsilon > 0$, there exists an N_2 such that $|d_n - D| < \dfrac{\epsilon}{2(|C| + 1)}$ for $n > N_2$. Similarly, there exists an N_3 such that for $n > N_3$, $|c_n - C| < \dfrac{\epsilon}{2|D| + 1}$.

(c) When n is greater than the largest of N_1, N_2, and N_3,

$$\begin{aligned}
|c_n d_n - CD| &\le |c_n||d_n - D| + |D||c_n - C| \\
&< (|C| + 1)\left[\frac{\epsilon}{2(|C| + 1)}\right] + |D|\left[\frac{\epsilon}{2|D| + 1}\right] \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \left(\text{since } \frac{|D|}{2|D| + 1} < \frac{1}{2}\right) \\
&= \epsilon.
\end{aligned}$$

This proves that $\lim_{n\to\infty} c_n d_n = CD$.

67. (a)
$$\begin{aligned}
\left|\frac{c_n}{d_n} - \frac{C}{D}\right| &= \left|\frac{Dc_n - Cd_n}{Dd_n}\right| = \left|\frac{D(c_n - C) + C(D - d_n)}{Dd_n}\right| \le \left|\frac{D(c_n - C)}{Dd_n}\right| + \left|\frac{C(D - d_n)}{Dd_n}\right| \\
&= \frac{|c_n - C|}{|d_n|} + \frac{|C||d_n - D|}{|D||d_n|}
\end{aligned}$$

(b) If $\lim_{n\to\infty} d_n = D$, then $\lim_{n\to\infty} |d_n| = |D|$. Certainly there exists an N_1 such that for $n > N_1$, $|d_n| > |D|/2$, else the terms would not have limit $|D|$.

Because $\lim_{n\to\infty} c_n = C$, given any $\epsilon > 0$, there exists an N_2 such that for $n > N_2$, $|c_n - C| < \epsilon|D|/4$. Similarly, there exists an N_3 such that for $n > N_3$, $|d_n - D| < \epsilon|D|^2/(4|C| + 1)$.

(c) When n is greater than the largest of N_1, N_2, and N_3,

$$\begin{aligned}
\left|\frac{c_n}{d_n} - \frac{C}{D}\right| &\le \frac{|c_n - C|}{|d_n|} + \frac{|C||d_n - D|}{|D||d_n|} < \frac{\epsilon|D|/4}{|D|/2} + \frac{|C|\epsilon|D|^2/(4|C| + 1)}{|D||D|/2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \left(\text{since } \frac{2|C|}{4|C| + 1} < \frac{1}{2}\right) \\
&= \epsilon.
\end{aligned}$$

68. The first ten terms of the sequence with corresponding values of n written above them are

$$\begin{array}{cccccccccc}
n=1 & n=2 & n=3 & n=4 & n=5 & n=6 & n=7 & n=8 & n=9 & n=10 \\
1 & 2 & \frac{3}{2} & \frac{7}{4} & \frac{13}{8} & \frac{27}{16} & \frac{53}{32} & \frac{107}{64} & \frac{213}{128} & \frac{427}{256}
\end{array}$$

The denominators are clearly powers of 2, and for $n \ge 2$ they are 2^{n-2}. Inspection of the numerators indicates that the n^{th} numerator is twice the $(n-1)^{\text{th}}$ numerator with 1 added if n is even and 1 subtracted if n is odd. Let us denote these numerators by d_n for $n \ge 2$. Then,

$$\begin{aligned}
d_3 &= 3 = 2^2 - 1, \\
d_4 &= 7 = 2d_3 + 1 = 2(2^2 - 1) + 1 = 2^3 - 2 + 1, \\
d_5 &= 13 = 2d_4 - 1 = 2(2^3 - 2 + 1) - 1 = 2^4 - 2^2 + 2 - 1, \\
d_6 &= 27 = 2d_5 + 1 = 2(2^4 - 2^2 + 2 - 1) + 1 = 2^5 - 2^3 + 2^2 - 2 + 1, \\
d_7 &= 53 = 2d_6 - 1 = 2(2^5 - 2^3 + 2^2 - 2 + 1) - 1 = 2^6 - 2^4 + 2^3 - 2^2 + 2 - 1.
\end{aligned}$$

The pattern emerging is that

$$d_n = 2^{n-1} - 2^{n-3} + 2^{n-4} - \cdots + (-1)^n.$$

If we multiply this by 2,

$$2d_n = 2^n - 2^{n-2} + 2^{n-3} - \cdots + 2(-1)^n,$$

and then add it to d_n,

$$3d_n = 2^n + 2^{n-1} - 2^{n-2} + (-1)^n.$$

Thus,

$$d_n = \frac{1}{3}\left[2^n + 2^{n-1} - 2^{n-2} + (-1)^n\right] = \frac{1}{3}\left[5 \cdot 2^{n-2} + (-1)^n\right].$$

Finally then, for $n \geq 2$,

$$c_n = \frac{d_n}{2^{n-2}} = \frac{1}{3 \cdot 2^{n-2}}\left[5 \cdot 2^{n-2} + (-1)^n\right] = \frac{5}{3} + \frac{(-1)^n}{3 \cdot 2^{n-2}}.$$

This formula also gives $c_1 = 1$.

EXERCISES 10.9

1. Since $\lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$, the series diverges by the n^{th} term test.

2. $\sum_{n=1}^{\infty} \frac{2^n}{5^{n+1}} = \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$, a geometric series with sum $\frac{1}{5}\left(\frac{2/5}{1-2/5}\right) = \frac{2}{15}$.

3. Since $\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{2}\right) = 0 - 1 + 0 + 1 + 0 - 1 + 0 + 1 + \cdots$, terms do not approach zero, and the series diverges by the n^{th} term test.

4. Since $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$ (see expression 1.68), the series diverges by the n^{th} term test.

5. This is a geometric series with common ratio 49/9, and therefore the series diverges.

6. $\sum_{n=1}^{\infty} \frac{7^{n+3}}{3^{2n-2}} = \frac{7^3}{3^{-2}}\sum_{n=1}^{\infty}\left(\frac{7}{9}\right)^n$ is a geometric series with sum $7^3(3)^2\left(\frac{7/9}{1-7/9}\right) = \frac{21\,609}{2}$.

7. Since $\lim_{n\to\infty}\sqrt{\frac{n^2-1}{n^2+1}} = 1$, the series diverges by the n^{th} term test.

8. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n$ is a geometric series with sum $\frac{-1/2}{1+1/2} = -\frac{1}{3}$.

9. Since terms of the series become arbitrarily large as n increases, the series diverges by the n^{th} term test.

10. Since $\lim_{n\to\infty} \mathrm{Tan}^{-1} n = \frac{\pi}{2}$, the series diverges by the n^{th} term test.

11. $0.666\,666\ldots = 0.6 + 0.06 + 0.006 + \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots = \frac{6/10}{1-1/10} = \frac{2}{3}$

12.
$$0.131\,313\,131\ldots = 0.13 + 0.001\,3 + 0.000\,013 + \cdots = \frac{13}{100} + \frac{13}{10\,000} + \frac{13}{1\,000\,000} + \cdots$$
$$= \frac{13/100}{1-1/100} = \frac{13}{99}$$

13. $1.347\,346\,346\,346\ldots = 1.347 + 0.000\,346 + 0.000\,000\,346 + \cdots = \dfrac{1347}{1000} + \dfrac{346}{10^6} + \dfrac{346}{10^9} + \cdots$

$$= \frac{1347}{1000} + \frac{346/10^6}{1 - 1/10^3} = \frac{1\,345\,999}{999\,000}$$

14. $43.020\,502\,050\,205\ldots = 43 + 0.020\,5 + 0.000\,002\,05 + \cdots = 43 + \dfrac{205}{10^4} + \dfrac{205}{10^8} + \cdots$

$$= 43 + \frac{205/10^4}{1 - 1/10^4} = \frac{430\,162}{9999}$$

15. If $\sum c_n$ and $\sum d_n$ converge, then $\sum (c_n + d_n)$ converges.
Proof: Let $\{C_n\}$ and $\{D_n\}$ be the sequences of partial sums for $\sum c_n$ and $\sum d_n$ with limits C and D. The sequence of partial sums for $\sum (c_n + d_n)$, is $\{C_n + D_n\}$. According to part (ii) of Theorem 10.10, this sequence has limit $C + D$. Consequently, $\sum (c_n + d_n)$ converges to $C + D$.

16. If $\sum c_n$ converges and $\sum d_n$ diverges, then $\sum (c_n + d_n)$ diverges.
Proof: Assume to the contrary that $\sum (c_n + d_n)$ converges. Let $\{C_n\}$ and $\{D_n\}$ be the sequences of partial sums for $\sum c_n$ and $\sum d_n$. It follows that $\lim_{n\to\infty} C_n$ exists, call it C, but $\lim_{n\to\infty} D_n$ does not exist. $\{C_n + D_n\}$ is the sequence of partial sums for $\sum (c_n + d_n)$, and by assumption, it has a limit, call it E. But then according to part (ii) of Theorem 10.10, the sequence $\{(C_n + D_n) - C_n\} = \{D_n\}$ must have limit $E - C$, a contradiction. Consequently, our assumption that $\sum (c_n + d_n)$ converges must be incorrect.

17. If $\sum c_n$ and $\sum d_n$ diverge, then $\sum (c_n + d_n)$ may converge or diverge.
Proof: We give an example of each situation. The series $\sum n$ and $\sum (-n)$ both diverge, but their sum $\sum (n - n) = \sum 0$ has sum 0. On the other hand, the sum of $\sum n$ and $\sum n$ is $\sum 2n$ which diverges.

18. Since $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n}$ are both geometric series with sums

$$\sum_{n=1}^{\infty} \frac{2^n}{4^n} = \frac{1/2}{1 - 1/2} = 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \frac{3/4}{1 - 3/4} = 3,$$

then, by Exercise 15, $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = 1 + 3 = 4.$

19. Since $\displaystyle\sum_{n=1}^{\infty} (3/2)^n$ is a divergent geometric series, and $\displaystyle\sum_{n=1}^{\infty} (1/2)^n$ is a convergent geometric series, it follows from Exercise 16, that the given series diverges. (It also diverges by the n^{th} term test.)

20. Since $\displaystyle\lim_{n\to\infty} \frac{n^2 + 2^{2n}}{4^n} = \lim_{n\to\infty} \left(\frac{n^2}{4^n} + 1\right) = 1$, the series diverges by the n^{th} term test.

21. Since $\displaystyle\lim_{n\to\infty} \frac{2^n + 4^n - 8^n}{2^{3n}} = \lim_{n\to\infty} \left(\frac{1}{2^{2n}} + \frac{1}{2^n} - 1\right) = -1$, the series diverges by the n^{th} term test.

22. Since $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, the n^{th} partial sum of the series is

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Since $\displaystyle\lim_{n\to\infty} S_n = 1$, it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

23. The total distance travelled is $20 + \displaystyle\sum_{n=1}^{\infty} 40(0.99)^n$. The series is geometric with sum $20 + \dfrac{40(0.99)}{1 - 0.99} = 3980$ m.

24. The total time taken to come to rest is

$$\sqrt{\frac{40}{9.81}}+t_1+t_2+t_3+\cdots=\sqrt{\frac{40}{9.81}}+\sum_{n=1}^{\infty}t_n=\sqrt{\frac{40}{9.81}}+\sum_{n=1}^{\infty}\frac{4}{\sqrt{0.981}}(0.99)^{n/2}$$
$$=\sqrt{\frac{40}{9.81}}+\frac{4\sqrt{0.99}/\sqrt{0.981}}{1-\sqrt{0.99}}=804 \text{ s.}$$

25. The total distance run by the dog is $\dfrac{2}{3}+\displaystyle\sum_{n=1}^{\infty}\frac{8}{3^{n+1}}=\frac{2}{3}+\frac{8/9}{1-1/3}=2$ km.

We could also have reasoned this without series. Since the dog runs twice as fast as the farmer, and the farmer walks 1 km, the dog must run 2 km.

26. According to Exercise 10.1–61,

$$A_n=\frac{\sqrt{3}P^2}{36}\left(1+\frac{1}{3}+\frac{4}{3^3}+\frac{4^2}{3^5}+\cdots+\frac{4^{n-1}}{3^{2n-1}}\right)\quad\text{(a finite geometric series after first term)}$$
$$=\frac{\sqrt{3}P^2}{36}\left\{1+\frac{(1/3)[1-(4/9)^n]}{1-4/9}\right\}\quad\text{(using 10.39a)}$$
$$=\frac{\sqrt{3}P^2}{180}\left[8-3\left(\frac{4}{9}\right)^n\right].$$

Thus, $\displaystyle\lim_{n\to\infty}A_n=\frac{\sqrt{3}P^2}{180}(8)=\frac{2\sqrt{3}P^2}{45}$.

27. The inequality is certainly true for $x \ge 0$ and any n. To discuss the case when $x<0$, we sum the geometric series

$$1+x+x^2+\cdots+x^n=\frac{1-x^{n+1}}{1-x}.$$

When $x<0$ and n is even, then $1-x^{n+1}>0$ and $1-x>0$. Hence, $(1-x^{n+1})/(1-x)>0$. When n is odd, and $-1\le x<0$, then $1-x^{n+1}\ge 0$ and $1-x>0$. Hence, $(1-x^{n+1})/(1-x)>0$. Finally, when n is odd, and $x<-1$, then $1-x^{n+1}<0$ and $1-x>0$. Hence, $(1-x^{n+1})/(1-x)<0$. Consequently, the inequality is valid for all x when n is even, and for $x\ge -1$ when n is odd.

28. (a) If we subtract $S_n=1+r+r^2+\cdots+r^{n-1}$ from $T_n=1+2r+3r^2+\cdots+nr^{n-1}$, we obtain

$$T_n-S_n=r+2r^2+3r^3+\cdots+(n-1)r^{n-1}=r[1+2r+3r^2+\cdots+(n-1)r^{n-2}]=r(T_n-nr^{n-1}).$$

When we solve this for T_n and substitute for S_n,

$$T_n=\frac{S_n-nr^n}{1-r}=\frac{\dfrac{1-r^n}{1-r}-nr^n}{1-r}=\frac{1-r^n-nr^n+nr^{n+1}}{(1-r)^2}=\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}.$$

If we now take limits as $n\to\infty$, we obtain

$$\sum_{n=1}^{\infty}nr^{n-1}=\lim_{n\to\infty}\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}=\frac{1}{(1-r)^2},\quad\text{provided } |r|<1.$$

(b) If we set $S(r)=\displaystyle\sum_{n=1}^{\infty}nr^{n-1}$, and integrate with respect to r,

$$\int S(r)\,dr+C=\sum_{n=1}^{\infty}r^n=\frac{r}{1-r}.$$

Differentiation now gives $S(r)=\dfrac{(1-r)(1)-r(-1)}{(1-r)^2}=\dfrac{1}{(1-r)^2}$.

29. $\dfrac{1}{2}+\dfrac{2}{2^2}+\dfrac{3}{2^3}+\dfrac{4}{2^4}+\cdots=\dfrac{1}{2}\left(1+\dfrac{2}{2}+\dfrac{3}{2^2}+\dfrac{4}{2^3}+\cdots\right)=\dfrac{1}{2}\left[\dfrac{1}{(1-1/2)^2}\right]=2$

30. $\dfrac{2}{5}+\dfrac{4}{25}+\dfrac{6}{125}+\dfrac{8}{625}+\cdots=\dfrac{2}{5}\left(1+\dfrac{2}{5}+\dfrac{3}{5^2}+\dfrac{4}{5^3}+\cdots\right)=\dfrac{2/5}{(1-1/5)^2}=\dfrac{5}{8}$

31. $\dfrac{2}{3}+\dfrac{3}{27}+\dfrac{4}{243}+\dfrac{5}{2187}+\cdots=3\left(1+\dfrac{2}{9}+\dfrac{3}{81}+\dfrac{4}{729}+\cdots\right)-3=3\left[\dfrac{1}{(1-1/9)^2}\right]-3=\dfrac{51}{64}$

32. $\dfrac{12}{5}+\dfrac{48}{25}+\dfrac{192}{125}+\dfrac{768}{625}+\cdots=\dfrac{12}{5}\left(1+\dfrac{4}{5}+\dfrac{16}{25}+\dfrac{64}{125}+\cdots\right)=\dfrac{12/5}{1-4/5}=12$

33. The probability that the first person wins on the first toss is $1/2$. The probability that the first person wins on the second toss is the product of the following three probabilities:

probability that first person throws a tail on the first toss $= 1/2$;
probability that second person throws a tail on first toss $= 1/2$;
probability that first person throws a head on second toss $= 1/2$.

The resultant probabilty is $(1/2)(1/2)(1/2) = 1/2^3$. The probability that the first person wins on the third toss is the product of the following five probabilities:

probability that first person throws a tail on the first toss $= 1/2$;
probability that second person throws a tail on first toss $= 1/2$;
probability that first person throws a tail on second toss $= 1/2$.
probability that second person throws a tail on the second toss $= 1/2$;
probability that first person throws a head on third toss $= 1/2$;

The resultant probabilty is $1/2^5$.
Continuation of this process leads to the following probability that the first person to toss wins

$$\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^7}+\cdots=\frac{1/2}{1-1/4}=\frac{2}{3}.$$

34. The probability that the first person wins on the first toss is $1/6$. The probability that the first person wins on the second toss is the product of the following three probabilities:

probability that first person does not throw a six on the first toss $= 5/6$;
probability that second person does not throw a six on first toss $= 5/6$;
probability that first person throws a six on second toss $= 1/6$.

The resultant probabilty is $(5/6)(5/6)(1/6) = 5^2/6^3$. The probability that the first person wins on the third toss is the product of the following five probabilities:

probability that first person does not throw a six on the first toss $= 5/6$;
probability that second person does not throw a six on first toss $= 5/6$;
probability that first person does not throw a six on second toss $= 5/6$.
probability that second person does not throw a six on the second toss $= 5/6$;
probability that first person throws a six on third toss $= 1/6$;

The resultant probabilty is $5^4/6^5$.
Continuation of this process leads to the following probability that the first person to toss wins

$$\frac{1}{6}+\frac{5^2}{6^3}+\frac{5^4}{6^5}+\frac{5^6}{6^7}+\cdots=\frac{1/6}{1-25/36}=\frac{6}{11}.$$

35. Since the radius of convergence of the series is $R=\lim\limits_{n\to\infty}\left|\dfrac{1/2^n}{1/2^{n+1}}\right|=2$, the open interval of convergence is $-2<x<2$. At $x=2$, the power series reduces to $\sum_{n=0}^{\infty}1$ which diverges by the n^{th} term test. At $x=-2$, it reduces to $\sum_{n=0}^{\infty}(-1)^n$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-2<x<2$.

36. Since the radius of convergence of the series is $R=\lim\limits_{n\to\infty}\left|\dfrac{n^2 3^n}{(n+1)^2 3^{n+1}}\right|=1/3$, the open interval of convergence is $-1/3<x<1/3$. At $x=1/3$, the power series reduces to $\sum_{n=1}^{\infty}n^2$ which diverges by the n^{th} term test. At $x=-1/3$, it reduces to $\sum_{n=1}^{\infty}(-1)^n n^2$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-1/3<x<1/3$.

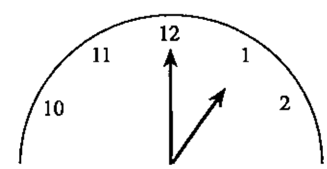
37. Since the radius of convergence of the series is $R = \lim_{n\to\infty} \left| \dfrac{2^n \left(\dfrac{n-1}{n+1}\right)^2}{2^{n+1}\left(\dfrac{n}{n+2}\right)^2} \right| = 1/2$, the open interval of convergence is $7/2 < x < 9/2$. At $x = 9/2$, the power series reduces to $\sum_{n=2}^{\infty} (n-1)^2/(n+1)^2$ which diverges by the n^{th} term test. At $x = 7/2$, it reduces to $\sum_{n=2}^{\infty} (-1)^n(n-1)^2/(n+1)^2$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $7/2 < x < 9/2$.

38. If we set $y = x^3$, the series becomes $\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n y^n$. Since the radius of convergence of this series is $R_y = \lim_{n\to\infty} \left| \dfrac{(-1)^n}{(-1)^{n+1}} \right| = 1$, the radius of convergence of the given series is $R_x = 1$. The open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\sum_{n=0}^{\infty} (-1)^n$ which diverges by the n^{th} term test. At $x = -1$, it reduces to $\sum_{n=0}^{\infty} 1$ which also diverges by the n^{th} term test. The interval of convergence for the series is therefore $-1 < x < 1$.

39. While Achilles makes up the head start L, the tortoise moves a further distance L/c. While Achilles makes up this distance, the tortoise moves a further distance $(L/c)/c = L/c^2$. Continuation of this process gives the following distance traveled by Achilles in catching the tortoise

$$L + \frac{L}{c} + \frac{L}{c^2} + \frac{L}{c^3} + \cdots = \frac{L}{1 - 1/c} = \frac{cL}{c-1}.$$

40. (a) The minute hand moves 12 times as fast as the hour hand. While the minute hand moves through the angle $\pi/6$ radians from 12 at 1:00 to 1 at 1:05, the hour hand moves a further $(\pi/6)/12$ radians. While the minute hand moves through this angle, the hour hand moves through a further angle $[(\pi/6)/12]/12 = (\pi/6)/12^2$. Continuation of this process leads to the following angle traveled by the minute hand in catching the hour hand

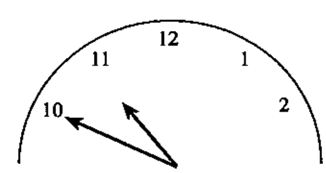


$$\frac{\pi}{6} + \frac{\pi/6}{12} + \frac{\pi/6}{12^2} + \frac{\pi/6}{12^3} + \cdots = \frac{\pi/6}{1 - 1/12} = \frac{2\pi}{11}.$$

This angle represents $\dfrac{2\pi}{11}\left(\dfrac{60}{2\pi}\right) = \dfrac{60}{11}$ minutes after 1:00.

(b) If we take time $t = 0$ at 1:00, the angle θ through which the minute hand moves in time t (in minutes) is $\theta = 2\pi t/60$. The angle ϕ that the hour hand makes with the vertical is $\phi = 2\pi t/720 + \pi/6$. These angles will be the same when $\dfrac{2\pi t}{60} = \dfrac{2\pi t}{720} + \dfrac{\pi}{6}$, the solution of which is $60/11$ minutes.

41. (a) The minute hand moves 12 times as fast as the hour hand. While the minute hand moves through the angle $5\pi/3$ radians from 12 at 10:00 to 10 at 10:50, the hour hand moves a further $(5\pi/3)/12$ radians. While the minute hand moves through this angle, the hour hand moves through a further angle $[(5\pi/3)/12]/12 = (5\pi/3)/12^2$. Continuation of this process leads to the following angle traveled by the minute hand in catching the hour hand



$$\frac{5\pi}{3} + \frac{5\pi/3}{12} + \frac{5\pi/3}{12^2} + \frac{5\pi/3}{12^3} + \cdots = \frac{5\pi/3}{1 - 1/12} = \frac{20\pi}{11}.$$

This angle represents $\dfrac{20\pi}{11}\left(\dfrac{60}{2\pi}\right) = \dfrac{600}{11}$ minutes after 10:00.

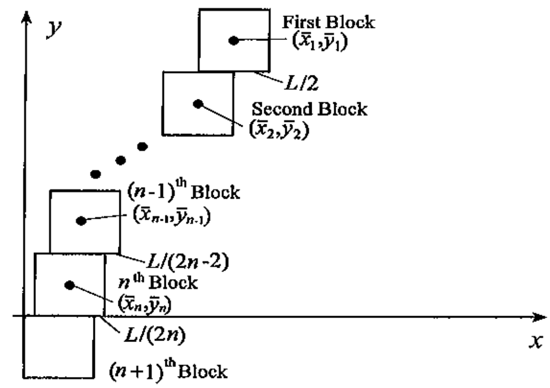
(b) If we take time $t = 0$ at 10:00, the angle θ through which the minute hand moves in time t (in minutes) is $\theta = 2\pi t/60$. The angle ϕ that the hour hand makes with the vertical is $\phi = 2\pi t/720 + 5\pi/3$. These angles will be the same when $\dfrac{2\pi t}{60} = \dfrac{2\pi t}{720} + \dfrac{5\pi}{3}$, the solution of which is $600/11$ minutes.

42. Suppose the length of each block is L.
Taking the density of the blocks as unity, the mass of the top n blocks is nL^3.
The first moment of the n^{th} block about the y-axis is

$$L^3\overline{x}_n = L^3\left(\frac{L}{2}+\frac{L}{2n}\right) = \frac{L^4}{2}\left(1+\frac{1}{n}\right).$$

The first moment of the $(n-1)^{\text{th}}$ block about the y-axis is

$$L^3\overline{x}_{n-1} = L^3\left[\frac{L}{2}+\frac{L}{2n}+\frac{L}{2(n-1)}\right] = \frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}\right).$$



Continuing in this way, the moment of the first block about the y-axis is

$$L^3\overline{x}_1 = L^3\left[\frac{L}{2}+\frac{L}{2n}+\frac{L}{2(n-1)}+\cdots+\frac{L}{2}\right] = \frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right).$$

The x-coordinate of the centre of mass of the top n blocks is therefore

$$\begin{aligned}\overline{x} &= \frac{1}{nL^3}\left[\frac{L^4}{2}\left(1+\frac{1}{n}\right)+\frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}\right)+\cdots+\frac{L^4}{2}\left(1+\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right)\right]\\ &= \frac{L}{2n}\left[n(1)+n\left(\frac{1}{n}\right)+(n-1)\left(\frac{1}{n-1}\right)+\cdots+2\left(\frac{1}{2}\right)+1(1)\right] = \frac{L}{2n}(2n) = L.\end{aligned}$$

Thus, the centre of mass of the top n blocks is over the edge of the $(n+1)^{\text{th}}$ block. They will not tip, but they are in a state of precarious equilibrium.

The right edge of the top block sticks out the following distance over the right edge of the $(n+1)^{\text{th}}$ block

$$\frac{L}{2}+\frac{L}{4}+\frac{L}{6}+\cdots+\frac{L}{2n} = \frac{L}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

This is $L/2$ times the n^{th} partial sum of the harmonic series which we know becomes arbitrarily large as n increases. Hence, the top n blocks can be made to protrude arbitrarily far over the $(n+1)^{\text{th}}$ block.

43. Let $\{S_n\}$ be the sequence of partial sums of the given series. It converges to the sum of the series, call it S. If terms of the series are grouped together, then the sequence of partial sums of the new series, call it $\{T_n\}$, is a subsequence of $\{S_n\}$. But every subseqeuence of a convergent series must converge to the same limit as the sequence. Thus, $\{T_n\}$ converges to S also, and the grouped series has sum S.

44. To verify this, we first write the Laplace transform as an infinite series of integrals

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \sum_{n=0}^{\infty}\int_{np}^{(n+1)p} e^{-st}f(t)\,dt.$$

If we change variables of integration in the n^{th} term with $u = t-np$, then

$$F(s) = \sum_{n=0}^{\infty}\int_0^p e^{-s(u+np)}f(u+np)\,du = \sum_{n=0}^{\infty}e^{-nps}\int_0^p e^{-su}f(u)\,du = \left(\int_0^p e^{-su}f(u)\,du\right)\left(\sum_{n=0}^{\infty}e^{-nps}\right).$$

Since the series is geometric with common ratio e^{-ps},

$$F(s) = \int_0^p e^{-su}f(u)\,du\left[\frac{1}{1-e^{-ps}}\right] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt.$$

45. (a) When V is the voltage across the capacitor, and resistor R_2 (they are in parallel), the currents through these devices are $i_C = CdV/dt$ and $i_{R_2} = V/R_2$. The current through R_1 must be the sum of these, $i_{R_1} = V/R_2 + CdV/dt$. The voltage across R_1 is therefore $R_1(V/R_2 + CdV/dt)$, and it follows that for $V_{\text{in}} = \overline{V}$,

$$\overline{V} = V + R_1\left(\frac{V}{R_2} + C\frac{dV}{dt}\right) \implies \frac{dV}{dt} + \tau V = \alpha\overline{V},$$

where $\tau = (R_1 + R_2)/(R_1R_2C)$ and $\alpha = 1/(R_1C)$.
(b) If we multiply the differential equation by $e^{\tau t}$, the left side becomes the derivative of a product,

$$e^{\tau t}\frac{dV}{dt} + \tau e^{\tau t}V = \alpha\overline{V}e^{\tau t} \Longrightarrow \frac{d}{dt}(Ve^{\tau t}) = \alpha\overline{V}e^{\tau t} \Longrightarrow Ve^{\tau t} = \frac{\alpha\overline{V}}{\tau}e^{\tau t} + D \Longrightarrow V = \frac{\alpha\overline{V}}{\tau} + De^{-\tau t}.$$

Using the condition that $\displaystyle\lim_{t\to 2(n-1)T^+} V(t) = V_{n-1}$, we obtain

$$V_{n-1} = \frac{\alpha\overline{V}}{\tau} + De^{-2\tau(n-1)T} \implies D = \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{2\tau(n-1)T}.$$

Hence, for $2(n-1)T < t < (2n-1)T$,

$$V(t) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{2\tau(n-1)T}e^{-\tau t} = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau[t-2(n-1)T]}.$$

At $t = (2n-1)T$,

$$V\big((2n-1)T\big) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau[(2n-1)T-2(n-1)T]} = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}.$$

(c) When $V_{\text{in}} = 0$, the rectifier prevents the charge that has been stored in the capacitor from flowing back through R_1; it simply discharges itself through R_2. Consequently, $dV/dt + \sigma V = 0$ where $\sigma = 1/(R_2C)$.
(d) We separate the differential equation:

$$\frac{dV}{V} = -\sigma\,dt \implies \ln|V| = -\sigma t + D \implies V(t) = Ee^{-\sigma t}.$$

If we now use the fact that $\displaystyle\lim_{t\to(2n-1)T^+} V(t) = \frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}$, we obtain

$$\frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T} = Ee^{-\sigma(2n-1)T} \implies E = \frac{\alpha\overline{V}}{\tau}e^{\sigma(2n-1)T} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-[\tau-\sigma(2n-1)]T}.$$

Hence, for $(2n-1)T < t < 2nT$, we have $\quad V(t) = \left[\dfrac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \dfrac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}\right]e^{-\sigma[t-(2n-1)T]}.$
(e) When the function in (d) is evaluated at $t = 2nT$, its value is V_n; that is,

$$V_n = \left[\frac{\alpha\overline{V}}{\tau} + \left(V_{n-1} - \frac{\alpha\overline{V}}{\tau}\right)e^{-\tau T}\right]e^{-\sigma T} = pV_{n-1} + q,$$

where $p = e^{-T(\tau+\sigma)}$ and $q = (\alpha\overline{V}/\tau)(1 - e^{-\tau T})e^{-\sigma T}$. If we iterate this recursive definition,

$$V_1 = pV_0 + q, \quad V_2 = pV_1 + q = p^2V_0 + q(p+1), \quad V_3 = pV_2 + q = p^3V_0 + q(p^2 + p + 1).$$

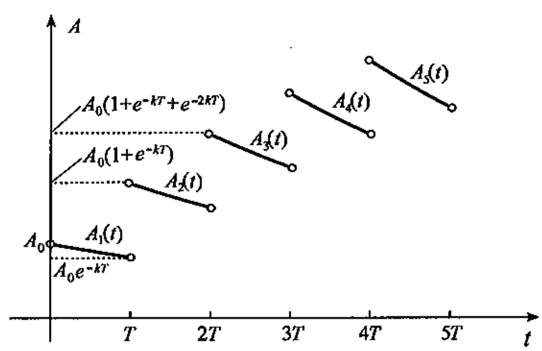
The pattern emerging is $V_n = p^nV_0 + q(1 + p + p^2 + \cdots + p^{n-1}) = p^nV_0 + \dfrac{q(1-p^n)}{1-p}$.
Since $V_0 = 0$, if the voltage across the capacitor is zero at time $t = 0$, we have

$$V_n = \frac{q(1-p^n)}{1-p} = \frac{\alpha\overline{V}}{\tau}(1 - e^{-\tau T})e^{-\sigma T}\left[\frac{1 - e^{-nT(\tau+\sigma)}}{1 - e^{-T(\tau+\sigma)}}\right].$$

46. (a) After time t, the amount of the first injection remaining is A_0e^{-kt}; the amount of the second injection remaining is $A_0e^{-k(t-T)}$; the amount of the third injection remaining is $A_0e^{-k(t-2T)}$; etc. At time t between the n^{th} and $(n+1)^{\text{th}}$ injection, the total amount remaining is

$$\begin{aligned}A_n(t) &= A_0e^{-kt} + A_0e^{-k(t-T)} + \cdots + A_0e^{-k[t-(n-1)T]}\\ &= A_0e^{-kt}\left[1 + e^{kT} + e^{2kT} + \cdots + e^{(n-1)kT}\right]\\ &= A_0e^{-kt}\left[\frac{1-(e^{kT})^n}{1-e^{kT}}\right] \qquad \text{(using 10.39a)}\\ &= A_0e^{-kt}\left[\frac{1-e^{knT}}{1-e^{kT}}\right] \qquad (n-1)T < t < nT.\end{aligned}$$

(b)



(c) $$\lim_{n\to\infty} A_n[(n-1)T] = \lim_{n\to\infty} A_0e^{-k(n-1)T}\left[\frac{1-e^{knT}}{1-e^{kT}}\right]$$
$$= \frac{A_0e^{kT}}{1-e^{kT}}\lim_{n\to\infty}(e^{-knT}-1) = \frac{-A_0e^{kT}}{1-e^{kT}} = \frac{A_0}{1-e^{-kT}}$$

EXERCISES 10.10

1. Since $l = \lim_{n\to\infty} \dfrac{\frac{1}{2n+1}}{\frac{1}{2n}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

2. Since $l = \lim_{n\to\infty} \dfrac{\frac{1}{4n-3}}{\frac{1}{4n}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{4n} = \dfrac{1}{4}\sum_{n=1}^{\infty}\dfrac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

3. Since $l = \lim_{n\to\infty} \dfrac{\frac{1}{2n^2+4}}{\frac{1}{2n^2}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{2n^2} = \dfrac{1}{2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

4. Since $l = \lim_{n\to\infty} \dfrac{\frac{1}{5n^2-3n-1}}{\frac{1}{5n^2}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{5n^2} = \dfrac{1}{5}\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

5. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{1}{n^3-1}}{\dfrac{1}{n^3}} = 1$, and $\sum_{n=2}^{\infty} \dfrac{1}{n^3}$ converges, so also does the given series (by the limit comparison test).

6. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n^2}{n^4-6n^2+5}}{\dfrac{1}{n^2}} = 1$, and $\sum_{n=4}^{\infty} \dfrac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

7. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{1}{(2n-1)(2n+1)}}{\dfrac{1}{4n^2}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{4n^2} = \dfrac{1}{4}\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

8. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n-5}{n^2+3n-2}}{\dfrac{1}{n}} = 1$, and $\sum_{n=6}^{\infty} \dfrac{1}{n}$ diverges, so also does the series $\sum_{n=6}^{\infty} \dfrac{n-5}{n^2+3n-2}$ (by the limit comparison test). The given series therefore diverges also.

9. Since $\dfrac{1}{\ln n} > \dfrac{1}{n}$, and the harmonic series diverges, so also does the given series (by the comparison test).

10. The function $f(x) = x^2e^{-2x}$ is positive and continuous. Since $f'(x) = 2xe^{-2x} - 2x^2e^{-2x} = 2xe^{-2x}(1-x)$, the function is decreasing for $x \geq 1$. Integrating by parts, and understanding that limits must be taken for the infinite limit,

$$\begin{aligned}\int_1^{\infty} x^2e^{-2x}\,dx &= \left\{\frac{x^2e^{-2x}}{-2}\right\}_1^{\infty} - \int_1^{\infty} 2x\left(\frac{e^{-2x}}{-2}\right)dx = \frac{e^{-2}}{2} + \left\{\frac{xe^{-2x}}{-2}\right\}_1^{\infty} - \int_1^{\infty} \frac{e^{-2x}}{-2}dx \\ &= \frac{e^{-2}}{2} + \frac{e^{-2}}{2} - \left\{\frac{e^{-2x}}{4}\right\}_1^{\infty} = \frac{5}{4e^2}.\end{aligned}$$

Since the integral converges, so also does the series $\sum_{n=1}^{\infty} n^2e^{-2n}$ (by the integral test).

11. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{\sqrt{n^2+2n-3}}{n^2+5}}{\dfrac{1}{n}} = 1$, and $\sum_{n=2}^{\infty} \dfrac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

12. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{\sqrt{n+5}}{n^3+3}}{\dfrac{1}{n^{5/2}}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{n^{5/2}}$ converges, so also does the given series (by the limit comparison test).

13. Since $l = \lim_{n\to\infty} \dfrac{\sqrt{\dfrac{n^2+2n+3}{2n^4-n}}}{\dfrac{1}{\sqrt{2}n}} = 1$, and $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{2}n} = \dfrac{1}{\sqrt{2}}\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so also does the given series (by the limit comparison test).

14. Since $\dfrac{1}{n^2\ln n} < \dfrac{1}{n^2}$ for $n \geq 3$, and $\sum_{n=3}^{\infty} \dfrac{1}{n^2}$ converges, so also does $\sum_{n=3}^{\infty} \dfrac{1}{n^2\ln n}$ (by the comparison test). The original series therefore converges also.

15. Since $\dfrac{1}{2^n}\sin\left(\dfrac{\pi}{n}\right) < \dfrac{1}{2^n}$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a convergent geometric series, the original series converges (by the comparison test).

16. Since $l = \displaystyle\lim_{n\to\infty}\frac{\dfrac{\sqrt{n^2+1}}{n^3}\mathrm{Tan}^{-1}n}{\dfrac{1}{n^2}\left(\dfrac{\pi}{2}\right)} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{\pi}{2n^2} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so also does the given series (by the limit comparison test).

17. Since $l = \displaystyle\lim_{n\to\infty}\frac{\dfrac{2^n+n}{3^n+1}}{(2/3)^n} = 1$, and $\displaystyle\sum_{n=1}^{\infty}(2/3)^n$ is a convergent geometric series, the given series converges (by the limit comparison test).

18. Since $\dfrac{1+\ln^2 n}{n\ln^2 n} = \dfrac{1}{n\ln^2 n} + \dfrac{1}{n} > \dfrac{1}{n}$, and $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ diverges, so also does the given series (by the comparison test).

19. Since $l = \displaystyle\lim_{n\to\infty}\frac{\dfrac{1+1/n}{e^n}}{\dfrac{1}{e^n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e^n}$ is a convergent geometric series, the given series converges (by the limit comparison test).

20. Since $\dfrac{\ln(n+1)}{n+1} > \dfrac{1}{n+1}$ for $n \geq 2$, and $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n+1}$ diverges (harmonic series with two terms missing), so also does $\displaystyle\sum_{n=2}^{\infty}\frac{\ln(n+1)}{n+1}$ (by the comparison test). It follows that the given series diverges also.

21. The function $f(x) = xe^{-x^2}$ is positive and continuous. Since $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2}$, the function is decreasing for $x \geq 1$. Since

$$\int_1^{\infty} xe^{-x^2}\,dx = \left\{\frac{e^{-x^2}}{-2}\right\}_1^{\infty} = \frac{1}{2e},$$

the improper integral converges. So also therefore does the series (by the integral test).

22. The function $1/[x(\ln x)^{1/3}]$ is positive, continuous, and decreasing for $x \geq 2$. Since

$$\int_2^{\infty}\frac{1}{x(\ln x)^{1/3}}dx = \left\{\frac{3}{2}(\ln x)^{2/3}\right\}_2^{\infty} = \infty,$$

the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1/3}}$ diverges (by the integral test).

23. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}x^{2n} = \sum_{n=1}^{\infty}\frac{1}{n}y^n$. Since the radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{1/n}{1/(n+1)}\right| = 1$, the radius of convergence of the original series is $R_x = \sqrt{R_y} = 1$. The open interval of convergence is $-1 < x < 1$. At the end points $x = \pm 1$, the series reduces to the harmonic series which diverges. The interval of convergence is therefore $-1 < x < 1$.

24. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}x^{2n} = \sum_{n=1}^{\infty}\frac{1}{n^2}y^n$. Since the radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{1/n^2}{1/(n+1)^2}\right| = 1$, the radius of convergence of the original series is $R_x = \sqrt{R_y} = 1$.

The open interval of convergence is $-1 < x < 1$. At the end points $x = \pm 1$, the series reduces to the convergent series $\sum_{n=1}^{\infty} 1/n^2$. The interval of convergence is therefore $-1 \le x \le 1$.

25. If we set $y = (x-1)^4$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-1)^{4n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} y^n$. Since the radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty} \left| \frac{1/(n2^n)}{1/[(n+1)2^{n+1}]} \right| = 2$, the radius of convergence of the original series is $R_x = {R_y}^{1/4} = 2^{1/4}$. The open interval of convergence is $1 - 2^{1/4} < x < 1 + 2^{1/4}$. At the end points $x = 1 \pm 2^{1/4}$, the series reduces to the harmonic series which diverges. The interval of convergence is therefore $1 - 2^{1/4} < x < 1 + 2^{1/4}$.

26. If we set $y = (x+1)^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{n3^n}{(n+1)^3}(x+2)^{2n} = \sum_{n=0}^{\infty} \frac{n3^n}{(n+1)^3} y^n$. Since the radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty} \left| \frac{n3^n/(n+1)^3}{(n+1)3^{n+1}/(n+2)^3} \right| = 1/3$, the radius of convergence of the original series is $R_x = \sqrt{R_y} = 1/\sqrt{3}$. The open interval of convergence is $-2 - 1/\sqrt{3} < x < -2 + 1/\sqrt{3}$. At the end points $x = -2 \pm 1/\sqrt{3}$, the series reduces to $\sum_{n=0}^{\infty} n/(n+1)^3$. Since $l = \displaystyle\lim_{n\to\infty} \frac{n/(n+1)^3}{1/n^2} = 1$, and $\sum_{n=1}^{\infty} 1/n^2$ converges, so also does $\sum_{n=0}^{\infty} n/(n+1)^3$ (by the limit comparison test). The interval of convergence is therefore $-2 - 1/\sqrt{3} \le x \le -2 + 1/\sqrt{3}$.

27. When $p = 1$, the series is $\sum_{n=2}^{\infty} 1/(n \ln n)$. The function $f(x) = 1/(x \ln x)$ is positive, continuous and decreasing function for $x \ge 2$. Since

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \left\{ \ln |\ln x| \right\}_2^{\infty} = \infty,$$

the series $\sum_{n=2}^{\infty} 1/(n \ln n)$ diverges (by the integral test). When $p < 1$, then $1/(n^p \ln n) > 1/(n \ln n)$, and therefore the given series diverges for $p < 1$ also. When $p > 1$, then $1/(n^p \ln n) < 1/n^p$ for $n \ge 3$. Since $\sum_{n=3}^{\infty} 1/n^p$ converges for $p > 1$, it follows that the given series converges for $p > 1$.

28. When $p = 1$, the function $1/(x \ln x)$ is positive, continuous, and decreasing for $x \ge 2$. Since

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \left\{ \ln |\ln x| \right\}_2^{\infty} = \infty,$$

it follows by the integral test that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

When $p < 1$, $\displaystyle\frac{1}{n(\ln n)^p} > \frac{1}{n \ln n}$, and therefore $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges when $p < 1$ (by the comparison test).

For $p > 1$, the function $1/[x(\ln x)^p]$ is positive, continuous, and decreasing for $x \ge 2$. Since

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \left\{ \frac{1}{(1-p)(\ln x)^{p-1}} \right\}_2^{\infty} = \frac{1}{(p-1)(\ln 2)^{p-1}},$$

the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges (by the integral test).

29. When $p = 1$, the series is $\sum_{n=2}^{\infty} 1/\ln n$. Since $1/\ln n > 1/n$, and $\sum_{n=2}^{\infty} 1/n$ diverges, so also does $\sum_{n=2}^{\infty} 1/\ln n$ (by the comparison test). Because $1/(\ln n)^p > 1/\ln n$ when $p < 1$, it follows that the given series diverges for all $p \le 1$. When $p > 1$, we set $f(x) = 1/(\ln x)^p$, a positive, continuous, decreasing function for $x \ge 2$. If we set $u = 1/(\ln x)^p$, $du = -p/[x(\ln x)^{p+1}]\,dx$, $dv = dx$, and $v = x$, then integration by parts gives

$$\int_2^{\infty} \frac{1}{(\ln x)^p} dx = \left\{ \frac{x}{(\ln x)^p} \right\}_2^{\infty} + p \int_2^{\infty} \frac{1}{(\ln x)^{p+1}} dx$$

$$= \lim_{x\to\infty}\left[\frac{x}{(\ln x)^p}\right] - \frac{2}{(\ln 2)^p} + p\int_2^\infty \frac{1}{(\ln x)^{p+1}}dx.$$

To evaluate $L = \lim_{x\to\infty}\frac{x}{(\ln x)^p}$, we take logarithms, $\ln L = \lim_{x\to\infty}[\ln x - p\ln(\ln x)]$. Consider instead

$$\lim_{x\to\infty}\frac{\ln x}{p\ln(\ln x)} = \lim_{x\to\infty}\frac{1/x}{\frac{p}{x\ln x}} = \lim_{x\to\infty}\frac{\ln x}{p} = \infty.$$

This implies that $\ln L = \lim_{x\to\infty}[\ln x - p\ln(\ln x)] = \infty$ also. Thus, $L = \lim_{x\to\infty}\frac{x}{(\ln x)^p} = \infty$. Since the $\int_2^\infty \frac{1}{(\ln x)^{p+1}}dx$ must be positive or diverge to "infinity", it follows that $\int_2^\infty \frac{1}{(\ln x)^p}dx = \infty$. The series $\sum_{n=2}^\infty 1/(\ln n)^p$ therefore diverges for $p > 1$ also. The series therefore diverges for all p.

EXERCISES 10.11

1. Since $L = \lim_{n\to\infty}\dfrac{\frac{e^{n+1}}{(n+1)^4}}{\frac{e^n}{n^4}} = e$, the series diverges (by the limit ratio test).

2. Since $L = \lim_{n\to\infty}\dfrac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{1}{n+1} = 0$, the series converges (by the limit ratio test).

3. Since $L = \lim_{n\to\infty}\dfrac{\frac{(n+1)^3}{2^{n+1}}}{\frac{n^3}{2^n}} = \frac{1}{2}$, the series converges (by the limit ratio test).

4. Since $R = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty}\frac{1}{n} = 0$, the series converges (by the limit root test).

5. Since $L = \lim_{n\to\infty}\dfrac{\frac{n(n-1)}{(n+1)^2 2^{n+1}}}{\frac{(n-1)(n-2)}{n^2 2^n}} = \frac{1}{2}$, the series converges (by the limit ratio test).

6. Since $L = \lim_{n\to\infty}\dfrac{\frac{(2n+2)!}{[(n+1)!]^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2} = 4 > 1$, the series diverges (by the limit ratio test).

7. Since $l = \lim_{n\to\infty}\dfrac{\frac{\sqrt{n+1}}{n^{n+1/2}}}{\frac{1}{n^n}} = \lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n}} = 1$, the given series converges if and only if the series $\sum_{n=1}^\infty 1/n^n$ converges. Because $R = \lim_{n\to\infty}(1/n^n)^{1/n} = \lim_{n\to\infty}(1/n) = 0$. the series $\sum_{n=1}^\infty 1/n^n$ converges (by the limit root test). Thus, the original series converges also.

8. Since $L = \lim_{n\to\infty}\dfrac{\frac{3^{-(n+1)}+2^{-(n+1)}}{4^{-(n+1)}+5^{-(n+1)}}}{\frac{3^{-n}+2^{-n}}{4^{-n}+5^{-n}}} = \lim_{n\to\infty}\left\{\frac{2^{-(n+1)}[1+(3/2)^{-(n+1)}]}{4^{-(n+1)}[1+(5/4)^{-(n+1)}]}\frac{4^{-n}[1+(5/4)^{-n}]}{2^{-n}[1+(3/2)^{-n}]}\right\} = 2$, the series diverges (by the limit ratio test).

9. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{e^{-n-1}}{\sqrt{n+1+\pi}}}{\dfrac{e^{-n}}{\sqrt{n+\pi}}} = \dfrac{1}{e}$, the series converges (by the limit ratio test).

10. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{2\cdot 4\cdots(2n+2)}{4\cdot 7\cdots(3n+4)}}{\dfrac{2\cdot 4\cdots(2n)}{4\cdot 7\cdots(3n+1)}} = \lim_{n\to\infty} \dfrac{2n+2}{3n+4} = \dfrac{2}{3}$, the series converges (by the limit ratio test).

11. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^n}{3^n\, n!}}{\dfrac{n^{n-1}}{3^{n-1}(n-1)!}} = \lim_{n\to\infty} \dfrac{1}{3}\left(\dfrac{n+1}{n}\right)^n = \dfrac{e}{3} < 1$, the series converges (by the limit ratio test).

12. Since $L = \lim_{n\to\infty} \dfrac{(n+1)(3/4)^{n+1}}{n(3/4)^n} = \dfrac{3}{4}$, the series converges (by the limit ratio test).

13. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{1+1/(n+1)}{e^{n+1}}}{\dfrac{1+1/n}{e^n}} = \dfrac{1}{e}$, the series converges (by the limit ratio test).

14. Since $\dfrac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}\left(\dfrac{1}{n^2}\right) = \left(\dfrac{2}{3}\right)\left(\dfrac{4}{5}\right)\cdots\left(\dfrac{2n}{2n+1}\right)\left(\dfrac{1}{n^2}\right) < \dfrac{1}{n^2}$, and $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, so also does the original series (by the comparison test).

15. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n^n}{(n+1)^{n+1}}}{\dfrac{1}{en}} = e \lim_{n\to\infty} \left(\dfrac{n}{n+1}\right)\left(\dfrac{n}{n+1}\right)^n = 1$, and $\sum_{n=1}^{\infty} 1/(en) = (1/e)\sum_{n=1}^{\infty}(1/n)$ diverges, so also does the original series (by the limit comparison test).

16. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^n}{n^{n+1}}}{\dfrac{e}{n}} = \lim_{n\to\infty} \dfrac{1}{e}\left(\dfrac{n+1}{n}\right)^n = 1$, and $\sum_{n=1}^{\infty} \dfrac{e}{n} = e\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so also does the original series (by the limit comparison test).

17. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^4+3}{5^{(n+1)/2}}}{\dfrac{n^4+3}{5^{n/2}}} = \dfrac{1}{\sqrt{5}}$, the series converges (by the limit ratio test).

18. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{2^{n+1}+(n+1)^2 3^{n+1}}{4^{n+1}}}{\dfrac{2^n+n^2 3^n}{4^n}} = \lim_{n\to\infty} \dfrac{1}{4}\left[\dfrac{2^{n+1}+(n+1)^2 3^{n+1}}{2^n+n^2 3^n}\right] = \lim_{n\to\infty} \dfrac{1}{4}\left[\dfrac{\dfrac{2^{n+1}}{3^{n+1}}+(n+1)^2}{\dfrac{2^n}{3^{n+1}}+\dfrac{n^2}{3}}\right]$ $= 3/4$, the series converges (by the limit ratio test).

19. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^2 2^{n+1}-(n+1)}{(n+1)^3+1}}{\dfrac{n^2 2^n - n}{n^3+1}} = 2$, the series diverges (by the limit ratio test).

20. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(2n+2)!5^{2n+2}}{(3n+3)!}}{\dfrac{(2n)!5^{2n}}{(3n)!}} = \lim_{n\to\infty} \dfrac{25(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} = 0$, the series converges (by the limit ratio test).

21. The series diverges for $a = 0, 1$ by the n^{th} term test. When $a = 2$, the series converges by the limit ratio test since $L = \lim_{n\to\infty} \dfrac{\dfrac{[(n+1)!]^2}{(2n+2)!}}{\dfrac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \dfrac{(n+1)^2}{(2n+2)(2n+1)} = \dfrac{1}{4}$. When $a > 2$, terms of the series are less than those when $a = 2$, and the series therefore converges in these cases also.

EXERCISES 10.12

1. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n}{n^3+1}}{\dfrac{1}{n^2}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so also does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ (by the limit comparison test). The original series therefore converges absolutely.

2. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n}{n^2+1}}{\dfrac{1}{n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so also does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (by the limit comparison test). The original series does not converge absolutely. Because the sequence $\{n/(n^2+1)\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

3. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the original series converges absolutely.

4. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^3}{3^{n+1}}}{\dfrac{n^3}{3^n}} = \dfrac{1}{3}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges (by the limit ratio test). The original series therefore converges absolutely.

5. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the original series does not converge absolutely. Because the sequence $\{1/\sqrt{n}\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

6. Since $\lim_{n\to\infty} (-1)^n \dfrac{3^n}{n^3}$ does not exist, the series diverges (by the n^{th} term test).

7. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{n}{n^2+n+1}}{\dfrac{1}{n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so also does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$ (by the limit comparison test). The original series does not converge absolutely. Because the sequence $\{n/(n^2+n+1)\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

8. Consider the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{n\sin(n\pi/4)}{2^n}\right|$. Since $\left|\dfrac{n\sin(n\pi/4)}{2^n}\right| \le \dfrac{n}{2^n}$, the series of absolute values converges if $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges. Because $L = \lim_{n\to\infty} \dfrac{\dfrac{n+1}{2^{n+1}}}{\dfrac{n}{2^n}} = \dfrac{1}{2}$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges (by the limit ratio test). Consequently, the original series converges absolutely.

9. Since $\lim_{n\to\infty} \left(\dfrac{n}{n+1}\right) = 1$, it follows that $\lim_{n\to\infty} (-1)^{n+1} \left(\dfrac{n}{n+1}\right)$ does not exist, and the series diverges (by the n^{th} term test).

10. Consider the series $\sum_{n=1}^{\infty} \frac{\sqrt{3n-2}}{n}$. Since $l = \lim_{n\to\infty} \frac{\frac{\sqrt{3n-2}}{n}}{\frac{\sqrt{3}}{\sqrt{n}}} = 1$, and $\sum_{n=1}^{\infty} \frac{\sqrt{3}}{\sqrt{n}}$ diverges so also does the series $\sum_{n=1}^{\infty} \frac{\sqrt{3n-2}}{n}$ (by the limit comparison test). The original series does not converge absolutely. Because the sequence $\{\sqrt{3n-2}/n\}$ is nonincreasing and has limit zero, the original series converges conditionally (by the alternating series test).

11. Since $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$, it follows that $\lim_{n\to\infty} (-1)^{n+1} \left(\frac{n}{n+1}\right)^n$ does not exist, and the series diverges (by the n^{th} term test).

12. Consider the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{n^2+5}$. Since $l = \lim_{n\to\infty} \frac{\frac{\sqrt{n^2+3}}{n^2+5}}{\frac{1}{n}} = 1$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so also does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{n^2+5}$ (by the limit comparison test). The original series does not therefore converge absolutely. Because the sequence $\{\sqrt{n^2+3}/(n^2+5)\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

13. Consider the series $\sum_{n=2}^{\infty} \frac{\ln n}{n}$. Since $(\ln n)/n > 1/n$ for $n \geq 3$, and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so also does $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ (by the comparison test). The original series does not converge absolutely. Because the sequence $\{(\ln n)/n\}$ is decreasing and has limit zero, the original series converges conditionally (by the alternating series test).

14. Consider the series $\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi/10)\text{Cot}^{-1}n}{n^3+5n}\right|$. Since $\left|\frac{\cos(n\pi/10)\text{Cot}^{-1}n}{n^3+5n}\right| < \frac{\pi/4}{n^3}$, and the series $\sum_{n=1}^{\infty} \frac{\pi/4}{n^3} = \frac{\pi}{4}\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so also does the series of absolute values (by the comparison test). The original series therefore converges absolutely.

15. Since the radius of convergence of the series is $\lim_{n\to\infty} \left|\frac{1/n}{1/(n+1)}\right| = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to the harmonic series which diverges. At $x = -1$, it reduces to the negative of the alternating harmonic series which converges conditionally. The interval of convergence is therefore $-1 \leq x < 1$.

16. With radius of convergence $\lim_{n\to\infty} \left|\frac{1/(n+1)^2}{1/(n+2)^2}\right| = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to a p-series $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges. At $x = -1$, it reduces to $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ which converges absolutely. The interval of convergence is therefore $-1 \leq x \leq 1$.

17. Since the radius of convergence of the series is $\lim_{n\to\infty} \left|\frac{\frac{1}{n2^n}}{\frac{1}{(n+1)2^{n+1}}}\right| = 2$, the open interval of convergence is $-1 < x < 3$. At $x = 3$, the power series reduces to the harmonic series which diverges. At $x = -1$, it reduces to the negative of the alternating harmonic series which converges conditionally. The interval of convergence is therefore $-1 \leq x < 3$.

18. With radius of convergence $\lim_{n\to\infty}\left|\frac{1/\sqrt{n}}{1/\sqrt{n+1}}\right| = 1$, the open interval of convergence is $-3 < x < -1$. At $x = -1$, the power series reduces to a p-series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which diverges. At $x = -3$, it reduces to $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ which converges conditionally. The interval of convergence is therefore $-3 \le x < -1$.

19. Since the radius of convergence of the series is $\lim_{n\to\infty}\left|\frac{\frac{(n-1)2^n}{n^2+1}}{\frac{n2^{n+1}}{(n+1)^2+1}}\right| = \frac{1}{2}$, the open interval of convergence is $-1/2 < x < 1/2$. At $x = 1/2$, the power series reduces to $\sum_{n=0}^{\infty}\frac{n-1}{n^2+1}$. Since $l = \lim_{n\to\infty}\frac{\frac{n-1}{n^2+1}}{\frac{1}{n}} = 1$, and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, the series diverges (by the limit comparison test). At $x = -1/2$, the power series reduces to $\sum_{n=0}^{\infty}\frac{(n-1)(-1)^n}{n^2+1}$. This series does not converge absolutely. Since the sequence $\{(n-1)/(n^2+1)\}$ is decreasing for $n \ge 2$ and has limit zero, it follows that the series $\sum_{n=0}^{\infty}\frac{(n-1)(-1)^n}{n^2+1}$ converges conditionally. The interval of convergence is therefore $-1/2 \le x < 1/2$.

20. If we set $y = x^3$, the series becomes $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}x^{3n+1} = y^{1/3}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}y^n$. Since the radius of convergence of this series is $R_y = \lim_{n\to\infty}\left|\frac{1/\sqrt{n+1}}{1/\sqrt{n+2}}\right| = 1$, it follows that $R_x = \sqrt{R_y} = 1$ also. The open interval of convergence is therefore $-1 < x < 1$. At $x = 1$, the power series reduces to a p-series $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which diverges. At $x = -1$, it reduces to $\sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$. This series does not converge absolutely but it does converge conditionally. Hence, the interval of convergence is $-1 \le x < 1$.

21. Since the radius of convergence of the series is $\lim_{n\to\infty}\left|\frac{1/\ln n}{1/\ln(n+1)}\right| = \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty}\frac{1/(n+1)}{1/n} = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\sum_{n=2}^{\infty}\frac{1}{\ln n}$. This series diverges by the comparison test since $1/\ln n > 1/n$. At $x = -1$, the power series reduces to $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ which converges conditionally. The interval of convergence is therefore $-1 \le x < 1$.

22. With radius of convergence $\lim_{n\to\infty}\left|\frac{\frac{1}{n^2\ln n}}{\frac{1}{(n+1)^2\ln(n+1)}}\right| = \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty}\frac{1/(n+1)}{1/n} = 1$, the open interval of convergence of the series is $1 < x < 3$. At $x = 3$, the power series reduces to $\sum_{n=2}^{\infty}\frac{1}{n^2\ln n}$. This series converges by the comparison test since $1/(n^2\ln n) < 1/n^2$ for $n \ge 3$. At $x = 1$, the power series reduces to $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2\ln n}$ which converges absolutely. The interval of convergence is therefore $1 \le x \le 3$.

23. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(nx)|}{n^2}$. Since $|\sin(nx)/n^2| < 1/n^2$, the series converges (by the comparison test). Hence, the original series converges absolutely for all x.

24. If $\sum_{n=1}^{\infty} c_n$ converges absolutely, then $\sum_{n=1}^{\infty} |c_n|$ converges, and then $\lim_{n\to\infty} |c_n| = 0$. It follows that for n greater than or equal to some integer N, $0 < |c_n| < 1$. For such n, we have $|c_n|^p < |c_n|$. Consequently, $\sum_{n=N}^{\infty} |c_n^p| = \sum_{n=N}^{\infty} |c_n|^p$ converges (by the comparison test). Hence, $\sum_{n=1}^{\infty} |c_n|^p$ converges also.

25. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{n+1}}$. Since $l = \displaystyle\lim_{n\to\infty} \frac{\dfrac{n^n}{(n+1)^{n+1}}}{\dfrac{1}{en}} = e \lim_{n\to\infty} \left(\frac{n}{n+1}\right)\left(\frac{n}{n+1}\right)^n = 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{en} = \frac{1}{e}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so also does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{n+1}}$ (by the limit comparison test). Since the sequence $\{n^n/(n+1)^{n+1}\}$ is decreasing and has limit 0 (it looks like $1/(en)$ for large n), the original series converges conditionally.

EXERCISES 10.13

1. We obtain this result by setting $x = 2$ in the Maclaurin series for e^x (see Example 10.10).

2. We obtain this result by setting $x = 1$ in the Maclaurin series for $\sin x$ (see Example 10.9).

3. We obtain this result by setting $x = 3$ in the Maclaurin series for $\cos x$ (see Example 10.21).

4. If we set $x = -1$ in the Maclaurin series for e^x (Example 10.10),

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \qquad \Longrightarrow \qquad \frac{1}{e} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}.$$

5. This is a geometric series with sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}} = \frac{-1/4}{1+1/4} = -\frac{1}{5}.$$

6. If we set $x = 2$ in the Maclaurin series for $\cos x$ (see example 10.21),

$$\cos 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} = 1 - \frac{2^2}{2!} + \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n}.$$

Consequently,

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(2n)!} 2^{2n+3} = -8(\cos 2 - 1 + 2) = -8(1 + \cos 2).$$

7. If we set $x = 1/3$ in the Taylor series for $\ln x$ about $x = 1$ (see Example 10.22),

$$\ln(1/3) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(-\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{-2^n}{n3^n} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \frac{2^n}{n3^n} = -\ln(1/3) = \ln 3.$$

8. If we set $x = 1/2$ in the Taylor series for $\ln x$ about $x = 1$ (see Example 10.22),

$$\ln(1/2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{-1}{n2^n} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \frac{1}{n2^n} = -\ln(1/2) = \ln 2.$$

9. If we set $x = 1/3$ in the Maclaurin series for $\sin x$ (see example 10.9),

$$\sin(1/3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{3}\right)^{2n+1} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)!}.$$

Consequently,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{2n}(2n+1)!} = 3\left[\sin\left(\frac{1}{3}\right) - \frac{1}{3}\right] = 3\sin\left(\frac{1}{3}\right) - 1.$$

10. The series $\sum_{n=1}^{\infty} nx^n$ converges for $-1 < x < 1$. If we set $S(x) = \sum_{n=1}^{\infty} nx^n$, then $x^{-1}S(x) = \sum_{n=1}^{\infty} nx^{n-1}$. Term-by-term integration gives $\int \frac{S(x)}{x}dx + C = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$, since the series is geometric. We now differentiate with respect to x, getting $\frac{S(x)}{x} = \frac{(1-x)(1) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$.
Hence, $S(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$, and if we set $x = 1/2$, we obtain $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2.$

11. If we integrate the geometric series $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$, which converges for $-1 < x < 1$, term-by term,

$$\text{Tan}^{-1}x + C = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots .$$

Substitution of $x = 0$ yields $C = 0$, and therefore $\text{Tan}^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$. The open interval of convergence of this series is $-1 < x < 1$. At $x = \pm 1$, the Taylor series reduces to $\pm\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, which converge conditionally. According to Theorem 10.20, we may therefore write that

$$\text{Tan}^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}, \quad -1 \le x \le 1.$$

When we set $x = 1$,

$$\text{Tan}^{-1}(1) = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} - 1.$$

12. Term-by-term differentiation of $\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, $|x| < 1$ gives

$$\frac{-2x}{(1+x^2)^2} = \sum_{n=0}^{\infty} 2n(-1)^n x^{2n-1} \quad \Longrightarrow \quad \frac{x}{(1+x^2)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}x^{2n-1}.$$

If we set $x = 1/3$, then $\frac{1/3}{(1+1/9)^2} = \sum_{n=1}^{\infty} n(-1)^{n-1}\frac{1}{3^{2n-1}}$. Hence

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{3^{2n}} = -\frac{1}{3}\left[\frac{1/3}{(1+1/9)^2}\right] = -\frac{9}{100}.$$

13. The sum of the first ten terms is $\displaystyle\sum_{n=2}^{11} \frac{n^2}{(n^3+1)^4} = 0.000\,625\,322$. According to expression 10.48, the error in approximating the sum of this series with this value is less than

$$\int_{11}^{\infty} \frac{x^2}{(x^3+1)^4}\,dx = \left\{\frac{1}{-9(x^3+1)^3}\right\}_{11}^{\infty} = \frac{1}{9(12)^3}.$$

14. The sum of the first five terms is $\displaystyle\sum_{n=1}^{5} \frac{n}{e^{3n}} = 0.055\,140\,9$. According to expression 10.48, the error in approximating the sum of this series with this value is less than

$$\int_5^{\infty} xe^{-3x}\,dx = \left\{-\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x}\right\}_5^{\infty} = \frac{5}{3}e^{-15} + \frac{1}{9}e^{-15} = \frac{16}{9}e^{-15} < 6 \times 10^{-7}.$$

15. According to expression 10.48, the error in approximating the sum of this series after the 100$^{\text{th}}$ term is less than

$$\int_{100}^{\infty} \frac{1}{x^2+1}\,dx = \left\{\mathrm{Tan}^{-1}x\right\}_{100}^{\infty} = \frac{\pi}{2} - \mathrm{Tan}^{-1}(100) < 0.01.$$

16. According to expression 10.48, the error in approximating the sum of this series after the 20$^{\text{th}}$ term is less than

$$\int_{20}^{\infty} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = \left\{\cos\left(\frac{1}{x}\right)\right\}_{20}^{\infty} = 1 - \cos\left(\frac{1}{20}\right) < 0.001\,25.$$

17. The sum of the first 3 three terms is $-0.012\,710$. Since the series is alternating and absolute values of the terms are decreasing and have limit zero, the maximum error is the absolute value of the 4$^{\text{th}}$ term, $1/(5^3 \cdot 3^5) < 3.3 \times 10^{-5}$.

18. The sum of the first 20 terms is $-0.947\,030$. Since the series is alternating and absolute values of the terms are decreasing and have limit zero, the maximum error is the absolute value of the 21$^{\text{st}}$ term, $1/21^4 < 5.2 \times 10^{-6}$.

19. The sum of the first 5 terms is 1.291 26. The error in using this to approximate the sum of the series is

$$\sum_{n=6}^{\infty} \frac{1}{n^n} = \frac{1}{6^6} + \frac{1}{7^7} + \frac{1}{8^8} + \cdots < \frac{1}{6^6} + \frac{1}{6^7} + \frac{1}{6^8} + \cdots = \frac{1/6^6}{1-1/6} = \frac{1}{5(6^6)}.$$

20. The sum of the first 10 terms is 0.693 065. The error in using this to approximate the sum of the series is

$$\begin{aligned}\sum_{n=11}^{\infty} \frac{1}{n2^n} &= \frac{1}{11 \cdot 2^{11}} + \frac{1}{12 \cdot 2^{12}} + \frac{1}{13 \cdot 2^{13}} + \cdots \\ &< \frac{1}{11 \cdot 2^{11}} + \frac{1}{11 \cdot 2^{12}} + \frac{1}{11 \cdot 2^{13}} + \cdots = \frac{1/(11 \cdot 2^{11})}{1-1/2} = \frac{1}{11 \cdot 2^{10}} < 9 \times 10^{-5}.\end{aligned}$$

21. The sum of the first fifteen terms is 0.434 732. The error in using this approximation for the sum of the series is

$$\begin{aligned}\sum_{n=16}^{\infty} \frac{1}{2^n}\sin\left(\frac{\pi}{n}\right) &= \frac{1}{2^{16}}\sin\left(\frac{\pi}{16}\right) + \frac{1}{2^{17}}\sin\left(\frac{\pi}{17}\right) + \frac{1}{2^{18}}\sin\left(\frac{\pi}{18}\right) + \cdots \\ &< \frac{1}{2^{16}}\sin\left(\frac{\pi}{16}\right) + \frac{1}{2^{17}}\sin\left(\frac{\pi}{16}\right) + \frac{1}{2^{18}}\sin\left(\frac{\pi}{16}\right) + \cdots \\ &= \frac{(1/2^{16})\sin(\pi/16)}{1-1/2} = \frac{1}{2^{15}}\sin\left(\frac{\pi}{16}\right) < 6 \times 10^{-6}.\end{aligned}$$

22. The sum of the first 20 terms is 1.06749. The error in using this to approximate the sum of the series is

$$\sum_{n=22}^{\infty} \frac{2^n - 1}{3^n + n} = \frac{2^{22} - 1}{3^{22} + 22} + \frac{2^{23} - 1}{3^{23} + 23} + \cdots < \frac{2^{22}}{3^{22}} + \frac{2^{23}}{3^{23}} + \cdots = \frac{(2/3)^{22}}{1 - 2/3} < 4.01 \times 10^{-4}.$$

23. The sum of the first 20 terms is 1.35166. The error in using this to approximate the sum of the series is

$$\begin{aligned}\sum_{n=22}^{\infty} \frac{2^n + 1}{3^n + n} &= \frac{2^{22} + 1}{3^{22} + 22} + \frac{2^{23} + 1}{3^{23} + 23} + \frac{2^{24} + 1}{3^{24} + 24} + \cdots < \frac{2^{22} + 1}{3^{22}} + \frac{2^{23} + 1}{3^{23}} + \frac{2^{24} + 1}{3^{24}} + \cdots \\ &= \left[\left(\frac{2}{3}\right)^{22} + \left(\frac{2}{3}\right)^{23} + \left(\frac{2}{3}\right)^{24} + \cdots\right] + \left(\frac{1}{3^{22}} + \frac{1}{3^{23}} + \frac{1}{3^{24}} + \cdots\right) \\ &= \frac{(2/3)^{22}}{1 - 2/3} + \frac{1/3^{22}}{1 - 1/3} < 4.01 \times 10^{-4}.\end{aligned}$$

24. The sum of the first 100 terms is -0.6881722. Since the series is alternating and absolute values of the terms are decreasing with limit zero, the maximum error is the absolute value of the 101st term, $1/101$.

25. When this alternating series (with absolute values of terms decreasing and approaching zero) is truncated after N terms, the maximum error is the absolute value of the next term, $1/(N+1)^2$. It is less than 10^{-4} if $1/(N+1)^2 < 10^{-4}$. This occurs for $N \geq 100$.

26. When this series is truncated after N terms, the error is

$$\begin{aligned}\sum_{n=N+1}^{\infty} \frac{1}{n^2 4^n} &= \frac{1}{(N+1)^2 4^{N+1}} + \frac{1}{(N+2)^2 4^{N+2}} + \cdots < \frac{1}{(N+1)^2 4^{N+1}} + \frac{1}{(N+1)^2 4^{N+2}} + \cdots \\ &= \frac{1}{(N+1)^2 4^{N+1}}\left(\frac{1}{1 - 1/4}\right) = \frac{1}{3(N+1)^2 4^N}.\end{aligned}$$

This quantity is less than 10^{-4} if $3(N+1)^2 4^N > 10^4$. This occurs for $N \geq 4$.

27. When this series is truncated after N terms, the error is

$$\begin{aligned}\sum_{n=N+1}^{\infty} \frac{2^n}{n!} &= \frac{2^{N+1}}{(N+1)!} + \frac{2^{N+2}}{(N+2)!} + \frac{2^{N+3}}{(N+3)!} + \cdots \\ &< \frac{2^{N+1}}{(N+1)!} + \frac{2^{N+2}}{(N+1)!(N+1)} + \frac{2^{N+3}}{(N+1)!(N+1)^2} + \cdots = \frac{2^{N+1}/(N+1)!}{1 - 2/(N+1)}.\end{aligned}$$

This is less than 10^{-4} when $N \geq 10$.

28. (a) Using 10.48, the error is less than $\displaystyle\int_{10}^{\infty} e^{-x} \sin^2 x \, dx$. By writing the integrand in the form $e^{-x}(1 - \cos 2x)/2$, and using integration by parts, we obtain the following antiderivative,

$$\begin{aligned}\int_{10}^{\infty} e^{-x} \sin^2 x \, dx &= \left\{-\frac{1}{5} e^{-x}(2 + \sin^2 x + 2 \sin x \cos x)\right\}_{10}^{\infty} \\ &= \frac{e^{-10}}{5}(2 + \sin^2 10 + 2 \sin 10 \cos 10) < 2.92 \times 10^{-5}.\end{aligned}$$

(b) The error in using ten terms to approximate the sum is

$$\begin{aligned}\sum_{n=11}^{\infty} e^{-n} \sin^2 n &= e^{-11} \sin^2 11 + e^{-12} \sin^2 12 + e^{-13} \sin^2 13 + \cdots \\ &< e^{-11} + e^{-12} + e^{-13} + \cdots = \frac{e^{-11}}{1 - 1/e} < 2.65 \times 10^{-5}.\end{aligned}$$

The error in part (b) is better.

29. $$\int_0^1 \frac{\sin x}{x}dx = \int_0^1 \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)dx = \int_0^1\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)dx$$
$$= \left\{x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots\right\}_0^1 = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = 1, \qquad S_2 = S_1 - \frac{1}{3\cdot 3!} = 0.944\,44,$$
$$S_3 = S_2 + \frac{1}{5\cdot 5!} = 0.946\,11, \qquad S_4 = S_3 - \frac{1}{7\cdot 7!} = 0.946\,08.$$

Consequently, to three decimals the value of the integral is 0.946.

30. $$\int_0^{1/2}\cos(x^2)\,dx = \int_0^{1/2}\left[1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots\right]dx = \int_0^{1/2}\left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right]dx$$
$$= \left\{x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \cdots\right\}_0^{1/2} = \frac{1}{2} - \frac{1}{5\cdot 2^5\cdot 2!} + \frac{1}{9\cdot 2^9\cdot 4!} - \frac{1}{13\cdot 2^{13}\cdot 6!} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = \frac{1}{2}, \qquad S_2 = S_1 - \frac{1}{5\cdot 2^5\cdot 2!} = 0.496\,78, \qquad S_3 = S_2 + \frac{1}{9\cdot 2^9\cdot 4!} = 0.496\,88.$$

Consequently, to three decimals the value of the integral is 0.497.

31. $$\int_0^{2/3}\frac{1}{x^4+1}dx = \int_0^{2/3}(1 - x^4 + x^8 - x^{12} + \cdots)\,dx = \left\{x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots\right\}_0^{2/3}$$
$$= \frac{2}{3} - \frac{(2/3)^5}{5} + \frac{(2/3)^9}{9} - \frac{(2/3)^{13}}{13} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = \frac{2}{3}, \qquad S_2 = S_1 - \frac{(2/3)^5}{5} = 0.640\,32,$$
$$S_3 = S_2 + \frac{(2/3)^9}{9} = 0.643\,22, \qquad S_4 = S_3 - \frac{(2/3)^{13}}{13} = 0.642\,82.$$

Consequently, to three decimals the value of the integral is 0.643.

32. $$\int_{-1}^1 x^{11}\sin x\,dx = 2\int_0^1 x^{11}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)dx = 2\int_0^1\left(x^{12} - \frac{x^{14}}{3!} + \frac{x^{16}}{5!} - \cdots\right)dx$$
$$= 2\left\{\frac{x^{13}}{13} - \frac{x^{15}}{15\cdot 3!} + \frac{x^{17}}{17\cdot 5!} - \cdots\right\}_0^1 = 2\left(\frac{1}{13} - \frac{1}{15\cdot 3!} + \frac{1}{17\cdot 5!} - \frac{1}{19\cdot 7!} + \cdots\right).$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = \frac{2}{13}, \qquad S_2 = S_1 - \frac{2}{15\cdot 3!} = 0.131\,62,$$
$$S_3 = S_2 + \frac{2}{17\cdot 5!} = 0.132\,60, \qquad S_4 = S_3 - \frac{2}{19\cdot 7!} = 0.132\,58.$$

Consequently, to three decimals the value of the integral is 0.133.

33. $$\int_0^{1/2}\frac{1}{\sqrt{1+x^3}}dx = \int_0^{1/2}\left[1 - \frac{x^3}{2} + \frac{(-1/2)(-3/2)}{2!}(x^3)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(x^3)^3 + \cdots\right]dx$$
$$= \int_0^{1/2}\left(1 - \frac{x^3}{2} + \frac{3x^6}{2^2\cdot 2!} - \frac{3\cdot 5x^9}{2^3\cdot 3!} + \cdots\right)dx$$
$$= \left\{x - \frac{x^4}{4\cdot 2} + \frac{3x^7}{7\cdot 2^2\cdot 2!} - \frac{3\cdot 5x^{10}}{10\cdot 2^3\cdot 3!} + \cdots\right\}_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{4\cdot 2\cdot 2^4} + \frac{3}{7\cdot 2^2\cdot 2!\cdot 2^7} - \frac{3\cdot 5}{10\cdot 2^3\cdot 3!\cdot 2^{10}} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = \frac{1}{2}, \qquad S_2 = S_1 - \frac{1}{4\cdot 2\cdot 2^4} = 0.492\,19,$$
$$S_3 = S_2 + \frac{3}{7\cdot 2^2\cdot 2!\cdot 2^7} = 0.492\,61, \qquad S_4 = S_3 - \frac{3\cdot 5}{10\cdot 2^3\cdot 3!\cdot 2^{10}} = 0.492\,58.$$

Consequently, to three decimals the value of the integral is 0.493.

34. $$\int_0^{0.3} e^{-x^2}\,dx = \int_0^{0.3}\left[1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots\right]dx = \int_0^{0.3}\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right]dx$$
$$= \left\{x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right\}_0^{0.3} = 0.3 - \frac{(0.3)^3}{3} + \frac{(0.3)^5}{5\cdot 2!} - \frac{(0.3)^7}{7\cdot 3!} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = 0.3, \qquad S_2 = S_1 - \frac{(0.3)^3}{3} = 0.291\,00, \qquad S_3 = S_2 + \frac{(0.3)^5}{5\cdot 2!} = 0.291\,24.$$

Consequently, to three decimals the value of the integral is 0.291.

35. Using the series from Example 10.23,

$$\int_{-0.1}^{0} \frac{1}{x-1}\ln(1-x)\,dx = \int_{-0.1}^{0}\left[x + \left(1+\frac{1}{2}\right)x^2 + \left(1+\frac{1}{2}+\frac{1}{3}\right)x^3 + \cdots\right]dx$$
$$= \left\{\frac{x^2}{2} + \left(1+\frac{1}{2}\right)\frac{x^3}{3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4} + \cdots\right\}_{-0.1}^{0}$$
$$= -\frac{1}{2\cdot 10^2} + \left(1+\frac{1}{2}\right)\frac{1}{3\cdot 10^3} - \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4\cdot 10^4} + \cdots .$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = -0.005\,00, \qquad S_2 = S_1 + \left(1+\frac{1}{2}\right)\frac{1}{3\cdot 10^3} = -0.004\,50.$$

Consequently, to three decimals the value of the integral is -0.005.

36. $$\int_0^{1/2}\frac{1}{x^6-3x^3-4}dx = \int_0^{1/2}\frac{1}{(x^3-4)(x^3+1)}dx = \int_0^{1/2}\left(\frac{1/5}{x^3-4} + \frac{-1/5}{x^3+1}\right)dx$$
$$= \frac{1}{5}\int_0^{1/2}\left[\frac{-1}{4(1-x^3/4)} - \frac{1}{1+x^3}\right]dx$$
$$= -\frac{1}{5}\int_0^{1/2}\left[\frac{1}{4}\left(1 + \frac{x^3}{4} + \frac{x^6}{4^2} + \frac{x^9}{4^3} + \cdots\right) + \left(1 - x^3 + x^6 - x^9 + \cdots\right)\right]dx$$
$$= -\frac{1}{5}\int_0^{1/2}\left[\frac{5}{4} - \left(1 - \frac{1}{4^2}\right)x^3 + \left(1 + \frac{1}{4^3}\right)x^6 - \cdots\right]dx$$
$$= -\frac{1}{5}\left\{\frac{5x}{4} - \frac{1}{4}\left(1-\frac{1}{4^2}\right)x^4 + \frac{1}{7}\left(1+\frac{1}{4^3}\right)x^7 - \cdots\right\}_0^{1/2}$$
$$= \frac{1}{5}\left[-\frac{5}{4}\left(\frac{1}{2}\right) + \frac{1}{4}\left(1-\frac{1}{4^2}\right)\left(\frac{1}{2}\right)^4 - \frac{1}{7}\left(1+\frac{1}{4^3}\right)\left(\frac{1}{2}\right)^7 + \cdots\right].$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = -\frac{1}{8}, \quad S_2 = S_1 + \frac{1}{20\cdot 2^4}\left(1 - \frac{1}{4^2}\right) = -0.122\,07, \quad S_3 = S_2 - \frac{1}{35\cdot 2^7}\left(1 + \frac{1}{4^3}\right) = -0.122\,30.$$

Consequently, to three decimals the value of the integral is -0.122.

37. $$\int_0^{2\pi} \frac{1-\cos\theta}{\theta}d\theta = \int_0^{2\pi}\frac{1}{\theta}\left[1 - \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)\right]d\theta = \int_0^{2\pi}\left[\frac{\theta}{2!} - \frac{\theta^3}{4!} + \frac{\theta^5}{6!} - \cdots\right]d\theta$$
$$= \left\{\frac{\theta^2}{2\cdot 2!} - \frac{\theta^4}{4\cdot 4!} + \frac{\theta^6}{6\cdot 6!} - \cdots\right\}_0^{2\pi} = \frac{(2\pi)^2}{2\cdot 2!} - \frac{(2\pi)^4}{4\cdot 4!} + \frac{(2\pi)^6}{6\cdot 6!} - \cdots$$

This is a convergent alternating series. To find a two-decimal approximation, we calculate partial sums until two successive sums agree to two decimals:

$$S_1 = \frac{(2\pi)^2}{2\cdot 2!} = 9.8696, \qquad S_2 = S_1 - \frac{(2\pi)^4}{4\cdot 4!} = -6.3652, \qquad S_3 = S_2 + \frac{(2\pi)^6}{6\cdot 6!} = 7.7876,$$
$$S_4 = S_3 - \frac{(2\pi)^8}{8\cdot 8!} = 0.3470, \qquad S_5 = S_4 + \frac{(2\pi)^{10}}{10\cdot 10!} = 2.9896, \qquad S_6 = S_5 - \frac{(2\pi)^{12}}{12\cdot 12!} = 2.3310,$$
$$S_7 = S_6 + \frac{(2\pi)^{14}}{14\cdot 14!} = 2.4534, \qquad S_8 = S_7 - \frac{(2\pi)^{16}}{16\cdot 16!} = 2.4358, \qquad S_9 = S_8 + \frac{(2\pi)^{18}}{18\cdot 18!} = 2.4378.$$

Consequently, to two decimals the value of the integral is 2.44.

38. If we replace the integrand by its Maclaurin series, we have

$$\frac{\sqrt{\pi}}{2}\text{erf}(1) = \int_0^1\left[1 - t^2 + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots\right]dt = \left\{t - \frac{t^3}{3} + \frac{t^5}{5\cdot 2!} - \frac{t^7}{7\cdot 3!} + \cdots\right\}_0^1$$
$$= 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \cdots.$$

This is a convergent alternating series. To find a three-decimal approximation, we calculate partial sums until two successive sums agree to three decimals:

$$S_1 = 2/\sqrt{\pi} = 1.128\,38, \qquad S_2 = S_1 - (2/\sqrt{\pi})/3 = 0.7552\,25,$$
$$S_3 = S_2 + (2/\sqrt{\pi})/(5\cdot 2!) = 0.865\,09, \qquad S_4 = S_3 - (2/\sqrt{\pi})/(7\cdot 3!) = 0.838\,23,$$
$$S_5 = S_4 + (2/\sqrt{\pi})/(9\cdot 4!) = 0.843\,45, \qquad S_6 = S_5 - (2/\sqrt{\pi})/(11\cdot 5!) = 0.843\,31.$$

Consequently, to three decimals the value of the integral is 0.843.

39. (a) $S = 3.125\,100\,1 - 0.000\,090\,18\left(\dfrac{1}{1-1/10}\right) = 3.124\,999\,9.$

(b) $S_1 = 3.125\,100\,1$

$S_2 = 3.125\,100\,1 - 0.000\,090\,18 = 3.125\,009\,92$

$S_3 = 3.125\,100\,1 - 0.000\,090\,18(1 + 1/10) = 3.125\,000\,902$

$S_4 = 3.125\,100\,1 - 0.000\,090\,18(1 + 1/10 + 1/100) = 3.125\,000\,000\,2$

(c) $E_1 = 3.125\,100\,1 - 3.124\,999\,9 = 0.000\,100\,2$

$E_2 = 3.125\,009\,92 - 3.124\,999\,9 = 0.000\,010\,02$

$E_3 = 3.125\,000\,902 - 3.124\,999\,9 = 0.000\,001\,002$

$E_4 = 3.125\,000\,000\,2 - 3.124\,999\,9 = 0.000\,000\,100\,2$

Clearly the approximations get better as n increases in S_n.

(d) To three decimals, S_1, S_2, S_3, and S_4 predict an approximation of 3.125. To three decimals, S is also rounded to 3.125. On the other hand, to four decimals, S_1 predicts 3.1251, and S_2, S_3, and S_4 predict 3.1250. Rounded to four decimals, S is 3.1250.

REVIEW EXERCISES

1. The first five terms are $-1/10$, $-1/6$, $-3/28$, $-1/40$, and $1/18$. The sequence is not therefore monotonic. Since all further terms are positive, a lower bound is $V = -1$. Because

$$1 - \frac{n^2 - 5n + 3}{n^2 + 5n + 4} = \frac{10n + 1}{n^2 + 5n + 4} > 0,$$

$U = 1$ is an upper bound for the sequence. The limit of the sequence is 1.

2. The first three terms are $c_1 = 1$, $c_2 = 1/\sqrt{2}$, and $c_3 = \sqrt{3/8}$. The sequence appears to be decreasing; that is, $c_{n+1} < c_n$. This is true for $n = 1$. Suppose k is some integer for which $c_{k+1} < c_k$. Then

$${c_{k+1}}^2 + 1 < {c_k}^2 + 1 \qquad \Longrightarrow \qquad \frac{1}{2}\sqrt{{c_{k+1}}^2 + 1} < \frac{1}{2}\sqrt{{c_k}^2 + 1}.$$

In other words, $c_{k+2} < c_{k+1}$, and therefore by mathematical induction, the sequence is decreasing. It follows that $U = c_1 = 1$ is an upper bound, and clearly $V = 0$ is a lower bound. By Theorem 10.7 we conclude that the sequence has a limit L, and to find L we set

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{2}\sqrt{{c_n}^2 + 1}.$$

This equation implies that $L = (1/2)\sqrt{L^2 + 1}$, the solution of which is $L = 1/\sqrt{3}$.

3. Since $\text{Tan}^{-1}(1/n)$ decreases as n increases, and n^2+1 increases, it follows that the sequence is decreasing. The first term $\pi/8$ is an upper bound and 0 is clearly a lower bound. The limit of the sequence is 0.

4. First we note that because the first term of the sequence is 7, and all other terms are at least 15, there is no difficulty with the square root. The first three terms are $c_1 = 7$, $c_2 = 15+\sqrt{5}$, and $c_3 = 15+\sqrt{13 + \sqrt{5}}$. These terms are increasing, $c_3 > c_2 > c_1$. Suppose k is some integer for which $c_{k+1} > c_k$. Then

$$\sqrt{c_{k+1} - 2} > \sqrt{c_k - 2} \qquad \Longrightarrow \qquad 15 + \sqrt{c_{k+1} - 2} > 15 + \sqrt{c_k - 2}.$$

But this means that $c_{k+2} > c_{k+1}$, and therefore by mathematical induction, the sequence is increasing. A lower bound must be $V = c_1 = 7$. The first three terms are less than 100. Suppose k is some integer for which $c_k < 100$. Then $c_{k+1} = 15 + \sqrt{c_k - 2} < 15 + \sqrt{100 - 2} < 100$, and by mathematical induction, an upper bound is $U = 100$. By Theorem 10.7, the sequence has a limit L, and to find L we set

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} (15 + \sqrt{c_n - 2}).$$

This equation implies that $L = 15 + \sqrt{L - 2}$, the solution of which is $L = (31 + \sqrt{53})/2$.

5. The first four terms of the sequence are $c_1 = 6$, $c_2 = 19/3 = 6.333$, $c_3 = 6.316$, and $c_4 = 6.317$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \left(6 + \frac{2}{c_n}\right) - \left(6 + \frac{2}{c_{n-1}}\right) = -\frac{2(c_n - c_{n-1})}{c_n c_{n-1}}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{2|c_n - c_{n-1}|}{c_n c_{n-1}}.$$

Since all terms of the sequence are greater than 6 (the recursive definition makes this clear), it follows that

$$|c_{n+1} - c_n| < \frac{2|c_n - c_{n-1}|}{(6)(6)} = \frac{|c_n - c_{n-1}|}{18}.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} 6 + \frac{2}{c_n} \quad \Longrightarrow \quad L = 6 + \frac{2}{L}.$$

Of the two solutions $3 \pm \sqrt{11}$ of this equation, only $L = 3 + \sqrt{11}$ is positive.

6. The first four terms of the sequence are $c_1 = 6$, $c_2 = 1/29 = 0.034$, $c_3 = 0.195$, and $c_4 = 0.173$. They are oscillating. To show that the entire sequence oscillates, we calculate

$$c_{n+1} - c_n = \frac{1}{5+4c_n} - \frac{1}{5+4c_{n-1}} = -\frac{4(c_n - c_{n-1})}{(5+4c_n)(5+4c_{n-1})}.$$

Since all terms of the sequence are positive, the denominator of this expression is positive. It follows that $c_{n+1} - c_n$ has the opposite sign of $c_n - c_{n-1}$, and the sequence oscillates. To verify properties 2 and 3 of Theorem 10.8, we take absolute values in the above equation,

$$|c_{n+1} - c_n| = \frac{4|c_n - c_{n-1}|}{(5+4c_n)(5+4c_{n-1})} < \frac{4|c_n - c_{n-1}|}{(5)(5)} = \frac{4}{25}|c_n - c_{n-1}|.$$

This shows that the $|c_{n+1} - c_n|$ decrease and have limit 0. By Theorem 10.8 the sequence has a limit L that can be obtained by taking limits on both sides of the recursive definition,

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{5+4c_n} \quad \Longrightarrow \quad L = \frac{1}{5+4L}.$$

Of the two solutions $(-5 \pm \sqrt{41})/8$ of this equation, only $L = (\sqrt{41} - 5)/8$ is positive.

7. If we rewrite the equation in the form $f(x) = x^3 + 7x^2 + 6x - 25 = 0$, then, with an initial approximation $x_1 = 1.5$, Newton's iterative procedure defines further approximations by

$$x_{n+1} = x_n - \frac{x_n^3 + 7x_n^2 + 6x_n - 25}{3x_n^2 + 14x_n + 6}.$$

Iteration gives $x_2 = 1.407$, $x_3 = 1.404\,32$, $x_4 = 1.404\,31$, and $x_5 = 1.404\,31$. Since $f(1.404\,305) = -2.7 \times 10^{-4}$ and $f(1.404\,315) = 4.5 \times 10^{-5}$, the root to five decimal places is 1.404 31. The method of successive approximations defines the sequence $x_1 = 1.5$, $x_{n+1} = \left(\dfrac{x_n + 5}{x_n + 4}\right)^2$. Iteration gives $x_2 = 1.40$, $x_3 = 1.405$, $x_4 = 1.404\,3$, $x_5 = 1.404\,31$, and $x_6 = 1.404\,31$. If we define $g(x) = x - (x+5)^2/(x+4)^2$, then $g(1.404305) = -9.3 \times 10^{-6}$ and $g(1.404\,315) = 1.5 \times 10^{-6}$. The root is 1.404 31 accurate to five decimal places.

8. When $|k| < 1$, the sequence converges to 0, and when $k = \pm 1$, it converges to 1. It diverges for all other values.

9. The first four terms of the sequence are $c_1 = 1$, $c_2 = \sqrt{2}$, $c_3 = \sqrt{3}$, and $c_4 = \sqrt{4}$. Just the way these were calculated makes it clear that an explicit formula for the terms is $c_n = \sqrt{n}$. This could be proved by mathematical induction.

10. Since $f'(x) = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$, this function is decreasing ($f'(x) \le 0$) for $x \ge e$. It follows that the sequence $\{\ln(n)/n\}$ is decreasing for $n \ge 3$.

11. Since $l = \displaystyle\lim_{n\to\infty} \frac{\dfrac{n^2 - 3n + 2}{n^3 + 4n}}{\dfrac{1}{n}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} 1/n$ diverges, so also does the original series (by the limit comparison test).

12. Since $l = \displaystyle\lim_{n\to\infty} \frac{\dfrac{n^2 + 5n + 3}{n^4 - 2n + 5}}{\dfrac{1}{n^2}} = 1$, and $\displaystyle\sum_{n=1}^{\infty} 1/n^2$ converges, so also does the original series (by the limit comparison test).

13. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{5^{2n+2}}{(n+1)!}}{\dfrac{5^{2n}}{n!}} = \lim_{n\to\infty} \dfrac{25}{n+1} = 0$, the series converges (by the limit ratio test).

14. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^2+3}{(n+1)3^{n+1}}}{\dfrac{n^2+3}{n3^n}} = \dfrac{1}{3}$, the series converges (by the limit ratio test).

15. Since $\dfrac{(\ln n)^2}{\sqrt{n}} > \dfrac{1}{\sqrt{n}}$ for $n \geq 3$, and $\sum_{n=3}^{\infty} 1/\sqrt{n}$ diverges, so also does the original series (by the comparison test).

16. Consider the series of absolute values $\sum_{n=1}^{\infty} \left(\dfrac{n+1}{n^2}\right)$. Since $l = \lim_{n\to\infty} \dfrac{(n+1)/n^2}{1/n} = 1$, and $\sum_{n=1}^{\infty} 1/n$ diverges, the original series does not converge absolutely. Because the sequence $\{(n+1)/n^2\}$ is decreasing and has limit 0, the original series converges conditionally (by the alternating series test).

17. Consider the series of absolute values $\sum_{n=1}^{\infty} \dfrac{n+1}{n^3}$. Since it is the term-by-term addition of the convergent $p = 2$ and $p = 3$ series, it converges also. The original series therefore converges absolutely.

18. Since $\lim_{n\to\infty} \mathrm{Cos}^{-1}(1/n) = \pi/2$, the series diverges (by the n^{th} term test).

19. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{1}{n}\mathrm{Cos}^{-1}\left(\dfrac{1}{n}\right)}{\dfrac{1}{n}} = \dfrac{\pi}{2}$, and $\sum_{n=1}^{\infty} 1/n$ diverges, so also does the original series (by the limit comparison test).

20. Since $l = \lim_{n\to\infty} \dfrac{\dfrac{1}{n^2}\mathrm{Cos}^{-1}(1/n)}{\dfrac{1}{n^2}} = \dfrac{\pi}{2}$, and $\sum_{n=1}^{\infty} 1/n^2$ converges, so also does the original series (by the limit comparison test).

21. Since the series simplifies to $\sum_{n=1}^{\infty} 2^n$, it diverges (by the n^{th} term test).

22. Since $L = \lim_{n\to\infty} \dfrac{\dfrac{3\cdot 6\cdot 9\cdots(3n+3)}{(2n+2)!}}{\dfrac{3\cdot 6\cdot 9\cdots(3n)}{(2n)!}} = \lim_{n\to\infty} \dfrac{3n+3}{(2n+2)(2n+1)} = 0$, the series converges (by the limit ratio test).

23. Since $\lim_{n\to\infty} \sqrt{\dfrac{n^2+1}{n^2+5}} = 1$, the series diverges (by the n^{th} term test).

24. Because $\lim_{n\to\infty} (-1)^{n+1}\left(1+\dfrac{1}{n}\right)^3$ does not exist, the series diverges (by the n^{th} term test).

25. Consider the series of absolute values $\sum_{n=1}^{\infty} \dfrac{1}{n^2}|\sin n|$. Since $(1/n^2)|\sin n| \leq 1/n^2$ and $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, so also does the series of absolute values (by the comparison test). The original series therefore converges absolutely.

26. Since this is a geometric series with common ratio $10/125 = 2/25$, the series converges.

27. Consider the series of absolute values $\sum_{n=1}^{\infty} \frac{\ln n}{n}$. Since $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$, and $\sum_{n=3}^{\infty} 1/n$ diverges, so also does the series of absolute values. The original series does not therefore converge absolutely. Since the sequence $\{(\ln n)/n\}$ is decreasing (for $n \geq 2$) with limit 0, the original series converges conditionally (by the alternating series test).

28. This is a geometric series with common ratio $1/e^{\pi}$, and therefore it converges.

29. Since the series is the sum of two convergent geometric series $\sum_{n=1}^{\infty} (2/3)^n$ and $\sum_{n=1}^{\infty} (1/6)^n$, it must converge.

30. Consider the series of absolute values $\sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{n}} \cos(n\pi) \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Since this series diverges, the original series does not converge absolutely. Because the sequence $\{1/\sqrt{n}\}$ is decreasing with limit 0, the series converges conditionally (by the alternating series test).

31. Since the radius of convergence of the series is $R = \lim_{n\to\infty} \left| \dfrac{\dfrac{n+1}{n^2+1}}{\dfrac{n+2}{(n+1)^2+1}} \right| = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$. Since $l = \lim_{n\to\infty} \frac{(n+1)/(n^2+1)}{1/n}$ $= 1$, and the harmonic series diverges, so also does the power series at $x = 1$. At $x = -2$, the power series reduces to $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n^2+1}$. This series does not converge absolutely, but because the sequence $\{(n+1)/(n^2+1)\}$ is decreasing, with limit 0, the series converges conditionally. The interval of convergence is therefore $-1 \leq x < 1$.

32. Since the radius of convergence of the series is $R = \lim_{n\to\infty} \left| \dfrac{\dfrac{1}{n^2 2^n}}{\dfrac{1}{(n+1)^2 2^{n+1}}} \right| = 2$, the open interval of convergence is $-2 < x < 2$. At $x = 2$, the power series reduces to $\sum_{n=1}^{\infty} 1/n^2$ which converges. At $x = -2$, it becomes $\sum_{n=1}^{\infty} (-1)^n/n^2$ which converges absolutely. The interval of convergence is therefore $-2 \leq x \leq 2$.

33. Since the radius of convergence of the series is $R = \lim_{n\to\infty} \left| \frac{(n+1)^3}{(n+2)^3} \right| = 1$, the open interval of convergence is $-1 < x < 1$. At $x = 1$, the power series reduces to $\sum_{n=1}^{\infty} (n+1)^3$ which diverges. At $x = -1$, it becomes $\sum_{n=1}^{\infty} (-1)^n(n+1)^3$ which also diverges. The interval of convergence is therefore $-1 < x < 1$.

34. Since $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{1/n^n}} = \infty$, the series converges for all x.

35. Since this is a geometric series with common ratio $(x-2)/4$ it converges for $|(x-2)/4| < 1 \implies -2 < x < 6$.

36. Since the radius of convergence of the series is $R = \lim_{n\to\infty} \left| \dfrac{\sqrt{\dfrac{n+1}{n-1}}}{\sqrt{\dfrac{n+2}{n}}} \right| = 1$, the open interval of convergence is $-4 < x < -2$. At $x = -2$, the power series reduces to $\sum_{n=2}^{\infty} \sqrt{(n+1)/(n-1)}$ which diverges (by the n^{th} term test). At $x = -4$, it becomes $\sum_{n=2}^{\infty} (-1)^n\sqrt{(n+1)/(n-1)}$ which also diverges. The interval of convergence is therefore $-4 < x < -2$.

37. If we set $y = x^2$, the series becomes $\displaystyle\sum_{n=1}^{\infty} n3^n x^{2n} = \sum_{n=1}^{\infty} n3^n y^n$. Since the radius of convergence of this series is $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{n3^n}{(n+1)3^{n+1}}\right| = \frac{1}{3}$, the radius of convergence of the x-series is $R_x = \sqrt{R_y} = 1/\sqrt{3}$. The open interval of convergence is $-1/\sqrt{3} < x < 1/\sqrt{3}$. At $x = \pm 1/\sqrt{3}$, the power series becomes $\displaystyle\sum_{n=1}^{\infty} n$ which diverges. The interval of convergence is therefore $-1/\sqrt{3} < x < 1/\sqrt{3}$.

38. If we set $y = x^3$, then $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n}x^{3n} = \sum_{n=1}^{\infty}\frac{2^n}{n}y^n$. Since $R_y = \displaystyle\lim_{n\to\infty}\left|\frac{\frac{2^n}{n}}{\frac{2^{n+1}}{n+1}}\right| = \frac{1}{2}$, the radius of convergence of the original series is $R_x = {R_y}^{1/3} = 2^{-1/3}$. The open interval of convergence is $-2^{-1/3} < x < 2^{-1/3}$. At $x = 2^{-1/3}$, the power series reduces to the divergent harmonic series. At $x = -2^{-1/3}$, it becomes the alternating harmonic series which converges conditionally. The interval of convergence is therefore $-2^{-1/3} \le x < 2^{-1/3}$.

39. Using the binomial expansion 10.33b,

$$\begin{aligned}
\sqrt{1+x^2} = (1+x^2)^{1/2} &= 1 + \frac{x^2}{2} + \frac{(1/2)(-1/2)}{2!}(x^2)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}(x^2)^3 + \cdots \\
&= 1 + \frac{x^2}{2} - \frac{1}{2^2\cdot 2!}x^4 + \frac{3}{2^3\cdot 3!}x^6 - \frac{3\cdot 5}{2^4\cdot 4!}x^8 + \cdots \\
&= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{2^n n!}x^{2n}, \\
&= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}[1\cdot 2\cdot 3\cdots(2n-3)(2n-2)]}{2^n n![2\cdot 4\cdot 6\cdots(2n-2)]}x^{2n}, \\
&= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}(2n-2)!}{2^{2n-1}n!(n-1)!}x^{2n} \\
&= 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-2)!}{2^{2n-1}n!(n-1)!}x^{2n} \quad \text{valid for } -1 \le x \le 1.
\end{aligned}$$

40. $f(x) = e^5 e^x = e^5\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}x^n = \sum_{n=0}^{\infty}\frac{e^5}{n!}x^n, \quad -\infty < x < \infty$

41. Since $\cos(x+\pi/4) = (1/\sqrt{2})(\cos x - \sin x)$, we may subtract the Maclaurin series for $\cos x$ and $\sin x$ and multiply by $1/\sqrt{2}$,

$$\cos(x+\pi/4) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2}(2n)!}x^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\sqrt{2}(2n+1)!}x^{2n+1}, \quad -\infty < x < \infty.$$

42. When we integrate $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$, $|x| < 1$, we obtain

$$\ln|1+x| = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} + C.$$

Substitution of $x = 0$ gives $C = 0$, and therefore $\ln|1+x| = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$. At $x = 1$, the series reduces to $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ which converges conditionally. The interval of convergence is therefore $-1 < x \le 1$. We now replace x by $2x$ and at the same time multiply by x,

$$x\ln(1+2x) = x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(2x)^n = \sum_{n=1}^{\infty}\frac{2^n(-1)^{n-1}}{n}x^{n+1} = \sum_{n=2}^{\infty}\frac{2^{n-1}(-1)^n}{n-1}x^n,$$

valid for $-1/2 < x \le 1/2$.

43. $\sin x = \sin[(x-\pi/4)+\pi/4] = \dfrac{1}{\sqrt{2}}\sin(x-\pi/4) + \dfrac{1}{\sqrt{2}}\cos(x-\pi/4)$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2}(2n+1)!}\left(x-\frac{\pi}{4}\right)^{2n+1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2}(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}, \quad -\infty < x < \infty$$

44. Partial fractions give $f(x) = \dfrac{x}{x^2+4x+3} = \dfrac{3/2}{x+3} - \dfrac{1/2}{x+1}$. Since

$$\frac{1}{x+1} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1,$$

and

$$\frac{1}{x+3} = \frac{1}{3(1+x/3)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}x^n, \quad |x| < 3,$$

it follows that

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2\cdot 3^n}x^n + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2}x^n = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2}\left(1-\frac{1}{3^n}\right)x^n, \quad |x| < 1.$$

45. $e^x = e^{3+(x-3)} = e^3\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}(x-3)^n = \sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n, \quad -\infty < x < \infty$

46. According to Exercise 42, $\quad \ln(1+x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$, $-1 < x \le 1$. With this,

$$f(x) = x\ln(1+x) + \ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{n+1} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

$$= \sum_{n=2}^{\infty}\frac{(-1)^n}{n-1}x^n + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n = x + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{-1}{n-1}+\frac{1}{n}\right)x^n$$

$$= x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}x^n, \quad -1 < x \le 1.$$

47. $x^3e^{x^2} = x^3\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}(x^2)^n = \sum_{n=0}^{\infty}\frac{1}{n!}x^{2n+3}, \quad -\infty < x < \infty$

48. $e^{-x^2} = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}(-x^2)^n = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$

When this alternating series is truncated after the term in x^{2n}, the absolute value of the maximum possible error for given x is $|x|^{2n+2}/(n+1)!$. For $0 \le x \le 2$, the error is a maximum when $x = 2$, namely $2^{2n+2}/(n+1)!$. This error is less than 10^{-5} if $(n+1)!/2^{2n+2} > 10^5$. The smallest integer n for which this holds is $n = 18$. The terms that should be used are therefore $e^{-x^2} \approx \displaystyle\sum_{n=0}^{18}\frac{(-1)^n}{n!}x^{2n}$.

49. If we substitute $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation,

$$0 = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} -4a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} -4a_n x^n$$
$$= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 4a_n]x^n.$$

When we equate coefficients to zero, we obtain the recursive formula

$$a_{n+2} = \frac{4a_n}{(n+2)(n+1)}, \quad n \geq 0.$$

Iteration gives

$$a_2 = \frac{4a_0}{2 \cdot 1} = \frac{4a_0}{2!}, \quad a_4 = \frac{4a_2}{4 \cdot 3} = \frac{4^2 a_0}{4!}, \quad a^6 = \frac{4a_4}{6 \cdot 5} = \frac{4^3 a_0}{6!}, \cdots$$

$$a_3 = \frac{4a_1}{3 \cdot 2} = \frac{4a_1}{3!}, \quad a_5 = \frac{4a_3}{5 \cdot 4} = \frac{4^2 a_1}{5!}, \quad a^7 = \frac{4a_5}{7 \cdot 6} = \frac{4^3 a_1}{7!}, \cdots$$

The solution is therefore

$$y = a_0\left(1 + \frac{4x^2}{2!} + \frac{4^2x^4}{4!} + \frac{4^3x^6}{6!} + \cdots\right) + a_1\left(x + \frac{4x^3}{3!} + \frac{4^2x^5}{5!} + \frac{4^3x^7}{7!} + \cdots\right)$$
$$= a_0 \sum_{n=0}^{\infty} \frac{4^n}{(2n)!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!}x^{2n+1}.$$

Both series converge for $-\infty < x < \infty$.

50.
$$\sqrt{1+\sin x} = \sqrt{[\cos^2(x/2) + \sin^2(x/2)] + 2\sin(x/2)\cos(x/2)}$$
$$= \sqrt{[\cos(x/2) + \sin(x/2)]^2} = \cos(x/2) + \sin(x/2)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{x}{2}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{x}{2}\right)^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!}x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!}x^{2n+1}$$

When x is not confined to $-\pi/2 \leq x \leq \pi/2$, we must write $\sqrt{1+\sin x} = |\cos(x/2) + \sin(x/2)|$.

51. The numbers obtained are

0.5403, 0.8576, 0.6543 0.7935, 0.7014, 0.7640, 0.7221, 0.7504, 0.7314, 0.7442, 0.7356.

They seem to be converging to some limit. We are actually calculating the numbers in the recursively defined sequence $x_{n+1} = \cos x_n$. This implies that the numbers are converging to the root of the equation $x = \cos x$.

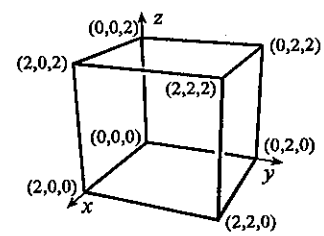
CHAPTER 11

EXERCISES 11.1

2. Length $= \sqrt{(1+3)^2 + (-2-2)^2 + (5-4)^2} = \sqrt{33}$

3. We find the squares of the lengths of the lines joining $P(2,0,4\sqrt{2})$, $Q(3,-1,5\sqrt{2})$, and $R(4,-2,4\sqrt{2})$: $\|PQ\|^2 = (1)^2 + (-1)^2 + (\sqrt{2})^2 = 4$, $\|PR\|^2 = (2)^2 + (-2)^2 = 8$, $\|QR\|^2 = (1)^2 + (-1)^2 + (-\sqrt{2})^2 = 4$. Since $\|PQ\| = \|QR\|$, the triangle is isosceles, and because $\|PR\|^2 = \|PQ\|^2 + \|QR\|^2$, the triangle is right angled.

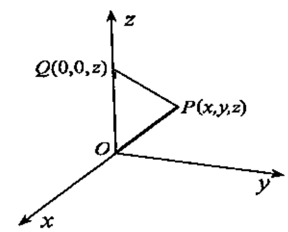
4. The diagram to the right indicates the vertices of the cube.



5. If we draw a line from $P(x,y,z)$ perpendicular to the z-axis, the coordinates of Q are $(0,0,z)$. The length of the perpendicular is

$$\begin{aligned}\|PQ\| &= \sqrt{\|OP\|^2 - \|OQ\|^2}\\ &= \sqrt{x^2+y^2+z^2-z^2}\\ &= \sqrt{x^2+y^2}.\end{aligned}$$

Similar derivations give distances to the x- and y-axes.



6. (a) $\sqrt{2^2+3^2+(-4)^2} = \sqrt{29}$ (b) $\sqrt{3^2+(-4)^2} = 5$ (c) $\sqrt{2^2+(-4)^2} = 2\sqrt{5}$ (d) $\sqrt{2^2+3^2} = \sqrt{13}$

7. (a) $\sqrt{1^2+(-5)^2+(-6)^2} = \sqrt{62}$ (b) $\sqrt{(-5)^2+(-6)^2} = \sqrt{61}$ (c) $\sqrt{1^2+(-6)^2} = \sqrt{37}$ (d) $\sqrt{1^2+(-5)^2} = \sqrt{26}$

8. (a) $\sqrt{4^2+3^2} = 5$ (b) $\sqrt{3^2} = 3$ (c) $\sqrt{4^2} = 4$ (d) $\sqrt{4^2+3^2} = 5$

9. (a) $\sqrt{(-2)^2+1^2+(-3)^2} = \sqrt{14}$ (b) $\sqrt{1^2+(-3)^2} = \sqrt{10}$ (c) $\sqrt{(-2)^2+(-3)^2} = \sqrt{13}$ (d) $\sqrt{(-2)^2+1^2} = \sqrt{5}$

10. The lengths of the lines joining $P(1,3,5)$, $Q(-2,0,3)$, and $R(7,9,9)$ are

$$\|PQ\| = \sqrt{(-3)^2+(-3)^2+(-2)^2} = \sqrt{22}, \quad \|QR\| = \sqrt{(9)^2+(9)^2+(6)^2} = 3\sqrt{22},$$

$$\|PR\| = \sqrt{(6)^2+(6)^2+(4)^2} = 2\sqrt{22}.$$

Since $\|QR\| = \|PQ\| + \|PR\|$, the three points are collinear.

11. The coordinates of the point can be taken in the form $(x, 3x, 0)$. The fact that the point is equidistant from $(1,3,2)$ and $(2,4,5)$ is expressed as $(x-1)^2 + (3x-3)^2 + (-2)^2 = (x-2)^2 + (3x-4)^2 + (-5)^2$. The only solution of this equation is $x = 31/8$, and therefore the required point is $(31/8, 93/8, 0)$.

12. A point $P(x,y,z)$ is equidistant from $(-3,0,4)$ and $(2,1,5)$ if and only if $(x+3)^2 + (y-0)^2 + (z-4)^2 = (x-2)^2 + (y-1)^2 + (z-5)^2$ and this equation reduces to $10x + 2y + 2z = 5$. The equation should describe a plane.

13. (a) If the third vertex is on the z-axis, its coordinates must be $P(0,0,z)$. Because this point is equidistant from $Q(\sqrt{3}-3, 2+2\sqrt{3}, 2\sqrt{3}-1)$ and $R(2\sqrt{3}, 4, \sqrt{3}-2)$, we can write that $(\sqrt{3}-3)^2 + (2+2\sqrt{3})^2 + (2\sqrt{3}-1-z)^2 = (2\sqrt{3})^2 + (4)^2 + (\sqrt{3}-2-z)^2$. The solution of this equation is $z = \sqrt{3}$. Since $\|PQ\| = \|QR\| = 4\sqrt{2}$, the triangle is equilateral.
(b) If the third vertex is on the x-axis, its coordinates must be $P(x,0,0)$. For P to be equidistant from Q and R, we can write that $(\sqrt{3}-3-x)^2 + (2+2\sqrt{3})^2 + (2\sqrt{3}-1)^2 = (2\sqrt{3}-x)^2 + (4)^2 + (\sqrt{3}-2)^2$. The solution of this equation is $x = -1$. Since $\|PQ\| \neq \|QR\|$, the triangle is isosceles but not equilateral.

14. Because $\|OJ\| = \|KA\|$, the x and y coordinates of A and J are the same, namely, $\sqrt{2}$ and $\sqrt{2}$. The coordinates of A are therefore $(\sqrt{2}, \sqrt{2}, 5)$. The length of BD is

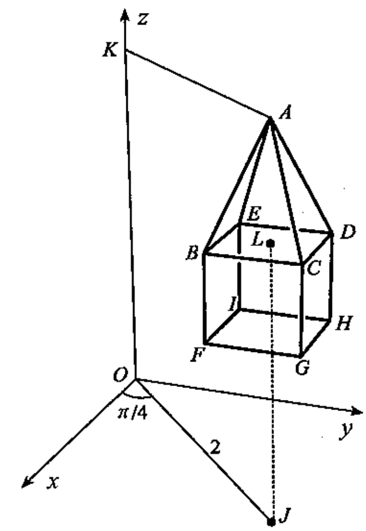
$\sqrt{\|BC\|^2 + \|CD\|^2} = \sqrt{(1/2)^2 + (1/2)^2} = \sqrt{2}/2.$

The length of AL is

$\sqrt{\|AD\|^2 - \|LD\|^2} = \sqrt{(3/4)^2 - (\sqrt{2}/4)^2} = \sqrt{7}/4.$

Consequently, the coordinates of the remaining corners are

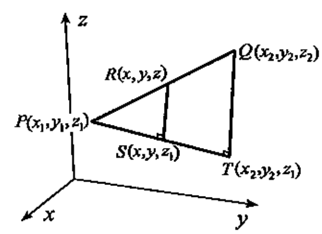
$$B(\sqrt{2}+1/4, \sqrt{2}-1/4, 5-\sqrt{7}/4),$$
$$C(\sqrt{2}+1/4, \sqrt{2}+1/4, 5-\sqrt{7}/4),$$
$$D(\sqrt{2}-1/4, \sqrt{2}+1/4, 5-\sqrt{7}/4),$$
$$E(\sqrt{2}-1/4, \sqrt{2}-1/4, 5-\sqrt{7}/4),$$
$$F(\sqrt{2}+1/4, \sqrt{2}-1/4, 9/2-\sqrt{7}/4),$$
$$G(\sqrt{2}+1/4, \sqrt{2}+1/4, 9/2-\sqrt{7}/4),$$
$$H(\sqrt{2}-1/4, \sqrt{2}+1/4, 9/2-\sqrt{7}/4),$$
$$I(\sqrt{2}-1/4, \sqrt{2}-1/4, 9/2-\sqrt{7}/4).$$



15. Because triangles PTQ and PSR are similar, ratios of corresponding sides are equal,

$$\frac{\|PQ\|}{\|PR\|} = \frac{\|QT\|}{\|RS\|} = \frac{z_2 - z_1}{z - z_1}.$$

Thus, $2 = (z_2 - z_1)/(z - z_1)$, from which $z = (z_1 + z_2)/2$. A similar proof gives the corresponding formulas for the x- and y-coordinates of R.



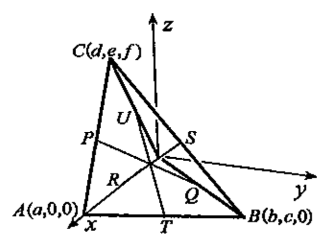
16. (a) According to Exercise 15, the required coordinates are $\left(\frac{1+3}{2}, \frac{-1+2}{2}, \frac{-3-4}{2}\right) = \left(2, \frac{1}{2}, -\frac{7}{2}\right)$.

(b) If the coordinates of R are (x, y, z), then, because Q is midway between P and R, $(3, 2, -4) = \left(\frac{x+1}{2}, \frac{y-1}{2}, \frac{z-3}{2}\right)$. When coordinates are equated, we obtain $x = 5$, $y = 5$, and $z = -5$.

17. The coordinates of the midpoints of the six sides of the tetrahedron have coordinates

$$P\left(\frac{a+d}{2}, \frac{e}{2}, \frac{f}{2}\right), \qquad Q\left(\frac{b}{2}, \frac{c}{2}, 0\right), \qquad R\left(\frac{a}{2}, 0, 0\right),$$
$$S\left(\frac{b+d}{2}, \frac{c+e}{2}, \frac{f}{2}\right), \quad T\left(\frac{a+b}{2}, \frac{c}{2}, 0\right), \quad U\left(\frac{d}{2}, \frac{e}{2}, \frac{f}{2}\right).$$

Midpoints of PQ, RS, and TU all have the same coordinates, $\left(\frac{a+b+d}{4}, \frac{e+c}{4}, \frac{f}{4}\right)$. This proves the required result.



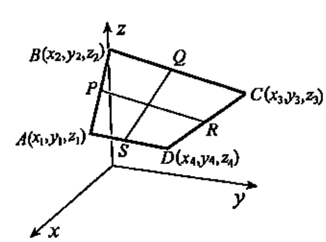
18. If coordinates of the vertices are as shown in the figure, then coordinates of the midpoints of the sides are

$$P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right), \quad Q\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right),$$
$$R\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}, \frac{z_3+z_4}{2}\right), \quad S\left(\frac{x_4+x_1}{2}, \frac{y_4+y_1}{2}, \frac{z_4+z_1}{2}\right).$$

Midpoints of the line segments PR and QS both have coordinates

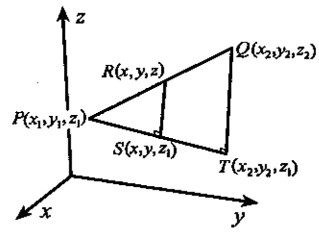
$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right),$$

and the line segments therefore intersect in this point.



19. Because triangles PTQ and PSR are similar, ratios of corresponding sides are equal,

$$\frac{\|PQ\|}{\|PR\|} = \frac{\|QT\|}{\|RS\|} = \frac{z_2 - z_1}{z - z_1}.$$



If we subtract 1 from each side of this equation,

$$\frac{\|PQ\|}{\|PR\|} - 1 = \frac{z_2 - z_1}{z - z_1} - 1$$

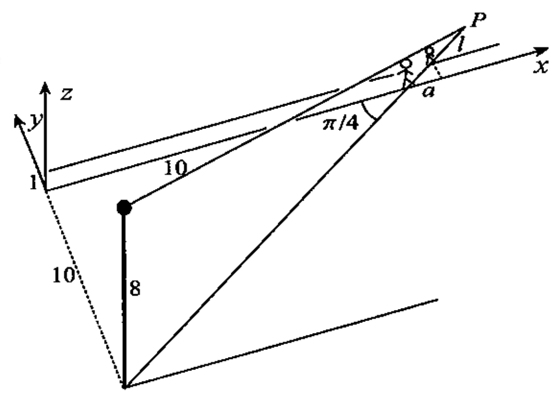
$$\Longrightarrow \quad \frac{\|PQ\| - \|PR\|}{\|PR\|} = \frac{z_2 - z}{z - z_1}$$

$$\Longrightarrow \quad \frac{r_2}{r_1} = \frac{z_2 - z}{z - z_1}.$$

Thus, $r_2 z - r_2 z_1 = r_1 z_2 - r_1 z \Longrightarrow z = \dfrac{r_1 z_2 + r_2 z_1}{r_1 + r_2}$. A similar proof gives the corresponding formulas for the x- and y-coordinates of R.

20. The x-coordinate of the tip of the shadow is $10 + a = 10 + 1 = 11$. The y-coordinate is 1. Let P be the point where the tip of the shadow would fall on the ground were the building not there. If z is the z-coordinate of the tip of the shadow on the wall, then from similar triangles, we may write

$$\frac{8}{2} = \frac{11\sqrt{2} + l}{\sqrt{2} + l} \quad \text{and} \quad \frac{8}{z} = \frac{11\sqrt{2} + l}{l}.$$



From the second of these, we obtain $l = \dfrac{11\sqrt{2}z}{8 - z}$, which substituted into the first gives

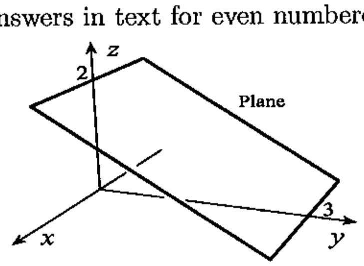
$$4\left(\sqrt{2} + \frac{11\sqrt{2}z}{8 - z}\right) = 11\sqrt{2} + \frac{11\sqrt{2}z}{8 - z}.$$

The solution of this equation is $z = 7/5$.

EXERCISES 11.2

See answers in text for even numbered exercises.

1.



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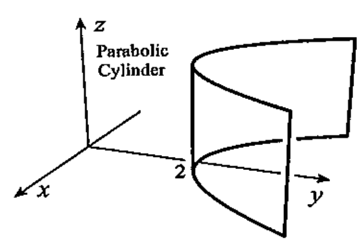

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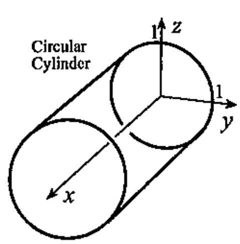

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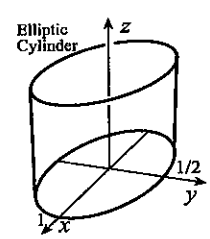

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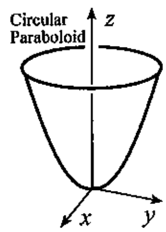

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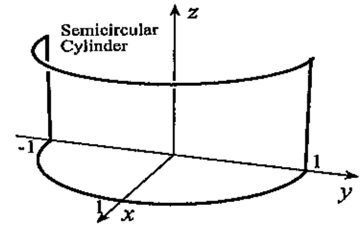

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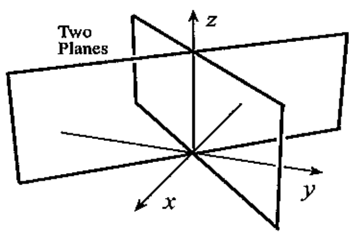

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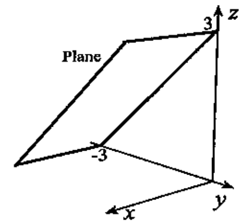

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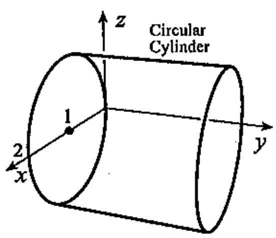

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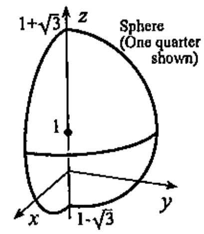

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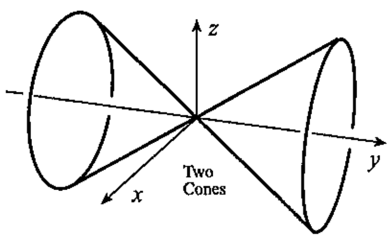

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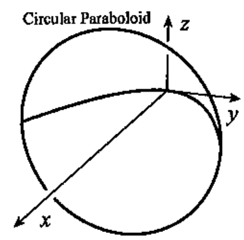

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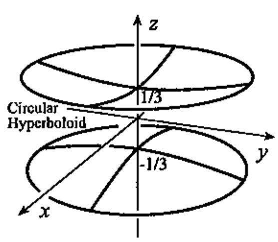

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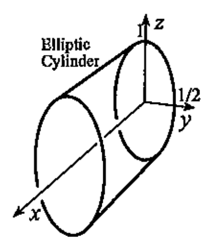

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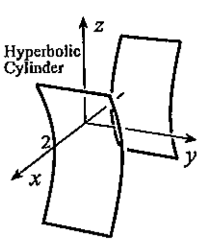

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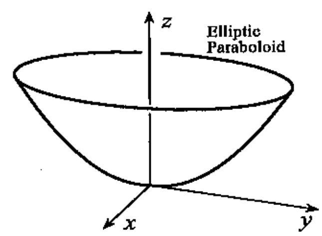

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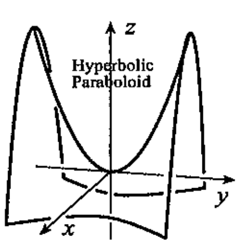

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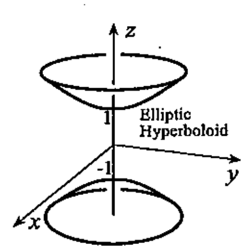

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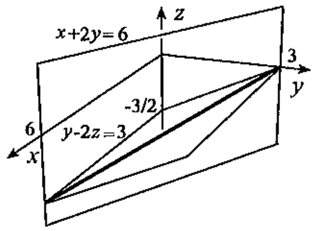

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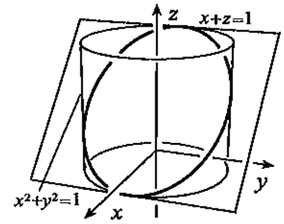

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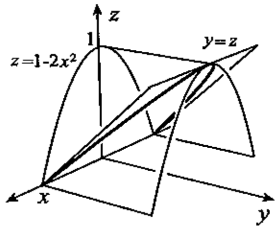

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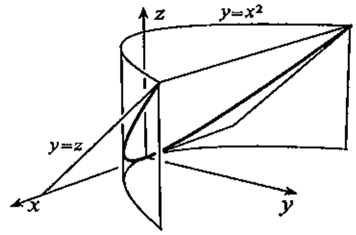

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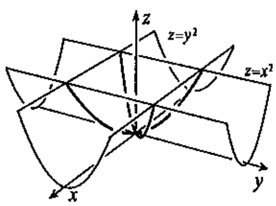
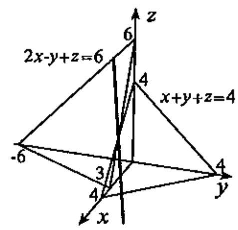
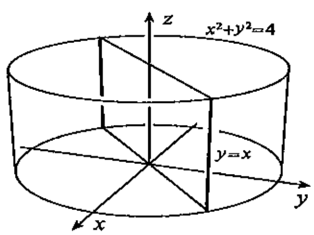


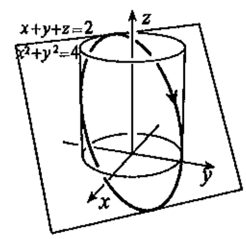
47. $x - 2y = 2$, $z = 0$;
$3y + z = 2$, $x = 0$;
$3x + 2z = 10$, $y = 0$



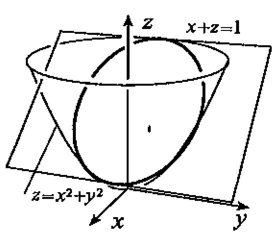
49. $x = \pm\sqrt{2}$, $y = \pm\sqrt{2}$, $z = 0$;
$y = \pm\sqrt{2}$, $x = 0$;
$x = \pm\sqrt{2}$, $y = 0$



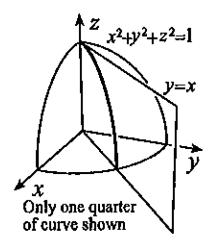
51. $x^2 + y^2 = 4$, $z = 0$;
$(2 - y - z)^2 + y^2 = 4$, $x = 0$;
$(2 - x - z)^2 + x^2 = 4$, $y = 0$



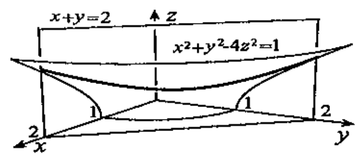
53. $(x + 1/2)^2 + y^2 = 5/4$, $z = 0$;
$y^2 + (z - 3/2)^2 = 5/4$, $x = 0$;
$x + z = 1$, $y = 0$



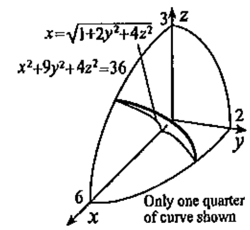
55. $y = x$, $z = 0$, $-1/\sqrt{2} \le x \le 1/\sqrt{2}$;
$2y^2 + z^2 = 1$, $x = 0$;
$2x^2 + z^2 = 1$, $y = 0$



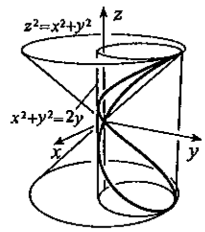
57. $4z^2 - 2(x-1)^2 = 1,\ y = 0$



59. $11y^2 + 8z^2 = 35,\ x = 0$



61. $z^2 = 2 \pm 2\sqrt{1-x^2},\ y = 0$



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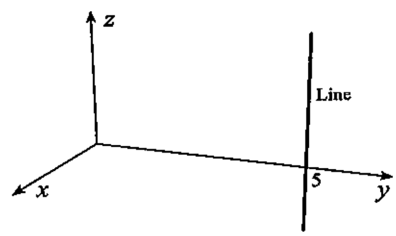

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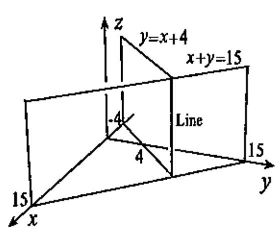

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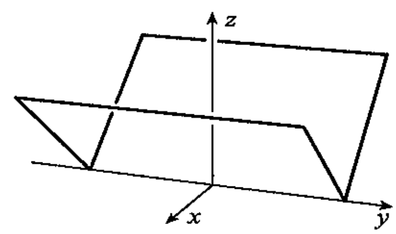

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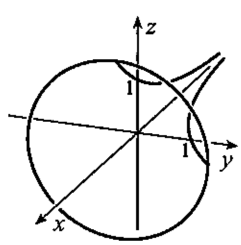

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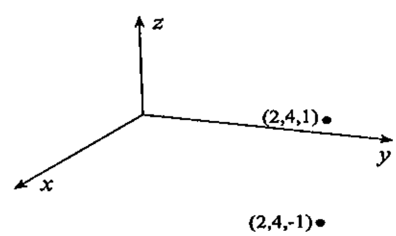


EXERCISES 11.3

1. $3\mathbf{u} - 2\mathbf{v} = 3(1,3,6) - 2(-2,0,4) = (7,9,10)$

2. $2\mathbf{w} + 3\mathbf{v} = 2(4,3,-2) + 3(-2,0,4) = (2,6,8)$

3. $\mathbf{w} - 3\mathbf{u} - 3\mathbf{v} = (4,3,-2) - 3(1,3,6) - 3(-2,0,4) = (7,-6,-32)$

4. $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{(-2,0,4)}{\sqrt{20}} = \left(-\dfrac{1}{\sqrt{5}}, 0, \dfrac{2}{\sqrt{5}}\right)$

5. $2\hat{\mathbf{w}} - 3\mathbf{v} = \dfrac{2(4,3,-2)}{\sqrt{29}} - 3(-2,0,4) = \left(\dfrac{8}{\sqrt{29}} + 6, \dfrac{6}{\sqrt{29}}, \dfrac{-4}{\sqrt{29}} - 12\right)$

6. $|\mathbf{v}|\mathbf{v} - 2|\hat{\mathbf{v}}|\mathbf{w} = \sqrt{20}(-2,0,4) - 2(1)(4,3,-2) = (-4\sqrt{5} - 8, -6, 8\sqrt{5} + 4)$

7. $(15 - 2|\mathbf{w}|)(\mathbf{u} + \mathbf{v}) = (15 - 2\sqrt{29})[(1,3,6) + (-2,0,4)] = (-15 + 2\sqrt{29}, 45 - 6\sqrt{29}, 150 - 20\sqrt{29})$

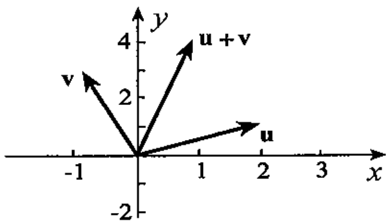
8. $|3\mathbf{u}|\mathbf{v} - |-2\mathbf{v}|\mathbf{u} = 3\sqrt{46}(-2,0,4) - 2\sqrt{20}(1,3,6) = (-6\sqrt{46} - 4\sqrt{5}, -12\sqrt{5}, 12\sqrt{46} - 24\sqrt{5})$

9. $|2\mathbf{u} + 3\mathbf{v} - \mathbf{w}|\hat{\mathbf{w}} = |(2(1,3,6) + 3(-2,0,4) - (4,3,-2)|\hat{\mathbf{w}}$

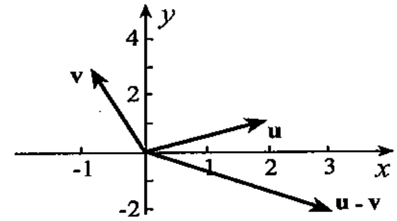
$$= |(-8,3,26)|\hat{\mathbf{w}} = \sqrt{(-8)^2 + 3^2 + 26^2}\,\frac{(4,3,-2)}{\sqrt{29}} = (4\sqrt{749/29}, 3\sqrt{749/29}, -2\sqrt{749/29})$$

10. $\dfrac{\mathbf{v}-\mathbf{w}}{|\mathbf{v}+\mathbf{w}|} = \dfrac{\mathbf{v}-\mathbf{w}}{|(-2,0,4)+(4,3,-2)|} = \dfrac{(-2,0,4)-(4,3,-2)}{|(2,3,2)|} = \dfrac{(-6,-3,6)}{\sqrt{17}} = \left(-\dfrac{6}{\sqrt{17}}, -\dfrac{3}{\sqrt{17}}, \dfrac{6}{\sqrt{17}}\right)$

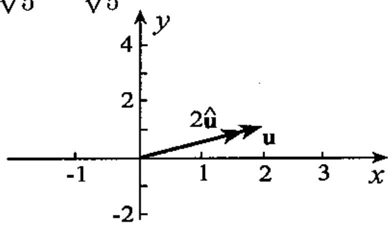
11. $\mathbf{u}+\mathbf{v} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}}$



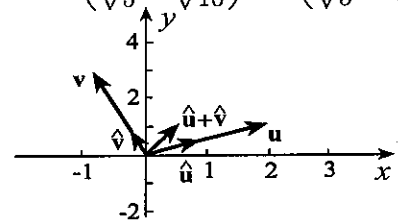
12. $\mathbf{u}-\mathbf{v} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$



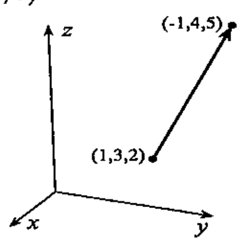
13. $2\hat{\mathbf{u}} = \dfrac{4}{\sqrt{5}}\hat{\mathbf{i}} + \dfrac{2}{\sqrt{5}}\hat{\mathbf{j}}$



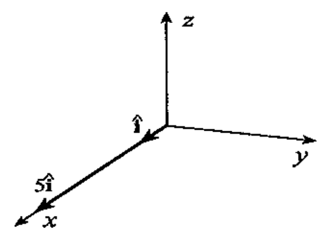
14. $\hat{\mathbf{v}} + \hat{\mathbf{u}} = \left(\dfrac{2}{\sqrt{5}} - \dfrac{1}{\sqrt{10}}\right)\hat{\mathbf{i}} + \left(\dfrac{1}{\sqrt{5}} + \dfrac{3}{\sqrt{10}}\right)\hat{\mathbf{j}}$



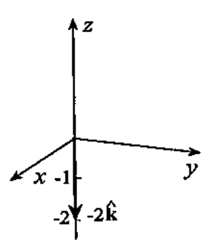
15. $(-2,1,3)$



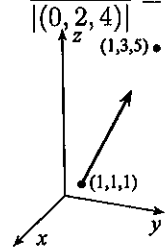
16. $5\hat{\mathbf{i}} = (5,0,0)$



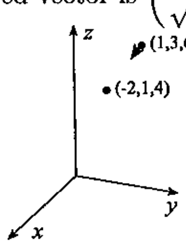
17. $-2\hat{\mathbf{k}} = (0,0,-2)$



18. The vector from $(1,1,1)$ to $(1,3,5)$ is $(1,3,5)-(1,1,1) = (0,2,4)$. The vector of length 3 in this direction is
$$\frac{3(0,2,4)}{|(0,2,4)|} = \left(0, \frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right)$$

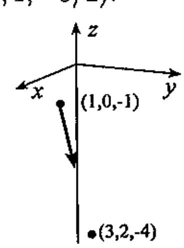


19. A vector in the correct direction is $(1,3,6)-(-2,1,4) = (3,2,2)$. The required vector is $\left(\dfrac{3}{\sqrt{17}}, \dfrac{2}{\sqrt{17}}, \dfrac{2}{\sqrt{17}}\right)$

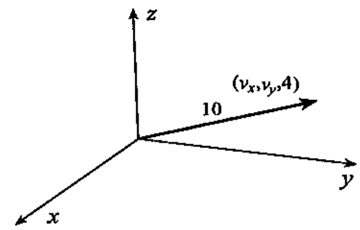


20. The vector from $(1,0,-1)$ to $(3,2,-4)$ is $(2,2,-3)$. A vector half as long is

$(1,1,-3/2)$.



21. If components of the vector are $(v_x, v_x, 4)$, the fact that its length is 10 requires $10 = \sqrt{2v_x{}^2 + 16} \implies v_x = \pm\sqrt{42}$. The required vector is $(\sqrt{42}, \sqrt{42}, 4)$.



22. Let (v_x, v_y, v_z) be the components of the vector, and draw perpendiculars from the tip P of the vector to the x- and y-axes. Since triangle OPQ is right-angled at Q, $\|OQ\|/\|OP\| = \cos(\pi/4)$. Consequently,

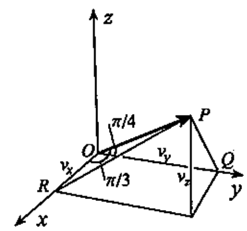
$$v_y = \|OQ\| = \|OP\|\cos(\pi/4) = \frac{5}{2}\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{2\sqrt{2}}.$$

Similarly, $v_x = \dfrac{5}{2}\cos(\pi/3) = \dfrac{5}{2}\left(\dfrac{1}{2}\right) = \dfrac{5}{4}$.

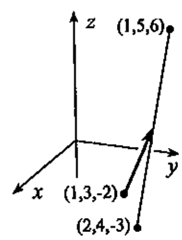
Finally, because the length of the vector is 5/2,

$$\frac{5}{2} = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{25/16 + 25/8 + v_z^2}.$$

Thus, $v_z{}^2 = \dfrac{25}{4} - \dfrac{25}{16} - \dfrac{25}{8} = \dfrac{25}{16} \implies v_z = 5/4$, and the vector has components $(5/4, 5\sqrt{2}/4, 5/4)$.



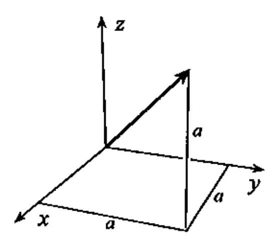
23. Since the midpoint of the line segment joining $(2, 4, -3)$ and $(1, 5, 6)$ has coordinates $(3/2, 9/2, 3/2)$, the required vector is $(3/2, 9/2, 3/2) - (1, 3, -2) = (1/2, 3/2, 7/2)$.



24. If the vector makes equal angles with the positive coordinate axes, its components must all be equal. If we take its components as (a, a, a), then the fact that its length is 2 requires

$$2 = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}a \implies a = 2/\sqrt{3}.$$

The required vector is $(2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3})$.

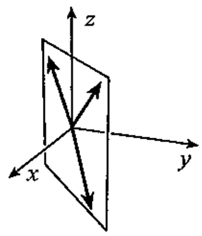


25. If coordinates of S are (x, y, z), the requirement $\mathbf{PQ} = \mathbf{RS}$ is expressed as $(-3, -1, 5) = (x-6, y-5, z+2)$. When we equate components, $x = 3$, $y = 4$, and $z = 3$.

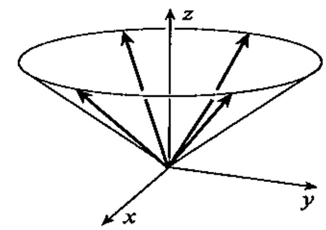
26. If we set $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, then

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_x + v_x, u_y + v_y, u_z + v_z) = (v_x + u_x, v_y + u_y, v_z + u_z) = \mathbf{v} + \mathbf{u};\\
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= (u_x + v_x, u_y + v_y, u_z + v_z) + (w_x, w_y, w_z)\\
&= (u_x + v_x + w_x, u_y + v_y + w_y, u_z + v_z + w_z)\\
&= (u_x, u_y, u_z) + (v_x + w_x, v_y + w_y, v_z + w_z) = \mathbf{u} + (\mathbf{v} + \mathbf{w});\\
\lambda(\mathbf{u} + \mathbf{v}) &= \lambda(u_x + v_x, u_y + v_y, u_z + v_z)\\
&= \big(\lambda(u_x + v_x), \lambda(u_y + v_y), \lambda(u_z + v_z)\big)\\
&= (\lambda u_x + \lambda v_x, \lambda u_y + \lambda v_y, \lambda u_z + \lambda v_z)\\
&= (\lambda u_x, \lambda u_y, \lambda u_z) + (\lambda v_x, \lambda v_y, \lambda v_z) = \lambda\mathbf{u} + \lambda\mathbf{v};\\
(\lambda + \mu)\mathbf{v} &= \big((\lambda + \mu)v_x, (\lambda + \mu)v_y, (\lambda + \mu)v_z\big)\\
&= (\lambda v_x + \mu v_x, \lambda v_y + \mu v_y, \lambda v_z + \mu v_z)\\
&= (\lambda v_x, \lambda v_y, \lambda v_z) + (\mu v_x, \mu v_y, \mu v_z) = \lambda\mathbf{v} + \mu\mathbf{v}.
\end{aligned}$$

27. All vectors lie in the plane $y = x$.



28. All vectors lie on the cone $z = \sqrt{x^2 + y^2}$.



29. When we equate components of $(5, -18, -32) = \mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v} = \lambda(3, 2, -4) + \mu(1, 6, 5) = (3\lambda + \mu, 2\lambda + 6\mu, -4\lambda + 5\mu)$, we obtain $3\lambda + \mu = 5$, $2\lambda + 6\mu = -18$, and $-4\lambda + 5\mu = -32$. The solution of these is $\lambda = 3$ and $\mu = -4$.

30. The slope of the curve $y = f(x) = x^2$ at $(2, 4)$ is $f'(2) = 4$. A vector along the tangent line is $(1, 4)$. A vector of length 3 along this line is $\mathbf{T} = \dfrac{3(1,4)}{\sqrt{1^2 + 4^2}} = \left(\dfrac{3}{\sqrt{17}}, \dfrac{12}{\sqrt{17}}\right)$.

31. The force due to the 3-coulomb charge at $(1, 1, 2)$ is

$$\mathbf{F}_1 = \frac{(2)(3)}{4\pi\epsilon_0(1+1+4)} \frac{(-1,-1,-2)}{\sqrt{6}} = \frac{(-1,-1,-2)}{4\sqrt{6}\pi\epsilon_0} \text{ N}.$$

The force due to the 3-coulomb charge at $(2, -1, -2)$ is

$$\mathbf{F}_2 = \frac{(2)(3)}{4\pi\epsilon_0(4+1+4)} \frac{(-2,1,2)}{3} = \frac{(-2,1,2)}{18\pi\epsilon_0} \text{ N}.$$

The resultant force due to both charges is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \frac{(-1,-1,-2)}{4\sqrt{6}\pi\epsilon_0} + \frac{(-2,1,2)}{18\pi\epsilon_0} = \frac{1}{4\pi\epsilon_0}\left(\frac{-1}{\sqrt{6}} - \frac{4}{9}, \frac{-1}{\sqrt{6}} + \frac{2}{9}, \frac{-2}{\sqrt{6}} + \frac{4}{9}\right) \text{ N}.$$

32. The force due to the mass at $(5, 1, 3)$ is

$$\mathbf{F}_1 = \frac{G(5)(10)}{3^2 + (-1)^2 + 1^2} \frac{(3,-1,1)}{\sqrt{3^2 + (-1)^2 + 1^2}} = \frac{50G}{11\sqrt{11}}(3,-1,1) \text{ N}.$$

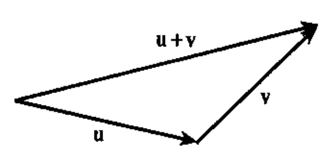
The force due to the mass at $(-1, 2, 1)$ is

$$\mathbf{F}_2 = \frac{G(5)(10)}{(-3)^2 + 0^2 + (-1)^2} \frac{(-3,0,-1)}{\sqrt{(-3)^2 + (-1)^2}} = \frac{5G}{\sqrt{10}}(-3,0,-1) \text{ N}.$$

The resultant force due to both masses is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \frac{50G(3,-1,1)}{11\sqrt{11}} + \frac{5G(-3,0,-1)}{\sqrt{10}} = 5G\left(\frac{30}{11\sqrt{11}} - \frac{3}{\sqrt{10}}, \frac{-10}{11\sqrt{11}}, \frac{10}{11\sqrt{11}} - \frac{1}{\sqrt{10}}\right) \text{ N}.$$

33. Since the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ form the sides of a triangle, the triangle inequality simply agrees with the fact that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. To prove it algebraically, we set $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$, then



$$|\mathbf{u}| + |\mathbf{v}| \geq |\mathbf{u} + \mathbf{v}| \quad \Longleftrightarrow \quad (|\mathbf{u}| + |\mathbf{v}|)^2 \geq |\mathbf{u} + \mathbf{v}|^2$$

$$\Longleftrightarrow \quad \left(\sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2} + \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}\right)^2 \geq (u_x + v_x)^2 + (u_y + v_y)^2 + (u_z + v_z)^2$$

$$\iff \quad 2\sqrt{u_x{}^2 + u_y{}^2 + u_z{}^2}\sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2} \geq 2(u_xv_x + u_yv_y + u_zv_z)$$

$$\iff \quad (u_x{}^2 + u_y{}^2 + u_z{}^2)(v_x{}^2 + v_y{}^2 + v_z{}^2) \geq (u_xv_x + u_yv_y + u_zv_z)^2$$

$$\iff \quad (u_xv_y - u_yv_x)^2 + (u_yv_z - u_zv_y)^2 + (u_xv_z - u_zv_x)^2 \geq 0$$

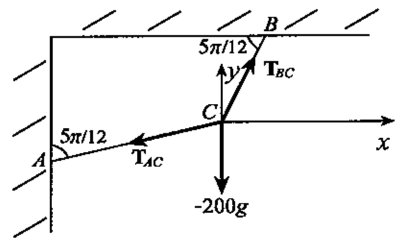
and this is clearly always true.

34. Let T_{AC} and T_{BC} be tensions in cables AC and BC. For equilibrium when both cables are taut, x- and y-components of all forces acting at C must sum to zero:

$0 = T_{BC}\cos 5\pi/12 - T_{AC}\sin 5\pi/12,$

$0 = T_{BC}\sin 5\pi/12 - T_{AC}\cos 5\pi/12 - 200g,$

where $g = 9.81$. When these are solved for T_{AC} and T_{BC}, the result, (to the nearest newton), is $T_{AC} = 586$ N and $T_{BC} = 2188$ N.

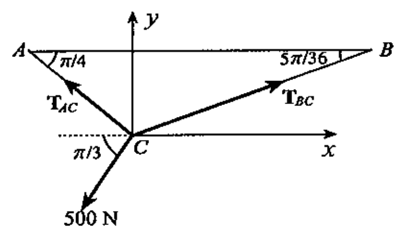


35. Let T_{AC} and T_{BC} be tensions in cables AC and BC. For equilibrium when both cables are taut, x- and y-components of all forces acting at C must sum to zero:

$0 = -T_{AC}\cos \pi/4 + T_{BC}\cos 5\pi/36 - 500\cos \pi/3,$

$0 = T_{AC}\sin \pi/4 + T_{BC}\sin 5\pi/36 - 500\sin \pi/3.$

When these are solved for T_{AC} and T_{BC}, the result, (to the nearest newton), is $T_{AC} = 305$ N and $T_{BC} = 514$ N.



36. For equilibrium when both cables are taut, x- and y-components of all forces acting at C must sum to zero:

$0 = 750\cos 5\pi/36 - 600\cos \pi/4 - |\mathbf{F}|\cos\theta,$

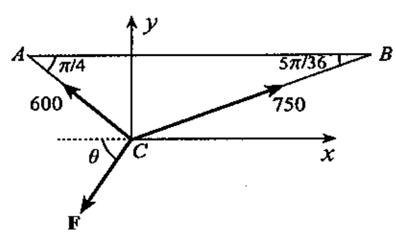
$0 = 750\sin 5\pi/36 + 600\sin \pi/4 - |\mathbf{F}|\sin\theta.$

When we write

$|\mathbf{F}|\sin\theta = 750\sin 5\pi/36 + 600/\sqrt{2},$

$|\mathbf{F}|\cos\theta = 750\cos 5\pi/36 - 600/\sqrt{2},$

and divide one equation by the other,



$$\tan\theta = \frac{750\sin 5\pi/36 + 600/\sqrt{2}}{750\cos 5\pi/36 - 600/\sqrt{2}} \quad \Longrightarrow \quad \theta = 1.24 \text{ radians.}$$

This in turn implies that $|\mathbf{F}| = 784$ N.

37. Assuming that there is no friction in the pulleys, the tension is the same at all points in the rope. For equilibrium, the sum of the x- and y-components of all forces acting on the pulley at O must be zero. If we assume that the two ropes from O to A are parallel, then

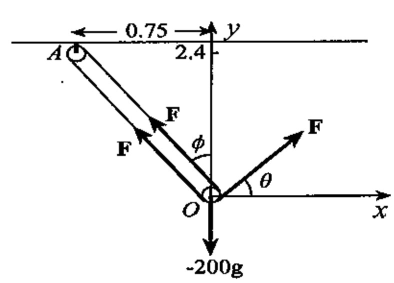
$0 = |\mathbf{F}|\cos\theta - 2|\mathbf{F}|\sin\phi,$

$0 = |\mathbf{F}|\sin\theta + 2|\mathbf{F}|\cos\phi - 200g,$

where $g = 9.81$. Since $\tan\phi = 0.75/2.4 = 5/16$, it follows that $\sin\phi = 5/\sqrt{281}$ and $\cos\phi = 16/\sqrt{281}$. From the first equation above,

$\cos\theta = 2\sin\phi = 10/\sqrt{281} \Longrightarrow \theta = \pm 0.9316$ radians.

When $\theta = 0.9316$, the second equation gives $|\mathbf{F}| = 200g/(\sin\theta + 2\cos\phi) = 724$ N. When $\theta = -0.9316$, we obtain $|\mathbf{F}| = 1773$ N.

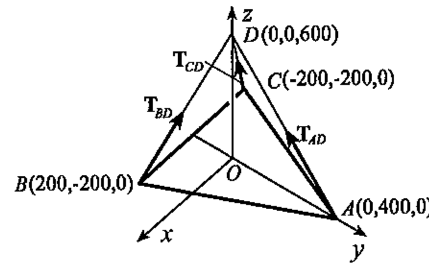


38. Let T_{AD}, T_{BD}, and T_{CD} denote tensions in the wires. The sum of all forces acting on the plate must be zero. There is the weight of the plate $\mathbf{W} = (0, 0, -16g)$, where $g = 9.81$, and tensions in the wires,

$$\mathbf{T}_{AD} = T_{AD}\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = \frac{T_{AD}(0,-400,600)}{\sqrt{400^2+600^2}} = \frac{T_{AD}(0,-2,3)}{\sqrt{13}},$$

$$\mathbf{T}_{BD} = T_{BD}\left(\frac{\mathbf{BD}}{|\mathbf{BD}|}\right) = \frac{T_{BD}(-200,200,600)}{\sqrt{200^2+200^2+600^2}} = \frac{T_{BD}(-1,1,3)}{\sqrt{11}},$$

$$\mathbf{T}_{CD} = T_{CD}\left(\frac{\mathbf{CD}}{|\mathbf{CD}|}\right) = \frac{T_{CD}(200,200,600)}{\sqrt{200^2+200^2+600^2}} = \frac{T_{CD}(1,1,3)}{\sqrt{11}}.$$



Hence, $\mathbf{0} = (0,0,-16g) + \dfrac{T_{AD}(0,-2,3)}{\sqrt{13}} + \dfrac{T_{BD}(-1,1,3)}{\sqrt{11}} + \dfrac{T_{CD}(1,1,3)}{\sqrt{11}}$. When we equate components to zero,

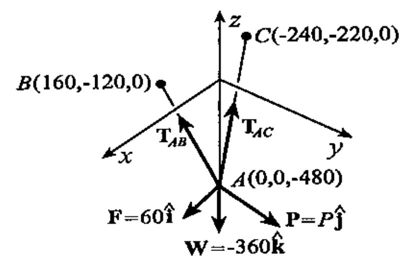
$$0 = -\frac{T_{BD}}{\sqrt{11}} + \frac{T_{CD}}{\sqrt{11}}, \quad 0 = -\frac{2T_{AD}}{\sqrt{13}} + \frac{T_{BD}}{\sqrt{11}} + \frac{T_{CD}}{\sqrt{11}}, \quad 0 = -16g + \frac{3T_{AD}}{\sqrt{13}} + \frac{3T_{BD}}{\sqrt{11}} + \frac{3T_{CD}}{\sqrt{11}}.$$

These can be solved for $T_{AD} = 16\sqrt{13}g/9$ N and $T_{BD} = T_{CD} = 16\sqrt{11}g/9$ N.

39. Let T_{AB} and T_{AC} denote tensions in the cables. The sum of all forces acting at A must be zero. There is $\mathbf{F} = 60\hat{\mathbf{i}}$, $\mathbf{P} = P\hat{\mathbf{j}}$, $\mathbf{W} = -360\hat{\mathbf{k}}$,

$$\mathbf{T}_{AB} = T_{AB}\left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right) = \frac{T_{AB}(160,-120,480)}{\sqrt{160^2+120^2+480^2}} = \frac{T_{AB}(4,-3,12)}{13},$$

$$\mathbf{T}_{AC} = T_{AC}\left(\frac{\mathbf{AC}}{|\mathbf{AC}|}\right) = \frac{T_{AC}(-240,-220,480)}{\sqrt{240^2+220^2+480^2}} = \frac{T_{AC}(-12,-11,24)}{29}.$$



Hence, $\mathbf{0} = 60\hat{\mathbf{i}} + P\hat{\mathbf{j}} - 360\hat{\mathbf{k}} + \dfrac{T_{AB}(4,-3,12)}{13} + \dfrac{T_{AC}(-12,-11,24)}{29}$. When we equate components to zero,

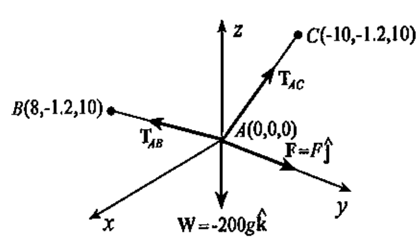
$$0 = 60 + \frac{4T_{AB}}{13} - \frac{12T_{AC}}{29}, \quad 0 = P - \frac{3T_{AB}}{13} - \frac{11T_{AC}}{29}, \quad 0 = -360 + \frac{12T_{AB}}{13} + \frac{24T_{AC}}{29}.$$

These can be solved for $T_{AB} = 156$ N, $T_{AC} = 261$ N, and $P = 135$ N.

40. Let T_{AB} and T_{AC} denote tensions in the cables. The sum of all forces acting at A must be zero. There is $\mathbf{F} = F\hat{\mathbf{j}}$, $\mathbf{W} = -200g\hat{\mathbf{k}}$ $(g = 9.81)$,

$$\mathbf{T}_{AB} = T_{AB}\left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right) = \frac{T_{AB}(8,-1.2,10)}{\sqrt{8^2+1.2^2+10^2}} = \frac{T_{AB}(20,-3,25)}{\sqrt{1034}},$$

$$\mathbf{T}_{AC} = T_{AC}\left(\frac{\mathbf{AC}}{|\mathbf{AC}|}\right) = \frac{T_{AC}(-10,-1.2,10)}{\sqrt{10^2+1.2^2+10^2}} = \frac{T_{AC}(-25,-3,25)}{\sqrt{1259}}.$$

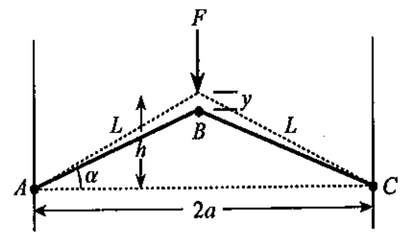


Hence, $\mathbf{0} = F\hat{\mathbf{j}} - 200g\hat{\mathbf{k}} + \dfrac{T_{AB}(20,-3,25)}{\sqrt{1034}} + \dfrac{T_{AC}(-25,-3,25)}{\sqrt{1259}}$. When we equate components to zero,

$$0 = \frac{20T_{AB}}{\sqrt{1034}} - \frac{25T_{AC}}{\sqrt{1259}}, \quad 0 = F - \frac{3T_{AB}}{\sqrt{1034}} - \frac{3T_{AC}}{\sqrt{1259}}, \quad 0 = -200g + \frac{25T_{AB}}{\sqrt{1034}} + \frac{25T_{AC}}{\sqrt{1259}}.$$

These can be solved for $T_{AB} = 40\sqrt{1034}g/9$ N, $T_{AC} = 32\sqrt{1259}g/9$ N, and $F = 24g$ N.

41. If we let P be the magnitude of the force in each bar when B is deflected downward by an amount y, then vertical components of forces acting at B give $2P\sin\alpha - F = 0 \Longrightarrow F = 2P\sin\alpha$. From equation 7.43, $P = (AE/L)[L - \sqrt{(h-y)^2 + a^2}]$, and therefore



$$F = \frac{2AE}{L}[L - \sqrt{(h-y)^2 + a^2}]\frac{h-y}{\sqrt{(h-y)^2 + a^2}} = \frac{2AE}{L}(h-y)\left[\frac{L}{\sqrt{(h-y)^2 + a^2}} - 1\right]$$

$$= \frac{2AE}{L}(h-y)\left[\frac{L}{\sqrt{y^2 - 2hy + L^2}} - 1\right].$$

42. Since $0\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$, the vectors are linearly dependent.

43. If we set $\mathbf{0} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = a(1,1,1) + b(2,1,3) + c(1,6,4)$
$= (a + 2b + c, a + b + 6c, a + 3b + 4c)$,

then $a+2b+c = 0$, $a+b+6c = 0$, $a+3b+4c = 0$. Since the only solution of this system is $a = b = c = 0$, the vectors are linearly independent.

44. If we set $\mathbf{0} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = a(-1,3,-5) + b(2,4,-1) + c(3,11,-7)$
$= (-a + 2b + 3c, 3a + 4b + 11c, -5a - b - 7c)$,

then $-a + 2b + 3c = 0$, $3a + 4b + 11c = 0$, $-5a - b - 7c = 0$. This system of equations has an infinite number of solutions representable in the form $a = -c$, $b = -2c$, where c is arbitrary. The vectors are therefore linearly dependent.

45. If we set $\mathbf{0} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = a(4,2,6) + b(1,3,-2) + c(7,1,4)$
$= (4a + b + 7c, 2a + 3b + c, 6a - 2b + 4c)$,

then $4a + b + 7c = 0$, $2a + 3b + c = 0$, $6a - 2b + 4c = 0$. Since the only solution of this system is $a = b = c = 0$, the vectors are linearly independent.

46. If the coordinates of A, B, and C are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) respectively, then

$$D = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) \quad \text{and} \quad E = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}, \frac{z_1 + z_3}{2}\right).$$

Thus, $\mathbf{DE} = \left(\dfrac{x_3 - x_2}{2}, \dfrac{y_3 - y_2}{2}, \dfrac{z_3 - z_2}{2}\right)$ and $\mathbf{BC} = (x_3 - x_2, y_3 - y_2, z_3 - z_2)$. Clearly, $\mathbf{DE} = (1/2)\mathbf{BC}$, and the result is therefore verified.

47. The coordinates of the midpoint D of side BC are $\big((x_2+x_3)/2, (y_2+y_3)/2, (z_2+z_3)/2\big)$. With the result of Exercise 11.1-19, the point on AD two-thirds of the way from A to D has coordinates

$$\left(\frac{2}{3}\left(\frac{x_2 + x_3}{2}\right) + \frac{x_1}{3}, \frac{2}{3}\left(\frac{y_2 + y_3}{2}\right) + \frac{y_1}{3}, \frac{2}{3}\left(\frac{z_2 + z_3}{2}\right) + \frac{z_1}{3}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

The points that divide medians from B and C in the same ratio have the same coordinates. It follows therefore that all three medians intersect in this point.

48. If we take x- and y-components of the vector equation $M(\overline{x}, \overline{y}) = \sum_{i=1}^{n} m_i\mathbf{r}_i = \sum_{i=1}^{n} m_i(x_i, y_i)$, we obtain

$$M\overline{x} = \sum_{i=1}^{n} m_i x_1 \quad \text{and} \quad M\overline{y} = \sum_{i=1}^{n} m_i y_i.$$ These are equations 7.31 and 7.32.

49. If $\mathbf{w} = w_x\hat{\mathbf{i}} + w_y\hat{\mathbf{j}}$ is any vector in the xy-plane, then for

$$\mathbf{w} = w_x\hat{\mathbf{i}} + w_y\hat{\mathbf{j}} = \lambda\mathbf{u} + \rho\mathbf{v} = (\lambda u_x + \rho v_x)\hat{\mathbf{i}} + (\lambda u_y + \rho v_y)\hat{\mathbf{j}}$$

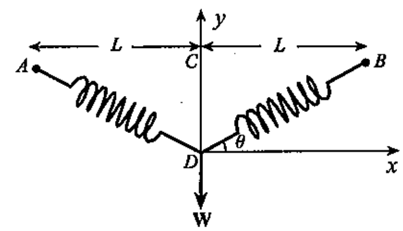
we must have $w_x = \lambda u_x + \rho v_x$, $w_y = \lambda u_y + \rho v_y$. The solution of these is

$$\lambda = \frac{w_x v_y - v_x w_y}{u_x v_y - u_y v_x}, \qquad \rho = \frac{u_x w_y - w_x u_y}{u_x v_y - u_y v_x}.$$

These are seen to exist provided $u_x v_y - u_y v_x \neq 0$.

50. (a) By summing vertical components of forces acting at D,

$$\begin{aligned} 0 &= -W + 2k(\sqrt{L^2+y^2} - L)\sin\theta \\ &= -W + 2k(\sqrt{L^2+y^2} - L)\frac{y}{\sqrt{L^2+y^2}}. \end{aligned}$$



Hence, $\quad W = 2ky\left(1 - \dfrac{L}{\sqrt{L^2+y^2}}\right)$.

(b) When y is very much less than L, we use the binomial expansion to write,

$$W = 2ky\left[1 - \frac{1}{\sqrt{1+(y/L)^2}}\right] = 2ky\left\{1 - \left[1 - \frac{1}{2}\left(\frac{y}{L}\right)^2 + \cdots\right]\right\}.$$

If we neglect higher order terms, $\quad W \approx 2ky\left(\dfrac{y^2}{2L^2}\right) = \dfrac{ky^3}{L^2}$.

51. If we denote the position of the sleeve by P, a distance x from O, the magnitude of the force on the sleeve due to spring one is $k_1(\sqrt{x^2+l^2} - l)$. The direction of this force is along $\mathbf{PA}$, and a unit vector in this direction is

$$\widehat{\mathbf{PA}} = \frac{\mathbf{PA}}{|\mathbf{PA}|} = \frac{(0-x, l-0)}{\sqrt{x^2+l^2}} = \frac{(-x,l)}{\sqrt{x^2+l^2}}.$$

Consequently, the force due to spring one is

$$\mathbf{F}_1 = k_1(\sqrt{x^2+l^2} - l)\frac{(-x,l)}{\sqrt{x^2+l^2}} = k_1\left(1 - \frac{l}{\sqrt{x^2+l^2}}\right)(-x,l).$$

Similarly, the force due to spring two is

$$\mathbf{F}_2 = k_2(\sqrt{(L-x)^2+l^2} - l)\frac{(L-x,l)}{\sqrt{(L-x)^2+l^2}} = k_2\left(1 - \frac{l}{\sqrt{(L-x)^2+l^2}}\right)(L-x,l).$$

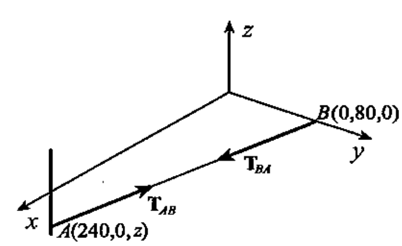
The resultant force on the sleeve is therefore

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = k_1\left(1 - \frac{l}{\sqrt{x^2+l^2}}\right)(-x,l) + k_2\left(1 - \frac{l}{\sqrt{(L-x)^2+l^2}}\right)(L-x,l).$$

52. If coordinates of A are denoted by $(240,0,z)$, then, $440 = \sqrt{240^2+80^2+z^2} \Longrightarrow z = -360$. Let the tension in the cable be T. Because there is no friction between collar A and the rod, the reaction of the rod $\mathbf{R}_A = N_x\hat{\mathbf{i}} + N_y\hat{\mathbf{j}}$ has no z-component.

Similarly, the reaction of the rod on collar B has no y-component, $\mathbf{R}_B = M_x\hat{\mathbf{i}} + M_z\hat{\mathbf{k}}$. When we sum forces on the collars and equate them to zero,

$$\mathbf{0} = -450\hat{\mathbf{k}} + \mathbf{R}_A + \mathbf{T}_{AB}, \quad \mathbf{0} = P\hat{\mathbf{j}} + \mathbf{R}_B + \mathbf{T}_{BA},$$



where $\mathbf{T}_{AB}$ and $\mathbf{T}_{BA}$ are tensions in the wire. Since

$$\mathbf{T}_{AB} = T\left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right) = \frac{T(-240,80,360)}{\sqrt{240^2+80^2+360^2}} = \frac{T(-6,2,9)}{11},$$

we obtain

$$\mathbf{0} = -450\hat{\mathbf{k}} + (N_x\hat{\mathbf{i}} + N_y\hat{\mathbf{j}}) + \frac{T(-6,2,9)}{11}, \qquad \mathbf{0} = P\hat{\mathbf{j}} + (M_x\hat{\mathbf{i}} + M_z\hat{\mathbf{k}}) - \frac{T(-6,2,9)}{11}.$$

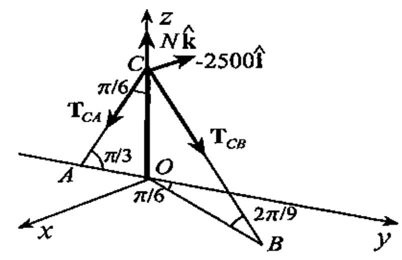
When we equate components,

$$0 = N_x - \frac{6T}{11}, \quad 0 = N_y + \frac{2T}{11}, \quad 0 = -450 + \frac{9T}{11}, \quad 0 = M_x + \frac{6T}{11}, \quad 0 = P - \frac{2T}{11}, \quad 0 = M_z - \frac{9T}{11}.$$

These give $T = 550$ N and $P = 100$ N.

53. Suppose $N\hat{\mathbf{k}}$ is the reaction of the pole at C. When we equate the sum of the forces acting at C to zero,

$$\begin{aligned}\mathbf{0} &= -2500\hat{\mathbf{i}} + N\hat{\mathbf{k}} + \mathbf{T}_{CA} + \mathbf{T}_{CB}\\ &= -2500\hat{\mathbf{i}} + N\hat{\mathbf{k}} + T_{CA}\left(-\frac{\hat{\mathbf{j}}}{2} - \frac{\sqrt{3}\hat{\mathbf{k}}}{2}\right)\\ &\quad + T_{CB}\left(\frac{1}{2}\cos 2\pi/9, \frac{\sqrt{3}}{2}\cos 2\pi/9, -\sin 2\pi/9\right).\end{aligned}$$



We now equate components,

$$0 = -2500 + \frac{T_{CB}}{2}\cos 2\pi/9, \quad 0 = -\frac{T_{CA}}{2} + \frac{\sqrt{3}T_{CB}}{2}\cos 2\pi/9, \quad 0 = N - \frac{\sqrt{3}T_{CA}}{2} - T_{CB}\sin 2\pi/9.$$

These can be solved for $T_{CA} = 8660$ N, $T_{CB} = 6527$ N, and $N = 11\,695$ N.

54. (a) Since $i = \phi$, the z-component of $\hat{\mathbf{v}}$ must be the negative of the z-components of $\hat{\mathbf{u}}$; that is, $v_z = -u_z$. As a result, $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ both have the same projections in the xy-plane, and hence they have the same x- and y-components. The components of $\hat{\mathbf{v}}$ are therefore $(u_x, u_y, -u_z)$.
(b) The z-components of $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ are $u_z = -\cos i$ and $w_z = -\cos\theta$, and therefore

$$w_z = -\sqrt{1-\sin^2\theta} = -\sqrt{1 - \frac{{n_1}^2}{{n_2}^2}\sin^2 i} = -\sqrt{1 - \frac{{n_1}^2}{{n_2}^2}(1-\cos^2 i)} = -\sqrt{1 - \frac{{n_1}^2}{{n_2}^2}(1-{u_z}^2)}.$$

The projections of $\hat{\mathbf{u}}$ and $\hat{\mathbf{w}}$ in the xy-plane must be in the same direction so that we may write $(w_x, w_y, 0) = \lambda(u_x, u_y, 0)$. In addition, the lengths of these vectors must be

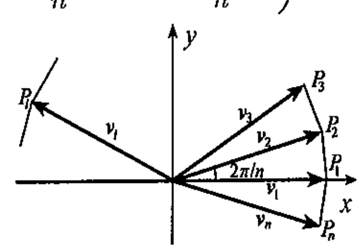
$$\sqrt{{w_x}^2 + {w_y}^2} = \sin\theta, \qquad \sqrt{{u_x}^2 + {u_y}^2} = \sin i.$$

Because $n_1 \sin i = n_2 \sin\theta$, it follows that ${n_1}^2({u_x}^2 + {u_y}^2) = {n_2}^2({w_x}^2 + {w_y}^2) = {n_2}^2(\lambda^2 {u_x}^2 + \lambda^2 {u_y}^2)$.
This equation implies that $\lambda = n_1/n_2$. Consequently, $\hat{\mathbf{w}} = \left(\frac{n_1}{n_2}u_x, \frac{n_1}{n_2}u_y, -\sqrt{1 - \frac{{n_1}^2}{{n_2}^2}(1-{u_z}^2)}\right)$.

55. Suppose that the length of each vector is a and the angle between any two successive vectors is $2\pi/n$. The i$^{\text{th}}$ vector $\mathbf{v}_i$ in the diagram has components $\quad \mathbf{v}_i = a\left(\cos\frac{2\pi(i-1)}{n}, \sin\frac{2\pi(i-1)}{n}\right)$.

The sum of these vectors is

$$\begin{aligned}\sum_{i=1}^{n} \mathbf{v}_i &= a\left(\sum_{i=1}^{n}\cos\frac{2\pi(i-1)}{n}, \sum_{i=1}^{n}\sin\frac{2\pi(i-1)}{n}\right)\\ &= a\left(\sum_{i=0}^{n-1}\cos\frac{2\pi i}{n}, \sum_{i=0}^{n-1}\sin\frac{2\pi i}{n}\right).\end{aligned}$$



According to Exercises 6.3–12 and 13,

$$\sum_{i=1}^{m} \sin i\theta = \frac{\sin \dfrac{(m+1)\theta}{2} \sin \dfrac{m\theta}{2}}{\sin \dfrac{\theta}{2}} \quad \text{and} \quad \sum_{i=1}^{m} \cos i\theta = \frac{\cos \dfrac{(m+1)\theta}{2} \sin \dfrac{m\theta}{2}}{\sin \dfrac{\theta}{2}}.$$

With these,

$$\sum_{i=0}^{n-1} \sin \frac{2\pi i}{n} = 0 + \frac{\sin \left[\dfrac{n}{2}\left(\dfrac{2\pi}{n}\right)\right] \sin \left[\dfrac{n-1}{2}\left(\dfrac{2\pi}{n}\right)\right]}{\sin \left[\dfrac{1}{2}\left(\dfrac{2\pi}{n}\right)\right]} = 0,$$

and

$$\begin{aligned}\sum_{i=0}^{n-1} \cos \frac{2\pi i}{n} &= 1 + \frac{\cos \left[\dfrac{n}{2}\left(\dfrac{2\pi}{n}\right)\right] \sin \left[\dfrac{n-1}{2}\left(\dfrac{2\pi}{n}\right)\right]}{\sin \left[\dfrac{1}{2}\left(\dfrac{2\pi}{n}\right)\right]} = 1 - \frac{\sin \dfrac{(n-1)\pi}{n}}{\sin \dfrac{\pi}{n}} \\ &= 1 - \frac{\sin \pi \cos \dfrac{\pi}{n} - \cos \pi \sin \dfrac{\pi}{n}}{\sin \dfrac{\pi}{n}} = 0.\end{aligned}$$

This completes the proof.

EXERCISES 11.4

1. $\mathbf{u} \cdot \mathbf{v} = (2)(0) + (-3)(1) + (1)(-1) = -4$
2. $(\mathbf{v} \cdot \mathbf{w})\mathbf{u} = [(0)(6) + (1)(-2) + (-1)(3)](2, -3, 1) = (-10, 15, -5)$
3. $(2\mathbf{u} - 3\mathbf{v}) \cdot \mathbf{w} = (4, -9, 5) \cdot (6, -2, 3) = (4)(6) + (-9)(-2) + (5)(3) = 57$
4. $2\hat{\mathbf{i}} \cdot \hat{\mathbf{u}} = (2, 0, 0) \cdot \dfrac{(2, -3, 1)}{\sqrt{4+9+1}} = \dfrac{4}{\sqrt{14}}$
5. $|2\mathbf{u}|\mathbf{v} \cdot \mathbf{w} = 2\sqrt{4+9+1}[(0)(6) + (1)(-2) + (-1)(3)] = -10\sqrt{14}$
6. $(3\mathbf{u} - 4\mathbf{w}) \cdot (2\hat{\mathbf{i}} + 3\mathbf{u} - 2\mathbf{v}) = (-18, -1, -9) \cdot (8, -11, 5) = -144 + 11 - 45 = -178$
7. $\mathbf{w} \cdot \hat{\mathbf{w}} = (6, -2, 3) \cdot \dfrac{(6, -2, 3)}{7} = 7$
8. $\dfrac{(105\mathbf{u} + 240\mathbf{v}) \cdot (105\mathbf{u} + 240\mathbf{v})}{|105\mathbf{u} + 240\mathbf{v}|^2} = \dfrac{|105\mathbf{u} + 240\mathbf{v}|^2}{|105\mathbf{u} + 240\mathbf{v}|^2} = 1$
9. $|\mathbf{u} - \mathbf{v} + \hat{\mathbf{k}}|(\hat{\mathbf{j}} + \mathbf{w}) \cdot \hat{\mathbf{k}} = |(2, -4, 3)|(6, -1, 3) \cdot (0, 0, 1) = 3\sqrt{29}$
10. $\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} - (\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} - \mathbf{u} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{v} = 0$
11. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 0 \\ -2 & -3 & 5 \end{vmatrix} = 10\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$
12. $(-3\mathbf{u}) \times (2\mathbf{v}) = -6(\mathbf{u} \times \mathbf{v}) = -6\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 4 \\ -1 & 2 & 0 \end{vmatrix} = -6[-8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}] = 48\hat{\mathbf{i}} + 24\hat{\mathbf{j}} - 42\hat{\mathbf{k}}$
13. Using the result of Exercise 11, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (3, 1, 4) \cdot (10, 5, 7) = 63$.
14. $$\begin{aligned}\hat{\mathbf{u}} \times \hat{\mathbf{w}} &= \frac{\mathbf{u}}{|\mathbf{u}|} \times \frac{\mathbf{w}}{|\mathbf{w}|} = \frac{1}{|\mathbf{u}||\mathbf{w}|}\mathbf{u} \times \mathbf{w} = \frac{1}{\sqrt{9+1+16}\sqrt{4+9+25}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 4 \\ -2 & -3 & 5 \end{vmatrix} \\ &= \frac{1}{\sqrt{26}\sqrt{38}}(17\hat{\mathbf{i}} - 23\hat{\mathbf{j}} - 7\hat{\mathbf{k}}) = \frac{1}{2\sqrt{247}}(17\hat{\mathbf{i}} - 23\hat{\mathbf{j}} - 7\hat{\mathbf{k}})\end{aligned}$$

15. $((3\mathbf{u})\times\mathbf{w})+(\mathbf{u}\times\mathbf{v}) = 3\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -2 & -3 & 5\end{vmatrix} + \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -1 & 2 & 0\end{vmatrix}$

$= 3(17,-23,-7)+(-8,-4,7) = (43,-73,-14)$

16. $\mathbf{u}\times(3\mathbf{v}-\mathbf{w}) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -1 & 9 & -5\end{vmatrix} = -41\hat{\mathbf{i}}+11\hat{\mathbf{j}}+28\hat{\mathbf{k}}$

17. Since $\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -1 & 2 & 0\end{vmatrix} = (-8,-4,7)$, we find $\dfrac{\mathbf{w}\times\mathbf{u}}{|\mathbf{u}\times\mathbf{v}|} = \dfrac{1}{\sqrt{129}}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -2 & -3 & 5\\ 3 & 1 & 4\end{vmatrix} = \dfrac{(-17,23,7)}{\sqrt{129}}$.

18. $\mathbf{u}\times\mathbf{w} - \mathbf{u}\times\mathbf{v}+\mathbf{u}\times(2\mathbf{u}+\mathbf{v}) = \mathbf{u}\times(\mathbf{w}-\mathbf{v}+2\mathbf{u}+\mathbf{v}) = \mathbf{u}\times(\mathbf{w}+2\mathbf{u})$

$= \mathbf{u}\times\mathbf{w} \qquad (\text{since } \mathbf{u}\times\mathbf{u}=\mathbf{0})$

$= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -2 & -3 & 5\end{vmatrix} = 17\hat{\mathbf{i}}-23\hat{\mathbf{j}}-7\hat{\mathbf{k}}$

19. $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ -1 & 2 & 0\end{vmatrix}\times\mathbf{w} = (-8,-4,7)\times\mathbf{w} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -8 & -4 & 7\\ -2 & -3 & 5\end{vmatrix} = (1,26,16)$

20. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{u}\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -1 & 2 & 0\\ -2 & -3 & 5\end{vmatrix} = \mathbf{u}\times(10,5,7) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 1 & 4\\ 10 & 5 & 7\end{vmatrix} = (-13,19,5)$

21. Since $(1,2)\cdot(3,5) = 13$, the vectors are not perpendicular.

22. Since $(2,4)\cdot(-8,4) = -16+16 = 0$, the vectors are perpendicular.

23. Since $(1,3,6)\cdot(-2,1,-4) = -23$, the vectors are not perpendicular.

24. Since $(2,3,-6)\cdot(-6,6,1) = -12+18-6 = 0$, the vectors are perpendicular.

25. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(3,4)\cdot(2,-5)}{|(3,4)|\,|(2,-5)|}\right] = \mathrm{Cos}^{-1}\left(\frac{-14}{5\sqrt{29}}\right) = 2.12 \text{ radians.}$$

26. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(1,6)\cdot(-4,7)}{|(1,6)|\,|(-4,7)|}\right] = \mathrm{Cos}^{-1}\left(\frac{38}{\sqrt{37}\sqrt{65}}\right) = 0.684 \text{ radians.}$$

27. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(4,2,3)\cdot(1,5,6)}{|(4,2,3)|\,|(1,5,6)|}\right] = \mathrm{Cos}^{-1}\left(\frac{32}{\sqrt{29}\sqrt{62}}\right) = 0.716 \text{ radians.}$$

28. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(3,1,-1)\cdot(-2,1,4)}{|(3,1,-1)|\,|(-2,1,4)|}\right] = \mathrm{Cos}^{-1}\left(\frac{-9}{\sqrt{11}\sqrt{21}}\right) = 2.20 \text{ radians.}$$

29. If θ is the angle between the vectors, then

$$\theta = \mathrm{Cos}^{-1}\left[\frac{(2,0,5)\cdot(0,3,0)}{|(2,0,5)|\,|(0,3,0)|}\right] = \mathrm{Cos}^{-1}(0) = \frac{\pi}{2} \text{ radians.}$$

30. Since $(-2,-6,4) = -2(1,3,-2)$, the vectors are in opposite directions, and $\theta = \pi$.

31. One such vector is $\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 3 & 5\\ -2 & 1 & 4\end{vmatrix} = (7,-14,7)$. All such vectors can be represented in the form $\lambda(1,-2,1)$.

32. One such vector is $\hat{\mathbf{j}} \times (1,-1,-9) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & -1 & -9 \end{vmatrix} = (-9,0,-1)$. All such vectors can be represented by $\lambda(9,0,1)$.

33. Two of the sides of the triangle are represented by the vectors $(6,1,-1)$ and $(-5,2,1)$. A vector perpendicular to the triangle is therefore $(6,1,-1) \times (-5,2,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & 1 & -1 \\ -5 & 2 & 1 \end{vmatrix} = (3,-1,17)$. All vectors perpendicular to the triangle are of the form $\lambda(3,-1,17)$.

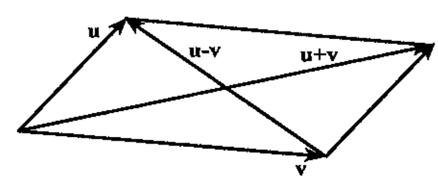
34. If we set $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, then

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_x v_x + u_y v_y + u_z v_z = v_x u_x + v_y u_y + v_z u_z = \mathbf{v} \cdot \mathbf{u}, \\ \mathbf{u} \cdot (\lambda \mathbf{v} + \rho \mathbf{w}) &= (u_x, u_y, u_z) \cdot (\lambda v_x + \rho w_x, \lambda v_y + \rho w_y, \lambda v_z + \rho w_z) \\ &= u_x(\lambda v_x + \rho w_x) + u_y(\lambda v_y + \rho w_y) + u_z(\lambda v_z + \rho w_z) \\ &= \lambda(u_x v_x + u_y v_y + u_z v_z) + \rho(u_x w_x + u_y w_y + u_z w_z) = \lambda(\mathbf{u} \cdot \mathbf{v}) + \rho(\mathbf{u} \cdot \mathbf{w}). \end{aligned}$$

35. With equation 11.23,

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= 2\mathbf{u} \cdot \mathbf{u} + 2\mathbf{v} \cdot \mathbf{v} = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2. \end{aligned}$$

In the diagram, $|\mathbf{u}+\mathbf{v}|$ and $|\mathbf{u}-\mathbf{v}|$ are the lengths of the diagonals of the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as coterminal sides. The law states that the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the four sides.



36. If α is the angle between $\mathbf{v}$ and $\hat{\mathbf{i}}$, then $\mathbf{v} \cdot \hat{\mathbf{i}} = |\mathbf{v}||\hat{\mathbf{i}}| \cos\alpha \implies \cos\alpha = \dfrac{\mathbf{v} \cdot \hat{\mathbf{i}}}{|\mathbf{v}|}$.

Since the angle between two vectors always lies between 0 and π, and such angles coincide with the principal values of the inverse cosine function, we may write that

$$\alpha = \mathrm{Cos}^{-1}\left(\frac{\mathbf{v} \cdot \hat{\mathbf{i}}}{|\mathbf{v}|}\right) = \mathrm{Cos}^{-1}\left(\frac{v_x}{|\mathbf{v}|}\right).$$

Similar discussions lead to the formulas for β and γ.

37. $\alpha = \mathrm{Cos}^{-1}\left(\dfrac{1}{\sqrt{1+4+9}}\right) = 1.30$; $\beta = \mathrm{Cos}^{-1}\left(\dfrac{2}{\sqrt{14}}\right) = 1.01$; $\gamma = \mathrm{Cos}^{-1}\left(\dfrac{-3}{\sqrt{14}}\right) = 2.50$.

38. $\alpha = \mathrm{Cos}^{-1} 0 = \pi/2$; $\beta = \mathrm{Cos}^{-1}\left(\dfrac{1}{\sqrt{1+9}}\right) = 1.25$; $\gamma = \mathrm{Cos}^{-1}\left(\dfrac{-3}{\sqrt{10}}\right) = 2.82$.

39. $\alpha = \mathrm{Cos}^{-1}\left(\dfrac{-1}{\sqrt{1+4+36}}\right) = 1.73$; $\beta = \mathrm{Cos}^{-1}\left(\dfrac{-2}{\sqrt{41}}\right) = 1.89$; $\gamma = \mathrm{Cos}^{-1}\left(\dfrac{6}{\sqrt{41}}\right) = 0.36$.

40. $\alpha = \mathrm{Cos}^{-1}\left(\dfrac{-2}{\sqrt{4+9+16}}\right) = 1.95$; $\beta = \mathrm{Cos}^{-1}\left(\dfrac{3}{\sqrt{29}}\right) = 0.980$; $\gamma = \mathrm{Cos}^{-1}\left(\dfrac{4}{\sqrt{29}}\right) = 0.734$.

41. If we set $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, then

$$-(\mathbf{v} \times \mathbf{u}) = -\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = -[(v_y u_z - v_z u_y)\hat{\mathbf{i}} + (v_z u_x - v_x u_z)\hat{\mathbf{j}} + (v_x u_y - v_y u_x)\hat{\mathbf{k}}$$

and this is the same as $\mathbf{u} \times \mathbf{v}$ (see equation 11.28). Property 11.31b is verified when we compare components of

$$\mathbf{u}\times(\lambda\mathbf{v}+\rho\mathbf{w}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ \lambda v_x+\rho w_x & \lambda v_y+\rho w_y & \lambda v_z+\rho w_z \end{vmatrix}$$
$$= [u_y(\lambda v_z+\rho w_z) - u_z(\lambda v_y+\rho w_y)]\hat{\mathbf{i}} + [u_z(\lambda v_x+\rho w_x) - u_x(\lambda v_z+\rho w_z)]\hat{\mathbf{j}} + [u_x(\lambda v_y+\rho w_y) - u_y(\lambda v_x+\rho w_x)]\hat{\mathbf{k}}$$

and

$$\lambda\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} + \rho\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ w_x & w_y & w_z \end{vmatrix} = \lambda[(u_yv_z-u_zv_y)\hat{\mathbf{i}} + (u_zv_x-u_xv_z)\hat{\mathbf{j}} + (u_xv_y-u_yv_x)\hat{\mathbf{k}}] + \rho[(u_yw_z-u_zw_y)\hat{\mathbf{i}} + (u_zw_x-u_xw_z)\hat{\mathbf{j}} + (u_xw_y-u_yw_x)\hat{\mathbf{k}}]$$

42. One example serves to show this. See Exercises 19 and 20 for an example.

43. (a) Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, and ϕ be the angle between $\mathbf{u}$ and $\mathbf{w}$. Because $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\mathbf{w}$,

$$|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{w}|\cos\phi \qquad \text{and} \qquad |\mathbf{u}||\mathbf{v}|\sin\theta = |\mathbf{u}||\mathbf{w}|\sin\phi,$$

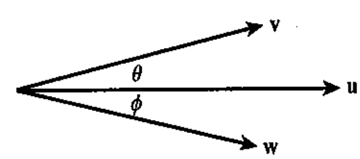
or,

$$|\mathbf{v}|\cos\theta = |\mathbf{w}|\cos\phi \qquad \text{and} \qquad |\mathbf{v}|\sin\theta = |\mathbf{w}|\sin\phi.$$

If these equations are squared and added,

$$|\mathbf{v}|^2\cos^2\theta + |\mathbf{v}|^2\sin^2\theta = |\mathbf{w}|^2\cos^2\phi + |\mathbf{w}|^2\sin^2\phi \implies |\mathbf{v}|^2 = |\mathbf{w}|^2.$$

Thus, $|\mathbf{v}|=|\mathbf{w}|$, and it follows that $\cos\theta=\cos\phi$ and $\sin\theta=\sin\phi$. These require $\theta=\phi$, and therefore the angle between $\mathbf{u}$ and $\mathbf{v}$ is the same as that between $\mathbf{u}$ and $\mathbf{w}$. Finally, if all three vectors are placed at the same point (see figure), $\mathbf{v}$ and $\mathbf{w}$ cannot lie on opposite sides of $\mathbf{u}$ for the right-hand rule would then give $\mathbf{u}\times\mathbf{v}=-\mathbf{u}\times\mathbf{w}$. Thus, $\mathbf{v}=\mathbf{w}$.



(b) This is a matter of logic. If $\mathbf{v}=\mathbf{w}$, then certainly $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\mathbf{w}$. Consequently, if one of $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\mathbf{w}$, is not satisfied, then $\mathbf{v}$ cannot be equal to $\mathbf{w}$.

44. (a) $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = (6,-1,0)\cdot\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & 4 \\ -2 & -1 & 4 \end{vmatrix} = (6,-1,0)\cdot(16,-12,5) = 108$

(b) This is verified when we compare for general $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$,

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = (u_x,u_y,u_z)\cdot\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$
$$= (u_x,u_y,u_z)\cdot[(v_yw_z-v_zw_y)\hat{\mathbf{i}} + (v_zw_x-v_xw_z)\hat{\mathbf{j}} + (v_xw_y-v_yw_x)\hat{\mathbf{k}}]$$
$$= u_x(v_yw_z-v_zw_y) + u_y(v_zw_x-v_xw_z) + u_z(v_xw_y-v_yw_x)$$

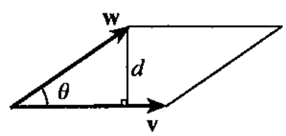
and

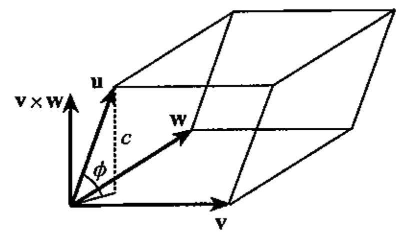
$$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}\cdot(w_x,w_y,w_z)$$
$$= [(u_yv_z-u_zv_y)\hat{\mathbf{i}} + (u_zv_x-u_xv_z)\hat{\mathbf{j}} + (u_xv_y-u_yv_x)\hat{\mathbf{k}}]\cdot(w_x,w_y,w_z)$$
$$= (u_yv_z-u_zv_y)w_x + (u_zv_x-u_xv_z)w_y + (u_xv_y-u_yv_x)w_z.$$

(c) The volume of the parallelepiped is the area of one of the parallelogram sides multiplied by the perpendicular distance to the opposite side. The area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$ in the left figure below is $|\mathbf{v}|d = |\mathbf{v}||\mathbf{w}|\sin\theta = |\mathbf{v}\times\mathbf{w}|$. The perpendicular distance between the parallel sides, one of which contains $\mathbf{v}$ and $\mathbf{w}$ (right figure), is $c = |\mathbf{u}|\sin\phi$. The volume of the parallelepiped is

$$(|\mathbf{u}|\sin\phi)|\mathbf{v}\times\mathbf{w}| = |\mathbf{u}||\mathbf{v}\times\mathbf{w}|\cos(\pi/2-\phi) = |\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}|.$$

The absolute values are included to take care of the possibility that the scalar product could be negative.





(d) This result follows from the fact that the three vectors are coplanar if and only if the volume of the parallelepiped with the vectors as coterminal sides is zero.

45. If $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, $\mathbf{w} = (w_x, w_y, w_z)$, and $\mathbf{r} = (r_x, r_y, r_z)$, then

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{r}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ w_x & w_y & w_z\\ r_x & r_y & r_z\end{vmatrix}\\ &= [(u_yv_z-u_zv_y)\hat{\mathbf{i}} + (u_zv_x-u_xv_z)\hat{\mathbf{j}} + (u_xv_y-u_yv_x)\hat{\mathbf{k}}]\cdot\\ &\qquad [(w_yr_z-w_zr_y)\hat{\mathbf{i}} + (w_zr_x-w_xr_z)\hat{\mathbf{j}} + (w_xr_y-w_yr_x)\hat{\mathbf{k}}]\\ &= (u_yv_z-u_zv_y)(w_yr_z-w_zr_y) + (u_zv_x-u_xv_z)(w_zr_x-w_xr_z)\\ &\qquad + (u_xv_y-u_yv_x)(w_xr_y-w_yr_x),\end{aligned}$$

and

$$\begin{aligned}(\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{r}) - (\mathbf{u}\cdot\mathbf{r})(\mathbf{v}\cdot\mathbf{w}) &= (u_xw_x+u_yw_y+u_zw_z)(v_xr_x+v_yr_y+v_zr_z)\\ &\qquad - (u_xr_x+u_yr_y+u_zr_z)(v_xw_x+v_yw_y+v_zw_z).\end{aligned}$$

When these two scalar expressions are expanded, they are identical.

46. If $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$, then

$$\begin{aligned}\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ u_x & u_y & u_z\\ v_yw_z-v_zw_y & v_zw_x-v_xw_z & v_xw_y-v_yw_x\end{vmatrix}\\ &= [u_y(v_xw_y-v_yw_x) - u_z(v_zw_x-v_xw_z)]\hat{\mathbf{i}} + [u_z(v_yw_z-v_zw_y) - u_x(v_xw_y-v_yw_x)]\hat{\mathbf{j}}\\ &\qquad + [u_x(v_zw_x-v_xw_z) - u_y(v_yw_z-v_zw_y)]\hat{\mathbf{k}},\end{aligned}$$

and

$$\begin{aligned}(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} &= (u_xw_x+u_yw_y+u_zw_z)(v_x,v_y,v_z) - (u_xv_x+u_yv_y+u_zv_z)(w_x,w_y,w_z)\\ &= [v_x(u_xw_x+u_yw_y+u_zw_z) - w_x(u_xv_x+u_yv_y+u_zv_z)]\hat{\mathbf{i}}\\ &\qquad + [v_y(u_xw_x+u_yw_y+u_zw_z) - w_y(u_xv_x+u_yv_y+u_zv_z)]\hat{\mathbf{j}}\\ &\qquad + [v_z(u_xw_x+u_yw_y+u_zw_z) - w_z(u_xv_x+u_yv_y+u_zv_z)]\hat{\mathbf{k}}.\end{aligned}$$

Comparison of components shows that these vectors are identical.

47. If we cross $\mathbf{PQ} + \mathbf{QR} + \mathbf{RP} = \mathbf{0}$ with $\mathbf{PQ}$,

$$\mathbf{0} = (\mathbf{PQ} \times \mathbf{PQ}) + (\mathbf{PQ} \times \mathbf{QR}) + (\mathbf{PQ} \times \mathbf{RP}) \quad \Longrightarrow \quad \mathbf{PQ} \times \mathbf{QR} = -\mathbf{PQ} \times \mathbf{RP}.$$

When we take lengths (using 11.33),

$$|\mathbf{PQ}||\mathbf{QR}| \sin(\pi - B) = |\mathbf{PQ}||\mathbf{RP}| \sin(\pi - A) \quad \Longrightarrow \quad \frac{\sin B}{b} = \frac{\sin A}{a}.$$

Similarly, crossing $\mathbf{PQ} + \mathbf{QR} + \mathbf{RP} = \mathbf{0}$ with $\mathbf{QR}$ gives $\dfrac{\sin C}{c} = \dfrac{\sin A}{a}$.

48. If we let $\mathbf{w} = (|\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v})/\big||\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}\big|$, then $\mathbf{w}$ is clearly a unit vector, since it is a vector divided by its own length. If we let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$. If we let ϕ be the angle between $\mathbf{w}$ and $\mathbf{u}$, then

$$\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}||\mathbf{w}| \cos\phi \quad \Longrightarrow \quad \cos\phi = \frac{|\mathbf{v}|(\mathbf{u} \cdot \mathbf{u}) + |\mathbf{u}|(\mathbf{v} \cdot \mathbf{u})}{|\mathbf{u}|(\big||\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}\big|)} = \frac{|\mathbf{v}||\mathbf{u}|^2 + |\mathbf{u}|(\mathbf{v} \cdot \mathbf{u})}{|\mathbf{u}|(\big||\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}\big|)} = \frac{|\mathbf{v}||\mathbf{u}| + (\mathbf{v} \cdot \mathbf{u})}{\big||\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}\big|}.$$

Now,

$$\begin{aligned}(|\mathbf{u}||\mathbf{v}| + \mathbf{v} \cdot \mathbf{u})^2 &= |\mathbf{u}|^2|\mathbf{v}|^2 + 2|\mathbf{u}||\mathbf{v}|\mathbf{v} \cdot \mathbf{u} + (\mathbf{v} \cdot \mathbf{u})^2 \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 + 2|\mathbf{u}|^2|\mathbf{v}|^2 \cos\theta + |\mathbf{u}|^2|\mathbf{v}|^2 \cos^2\theta \\ &= |\mathbf{u}|^2|\mathbf{v}|^2(1 + \cos\theta)^2,\end{aligned}$$

and

$$\begin{aligned}\big||\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}\big|^2 &= (|\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}) \cdot (|\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}) \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 + |\mathbf{v}|^2|\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}|(\mathbf{u} \cdot \mathbf{v}) \\ &= 2|\mathbf{u}|^2|\mathbf{v}|^2(1 + \cos\theta).\end{aligned}$$

It follows that

$$2\cos^2\phi - 1 = \frac{2[|\mathbf{u}|^2|\mathbf{v}|^2(1 + \cos\theta)^2]}{2|\mathbf{u}|^2|\mathbf{v}|^2(1 + \cos\theta)} - 1 = \cos\theta.$$

In other words $\phi = \theta/2$.

EXERCISES 11.5

1. The equation of the plane is $0 = (4, 3, -2) \cdot (x - 1, y + 1, z - 3) = 4x + 3y - 2z + 5$.

2. A vector normal to the plane is $(4, 2, 3) - (2, 1, 5) = (2, 1, -2)$. The equation of the plane is therefore $0 = (2, 1, -2) \cdot (x - 2, y - 1, z - 5) = 2x + y - 2z + 5$.

3. Since $(3, 3, 4)$ and $(0, 1, 1)$ are vectors that lie in the plane, a vector normal to the plane is

$$(3, 3, 4) \times (0, 1, 1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 3 & 4 \\ 0 & 1 & 1 \end{vmatrix} = (-1, -3, 3), \quad \text{as is} \quad (1, 3, -3).$$

The equation of the plane is therefore $0 = (1, 3, -3) \cdot (x - 1, y - 3, z - 2) = x + 3y - 3z - 4$.

4. One vector in the plane is $(3, 4, 1)$. Since $(1, -5, -2)$ is a second point in the plane, a second vector in the plane is $(2, -4, 3) - (1, -5, -2) = (1, 1, 5)$. A vector normal to the plane is

$$(3, 4, 1) \times (1, 1, 5) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 4 & 1 \\ 1 & 1 & 5 \end{vmatrix} = (19, -14, -1).$$

The equation of the plane is $0 = (19, -14, -1) \cdot (x - 2, y + 4, z - 3) = 19x - 14y - z - 91$.

5. Two lines determine a plane only if they are parallel or they intersect. Since the lines are obviously not parallel, they must intersect. To confirm this, we substitute $x = t$ and $y = 2t$ into $x = 2y$ giving $t = 4t$ or $t = 0$. This gives the point $(0, 0, -1)$, which satisfies equations for both lines. Two vectors that lie in the plane are $(1, 1/2, 4)$ and $(1, 2, 6)$. A vector normal to the plane is

$$(2,1,8)\times(1,2,6)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 1 & 8\\ 1 & 2 & 6\end{vmatrix}=(-10,-4,3),\quad \text{as is}\quad (10,4,-3).$$

The equation of the plane is therefore $0 = (10, 4, -3)\cdot(x - 0, y - 0, z + 1) = 10x + 4y - 3z - 3$.

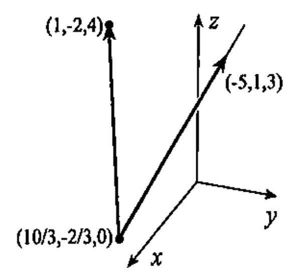
6. Two lines determine a plane only if they are parallel or they intersect. Since these lines are parallel (a vector along each is $(3, 4, 1)$), they determine a plane. Since $(1, 0, -2)$ and $(-1, 2, -5)$ are points on the lines, a second vector in the plane is $(1, 0, -2) - (-1, 2, -5) = (2, -2, 3)$. A vector normal to the plane is

$$(3,4,1)\times(2,-2,3)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 4 & 1\\ 2 & -2 & 3\end{vmatrix}=(14,-7,-14),\quad \text{as is}\quad (2,-1,-2).$$

The equation of the plane is $0 = (2, -1, -2)\cdot(x - 1, y, z + 2) = 2x - y - 2z - 6$.

7. The vectors $(1, -1, 2)$ and $(2, 1, 3)$ are normal to the planes $x - y + 2z = 4$ and $2x + y + 3z = 6$, respectively. A vector along the given line is

$$(1,-1,2)\times(2,1,3)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & -1 & 2\\ 2 & 1 & 3\end{vmatrix}=(-5,1,3).$$



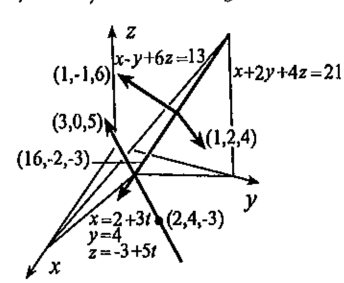
Since $(10/3, -2/3, 0)$ (a point on the given line) and $(1, -2, 4)$ are points on the required plane, a second vector in this plane is $(10/3, -2/3, 0) - (1, -2, 4) = (7/3, 4/3, -4)$, as is $(7, 4, -12)$. A vector normal to the required plane is

$$(7,4,-12)\times(-5,1,3)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 7 & 4 & -12\\ -5 & 1 & 3\end{vmatrix}=(24,39,27),\quad \text{as is}\quad (8,13,9).$$

The equation of the plane is therefore $0 = (8, 13, 9)\cdot(x - 1, y + 2, z - 4) = 8x + 13y + 9z - 18$.

8. Two lines determine a plane only if they are parallel or they intersect. The vectors $(1, 2, 4)$ and $(1, -1, 6)$ are normal to the planes $x + 2y + 4z = 21$ and $x - y + 6z = 13$, respectively. A vector along the line determined by these planes is

$$(1,2,4)\times(1,-1,6)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 2 & 4\\ 1 & -1 & 6\end{vmatrix}=(16,-2,-3).$$



Since $(3, 0, 5)$ is a vector along the second line, the given lines are not parallel. To ensure that the lines intersect, we substitute $x = 2 + 3t$, $y = 4$, and $z = -3 + 5t$ into $x + 2y + 4z = 21$ giving $(2+3t)+8+4(-3+5t) = 21 \Longrightarrow t = 1$. This gives the point $(5, 4, 2)$, which also satisfies $x - y + 6z = 13$. The lines therefore intersect at this point. A vector normal to the required plane is

$$(16,-2,-3)\times(3,0,5)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 16 & -2 & -3\\ 3 & 0 & 5\end{vmatrix}=(-10,-89,6).$$

The equation of the plane is therefore $0 = (10, 89, -6)(x - 2, y - 4, z + 3) = 10x + 89y - 6z - 394$.

9. Two lines determine a plane only if they are parallel or they intersect. Since the vectors $(3,4,0)$ and $(1,2,1)$ are normal to the planes $3x+4y=-6$ and $x+2y+z=2$, respectively, a vector along their line of intersection is

$$(3,4,0)\times(1,2,1)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 4 & 0\\ 1 & 2 & 1\end{vmatrix}=(4,-3,2).$$

Similarly, a vector along the other line is

$$(0,2,3)\times(3,-2,-9)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 2 & 3\\ 3 & -2 & -9\end{vmatrix}=(-12,9,-6),\quad \text{as is}\quad (4,-3,2).$$

Consequently, the two lines are parallel. Since $(-6,3,2)$ and $(-11,8,1)$ are points on the two lines, $(-6,3,2)-(-11,8,1)=(5,-5,1)$ is a vector that also lies in the required plane. A vector normal to this plane is

$$(4,-3,2)\times(5,-5,1)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 4 & -3 & 2\\ 5 & -5 & 1\end{vmatrix}=(7,6,-5).$$

The equation of the required plane is therefore $0=(7,6,-5)\cdot(x+6,y-3,z-2)=7x+6y-5z+34$.

10. (a) Since the vectors $(1,1,-4)$ and $(2,3,5)$ are normal to the planes $x+y-4z=6$ and $2x+3y+5z=10$, respectively, a vector along their line of intersection is

$$(1,1,-4)\times(2,3,5)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 1 & -4\\ 2 & 3 & 5\end{vmatrix}=(17,-13,1).$$

Since the plane is to be perpendicular to the xy-plane, $\hat{\mathbf{k}}$ must also lie in the plane, and a vector normal to the required plane is

$$(17,-13,1)\times\hat{\mathbf{k}}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 17 & -13 & 1\\ 0 & 0 & 1\end{vmatrix}=(-13,-17,0).$$

Since a point on the plane is $(8,-2,0)$, the equation of the plane is $0=(13,17,0)\cdot(x-8,y+2,z)=13x+17y-70$. Similar derivations give the other two equations.

11. The angle between the normals $(1,-2,4)$ and $(2,1,-1)$ to the planes is

$$\theta=\operatorname{Cos}^{-1}\left[\frac{(1,-2,4)\cdot(2,1,-1)}{|(1,-2,4)||(2,1,-1)|}\right]=\operatorname{Cos}^{-1}\left(\frac{-4}{\sqrt{21}\sqrt{6}}\right)=1.935\text{ radians}.$$

The acute angle between the planes is therefore 1.206 radians.

12. A vector equation is $\mathbf{r}=(x,y,z)=(1,-1,3)+t(2,4,-3)$. By equating components we obtain parametric equations $x=1+2t$, $y=-1+4t$, $z=3-3t$, and by solving each for t, symmetric equations are $\dfrac{x-1}{2}=\dfrac{y+1}{4}=\dfrac{z-3}{-3}$.

13. A vector equation is $\mathbf{r}=(x,y,z)=(-1,3,6)+t(2,-3,0)$. By equating components we obtain parametric equations $x=-1+2t$, $y=3-3t$, $z=6$. By solving the first and second for t, partial symmetric equations are $\dfrac{x+1}{2}=\dfrac{y-3}{-3}$, $z=6$.

14. Since a vector along the line is $(3,5,-5)$, a vector equation for the line is $\mathbf{r}=(x,y,z)=(2,-3,4)+t(3,5,-5)$. By equating components we obtain parametric equations $x=2+3t$, $y=-3+5t$, $z=4-5t$, and by solving each for t, symmetric equations are $\dfrac{x-2}{3}=\dfrac{y+3}{5}=\dfrac{z-4}{-5}$.

15. Since a vector along the line is $(0,6,6)$, a vector equation for the line is $\mathbf{r} = (x,y,z) = (-2,3,3)+t(0,1,1)$. By equating components we obtain parametric equations $x = -2$, $y = 3+t$, $z = 3+t$. By solving the second and third for t, partial symmetric equations are $y = z$, $x = -2$.

16. Since a vector along the line is $(0,0,1)$, a vector equation for the line is $\mathbf{r} = (x,y,z) = (1,3,4)+t(0,0,1)$. By equating components we obtain parametric equations $x = 1$, $y = 3$, $z = 4+t$. Symmetric equations do not exist.

17. Since $(5,3,-2)$ is a vector along the line, a vector equation for the line is $\mathbf{r} = (x,y,z) = (1,-3,5) + t(5,3,-2)$. By equating components we obtain parametric equations $x = 1+5t$, $y = -3+3t$, $z = 5-2t$, and by solving each for t, symmetric equations are $\dfrac{x-1}{5} = \dfrac{y+3}{3} = \dfrac{z-5}{-2}$.

18. Since a vector along the line is $(1,0,-2)$, a vector equation for the line is $\mathbf{r} = (x,y,z) = (2,0,3) + u(1,0,-2)$. By equating components we obtain parametric equations $x = 2+u$, $y = 0$, $z = 3-2u$, and by solving the first and last for u, partial symmetric equations are $x-2 = \dfrac{z-3}{-2}$, $y = 0$.

19. The given lines intersect at the point $(1,-4,2)$. Since a vector along the line is $(1,3,-2)-(2,-2,1) = (-1,5,-3)$, a vector equation for the line is $\mathbf{r} = (x,y,z) = (1,-4,2)+t(1,-5,3)$. By equating components we obtain parametric equations $x = 1+t$, $y = -4-5t$, $z = 2+3t$, and by solving each for t, symmetric equations are $x-1 = \dfrac{y+4}{-5} = \dfrac{z-2}{3}$.

20. Parametric equations for the line are $x = t$, $y = 2t-5$, $z = 10-3(t)-4(2t-5) = 30-11t$. By solving each for t, symmetric equations are $x = \dfrac{y+5}{2} = \dfrac{z-30}{-11}$. A vector equation is $\mathbf{r} = (x,y,z) = (0,-5,30)+t(1,2,-11)$.

21. Since the vectors $(1,1,0)$ and $(2,-1,1)$ are normal to the planes $x+y = 3$ and $2x-y+z = -2$, respectively , a vector along the given line is $(1,1,0)\times(2,-1,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = (1,-1,-3)$. A vector equation for the line is $\mathbf{r} = (x,y,z) = (-2,3,1)+t(1,-1,-3)$. By equating components we obtain parametric equations $x = -2+t$, $y = 3-t$, $z = 1-3t$, and by solving each for t, symmetric equations are $x+2 = \dfrac{y-3}{-1} = \dfrac{z-1}{-3}$.

22. Parametric equations for the line are $x = 3+2t$, $y = 2+t$, $z = -1+4t$. When these values are substituted into $x-y+2z$, the result is $x-y+2z = (3+2t)-(2+t)+2(-1+4t) = -1+9t \not\equiv -1$. Consequently, the line does not lie in the plane.

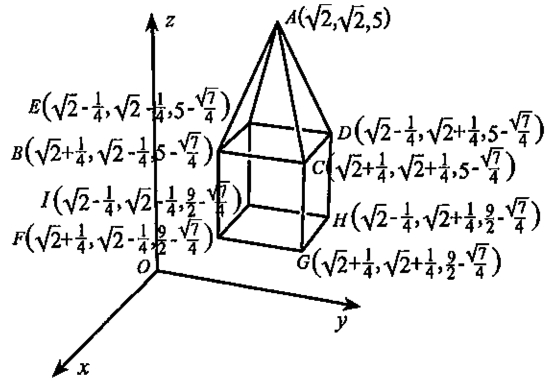
23. Since the slope of the line is $-A/B$, a vector along the line is $(B,-A)$. Because the scalar product of this vector with (A,B) is zero, (A,B) is perpendicular to the line.

24. The equation of every plane is of the form $Ax+By+Cz+D = 0$. Because its intercept with the x-axis is a, it follows that $a = -D/A$ or $A = -D/a$. Similarly, $B = -D/b$ and $C = -D/c$. Thus, the equation of the plane is $-\dfrac{D}{a}x-\dfrac{D}{b}y-\dfrac{D}{c}z+D = 0 \quad\Longrightarrow\quad \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1$.

25. The bottom face has equation $z = 0$. A vector normal to the left face is $(a,0,0)\times(d,e,f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a & 0 & 0 \\ d & e & f \end{vmatrix}$ $= (0,-af,ae)$, as is $(0,f,-e)$. The equation of this face is $0 = (0,f,-e)\cdot(x,y,z) = fy-ez$. A vector normal to the right face is $(b,c,0)\times(d,e,f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b & c & 0 \\ d & e & f \end{vmatrix} = (cf,-bf,be-cd)$. The equation of this face is $0 = (cf,-bf,be-cd)\cdot(x,y,z) = cfx-bfy+(be-cd)z$. A vector normal to the front face is $(b-a,c,0)\times(d-b,e-c,f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b-a & c & 0 \\ d-b & e-c & f \end{vmatrix} = (cf,f(a-b),be-cd-ae+ac)$. The equation of this face is $0 = (cf,f(a-b),be-cd-ae+ac)\cdot(x-a,y,z) = cfx+f(a-b)y+(be-cd-ae+ac)z-acf$.

26. In Exercise 11.1–14, the coordinates of the corners of the birdhouse were determined as shown. The equation of face:

$FGHI$ is $z = 9/2 - \sqrt{7}/4$;
$BFIE$ is $y = \sqrt{2} - 1/4$;
$CGHD$ is $y = \sqrt{2} + 1/4$;
$BFCG$ is $x = \sqrt{2} + 1/4$;
$EIHD$ is $x = \sqrt{2} - 1/4$.



A vector normal to face ABC is

$$(4\mathbf{AB}) \times (4\mathbf{AC}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & -\sqrt{7} \\ 1 & 1 & -\sqrt{7} \end{vmatrix} = (2\sqrt{7}, 0, 2).$$

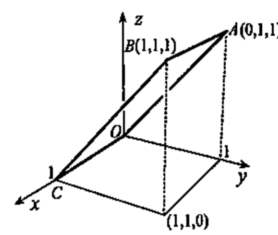
The equation of this face is therefore

$$\begin{aligned} 0 &= (\sqrt{7}, 0, 1) \cdot (x - \sqrt{2}, y - \sqrt{2}, z - 5) \\ &= \sqrt{7}x + z - 5 - \sqrt{14}. \end{aligned}$$

Similarly, equations for faces ACD, ADE, and AEB are $\sqrt{7}y + z - \sqrt{14} - 5 = 0$, $\sqrt{7}x - z = \sqrt{14} - 5$, and $\sqrt{7}y - z = \sqrt{14} - 5$.

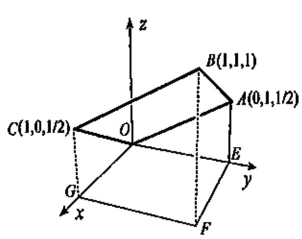
27. Since $\mathbf{P_1P_2}$ and $\mathbf{P_1P_3}$ are vectors that lie in the plane, $\mathbf{P_1P_2} \times \mathbf{P_1P_3}$ is a vector normal to the plane. Since $\mathbf{P_1P}$ is a vector in the plane for any point $P(x, y, z)$ in the plane, it follows that $\mathbf{P_1P} \cdot \mathbf{P_1P_2} \times \mathbf{P_1P_3} = 0$, and this must be the equation of the plane.

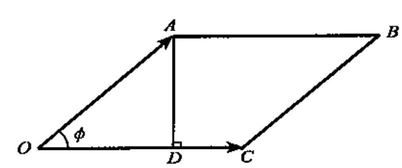
28. (a) Since the coordinates of A and B are $(0, 1, 1)$ and $(1, 1, 1)$, it follows that $\mathbf{AB} = (1, 0, 0)$ and $\mathbf{CB} = (0, 1, 1)$. Because $\mathbf{AB} = \mathbf{OC}$ and $\mathbf{CB} = \mathbf{OA}$, $OCBA$ is a rectangle with area $|\mathbf{OA}||\mathbf{OC}| = \sqrt{2}(1)$.



(b) The coordinates of the points A, B and C on the plane $x + y - 2z = 0$ directly above E, F, and D are $(0, 1, 1/2)$, $(1, 1, 1)$, and $(1, 0, 1/2)$, respectively (left figure below). Since $\mathbf{OA} = (0, 1, 1/2)$ is parallel to $\mathbf{CB} = (0, 1, 1/2)$, and $\mathbf{OC} = (1, 0, 1/2)$ is parallel to $\mathbf{AB} = (1, 0, 1/2)$, it follows that $OABC$ is a parallelogram with area (right figure below)

$$|\mathbf{OC}|(AD) = |\mathbf{OC}||\mathbf{OA}|\sin\phi = |\mathbf{OC} \times \mathbf{OA}| = \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{matrix} \right\| = |(-1/2, -1/2, 1)| = \frac{\sqrt{6}}{2}.$$





(c) The coordinates of P, Q, R, and S, points in the plane $Ax + By + Cz + D = 0$ directly above O, G, F, and E are

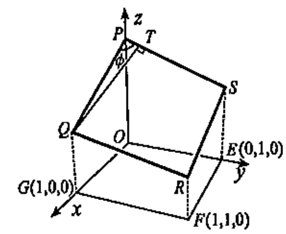
$$P(0, 0, -D/C),\ Q(1, 0, -(D + A)/C),\ R(1, 1, -(D + A + B)/C),\ S(0, 1, -(D + B)/C).$$

Consequently,

$$\mathbf{PQ} = (1, 0, -A/C),\ \mathbf{QR} = (0, 1, -B/C),\ \mathbf{PS} = (0, 1, -B/C),\ \mathbf{SR} = (1, 0, -A/C).$$

Because $\mathbf{PQ} = \mathbf{SR}$ and $\mathbf{PS} = \mathbf{QR}$, $PQRS$ is a parallelogram with area

$$|\mathbf{PS}|(QT) = |\mathbf{PS}||\mathbf{PQ}|\sin\phi = |\mathbf{PQ}\times\mathbf{PS}| = \left\|\begin{array}{ccc}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 0 & -A/C\\ 0 & 1 & -B/C\end{array}\right\|$$
$$= |(A/C, B/C, 1)|$$
$$= \sqrt{A^2/C^2 + B^2/C^2 + 1}$$
$$= \frac{\sqrt{A^2+B^2+C^2}}{|C|}.$$



The acute angle between the xy-plane and the plane $Ax+By+Cz+D=0$ is defined as the acute angle between their normals (see Exercise 11). Normals to these planes are $\hat{\mathbf{k}}$ and (A, B, C). If θ is the angle between these vectors, then

$$\hat{\mathbf{k}}\cdot(A,B,C) = |\hat{\mathbf{k}}||(A,B,C)|\cos\theta = (1)\sqrt{A^2+B^2+C^2}\cos\theta.$$

Consequently, $\cos\theta = C/\sqrt{A^2+B^2+C^2}$. If $C > 0$, then θ is acute and $\theta = \gamma$ where

$$\cos\gamma = \frac{C}{\sqrt{A^2+B^2+C^2}}.$$

If $C < 0$, then θ is obtuse and $\gamma = \pi - \theta$ where

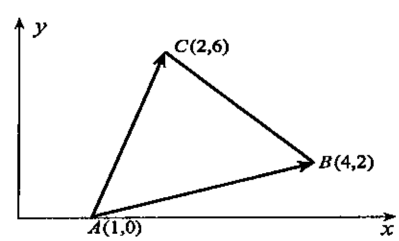
$$\cos(\pi-\gamma) = \frac{C}{\sqrt{A^2+B^2+C^2}} \quad\Longrightarrow\quad \cos\gamma = \frac{-C}{\sqrt{A^2+B^2+C^2}}.$$

Thus, $\cos\gamma = \dfrac{|C|}{\sqrt{A^2+B^2+C^2}}$ or $\sec\gamma = \dfrac{\sqrt{A^2+B^2+C^2}}{|C|}$.

EXERCISES 11.6

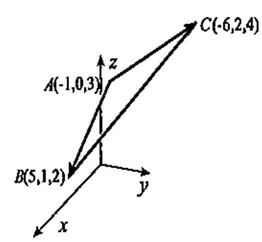
1. The area of the triangle is

$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left\|\begin{array}{ccc}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 2 & 0\\ 1 & 6 & 0\end{array}\right\|$$
$$= \frac{1}{2}|(0,0,16)| = 8.$$



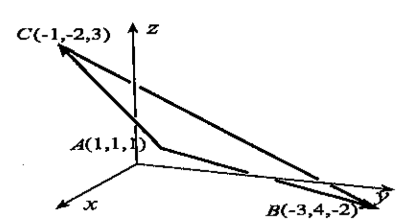
2. The area of the triangle is

$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left\|\begin{array}{ccc}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 6 & 1 & -1\\ -5 & 2 & 1\end{array}\right\|$$
$$= \frac{1}{2}|(3,-1,17)| = \frac{\sqrt{299}}{2}.$$



3. The area of the triangle is

$$\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}\left\|\begin{array}{ccc}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -4 & 3 & -3\\ -2 & -3 & 2\end{array}\right\|$$
$$= \frac{1}{2}|(-3,14,18)| = \frac{\sqrt{529}}{2}.$$

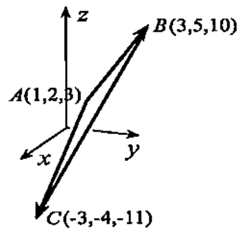


4. The area of the triangle is

$$\frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{1}{2}\left\|\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 7 \\ -4 & -6 & -14 \end{vmatrix}\right\|$$

$$= \frac{1}{2}|(0,0,0)| = 0.$$

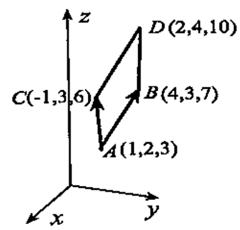
The points must be collinear.



5. The area of the parallelogram is

$$|\mathbf{AB} \times \mathbf{AC}| = \left\|\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 4 \\ -2 & 1 & 3 \end{vmatrix}\right\|$$

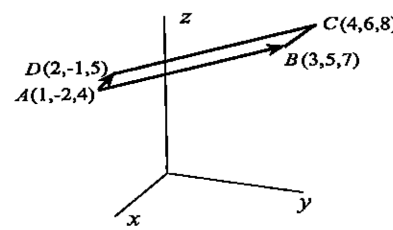
$$= |(-1,-17,5)| = 3\sqrt{35}.$$



6. The area of the parallelogram is

$$|\mathbf{AB} \times \mathbf{AD}| = \left\|\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 7 & 3 \\ 1 & 1 & 1 \end{vmatrix}\right\|$$

$$= |(4,1,-5)| = \sqrt{42}.$$



7. The component is $\mathbf{v} \cdot \dfrac{(1,2,-3)}{\sqrt{14}} = -\dfrac{16}{\sqrt{14}}$.

8. Since $(4,-3,2)-(-1,2,3)=(5,-5,-1)$ is a vector in the required direction, the component is $\mathbf{v} \cdot \dfrac{(5,-5,-1)}{\sqrt{51}} = \dfrac{22}{\sqrt{51}}$.

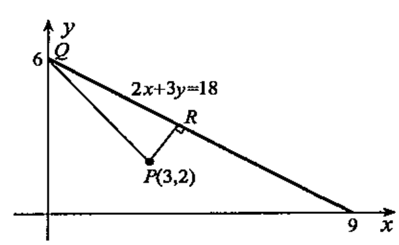
9. Since a vector normal to the plane is $(1,1,2)$, the component is $\mathbf{v} \cdot \dfrac{(1,1,2)}{\sqrt{6}} = \dfrac{3}{\sqrt{6}}$.

10. Since a vector along the line is $(-1,2,-3)$, the component is $\mathbf{v} \cdot \dfrac{(1,-2,3)}{\sqrt{14}} = \dfrac{18}{\sqrt{14}}$.

11. The required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since the slope of the line is $-2/3$, a vector along it is $(-3,2)$, and a vector in the same direction as $\mathbf{PR}$ is $(2,3)$. Consequently,

$$d = \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right| = \left|(-3,4) \cdot \frac{(2,3)}{\sqrt{4+9}}\right| = \frac{6}{\sqrt{13}}.$$

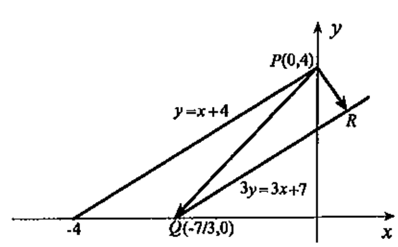
We could also have used formula 1.16.



12. Since the lines are parallel, the required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since the slope of the line is 1, a vector along it is $(1,1)$, and a vector in the same direction as $\mathbf{PR}$ is $(1,-1)$. Consequently,

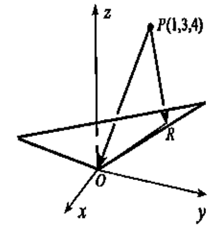
$$d = \left|\mathbf{PQ} \cdot \widehat{\mathbf{PR}}\right| = \left|(-7/3,-4) \cdot \frac{(1,-1)}{\sqrt{1+1}}\right| = \frac{5}{3\sqrt{2}}.$$

We could also have used formula 1.16.



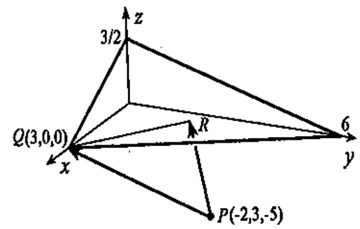
13. The required distance d is the component of $\mathbf{PO}$ along $\mathbf{PR}$. Since a vector in the same direction as $\mathbf{PR}$ is $(1,1,-2)$,

$$d = \left|\mathbf{PO}\cdot\widehat{\mathbf{PR}}\right| = \left|(-1,-3,-4)\cdot\frac{(1,1,-2)}{\sqrt{1+1+4}}\right| = \frac{4}{\sqrt{6}}.$$

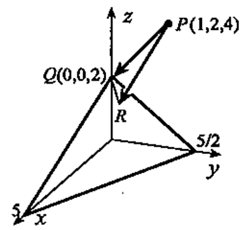


14. The required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since a vector in the same direction as $\mathbf{PR}$ is $(2,1,4)$,

$$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(5,-3,5)\cdot\frac{(2,1,4)}{\sqrt{4+1+16}}\right| = \frac{27}{\sqrt{21}}.$$



15. First we confirm that the line is parallel to the plane (else the distance is zero). Since a vector along the line is $(-1,3,-2)$ and $(-1,3,-2)\cdot(2,4,5)=0$, the line is indeed parallel to the plane. Since a point on the line is $P(1,2,4)$, the required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since a vector in the same direction as $\mathbf{PR}$ is $(-2,-4,-5)$,

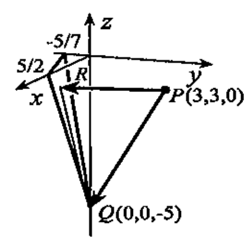


$$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(-1,-2,-2)\cdot\frac{(-2,-4,-5)}{\sqrt{4+16+25}}\right| = \frac{20}{3\sqrt{5}}.$$

16. First we confirm that the line is parallel to the plane (else the distance is zero). Since a vector along the line is

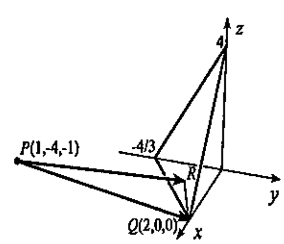
$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = (2,1,-3),$$

and $(2,1,-3)\cdot(2,-7,-1)=0$, the line is indeed parallel to the plane. Since a point on the line is $P(3,3,0)$, the required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since a vector in the same direction as $\mathbf{PR}$ is $(2,-7,-1)$,



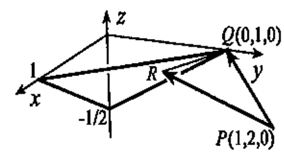
$$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(-3,-3,-5)\cdot\frac{(2,-7,-1)}{\sqrt{4+49+1}}\right| = \frac{20}{3\sqrt{6}}.$$

17. First we confirm that the line is parallel to the plane (else the distance is zero). Since a vector along the line is $(1,1/3,-1)$ and $(1,1/3,-1)\cdot(2,-3,1)=0$, the line is indeed parallel to the plane. Since a point on the line is $P(1,-4,-1)$, the required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since a vector in the same direction as $\mathbf{PR}$ is $(-2,3,-1)$,



$$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(1,4,1)\cdot\frac{(-2,3,-1)}{\sqrt{4+9+1}}\right| = \frac{9}{\sqrt{14}}.$$

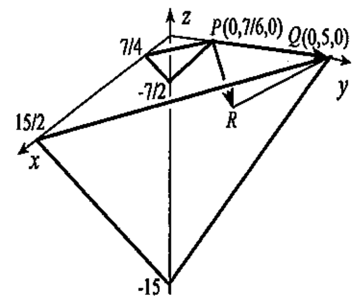
18. First we confirm that the line is parallel to the plane (else the distance is zero). Since a vector along the line is $(-6,4,-1)$ and $(-6,4,-1)\cdot(1,1,-2)=0$, the line is indeed parallel to the plane. Since a point on the line is $P(1,2,0)$, the required distance is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since a vector in the same direction as $\mathbf{PR}$ is $(-1,-1,2)$,



$$d=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|=\left|(-1,-1,0)\cdot\frac{(-1,-1,2)}{\sqrt{6}}\right|=\frac{2}{\sqrt{6}}.$$

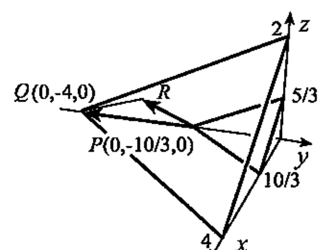
19. The required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since a vector in the same direction as $\mathbf{PR}$ is $(2,3,-1)$,

$$\begin{aligned}d&=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|\\&=\left|(0,23/6,0)\cdot\frac{(2,3,-1)}{\sqrt{4+9+1}}\right|=\frac{23}{2\sqrt{14}}.\end{aligned}$$



20. The required distance d is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. Since a vector in the same direction as $\mathbf{PR}$ is $(1,-1,2)$,

$$\begin{aligned}d&=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|\\&=\left|(0,-2/3,0)\cdot\frac{(1,-1,2)}{\sqrt{1+1+4}}\right|=\frac{2}{3\sqrt{6}}.\end{aligned}$$

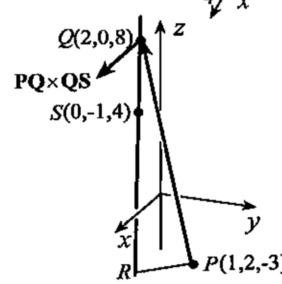


21. The required distance is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. A vector perpendicular to $\mathbf{PQ}$ and $\mathbf{QS}$ is

$$\mathbf{PQ}\times\mathbf{QS}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\1&-2&11\\-2&-1&-4\end{vmatrix}=(19,-18,-5).$$

A vector in direction $\mathbf{PR}$ is therefore

$$(\mathbf{PQ}\times\mathbf{QS})\times\mathbf{SQ}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\19&-18&-5\\2&1&4\end{vmatrix}=(-67,-86,55).$$



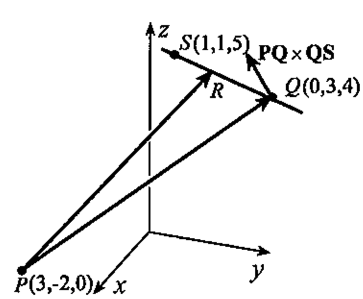
Thus, $$d=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|=\left|(1,-2,11)\cdot\frac{(-67,-86,55)}{\sqrt{(-67)^2+(-86)^2+(55)^2}}\right|=\frac{710}{\sqrt{14\,910}}.$$

22. The required distance is the component of $\mathbf{PQ}$ along $\mathbf{PR}$. A vector perpendicular to $\mathbf{PQ}$ and $\mathbf{QS}$ is

$$\mathbf{PQ}\times\mathbf{QS}=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\-3&5&4\\1&-2&1\end{vmatrix}=(13,7,1).$$

A vector in direction $\mathbf{PR}$ is therefore

$$\begin{aligned}(\mathbf{PQ}\times\mathbf{QS})\times\mathbf{SQ}&=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\13&7&1\\-1&2&-1\end{vmatrix}\\&=(-9,12,33)\quad\text{or}\quad(-3,4,11).\end{aligned}$$



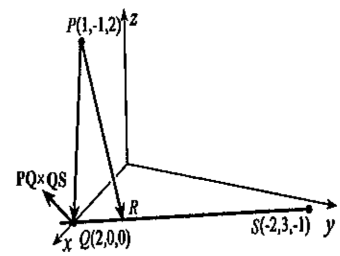
Thus, $$d=\left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right|=\left|(-3,5,4)\cdot\frac{(-3,4,11)}{\sqrt{(-3)^2+(4)^2+(11)^2}}\right|=\frac{73}{\sqrt{146}}=\frac{\sqrt{146}}{2}.$$

23. The required distance is the component of **PQ** along **PR**. A vector perpendicular to **PQ** and **QS** is

$$\mathbf{PQ}\times\mathbf{QS} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -2 \\ -4 & 3 & -1 \end{vmatrix} = (5, 9, 7).$$

A vector in direction **PR** is therefore

$$(\mathbf{PQ}\times\mathbf{QS})\times\mathbf{SQ} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 9 & 7 \\ 4 & -3 & 1 \end{vmatrix} = (30, 23, -51).$$

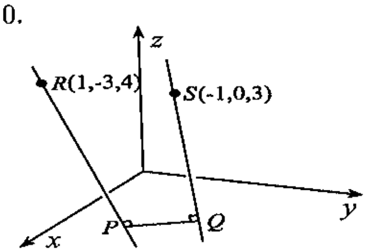


Thus, $d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(1,1,-2)\cdot\dfrac{(30,23,-51)}{\sqrt{(30)^2+(23)^2+(-51)^2}}\right| = \dfrac{155}{\sqrt{4030}}.$

24. Since the point $(1, 3, 3)$ is on the line, the minimum distance is 0.

25. Following Example 11.24, the distance is the component of **RS** along **PQ**. A vector in direction **PQ** is

$$(2,3,-1)\times(1,2,-2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & 2 & -2 \end{vmatrix} = (-4, 3, 1).$$

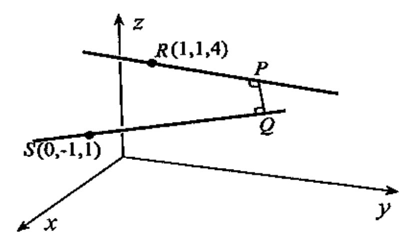


Consequently,

$$d = \left|\mathbf{RS}\cdot\widehat{\mathbf{PQ}}\right| = \left|(-2,3,-1)\cdot\frac{(-4,3,1)}{\sqrt{26}}\right| = \frac{16}{\sqrt{26}}.$$

26. Following Example 11.24, the distance is the component of **RS** along **PQ**. A vector in direction **PQ** is

$$(1,3,2)\times(2,-1,2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & 2 \\ 2 & -1 & 2 \end{vmatrix} = (8, 2, -7).$$

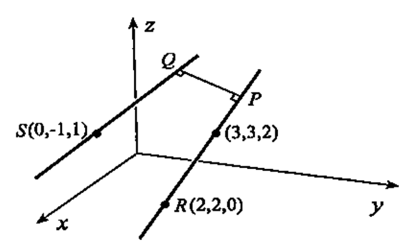


Consequently,

$$d = \left|\mathbf{RS}\cdot\widehat{\mathbf{PQ}}\right| = \left|(-1,-2,-3)\cdot\frac{(8,2,-7)}{\sqrt{117}}\right| = \frac{9}{\sqrt{117}}.$$

27. Following Example 11.24, the distance is the component of **RS** along **PQ**. A vector in direction **PQ** is

$$(1,1,2)\times(1,2,3) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = (-1, -1, 1).$$

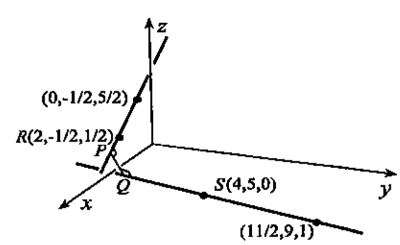


Consequently,

$$d = \left|\mathbf{RS}\cdot\widehat{\mathbf{PQ}}\right| = \left|(-2,-3,1)\cdot\frac{(-1,-1,1)}{\sqrt{3}}\right| = \frac{6}{\sqrt{3}}.$$

28. Following Example 11.24, the distance is the component of **RS** along **PQ**. A vector in direction **PQ** is

$$(-2,0,2)\times(3,8,2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 0 & 2 \\ 3 & 8 & 2 \end{vmatrix} = (-16, 10, -16).$$



Consequently,

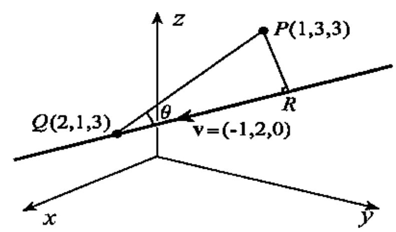
$$d = \left|\mathbf{RS}\cdot\widehat{\mathbf{PQ}}\right| = \left|(2,11/2,-1/2)\cdot\frac{(-8,5,-8)}{\sqrt{153}}\right| = \frac{31}{2\sqrt{153}}.$$

29. Since $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$, and $\mathbf{v} = (1, 2, 3)$,

$$|\mathbf{PR}| = |\mathbf{PQ}|\sin\theta = |\mathbf{PQ}||\hat{\mathbf{v}}|\sin\theta = |\mathbf{PQ} \times \hat{\mathbf{v}}| = \frac{1}{|\mathbf{v}|}|\mathbf{PQ} \times \mathbf{v}|$$
$$= \frac{1}{\sqrt{14}}\left\|\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -1 & -2 \\ 1 & 2 & 3 \end{vmatrix}\right\| = \frac{1}{\sqrt{14}}|(1, -2, 1)| = \frac{\sqrt{6}}{\sqrt{14}} = \sqrt{\frac{3}{7}} = \frac{\sqrt{21}}{7}.$$

30. A point on the line is $Q(2, 1, 3)$ and a vector along the line is $\mathbf{v} = (-1, 2, 0)$. Using the technique of Exercise 29,

$$|\mathbf{PR}| = |\mathbf{PQ}|\sin\theta = |\mathbf{PQ}||\hat{\mathbf{v}}|\sin\theta$$
$$= |\mathbf{PQ} \times \hat{\mathbf{v}}| = \frac{1}{|\mathbf{v}|}|\mathbf{PQ} \times \mathbf{v}|$$
$$= \frac{1}{\sqrt{5}}\left\|\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & 0 \\ -1 & 2 & 0 \end{vmatrix}\right\|$$
$$= \frac{1}{\sqrt{5}}|(0, 0, 0)| = 0.$$



Hence the point is on the line.

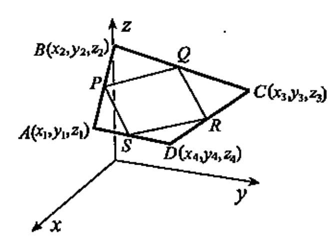
31. If $P(x^*, y^*, z^*)$ is any point in the first plane, then according to equation 11.41, the distance from P to the second plane is

$$\frac{|Ax^* + By^* + Cz^* + D_2|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|-D_1 + D_2|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}},$$

(since $Ax^* + By^* + Cz^* + D_1 = 0$).

32. If coordinates of the vertices are as shown in the figure, then coordinates of the midpoints of the sides are

$$P\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right), \quad Q\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right),$$
$$R\left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2}, \frac{z_3 + z_4}{2}\right), \quad S\left(\frac{x_4 + x_1}{2}, \frac{y_4 + y_1}{2}, \frac{z_4 + z_1}{2}\right).$$



Since $\mathbf{PQ} = \left(\dfrac{x_3 - x_1}{2}, \dfrac{y_3 - y_1}{2}, \dfrac{z_3 - z_1}{2}\right) = \mathbf{SR}$, and similarly, $\mathbf{PS} = \mathbf{QR}$, it follows that $PQRS$ is a parallelogram.

33. The lines form a triangle if each pair of lines intersects in a single point. For the first pair of lines, we set $4x - 16 = 3y - 24$ and $2x + 10 = 3y + 6$. The solution of these equations is $x = -2$, $y = 0$. Both of the original lines then give $z = 1$, and these lines therefore intersect in the point $A(-2, 0, 1)$.

To determine whether the first and third lines intersect in a point, we set $x = 1$ and $y = 5 + t$ in $4x - 16 = 3y - 24$. The result is $t = -1$. This value of t in the third line gives the point $B(1, 4, -3)$, and this point also satisfies the equations for the first line.

A similar procedure gives $C(1, 2, 3)$ as the point of intersection of the second and third lines.

According to equation 11.42, the area of triangle ABC is

$$\frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{1}{2}\left\|\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 4 & -4 \\ 3 & 2 & 2 \end{vmatrix}\right\| = \frac{1}{2}|(16, -18, -6)| = \sqrt{154}.$$

34. Since $\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = 1/2 - 1/2 = 0$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are perpendicular. The components of $\mathbf{u}$ along $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are

$$\lambda = \mathbf{u} \cdot \hat{\mathbf{v}} = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}} \qquad \text{and} \qquad \rho = \mathbf{u} \cdot \hat{\mathbf{w}} = \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

35. Since $\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = 2/5 - 2/5 = 0$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are perpendicular. The components of $\mathbf{u}$ along $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are

$$\lambda = \mathbf{u} \cdot \hat{\mathbf{v}} = \frac{3+4}{\sqrt{5}} = \frac{7}{\sqrt{5}} \qquad \text{and} \qquad \rho = \mathbf{u} \cdot \hat{\mathbf{w}} = \frac{6-2}{\sqrt{5}} = \frac{4}{\sqrt{5}}.$$

36. Since $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = (1/\sqrt{70})(-2+2+0) = 0$, $\hat{\mathbf{u}} \cdot \hat{\mathbf{w}} = [1/(5\sqrt{14})](6-6+0) = 0$, and $\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = [1/(14\sqrt{5})](-3-12+15) = 0$, the three unit vectors are mutually perpendicular. The components of $\mathbf{r}$ along $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ are

$$\mathbf{r} \cdot \hat{\mathbf{u}} = \frac{1}{\sqrt{5}}(2+3) = \sqrt{5};\ \mathbf{r} \cdot \hat{\mathbf{v}} = \frac{1}{\sqrt{14}}(-1+6-12) = -\frac{\sqrt{14}}{2};\ \mathbf{r} \cdot \hat{\mathbf{w}} = \frac{1}{\sqrt{70}}(3-18-20) = -\frac{\sqrt{70}}{2}.$$

37. Since $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = (1/\sqrt{18})(1+1-2) = 0$, $\hat{\mathbf{u}} \cdot \hat{\mathbf{w}} = [1/(\sqrt{6})](1-1) = 0$, and $\hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = [1/\sqrt{12}](1-1) = 0$, the three unit vectors are mutually perpendicular. The components of $\mathbf{r}$ along $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ are

$$\mathbf{r} \cdot \hat{\mathbf{u}} = \frac{1}{\sqrt{3}}(2-1) = \frac{1}{\sqrt{3}};\ \mathbf{r} \cdot \hat{\mathbf{v}} = \frac{1}{\sqrt{6}}(2+2) = \frac{4}{\sqrt{6}};\ \mathbf{r} \cdot \hat{\mathbf{w}} = \frac{1}{\sqrt{2}}(2) = \sqrt{2}.$$

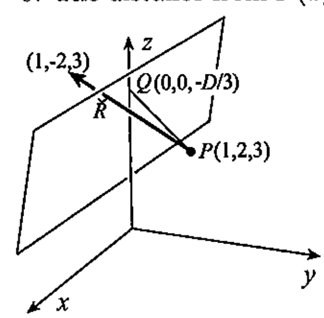
38. Since $\mathbf{v} \cdot \mathbf{w} = 6 - 6 = 0$, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular. Because $\mathbf{u} \cdot \mathbf{v} = \lambda \mathbf{v} \cdot \mathbf{v} + \rho \mathbf{w} \cdot \mathbf{v} = \lambda |\mathbf{v}|^2$, it follows that $\lambda = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} = \dfrac{3-6}{1+9} = -\dfrac{3}{10}$. Similarly, $\rho = \dfrac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{w}|^2} = \dfrac{18+4}{36+4} = \dfrac{11}{20}$.

39. Since $\mathbf{u} \cdot \mathbf{v} = 1 - 1 = 0$, $\mathbf{v} \cdot \mathbf{w} = -1 + 2 - 1 = 0$ and $\mathbf{u} \cdot \mathbf{w} = -1 + 1 = 0$ the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular. Because $\mathbf{r} \cdot \mathbf{u} = \lambda \mathbf{u} \cdot \mathbf{u} + \rho \mathbf{v} \cdot \mathbf{u} + \mu \mathbf{w} \cdot \mathbf{u} = \lambda |\mathbf{u}|^2$, it follows that $\lambda = \dfrac{\mathbf{r} \cdot \mathbf{u}}{|\mathbf{u}|^2} = \dfrac{-2+4}{1+1} = 1$. Similarly, $\rho = \dfrac{\mathbf{r} \cdot \mathbf{v}}{|\mathbf{v}|^2} = \dfrac{-2-3-4}{1+1+1} = -3$, and $\mu = \dfrac{\mathbf{r} \cdot \mathbf{w}}{|\mathbf{w}|^2} = \dfrac{2-6+4}{1+4+1} = 0$.

40. The equation of the plane must be of the form $x - 2y + 3z + D = 0$. The distance from $P(1,2,3)$ to this plane is the projection of $\mathbf{PQ}$ along $\mathbf{PR}$, and hence

$$\begin{aligned} 2 &= |\mathbf{PQ} \cdot \widehat{\mathbf{PR}}| \\ &= \left| (-1, -2, -D/3 - 3) \cdot \frac{(1,-2,3)}{\sqrt{1+4+9}} \right| \\ &= \frac{1}{\sqrt{14}} |D + 6|. \end{aligned}$$

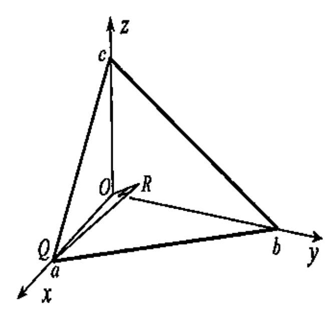
When this equation is solved, $D = -6 \pm 2\sqrt{14}$, and the two possible planes are $x - 2y + 3z = 6 \pm 2\sqrt{14}$.



41. The equation of the plane is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. The distance p from the origin to the plane is the component of $\mathbf{OQ}$ along $\mathbf{OR}$,

$$\begin{aligned} p &= |\mathbf{OQ} \cdot \widehat{\mathbf{OR}}| \\ &= \left| (a, 0, 0) \cdot \frac{(1/a, 1/b, 1/c)}{\sqrt{1/a^2 + 1/b^2 + 1/c^2}} \right| \\ &= \frac{1}{\sqrt{1/a^2 + 1/b^2 + 1/c^2}}. \end{aligned}$$

When this equation is squared and inverted, the required result is obtained.



EXERCISES 11.7

1. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 0 \\ 2 & 3 & -4 \end{vmatrix} = (4, 8, 8)$

2. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 0 & 5 \\ 1 & 2 & 0 \end{vmatrix} = (-10, 5, -2)$

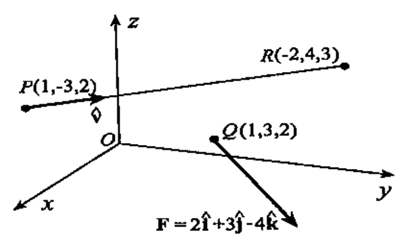
3. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -3 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (-9, -3, -3)$

4. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & -1 & -1 \\ 3 & -1 & 4 \end{vmatrix} = (-5, 1, 4)$

5. $\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 1 & 3 \\ 6 & 0 & 0 \end{vmatrix} = (0, 18, -6)$

6. (a) Since $\mathbf{v} = (-3, 7, 1)$ is a vector along the line through $P(1, -3, 2)$ and $R(-2, 4, 3)$, the moment about the line is

$$\begin{aligned} \text{Moment } &= \mathbf{PQ} \times \mathbf{F} \cdot (\pm\hat{\mathbf{v}}) \\ &= \pm(0, 6, 0) \times (2, 3, -4) \cdot \frac{(-3, 7, 1)}{\sqrt{59}} \\ &= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 6 & 0 \\ 2 & 3 & -4 \end{vmatrix} \cdot \frac{(-3, 7, 1)}{\sqrt{59}} \\ &= \pm(-24, 0, -12) \cdot \frac{(-3, 7, 1)}{\sqrt{59}} = \pm \frac{60}{\sqrt{59}}. \end{aligned}$$



(b) Since the moment of $\mathbf{F}$ about O is

$$\mathbf{M} = \mathbf{OQ} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & 2 \\ 2 & 3 & -4 \end{vmatrix} = (-18, 8, -3),$$

moments about the x-, y-, and z-axes are -18, 8, and -3, respectively.
(c) We use the point $S(2, 4, 1)$ on the line. If $\hat{\mathbf{v}}$ is a unit vector along the line, the moment of $\mathbf{F}$ about the line is

$$\begin{aligned} \text{Moment} &= \mathbf{SQ} \times \mathbf{F} \cdot \hat{\mathbf{v}} = (-1, -1, 1) \times (2, 3, -4) \cdot \frac{\pm(3, -2, 5)}{\sqrt{38}} \\ &= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & -1 & 1 \\ 2 & 3 & -4 \end{vmatrix} \cdot \frac{(3, -2, 5)}{\sqrt{38}} = \pm(1, -2, -1) \cdot \frac{(3, -2, 5)}{\sqrt{38}} = \pm \frac{2}{\sqrt{38}}. \end{aligned}$$

7. Since $P(3, -1, 0)$ is a point on the line and $\mathbf{v} = (2, 1, 4)$ is a vector along the line, the moment of $\mathbf{F}$ at $Q(-2, 3, 1)$ is

$$\begin{aligned} \text{Moment} &= \pm\mathbf{PQ} \times \mathbf{F} \cdot \hat{\mathbf{v}} = \pm(-5, 4, 1) \times (6, -5, 1) \cdot \frac{(2, 1, 4)}{\sqrt{21}} \\ &= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 & 4 & 1 \\ 6 & -5 & 1 \end{vmatrix} \cdot \frac{(2, 1, 4)}{\sqrt{21}} = \pm(9, 11, 1) \cdot \frac{(2, 1, 4)}{\sqrt{21}} = \pm \frac{33}{\sqrt{21}}. \end{aligned}$$

8. Since $P(0, 0, 1)$ is a point on the line and $\mathbf{v} = (1, 1, 1)$ is a vector along the line, the moment of $\mathbf{F}$ at $Q(6, -2, 1)$ is

$$\begin{aligned} \text{Moment} &= \pm\mathbf{PQ} \times \mathbf{F} \cdot \hat{\mathbf{v}} = \pm(6, -2, 0) \times (4, 0, -2) \cdot \frac{(1, 1, 1)}{\sqrt{3}} \\ &= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & -2 & 0 \\ 4 & 0 & -2 \end{vmatrix} \cdot \frac{(1, 1, 1)}{\sqrt{3}} = \pm(4, 12, 8) \cdot \frac{(1, 1, 1)}{\sqrt{3}} = \pm 8\sqrt{3}. \end{aligned}$$

9. A point on the line is $P(-4,-2,4)$, and a vector along the line is

$$\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (-5,-2,3).$$

The moment of $\mathbf{F}$ at $Q(-1,-1,-2)$ is

$$\text{Moment} = \pm \mathbf{PQ} \times \mathbf{F} \cdot \hat{\mathbf{v}} = \pm(3,1,-6) \times (1,1,-1) \cdot \frac{(-5,-2,3)}{\sqrt{38}}$$
$$= \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & -6 \\ 1 & 1 & -1 \end{vmatrix} \cdot \frac{(-5,-2,3)}{\sqrt{38}} = \pm(5,-3,2) \cdot \frac{(-5,-2,3)}{\sqrt{38}} = \pm \frac{13}{\sqrt{38}}.$$

10. Moments about the axes are M_x, M_y, and M_z.

11. The moment of $\mathbf{F}$ about ℓ is $\pm(\mathbf{PQ} \times \mathbf{F} \cdot \hat{\mathbf{v}})$, where $\hat{\mathbf{v}}$ is a unit vector along ℓ. Since $\mathbf{PQ} \times \mathbf{F}$ is perpendicular to $\hat{\mathbf{v}}$, the scalar product is zero.

12. Since the vector $\mathbf{PQ}$ in equation 11.46 is equal to zero, the moment is zero.

13. The moment of $\mathbf{F}$ about ℓ is

$$\begin{aligned} \mathrm{M} &= \mathbf{PQ} \times \mathbf{F} \cdot \hat{\mathbf{v}} = \hat{\mathbf{v}} \cdot \mathbf{PQ} \times \mathbf{F} \\ &= (v_x, v_y, v_z) \cdot (x_0 - x_1, y_0 - y_1, z_0 - z_1) \times (F_x, F_y, F_z) \\ &= (v_x, v_y, v_z) \cdot \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_0 - x_1 & y_0 - y_1 & z_0 - z_1 \\ F_x & F_y & F_z \end{vmatrix} \\ &= (v_x, v_y, v_z) \cdot \left(F_z(y_0 - y_1) - F_y(z_0 - z_1), F_x(z_0 - z_1) - F_z(x_0 - x_1), F_y(x_0 - x_1) - F_x(y_0 - y_1)\right) \\ &= v_x[F_z(y_0 - y_1) - F_y(z_0 - z_1)] + v_y[F_x(z_0 - z_1) - F_z(x_0 - x_1)] + v_z[F_y(x_0 - x_1) - F_x(y_0 - y_1)]. \end{aligned}$$

The same result is obtained by expanding the determinant along the first row.

14. When the sleeve is at D, the magnitude of $\mathbf{F}$ is $|\mathbf{F}| = k[\sqrt{(1-x)^2 + 1/4} - l]$, and therefore $\mathbf{F} = k[\sqrt{(1-x)^2 + 1/4} - l]\dfrac{(1-x), 1/2)}{\sqrt{(1-x)^2 + 1/4}}$. For a small displacement dx at position x, the amount of work done by $\mathbf{F}$ is approximately

$$\mathbf{F} \cdot (dx\, \hat{\mathbf{i}}) = k[\sqrt{(1-x)^2 + 1/4} - l]\frac{1-x}{\sqrt{(1-x)^2 + 1/4}}dx = k(1-x)\left[1 - \frac{l}{\sqrt{(1-x)^2 + 1/4}}\right]dx.$$

The total work done between B and C is therefore

$$\begin{aligned} W &= \int_0^1 k(1-x)\left[1 - \frac{l}{\sqrt{(1-x)^2 + 1/4}}\right]dx = k\int_0^1 \left[1 - x - \frac{l(1-x)}{\sqrt{(1-x)^2 + 1/4}}\right]dx \\ &= k\left\{x - \frac{x^2}{2} + l\sqrt{(1-x)^2 + 1/4}\right\}_0^1 = \frac{k}{2}[1 + l(1 - \sqrt{5})] \text{ J}. \end{aligned}$$

15. The work done by the resultant force of q_1 and q_2 is the sum of the individual amounts of work done by q_1 and q_2 separately. The force on q_3 due to q_1 when it is at position x on the x-axis is

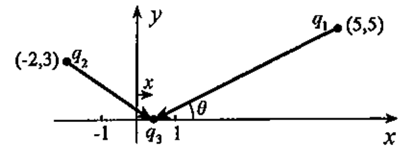
$$\mathbf{F}_1 = \frac{q_1 q_3}{4\pi\epsilon_0[(x-5)^2 + 25]} \frac{(x-5, -5)}{\sqrt{(x-5)^2 + 25}} = \frac{q_1 q_3 (x-5, -5)}{4\pi\epsilon_0[(x-5)^2 + 25]^{3/2}}.$$

For a small displacement dx at position x, the amount of work done by $\mathbf{F}_1$ is

$$\mathbf{F}_1 \cdot (dx\hat{\mathbf{i}}) = \frac{q_1 q_3 (x-5)}{4\pi\epsilon_0[(x-5)^2 + 25]^{3/2}}dx.$$

The work done by this force as q_3 moves from $x = 1$ to $x = -1$ is

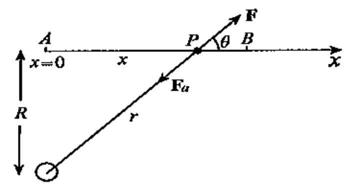
$$W_1 = \int_1^{-1} \frac{q_1 q_3 (x-5)}{4\pi\epsilon_0[(x-5)^2+25]^{3/2}}\,dx = \frac{q_1q_3}{4\pi\epsilon_0}\left\{\frac{-1}{\sqrt{(x-5)^2+25}}\right\}_1^{-1}$$
$$= \frac{q_1q_3}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{41}} - \frac{1}{\sqrt{61}}\right).$$



Similarly, the work done by q_2 is $W_2 = \dfrac{q_2q_3}{4\pi\epsilon_0}\left(\dfrac{\sqrt{2}}{6} - \dfrac{1}{\sqrt{10}}\right)$.

The total work is $W_1 + W_2$.

16. At position P, the magnitude of the force $\mathbf{F}_a$ of attraction of the asteroid on the rocket is $|\mathbf{F}_a| = \dfrac{GMm}{x^2+R^2}$, and therefore
$\mathbf{F}_a = \dfrac{GMm}{x^2+R^2}\dfrac{(-x,-R)}{\sqrt{x^2+R^2}}$. For a small displacement dx at P, the work done by an equal and opposite force $\mathbf{F}$ is approximately



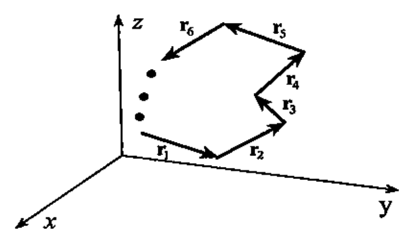
$$\mathbf{F}\cdot(dx\,\hat{\mathbf{i}}) = \frac{GMm}{x^2+R^2}\frac{x}{\sqrt{x^2+R^2}}dx$$
$$= \frac{GMmx}{(x^2+R^2)^{3/2}}dx.$$

The total work done between A and B is therefore

$$W = \int_0^R \frac{GMmx}{(x^2+R^2)^{3/2}}dx = GMm\left\{-\frac{1}{\sqrt{x^2+R^2}}\right\}_0^R = \frac{GMm}{\sqrt{2}R}(\sqrt{2}-1).$$

17. Suppose the sides of the polygon are denoted by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ so that $\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_n = \mathbf{0}$. The work done by a constant force $\mathbf{F}$ as an object moves around the polygon is

$$\mathbf{F}\cdot\mathbf{r}_1 + \mathbf{F}\cdot\mathbf{r}_2 + \cdots + \mathbf{F}\cdot\mathbf{r}_n$$
$$= \mathbf{F}\cdot(\mathbf{r}_1 + \cdots + \mathbf{r}_n) = \mathbf{F}\cdot\mathbf{0} = 0.$$

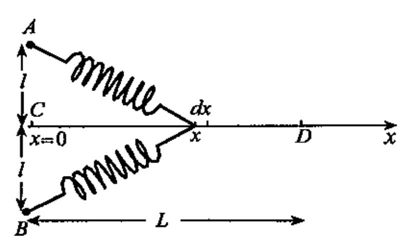


18. The work to stretch the two springs is three times the work to stretch the upper spring. The force necessary to hold the upper spring at position x on the x-axis against the upper spring is

$$\mathbf{F} = k(\sqrt{x^2+l^2} - l)\frac{(x,-l)}{\sqrt{x^2+l^2}}.$$

The work done by this force in moving the end of the spring a small amount dx along the x-axis is

$$\mathbf{F}\cdot(dx\hat{\mathbf{i}}) = k(\sqrt{x^2+l^2}-l)\frac{x}{\sqrt{x^2+l^2}}dx$$
$$= k\left(x - \frac{lx}{\sqrt{x^2+l^2}}\right)dx.$$

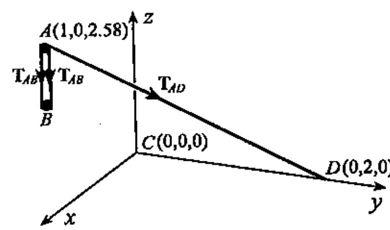


The total work done in stretching both springs is therefore

$$W = 3\int_0^L k\left(x - \frac{lx}{\sqrt{x^2+l^2}}\right)dx = 3k\left\{\frac{x^2}{2} - l\sqrt{x^2+l^2}\right\}_0^L = 3k\left(\frac{L^2}{2} - l\sqrt{L^2+l^2} + l^2\right).$$

19. Tensions in the ropes joining A to the boat B are the same, namely $\mathbf{T}_{AB} = -410\hat{\mathbf{k}}$. The tension in rope AD is

$$\mathbf{T}_{AD} = 410\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = \frac{410(-1,2,-2.58)}{\sqrt{1^2+2^2+2.58^2}} = \frac{410(-1,2,-2.58)}{\sqrt{11.6564}}.$$



The moment of the sum of these tensions about C is

$$\mathbf{M} = \mathbf{CA}\times(2\mathbf{T}_{AB}+\mathbf{T}_{AD}) = (1,0,2.58)\times\left[2(-410\hat{\mathbf{k}}) + \frac{410(-1,2,-2.58)}{\sqrt{11.6564}}\right]$$

$$= \frac{410}{\sqrt{11.6564}}(1,0,2.58)\times(-1,2,-2\sqrt{11.6564}-2.58) = \frac{410}{\sqrt{11.6564}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 2.58 \\ -1 & 2 & -2\sqrt{11.6564}-2.58 \end{vmatrix}$$

$$= \frac{410}{\sqrt{11.6564}}(-5.16, 2\sqrt{11.6564}, 2) = (-619.7, 820, 240.2)\ \text{N·m}.$$

20. Suppose we denote the components of $\mathbf{F}$ by $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$. Then $F_x^2 + F_y^2 + F_z^2 = 200^2$. Since the angle between $\mathbf{F}$ and $\hat{\mathbf{i}}$ is $\pi/3$ radians,

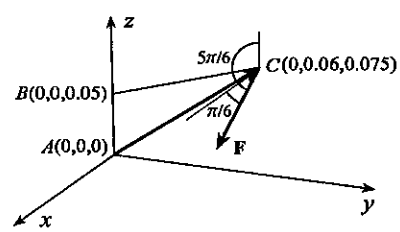
$F_x = \mathbf{F}\cdot\hat{\mathbf{i}} = |\mathbf{F}||\hat{\mathbf{i}}|\cos\pi/3 = 200(1/2) = 100.$

Similarly,

$F_z = \mathbf{F}\cdot\hat{\mathbf{k}} = |\mathbf{F}||\hat{\mathbf{k}}|\cos 5\pi/6 = 200(-\sqrt{3}/2) = -100\sqrt{3}.$

It follows that

$100^2 + F_y^2 + 3(100)^2 = 200^2 \Longrightarrow F_y = 0.$

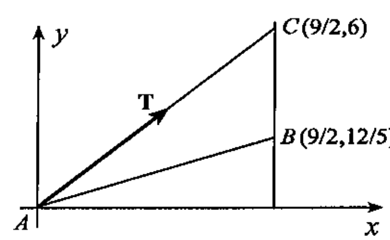


The moment of $\mathbf{F}$ about A is

$$\mathbf{M} = (0,0.06,0.075)\times(100,0,-100\sqrt{3}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0.06 & 0.075 \\ 100 & 0 & -100\sqrt{3} \end{vmatrix} = (-6\sqrt{3}, 15/2, -6)\ \text{N·m}.$$

21. Since $\mathbf{BA} = (-9/2, -12/5)$, and

$$\mathbf{T} = 1500\left(\frac{\mathbf{AC}}{|\mathbf{AC}|}\right) = 1500\left[\frac{(9/2,6)}{\sqrt{81/4+36}}\right] = 300(3,4),$$



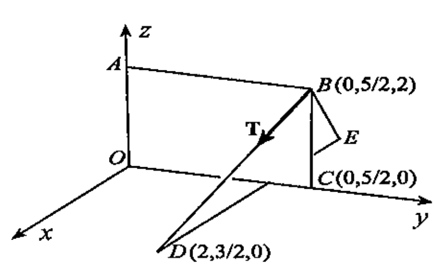
the moment of $\mathbf{T}$ about B is

$$\mathbf{M} = \mathbf{BA}\times\mathbf{T} = (-9/2,-12/5)\times 300(3,4) = 300\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -9/2 & -12/5 & 0 \\ 3 & 4 & 0 \end{vmatrix} = (0,0,-3240).$$

The magnitude of this moment is 3240 N·m.

22. (a) Since $\mathbf{T} = 900\left(\frac{\mathbf{BD}}{|\mathbf{BD}|}\right) = \frac{900(2,-1,-2)}{3}$, the moment of $\mathbf{T}$ at B about O is

$$\mathbf{M} = \mathbf{OB}\times\mathbf{T} = (0,5/2,2)\times 300(2,-1,-2) = 300\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 5/2 & 2 \\ 2 & -1 & -2 \end{vmatrix} = 300(-3,4,-5)\ \text{N·m}.$$



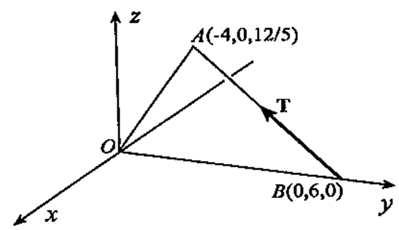
(b) The moment about the z-axis is -1500 N·m.

(c) Since the moment of $\mathbf{T}$ about B is zero, so also is the moment about any line through B (see Exercise 12).

23. Since $\mathbf{T} = 1900\left(\dfrac{\mathbf{BA}}{|\mathbf{BA}|}\right) = \dfrac{1900(-4,-6,12/5)}{\sqrt{16+36+144/25}}$
$= 100(-10,-15,6)$,

the moment of $\mathbf{M}$ at B about O is

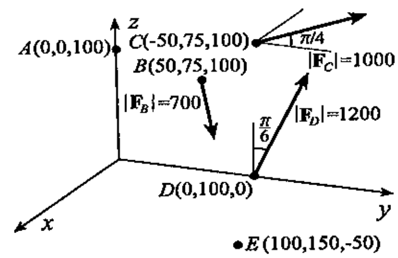
$$\mathbf{M} = \mathbf{OB}\times\mathbf{T} = (0,6,0)\times 100(-10,-15,6)$$
$$= 100\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 6 & 0 \\ -10 & -15 & 6 \end{vmatrix} = 1200(3,0,5) \text{ N·m.}$$



24. Components of the forces acting at B, C, and D are

$$\mathbf{F}_B = 700\left(\frac{\mathbf{BE}}{|\mathbf{BE}|}\right) = \frac{700(50,75,-150)}{\sqrt{50^2+75^2+150^2}} = 100(2,3,-6),$$
$$\mathbf{F}_C = 1000\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) = 500\sqrt{2}(-1,1,0),$$
$$\mathbf{F}_D = 1200\left(0,\frac{1}{2},\frac{\sqrt{3}}{2}\right) = 600(0,1,\sqrt{3}).$$

The sum of the moments of these forces about A is

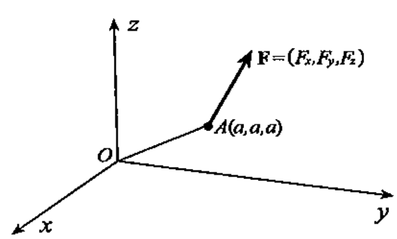


$$\begin{aligned}
\mathbf{M} &= \mathbf{AB}\times\mathbf{F}_B + \mathbf{AC}\times\mathbf{F}_C + \mathbf{AD}\times\mathbf{F}_D \\
&= (50,75,0)\times 100(2,3,-6) + (-50,75,0)\times 500\sqrt{2}(-1,1,0) + (0,100,-100)\times 600(0,1,\sqrt{3}) \\
&= 100\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 50 & 75 & 0 \\ 2 & 3 & -6 \end{vmatrix} + 500\sqrt{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -50 & 75 & 0 \\ -1 & 1 & 0 \end{vmatrix} + 600\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 100 & -100 \\ 0 & 1 & \sqrt{3} \end{vmatrix} \\
&= 100(-450,300,0) + 500\sqrt{2}(0,0,25) + 600(100\sqrt{3}+100,0,0) \\
&= 500(120\sqrt{3}+120-90,60,25\sqrt{2}) = (60\sqrt{3}+15,30,25/\sqrt{2}) \text{ N·m.}
\end{aligned}$$

25. If $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$, then the moment of $\mathbf{F}$ about O is

$$\mathbf{M} = \mathbf{OA}\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a & a & a \\ F_x & F_y & F_z \end{vmatrix}$$
$$= a(F_z - F_y)\hat{\mathbf{i}} + a(F_x - F_z)\hat{\mathbf{j}} + a(F_y - F_x)\hat{\mathbf{k}}.$$

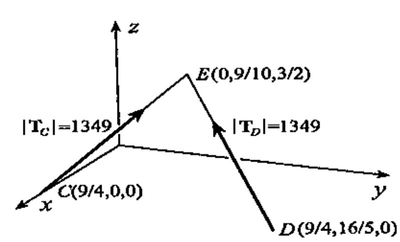
The sum of the moments of $\mathbf{F}$ about the coordinate axes is the sum of the components of this moment, and this is clearly seen to be zero.



26. Tensions in the cable exerted on C and D are

$$\mathbf{T}_C = 1349\left(\frac{\mathbf{CE}}{|\mathbf{CE}|}\right) = \frac{1349(-9/4,9/10,3/2)}{\sqrt{81/16+81/100+9/4}}$$
$$= 71(-15,6,10),$$
$$\mathbf{T}_D = 1349\left(\frac{\mathbf{DE}}{|\mathbf{DE}|}\right) = \frac{1349(-9/4,-23/10,3/2)}{\sqrt{81/16+529/100+9/4}}$$
$$= 19(-45,-46,30).$$

The moment of $\mathbf{T}_C$ about O is



$$\mathbf{M}_C = \mathbf{OC}\times\mathbf{T}_C = (9/4,0,0)\times 71(-15,6,10) = 71\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 9/4 & 0 & 0 \\ -15 & 6 & 10 \end{vmatrix} = 71(0,-45/2,27/2).$$

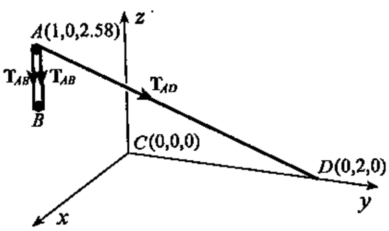
Moments of $\mathbf{T}_C$ about the coordinate axes are therefore 0, $-3195/2$ N·m, and $1917/2$ N·m. The moment of $\mathbf{T}_D$ about O is

$$\mathbf{M}_D = \mathbf{OD} \times \mathbf{T}_D = (9/4, 16/5, 0) \times 19(-45, -46, 30) = 19 \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 9/4 & 16/5 & 0 \\ -45 & -46 & 30 \end{vmatrix} = 19(96, -135/2, 81/2).$$

Moments of $\mathbf{T}_D$ about the coordinate axes are therefore 1824 N·m, $-2565/2$ N·m, and $1539/2$ N·m.

27. If T is the tension in the ropes , then $\mathbf{T}_{AB} = -T\hat{\mathbf{k}}$. The tension in rope AD is

$$\mathbf{T}_{AD} = T\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = \frac{T(-1, 2, -2.58)}{\sqrt{1^2 + 2^2 + 2.58^2}} = \frac{T(-1, 2, -2.58)}{\sqrt{11.6564}}.$$



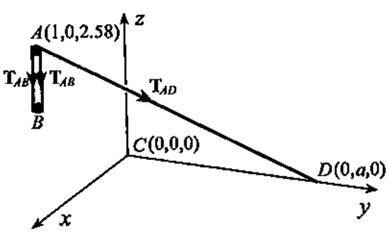
The moment of the sum of these tensions about C is

$$\begin{aligned}\mathbf{M} &= \mathbf{CA} \times (2\mathbf{T}_{AB} + \mathbf{T}_{AD}) = (1, 0, 2.58) \times \left[2(-T\hat{\mathbf{k}}) + \frac{T(-1, 2, -2.58)}{\sqrt{11.6564}}\right] \\ &= \frac{T}{\sqrt{11.6564}}(1, 0, 2.58) \times (-1, 2, -2\sqrt{11.6564} - 2.58) = \frac{T}{\sqrt{11.6564}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 2.58 \\ -1 & 2 & -2\sqrt{11.6564} - 2.58 \end{vmatrix} \\ &= \frac{T}{\sqrt{11.6564}}(-5.16, 2\sqrt{11.6564}, 2) \text{ N·m}.\end{aligned}$$

Since the absolute value of the x-component must be less than 375 N·m, it follows that $5.16T/\sqrt{11.6564} \leq 375 \Longrightarrow T \leq 248$ N.

28. Since tension in the ropes is 300 N, $\mathbf{T}_{AB} = -300\hat{\mathbf{k}}$. The tension in rope AD is

$$\mathbf{T}_{AD} = 300\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right) = \frac{300(-1, a, -2.58)}{\sqrt{1^2 + a^2 + 2.58^2}} = \frac{300(-1, a, -2.58)}{\sqrt{a^2 + 7.6564}}.$$



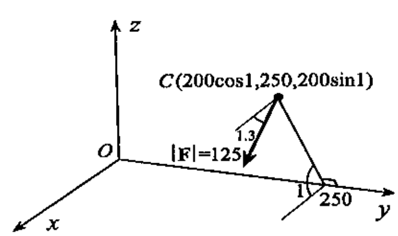
The moment of the sum of these tensions about C is

$$\begin{aligned}\mathbf{M} &= \mathbf{CA} \times (2\mathbf{T}_{AB} + \mathbf{T}_{AD}) = (1, 0, 2.58) \times \left[2(-300\hat{\mathbf{k}}) + \frac{300(-1, a, -2.58)}{\sqrt{a^2 + 7.6564}}\right] \\ &= \frac{300}{\sqrt{a^2 + 7.6564}}(1, 0, 2.58) \times (-1, a, -2\sqrt{a^2 + 7.6564} - 2.58) \\ &= \frac{300}{\sqrt{a^2 + 7.6564}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 2.58 \\ -1 & a & -2\sqrt{a^2 + 7.6564} - 2.58 \end{vmatrix} = \frac{300}{\sqrt{a^2 + 7.6564}}(-2.58a, 2\sqrt{a^2 + 7.6564}, a) \text{ N·m}.\end{aligned}$$

Since the absolute value of the x-component must be less than 375 N·m, it follows that $300(2.58a)/\sqrt{a^2 + 7.6564} \leq 375$. This implies that $a \leq 1.532$ m.

29. Since components of $\mathbf{F}$ are $\mathbf{F} = 125(\cos 1.3, 0, -\sin 1.3)$, its moment about O is

$$\begin{aligned}\mathbf{M} &= \mathbf{OC} \times \mathbf{F} \\ &= (200\cos 1, 250, 200\sin 1) \times 125(\cos 1.3, 0, -\sin 1.3) \\ &= 125 \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 200\cos 1 & 250 & 200\sin 1 \\ \cos 1.3 & 0 & -\sin 1.3 \end{vmatrix} \\ &= 125(-250\sin 1.3, 200\sin 1\cos 1.3 + 200\cos 1\sin 1.3, -250\cos 1.3).\end{aligned}$$



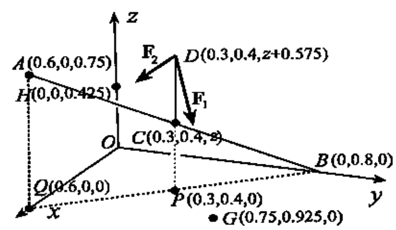
Moments about the coordinate axes (in N·m) are components of this vector divided by 1000, namely, -30.1 N·m, 18.6 N·m, and -8.36 N·m.

30. If we denote the z-coordinate of C by z, then from similar triangles AQB and CPB,

$$\frac{\|PC\|}{\|AQ\|}=\frac{\|PB\|}{\|QB\|},$$

from which

$$\frac{z}{0.75}=\frac{\sqrt{0.3^2+0.4^2}}{\sqrt{0.6^2+0.8^2}} \quad\Longrightarrow\quad z=0.375.$$



The z-coordinate of D is then $0.375+0.575=0.95$.
Components of $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\mathbf{F}_1=1175\left(\frac{\mathbf{DG}}{|\mathbf{DG}|}\right)=\frac{1175(0.45,0.525,-0.95)}{\sqrt{0.45^2+0.525^2+0.95^2}}=1000(0.45,0.525,-0.95),$$

$$\mathbf{F}_2=870\left(\frac{\mathbf{DH}}{|\mathbf{DH}|}\right)=\frac{870(-0.3,-0.4,-0.525)}{\sqrt{0.3^2+0.4^2+0.525^2}}=-1200(0.3,0.4,0.525).$$

Moments of these forces about C are

$$\mathbf{M}_1=\mathbf{CD}\times\mathbf{F}_1=(0,0,0.575)\times 1000(0.45,0.525,-0.95)=1000\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 0 & 0.575\\ 0.45 & 0.525 & -0.95\end{vmatrix}$$

$$=(-301.875,258.75,0),$$

$$\mathbf{M}_2=\mathbf{CD}\times\mathbf{F}_2=(0,0,0.575)\times(-1200)(0.3,0.4,0.525)=-1200\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 0 & 0.575\\ 0.3 & 0.4 & 0.525\end{vmatrix}$$

$$=-1200(-0.23,0.1725,0).$$

Moments of these forces about AB are

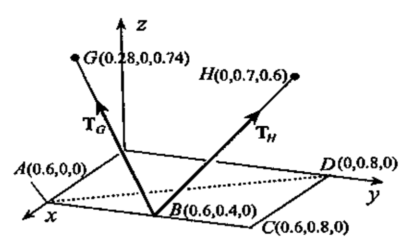
$$\mathbf{M}_1\cdot\left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right)=(-301.875,258.75,0)\cdot\frac{(-0.6,0.8,-0.75)}{\sqrt{0.6^2+0.8^2+0.75^2}}=310.5\text{ N}\cdot\text{m},$$

$$\mathbf{M}_2\cdot\left(\frac{\mathbf{AB}}{|\mathbf{AB}|}\right)=-1200(-0.23,0.1725,0)\cdot\frac{(-0.6,0.8,-0.75)}{\sqrt{0.6^2+0.8^2+0.75^2}}=-264.96\text{ N}\cdot\text{m}.$$

31. Tensions in BH and BG are

$$\mathbf{T}_H=1125\left(\frac{\mathbf{BH}}{|\mathbf{BH}|}\right)=\frac{1125(-0.6,0.3,0.6)}{\sqrt{0.6^2+0.3^2+0.6^2}}$$
$$=1250(-0.6,0.3,0.6),$$

$$\mathbf{T}_G=1125\left(\frac{\mathbf{BG}}{|\mathbf{BG}|}\right)=\frac{1125(-0.32,-0.4,0.74)}{\sqrt{0.32^2+0.4^2+0.74^2}}$$
$$=1250(-0.32,-0.4,0.74).$$



Moments of these forces about A are

$$\mathbf{M}_H=\mathbf{AB}\times\mathbf{T}_H=1250\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 0.4 & 0\\ -0.6 & 0.3 & 0.6\end{vmatrix}=1250(0.24,0,0.24),$$

$$\mathbf{M}_G=\mathbf{AB}\times\mathbf{T}_G=1250\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 0.4 & 0\\ -0.32 & -0.4 & 0.74\end{vmatrix}=1250(0.296,0,0.128).$$

Moments of these forces about diagonal AD are

$$\mathbf{M}_H\cdot\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right)=1250(0.24,0,0.24)\cdot\frac{(-0.6,0.8,0)}{\sqrt{0.6^2+0.8^2}}=-180\text{ N}\cdot\text{m},$$

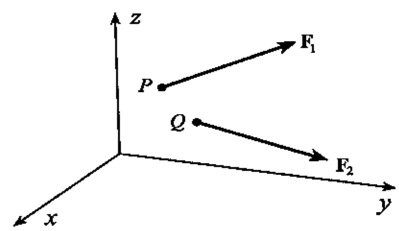
$$\mathbf{M}_G\cdot\left(\frac{\mathbf{AD}}{|\mathbf{AD}|}\right)=1250(0.296,0,0.128)\cdot\frac{(-0.6,0.8,0)}{\sqrt{0.6^2+0.8^2}}=-222\text{ N}\cdot\text{m}.$$

32. The moment of $\mathbf{F}_1$ about the line of action of $\mathbf{F}_2$ is

$$\mathbf{QP} \times \mathbf{F}_1 \cdot \left(\frac{\mathbf{F}_2}{|\mathbf{F}_2|}\right),$$

and the moment of $\mathbf{F}_2$ about the line of action of $\mathbf{F}_1$ is

$$\mathbf{PQ} \times \mathbf{F}_2 \cdot \left(\frac{\mathbf{F}_1}{|\mathbf{F}_1|}\right).$$



If $\mathbf{PQ} = (a, b, c)$, $\mathbf{F}_1 = (F_x, F_y, F_z)$, and $\mathbf{F}_2 = (G_x, G_y, G_z)$, then

$$\mathbf{PQ} \times \mathbf{F}_2 \cdot \mathbf{F}_1 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a & b & c \\ G_x & G_y & G_z \end{vmatrix} \cdot (F_x, F_y, F_z) = F_x(bG_z - cG_y) + F_y(cG_x - aG_z) + F_z(aG_y - bG_x),$$

and

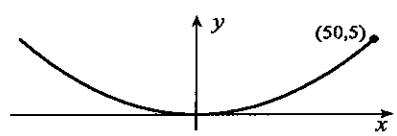
$$\mathbf{QP} \times \mathbf{F}_1 \cdot \mathbf{F}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a & -b & -c \\ F_x & F_y & F_z \end{vmatrix} \cdot (G_x, G_y, G_z) = G_x(-bF_z + cF_y) + G_y(-cF_x + aF_z) + G_z(-aF_y + bF_x).$$

Hence, $\mathbf{PQ} \times \mathbf{F}_2 \cdot \mathbf{F}_1 = \mathbf{QP} \times \mathbf{F}_1 \cdot \mathbf{F}_2$, and because $|\mathbf{F}_1| = |\mathbf{F}_2|$, the two moments are equal.

EXERCISES 11.8

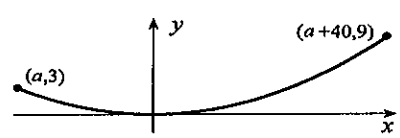
1. In the coordinate system shown, the equation of the cable is $\quad y(x) = \dfrac{1000x^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(50, 5)$ is a point on the cable,

$$5 = \frac{500(50)^2}{T_0} \quad \Longrightarrow \quad T_0 = 250\,000.$$



According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta}$ $= T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a minimum when slope is a minimum, namely at $x = 0$. In other words, $T_0 = 250\,000$ N is the minimum tension in the cable.

2. In the coordinate system shown, the equation of the cable is $\quad y(x) = \dfrac{1100x^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(a, 3)$, and $(a + 40, 9)$ are points on the cable,

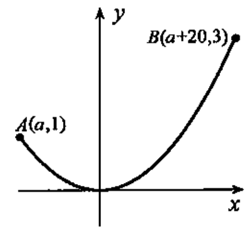


$$3 = \frac{550a^2}{T_0}, \qquad 9 = \frac{550(a + 40)^2}{T_0}.$$

These imply that $a = 20(1 - \sqrt{3})$ and $T_0 = 3.93 \times 10^4$. According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta} = T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a minimum when slope is a minimum, namely at $x = 0$. In other words, $T_0 = 3.93 \times 10^4$ N is the minimum tension in the cable. Maximum tension is at $x = a + 40 = 60 - 20\sqrt{3}$ where slope is greatest; i.e., maximum tension is

$$T_0\sqrt{1 + [y'(60 - 20\sqrt{3})]^2} = T_0\sqrt{1 + \left[\frac{1100(60 - 20\sqrt{3})}{T_0}\right]^2} = 4.82 \times 10^4 \text{ N}.$$

3. In the coordinate system shown, the equation of the cable is $y(x) = \dfrac{wx^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(a, 1)$, and $(a + 20, 3)$ are points on the cable,



$$1 = \frac{wa^2}{2T_0}, \qquad 3 = \frac{w(a+20)^2}{2T_0}.$$

The first implies that $a = -\sqrt{2T_0/w}$, which substituted into the second gives

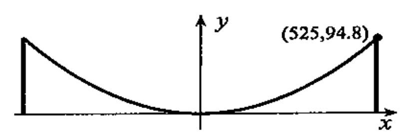
$$6T_0 = w(-\sqrt{2T_0/w} + 20)^2 \quad \Longrightarrow \quad T_0 + 10\sqrt{2w}\sqrt{T_0} - 100w = 0.$$

This is a quadratic in $\sqrt{T_0}$ with solutions

$$\sqrt{T_0} = \frac{-10\sqrt{2w} \pm \sqrt{200w + 400w}}{2} = 5\sqrt{w}(\sqrt{6} - \sqrt{2}) \quad \Longrightarrow \quad T_0 = 100(2 - \sqrt{3})w.$$

According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta}$ $= T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a minimum when slope is a minimum, namely at $x = 0$. In other words, $T_0 = 100(2 - \sqrt{3})w$ is the minimum tension in the cable.

4. In the coordinate system shown, the equation of the cable is $y(x) = \dfrac{wx^2}{2T_0}$, where T_0 is the tension in the cable at $x = 0$. Since $(525, 94.8)$ is a point on the cable,



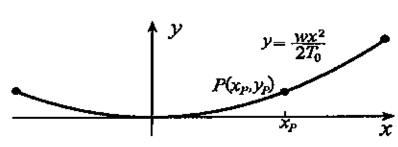
$$94.8 = \frac{142\,000(525)^2}{2T_0}.$$

This implies that $T_0 = 2.06 \times 10^8$. According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta} = T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a minimum when slope is a minimum, namely at $x = 0$. In other words, $T_0 = 2.06 \times 10^8$ N is the minimum tension in the cable. Maximum tension is at $x = 525$ where slope is greatest; i.e., maximum tension is

$$T_0\sqrt{1 + [y'(525)]^2} = T_0\sqrt{1 + \left[\frac{w(525)}{T_0}\right]^2} = 2.19 \times 10^8 \text{ N}.$$

5. (a) With the coordinate system shown, the equation of the cable is $y = wx^2/(2T_0)$. The length of the cable from $x = 0$ to an arbitrary point $P(x_P, y_P)$ is

$$L(P) = \int_0^{x_P} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{x_P} \sqrt{1 + \frac{w^2x^2}{T_0^2}}\,dx.$$



If we set $x = (T_0/w)\tan\theta$, $dx = (T_0/w)\sec^2\theta\,d\theta$, and $\overline{\theta} = \mathrm{Tan}^{-1}(wx_P/T_0)$, then

$$\begin{aligned} L(P) &= \int_0^{\overline{\theta}} \sec\theta\left(\frac{T_0}{w}\right)\sec^2\theta\,d\theta = \frac{T_0}{w}\int_0^{\overline{\theta}} \sec^3\theta\,d\theta \\ &= \frac{T_0}{2w}\left\{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right\}_0^{\overline{\theta}} \qquad \text{(see Example 8.9)} \\ &= \frac{T_0}{2w}(\sec\overline{\theta}\tan\overline{\theta} + \ln|\sec\overline{\theta} + \tan\overline{\theta}|) \\ &= \frac{T_0}{2w}\left[\frac{\sqrt{T_0^2 + w^2x_P^2}}{T_0}\left(\frac{wx_P}{T_0}\right) + \ln\left(\frac{\sqrt{T_0^2 + w^2x_P^2}}{T_0} + \frac{wx_P}{T_0}\right)\right] \end{aligned}$$

$$= \frac{x_P}{2}\sqrt{1+\left(\frac{wx_P}{T_0}\right)^2} + \frac{T_0}{2w}\ln\left[\sqrt{1+\left(\frac{wx_P}{T_0}\right)^2} + \frac{wx_P}{T_0}\right].$$

(b) If we expand the root by the binomial expansion,

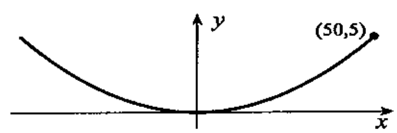
$$\begin{aligned}
L(P) &= \int_0^{x_P}\left[1+\frac{1}{2}\left(\frac{w^2x^2}{T_0^2}\right)+\frac{(1/2)(-1/2)}{2}\left(\frac{w^2x^2}{T_0^2}\right)^2+\cdots\right]dx \\
&= \left\{x+\frac{w^2x^3}{6T_0^2}-\frac{w^4x^5}{40T_0^4}+\cdots\right\}_0^{x_P}, \qquad \text{valid for } w^2x^2/T_0^2<1 \\
&= x_P+\frac{w^2x_P^3}{6T_0^2}-\frac{w^4x_P^5}{40T_0^4}+\cdots, \qquad \text{valid for } x_P<T_0/w.
\end{aligned}$$

Since $y_p = wx_P^2/(2T_0)$, this can be expressed in the form

$$L(P) = x_P\left(1+\frac{w^2x_P^2}{6T_0^2}-\frac{w^4x_P^4}{40T_0^4}+\cdots\right) = x_P\left[1+\frac{2}{3}\left(\frac{y_P}{x_P}\right)^2-\frac{2}{5}\left(\frac{y_P}{x_P}\right)^4+\cdots\right].$$

This is valid for $x_P < T_0/w = x_P^2/(2y_P) \Longrightarrow y_P/x_P < 1/2$.

6. In the coordinate system shown, the equation of the cable is $y(x) = \dfrac{1000x^2}{2T_0}$, where T_0 is the tension in the cable at $x=0$. Since $(50, 5)$ is a point on the cable,



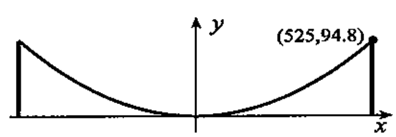
$$5 = \frac{500(50)^2}{T_0} \quad \Longrightarrow \quad T_0 = 250\,000.$$

Using the formula in Exercise 5(a), the length of the cable is

$$2\left\{\frac{50}{2}\sqrt{1+\left[\frac{1000(50)}{250\,000}\right]^2}+\frac{250\,000}{2(1000)}\ln\left[\sqrt{1+\left(\frac{1000(50)}{250\,000}\right)^2}+\frac{1000(50)}{250\,000}\right]\right\} = 100.663 \text{ m}.$$

With the two-term approximation in part (b), the length is $2(50)[1+(2/3)(5/50)^2] = 100.667$ m.

7. In the coordinate system shown, the equation of the cable is $y(x) = wx^2(2T_0)$, where T_0 is the tension in the cable at $x = 0$. Since $(525, 94.8)$ is a point on the cable,



$$94.8 = \frac{142\,000(525)^2}{2T_0} \quad \Longrightarrow \quad T_0 = 2.06\times 10^8.$$

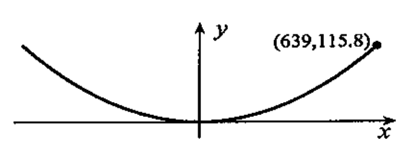
Using the formula in Exercise 5(a), the length of the cable is

$$2\left\{\frac{525}{2}\sqrt{1+\left[\frac{142\,000(525)}{2.06\times10^8}\right]^2}+\frac{2.06\times10^8}{2(142\,000)}\ln\left[\sqrt{1+\left(\frac{142\,000(525)}{2.06\times10^8}\right)^2}+\frac{142\,000(525)}{2.06\times10^8}\right]\right\} = 1072.5 \text{ m}.$$

With the two-term approximation in part (b), the length is $2(525)[1+(2/3)(94.8/525)^2] = 1072.8$ m.

8. When the sag is 115.8 m, the length is approximated by

$$2(639)\left[1+\frac{2}{3}\left(\frac{115.8}{639}\right)^2\right] = 1305.98.$$

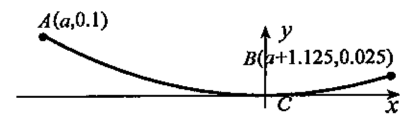


When the sag is 118.2, we find

$$2(639)\left[1+\frac{2}{3}\left(\frac{118.2}{639}\right)^2\right] = 1307.15.$$

The difference is 1.17 m.

9. (a) In the coordinate system shown, the equation of the curve for the paper is $y = wx^2/(2T_0)$, where T_0 is the tension at C. With points $A(a, 0.1)$, and $B(a + 1.125, 0.025)$ on the parabola, we must have



$$0.1 = \frac{0.3(9.81)a^2}{2T_0}, \qquad 0.025 = \frac{0.3(9.81)(a + 1.125)^2}{2T_0}.$$

These imply that

$$2T_0 = 3(9.81)a^2, \quad T_0 = 6(9.81)(a + 1.125)^2 \quad \Longrightarrow \quad \frac{3a^2}{2} = 6(a + 1.125)^2 \quad \Longrightarrow \quad 3a^2 + 9a + 5.0625 = 0.$$

Solutions of this quadratic are $a = -0.75, -2.25$, only the first being acceptable. Hence, C is 75 cm to the right of A.

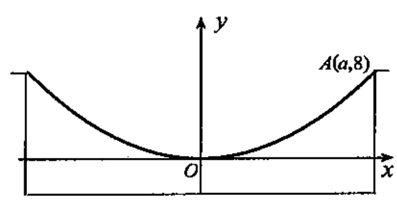
(b) According to the first of equations 11.47, tension in the cable at any point is $T = T_0 \sec\theta = T_0\sqrt{1 + \tan^2\theta} = T_0\sqrt{1 + (dy/dx)^2}$. This means that tension is a maximum when slope is a maximum, namely at A. With $T_0 = 3(9.81)(-0.75)^2/2$, maximum tension is

$$T_0\sqrt{1 + [y'(-0.75)]^2} = T_0\sqrt{1 + \left[\frac{w(-0.75)}{T_0}\right]^2} = 8.57 \text{ N}.$$

10. Using equation 11.51, the equation of the rope in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$

Since $A(a, 8)$ is on the curve,



$$8 = \frac{T_0}{w}\left[\cosh\left(\frac{wa}{T_0}\right) - 1\right].$$

According to Example 11.32, the length of one-half the rope is

$$20 = \frac{T_0}{w}\sinh\left(\frac{wa}{T_0}\right).$$

Finally, Example 11.33 indicates that maximum tension is at A, and therefore $350 = T_0 + 8w$. The first two equations give

$$1 = \cosh^2\left(\frac{wa}{T_0}\right) - \sinh^2\left(\frac{wa}{T_0}\right) = \left(\frac{8w}{T_0} + 1\right)^2 - \left(\frac{20w}{T_0}\right)^2 \quad \Longrightarrow \quad \frac{16w}{T_0^2}(T_0 - 21w) = 0.$$

Thus, $T_0 = 21w$, and when this is substituted into $T_0 + 8w = 350$, we obtain $29w = 350 \Longrightarrow w = 350/29$. The mass of the rope is $40(350/29)/9.81 = 49.2$ kg. With $T_0 = 350 - 8(350/29) = 7350/29$,

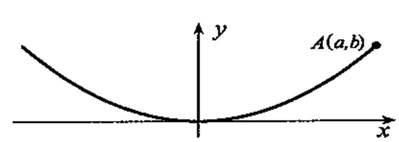
$$20 = \frac{7350/29}{350/29}\sinh\left(\frac{350a/29}{7350/29}\right) = 21\,\sinh\left(\frac{a}{21}\right) \quad \Longrightarrow \quad a = 21\,\mathrm{Sinh}^{-1}\left(\frac{20}{21}\right) = 17.79.$$

Hence, the horizontal distance between the buildings is $2a = 35.6$ m.

11. Using equation 11.51, the equation of the tape in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right],$$

where $w = 1.6(9.81)/50$. Since $A(a, b)$ is on the curve,



$$b = \frac{T_0}{w}\left[\cosh\left(\frac{wa}{T_0}\right) - 1\right].$$

According to Example 11.32, the length of one-half the tape is $25 = \dfrac{T_0}{w}\sinh\left(\dfrac{wa}{T_0}\right)$. Finally, Example

11.33 indicates that maximum tension is at A, and therefore $60 = T_0 + wb$. The first two equations give

$$1 = \cosh^2\left(\frac{wa}{T_0}\right) - \sinh^2\left(\frac{wa}{T_0}\right) = \left(\frac{wb}{T_0} + 1\right)^2 - \left(\frac{25w}{T_0}\right)^2.$$

Substituting $wb = 60 - T_0$ gives

$$1 = \left(\frac{60 - T_0}{T_0} + 1\right)^2 - \left(\frac{25w}{T_0}\right)^2 = \left(\frac{60}{T_0}\right)^2 - \left[\frac{25(1.6)(9.81)}{50T_0}\right]^2 \quad \Longrightarrow \quad T_0 = \sqrt{60^2 - (9.81)^2(0.8)^2}.$$

It now follows that

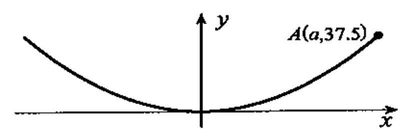
$$a = \frac{T_0}{w}\mathrm{Sinh}^{-1}\left(\frac{25w}{T_0}\right) = \frac{\sqrt{60^2 - (9.81)^2(0.8)^2}}{1.6(9.81)/50}\mathrm{Sinh}^{-1}\left[\frac{25(1.6)(9.81)}{50\sqrt{60^2 - (9.81)^2(0.8)^2}}\right] = 24.928.$$

The span of the tape is therefore $2a = 49.86$ m.

12. Using equation 11.51, the equation of the cable in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right],$$

where $w = 4(9.81)$. Since $A(a, 37.5)$ is on the curve,



$$37.5 = \frac{T_0}{w}\left[\cosh\left(\frac{wa}{T_0}\right) - 1\right].$$

According to Example 11.32, the length of one-half the cable is $75 = \dfrac{T_0}{w}\sinh\left(\dfrac{wa}{T_0}\right)$. From these,

$$1 = \cosh^2\left(\frac{wa}{T_0}\right) - \sinh^2\left(\frac{wa}{T_0}\right) = \left(\frac{37.5w}{T_0} + 1\right)^2 - \left(\frac{75w}{T_0}\right)^2 \quad \Longrightarrow \quad \frac{75w}{T_0^2}\left(T_0 - \frac{225w}{4}\right) = 0.$$

Thus, $T_0 = 225w/4$, and this in turn implies that

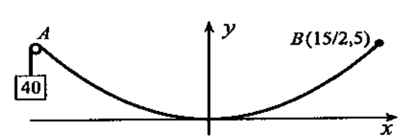
$$75 = \frac{225}{4}\sinh\left(\frac{4a}{225}\right) \quad \Longrightarrow \quad a = \frac{225}{4}\mathrm{Sinh}^{-1}\left(\frac{4}{3}\right) = 61.797.$$

The span is therefore $2a = 123.6$ m. Maximum tension is at A where, according to Example 11.33, $T = 225(4)(9.81)/4 + 4(9.81)(37.5) = 3679$ N.

13. Using equation 11.51, the equation of the cable in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$

Since $B(15/2, 5)$ is on the curve,



$$5 = \frac{T_0}{w}\left[\cosh\left(\frac{15w}{2T_0}\right) - 1\right].$$

Since tension at A must be $40g$ N, it follows from Example 11.33 that $40g = T_0 + 5w$. These combine to give

$$5w = (40g - 5w)\left[\cosh\left(\frac{15w}{80g - 10w}\right) - 1\right].$$

When this is solved numerically, the result is $w = 34.672$. Mass per unit length of the cable is therefore $34.672/9.81 = 3.53$ kg/m. According to Example 11.32 length of the cable from A to B is

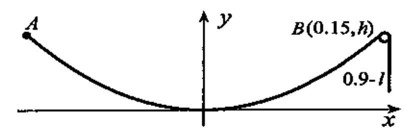
$$\frac{2T_0}{w}\sinh\left(\frac{15w}{2T_0}\right) = \frac{2[40g - 5(34.672)]}{34.672}\sinh\left[\frac{15(34.672)}{2[40g - 5(34.672)]}\right] = 18.78 \text{ m}.$$

14. Using equation 11.51, the equation of the cord in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$

Since $B(0.15, h)$ is on the curve,

$$h = \frac{T_0}{w}\left[\cosh\left(\frac{0.15w}{T_0}\right) - 1\right].$$



According to Example 11.32, the length of cord between A and B is

$$l = \frac{2T_0}{w}\sinh\left(\frac{0.15w}{T_0}\right).$$

Since the cord is in equilibrium, the maximum tension in the cord between A and B must be equal to the force of gravity on that part of the cord hanging to the right of B. Since maximum tension between A and B is at B, we obtain $T_0 + wh = (0.9 - l)w$. The first two equations give

$$1 = \cosh^2\left(\frac{0.15w}{T_0}\right) - \sinh^2\left(\frac{0.15w}{T_0}\right) = \left(\frac{wh}{T_0} + 1\right)^2 - \left(\frac{wl}{2T_0}\right)^2 \quad \Longrightarrow \quad 4wh^2 + 8T_0h - wl^2 = 0.$$

When we substitute $l = 0.9 - (T_0 + wh)/w$,

$$4wh^2 + 8T_0h - w\left(0.9 - \frac{T_0 + wh}{w}\right)^2 = 0 \quad \Longrightarrow \quad \frac{3wh^2}{T_0} + h\left(6 + \frac{1.8w}{T_0}\right) - \frac{w}{T_0}\left(0.9 - \frac{T_0}{w}\right)^2 = 0.$$

Solutions of this quadratic equation in h are

$$\begin{aligned} h &= \frac{-(6 + 1.8w/T_0) \pm \sqrt{(6 + 1.8w/T_0)^2 + 12(w^2/T_0^2)(0.9 - T_0/w)^2}}{6w/T_0} \\ &= \frac{-(6 + 1.8w/T_0) \pm \sqrt{48 + 12.96w^2/T_0^2}}{6w/T_0}. \end{aligned}$$

When we choose the positive root and substitute into the equation $h = (T_0/w)[\cosh(0.15w/T_0) - 1]$,

$$-\frac{T_0}{w} - 0.3 + \frac{1}{6}\sqrt{12.96 + \frac{48T_0^2}{w^2}} = \frac{T_0}{w}\left[\cosh\left(\frac{0.15w}{T_0}\right) - 1\right],$$

which simplifies to

$$-1.8 + \sqrt{12.96 + \frac{48T_0^2}{w^2}} = \frac{6T_0}{w}\cosh\left(\frac{0.15w}{T_0}\right).$$

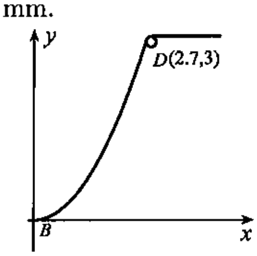
If we set $z = w/T_0$, solutions of $-1.8z + \sqrt{12.96z^2 + 48} - 6\cosh(0.15z) = 0$ are $z = 0.83939, 1.7331$. These give $h = 0.00946, 0.0196$. Hence, the smaller sag is 9.46 mm.

15. Using equation 11.51, the equation of the cable in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$

Since $D(2.7, 3)$ is on the curve,

$$3 = \frac{T_0}{w}\left[\cosh\left(\frac{2.7w}{T_0}\right) - 1\right].$$

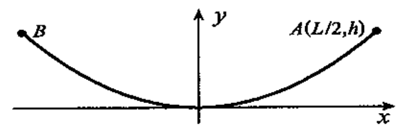


When w is set equal to 2.7(9.81), the equation can be solved numerically for $T_0 = 41.145$. Maximum tension occurs at D, and this tension is equal to the required force F. According to Example 11.33,

$$F = T = T_0 + wh = T_0 + 3w = 41.145 + 3(2.7)(9.81) = 120.6 \text{ N}.$$

16. Using equation 11.51, the equation of the cable in the coordinate system shown is

$$y = \frac{T_0}{w}\left[\cosh\left(\frac{wx}{T_0}\right) - 1\right].$$



If h is the sag, then

$$h = \frac{T_0}{w}\left[\cosh\left(\frac{wL}{2T_0}\right) - 1\right].$$

Maximum tension occurs at A and according to Example 11.33 is given by

$$T = T_0 + wh = T_0 + T_0\left[\cosh\left(\frac{wL}{2T_0}\right) - 1\right] = T_0\cosh\left(\frac{wL}{2T_0}\right).$$

Critical points of $T(T_0)$ are defined

$$0 = \frac{dT}{dT_0} = \cosh\left(\frac{wL}{2T_0}\right) - \frac{wL}{2T_0}\sinh\left(\frac{wL}{2T_0}\right) \implies \frac{wL}{2T_0}\tanh\left(\frac{wL}{2T_0}\right) = 1.$$

This equation can be solved numerically for $wL/(2T_0) = 1.200$. It now follows that

$$h = \frac{L}{2.4}[\cosh(1.200) - 1] \implies \frac{h}{L} = \frac{\cosh(1.200) - 1}{2.4} = 0.338.$$

EXERCISES 11.9

1. $t \geq 1$ **2.** All real t **3.** $-1 \leq t \leq 1$

4. $t > -4$ **5.** All real t **6.** $\dfrac{d\mathbf{u}}{dt} = \hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

7. $$\frac{d}{dt}[f(t)\mathbf{v}(t)] = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t) = 2t(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}) + (t^2+3)(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})$$
$$= 2t\hat{\mathbf{i}} - 6(t^2+1)\hat{\mathbf{j}} + 6t(2t^2+3)\hat{\mathbf{k}}$$

8. $$\frac{d}{dt}[g(t)\mathbf{u}(t)] = g'(t)\mathbf{u}(t) + g(t)\mathbf{u}'(t) = (6t^2-3)(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}) + (2t^3-3t)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$
$$= 2t(4t^2-3)\hat{\mathbf{i}} + t^2(9-10t^2)\hat{\mathbf{j}} + 4t(4t^2-3)\hat{\mathbf{k}}$$

9. $$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & -t^2 & 2t \\ 0 & -2 & 6t \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2t & 2 \\ 1 & -2t & 3t^2 \end{vmatrix}$$
$$= [(-6t^3+4t)\hat{\mathbf{i}} - 6t^2\hat{\mathbf{j}} - 2t\hat{\mathbf{k}}] + [(-6t^3+4t)\hat{\mathbf{i}} + (2-3t^2)\hat{\mathbf{j}}]$$
$$= 4t(2-3t^2)\hat{\mathbf{i}} + (2-9t^2)\hat{\mathbf{j}} - 2t\hat{\mathbf{k}}$$

10. $$\frac{d}{dt}(\mathbf{u}\times t\mathbf{v}) = \frac{d}{dt}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & -t^2 & 2t \\ t & -2t^2 & 3t^3 \end{vmatrix} = \frac{d}{dt}[(4t^3-3t^5)\hat{\mathbf{i}} + (2t^2-3t^4)\hat{\mathbf{j}} - t^3\hat{\mathbf{k}}]$$
$$= (12t^2-15t^4)\hat{\mathbf{i}} + (4t-12t^3)\hat{\mathbf{j}} - 3t^2\hat{\mathbf{k}}$$

11. $$\frac{d}{dt}(2\mathbf{u}\cdot\mathbf{v}) = 2\frac{d}{dt}(t + 2t^3 + 6t^3) = 2(1+24t^2)$$

12. $$\frac{d}{dt}(3\mathbf{u}+4\mathbf{v}) = 3\frac{d\mathbf{u}}{dt} + 4\frac{d\mathbf{v}}{dt} = 3(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + 4(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}}) = 3\hat{\mathbf{i}} - (6t+8)\hat{\mathbf{j}} + (6+24t)\hat{\mathbf{k}}$$

13. $$\int \mathbf{u}(t)\,dt = \int (t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})\,dt = \frac{t^2}{2}\hat{\mathbf{i}} - \frac{t^3}{3}\hat{\mathbf{j}} + t^2\hat{\mathbf{k}} + \mathbf{C}$$

14. $$\frac{d}{dt}[f(t)\mathbf{u} + g(t)\mathbf{v}] = f'(t)\mathbf{u} + f(t)\mathbf{u}'(t) + g'(t)\mathbf{v} + g(t)\mathbf{v}'(t)$$
$$= 2t(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}) + (t^2+3)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$
$$+(6t^2-3)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}) + (2t^3-3t)(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})$$
$$= 9t^2\hat{\mathbf{i}} + (6t-20t^3)\hat{\mathbf{j}} + (6-21t^2+30t^4)\hat{\mathbf{k}}$$

15. $\displaystyle\int 4\mathbf{v}(t)\,dt = 4\int (\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}})\,dt = 4(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}) + \mathbf{C}$

16. $$\frac{d}{dt}[t(\mathbf{u}\times\mathbf{v})] = \frac{d}{dt}\left[t\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ t & -t^2 & 2t\\ 1 & -2t & 3t^2\end{vmatrix}\right] = \frac{d}{dt}[(4t^3 - 3t^5)\hat{\mathbf{i}} + (2t^2 - 3t^4)\hat{\mathbf{j}} - t^3\hat{\mathbf{k}}$$
$$= (12t^2 - 15t^4)\hat{\mathbf{i}} + (4t - 12t^3)\hat{\mathbf{j}} - 3t^2\hat{\mathbf{k}}$$

17. $\displaystyle\int f(t)\mathbf{u}(t)\,dt = \int (t^2+3)(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})\,dt = (t^4/4 + 3t^2/2)\hat{\mathbf{i}} - (t^5/5 + t^3)\hat{\mathbf{j}} + (t^4/2 + 3t^2)\hat{\mathbf{k}} + \mathbf{C}$

18. $$\int [3g(t)\mathbf{v}(t) + \mathbf{u}(t)]\,dt = \int [3(2t^3 - 3t)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}) + (t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})]\,dt$$
$$= \int [(6t^3 - 8t)\hat{\mathbf{i}} + (-12t^4 + 17t^2)\hat{\mathbf{j}} + (18t^5 - 27t^3 + 2t)\hat{\mathbf{k}}]\,dt$$
$$= \left(\frac{3t^4}{2} - 4t^2\right)\hat{\mathbf{i}} + \left(-\frac{12t^5}{5} + \frac{17t^3}{3}\right)\hat{\mathbf{j}} + \left(3t^6 - \frac{27t^4}{4} + t^2\right)\hat{\mathbf{k}} + \mathbf{C}$$

19. $\displaystyle\int [f(t)\mathbf{u}\cdot\mathbf{v}]\,dt = \int (t^2+3)(t + 2t^3 + 6t^3)\,dt = \int (8t^5 + 25t^3 + 3t)\,dt = \frac{4t^6}{3} + \frac{25t^4}{4} + \frac{3t^2}{2} + C$

20. $$\mathbf{u}\times\frac{d\mathbf{v}}{dt} - f(t)\mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\mathbf{v} = \mathbf{u}\times(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}}) - f(t)\mathbf{u}\cdot(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})\mathbf{v}$$
$$= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ t & -t^2 & 2t\\ 0 & -2 & 6t\end{vmatrix} - (t^2+3)(t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})\cdot(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})\mathbf{v}$$
$$= (-6t^3 + 4t)\hat{\mathbf{i}} - 6t^2\hat{\mathbf{j}} - 2t\hat{\mathbf{k}} - (t^2+3)(2t^2 + 12t^2)(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}})$$
$$= (-14t^4 - 6t^3 - 42t^2 + 4t)\hat{\mathbf{i}} + (28t^5 + 84t^3 - 6t^2)\hat{\mathbf{j}} + (-42t^6 - 126t^4 - 2t)\hat{\mathbf{k}}$$

21. $$\mathbf{u}\cdot\frac{d\mathbf{v}}{dt} - \mathbf{v}\cdot\int \mathbf{u}(t)\,dt = (t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})\cdot(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})$$
$$-(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}})\cdot\left(\frac{t^2}{2}\hat{\mathbf{i}} - \frac{t^3}{3}\hat{\mathbf{j}} + t^2\hat{\mathbf{k}} + a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}\right)$$
$$= 2t^2 + 12t^2 - \frac{t^2}{2} - \frac{2t^4}{3} - 3t^4 - a + 2bt - 3ct^2 = -\frac{11t^4}{3} + C_1 + C_2 t + C_3 t^2$$

22. If $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$, then
$$\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d}{dt}[(u_x+v_x)\hat{\mathbf{i}} + (u_y+v_y)\hat{\mathbf{j}} + (u_z+v_z)\hat{\mathbf{k}}] = \left[\frac{d}{dt}(u_x+v_x)\right]\hat{\mathbf{i}} + \left[\frac{d}{dt}(u_y+v_y)\right]\hat{\mathbf{j}} + \left[\frac{d}{dt}(u_z+v_z)\right]\hat{\mathbf{k}}$$
$$= \left(\frac{du_x}{dt}\hat{\mathbf{i}} + \frac{du_y}{dt}\hat{\mathbf{j}} + \frac{du_z}{dt}\hat{\mathbf{k}}\right) + \left(\frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}\right) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}.$$

23. If we set $\mathbf{u} = u_x\hat{\mathbf{i}} + u_y\hat{\mathbf{j}} + u_z\hat{\mathbf{k}}$ and $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, then
$$\frac{d}{dt}(f\mathbf{v}) = \frac{d}{dt}[f(t)v_x\hat{\mathbf{i}} + f(t)v_y\hat{\mathbf{j}} + f(t)v_z\hat{\mathbf{k}}]$$
$$= \left[f'(t)v_x + f(t)\frac{dv_x}{dt}\right]\hat{\mathbf{i}} + \left[f'(t)v_y + f(t)\frac{dv_y}{dt}\right]\hat{\mathbf{j}} + \left[f'(t)v_z + f(t)\frac{dv_z}{dt}\right]\hat{\mathbf{k}}$$
$$= f'(t)(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) + f(t)\left(\frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}\right) = f'(t)\mathbf{v}(t) + f(t)\frac{d\mathbf{v}}{dt},$$
$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \frac{d}{dt}(u_xv_x + u_yv_y + u_zv_z) = u_x\frac{dv_x}{dt} + v_x\frac{du_x}{dt} + u_y\frac{dv_y}{dt} + v_y\frac{du_y}{dt} + u_z\frac{dv_z}{dt} + v_z\frac{du_z}{dt}$$
$$= (u_x\hat{\mathbf{i}} + u_y\hat{\mathbf{j}} + u_z\hat{\mathbf{k}})\cdot\left(\frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}\right) + \left(\frac{du_x}{dt}\hat{\mathbf{i}} + \frac{du_y}{dt}\hat{\mathbf{j}} + \frac{du_z}{dt}\hat{\mathbf{k}}\right)\cdot(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}})$$
$$= \mathbf{u}\cdot\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v},$$

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \frac{d}{dt}[(u_yv_z - u_zv_y)\hat{\mathbf{i}} + (u_zv_x - u_xv_z)\hat{\mathbf{j}} + (u_xv_y - u_yv_x)\hat{\mathbf{k}}]$$
$$= \left(u_y\frac{dv_z}{dt} + v_z\frac{du_y}{dt} - u_z\frac{dv_y}{dt} - v_y\frac{du_z}{dt}\right)\hat{\mathbf{i}} + \left(u_z\frac{dv_x}{dt} + v_x\frac{du_z}{dt} - u_x\frac{dv_z}{dt} - v_z\frac{du_x}{dt}\right)\hat{\mathbf{j}}$$
$$+ \left(u_x\frac{dv_y}{dt} + v_y\frac{du_x}{dt} - u_y\frac{dv_x}{dt} - v_x\frac{du_y}{dt}\right)\hat{\mathbf{k}}$$
$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_x & u_y & u_z \\ \dfrac{dv_x}{dt} & \dfrac{dv_y}{dt} & \dfrac{dv_z}{dt} \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{du_x}{dt} & \dfrac{du_y}{dt} & \dfrac{du_z}{dt} \\ v_x & v_y & v_z \end{vmatrix} = \mathbf{u}\times\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}\times\mathbf{v}.$$

24. Using equation 11.59b, $\frac{d}{dt}[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = \mathbf{u}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{w}) + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w}$, and now using equation 11.59c,

$$\frac{d}{dt}[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = \mathbf{u}\cdot\left(\mathbf{v}\times\frac{d\mathbf{w}}{dt} + \frac{d\mathbf{v}}{dt}\times\mathbf{w}\right) + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\cdot\mathbf{v}\times\frac{d\mathbf{w}}{dt} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}\times\mathbf{w} + \frac{d\mathbf{u}}{dt}\cdot\mathbf{v}\times\mathbf{w}.$$

25. If $\mathbf{v}$ has constant length, then $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 =$ constant. Differentiation with 11.59b gives

$$0 = \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = 2\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right).$$

But this implies that $\mathbf{v}$ and $d\mathbf{v}/dt$ are perpendicular.

26. If we set $\mathbf{v} = v_x(s)\hat{\mathbf{i}} + v_y(s)\hat{\mathbf{j}} + v_z(s)\hat{\mathbf{k}}$, then

$$\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}} = \frac{dv_x}{ds}\frac{ds}{dt}\hat{\mathbf{i}} + \frac{dv_y}{ds}\frac{ds}{dt}\hat{\mathbf{j}} + \frac{dv_z}{ds}\frac{ds}{dt}\hat{\mathbf{k}} = \left(\frac{dv_x}{ds}\hat{\mathbf{i}} + \frac{dv_y}{ds}\hat{\mathbf{j}} + \frac{dv_z}{ds}\hat{\mathbf{k}}\right)\frac{ds}{dt} = \frac{d\mathbf{v}}{ds}\frac{ds}{dt}.$$

27. Suppose first of all that the limit satisfies the definition of the exercise. Then given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|\mathbf{v}(t) - \mathbf{V}| < \epsilon$ whenever $0 < |t - t_0| < \delta$. If components of $\mathbf{v}$ and $\mathbf{V}$ are denoted by $\mathbf{v} = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}}$ and $\mathbf{V} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}}$, then for $0 < |t - t_0| < \delta$,

$$|\mathbf{v}(t) - \mathbf{V}| = \sqrt{[v_x(t) - V_x]^2 + [v_y(t) - V_y]^2 + [v_z(t) - V_z]^2} < \epsilon.$$

But this implies that $|v_x(t) - V_x| < \epsilon$, $|v_y(t) - V_y| < \epsilon$, and $|v_z(t) - V_z| < \epsilon$ for $0 < |t - t_0| < \delta$. But this means that

$$\lim_{t\to t_0} v_x(t) = V_x, \quad \lim_{t\to t_0} v_y(t) = V_y, \quad \lim_{t\to t_0} v_z(t) = V_z.$$

Thus,

$$\lim_{t\to t_0}\mathbf{v}(t) = \mathbf{V} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}} = \left[\lim_{t\to t_0} v_x(t)\right]\hat{\mathbf{i}} + \left[\lim_{t\to t_0} v_y(t)\right]\hat{\mathbf{j}} + \left[\lim_{t\to t_0} v_z(t)\right]\hat{\mathbf{k}};$$

that is, 11.54 is satisfied. Conversely, suppose 11.54 is satisfied and we denote the component limits by

$$V_x = \lim_{t\to t_0} v_x(t), \quad V_y = \lim_{t\to t_0} v_y(t), \quad V_z = \lim_{t\to t_0} v_z(t).$$

Given any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$|v_x(t) - V_x| < \frac{\epsilon}{\sqrt{3}} \qquad \text{whenever } 0 < |t - t_0| < \delta_1;$$

a $\delta_2 > 0$ such that

$$|v_y(t) - V_y| < \frac{\epsilon}{\sqrt{3}} \qquad \text{whenever } 0 < |t - t_0| < \delta_2;$$

and a $\delta_3 > 0$ such that

$$|v_z(t) - V_z| < \frac{\epsilon}{\sqrt{3}} \qquad \text{whenever } 0 < |t - t_0| < \delta_3.$$

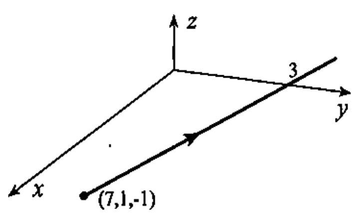
If we set $\mathbf{V} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}}$, and choose δ as the smallest of δ_1, δ_2, and δ_3, then whenever $0 < |t - t_0| < \delta$,

$$|\mathbf{v}(t) - \mathbf{V}| = \sqrt{[v_x(t) - V_x]^2 + [v_y(t) - V_y]^2 + [v_z(t) - V_z]^2} < \sqrt{\epsilon^2/3 + \epsilon^2/3 + \epsilon^2/3} = \epsilon;$$

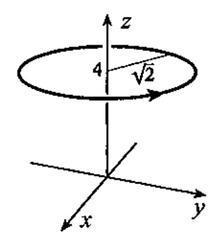
that is, the limit satisfies the definition of the exercise.

EXERCISES 11.10

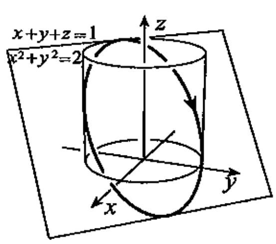
1. If we set $z = t$, then $y = 2t + 3$, and $x = 6 - 2(2t + 3) - 3t = -7t$. These are parametric equations for the curve. A vector representation is $\mathbf{r} = -7t\hat{\mathbf{i}} + (2t + 3)\hat{\mathbf{j}} + t\hat{\mathbf{k}}$.



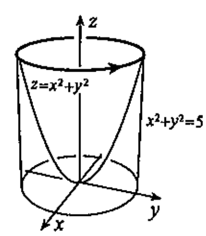
2. If we set $x = \sqrt{2}\cos t$, then $y = \pm\sqrt{2}\sin t$. For y to increase in the first octant, we choose $y = \sqrt{2}\sin t$. Parametric and vector equations are therefore $x = \sqrt{2}\cos t$, $y = \sqrt{2}\sin t$, $z = 4$, $\mathbf{r} = \sqrt{2}\cos t\,\hat{\mathbf{i}} + \sqrt{2}\sin t\,\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $0 \le t < 2\pi$.



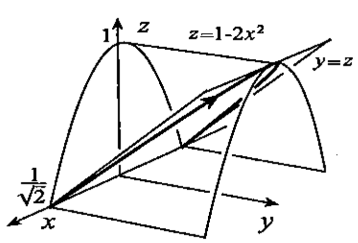
3. If we set $x = \sqrt{2}\cos t$, then $y = \pm\sqrt{2}\sin t$, and for correct direction we choose $y = -\sqrt{2}\sin t$. Parametric and vector equations are therefore $x = \sqrt{2}\cos t$, $y = -\sqrt{2}\sin t$, $z = 1 + \sqrt{2}(\sin t - \cos t)$, $\mathbf{r} = \sqrt{2}\cos t\,\hat{\mathbf{i}} - \sqrt{2}\sin t\,\hat{\mathbf{j}}$ $+(1 + \sqrt{2}\sin t - \sqrt{2}\cos t)\hat{\mathbf{k}}$, $0 \le t < 2\pi$



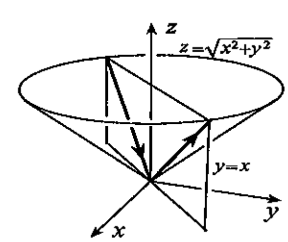
4. If we set $x = \sqrt{5}\cos t$, then $y = \pm\sqrt{5}\sin t$, and for correct direction we choose $y = \sqrt{5}\sin t$. Parametric and vector equations are therefore $x = \sqrt{5}\cos t$, $y = \sqrt{5}\sin t$, $z = 5$, $\mathbf{r} = \sqrt{5}\cos t\,\hat{\mathbf{i}} + \sqrt{5}\sin t\,\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, $0 \le t < 2\pi$.



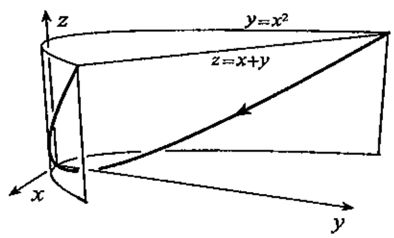
5. If we set $x = -t$ so that x decreases along the curve, then parametric and vector equations for the curve are $x = -t$, $y = 1 - 2t^2$, $z = 1 - 2t^2$, and $\mathbf{r} = -t\hat{\mathbf{i}} + (1 - 2t^2)(\hat{\mathbf{j}} + \hat{\mathbf{k}})$.



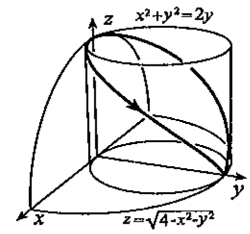
6. If we choose x as parameter by setting $x = t$, then $y = t$ and $z = \sqrt{t^2 + t^2} = \sqrt{2}|t|$. Parametric and vector equations are $x = t$, $y = t$, $z = \sqrt{2}|t|$, $\mathbf{r} = t(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \sqrt{2}|t|\hat{\mathbf{k}}$.



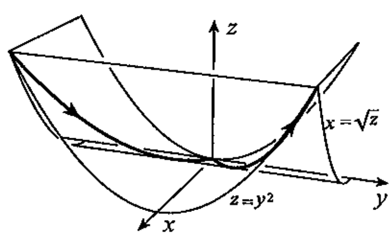
7. If we set $x = t$ so that x increases along the curve, then parametric and vector equations for the curve are $x = t$, $y = t^2$, $z = t + t^2$, and $\mathbf{r} = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + (t + t^2)\hat{\mathbf{k}}$.



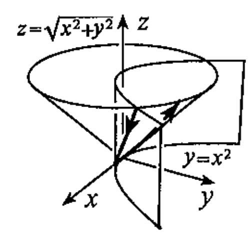
8. If we set $x = \cos t$, then $y = 1 \pm \sin t$, and for correct direction we choose $y = 1 + \sin t$. Parametric and vector equations are therefore
$x = \cos t$, $y = 1 + \sin t$,
$z = \sqrt{4 - \cos^2 t - (1 + \sin t)^2} = \sqrt{2 - 2\sin t}$,
$\mathbf{r} = \cos t\,\hat{\mathbf{i}} + (1 + \sin t)\hat{\mathbf{j}} + \sqrt{2 - 2\sin t}\,\hat{\mathbf{k}}$,
$0 \le t < 2\pi$.



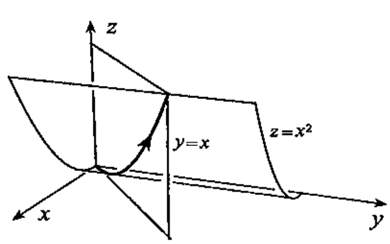
9. If we set $y = t$, then parametric and vector equations for the curve are $x = |t|$, $y = t$, $z = t^2$, and $\mathbf{r} = |t|\hat{\mathbf{i}} + t\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$.



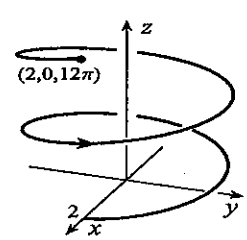
10. If we set $x = -t$ (so that x decreases as t increases), parametric and vector equations are $x = -t$, $y = t^2$, $z = \sqrt{t^2 + t^4}$, $\mathbf{r} = -t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + \sqrt{t^2 + t^4}\,\hat{\mathbf{k}}$.



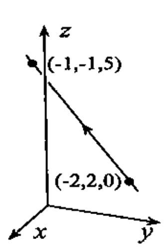
11. The curve is the first octant intersection of the plane $y = x$ and the parabolic cylinder $z = x^2$.



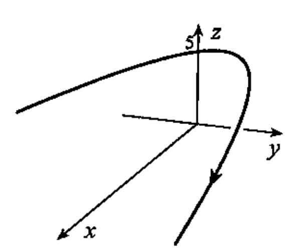
12. Since $x^2 + y^2 = 4$, the curve is two turns of a helix that rises a distance of 6π in each turn.



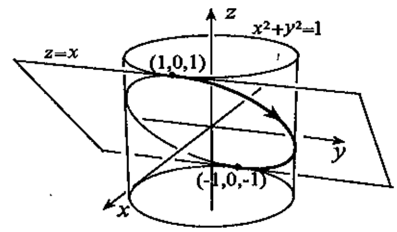
13. This is a straight line through the points $(-2, 2, 0)$ and $(-1, -1, 5)$.



14. Since $x = t^2 - t = y^2 - y$, the curve is a parabola in the plane $z = 5$.



15. The curve is half the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = x$.



EXERCISES 11.11

1. Since $\mathbf{r} = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + t\hat{\mathbf{k}}$, a tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = \cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$. A unit tangent vector is $\hat{\mathbf{T}} = \dfrac{\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{2}}$.

2. Since $\mathbf{r} = t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$, a tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}$. A unit tangent vector is $\hat{\mathbf{T}} = \dfrac{\hat{\mathbf{i}} + 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}}{\sqrt{1+4t^2+9t^4}}$.

3. Since $\mathbf{r} = (t-1)^2\hat{\mathbf{i}} + (t+1)^2\hat{\mathbf{j}} - t\hat{\mathbf{k}}$, a tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = 2(t-1)\hat{\mathbf{i}} + 2(t+1)\hat{\mathbf{j}} - \hat{\mathbf{k}}$. A unit tangent vector is $\hat{\mathbf{T}} = \dfrac{2(t-1)\hat{\mathbf{i}} + 2(t+1)\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{4(t-1)^2+4(t+1)^2+1}} = \dfrac{2(t-1)\hat{\mathbf{i}} + 2(t+1)\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{8t^2+9}}$.

4. Since x decreases along the curve, we set $x = -t$ for parametric equations, in which case $y = 5 + t$, $z = t^2 - 5 - t$. A vector equation for the curve is $\mathbf{r} = -t\hat{\mathbf{i}} + (5+t)\hat{\mathbf{j}} + (t^2 - t - 5)\hat{\mathbf{k}}$, $-5 \le t \le 0$. A tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + (2t-1)\hat{\mathbf{k}}$, and a unit tangent vector is

$$\hat{\mathbf{T}} = \frac{-\hat{\mathbf{i}} + \hat{\mathbf{j}} + (2t-1)\hat{\mathbf{k}}}{\sqrt{1+1+(2t-1)^2}} = \frac{-\hat{\mathbf{i}} + \hat{\mathbf{j}} + (2t-1)\hat{\mathbf{k}}}{\sqrt{4t^2-4t+3}}.$$

5. If we set $x = 2\cos t$, then $y = 2\sin t$ and $z = 4 - 2\cos t - 2\sin t$. A vector representation for the curve is $\mathbf{r} = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + (4 - 2\cos t - 2\sin t)\hat{\mathbf{k}}$. A tangent vector is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = -2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}} + (2\sin t - 2\cos t)\hat{\mathbf{k}}$, and a unit tangent vector is

$$\hat{\mathbf{T}} = \frac{-2\sin t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}} + (2\sin t - 2\cos t)\hat{\mathbf{k}}}{\sqrt{4\sin^2 t + 4\cos^2 t + (2\sin t - 2\cos t)^2}} = \frac{-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + (\sin t - \cos t)\hat{\mathbf{k}}}{\sqrt{2 - \sin 2t}}.$$

6. Since $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = -4\sin t\,\hat{\mathbf{i}} + 6\cos t\,\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$, the unit tangent vector at $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$ is

$$\hat{\mathbf{T}}(\pi/4) = \frac{-2\sqrt{2}\hat{\mathbf{i}} + 3\sqrt{2}\hat{\mathbf{j}} + \sqrt{2}\hat{\mathbf{k}}}{\sqrt{8+18+2}} = \frac{-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{14}}.$$

7. Since $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = -5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, the unit tangent vector at every point on the line is $\hat{\mathbf{T}} = \dfrac{-5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{42}}$.

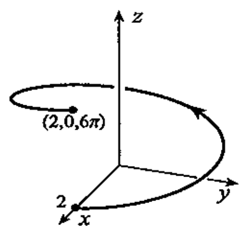
8. With $x = \sqrt{2}\cos t$, $y = -\sqrt{2}\sin t$, $z = \sqrt{2}$, $0 \le t < 2\pi$, a tangent vector at $(1, 1, \sqrt{2})$ is $\mathbf{T}(7\pi/4) = (-\sqrt{2}\sin t, -\sqrt{2}\cos t, 0)_{|t=7\pi/4} = (1, -1, 0)$. Hence, $\hat{\mathbf{T}}(7\pi/4) = (1, -1, 0)/\sqrt{2}$.

9. With $x = t^2 + 1$, $y = t$, $z = t^2 + 6$, a tangent vector is $\dfrac{d\mathbf{r}}{dt} = 2t\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2t\hat{\mathbf{k}}$. A unit tangent vector at $(5, 2, 10)$ is $\hat{\mathbf{T}}(2) = \dfrac{4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}}{\sqrt{33}}$.

10. If we set $x = 2\cos t$, then $y = 1 \pm 2\sin t$. For z to decrease when y is negative, we choose $y = 1 - 2\sin t$, and then $\mathbf{r} = 2\cos t\,\hat{\mathbf{i}} + (1 - 2\sin t)\hat{\mathbf{j}} + 2\cos t\,\hat{\mathbf{k}}$, $0 \le t < 2\pi$. Since $\mathbf{T} = d\mathbf{r}/dt = -2\sin t\,\hat{\mathbf{i}} - 2\cos t\,\hat{\mathbf{j}} - 2\sin t\,\hat{\mathbf{k}}$, $\mathbf{T}(0) = -2\hat{\mathbf{j}}$. Consequently, $\hat{\mathbf{T}}(0) = -\hat{\mathbf{j}}$.

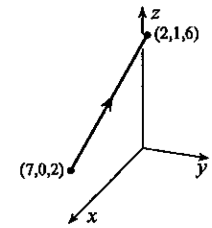
11. With equation 11.78,

$$L = \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 9}\,dt$$
$$= \sqrt{13}\Big\{t\Big\}_0^{2\pi} = 2\sqrt{13}\pi.$$



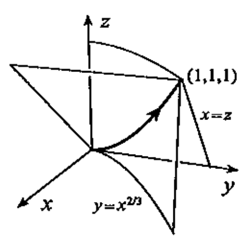
12. Since this is a straight line segment from $(7,0,2)$ to $(2,1,6)$, its length is

$$\sqrt{5^2 + (-1)^2 + (-4)^2} = \sqrt{42}.$$



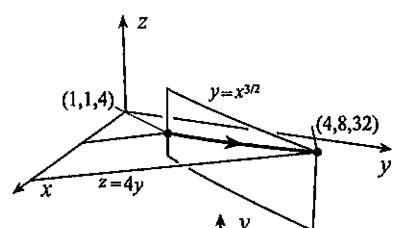
13. With equation 11.78,

$$L = \int_0^1 \sqrt{(3t^2)^2 + (2t)^2 + (3t^2)^2}\,dt$$
$$= \int_0^1 t\sqrt{4 + 18t^2}\,dt.$$
$$= \left\{\frac{(4+18t^2)^{3/2}}{54}\right\}_0^1 = \frac{11\sqrt{22}-4}{27}.$$

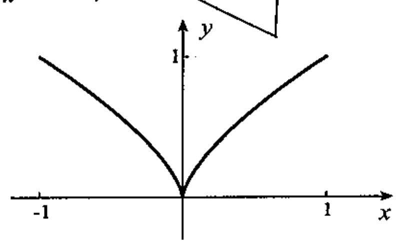


14. With equation 11.78,

$$L = \int_1^4 \sqrt{1^2 + \left(\frac{3\sqrt{t}}{2}\right)^2 + (6\sqrt{t})^2}\,dt$$
$$= \int_1^4 \sqrt{1 + 153t/4}\,dt$$
$$= \left\{\frac{8}{459}\left(1 + \frac{153t}{4}\right)^{3/2}\right\}_1^4$$
$$= \frac{616\sqrt{616} - 157\sqrt{157}}{459}.$$



15. There may be a corner at a point at which all derivatives vanish. For example, $d\mathbf{r}/dt = \mathbf{0}$ when $t = 0$ for the curve $x = t^3$, $y = t^2$, $z = 0$. The figure shows that the curve reverses direction at $(0,0)$.



16. Since a tangent vector is

$$\mathbf{T} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} = (-\sin t + \sin t + t\cos t)\hat{\mathbf{i}} + (\cos t - \cos t + t\sin t)\hat{\mathbf{j}} = t\cos t\,\hat{\mathbf{i}} + t\sin t\,\hat{\mathbf{j}},$$

a unit tangent vector is $\hat{\mathbf{T}} = \dfrac{t\cos t\,\hat{\mathbf{i}} + t\sin t\,\hat{\mathbf{j}}}{\sqrt{t^2\cos^2 t + t^2\sin^2 t}} = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$.

17. This follows from the fact that $\mathbf{r}(t+h) - \mathbf{r}(t)$ in limit 11.65 always points in the direction in which t increases along the curve (see Figure 11.102).

18. (a) If we use equation 11.66, $\mathbf{T}(0) = (2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}})_{|t=0} = \mathbf{0}$.

(b) A tangent vector at any point except $t = 0$ is $\mathbf{T} = 2t\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}} + 2t\hat{\mathbf{k}} = t(2\hat{\mathbf{i}} + 3t\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$. It follows then that $\mathbf{S}(t) = 2\hat{\mathbf{i}} + 3t\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ must also be tangent at any point (except possibly when $t = 0$). The only way $\mathbf{S}(t)$ can assign a tangent vector to the curve continuously is for $\mathbf{S}(0) = \lim_{t\to 0}\mathbf{S}(t) = 2\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$.

EXERCISES 11.12

1. From $\hat{\mathbf{T}} = \dfrac{(\cos t, -\sin t, 1)}{\sqrt{2}}$, a vector in the direction of $\hat{\mathbf{N}}$ is $\mathbf{N} = \dfrac{d\hat{\mathbf{T}}}{dt} = \dfrac{1}{\sqrt{2}}(-\sin t, -\cos t, 0)$. Consequently, the principal normal is $\hat{\mathbf{N}} = (-\sin t, -\cos t, 0)$. The binormal is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{2}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos t & -\sin t & 1 \\ -\sin t & -\cos t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}(\cos t, -\sin t, -1).$$

2. From $\hat{\mathbf{T}} = \dfrac{(1, 2t, 3t^2)}{\sqrt{1+4t^2+9t^4}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned} \mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(8t+36t^3)}{2(1+4t^2+9t^4)^{3/2}}(1, 2t, 3t^2) + \frac{(0, 2, 6t)}{\sqrt{1+4t^2+9t^4}} \\ &= \frac{1}{(1+4t^2+9t^4)^{3/2}}\left[-(4t+18t^3)(1, 2t, 3t^2) + (1+4t^2+9t^4)(0, 2, 6t)\right] \\ &= \frac{1}{(1+4t^2+9t^4)^{3/2}}(-4t-18t^3, 2-18t^4, 6t+12t^3). \end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(-2t-9t^3, 1-9t^4, 3t+6t^3)}{\sqrt{(2t+9t^3)^2+(1-9t^4)^2+(3t+6t^3)^2}} = \frac{(-2t-9t^3, 1-9t^4, 3t+6t^3)}{\sqrt{1+13t^2+54t^4+117t^6+81t^8}}.$$

The direction of the binormal is

$$\begin{aligned} \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 3t^2 \\ -2t-9t^3 & 1-9t^4 & 3t+6t^3 \end{vmatrix} \\ &= (6t^2+12t^4-3t^2+27t^6)\hat{\mathbf{i}} + (-6t^3-27t^5-3t-6t^3)\hat{\mathbf{j}} + (1-9t^4+4t^2+18t^4)\hat{\mathbf{k}} \\ &= (1+4t^2+9t^4)(3t^2\hat{\mathbf{i}} - 3t\hat{\mathbf{j}} + \hat{\mathbf{k}}). \end{aligned}$$

Thus, $\hat{\mathbf{B}} = \dfrac{(3t^2, -3t, 1)}{\sqrt{9t^4+9t^2+1}}$.

3. From $\hat{\mathbf{T}} = \dfrac{(2t-2, 2t+2, -1)}{\sqrt{(2t-2)^2+(2t+2)^2+1}} = \dfrac{(2t-2, 2t+2, -1)}{\sqrt{8t^2+9}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned} \mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-8t}{(8t^2+9)^{3/2}}(2t-2, 2t+2, -1) + \frac{(2, 2, 0)}{\sqrt{8t^2+9}} \\ &= \frac{1}{(8t^2+9)^{3/2}}\left[-8t(2t-2, 2t+2, -1) + (8t^2+9)(2, 2, 0)\right] \\ &= \frac{1}{(8t^2+9)^{3/2}}(16t+18, -16t+18, 8t). \end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(8t+9, -8t+9, 4t)}{\sqrt{(8t+9)^2+(-8t+9)^2+16t^2}} = \frac{(9+8t, 9-8t, 4t)}{3\sqrt{18+16t^2}}.$$

The direction of the binormal is

$$\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2t-2 & 2t+2 & -1 \\ 9+8t & 9-8t & 4t \end{vmatrix} = (8t^2+9, -8t^2-9, -32t^2-36) = (8t^2+9)(1, -1, -4).$$

Thus, $\hat{\mathbf{B}} = (1, -1, -4)/\sqrt{18} = (1, -1, -4)/(3\sqrt{2})$.

4. With parametric equations $x = -t$, $y = 5 + t$, $z = t^2 - t - 5$, (see Exercise 11.11–4),

$$\hat{\mathbf{T}} = \frac{(-1, 1, 2t-1)}{\sqrt{1 + 1 + (2t-1)^2}} = \frac{(-1, 1, 2t-1)}{\sqrt{4t^2 - 4t + 3}}.$$

A vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(4t-2)}{(4t^2 - 4t + 3)^{3/2}}(-1, 1, 2t-1) + \frac{(0, 0, 2)}{\sqrt{4t^2 - 4t + 3}} \\ &= \frac{2}{(4t^2 - 4t + 3)^{3/2}}\left[-(2t-1)(-1, 1, 2t-1) + (4t^2 - 4t + 3)(0, 0, 1)\right] \\ &= \frac{2}{(4t^2 - 4t + 3)^{3/2}}(2t-1, 1-2t, 2).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(2t-1, 1-2t, 2)}{\sqrt{(2t-1)^2 + (1-2t)^2 + 4}} = \frac{(2t-1, 1-2t, 2)}{\sqrt{8t^2 - 8t + 6}}.$$

The direction of the binormal is

$$\begin{aligned}\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 2t-1 \\ 2t-1 & 1-2t & 2 \end{vmatrix} &= (2 + 4t^2 - 4t + 1)\hat{\mathbf{i}} + (4t^2 - 4t + 1 + 2)\hat{\mathbf{j}} + (-1 + 2t - 2t + 1)\hat{\mathbf{k}} \\ &= (3 - 4t + 4t^2)(\hat{\mathbf{i}} + \hat{\mathbf{j}}).\end{aligned}$$

Thus, $\hat{\mathbf{B}} = (\hat{\mathbf{i}} + \hat{\mathbf{j}})/\sqrt{2}$.

5. With parametric equations $x = 2\cos t$, $y = 2\sin t$, $z = 2\cos t$, $0 \le t \le \pi$,

$$\hat{\mathbf{T}} = \frac{(-2\sin t, 2\cos t, -2\sin t)}{\sqrt{4\sin^2 t + 4\cos^2 t + 4\sin^2 t}} = \frac{(-\sin t, \cos t, -\sin t)}{\sqrt{1 + \sin^2 t}}.$$

A vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-\sin t\cos t}{(1 + \sin^2 t)^{3/2}}(-\sin t, \cos t, -\sin t) + \frac{(-\cos t, -\sin t, -\cos t)}{\sqrt{1 + \sin^2 t}} \\ &= \frac{1}{(1 + \sin^2 t)^{3/2}}\left[-\sin t\cos t(-\sin t, \cos t, -\sin t) + (1 + \sin^2 t)(-\cos t, -\sin t, -\cos t)\right] \\ &= \frac{1}{(1 + \sin^2 t)^{3/2}}(-\cos t, -2\sin t, -\cos t).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(-\cos t, -2\sin t, -\cos t)}{\sqrt{\cos^2 t + 4\sin^2 t + \cos^2 t}} = -\frac{(\cos t, 2\sin t, \cos t)}{\sqrt{2 + 2\sin^2 t}}.$$

The direction of the binormal is

$$\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin t & \cos t & -\sin t \\ -\cos t & -2\sin t & -\cos t \end{vmatrix} = (-\cos^2 t - 2\sin^2 t, 0, 2\sin^2 t + \cos^2 t) = (\cos^2 t + 2\sin^2 t)(-1, 0, 1).$$

Thus, $\hat{\mathbf{B}} = (-1, 0, 1))/\sqrt{2}$.

6. From $\hat{\mathbf{T}} = \dfrac{(-4\sin t, 6\cos t, 2\cos t)}{\sqrt{16\sin^2 t + 36\cos^2 t + 4\cos^2 t}} = \dfrac{(-2\sin t, 3\cos t, \cos t)}{\sqrt{4+6\cos^2 t}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{6\cos t\,\sin t}{(4+6\cos^2 t)^{3/2}}(-2\sin t, 3\cos t, \cos t) + \frac{(-2\cos t, -3\sin t, -\sin t)}{\sqrt{4+6\cos^2 t}}.$$

At $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$, we may take $t = \pi/4$, in which case

$$\mathbf{N}(\pi/4) = \frac{3}{7\sqrt{7}}(-\sqrt{2}, 3/\sqrt{2}, 1/\sqrt{2}) + \frac{(-\sqrt{2}, -3/\sqrt{2}, -1/\sqrt{2})}{\sqrt{7}} = -\frac{4}{7\sqrt{14}}(5,3,1).$$

Hence, the principal normal at $(2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$ is $\hat{\mathbf{N}} = -\dfrac{(5,3,1)}{\sqrt{35}}$. Since a tangent vector at the point is $\mathbf{T}(\pi/4) = (-\sqrt{2}, 3/\sqrt{2}, 1/\sqrt{2}) = (-2,3,1)/\sqrt{2}$, the direction of the binormal at the point is

$$\mathbf{B}(\pi/4) = (-2,3,1) \times [-(5,3,1)] = -\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 3 & 1 \\ 5 & 3 & 1 \end{vmatrix} = -(0,7,-21).$$

Thus, $\hat{\mathbf{B}}(\pi/4) = (0,-1,3)/\sqrt{10}$.

7. From $\hat{\mathbf{T}} = \dfrac{(-5,1,12t^2)}{\sqrt{26+144t^4}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{-576t^3}{2(26+144t^4)^{3/2}}(-5,1,12t^2) + \frac{(0,0,24t)}{\sqrt{26+144t^4}}.$$

At $(7,0,2)$, $t=-1$, in which case

$$\mathbf{N}(-1) = \frac{576}{2(170)^{3/2}}(-5,1,12) + \frac{(0,0,-24)}{\sqrt{170}} = \frac{48(-30,6,-13)}{170^{3/2}}.$$

Hence, the principal normal at $(7,0,2)$ is $\hat{\mathbf{N}} = \dfrac{(-30,6,-13)}{\sqrt{1105}}$. Since a tangent vector at the point is $\mathbf{T}(-1) = (-5,1,12)$, the direction of the binormal at the point is

$$\mathbf{B}(-1) = (-5,1,12) \times [(-30,6,-13)] = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 & 1 & 12 \\ -30 & 6 & -13 \end{vmatrix} = (-85,-425,0).$$

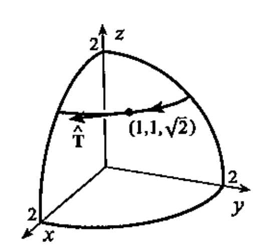
Thus, $\hat{\mathbf{B}}(-1) = -(1,5,0)/\sqrt{26}$.

8. This curve is the circle $x^2+y^2=2$ in the plane $z=\sqrt{2}$. The unit tangent and principal normal vectors at $(1,1,\sqrt{2})$ are

$$\hat{\mathbf{T}} = \frac{(1,-1,0)}{\sqrt{2}}, \qquad \hat{\mathbf{N}} = \frac{(-1,-1,0)}{\sqrt{2}}.$$

The binormal is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ -1 & -1 & 0 \end{vmatrix}$$
$$= \frac{1}{2}(0,0,-2) = (0,0,-1).$$



9. With parametric equations $x = t^2+1$, $y=t$, $z=t^2+6$, a unit tangent vector to the curve is $\hat{\mathbf{T}} = \dfrac{(2t,1,2t)}{\sqrt{8t^2+1}}$. A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{-8t}{(1+8t^2)^{3/2}}(2t,1,2t) + \frac{(2,0,2)}{\sqrt{1+8t^2}}.$$

At $(5,2,10)$, $t=2$, in which case

$$\mathbf{N}(2) = \frac{-16}{33\sqrt{33}}(4,1,4) + \frac{(2,0,2)}{\sqrt{33}} = \frac{2(1,-8,1)}{33\sqrt{33}}.$$

Hence, the principal normal at $(5,2,10)$ is $\hat{\mathbf{N}} = \dfrac{(1,-8,1)}{\sqrt{66}}$. Since a tangent vector at the point is $\mathbf{T}(2) = (4,1,4)$, the direction of the binormal at the point is

$$\mathbf{B}(2) = (4,1,4)\times(1,-8,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 1 & 4 \\ 1 & -8 & 1 \end{vmatrix} = (33,0,-33).$$

Thus, $\hat{\mathbf{B}}(2) = (1,0,-1)/\sqrt{2}$.

10. With the parametric equations $x = 2\cos t$, $y = 1 - 2\sin t$, $z = 2\cos t$, (see Exercise 11.11–10),

$$\hat{\mathbf{T}} = \frac{(-2\sin t, -2\cos t, -2\sin t)}{\sqrt{4\sin^2 t + 4\cos^2 t + 4\sin^2 t}} = -\frac{(\sin t, \cos t, \sin t)}{\sqrt{1+\sin^2 t}}.$$

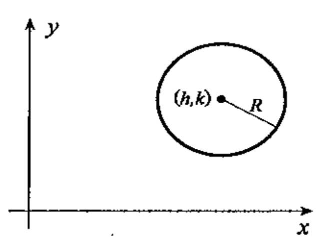
A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{\sin t\,\cos t}{(1+\sin^2 t)^{3/2}}(\sin t, \cos t, \sin t) - \frac{(\cos t, -\sin t, \cos t)}{\sqrt{1+\sin^2 t}}.$$

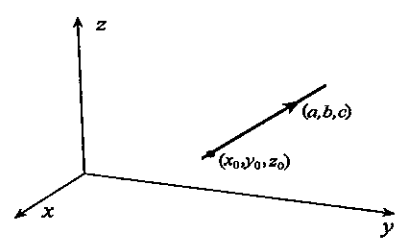
At $(2,1,2)$, $t=0$ and $\mathbf{N}(0) = -(1,0,1)$. Thus, the principal normal at $(2,1,2)$ is $\hat{\mathbf{N}}(0) = -(1,0,1)/\sqrt{2}$. Since $\hat{\mathbf{T}}(0) = -(0,1,0)$, the binormal at $(2,1,2)$ is

$$\hat{\mathbf{B}}(0) = -(0,1,0)\times\frac{-(1,0,1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}}(1,0,-1).$$

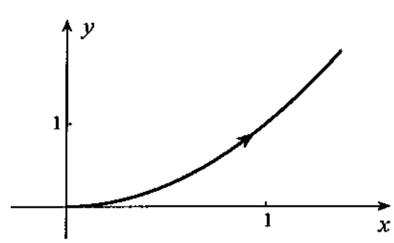
11. For a circle, the radius of curvature is the radius $\rho = R$ of the circle, and the curvature is $\kappa = 1/R$.



12. This is a straight line in space for which $\ddot{\mathbf{r}} = \mathbf{0}$, and therefore $\kappa = 0$. Its radius of curvature is undefined.



13.
$$\kappa(t) = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(1,2t,0)\times(0,2,0)|}{|(1,2t,0)|^3}$$
$$= \frac{1}{(1+4t^2)^{3/2}}\left\|\begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{matrix}\right\|$$
$$= \frac{|(0,0,2)|}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}$$
and $\rho(t) = \dfrac{1}{\kappa} = \dfrac{(1+4t^2)^{3/2}}{2}$.



14. With $\dot{\mathbf{r}} = (e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t, 1)$ and $\ddot{\mathbf{r}} = (-2e^t\sin t, 2e^t\cos t, 0)$, we obtain

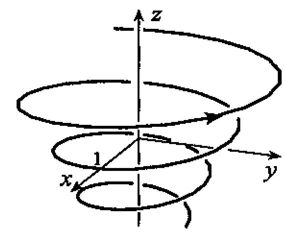
$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 1 \\ -2e^t\sin t & 2e^t\cos t & 0 \end{vmatrix} = (-2e^t\cos t, -2e^t\sin t, 2e^{2t}).$$

Consequently,

$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{\sqrt{4e^{2t}\cos^2 t + 4e^{2t}\sin^2 t + 4e^{4t}}}{[(e^t\cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + 1]^{3/2}}$$

$$= \frac{2e^t\sqrt{1+e^{2t}}}{(1+2e^{2t})^{3/2}},$$

and $\rho(t) = \dfrac{1}{\kappa} = \dfrac{(1+2e^{2t})^{3/2}}{2e^t\sqrt{1+e^{2t}}}$.

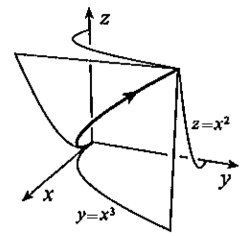


15. $\kappa(t) = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \dfrac{|(1, 3t^2, 2t) \times (0, 6t, 2)|}{|(1, 3t^2, 2t)|^3}$

$$= \frac{1}{(1+4t^2+9t^4)^{3/2}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3t^2 & 2t \\ 0 & 6t & 2 \end{matrix} \right\|$$

$$= \frac{|(-6t^2, -2, 6t)|}{(1+4t^2+9t^4)^{3/2}} = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}$$

and $\rho(t) = \dfrac{1}{\kappa} = \dfrac{(1+4t^2+9t^4)^{3/2}}{2\sqrt{1+9t^2+9t^4}}$.

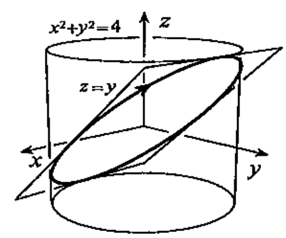


16. Since $\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2\sin t & 2\cos t & 2\cos t \\ -2\cos t & -2\sin t & -2\sin t \end{vmatrix} = (0, -4, 4)$,

$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{\sqrt{16+16}}{(4\sin^2 t + 4\cos^2 t + 4\cos^2 t)^{3/2}}$$

$$= \frac{1}{\sqrt{2}(1+\cos^2 t)^{3/2}},$$

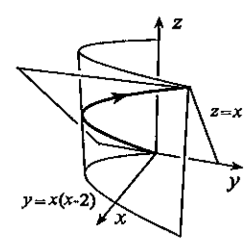
and $\rho(t) = \dfrac{1}{\kappa} = \sqrt{2}(1+\cos^2 t)^{3/2}$.



17. $\kappa(t) = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \dfrac{|(1, 2t, 1) \times (0, 2, 0)|}{|(1, 2t, 1)|^3} = \dfrac{1}{(2+4t^2)^{3/2}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 1 \\ 0 & 2 & 0 \end{matrix} \right\|$

$$= \frac{|(-2, 0, 2)|}{(2+4t^2)^{3/2}} = \frac{1}{(1+2t^2)^{3/2}}$$

and $\rho(t) = \dfrac{1}{\kappa} = (1+2t^2)^{3/2}$.

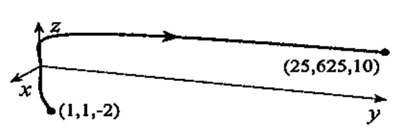


18. $\kappa(t) = \dfrac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \dfrac{|(2t, 4t^3, 2) \times (2, 12t^2, 0)|}{|(2t, 4t^3, 2)|^3} = \dfrac{1}{(4t^2+16t^6+4)^{3/2}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2t & 4t^3 & 2 \\ 2 & 12t^2 & 0 \end{matrix} \right\|$

$$= \frac{|(-24t^2, 4, 16t^3)|}{8(1+t^2+4t^6)^{3/2}} = \frac{4\sqrt{36t^4+1+16t^6}}{8(1+t^2+4t^6)^{3/2}}$$

$$= \frac{\sqrt{1+36t^4+16t^6}}{2(1+t^2+4t^6)^{3/2}},$$

and $\rho(t) = \dfrac{1}{\kappa} = \dfrac{2(1+t^2+4t^6)^{3/2}}{\sqrt{1+36t^4+16t^6}}$.



19. The curvature is a maximum at the points $(\pm a, 0)$ and a minumum at $(0, \pm b)$.

20. For a smooth curve $\mathbf{r} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$,

$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(\dot{x}, \dot{y}, 0) \times (\ddot{x}, \ddot{y}, 0)|}{|(\dot{x}, \dot{y}, 0)|^3} = \frac{1}{(\dot{x}^2+\dot{y}^2)^{3/2}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{matrix} \right\|$$

$$= \frac{1}{(\dot{x}^2 + \dot{y}^2)^{3/2}}|(0, 0, \dot{x}\ddot{y} - \ddot{x}\dot{y})| = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|\dot{y}\ddot{x} - \dot{x}\ddot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

21. According to equation 11.90, curvature is identically equal to zero when $0 = \kappa(s) = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \left|\frac{d^2\mathbf{r}}{ds^2}\right|$. But for $d^2\mathbf{r}/ds^2 = \mathbf{0}$, we must have $\mathbf{r} = \mathbf{C}s + \mathbf{D}$, where $\mathbf{C}$ and $\mathbf{D}$ are constant vectors. This defines a straight line.

22. Suppose (x, y) is a point of inflection at which $f''(x) = 0$. Because

$$f''(x) = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3} \qquad \text{(see Exercise 9.1–38)},$$

it follows from Exercise 20 that at such a point of inflection, $\kappa = 0$. If (x, y) is a point of inflection at which $f''(x)$ does not exist, then neither does the curvature.

23. (a) $\hat{\mathbf{T}} = \dfrac{(1, 2t)}{\sqrt{1 + 4t^2}}$ According to Example 11.47,

$$\hat{\mathbf{N}} = \operatorname{sgn}[(2t)(0) - (1)(2)]\frac{(2t, -1)}{\sqrt{1 + 4t^2}} = \frac{(-2t, 1)}{\sqrt{1 + 4t^2}}.$$

The direction of the binormal is $\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 0 \\ -2t & 1 & 0 \end{vmatrix} = (0, 0, 1 + 4t^2)$. Thus, $\hat{\mathbf{B}} = (0, 0, 1)$.

(b) $F_T = \mathbf{F} \cdot \hat{\mathbf{T}} = (t^2, t^4) \cdot \dfrac{(1, 2t)}{\sqrt{1 + 4t^2}} = \dfrac{t^2 + 2t^5}{\sqrt{1 + 4t^2}}$; $F_N = \mathbf{F} \cdot \hat{\mathbf{N}} = (t^2, t^4) \cdot \dfrac{(-2t, 1)}{\sqrt{1 + 4t^2}} = \dfrac{t^4 - 2t^3}{\sqrt{1 + 4t^2}}$

(c) $\mathbf{F} = F_T\hat{\mathbf{T}} + F_N\hat{\mathbf{N}} = \dfrac{t^2 + 2t^5}{\sqrt{1 + 4t^2}}\hat{\mathbf{T}} + \dfrac{t^4 - 2t^3}{\sqrt{1 + 4t^2}}\hat{\mathbf{N}}$

24. (a) $\hat{\mathbf{T}} = \dfrac{(-2\sin t, 2\cos t)}{\sqrt{4\sin^2 t + 4\cos^2 t}} = (-\sin t, \cos t)$ According to Example 11.47,

$$\hat{\mathbf{N}} = \operatorname{sgn}[(2\cos t)(-2\cos t) - (-2\sin t)(-2\sin t)]\frac{(2\cos t, 2\sin t)}{\sqrt{4\sin^2 t + 4\cos^2 t}} = -(\cos t, \sin t).$$

The binormal is $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = (0, 0, 1)$.

(b)
$$\begin{aligned} F_T = \mathbf{F} \cdot \hat{\mathbf{T}} &= (x^2, y^2) \cdot (-\sin t, \cos t) = -x^2 \sin t + y^2 \cos t \\ &= -(2\cos t)^2 \sin t + (2\sin t)^2 \cos t = 2\sin 2t(\sin t - \cos t) \\ F_N = \mathbf{F} \cdot \hat{\mathbf{N}} &= (x^2, y^2) \cdot (-\cos t, -\sin t) = -x^2 \cos t - y^2 \sin t \\ &= -(2\cos t)^2 \cos t - (2\sin t)^2 \sin t = -4(\cos^3 t + \sin^3 t) \end{aligned}$$

(c) $\mathbf{F} = F_T\hat{\mathbf{T}} + F_N\hat{\mathbf{N}} = 2\sin 2t(\sin t - \cos t)\hat{\mathbf{T}} - 4(\cos^3 t + \sin^3 t)\hat{\mathbf{N}}$

25.
$$F_T = \mathbf{F} \cdot \hat{\mathbf{T}} = (t^2, 2t, -3) \cdot \frac{(1, 2t, 2t)}{\sqrt{1 + 8t^2}} = \frac{5t^2 - 6t}{\sqrt{1 + 8t^2}}$$

$$F_N = \mathbf{F} \cdot \hat{\mathbf{N}} = (t^2, 2t, -3) \cdot \frac{(-4t, 1, 1)}{\sqrt{2 + 16t^2}} = \frac{-4t^3 + 2t - 3}{\sqrt{2 + 16t^2}}$$

$$F_B = \mathbf{F} \cdot \hat{\mathbf{B}} = (t^2, 2t, -3) \cdot \frac{(0, -1, 1)}{\sqrt{2}} = \frac{-2t - 3}{\sqrt{2}}$$

$$\mathbf{F} = F_T\hat{\mathbf{T}} + F_N\hat{\mathbf{N}} + F_B\hat{\mathbf{B}} = \frac{5t^2 - 6t}{\sqrt{1 + 8t^2}}\hat{\mathbf{T}} + \frac{-4t^3 + 2t - 3}{\sqrt{2 + 16t^2}}\hat{\mathbf{N}} - \frac{2t + 3}{\sqrt{2}}\hat{\mathbf{B}}$$

26. $\hat{\mathbf{T}} = \dfrac{(-\sin t, \cos t, 1)}{\sqrt{\sin^2 t + \cos^2 t + 1}} = \dfrac{(-\sin t, \cos t, 1)}{\sqrt{2}}$ A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{(-\cos t, -\sin t, 0)}{\sqrt{2}},$$

and therefore the principal normal is $\hat{\mathbf{N}} = -(\cos t, \sin t, 0)$.

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{2}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1).$$

$$\begin{aligned} F_T = \mathbf{F} \cdot \hat{\mathbf{T}} &= (x, xy^2, 1) \cdot \frac{(-\sin t, \cos t, 1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(-x\sin t + xy^2 \cos t + 1) \\ &= \frac{1}{\sqrt{2}}(-\cos t\,\sin t + \cos^2 t\,\sin^2 t + 1) \\ F_N = \mathbf{F} \cdot \hat{\mathbf{N}} &= (x, xy^2, 1) \cdot (-\cos t, -\sin t, 0) = -(x\cos t + xy^2 \sin t) \\ &= -(\cos^2 t + \cos t\,\sin^3 t) \\ F_B = \mathbf{F} \cdot \hat{\mathbf{B}} &= (x, xy^2, 1) \cdot \frac{(\sin t, -\cos t, 1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x\sin t - xy^2 \cos t + 1) \\ &= \frac{1}{\sqrt{2}}(\cos t\,\sin t - \cos^2 t\,\sin^2 t + 1) \end{aligned}$$

$$\begin{aligned} \mathbf{F} = F_T\hat{\mathbf{T}} + F_N\hat{\mathbf{N}} + F_B\hat{\mathbf{B}} &= \frac{1}{\sqrt{2}}(1 - \cos t\,\sin t + \cos^2 t\,\sin^2 t)\hat{\mathbf{T}} \\ &\quad - \cos t(\cos t + \sin^3 t)\hat{\mathbf{N}} + \frac{1}{\sqrt{2}}(1 + \cos t\,\sin t - \cos^2 t\,\sin^2 t)\hat{\mathbf{B}} \end{aligned}$$

27. (a) Yes
(b) Since the circle of curvature is in the plane of $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ (see Figure 11.116 and Definition 11.20), and the centre is along $\hat{\mathbf{N}}$, it follows that $\hat{\mathbf{T}}$, the unit tangent vector to the curve, must also be tangent to the circle of curvature.
(c) They have the same curvature if and only if they have the same radius of curvature. The radius of curvature of the curve is ρ. The radius of curvature of a circle is its radius, and the radius of the circle of curvature is ρ.

28. If ϕ is the angle shown, then $\hat{\mathbf{T}} = (\cos\phi, \sin\phi)$. Consequently,

$$\kappa = \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \left|\left(-\sin\phi\frac{d\phi}{ds}, \cos\phi\frac{d\phi}{ds}\right)\right| = \left|\frac{d\phi}{ds}\right|.$$

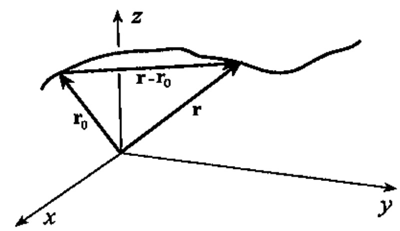
29. Since $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ form a set of three mutually perpendicular vectors at each point on a curve, there is no question that $\hat{\mathbf{N}}$ must be in the same direction as, or opposite to, $\hat{\mathbf{B}} \times \hat{\mathbf{T}}$; that is, $\hat{\mathbf{N}} = \lambda(\hat{\mathbf{B}} \times \hat{\mathbf{T}})$. The scalar product of this result with $\hat{\mathbf{N}}$ gives $1 = \hat{\mathbf{N}} \cdot \hat{\mathbf{N}} = \lambda(\hat{\mathbf{N}} \cdot \hat{\mathbf{B}} \times \hat{\mathbf{T}})$. But the scalar triple product $\hat{\mathbf{N}} \cdot \hat{\mathbf{B}} \times \hat{\mathbf{T}}$ is the volume of the rectangular parallelepiped formed by the vectors $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{T}}$ (see Exercise 11.4–44). Since this volume is 1, we obtain $\lambda = 1$. Thus, $\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}}$, and

$$\frac{d\hat{\mathbf{N}}}{ds} = \frac{d\hat{\mathbf{B}}}{ds} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \frac{d\hat{\mathbf{T}}}{ds} = -\tau\hat{\mathbf{N}} \times \hat{\mathbf{T}} + \hat{\mathbf{B}} \times \kappa\hat{\mathbf{N}} = -\tau(\hat{\mathbf{N}} \times \hat{\mathbf{T}}) + \kappa(\hat{\mathbf{B}} \times \hat{\mathbf{N}}).$$

Now $\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$ so that $-\hat{\mathbf{B}} = \hat{\mathbf{N}} \times \hat{\mathbf{T}}$. By a similar proof to the above we can show that $\hat{\mathbf{T}} = \hat{\mathbf{N}} \times \hat{\mathbf{B}}$. Hence, $\dfrac{d\hat{\mathbf{N}}}{ds} = -\tau(-\hat{\mathbf{B}}) + \kappa(-\hat{\mathbf{T}}) = \tau\hat{\mathbf{B}} - \kappa\hat{\mathbf{T}}$.

30. According to the second Frenet-Serret formula, torsion along a curve vanishes if and only if the binormal vector is constant. We therefore prove that a curve lies in a plane if and only if its binormal is constant. If a curve lies in a plane, then $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ will both lie in the plane. It follows that $\hat{\mathbf{B}}$ will be perpendicular to the plane, and will always therefore be in the same direction. Since its length is one, $\hat{\mathbf{B}}$ must be a constant vector.

Conversely, suppose now that $\hat{\mathbf{B}}$ is constant along a curve. Let $\mathbf{r}(s)$ denote the position vector for points along the curve where s is length along the curve measured from some initial point denoted by $\mathbf{r}_0 = \mathbf{r}(0)$. Consider



$$\frac{d}{ds}[\hat{\mathbf{B}} \cdot (\mathbf{r} - \mathbf{r}_0)] = \frac{d\hat{\mathbf{B}}}{ds} \cdot (\mathbf{r} - \mathbf{r}_0) + \hat{\mathbf{B}} \cdot \frac{d\mathbf{r}}{ds}$$
$$= 0 + \hat{\mathbf{B}} \cdot \hat{\mathbf{T}} = 0.$$

This implies that $\hat{\mathbf{B}} \cdot (\mathbf{r} - \mathbf{r}_0)$ is constant along the curve. Since it has value 0 at $\mathbf{r} = \mathbf{r}_0$, it has value 0 everywhere. This means that $\mathbf{r} - \mathbf{r}_0$ is perpendicular to $\hat{\mathbf{B}}$ for every point on the curve. In other words, the curve must lie in the plane containing $\mathbf{r}_0$ and perpendicular to $\hat{\mathbf{B}}$.

EXERCISES 11.13

1. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{t}{\sqrt{t^2+1}}\hat{\mathbf{i}} + \dfrac{2t^2+1}{\sqrt{t^2+1}}\hat{\mathbf{j}} = \dfrac{t\hat{\mathbf{i}} + (2t^2+1)\hat{\mathbf{j}}}{\sqrt{t^2+1}}$; $\quad |\mathbf{v}| = \dfrac{1}{\sqrt{t^2+1}}\sqrt{t^2 + (2t^2+1)^2} = \sqrt{\dfrac{4t^4+5t^2+1}{t^2+1}}$

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{1}{(t^2+1)^{3/2}}\hat{\mathbf{i}} + \dfrac{2t^3+3t}{(t^2+1)^{3/2}}\hat{\mathbf{j}} = \dfrac{\hat{\mathbf{i}} + (2t^3+3t)\hat{\mathbf{j}}}{(t^2+1)^{3/2}}$

2. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \left(1 - \dfrac{1}{t^2}\right)\hat{\mathbf{i}} + \left(1 + \dfrac{1}{t^2}\right)\hat{\mathbf{j}} = \dfrac{1}{t^2}[(t^2-1)\hat{\mathbf{i}} + (t^2+1)\hat{\mathbf{j}}]$

$|\mathbf{v}| = \dfrac{1}{t^2}\sqrt{(t^2-1)^2 + (t^2+1)^2} = \dfrac{\sqrt{2t^4+2}}{t^2}$; $\quad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{2}{t^3}\hat{\mathbf{i}} - \dfrac{2}{t^3}\hat{\mathbf{j}} = \dfrac{2}{t^3}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$

3. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \cos t\,\hat{\mathbf{i}} - 3\sin t\,\hat{\mathbf{j}} + \cos t\,\hat{\mathbf{k}}$; $\quad |\mathbf{v}| = \sqrt{\cos^2 t + 9\sin^2 t + \cos^2 t} = \sqrt{2 + 7\sin^2 t}$;

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -\sin t\,\hat{\mathbf{i}} - 3\cos t\,\hat{\mathbf{j}} - \sin t\,\hat{\mathbf{k}}$

4. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 2t\hat{\mathbf{i}} + 2e^t(t+1)\hat{\mathbf{j}} - \dfrac{2}{t^3}\hat{\mathbf{k}}$; $\quad |\mathbf{v}| = \sqrt{4t^2 + 4e^{2t}(t+1)^2 + 4/t^6} = 2\sqrt{t^2 + e^{2t}(t+1)^2 + 1/t^6}$;

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 2\hat{\mathbf{i}} + 2e^t(t+2)\hat{\mathbf{j}} + \dfrac{6}{t^4}\hat{\mathbf{k}}$

5. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -2te^{-t^2}\hat{\mathbf{i}} + (\ln t + 1)\hat{\mathbf{j}}$; $\quad |\mathbf{v}| = \sqrt{4t^2e^{-2t^2} + (\ln t + 1)^2}$; $\quad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 2e^{-t^2}(2t^2-1)\hat{\mathbf{i}} + (1/t)\hat{\mathbf{j}}$

6. If $\mathbf{a} = 3t^2\hat{\mathbf{i}} + (t+1)\hat{\mathbf{j}} - 4t^3\hat{\mathbf{k}}$, then $\mathbf{v} = t^3\hat{\mathbf{i}} + \left(\dfrac{t^2}{2} + t\right)\hat{\mathbf{j}} - t^4\hat{\mathbf{k}} + \mathbf{C}$. Since $\mathbf{v}(0) = \mathbf{0}$, it follows that $\mathbf{C} = \mathbf{0}$, and $\mathbf{v} = t^3\hat{\mathbf{i}} + \left(\dfrac{t^2}{2} + t\right)\hat{\mathbf{j}} - t^4\hat{\mathbf{k}}$. Integration now gives $\mathbf{r} = \dfrac{t^4}{4}\hat{\mathbf{i}} + \left(\dfrac{t^3}{6} + \dfrac{t^2}{2}\right)\hat{\mathbf{j}} - \dfrac{t^5}{5}\hat{\mathbf{k}} + \mathbf{D}$. Since $\mathbf{r}(0) = (1, 2, -1)$, we find $(1, 2, -1) = \mathbf{D}$, and $\mathbf{r} = \left(\dfrac{t^4}{4} + 1\right)\hat{\mathbf{i}} + \left(\dfrac{t^3}{6} + \dfrac{t^2}{2} + 2\right)\hat{\mathbf{j}} - \left(\dfrac{t^5}{5} + 1\right)\hat{\mathbf{k}}$.

7. If $\mathbf{a} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}/(t+1)^3$, then $\mathbf{v} = 3t\hat{\mathbf{i}} - \dfrac{\hat{\mathbf{j}}}{2(t+1)^2} + \mathbf{C}$. Since $\mathbf{v}(0) = \mathbf{0}$, it follows that $\mathbf{0} = -\hat{\mathbf{j}}/2 + \mathbf{C}$, and $\mathbf{v} = 3t\hat{\mathbf{i}} - \dfrac{\hat{\mathbf{j}}}{2(t+1)^2} + \dfrac{\hat{\mathbf{j}}}{2}$. Integration now gives $\mathbf{r} = \dfrac{3t^2}{2}\hat{\mathbf{i}} + \dfrac{\hat{\mathbf{j}}}{2(t+1)} + \dfrac{t\hat{\mathbf{j}}}{2} + \mathbf{D}$. Since $\mathbf{r}(0) = (1, 2, -1)$, we find $(1, 2, -1) = \hat{\mathbf{j}}/2 + \mathbf{D} \implies \mathbf{D} = (1, 3/2, -1)$. Thus, $\mathbf{r} = \left(\dfrac{3t^2}{2} + 1\right)\hat{\mathbf{i}} + \left(\dfrac{1}{2t+2} + \dfrac{t}{2} + \dfrac{3}{2}\right)\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

8. The velocity and acceleration are $\mathbf{v} = \hat{\mathbf{i}} + 2t\hat{\mathbf{j}}$ and $\mathbf{a} = 2\hat{\mathbf{j}}$. According to equation 11.112b, the tangential component of acceleration is

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}\sqrt{1+4t^2} = \frac{4t}{\sqrt{1+4t^2}}.$$

According to equation 11.113, the normal component of acceleration is

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{4 - \frac{16t^2}{1+4t^2}} = \sqrt{\frac{4}{1+4t^2}} = \frac{2}{\sqrt{1+4t^2}}.$$

9. The velocity and acceleration are $\mathbf{v} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{a} = -\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}}$. According to equation 11.112b, the tangential component of acceleration is

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}\sqrt{\sin^2 t + \cos^2 t + 1} = 0.$$

According to equation 11.113, the normal component of acceleration is

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{\cos^2 t + \sin^2 t} = 1.$$

10. From equation 11.112b, $a_N = |\mathbf{v}|\left|\dfrac{d\hat{\mathbf{T}}}{dt}\right| = |\mathbf{v}|\left|\dfrac{d\hat{\mathbf{T}}}{ds}\right|\left|\dfrac{ds}{dt}\right| = |\mathbf{v}|\kappa|\mathbf{v}| = \dfrac{|\mathbf{v}|^2}{\rho}.$

11. (a) $\text{KE} = \dfrac{1}{2}m|\mathbf{v}|^2 = \dfrac{1}{2}\left(\dfrac{2}{1000}\right)\left(\dfrac{4t^4+5t^2+1}{t^2+1}\right) = \dfrac{4t^4+5t^2+1}{1000(t^2+1)}$ J

(b) $\text{KE} = \dfrac{1}{2}m|\mathbf{v}|^2 = \dfrac{1}{2}\left(\dfrac{2}{1000}\right)\left(\dfrac{2t^4+2}{t^4}\right) = \dfrac{t^4+1}{500t^4}$ J

(c) $\text{KE} = \dfrac{1}{2}m|\mathbf{v}|^2 = \dfrac{1}{2}\left(\dfrac{2}{1000}\right)(2+7\sin^2 t) = \dfrac{2+7\sin^2 t}{1000}$ J

(d) $\text{KE} = \dfrac{1}{2}m|\mathbf{v}|^2 = \dfrac{1}{2}\left(\dfrac{2}{1000}\right)4\left[t^2 + e^{2t}(t+1)^2 + \dfrac{1}{t^6}\right] = \dfrac{1}{250}\left[t^2 + e^{2t}(t+1)^2 + \dfrac{1}{t^6}\right]$ J

(e) $\text{KE} = \dfrac{1}{2}m|\mathbf{v}|^2 = \dfrac{1}{2}\left(\dfrac{2}{1000}\right)\left[4t^2e^{-2t^2} + (\ln t+1)^2\right] = \dfrac{4t^2e^{-t^2} + (\ln t+1)^2}{1000}$ J

12. (a) If $d^2y/dt^2 = 2$, then $dy/dt = 2t + C$. Since $y'(0) = 0$, $C = 0$, and $dy/dt = 2t$. Thus, $y(t) = t^2 + D$. Since $y(0) = 0$, $D = 0$, and $y(t) = t^2$. Consequently, $4t^2 = x^2$, from which $x = 2t$. Thus, $dx/dt = 2$ and $d^2x/dt^2 = 0$.

(b) If $d^2x/dt^2 = 24t^2$, then $dx/dt = 8t^3 + C$. Since $x'(0) = 0$, $C = 0$ and $dx/dt = 8t^3$. Hence, $x(t) = 2t^4 + D$. Since $x(0) = 0$, $D = 0$ and $x(t) = 2t^4$. From $4y = x^2$, we obtain $y = x^2/4 = t^8$. Differentiation now gives $dy/dt = 8t^7$ and $d^2y/dt^2 = 56t^6$.

13. Velocity and displacement will be parallel if for some value of λ,

$$\mathbf{v} = \lambda\mathbf{r} \quad\Longrightarrow\quad \mathbf{v} = \hat{\mathbf{i}} + (3t^2 - 6t + 2)\hat{\mathbf{j}} = \lambda[t\hat{\mathbf{i}} + (t^3 - 3t^2 + 2t)\hat{\mathbf{j}}].$$

When we equate components, $1 = \lambda t$, $3t^2 - 6t + 2 = \lambda(t^3 - 3t^2 + 2t)$. Substituting $\lambda = 1/t$ into the second leads to the equation $2t^2 - 3t = 0$ with solutions $t = 0, 3/2$. Since $\mathbf{r}(0) = \mathbf{0}$, we cannot discuss parallelism at $t = 0$. The position of the particle at $t = 3/2$ is $(3/2, -3/8)$.

14. If $dx/dt = 5$, then $dy/dt = 3x^2\,dx/dt - 2dx/dt = 5(3x^2 - 2)$. The acceleration of the particle is therefore

$$\mathbf{a} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} = (0)\hat{\mathbf{i}} + 5\left(6x\frac{dx}{dt}\right)\hat{\mathbf{j}} = 30x(5)\hat{\mathbf{j}} = 150x\hat{\mathbf{j}}.$$

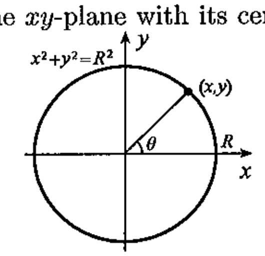
15. Antidifferentiation of the acceleration gives $\mathbf{v}(t) = -t^5\hat{\mathbf{i}} - \left(\dfrac{t^4}{2} + t\right)\hat{\mathbf{j}} + \mathbf{C}$. Since $\mathbf{v}(0) = \mathbf{0}$, it follows that $\mathbf{C} = \mathbf{0}$. The speed of the particle at $t = 2$ is $|\mathbf{v}(2)| = \sqrt{2^{10} + 100} = 2\sqrt{281}$.

16. Parametric equations for the particle's path are $x = h + R\cos\theta$, $y = k + R\sin\theta$, in which case $\mathbf{v} = \left(-R\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(R\cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}}$. Thus, $|\mathbf{v}| = \sqrt{R^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + R^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2} = R\frac{d\theta}{dt} = R\omega$.

17. If we choose $t \geq 0$ and $t = 0$ when the particle is at the point $(2, 0)$, its position can be described by $x = 2\cos(4\pi t)$, $y = 2\sin(4\pi t)$. It is at the point $(1, -\sqrt{3})$ when $1 = 2\cos(4\pi t)$, and $-\sqrt{3} = 2\sin(4\pi t)$. This happens for the first time at $t = 5/12$ s. The velocity of the particle at this time is

$$\mathbf{v}(5/12) = -8\pi\sin\left(\frac{5\pi}{3}\right)\hat{\mathbf{i}} + 8\pi\cos\left(\frac{5\pi}{3}\right)\hat{\mathbf{j}} = 4\pi(\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{j}}) \text{ m/s}.$$

18. (a) We choose a coordinate system so that the circle lies in the xy-plane with its centre at the origin. Then, $x = R\cos\theta$, $y = R\sin\theta$, and



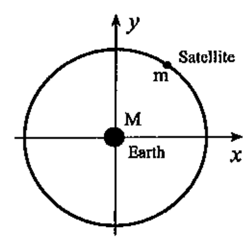
$$\mathbf{v} = \left(-R\sin\theta\frac{d\theta}{dt}, R\cos\theta\frac{d\theta}{dt}\right) = \omega R(-\sin\theta, \cos\theta),$$

where $\omega = d\theta/dt$. Since $|\mathbf{v}| = \omega R$ (see Exercise 16), and $|\mathbf{v}|$ is constant, so also is ω. Hence,

$$\mathbf{a} = \omega R\left(-\cos\theta\frac{d\theta}{dt}, -\sin\theta\frac{d\theta}{dt}\right) = -\omega^2 R(\cos\theta, \sin\theta),$$

and $|\mathbf{a}| = \omega^2 R = \left(\frac{|\mathbf{v}|^2}{R^2}\right)R = \frac{|\mathbf{v}|^2}{R}$.

(b) According to Newton's universal law of gravitation, the magnitude of the force of the earth on the satellite is



$$|\mathbf{F}| = \frac{GMm}{r^2}.$$

According to Newton's second law, $\mathbf{F} = m\mathbf{a}$, and therefore

$$m|\mathbf{a}| = \frac{GMm}{r^2} \implies |\mathbf{a}| = \frac{GM}{r^2}.$$

But from part (a), $|\mathbf{a}| = |\mathbf{v}|^2/r$, and therefore $\frac{|\mathbf{v}|^2}{r} = \frac{GM}{r^2}$, from which

$$|\mathbf{v}| = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67 \times 10^{-11}(4/3)\pi(6370 \times 10^3)^3(5.52 \times 10^3)}{6570 \times 10^3}} = 7.79 \times 10^3.$$

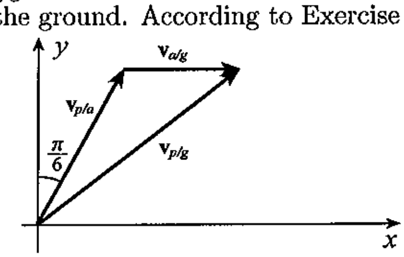
The speed of the satellite is therefore 7.79 km/s.

19. Differentiation of $\mathbf{OP}_1 + \mathbf{P}_1\mathbf{P}_2 = \mathbf{OP}_2$ gives

$$\frac{d\mathbf{OP}_1}{dt} + \frac{d\mathbf{P}_1\mathbf{P}_2}{dt} = \frac{d\mathbf{OP}_2}{dt} \implies \mathbf{v}_{P_1/O} + \mathbf{v}_{P_2/P_1} = \mathbf{v}_{P_2/O}.$$

Since $\mathbf{v}_{P_2/O} = -\mathbf{v}_{O/P_2}$ and $\mathbf{v}_{P_2/P_1} = -\mathbf{v}_{P_1/P_2}$, the above equation can indeed be expressed in the form $\mathbf{v}_{P_1/O} + \mathbf{v}_{O/P_2} = \mathbf{v}_{P_1/P_2}$.

20. Let $\mathbf{v}_{\mathrm{p/a}}$ be the velocity of the plane with respect to the air, $\mathbf{v}_{\mathrm{a/g}}$ the velocity of the air with respect to the ground, and $\mathbf{v}_{\mathrm{p/g}}$ the velocity of the plane with respect to the ground. According to Exercise 19,



$$\mathbf{v}_{\mathrm{p/a}} + \mathbf{v}_{\mathrm{a/g}} = \mathbf{v}_{\mathrm{p/g}},$$

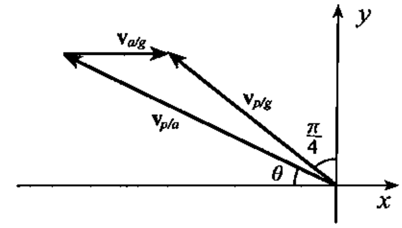
where $\mathbf{v}_{\mathrm{p/a}} = 650[\cos(\pi/3)\hat{\mathbf{i}} + \sin(\pi/3)\hat{\mathbf{j}}] = 325(\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}})$, and $\mathbf{v}_{\mathrm{a/g}} = 40\hat{\mathbf{i}}$. Thus,

$$\mathbf{v}_{\mathrm{p/g}} = 325(\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}}) + 40\hat{\mathbf{i}} = 365\hat{\mathbf{i}} + 325\sqrt{3}\hat{\mathbf{j}} \text{ km/hr},$$

and $|\mathbf{v}_{\mathrm{p/g}}| = \sqrt{365^2 + (325\sqrt{3})^2} = 670.9$ km/hr.

21. Let $\mathbf{v}_{\mathrm{p/a}}$ be the velocity of the plane with respect to the air, $\mathbf{v}_{\mathrm{a/g}}$ the velocity of the air with respect to the ground, and $\mathbf{v}_{\mathrm{p/g}}$ the velocity of the plane with respect to the ground. According to Exercise 19, $\mathbf{v}_{\mathrm{p/a}} + \mathbf{v}_{\mathrm{a/g}} = \mathbf{v}_{\mathrm{p/g}}$, where

$$\mathbf{v}_{\mathrm{p/a}} = 600[-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}], \qquad \mathbf{v}_{\mathrm{a/g}} = 50\hat{\mathbf{i}},$$
$$\mathbf{v}_{\mathrm{p/g}} = v[-\cos(\pi/4)\hat{\mathbf{i}} + \sin(\pi/4)\hat{\mathbf{j}}] = \frac{v}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$



where v is the speed of the plane with respect to the ground. When we substitute these into the above equation

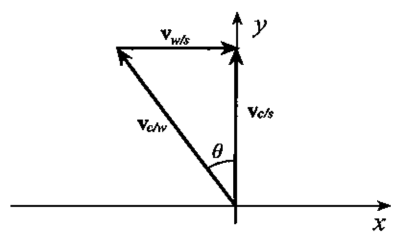
$$600(-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + 50\hat{\mathbf{i}} = \frac{v}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$

and equate components, $-600\cos\theta + 50 = -\dfrac{v}{\sqrt{2}}$, $600\sin\theta = \dfrac{v}{\sqrt{2}}$. Eliminating v leads to the equation $\cos\theta - \sin\theta = 1/12$, which we square

$$\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta = \frac{1}{144} \quad\Longrightarrow\quad \sin 2\theta = \frac{143}{144}.$$

The appropriate angle satisfying this equation (between 0 and $\pi/4$) is 0.726 radians. The plane should therefore take the bearing of west 0.726 radians north. Ground speed of the plane is $600\sqrt{2}\sin(0.726)$, which when divided into 1000 results in a trip time of 1.8 hours.

22. Let us take the positive x-direction as the direction in which the river flows. Then the velocity of the water relative to the shore is $\mathbf{v}_{\mathrm{w/s}} = 3\hat{\mathbf{i}}$. If the canoe points in direction θ as shown, $\mathbf{v}_{\mathrm{c/w}} = -4\sin\theta\,\hat{\mathbf{i}} + 4\cos\theta\,\hat{\mathbf{j}}$. If v is the speed of the canoe with respect to the shore, then $\mathbf{v}_{\mathrm{c/s}} = v\hat{\mathbf{j}}$. Since $\mathbf{v}_{\mathrm{c/s}} = \mathbf{v}_{\mathrm{c/w}} + \mathbf{v}_{\mathrm{w/s}}$, we obtain $v\,\hat{\mathbf{j}} = -4\sin\theta\,\hat{\mathbf{i}} + 4\cos\theta\,\hat{\mathbf{j}} + 3\hat{\mathbf{i}}$. When we equate components, $-4\sin\theta + 3 = 0$ and $4\cos\theta = v$. Consequently, $\theta = \mathrm{Sin}^{-1}(3/4)$ radians, and $v = 4\sqrt{1 - 9/16} = \sqrt{7}$ km/hr. To cross the river takes $(200/1000)/v = 1/(5\sqrt{7})$ hr or $12/\sqrt{7}$ min.

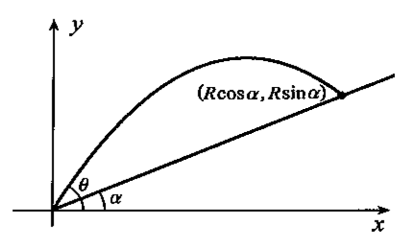


23. (a) The acceleration of the cannon ball is $\mathbf{a} = -g\hat{\mathbf{j}}$ so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take as $t = 0$ the time when the cannon ball leaves the cannon, then when the cannon ball is fired at angle θ,

$$\mathbf{v}(0) = S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}),$$

from which

$$S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = \mathbf{C}.$$



Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + St(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (St\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + St\sin\theta\right)\hat{\mathbf{j}}.$$

The cannon ball strikes the ground at position $(R\cos\alpha, R\sin\alpha)$. When we equate $R\cos\alpha$ and $R\sin\alpha$ to the x- and y-components of the displacement vector,

$$R\cos\alpha = St\cos\theta, \qquad R\sin\alpha = -\frac{1}{2}gt^2 + St\sin\theta.$$

Substituting $t = R\cos\alpha/(S\cos\theta)$ into the second gives

$$R\sin\alpha = -\frac{1}{2}g\left(\frac{R\cos\alpha}{S\cos\theta}\right)^2 + S\left(\frac{R\cos\alpha}{S\cos\theta}\right)\sin\theta.$$

When this is solved for R, the required expression is obtained.
(b) The function $R(\theta)$ must be maximized on the interval $\alpha \le \theta \le \pi/2$. For critical points of the function, we solve

$$0 = R'(\theta) = \frac{2S^2}{g\cos^2\alpha}[-\sin\theta\sin(\theta-\alpha)+\cos\theta\cos(\theta-\alpha)] = \frac{2S^2}{g\cos^2\alpha}\cos(2\theta-\alpha).$$

The only solution of this equation in the specified interval is $\theta = \pi/4+\alpha/2$. Since $\theta=\alpha$ and $\theta=\pi/2$ leads to $R=0$, maximum range is obtained for $\theta = \pi/4+\alpha/2$.

24. If its constant acceleration is $\mathbf{a}$, then $\mathbf{v} = \mathbf{a}t+\mathbf{C}$. If the particle starts at $t=0$, then $\mathbf{v}(0)=\mathbf{0}$, so that $\mathbf{C}=\mathbf{0}$, and $\mathbf{v}=\mathbf{a}t$. Integration now gives $\mathbf{r} = \mathbf{a}t^2/2+\mathbf{D}$. Since $\mathbf{r}(0) = (1,2,3)$, $(1,2,3)=\mathbf{D}$, and $\mathbf{r} = \mathbf{a}t^2/2+(1,2,3)$. For the particle to be at $(4,5,7)$ when $t=2$,

$$(4,5,7) = \frac{1}{2}\mathbf{a}(2)^2+(1,2,3) \quad\Longrightarrow\quad \mathbf{a} = \frac{1}{2}[(4,5,7)-(1,2,3)] = \left(\frac{3}{2},\frac{3}{2},2\right).$$

25. Since $\hat{\mathbf{T}} = \dfrac{(2t,4t,2t+5)}{\sqrt{4t^2+16t^2+(2t+5)^2}} = \dfrac{(2t,4t,2t+5)}{\sqrt{24t^2+20t+25}}$,

$$\begin{aligned}\frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(24t+10)}{(24t^2+20t+25)^{3/2}}(2t,4t,2t+5)+\frac{(2,4,2)}{\sqrt{24t^2+20t+5}}\\ &= \frac{1}{(24t^2+20t+25)^{3/2}}[-(24t+10)(2t,4t,2t+5)+(24t^2+20t+25)(2,4,2)]\\ &= \frac{10(2t+5,4t+10,-10t)}{(24t^2+20t+25)^{3/2}}.\end{aligned}$$

According to 11.112b,

$$\begin{aligned}a_N &= |(2t,4t,2t+5)|\left|\frac{10(2t+5,4t+10,-10t)}{(24t^2+20t+25)^{3/2}}\right|\\ &= \sqrt{24t^2+20t+25}\left[\frac{10\sqrt{(2t+5)^2+(4t+10)^2+100t^2}}{(24t^2+20t+25)^{3/2}}\right]\\ &= \frac{10\sqrt{5}}{\sqrt{24t^2+20t+25}}.\end{aligned}$$

Since $\mathbf{v} = (2t,4t,2t+5)$ and $\mathbf{a} = (2,4,2)$,

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}\sqrt{24t^2+20t+25} = \frac{24t+10}{\sqrt{24t^2+20t+25}}.$$

Equation 11.113 gives $a_N = \sqrt{24-\dfrac{(24t+10)^2}{24t^2+20t+25}} = \dfrac{10\sqrt{5}}{\sqrt{24t^2+20t+25}}$.

26. Since $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{-t}{\sqrt{1-t^2}}\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{a} = \left[\dfrac{-1}{\sqrt{1-t^2}}-\dfrac{t^2}{(1-t^2)^{3/2}}\right]\hat{\mathbf{i}} = \dfrac{-1}{(1-t^2)^{3/2}}\hat{\mathbf{i}}$, acceleration and velocity are perpendicular if, and when, $0 = \mathbf{v}\cdot\mathbf{a} = \left[\dfrac{-t}{\sqrt{1-t^2}}\right]\left[\dfrac{-1}{(1-t^2)^{3/2}}\right]$. This only occurs at $t=0$ when the particle is at position $(3,0)$.

27. (a)(i) When $\mathbf{a}$ is constant, integration of $d\mathbf{v}/dt = \mathbf{a}$ gives $\mathbf{v} = \mathbf{a}t+\mathbf{C}$. If we take $t=0$ as initial time, then $\mathbf{v}(0)=\mathbf{0}$, and this implies that $\mathbf{C}=\mathbf{0}$. Integration of $d\mathbf{r}/dt = \mathbf{a}t$ gives $\mathbf{r} = (1/2)\mathbf{a}t^2+\mathbf{D}$. When we equate components of this equation,

$$x = \frac{1}{2}a_xt^2+D_x, \quad y = \frac{1}{2}a_yt^2+D_y, \quad z = \frac{1}{2}a_zt^2+D_z.$$

Solving each for $t^2/2$ and equating results gives $\dfrac{x-D_x}{a_x}=\dfrac{y-D_y}{a_y}=\dfrac{z-D_z}{a_z}$. These are symmetric equations for a line.

(ii) If the acceleration is constant, then (as in part (a)), $\mathbf{v}=\mathbf{a}t+\mathbf{C}$. If $\mathbf{v}(0)=\mathbf{V}$, then $\mathbf{v}=\mathbf{a}t+\mathbf{V}$. Integration of this gives $\mathbf{r}=(1/2)\mathbf{a}t^2+\mathbf{V}t+\mathbf{D}$. When we equate components of this equation,

$$x=\frac{1}{2}a_xt^2+V_xt+D_x,\quad y=\frac{1}{2}a_yt^2+V_yt+D_y,\quad z=\frac{1}{2}a_zt^2+V_zt+D_z.$$

Since $\mathbf{V}$ and $\mathbf{a}$ are parallel, we can say that $\mathbf{a}=\lambda\mathbf{V}$; that is, $a_x=\lambda V_x$, $a_y=\lambda V_y$, and $a_z=\lambda V_z$. Thus,

$$x=\frac{1}{2}\lambda V_xt^2+V_xt+D_x,\quad y=\frac{1}{2}\lambda V_yt^2+V_yt+D_y,\quad z=\frac{1}{2}\lambda V_zt^2+V_zt+D_z.$$

Solving each of these for $\lambda t^2/2+t$ and equating results gives $\dfrac{x-D_x}{V_x}=\dfrac{y-D_y}{V_y}=\dfrac{z-D_z}{V_z}$. These are symmetric equations for a line.

(b) No. A stone thrown at an angle to the ground is subjected to the constant acceleration due to gravity, but it follows a parabolic path.

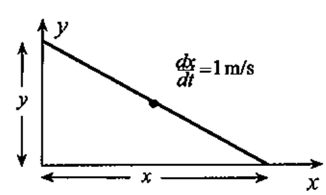
28. Because the acceleration $\mathbf{a}$ is constant, $\mathbf{v}=\mathbf{a}t+\mathbf{C}$. Since $\mathbf{v}(t_0)=\mathbf{v}_0$, it follows that $\mathbf{v}_0=\mathbf{a}t_0+\mathbf{C}$, and $\mathbf{v}=\mathbf{a}t+\mathbf{v}_0-\mathbf{a}t_0=\mathbf{a}(t-t_0)+\mathbf{v}_0$. Integration now gives $\mathbf{r}=\mathbf{a}(t-t_0)^2/2+\mathbf{v}_0t+\mathbf{D}$. Since $\mathbf{r}(t_0)=\mathbf{r}_0$, we obtain $\mathbf{r}_0=\mathbf{v}_0t_0+\mathbf{D}$, and therefore

$$\mathbf{r}=\frac{1}{2}\mathbf{a}(t-t_0)^2+\mathbf{v}_0t+\mathbf{r}_0-\mathbf{v}_0t_0=\frac{1}{2}\mathbf{a}(t-t_0)^2+\mathbf{v}_0(t-t_0)+\mathbf{r}_0.$$

29. (a) Coordinates of the middle point of the ladder are $(x/2,y/2)$. Differentiation of $x^2+y^2=64$ with respect to time t gives

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0\implies\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}.$$

When $x=3$ and $y=\sqrt{55}$, we find that $\dfrac{dy}{dt}=-\dfrac{3}{\sqrt{55}}$.



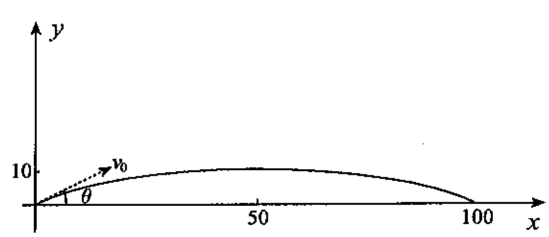
The velocity of the midpoint of the ladder is therefore $\dfrac{1}{2}\left(\hat{\mathbf{i}}-\dfrac{3\hat{\mathbf{j}}}{\sqrt{55}}\right)$ m/s. Since dx/dt is constant, the x-component of the acceleration is always 0. Furthermore,

$$\frac{d^2y}{dt^2}=-\left[\frac{y\left(\dfrac{dx}{dt}\right)-x\left(\dfrac{dy}{dt}\right)}{y^2}\right]\frac{dx}{dt}.$$

When $x=3$, $\dfrac{d^2y}{dt^2}=-\left[\dfrac{\sqrt{55}(1)-3(-3/\sqrt{55})}{55}\right](1)=-\dfrac{64}{55^{3/2}}$. Thus, the acceleration of the midpoint of the ladder is $-32\hat{\mathbf{j}}/55^{3/2}$ m/s^2.

(b) The limit of dy/dt as y approaches 0 is "infinity"; that is, the midpoint of the ladder strikes the floor infinitely fast.

30. The acceleration of the arrow is $\mathbf{a}=-g\hat{\mathbf{j}}$, so that $\mathbf{v}=-gt\hat{\mathbf{j}}+\mathbf{C}$. If we take $t=0$ to be the time when the arrow leaves the bow, then when the bow is held at angle θ, and the initial speed of the arrow is v_0, $\mathbf{v}(0)=v_0(\cos\theta\,\hat{\mathbf{i}}+\sin\theta\,\hat{\mathbf{j}})$. This gives $v_0(\cos\theta\,\hat{\mathbf{i}}+\sin\theta\,\hat{\mathbf{j}})=\mathbf{C}$. Integration of $\mathbf{v}=-gt\hat{\mathbf{j}}+\mathbf{C}$ gives $\mathbf{r}=-gt^2\hat{\mathbf{j}}/2+\mathbf{C}t+\mathbf{D}$.



If the arrow starts from the origin, then $\mathbf{r}(0)=\mathbf{0}$, from which $\mathbf{D}=\mathbf{0}$, and therefore

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + v_0t(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (v_0t\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + v_0t\sin\theta\right)\hat{\mathbf{j}}.$$

If T is the time for the arrow to reach maximum height, we can say that $\mathbf{r}(T) = 50\hat{\mathbf{i}} + 10\hat{\mathbf{j}}$ and $\mathbf{r}(2T) = 100\hat{\mathbf{i}}$. These imply that

$$50\hat{\mathbf{i}} + 10\hat{\mathbf{j}} = (v_0T\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gT^2 + v_0T\sin\theta\right)\hat{\mathbf{j}}, \quad 100\hat{\mathbf{i}} = [v_0(2T)\cos\theta]\hat{\mathbf{i}} + \left[-\frac{1}{2}g(2T)^2 + v_0(2T)\sin\theta\right]\hat{\mathbf{j}}.$$

When we equate components, we obtain four equations, three of which are independent,

$$50 = v_0T\cos\theta, \quad 10 = -\frac{1}{2}gT^2 + v_0T\sin\theta, \quad 0 = -2gT^2 + 2v_0T\sin\theta.$$

We eliminate T and solve for v_0 and θ. The result is $v_0 = 37.7$ m/s and $\theta = 0.381$ radians.

31. If we let T be the tension in the cord and apply Newton's second law to each of the masses, we obtain

$$M\frac{dv}{dt} = T - \frac{\mu vA}{h}, \qquad m\frac{dv}{dt} = mg - T.$$

Elimination of T gives

$$M\frac{dv}{dt} = mg - m\frac{dv}{dt} - \frac{\mu vA}{h} \quad\Longrightarrow\quad (m+M)\frac{dv}{dt} = mg - \frac{\mu vA}{h} \quad\Longrightarrow\quad \frac{dv}{mg - \mu Av/h} = \frac{1}{m+M}dt.$$

This is a separated differential equation with solutions defined implicitly by

$$-\frac{h}{\mu A}\ln\left|mg - \frac{\mu Av}{h}\right| = \frac{t}{m+M} + C \qquad\Longrightarrow\qquad \ln\left|mg - \frac{\mu Av}{h}\right| = -\frac{\mu At}{h(m+M)} - \frac{\mu AC}{h}.$$

Exponentiation gives

$$mg - \frac{\mu Av}{h} = Ee^{-\mu At/(hm+hM)}, \qquad \text{where } E = \pm e^{-\mu AC/h}.$$

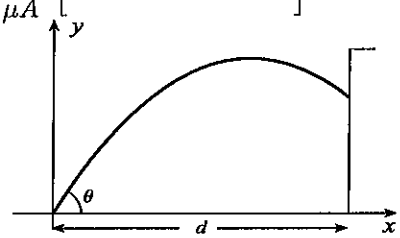
Since $v(0) = 0$, it follows that $mg = E$, and therefore

$$mg - \frac{\mu Av}{h} = mge^{-\mu At/(hm+hM)} \qquad\Longrightarrow\qquad v = \frac{hmg}{\mu A}\left[1 - e^{-\mu At/(hm+hM)}\right].$$

32. The acceleration of any droplet of water is $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take as $t = 0$ the time when the droplet leaves the nozzle, then when the hose is held at angle θ,

$$\mathbf{v}(0) = S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}),$$

from which $S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = \mathbf{C}$. Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and



$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + St(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (St\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + St\sin\theta\right)\hat{\mathbf{j}}.$$

The droplet strikes the building where $x = d$, in which case $d = St\cos\theta$. This equation implies that $t = d/(S\cos\theta)$, and this is the time that it strikes the building if it is fired at angle θ. The height it reaches on the building when it is fired at angle θ is therefore

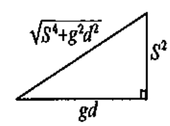
$$y(\theta) = -\frac{1}{2}g\left(\frac{d}{S\cos\theta}\right)^2 + S\sin\theta\left(\frac{d}{S\cos\theta}\right) = -\frac{gd^2}{2S^2\cos^2\theta} + d\tan\theta.$$

The problem then is to maximize $y(\theta)$ considering those values of θ which guarantee that the droplet does strike the building. There is a smallest value, say α, below which the droplet does not reach the wall, and a largest value, say β, beyond which the droplet also fails to reach the wall (and these values depend on d and S). Thus, we should maximize $y(\theta)$ for $\alpha \le \theta \le \beta$ where $y(\alpha) = y(\beta) = 0$. For critical values of $y(\theta)$, we solve

$$0 = \frac{dy}{d\theta} = \frac{gd^2}{S^2\cos^3\theta}(-\sin\theta) + d\sec^2\theta = \frac{d(-gd\sin\theta + S^2\cos\theta)}{S^2\cos^3\theta}.$$

Thus, $\tan\theta = S^2/(gd)$. We accept from this equation only the acute angle, and this must be the value which maximizes $y(\theta)$, so that maximum $y(\theta)$ is

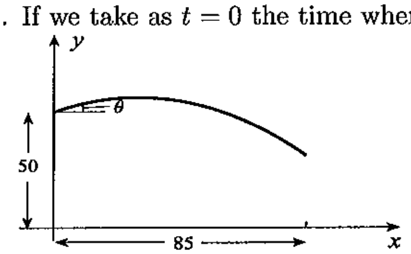
$$\frac{-gd^2}{2S^2}\left(\frac{S^4 + g^2d^2}{g^2d^2}\right) + \frac{dS^2}{gd} = \frac{S^4 - g^2d^2}{2gS^2}.$$



Notice that this result also implies that the water reaches the wall if, and only if,

$$S^4 - g^2d^2 > 0 \quad \text{or} \quad S^2 > gd.$$

33. The acceleration of the stone is $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take as $t = 0$ the time when the stone is thrown, then when it is thrown at angle θ,



$$\mathbf{v}(0) = 25(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}),$$

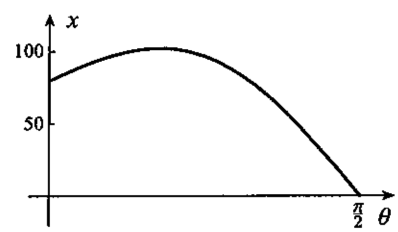
from which

$$25(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = \mathbf{C}.$$

Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. Since $\mathbf{r}(0) = 50\hat{\mathbf{j}}$, it follows that $\mathbf{D} = 50\hat{\mathbf{j}}$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + 25t(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + 50\hat{\mathbf{j}} = (25t\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + 25t\sin\theta + 50\right)\hat{\mathbf{j}}.$$

The stone hits water level when $y = 0$ in which case



$$0 = -\frac{1}{2}gt^2 + 25t\sin\theta + 50.$$

from which

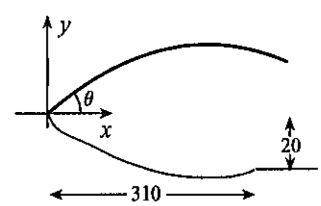
$$t = \frac{-25\sin\theta \pm \sqrt{625\sin^2\theta + 100g}}{-g}.$$

Using the positive solution, the x-coordinate of the stone at water level is

$$x = 25\cos\theta\left(\frac{25\sin\theta + \sqrt{625\sin^2\theta + 100g}}{g}\right).$$

A plot of this function shows that there are indeed angles for which $x \ge 85$.

34. The acceleration of the ball after it leaves the tee is $\mathbf{a} = -g\hat{\mathbf{j}}$ so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take $t = 0$ at the instant the ball is struck, and it begins at angle θ, then



$$\mathbf{v}(0) = S(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})$$

where $S > 0$ is its initial speed off the tee. This implies that $\mathbf{C} = \mathbf{v}(0)$. Integration gives

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \mathbf{C}t + \mathbf{D}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + St(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (St\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + St\sin\theta\right)\hat{\mathbf{j}}.$$

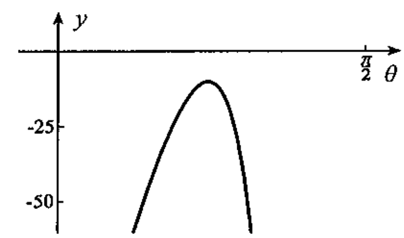
According to Exercise 23, maximum range along a level fairway is attained for $\theta = \pi/4$, and in this case

$$R = \frac{2S^2(1/\sqrt{2})(1/\sqrt{2})}{g} = \frac{S^2}{g}.$$

Since maximum range is 300 m, $300 = S^2/g$, or, $S = \sqrt{300g} = 54.25$ m/s. In other words, maximum speed of the ball off the tee is 54.25 m/s. The ball covers a horizontal displacement of 310 m when $310 = (54.25)t\cos\theta$, or $t = (310/54.25)\sec\theta$. The y-displacement at this instant is

$$y = -\frac{1}{2}g\left(\frac{310}{54.25}\sec\theta\right)^2 + 54.25\left(\frac{310}{54.25}\sec\theta\right)\sin\theta$$

$$= -\frac{(310)(155)g}{(54.25)^2}\sec^2\theta + 310\tan\theta.$$



A plot of this function shows that there are angles for which $y = -20$.

35. (a) The acceleration of the projectile is $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take $t = 0$ when the projectile begins its trajectory, then $\mathbf{v}(0) = v(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})$, and this implies that $\mathbf{C} = \mathbf{v}(0)$. Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. Since $\mathbf{r}(0) = h\hat{\mathbf{j}}$, it follows that $\mathbf{D} = h\hat{\mathbf{j}}$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + vt(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + h\hat{\mathbf{j}} = vt\cos\theta\,\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + vt\sin\theta + h\right)\hat{\mathbf{j}}.$$

The projectile hits the ground when $y = 0$, in which case

$$0 = -\frac{1}{2}gt^2 + vt\sin\theta + h \quad\Longrightarrow\quad t = \frac{-v\sin\theta \pm \sqrt{v^2\sin^2\theta + 2gh}}{-g}.$$

Using the positive solution, the range of the projectile is

$$R = v\cos\theta\left(\frac{v\sin\theta + \sqrt{v^2\sin^2\theta + 2gh}}{g}\right) = \frac{v^2\cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v^2}}\right).$$

(b) For critical points of $R(\theta)$, we solve

$$0 = \frac{dR}{d\theta} = \frac{v^2}{g}\left(-\sin^2\theta + \cos^2\theta - \sin\theta\sqrt{\sin^2\theta + \frac{2gh}{v^2}} + \frac{\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + 2gh/v^2}}\right).$$

From this equation,

$$(\sin^2\theta - \cos^2\theta)\sqrt{\sin^2\theta + \frac{2gh}{v^2}} = \sin\theta\cos^2\theta - \sin\theta\left(\sin^2\theta + \frac{2gh}{v^2}\right) = \sin\theta(\cos^2\theta - \sin^2\theta) - \frac{2gh}{v^2}\sin\theta,$$

from which $-\cos 2\theta\sqrt{\sin^2\theta + \dfrac{2gh}{v^2}} = \sin\theta\cos 2\theta - \dfrac{2gh}{v^2}\sin\theta$. Squaring gives

$$\cos^2 2\theta\left(\sin^2\theta + \frac{2gh}{v^2}\right) = \sin^2\theta\cos^2 2\theta - \frac{4gh}{v^2}\sin^2\theta\cos 2\theta + \frac{4g^2h^2}{v^4}\sin^2\theta,$$

from which

$$\cos^2 2\theta = \frac{2gh}{v^2}\sin^2\theta - 2\sin^2\theta\cos 2\theta = \left(\frac{2gh}{v^2} - 2\cos 2\theta\right)\left(\frac{1 - \cos 2\theta}{2}\right) = \frac{gh}{v^2} - \cos 2\theta - \frac{gh}{v^2}\cos 2\theta + \cos^2 2\theta.$$

Thus, $\cos 2\theta = \dfrac{gh/v^2}{1 + gh/v^2} = \dfrac{gh}{v^2 + gh}$. The only solution of this equation in the interval $0 < \theta < \pi/2$ is $\theta = \dfrac{1}{2}\mathrm{Cos}^{-1}\left(\dfrac{gh}{v^2 + gh}\right)$. It is geometrically clear that there is an angle between $\theta = 0$ and $\theta = \pi/2$ that maximizes R, and since only one critical point has been obtained, it must maximize R.

(c) For $v = 13.7$ and $h = 2.25$, $\theta = \dfrac{1}{2}\mathrm{Cos}^{-1}\left[\dfrac{9.81(2.25)}{13.7^2 + 9.81(2.25)}\right] = 0.733$ radians.

(d) The height of the projectile $y = -gt^2/2 + vt\sin\theta + h$ is a maximum when $0 = -gt + v\sin\theta \Longrightarrow t = (v/g)\sin\theta$; that is, maximum height is $-\dfrac{g}{2}\left(\dfrac{v}{g}\sin\theta\right)^2 + v\sin\theta\left(\dfrac{v}{g}\sin\theta\right) + h = h + \dfrac{v^2}{2g}\sin^2\theta$. From the formula for the range R in part (a), $\dfrac{gR}{v^2\cos\theta} - \sin\theta = \sqrt{\sin^2\theta + \dfrac{2gh}{v^2}}$. When this is squared,

$$\frac{g^2R^2}{v^4\cos^2\theta} - \frac{2gR\tan\theta}{v^2} + \sin^2\theta = \sin^2\theta + \frac{2gh}{v^2} \quad\Longrightarrow\quad \frac{v^2}{2g} = \frac{R^2}{4\cos^2\theta(h + R\tan\theta)}.$$

Maximum height is therefore $h + \dfrac{R^2\sin^2\theta}{4\cos^2\theta(h + R\tan\theta)} = h + \dfrac{R^2\tan^2\theta}{4(h + R\tan\theta)}$.

36. The acceleration of the projectile is $\mathbf{a} = -g\hat{\mathbf{j}}$, integration of which gives $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If $\mathbf{v}_0 = v_0\cos(\alpha+\beta)\,\hat{\mathbf{i}} + v_0\sin(\alpha+\beta)\,\hat{\mathbf{j}}$ is the initial velocity of the projectile at time $t = 0$ when it leaves the cannon, then $\mathbf{v}_0 = \mathbf{C}$, and $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{v}_0$. Integration gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{v}_0 t + \mathbf{D}$. Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and $\mathbf{r}(t) = -gt^2\hat{\mathbf{j}}/2 + \mathbf{v}_0 t$. In component form,

$$x = v_0\cos(\alpha+\beta)\,t, \qquad y = -\frac{1}{2}gt^2 + v_0\sin(\alpha+\beta)\,t.$$

The projectile strikes the inclined plane at a point satisfying $y = x\tan\alpha$, and this implies that

$$-\frac{1}{2}gt^2 + v_0\sin(\alpha+\beta)\,t = v_0\cos(\alpha+\beta)\tan\alpha\, t \quad\Longrightarrow\quad t = \frac{2v_0}{g}[\sin(\alpha+\beta) - \cos(\alpha+\beta)\tan\alpha].$$

Since the projectile strikes the ground horizontally, the y-component of velocity at the point of impact must be zero,

$$0 = -gt + v_0\sin(\alpha+\beta) \quad\Longrightarrow\quad t = \frac{v_0}{g}\sin(\alpha+\beta).$$

When we equate these two expressions for t,

$$\frac{2v_0}{g}[\sin(\alpha+\beta) - \cos(\alpha+\beta)\tan\alpha] = \frac{v_0}{g}\sin(\alpha+\beta) \quad\Longrightarrow\quad \sin(\alpha+\beta) = 2\cos(\alpha+\beta)\tan\alpha.$$

Multiplication by $\cos\alpha$ gives

$$(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\cos\alpha = 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\sin\alpha,$$

from which $\sin\alpha\cos\alpha\cos\beta = (\cos^2\alpha + 2\sin^2\alpha)\sin\beta$. This can be solved for

$$\tan\beta = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha + 2\sin^2\alpha} = \frac{(1/2)\sin 2\alpha}{(1+\cos 2\alpha)/2 + (1-\cos 2\alpha)} = \frac{\sin 2\alpha}{3 - \cos 2\alpha},$$

and therefore $\beta = \mathrm{Tan}^{-1}\left(\dfrac{\sin 2\alpha}{3 - \cos 2\alpha}\right)$.

37. From equation 11.108,

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} = \mathbf{r}\times\left[\frac{d}{dt}(m\mathbf{v})\right] = \frac{d}{dt}[\mathbf{r}\times(m\mathbf{v})] - \frac{d\mathbf{r}}{dt}\times(m\mathbf{v}) = \frac{d\mathbf{H}}{dt} - \mathbf{v}\times m\mathbf{v} = \frac{d\mathbf{H}}{dt}$$

since $\mathbf{v}\times m\mathbf{v} = \mathbf{0}$.

38. (a) Since length around the circumference of the tire is given by $R\theta$ and the time rate of change of this quantity is the speed of the centre of the tire, it follows that $S = R(d\theta/dt)$. Antidifferentiation gives $\theta = St/R + C$. Since $\theta = 0$ when $t = 0$, it follows that $C = 0$ and $\theta = St/R$.

(b) Since $x = R(\theta - \sin\theta)$ and $y = R(1 - \cos\theta)$,

$$\mathbf{v} = R\left(\frac{d\theta}{dt} - \cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(R\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}} = S[(1-\cos\theta)\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}]$$

$$|\mathbf{v}| = S\sqrt{(1-\cos\theta)^2 + \sin^2\theta} = S\sqrt{2 - 2\cos\theta}$$

$$\mathbf{a} = S\left(\sin\theta\frac{d\theta}{dt}\hat{\mathbf{i}} + \cos\theta\frac{d\theta}{dt}\hat{\mathbf{j}}\right) = \frac{S^2}{R}(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}).$$

(c) $$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{S\sin\theta}{\sqrt{2-2\cos\theta}}\frac{d\theta}{dt} = \frac{S^2\sin\theta}{R\sqrt{2-2\cos\theta}}$$

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T{}^2} = \sqrt{\frac{S^4}{R^2} - \frac{S^4\sin^2\theta}{R^2(2-2\cos\theta)}} = \frac{S^2}{R}\sqrt{(1-\cos\theta)/2}$$

39. (a) Since length around the circumference of the tire is given by $R\theta$ and the time rate of change of this quantity is the speed of the centre of the tire, it follows that $S = R(d\theta/dt)$. Antidifferentiation gives $\theta = St/R + C$. Since $\theta = 0$ when $t = 0$, it follows that $C = 0$ and $\theta = St/R$. Since $x = R\theta - b\sin\theta$ and $y = R - b\cos\theta$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(R\frac{d\theta}{dt} - b\cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(b\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}} = \frac{S}{R}\left[(R - b\cos\theta)\hat{\mathbf{i}} + b\sin\theta\hat{\mathbf{j}}\right]$$

$$|\mathbf{v}| = \frac{S}{R}\sqrt{(R-b\cos\theta)^2 + b^2\sin^2\theta} = \frac{S}{R}\sqrt{R^2 - 2Rb\cos\theta + b^2}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{S}{R}\left[\left(b\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(b\cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{j}}\right] = \frac{bS^2}{R^2}(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}).$$

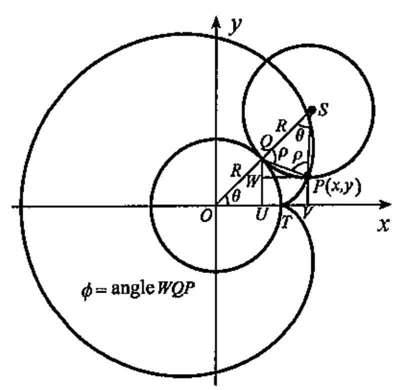
(b) $$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{S}{R}\frac{2Rb\sin\theta(d\theta/dt)}{2\sqrt{R^2 - 2Rb\cos\theta + b^2}} = \frac{bS^2}{R}\frac{\sin\theta}{\sqrt{R^2 - 2Rb\cos\theta + b^2}}$$

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \left(\frac{b^2S^4}{R^4} - \frac{b^2S^4}{R^2}\frac{\sin^2\theta}{R^2 - 2Rb\cos\theta + b^2}\right)^{1/2} = \frac{bS^2}{R^2}\left(1 - \frac{R^2\sin^2\theta}{R^2 - 2Rb\cos\theta + b^2}\right)^{1/2}$$

$$= \frac{bS^2}{R^2}\left[\frac{R^2 - 2Rb\cos\theta + b^2 - R^2(1-\cos^2\theta)}{R^2 - 2Rb\cos\theta + b^2}\right]^{1/2}$$

$$= \frac{bS^2}{R^2}\left[\frac{(-b + R\cos\theta)^2}{R^2 - 2Rb\cos\theta + b^2}\right]^{1/2} = \frac{bS^2}{R^2}\frac{|-b + R\cos\theta|}{\sqrt{R^2 - 2Rb\cos\theta + b^2}}$$

40. (a) $$x = R + \|TV\| = R + \|UV\| - \|UT\|$$
$$= R + \|PQ\|\sin\phi - (R - \|OU\|)$$
$$= \|PQ\|\sin\phi + R\cos\theta.$$



When $(\pi/2 - \theta) + \phi + \rho = \pi$, and $\theta + 2\rho = \pi$, are solved for ρ, and results are equated, we obtain $\phi = 3\theta/2$. Hence, with $\|PQ\| = 2R\sin(\theta/2)$,

$$x = 2R\sin(\theta/2)\sin(3\theta/2) + R\cos\theta$$
$$= R(-\cos 2\theta + \cos\theta + \cos\theta)$$
$$= R(2\cos\theta - \cos 2\theta).$$

Furthermore,

$$y = \|UQ\| - \|QW\| = R\sin\theta - \|PQ\|\cos\phi$$
$$= R\sin\theta - 2R\sin(\theta/2)\cos(3\theta/2)$$
$$= R(\sin\theta + \sin\theta - \sin 2\theta)$$
$$= R(2\sin\theta - \sin 2\theta).$$

(b) When the point of contact has moved so that it makes angle θ with the positive x-axis, length along the stationary circle from $(R, 0)$ to the point of contact is $R\theta$. Since this length changes at a rate of $Rd\theta/dt = S$, it follows that $\theta = St/R$.

(c) $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = R(-2\sin\theta + 2\sin 2\theta)\frac{d\theta}{dt}\hat{\mathbf{i}} + R(2\cos\theta - 2\cos 2\theta)\frac{d\theta}{dt}\hat{\mathbf{j}}$$
$$= 2S(\sin 2\theta - \sin\theta)\hat{\mathbf{i}} + 2S(\cos\theta - \cos 2\theta)\hat{\mathbf{j}};$$

$$
\begin{aligned}
|\mathbf{v}| &= 2S\sqrt{(\sin 2\theta - \sin\theta)^2 + (\cos\theta - \cos 2\theta)^2} = 2S\sqrt{2 - 2\sin 2\theta \sin\theta - 2\cos 2\theta\cos\theta} \\
&= 2\sqrt{2}S\sqrt{1-\cos\theta};
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a} &= 2S(2\cos 2\theta - \cos\theta)\frac{d\theta}{dt}\hat{\mathbf{i}} + 2S(-\sin\theta + 2\sin 2\theta)\frac{d\theta}{dt}\hat{\mathbf{j}} \\
&= \frac{2S^2}{R}[(2\cos 2\theta - \cos\theta)\hat{\mathbf{i}} + (2\sin 2\theta - \sin\theta)\hat{\mathbf{j}}].
\end{aligned}
$$

(d) $a_T = \dfrac{d}{dt}|\mathbf{v}| = \dfrac{\sqrt{2}S\sin\theta}{\sqrt{1-\cos\theta}}\dfrac{d\theta}{dt} = \dfrac{\sqrt{2}S^2\sin\theta}{R\sqrt{1-\cos\theta}};$

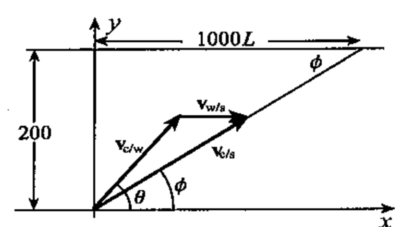
$$
\begin{aligned}
a_N &= \sqrt{|\mathbf{a}|^2 - a_T{}^2} = \sqrt{\frac{4S^4}{R^2}\left[(2\cos 2\theta - \cos\theta)^2 + (2\sin 2\theta - \sin\theta)^2\right] - \frac{2S^4\sin^2\theta}{R^2(1-\cos\theta)}} \\
&= \frac{S^2}{R}\sqrt{4[5 - 4\cos 2\theta\cos\theta - 4\sin 2\theta\sin\theta] - \frac{2\sin^2\theta}{1-\cos\theta}} \\
&= \frac{S^2}{R}\sqrt{20 - 16\cos\theta - \frac{2\sin^2\theta}{1-\cos\theta}} = \frac{S^2}{R}\sqrt{\frac{20 - 36\cos\theta + 16\cos^2\theta - 2(1-\cos^2\theta)}{1-\cos\theta}} \\
&= \frac{S^2}{R}\sqrt{\frac{18\cos^2\theta - 36\cos\theta + 18}{1-\cos\theta}} = \frac{3\sqrt{2}S^2}{R}\sqrt{1-\cos\theta}
\end{aligned}
$$

41. If $\mathbf{r}$ and $\dfrac{d\mathbf{r}}{dt}$ are always perpendicular, then $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt} = 0$. It follows that

$$
\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \mathbf{r}\cdot\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{r} = 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0,
$$

and this implies that $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{r} = $ constant. But this means that the tip of $\mathbf{r}$ lies on a sphere.

42. Let us take the positive x-direction as the direction in which the river flows. Then the velocity of the water with respect to the shore is $\mathbf{v}_{w/s} = 3\hat{\mathbf{i}}$. If the canoe points in direction θ shown, $\mathbf{v}_{c/w} = 2\cos\theta\,\hat{\mathbf{i}} + 2\sin\theta\,\hat{\mathbf{j}}$. If v is the speed of the canoe with respect to the shore, then $\mathbf{v}_{c/s} = v\cos\phi\,\hat{\mathbf{i}} + v\sin\phi\,\hat{\mathbf{j}}$. Since $\mathbf{v}_{c/s} = \mathbf{v}_{c/w} + \mathbf{v}_{w/s}$, we obtain



$$
v\cos\phi\,\hat{\mathbf{i}} + v\sin\phi\,\hat{\mathbf{j}} = 2\cos\theta\,\hat{\mathbf{i}} + 2\sin\theta\,\hat{\mathbf{j}} + 3\hat{\mathbf{i}}.
$$

When we equate components, $v\cos\phi = 2\cos\theta + 3 = 0$ and $v\sin\phi = 2\sin\theta$. These imply that

$$
4 = 4\cos^2\theta + 4\sin^2\theta = (v\cos\phi - 3)^2 + (v\sin\phi)^2 = v^2 - 6v\cos\phi + 9.
$$

Thus, $v^2 - 6v\cos\phi + 5 = 0$, the solutions of which are $v = 3\cos\phi \pm \sqrt{9\cos^2\phi - 5}$. Clearly v exists only if $9\cos^2\phi - 5 \geq 0 \implies \cos\phi \geq \sqrt{5}/3$. But from the figure, $\cos\phi = 5L/\sqrt{1+25L^2}$, and therefore

$$
\frac{5L}{\sqrt{1+25L^2}} \geq \frac{\sqrt{5}}{3} \quad \Longrightarrow \quad L \geq \frac{\sqrt{5}}{10}.
$$

When $L = \sqrt{5}/10$, there is one solution for v, namely, $v = 3\cos\phi$, in which case

$$
2\sin\theta = 3\cos\phi\sin\phi = 3\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right) \quad \Longrightarrow \quad \theta = \mathrm{Sin}^{-1}\left(\frac{\sqrt{5}}{3}\right) = 0.841 \text{ radians}.
$$

When $L > \sqrt{5}/10$, there are two solutions for v. The larger one $v = 3\cos\phi + \sqrt{9\cos^2\phi - 5}$ gives the shorter travel time. For this choice,

$$2\sin\theta = \sin\phi(3\cos\phi + \sqrt{9\cos^2\phi - 5}) = \frac{1}{\sqrt{1+25L^2}}\left(\frac{15L}{\sqrt{1+25L^2}} + \sqrt{\frac{225L^2}{1+25L^2} - 5}\right)$$
$$= \frac{15L + \sqrt{100L^2 - 5}}{1+25L^2}.$$

Therefore, $\theta = \mathrm{Sin}^{-1}\left(\dfrac{15L + \sqrt{100L^2-5}}{2+50L^2}\right)$.

43. If $\mathbf{F}_i$ is the force on m_i, then $\mathbf{F}_i = m_i\mathbf{a}_i$ for $i = 1, \ldots, n$, and $\mathbf{F} = \sum_{i=1}^{n}\mathbf{F}_i = \sum_{i=1}^{n} m_i\mathbf{a}_i$. The centre of mass $\overline{\mathbf{r}}$ of the system is given by $\overline{\mathbf{r}} = \dfrac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i$. Therefore,

$$\overline{\mathbf{a}} = \frac{d^2\overline{\mathbf{r}}}{dt^2} = \frac{1}{M}\sum_{i=1}^{n} m_i\frac{d^2\mathbf{r}_i}{dt^2} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{a}_i,$$

and $\mathbf{F} = M\overline{\mathbf{a}}$.

44. If we express the acceleration $\mathbf{a}$ of the particle in form 11.112a, write the force in the form $\mathbf{F} = \lambda(t)\mathbf{T}$, and substitute these into 11.109

$$\lambda(t)\mathbf{T} = m(a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}}) \quad\Longrightarrow\quad a_N = 0.$$

But then 11.112b implies that $|\mathbf{v}|\left|\dfrac{d\hat{\mathbf{T}}}{dt}\right| = 0$, from which $\dfrac{d\hat{\mathbf{T}}}{dt} = 0$, and $\hat{\mathbf{T}}$ is a constant vector. But this means that the trajectory is a straight line.

45. (a) Using formula 9.16, $A(t) = \displaystyle\int_{\theta_0}^{\theta(t)} \frac{1}{2}r^2\,d\theta$. If we differentiate this with respect to t using equation 6.19,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

(b) If $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ is a unit vector pointing to the planet at any given time, then differentiation of $1 = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}$ with respect to time gives

$$0 = \hat{\mathbf{r}}\cdot\frac{d\hat{\mathbf{r}}}{dt} + \frac{d\hat{\mathbf{r}}}{dt}\cdot\hat{\mathbf{r}} = 2\left(\hat{\mathbf{r}}\cdot\frac{d\hat{\mathbf{r}}}{dt}\right).$$

This implies that $\hat{\mathbf{r}}$ and $d\hat{\mathbf{r}}/dt$ are perpendicular at any given time. If we write $\mathbf{r} = r\,\hat{\mathbf{r}}$, where r is therefore the length of $\mathbf{r}$, then $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = r\dfrac{d\hat{\mathbf{r}}}{dt} + \dfrac{dr}{dt}\hat{\mathbf{r}}$. When we take magnitudes of 11.115 and substitute this expression for $\mathbf{v}$,

$$|\mathbf{C}| = |\mathbf{r}\times\mathbf{v}| = \left|r\,\hat{\mathbf{r}}\times\left(r\frac{d\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\hat{\mathbf{r}}\right)\right| = \left|r^2\hat{\mathbf{r}}\times\frac{d\hat{\mathbf{r}}}{dt} + r\frac{dr}{dt}\hat{\mathbf{r}}\times\hat{\mathbf{r}}\right| = r^2\left|\frac{d\hat{\mathbf{r}}}{dt}\right|,$$

since $\hat{\mathbf{r}}\times\hat{\mathbf{r}} = \mathbf{0}$ and $\hat{\mathbf{r}}$ and $d\hat{\mathbf{r}}/dt$ are perpendicular. Since $d\hat{\mathbf{r}}/dt = (-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})d\theta/dt$,

$$|\mathbf{C}| = r^2\left|(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})\frac{d\theta}{dt}\right| \quad\Longrightarrow\quad \frac{d\theta}{dt} = \frac{|\mathbf{C}|}{r^2}.$$

(c) When we combine the results in (a) and (b),

$$\frac{dA}{dt} = \frac{1}{2}r^2\left(\frac{|\mathbf{C}|}{r^2}\right) = \frac{|\mathbf{C}|}{2}.$$

This implies that $\mathbf{A}(t) = \dfrac{|\mathbf{C}|}{2}t + k$, where k is a constant. Since $A(t)$ is a linear function, its changes in time intervals of the same length are the same, thus verifying Kepler's second law.

46. (a) Using the formula in part (c) of Exercise 45, the area swept out by the plane in one revolution is

$$A(t+P) - A(t) = \left[\frac{|\mathbf{C}|}{2}(t+P) + k\right] - \left[\frac{|\mathbf{C}|}{2}t + k\right] = \frac{P|\mathbf{C}|}{2}.$$

But the area of the ellipse is πab, so that $\dfrac{P|\mathbf{C}|}{2} = \pi ab \Longrightarrow P = \dfrac{2\pi ab}{|\mathbf{C}|}$.

(b) If we change 11.116 to polar coordinates by setting $x = r\cos\theta$ and $y = r\sin\theta$,

$$\sqrt{x^2+y^2} = \frac{\epsilon d}{1 + \dfrac{\epsilon x}{\sqrt{x^2+y^2}}} \quad \Longrightarrow \quad \sqrt{x^2+y^2} = \epsilon(d-x).$$

Squaring gives

$$x^2 + y^2 = \epsilon^2(d-x)^2 \quad \Longrightarrow \quad x^2 + \frac{2d\epsilon^2 x}{1-\epsilon^2} + \frac{y^2}{1-\epsilon^2} = \frac{\epsilon^2 d^2}{1-\epsilon^2}.$$

Completing the square on the x-terms results in

$$\left(x + \frac{d\epsilon^2}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{\epsilon^2 d^2}{1-\epsilon^2} + \frac{d^2\epsilon^4}{(1-\epsilon^2)^2} = \left(\frac{\epsilon d}{1-\epsilon^2}\right)^2.$$

This allows us to identify $a = \epsilon d/(1-\epsilon^2)$ and $b = \epsilon d/\sqrt{1-\epsilon^2}$. With $\epsilon = |\mathbf{b}|/(GM)$ and $d = |\mathbf{C}|^2/|\mathbf{b}|$,

$$a = \frac{|\mathbf{C}|^2/(GM)}{1 - |\mathbf{b}|^2/(G^2M^2)} = \frac{|\mathbf{C}|^2 GM}{G^2M^2 - |\mathbf{b}|^2}, \quad b = \frac{|\mathbf{C}|^2/(GM)}{\sqrt{1 - |\mathbf{b}|^2/(G^2M^2)}} = \frac{|\mathbf{C}|^2}{\sqrt{G^2M^2 - |\mathbf{b}|^2}}.$$

It follows therefore that $\dfrac{b^2}{a} = \dfrac{|\mathbf{C}|^4}{G^2M^2 - |\mathbf{b}|^2}\dfrac{G^2M^2 - |\mathbf{b}|^2}{|\mathbf{C}|^2 GM} = \dfrac{|\mathbf{C}|^2}{GM}$, and hence,

$$P^2 = \frac{4\pi^2 a^2 b^2}{|\mathbf{C}|^2} = 4\pi^2 a^2\left(\frac{a}{GM}\right) = \frac{4\pi^2 a^3}{GM}.$$

47. The first point in Consulting Project 18 at which use is made of the initial speed of the ice is in the equation $-9.81\sin\theta = (a/2)(d\theta/dt)^2 + C$. With $d\theta/dt = v_0/a$ when $\theta = \pi/2$, we find that $C = -9.81 - v_0^2/(2a)$. Hence,

$$-9.81\sin\theta = \frac{a}{2}\left(\frac{d\theta}{dt}\right)^2 - 9.81 - \frac{v_0^2}{2a} \quad \Longrightarrow \quad \left(\frac{d\theta}{dt}\right)^2 = \frac{19.62}{a}(1 - \sin\theta) + \frac{v_0^2}{a^2}.$$

If we now substitute this into the equation $-N + 9.81m\sin\theta = ma\left(\dfrac{d\theta}{dt}\right)^2$, we obtain

$$-N + 9.81m\sin\theta = 19.62m(1-\sin\theta) + \frac{mv_0^2}{a} \quad \Longrightarrow \quad N = 9.81m(3\sin\theta - 2) - \frac{mv_0^2}{a}.$$

We find that $N = 0$ when

$$9.81m(3\sin\theta - 2) - \frac{mv_0^2}{a} = 0 \quad \Longrightarrow \quad \theta = \mathrm{Sin}^{-1}\left(\frac{2}{3} + \frac{v_0^2}{3(9.81)a}\right).$$

This will be valid provided $\dfrac{2}{3} + \dfrac{v_0^2}{3(9.81)a} \le 1 \Longrightarrow v_0 \le \sqrt{9.81a}$.

REVIEW EXERCISES

1. $2\mathbf{u} - 3\mathbf{w} + \mathbf{r} = 2(1,3,-2) - 3(0,2,1) + (2,0,-1) = (4,0,-8)$

2. $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 4 & -1\\ 0 & 2 & 1\end{vmatrix} = (1,3,-2)\cdot(6,-2,4) = 6-6-8 = -8$

3. $(3\mathbf{u}\times 4\mathbf{v}) - \mathbf{w} = 12\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 3 & -2\\ 2 & 4 & -1\end{vmatrix} - \mathbf{w} = 12(5,-3,-2) - (0,2,1) = (60,-38,-25)$

4. $3\mathbf{u}\times(4\mathbf{v}-\mathbf{w}) = 3\mathbf{u}\times(8,14,-5) = 3\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 3 & -2\\ 8 & 14 & -5\end{vmatrix} = 3(13,-11,-10) = (39,-33,-30)$

5. $|\mathbf{u}|\mathbf{v} - |\mathbf{v}|\mathbf{r} = \sqrt{14}(2,4,-1) - \sqrt{21}(2,0,-1) = (2\sqrt{14}-2\sqrt{21}, 4\sqrt{14}, -\sqrt{14}+\sqrt{21})$

6. $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{r}-\mathbf{w}) = (3,7,-3)\cdot(2,-2,-2) = 6-14+6 = -2$

7. $(\mathbf{u}+\mathbf{v})\times(\mathbf{r}-\mathbf{w}) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 7 & -3\\ 2 & -2 & -2\end{vmatrix} = (-20,0,-20)$

8. $(\mathbf{u}\times\mathbf{v})\times(\mathbf{r}\times\mathbf{w}) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 3 & -2\\ 2 & 4 & -1\end{vmatrix}\times\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & 0 & -1\\ 0 & 2 & 1\end{vmatrix} = (5,-3,-2)\times(2,-2,4)$

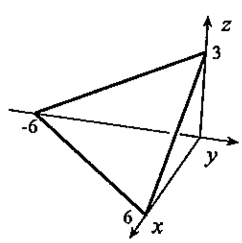
$$= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 5 & -3 & -2\\ 2 & -2 & 4\end{vmatrix} = (-16,-24,-4)$$

9. $(\mathbf{u}\cdot\mathbf{v})\mathbf{r} - 3(\mathbf{v}\cdot\mathbf{w})\mathbf{u} = 16(2,0,-1) - 3(7)(1,3,-2) = (11,-63,26)$

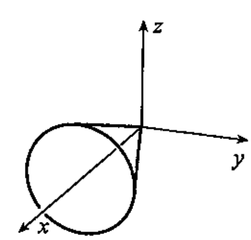
10. $\dfrac{2\mathbf{r}}{\mathbf{v}\cdot\mathbf{w}} + 3(\mathbf{v}+\mathbf{u}) = \dfrac{2}{8-1}(2,0,-1) + 3(3,7,-3) = \left(\dfrac{67}{7}, 21, -\dfrac{65}{7}\right)$

For Exercises 12, 14, 16, 18, 20, 22, 24, and 26, see answers in text.

11.



13.



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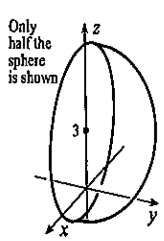

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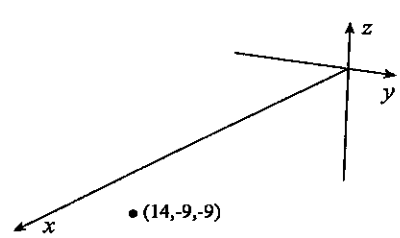

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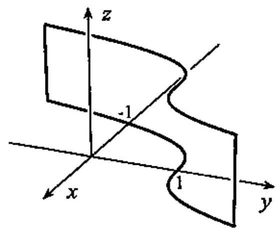

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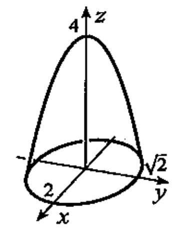

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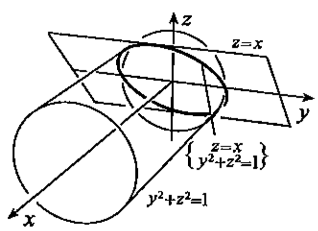

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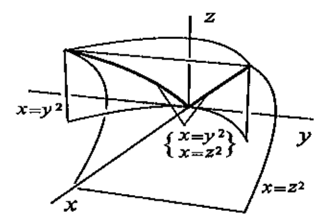


27. Parametric equations for the line are $x = -2 + 3t$, $y = 3 - 5t$, $z = 4t$.

28. Since a vector along the line is $(5, -2, 1)$, parametric equations for the line are $x = 6 + 5t$, $y = 6 - 2t$, $z = 2 + t$.

29. Since a vector along the line is $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 2 & 3 & 6 \end{vmatrix} = (-6, -6, 5)$ as is $(6, 6, -5)$, parametric equations are $x = 6t$, $y = 6t$, $z = -5t$.

30. Since $(1, 3, 2)$ is a point on the required line, parametric equations must be of the form

$$x = 1 + au, \ \ y = 3 + bu, \ \ z = 2 + cu.$$

Because the line must be perpendicular to the given line,

$$0 = (a, b, c) \cdot (1, -2, 1) = a - 2b + c.$$

Finally, since the lines must intersect, we set

$$t + 2 = 1 + au, \ \ 3 - 2t = 3 + bu, \ \ 4 + t = 2 + cu.$$

When the first two of these are solved for t and u,

$$u = \frac{2}{b + 2a} \qquad \text{and} \qquad t = \frac{-b}{b + 2a}.$$

Substitution into the third gives $4a + b - 2c = 0$. When this is combined with $a - 2b + c = 0$, the result is $b = 2a$ and $c = 3a$. Consequently, parametric equations for the required line are $x = 1 + au$, $y = 3 + 2au$, $z = 2 + 3au$, or, $x = 1 + v$, $y = 3 + 2v$, $z = 2 + 3v$ (where we have set $v = au$).

31. Since $(2, -1, 0) - (1, 3, 2) = (1, -4, -2)$ and $(6, 1, 3) - (1, 3, 2) = (5, -2, 1)$ are vectors in the plane, a normal to the plane is $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -4 & -2 \\ 5 & -2 & 1 \end{vmatrix} = (-8, -11, 18)$ as is $(8, 11, -18)$. The equation of the plane is $0 = 8(x - 1) + 11(y - 3) - 18(z - 2) = 8x + 11y - 18z - 5$.

32. Since parametric equations for the line are $x = 4 - t$, $y = t$, $z = t$, a vector along the line (and therefore normal to the plane is $(-1, 1, 1)$. The equation of the plane is

$$0 = (-1, 1, 1) \cdot (x - 1, y - 2, z + 1) = -x + y + z.$$

33. Since two points on the line are $(2,0,1)$ and $(6,-3,-6)$, two vectors in the plane are $(2,2,2)-(2,0,1)=(0,2,1)$ and $(2,2,2)-(6,-3,-6)=(-4,5,8)$. A normal to the plane is $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & 1 \\ -4 & 5 & 8 \end{vmatrix} = (11,-4,8)$. The equation of the plane is $0 = 11(x-2) - 4(y-2) + 8(z-2) = 11x - 4y + 8z - 30$.

34. Since the lines are not parallel, they determine a plane only if they intersect. To confirm this, we set $3t = 1 + 2t$ and $3t = 4 - t$. These both give $t = 1$ leading to the point of intersection $(3,3,3)$. A vector normal to the plane is

$$(3,2,-1)\times(1,1,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (3,-4,1).$$

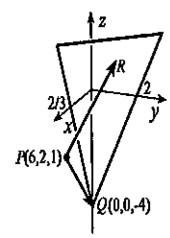
The equation of the plane is

$$0 = (3,-4,1)\cdot(x-3, y-3, z-3) = 3x - 4y + z.$$

35. $\sqrt{25+1+9} = \sqrt{35}$

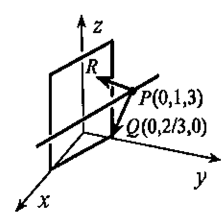
36. The required distance d is the projection of **PQ** on the direction **PR** normal to the plane:

$$\begin{aligned} d &= \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| \\ &= \left|(-6,-2,-5)\cdot\frac{(-6,-2,1)}{\sqrt{36+4+1}}\right| \\ &= 35/\sqrt{41}. \end{aligned}$$



37. The distance is zero unless the line and plane are parallel. A vector along the line is $(1,-1,1)\times(2,1,1) = (-2,1,3)$. Since $(-2,1,3)\cdot(1,-1,0) = -3 \neq 0$, the line plane are not parallel; they intersect. The minimum distance is therefore 0.

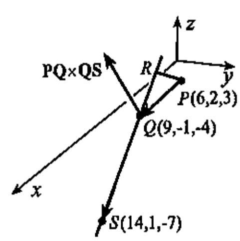
38. The distance is zero unless the line and plane are parallel. A vector along the line is $(1,-1,1)\times(2,1,1) = (-2,1,3)$. Since $(-2,1,3)\cdot(3,6,0) = 0$, the line and plane are parallel. The required distance d is the projection of **PQ** on the direction **PR** normal to the plane:



$$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(0,-1/3,-3)\cdot\frac{(-3,-6,0)}{\sqrt{9+36}}\right| = \frac{2}{3\sqrt{5}}.$$

39. The required distance is the component of **PQ** along **PR**. A vector perpendicular to **PQ** and **QS** is

$$\mathbf{PQ}\times\mathbf{QS} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -3 & -7 \\ 5 & 2 & -3 \end{vmatrix} = (23,-26,21).$$



A vector in direction **PR** is therefore

$$(\mathbf{PQ}\times\mathbf{QS})\times\mathbf{SQ} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 23 & -26 & 21 \\ -5 & -2 & 3 \end{vmatrix} = (-36,-174,-176).$$

Thus,

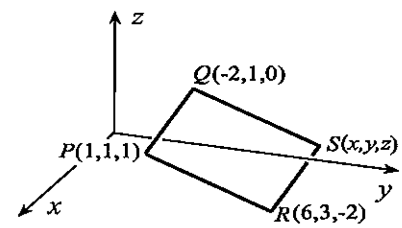
$$d = \left|\mathbf{PQ}\cdot\widehat{\mathbf{PR}}\right| = \left|(3,-3,-7)\cdot\frac{(-18,-87,-88)}{\sqrt{(-18)^2+(-87)^2+(-88)^2}}\right| = \frac{823}{\sqrt{15637}}.$$

40. According to equation 11.42,

$$\begin{aligned}\text{Area } &= \frac{1}{2}|[(-2,1,0)-(1,1,1)]\times[(6,3,-2)-(1,1,1)]| \\ &= \frac{1}{2}|(-3,0,-1)\times(5,2,-3)| = \frac{1}{2}\left\|\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -3 & 0 & -1\\ 5 & 2 & -3\end{vmatrix}\right\| = \frac{1}{2}|(2,-14,-6)| = \sqrt{59}.\end{aligned}$$

41. One such parallelogram is shown in the figure. The coordinates of S are

$$\begin{aligned}(x,y,z) &= \mathbf{OR}+\mathbf{RS} = \mathbf{OR}+\mathbf{PQ}\\ &= (6,3,-2)+(-3,0,-1)\\ &= (3,3,-3).\end{aligned}$$



The area of the parallelogram is

$$\text{area} = |\mathbf{PQ}\times\mathbf{PR}| = \left\|\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ -3 & 0 & -1\\ 5 & 2 & -3\end{vmatrix}\right\| = |(2,-14,-6)| = 2\sqrt{59}.$$

Similar procedures give two additional vertices $(-7,-1,3)$ and $(9,3,-1)$ with equal areas.

42. From $\mathbf{T} = \dfrac{d\mathbf{r}}{dt} = (2\cos t, -2\sin t, 1)$, we obtain the unit tangent vector

$$\hat{\mathbf{T}} = \frac{(2\cos t, -2\sin t, 1)}{\sqrt{4\cos^2 t + 4\sin^2 t + 1}} = \frac{(2\cos t, -2\sin t, 1)}{\sqrt{5}}.$$

A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{1}{\sqrt{5}}(-2\sin t, -2\cos t, 0) = -\frac{2}{\sqrt{5}}(\sin t, \cos t, 0),$$

and therefore $\hat{\mathbf{N}} = -(\sin t, \cos t, 0)$. The binormal is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}}\times\hat{\mathbf{N}} = -\frac{1}{\sqrt{5}}\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2\cos t & -2\sin t & 1\\ \sin t & \cos t & 0\end{vmatrix} = \frac{1}{\sqrt{5}}(\cos t, -\sin t, -2).$$

43. From $\hat{\mathbf{T}} = \dfrac{(3t^2, 4t, 1)}{\sqrt{9t^4+16t^2+1}}$, a vector in the direction of $\hat{\mathbf{N}}$ is

$$\begin{aligned}\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(18t^3+16t)}{(9t^4+16t^2+1)^{3/2}}(3t^2,4t,1) + \frac{(6t,4,0)}{\sqrt{9t^4+16t^2+1}}\\ &= \frac{1}{(9t^4+16t^2+1)^{3/2}}\left[-(18t^3+16t)(3t^2,4t,1)+(9t^4+16t^2+1)(6t,4,0)\right]\\ &= \frac{1}{(9t^4+16t^2+1)^{3/2}}(48t^3+6t, -36t^4+4, -18t^3-16t).\end{aligned}$$

Consequently, the principal normal is

$$\hat{\mathbf{N}} = \frac{(24t^3+3t, 2-18t^4, -9t^3-8t)}{\sqrt{(24t^3+3t)^2+(2-18t^4)^2+(-9t^3-8t)^2}} = \frac{(24t^3+3t, 2-18t^4, -9t^3-8t)}{\sqrt{324t^8+657t^6+216t^4+73t^2+4}}.$$

The direction of the binormal is

$$\begin{aligned}\mathbf{B} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3t^2 & 4t & 1\\ 24t^3+3t & 2-18t^4 & -9t^3-8t\end{vmatrix}\\ &= (-18t^4-32t^2-2)\hat{\mathbf{i}} + (27t^5+48t^3+3t)\hat{\mathbf{j}} + (-54t^6-96t^4-6t^2)\hat{\mathbf{k}}\\ &= (9t^4+16t^2+1)(-2\hat{\mathbf{i}}+3t\hat{\mathbf{j}}-6t^2\hat{\mathbf{k}}).\end{aligned}$$

Thus, $\hat{\mathbf{B}} = \dfrac{(-2, 3t, -6t^2)}{\sqrt{4 + 9t^2 + 36t^4}}$.

44. $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = (1, 2t, 2t) \qquad |\mathbf{v}| = \sqrt{1 + 4t^2 + 4t^2} = \sqrt{1 + 8t^2} \qquad \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = (0, 2, 2)$

The normal component of velocity is always zero, and the tangential component of velocity is speed $|\mathbf{v}| = \sqrt{1 + 8t^2}$.

$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{8t}{\sqrt{1 + 8t^2}} \qquad a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \left(8 - \frac{64t^2}{1 + 8t^2}\right)^{1/2} = \frac{2\sqrt{2}}{\sqrt{1 + 8t^2}}$$

45. $\mathrm{W} = \displaystyle\int_1^4 \mathbf{F} \cdot dx\hat{\mathbf{i}} = \int_1^4 \frac{2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}}{x^2} \cdot dx\hat{\mathbf{i}} = \int_1^4 \frac{2}{x^2} dx = \left\{-\frac{2}{x}\right\}_1^4 = \frac{3}{2}$

46. (a) The acceleration of the ball after it leaves the table is $\mathbf{a} = -g\hat{\mathbf{j}}$ where $g = 9.81$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take $t = 0$ at the instant the ball leaves the table, $\mathbf{v}(0) = \hat{\mathbf{i}}/2$. This implies that $\mathbf{C} = \hat{\mathbf{i}}/2$, and

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \frac{t}{2}\hat{\mathbf{i}} + \mathbf{D}.$$

Since $\mathbf{r}(0) = \mathbf{0}$, it follows that $\mathbf{D} = \mathbf{0}$, and $\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \frac{t}{2}\hat{\mathbf{i}}$. The ball strikes the floor when the y-component of $\mathbf{r}$ is -1,

$$-1 = -\frac{1}{2}gt^2, \qquad \text{or,} \qquad t = \sqrt{\frac{2}{g}} = \sqrt{\frac{2}{9.81}}.$$

At this instant, $\mathbf{v} = -g\sqrt{\dfrac{2}{9.81}}\hat{\mathbf{j}} + \dfrac{1}{2}\hat{\mathbf{i}} = \dfrac{1}{2}\hat{\mathbf{i}} - \sqrt{19.62}\hat{\mathbf{j}}$, and therefore its speed when it strikes the floor is

$$|\mathbf{v}| = \sqrt{1/4 + 19.62} = \sqrt{19.87}\ \text{m/s}.$$

(b) Its displacement vector when it strikes the floor is

$$\mathbf{r} = -\frac{1}{2}g\left(\frac{2}{g}\right)\hat{\mathbf{j}} + \frac{1}{2}\sqrt{\frac{2}{g}}\hat{\mathbf{i}} = 0.226\hat{\mathbf{i}} - \hat{\mathbf{j}}.$$

(c) After the rebound, the acceleration of the ball is once again $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we redefine $t = 0$ at the time of the first bounce, then

$$\mathbf{C} = \mathbf{v}(0) = \frac{4}{5}\left(\frac{1}{2}\hat{\mathbf{i}} + \sqrt{19.62}\hat{\mathbf{j}}\right).$$

Integration now gives

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + \mathbf{C}t + \mathbf{D}.$$

If we redefine $(x, y) = (0, 0)$ at the position of the first bounce, then $\mathbf{D} = \mathbf{0}$. The ball hits the floor for the second time when the y-component of $\mathbf{r}$ is zero,

$$0 = -\frac{1}{2}gt^2 + \frac{4}{5}\sqrt{19.62}t \qquad \text{or} \qquad t = \frac{16}{5\sqrt{19.62}}.$$

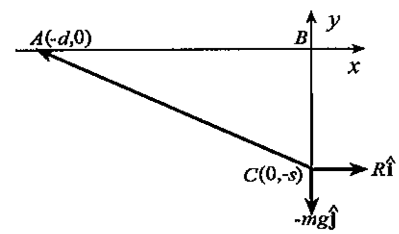
At this instant, the x-component of its displacement is

$$r_x = \frac{2}{5}\left(\frac{16}{5\sqrt{19.62}}\right) = 0.289.$$

Thus, the second bounce takes place 0.515 m from the point on the floor directly below the point it left the table.

47. With the coordinate system shown, the force of gravity on the sleeve is $-mg\hat{\mathbf{j}}$. The spring force is

$$k(\sqrt{d^2+s^2}-d)\frac{-d\hat{\mathbf{i}}+s\hat{\mathbf{j}}}{\sqrt{d^2+s^2}}.$$



The reaction of the rod on the sleeve is $R\hat{\mathbf{i}}$, there being no friction. At equilibrium,

$$k(\sqrt{d^2+s^2}-d)\frac{-d\hat{\mathbf{i}}+s\hat{\mathbf{j}}}{\sqrt{d^2+s^2}}-mg\hat{\mathbf{j}}+R\hat{\mathbf{i}}=\mathbf{0}.$$

The horizontal component of this equation determines R, and the vertical component gives

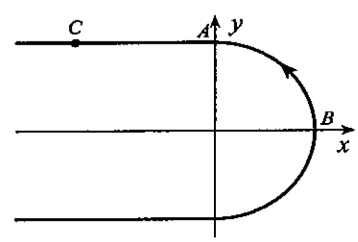
$$\frac{ks(\sqrt{d^2+s^2}-d)}{\sqrt{d^2+s^2}}-mg=0.$$

Thus, $(ks-mg)\sqrt{d^2+s^2}=kds$, and this is the required equation.

48. In Example 11.26, it was shown that $|\mathbf{F}|=k[\sqrt{(1-x)^2+1/4}-1/2]$. A vector in the direction of $\mathbf{F}$ is $(1,1/2)-(x,0)=(1-x,1/2)$, and therefore

$$\begin{aligned}\mathbf{F}&=k[\sqrt{(1-x)^2+1/4}-1/2]\frac{(1-x,1/2)}{\sqrt{(1-x)^2+1/4}}\\&=k(1-x)\left[1-\frac{1}{\sqrt{4(1-x)^2+1}}\right]\hat{\mathbf{i}}+\frac{k}{2}\left[1-\frac{1}{\sqrt{4(1-x)^2+1}}\right]\hat{\mathbf{j}}\text{ N.}\end{aligned}$$

49. Suppose s is the constant speed of the train, and we take $t=0$ when the train passes through B. Along BA, $x=R\cos\omega t$, $y=R\sin\omega t$, where $\omega=s/R$. The train passes through A at $t=\pi R/(2s)$. The acceleration of the train along BA is



$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=(-R\omega^2\cos\omega t)\hat{\mathbf{i}}+(-R\omega^2\sin\omega t)\hat{\mathbf{j}}.$$

Consequently, $\displaystyle\lim_{t\to\pi R/(2s)^-}\mathbf{a}(t)=-R\omega^2\hat{\mathbf{j}}$.

Along AC, $x=-s[t-\pi R/(2s)]$, $y=R$, so that acceleration on this part of the track is $\mathbf{a}=\mathbf{0}$. Hence $\mathbf{a}$ is discontinuous at A.

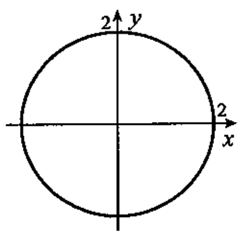
CHAPTER 12

EXERCISES 12.1

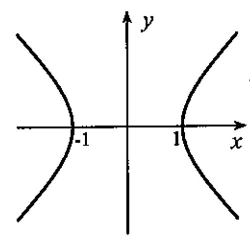
1. (a) $f(1,2) = 1^3(2) + 1\sin 2 = 2 + \sin 2$
(b) $f(-2,-2) = (-2)^3(-2) - 2\sin(-2) = 16 + 2\sin 2$
(c) $f(x^2+y, x-y^2) = (x^2+y)^3(x-y^2) + (x^2+y)\sin(x-y^2)$
(d) $f(x+h,y) - f(x,y) = [(x+h)^3y + (x+h)\sin y] - [x^3y + x\sin y] = y(3x^2h + 3xh^2 + h^3) + h\sin y$

2. $f(a+b, a-b, ab) = (a+b)^2(a-b)^2 - (a+b)^4 + 4ab(a+b)^2 = (a+b)^2[(a-b)^2 - (a+b)^2 + 4ab] = 0$

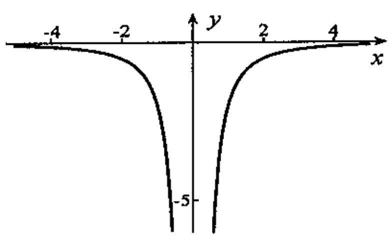
3. For $4 - x^2 - y^2 \geq 0$, we must take $x^2 + y^2 \leq 4$. This inequality describes all points inside and on the circle $x^2 + y^2 = 4$.



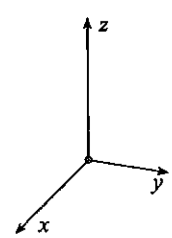
4. For $1 - x^2 + y^2 > 0$, we must take $x^2 - y^2 < 1$. This inequality describes all points between, but not on, the branches of the hyperbola $x^2 - y^2 = 1$



5. For $-1 \leq x^2y + 1 \leq 1$, we require $-2 \leq x^2y \leq 0$ or $-2/x^2 \leq y \leq 0$. Points are below the x-axis and above the curve $y = -2/x^2$. Points on the boundary are also included.



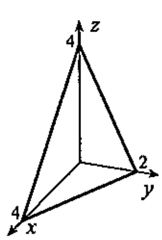
6. This function is defined for all points in space except the origin $(0,0,0)$.



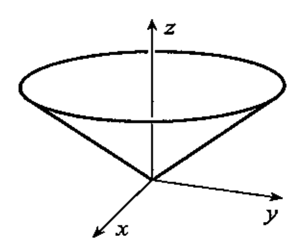
7. For $f(x,y) = \dfrac{12xy - x^2y^2}{2(x+y)} = \dfrac{xy(12-xy)}{2(x+y)} = 0$, we set $xy(12-xy) = 0$. This is satisfied if $x = 0$ or $y = 0$ or $12 - xy = 0$. Thus, the function is equal to zero for all points on the x- and y-axes (except $(0,0)$), and all points on the hyperbola $xy = 12$. The largest domain of the function is all points in the xy-plane except those on the line $y = -x$.

For Exercises 8, 10, 12, 14, 16, 18, and 20, see answers in text.

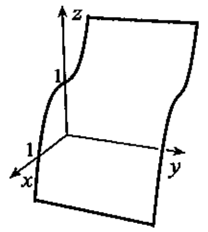
9.



11.



13.



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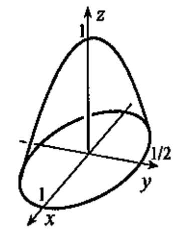

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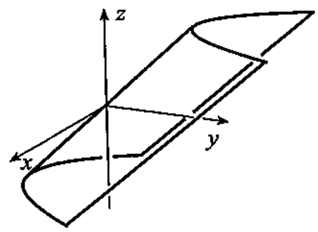

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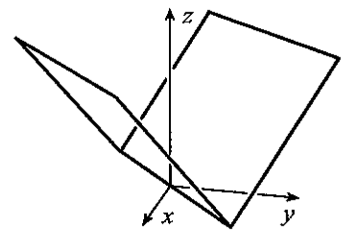

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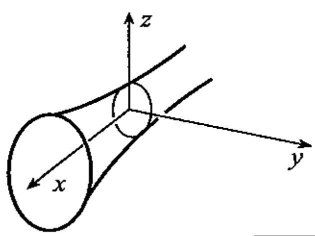
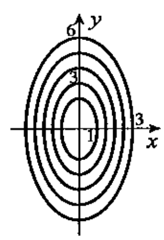
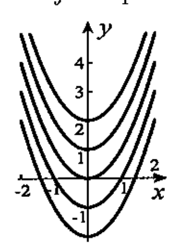


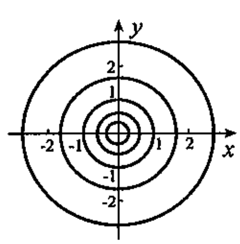
22. Level curves are defined by $4 - \sqrt{4x^2 + y^2} = C$, or, $4x^2 + y^2 = (C - 4)^2$. They are ellipses.



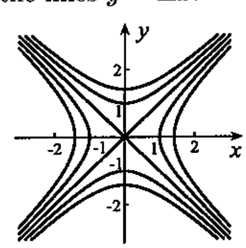
23. Level curves are defined by $y - x^2 = C$, or $y = x^2 + C$. They are parabolas.



24. Level curves are defined by $\ln(x^2 + y^2) = C$, or, $x^2 + y^2 = e^C$. They are circles.



25. Level curves are defined by $x^2 - y^2 = C$. They are hyperbolas, except when $C = 0$ when they are the lines $y = \pm x$.

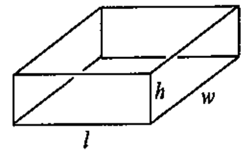


26. The volume of the box is $V = lwh$, where h is its height. Since $30 = 2hl + 2wh + 2wl$, it follows that $h = (15 - wl)/(l + w)$, and therefore $V = lw(15 - wl)/(l + w)$.

27. (a) The cost in cents is

$C = 125(2wh + 2lh + lw) + 475lw = 600lw + 250h(l + w)$.

(b) If $V = 1000 = lwh$, then $h = 1000/(lw)$, and



$$C = 600lw + 250\left(\frac{1000}{lw}\right)(l + w)$$
$$= 600lw + \frac{250\,000(l + w)}{lw}.$$

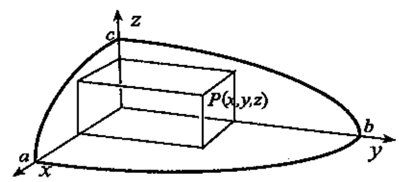
(c) If we add the cost of welding to the functions in (a) and (b) we obtain

$$C = 600lw + 250h(l + w) + 750(4l + 4h + 4w) = 600lw + 250h(l + w) + 3000(l + h + w),$$

and

$$C = 600lw + \frac{250\,000(l + w)}{lw} + 3000\left(l + \frac{1000}{lw} + w\right).$$

28. If P is the corner of the ellipsoid in the first octant, then

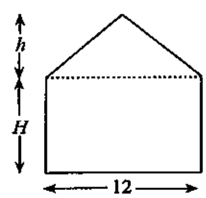


$$V = 8xyz = 8cxy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

When P is not in the first octant,

$$V = 8c|xy|\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

29. (a) The volume of the silo is



$V = \pi(6)^2H + \frac{1}{3}\pi(6)^2h = 12\pi(h + 3H)$ m^3.

(b) If $200 = 2\pi(6)H + \pi(6)\sqrt{36 + h^2}$, then $H = (100 - 3\pi\sqrt{36 + h^2})/(6\pi)$, and

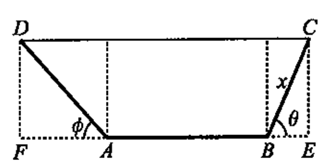
$$V = 12\pi\left(h + \frac{100 - 3\pi\sqrt{36 + h^2}}{2\pi}\right) \text{ m}^3.$$

30. (a) $D(0.35, 9.0) = 0.35 + 0.9 + \dfrac{81\cos 0.35}{9.81}\left[\sin 0.35 + \sqrt{\sin^2 0.35 + \dfrac{2(9.81)(0.5)}{81}}\right] = 7.70$ m

(b) Since $D(0.35, 9.9) = 0.35 + 0.9 + \dfrac{(9.9)^2\cos 0.35}{9.81}\left[\sin 0.35 + \sqrt{\sin^2 0.35 + \dfrac{2(9.81)(0.5)}{(9.9)^2}}\right] = 8.847$, the percentage increase is $100(8.847 - 7.70)/7.70 = 14.9$.

(c) Since $D(0.385, 9.0) = 0.35 + 0.9 + \dfrac{81\cos 0.385}{9.81}\left[\sin 0.385 + \sqrt{\sin^2 0.385 + \dfrac{2(9.81)(0.5)}{81}}\right] = 8.042$, the percentage increase is $100(8.042 - 7.70)/7.70 = 4.4$.

31. Since

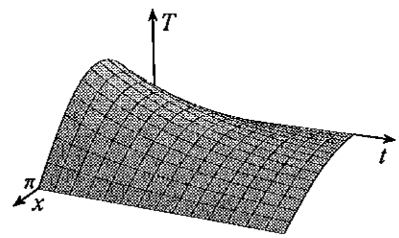


$$\begin{aligned}
\|DF\| &= \|CE\| = x\sin\theta,\\
\|AF\| &= \|DF\|\cot\phi = x\sin\theta\,\cot\phi,\\
\|AB\| &= 1 - \|BC\| - \|AD\|\\
&= 1 - x - x\sin\theta\,\csc\phi,
\end{aligned}$$

the cross-sectional area is

$$\begin{aligned}
\text{Area} &= \|AB\|\|CE\| + \frac{1}{2}\|BE\|\|CE\| + \frac{1}{2}\|FA\|\|DF\|\\
&= \|CE\|\left(\|AB\| + \frac{1}{2}\|BE\| + \frac{1}{2}\|FA\|\right)\\
&= x\sin\theta\left[(1 - x - x\sin\theta\,\csc\phi) + \frac{1}{2}(x\cos\theta) + \frac{1}{2}(x\sin\theta\,\cot\phi)\right].
\end{aligned}$$

32. (a)



(b) The intersection curve with $x = x_0$ is a graphical history of temperature at position x_0 as a function of time. The intersection curve with $t = t_0$ is a graph of the temperature distribution throughout the rod at time t_0.

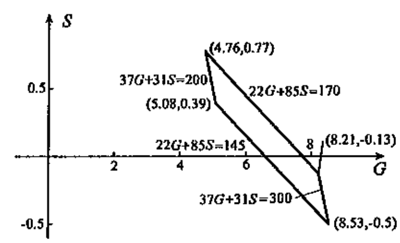
33. The cost per day in cents is $C = f(G, S) = \frac{2750}{1000}(11) + \frac{11\,000}{1000}G + \frac{17\,500}{1000}S = 30.25 + 11G + 17.5S$. Because the cow must have between 9.5 and 11.5 kg of digestive material,

$$9.5 \le 11\left(\frac{1}{2}\right) + G\left(\frac{74}{100}\right) + S\left(\frac{62}{100}\right) \le 11.5 \quad \Longrightarrow \quad 200 \le 37G + 31S \le 300.$$

Because the cow must have between 1.9 and 2.0 kg of protein,

$$1.9 \le 11\left(\frac{12}{100}\right) + G\left(\frac{8.8}{100}\right) + S\left(\frac{34}{100}\right) \le 2.0 \quad \Longrightarrow \quad 145 \le 22G + 85S \le 170.$$

The domain of $f(G, S)$ therefore consists of all non-negative values of G and S satisfying these two inequalities. It is the points in the first quadrant of the parallelogram to the right.



34. If x and y are the numbers of computers of models A and B, then the cost of the 100 computers is

$$C = f(x, y) = 1300x + 1200y + 1000(100 - x - y) = 100\,000 + 300x + 200y.$$

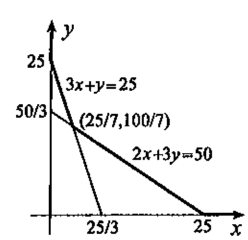
Because the computers must have at least 2000 MB of memory,

$$64x + 32y + 16(100 - x - y) \ge 2000 \quad \Longrightarrow 3x + y \ge 25.$$

Because the computers must have at least 150 GB of disk space,

$$3x + 4y + (100 - x - y) \ge 150 \quad \Longrightarrow \quad 2x + 3y \ge 50.$$

The domain of $f(x, y)$ therefore consists of all non-negative values of x and y satisfying these two inequalities. It is the points in the first quadrant above the lines in the figure to the right.



EXERCISES 12.2

1. $\lim\limits_{(x,y)\to(2,-3)} \dfrac{x^2 - 1}{x + y} = -3$

2. $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^3 + 2y^3}{x^3 + 4y^3} = \dfrac{3}{5}$

3. $\lim\limits_{(x,y)\to(3,2)} \dfrac{2x - 3y}{x + y} = 0$

4. $\lim\limits_{(x,y,z)\to(2,3,-1)} \dfrac{xyz}{x^2 + y^2 + z^2} = -\dfrac{3}{7}$

5. $\lim_{(x,y)\to(1,0)} \frac{x}{y}$ does not exist.

6. $\lim_{(x,y,z)\to(0,\pi/2,1)} \mathrm{Tan}^{-1}\left(\frac{x}{yz}\right) = 0$

7. $\lim_{(x,y,z)\to(0,\pi/2,1)} \mathrm{Tan}^{-1}\left(\frac{yz}{x}\right)$ does not exist since it depends on whether $x \to 0^-$ or $x \to 0^+$.

8. $\lim_{(x,y,z)\to(0,\pi/2,1)} \mathrm{Tan}^{-1}\left|\frac{yz}{x}\right| = \frac{\pi}{2}$

9. $\lim_{(x,y)\to(3,4)} \frac{|x^2-y^2|}{x^2-y^2} = \frac{|9-16|}{9-16} = -1$

10. $\lim_{(x,y)\to(3,4)} \frac{|x^2+y^2|}{x^2+y^2} = \frac{|9+16|}{9+16} = 1$

11. $\lim_{(x,y)\to(2,1)} \frac{x^2-y^2}{x-y} = 3$

12. $\lim_{(x,y)\to(2,2)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(2,2)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y)\to(2,2)} (x+y) = 4$

13. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y)\to(0,0)} (x+y) = 0$

14. If we approach $(0,0)$ along the line $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{x\to 0} \frac{x-mx}{x+mx} = \lim_{x\to 0} \frac{1-m}{1+m} = \frac{1-m}{1+m}.$$

Since this result depends on m, the original limit does not exist.

15. $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2-y^2}{y^2+z^2+1} = 0$

16. $\lim_{(x,y)\to(2,1)} \frac{(x-2)^2(y+1)}{x-2} = \lim_{(x,y)\to(2,1)} (x-2)(y+1) = 0$

17. $\lim_{(x,y,z)\to(1,1,1)} |2x-y-z| = 0$

18. If we approach $(0,0)$ along the line $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{3x^3-y^3}{2x^3+4y^3} = \lim_{x\to 0} \frac{3x^3-m^3x^3}{2x^3+4m^3x^3} = \lim_{x\to 0} \frac{3-m^3}{2+4m^3} = \frac{3-m^3}{2+4m^3}.$$

Since this result depends on m, the original limit does not exist.

19. $\lim_{(x,y)\to(0,0)} \mathrm{Sec}^{-1}\left(\frac{-1}{x^2+y^2}\right) = -\frac{\pi}{2}$

20. $\lim_{(x,y)\to(0,0)} \mathrm{Sec}^{-1}(x^2+y^2)$ does not exist.

21. The function is discontinuous at all points on the line $y = -x$.

22. The function is discontinuous at $(0,0)$.

23. The function is discontinuous at all points on the circle $x^2+y^2 = 1$.

24. The function is discontinuous when $x = 0$, or $y = 0$, or $z = 0$.

25. The function has no discontinuities.

26. Since $f(x,y) = \frac{x+y}{xy(x+y)}$, the function is discontinuous when $x = 0$, or $y = 0$, or $x+y = 0$.

27. $$\lim_{(x,y)\to(a,a)} \left[\cos(x+y) - \sqrt{1-\sin^2(x+y)}\right] = \lim_{(x,y)\to(a,a)} [\cos(x+y) - |\cos(x+y)|]$$
$$= \cos 2a - |\cos 2a| = \begin{cases} 0, & 0 \le a \le \pi/4 \\ 2\cos 2a, & \pi/4 < a \le \pi/2. \end{cases}$$

28. If we approach $(0,0)$ along parabola $y = ax^2$,

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^2}{x^4-y^2} = \lim_{x\to 0} \frac{x^4+a^2x^4}{x^4-a^2x^4} = \lim_{x\to 0} \frac{1+a^2}{1-a^2} = \frac{1+a^2}{1-a^2}.$$

Since this result depends on a, the original limit does not exist.

29. If (x, y) is made to approach $(0,0)$ along the cubic curve $y = ax^3$,

$$\lim_{(x,y)\to(0,0)} \frac{x^6 - 2y^2}{3x^6 + y^2} = \lim_{x\to 0} \frac{x^6 - 2a^2x^6}{3x^6 + a^2x^6} = \lim_{x\to 0} \frac{1 - 2a^2}{3 + a^2} = \frac{1 - 2a^2}{3 + a^2}.$$

Since this result depends on a, the original limit does not exist.

30. If (x, y) is made to approach $(1,0)$ along the straight line $y = m(x-1)$,

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 + y^2}{3(x-1)^2 - 2y^2} = \lim_{x\to 1} \frac{(x-1)^2 + m^2(x-1)^2}{3(x-1)^2 - 2m^2(x-1)^2} = \lim_{x\to 1} \frac{1 + m^2}{3 - 2m^2} = \frac{1 + m^2}{3 - 2m^2}.$$

Since this result depends on m, the original limit does not exist.

31. If (x, y) is made to approach $(0,-2)$ along the straight line $y + 2 = mx$,

$$\lim_{(x,y)\to(0,-2)} \frac{x^3 + 4(y+2)^3}{3x^3 - (y+2)^3} = \lim_{x\to 0} \frac{x^3 + 4m^3x^3}{3x^3 - m^3x^3} = \lim_{x\to 0} \frac{1 + 4m^3}{3 - m^3} = \frac{1 + 4m^3}{3 - m^3}.$$

Since this result depends on m, the original limit does not exist.

32. If (x, y) is made to approach $(1,1)$ along the straight line $y - 1 = m(x-1)$,

$$\begin{aligned}\lim_{(x,y)\to(1,1)} \frac{x^2 - 2x - y^2 + 2y}{x^2 - 2x + y^2 - 2y + 2} &= \lim_{(x,y)\to(1,1)} \frac{(x-1)^2 - (y-1)^2}{(x-1)^2 + (y-1)^2} = \lim_{x\to 1} \frac{(x-1)^2 - m^2(x-1)^2}{(x-1)^2 + m^2(x-1)^2} \\ &= \lim_{x\to 1} \frac{1 - m^2}{1 + m^2} = \frac{1 - m^2}{1 + m^2}.\end{aligned}$$

Since this result depends on m, the original limit does not exist.

33. If (x, y) is made to approach $(1,1)$ along the vertical line $x = 1$, then

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - 2x + y^2 + 2y - 2}{x^2 - y^2 - 2x + 2y} = \lim_{y\to 1} \frac{1 - 2 + y^2 + 2y - 2}{1 - y^2 - 2 + 2y} = \lim_{y\to 1} \frac{y^2 + 2y - 3}{-y^2 + 2y - 1} = \lim_{y\to 1} \frac{y+3}{1-y}.$$

Since this limit does not exist, neither does the original limit.

34.
$$\begin{aligned}\lim_{(x,y)\to(1,0)} \frac{\sqrt{x+y} - \sqrt{x-y}}{y} &= \lim_{(x,y)\to(1,0)} \left(\frac{\sqrt{x+y} - \sqrt{x-y}}{y} \frac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} + \sqrt{x-y}} \right) \\ &= \lim_{(x,y)\to(1,0)} \frac{(x+y) - (x-y)}{y(\sqrt{x+y} + \sqrt{x-y})} = \lim_{(x,y)\to(1,0)} \frac{2}{\sqrt{x+y} + \sqrt{x-y}} = 1\end{aligned}$$

35. If we set $\theta = x^2 + y^2$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

36. (a) If we set $z = x - y$, then $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\sin(x-y)}{x-y} = \lim_{z\to 0} \frac{\sin z}{z} = 1.$

The function is not continuous at $(1,1)$ (being undefined there).

(b) Since the value of the function along $y = x$ is always equal to 1, its limit as $(x,y) \to (1,1)$ along $y = x$ is also 1. Consequently the limit of the function as $(x,y) \to (1,1)$ is still 1. The function is now continuous at $(1,1)$.

37. $f(x,y,z)$ has limit L as (x,y,z) approaches (x_0, y_0, z_0) if given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x,y,z) - L| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta$ and (x,y,z) is in the domain of definition of $f(x,y,z)$.

38. False. In calculating the limit, we consider only those values of (x,y) where $f(x,y)$ is defined. The function in Exercise 36(a) is a counterexample.

39. (a) Since $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0 = 0$, and this is the value of $f(0,0)$, the function $f(x,0)$ is continuous at $x=0$. Similarly, $f(0,y)$ is continuous at $y=0$.

(b) If we approach $(0,0)$ along straight line $y = mx$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4} = \lim_{x\to 0} \frac{x^2(mx)^2}{x^4+(mx)^4} = \lim_{x\to 0} \frac{m^2}{1+m^4} = \frac{m^2}{1+m^4}.$$

Since this result depends on m, the limit of $f(x,y)$ as $(x,y)\to(0,0)$ does not exist. The function is therefore discontinuous at $(0,0)$.

40. (a) Suppose $\epsilon > 0$ is given. We must find a $\delta > 0$ such that whenever $0 < x^2+y^2 < \delta^2$,

$$\epsilon > |(xy+5)-5| = |xy|.$$

If we choose $\delta = \sqrt{\epsilon}$, then for $x^2+y^2 < \delta^2 = \epsilon$, it must certainly be true that $|x| < \sqrt{\epsilon}$ and $|y| < \sqrt{\epsilon}$, and hence

$$|xy| = |x||y| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

(b) Suppose $\epsilon > 0$ is given. We must find a $\delta > 0$ such that whenever $0 < (x-1)^2+(y-1)^2 < \delta^2$,

$$\begin{aligned}\epsilon &> |(x^2+2xy+5)-8| = |x^2+2xy-3|\\ &= |(x-1)^2+2(x-1)(y-1)-6+4x+2y|\\ &= |(x-1)^2+2(x-1)(y-1)+4(x-1)+2(y-1)|.\end{aligned}$$

Consider the quadratic $Q(z) = 3z^2+6z-\epsilon$. It is equal to zero when $z = (-6\pm\sqrt{36+12\epsilon})/6 = -1\pm\sqrt{1+\epsilon/3}$. Consequently, we can say that $3z^2+6z<\epsilon$ when $0<z<-1+\sqrt{1+\epsilon/3}$. If we now choose $\delta = -1+\sqrt{1+\epsilon/3}$, then for $0<(x-1)^2+(y-1)^2<\delta^2$, we have $|x-1|<\delta = -1+\sqrt{1+\epsilon/3}$ and $|y-1|<\delta = -1+\sqrt{1+\epsilon/3}$. Furthermore,

$$|(x-1)^2+2(x-1)(y-1)+4(x-1)+2(y-1)| < \delta^2+2\delta\delta+4\delta+2\delta = 3\delta^2+6\delta < \epsilon.$$

EXERCISES 12.3

1. $\dfrac{\partial f}{\partial x} = 3x^2y^2+2y;\quad \dfrac{\partial f}{\partial y} = 2x^3y+2x$

2. $\dfrac{\partial f}{\partial x} = 3y-16x^3y^4;\quad \dfrac{\partial f}{\partial y} = 3x-16x^4y^3$

3. $\dfrac{\partial f}{\partial x} = 4x^3/y^3;\quad \dfrac{\partial f}{\partial y} = -3x^4/y^4$

4. $\dfrac{\partial f}{\partial x} = \dfrac{(x+y)(1)-x(1)}{(x+y)^2} - \dfrac{1}{y} = \dfrac{y^2-(x+y)^2}{y(x+y)^2} = \dfrac{-x(x+2y)}{y(x+y)^2};$

$\dfrac{\partial f}{\partial y} = -\dfrac{x}{(x+y)^2} + \dfrac{x}{y^2} = \dfrac{-xy^2+x(x+y)^2}{y^2(x+y)^2} = \dfrac{x^2(x+2y)}{y^2(x+y)^2}$

5. $\dfrac{\partial f}{\partial x} = \dfrac{(2x^2+y)(1)-x(4x)}{(2x^2+y)^2} = \dfrac{y-2x^2}{(2x^2+y)^2};\quad \dfrac{\partial f}{\partial y} = -\dfrac{x}{(2x^2+y)^2}$

6. $\dfrac{\partial f}{\partial x} = y\cos(xy);\quad \dfrac{\partial f}{\partial y} = x\cos(xy)$

7. $\dfrac{\partial f}{\partial x} = \cos(x+y) - x\sin(x+y);\quad \dfrac{\partial f}{\partial y} = -x\sin(x+y)$

8. $\dfrac{\partial f}{\partial x} = \dfrac{1}{2}(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+y^2}};\quad \dfrac{\partial f}{\partial y} = \dfrac{1}{2}(x^2+y^2)^{-1/2}(2y) = \dfrac{y}{\sqrt{x^2+y^2}}$

9. $\dfrac{\partial f}{\partial x} = \sqrt{x^2-y^2} + \dfrac{x^2}{\sqrt{x^2-y^2}} = \dfrac{2x^2-y^2}{\sqrt{x^2+y^2}};\quad \dfrac{\partial f}{\partial y} = \dfrac{-xy}{\sqrt{x^2+y^2}}$

10. $\dfrac{\partial f}{\partial x} = \sec^2(2x^2+y^2)(4x);\quad \dfrac{\partial f}{\partial y} = \sec^2(2x^2+y^2)(2y)$

11. $\dfrac{\partial f}{\partial x} = e^{x+y}$; $\quad \dfrac{\partial f}{\partial y} = e^{x+y}$

12. $\dfrac{\partial f}{\partial x} = e^{xy}(y)$; $\quad \dfrac{\partial f}{\partial y} = e^{xy}(x)$

13. $\dfrac{\partial f}{\partial x} = ye^{xy} + xye^{xy}(y) = y(1+xy)e^{xy}$; $\quad \dfrac{\partial f}{\partial y} = xe^{xy} + xye^{xy}(x) = x(1+xy)e^{xy}$

14. $\dfrac{\partial f}{\partial x} = \dfrac{1}{x^2+y^2}(2x)$; $\quad \dfrac{\partial f}{\partial y} = \dfrac{1}{x^2+y^2}(2y)$

15. $\dfrac{\partial f}{\partial x} = \ln(xy) + \dfrac{x+1}{x}$; $\quad \dfrac{\partial f}{\partial y} = \dfrac{x+1}{y}$

16. $\dfrac{\partial f}{\partial x} = \cos(ye^x)(ye^x)$; $\quad \dfrac{\partial f}{\partial y} = \cos(ye^x)(e^x)$

17. $\dfrac{\partial f}{\partial x} = \dfrac{1/y}{1+(x/y)^2} = \dfrac{y}{x^2+y^2}$; $\quad \dfrac{\partial f}{\partial y} = \dfrac{-x/y^2}{1+(x/y)^2} = \dfrac{-x}{x^2+y^2}$

18. $\dfrac{\partial f}{\partial x} = \dfrac{1}{3}[1-\cos^3(x^2y)]^{-2/3}[-3\cos^2(x^2y)][-\sin(x^2y)](2xy) = \dfrac{2xy\cos^2(x^2y)\sin(x^2y)}{[1-\cos^3(x^2y)]^{2/3}}$;

$\dfrac{\partial f}{\partial y} = \dfrac{1}{3}[1-\cos^3(x^2y)]^{-2/3}[-3\cos^2(x^2y)][-\sin(x^2y)](x^2) = \dfrac{x^2\cos^2(x^2y)\sin(x^2y)}{[1-\cos^3(x^2y)]^{2/3}}$

19. $\dfrac{\partial f}{\partial x} = \dfrac{\cos x}{\cos y}$; $\quad \dfrac{\partial f}{\partial y} = -\dfrac{\sin x}{\cos^2 y}(-\sin y) = \dfrac{\sin x \sin y}{\cos^2 y}$

20. $\dfrac{\partial f}{\partial x} = \dfrac{1}{\sec\sqrt{x+y}}\sec\sqrt{x+y}\,\tan\sqrt{x+y}\,\dfrac{1}{2\sqrt{x+y}} = \dfrac{\tan\sqrt{x+y}}{2\sqrt{x+y}}$;

$\dfrac{\partial f}{\partial y} = \dfrac{1}{\sec\sqrt{x+y}}\sec\sqrt{x+y}\,\tan\sqrt{x+y}\,\dfrac{1}{2\sqrt{x+y}} = \dfrac{\tan\sqrt{x+y}}{2\sqrt{x+y}}$

21. $\dfrac{\partial f}{\partial x} = yze^{x^2+y^2} + xyze^{x^2+y^2}(2x) = yz(1+2x^2)e^{x^2+y^2}$

22. $\dfrac{\partial f}{\partial z} = \dfrac{1}{1+\dfrac{1}{(x^2+z^2)^2}}\dfrac{-1}{(x^2+z^2)^2}(2z) = \dfrac{-2z}{(x^2+z^2)^2+1}$

23. Since $\dfrac{\partial f}{\partial y} = x(x^2+y^2+z^2)^{1/3} + (xy/3)(x^2+y^2+z^2)^{-2/3}(2y)$, the partial derivative at $(1,1,0)$ is $2^{1/3} + (1/3)2^{-2/3}(2) = 2^{7/3}/3$.

24. Since $\dfrac{\partial f}{\partial x} = \dfrac{-zt}{(x^2+y^2-t^2)^2}(2x)$, we find $\dfrac{\partial f}{\partial x}_{|(1,-1,1,-1)} = \dfrac{-(1)(-1)(2)}{(1+1-1)^2} = 2$

25.
$$\frac{\partial f}{\partial t} = -\frac{2x}{t^3}\sqrt{t^2-y^2} + \frac{xt}{t^2\sqrt{t^2-y^2}} + \frac{x/y}{(t/3)\sqrt{t^2/9-1}}\left(\frac{1}{3}\right)$$
$$= \frac{-2x(t^2-y^2)+xt^2}{t^3\sqrt{t^2-y^2}} + \frac{3x}{ty\sqrt{t^2-9}} = \frac{x(2y^2-t^2)}{t^3\sqrt{t^2-y^2}} + \frac{3x}{ty\sqrt{t^2-9}}$$

26. $\dfrac{\partial f}{\partial x} = \dfrac{-1}{1+(1+x+y+z)^2}$

27. Since $xyt = 6$, the function and its derivatives are not defined at $(1,2,3)$.

28. $\dfrac{\partial f}{\partial x} = \dfrac{3x^2}{y} + \sin(yz/x) + x\cos(yz/x)\left(-\dfrac{yz}{x^2}\right) = \dfrac{3x^2}{y} + \sin(yz/x) - \dfrac{yz}{x}\cos(yz/x)$

29. The derivative is 0.

30.
$$\frac{\partial f}{\partial z} = z\mathrm{Sin}^{-1}\left(\frac{x}{z}\right) + \frac{z^2}{2}\frac{1}{\sqrt{1-(x/z)^2}}\left(-\frac{x}{z^2}\right) + \frac{x}{2}\left(\frac{1}{2}\right)(z^2-x^2)^{-1/2}(2z)$$
$$= z\mathrm{Sin}^{-1}\left(\frac{x}{z}\right) - \frac{x}{2}\frac{|z|}{\sqrt{z^2-x^2}} + \frac{xz}{2\sqrt{z^2-x^2}}$$
$$= \begin{cases} z\mathrm{Sin}^{-1}(x/z), & z > 0 \\ z\mathrm{Sin}^{-1}(x/z) + xz/\sqrt{z^2-x^2}, & z < 0 \end{cases}$$

31. $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=x\left[\frac{(x-y)(3x^2y)-x^3y}{(x-y)^2}\right]+y\left[\frac{(x-y)(x^3)-x^3y(-1)}{(x-y)^2}\right]$$
$$=\frac{3x^4y-3x^3y^2-x^4y+x^4y-x^3y^2+x^3y^2}{(x-y)^2}=\frac{3x^4y-3x^3y^2}{(x-y)^2}=\frac{3x^3y(x-y)}{(x-y)^2}=3f(x,y)$$

32. Since $f(x,y,z)=\dfrac{x^3}{yz}+\dfrac{y^3}{xz}+\dfrac{z^3}{xy}$,
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=x\left(\frac{3x^2}{yz}-\frac{y^3}{x^2z}-\frac{z^3}{x^2y}\right)+y\left(-\frac{x^3}{y^2z}+\frac{3y^2}{xz}-\frac{z^3}{xy^2}\right)+z\left(-\frac{x^3}{yz^2}-\frac{y^3}{xz^2}+\frac{3z^2}{xy}\right)$$
$$=\frac{x^3}{yz}+\frac{y^3}{xz}+\frac{z^3}{xy}=f(x,y,z).$$

33. $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=x\left[2x\cos\left(\frac{y+z}{x}\right)-(x^2+y^2)\left(-\frac{y+z}{x^2}\right)\sin\left(\frac{y+z}{x}\right)\right]$$
$$+y\left[2y\cos\left(\frac{y+z}{x}\right)-(x^2+y^2)\left(\frac{1}{x}\right)\sin\left(\frac{y+z}{x}\right)\right]$$
$$+z\left[-(x^2+y^2)\left(\frac{1}{x}\right)\sin\left(\frac{y+z}{x}\right)\right]$$
$$=(2x^2+2y^2)\cos\left(\frac{y+z}{x}\right)+\left[\frac{(x^2+y^2)(y+z)}{x}-\frac{y(x^2+y^2)}{x}-\frac{z(x^2+y^2)}{x}\right]\sin\left(\frac{y+z}{x}\right)$$
$$=2f(x,y,z)$$

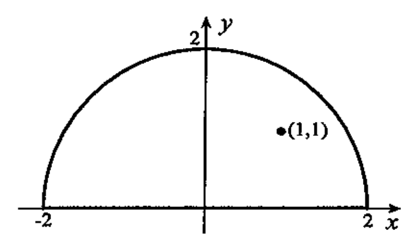
34. (a) This is the normal way to calculate the derivative.

(b) This would lead to an answer of zero. We must always differentiate with respect to a variable and **then** set that variable equal to its prescribed value.

(c) This is acceptable since variables other than the one with respect to which differentiation is being performed can be specified either before or after differentiation.

(d) Same as (b)

35. (a) $T_x(1,1)=(32x-24y)_{|(1,1)}=8$
(b) $T_y(1,1)=(-24x+80y)_{|(1,1)}=56$
(c) $T_x(1,0)=(32x-24y)_{|(1,0)}=32$
(d) $T_y(1,0)$ is not defined
(e) $T_x(0,2)$ is not defined
(f) $T_y(0,2)$ is not defined



36. The derivative vanishes if

$$0=\frac{\partial F}{\partial x}=-\frac{2AE}{L}\left(\frac{L}{\sqrt{L^2-2hx+x^2}}-1\right)+2AE\left(\frac{h-x}{L}\right)\left[\frac{-(L/2)(-2h+2x)}{(L^2-2hx+x^2)^{3/2}}\right]$$
$$=\frac{2AE}{L}\left[\frac{-L}{\sqrt{L^2-2hx+x^2}}+1+\frac{L(h-x)^2}{(L^2-2hx+x^2)^{3/2}}\right]$$
$$=\frac{2AE}{L(L^2-2hx+x^2)^{3/2}}\left[-L(L^2-2hx+x^2)+(L^2-2hx+x^2)^{3/2}+L(h-x)^2\right].$$

This implies that

$$(L^2-2hx+x^2)^{3/2}=L(L^2-2hx+x^2)-L(h^2-2hx+x^2)=L(L^2-h^2).$$

Consequently,

$$x^2-2hx+L^2=(L^3-Lh^2)^{2/3}\quad\Longrightarrow\quad x^2-2hx+L^2-(L^3-Lh^2)^{2/3}=0.$$

Solutions of this quadratic are

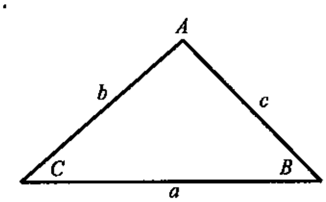
$$x = \frac{2h \pm \sqrt{4h^2 - 4L^2 + 4(L^3 - Lh^2)^{2/3}}}{2} = h \pm (L^3 - Lh^2)^{1/3}\sqrt{1 - \frac{L^2 - h^2}{(L^3 - Lh^2)^{2/3}}}$$
$$= h \pm (L^3 - Lh^2)^{1/3}\sqrt{1 - \frac{L^2 - h^2}{L^{2/3}(L^2 - h^2)^{2/3}}} = h \pm (L^3 - Lh^2)^{1/3}\sqrt{1 - \frac{(L^2 - h^2)^{1/3}}{L^{2/3}}}$$
$$= h \pm (L^3 - Lh^2)^{1/3}\sqrt{1 - \left(1 - \frac{h^2}{L^2}\right)^{1/3}}.$$

Choosing the negative sign gives the required solution.

37. For $f_x(x,y) = 2x - 3y$, the function must be of the form $f(x,y) = x^2 - 3xy + g(y)$ for some function $g(y)$. If we substitute this into the second condition, $-3x + g'(y) = 3x + 4y$ or $g'(y) = 4y + 6x$. This is contradictory, stating that a function of y depends on x.

38. (a) From the cosine law $a = \sqrt{b^2 + c^2 - 2bc\cos A}$,

$$a_A(b,c,A) = \frac{bc\sin A}{\sqrt{b^2 + c^2 - 2bc\cos A}}.$$



(b) From the cosine law in part (a),

$A = \mathrm{Cos}^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right)$, and therefore,

$$A_a(a,b,c) = \frac{-1}{\sqrt{1 - (b^2 + c^2 - a^2)^2/(4b^2c^2)}}\left(-\frac{a}{bc}\right)$$
$$= \frac{2a}{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}$$

(c) From the cosine law in part (a), $a_b(b,c,A) = \dfrac{b - c\cos A}{\sqrt{b^2 + c^2 - 2bc\cos A}}$.

(d) From the function in part (b),

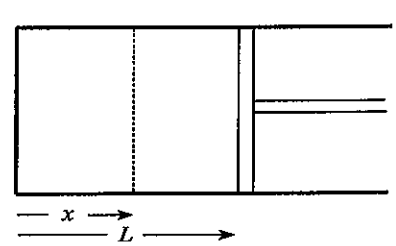
$$A_b(a,b,c) = \frac{-1}{\sqrt{1 - (b^2 + c^2 - a^2)^2/(4b^2c^2)}}\left[\frac{2bc(2b) - (b^2 + c^2 - a^2)(2c)}{4b^2c^2}\right]$$
$$= \frac{-2bc}{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}\left[\frac{2c(b^2 + a^2 - c^2)}{4b^2c^2}\right] = \frac{c^2 - a^2 - b^2}{b\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}$$

39. (a) $\dfrac{\partial\rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) + \dfrac{\partial}{\partial z}(\rho w) = 0 + \rho(4xt - yt) + \rho(-4xt + 2yt) + \rho(-yt) = 0$

(b)
$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = z + \frac{\partial}{\partial x}(x^3y^2 + xyt + x^2yzt + zt^2)$$
$$+ \frac{\partial}{\partial y}(xy^3z - 2xyt^2 + y^2z^2t - 2zt^3) + \frac{\partial}{\partial z}(5x^2y + 2xyz + 5xzt + 2z^2t)$$
$$= z + (3x^2y^2 + yt + 2xyzt) + (3xy^2z - 2xt^2 + 2yz^2t) + (2xy + 5xt + 4zt) \neq 0$$

40. Let x be the distance from the end of the cylinder to any cross section in the cylinder and let L be the distance from the end of the cylinder to the piston (figure to the right). The x-component (the only component) of the velocity of gas at position x in the cylinder is

$$u = \frac{12x}{L} = \frac{12x}{0.15 + 12t}.$$



Since $\rho = \rho(t)$, the equation of continuity gives

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \qquad \Longrightarrow \qquad \frac{d\rho}{dt} = -\rho\frac{\partial u}{\partial x} = -\frac{12\rho}{0.15 + 12t} \qquad \Longrightarrow \qquad \frac{1}{\rho}d\rho = -\frac{12}{0.15+12t}dt.$$

This is a separated differential equation with solutions defined implicitly by

$$\ln\rho = -\ln(0.15 + 12t) + C \qquad \Longrightarrow \qquad \rho = \frac{D}{0.15+12t}, \quad (D = e^C).$$

Since $\rho(0) = 18$, it follows that $18 = D/0.15 \Longrightarrow D = 2.7$. Consequently, $\rho(t) = \dfrac{2.7}{0.15+12t}$ kg/m^3.

41. (a) $\dfrac{\partial u}{\partial x} = -3y^2 + 3x^2 \qquad \dfrac{\partial u}{\partial y} = -6xy + 1 \qquad \dfrac{\partial v}{\partial x} = 6xy - 1 \qquad \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

(b) $\dfrac{\partial u}{\partial x} = \dfrac{(x^2+y^2)(2x+1) - (x^2+x+y^2)(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{(x^2+y^2)(2y) - (x^2+x+y^2)(2y)}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2}$

$\dfrac{\partial v}{\partial x} = \dfrac{2xy}{(x^2+y^2)^2} \qquad \dfrac{\partial v}{\partial y} = \dfrac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

(c) $\dfrac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = e^x[(x+1)\cos y - y\sin y]$

$\dfrac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) = -e^x[(x+1)\sin y + y\cos y]$

$\dfrac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x\sin y = e^x[(x+1)\sin y + y\cos y]$

$\dfrac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y) = e^x[(x+1)\cos y - y\sin y]$

42. (a) $\dfrac{\partial u}{\partial r} = \dfrac{(1+r^2+2r\cos\theta)(2r+\cos\theta) - (r^2+r\cos\theta)(2r+2\cos\theta)}{(1+r^2+2r\cos\theta)^2} = \dfrac{2r + r^2\cos\theta + \cos\theta}{(1+r^2+2r\cos\theta)^2}$;

$\dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = \dfrac{1}{r}\left[\dfrac{(1+r^2+2r\cos\theta)(r\cos\theta) - r\sin\theta(-2r\sin\theta)}{(1+r^2+2r\cos\theta)^2}\right] = \dfrac{2r+r^2\cos\theta+\cos\theta}{(1+r^2+2r\cos\theta)^2}$;

$\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = \dfrac{1}{r}\left[\dfrac{(1+r^2+2r\cos\theta)(-r\sin\theta) - (r^2+r\cos\theta)(-2r\sin\theta)}{(1+r^2+2r\cos\theta)^2}\right] = \dfrac{(r^2-1)\sin\theta}{(1+r^2+2r\cos\theta)^2}$;

$\dfrac{\partial v}{\partial r} = \dfrac{(1+r^2+2r\cos\theta)(\sin\theta) - r\sin\theta(2r+2\cos\theta)}{(1+r^2+2r\cos\theta)^2} = \dfrac{(1-r^2)\sin\theta}{(1+r^2+2r\cos\theta)^2}$

(b) $\dfrac{\partial u}{\partial r} = \dfrac{1}{2\sqrt{r}}\cos(\theta/2), \qquad \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = \dfrac{1}{\sqrt{r}}\cos(\theta/2)\left(\dfrac{1}{2}\right),$

$\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{1}{\sqrt{r}}\sin(\theta/2)\left(\dfrac{1}{2}\right), \qquad \dfrac{\partial v}{\partial r} = \dfrac{1}{2\sqrt{r}}\sin(\theta/2)$

(c) $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}, \quad \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = \dfrac{1}{r}, \quad \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = 0, \quad \dfrac{\partial v}{\partial r} = 0$

EXERCISES 12.4

1. $\nabla f = (2xy+z)\hat{\mathbf{i}} + (x^2+z^2)\hat{\mathbf{j}} + (x+2yz)\hat{\mathbf{k}}$

2. $\nabla f = 2xyz\hat{\mathbf{i}} + x^2z\hat{\mathbf{j}} + x^2y\hat{\mathbf{k}}$

3. $\nabla f = (2xy/z - 2z^6)\hat{\mathbf{i}} + (x^2/z)\hat{\mathbf{j}} - (x^2y/z^2 + 12xz^5)\hat{\mathbf{k}}$

4. $\nabla f = (2xy+y^2)\hat{\mathbf{i}} + (x^2+2xy)\hat{\mathbf{j}}$

5. $\nabla f = \cos(x+y)\,\hat{\mathbf{i}} + \cos(x+y)\,\hat{\mathbf{j}}$

6. $\nabla f = \dfrac{yz}{1+(xyz)^2}\hat{\mathbf{i}} + \dfrac{xz}{1+(xyz)^2}\hat{\mathbf{j}} + \dfrac{xy}{1+(xyz)^2}\hat{\mathbf{k}} = \dfrac{1}{1+(xyz)^2}(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})$

7. $\nabla f = \dfrac{-y/x^2}{1+y^2/x^2}\hat{\mathbf{i}} + \dfrac{1/x}{1+y^2/x^2}\hat{\mathbf{j}} = \dfrac{-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}}{x^2+y^2}$

8. $\nabla f = e^{x+y+z}\hat{\mathbf{i}} + e^{x+y+z}\hat{\mathbf{j}} + e^{x+y+z}\hat{\mathbf{k}} = e^{x+y+z}(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$

9. $\nabla f = \dfrac{-2x}{(x^2+y^2)^2}\hat{\mathbf{i}} - \dfrac{2y}{(x^2+y^2)^2}\hat{\mathbf{j}} = -\dfrac{2(x\hat{\mathbf{i}}+y\hat{\mathbf{j}})}{(x^2+y^2)^2}$

10. $\nabla f = \dfrac{-x}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{i}} + \dfrac{-y}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{j}} + \dfrac{-z}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{k}} = -\dfrac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}$

11. Since $\nabla f = (y+1)\hat{\mathbf{i}} + (x+1)\hat{\mathbf{j}}$, the gradient at $(1,3)$ is $4\hat{\mathbf{i}}+2\hat{\mathbf{j}}$.

12. Since $\nabla f = -\sin(x+y+z)(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$, the gradient at the point $(-1,1,1)$ is $= -(\sin 1)(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$.

13. Since $\nabla f = 2(x^2+y^2+z^2)(2x\hat{\mathbf{i}}+2y\hat{\mathbf{j}}+2z\hat{\mathbf{k}})$, the gradient at $(0,3,6)$ is $4(45)(3\hat{\mathbf{j}}+6\hat{\mathbf{k}}) = 540(\hat{\mathbf{j}}+2\hat{\mathbf{k}})$.

14. $\nabla f_{|(2,2)} = e^{-x^2-y^2}(-2x\hat{\mathbf{i}}-2y\hat{\mathbf{j}})_{|(2,2)} = -2e^{-8}(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}) = -4e^{-8}(\hat{\mathbf{i}}+\hat{\mathbf{j}})$

15. Since $\nabla f = \left[y\ln(x+y) + \dfrac{xy}{x+y}\right]\hat{\mathbf{i}} + \left[x\ln(x+y) + \dfrac{xy}{x+y}\right]\hat{\mathbf{j}}$, the gradient at $(4,-2)$ is $(-2\ln 2 - 4)\hat{\mathbf{i}} + (4\ln 2 - 4)\hat{\mathbf{j}} = -2(2+\ln 2)\hat{\mathbf{i}} + 4(\ln 2 - 1)\hat{\mathbf{j}}$.

16. $\nabla F = A\hat{\mathbf{i}} + B\hat{\mathbf{j}} + C\hat{\mathbf{k}}$ But according to Theorem 11.4, this vector is perpendicular to the plane.

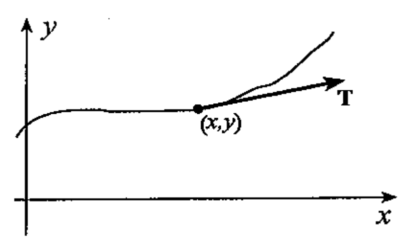
17. Since $\nabla F = (2,3,-2)$ is perpendicular to the plane $2x+3y-2z+4=0$ and $\nabla G = (1,-1,3)$ is perpendicular to the plane $x-y+3z+6=0$, a vector along the line of intersection of the planes is $\nabla F \times \nabla G = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -2 \\ 1 & -1 & 3 \end{vmatrix} = (7,-8,-5)$. Since $(-22/5, 8/5, 0)$ is a point on the line, parametric equations for the line are $x = -22/5 + 7t$, $y = 8/5 - 8t$, $z = -5t$.

18.
$$\nabla(fg) = \frac{\partial}{\partial x}(fg)\hat{\mathbf{i}} + \frac{\partial}{\partial y}(fg)\hat{\mathbf{j}} + \frac{\partial}{\partial z}(fg)\hat{\mathbf{k}} = \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\hat{\mathbf{i}} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\hat{\mathbf{j}} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\hat{\mathbf{k}}$$
$$= f\left(\frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}\right) + g\left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) = f\,\nabla g + g\,\nabla f$$
It looks like the product rule for differentiation.

19. (a) If we set $y = x$ in the quotient $\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}} = \dfrac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$, and take the limit as $x \to 0^+$, we obtain $\lim\limits_{x\to 0^+}\sqrt{x^2+x^2} = 0$.

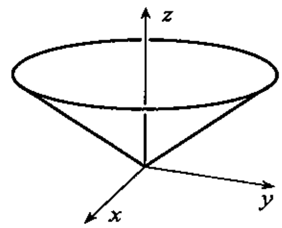
(b) If we set $y = x$ in the quotient $\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}} = \dfrac{2x^3-3y}{\sqrt{x^2+y^2}}$, and take the limit as $x \to 0^+$, we obtain $\lim\limits_{x\to 0^+}\dfrac{2x^3-3x}{\sqrt{x^2+x^2}} = \lim\limits_{x\to 0^+}\dfrac{2x^2-3}{\sqrt{2}} = -\dfrac{3}{\sqrt{2}}$.

20. The slope dy/dx of the tangent line at any point (x,y) on this curve is defined by
$$3x^2 + y + x\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0,$$
or,
$$\frac{dy}{dx} = -\frac{3x^2+y}{x+4y^3}.$$

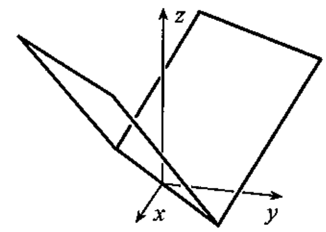


It follows that a vector along the tangent line at (x,y) is $\mathbf{T} = (x+4y^3, -3x^2-y)$. A vector perpendicular to $\mathbf{T}$ is $\mathbf{N} = (3x^2+y, x+4y^3)$. But $\nabla F = (3x^2+y)\hat{\mathbf{i}} + (x+4y^3)\hat{\mathbf{j}}$, and therefore $\nabla F = \mathbf{N}$.

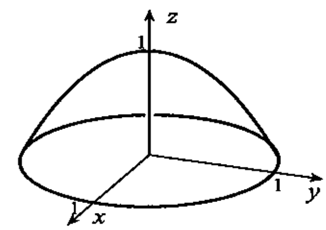
21. The gradient
$$\nabla F = \frac{-x}{\sqrt{x^2+y^2}}\hat{\mathbf{i}} - \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{j}} + \hat{\mathbf{k}},$$
is undefined at the vertex of the cone.



22. Since $|x-y|$ fails to have derivatives with respect to x and y when $y=x$, it follows that ∇F is not defined at the points $(x,x,0)$.



23. $\nabla f = -2x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}}$ The gradient is equal to $\mathbf{0}$ at the point $(0,0)$. This is the highest point on the surface $z = 1 - x^2 - y^2$.



24. If $\nabla f = (2xy-y)\hat{\mathbf{i}} + (x^2-x)\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x} = 2xy - y$ and $\dfrac{\partial f}{\partial y} = x^2 - x$. From the first equation, we can say that $f(x,y) = x^2y - xy + \phi(y)$, where $\phi(y)$ is any differentiable function of y. To determine $\phi(y)$ we substitute this expression for $f(x,y)$ into the second equation, $x^2 - x + d\phi/dy = x^2 - x$. Consequently, $d\phi/dy = 0$, and therefore $\phi(y) = C$, a constant. Thus, $f(x,y) = x^2y - xy + C$.

25. If $\nabla f = (2x/y+1)\hat{\mathbf{i}} + (-x^2/y^2+2)\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x} = 2x/y + 1$ and $\dfrac{\partial f}{\partial y} = -x^2/y^2 + 2$. From the first equation, we can say that $f(x,y) = x^2/y + x + \phi(y)$, where $\phi(y)$ is any differentiable function of y. To determine $\phi(y)$ we substitute this expression for $f(x,y)$ into the second equation, $-x^2/y^2 + d\phi/dy = -x^2/y^2 + 2$. Consequently, $d\phi/dy = 2$, and therefore $\phi(y) = 2y + C$, a constant. Thus, $f(x,y) = x^2/y + x + 2y + C$.

26. If $\nabla f = yz\hat{\mathbf{i}} + (xz+2yz)\hat{\mathbf{j}} + (xy+y^2)\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x} = yz$, $\dfrac{\partial f}{\partial y} = xz + 2yz$, $\dfrac{\partial f}{\partial z} = xy + y^2$. From the first, $f(x,y,z) = xyz + \phi(y,z)$, where $\phi(y,z)$ is any function with first partial derivatives. Substitution of this into the second equation gives

$$xz + \frac{\partial \phi}{\partial y} = xz + 2yz \quad \Longrightarrow \quad \frac{\partial \phi}{\partial y} = 2yz.$$

Consequently, $\phi(y,z) = y^2z + \psi(z)$, and $f(x,y,z) = xyz + y^2z + \psi(z)$. Substitution of this into the third equation requires $\psi(z)$ to satisfy

$$xy + y^2 + \frac{d\psi}{dz} = xy + y^2 \quad \Longrightarrow \quad \frac{d\psi}{dz} = 0.$$

Thus, $\psi(z) = C$, and $f(x,y,z) = xyz + y^2z + C$.

27. If $\nabla f = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})/\sqrt{x^2+y^2+z^2}$, then

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

From the first, $f(x,y,z) = \sqrt{x^2+y^2+z^2} + \phi(y,z)$, where $\phi(y,z)$ is any function with first partial derivatives. Substitution of this into the second equation gives

$$\frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{\partial \phi}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \quad \Longrightarrow \quad \frac{\partial \phi}{\partial y} = 0.$$

Consequently, $\phi(y,z) = \psi(z)$, and $f(x,y,z) = \sqrt{x^2+y^2+z^2} + \psi(z)$. Substitution of this into the third equation requires $\psi(z)$ to satisfy

$$\frac{z}{\sqrt{x^2+y^2+z^2}} + \frac{d\psi}{dz} = \frac{z}{\sqrt{x^2+y^2+z^2}} \implies \frac{d\psi}{dz} = 0.$$

Thus, $\psi(z) = C$, and $f(x,y,z) = \sqrt{x^2+y^2+z^2} + C$.

28. If $\nabla f = \nabla g$, then $\dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial x}$ and $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial y}$. The first requires $f(x,y) = g(x,y) + \phi(y)$, which substituted into the second gives

$$\frac{\partial g}{\partial y} + \frac{d\phi}{dy} = \frac{\partial g}{\partial y} \qquad \text{or} \qquad \frac{d\phi}{dy} = 0.$$

Thus, $\phi(y) = C$, a constant, and $f(x,y) = g(x,y) + C$.

29. If $\nabla f = \mathbf{0}$, then $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$, $\dfrac{\partial f}{\partial z} = 0$. From the first, $f(x,y,z) = \phi(y,z)$, where $\phi(y,z)$ is any function with first partial derivatives. Substitution of this function into the second equation gives $\dfrac{\partial \phi}{\partial y} = 0 \implies \phi(y,z) = \psi(z)$. Substitution of this into the third equation requires $\psi(z)$ to satisfy $d\psi/dz = 0$, from which $\psi(z) = C$, and $f(x,y,z) = C$.

30. To find the slope dy/dx of the tangent line to C at any point (x,y), we implicitly differentiate $F(x,y) = 0$ with respect to x. This can be accomplished by differentiating $F(x,y)$ partially with respect to x, and adding to this the partial derivative with respect to y multiplied by dy/dx,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Thus, $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}$, and it follows that a tangent vector to C at (x,y) is $\left(\dfrac{\partial F}{\partial y}, -\dfrac{\partial F}{\partial x}\right)$. A vector perpendicular to C must therefore be $\left(\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}\right)$. Since this is ∇F, the proof is complete.

EXERCISES 12.5

1. From $\dfrac{\partial f}{\partial x} = 2xy^2 - 6x^2y$, we obtain $\dfrac{\partial^2 f}{\partial x^2} = 2y^2 - 12xy$.

2. From $\dfrac{\partial f}{\partial y} = -\dfrac{2x}{y^2} + 12x^3y^3$ and $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{4x}{y^3} + 36x^3y^2$, we obtain $\dfrac{\partial^3 f}{\partial y^3} = -\dfrac{12x}{y^4} + 72x^3y$.

3. From $\dfrac{\partial f}{\partial z} = xy\cos(xyz)$, we obtain $\dfrac{\partial^2 f}{\partial z^2} = -x^2y^2\sin(xyz)$.

4. Since $\dfrac{\partial f}{\partial z} = xye^{x+y+z} + xyze^{x+y+z} = xy(1+z)e^{x+y+z}$,

$$\frac{\partial^2 f}{\partial y \partial z} = x(1+z)e^{x+y+z} + xy(1+z)e^{x+y+z} = x(1+z)(1+y)e^{x+y+z}.$$

5. From $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$, we obtain $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{-xy}{(x^2+y^2)^{3/2}}$.

6. Since $\dfrac{\partial f}{\partial y} = e^{x+y} + \dfrac{2x^2}{y^3}$, we find $\dfrac{\partial^2 f}{\partial x \partial y} = e^{x+y} + \dfrac{4x}{y^3}$, and $\dfrac{\partial^3 f}{\partial x^2 \partial y} = e^{x+y} + \dfrac{4}{y^3}$.

7. From $\dfrac{\partial f}{\partial y} = 9x^3y^2 + \dfrac{3x}{y^2}$, and $\dfrac{\partial^2 f}{\partial y^2} = 18x^3y - \dfrac{6x}{y^3}$, we obtain $\dfrac{\partial^3 f}{\partial y^3} = 18x^3 + \dfrac{18x}{y^4}$. At $(1,3)$, the derivative is $164/9$.

8. Since $\dfrac{\partial f}{\partial z} = 2x^2z + 2y^2z$, we have $\dfrac{\partial^2 f}{\partial y \partial z} = 4yz$, and $\dfrac{\partial^3 f}{\partial x \partial y \partial z} = 0$. This must also be its value at $(1, 0, -1)$.

9. From $\dfrac{\partial f}{\partial x} = \dfrac{-x}{\sqrt{1 - x^2 - y^2}}$, we obtain $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{-1}{\sqrt{1 - x^2 - y^2}} - \dfrac{x^2}{(1 - x^2 - y^2)^{3/2}}$, and this simplifies to $(y^2 - 1)/(1 - x^2 - y^2)^{3/2}$.

10. Since $\dfrac{\partial f}{\partial z} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}} \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{z}{x^2 + y^2 + z^2}$,

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{x^2 + y^2 + z^2} - \frac{2z^2}{(x^2 + y^2 + z^2)^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2}.$$

11. From $\dfrac{\partial f}{\partial y} = x^2e^y + 2ye^x$, we obtain $\dfrac{\partial^2 f}{\partial x \partial y} = 2xe^y + 2ye^x$ and $\dfrac{\partial^3 f}{\partial x^2 \partial y} = 2e^y + 2ye^x$.

12. From $\dfrac{\partial f}{\partial x} = \dfrac{1}{1 + (y/x)^2}\left(\dfrac{-y}{x^2}\right) = \dfrac{x^2}{x^2 + y^2}\left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x^2 + y^2}$, we obtain $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2xy}{(x^2 + y^2)^2}$.

13. From $\dfrac{\partial f}{\partial y} = -2y \csc^2(x^2 + y^2 + z^2)$, we obtain

$$\frac{\partial^2 f}{\partial y^2} = -2\csc^2(x^2 + y^2 + z^2) + 8y^2 \csc^2(x^2 + y^2 + z^2)\cot(x^2 + y^2 + z^2),$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial^2 y} = {} & 8x\csc^2(x^2 + y^2 + z^2)\cot(x^2 + y^2 + z^2) - 32xy^2\csc^2(x^2 + y^2 + z^2)\cot^2(x^2 + y^2 + z^2) \\ & - 16xy^2\csc^4(x^2 + y^2 + z^2).\end{aligned}$$

14. From $\dfrac{\partial f}{\partial y} = \dfrac{1}{\sqrt{1 - (x^2 + y^2)^{-2}}} \dfrac{-2y}{(x^2 + y^2)^2} = \dfrac{x^2 + y^2}{\sqrt{(x^2 + y^2)^2 - 1}} \dfrac{-2y}{(x^2 + y^2)^2} = \dfrac{-2y}{(x^2 + y^2)\sqrt{(x^2 + y^2)^2 - 1}}$,
we obtain $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{4xy}{(x^2 + y^2)^2\sqrt{(x^2 + y^2)^2 - 1}} + \dfrac{y(2)(x^2 + y^2)(2x)}{(x^2 + y^2)[(x^2 + y^2)^2 - 1]^{3/2}}$.
Thus, $\dfrac{\partial^2 f}{\partial x \partial y}_{|(-2,-2)} = \dfrac{16}{8^2\sqrt{8^2 - 1}} + \dfrac{(-2)(2)(8)(-4)}{8[8^2 - 1]^{3/2}} = \dfrac{127}{756\sqrt{7}}$.

15. Three derivatives with respect to y eliminates the first term, and then derivatives with respect to x eliminate the second term; that is, the derivative is 0.

16. $\dfrac{\partial f}{\partial x} = 8x^7y^9z^{10}$; $\quad \dfrac{\partial^2 f}{\partial x^2} = 56x^6y^9z^{10}$; $\quad \cdots$; $\quad \dfrac{\partial^8 f}{\partial x^8} = 8!y^9z^{10}$

17. The derivative is 0.

18. $\dfrac{\partial f}{\partial x} = -\sin(x + y^3)$; $\dfrac{\partial^2 f}{\partial x^2} = -\cos(x + y^3)$; $\dfrac{\partial^3 f}{\partial x^3} = \sin(x + y^3)$; $\dfrac{\partial^4 f}{\partial x^3 \partial y} = 3y^2\cos(x + y^3)$

19. From $\dfrac{\partial f}{\partial t} = \dfrac{-t}{\sqrt{x^2 + y^2 + z^2 - t^2}}$, we obtain $\dfrac{\partial^2 f}{\partial z \partial t} = \dfrac{zt}{(x^2 + y^2 + z^2 - t^2)^{3/2}}$, and

$$\frac{\partial^3 f}{\partial y \partial z \partial t} = \frac{-3yzt}{(x^2 + y^2 + z^2 - t^2)^{5/2}}, \qquad \frac{\partial^4 f}{\partial x \partial y \partial z \partial t} = \frac{15xyzt}{(x^2 + y^2 + z^2 - t^2)^{7/2}}.$$

20. Since $\dfrac{\partial f}{\partial y} = \dfrac{x}{xy\sqrt{x^2y^2 - 1}} = \dfrac{1}{y\sqrt{x^2y^2 - 1}}$, we find $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{-xy^2}{y(x^2y^2 - 1)^{3/2}} = \dfrac{-xy}{(x^2y^2 - 1)^{3/2}}$.

21.
$$\begin{aligned} x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= x\left[2x + y + y\cos\left(\frac{x}{y}\right)\right] + y\left[x + 2y\sin\left(\frac{x}{y}\right) - x\cos\left(\frac{x}{y}\right)\right] \\ &= 2\left[x^2 + xy + y^2\sin\left(\frac{x}{y}\right)\right] = 2f(x, y)\end{aligned}$$

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=x^2\left[2-\sin\left(\frac{x}{y}\right)\right]+2xy\left[1+\cos\left(\frac{x}{y}\right)+\frac{x}{y}\sin\left(\frac{x}{y}\right)\right]$$
$$+y^2\left[2\sin\left(\frac{x}{y}\right)-\frac{2x}{y}\cos\left(\frac{x}{y}\right)-\frac{x^2}{y^2}\sin\left(\frac{x}{y}\right)\right]$$
$$=2x^2+2xy+(-x^2+2x^2+2y^2-x^2)\sin\left(\frac{x}{y}\right)+(2xy-2xy)\cos\left(\frac{x}{y}\right)$$
$$=2\left[x^2+xy+y^2\sin\left(\frac{x}{y}\right)\right]=2f(x,y)$$

22. $\dfrac{\partial u}{\partial x}=1+ze^{y/x}(-y/x^2)=1-(yz/x^2)e^{y/x};\quad \dfrac{\partial u}{\partial y}=1+ze^{y/x}(1/x)=1+(z/x)e^{y/x};$

$\dfrac{\partial u}{\partial z}=e^{y/x};\quad \dfrac{\partial^2 u}{\partial x^2}=2(yz/x^3)e^{y/x}-(yz/x^2)e^{y/x}(-y/x^2)=[yz(2x+y)/x^4]e^{y/x};$

$\dfrac{\partial^2 u}{\partial x\partial y}=-(z/x^2)e^{y/x}-(yz/x^2)e^{y/x}(1/x)=-[z(x+y)/x^3]e^{y/x};$

$\dfrac{\partial^2 u}{\partial y^2}=(z/x)e^{y/x}(1/x)=(z/x^2)e^{y/x};\quad \dfrac{\partial^2 u}{\partial y\partial z}=(1/x)e^{y/x};\quad \dfrac{\partial^2 u}{\partial z^2}=0;$

$\dfrac{\partial^2 u}{\partial x\partial z}=e^{y/x}(-y/x^2)=-(y/x^2)e^{y/x}.$

Thus,

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2yz\frac{\partial^2 u}{\partial y\partial z}+2xz\frac{\partial^2 u}{\partial x\partial z}$$
$$=\frac{yz(2x+y)}{x^2}e^{y/x}+\frac{y^2z}{x^2}e^{y/x}+0-\frac{2yz(x+y)}{x^2}e^{y/x}+\frac{2yz}{x}e^{y/x}-\frac{2yz}{x}e^{y/x}=0.$$

23. $\dfrac{\partial f}{\partial x}=2x+2y;\quad \dfrac{\partial f}{\partial y}=-2y+2x+1;\quad \dfrac{\partial^2 f}{\partial x^2}=2;\quad \dfrac{\partial^2 f}{\partial y^2}=-2$ Since $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$, and second partial derivatives are continuous, $f(x,y)$ is harmonic in the entire xy-plane.

24. From $\dfrac{\partial f}{\partial x}=\dfrac{2x}{x^2+y^2}$, we obtain $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{(x^2+y^2)(2)-2x(2x)}{(x^2+y^2)^2}=\dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$, and similarly, $\dfrac{\partial^2 f}{\partial y^2}=\dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$. Since $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$, and second partial derivatives are continuous except at $(0,0)$, the function $f(x,y)$ is harmonic in any region not containing $(0,0)$.

25. $\dfrac{\partial f}{\partial x}=3x^2y^2-3y;\quad \dfrac{\partial f}{\partial y}=2x^3y-3x;\quad \dfrac{\partial^2 f}{\partial x^2}=6xy^2;\quad \dfrac{\partial^2 f}{\partial y^2}=2x^3$ Since $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=6xy^2+2x^3$, the function is not harmonic.

26. $\dfrac{\partial f}{\partial x}=6xyz+y;\ \dfrac{\partial^2 f}{\partial x^2}=6yz;\ \dfrac{\partial f}{\partial y}=3x^2z-3y^2z+x;\ \dfrac{\partial^2 f}{\partial y^2}=-6yz;\ \dfrac{\partial f}{\partial z}=3x^2y-y^3;\ \dfrac{\partial^2 f}{\partial z^2}=0$

Since $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}=6yz-6yz+0=0$, and all second partial derivatives are continuous, $f(x,y,z)$ is harmonic in all space.

27. From $\dfrac{\partial f}{\partial x}=\dfrac{-x}{(x^2+y^2+z^2)^{3/2}},\quad \dfrac{\partial^2 f}{\partial x^2}=\dfrac{-1}{(x^2+y^2+z^2)^{3/2}}+\dfrac{3x^2}{(x^2+y^2+z^2)^{5/2}}=\dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$. With similar results for second derivatives with respect to y and z,

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}=0.$$

Since second partial derivatives are continuous except at $(0,0,0)$, the function is harmonic in any region not containing $(0,0,0)$.

28. From $\dfrac{\partial f}{\partial x} = 3x^2y^3z^3$, we find $\dfrac{\partial^2 f}{\partial x^2} = 6xy^3z^3$. Similarly, $\dfrac{\partial^2 f}{\partial y^2} = 6x^3yz^2$ and $\dfrac{\partial^2 f}{\partial z^2} = 6x^3y^3z$. Since $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 6xyz(y^2z^2 + x^2z^2 + x^2y^2)$, and this is not zero in any region of space, the function is not harmonic.

29. According to Example 12.12, potential at (x, y, z) due to point charges q_i at points (x_i, y_i, z_i) is $V(x, y, z) = \displaystyle\sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0 r_i} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$, where $r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}$. Now,

$$\frac{\partial}{\partial x}\left(\frac{1}{r_i}\right) = \frac{-(x-x_i)}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{3/2}}, \text{ so that}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r_i}\right) &= \frac{-1}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{3/2}} + \frac{3(x-x_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}}\\ &= \frac{2(x-x_i)^2 - (y-y_i)^2 - (z-z_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}}.\end{aligned}$$

With similar results for derivatives of $1/r_i$ with respect to y and z,

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{1}{r_i}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r_i}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r_i}\right) &= \frac{2(x-x_i)^2 - (y-y_i)^2 - (z-z_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}}\\ &\quad + \frac{2(y-y_i)^2 - (x-x_i)^2 - (z-z_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}}\\ &\quad + \frac{2(z-z_i)^2 - (x-x_i)^2 - (y-y_i)^2}{[(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2]^{5/2}} = 0.\end{aligned}$$

It follows that each term in the sum representing $V(x, y, z)$ satisfies Laplace's equation, so $V(x, y, z)$ itself does.

30. From $\dfrac{\partial V}{\partial x} = GM\left[\dfrac{-x}{(x^2+y^2+z^2)^{3/2}}\right]$, we obtain

$$\frac{\partial^2 V}{\partial x^2} = GM\left[\frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right] = GM\left[\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}\right].$$

Similarly, $\dfrac{\partial^2 V}{\partial y^2} = GM\left[\dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}\right]$ and $\dfrac{\partial^2 V}{\partial z^2} = GM\left[\dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}\right]$, and therefore

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(except at } (0,0,0)\text{)}.$$

31. As long as C and D are constants,

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} &= [-9\pi^2 C(e^{3\pi y} - e^{-3\pi y})\sin(3\pi x) - 16\pi^2 D(e^{4\pi y} - e^{-4\pi y})\sin(4\pi x)]\\ &\quad + [9\pi^2 C(e^{3\pi y} - e^{-3\pi y})\sin(3\pi x) + 16\pi^2 D(e^{4\pi y} - e^{-4\pi y})\sin(4\pi x)] = 0.\end{aligned}$$

Since second partial derivatives are continuous, $T(x, y)$ is harmonic in the plate. It is obvious that $T(x, y)$ satisfies $T(0, y) = T(1, y) = T(x, 0) = 0$. Finally

$$\begin{aligned}T(x, 1) &= C(e^{3\pi} - e^{-3\pi})\sin(3\pi x) + D(e^{4\pi} - e^{-4\pi})\sin(4\pi x)\\ &= \sin(3\pi x) - 2\sin(4\pi x).\end{aligned}$$

32. Suppose the cross-sectional area of the rod is A and its density is ρ. Then $F(x) = \rho g A(L - x)$, and $y(x)$ must satisfy

$$0 = E\frac{d^2y}{dx^2} + \rho g A(L-x) \implies \frac{d^2y}{dx^2} = -\frac{\rho g A}{E}(L-x) \implies \frac{dy}{dx} = \frac{\rho g A}{2E}(L-x)^2 + C.$$

Since $y'(L) = 0$, it follows that $C = 0$. Thus, $\dfrac{dy}{dx} = \dfrac{\rho g A}{2E}(L-x)^2$, and a second integration gives $y = -\dfrac{\rho g A}{6E}(L-x)^3 + D$. The condition $y(0) = 0$ implies that $D = \dfrac{\rho g A L^3}{6E}$, and therefore displacements of cross sections are given by $y(x) = -\dfrac{\rho g A}{6E}(L-x)^3 + \dfrac{\rho g A L^3}{6E}$. Since $y(L) = \dfrac{\rho g A L^3}{6E}$, it follows that the length of the bar is $L + \dfrac{\rho g A L^3}{6E}$.

33. Two integrations of $Ey'' = 0$ give $y = Ax + B$. The conditions $y(0) = 0$ and $y'(L) = F/E$ imply that $y = Fx/E$. Since $y(L) = FL/E$, the length of the bar is $L + FL/E = L(1 + F/E)$.

34. (a) Since second partial derivatives are $\dfrac{\partial^2 y}{\partial x^2} = -\lambda^2 y$ and $\dfrac{\partial^2 y}{\partial t^2} = -c^2\lambda^2 y$, it follows that $y(x,t)$ does indeed satisfy equation 12.13a.

(b) The condition $y(0,t) = 0$ implies that $0 = B(C\sin c\lambda t + D\cos c\lambda t)$ for all t. This requires $B = 0$. With $B = 0$, condition $y_x(L,t) = 0$ implies that $0 = A\lambda\cos\lambda L(C\sin c\lambda t + D\cos c\lambda t)$. Since A cannot be zero, nor can λ, and the term in brackets cannot be equal to 0 for all t, we must set $\cos\lambda L = 0$. But this implies that $\lambda L = (2n-1)\pi/2$, where n is an integer; that is, $\lambda = (2n-1)\pi/(2L)$.

(c) With $B = 0$ and $g(x) = 0$, initial condition 12.13c requires $0 = y_t(x,0) = A\sin\lambda x(Cc\lambda) \implies C = 0$.

(d) With $B = 0$ and $f(x) = 0$, initial condition 12.13b requires $0 = y(x,0) = A\sin\lambda x(D) \implies D = 0$.

35. (a) Since second partial derivatives are $\dfrac{\partial^2 y}{\partial x^2} = -\lambda^2 y$ and $\dfrac{\partial^2 y}{\partial t^2} = -c^2\lambda^2 y$, it follows that $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$.

(b) The condition $y(0,t) = 0$ implies that $0 = B(C\sin c\lambda t + D\cos c\lambda t)$ for all t. This requires $B = 0$. With $B = 0$, condition $y(L,t) = 0$ implies that $0 = A\sin\lambda L(C\sin c\lambda t + D\cos c\lambda t)$. Since $A \neq 0$ and the term in brackets cannot be equal to 0 for all t, we must set $\sin\lambda L = 0$. But this implies that $\lambda L = n\pi$, where n is an integer; that is, $\lambda = n\pi/L$.

(c) With $B = 0$ and $g(x) = 0$, initial condition 12.13c requires $0 = y_t(x,0) = A\sin\lambda x(Cc\lambda) \implies C = 0$.

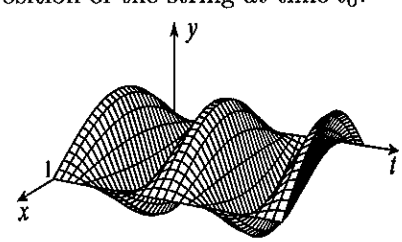
(d) With $B = 0$ and $f(x) = 0$, initial condition 12.13b requires $0 = y(x,0) = A\sin\lambda x(D) \implies D = 0$.

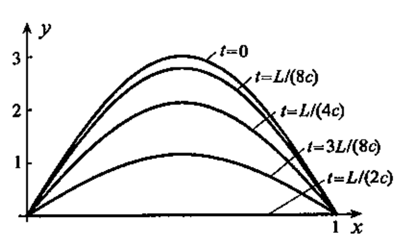
36. With $F(x) = -9.81\rho$, a constant, two integrations of $d^2y/dx^2 = 9.81\rho/T$ give $y(x) = 4.905\rho x^2/T + Ax + B$. The conditions $y(0) = 0 = y(L)$ require $B = 0$ and $A = -4.905\rho L/T$. Thus, $y(x) = 4.905\rho x(x-L)/T$, a parabola. In Exercise 35 it was assumed that that the string experiences only small displacements. This results in a constant force for gravity. No such assumption is made in Example 3.39.

37. Two integrations of $d^2y/dx^2 = 9.81\rho/T$ give $y(x) = 4.905\rho x^2/T + Ax + B$. The conditions $y'(0) = 0 = y(L)$ require $A = 0$ and $B = -4.905\rho L^2/T$. Thus, $y(x) = 4.905\rho(x^2 - L^2)/T$, a parabola.

38. (a) Since $\dfrac{\partial^2 y}{\partial t^2} = -(3\pi^2c^2/L^2)\sin(\pi x/L)\cos(\pi ct/L)$ and $\dfrac{\partial^2 y}{\partial x^2} = -(3\pi^2/L^2)\sin(\pi x/L)\cos(\pi ct/L)$, it follows that $y(x,t)$ satisfies the partial differential equation. It is straightforward to check that it satisfies the remaining conditions.

(b) A plot of the surface is shown to the left below. A cross section of the surface with a plane $x = x_0$ gives a graphical history of the displacement of the point x_0 in the string. A cross section $t = t_0$ gives the position of the string at time t_0.





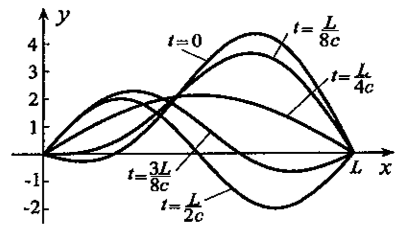
(c) Plots are shown to the right above.

39. Since

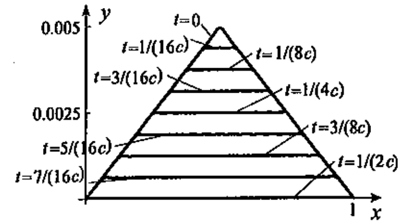
$$\frac{\partial^2 y}{\partial t^2} = -(3\pi^2 c^2/L^2)\sin(\pi x/L)\cos(\pi ct/L) + (8\pi^2 c^2/L^2)\sin(2\pi x/L)\cos(2\pi ct/L),$$

$$\frac{\partial^2 y}{\partial x^2} = -(3\pi^2/L^2)\sin(\pi x/L)\cos(\pi ct/L) + (8\pi^2/L^2)\sin(2\pi x/L)\cos(2\pi ct/L),$$

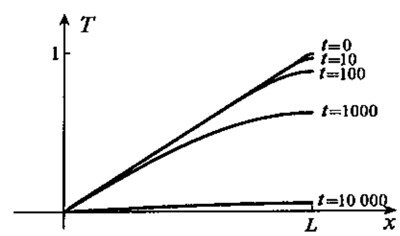
it follows that $y(x,t)$ satisfies the partial differential equation. It is straightforward to check that it satisfies the remaining conditions. Plots are shown to the right.



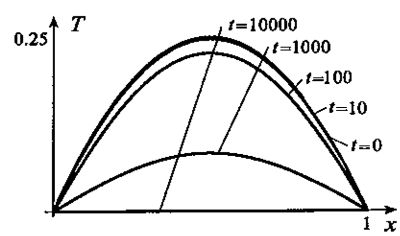
40. (a) If differentiation and summation operations can be interchanged, it is a matter of showing that the function $z(x,t) = \sin(2n-1)\pi x \cos(2n-1)\pi ct$ satisfies the condition for any positive integer n. Since $\partial^2 z/\partial t^2 = -(2n-1)^2\pi^2 c^2 z(x,t)$ and $\partial^2 z/\partial x^2 = -(2n-1)^2\pi^2 z(x,t)$, it follows that $z(x,t)$ satisfies the partial differential equation. It is obvious that $z(x,t)$ satisfies the boundary conditions $z(0,t) = z(L,t) = 0$ and the initial condition $z_t(x,0) = 0$.
(b) Plots are shown to the right.



41. (a) Since $\partial T/\partial t = -k\lambda^2 T$ and $\partial^2 T/\partial x^2 = -\lambda^2 T$, it follows that $T(x,t)$ does indeed satisfy the heat conduction equation.
(b) The boundary condition $T(0,t) = 0$ requires $0 = B$. With $B = 0$, the condition $T_x(L,t) = 0$ necessitates $0 = A\lambda\cos\lambda L e^{-k\lambda^2 t}$. Since A cannot be equal to zero, we must set $\cos\lambda L = 0$ and this requires $\lambda L = (2n-1)\pi/2 \implies \lambda = (2n-1)\pi/(2L)$, where n is an integer.
(c) If differentiation and summation operations can be interchanged, it is a matter of showing that the function $z(x,t) = e^{-(2n-1)^2\pi^2 kt/(4L^2)}\sin[(2n-1)\pi x/(2L)]$ satisfies the partial differential equation and boundary conditions. Since this is the function in part (a) with λ as determined in part (b), it must indeed satisfy the heat conduction equation and boundary conditions.
(d) Plots are shown to the right. Notice that the $t = 0$ plot approximates $f(x) = x$. Each plot passes through the origin and the slope at $x = L$ is zero, reflecting the boundary conditions.



42. (a) The boundary condition $T(0,t) = 0$ requires $0 = B$. With $B = 0$, the condition $T(L,t) = 0$ necessitates $0 = A\sin\lambda L e^{-k\lambda^2 t}$. Since A cannot be equal to zero, we must set $\sin\lambda L = 0$ and this requires $\lambda L = n\pi \implies \lambda = n\pi/L$, where n is an integer.
(b) Plots are shown to the right. The $t = 0$ plot does appear to be the parabola $x(1-x)$.



43. Two integrations of the differential equation give $T(x) = Ax + B$. The boundary conditions require $T_0 = T(0) = B$ and $T_L = T(L) = AL + B$. Thus, $T(x) = T_0 + (T_L - T_0)x/L$.

44. When $F(x)$ has constant value F, two integrations of the differential equation give $T(x) = -Fx^2/(2k) + Ax + B$. The boundary conditions require $T_0 = T(0) = B$ and $T_L = T(L) = -FL^2/(2k) + AL + B$. These give $T(x) = T_0 + (T_L - T_0)x/L + Fx(L-x)/(2k)$.

45. Two integrations of $d^2y/dx^2 = -x/E$ give $y(x) = -x^3/(6E) + Ax + B$. The boundary conditions give $0 = y(0) = B$ and $0 = y'(L) = A$. Thus, $y(x) = -x^3/(6E)$.

46. Integration of $\dfrac{d^2y}{dx^2} = -\dfrac{\tau(x)}{E}$ gives $\dfrac{dy}{dx} = -\dfrac{1}{E}\displaystyle\int \tau(x)\,dx + C$. To incorporate the boundary condtion at $x = L$, we write this in the form

$$y'(x) = -\frac{1}{E}\int_0^x \tau(u)\,du + C.$$

The condition $y'(L) = 0$ now implies that $0 = -\dfrac{1}{E}\displaystyle\int_0^L \tau(u)\,du + C$. This gives the indicated value for C. A second integration gives

$$y(x) = Cx - \frac{1}{E}\int\left[\int_0^x \tau(u)\,du\right]dx + D.$$

To incorporate the boundary condition at $x = 0$, we write this in the form

$$y(x) = Cx - \frac{1}{E}\int_0^x\left[\int_0^v \tau(u)\,du\right]dv + D.$$

The condition $y(0) = 0$ now implies that $D = 0$, and therefore

$$y(x) = Cx - \frac{1}{E}\int_0^x\int_0^v \tau(u)\,du\,dv.$$

47. Using the Cauchy-Riemann equations from Exercise 41 in Section 12.3,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}.$$

This shows that $u(x,y)$ satisfies Laplace's equation. Since second partial derivatives are assumed continuous, $u(x,y)$ is harmonic. A similar proof shows that $v(x,y)$ is harmonic.

48. (a) Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$, the function is harmonic.

(b) According to Exercise 41 of Section 12.3, $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 2y$. From the first $v(x,y) = 2xy + \phi(x)$, which substituted into the second gives $2y + d\phi/dx = 2y$. Thus, $\phi(x) = C$, a constant, and $v(x,y) = 2xy + C$.

49. (a) Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^x\cos y - e^x\cos y = 0$, the function is harmonic.

(b) According to Exercise 41 of Section 12.3, $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = e^x\cos y + 1$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = e^x\sin y$. From the first $v(x,y) = e^x\sin y + y + \phi(x)$, which substituted into the second gives $e^x\sin y + d\phi/dx = e^x\sin y$. Thus, $\phi(x) = C$, a constant, and $v(x,y) = e^x\sin y + y + C$.

50. Since $\dfrac{\partial f}{\partial x} = n(x^2+y^2+z^2)^{n-1}(2x)$, we find

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= 2n(x^2+y^2+z^2)^{n-1} + 2nx(n-1)(x^2+y^2+z^2)^{n-2}(2x)\\ &= 2n(x^2+y^2+z^2)^{n-2}[x^2+y^2+z^2+2(n-1)x^2]\\ &= 2n(x^2+y^2+z^2)^{n-2}[(2n-1)x^2+y^2+z^2].\end{aligned}$$

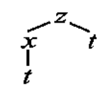
Similarly, $\dfrac{\partial^2 f}{\partial y^2} = 2n(x^2+y^2+z^2)^{n-2}[x^2+(2n-1)y^2+z^2]$, and
$\dfrac{\partial^2 f}{\partial z^2} = 2n(x^2+y^2+z^2)^{n-2}[x^2+y^2+(2n-1)z^2]$. The function satisfies Laplace's equation if

$$\begin{aligned}0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= 2n(x^2+y^2+z^2)^{n-2}[(2n+1)x^2+(2n+1)y^2+(2n+1)z^2]\\ &= 2n(2n+1)(x^2+y^2+z^2)^{n-1}.\end{aligned}$$

Thus, $n = 0$ or $n = -1/2$. When $n = 0$, the function is equal to 1, and it is harmonic in all space. When $n = -1/2$, the function is $1/\sqrt{x^2+y^2+z^2}$, and it is harmonic in any region not containing the origin.

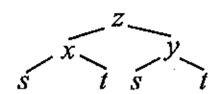
EXERCISES 12.6

1. In general, $\displaystyle \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial t}$, and specifically,



$$\frac{dz}{dt} = \left[\frac{(x+t)(t^2) - xt^2}{(x+t)^2}\right](3e^{3t}) + \left[\frac{(x+t)(2xt) - xt^2}{(x+t)^2}\right] = \frac{3t^3e^{3t} + xt(t+2x)}{(x+t)^2}.$$

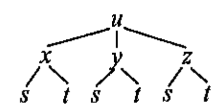
2. In general, $\displaystyle \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$, and specifically,



$$\frac{\partial z}{\partial t} = \left(2xe^y + \frac{y}{x}\right)(-s^2\sin t) + (x^2e^y + \ln x)\left[\frac{4(2t)}{(t^2+2s)\sqrt{(t^2+2s)^2-1}}\right].$$

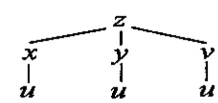
3. In general, $\displaystyle \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$, and specifically,

$$\frac{\partial u}{\partial s} = \frac{x}{\sqrt{x^2+y^2+z^2}}(2t) + \frac{y}{\sqrt{x^2+y^2+z^2}}(2s) + \frac{z}{\sqrt{x^2+y^2+z^2}}(t)$$
$$= \frac{2xt + 2ys + zt}{\sqrt{x^2+y^2+z^2}}.$$



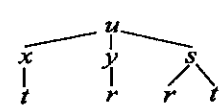
4. In general, $\displaystyle \frac{dz}{du} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du} + \frac{\partial z}{\partial v}\frac{dv}{du}$, and specifically,

$$\frac{dz}{du} = (2xyv^3)(3u^2+2) + (x^2v^3)\left(\frac{2u}{u^2+1}\right) + (3x^2yv^2)(ue^u + e^u)$$
$$= xv^2\left[2yv(3u^2+2) + \frac{2xuv}{u^2+1} + 3xye^u(u+1)\right].$$

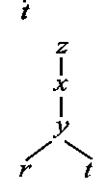


5. In general, $\displaystyle \frac{\partial u}{\partial r} = \frac{\partial u}{\partial y}\frac{dy}{dr} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial r}$, and specifically,

$$\frac{\partial u}{\partial r} = \left(\frac{ys}{\sqrt{x^2+y^2s}}\right)\left[\frac{2r}{\sqrt{1-(r^2+5)^2}}\right] + \left(\frac{y^2}{2\sqrt{x^2+y^2s}}\right)[t\sec^2(rt)]$$
$$= \frac{y}{2\sqrt{x^2+y^2s}}\left[\frac{4rs}{\sqrt{1-(r^2+5)^2}} + yt\sec^2(rt)\right].$$

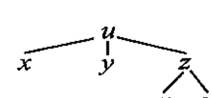


6. In general, $\displaystyle \frac{\partial z}{\partial t} = \frac{dz}{dx}\frac{dx}{dy}\frac{\partial y}{\partial t}$, and specifically,



$$\frac{\partial z}{\partial t} = (3^{x+2}\ln 3)(2y)[-\csc(r^2+t)\cot(r^2+t)].$$

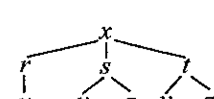
7. In general, $\displaystyle \frac{\partial u}{\partial x} = \left.\frac{\partial u}{\partial x}\right)_{y,z} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}$, and specifically,



$$\frac{\partial u}{\partial x} = \frac{-xy}{(x^2+y^2+z^2)^{3/2}} - \frac{yz}{(x^2+y^2+z^2)^{3/2}}\left(\frac{1}{y}\right) = \frac{-(xy+z)}{(x^2+y^2+z^2)^{3/2}}$$

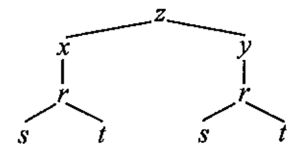
8. In general, $\displaystyle \frac{\partial x}{\partial y} = \frac{\partial x}{\partial r}\frac{dr}{dy} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial x}{\partial t}\frac{\partial t}{\partial y}$, and specifically,

$$\frac{\partial x}{\partial y} = (2rs^2t^2)(-5y^{-6}) + (2r^2st^2)\left[\frac{-2y}{(y^2+z^2)^2}\right] + (2r^2s^2t)\left(\frac{-2}{y^3}\right)$$
$$= -2rst\left[\frac{5st}{y^6} + \frac{2rty}{(y^2+z^2)^2} + \frac{2rs}{y^3}\right].$$



9. In general, $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dr}\dfrac{\partial r}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dr}\dfrac{\partial r}{\partial t}$, and specifically,

$$\frac{\partial z}{\partial t} = e^{x+y}(2)\left[\ln(s^2+t^2) + \frac{2t^2}{s^2+t^2}\right] + e^{x+y}(2)\left[\ln(s^2+t^2) + \frac{2t^2}{s^2+t^2}\right]$$
$$= 4e^{x+y}\left[\ln(s^2+t^2) + \frac{2t^2}{s^2+t^2}\right].$$

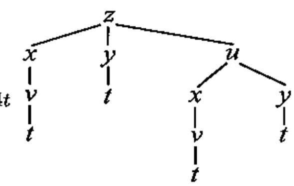


10. In general,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dv}\frac{dv}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}\frac{dx}{dv}\frac{dv}{dt} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}\frac{dy}{dt}$$
$$= \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial x}\right)\frac{dx}{dv}\frac{dv}{dt} + \left(\frac{\partial z}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}\right)\frac{dy}{dt},$$

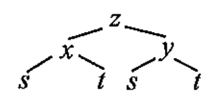
and specifically,

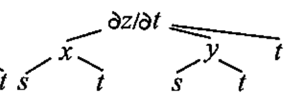
$$\frac{dz}{dt} = \left\{2x + 2u\left[\frac{-2x}{(x^2-y^2)^2}\right]\right\}(3v^2-6v)e^t + \left\{2y + 2u\left[\frac{2y}{(x^2-y^2)^2}\right]\right\}4e^{4t}$$
$$= 6xve^t(v-2)\left[1 - \frac{2u}{(x^2-y^2)^2}\right] + 8ye^{4t}\left[1 + \frac{2u}{(x^2-y^2)^2}\right].$$



11. From $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$

$$= (2xy^2 + e^y)(2t) + (2x^2y + xe^y)(-2t)$$
$$= 2t(2xy^2 - 2x^2y + e^y - xe^y),$$

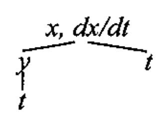




$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)_{x,y}$$
$$= 2t(2y^2 - 4xy - e^y)(2t) + 2t(4xy - 2x^2 + e^y - xe^y)(-2t) + 2(2xy^2 - 2x^2y + e^y - xe^y)$$
$$= 4t^2[2(x^2 + y^2 - 4xy) + (x-2)e^y] + 2[2(xy^2 - x^2y) + (1-x)e^y].$$

12. From $\dfrac{dx}{dt} = \dfrac{\partial x}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial x}{\partial t} = (2y+t)(t^2e^t + 2te^t) + (y - 2t)$

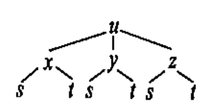
$$= te^t(t^2 + 2yt + 4y + 2t) + y - 2t,$$

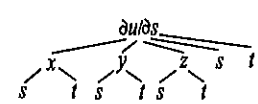


$$\frac{d^2x}{dt^2} = \frac{\partial}{\partial y}\left(\frac{dx}{dt}\right)\frac{dy}{dt} + \frac{\partial}{\partial t}\left(\frac{dx}{dt}\right)$$
$$= [te^t(2t+4) + 1](t^2e^t + 2te^t) + [te^t(2t + 2y + 2) + (te^t + e^t)(t^2 + 2yt + 4y + 2t) - 2]$$
$$= 2t^2(t+2)^2e^{2t} + (6t^2 + 6t + 8ty + t^3 + 2yt^2 + 4y)e^t - 2.$$

13. From $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial s}$

$$= (2x + yz)(2s) + (2y + xz)(2s) + (2z + xy)(t)$$
$$= 2s(2x + 2y + xz + yz) + t(2z + xy),$$

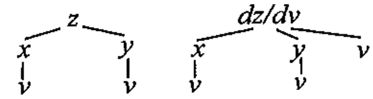




$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial s}\right)\frac{\partial y}{\partial s} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial s}\right)\frac{\partial z}{\partial s} + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right)_{x,y,z,t}$$
$$= (4s + 2sz + ty)(2s) + (4s + 2sz + tx)(2s) + (2sx + 2sy + 2t)(t) + 2(2x + 2y + xz + yz)$$
$$= 8s^2(z+2) + 2(x+y)(2st + z + 2) + 2t^2.$$

14. From $\dfrac{dz}{dv} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dv} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dv}$

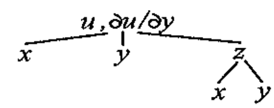
$$= y\cos(xy)(-3\sin v) + x\cos(xy)(4\cos v)$$
$$= \cos(xy)(-3y\sin v + 4x\cos v),$$



$$\frac{d^2z}{dv^2} = \frac{\partial}{\partial x}\left(\frac{dz}{dv}\right)\frac{dx}{dv} + \frac{\partial}{\partial y}\left(\frac{dz}{dv}\right)\frac{dy}{dv} + \frac{\partial}{\partial v}\left(\frac{dz}{dv}\right)$$
$$= [-y\sin(xy)(-3y\sin v + 4x\cos v) + 4\cos v\ \cos(xy)](-3\sin v)$$
$$+[-x\sin(xy)(-3y\sin v + 4x\cos v) - 3\sin v\ \cos(xy)](4\cos v)$$
$$+\cos(xy)(-3y\cos v - 4x\sin v)$$
$$= (24xy\sin v\ \cos v - 16x^2\cos^2 v - 9y^2\sin^2 v)\sin(xy)$$
$$-(24\sin v\ \cos v + 3y\cos v + 4x\sin v)\cos(xy)$$
$$= -(3y\sin v - 4x\cos v)^2\sin(xy) - (24\sin v\ \cos v + 3y\cos v + 4x\sin v)\cos(xy).$$

15. From $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial y} + \left.\dfrac{\partial u}{\partial y}\right)_{x,z}$

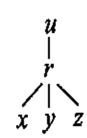
$$= \frac{-yz}{(x^2+y^2+z^2)^{3/2}}\left(\frac{-x}{y^2}\right) + \frac{1}{\sqrt{x^2+y^2+z^2}} + \frac{-y^2}{(x^2+y^2+z^2)^{3/2}}$$
$$= \frac{xz}{y(x^2+y^2+z^2)^{3/2}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}},$$



$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{y,z}$$
$$= \left[\frac{x}{y(x^2+y^2+z^2)^{3/2}} - \frac{3xz^2}{y(x^2+y^2+z^2)^{5/2}} + \frac{2z}{(x^2+y^2+z^2)^{3/2}} - \frac{3z(x^2+z^2)}{(x^2+y^2+z^2)^{5/2}}\right]\left(\frac{1}{y}\right)$$
$$+ \frac{z}{y(x^2+y^2+z^2)^{3/2}} - \frac{3x^2z}{y(x^2+y^2+z^2)^{5/2}} + \frac{2x}{(x^2+y^2+z^2)^{3/2}} - \frac{3x(x^2+z^2)}{(x^2+y^2+z^2)^{5/2}}.$$

This simplifies to $\dfrac{x(x^2+y^2+z^2)(1-y^2) + 3(xy+z)(y^3-xz)}{y^2(x^2+y^2+z^2)^{5/2}}$.

16. From the schematic, $\dfrac{\partial u}{\partial x} = \dfrac{du}{dr}\dfrac{\partial r}{\partial x} = \dfrac{du}{dr}\dfrac{x}{\sqrt{x^2+y^2+z^2}}$,



and similarly,

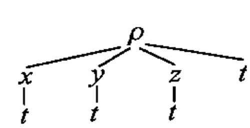
$$\frac{\partial u}{\partial y} = \frac{du}{dr}\frac{y}{\sqrt{x^2+y^2+z^2}},\quad \frac{\partial u}{\partial z} = \frac{du}{dr}\frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Consequently,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{du}{dr}\right)^2\left(\frac{x^2}{x^2+y^2+z^2} + \frac{y^2}{x^2+y^2+z^2} + \frac{z^2}{x^2+y^2+z^2}\right) = \left(\frac{du}{dr}\right)^2.$$

17. (a) $\dfrac{d\rho}{dt} = \dfrac{\partial\rho}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial\rho}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial\rho}{\partial z}\dfrac{dz}{dt} + \dfrac{\partial\rho}{\partial t}$

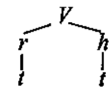
(b) A fixed observer in the gas measures the rate of change of ρ as $\partial\rho/\partial t$. An observer moving with the gas measures the rate of change of ρ as $d\rho/dt$.



18. The volume V of a right circular cone in terms of its radius r and height h is $V = \pi r^2 h/3$. From the schematic,

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2}{3}\pi r h\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}.$$

When $r = 10$ and $h = 20$,

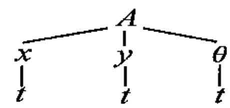


$$\frac{dV}{dt} = \frac{2}{3}\pi(10)(20)(1) + \frac{1}{3}\pi(10)^2(-2) = \frac{200\pi}{3}\ \mathrm{cm}^3/\mathrm{min}.$$

Multivariable calculus is not needed since $V = \pi r^2 h/3$ can be differentiated with respect to t using the product and power rules, $\dfrac{dV}{dt} = \dfrac{1}{3}\pi\left(2r\dfrac{dr}{dt}h + r^2\dfrac{dh}{dt}\right)$, and this is the same result as above.

19.

$$\begin{aligned}\frac{dA}{dt} &= \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}\\ &= \frac{1}{2}y\sin\theta\frac{dx}{dt} + \frac{1}{2}x\sin\theta\frac{dy}{dt} + \frac{1}{2}xy\cos\theta\frac{d\theta}{dt}\end{aligned}$$

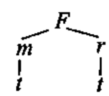


At the instant in question,

$$\frac{dA}{dt} = \frac{1}{2}(2)\sin(1/3)\left(\frac{1}{2}\right) + \frac{1}{2}(1)\sin(1/3)\left(\frac{1}{2}\right) + \frac{1}{2}(1)(2)\cos(1/3)\left(\frac{-1}{10}\right) = 0.151\ \mathrm{m}^2/\mathrm{s}.$$

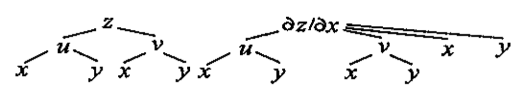
20. Newton's universal law of gravitation states that $F = GMm/r^2$, where $G = 6.67 \times 10^{-11}$, $M =$ mass of earth, $m =$ mass of rocket, and $r =$ distance from centre of earth to rocket.

$$\begin{aligned}\frac{dF}{dt} &= \frac{\partial F}{\partial m}\frac{dm}{dt} + \frac{\partial F}{\partial r}\frac{dr}{dt} = \frac{GM}{r^2}\frac{dm}{dt} - \frac{2GMm}{r^3}\frac{dr}{dt}\\ &= \frac{GM}{r^3}\left(r\frac{dm}{dt} - 2m\frac{dr}{dt}\right).\end{aligned}$$



With $M = (4/3)\pi(\text{radius of earth})^3(\text{density of earth}) = (4/3)\pi(6.37 \times 10^6)^3(5.52 \times 10^3)$, we find that when $r = 6.47 \times 10^6$,

$$\begin{aligned}\frac{dF}{dt} &= 6.67 \times 10^{-11}\left(\frac{4}{3}\right)\pi\frac{(6.37 \times 10^6)^3(5.52 \times 10^3)}{(6.47 \times 10^6)^3}\left[6.47 \times 10^6(-50) - 2(12 \times 10^6)(2 \times 10^3)\right]\\ &= -7.11 \times 10^4\ \mathrm{N/s}.\end{aligned}$$



21. From $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}$,

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)_{u,v,y}\\ &= \left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u\partial v}\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial^2 z}{\partial v\partial u}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x}\\ &\quad + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}\\ &= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}.\end{aligned}$$

22. (a) Since $f(tx, ty) = (tx)^2 + (tx)(ty) + 3(ty)^2 = t^2(x^2 + xy + 3y^2) = t^2 f(x, y)$, the function is positively homogeneous of degree 2.

(b) Since $f(tx, ty) = (tx)^2(ty) + (tx)(ty) - 2(tx)(ty)^2 = t^2(tx^2y + xy - 2txy^2)$, the function is not homogeneous.

(c) Since $f(tx, ty, tz) = (tx)^2\sin[ty/(tz)] + (ty)^2 + (ty)^3/(tz) = t^2[x^2\sin(y/z) + y^2 + y^3/z]$, the function is positively homogeneous of degree 2.

(d) Since $f(tx, ty, tz) = (tx)e^{ty/(tz)} - (tx)(ty)(tz) = t(xe^{y/z} - t^2xyz)$, the function is not homogeneous.

(e) Since $f(ux, uy, uz, ut) = (ux)^4 + (uy)^4 + (uz)^4 + (ut)^4 - (ux)(uy)(uz)(ut) = u^4(x^4 + y^4 + z^4 + t^4 - xyzt)$, the function is positively homogeneous of degree 4.

(f) Since $f(ux, uy, uz, ut) = e^{u^2x^2+u^2y^2}[(uz)^2 + (ut)^2] = u^2(z^2+t^2)e^{u^2(x^2+y^2)}$, the function is not homogeneous.

(g) Since $f(tx, ty, tz) = \cos(t^2xy)\sin(t^2yz)$, the function is not homogeneous.

(h) Since $f(tx, ty) = \sqrt{t^2x^2+t^2xy+t^2y^2}e^{ty/(tx)}(2t^2x^2 - 3t^2y^2) = t^3\sqrt{x^2+xy+y^2}e^{y/x}(2x^2-3y^2)$, the function is positively homogeneous of degree 3.

23.
$$\frac{\partial V}{\partial r} = \frac{1}{\pi\left[1+\left(\dfrac{2Rr\sin\theta}{R^2-r^2}\right)^2\right]}\left[\frac{(R^2-r^2)(2R\sin\theta)-2Rr\sin\theta(-2r)}{(R^2-r^2)^2}\right] = \frac{2R(R^2+r^2)\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{4Rr\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]} - \frac{2R(R^2+r^2)\sin\theta[-4r(R^2-r^2)+8R^2r\sin^2\theta]}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]^2}$$

$$\frac{\partial V}{\partial \theta} = \frac{1}{\pi\left[1+\left(\dfrac{2Rr\sin\theta}{R^2-r^2}\right)^2\right]}\left(\frac{2Rr\cos\theta}{R^2-r^2}\right) = \frac{2Rr(R^2-r^2)\cos\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]}$$

$$\frac{\partial^2 V}{\partial \theta^2} = \frac{-2Rr(R^2-r^2)\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]} - \frac{2Rr(R^2-r^2)\cos\theta(8R^2r^2\sin\theta\cos\theta)}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]^2}$$

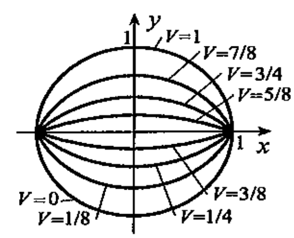
When these are substituted into the left side of equation 12.24, and considerable algebra is performed, the result is zero.

(b) If we solve the equation for r in terms of V and θ, the result is

$$r = \frac{-R\sin\theta + \sqrt{R^2\sin^2\theta + R^2\tan^2[\pi(2V-1)/2]}}{\tan[\pi(2V-1)/2]}.$$

When we set $R = 1$,

$$r = \frac{-\sin\theta + \sqrt{\sin^2\theta + \tan^2[\pi(2V-1)/2]}}{\tan[\pi(2V-1)/2]}.$$



This is plotted to the right for $V = 1/8,\ 1/4,\ 3/8,\ 5/8,\ 3/4,\ 7/8$. For $V = 1/2$, the equation in part (a) requires $\sin\theta = 0$, from which $\theta = 0$ or $\theta = \pi$, the x-axis.

24.
$$\frac{\partial V}{\partial r} = \frac{V_1-V_2}{\pi\left[1+\left(\dfrac{2Rr\sin\theta}{R^2-r^2}\right)^2\right]}\left[\frac{(R^2-r^2)(2R\sin\theta)-2Rr\sin\theta(-2r)}{(R^2-r^2)^2}\right] = \frac{2(V_1-V_2)R(R^2+r^2)\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{4(V_1-V_2)Rr\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]} - \frac{2(V_1-V_2)R(R^2+r^2)\sin\theta[-4r(R^2-r^2)+8R^2r\sin^2\theta]}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]^2}$$

$$\frac{\partial V}{\partial \theta} = \frac{V_1-V_2}{\pi\left[1+\left(\dfrac{2Rr\sin\theta}{R^2-r^2}\right)^2\right]}\left(\frac{2Rr\cos\theta}{R^2-r^2}\right) = \frac{2(V_1-V_2)Rr(R^2-r^2)\cos\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]}$$

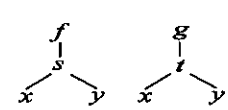
$$\frac{\partial^2 V}{\partial \theta^2} = \frac{-2(V_1-V_2)Rr(R^2-r^2)\sin\theta}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]} - \frac{2(V_1-V_2)Rr(R^2-r^2)\cos\theta(8R^2r^2\sin\theta\cos\theta)}{\pi[(R^2-r^2)^2+4R^2r^2\sin^2\theta]^2}$$

When these are substituted into the left side of equation 12.24, and considerable algebra is performed, the result is zero.

25. If we set $s = x^2 - y^2$ and $t = xy$, then

$$\nabla f(x^2-y^2) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} = \frac{df}{ds}\frac{\partial s}{\partial x}\hat{\mathbf{i}} + \frac{df}{ds}\frac{\partial s}{\partial y}\hat{\mathbf{j}} = f'(s)(2x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}}),$$

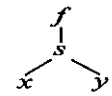
$$\nabla g(xy) = \frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} = \frac{dg}{dt}\frac{\partial t}{\partial x}\hat{\mathbf{i}} + \frac{dg}{dt}\frac{\partial t}{\partial y}\hat{\mathbf{j}} = g'(t)(y\hat{\mathbf{i}} + x\hat{\mathbf{j}}),$$



Consequently, $\nabla f(x^2-y^2)\cdot\nabla g(xy) = f'(s)g'(t)(2xy - 2xy) = 0$.

26. If we set $s = x - y$, then

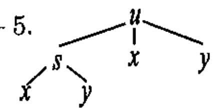
$$\frac{\partial f}{\partial y} = \frac{df}{ds}\frac{\partial s}{\partial y} = \frac{df}{ds}(-1) \quad \text{and} \quad \frac{\partial f}{\partial x} = \frac{df}{ds}\frac{\partial s}{\partial x} = \frac{df}{ds}(1).$$



Hence, $\dfrac{\partial f}{\partial y} = -\dfrac{\partial f}{\partial x}$.

27. If we set $s = 4x - 3y$, then

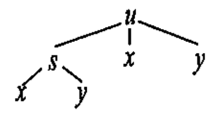
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \left.\frac{\partial u}{\partial x}\right)_{s,y} = f'(s)(4) - 5, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \left.\frac{\partial u}{\partial y}\right)_{s,x} = f'(s)(-3) + 5.$$



Hence, $3\dfrac{\partial u}{\partial x} + 4\dfrac{\partial u}{\partial y} = 3[4f'(s) - 5] + 4[-3f'(s) + 5] = 5.$

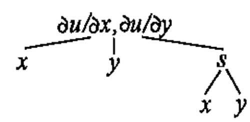
28. If we set $s = x + y$, then $u = xf(s) + yg(s)$. From the schematic,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \left.\frac{\partial u}{\partial x}\right)_{y,s} \\ &= \left[x\frac{df}{ds} + y\frac{dg}{ds}\right](1) + f(s) = xf'(s) + yg'(s) + f(s),\end{aligned}$$



and similarly, $\dfrac{\partial u}{\partial y} = xf'(s) + yg'(s) + g(s)$. The schematic for $\partial u/\partial x$ and $\partial u/\partial y$ gives

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)_{y,s} \\ &= [xf''(s) + yg''(s) + f'(s)](1) + f'(s); \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{y,s} \\ &= [xf''(s) + yg''(s) + g'(s)](1) + f'(s); \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\frac{\partial s}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)_{x,s} = [xf''(s) + yg''(s) + g'(s)](1) + g'(s);\end{aligned}$$

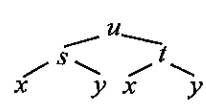


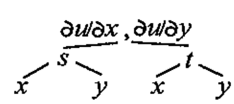
Thus, $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

29. If we set $s = x - y$, $t = x + y$, and $u = f(s) + g(t)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = f'(s) + g'(t), \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -f'(s) + g'(t),$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\frac{\partial t}{\partial x} = f''(s) + g''(t), \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right)\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\frac{\partial t}{\partial y} = f''(s) + g''(t),\end{aligned}$$

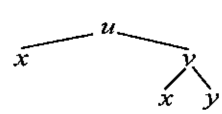




Thus, $\partial^2 u/\partial x^2 = \partial^2 u/\partial y^2$.

30. From the schematic for $u = x^2 f(v)$, $v = y/x$,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \left.\frac{\partial u}{\partial x}\right)_v + \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} = 2xf(v) + x^2 f'(v)\left(\frac{-y}{x^2}\right), \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} = x^2 f'(v)\left(\frac{1}{x}\right).\end{aligned}$$



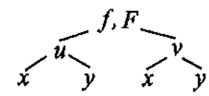
Hence,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x[2xf(v) - yf'(v)] + y[xf'(v)] = 2x^2 f(v) = 2u.$$

This is also an immediate consequence of Euler's theorem since u is positively homogeneous of degree 2.

31. The schematic to the right describes the functional situation $f(x,y) = F[u(x,y), v(x,y)]$ where $u = u(x,y) = (x+y)/2$ and $v = v(x,y) = (x-y)/2$. It gives

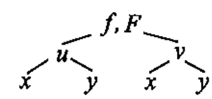
$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v},$$
$$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial F}{\partial v}.$$



Hence,

$$0 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v}\right)^2 + \left(\frac{1}{2}\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial F}{\partial v}\right)^2 = \frac{1}{2}\left[\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2\right].$$

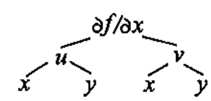
32. The schematic to the right describes the functional situation $f(x,y) = F[u(x,y), v(x,y)]$ where $u = u(x,y) = (x+y)/2$ and $v = v(x,y) = (x-y)/2$. It gives



$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v}.$$

The schematic for $\partial f/\partial x$ leads to

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x}\\ &= \left(\frac{1}{2}\frac{\partial^2 F}{\partial u^2} + \frac{1}{2}\frac{\partial^2 F}{\partial u\partial v}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\frac{\partial^2 F}{\partial v\partial u} + \frac{1}{2}\frac{\partial^2 F}{\partial v^2}\right)\left(\frac{1}{2}\right)\\ &= \frac{1}{4}\left(\frac{\partial^2 F}{\partial u^2} + 2\frac{\partial^2 F}{\partial u\partial v} + \frac{\partial^2 F}{\partial v^2}\right).\end{aligned}$$



Similarly, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{1}{4}\left(\dfrac{\partial^2 F}{\partial u^2} - 2\dfrac{\partial^2 F}{\partial u\partial v} + \dfrac{\partial^2 F}{\partial v^2}\right)$. Hence, $\quad 0 = \dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 F}{\partial u\partial v}$.

33. Since the transformation $x = u\cos v$, $y = u\sin v$ is that due to polar coordinates with u and v replacing r and θ, we use the results of Example 12.19 to write that

$$\frac{\partial f}{\partial x} = \cos v\frac{\partial F}{\partial u} - \frac{\sin v}{u}\frac{\partial F}{\partial v}.$$

A similar derivation gives $\dfrac{\partial f}{\partial y} = \sin v\dfrac{\partial F}{\partial u} + \dfrac{\cos v}{u}\dfrac{\partial F}{\partial v}$. Hence,

$$\begin{aligned}0 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 &= \left(\cos v\frac{\partial F}{\partial u} - \frac{\sin v}{u}\frac{\partial F}{\partial v}\right)^2 + \left(\sin v\frac{\partial F}{\partial u} + \frac{\cos v}{u}\frac{\partial F}{\partial v}\right)^2\\ &= \left(\frac{\partial F}{\partial u}\right)^2 + \frac{1}{u^2}\left(\frac{\partial F}{\partial v}\right)^2.\end{aligned}$$

34. The biharmonic equation can be expressed in the form

$$0 = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\right)V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 V.$$

But according to Example 12.19, the operator equivalent to $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in polar coordinates is

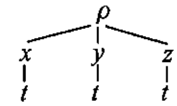
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Hence, the biharmonic equation in polar coordinates is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}\right) = 0.$$

35. $\dfrac{d\rho}{dt} = \dfrac{\partial\rho}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial\rho}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial\rho}{\partial z}\dfrac{dz}{dt}$

$= \left(\dfrac{6x}{z^2+5}\right)(2t) + \left(\dfrac{2y}{z^2+5}\right)(9t^2) + \left[-\dfrac{2z(3x^2+y^2)}{(z^2+5)^2}\right](2).$

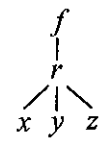


When $t = 2$, we obtain $x = 4$, $y = 25$, $z = 9$, and

$$\frac{d\rho}{dt} = \left(\frac{24}{86}\right)(4) + \left(\frac{50}{86}\right)(36) + \left[-\frac{2(9)(673)}{86^2}\right](2) = 18.77\ \text{kg/m}^3\text{/s}.$$

36. From the schematic,

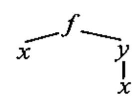
$\dfrac{\partial f}{\partial x} = \dfrac{df}{dr}\dfrac{\partial r}{\partial x} = \dfrac{xf'(r)}{\sqrt{x^2+y^2+z^2}} = \dfrac{xf'(r)}{r}.$



Similarly, $\dfrac{\partial f}{\partial y} = \dfrac{yf'(r)}{r}$ and $\dfrac{\partial f}{\partial z} = \dfrac{zf'(r)}{r}$. Thus,

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} = \frac{xf'(r)}{r}\hat{\mathbf{i}} + \frac{yf'(r)}{r}\hat{\mathbf{j}} + \frac{zf'(r)}{r}\hat{\mathbf{k}} = \frac{f'(r)}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}).$$

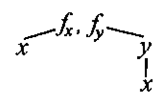
37. If $f(x,y) = 0$ defines y as a function of x, say $y(x)$, then $f[x, y(x)] \equiv 0$ for all x in the domain of the function. The derivative of this function must therefore be zero. To differentiate it with respect to x, we use the schematic to the right,



$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

This equation can be solved for $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{f_x}{f_y}$. Differentiation of this equation gives

$$\frac{d^2y}{dx^2} = -\frac{f_y\dfrac{d}{dx}(f_x) - f_x\dfrac{d}{dx}(f_y)}{{f_y}^2}.$$



Now the schematic to the right gives

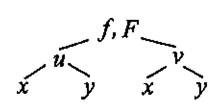
$$\frac{d}{dx}(f_x) = \frac{\partial}{\partial x}(f_x) + \frac{\partial}{\partial y}(f_x)\frac{dy}{dx} = f_{xx} + f_{xy}\frac{dy}{dx} = f_{xx} - \frac{f_{xy}f_x}{f_y},$$

and $\dfrac{d}{dx}(f_y) = \dfrac{\partial}{\partial x}(f_y) + \dfrac{\partial}{\partial y}(f_y)\dfrac{dy}{dx} = f_{yx} + f_{yy}\dfrac{dy}{dx} = f_{xy} - \dfrac{f_{yy}f_x}{f_y}.$

Therefore, $\dfrac{d^2y}{dx^2} = -\dfrac{f_y\left(f_{xx} - \dfrac{f_{xy}f_x}{f_y}\right) - f_x\left(f_{xy} - \dfrac{f_{yy}f_x}{f_y}\right)}{{f_y}^2} = -\dfrac{f_{xx}{f_y}^2 - 2f_{xy}f_xf_y + f_{yy}{f_x}^2}{{f_y}^3}.$

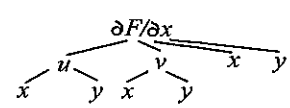
38. If we set $u = x^2 - y^2$ and $v = 2xy$, the schematic to the right gives

$\dfrac{\partial F}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} = 2x\dfrac{\partial f}{\partial u} + 2y\dfrac{\partial f}{\partial v}.$



The schematic for $\partial F/\partial x$ now gives

$$\begin{aligned}\frac{\partial^2 F}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial x}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)_{u,v,y} \\ &= \left(2x\frac{\partial^2 f}{\partial u^2} + 2y\frac{\partial^2 f}{\partial u\partial v}\right)(2x) + \left(2x\frac{\partial^2 f}{\partial v\partial u} + 2y\frac{\partial^2 f}{\partial v^2}\right)(2y) + 2\frac{\partial f}{\partial u} \\ &= 4x^2\frac{\partial^2 f}{\partial u^2} + 8xy\frac{\partial^2 f}{\partial u\partial v} + 4y^2\frac{\partial^2 f}{\partial v^2} + 2\frac{\partial f}{\partial u}.\end{aligned}$$



A similar derivation gives $\dfrac{\partial^2 F}{\partial y^2} = 4y^2\dfrac{\partial^2 f}{\partial u^2} - 8xy\dfrac{\partial^2 f}{\partial u \partial v} + 4x^2\dfrac{\partial^2 f}{\partial v^2} - 2\dfrac{\partial f}{\partial u}$. Hence,

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 4(x^2+y^2)\frac{\partial^2 f}{\partial u^2} + 4(x^2+y^2)\frac{\partial^2 f}{\partial v^2} = 4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) = 0,$$

since $f(u,v)$ is harmonic.

39. (a) From $\dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2}$, we obtain $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$. Similarly, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$. Consequently, $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

(b) If we change to polar coordinates, and set $F(r) = f(r\cos\theta, r\sin\theta) = \ln r^2 = 2\ln r$, then $\dfrac{\partial F}{\partial r} = \dfrac{2}{r}$, $\dfrac{\partial^2 F}{\partial r^2} = -\dfrac{2}{r^2}$, and $\dfrac{\partial^2 F}{\partial \theta^2} = 0$. Hence, $\dfrac{\partial^2 F}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial F}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial \theta^2} = -\dfrac{2}{r^2} + \dfrac{1}{r}\left(\dfrac{2}{r}\right) = 0$.

40. In the proof of Theorem 12.3, differentiation of equation 12.25 with respect to t gave

$$nt^{n-1}f(x,y,z) = x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} + z\frac{\partial f}{\partial w}.$$

Since the same schematic in the theorem applies to $\partial f/\partial u$, $\partial f/\partial v$, and $\partial f/\partial w$, differentiation of this equation with respect to t gives

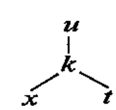
$$\begin{aligned}
n(n-1)t^{n-2}f(x,y,z) &= x\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial u}\right) + y\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial v}\right) + z\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial w}\right)\\
&= x\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial u}\right)\frac{\partial w}{\partial t}\right]\\
&\quad + y\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial v}\right)\frac{\partial w}{\partial t}\right]\\
&\quad + z\left[\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial w}\right)\frac{\partial u}{\partial t} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial w}\right)\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left(\frac{\partial f}{\partial w}\right)\frac{\partial w}{\partial t}\right]\\
&= x\left[x\frac{\partial^2 f}{\partial u^2} + y\frac{\partial^2 f}{\partial v \partial u} + z\frac{\partial^2 f}{\partial w \partial u}\right] + y\left[x\frac{\partial^2 f}{\partial u \partial v} + y\frac{\partial^2 f}{\partial v^2} + z\frac{\partial^2 f}{\partial w \partial v}\right]\\
&\quad + z\left[x\frac{\partial^2 f}{\partial u \partial w} + y\frac{\partial^2 f}{\partial v \partial w} + z\frac{\partial^2 f}{\partial w^2 u}\right].
\end{aligned}$$

If we set $t = 1$, we obtain the identity

$$n(n-1)f(x,y,z) = x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + z^2\frac{\partial^2 f}{\partial z^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + 2yz\frac{\partial^2 f}{\partial y \partial z} + 2zx\frac{\partial^2 f}{\partial x \partial z}.$$

41. (a) Since $\dfrac{\partial u}{\partial x} = \dfrac{du}{dk}\dfrac{\partial k}{\partial x}$ and $\dfrac{\partial u}{\partial t} = \dfrac{du}{dk}\dfrac{\partial k}{\partial t}$, it follows that

$$u\frac{du}{dk}\frac{\partial k}{\partial x} + \frac{du}{dk}\frac{\partial k}{\partial t} = -c^2k^n\frac{\partial k}{\partial x} \quad \text{or} \quad \frac{du}{dk}\left(u\frac{\partial k}{\partial x} + \frac{\partial k}{\partial t}\right) = -c^2k^n\frac{\partial k}{\partial x}.$$



(b) If we use the equation of continuity in the form $u\dfrac{\partial k}{\partial x} + \dfrac{\partial k}{\partial t} = -k\dfrac{\partial u}{\partial x}$ to substitute into the equation in part (a),

$$-c^2k^n\frac{\partial k}{\partial x} = \frac{du}{dk}\left(-k\frac{\partial u}{\partial x}\right) = -k\frac{du}{dk}\left(\frac{du}{dk}\frac{\partial k}{\partial x}\right).$$

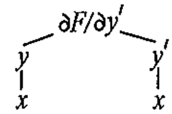
Thus, $c^2k^{n-1} = \left(\dfrac{du}{dk}\right)^2$ or $\dfrac{du}{dk} = \pm ck^{(n-1)/2}$. Since $u(k)$ must be a decreasing function, it follows that $\dfrac{du}{dk} = -ck^{(n-1)/2}$.

(c) Integration gives $u(k) = -\dfrac{2c}{n+1}k^{(n+1)/2} + E$.

42. (a) First we calculate that $\quad \dfrac{\partial F}{\partial y} = -\dfrac{\sqrt{1+(y')^2}}{2y^{3/2}}$ and $\dfrac{\partial F}{\partial y'} = \dfrac{y'}{\sqrt{y}\sqrt{1+(y')^2}}$.

From the schematic,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{d(y')}{dx}\\
&= \left[\frac{-y'}{2y^{3/2}\sqrt{1+(y')^2}}\right]y' + \left[\frac{1}{\sqrt{y}\sqrt{1+(y')^2}} - \frac{(y')^2}{\sqrt{y}[1+(y')^2]^{3/2}}\right]y''\\
&= \frac{-(y')^2[1+(y')^2] + 2yy''[1+(y')^2] - 2y(y')^2y''}{2y^{3/2}[1+(y')^2]^{3/2}}\\
&= \frac{2yy'' - (y')^2 - (y')^4}{2y^{3/2}[1+(y')^2]^{3/2}}.
\end{aligned}$$



Consequently, $y(x)$ must satisfy

$$\begin{aligned}
0 &= \frac{2yy'' - (y')^2 - (y')^4}{2y^{3/2}[1+(y')^2]^{3/2}} + \frac{\sqrt{1+(y')^2}}{2y^{3/2}}\\
&= \frac{2yy'' - (y')^2 - (y')^4 + [1+(y')^2]^2}{2y^{3/2}[1+(y')^2]^{3/2}} = \frac{2yy'' + (y')^2 + 1}{2y^{3/2}[1+(y')^2]^{3/2}}.
\end{aligned}$$

Thus, $2yy'' + (y')^2 + 1 = 0$.

(b) Since $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\sin\theta}{a(1-\cos\theta)} = \dfrac{\sin\theta}{1-\cos\theta}$, and

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)\frac{d\theta}{dx} = \frac{\dfrac{d}{d\theta}\left(\dfrac{\sin\theta}{1-\cos\theta}\right)}{dx/d\theta} = \frac{\dfrac{(1-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}}{a(1-\cos\theta)}\\
&= \frac{\cos\theta - 1}{a(1-\cos\theta)^3} = \frac{-1}{a(1-\cos\theta)^2},
\end{aligned}$$

it follows that

$$\begin{aligned}
2yy'' + (y')^2 + 1 &= \frac{-2a(1-\cos\theta)}{a(1-\cos\theta)^2} + \left(\frac{\sin\theta}{1-\cos\theta}\right)^2 + 1 = \frac{-2}{1-\cos\theta} + \frac{\sin^2\theta}{(1-\cos\theta)^2} + 1\\
&= \frac{-2(1-\cos\theta) + \sin^2\theta + (1 - 2\cos\theta + \cos^2\theta)}{(1-\cos\theta)^2} = 0.
\end{aligned}$$

(c) Suppose that the bead starts at any other point (x_1, y_1) on the cycloid corresponding to parameter value θ_1. Its velocity at any point (x, y) is given by

$$mg(y - y_1) = \frac{1}{2}mv^2 \quad \Longrightarrow \quad v = \sqrt{2g(y-y_1)}.$$

The time to travel from (x_1, y_1) to (x_0, y_0) is

$$\begin{aligned}
t &= \int_{x_1}^{x_0}\frac{\sqrt{(dx)^2 + (dy)^2}}{\sqrt{2g(y-y_1)}} = \frac{1}{\sqrt{2g}}\int_{\theta_1}^{\pi}\frac{1}{\sqrt{a(\cos\theta_1 - \cos\theta)}}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta\\
&= \frac{1}{\sqrt{2ga}}\int_{\theta_1}^{\pi}\sqrt{\frac{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}{\cos\theta_1 - \cos\theta}}\,d\theta = \sqrt{\frac{a}{2g}}\int_{\theta_1}^{\pi}\sqrt{\frac{2 - 2\cos\theta}{\cos\theta_1 - \cos\theta}}\,d\theta\\
&= \sqrt{\frac{a}{g}}\int_{\theta_1}^{\pi}\sqrt{\frac{1 - [1 - 2\sin^2(\theta/2)]}{[2\cos^2(\theta_1/2) - 1] - [2\cos^2(\theta/2) - 1]}}\,d\theta = \sqrt{\frac{a}{g}}\int_{\theta_1}^{\pi}\frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_1/2) - \cos^2(\theta/2)}}\,d\theta.
\end{aligned}$$

If we set $u = \cos(\theta/2)$ and $du = -(1/2)\sin(\theta/2)\,d\theta$, then

$$t = \sqrt{\frac{a}{g}}\int_{\cos(\theta_1/2)}^{0} \frac{-2}{\sqrt{\cos^2(\theta_1/2) - u^2}}du = -2\sqrt{\frac{a}{g}}\left\{\mathrm{Sin}^{-1}\left(\frac{u}{\cos(\theta_1/2)}\right)\right\}_{\cos(\theta_1/2)}^{0} = 2\sqrt{\frac{a}{g}}\left(\frac{\pi}{2}\right) = \pi\sqrt{\frac{a}{g}}.$$

Since this is independent of θ_1, the time taken is the same for all starting points.

43. (a) Newton's second law applied to the two masses gives

$$m\frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1), \qquad m\frac{d^2x_2}{dt^2} = -kx_2 - k(x_2 - x_1),$$

or, $m\ddot{x}_1 = k(x_2 - 2x_1)$, $m\ddot{x}_2 = k(x_1 - 2x_2)$.

(b) Kinetic energy of a mass is one-half the product of its mass and the square of its speed. Potential energy in a spring is one-half the product of the spring constant and the square of its stretch (or compression). When the masses are displaced as in part (a) then,

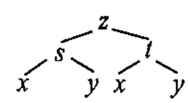
$$L = \frac{1}{2}m\dot{x}_1{}^2 + \frac{1}{2}m\dot{x}_2{}^2 - \frac{1}{2}kx_1{}^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}kx_2{}^2 = \frac{m}{2}(\dot{x}_1{}^2 + \dot{x}_2{}^2) - k(x_1{}^2 + x_2{}^2 - x_1x_2).$$

(c) Since $\dfrac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$, $\quad\dfrac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$, $\quad\dfrac{\partial L}{\partial x_1} = -k(2x_1 - x_2)$, $\quad\dfrac{\partial L}{\partial x_2} = -k(2x_2 - x_1)$,

the Euler-Lagrange equations give

$$0 = m\ddot{x}_1 + k(2x_1 - x_2) \qquad \text{and} \qquad 0 = m\ddot{x}_2 + k(2x_2 - x_1).$$

44. The schematic gives $\quad\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial z}{\partial t}\dfrac{\partial t}{\partial x} = a\dfrac{\partial z}{\partial s} + c\dfrac{\partial z}{\partial t}.$



Since the same schematic applies to $\partial z/\partial x$,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial t}{\partial x} = \left(a\frac{\partial^2 z}{\partial s^2} + c\frac{\partial^2 z}{\partial s\partial t}\right)a + \left(a\frac{\partial^2 z}{\partial t\partial s} + c\frac{\partial^2 z}{\partial t^2}\right)c$$
$$= a^2\frac{\partial^2 z}{\partial s^2} + 2ac\frac{\partial^2 z}{\partial s\partial t} + c^2\frac{\partial^2 z}{\partial t^2}.$$

Similarly, $\dfrac{\partial^2 z}{\partial y^2} = b^2\dfrac{\partial^2 z}{\partial s^2} + 2bd\dfrac{\partial^2 z}{\partial s\partial t} + d^2\dfrac{\partial^2 z}{\partial t^2}$ and $\dfrac{\partial^2 z}{\partial x\partial y} = ab\dfrac{\partial^2 z}{\partial s^2} + (bc + ad)\dfrac{\partial^2 z}{\partial s\partial t} + cd\dfrac{\partial^2 z}{\partial t^2}$. Substituting these into the partial differential equation for $z(x, y)$ gives

$$p\left(a^2\frac{\partial^2 z}{\partial s^2} + 2ac\frac{\partial^2 z}{\partial s\partial t} + c^2\frac{\partial^2 z}{\partial t^2}\right) + q\left(ab\frac{\partial^2 z}{\partial s^2} + (bc + ad)\frac{\partial^2 z}{\partial s\partial t} + cd\frac{\partial^2 z}{\partial t^2}\right)$$
$$+ r\left(b^2\frac{\partial^2 z}{\partial s^2} + 2bd\frac{\partial^2 z}{\partial s\partial t} + d^2\frac{\partial^2 z}{\partial t^2}\right) = F\left(x(s,t), y(s,t), z, a\frac{\partial z}{\partial s} + c\frac{\partial z}{\partial t}, b\frac{\partial z}{\partial s} + d\frac{\partial z}{\partial t}\right).$$

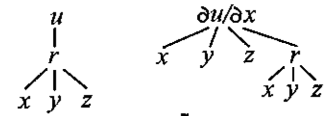
If we define $P = pa^2 + qab + rb^2$, $Q = 2pac + q(bc + ad) + 2rbd$, and $R = pc^2 + qcd + rd^2$, then this equation can be expressed in the form

$$P\frac{\partial^2 z}{\partial s^2} + Q\frac{\partial^2 z}{\partial s\partial t} + R\frac{\partial^2 z}{\partial t^2} = G\left(s, t, z, \frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}\right),$$

where

$$\begin{aligned} Q^2 - 4PR &= [2pac + q(bc + ad) + 2rbd]^2 - 4(pa^2 + qab + rb^2)(pc^2 + qcd + rd^2) \\ &= 4p^2a^2c^2 + q^2(b^2c^2 + 2abcd + a^2d^2) + 4r^2b^2d^2 + 4pqac(bc + ad) + 4qrbd(bc + ad) + 8prabcd \\ &\quad - 4pa^2(pc^2 + qcd + rd^2) - 4qab(pc^2 + qcd + rd^2) - 4rb^2(pc^2 + qcd + rd^2) \\ &= q^2(a^2d^2 - 2abcd + b^2c^2) - 4pr(a^2d^2 - 2abcd + b^2c^2) = (q^2 - 4pr)(ad - bc)^2. \end{aligned}$$

45. Since $\dfrac{\partial u}{\partial x} = \dfrac{du}{dr}\dfrac{\partial r}{\partial x} = \dfrac{du}{dr}\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, it follows that



$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)_{y,z,r} + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\frac{\partial r}{\partial x}\\
&= \frac{du}{dr}\left[\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}}\right] + \left[\frac{x}{\sqrt{x^2+y^2+z^2}}\frac{d^2u}{dr^2}\right]\frac{x}{\sqrt{x^2+y^2+z^2}}\\
&= \frac{1}{(x^2+y^2+z^2)^{3/2}}\left[(x^2+y^2+z^2-x^2)\frac{du}{dr} + x^2\sqrt{x^2+y^2+z^2}\frac{d^2u}{dr^2}\right]\\
&= \frac{1}{r^3}\left[(y^2+z^2)\frac{du}{dr} + rx^2\frac{d^2u}{dr^2}\right].
\end{aligned}$$

With similar results for $\partial^2 u/\partial y^2$ and $\partial^2 u/\partial z^2$, Laplace's equation becomes

$$\begin{aligned}
0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= \frac{1}{r^3}\Big[(y^2+z^2)\frac{du}{dr} + rx^2\frac{d^2u}{dr^2} + (x^2+z^2)\frac{du}{dr} + ry^2\frac{d^2u}{dr^2}\\
&\qquad + (x^2+y^2)\frac{du}{dr} + rz^2\frac{d^2u}{dr^2}\Big]\\
&= \frac{1}{r^3}\left[2(x^2+y^2+z^2)\frac{du}{dr} + r(x^2+y^2+z^2)\frac{d^2u}{dr^2}\right]\\
&= \frac{2}{r}\frac{du}{dr} + \frac{d^2u}{dr^2}.
\end{aligned}$$

When this equation is multiplied by r^2, $0 = 2r\dfrac{du}{dr} + r^2\dfrac{d^2u}{dr^2} = \dfrac{d}{dr}\left(r^2\dfrac{du}{dr}\right)$. Consequently,

$$r^2\frac{du}{dr} = E = \text{constant} \quad\Longrightarrow\quad \frac{du}{dr} = \frac{E}{r^2}.$$

Integration gives $u = -\dfrac{E}{r} + D = \dfrac{C}{r} + D = \dfrac{C}{\sqrt{x^2+y^2+z^2}} + D$.

EXERCISES 12.7

1. If we set $F(x,y) = x^3y^2 - 2xy + 5$, then $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\dfrac{F_x}{F_y} = -\dfrac{3x^2y^2-2y}{2x^3y-2x} = \dfrac{2y-3x^2y^2}{2x^3y-2x}$.

2. If we set $F(x,y) = (x+y)^2 - 2x$, then $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\dfrac{F_x}{F_y} = -\dfrac{2(x+y)-2}{2(x+y)} = \dfrac{1-x-y}{x+y}$.

3. If we set $F(x,y) = x^2 - xy - 4y^3 - 2e^{xy} - 6$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\frac{F_x}{F_y} = -\frac{2x-y-2ye^{xy}}{-x-12y^2-2xe^{xy}} = \frac{2x-y-2ye^{xy}}{x+12y^2+2xe^{xy}}.$$

4. If we set $F(x,y) = \sin(x+y) + y^2 - 12x^2 - y$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\frac{F_x}{F_y} = -\frac{\cos(x+y)-24x}{\cos(x+y)+2y-1} = \frac{24x-\cos(x+y)}{\cos(x+y)+2y-1}.$$

5. If we set $F(x,y,z) = x^2\sin z - ye^z - 2x$, then $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_x}{F_z} = -\dfrac{2x\sin z - 2}{x^2\cos z - ye^z} = \dfrac{2(1 - x\sin z)}{x^2\cos z - ye^z}$,

and $\dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_y}{F_z} = -\dfrac{-e^z}{x^2\cos z - ye^z} = \dfrac{e^z}{x^2\cos z - ye^z}$.

6. If we set $F(x,y,z) = x^2z^2 + yz + 3x - 4$, then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} = -\frac{2xz^2+3}{2x^2z+y}; \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_y}{F_z} = -\frac{z}{2x^2z+y}.$$

7. If we set $F(x,y,z) = z\sin^2 y + y\sin^2 x - z^3$, then $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_x}{F_z} = -\dfrac{2y\sin x\cos x}{\sin^2 y - 3z^2} = \dfrac{2y\sin x\cos x}{3z^2 - \sin^2 y}$,

and $\dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_y}{F_z} = -\dfrac{2z\sin y\cos y + \sin^2 x}{\sin^2 y - 3z^2} = \dfrac{2z\sin y\cos y + \sin^2 x}{3z^2 - \sin^2 y}$.

8. If we set $F(x,y,z) = \mathrm{Tan}^{-1}(yz) - xz$, then $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_x}{F_z} = -\dfrac{-z}{\dfrac{y}{1+y^2z^2} - x} = \dfrac{z(1+y^2z^2)}{y - x(1+y^2z^2)}$,

and $\dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\dfrac{F_y}{F_z} = -\dfrac{\dfrac{z}{1+y^2z^2}}{\dfrac{y}{1+y^2z^2} - x} = \dfrac{z}{x(1+y^2z^2) - y}$.

9. If we set $F(x,y,u,v) = x^2 - y^2 + u^2 + 2v^2 - 1$ and $G(x,y,u,v) = x^2 + y^2 - u^2 - v^2 - 2$, then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2x & 4v \\ 2x & -2v \end{vmatrix}}{\begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix}} = \frac{3x}{u},$$

and $\dfrac{\partial v}{\partial y} = -\dfrac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\dfrac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{4uv} = -\dfrac{\begin{vmatrix} 2u & -2y \\ -2u & 2y \end{vmatrix}}{4uv} = 0.$

10. If we set $F(x,t,z) = \sin(x+t) - \sin(x-t) - z$, then

$$\frac{\partial x}{\partial t} = -\frac{\dfrac{\partial(F)}{\partial(t)}}{\dfrac{\partial(F)}{\partial(x)}} = -\frac{F_t}{F_x} = -\frac{\cos(x+t) + \cos(x-t)}{\cos(x+t) - \cos(x-t)} = \frac{\cos(x+t) + \cos(x-t)}{\cos(x-t) - \cos(x+t)}.$$

11. If we set $F(x,r,\phi,\theta) = r\sin\phi\cos\theta - x$, $G(y,r,\phi,\theta) = r\sin\phi\sin\theta - y$, and $H(z,r,\phi) = r\cos\phi - z$, then

$$\frac{\partial\phi}{\partial x} = -\frac{\dfrac{\partial(F,G,H)}{\partial(r,x,\theta)}}{\dfrac{\partial(F,G,H)}{\partial(r,\phi,\theta)}} = -\frac{\begin{vmatrix} F_r & F_x & F_\theta \\ G_r & G_x & G_\theta \\ H_r & H_x & H_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_\phi & F_\theta \\ G_r & G_\phi & G_\theta \\ H_r & H_\phi & H_\theta \end{vmatrix}} = -\frac{\begin{vmatrix} \sin\phi\cos\theta & -1 & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & 0 & r\sin\phi\cos\theta \\ \cos\phi & 0 & 0 \end{vmatrix}}{\begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}}$$

$$= \frac{r\sin\phi\cos\phi\cos\theta}{\cos\phi(r^2\sin\phi\cos\phi) + r\sin\phi(r\sin^2\phi)} = \frac{r\sin\phi\cos\phi\cos\theta}{r^2\sin\phi} = \frac{\cos\phi\cos\theta}{r}.$$

12. If we set $F(x,y,z) = x^2 + y^2 - z^2 + 2xy - 1$ and $G(x,y) = x^3 + y^3 - 5y - 4$, then

$$\frac{dz}{dx} = -\frac{\dfrac{\partial(F,G)}{\partial(y,x)}}{\dfrac{\partial(F,G)}{\partial(y,z)}} = -\frac{\begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} = -\frac{\begin{vmatrix} 2y+2x & 2x+2y \\ 3y^2-5 & 3x^2 \end{vmatrix}}{\begin{vmatrix} 2y+2x & -2z \\ 3y^2-5 & 0 \end{vmatrix}}$$

$$= -\frac{3x^2(2y+2x) - (2x+2y)(3y^2-5)}{2z(3y^2-5)} = \frac{(x+y)(3x^2-3y^2+5)}{z(5-3y^2)}.$$

13. If we set $F(x,y,u,v,w) = xyu+vw-4$, $G(y,u,v) = y^2+u^2-u^2v-y$, and $H(x,y,u,v,w) = yw+xu+v+4$, then

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G,H)}{\partial(y,v,w)}}{\dfrac{\partial(F,G,H)}{\partial(u,v,w)}} = -\frac{\begin{vmatrix} F_y & F_v & F_w \\ G_y & G_v & G_w \\ H_y & H_v & H_w \end{vmatrix}}{\begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}} = -\frac{\begin{vmatrix} xu & w & v \\ 2y-1 & -u^2 & 0 \\ w & 1 & y \end{vmatrix}}{\begin{vmatrix} xy & w & v \\ 2u(1-v) & -u^2 & 0 \\ x & 1 & y \end{vmatrix}}$$

$$= -\frac{v[(2y-1)+wu^2] + y[-xu^3 - w(2y-1)]}{v[2u(1-v)+xu^2] + y[-xyu^2 - 2uw(1-v)]} = \frac{(2y-1)(yw-v) + u^2(xyu-vw)}{2u(1-v)(v-yw) + xu^2(v-y^2)}.$$

14. If we set $F(x,y,z,u,v) = x^2 - y\cos(uv) + z^2$, $G(x,y,z,u,v) = x^2 + y^2 - \sin(uv) + 2z^2 - 2$, and $H(x,y,z,u,v) = xy - \sin u\,\cos v + z$, then

$$\frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(F,G,H)}{\partial(u,y,z)}}{\dfrac{\partial(F,G,H)}{\partial(x,y,z)}} = -\frac{\begin{vmatrix} F_u & F_y & F_z \\ G_u & G_y & G_z \\ H_u & H_y & H_z \end{vmatrix}}{\begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix}} = -\frac{\begin{vmatrix} yv\sin(uv) & -\cos(uv) & 2z \\ -v\cos(uv) & 2y & 4z \\ -\cos u\,\cos v & x & 1 \end{vmatrix}}{\begin{vmatrix} 2x & -\cos(uv) & 2z \\ 2x & 2y & 4z \\ y & x & 1 \end{vmatrix}}.$$

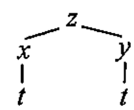
When $x = y = 1$, $u = \pi/2$, $v = z = 0$, $\quad \dfrac{\partial x}{\partial u} = -\dfrac{\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}} = 0.$

15. $$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = \left[-\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}}\right]\left[-\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(x)}}\right]\left[-\frac{\dfrac{\partial(F)}{\partial(z)}}{\dfrac{\partial(F)}{\partial(y)}}\right] = -1$$

16. $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = e^x \cos y \dfrac{dx}{dt} - e^x \sin y \dfrac{dy}{dt}$ If we set $F(x,t) = x^3 + e^x - t^2 - t - 1$ and

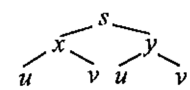
$G(y,t) = yt^2 + y^2t - t + y$, then $\dfrac{dx}{dt} = -\dfrac{\dfrac{\partial(F)}{\partial(t)}}{\dfrac{\partial(F)}{\partial(x)}} = -\dfrac{F_t}{F_x} = -\dfrac{-2t-1}{3x^2+e^x} = \dfrac{2t+1}{3x^2+e^x}$, and

$$\frac{dy}{dt} = -\frac{\dfrac{\partial(G)}{\partial(t)}}{\dfrac{\partial(G)}{\partial(y)}} = -\frac{G_t}{G_y} = -\frac{2yt+y^2-1}{t^2+2yt+1} = \frac{1-2yt-y^2}{1+2yt+t^2}.$$



Consequently, $\dfrac{dz}{dt} = e^x \cos y \left(\dfrac{2t+1}{3x^2+e^x}\right) - e^x \sin y \left(\dfrac{1-2yt-y^2}{1+2yt+t^2}\right)$.

17. The chain rule gives $\dfrac{\partial s}{\partial u} = \dfrac{\partial s}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial s}{\partial y}\dfrac{\partial y}{\partial u} = 2x\dfrac{\partial x}{\partial u} + 2y\dfrac{\partial y}{\partial u}$.
If we set $F(x,y,u) = u - x^2 + y^2$ and
$G(x,y,v) = v - x^2 + y$, then

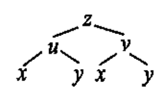


$$\frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & 2y \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} -2x & 2y \\ -2x & 1 \end{vmatrix}} = -\frac{1}{-2x+4xy} = \frac{1}{2x(1-2y)},$$

and $$\frac{\partial y}{\partial u} = -\frac{\dfrac{\partial(F,G)}{\partial(x,u)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{-2x+4xy} = -\frac{\begin{vmatrix} -2x & 1 \\ -2x & 0 \end{vmatrix}}{2x(2y-1)} = \frac{-2x}{2x(2y-1)} = \frac{1}{1-2y}.$$

Thus, $\dfrac{\partial s}{\partial u} = \dfrac{2x}{2x(1-2y)} + \dfrac{2y}{1-2y} = \dfrac{1+2y}{1-2y}$.

18. $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = [3u^2v + v\cos(uv)]\dfrac{\partial u}{\partial y} + [u^3 + u\cos(uv)]\dfrac{\partial v}{\partial y}$
If we set $F(x,u,v) = e^u \cos v - x$, $G(y,u,v) = e^u \sin v - y$, then



$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 0 & -e^u \sin v \\ -1 & e^u \cos v \end{vmatrix}}{\begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}} = \frac{e^u \sin v}{e^{2u}} = e^{-u}\sin v,$$

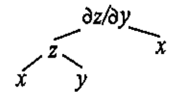
and $$\frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{e^{2u}} = -\frac{\begin{vmatrix} e^u \cos v & 0 \\ e^u \sin v & -1 \end{vmatrix}}{e^{2u}} = e^{-u}\cos v.$$

Thus, $\dfrac{\partial z}{\partial y} = [3u^2v + v\cos(uv)]e^{-u}\sin v + [u^3 + u\cos(uv)]e^{-u}\cos v$.

19. If we define $F(x,y,z)=z^3-xz-y$, then $\dfrac{\partial z}{\partial y}=-\dfrac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}}=-\dfrac{F_y}{F_z}=-\dfrac{-1}{3z^2-x}=\dfrac{1}{3z^2-x}.$

The chain rule now gives

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)_z=-\frac{6z}{(3z^2-x)^2}\frac{\partial z}{\partial x}+\frac{1}{(3z^2-x)^2}.$$



Since $\dfrac{\partial z}{\partial x}=-\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}}=-\dfrac{F_x}{F_z}=-\dfrac{-z}{3z^2-x}=\dfrac{z}{3z^2-x},$

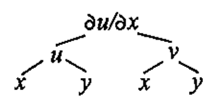
we have $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{-6z^2}{(3z^2-x)^3}+\dfrac{1}{(3z^2-x)^2}=-\dfrac{3z^2+x}{(3z^2-x)^3}.$

20. If we set $F(x,u,v)=x-u^2+v^2$, $G(y,u,v)=y-2uv$, then

$$\frac{\partial u}{\partial x}=-\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}=-\frac{\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}=-\frac{\begin{vmatrix}1 & 2v\\ 0 & -2u\end{vmatrix}}{\begin{vmatrix}-2u & 2v\\ -2v & -2u\end{vmatrix}}=\frac{2u}{4u^2+4v^2}=\frac{u}{2(u^2+v^2)}.$$

The chain rule now gives

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial x}\\ &=\left[\frac{2(u^2+v^2)-u(4u)}{4(u^2+v^2)^2}\right]\frac{\partial u}{\partial x}+\left[\frac{-2uv}{2(u^2+v^2)^2}\right]\frac{\partial v}{\partial x}.\end{aligned}$$



Since $\dfrac{\partial v}{\partial x}=-\dfrac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}=-\dfrac{\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}}{4(u^2+v^2)}=-\dfrac{\begin{vmatrix}-2u & 1\\ -2v & 0\end{vmatrix}}{4(u^2+v^2)}=\dfrac{-2v}{4(u^2+v^2)}=\dfrac{-v}{2(u^2+v^2)},$

we obtain $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{v^2-u^2}{2(u^2+v^2)^2}\dfrac{u}{2(u^2+v^2)}-\dfrac{uv}{(u^2+v^2)^2}\dfrac{-v}{2(u^2+v^2)}=\dfrac{3uv^2-u^3}{4(u^2+v^2)^3}.$

21. (a) If we define $F(x,y,z)=z^4x+y^3z+9x^3-2$, then

$$\frac{\partial z}{\partial x}=-\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}}\quad\text{and}\quad\frac{\partial x}{\partial z}=-\frac{\dfrac{\partial(F)}{\partial(z)}}{\dfrac{\partial(F)}{\partial(x)}}.$$

Therefore, $\partial z/\partial x$ and $\partial x/\partial z$ are reciprocals.

(b) If we define $F(x,y,z)=z^4x+y^3z+9x^3-2$ and $G(x,y,z)=x^2y+xz-1$, then

$$\frac{dz}{dx}=-\frac{\dfrac{\partial(F,G)}{\partial(y,x)}}{\dfrac{\partial(F,G)}{\partial(y,z)}}\quad\text{and}\quad\frac{dx}{dz}=-\frac{\dfrac{\partial(F,G)}{\partial(y,z)}}{\dfrac{\partial(F,G)}{\partial(y,x)}}.$$

Thus, dz/dx and dx/dz are reciprocals.

(c) If we define $F(x,y,u,v)=u^2-v-3x-y$ and $G(x,y,u,v)=u-2v^2-x+2y$, then

$$\frac{\partial u}{\partial x}=-\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}\quad\text{and}\quad\frac{\partial x}{\partial u}=-\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}}.$$

These are not reciprocals.

22. If we set $F(x,y,s,t) = x^2 - 2y^2s^2t - 2st^2 - 1$ and $G(x,y,s,t) = x^2 + 2y^2s^2t + 5st^2 - 1$, then

$$\frac{\partial t}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(s,y)}}{\dfrac{\partial(F,G)}{\partial(s,t)}} = -\frac{\begin{vmatrix} F_s & F_y \\ G_s & G_y \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}} = -\frac{\begin{vmatrix} -4y^2st - 2t^2 & -4ys^2t \\ 4y^2st + 5t^2 & 4ys^2t \end{vmatrix}}{\begin{vmatrix} -4y^2st - 2t^2 & -2y^2s^2 - 4st \\ 4y^2st + 5t^2 & 2y^2s^2 + 10st \end{vmatrix}}$$

$$= -\frac{4ys^2t(-4y^2st - 2t^2 + 4y^2st + 5t^2)}{4y^2st(-2y^2s^2 - 10st + 2y^2s^2 + 4st) + t^2(-4y^2s^2 - 20st + 10y^2s^2 + 20st)}$$

$$= \frac{-12ys^2t^3}{-24y^2s^2t^2 + 6y^2s^2t^2} = \frac{2t}{3y}.$$

Thus, $\dfrac{\partial^2 t}{\partial y^2} = \dfrac{2}{3y}\dfrac{\partial t}{\partial y} - \dfrac{2t}{3y^2} = \dfrac{2}{3y}\left(\dfrac{2t}{3y}\right) - \dfrac{2t}{3y^2} = -\dfrac{2t}{9y^2}.$

23. (a) $$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\right)\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}\right)$$
$$- \left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\right)\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}\right)$$
$$= \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}\right) + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial y}\right)$$
$$- \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}\right) - \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\left(\frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x}\right)$$
$$= \left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right)\left(\frac{\partial s}{\partial x}\frac{\partial t}{\partial y} - \frac{\partial s}{\partial y}\frac{\partial t}{\partial x}\right)$$
$$= \frac{\partial(u,v)}{\partial(s,t)}\frac{\partial(s,t)}{\partial(x,y)}$$

(b) In part (a) we replace s and t by x and y in F, G, H, and I, and x and y by u and v in H and I,

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(u,v)} = 1 \quad \Longrightarrow \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}}.$$

24. If we set $F_i = \displaystyle\sum_{j=1}^{n} a_{ij}x_j - c_i$, for $i = 1, \ldots, m$, then

$$\frac{\partial x_i}{\partial x_j} = -\frac{\dfrac{\partial(F_1,\ldots,F_{i-1},F_i,F_{i+1},\ldots,F_m)}{\partial(x_1,\ldots,x_{i-1},x_j,x_{i+1},\ldots,x_m)}}{\dfrac{\partial(F_1,\ldots,F_{i-1},F_i,F_{i+1},\ldots,F_m)}{\partial(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_m)}}$$

$$= -\frac{\begin{vmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_{i-1} & \partial F_1/\partial x_j & \partial F_1/\partial x_{i+1} & \cdots & \partial F_1/\partial x_m \\ \vdots & & & \vdots & & & \vdots \\ \partial F_m/\partial x_1 & \cdots & \partial F_m/\partial x_{i-1} & \partial F_m/\partial x_j & \partial F_m/\partial x_{i+1} & \cdots & \partial F_m/\partial x_m \end{vmatrix}}{\begin{vmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_{i-1} & \partial F_1/\partial x_i & \partial F_1/\partial x_{i+1} & \cdots & \partial F_1/\partial x_m \\ \vdots & & & \vdots & & & \vdots \\ \partial F_m/\partial x_1 & \cdots & \partial F_m/\partial x_{i-1} & \partial F_m/\partial x_i & \partial F_m/\partial x_{i+1} & \cdots & \partial F_m/\partial x_m \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1j} & a_{1,i+1} & \cdots & a_{1m} \\ \vdots & & & \vdots & & & \vdots \\ a_{m1} & \cdots & a_{m,i-1} & a_{mj} & a_{m,i+1} & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1i} & a_{1,i+1} & \cdots & a_{1m} \\ \vdots & & & \vdots & & & \vdots \\ a_{m1} & \cdots & a_{m,i-1} & a_{mi} & a_{m,i+1} & \cdots & a_{mm} \end{vmatrix}} = -\frac{D_{ij}}{D}.$$

EXERCISES 12.8

1. The vector from $(1,2,3)$ to $(3,5,0)$ is $\mathbf{v} = (2,3,-3)$. At the point $(1,2,3)$,

$$D_{\mathbf{v}}f = \nabla f_{|(1,2,3)} \cdot \hat{\mathbf{v}} = (4x, -2y, 2z)_{|(1,2,3)} \cdot \frac{(2,3,-3)}{\sqrt{4+9+9}} = (4,-4,6) \cdot \frac{(2,3,-3)}{\sqrt{22}} = -\sqrt{22}.$$

2. The vector that joins $(3,2,1)$ to $(3,1,-1)$ is $\mathbf{v} = (0,-1,-2)$. At the point $(-1,1,-1)$,

$$D_{\mathbf{v}}f = \nabla f_{|(-1,1,-1)} \cdot \hat{\mathbf{v}} = (2xy+z, x^2, x)_{|(-1,1,-1)} \cdot \frac{(0,-1,-2)}{\sqrt{1+4}} = (-3,1,-1) \cdot \frac{(0,-1,-2)}{\sqrt{5}} = \frac{1}{\sqrt{5}}.$$

3. The vector from $(3,0)$ to $(-2,-4)$ is $\mathbf{v} = (-5,-4)$. At the point $(3,0)$,

$$D_{\mathbf{v}}f = \nabla f_{|(3,0)} \cdot \hat{\mathbf{v}} = (e^y, xe^y + 1)_{|(3,0)} \cdot \frac{(-5,-4)}{\sqrt{25+16}} = (1,4) \cdot \frac{(-5,-4)}{\sqrt{41}} = \frac{-21}{\sqrt{41}}.$$

4. The vector from $(1,1,1)$ to $(-1,-2,3)$ is $\mathbf{v} = (-2,-3,2)$. At the point $(1,1,1)$,

$$\begin{aligned} D_{\mathbf{v}}f = \nabla f_{|(1,1,1)} \cdot \hat{\mathbf{v}} &= \left[\frac{1}{xy+yz+xz}(y+z, x+z, x+y)\right]_{|(1,1,1)} \cdot \frac{(-2,-3,2)}{\sqrt{4+9+4}} \\ &= \frac{1}{3}(2,2,2) \cdot \frac{(-2,-3,2)}{\sqrt{17}} = \frac{-2}{\sqrt{17}}. \end{aligned}$$

5. A vector along the line is $\mathbf{v} = (1,2)$. At the point $(1,2)$,

$$D_{\mathbf{v}}f = \nabla f_{|(1,2)} \cdot \hat{\mathbf{v}} = \left(\frac{y}{1+x^2y^2}, \frac{x}{1+x^2y^2}\right)_{|(1,2)} \cdot \frac{(1,2)}{\sqrt{1+4}} = \frac{(2,1)}{5} \cdot \frac{(1,2)}{\sqrt{5}} = \frac{4}{5\sqrt{5}}.$$

6. A vector along $3x + 4y = -2$ in the direction of decreasing y is $\mathbf{v} = (4,-3)$. At the point $(2,-2)$,

$$D_{\mathbf{v}}f = \nabla f_{|(2,-2)} \cdot \hat{\mathbf{v}} = [\cos(x+y)(1,1)]_{|(2,-2)} \cdot \frac{(4,-3)}{\sqrt{16+9}} = (1,1) \cdot \frac{(4,-3)}{5} = \frac{1}{5}.$$

7. A vector along the line is $\mathbf{v} = (-1,-4,-2)$. At the point $(3,-1,-2)$,

$$\begin{aligned} D_{\mathbf{v}}f = \nabla f_{|(3,-1,-2)} \cdot \hat{\mathbf{v}} &= (3x^2y\sin z, x^3\sin z, x^3y\cos z)_{|(3,-1,-2)} \cdot \frac{(-1,-4,-2)}{\sqrt{1+16+4}} \\ &= (27\sin 2, -27\sin 2, -27\cos 2) \cdot \frac{(-1,-4,-2)}{\sqrt{21}} = \frac{81\sin 2 + 54\cos 2}{\sqrt{21}}. \end{aligned}$$

8. Since parametric equations for the line are $x = -2t - 1,\ y = t,\ z = \frac{1}{2}(2 + 2t + 1 + t) = \frac{3}{2} + \frac{3}{2}t$, a vector along the line in the direction of decreasing z is $\mathbf{v} = (4,-2,-3)$. At the point $(1,-1,0)$,

$$\begin{aligned} D_{\mathbf{v}}f = \nabla f_{|(1,-1,0)} \cdot \hat{\mathbf{v}} &= (2xy+z^2, x^2+2yz, y^2+2xz)_{|(1,-1,0)} \cdot \frac{(4,-2,-3)}{\sqrt{16+4+9}} \\ &= (-2,1,1) \cdot \frac{(4,-2,-3)}{\sqrt{29}} = \frac{-13}{\sqrt{29}}. \end{aligned}$$

9. Since the slope of the curve at $(1,1)$ is 2, a tangent vector at the point is $\mathbf{T} = (1,2)$. The required rate of change is therefore

$$D_{\mathbf{T}}f = \nabla f_{|(1,1)} \cdot \hat{\mathbf{T}} = (2,-3) \cdot \frac{(1,2)}{\sqrt{5}} = \frac{-4}{\sqrt{5}}.$$

10. Since the slope of the curve at $(-1,3)$ is -9, a tangent vector at the point is $\mathbf{T}=(-1,9)$. The required rate of change is therefore

$$D_{\mathbf{T}}f=\nabla f_{|(-1,3)}\cdot\hat{\mathbf{T}}=(2x,1)_{|(-1,3)}\cdot\hat{\mathbf{T}}=(-2,1)\cdot\frac{(-1,9)}{\sqrt{82}}=\frac{11}{\sqrt{82}}.$$

11. Since parametric equations for the curve are $x=t$, $y=t^2-1$, $z=-2t$, a tangent vector to the curve is $\mathbf{T}=(1,2t,-2)$. At the point $(1,0,-2)$, a tangent vector is $(1,2,-2)$, and

$$D_{\mathbf{T}}f=\nabla f_{|(1,0,-2)}\cdot\hat{\mathbf{T}}=(y,x,2z)_{|(1,0,-2)}\cdot\frac{(1,2,-2)}{\sqrt{1+4+4}}=(0,1,-4)\cdot\frac{(1,2,-2)}{3}=\frac{10}{3}.$$

12. Since parametric equations for the curve are $x=t$, $y=-\sqrt{t^2-3}$, $z=t$, a tangent vector at $(2,-1,2)$ is $\mathbf{T}(2)=\left(1,\dfrac{-t}{\sqrt{t^2-3}},1\right)_{|t=2}=(1,-2,1)$. At the point $(2,-1,2)$ then,

$$\begin{aligned}D_{\mathbf{T}}f=\nabla f_{|(2,-1,2)}\cdot\hat{\mathbf{T}}&=(2xy+y^3z,x^2+3xy^2z,xy^3)_{|(2,-1,2)}\cdot\frac{(1,-2,1)}{\sqrt{1+4+1}}\\&=(-6,16,-2)\cdot\frac{(1,-2,1)}{\sqrt{6}}=\frac{-40}{\sqrt{6}}.\end{aligned}$$

13. The function increases most rapidly in the direction

$$\nabla f_{|(1,1,-3)}=(4x^3yz-y^3,x^4z-3xy^2,x^4y+1)_{|(1,1,-3)}=(-13,-6,2).$$

The rate of change in this direction is $\sqrt{169+36+4}=\sqrt{209}$.

14. The function increases most rapidly in the direction

$$\nabla f_{|(2,1/2)}=(2y+1/x,2x+1/y)_{|(2,1/2)}=(3/2,6),$$

or, $(2/3)(3/2,6)=(1,4)$. The rate of change in this direction is $\sqrt{9/4+36}=\sqrt{153}/2$.

15. The function increases most rapidly in the direction

$$\nabla f_{|(1,-3,2)}=\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}(-x,-y,-z)\right]_{|(1,-3,2)}=\frac{1}{14\sqrt{14}}(-1,3,-2),$$

or, $(-1,3,-2)$. The rate of change in this direction is $\dfrac{1}{14\sqrt{14}}\sqrt{1+9+4}=\dfrac{1}{14}$.

16. The function increases most rapidly in the direction

$$\nabla f_{|(1,-3,2)}=\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z)\right]_{|(1,-3,2)}=\frac{1}{14\sqrt{14}}(1,-3,2),$$

or, $(1,-3,2)$. The rate of change in this direction is $\dfrac{1}{14\sqrt{14}}\sqrt{1+9+4}=\dfrac{1}{14}$.

17. The function increases most rapidly in the direction

$$\nabla f_{|(3,2,-4)}=\left(\frac{yz}{1+x^2y^2z^2},\frac{xz}{1+x^2y^2z^2},\frac{xy}{1+x^2y^2z^2}\right)_{|(3,2,-4)}=\frac{(-8,-12,6)}{577},$$

or, $(-4,-6,3)$. The rate of change in this direction is $\dfrac{1}{577}\sqrt{64+144+36}=2\sqrt{61}/577$.

18. The function increases most rapidly in the direction

$$\nabla f_{|(1,1)}=(ye^{xy}+xy^2e^{xy},xe^{xy}+x^2ye^{xy})_{|(1,1)}=(2e,2e),$$

or, $(1,1)$. The rate of change in this direction is $\sqrt{4e^2+4e^2}=2\sqrt{2}e$.

19. The rate of change is smallest in the direction $-\nabla f_{|(2,-1,3)}=-(yz,xz,xy)_{|(2,-1,3)}=(3,-6,2)$.

20. At the point $(1,-1)$ and in the direction $\hat{\mathbf{v}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$,

$$D_{\hat{\mathbf{v}}}f = \nabla f_{|(1,-1)} \cdot \hat{\mathbf{v}} = (2xy, x^2+3y^2)_{|(1,-1)} \cdot \hat{\mathbf{v}} = (-2,4)\cdot(a,b) = -2a+4b.$$

(a) The directional derivative vanishes if $0 = -2a+4b$. Because $\hat{\mathbf{v}}$ is a unit vector, we also know that $a^2+b^2=1$. Thus, $1=(2b)^2+b^2=5b^2$. This implies that $b=\pm 1/\sqrt{5}$ and $a=\pm 2/\sqrt{5}$. The required directions are therefore $\pm(2,1)$. This is to be expected since in a direction perpendicular to ∇f, the rate of change should be zero, and $\pm(2,1)$ are both perpendicular to $(-2,4)$.

(b) The rate of change is 1 if, and when, $1=-2a+4b$. Substitution from this equation into $a^2+b^2=1$ gives $1=\left(\dfrac{4b-1}{2}\right)^2+b^2 \Longrightarrow 20b^2-8b-3=0$. Solutions of this equation are $b=(2\pm\sqrt{19})/10$, and these give $a=(-1\pm 2\sqrt{19})/10$. The required directions are therefore $(-1\pm 2\sqrt{19}, 2\pm\sqrt{19})$.

(c) The rate of change is 20, if, and when $20=-2a+4b$. Substitution from this equation into $a^2+b^2=1$ gives $1=(2b-10)^2+b^2 \longrightarrow 5b^2-40b+99=0$. Since this quadratic equation has no real solutions, there are no directions in which the rate of change of the function is equal to 20. This is also clear from the fact that the maximum rate of change is $|\nabla f| = 2\sqrt{5}$.

21. At the point $(0,1,-2)$ and in the direction $\hat{\mathbf{v}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$,

$$D_{\hat{\mathbf{v}}}f = \nabla f_{|(0,1,-2)} \cdot \hat{\mathbf{v}} = (y,x,1)_{|(0,1,-2)} \cdot \hat{\mathbf{v}} = (1,0,1)\cdot(a,b,c) = a+c.$$

(a) The directional derivative vanishes if $0=a+c$. Because $\hat{\mathbf{v}}$ is a unit vector, we also know that $a^2+b^2+c^2=1$. Substituting $c=-a$ gives $1=2a^2+b^2 \longrightarrow b=\pm\sqrt{1-2a^2}$. Thus, the directional derivative vanishes in the directions $(a,\pm\sqrt{1-2a^2},-a)$.

(b) The rate of change is 1 if, and when, $1=a+c$. Because $\hat{\mathbf{v}}$ is a unit vector, we also know that $a^2+b^2+c^2=1$. Substituting $c=1-a$ gives $1=a^2+b^2+(1-a)^2 \Longrightarrow b=\pm\sqrt{2a(1-a)}$. The required directions are therefore $(a,\pm\sqrt{2a(1-a)},1-a)$.

(c) The rate of change is -20, if, and when $-20=a+c$. Substitution from this equation into $a^2+b^2+c^2=1$ gives $1=a^2+b^2+(-20-a)^2 \Longrightarrow b^2=-399-40a-2a^2$. Since the quadratic expression in a is always negative, there are no directions in which the rate of change of the function is equal to -20. This is also clear from the fact that the minimum rate of change is $-|\nabla f| = -\sqrt{2}$.

22. (a) Yes. In any direction perpendicular to the gradient of the function, its rate of change is zero.
(b) Not necessarily. If the gradient of the function at the point has length 2 say, then the maximum rate of change for all directions is 2. Hence, for no direction could it be equal to 3. On the other hand, if the length of the gradient is 4, then values of the rate of change would vary between -4 and 4 and hence there would exist directions in which it is equal to 3.

23. If s were not length along the line then df/ds would not measure the rate of change of f with respect to distance.

24. The distance to the origin is given by $d=\sqrt{x^2+y^2+z^2}$. The required derivative is

$$\begin{aligned}
D_{\mathbf{T}}d = \nabla d \cdot \hat{\mathbf{T}} &= \frac{1}{\sqrt{x^2+y^2+z^2}}(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}) \cdot \frac{(-2\sin t, 2\cos t, 3)}{\sqrt{4\sin^2 t+4\cos^2 t+9}} \\
&= \frac{1}{\sqrt{x^2+y^2+z^2}} \frac{-2x\sin t+2y\cos t+3z}{\sqrt{13}} \\
&= \frac{1}{\sqrt{4\cos^2 t+4\sin^2 t+9t^2}} \frac{-4\cos t\,\sin t+4\sin t\,\cos t+9t}{\sqrt{13}} \\
&= \frac{9t}{\sqrt{13}\sqrt{4+9t^2}}.
\end{aligned}$$

When $t=0$, the rate of change is 0. This is expected since the curve is a helix, and when $t=0$, the point $(2,0,0)$ is the closest point to the origin. Hence, the distance should have a minimum and its derivative should vanish.

25. Since $\mathbf{T} = (1, -2, 1)$ is a vector along the line, the rate of change of $f(x, y, z)$ with respect to distance travelled along the curve vanishes if

$$0 = D_{\mathbf{T}} f = \nabla f \cdot \hat{\mathbf{T}} = (2x + yz, xz, xy) \cdot \frac{(1, -2, 1)}{\sqrt{6}} \quad \Longrightarrow \quad 0 = 2x + yz - 2xz + xy.$$

If we substitute the parametric equations of the line into this equation,

$$0 = 2t + (1 - 2t)(t) - 2t(t) + t(1 - 2t) = -6t^2 + 4t = 2t(2 - 3t) \quad \Longrightarrow \quad t = 0 \text{ or } t = 2/3.$$

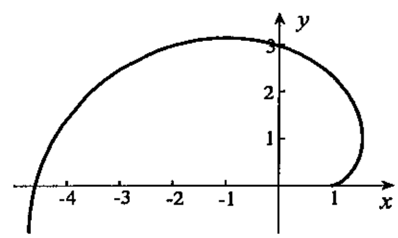
The required points are therfore $(0, 1, 0)$ and $(2/3, -1/3, 2/3)$.

26. A tangent vector along $C: \ x = t^2$, $y = t$, $z = t^2$ is $\mathbf{T} = (2t, 1, 2t)$. At any point on C,

$$D_{\mathbf{T}} f = \nabla f \cdot \hat{\mathbf{T}} = (2x, -2y, 2z) \cdot \frac{(2t, 1, 2t)}{\sqrt{1 + 8t^2}} = \frac{2(2xt - y + 2zt)}{\sqrt{1 + 8t^2}}.$$

For this derivative to vanish, $0 = 2xt - y + 2zt = 2t^3 - t + 2t^3 = t(4t^2 - 1)$. Thus, $t = 0, \pm 1/2$, and these values give the points $(0, 0, 0)$ and $(1/4, \pm 1/2, 1/4)$.

27. A plot of the involute is shown to the right. A tangent vector along the curve is $\mathbf{T} = (-\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t)$ $= (t\cos t, t\sin t)$. The rate of change of the distance $d = \sqrt{x^2 + y^2}$ from the origin to a point on the involute with respect to distance travelled along the curve is



$$D_{\mathbf{T}} d = \nabla d \cdot \hat{\mathbf{T}} = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \cdot (\cos t, \sin t) = \frac{x\cos t + y\sin t}{\sqrt{x^2 + y^2}}$$
$$= \frac{\cos t(\cos t + t\sin t) + \sin t(\sin t - t\cos t)}{(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2} = \frac{1}{1 + t^2}.$$

This rate of change is always positive.

28. It is the negative of the rate of change in the other direction.

29. The rate of change is zero (since the component of ∇f perpendicular to ∇f is zero).

30. Let (a, b) be the gradient of $f(x, y)$ at the point (x_0, y_0). Then,

$$3 = D_{\hat{\mathbf{i}}+2\hat{\mathbf{j}}} f_{|(x_0,y_0)} = \nabla f_{|(x_0,y_0)} \cdot \frac{(1, 2)}{\sqrt{5}} = (a, b) \cdot \frac{(1, 2)}{\sqrt{5}} = \frac{a + 2b}{\sqrt{5}}, \quad \text{and}$$
$$-1 = D_{-2\hat{\mathbf{i}}-\hat{\mathbf{j}}} f_{|(x_0,y_0)} = \nabla f_{|(x_0,y_0)} \cdot \frac{(-2, -1)}{\sqrt{5}} = (a, b) \cdot \frac{(-2, -1)}{\sqrt{5}} = \frac{-2a - b}{\sqrt{5}}.$$

These imply that $a = -\sqrt{5}/3$ and $b = 5\sqrt{5}/3$. The rate of change in direction $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ is therefore

$$\left(-\frac{\sqrt{5}}{3}, \frac{5\sqrt{5}}{3} \right) \cdot \frac{(2, 3)}{\sqrt{13}} = \frac{\sqrt{65}}{3}.$$

31. Let (a, b, c) be the gradient of $f(x, y)$ at the point (x_0, y_0, z_0). Then,

$$1 = D_{\hat{\mathbf{i}}+\hat{\mathbf{j}}} f_{|(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(1, 1, 0)}{\sqrt{2}} = (a, b, c) \cdot \frac{(1, 1, 0)}{\sqrt{2}} = \frac{a + b}{\sqrt{2}},$$
$$2 = D_{2\hat{\mathbf{i}}-\hat{\mathbf{k}}} f_{|(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(2, 0, -1)}{\sqrt{5}} = (a, b, c) \cdot \frac{(2, 0, -1)}{\sqrt{5}} = \frac{2a - c}{\sqrt{5}},$$
$$-3 = D_{\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}} f_{(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(1, -1, 1)}{\sqrt{3}} = (a, b, c) \cdot \frac{(1, -1, 1)}{\sqrt{3}} = \frac{a - b + c}{\sqrt{3}}.$$

These imply that $c = (\sqrt{2} - 3\sqrt{3} - 2\sqrt{5})/2$ and this is $\partial f / \partial z$ at the point.

32. Since $\nabla f = (3x^2y^2, 2x^3y)$, the first directional derivative at any point (x, y) in direction $\mathbf{v} = (1, -2)$ is

$$D_{\mathbf{v}}f = (3x^2y^2, 2x^3y) \cdot \frac{(1,-2)}{\sqrt{5}} = \frac{1}{\sqrt{5}}(3x^2y^2 - 4x^3y).$$

The second directional derivative is

$$\begin{aligned} D_{\mathbf{v}}(D_{\mathbf{v}}f)_{|(1,1)} &= \nabla\left[\frac{1}{\sqrt{5}}(3x^2y^2 - 4x^3y)\right]_{|(1,1)} \cdot \frac{(1,-2)}{\sqrt{5}} = \frac{1}{5}(6xy^2 - 12x^2y, 6x^2y - 4x^3)_{|(1,1)} \cdot (1,-2) \\ &= \frac{1}{5}(-6, 2) \cdot (1,-2) = -2. \end{aligned}$$

33. Since $\nabla f = (2x, 4y, 6z)$, the first directional derivative at any point (x, y, z) in direction $\mathbf{v} = (1, 1, -1)$ is

$$D_{\mathbf{v}}f = (2x, 4y, 6z) \cdot \frac{(1,1,-1)}{\sqrt{3}} = \frac{1}{\sqrt{3}}(2x + 4y - 6z).$$

The second directional derivative is

$$D_{\mathbf{v}}(D_{\mathbf{v}}f)_{|(-2,-1,3)} = \nabla\left[\frac{1}{\sqrt{3}}(2x + 4y - 6z)\right]_{|(-2,-1,3)} \cdot \frac{(1,1,-1)}{\sqrt{3}} = \frac{1}{3}(2,4,-6)_{|(-2,-1,3)} \cdot (1,1,-1) = 4.$$

34. Since a tangent vector to the curve at any point is $\mathbf{T} = R(1 - \cos\theta, \sin\theta)$, a unit tangent vector is

$$\hat{\mathbf{T}} = \frac{(1-\cos\theta, \sin\theta)}{\sqrt{(1-\cos\theta)^2 + \sin^2\theta}} = \frac{(1-\cos\theta, \sin\theta)}{\sqrt{2 - 2\cos\theta}}.$$

The rate of change of the distance $d = \sqrt{x^2 + y^2}$ from the origin to the stone is

$$D_{\hat{\mathbf{T}}}d = \nabla d \cdot \hat{\mathbf{T}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{\sqrt{x^2+y^2}} \cdot \hat{\mathbf{T}} = \frac{x(1-\cos\theta) + y\sin\theta}{\sqrt{x^2+y^2}\sqrt{2-2\cos\theta}}.$$

When $\theta = \pi/2$, we have $x = R(\pi/2 - 1)$, $y = R$, and

$$D_{\hat{\mathbf{T}}}d_{|\theta=\pi/2} = \frac{R(\pi/2-1)(1) + R(1)}{\sqrt{R^2(\pi/2-1)^2 + R^2}\sqrt{2}} = \frac{\pi}{\sqrt{2}\sqrt{8 - 4\pi + \pi^2}}.$$

When $\theta = \pi$, we have $x = \pi R$, $y = 2R$, and

$$D_{\hat{\mathbf{T}}}d_{|\theta=\pi} = \frac{\pi R(2) + 2R(0)}{\sqrt{\pi^2R^2 + 4R^2}\sqrt{4}} = \frac{\pi}{\sqrt{4+\pi^2}}.$$

(b) Since $\nabla(y) = \hat{\mathbf{j}}$, the rate of change of the y-coordinate is

$$D_{\hat{\mathbf{T}}}y = \hat{\mathbf{j}} \cdot \frac{(1-\cos\theta, \sin\theta)}{\sqrt{2-2\cos\theta}} = \frac{\sin\theta}{\sqrt{2-2\cos\theta}}.$$

When $\theta = \pi/2$, $D_{\hat{\mathbf{T}}}y_{|\theta=\pi/2} = 1/\sqrt{2}$, and when $\theta = \pi$, $D_{\hat{\mathbf{T}}}y_{|\theta=\pi} = 0$.

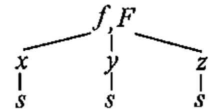
(c) Since $\nabla(x) = \hat{\mathbf{i}}$, the rate of change of the x-coordinate is

$$D_{\hat{\mathbf{T}}}x = \hat{\mathbf{i}} \cdot \frac{(1-\cos\theta, \sin\theta)}{\sqrt{2-2\cos\theta}} = \frac{1-\cos\theta}{\sqrt{2-2\cos\theta}}.$$

When $\theta = \pi/2$, $D_{\hat{\mathbf{T}}}x_{|\theta=\pi/2} = 1/\sqrt{2}$, and when $\theta = \pi$, $D_{\hat{\mathbf{T}}}x_{|\theta=\pi} = 2/\sqrt{4} = 1$.

35. If we apply the mean value theorem (or Taylor's remainder formula) to the function $F(s) = f(x_0 + v_x s, y_0 + v_y s, z_0 + v_z s)$, between $s = 0$ and an arbitrary value of s, we obtain $F(s) = F(0) + F'(c)s$, where $0 < c < s$. Substitution of this into the limit gives

$$D_{\mathbf{v}} f = \lim_{s \to 0^+} \frac{[F(0) + F'(c)s] - F(0)}{s} = \lim_{s \to 0^+} F'(c).$$



The schematic to the right shows that

$$F'(s) = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y + \frac{\partial f}{\partial z} v_z.$$

Consequently, the directional derivative at (x_0, y_0, z_0) is

$$D_{\mathbf{v}} f = \lim_{s \to 0^+} \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y + \frac{\partial f}{\partial z} v_z = \frac{\partial f}{\partial x}_{|(x_0,y_0,z_0)} v_x + \frac{\partial f}{\partial y}_{|(x_0,y_0,z_0)} v_y + \frac{\partial f}{\partial z}_{|(x_0,y_0,z_0)} v_z.$$

EXERCISES 12.9

1. Since slope of the tangent line at $(-2, 4, 0)$ is -4, equations of the tangent line are $y - 4 = -4(x + 2)$, $z = 0 \Longrightarrow 4x + y + 4 = 0$, $z = 0$.

2. Since a tangent vector at $(1, 1, 1)$ is $\frac{d\mathbf{r}}{dt}_{|t=1} = (1, 2t, 3t^2)_{|t=1} = (1, 2, 3)$, parametric equations for the tangent line are $x = 1 + u$, $y = 1 + 2u$, $z = 1 + 3u$.

3. Since a tangent vector at $(1, 0, 1)$ is $\frac{d\mathbf{r}}{dt}_{|t=0} = (-\sin t, \cos t, -\sin t)_{|t=0} = (0, 1, 0)$, parametric equations for the tangent line are $x = 1$, $y = u$, $z = 1$.

4. With parametric equations $x = t$, $y = t^2$, $z = t$, a tangent vector at the point $(-2, 4, -2)$ is $\frac{d\mathbf{r}}{dt}_{|t=-2} = (1, 2t, 1)_{|t=-2} = (1, -4, 1)$. Parametric equations for the tangent line are $x = -2 + u$, $y = 4 - 4u$, $z = -2 + u$.

5. With parametric equations $x = t$, $y = t^2$, $z = t^2 - t$, a tangent vector at the point $(1, 1, 0)$ is $\frac{d\mathbf{r}}{dt}_{|t=1} = (1, 2t, 2t - 1)_{|t=1} = (1, 2, 1)$. Parametric equations for the tangent line are $x = 1 + u$, $y = 1 + 2u$, $z = u$.

6. Since a tangent vector at $(1, 5, 1)$ is $\frac{d\mathbf{r}}{dt}_{|t=1} = (-2t, 2, 1)_{|t=1} = (-2, 2, 1)$, parametric equations for the tangent line are $x = 1 - 2u$, $y = 5 + 2u$, $z = 1 + u$.

7. Since a tangent vector at $(\sqrt{2}, -3/\sqrt{2}, 5)$ is $\frac{d\mathbf{r}}{dt}_{|t=-\pi/4} = (-2\sin t, 3\cos t, 0)_{|t=-\pi/4} = (\sqrt{2}, 3/\sqrt{2}, 0)$. Since $(2, 3, 0)$ is also a tangent vector, parametric equations for the tangent line are $x = \sqrt{2} + 2u$, $y = -3/\sqrt{2} + 3u$, $z = 5$.

8. Because a vector normal to the curve at $(1, 4)$ is

$$\nabla(x^2y^3 + xy - 68)_{|(1,4)} = (2xy^3 + y, 3x^2y^2 + x)_{|(1,4)} = (132, 49),$$

a vector tangent to the curve is $(49, -132)$. The slope of the tangent line is is therefore $-132/49$, and its equations are $y - 4 = -\frac{132}{49}(x - 1)$, $z = 0 \Longrightarrow 132x + 49y = 328$, $z = 0$.

9. Since the curve is a straight line, the tangent line is the line itself.

10. Since a tangent vector at $(1, 0, 0)$ is

$$\frac{d\mathbf{r}}{dt}_{|t=0} = (-e^{-t}\cos t - e^{-t}\sin t, -e^{-t}\sin t + e^{-t}\cos t, 1)_{|t=0} = (-1, 1, 1),$$

parametric equations for the tangent line are $x = 1 - u$, $y = u$, $z = u$.

11. Since a tangent vector at $(2,-6,2)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=-1} = (2t, 2, 3t^2)_{|t=-1} = (-2,2,3)$, parametric equations for the tangent line are $x = 2-2u$, $y = -6+2u$, $z = 2+3u$.

12. Since normals to the surfaces at $(2,-\sqrt{5},-1)$ are $\nabla(y^2+z^2-6)_{|(2,-\sqrt{5},-1)} = (0,2y,2z)_{|(2,-\sqrt{5},-1)} = (0,-2\sqrt{5},-2)$ and $\nabla(x+z-1)_{|(2,-\sqrt{5},-1)} = (1,0,1)$, a vector along the tangent line is

$$(0,\sqrt{5},1)\times(1,0,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & \sqrt{5} & 1 \\ 1 & 0 & 1 \end{vmatrix} = (\sqrt{5},1,-\sqrt{5}).$$

Because $(1,1/\sqrt{5},-1)$ is also a tangent vector, parametric equations for the tangent line are $x = 2+t$, $y = -\sqrt{5}+t/\sqrt{5}$, $z = -1-t$.

13. Since normals to the surfaces at $(1,1,-\sqrt{2})$ are $\nabla(x^2+y^2+z^2-4)_{|(1,1,-\sqrt{2})} = (2x,2y,2z)_{|(1,1,-\sqrt{2})} = (2,2,-2\sqrt{2})$ and $\nabla(x^2+y^2-z^2)_{|(1,1,-\sqrt{2})} = (2x,2y,-2z)_{|(1,1,-\sqrt{2})} = (2,2,2\sqrt{2})$, a vector along the tangent line is

$$(1,1,-\sqrt{2})\times(1,1,\sqrt{2}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \end{vmatrix} = (2\sqrt{2},-2\sqrt{2},0).$$

Because $(1,-1,0)$ is also a tangent vector, parametric equations for the tangent line are $x = 1+u$, $y = 1-u$, $z = -\sqrt{2}$.

14. Because a tangent vector at $(4,1,\sqrt{17})$ is $\dfrac{d\mathbf{r}}{dt}_{|t=4} = (1,0,t/\sqrt{1+t^2})_{|t=4} = (1,0,4/\sqrt{17})$, as is the vector $(\sqrt{17},0,4)$, parametric equations for the tangent line are $x = 4+\sqrt{17}u$, $y = 1$, $z = \sqrt{17}+4u$.

15. A tangent vector at $(2,2,2)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=0} = (-\sin t, -\cos t, 1/(2\sqrt{4+t}))_{|t=0} = (0,-1,1/4)$. Since $(0,-4,1)$ is also a tangent vector, parametric equations for the tangent line are $x = 2$, $y = 2-4u$, $z = 2+u$.

16. With parametric equations $x = t^2+t^3$, $y = t-t^4$, $z = t$, a tangent vector at the point $(12,-14,2)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=2} = (2t+3t^2, 1-4t^3, 1)_{|t=2} = (16,-31,1)$. Parametric equations for the tangent line are $x = 12+16u$, $y = -14-31u$, $z = 2+u$.

17. With parametric equations $x = t^2+3t^3-2t+5$, $y = t$, $z = 0$, a tangent vector at $(7,1,0)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=1} = (2t+9t^2-2, 1, 0)_{|t=1} = (9,1,0)$. Parametric equations for the tangent line are $x = 7+9u$, $y = 1+u$, $z = 0$.

18. Since normals to the surfaces at $(0,1,1)$ are $\nabla(2x^2+y^2+2y-3)_{|(0,1,1)} = (4x, 2y+2, 0)_{|(0,1,1)} = (0,4,0)$ and $\nabla(x-z+1)_{|(0,1,1)} = (1,0,-1)$, a vector along the tangent line is

$$(0,1,0)\times(1,0,-1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = (-1,0,-1),$$

as is $(1,0,1)$. Parametric equations for the tangent line are $x = t$, $y = 1$, $z = 1+t$.

19. A tangent vector at the point $(1,1,\sqrt{2})$ is $\dfrac{d\mathbf{r}}{dt}_{|t=1} = (2t, 1, (1+4t^3)/(2\sqrt{t+t^4}))_{|t=1} = (2,1,5/(2\sqrt{2}))$. Since $(8,4,5\sqrt{2})$ is also a tangent vector, parametric equations for the tangent line are $x = 1+8u$, $y = 1+4u$, $z = \sqrt{2}+5\sqrt{2}u$.

20. Since a tangent vector at $(0,2\pi,4\pi)$ is $\dfrac{d\mathbf{r}}{dt}_{|t=2\pi} = (\sin t + t\cos t, \cos t - t\sin t, 2)_{|t=2\pi} = (2\pi,1,2)$, parametric equations for the tangent line are $x = 2\pi u$, $y = 2\pi+u$, $z = 4\pi+2u$.

21. Since a normal to the tangent plane is

$$\nabla(\sqrt{x^2+y^2}-z)_{|(1,1,\sqrt{2})} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)_{|(1,1,\sqrt{2})} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1\right),$$

as is $(1,1,-\sqrt{2})$, the equation of the tangent plane is

$$0 = (1,1,-\sqrt{2})\cdot(x-1, y-1, z-\sqrt{2}) = x+y-\sqrt{2}z.$$

22. Since a normal to the tangent plane is

$$\nabla(x-x^2+y^3z)_{|(2,-1,-2)} = (1-2x, 3y^2z, y^3)_{|(2,-1,-2)} = (-3,-6,-1),$$

as is $(3,6,1)$, the equation of the tangent plane is

$$0 = (3,6,1)\cdot(x-2, y+1, z+2) = 3x+6y+z+2.$$

23. Since a normal to the tangent plane is

$$\nabla(x^2y+y^2z+z^2x+3)_{|(2,-1,-1)} = (2xy+z^2, x^2+2yz, y^2+2zx)_{|(2,-1,-1)} = (-3,6,-3),$$

as is $(1,-2,1)$, the equation of the tangent plane is

$$0 = (1,-2,1)\cdot(x-2, y+1, z+1) = x-2y+z-3.$$

24. Because the surface is a plane, the tangent plane is the surface itself, $x+y+z=4$.

25. Since a normal to the tangent plane is

$$\nabla(y\sin(\pi z/2)-x)_{|(-1,-1,1)} = (-1, \sin(\pi z/2), (\pi y/2)\cos(\pi z/2))_{|(-1,-1,1)} = (-1,1,0),$$

as is $(1,-1,0)$, the equation of the tangent plane is $0 = (1,-1,0)\cdot(x+1, y+1, z-1) = x-y$.

26. Since a normal to the tangent plane is $\nabla(x^2+y^2+2y-1)_{|(1,0,3)} = (2x, 2y+2, 0)_{|(1,0,3)} = (2,2,0)$, as is $(1,1,0)$, the equation of the tangent plane is $0 = (1,1,0)\cdot(x-1, y, z-3) = x+y-1$.

27. A tangent vector to the curve at $(2,2,1)$ is $(2t^2, 4t, 3)_{|t=1} = (2,4,3)$. A normal vector to the surface at the point is $(2x, 4y, 6z)_{|(2,2,1)} = (4,8,6)$. Since the vectors are in the same direction, the curve intersects the surface at right angles.

28. A vector tangent to the curve at $(1,1,1)$ is

$$\begin{aligned}\mathbf{T} &= \nabla(x^2-y^2+z^2-1)_{|(1,1,1)} \times \nabla(xy+xz-2)_{|(1,1,1)} = (2x,-2y,2z)_{|(1,1,1)} \times (y+z, x, x)_{|(1,1,1)} \\ &= (2,-2,2)\times(2,1,1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = (-4,2,6).\end{aligned}$$

A vector normal to the surface at $(1,1,1)$ is

$$\mathbf{n} = \nabla(xyz-x^2-6y+6)_{|(1,1,1)} = (yz-2x, xz-6, xy)_{|(1,1,1)} = (-1,-5,1).$$

Since $\mathbf{T}\cdot\mathbf{n} = 4-10+6 = 0$, the vectors are perpendicular, and the curve is tangent to the surface at $(1,1,1)$.

29. Since a vector normal to the surface at (x_0, y_0, z_0) is

$$\nabla[z-f(x,y)]_{|(x_0,y_0,z_0)} = (f_x(x_0,y_0), f_y(x_0,y_0), -1),$$

the equation of the tangent plane at the point is

$$0 = (f_x(x_0,y_0), f_y(x_0,y_0), -1)\cdot(x-x_0, y-y_0, z-z_0) \implies z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

30. Since a vector along the curve is $\mathbf{T} = \nabla(x+y+z-4) \times \nabla(x-y+z-2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = (2,0,-2)$,

we find that at $(3,1,0)$,

$$D_{\hat{\mathbf{T}}}f = \nabla f_{|(3,1,0)} \cdot \hat{\mathbf{T}} = (4x, 2yz^2, 2y^2z)_{|(3,1,0)} \cdot \frac{(1,0,-1)}{\sqrt{2}} = (12,0,0) \cdot \frac{(1,0,-1)}{\sqrt{2}} = \frac{12}{\sqrt{2}} = 6\sqrt{2}.$$

31. Since a vector perpendicular to the surface at the point $(1,-2,5)$ is

$$\mathbf{n} = \nabla(x^2+y^2-z)_{|(1,-2,5)} = (2x, 2y, -1)_{|(1,-2,5)} = (2,-4,-1),$$

the required derivative is

$$\pm D_{\hat{\mathbf{n}}}f = \pm\nabla f_{|(1,-2,5)} \cdot \hat{\mathbf{n}} = \pm(yz+y+z, xz+x+z, xy+x+y)_{|(1,-2,5)} \cdot \frac{(2,-4,-1)}{\sqrt{21}}$$
$$= \pm(-7, 11, -3) \cdot \frac{(2,-4,-1)}{\sqrt{21}} = \frac{\pm 55}{\sqrt{21}}.$$

32. Since $f(x,y,z) = 0$ everywhere on the curve, its directional derivative must also be zero.

33. Since the slope of the tangent line to the curve is given by $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\dfrac{F_x}{F_y}$, a vector along the tangent line is $(F_y, -F_x)$. Since the scalar product of this vector with $\nabla F = (F_x, F_y)$ is zero, the gradient ∇F is perpendicular to the curve.

34. A vector normal to the surface at (x_0, y_0, z_0) is $\nabla\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1\right)_{|(x_0,y_0,z_0)} = \left(\dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2}\right)$. Hence, the equation of the tangent plane is

$$0 = \left(\frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2}\right) \cdot (x-x_0, y-y_0, z-z_0) = \frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right).$$

Since (x_0, y_0, z_0) is on the ellipsoid, $x_0^2/a^2 + y_0^2/b^2 + z_0^2/c^2 = 1$, and the equation of the plane reduces to $x_0x/a^2 + y_0y/b^2 + z_0z/c^2 = 1$.

35. A normal vector to the surface at any point is $\nabla(9x^2 - 4y^2 - 36z) = (18x, -8y, -36)$. The tangent plane to the surface is parallel to the plane $x+y+z=4$ if this vector is a multiple of the normal $(1,1,1)$ to the plane,

$$(18x, -8y, -36) = \lambda(1,1,1) \quad \Longrightarrow \quad 18x = \lambda,\ -8y = \lambda,\ -36 = \lambda.$$

Thus, $x = -2$ and $y = 9/2$, and the only point is $(-2, 9/2, -5/4)$.

36. A normal vector to the surface is $\nabla(4x^2+4y^2-z^2) = (8x, 8y, -2z)$, as is $(4x, 4y, -z)$. The tangent plane is parallel to $x - y + 2z = 3$, which has normal $(1,-1,2)$, if, and where, $(4x, 4y, -z) = \lambda(1,-1,2)$ for some λ. This requires $4x = \lambda$, $4y = -\lambda$, and $-z = 2\lambda$. Substitution of $x = \lambda/4$, $y = -\lambda/4$, and $z = -2\lambda$ into the equation of the surface gives $4\lambda^2 = 4\left(\dfrac{\lambda^2}{16}\right) + 4\left(\dfrac{\lambda^2}{16}\right)$. The only solution of this equation is $\lambda = 0$. But this implies that $x = y = z = 0$, and this is unacceptable since there is no tangent plane to the surface at $(0,0,0)$.

37. A tangent vector to the curve is $(dx/dt, dy/dt, dz/dt)$. But,

$$\frac{dx}{dt} = -\frac{\dfrac{\partial(F,G,H)}{\partial(t,y,z)}}{\dfrac{\partial(F,G,H)}{\partial(x,y,z)}}, \quad \frac{dy}{dt} = -\frac{\dfrac{\partial(F,G,H)}{\partial(x,t,z)}}{\dfrac{\partial(F,G,H)}{\partial(x,y,z)}}, \quad \frac{dz}{dt} = -\frac{\dfrac{\partial(F,G,H)}{\partial(x,y,t)}}{\dfrac{\partial(F,G,H)}{\partial(x,y,z)}}.$$

Hence a tangent vector is $\left(\dfrac{\partial(F,G,H)}{\partial(t,y,z)}, \dfrac{\partial(F,G,H)}{\partial(x,t,z)}, \dfrac{\partial(F,G,H)}{\partial(x,y,t)}\right)$. Symmetric equations for the tangent line at P are

$$\frac{x-x_0}{\left.\dfrac{\partial(F,G,H)}{\partial(t,y,z)}\right|_P} = \frac{y-y_0}{\left.\dfrac{\partial(F,G,H)}{\partial(x,t,z)}\right|_P} = \frac{z-z_0}{\left.\dfrac{\partial(F,G,H)}{\partial(x,y,t)}\right|_P}.$$

38. A normal vector to the paraboloid at any point $P(x,y,z)$ is

$$\mathbf{n} = \nabla(x^2+y^2-1-z) = (2x, 2y, -1).$$

This vector coincides with **OP** if $(x,y,z) = \lambda(2x,2y,-1)$, or, $x = 2\lambda x$, $y = 2\lambda y$, $z = -\lambda$. When these are combined with $z = x^2+y^2-1$, the points obtained are $(0,0,-1)$ and $x^2+y^2 = 1/2$, $z = -1/2$.

39. Since a normal vector to the surface is $\nabla(\sqrt{x}+\sqrt{y}+\sqrt{z}-\sqrt{a}) = \left(\dfrac{1}{2\sqrt{x}}, \dfrac{1}{2\sqrt{y}}, \dfrac{1}{2\sqrt{z}}\right)$, the equation of the tangent plane at any point (x_0,y_0,z_0) is

$$0 = \left(\frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}}\right)\cdot(x-x_0, y-y_0, z-z_0) = \frac{1}{\sqrt{x_0}}(x-x_0) + \frac{1}{\sqrt{y_0}}(y-y_0) + \frac{1}{\sqrt{z_0}}(z-z_0).$$

The x-intercept of this plane is given by

$$0 = \frac{1}{\sqrt{x_0}}(x-x_0) - \sqrt{y_0} - \sqrt{z_0} \quad\Longrightarrow\quad x = x_0 + \sqrt{x_0}(\sqrt{y_0}+\sqrt{z_0}) = \sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}).$$

With similar expressions for the y- and z-intercepts, their sum is

$$\begin{aligned}\sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}) + \sqrt{y_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}) + \sqrt{z_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})\\ = (\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})^2 = a.\end{aligned}$$

EXERCISES 12.10

1. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x+2y$, $0 = \dfrac{\partial f}{\partial y} = 2x+4y-6$. The only solution is $(-3,3)$.

$$\frac{\partial^2 f}{\partial x^2} = 2, \qquad \frac{\partial^2 f}{\partial x\partial y} = 2, \qquad \frac{\partial^2 f}{\partial y^2} = 4$$

Since $B^2 - AC = 4-8 = -4$ and $A = 2$, there is a relative minimum at $(-3,3)$.

2. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 3y-3x^2$, $0 = \dfrac{\partial f}{\partial y} = 3x-3y^2$. Solutions are $(0,0)$ and $(1,1)$.

$$\frac{\partial^2 f}{\partial x^2} = -6x, \qquad \frac{\partial^2 f}{\partial x\partial y} = 3, \qquad \frac{\partial^2 f}{\partial y^2} = -6y$$

At $(0,0)$, $B^2-AC = 9-0$, and therefore $(0,0)$ yields a saddle point. At $(1,1)$, $B^2-AC = 9-(-6)(-6) = -27$, and $A = -6$, and therefore $(1,1)$ gives a relative maximum.

3. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 3x^2-3$, $0 = \dfrac{\partial f}{\partial y} = 2y+2$. Solutions are $(\pm 1,-1)$.

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial x\partial y} = 0, \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

At $(-1,-1)$, $B^2-AC = 0+12$, and therefore $(-1,-1)$ yields a saddle point. At $(1,-1)$, $B^2-AC = 0-12$ and $A = 6$. Thus, $(1,-1)$ gives a relative minimum.

4. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2xy^2+3$, $0 = \dfrac{\partial f}{\partial y} = 2x^2y$. Because there are no solutions of these equations, there are no critical points.

5. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = y - 2x$, $0 = \dfrac{\partial f}{\partial y} = x + 2y$. The only solution is $(0, 0)$.

$$\frac{\partial^2 f}{\partial x^2} = -2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 1, \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

Since $B^2 - AC = 1 + 4$, $(0, 0)$ yields a saddle point.

6. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = \sin y$, $0 = \dfrac{\partial f}{\partial y} = x \cos y$. Solutions are $(0, n\pi)$, where n is an integer.

$$\frac{\partial^2 f}{\partial x^2} = 0, \qquad \frac{\partial^2 f}{\partial x \partial y} = \cos y, \qquad \frac{\partial^2 f}{\partial y^2} = -x \sin y$$

At $(0, n\pi)$, $B^2 - AC = [\cos(n\pi)]^2 - 0 > 0$, and therefore all critical points yield saddle points.

7. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = ye^{-(x^2+y^2)} - 2x^2ye^{-(x^2+y^2)} = y(1 - 2x^2)e^{-(x^2+y^2)},$$
$$0 = \frac{\partial f}{\partial y} = xe^{-(x^2+y^2)} - 2xy^2e^{-(x^2+y^2)} = x(1 - 2y^2)e^{-(x^2+y^2)}.$$

Solutions are $(0, 0)$, $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ and $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$.

$$\frac{\partial^2 f}{\partial x^2} = y(-4x)e^{-(x^2+y^2)} - 2xy(1 - 2x^2)e^{-(x^2+y^2)} = 2xy(2x^2 - 3)e^{-(x^2+y^2)},$$
$$\frac{\partial^2 f}{\partial x \partial y} = (1 - 2x^2)e^{-(x^2+y^2)} - 2y^2(1 - 2x^2)e^{-(x^2+y^2)} = (1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)},$$
$$\frac{\partial^2 f}{\partial y^2} = x(-4y)e^{-(x^2+y^2)} - 2xy(1 - 2y^2)e^{-(x^2+y^2)} = 2xy(2y^2 - 3)e^{-(x^2+y^2)},$$
$${f_{xy}}^2 - f_{xx}f_{yy} = [(1 - 2x^2)^2(1 - 2y^2)^2 - 4x^2y^2(2x^2 - 3)(2y^2 - 3)]e^{-2(x^2+y^2)}.$$

At $(0, 0)$, $B^2 - AC = 1$, and therefore $(0, 0)$ yields a saddle point.
At $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, $B^2 - AC = -4(1/2)(1/2)(-2)(-2)e^{-2} < 0$, and $A = 2(1/2)(-2)e^{-1} < 0$. These critical points give relative maxima.
At $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$, $B^2 - AC = -4(1/2)(1/2)(-2)(-2)e^{-2} < 0$, and $A = 2(-1/2)(-2)e^{-1} > 0$. These critical points give relative minima.

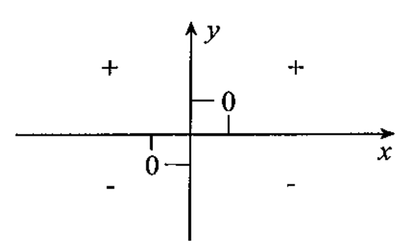
8. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x - 2y$, $0 = \dfrac{\partial f}{\partial y} = -2x + 2y$. All points on the line $y = x$ are critical. Because $f(x, y) = (x - y)^2$, it is clear that each of these points gives a relative minimum.

9. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = \dfrac{4x}{3(x^2 + y^2)^{1/3}}$, $0 = \dfrac{\partial f}{\partial y} = \dfrac{4y}{3(x^2 + y^2)^{1/3}}$. There are no solutions to these equations. Because the derivatives are undefined at $(0, 0)$, but $f(0, 0) = 0$, there is a critical point at $(0, 0)$. Since $f(x, y) > 0$ for all $(x, y) \neq (0, 0)$, there is a relative minimum at $(0, 0)$.

10. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = 4x^3y^3, \qquad 0 = \frac{\partial f}{\partial y} = 3x^4y^2.$$

Every point on the x- and y-axes is critical, and at each of these points $f(x, y) = 0$. The diagram to the right showing the sign of $f(x, y)$ in the four quadrants indicates that the points $(0, y)$ for $y > 0$ yield relative minima; $(0, y)$ for $y < 0$ yield relative maxima; and $(x, 0)$ yield saddle points.

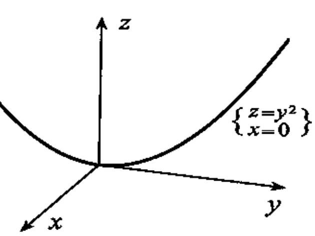


11. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2y^2 + 3y + 2xy^3$, $0 = \dfrac{\partial f}{\partial y} = 4xy + 3x + 3x^2y^2$. Solutions are $(0,0)$, $(0,-3/2)$ and $(-4/3, 3/2)$.

$$\frac{\partial^2 f}{\partial x^2} = 2y^3, \qquad \frac{\partial^2 f}{\partial x \partial y} = 4y + 3 + 6xy^2, \qquad \frac{\partial^2 f}{\partial y^2} = 4x + 6x^2y$$

At $(0,0)$, $B^2 - AC = 9 > 0$, so that $(0,0)$ yields a saddle point.
At $(0,-3/2)$, $B^2 - AC = (-3)^2 > 0$, so that $(0,-3/2)$ yields a saddle point.
At $(-4/3, 3/2)$, $B^2 - AC = (9)^2 - 2(27/8)(32/3) > 0$, so that $(-4/3, 3/2)$ also yields a saddle point.

12. If $x > 0$, then $\partial f/\partial x = 1$, and if $x < 0$, then $\partial f/\partial x = -1$. Consequently, there are no critical points at which $0 = \partial f/\partial x = \partial f/\partial y$. However $\partial f/\partial x$ does not exist when $x = 0$, and therefore every point on the y-axis is critical. They cannot give saddle points because $\partial f/\partial x$ does not exist at these points. Since the cross-section of the surface $z = |x| + y^2$ with the plane $x = 0$ is the parabola $z = y^2$, $x = 0$ (see figure to the right), no critical point except possible $(0,0)$ can yield a relative maximum or minimum. Finally, because $f(0,0) = 0$, and $f(x,y) > 0$ for $(x,y) \neq (0,0)$, it follows that $(0,0)$ gives a relative minimum.



13. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = -(1-y)(x+y-1) + (1-x)(1-y) = (1-y)(2-2x-y),$$
$$0 = \frac{\partial f}{\partial y} = -(1-x)(x+y-1) + (1-y)(1-x) = (1-x)(2-2y-x).$$

Solutions are $(1,1)$, $(0,1)$, $(1,0)$ and $(2/3, 2/3)$.

$$\frac{\partial^2 f}{\partial x^2} = 2(y-1), \qquad \frac{\partial^2 f}{\partial x \partial y} = -(2-2x-y) - (1-y) = -3 + 2x + 2y, \qquad \frac{\partial^2 f}{\partial y^2} = 2(x-1)$$

At $(1,1)$, $(0,1)$ and $(1,0)$, $B^2 - AC = 1$, so that each gives a saddle point. At $(2/3, 2/3)$, $B^2 - AC = (-1/3)^2 - 4(-1/3)(-1/3) = -1/3$ and $A = -2/3$. This critical point therefore yields a relative maximum.

14. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 4x^3 - 2x = 2x(2x^2 - 1)$, $0 = \dfrac{\partial f}{\partial y} = 4y^3 - 2y = 2y(2y^2 - 1)$. These give $x = 0, \pm 1/\sqrt{2}$ and $y = 0, \pm 1/\sqrt{2}$, and therefore critical points are $(0,0)$, $(0, \pm 1/\sqrt{2})$, $(\pm 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, \pm 1/\sqrt{2})$, $(-1/\sqrt{2}, \pm 1/\sqrt{2})$. With $f_{xx} = 12x^2 - 2$, $f_{xy} = 0$, and $f_{yy} = 12y^2 - 2$, we construct the following table.

	A	B	C	$B^2 - AC$	Classification
$(0,0)$	-2	0	-2	-4	relative maximum
$(0, \pm 1/\sqrt{2})$	-2	0	4	8	saddle points
$(\pm 1/\sqrt{2}, 0)$	4	0	-2	8	saddle points
$(1/\sqrt{2}, \pm 1/\sqrt{2})$	4	0	4	-16	relative minima
$(-1/\sqrt{2}, \pm 1/\sqrt{2})$	4	0	4	-16	relative minima

15. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x + 3$, $0 = \dfrac{\partial f}{\partial y} = 2y - 2$, $0 = \dfrac{\partial f}{\partial z} = -2z$. The only solution is $(-3/2, 1, 0)$.

16. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = 2xy^2z^2 + 2xt^2 + 3, \quad 0 = \frac{\partial f}{\partial y} = 2x^2yz^2, \quad 0 = \frac{\partial f}{\partial z} = 2x^2y^2z, \quad 0 = \frac{\partial f}{\partial t} = 2x^2t.$$

There are no solutions to these equations.

17. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = yz + 2xyz$, $0 = \dfrac{\partial f}{\partial y} = xz + x^2z - 1$, $0 = \dfrac{\partial f}{\partial z} = xy + x^2y$. Solutions of this equation are $\left(x, 0, \dfrac{1}{x^2 + x}\right)$, where x is any real number except 0 and -1.

18. For critical points we solve

$$0 = \frac{\partial f}{\partial x} = yze^{x^2+y^2+z^2} + 2x^2yze^{x^2+y^2+z^2} = yz(1 + 2x^2)e^{x^2+y^2+z^2},$$

$$0 = \frac{\partial f}{\partial y} = xz(1 + 2y^2)e^{x^2+y^2+z^2}, \qquad 0 = \frac{\partial f}{\partial z} = xy(1 + 2z^2)e^{x^2+y^2+z^2}.$$

All points on the coordinate axes satisfy these equations.

19. A function $f(x, y, z)$ has a relative maximum at (x_0, y_0, z_0) if there exists a sphere centred at (x_0, y_0, z_0) such that for all points inside this sphere, $f(x, y, z) \le f(x_0, y_0, z_0)$. A function $f(x, y, z)$ has a relative minimum at (x_0, y_0, z_0) if there exists a sphere centred at (x_0, y_0, z_0) such that for all points inside this sphere, $f(x, y, z) \ge f(x_0, y_0, z_0)$.

20. If $f(x, y)$ is harmonic in D, it has continuous second partial derivatives in D and $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ therein. For $f(x, y)$ to have a relative maximum or minimum at a point (x, y) in D, its first partial derivatives must vanish there. In addition,

$$0 > B^2 - AC = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(-\frac{\partial^2 f}{\partial x^2}\right) = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial x^2}\right)^2,$$

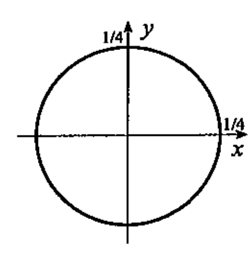
an impossibility if f_{xx} or f_{xy} does not vanish.

21. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = -8xy + 12x^3$, $0 = \dfrac{\partial f}{\partial y} = 2y - 4x^2$.
The only solution is $(0, 0)$. Since $B^2 - AC = (-8x)^2 - (-8y + 36x^2)(2) = 0$ at $(0, 0)$, the second derivative test fails. On the parabola $y = ax^2$, where a is a nonzero constant, function values are $f(x, ax^2) = (ax^2)^2 - 4x^2(ax^2) + 3x^4 = (a^2 - 4a + 3)x^4 = (a - 1)(a - 3)x^4$. Since this function takes on positive and negative values in every circle centred at the origin, it follows that the function must have a saddle point at $(0, 0)$.

22. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 4x^3 + 3y^2$, $0 = \dfrac{\partial f}{\partial y} = 6xy + 2y$. The only solutions are $(0, 0)$ and $(-1/3, \pm 2/9)$.

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 6y, \qquad \frac{\partial^2 f}{\partial y^2} = 6x + 2$$

At $(-1/3, \pm 2/9)$, $B^2 - AC = 36(4/81) - (12/9)(0) > 0$, and therefore $(-1/3, \pm 2/9)$ yield saddle points. At $(0, 0)$, $B^2 - AC = 0$ and the test fails. Now $f(0, 0) = 0$, and in the circle shown to the right, $f(x, y) = x^4 + y^2(1 + 3x) > 0$ (except at $(0, 0)$). This implies that $(0, 0)$ gives a relative minimum.



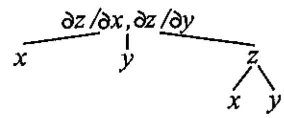
23. If we set $F(x,y,z) = 2x^2 + 3y^2 + z^2 - 12xy + 4xz - 35$, then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} = -\frac{4x - 12y + 4z}{2z + 4x} = \frac{6y - 2x - 2z}{z + 2x}, \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_y}{F_z} = -\frac{6y - 12x}{2z + 4x} = \frac{6x - 3y}{z + 2x}.$$

For critical points we solve $0 = 2x - 6y + 2z$ and $0 = 3y - 6x$. Certainly $x = 1$ and $y = 2$ satisfy the second of these. When these values are substituted into the first, $z = 5$ is obtained. Since $x = 1$, $y = 2$, and $z = 5$ also satisfy the original equation, it follows that $(1,2)$ is indeed a critical point. Consider now finding the second derivatives at the critical point. From the schematic,

$$\frac{\partial^2 z}{\partial x^2}_{|(1,2,5)} = \left[\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right]_{|(1,2,5)},$$

$$\frac{\partial^2 z}{\partial x \partial y}_{|(1,2,5)} = \left[\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right]_{|(1,2,5)},$$



$$\frac{\partial^2 z}{\partial y^2}_{|(1,2,5)} = \left[\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]_{|(1,2,5)}.$$

But at the critical point $(1,2,5)$, $\partial z/\partial x$ and $\partial z/\partial y$ are both zero. Hence,

$$\frac{\partial^2 z}{\partial x^2}_{|(1,2,5)} = \left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right]_{(1,2,5)} = \left[\frac{(z+2x)(-2) - (6y - 2x - 2z)(2)}{(z+2x)^2}\right]_{|(1,2,5)} = -\frac{2}{7},$$

$$\frac{\partial^2 z}{\partial x \partial y}_{|(1,2,5)} = \left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right]_{|(1,2,5)} = \left[\frac{6}{z+2x}\right]_{|(1,2,5)} = \frac{6}{7},$$

$$\frac{\partial^2 z}{\partial y^2}_{|(1,2,5)} = \left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]_{|(1,2,5)} = \left[\frac{-3}{z+2x}\right]_{|(1,2,5)} = -\frac{3}{7}.$$

Consequently, at $(1,2)$, $B^2 - AC = (6/7)^2 - (-2/7)(-3/7) > 0$, and the critical point yields a saddle point.

24. (a) For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x - yz$, $0 = \dfrac{\partial f}{\partial y} = 2y - xz$, $0 = \dfrac{\partial f}{\partial z} = 2z - xy$. Solutions are $(0,0,0)$, $(2,\pm2,\pm2)$, $(-2,\pm2,\mp2)$, $(2,\mp2,\mp2)$, and $(-2,\mp2,\pm2)$.

(b) The value of the function at $(0,0,0)$ is $f(0,0,0) = 0$. Since $(x-y)^2 \geq 0$, it follows that $x^2 + y^2 \geq 2xy$. Similarly, $y^2 + z^2 \geq 2yz$ and $x^2 + z^2 \geq 2xz$. Addition of these gives

$$2x^2 + 2y^2 + 2z^2 \geq 2(xy + yz + xz), \quad \text{or,} \quad x^2 + y^2 + z^2 \geq xy + yz + xz.$$

If $|z| < 1$, then $xy > xyz/3$. Similarly, if $|x| < 1$ and $|y| < 1$, then $yz > xyz/3$ and $xz > xyz/3$. Hence, for $|x| < 1$, $|y| < 1$, and $|z| < 1$,

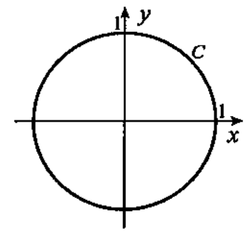
$$x^2 + y^2 + z^2 \geq xy + yz + xz \geq \frac{xyz}{3} + \frac{xyz}{3} + \frac{xyz}{3} = xyz.$$

In other words, for all points inside the sphere $x^2 + y^2 + z^2 = 1$, we can say that $x^2 + y^2 + z^2 - xyz \geq 0$. Thus, $f(x,y,z)$ must have a relative minimum at $(0,0,0)$.

EXERCISES 12.11

1. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 2x, \quad 0 = \frac{\partial f}{\partial y} = 3y^2.$$



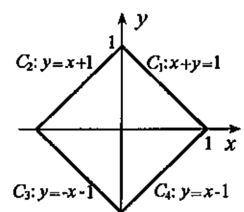
The only solution is $(0,0)$ at which $f(0,0) = \boxed{0}$. On C, $z = f(x,y)$ can be expressed in the form

$$z = F(y) = 1 - y^2 + y^3, \ -1 \le y \le 1.$$

For critical points of this function, we solve $0 = F'(y) = -2y + 3y^2$. Critical points are $y = 0$ and $y = 2/3$. Since $F(-1) = \boxed{-1}$, $F(0) = \boxed{1}$, $F(2/3) = \boxed{23/27}$, and $F(1) = \boxed{1}$, maximum and minimum values of $f(x,y)$ are 1 and -1.

2. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 2x + 1, \quad 0 = \frac{\partial f}{\partial y} = 6y + 1.$$



The only solution is $(-1/2, -1/6)$ at which $f(-1/2,-1/6) = \boxed{-1/3}$.

On C_1, $z = f(x,y)$ becomes

$$z = F(x) = x^2 + x + 3(1-x)^2 + (1-x)$$
$$= 4x^2 - 6x + 4, \ 0 \le x \le 1.$$

For critical points of this function, we solve $0 = F'(x) = 8x - 6$. The only critical point is $x = 3/4$ at which $F(3/4) = \boxed{7/4}$.

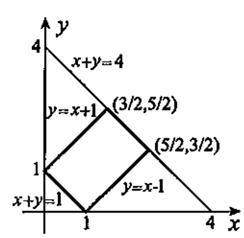
On C_2, $z = f(x,y)$ is $z = F(x) = x^2 + x + 3(x+1)^2 + (x+1) = 4x^2 + 8x + 4$, $-1 \le x \le 0$. For critical points of this function, we solve $0 = F'(x) = 8x + 8$. At the critical point $x = -1$, $F(-1) = \boxed{0}$.

On C_3, $z = f(x,y)$ is $z = F(x) = x^2 + x + 3(-x-1)^2 + (-x-1) = 4x^2 + 6x + 2$, $-1 \le x \le 0$. For critical points of this function, we solve $0 = F'(x) = 8x + 6$. At the critical point $x = -3/4$, $F(-3/4) = \boxed{-1/4}$.

On C_4, $z = f(x,y)$ is $z = F(x) = x^2 + x + 3(x-1)^2 + (x-1) = 4x^2 - 4x + 2$, $0 \le x \le 1$. For critical points we solve $0 = F'(x) = 8x - 4$. At the critical point $x = 1/2$, $F(1/2) = \boxed{1}$.

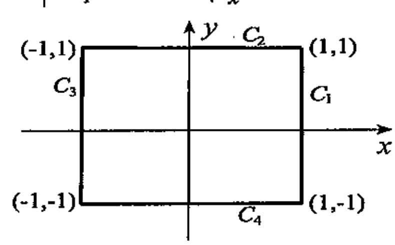
Finally, at the remaining three corners of R, $f(1,0) = \boxed{2}$, $f(0,1) = \boxed{4}$, $f(0,-1) = \boxed{2}$. Maximum and minimum values of $f(x,y)$ on R are therefore 4 and $-1/3$.

3. The function has no critical points inside C. When $f(x,y)$ is expressed in terms of one variable on each part of C, the resulting function is linear, and therefore has no critical points. It follows that maximum and minimum of the function must occur at the vertices of the rectangle. Since $f(1,0) = \boxed{3}$, $f(0,1) = \boxed{4}$, $f(3/2,5/2) = \boxed{29/2}$, and $f(5/2,3/2) = \boxed{27/2}$, maximum and minimum values are 29/2 and 3.



4. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 2xy + y^2, \quad 0 = \frac{\partial f}{\partial y} = x^2 + 2xy + 1.$$



Solutions are $(\pm 1/\sqrt{3}, \mp 2/\sqrt{3})$ at which

$$f(1/\sqrt{3}, -2/\sqrt{3}) = \boxed{-4\sqrt{3}/9}, \quad f(-1/\sqrt{3}, 2/\sqrt{3}) = \boxed{4\sqrt{3}/9}.$$

On C_1, $z = f(x,y)$ becomes

$$z = F(y) = y + y^2 + y = 2y + y^2, \quad -1 \le y \le 1.$$

For critical points of this function, we solve $0 = F'(y) = 2 + 2y$. The only critical point is $y = -1$ at which $F(-1) = \boxed{-1}$.

On C_2, $z = f(x,y)$ is $z = F(x) = x^2 + x + 1$, $-1 \le x \le 1$. For critical points of this function, we solve $0 = F'(x) = 2x + 1$. At the critical point $x = -1/2$, $F(-1/2) = \boxed{3/4}$.

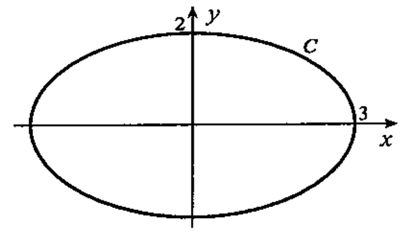
On C_3, $z = f(x,y)$ becomes $z = F(y) = y - y^2 + y = 2y - y^2$, $-1 \le y \le 1$. For critical points, we solve $0 = F'(y) = 2 - 2y$. At the critical point $y = 1$, $F(1) = \boxed{1}$.

On C_4, $z = f(x,y)$ is $z = F(x) = -x^2 + x - 1$, $-1 \le x \le 1$. For critical points we solve $0 = F'(x) = -2x + 1$. At the critical point $x = 1/2$, $F(1/2) = \boxed{-3/4}$.

Finally, we evaluate $f(x,y)$ at the remaining two corners, $f(1,1) = \boxed{3}$ and $f(-1,-1) = \boxed{-3}$. Maximum and minimum values of $f(x,y)$ on R are therefore 3 and -3.

5. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 6x + 2y, \quad 0 = \frac{\partial f}{\partial y} = 2x - 2y.$$



The only solution is $(0,0)$ at which $f(0,0) = \boxed{5}$.
On the edge of the ellipse we set

$$x = 3\cos t, \ y = 2\sin t, \ 0 \le t \le 2\pi,$$

in which case

$$f(x,y) = F(t) = 27\cos^2 t + 12\sin t\cos t - 4\sin^2 t + 5, \quad 0 \le t \le 2\pi.$$

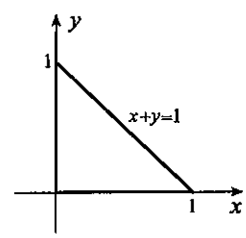
For critical points we solve

$$0 = F'(t) = -54\cos t\sin t + 12(\cos^2 t - \sin^2 t) - 8\sin t\cos t = -31\sin 2t + 12\cos 2t.$$

This implies that $\tan 2t = 12/31$, and the four solutions of this equation in the interval $0 \le t \le 2\pi$ are $(1/2)\mathrm{Tan}^{-1}(12/31)$, $(1/2)\mathrm{Tan}^{-1}(12/31) + \pi/2$, $(1/2)\mathrm{Tan}^{-1}(12/31) + \pi$, and $(1/2)\mathrm{Tan}^{-1}(12/31) + 3\pi/2$. When these are substituted into $F(t)$, two values result, namely $\boxed{33.12}$ and $\boxed{-0.12}$. These are maximum and minimum values of the function on the ellipse.

6. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 3x^2 - 3, \quad 0 = \frac{\partial f}{\partial y} = 2y + 2.$$



Solutions are $(\pm 1, -1)$ both of which are outside the region. On the edge $x = 0$,

$$f(0,y) = F(y) = y^2 + 2y, \quad 0 \le y \le 1.$$

It has a critical point when $0 = 2y + 2 \Longrightarrow y = -1$, which we reject.

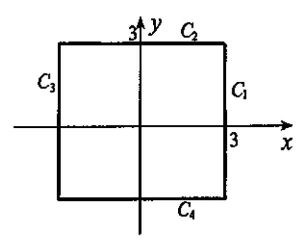
On $y = 0$, $f(x,0) = G(x) = x^3 - 3x$, $0 \le x \le 1$, which has critical points when $0 = 3x^2 - 3 \Longrightarrow x = \pm 1$. The value of $G(x)$ at $x = 1$ is $G(1) = \boxed{-2}$.

On $x + y = 1$, $f(x,y) = H(x) = x^3 - 3x + (1-x)^2 + 2(1-x) = x^3 + x^2 - 7x + 3$, $0 \le x \le 1$. It has critical points when $0 = H'(x) = 3x^2 + 2x - 7$. Neither of the solutions $x = (-1 \pm \sqrt{22})/3$ lie in the interval $0 \le x \le 1$.

We have evaluated $f(x,y)$ at vertex $(1,0)$ of the triangle. Its values at the remaining two vertices are $f(0,1) = \boxed{3}$ and $f(0,0) = \boxed{0}$. Maximum and minimum values are therefore 3 and -2.

7. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 3x^2 - 3, \quad 0 = \frac{\partial f}{\partial y} = 3y^2 - 12.$$



Solutions are $(\pm 1, \pm 2)$ and $(\pm 1, \mp 2)$ at which $f(x,y)$ has the values $\boxed{-16}$, $\boxed{20}$, $\boxed{16}$, and $\boxed{-12}$.

On C_1, $z = f(x,y)$ becomes

$$z = F(y) = y^3 - 12y + 20, \quad -3 \le y \le 3.$$

For critical points of this function, we solve $0 = F'(y) = 3y^2 - 12$. Critical points are $y = \pm 2$ at which $F(\pm 2) = \boxed{4}, \boxed{36}$.

On C_2, $z = f(x,y)$ is $z = F(x) = x^3 - 3x - 7$, $-3 \le x \le 3$. For critical points of this function, we solve $0 = F'(x) = 3x^2 - 3$. Critical points are $x = \pm 1$ at which $F(\pm 1) = \boxed{-9}, \boxed{-5}$.

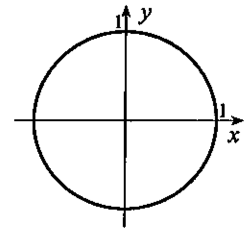
On C_3, $z = f(x,y)$ becomes $z = F(y) = y^3 - 12y - 16$, $-3 \le y \le 3$. For critical points, we solve $0 = F'(y) = 3y^2 - 12$. Critical points are $y = \pm 2$ at which $F(\pm 2) = \boxed{-32}, \boxed{0}$.

On C_4, $z = f(x,y)$ is $z = F(x) = x^3 - 3x + 11$, $-3 \le x \le 3$. For critical points we solve $0 = F'(x) = 3x^2 - 3$. Critical points are $x = \pm 1$ at which $F(\pm 1) = \boxed{9}, \boxed{13}$.

Finally, we evaluate $f(x,y)$ at the corners , $f(3,3) = \boxed{11}$, $f(-3,-3) = \boxed{-7}$, $f(3,-3) = \boxed{29}$, $f(-3,3) = \boxed{-25}$. The maximum and minimum values of $f(x,y)$ on R are therefore 36 and -32.

8. For critical points of $f(x,y)$, we solve

$$0 = \frac{\partial f}{\partial x} = 3x^2 - 3, \quad 0 = \frac{\partial f}{\partial y} = 3y^2 - 3.$$



The solutions $(\pm 1, 1)$ and $(\pm 1, -1)$ are exterior to the circle. On the edge of the circle we set

$$x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi,$$

in which case

$$f(x,y) = F(t) = \cos^3 t + \sin^3 t - 3\cos t - 3\sin t + 2, \quad 0 \le t \le 2\pi.$$

For critical points we solve

$$\begin{aligned} 0 = F'(t) &= -3\cos^2 t\,\sin t + 3\sin^2 t\,\cos t + 3\sin t - 3\cos t \\ &= 3\sin t\,\cos t(\sin t - \cos t) + 3(\sin t - \cos t) \\ &= 3(\sin t\,\cos t + 1)(\sin t - \cos t). \end{aligned}$$

Setting the factor $\sin t\,\cos t + 1 = 0$ leads to $\sin 2t = -2$, an impossibility. The other possibility is to set $\sin t - \cos t = 0$, which leads to $t = \pi/4$, and $t = 5\pi/4$. Since

$$F(0) = F(2\pi) = \boxed{0}, \quad F(\pi/4) = \boxed{\frac{2\sqrt{2} - 5}{\sqrt{2}}}, \quad F(5\pi/4) = \boxed{\frac{2\sqrt{2} + 5}{\sqrt{2}}},$$

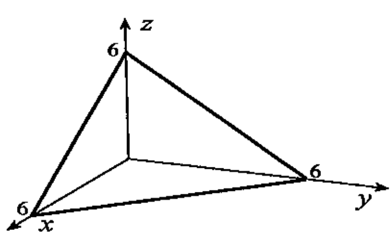
maximum and minimum values are $(2\sqrt{2} + 5)/\sqrt{2}$ and $(2\sqrt{2} - 5)/\sqrt{2}$.

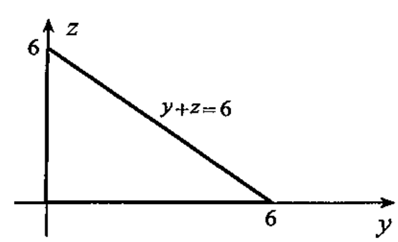
9. (a) If we set $x = 6 - y - z$, then $f(x,y,z) = F(y,z) = (6 - y - z)y^2z^3$. For critical points we solve

$$\begin{aligned} 0 = F_y &= -y^2z^3 + 2(6 - y - z)yz^3 = yz^3(12 - 3y - 2z), \\ 0 = F_z &= -y^2z^3 + 3(6 - y - z)y^2z^2 = y^2z^2(18 - 3y - 4z). \end{aligned}$$

The only solution inside the triangle in the right figure below is $y = 2$ and $z = 3$. Since $F(2,3) = \boxed{108}$, and values of the function become arbitrarily close to 0 as we approach the sides of the triangle, it follows that the function has maximum value 108, but no minimum.

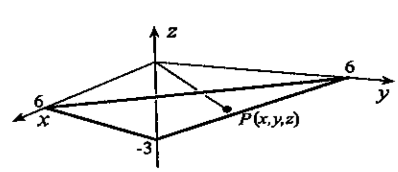
(b) When we include the edges of the plane (left figure below), we must find extreme values of $F(y,z)$ on the triangle in part (a), but now including its edges. Since the value of $F(y,z)$ is identically zero on the edges, it follows that maximum and minimum values now are 108 and 0.





10. The distance D from O to any point $P(x,y,z)$ on the plane is given by

$$\begin{aligned} D^2 &= x^2 + y^2 + z^2 \\ &= (6 - y + 2z)^2 + y^2 + z^2. \end{aligned}$$



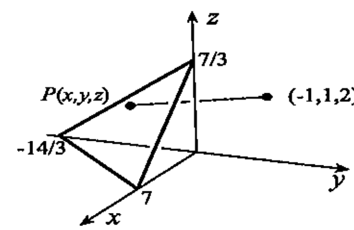
For critical points of D^2 we solve

$$0 = \frac{\partial D^2}{\partial y} = -2(6 - y + 2z) + 2y, \quad 0 = \frac{\partial D^2}{\partial z} = 4(6 - y + 2z) + 2z.$$

The only solution is $y = 1$, $z = -2$. Since y and z can take on all possible values, and D^2 becomes infinite for large values of y or z, the critical point must minimize D^2. The closest point is therefore $(1, 1, -2)$.

11. The distance D from $(-1, 1, 2)$ to any point $P(x, y, z)$ on the plane is given by

$$\begin{aligned} D^2 &= (x+1)^2 + (y-1)^2 + (z-2)^2 \\ &= \left(7 + \frac{3y}{2} - 3z + 1\right)^2 + (y-1)^2 + (z-2)^2 \\ &= \left(8 + \frac{3y}{2} - 3z\right)^2 + (y-1)^2 + (z-2)^2. \end{aligned}$$



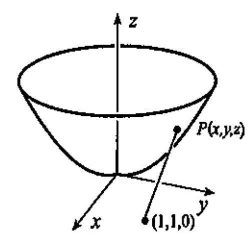
For critical points of D^2 we solve

$$0 = \frac{\partial D^2}{\partial y} = 2\left(8 + \frac{3y}{2} - 3z\right)\left(\frac{3}{2}\right) + 2(y-1), \quad 0 = \frac{\partial D^2}{\partial z} = 2\left(8 + \frac{3y}{2} - 3z\right)(-3) + 2(z-2).$$

The only solution is $y = 4/7$, $z = 20/7$. Since y and z can take on all possible values, and D^2 becomes infinite for large values of y or z, the critical point must minimize D^2. The shortest distance is therefore $D(4/7, 20/7) = 1$.

12. The distance D from $(1, 1, 0)$ to any point $P(x, y, z)$ on the surface is given by

$$\begin{aligned} D^2 &= (x-1)^2 + (y-1)^2 + z^2 \\ &= (x-1)^2 + (y-1)^2 + (x^2+y^2)^2. \end{aligned}$$



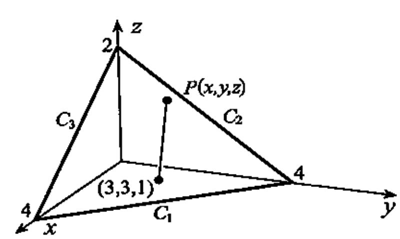
For critical points of D^2, we solve

$$\begin{aligned} 0 &= \frac{\partial D^2}{\partial x} = 2(x-1) + 4x(x^2+y^2), \\ 0 &= \frac{\partial D^2}{\partial y} = 2(y-1) + 4y(x^2+y^2). \end{aligned}$$

The only solution is $x = 1/2$, $y = 1/2$. Since x and y can take on all possible values, and D^2 becomes infinite for large values of x and y, the critical point must minimize D^2. The closest point is therefore $(1/2, 1/2, 1/2)$.

13. The distance D from $(3, 3, 1)$ to any point $P(x, y, z)$ on the plane is given by

$$\begin{aligned} D^2 &= (x-3)^2 + (y-3)^2 + (z-1)^2 \\ &= (4 - y - 2z - 3)^2 + (y-3)^2 + (z-1)^2 \\ &= (1 - y - 2z)^2 + (y-3)^2 + (z-1)^2. \end{aligned}$$



For critical points of D^2 we solve

$$0 = \frac{\partial D^2}{\partial y} = -2(1 - y - 2z) + 2(y-3), \quad 0 = \frac{\partial D^2}{\partial z} = -4(1 - y - 2z) + 2(z-1).$$

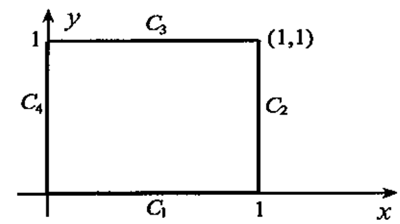
The only solution is $y = 7/3$, $z = -1/3$, which is unacceptable (not being in the first octant). Because of the proximity of $(3, 3, 1)$ to C_1, we can be sure that the closest point lies somewhere along C_1, not on C_2 or C_3. We therefore minimize

$$D^2(x, y, z) = F(y) = (1-y)^2 + (y-3)^2 + 1 = 2y^2 - 8y + 11, \quad 0 \le y \le 4.$$

For critical points we solve $0 = F'(y) = 4y - 8$. Since the solution is $y = 2$, the closest point is $(2, 2, 0)$.

14. For critical points of $V(x,y)$, we solve

$$0 = \frac{\partial V}{\partial x} = 48y - 96x^2, \quad 0 = \frac{\partial V}{\partial y} = 48x - 48y.$$



At the critical points $(0,0)$ and $(1/2,1/2)$,

$V(0,0) = \boxed{0}, \quad V(1/2,1/2) = \boxed{2}.$

On C_1, $V = -32x^3$, $0 \le x \le 1$. This function has a critical point at $x = 0$ corresponding to $(0,0)$.

On C_2, $V = 48y - 32 - 24y^2$, $0 \le y \le 1$. For critical points, $0 = dV/dy = 48 - 48y$. At the critical point $y = 1$, $V = \boxed{-8}$.

On C_3, $V = 48x - 32x^3 - 24$, $0 \le x \le 1$. For critical points, $0 = dV/dx = 48 - 96x^2$. At the critical point $x = 1/\sqrt{2}$, $V = \boxed{8(2\sqrt{2}-3)}$.

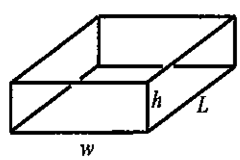
On C_4, $V = -24y^2$, $0 \le y \le 1$. The only critical point of this function is $y = 0$, corresponding to $(0,0)$.

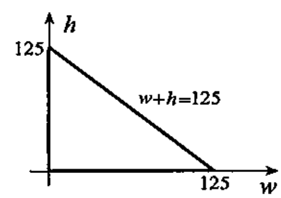
Finally, at the remaining two corners, $V(1,0) = \boxed{-32}$ and $V(0,1) = \boxed{-24}$. Maximum and minimum values of $V(x,y)$ are therefore 2 and -32.

15. The volume of the box is $V = Lwh$. Since $L + 2(w+h) \le 250$, we set $L = 250 - 2w - 2h$, in which case $V = wh(250 - 2w - 2h)$. This function must be maximized on the triangle in the right figure below. For critical points we solve

$$0 = \frac{\partial V}{\partial w} = 250h - 4wh - 2h^2, \qquad 0 = \frac{\partial V}{\partial h} = 250w - 2w^2 - 4wh.$$

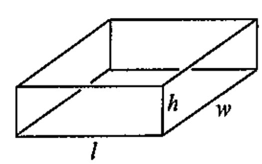
The only solution inside the triangle is $w = h = 125/3$. Since $V = 0$ on the edges of the triangle, it follows that this critical point must yield a maximum volume. Dimensions of the box are therefore $w = h = 125/3$ cm and $L = 250/3$ cm.





16. If k is the cost per square centimetre for lining the side of the tank, the total cost is

$C = k(2hw + 2hl) + 3k(wl).$



Because the tank must hold 1000 L,

$10^6 = lwh,$

and therefore

$$C = C(w,h) = k\left[2hw + 2h\left(\frac{10^6}{wh}\right) + 3w\left(\frac{10^6}{wh}\right)\right] = k\left[2hw + \frac{2\times 10^6}{w} + \frac{3\times 10^6}{h}\right].$$

This function must be minimized for all points in the first quadrant of the wh-plane. For critical points of $C(w,h)$,

$$0 = \frac{\partial C}{\partial w} = k\left[2h - \frac{2\times 10^6}{w^2}\right], \qquad 0 = \frac{\partial C}{\partial h} = k\left[2w - \frac{3\times 10^6}{h^2}\right].$$

The only critical point is $w = 100(2/3)^{1/3}$, $h = 100(3/2)^{2/3}$. Since C becomes infinite as $h \to 0$ or $w \to 0$, or h or w become infinite, it follows that the critical point must minimize C. The required dimensions are therefore $w = 100(2/3)^{1/3}$ cm, $h = 100(3/2)^{2/3}$ cm, $l = 100(2/3)^{1/3}$ cm.

17. The distance D from (x_1, y_1, z_1) to any point $P(x, y, z)$ on the plane is given by

$$D^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = (x-x_1)^2 + (y-y_1)^2 + \left(\frac{-D-Ax-By}{C} - z_1\right)^2$$
$$= (x-x_1)^2 + (y-y_1)^2 + \frac{1}{C^2}(D+Ax+By+Cz_1)^2.$$

For critical points of D^2 we solve

$$0 = \frac{\partial D^2}{\partial x} = 2(x-x_1) + \frac{2A}{C^2}(D+Ax+By+Cz_1), \quad 0 = \frac{\partial D^2}{\partial y} = 2(y-y_1) + \frac{2B}{C^2}(D+Ax+By+Cz_1),$$

getting $x = \dfrac{(B^2+C^2)x_1 - A(By_1+Cz_1+D)}{A^2+B^2+C^2}$, $\quad y = \dfrac{(A^2+C^2)y_1 - B(Ax_1+Cz_1+D)}{A^2+B^2+C^2}$. Since this is the only critical point, and distance becomes infinite as x and y take on large values, it follows that this critical point must minimize D^2. To find the minimum value we substitute these values for x and y into the formula for D^2,

$$D^2 = \left[\frac{(B^2+C^2)x_1 - A(By_1+Cz_1+D)}{A^2+B^2+C^2} - x_1\right]^2 + \left[\frac{(A^2+C^2)y_1 - B(Ax_1+Cz_1+D)}{A^2+B^2+C^2} - y_1\right]^2$$
$$+ \frac{1}{C^2}\left[D + \frac{A(B^2+C^2)x_1 - A^2(By_1+Cz_1+D)}{A^2+B^2+C^2} + \frac{B(A^2+C^2)y_1 - B^2(Ax_1+Cz_1+D)}{A^2+B^2+C^2} + Cz_1\right]^2.$$

This simplifies to $\dfrac{(Ax_1+By_1+Cz_1+D)^2}{A^2+B^2+C^2}$, and its square root gives the desired result.

18. If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$ are vertices of a triangle, and $P(x, y, z)$ is any other point, then the sum of the squares of the distances from P to A, B, and C is

$$\begin{aligned} D &= \|PA\|^2 + \|PB\|^2 + \|PC\|^2 \\ &= (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 + (x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2 \\ &\quad + (x-x_3)^2 + (y-y_3)^2 + (z-z_3)^2. \end{aligned}$$

For critical points of this function, we solve

$$\begin{aligned} 0 &= \frac{\partial D}{\partial x} = 2(x-x_1) + 2(x-x_2) + 2(x-x_3), \\ 0 &= \frac{\partial D}{\partial y} = 2(y-y_1) + 2(y-y_2) + 2(y-y_3), \\ 0 &= \frac{\partial D}{\partial z} = 2(z-z_1) + 2(z-z_2) + 2(z-z_3). \end{aligned}$$

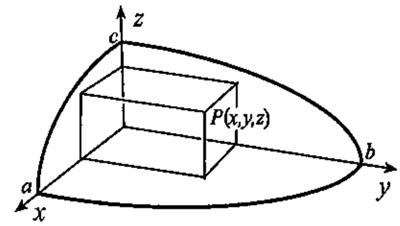
The solution is $x = \frac{1}{3}(x_1+x_2+x_3)$, $y = \frac{1}{3}(y_1+y_2+y_3)$, $z = \frac{1}{3}(z_1+z_2+z_3)$, the centroid of the triangle (see Exercise 43 in Section 7.7). Since D becomes infinite as the point (x, y) moves farther and farther away from the triangle, it follows that this one, and only one, critical point must minimize D.

19. The distance D from any point (x, y, z) on the curve to the origin is given by $D^2 = x^2 + y^2 + z^2$. Because every point on the curve satisfies $x^2 + y^2 = 1$, we can write that $D^2 = 1 + z^2$. Furthermore, the equations of the curve imply that $z^2 + xy = 0$, and therefore $D^2 = 1 - xy$. We now set $x = \cos t$ and $y = \sin t$, so that $D^2 = F(t) = 1 - \cos t \sin t$. Since $z^2 + xy = 0$ requires x and y to have opposite signs, we consider this function for values $\pi/2 \le t \le \pi$ and $-\pi/2 \le t \le 0$. Critical points of $F(t)$ are given by $0 = F'(t) = -\cos^2 t + \sin^2 t = -\cos 2t$. Acceptable solution are $t = 3\pi/4$ and $t = -\pi/4$. Since

$$F(-\pi/2) = 1, \quad F(-\pi/4) = 3/2, \quad F(0) = 1, \quad F(\pi/2) = 1, \quad F(3\pi/4) = 3/2, \quad F(\pi) = 1,$$

it follows that the points on the curve closest to the origin are $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$.

20. The volume obtained from a point $P(x,y,z)$ on that part of the ellipsoid in the first octant is
$$V = 8xyz = 8cxy\sqrt{1 - x^2/a^2 - y^2/b^2}.$$
This function must be maximized in the first quadrant portion R of the ellipse $x^2/a^2 + y^2/b^2 = 1$. For critical points of V, we solve



$$0 = \frac{\partial V}{\partial x} = 8cy\sqrt{1 - x^2/a^2 - y^2/b^2} - \frac{8cx^2y/a^2}{\sqrt{1 - x^2/a^2 - y^2/b^2}},$$
$$0 = \frac{\partial V}{\partial y} = 8cx\sqrt{1 - x^2/a^2 - y^2/b^2} - \frac{8cxy^2/b^2}{\sqrt{1 - x^2/a^2 - y^2/b^2}}.$$

The only solution of these equations inside R is $x = a/\sqrt{3}$, $y = b/\sqrt{3}$. Since $V = 0$ on the three parts of the boundary of R, it follows that V must be maximized at this critical point, and therefore the dimensions of the largest box are $2a/\sqrt{3} \times 2b/\sqrt{3} \times 2c/\sqrt{3}$.

21. On the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, in which case $f(x,y,z) = F(x,y) = xy\sqrt{1 - x^2 - y^2}$, where $x^2 + y^2 \le 1$. For critical points we solve

$$0 = \frac{\partial F}{\partial x} = y\sqrt{1 - x^2 - y^2} - \frac{x^2y}{\sqrt{1 - x^2 - y^2}}, \quad 0 = \frac{\partial F}{\partial y} = x\sqrt{1 - x^2 - y^2} - \frac{xy^2}{\sqrt{1 - x^2 - y^2}}.$$

Solutions are $x = \pm 1/\sqrt{3}$ and $y = \pm 1/\sqrt{3}$. At the critical points with these coordinates, values of $F(x,y)$ are $\pm\sqrt{3}/9$. On the edge $x^2 + y^2 = 1$ the value of the function is identically zero. Hence, maximum and minimum values on the upper hemisphere are $\pm\sqrt{3}/9$. The same values are obtained for the lower hemisphere.

22. We write $f(x,y) = F(x) = x^2 - (1 - x^2) = 2x^2 - 1$, $-1 \le x \le 1$. For critical points of $F(x)$, we solve $0 = F'(x) = 4x \implies x = 0$. Since $F(-1) = 1$, $F(0) = -1$, and $F(1) = 1$, maximum and minimum values are ± 1.

23. If we set $x = \cos t$, $y = \sin t$, then $f(x,y) = F(t) = |\cos t - \sin t|$, $0 \le t \le 2\pi$. For critical points of $F(t)$ we solve

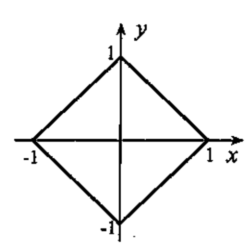
$$0 = F'(t) = \frac{|\cos t - \sin t|}{\cos t - \sin t}(-\sin t - \cos t).$$

The derivative is zero when $\sin t + \cos t = 0$, and the only angles in $0 \le t \le 2\pi$ satisfying this equation are $t = 3\pi/4$ and $t = 7\pi/4$. The derivative does not exist when $\cos t = \sin t$, and solutions of this are $t = \pi/4$ and $t = 5\pi/4$. Since

$$F(0) = 1,\ F(\pi/4) = 0,\ F(3\pi/4) = \sqrt{2},\ F(5\pi/4) = 0,\ F(7\pi/4) = \sqrt{2},\ F(2\pi) = 1,$$

maximum and minimum values are $\sqrt{2}$ and 0.

24. Since $y^2 = (1 - |x|)^2$ on the edges of the square, we can express $f(x,y)$ in terms of x alone,
$$f(x,y) = F(x) = x^2 - (1 - |x|)^2 = 2|x| - 1, \ -1 \le x \le 1.$$
There are no points at which the derivative of this function vanishes, but it does not exist at the critical point $x = 0$. Since $F(-1) = 1$, $F(0) = -1$, and $F(1) = 1$, maximum and minimum values are ± 1.

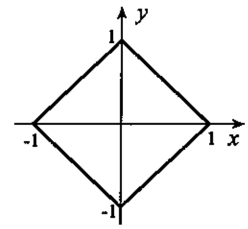


25. On the top half of the square $y = 1 - |x|$, in which case we can write that

$$f(x,y) = F(x) = \big|x - 2 + 2|x|\big|, \ -1 \le x \le 1.$$

For critical points of this function, we solve

$$0 = F'(x) = \frac{\big|x - 2 + 2|x|\big|}{x - 2 + 2|x|}\left(1 + \frac{2|x|}{x}\right).$$



There are no solutions of this equation, but the derivative does not exist at $x = 0$ and $x = 2/3$. We now calculate $F(-1) = 1$, $F(0) = 2$, $F(2/3) = 0$, and $F(1) = 1$. A similar analysis on the bottom half of the square leads to the same values. Hence, maximum and minimum values of $f(x,y)$ are 2 and 0.

26. We may write

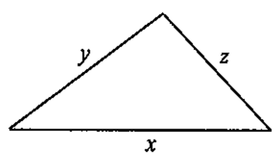
$$A^2 = \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right)\left(\frac{P}{2} + x + y - P\right) = \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right)\left(x + y - \frac{P}{2}\right).$$

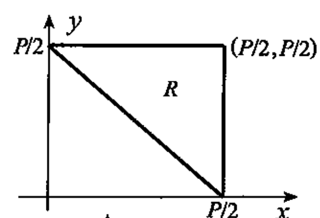
This function must be maximized for those points (x,y) in the triangle R shown to the right. For critical points of A^2,

$$0 = \frac{\partial(A^2)}{\partial x} = -\frac{P}{2}\left(\frac{P}{2} - y\right)\left(x + y - \frac{P}{2}\right) + \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right),$$

$$0 = \frac{\partial(A^2)}{\partial y} = -\frac{P}{2}\left(\frac{P}{2} - x\right)\left(x + y - \frac{P}{2}\right) + \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right).$$

The only solution of these equations inside R is $x = y = P/3$. Since $A^2 = 0$ on the boundary of R, it follows that A is maximized when $x = y = P/3$, in which case $z = P/3$ also.

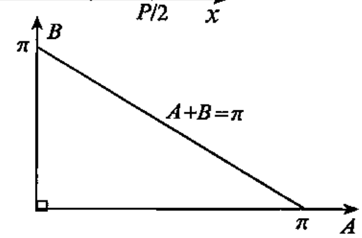




27. Since $A + B + C = \pi$, we may write that

$$f(A,B,C) = F(A,B) = \sin(A/2)\sin(B/2)\sin(\pi/2 - A/2 - B/2).$$

Consider finding the maximum value of this function on the triangle shown. Its minimum is zero (everywhere on the edge). For critical points we solve



$$0 = \frac{\partial F}{\partial A} = \frac{1}{2}\cos(A/2)\sin(B/2)\sin(\pi/2 - A/2 - B/2) - \frac{1}{2}\sin(A/2)\sin(B/2)\cos(\pi/2 - A/2 - B/2),$$

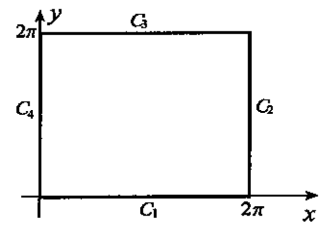
$$0 = \frac{\partial F}{\partial B} = \frac{1}{2}\sin(A/2)\cos(B/2)\sin(\pi/2 - A/2 - B/2) - \frac{1}{2}\sin(A/2)\sin(B/2)\cos(\pi/2 - A/2 - B/2).$$

From the second,

$$\begin{aligned} 0 &= \sin(A/2)[\sin(\pi/2 - A/2 - B/2)\cos(B/2) - \sin(B/2)\cos(\pi/2 - A/2 - B/2)] \\ &= \sin(A/2)\sin(\pi/2 - A/2 - B). \end{aligned}$$

Thus, $\pi/2 - A/2 - B = n\pi \implies A/2 + B = \pi/2 - n\pi$. Since A and B are angles in a triangle, it follows that $n = 0$, and $A/2 + B = \pi/2$. Similarly, from the first equation, we obtain $A + B/2 = \pi/2$, and together, these equations imply that $A = B = \pi/3$. Since these values yield a maximum value for the function of $F(\pi/3, \pi/3) = 1/8$, it follows that for all other angles $F(A,B) \le 1/8$.

28. Because the function is 2π-periodic in x and y, we need only show that the inequality is valid in the square $R: 0 \le x \le 2\pi$, $0 \le y \le 2\pi$. We shall show that maximum and minimum values of the function $f(x,y) = \cos x + \cos y + \sin x \sin y$ in R are ± 2. Critical points of $f(x,y)$ are given by



$$0 = \frac{\partial f}{\partial x} = -\sin x + \cos x \sin y, \qquad 0 = \frac{\partial f}{\partial y} = -\sin y + \sin x \cos y.$$

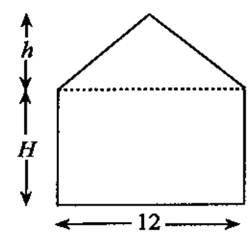
If we solve the first for $\sin x$ and substitute into the second,

$$0 = -\sin y + (\cos x \sin y)\cos y = \sin y(\cos x \cos y - 1).$$

For $\sin y$ to vanish, y must be 0, π, or 2π. The first and third of these are boundaries of R which will be treated later. If $y = \pi$, then $\sin x = 0$, from which x must be 0, π, or 2π. Again, the first and third of these are boundaries of R. In other words, we obtain a critical point of $f(x,y)$ in R to be (π,π) at which $f(\pi,\pi) = -2$. The other possibility for critical points interior to R is to set $\cos x \cos y = 1$. This can be true only if both x and y are equal to 1 or -1. This implies that both x and y must be equal to 0, π, or 2π, and we are led to the same critical point (π,π) inside R. On the boundary C_1, $f(x,y) = 1 + \cos x$, $0 \le x \le 2\pi$. The maximum and minimum values are 2 and 0. The same results are obtained on the remaining three boundaries C_2, C_3, and C_4. Thus, maximum and minimum values of $f(x,y)$ on the square are ± 2, and our proof is complete.

29. The volume of the silo is

$$V = \pi(6)^2 H + \frac{1}{3}\pi(6)^2 h = 12\pi(h + 3H).$$



Since area of the silo is $200 = 2\pi(6)H + \pi(6)\sqrt{36 + h^2}$, it follows that

$$\begin{aligned} V &= 12\pi\left[h + 3\left(\frac{200 - 6\pi\sqrt{36+h^2}}{12\pi}\right)\right] \\ &= 12\pi h + 600 - 18\pi\sqrt{36+h^2}, \quad 0 \le h \le \frac{\sqrt{4\times 10^4 - 36^2\pi^2}}{6\pi}. \end{aligned}$$

For critical points of V, we solve

$$0 = \frac{dV}{dh} = 12\pi - \frac{18\pi h}{\sqrt{36+h^2}}.$$

The only positive solution of this equation is $h = 12/\sqrt{5}$. Since

$$V(0) = 600 - 108\pi = 260.7, \quad V\left(\frac{12}{\sqrt{5}}\right) = 347.1, \quad V\left(\frac{\sqrt{4\times 10^4 - 36^2\pi^2}}{6\pi}\right) = 329.9,$$

V is maximized for $h = 12/\sqrt{5}$ m and $H = (50\sqrt{5} - 27\pi)/(3\sqrt{5}\pi)$ m.

30. If we set $P(x,y) = F(y) = kx^\alpha y^{1-\alpha} = k\left(\frac{C - By}{A}\right)^\alpha y^{1-\alpha}$, $0 \le y \le C/B$, then the minimum value of the function occurs at the end points. The maximum must occur at a critical point. For critical points we solve

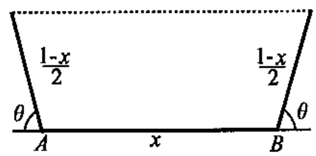
$$\begin{aligned} 0 = F'(y) &= k\left[\left(\frac{C-By}{A}\right)^\alpha (1-\alpha)y^{-\alpha} - \frac{\alpha B}{A}\left(\frac{C-By}{A}\right)^{\alpha-1} y^{1-\alpha}\right] \\ &= k\left(\frac{C-By}{A}\right)^{\alpha-1} y^{-\alpha}\left[\left(\frac{C-By}{A}\right)(1-\alpha) - \frac{\alpha B}{A}y\right]. \end{aligned}$$

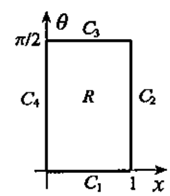
Ths solution is $y = C(1-\alpha)/B = C\beta/B$. The corresponding x-value is $x = C\alpha/A$.

31. If the length of AB is x, and the bends are at angle θ, then the area of the trapezoid is

$$F(x,\theta) = \frac{1}{2}\left(\frac{1-x}{2}\right)\sin\theta\left[2x + 2\left(\frac{1-x}{2}\right)\cos\theta\right] = \frac{1}{4}(1-x)\sin\theta[2x+(1-x)\cos\theta].$$

This function must be maximized for the region R of the $x\theta$-plane shown to the right below.





For critical points, we solve

$$0 = \frac{\partial F}{\partial x} = \frac{1}{4}\sin\theta[2-4x-2(1-x)\cos\theta],$$
$$0 = \frac{\partial F}{\partial \theta} = \frac{1}{4}(1-x)[2x\cos\theta + (1-x)(\cos^2\theta - \sin^2\theta)].$$

Since $\sin\theta = 0$ and $1-x=0$ correspond to edges of R, we set

$$1-2x-(1-x)\cos\theta = 0, \qquad 2x\cos\theta + (1-x)(\cos^2\theta - \sin^2\theta) = 0.$$

The first equation implies that $\cos\theta = (1-2x)/(1-x)$, and when this is substituted into the second equation,

$$0 = 2x\left(\frac{1-2x}{1-x}\right) + (1-x)\left[2\left(\frac{1-2x}{1-x}\right)^2 - 1\right].$$

This simplifies to $0 = 3x^2 - 4x + 1 = (3x-1)(x-1)$. Thus, $x = 1/3$, and from this $\theta = \pi/3$. The area of the trapezoid so formed is

$$F = \frac{1}{4}\left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\left[\frac{2}{3} + \frac{2}{3}\left(\frac{1}{2}\right)\right] = \boxed{\frac{\sqrt{3}}{12}}.$$

For values of x and θ along edges C_1 and C_2 of R, the area of the trapezoid is $\boxed{0}$. Along C_3,

$$F = \frac{1}{4}(1-x)2x = \frac{x(1-x)}{2}, \quad 0 \le x \le 1.$$

For critical points, $0 = dF/dx = (1-2x)/2$. At the critical point $x = 1/2$, $F(1/2) = \boxed{1/8}$.
Along C_4,

$$F = \frac{1}{4}\sin\theta\cos\theta = \frac{1}{8}\sin 2\theta, \quad 0 \le \theta \le \pi/2.$$

For critical points, $0 = dF/d\theta = (1/4)\cos 2\theta$. At the critical point $\theta = \pi/4$, $F(\pi/4) = \boxed{1/8}$.
Finally, at the four vertices of the rectangle,

$$F(0,0) = \boxed{0}, \quad F(1,0) = \boxed{0}, \quad F(1,\pi/2) = \boxed{0}, \quad F(0,\pi/2) = \boxed{0}.$$

Thus, area is maximized when $\theta = \pi/3$ and $x = 1/3$ m.

32. For critical points of $f(x,y,z)$ inside the sphere we solve

$$0=\frac{\partial f}{\partial x}=y+z,\quad 0=\frac{\partial f}{\partial y}=x,\quad 0=\frac{\partial f}{\partial z}=x.$$

For the line of critical points $(0,y,-y)$, $f(0,y,-y)=\boxed{0}$. On the boundary $S: x^2+y^2+z^2=1$ of the region,

$$f(x,y,z)=F(y,z)=\pm(y+z)\sqrt{1-y^2-z^2},\quad y^2+z^2\le 1.$$

For critical points of $F(y,z)$ inside $y^2+z^2=1$, we solve

$$0=\frac{\partial F}{\partial y}=\pm\sqrt{1-y^2-z^2}\mp\frac{y(y+z)}{\sqrt{1-y^2-z^2}},$$
$$0=\frac{\partial F}{\partial z}=\pm\sqrt{1-y^2-z^2}\mp\frac{z(y+z)}{\sqrt{1-y^2-z^2}}.$$

Solutions are $(y,z)=(\pm 1/2,\pm 1/2)$ at which $F(\pm 1/2,\pm 1/2)=\boxed{\pm 1/\sqrt{2}}$. On the boundary $C: y^2+z^2=1$ of S, $f(x,y,z)=\boxed{0}$. Consequently, maximum and minimum values of $f(x,y,z)$ are $\pm 1/\sqrt{2}$.

33. Since values of z are independent of those of x and y in the region, $f(x,y,z)$ is maximized when $z=1$. In other words, we should maximize and minimize $F(x,y)=x^2y$ on the circle $x^2+y^2\le 1$. For critical points of $F(x,y)$, we solve $0=\dfrac{\partial F}{\partial x}=2xy$, $0=\dfrac{\partial F}{\partial y}=x^2$. At the critical points $(0,y)$, $F(0,y)=\boxed{0}$. On the boundary $C: x^2+y^2=1$, we set $x=\cos t$, $y=\sin t$, in which case $F(x,y)=u(t)=\cos^2 t\sin t$, $0\le t\le 2\pi$. For critical points of $u(t)$, we solve $0=u'(t)=-2\cos t\sin^2 t+\cos^3 t=\cos t(-2\sin^2 t+\cos^2 t)$. Either $\cos t=0$ or $0=-2\sin^2 t+1-\sin^2 t=1-3\sin^2 t \Longrightarrow \sin t=\pm 1/\sqrt{3}$. Thus, $t=\pi/2, 3\pi/2, \mathrm{Sin}^{-1}(\pm 1/\sqrt{3}), \mathrm{Sin}^{-1}(\pm 1/\sqrt{3})+\pi$. Since

$$u(0)=\boxed{0},\quad u(\pi/2)=u(3\pi/2)=\boxed{0},\quad u(\mathrm{Sin}^{-1}(\pm 1/\sqrt{3}))=u(\mathrm{Sin}^{-1}(\mp 1/\sqrt{3})+\pi)=\boxed{\pm\frac{2\sqrt{3}}{9}},\quad u(2\pi)=\boxed{0},$$

maximum and minimum values of $f(x,y,z)$ are $\pm 2\sqrt{3}/9$.

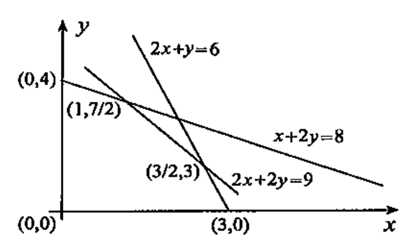
34. Let x and y be the numbers of X and Y produced per hour. Then the profit per hour is $P(x,y)=200x+300y$. But x and y must satisfy the following inequalities:

$$\frac{x}{8}+\frac{y}{4}\le 1,\quad \frac{x}{3}+\frac{y}{6}\le 1,\quad \frac{x}{9/2}+\frac{y}{9/2}\le 1.$$

These can be rewritten

$$x+2y\le 8,\quad 2x+y\le 6,\quad 2x+2y\le 9.$$

Points that satisfy these inequalities lie in the polygon shown to the right. Profit $P(x,y)$ does not have any critical points. In addition, when $P(x,y)$ is evaluated along the five edges of the polygon, linear functions are obtained. They do not have critical points. It follows that the maximum value of P must occur at one of the vertices of the polygon. Since $P(0,0)=0$, $P(0,4)=1200$, $P(1,7/2)=1250$, $P(3/2,3)=1200$, and $P(3,0)=600$, maximum profit occurs when 2 units of X and 7 units of Y are produced in a two-hour shift.



35. If G and S are the numbers of grams of grain and supplements per day, the cost for feeding the cow per day is

$$C = C(G,S) = \frac{2750}{1000}(11) + \frac{11\,000}{1000}G + \frac{17\,500}{1000}S = 30.25 + 11G + 17.5S.$$

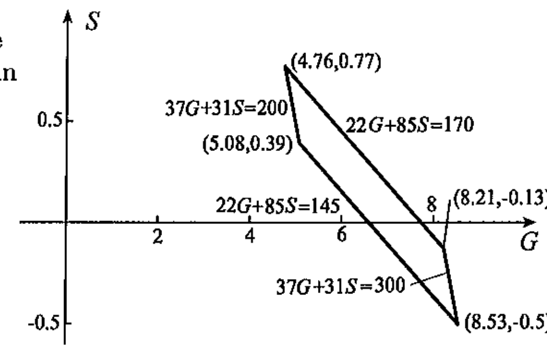
Because the cow's diet must contain between 9.5 and 11.5 kilograms of digestive material and between 1.9 and 2.0 kilograms of protein per day,

$$9.5 \le \frac{11}{2} + \frac{74}{100}G + \frac{62}{100}S \le 11.5, \qquad 1.9 \le \frac{12}{100}(11) + \frac{8.8}{100}G + \frac{34}{100}S \le 2.0,$$

and these inequalitities reduce to

$$200 \le 37G + 31S \le 300, \qquad 145 \le 22G + 85S \le 170.$$

We must therefore minimize $C(G,S)$ for those points (G,S) in the parallelogram R (actually only for those points in the first quadrant). Since $C(G,S)$ is linear in G and S, it has no critical points. On the four parts of the boundary of R, C is also linear, and therefore C has no critical points on these lines. It follows that C is minimized at one of the two corners of the parallelogram in the first quadrant. Since $C(4.76, 0.77) = 96.09$ and $C(5.08, 0.39) = 92.96$, C is minimized for $G = 5.08$ kg and $S = 0.39$ kg.



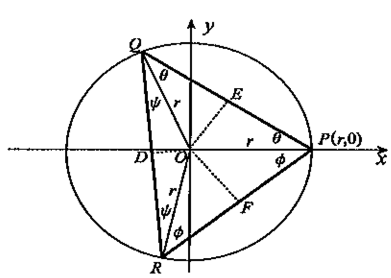
36. We choose one vertex of the triangle at $P(r, 0)$. For maximum area, one of the remaining vertices must have a positive y-coordinate and the other a negative y-coordinate. When vertices are denoted by Q and R, the area of ΔPQR is twice the area of ΔPOE plus twice the area of ΔQOD plus twice the area of ΔPOF,

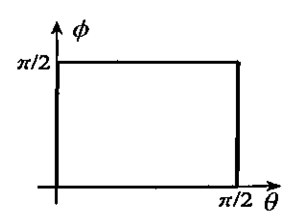
$$\begin{aligned} A &= 2\left(\frac{1}{2}\right)(r\sin\theta)(r\cos\theta) + 2\left(\frac{1}{2}\right)(r\sin\psi)(r\cos\psi) + 2\left(\frac{1}{2}\right)(r\sin\phi)(r\cos\phi) \\ &= \frac{r^2}{2}(\sin 2\theta + \sin 2\psi + \sin 2\phi). \end{aligned}$$

But, $2\theta + 2\phi + 2\psi = \pi$, so that

$$A(\theta,\phi) = \frac{r^2}{2}[\sin 2\theta + \sin 2\phi + \sin(\pi - 2\theta - 2\phi)] = \frac{r^2}{2}[\sin 2\theta + \sin 2\phi + \sin(2\theta + 2\phi)],$$

defined on the square $0 \le \theta \le \pi/2$ and $0 \le \phi \le \pi/2$.





For critical points of this function, we solve

$$0 = \frac{\partial A}{\partial \theta} = \frac{r^2}{2}[2\cos 2\theta + 2\cos(2\theta + 2\phi)], \quad 0 = \frac{\partial A}{\partial \phi} = \frac{r^2}{2}[2\cos 2\phi + 2\cos(2\theta + 2\phi)].$$

These imply that $\cos 2\theta = \cos 2\phi$ and therefore $\theta = \phi$. Then

$$0 = 2\cos 2\theta + 2\cos 4\theta = 2\cos 2\theta + 2(2\cos^2 2\theta - 1) = 2(2\cos 2\theta - 1)(\cos 2\theta + 1).$$

Thus, $\cos 2\theta = 1/2$ or $\cos 2\theta = -1$. The only solution in the interval $0 < \theta < \pi/2$ is $\theta = \pi/6$. The only critical point inside the square is $(\pi/6, \pi/6)$, and for this equilateral triangle

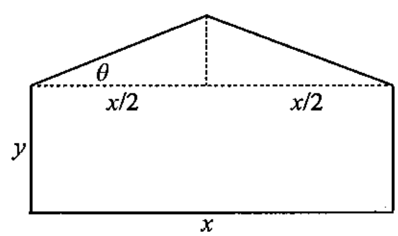
$$A = \frac{r^2}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}r^2}{4}.$$

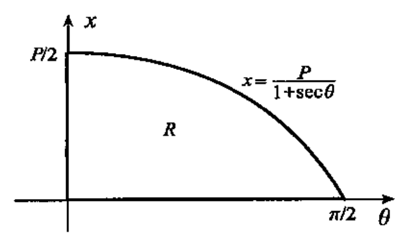
On the edge $\theta = 0$ of the square, area reduces to $r^2 \sin 2\phi$, and this function has maximum value r^2. The same situation occurs for $\phi = 0$. On the edges $\theta = \pi/2$ or $\phi = \pi/2$, area is zero. Hence, maximum area is $3\sqrt{3}r^2/4$ when the triangle is equilateral.

37. The area of the left figure is $A = xy + \frac{x}{2}\left(\frac{x}{2}\tan\theta\right)$. Since $x + 2y + 2\left(\frac{x}{2}\sec\theta\right) = P$, it follows that $y = (P - x - x\sec\theta)/2$, and

$$A(\theta, x) = \frac{x}{2}(P - x - x\sec\theta) + \frac{x^2}{4}\tan\theta = \frac{Px}{2} + \frac{x^2}{4}(\tan\theta - 2 - 2\sec\theta).$$

Now, θ varies from 0 to $\pi/2$. In order to guarantee nonnegativity of y, corresponding values of x must satisfy $P - x - x\sec\theta \geq 0$; that is, $x \leq \dfrac{P}{1 + \sec\theta}$. Thus, the region R of the θx-plane over which A must be maximized is shown to the right.





For critical points of $A(\theta, x)$ we solve

$$0 = \frac{\partial A}{\partial x} = \frac{P}{2} + \frac{x}{2}(\tan\theta - 2 - 2\sec\theta), \quad 0 = \frac{\partial A}{\partial \theta} = \frac{x^2}{4}(\sec^2\theta - 2\sec\theta\tan\theta).$$

The second implies that $x = 0$ or $\sec\theta = 2\tan\theta$. The only solution of the latter of these in the interval $0 < \theta < \pi/2$ is $\theta = \pi/6$. The corresponding value of x is $x = (2 - \sqrt{3})P$, and for these values $A(\pi/6, (2-\sqrt{3})P) = \boxed{P^2(2-\sqrt{3})/4}$. On the boundary $x = 0$ of R, A is identically equal to 0. On $\theta = 0$,

$$A(0, x) = F(x) = \frac{Px}{2} + \frac{x^2}{4}(-2 - 2) = \frac{Px}{2} - x^2, \quad 0 \leq x \leq P/2.$$

For critical points we solve $0 = P/2 - 2x \implies x = P/4$. At this value $F(P/4) = \boxed{P^2/16}$. On the boundary $x = P/(1 + \sec\theta)$,

$$A = G(\theta) = \frac{P^2}{4(1 + \sec\theta)^2}\tan\theta, \quad 0 \leq \theta < \pi/2.$$

For critical points we solve $0 = G'(\theta)$. Neglecting the $P^2/4$, we obtain

$$0 = \frac{(1 + \sec\theta)^2\sec^2\theta - 2\tan\theta(1 + \sec\theta)\sec\theta\tan\theta}{(1 + \sec\theta)^2} = \frac{\sec\theta(1 + \sec\theta)[\sec\theta(1 + \sec\theta) - 2\tan^2\theta]}{(1 + \sec\theta)^2}.$$

This implies that

$$0 = \sec\theta + \sec^2\theta - 2(\sec^2\theta - 1) = -\sec^2\theta + \sec\theta + 2 = -(\sec\theta - 2)(\sec\theta + 1).$$

Thus, $\sec\theta = 2 \implies \theta = \pi/3$. For this θ, $G(\pi/3) = \boxed{\sqrt{3}P^2/36}$. Finally, we evaluate $A(0, P/2) = \boxed{0}$. Maximum area is $P^2(2 - \sqrt{3})/4$ when $\theta = \pi/6$, $x = (2 - \sqrt{3})P$, and $y = (1 - \sqrt{3}/3)P/2$.

38. If x and y are the numbers of computers of models A and B, then the cost of the 100 computers is

$$C = f(x,y) = 1300x + 1200y + 1000(100 - x - y) = 100\,000 + 300x + 200y.$$

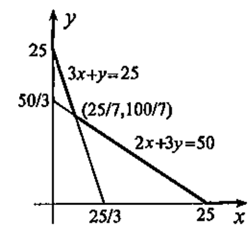
Because the computers must have at least 2000 MB of memory,

$$64x + 32y + 16(100 - x - y) \geq 2000 \quad \Longrightarrow 3x + y \geq 25.$$

Because the computers must have at least 150 GB of disk space,

$$3x + 4y + (100 - x - y) \geq 150 \quad \Longrightarrow \quad 2x + 3y \geq 50.$$

The domain of $f(x,y)$ therefore consists of all non-negative values of x and y satisfying these two inequalities. They are shown to the right. Since $f(x,y)$ is linear, there are no critical points inside the region. The function will also be linear on the boundaries so that no critical points occur there either. The function must be minimized at one of the three corners. We find $f(0,25) = 105\,000$, $f(25,0) = 107\,500$, and $f(25/7,100/7) = 103\,929$. We need to take either x or y, or both, to the next integer value. For $x = 4$, we require $y \geq 13$ and $3y \geq 42 \Longrightarrow y \geq 14$. For $y = 15$, we require $3x \geq 11$ and $2x \geq 5 \Longrightarrow x \geq 4$. Of these two, we should choose $x = 4$ and $y = 14$. Cost for these along with $z = 87$ is $f(4,13) = 104\,000$.



39. For critical points of $f(x,y)$ inside the polygon, we should solve $0 = f_x = c$ and $0 = f_y = d$. There are no solutions. On any edge of the polygon described by a straight line say $A_i x + B_i y = C_i$, $f(x,y)$ has value

$$F(x) = cx + \frac{d}{B_i}(C_i - A_i x) = \left(c - \frac{A_i d}{B_i}\right)x + \frac{dC_i}{B_i},$$

for some values of x representing the extent of the edge of the polygon. Since $F'(x) = c - A_i d/B_i \neq 0$, there are no critical points on the edge of the polygon. Since this is the case for all edges, it follows that the maximum value of $f(x,y)$ (and there must be one since the function is continuous on a closed polygon) must occur at one of the vertices of the polygon. If $c - A_i d/B_i$ should equal zero along some edge of the polygon, then $F(x)$ would be constant along that edge. If this value turned out to be a maximum for $f(x,y)$, then it would also be taken on at the ends of the edge and therefore at two vertices of the polygon. Once again the maximum is at a vertex of the polygon.

40. Since travel times through the media are distances divided by speeds, it follows that time from source to receiver is

$$t = \frac{2d_1 \sec\theta_1}{v_1} + \frac{2d_2 \sec\theta_2}{v_2} + \frac{s - 2d_1 \tan\theta_1 - 2d_2 \tan\theta_2}{v_3}.$$

For critical points of this function we solve

$$0 = \frac{\partial t}{\partial \theta_1} = \frac{2d_1 \sec\theta_1 \tan\theta_1}{v_1} - \frac{2d_1 \sec^2\theta_1}{v_3}, \quad 0 = \frac{\partial t}{\partial \theta_2} = \frac{2d_2 \sec\theta_2 \tan\theta_2}{v_2} - \frac{2d_2 \sec^2\theta_2}{v_3}.$$

These equation can be solved separately. If we divide the first by $2d_1 \sec\theta_1$,

$$\frac{\tan\theta_1}{v_1} = \frac{\sec\theta_1}{v_3} \quad \Longrightarrow \quad \sin\theta_1 = \frac{v_1}{v_3} \quad \Longrightarrow \quad \theta_1 = \mathrm{Sin}^{-1}\left(\frac{v_1}{v_3}\right).$$

Similarly, $\theta_2 = \mathrm{Sin}^{-1}\left(\dfrac{v_2}{v_3}\right)$.

EXERCISES 12.12

1. The constraint $x^2 + y^2 = 4$ defines a **closed** curve (a circle). We define the Lagrangian

$$L(x, y, \lambda) = x^2 + y + \lambda(x^2 + y^2 - 4).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = 1 + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 4.$$

Critical points (x, y) are $(0, \pm 2)$ and $(\pm\sqrt{15}/2, 1/2)$. Since $f(0, \pm 2) = \pm 2$, and $f(\pm\sqrt{15}/2, 1/2) = 17/4$, maximum and minimum values of $f(x, y)$ are $17/4$ and -2.

2. The constraint $x^2 + 2y^2 + 4z^2 = 9$ defines a **closed** surface (an ellipsoid). We define the Lagrangian

$$L(x, y, z, \lambda) = 5x - 2y + 3z + 4 + \lambda(x^2 + 2y^2 + 4z^2 - 9).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 5 + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = -2 + 4\lambda y, \quad 0 = \frac{\partial L}{\partial z} = 3 + 8\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + 2y^2 + 4z^2 - 9.$$

Critical points (x, y, z) are $(\pm 10/\sqrt{13}, \mp 2/\sqrt{13}, \pm 3/(2\sqrt{13}))$. Since $f(\pm 10/\sqrt{13}, \mp 2/\sqrt{13}, \pm 3/(2\sqrt{13})) = (8 \pm 9\sqrt{13})/2$, these are the maximum and minimum values of $f(x, y, z)$.

3. The constraint $(x-1)^2 + y^2 = 1$ defines a **closed** curve (a circle). We define the Lagrangian

$$L(x, y, \lambda) = x + y + \lambda[(x-1)^2 + y^2 - 1].$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 1 + 2\lambda(x-1), \quad 0 = \frac{\partial L}{\partial y} = 1 + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = (x-1)^2 + y^2 - 1.$$

Critical points (x, y) are $(1 \pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Since $f(1 \pm 1/\sqrt{2}, \pm 1/\sqrt{2}) = 1 \pm \sqrt{2}$, these are maximum and minimum values of $f(x, y)$.

4. The constraint $x^2 + y^2 + z^2 = 9$ is a **closed** surface (a sphere). We define the Lagrangian

$$L(x, y, z, \lambda) = x^3 + y^3 + z^3 + \lambda(x^2 + y^2 + z^2 - 9).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 3x^2 + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = 3y^2 + 2\lambda y, \quad 0 = \frac{\partial L}{\partial z} = 3z^2 + 2\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 9.$$

Critical points (x, y, z) are $(\pm 3, 0, 0)$, $(0, \pm 3, 0)$, $(0, 0, \pm 3)$, $(0, \pm 3/\sqrt{2}, \pm 3/\sqrt{2})$, $(\pm 3/\sqrt{2}, 0, \pm 3/\sqrt{2})$, $(\pm 3/\sqrt{2}, \pm 3/\sqrt{2}, 0)$, $(\pm\sqrt{3}, \pm\sqrt{3}, \pm\sqrt{3})$. Since $f(x, y, z) = \pm 27$ at the first six critical points, $f(x, y, z) = \pm 27/\sqrt{2}$ at the second set of six critical points, and $f(\pm\sqrt{3}, \pm\sqrt{3}, \pm\sqrt{3}) = \pm 9\sqrt{3}$, maximum and minimum values of $f(x, y, z)$ are ± 27.

5. The constraint $x^2 + 2y^2 + 3z^2 = 12$ defines a **closed** surface (an ellipsoid). We define the Lagrangian

$$L(x, y, z, \lambda) = xyz + \lambda(x^2 + 2y^2 + 3z^2 - 12).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = yz + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = xz + 4\lambda y, \quad 0 = \frac{\partial L}{\partial z} = xy + 6\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + 2y^2 + 3z^2 - 12.$$

Critical points (x, y, z) are $(0, 0, \pm 2)$, $(0, \pm\sqrt{6}, 0)$, $(\pm 2\sqrt{3}, 0, 0)$, $(2, \pm\sqrt{2}, \pm 2/\sqrt{3})$, $(2, \pm\sqrt{2}, \mp 2/\sqrt{3})$, $(-2, \pm\sqrt{2}, \pm 2/\sqrt{3})$, and $(-2, \pm\sqrt{2}, \mp 2/\sqrt{3})$. Since $f(x, y, z)$ has value 0 at the first six critical points and values $\pm 4\sqrt{6}/3$ at the other critical points, maximum and minimum values of $f(x, y, z)$ are $\pm 4\sqrt{6}/3$.

6. The constraints $x^2+y^2=1$, $z=y$ define a **closed** curve. We define the Lagrangian

$$L(x,y,z,\lambda,\mu)=x^2y+z+\lambda(x^2+y^2-1)+\mu(z-y).$$

For critical points of L, we solve

$$0=\frac{\partial L}{\partial x}=2xy+2\lambda x,\quad 0=\frac{\partial L}{\partial y}=x^2+2\lambda y-\mu,\quad 0=\frac{\partial L}{\partial z}=1+\mu,$$

$$0=\frac{\partial L}{\partial \lambda}=x^2+y^2-1,\quad 0=\frac{\partial L}{\partial \mu}=z-y.$$

Solutions (x,y,z) of these equations are $(0,\pm1,\pm1)$, $(1/\sqrt{3},\pm\sqrt{2/3},\pm\sqrt{2/3})$, and $(-1/\sqrt{3},\pm\sqrt{2/3},\pm\sqrt{2/3})$. Since $f(0,\pm1,\pm1)=\pm1$, $f(1/\sqrt{3},\pm\sqrt{2/3},\pm\sqrt{2/3})=\pm\sqrt{32/27}$, and $f(-1/\sqrt{3},\pm\sqrt{2/3},\pm\sqrt{2/3})=\pm\sqrt{32/27}$, maximum and minimum values are $\pm\sqrt{32/27}$.

7. Since all points must satisfy $x^2+y^2+z^2=2z$, we may replace the function $f(x,y,z)=x^2+y^2+z^2$ with $f(x,y,z)=2z$. The constraints define a **closed** curve. We define the Lagrangian

$$L(x,y,z,\lambda,\mu)=2z+\lambda(x^2+y^2+z^2-2z)+\mu(x+y+z-1).$$

For critical points of L, we solve

$$0=\frac{\partial L}{\partial x}=2\lambda x+\mu,\quad 0=\frac{\partial L}{\partial y}=2\lambda y+\mu,\quad 0=\frac{\partial L}{\partial z}=2+\lambda(2z-2)+\mu,$$

$$0=\frac{\partial L}{\partial \lambda}=x^2+y^2+z^2-2z,\quad 0=\frac{\partial L}{\partial \mu}=x+y+z-1.$$

Solutions (x,y,z) of these equations are $(\pm1/\sqrt{6},\pm1/\sqrt{6},1\mp\sqrt{6}/3)$. Since $f(\pm1/\sqrt{6},\pm1/\sqrt{6},1\mp\sqrt{6}/3)=2(1\mp\sqrt{6}/3)$, these are minimum and maximum values of $f(x,y,z)$.

8. The constraints $x^2+y^2=1$, $z=\sqrt{x^2+y^2}$ define the **closed** curve $x^2+y^2=1$, $z=1$, so that we write alternatively, $f(x,y,z)=F(x,y)=xy-x^2$ subject to $x^2+y^2=1$. We define the Lagrangian $L(x,y,\lambda)=xy-x^2+\lambda(x^2+y^2-1)$. For critical points of L, we solve

$$0=\frac{\partial L}{\partial x}=y-2x+2\lambda x,\quad 0=\frac{\partial L}{\partial y}=x+2\lambda y,\quad 0=\frac{\partial L}{\partial \lambda}=x^2+y^2-1.$$

Solutions (x,y) of these equations are $\left(\dfrac{\pm\sqrt{2-\sqrt{2}}}{2},\dfrac{\pm\sqrt{2+\sqrt{2}}}{2}\right)$, $\left(\dfrac{\pm\sqrt{2+\sqrt{2}}}{2},\dfrac{\mp\sqrt{2-\sqrt{2}}}{2}\right)$. Since values of $F(x,y)$ at these points are $(-1\pm\sqrt{2})/2$, maximum and minimum values are $(\sqrt{2}-1)/2$ and $-(\sqrt{2}+1)/2$.

9. The distance D from $(-1,1,2)$ to any point $P(x,y,z)$ is given by $D^2=(x+1)^2+(y-1)^2+(z-2)^2$. To minimize this function subject to the constraint $2x-3y+6z=14$, we define the Lagrangian $L(x,y,z,\lambda)=(x+1)^2+(y-1)^2+(z-2)^2+\lambda(2x-3y+6z-14)$. For critical points of $L(x,y,z,\lambda)$ we solve

$$0=\frac{\partial L}{\partial x}=2(x+1)+2\lambda,\quad 0=\frac{\partial L}{\partial y}=2(y-1)-3\lambda,\quad 0=\frac{\partial L}{\partial z}=2(z-2)+6\lambda,$$

$$0=\frac{\partial L}{\partial \lambda}=2x-3y+6z-14,$$

The only solution is $x=-5/7$, $y=4/7$, $z=20/7$. Since x, y and z can take on all possible values, and D^2 becomes infinite for large values of x, y or z, the critical point must minimize D^2. The shortest distance is therefore $D(-5/7,4/7,20/7)=1$.

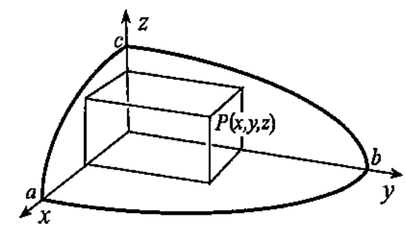
10. The distance D from $(1,1,0)$ to any point $P(x,y,z)$ is given by $D^2 = (x-1)^2 + (y-1)^2 + z^2$. This function must be minimized subject to the constraint $z = x^2 + y^2$. If we define the Lagrangian $L(x,y,z,\lambda) = (x-1)^2 + (y-1)^2 + z^2 + \lambda(x^2 + y^2 - z)$, its critical points are given by

$$0 = \frac{\partial L}{\partial x} = 2(x-1) + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = 2(y-1) + 2\lambda y, \quad 0 = \frac{\partial L}{\partial z} = 2z - \lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - z.$$

The solution for (x,y,z) is $(1/2, 1/2, 1/2)$. Since D^2 becomes infinite as x, y, and z become infinite, it follows that this point must minimize D^2 (or D).

11. The volume obtained from a point $P(x,y,z)$ on that part of the ellipsoid in the first octant is $V = 8xyz$. This function must be maximized for (x,y,z) satisfying $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. We define the Lagrangian

$L(x,y,z,\lambda) = 8xyz + \lambda(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1)$.



For critical points of $L(x,y,z,\lambda)$, we solve

$$0 = \frac{\partial L}{\partial x} = 8yz + \frac{2\lambda x}{a^2}, \quad 0 = \frac{\partial L}{\partial y} = 8xz + \frac{2\lambda y}{b^2}, \quad 0 = \frac{\partial L}{\partial z} = 8xy + \frac{2\lambda z}{c^2}, \quad 0 = \frac{\partial L}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

The only solution (x,y,z) of these equations with positive coordinates is $(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3})$. Since $V = 0$ on the three edges of the ellipsoid in the first octant, it follows that V must be maximized at this critical point, and therefore the dimensions of the largest box are $2a/\sqrt{3} \times 2b/\sqrt{3} \times 2c/\sqrt{3}$.

12. Since A is maximized when A^2 is maximized, we define the Lagrangian

$$L(x,y,z,\lambda) = \frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right)\left(\frac{P}{2} - z\right) + \lambda(x + y + z - P).$$

Critical points of L are given by

$$0 = \frac{\partial L}{\partial x} = -\frac{P}{2}\left(\frac{P}{2} - y\right)\left(\frac{P}{2} - z\right) + \lambda, \quad 0 = \frac{\partial L}{\partial y} = -\frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - z\right) + \lambda,$$

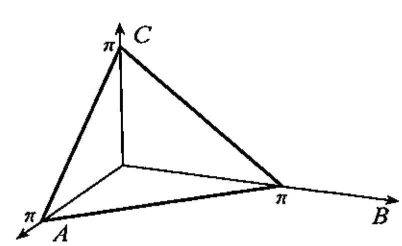
$$0 = \frac{\partial L}{\partial z} = -\frac{P}{2}\left(\frac{P}{2} - x\right)\left(\frac{P}{2} - y\right) + \lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = x + y + z - P.$$

The only solution of these equations is $x = y = z = P/3$; or, any two of x, y, and z equal to $P/2$ and the third equal to zero. In the latter case, the triangle has degenerated to a straight line with $A = 0$. Since this case represents the bounds for possible values of x, y, and z, it follows that $x = y = z = P/3$ must maximize A.

13. Consider finding the maximum value of the function $f(A,B,C) = \sin(A/2)\sin(B/2)\sin(C/2)$ subject to the constraint $A + B + C = \pi$ (because A, B and C are angles in a triangle). This is a plane in ABC-space, and we consider only that part R of the plane in the first octant. If we define the Lagrangian

$L(A,B,C,\lambda) = f(A,B,C) + \lambda(A + B + C - \pi)$,

its critical points are defined by



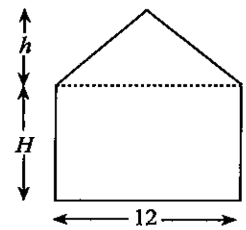
$$0 = \frac{\partial L}{\partial A} = \frac{1}{2}\cos(A/2)\sin(B/2)\sin(C/2) + \lambda, \quad 0 = \frac{\partial L}{\partial B} = \frac{1}{2}\sin(A/2)\cos(B/2)\sin(C/2) + \lambda,$$

$$0 = \frac{\partial L}{\partial C} = \frac{1}{2}\sin(A/2)\sin(B/2)\cos(C/2) + \lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = A + B + C - \pi.$$

The only solution of these equations for which all three angles are between 0 and π is $A = B = C = \pi/3$. On the three edges of R, the value of $f(A, B, C)$ is zero. It follows that $f(A, B, C)$ must have a maximum value at $A = B = C = \pi/3$, and this value is $f(\pi/3, \pi/3, \pi/3) = 1/8$. Hence, $\sin(A/2)\sin(B/2)\sin(C/2 \leq 1/8$ for all other values of A, B, and C.

14. The volume of the silo is

$V = \pi(6)^2H + \frac{1}{3}\pi(6)^2h = 12\pi(h + 3H)$.



Since the area of the silo must be 200 m^2,

$200 = 2\pi(6)H + \pi(6)\sqrt{36 + h^2}$.

We define the Lagrangian

$L(h, H, \lambda) = 12\pi(h + 3H) + \lambda(12\pi H + 6\pi\sqrt{36 + h^2} - 200)$.

Its critical points are given by

$$0 = \frac{\partial L}{\partial h} = 12\pi + \frac{6\pi\lambda h}{\sqrt{36 + h^2}}, \quad 0 = \frac{\partial L}{\partial H} = 36\pi + 12\pi\lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = 12\pi H + 6\pi\sqrt{36 + h^2} - 200.$$

The solution of these equations for h and H is $h = 12/\sqrt{5}$ and $H = (50\sqrt{5} - 27\pi)/(3\sqrt{5}\pi)$. Clearly $h \geq 0$ and the constraint requires $h \leq \sqrt{4 \times 10^4 - 36^2\pi^2}/(6\pi)$. Since

$$V_{|h=0} = 260.7, \quad V_{|h=12/\sqrt{5}} = 347.1, \quad V_{|h=\sqrt{4\times 10^4 - 36^2\pi^2}/(6\pi)} = 329.9,$$

it follows that V is maximized for $h = 12/\sqrt{5}$ m and $H = (50\sqrt{5} - 27\pi)/(3\sqrt{5}\pi)$ m.

15. We define the Lagrangian $L(x, y, \lambda) = x^2 - y^2 + \lambda(x^2 + y^2 - 1)$. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = -2y + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1.$$

Critical points (x, y) are $(\pm 1, 0)$ and $(0, \pm 1)$. Since $f(\pm 1, 0) = 1$, and $f(0, \pm 1) = -1$, maximum and minimum values of $f(x, y)$ are ± 1.

16. We define the Lagrangian $L(x, y, \lambda) = |x - y| + \lambda(x^2 + y^2 - 1)$. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = \frac{|x - y|}{x - y} + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = -\frac{|x - y|}{x - y} + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1.$$

Critical points (x, y) are $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$. The derivatives do not exist when $y = x$ and this leads to the additional critical points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Since $f(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}) = 0$ and $f(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}) = \sqrt{2}$, maximum and minimum values of $f(x, y)$ are $\sqrt{2}$ and 0.

17. We define the Lagrangian $L(x, y, \lambda) = x^2 - y^2 + \lambda(|x| + |y| - 1)$. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + \frac{\lambda|x|}{x}, \quad 0 = \frac{\partial L}{\partial y} = -2y + \frac{\lambda|y|}{y}, \quad 0 = \frac{\partial L}{\partial \lambda} = |x| + |y| - 1.$$

There are no solutions of these equations. Since the partial derivative with respect to x fails to exist at $x = 0$, and the derivative with respect to y does not exist at $y = 0$, critical points are $(0, \pm 1)$ and $(\pm 1, 0)$. Since $f(\pm 1, 0) = 1$, and $f(0, \pm 1) = -1$, maximum and minimum values of $f(x, y)$ are ± 1.

18. We define the Lagrangian $L(x, y, \lambda) = |x - 2y| + \lambda(|x| + |y| - 1)$. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = \frac{|x - 2y|}{x - 2y} + \frac{\lambda|x|}{x}, \quad 0 = \frac{\partial L}{\partial y} = -2\frac{|x - 2y|}{x - 2y} + \frac{\lambda|y|}{y}, \quad 0 = \frac{\partial L}{\partial \lambda} = |x| + |y| - 1.$$

There are no solutions of these equations. Since the partial derivative with respect to x fails to exist at $x = 0$ and when $x = 2y$, and the derivative with respect to y does not exist at $y = 0$ or when $x = 2y$, critical points are $(0, \pm 1)$, $(\pm 1, 0)$, and $(\pm 2/3, \pm 1/3)$. Since $f(\pm 1, 0) = 1$, $f(0, \pm 1) = 2$, and $f(\pm 2/3, \pm 1/3) = 0$, maximum and minimum values of $f(x, y)$ are 2 and 0.

19. The distance D from (x_1, y_1, z_1) to any point $P(x, y, z)$ is given by $D^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$. This function must be minimized subject to the constraint $Ax + By + Cz + D = 0$. If we define the Lagrangian $L(x, y, z, \lambda) = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 + \lambda(Ax + By + Cz + D)$, critical points are given by

$$0 = \frac{\partial L}{\partial x} = 2(x - x_1) + \lambda A, \quad 0 = \frac{\partial L}{\partial y} = 2(y - y_1) + \lambda B, \quad 0 = \frac{\partial L}{\partial z} = 2(z - z_1) + \lambda C,$$

$$0 = \frac{\partial L}{\partial \lambda} = Ax + By + Cz + D.$$

The only solution is

$$x = \frac{(B^2 + C^2)x_1 - A(By_1 + Cz_1 + D)}{A^2 + B^2 + C^2}, \quad y = \frac{(A^2 + C^2)y_1 - B(Ax_1 + Cz_1 + D)}{A^2 + B^2 + C^2},$$

$$z = \frac{(A^2 + B^2)z_1 - C(Ax_1 + By_1 + D)}{A^2 + B^2 + C^2}.$$

Since this is the only critical point, and distance becomes infinite as x, y and z take on large values, it follows that this critical point must minimize D^2. To find the minimum value we substitute these values for x, y, and z into the formula for D^2,

$$D^2 = \left[\frac{(B^2 + C^2)x_1 - A(By_1 + Cz_1 + D)}{A^2 + B^2 + C^2} - x_1\right]^2 + \left[\frac{(A^2 + C^2)y_1 - B(Ax_1 + Cz_1 + D)}{A^2 + B^2 + C^2} - y_1\right]^2$$
$$+ \left[\frac{(A^2 + B^2)z_1 - C(Ax_1 + By_1 + D)}{A^2 + B^2 + C^2} - z_1\right]^2.$$

This simplifies to $\dfrac{(Ax_1 + By_1 + Cz_1 + D)^2}{A^2 + B^2 + C^2}$, and its square root gives the desired result.

20. The distance D between points $P(x, y)$ on $F(x, y) = 0$ and $Q(X, Y)$ on $G(x, y) = 0$ is given by $D^2 = (x - X)^2 + (y - Y)^2$, where $F(x, y) = 0$ and $G(X, Y) = 0$. We define the Lagrangian

$$L(x, y, X, Y, \lambda, \mu) = (x - X)^2 + (y - Y)^2 + \lambda F(x, y) + \mu G(X, Y).$$

For critical points of this function,

$$0 = \frac{\partial L}{\partial x} = 2(x - X) + \lambda F_x, \quad 0 = \frac{\partial L}{\partial y} = 2(y - Y) + \lambda F_y, \quad 0 = \frac{\partial L}{\partial X} = -2(x - X) + \mu G_X,$$

$$0 = \frac{\partial L}{\partial Y} = -2(y - Y) + \mu G_Y, \quad 0 = \frac{\partial L}{\partial \lambda} = F(x, y), \quad 0 = \frac{\partial L}{\partial \mu} = G(X, Y).$$

If $P(x_0, y_0)$ and $Q(X_0, Y_0)$ are the points that minimize D^2, then they must satisfy these equations. In particular, the first four give

$$0 = 2(x_0 - X_0) + \lambda F_x(x_0, y_0), \quad 0 = 2(y_0 - Y_0) + \lambda F_y(x_0, y_0),$$

$$0 = -2(x_0 - X_0) + \mu G_X(X_0, Y_0), \quad 0 = -2(y_0 - Y_0) + \mu G_Y(X_0, Y_0).$$

From the first two equations, we obtain $\dfrac{y_0 - Y_0}{x_0 - X_0} = \dfrac{F_y(x_0, y_0)}{F_x(x_0, y_0)}$. But $(y_0 - Y_0)/(x_0 - X_0)$ is the slope of the line joining $P(x_0, y_0)$ and $Q(X_0, Y_0)$, and the slope of the tangent line to $F(x, y) = 0$ at (x_0, y_0) is $-F_x(x_0, y_0)/F_y(x_0, y_0)$. Hence, these lines are perpendicular. The last two equations indicate that PQ is perpendicular to the tangent line to $G(x, y) = 0$ at Q.

21. Production levels at the four plants are

$$x_1 = \frac{26(500)}{100} = 130, \quad x_2 = \frac{24(500)}{100} = 120, \quad x_3 = \frac{23(500)}{100} = 115, \quad x_4 = \frac{27(500)}{100} = 135.$$

Total cost is $\dfrac{500^2}{100} = 2500.$

22. The volume of a right circular cylinder is $V = \pi r^2 h$, and were there no constraints on r and h, this function would be considered for all points in the first quadrant of the rh-plane. However, r and h must satisfy a constraint that geometrically can be interpreted as a curve in the rh-plane. What we must do then is minimize $V = \pi r^2 h$, considering only those points (r, h) on the curve defined by the constraint. Clearly there is only one independent variable in the problem—either r or h, but not both. If we choose r as the independent variable, then we note from the constraint that as h becomes very large, r approaches $2.4048/\sqrt{k}$. Since there is no upper bound on r, we can state that the values of r to be considered in the minimization of V are $r > 2.4048/\sqrt{k}$.

To find critical points of V we introduce the Lagrangian

$$L(r, h, \lambda) = \pi r^2 h + \lambda\left[\left(\frac{2.4048}{r}\right)^2 + \left(\frac{\pi}{h}\right)^2 - k\right],$$

and first solve the equations

$$0 = \frac{\partial L}{\partial r} = 2\pi r h + \lambda\left[\frac{-2(2.4048)^2}{r^3}\right],$$
$$0 = \frac{\partial L}{\partial h} = \pi r^2 + \lambda\left(\frac{-2\pi^2}{h^3}\right),$$
$$0 = \frac{\partial L}{\partial \lambda} = \left(\frac{2.4048}{r}\right)^2 + \left(\frac{\pi}{h}\right)^2 - k.$$

If we solve each of the first two equations for λ and equate the resulting expressions, we have

$$\frac{\pi r^4 h}{2.4048^2} = \frac{r^2 h^3}{2\pi}.$$

Since neither r nor h can be zero, we divide by $r^2 h$:

$$\frac{\pi r^2}{2.4048^2} = \frac{h^2}{2\pi} \quad \Longrightarrow \quad r = \frac{2.4048h}{\sqrt{2}\pi}.$$

Substitution of this result into the constraint equation gives

$$\left(\frac{\sqrt{2}\pi}{h}\right)^2 + \left(\frac{\pi}{h}\right)^2 = k,$$

and this equation can be solved for $h = \pi\sqrt{3/k}$. This gives

$$r = \frac{2.4048}{\sqrt{2}\pi}\frac{\pi\sqrt{3}}{\sqrt{k}} = 2.4048\sqrt{3/(2k)}.$$

We have obtained therefore only one critical point (r, h) at which the derivatives of L vanish. The only values of r and h at which the derivatives of L do not exist are $r = 0$ and $h = 0$, but these must be rejected since the constraint requires both r and h to be positive.

To finish the problem we note that

$$\lim_{r\to\infty} V = \infty, \qquad \lim_{r\to 2.4048/\sqrt{k}^+} V = \lim_{h\to\infty} V = \infty.$$

It follows, therefore, that the single critical point at which $r = 2.4048\sqrt{3/(2k)}$ and $h = \pi\sqrt{3/k}$ must give the absolute minimum value of $V(r, h)$.

23. We find maximum and minimum values of $D^2 = x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 1$. Critical points of the Lagrangian $L(x, y, \lambda) = x^2 + y^2 + \lambda(x^2 + xy + y^2 - 1)$ are given by

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda(2x + y), \quad 0 = \frac{\partial L}{\partial y} = 2y + \lambda(x + 2y), \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + xy + y^2 - 1.$$

Solutions (x, y) are $(\pm 1, \mp 1)$ and $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$. Since $D^2(\pm 1, \mp 1) = 2$ and $D^2(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}) = 2/3$, the closest points are $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ and the farthest points are $(\pm 1, \mp 1)$.

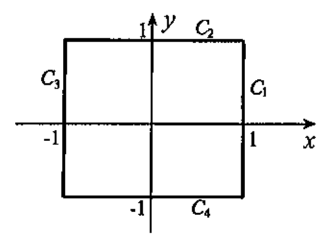
24. For critical points of $f(x, y)$, we solve $0 = \dfrac{\partial f}{\partial x} = 6x + 2y$, $0 = \dfrac{\partial f}{\partial y} = 2x - 2y$. At the only solution $(0, 0)$, $f(0, 0) = 5$. On the **closed** boundary $4x^2 + 9y^2 = 36$ of the region, we define the Lagrangian $L(x, y, \lambda) = 3x^2 + 2xy - y^2 + 5 + \lambda(4x^2 + 9y^2 - 36)$. For its critical points,

$$0 = \frac{\partial L}{\partial x} = 6x + 2y + 8\lambda x, \quad 0 = \frac{\partial L}{\partial y} = 2x - 2y + 18\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = 4x^2 + 9y^2 - 36.$$

The solutions (x, y) of these equations are $(\pm 0.55086, \mp 1.96600)$ and $(\pm 2.94899, \pm 0.36724)$. Since $f(\pm 0.55086, \mp 1.96600) = -0.12$ and $f(\pm 2.94899, \pm 0.36724) = 33.12$, maximum and minimum values of $f(x, y)$ are 33.12 and -0.12.

25. For critical points of $f(x, y)$ we solve

$$0 = \frac{\partial f}{\partial x} = 2xy + y^2, \; 0 = \frac{\partial f}{\partial y} = x^2 + 2xy + 1.$$



Solutions are $(\pm 1/\sqrt{3}, \mp 2/\sqrt{3})$ at which $f(\pm 1/\sqrt{3}, \mp 2/\sqrt{3}) = \boxed{\mp 4\sqrt{3}/9}$.

On C_1, we define the Lagrangian $L_1(x, y, \lambda) = x^2 y + xy^2 + y + \lambda(x - 1)$. Critical points are given by

$$0 = \frac{\partial L_1}{\partial x} = 2xy + y^2 + \lambda, \quad 0 = \frac{\partial L_1}{\partial y} = x^2 + 2xy + 1, \quad 0 = \frac{\partial L_1}{\partial \lambda} = x - 1.$$

The only solution is $(1, -1)$ at which $f(1, -1) = \boxed{-1}$.

On C_2, we define the Lagrangian $L_2(x, y, \lambda) = x^2 y + xy^2 + y + \lambda(y - 1)$. Critical points are given by

$$0 = \frac{\partial L_2}{\partial x} = 2xy + y^2, \quad 0 = \frac{\partial L_2}{\partial y} = x^2 + 2xy + 1 + \lambda, \quad 0 = \frac{\partial L_2}{\partial \lambda} = y - 1.$$

The only solution is $(-1/2, 1)$ at which $f(-1/2, 1) = \boxed{3/4}$.

On C_3, we define the Lagrangian $L_3(x, y, \lambda) = x^2 y + xy^2 + y + \lambda(x + 1)$. Critical points are given by

$$0 = \frac{\partial L_3}{\partial x} = 2xy + y^2 + \lambda, \quad 0 = \frac{\partial L_3}{\partial y} = x^2 + 2xy + 1, \quad 0 = \frac{\partial L_3}{\partial \lambda} = x + 1.$$

The only solution is $(-1, 1)$ at which $f(-1, 1) = \boxed{1}$.

On C_4, we define the Lagrangian $L_4(x, y, \lambda) = x^2 y + xy^2 + y + \lambda(y + 1)$. Critical points are given by

$$0 = \frac{\partial L_4}{\partial x} = 2xy + y^2, \quad 0 = \frac{\partial L_4}{\partial y} = x^2 + 2xy + 1 + \lambda, \quad 0 = \frac{\partial L_4}{\partial \lambda} = y + 1.$$

The only solution is $(1/2, -1)$ at which $f(1/2, -1) = \boxed{-3/4}$. The function must also be evaluated at the two remaining corners $f(1, 1) = \boxed{3}$ and $f(-1, -1) = \boxed{-3}$. Maximum and minimum values are ± 3.

26. For critical points of $f(x,y,z)$ we solve $0 = \dfrac{\partial f}{\partial x} = y + z$, $0 = \dfrac{\partial f}{\partial y} = x$, $0 = \dfrac{\partial f}{\partial z} = x$. For the line of critical points $(0, y, -y)$, we evaluate $f(0, y, -y) = \boxed{0}$. On the **closed** boundary $x^2 + y^2 + z^2 = 1$, we define the Lagrangian $L(x, y, z, \lambda) = xy + xz + \lambda(x^2 + y^2 + z^2 - 1)$. For critical points of L,

$$0 = \frac{\partial L}{\partial x} = y + z + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = x + 2\lambda y, \quad 0 = \frac{\partial L}{\partial z} = x + 2\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1.$$

The solutions of these equations for (x, y, z) are $(0, \pm 1/\sqrt{2}, \mp 1/\sqrt{2})$, $(1/\sqrt{2}, \pm 1/2, \pm 1/2)$, and $(-1/\sqrt{2}, \pm 1/2, \pm 1/2)$. Since $f(0, \pm 1/\sqrt{2}, \mp 1/\sqrt{2}) = \boxed{0}$, $f(1/\sqrt{2}, \pm 1/2, \pm 1/2) = \boxed{\pm 1/\sqrt{2}}$, and $f(-1/\sqrt{2}, \pm 1/2, \pm 1/2) = \boxed{\mp 1/\sqrt{2}}$, maximum and minimum values of $f(x, y, z)$ are $\pm 1/\sqrt{2}$.

27. Since z is always equal to 1 on the curve of intersection of the surfaces, we maximize the function $F(x, y) = x^2 y - xy^2$ subject to the constraint $x^2 + y^2 = 1$. Critical points of the Lagrangian $L(x, y, \lambda) = x^2 y - xy^2 + \lambda(x^2 + y^2 - 1)$ are given by

$$0 = \frac{\partial L}{\partial x} = 2xy - y^2 + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = x^2 - 2xy + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1.$$

Solutions for (x, y) are $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$, $\left(\sqrt{(3 \pm \sqrt{5})/6}, 1/[3\sqrt{3 \pm \sqrt{5})/6}]\right)$, and $\left(-\sqrt{(3 \pm \sqrt{5})/6}, -1/[3\sqrt{3 \pm \sqrt{5})/6}]\right)$. When these are substituted into $F(x, y)$, the largest value is $1/\sqrt{2}$.

28. The distance D from the origin to any point (x, y) on the ellipse is given by $D^2 = x^2 + y^2$, subject to $3x^2 + 4xy + 6y^2 = 140$. Ends of the major and minor axes maximize and minimize this function. To find these points we define the Lagrangian $L(x, y, \lambda) = x^2 + y^2 + \lambda(3x^2 + 4xy + 6y^2 - 140)$. Critical points are given by

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda(6x + 4y), \quad 0 = \frac{\partial L}{\partial y} = 2y + \lambda(4x + 12y), \quad 0 = \frac{\partial L}{\partial \lambda} = 3x^2 + 4xy + 6y^2 - 140.$$

When the first two are solved for λ and the expressions equated, $\dfrac{-x}{3x + 2y} = \dfrac{-y}{2x + 6y}$, which simplifies to $0 = 2x^2 + 3xy - 2y^2 = (2x - y)(x + 2y)$. Thus, $y = 2x$ or $x = -2y$. These lead to the four points $(\pm 2, \pm 4)$ and $(\pm 2\sqrt{14}, \mp\sqrt{14})$. Since $D^2(\pm 2, \pm 4) = 20$ and $D^2(\pm 2\sqrt{14} \mp \sqrt{14}) = 70$, the ends of the major axis are $(\pm 2\sqrt{14}, \mp\sqrt{14})$, and the ends of the minor axis are $(\pm 2, \pm 4)$.

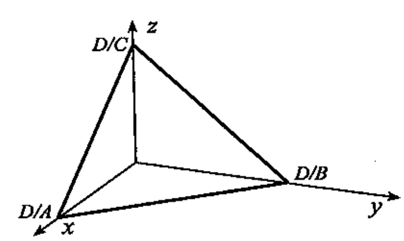
29. Critical points of the Lagrangian

$L(x, y, z, \lambda) = x^p y^q z^r + \lambda(Ax + By + Cz - D)$

are given by

$0 = \dfrac{\partial L}{\partial x} = px^{p-1} y^q z^r + \lambda A$, $0 = \dfrac{\partial L}{\partial y} = qx^p y^{q-1} z^r + \lambda B$,

$0 = \dfrac{\partial L}{\partial z} = rx^p y^q z^{r-1} + \lambda C$, $0 = \dfrac{\partial L}{\partial \lambda} = Ax + By + Cz - D$.



The only critical point has coordinates

$x = \dfrac{pD}{A(p + q + r)}$, $y = \dfrac{qD}{B(p + q + r)}$, $z = \dfrac{rD}{C(p + q + r)}$.

Since $f(x, y, z) = x^p y^q z^r$ vanishes along the edges of that part of the plane in the first octant, and is positive otherwise, it follows that this critical point must maximize the function.

30. (a) The distance D from the origin to any point (x, y) on the folium is given by

$$D^2(t) = x^2 + y^2 = \frac{9a^2t^2}{(1+t^3)^2} + \frac{9a^2t^4}{(1+t^3)^2} = \frac{9a^2(t^2+t^4)}{(1+t^3)^2}, \quad 0 \le t < \infty.$$

For critical points we solve

$$\begin{aligned} 0 = \frac{dD^2}{dt} &= 9a^2\left[\frac{(1+t^3)^2(2t+4t^3) - (t^2+t^4)2(1+t^3)(3t^2)}{(1+t^3)^4}\right] \\ &= \frac{9a^2}{(1+t^3)^3}(2t+4t^3+2t^4+4t^6-6t^4-6t^6) \\ &= \frac{-18a^2t(t-1)(t^4+t^3+3t^2+t+1)}{(1+t^3)^3}. \end{aligned}$$

Since the quartic polynomial has no positive solutions, the only critical points are $t = 0$ and $t = 1$. Since

$$D^2(0) = 0, \quad D^2(1) = \frac{9a^2}{2}, \quad \lim_{t\to\infty} D^2 = 0,$$

it follows that distance is maximized at the point $(3a/2, 3a/2)$.

(b) An implicit definition of the curve is

$$\begin{aligned} x^3 + y^3 &= \frac{27a^3t^3}{(1+t^3)^3} + \frac{27a^3t^6}{(1+t^3)^3} = \frac{27a^3t^3(1+t^3)}{(1+t^3)^3} \\ &= \frac{27a^3t^3}{(1+t^3)^2} = 3a\left(\frac{3at}{1+t^3}\right)\left(\frac{3at^2}{1+t^3}\right) = 3axy. \end{aligned}$$

To maximize $D^2 = x^2 + y^2$ subject to $x^3 + y^3 = 3axy$, we define the Lagrangian

$$L(x, y, \lambda) = x^2 + y^2 + \lambda(x^3 + y^3 - 3axy).$$

For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda(3x^2 - 3ay), \quad 0 = \frac{\partial L}{\partial y} = 2y + \lambda(3y^2 - 3ax), \quad 0 = \frac{\partial L}{\partial \lambda} = x^3 + y^3 - 3axy.$$

When x times the second equation is subtracted from y times the first,

$$\begin{aligned} 0 &= \lambda(3x^2y - 3xy^2 - 3ay^2 + 3ax^2) = 3\lambda[xy(x-y) + a(x-y)(x+y)] \\ &= 3\lambda(x-y)[xy + a(x+y)]. \end{aligned}$$

Thus, $y = x$ or $xy + a(x+y) = 0$. The second equation cannot be satisfied for positive x and y. The only critical point is obtained from $x^3 + x^3 - 3ax^2 = 0$. The solution is $x = 3a/2$. Since the curve is **closed**, the maximum value of D^2 must be at $(3a/2, 3a/2)$.

31. (a) We define the Lagrangian $L(x,y,z,\lambda,\mu) = x^2+y^2+z^2+\lambda(x^2-xy+y^2-z^2-1)+\mu(x^2+y^2-1)$. Its critical points are given by

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda(2x-y) + 2\mu x, \quad 0 = \frac{\partial L}{\partial y} = 2y + \lambda(-x+2y) + 2\mu y,$$

$$0 = \frac{\partial L}{\partial z} = 2z - 2\lambda z, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 - xy + y^2 - z^2 - 1, \quad 0 = \frac{\partial L}{\partial \mu} = x^2 + y^2 - 1.$$

Solutions of these equations are $(0,\pm 1,0)$, $(\pm 1,0,0)$, $(1/\sqrt{2},-1/\sqrt{2},\pm 1/\sqrt{2})$, and $(-1/\sqrt{2},1/\sqrt{2},\pm 1/\sqrt{2})$. Because the curve is closed (actually two closed curves), we evaluate

$$f(0,\pm 1,0) = 1, \; f(\pm 1,0,0) = 1, \; f(1/\sqrt{2},-1/\sqrt{2},\pm 1/\sqrt{2}) = 3/2, \; f(-1/\sqrt{2},1/\sqrt{2},\pm 1/\sqrt{2}) = 3/2,$$

and conclude that $(\pm 1,0,0)$ and $(0,\pm 1,0)$ are all equally close to the origin.
(b) By subtracting one constraint from the other, we obtain $z^2 = -xy$, so that we can write

$$f(x,y,z) = u(x,y) = x^2 + y^2 - xy = 1 - xy,$$

subject to $x^2+y^2 = 1$. Because x and y must have opposite signs ($z^2 = -xy$), we consider only those points on $x^2+y^2=1$ in the second and fourth quadrants. Critical points of the Lagrangian $L(x,y,\lambda) = 1 - xy + \lambda(x^2+y^2-1)$ are given by

$$0 = \frac{\partial L}{\partial x} = -y + 2\lambda x, \quad 0 = \frac{\partial L}{\partial y} = -x + 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1.$$

The only acceptable solutions (x,y) of these equations are $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$. We now evaluate $u(x,y)$ at these points and the ends of the two arcs of the circle:

$$u(\pm 1/\sqrt{2}, \mp 1/\sqrt{2}) = 3/2, \quad u(\pm 1, 0) = u(0,\pm 1) = 1,$$

and arrive at the same conclusion as in part (a).
(c) Since $y = \begin{cases} -\sqrt{1-x^2}, & x > 0 \\ \sqrt{1-x^2}, & x < 0 \end{cases}$, we may write

$$f(x,y,z) = u(x) = 1 - x(\mp\sqrt{1-x^2}) = 1 \pm x\sqrt{1-x^2},$$

(the positive being chosen when $0 < x \le 1$ and the negative when $-1 \le x < 0$). For critical points of these functions we solve

$$0 = u'(x) = \pm\sqrt{1-x^2} \mp \frac{x^2}{\sqrt{1-x^2}} = \frac{\pm(1-2x^2)}{\sqrt{1-x^2}}.$$

The critical points are $x = \pm 1/\sqrt{2}$. When we evaluate $u(x)$ at these points and the ends of the two arcs of the circle,

$$u(1/\sqrt{2}) = 3/2 = u(-1/\sqrt{2}), \quad u(-1) = u(0) = u(1) = 1,$$

and the same conclusion is in parts (a) and (b) is obtained.
(d) If we write $x = \cos t$ and $y = \sin t$, then

$$f(x,y,z) = u(t) = 1 - \cos t \sin t = 1 - \frac{1}{2}\sin 2t,$$

and this function must be minimized on $-\pi/2 \le t \le 0$, $\pi/2 \le t \le \pi$. For critical points we solve $0 = u'(t) = -\cos 2t$, and obtain $t = -\pi/4$ and $t = 3\pi/4$. When we evaluate $u(t)$ at these points and the ends of the two intervals, we find

$$u(-\pi/4) = 3/2 = u(3\pi/4), \quad u(-\pi/2) = u(0) = u(\pi/2) = u(\pi) = 1.$$

Once again the same points $(0,\pm 1,0)$ and $\pm 1,0,0)$ are obtained.

32. We must maximize and minimize the function $D^2 = x^2 + y^2 + z^2$ subject to the constraints $x^2 + y^2/4 + z^2/9 = 1$ and $x + y + z = 0$. We define the the Lagrangian

$$L(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 1\right) + \mu(x + y + z).$$

For critical points, we solve

$$0 = \frac{\partial L}{\partial x} = 2x + 2\lambda x + \mu, \quad 0 = \frac{\partial L}{\partial y} = 2y + \frac{\lambda y}{2} + \mu, \quad 0 = \frac{\partial L}{\partial z} = 2z + \frac{2\lambda z}{9} + \mu,$$

$$0 = \frac{\partial L}{\partial \lambda} = x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 1, \quad 0 = \frac{\partial L}{\partial \mu} = x + y + z.$$

The second equation subtracted from the first implies that $y = 4x(\lambda + 1)/(4 + \lambda)$. The third subtracted from the first gives $z = 9x(\lambda + 1)/(9 + \lambda)$. When these are substituted into the last equation

$$\begin{aligned} 0 &= x + \frac{4x(\lambda+1)}{4+\lambda} + \frac{9x(\lambda+1)}{9+\lambda} \\ &= x\left[\frac{(4+\lambda)(9+\lambda) + 4(\lambda+1)(9+\lambda) + 9(\lambda+1)(4+\lambda)}{(4+\lambda)(9+\lambda)}\right]. \end{aligned}$$

Since $x \neq 0$ (else $y = z = 0$), we set $0 = 14\lambda^2 + 98\lambda + 108 = 2(7\lambda^2 + 49\lambda + 54)$. Solutions are $\lambda = (-49 \pm \sqrt{889})/14$. Substitution of these results into the ellipsoid constraint gives

$$\begin{aligned} 1 &= x^2 + \frac{1}{4}\left[\frac{4x(\lambda+1)}{4+\lambda}\right]^2 + \frac{1}{9}\left[\frac{9x(\lambda+1)}{9+\lambda}\right]^2 \\ &= x^2 + \frac{4x^2(\lambda+1)^2}{(4+\lambda)^2} + \frac{9x^2(\lambda+1)^2}{(9+\lambda)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} x^2 &= \frac{(4+\lambda)^2(9+\lambda)^2}{(4+\lambda)^2(9+\lambda)^2 + 4(\lambda+1)^2(9+\lambda)^2 + 9(\lambda+1)^2(4+\lambda)^2}, \\ y^2 &= \frac{16(1+\lambda)^2(9+\lambda)^2}{(4+\lambda)^2(9+\lambda)^2 + 4(\lambda+1)^2(9+\lambda)^2 + 9(\lambda+1)^2(4+\lambda)^2}, \\ z^2 &= \frac{81(1+\lambda)^2(4+\lambda)^2}{(4+\lambda)^2(9+\lambda)^2 + 4(\lambda+1)^2(9+\lambda)^2 + 9(\lambda+1)^2(4+\lambda)^2}. \end{aligned}$$

For $\lambda = (-49 + \sqrt{889})/14$, we obtain $D = \sqrt{x^2 + y^2 + z^2} = 1.171$, and for $\lambda = (-49 - \sqrt{889})/14$, $D = 2.373$.

33. Geometrically, the constraint represents a cylinder in the z-direction, and therfore z is arbitrary. Because z appears in only the denominator, we should choose $z = 0$ in order to maximize the function. In other words, the maximum value of $f(x, y, z)$ is the maximum value of $F(x, y) = xy + x^2$ subject $x^2(4 - x^2) = y^2$. Critical points of the Lagrangian $L(x, y, \lambda) = xy + x^2 + \lambda(4x^2 - x^4 - y^2)$ are given by

$$0 = \frac{\partial L}{\partial x} = y + 2x + \lambda(8x - 4x^3), \quad 0 = \frac{\partial L}{\partial y} = x - 2\lambda y, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2(4 - x^2) - y^2.$$

The solutions of these equations are $(0, 0)$, $(\pm 1.80158, \pm 1.56470)$ and $(\pm 1.28188, \mp 1.96972)$. Since the curve is closed, we evaluate

$$F(0, 0) = 0, \; F(\pm 1.80158, \pm 1.56470) = 6.06, \; F(\pm 1.28188, \mp 1.96972) = -0.88,$$

and conclude that the maximum value of $f(x, y, z)$ is 6.06.

34. For the solution without Lagrange multipliers we express $f(x,y,z)$ as a function of one variable. We solve the constraint equations for $x^2 = 1 - 3y^2$ and $y^2 = (1-z)/2$, and substitute into $f(x,y,z)$:

$$\begin{aligned} f(x,y,z) &= 2(1-3y^2)y^2 + 2y^2z^2 + 3z = 2\left(1 - \frac{3}{2} + \frac{3z}{2}\right)\left(\frac{1}{2} - \frac{z}{2}\right) + 2z^2\left(\frac{1}{2} - \frac{z}{2}\right) + 3z \\ &= \frac{1}{2}(-2z^3 - z^2 + 10z - 1). \end{aligned}$$

Since $z = x^2 + y^2 = (1-3y^2) + y^2 = 1 - 2y^2$, and y must be restricted to $|y| \le 1/\sqrt{3}$, it follows that the only possible values for z are $1/3 \le z \le 1$. Thus, finding maximimum and minimum values of $f(x,y,z)$ subject to the two constraints is equivalent to finding maximimum and minimum values of

$$F(z) = \frac{1}{2}(-2z^3 - z^2 + 10z - 1), \quad \frac{1}{3} \le z \le 1.$$

For critical points of $F(z)$, we solve $0 = F'(z) = (1/2)(-6z^2 - 2z + 10) \implies z = (-1 \pm \sqrt{61})/6$. These must be rejected as not lying in the required interval, and maximum and minimum values of $F(z)$ must therefore occur at $z = 1/3$ and $z = 1$. Since $F(1/3) = 29/27$ and $F(1) = 3$, these are minimum and maximum values of $F(z)$.

To use Lagrange multipliers, we define the Lagrangian

$$L(x,y,z,\lambda,\mu) = 2x^2y^2 + 2y^2z^2 + 3z + \lambda(x^2 + y^2 - z) + \mu(x^2 + 3y^2 - 1).$$

For critical points we solve

$$0 = \frac{\partial L}{\partial x} = 4xy^2 + 2\lambda x + 2\mu x, \quad 0 = \frac{\partial L}{\partial y} = 4x^2y + 4yz^2 + 2\lambda y + 6\mu y,$$

$$0 = \frac{\partial L}{\partial z} = 4y^2z + 3 - \lambda, \quad 0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 - z, \quad 0 = \frac{\partial L}{\partial \mu} = x^2 + 3y^2 - 1.$$

If we choose $x = 0$ to satisfy the first equation, then the remaining equations imply that $y = \pm 1/\sqrt{3}$ and $z = 1/3$. If we choose $y = 0$ to satisfy the second equation, the remaining equations require $x = \pm 1$ and $z = 1$. The only other way to satisfy the first two equations is to set

$$2y^2 + \lambda + \mu = 0, \quad 2x^2 + 2z^2 + \lambda + 3\mu = 0.$$

If we multiply the first by three and subtract the second, we obtain

$$\begin{aligned} 0 &= 6y^2 - 2x^2 - 2z^2 + 2\lambda = 6y^2 - 2x^2 - 2(x^2 + y^2)^2 + 2\lambda \\ &= 6y^2 - 2x^2 - 2(x^4 + 2x^2y^2 + y^4) + 2\lambda \\ &= 6y^2 - 2(1-3y^2) - 2(1-3y^2)^2 - 4y^2(1-3y^2) - 2y^4 + 2\lambda \\ &= 2(\lambda - 2 + 10y^2 - 4y^4), \end{aligned}$$

But from the equation for $\partial L/\partial z$, we can also write

$$0 = 3 - \lambda + 4y^2(x^2 + y^2) = 3 - \lambda + 4y^2(1 - 3y^2 + y^2) = 3 - \lambda + 4y^2 - 8y^4.$$

These two equations in y and λ imply that

$$2 - 10y^2 + 4y^4 = 3 + 4y^2 - 8y^4 \quad \Longrightarrow \quad 12y^4 - 14y^2 - 1 = 0 \quad \Longrightarrow \quad y^2 = \frac{7 \pm \sqrt{61}}{12}.$$

We reject the negative solution. But substitution of this result into the constraint $x^2 + 3y^2 = 1$ requires x^2 to be negative. Consequently, only four critical points (x,y,z) are obtained: $(0, \pm 1/\sqrt{3}, 1/3)$ and $(\pm 1, 0, 1)$. Since the curve defined by the constraints is closed, we evaluate

$$f(0, \pm 1/\sqrt{3}, 1/3) = 29/27, \quad f(\pm 1, 0, 1) = 3.$$

These are minimum and maximum values.

EXERCISES 12.13

1. Least squares estimates for parameters a and b in a linear function $S = aM + b$ must satisfy equations similar to 12.71. For the tabular values these equations are

$$160\,080a + 1260b = 109\,624, \qquad 1260a + 10b = 856.$$

The solution is $a = 1.3394$ and $b = -83.164$.

2. Least squares estimates for parameters a and b in a linear function $P = aA + b$ must satisfy equations similar to 12.71. For the tabular values these equations are

$$1482a + 130b = 13\,663, \qquad 130a + 13b = 1315.$$

The solution is $a = 2.8187$ and $b = 72.967$.

3. Least squares estimates for a and b must satisfy the equations

$$\left(\sum_{i=1}^{11} t_i^2\right) a + \left(\sum_{i=1}^{11} t_i\right) b = \sum_{i=1}^{11} t_i\overline{S}_i, \qquad \left(\sum_{i=1}^{11} t_i\right) a + 11b = \sum_{i=1}^{11} \overline{S}_i,$$

where $\overline{S}_i$ are the values in the table.
(a) Using $t = 0$ in year zero, we obtain

$$41\,870\,521a + 21\,461b = 1\,843\,532.5, \qquad 21\,461a + 11b = 944.7,$$

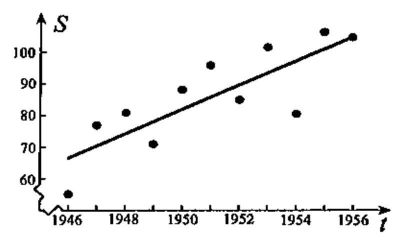
the solution of which is $a = 3.8436$ and $b = -7413.05$. The least-squares line is therefore $S = 3.8436t - 7413.05$.
(b) Using $t = 0$ in year 1946, the equations defining a and b are

$$385a + 55b = 5146.3, \quad 55a + 11b = 944.7,$$

the solution of which is $a = 3.8436$ and $b = 66.664$. The least squares line is therefore $S = 3.8436t + 66.664$. If we replace t by $t - 1946$, where t is now the actual year,

$$S(t) = 3.8436(t - 1946) + 66.664 = 3.8436t - 7412.98.$$

A plot of curve and data points is shown to the right.

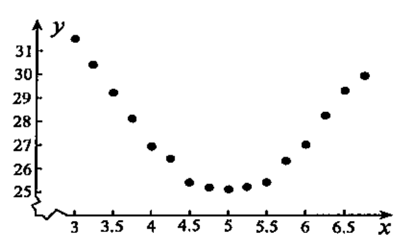


4. (a) The plot is to the right.
(b) For critical points of $S(a, b, c)$ we solve

$$0 = \frac{\partial S}{\partial a} = \sum_{i=1}^{16} 2(ax_i{}^2 + bx_i + c - \overline{y_i})(x_i{}^2),$$

$$0 = \frac{\partial S}{\partial b} = \sum_{i=1}^{16} 2(ax_i{}^2 + bx_i + c - \overline{y_i})(x_i),$$

$$0 = \frac{\partial S}{\partial c} = \sum_{i=1}^{16} 2(ax_i{}^2 + bx_i + c - \overline{y_i}).$$



These can be rewritten in the form

$$\left(\sum_{i=1}^{16} x_i{}^4\right) a + \left(\sum_{i=1}^{16} x_i{}^3\right) b + \left(\sum_{i=1}^{16} x_i{}^2\right) c = \sum_{i=1}^{16} x_i{}^2\overline{y_i},$$

$$\left(\sum_{i=1}^{16} x_i{}^3\right) a + \left(\sum_{i=1}^{16} x_i{}^2\right) b + \left(\sum_{i=1}^{16} x_i\right) c = \sum_{i=1}^{16} x_i\overline{y_i},$$

$$\left(\sum_{i=1}^{16} x_i{}^2\right) a + \left(\sum_{i=1}^{16} x_i\right) b + 16c = \sum_{i=1}^{16} \overline{y_i}.$$

(c) From the tabular values,

$$12\,117.5a + 2164.5b + 401.5c = 10\,979.4,\ 2164.5a + 401.5b + 78c = 2133.47,\ 401.5a + 78b + 16c = 439.4.$$

The solution is $a = 1.6653$, $b = -16.642$, and $c = 66.802$.

5. (a) Using the equations in Exercise 4, least squares estimates for parameters a, b, and c of the quadratic function $y = ax^2 + bx + c$ must satisfy

$$1442.9a + 410.688b + 121.04c = 5613.13, \quad 410.688a + 121.04b + 37.2c = 1559.86,$$

$$121.04a + 37.2b + 12c = 445.69.$$

The solution is $a = 5.9226$, $b = -5.5627$, and $c = -5.3543$.
(b) The value of $S(5.9226, -5.5627, -5.3543)$ is 1.5275.

6. The sum of the squares of the differences between observed and predicted values is

$$S = S(a,b) = \sum_{i=1}^{8} (a + bQ_i^2 - \overline{H}_i)^2,$$

where $(Q_i, \overline{H}_i)$ are the points in the table. For critical points of S, we solve

$$0 = \frac{\partial S}{\partial b} = \sum_{i=1}^{8} 2(a + bQ_i^2 - \overline{H}_i)(Q_i^2), \qquad 0 = \frac{\partial S}{\partial a} = \sum_{i=1}^{8} 2(a + bQ_i^2 - \overline{H}_i).$$

These can rewritten in the form

$$\left(\sum_{i=1}^{8} Q_i^4\right) b + \left(\sum_{i=1}^{8} Q_i^2\right) a = \sum_{i=1}^{n} Q_i^2 \overline{H}_i, \qquad \left(\sum_{i=1}^{8} Q_i^2\right) b + 8a = \sum_{i=1}^{8} \overline{H}_i.$$

From the table, these become

$$2.19185 \times 10^8 b + 34728.8a = 632647, \qquad 34728.8b + 8a = 197.2.$$

The solution is $a = 38.82$ and $b = 0.003265$; that is, the least squares quadratic is $H = 38.82 - 0.003265Q^2$.

7. By analogy with Exercise 4, least squares estimates for parameters a, b, c, and d must satisfy

$$\left(\sum_{i=1}^{12} {x_i}^6\right) a + \left(\sum_{i=1}^{12} {x_i}^5\right) b + \left(\sum_{i=1}^{12} {x_i}^4\right) c + \left(\sum_{i=1}^{12} {x_i}^3\right) d = \sum_{i=1}^{12} {x_i}^3 \overline{y_i},$$

$$\left(\sum_{i=1}^{12} {x_i}^5\right) a + \left(\sum_{i=1}^{12} {x_i}^4\right) b + \left(\sum_{i=1}^{12} {x_i}^3\right) c + \left(\sum_{i=1}^{12} {x_i}^2\right) d = \sum_{i=1}^{12} {x_i}^2 \overline{y_i},$$

$$\left(\sum_{i=1}^{12} {x_i}^4\right) a + \left(\sum_{i=1}^{12} {x_i}^3\right) b + \left(\sum_{i=1}^{12} {x_i}^2\right) c + \left(\sum_{i=1}^{12} x_i\right) d = \sum_{i=1}^{12} x_i \overline{y_i},$$

$$\left(\sum_{i=1}^{12} {x_i}^3\right) a + \left(\sum_{i=1}^{12} {x_i}^2\right) b + \left(\sum_{i=1}^{12} x_i\right) c + 12d = \sum_{i=1}^{12} \overline{y_i}.$$

From the tabular values,

$$19\,279.6a + 5214.9b + 1442.9c + 410.688d = 20\,660.5,\ 5214.9a + 1442.9b + 410.688c + 121.04d = 5613.13,$$

$$1442.9a + 410.688b + 121.04c + 37.2d = 1559.86,\ 410.688a + 121.04b + 37.2c + 12d = 445.69.$$

The solution is $a = -0.007\,693\,4$, $b = 5.9943$, $c = -5.7787$, and $d = -5.1442$.
(b) The value of $S(-0.007\,693\,4, 5.9943, -5.7787, -5.1442)$ is 1.5289. The fact that this is essentially the same as that in Exercise 5 indicates that the parabola fits the data as well as the cubic.

8. (a) When we take logarithms of y-values in the given table, we obtain

x	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
y	4.94	5.19	5.44	5.67	5.90	6.12	6.34	6.51	6.67	6.91	7.11

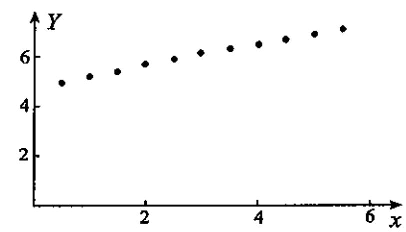
The plot of these to the right indicates that a straight line fit is indeed reasonable.
(b) Equations for a and B corresponding to 12.71 are

$$\left(\sum_{i=1}^{11} x_i^2\right) a + \left(\sum_{i=1}^{11} x_i\right) B = \sum_{i=1}^{11} x_i \overline{Y}_i,$$

$$\left(\sum_{i=1}^{11} x_i\right) a + (11)b = \sum_{i=1}^{11} \overline{Y}_i,$$

from which

$$126.5a + 33B = 212.149, \quad 33a + 11B = 66.795.$$



The solution of these is $a = 0.43$ and $B = 4.79$. Consequently, the least-squares estimates give

$$\ln y = Y = 4.79 + 0.43x.$$

When we take exponentials,

$$y = e^{4.79+0.43x} = 120.3e^{0.43x}.$$

9. The points in the plot of $\ln W$ against $\ln F$ in the left figure below are reasonably collinear so that

$$W = aF^b \quad \Longrightarrow \quad \ln W = \ln a + b \ln F$$

is an acceptable functional representation for $W(F)$. If we set $w = \ln W$, $A = \ln a$, and $f = \ln F$, then $w = A + bf$. Least squares estimates for b and A are defined by

$$\left(\sum_{i=1}^{10} f_i^2\right) b + \left(\sum_{i=1}^{10} f_i\right) A = \sum_{i=1}^{10} f_i \overline{w}_i, \qquad \left(\sum_{i=1}^{10} f_i\right) b + 10A = \sum_{i=1}^{10} \overline{w}_i.$$

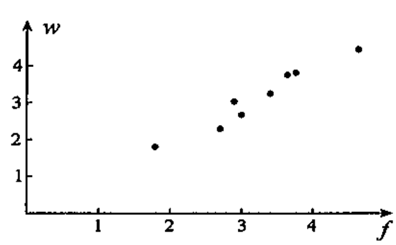
These become

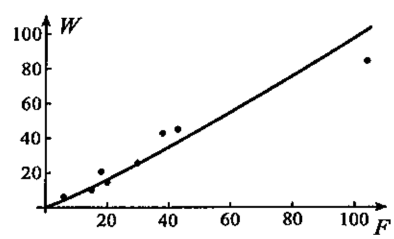
$$99.9544b + 30.5124A = 94.1115, \quad 30.1524b + 10A = 28.3150,$$

the solution of which is $b = 1.126$ and $A = -0.6034$. Thus,

$$w = -0.6034 + 1.126f \quad \Longrightarrow \quad \ln W = -0.6034 + 1.126 \ln F \quad \Longrightarrow \quad W = 0.547F^{1.126}.$$

A plot is shown in the right figure along with the original data points.





10. To find $y = f(v) = a - bv$, the number of kilometres per litre for a truck travelling at speed v, we use least squares for the 18 data points in the table. Equations for a and b are

$$-\left(\sum_{i=1}^{18} v_i^2\right) b + \left(\sum_{i=1}^{18} v_i\right) a = \sum_{i=1}^{18} v_i \overline{y}_i, \qquad -\left(\sum_{i=1}^{18} v_i\right) b + 18a = \sum_{i=1}^{18} \overline{y}_i.$$

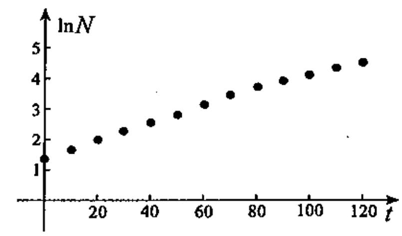
These become

$$-147\,000b + 1620a = 3379, \quad -1620b + 18a = 37.71.$$

The solution is $a = 3.2125$ and $b = 0.0124167$. We can now substitute these into the formula $v = a/(b + \sqrt{bp/w})$ in Exercise 59 of Section 4.7,

$$v = \frac{3.2125}{0.0124167 + \sqrt{\dfrac{0.0124167(0.6)}{20}}} = 101.3 \text{ kilometres per hour.}$$

11. If N denotes the population (in millions) and t is the year (taking $t = 0$ in 1790), then $N(t)$ can be approximated by an exponential if $\ln N$ can be aproximated by a straight line. The plot indicates that this is indeed the case, and we therefore set $N(t) = be^{at} \implies \ln N = at + b$. If we set $Y = \ln N$, then $Y = at + b$. Least squares estimates of a and b are then defined by equations similar to 12.71 with x replaced by t. They are



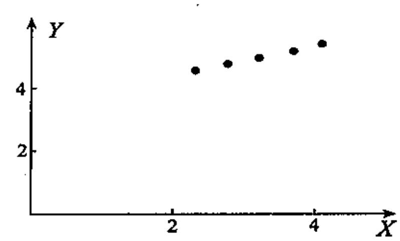
$$65\,000a + 780b = 2878.4809, \quad 780a + 13b = 39.845\,743.$$

The solution is $a = 0.026\,798\,699$ and $b = 1.457\,135\,2$. Consequently,

$$Y = \ln N = 0.026\,798\,699t + 1.457\,135\,2 \implies N = 4.293\,641\,5\,e^{0.026\,798\,699t}.$$

For $t = 0$ in year zero, we set $N = 4.293\,641\,5\,e^{0.026\,798\,699(t-1790)} = 6.308 \times 10^{-21}e^{0.0268t}$.

12. (a) The data points of Y against X are reasonably collinear.
(b) If a line $Y = aX + B$ is to fit the points in the plot, then a and B must satisfy equations similar to 12.71 where B replaces b. They are

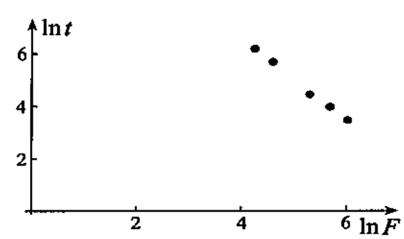


$$53.721\,796a + 16.077\,273B = 81.047\,068,$$
$$16.077\,273a + 5B = 24.904\,540.$$

The solution is $a = 0.477\,603\,27$ and $B = 3.445\,196\,4$.

Thus, $Y = \ln y = 0.477\,603\,27X + 3.445\,196\,4 = 0.477\,603\,27\ln x + 3.445\,196\,4$, or, $y = 31.35x^{0.4776}$.

13. The points in the plot of $\ln F$ against $\ln t$ are reasonably collinear so that $t = aF^b \implies \ln t = \ln a + b\ln F$ is indeed an acceptable functional representation.
If we set $T = \ln t$, $A = \ln a$, and $f = \ln F$, then $T = bf + A$. Least squares estimates for b and A are defined by



$$\left(\sum_{i=1}^{5} f_i^2\right) b + \left(\sum_{i=1}^{5} f_i\right) A = \sum_{i=1}^{5} f_i\overline{T}_i,$$
$$\left(\sum_{i=1}^{5} f_i\right) b + 5A = \sum_{i=1}^{5} \overline{T}_i.$$

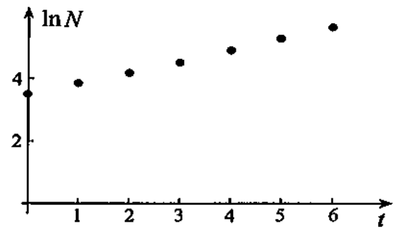
These become

$$135.760b + 25.847A = 118.9965, \qquad 25.847b + 5A = 23.6635,$$

the solution of which is $b = -1.551$ and $A = 12.75$. Thus, $T = -1.551f + 12.75$, and when we substitute $T = \ln t$ and $f = \ln F$,

$$\ln t = -1.551\ln F + 12.75 \implies t = \frac{e^{12.75}}{F^{1.551}} = \frac{3.45 \times 10^5}{F^{1.551}}.$$

14. (a) A plot is shown to the right.
(b) If we take logarithms of $N = be^{at}$, $\ln N = at + \ln b$, and define $n = \ln N$ and $B = \ln b$, then, $n = at + B$. Least-squares estimates for a and B are given by



$$\left(\sum_{i=1}^{7} t_i^2\right) a + \left(\sum_{i=1}^{7} t_i\right) B = \sum_{i=1}^{7} t_i \overline{n}_i,$$

$$\left(\sum_{i=1}^{7} t_i\right) a + 7B = \sum_{i=1}^{7} \overline{n}_i.$$

These become

$$91a + 21B = 105.2312, \qquad 21a + 7B = 31.7587,$$

the solution of which is $a = 0.35554$ and $B = 3.47034$. Therefore,

$$n = 0.35554t + 3.47034 \Longrightarrow \ln N = 0.35554t + 3.47034 \Longrightarrow N = e^{0.355554t+3.47034} = 32.1476e^{0.35554t}.$$

15. If we take logarithms of $PV^a = b$, we obtain

$$\ln P + a \ln V = \ln b \quad \Longrightarrow \quad \ln P = -a \ln V + \ln b.$$

If we set $p = \ln P$, $v = \ln V$, $A = -a$, and $B = \ln b$, then $p = Av + B$. Least-squares estimates of A and B must satisfy the equations

$$\left(\sum_{i=1}^{6} v_i^2\right) A + \left(\sum_{i=1}^{6} v_i\right) B = \sum_{i=1}^{6} v_i \overline{p}_i, \qquad \left(\sum_{i=1}^{6} v_i\right) A + 6B = \sum_{i=1}^{6} \overline{p}_i.$$

These become

$$121.9758A + 26.9295B = 89.3605, \qquad 26.9295A + 6B = 20.2570,$$

the solution of which is $A = -1.4043$ and $B = 9.6788$. Consequently,

$$p = -1.4043v + 9.6788 \Longrightarrow \ln P = -1.4043 \ln V + 9.6788 \Longrightarrow P = e^{-1.4043 \ln V + 9.6788} = 15975V^{-1.4043}.$$

16. (a) To find a least-squares estimate for b, we form the sum of squares $S(b) = \sum_{i=1}^{n} \left(\overline{y_i} - \dfrac{b}{{x_i}^2}\right)^2$. For critical points of $S(b)$, we solve

$$0 = \frac{dS}{db} = \sum_{i=1}^{n} 2\left(\overline{y_i} - \frac{b}{{x_i}^2}\right)\left(-\frac{1}{{x_i}^2}\right) = \sum_{i=1}^{n} \frac{b}{{x_i}^4} - \sum_{i=1}^{n} \frac{\overline{y_i}}{{x_i}^2}.$$

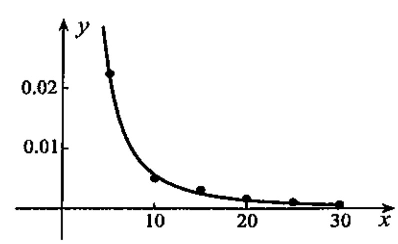
Thus, $b = \left(\sum_{i=1}^{n} \dfrac{\overline{y_i}}{{x_i}^2}\right) \Bigg/ \left(\sum_{i=1}^{n} \dfrac{1}{{x_i}^4}\right)$.

(b) For the data in the table,

$$b = \frac{9.669\,874 \times 10^{-4}}{1.729\,797\,7 \times 10^{-3}} = 0.559\,017\,65.$$

The least squares approximation for the curve is therefore $y = \dfrac{0.5590}{x^2}$.

(c) The curve and data are plotted to the right.



17. (a) If we set $Y = e^y = ax + b$, then least-squares estimates for a and b are defined by equations similar to 12.71,

$$139a + 27b = 352.831\,52, \quad 27a + 6b = 70.874\,602.$$

The solution is $a = 1.936\,903\,5$ and $b = 3.096\,368\,0$. Thus,

$$Y = e^y = 1.936\,903\,5x + 3.096\,368\,0 \quad \Longrightarrow \quad y = \ln(3.0964 + 1.9369x).$$

(b) If we attempt to do so, we define $S(a,b) = \sum_{i=1}^{n} [\overline{y_i} - \ln(b + ax_i)]^2$. For critical points of S, we then solve

$$0 = \frac{\partial S}{\partial a} = \sum_{i=1}^{n} 2[\overline{y_i} - \ln(b + ax_i)]\left(\frac{-x_i}{b + ax_i}\right), \quad 0 = \frac{\partial S}{\partial b} = \sum_{i=1}^{n} 2[\overline{y_i} - \ln(b + ax_i)]\left(\frac{-1}{b + ax_i}\right).$$

Unfortunately, we cannot solve these equations for a and b.

18. If we set $Y = 1/y = ax + b$, then least-squares estimates for a and b are given by equations similar to 12.71,

$$255a + 35b = 28.285\,127, \quad 35a + 5b = 3.979\,676\,5.$$

The solution is $a = 0.042\,739\,2$ and $b = 0.496\,761\,3$, and therefore

$$Y = \frac{1}{y} = 0.042\,739\,2x + 0.496\,761\,3 \quad \Longrightarrow \quad y = \frac{1}{0.04274x + 0.4968}.$$

19. If we take logarithms of $P/y = k(x/y)^q$, we obtain $\ln(P/y) = \ln k + q\ln(x/y)$. We now set $R = \ln(P/y)$, $K = \ln k$, and $Z = \ln(x/y)$, then $R = K + qZ$. Least-squares estimates for K and q must satisfy

$$\left(\sum_{i=1}^{7} Z_i^2\right) q + \left(\sum_{i=1}^{7} Z_i\right) K = \sum_{i=1}^{7} Z_i\overline{R}_i, \qquad \left(\sum_{i=1}^{7} Z_i\right) q + 7K = \sum_{i=1}^{7} \overline{R}_i.$$

These become

$$147.95q - 32.169K = 81.148, \qquad -32.169q + 7K = -17.643,$$

with solution $q = 0.59388$ and $K = 0.20878$. Consequently, $R = 0.20878 + 0.59388Z$, from which

$$\ln\left(\frac{P}{y}\right) = 0.20878 + 0.59388\ln\left(\frac{x}{y}\right) \Longrightarrow \frac{P}{y} = e^{0.20878}\left(\frac{x}{y}\right)^{0.59388} \Longrightarrow P = 1.232x^{0.59388}y^{0.40612}.$$

EXERCISES 12.14

1. If we set $z = f(x,y)$, then $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 2xy\,dx + (x^2 - \cos y)\,dy$.

2. If we set $z = f(x,y)$, then $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{y}{1 + x^2y^2}\,dx + \frac{x}{1 + x^2y^2}\,dy = \frac{1}{1 + x^2y^2}(y\,dx + x\,dy)$.

3. If we set $u = f(x,y,z)$, then $du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (yz - 3x^2e^z)\,dx + xz\,dy + (xy - x^3e^z)\,dz$.

4. If we set $u = f(x,y,z)$, then

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = [yz\cos(xyz) - 2xy^2z^2]\,dx + [xz\cos(xyz) - 2x^2yz^2]\,dy + [xy\cos(xyz) - 2x^2y^2z]\,dz.$$

5. If we set $u = f(x, y, z)$, then
$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \frac{2x}{x^2+y^2+z^2}dx + \frac{2y}{x^2+y^2+z^2}dy + \frac{2z}{x^2+y^2+z^2}dz = \frac{2}{x^2+y^2+z^2}(x\,dx + y\,dy + z\,dz).$$

6. If we set $z = f(x, y)$, then
$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{y}{\sqrt{1-x^2y^2}}\,dx + \frac{x}{\sqrt{1-x^2y^2}}\,dy = \frac{1}{\sqrt{1-x^2y^2}}(y\,dx + x\,dy).$$

7. If we set $z = f(x, y)$, then
$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \left[\frac{1}{\sqrt{1-(x+y)^2}} - \frac{1}{\sqrt{1-(x+y)^2}}\right]dx + \left[\frac{1}{\sqrt{1-(x+y)^2}} - \frac{1}{\sqrt{1-(x+y)^2}}\right]dy = 0.$$

8. If we set $u = f(x, y, z, t)$, then
$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt = (y+t)\,dx + (x+z)\,dy + (y+t)\,dz + (z+x)\,dt.$$

9. If we set $u = f(x, y, z, w)$, then
$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial w}dw = y\tan(zw)\,dx + x\tan(zw)\,dy + xyw\sec^2(zw)\,dz + xyz\sec^2(zw)\,dw.$$

10. If we set $u = f(x, y, z, t)$, then
$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt = 2xe^{x^2+y^2+z^2-t^2}dx + 2ye^{x^2+y^2+z^2-t^2}dy + 2ze^{x^2+y^2+z^2-t^2}dz - 2te^{x^2+y^2+z^2-t^2}dt = 2e^{x^2+y^2+z^2-t^2}(x\,dx + y\,dy + z\,dz - t\,dt)$$

11. If $V(r, h) = (1/3)\pi r^2 h$, then $dV = (2/3)\pi rh\,dr + (1/3)\pi r^2\,dh$. When $r = 10$, $h = 20$, $dr = 0.1$, and $dh = -0.3$, then
$$dV = \left(\frac{2}{3}\right)\pi(10)(20)(0.1) + \left(\frac{1}{3}\right)\pi(100)(-0.3) = 10.47 \text{ cm}^3.$$
The actual change is $V(10.1, 19.7) - V(10, 20) = (1/3)\pi(10.1)^2(19.7) - (1/3)\pi(10)^2(20) = 10.05 \text{ cm}^3$.

12. $dV = \dfrac{\partial V}{\partial a}da + \dfrac{\partial V}{\partial b}db = \dfrac{4}{3}\pi b^2\,da + \dfrac{8}{3}\pi ab\,db = \dfrac{4}{3}\pi b(b\,da + 2a\,db)$

Since the percentage changes in a and b are 1%, it follows that $100\dfrac{da}{a} = 100\dfrac{db}{b} = 1$, and therefore the approximate percentage change in V is
$$100\frac{dV}{V} = 100\frac{(4/3)\pi b(b\,da + 2a\,db)}{4\pi ab^2/3} = 100\left(\frac{da}{a} + 2\frac{db}{b}\right) = 1 + 2 = 3.$$

EXERCISES 12.15

1. By multiplying the Maclaurin series of $F(x)$ and $G(y)$,
$$F(x)G(y) = (a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1y + b_2y^2 + \cdots) = a_0b_0 + (a_1b_0x + a_0b_1y) + (a_2b_0x^2 + a_1b_1xy + a_0b_2y^2) + \cdots,$$
we get a series of form 12.76 with $c = d = 0$. It must therefore be the Taylor series about $(0, 0)$.

2. By setting $v_x = x - c$ and $v_y = y - d$ in the expression for $F'''(0)$,

$$F'''(0) = f_{xxx}(c,d)(x-c)^3 + 3f_{xxy}(c,d)(x-c)^2(y-d) + 3f_{xyy}(c,d)(x-c)(y-d)^2 + f_{yyy}(c,d)(y-d)^3.$$

When this is substituted into $F'''(0)t^3/3!$ and t is set equal to 1, the cubic terms are

$$\frac{1}{3!}[f_{xxx}(c,d)(x-c)^3 + 3f_{xxy}(c,d)(x-c)^2(y-d) + 3f_{xyy}(c,d)(x-c)(y-d)^2 + f_{yyy}(c,d)(y-d)^3].$$

3. Since $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$, it follows that $\cos(xy) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}y^{2n}$.

4. $$e^{2x-3y} = e^5 e^{2(x-1)-3(y+1)} = e^5 \sum_{n=0}^{\infty} \frac{1}{n!}[2(x-1)-3(y+1)]^n$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{e^5}{n!}\binom{n}{r}2^r(x-1)^r(-3)^{n-r}(y+1)^{n-r} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{e^5(-1)^{n-r}2^r3^{n-r}}{(n-r)!r!}(x-1)^r(y+1)^{n-r}$$

5. If we expand $\sqrt{1+x}$ with the binomial expansion 10.33b,

$$\begin{aligned} x^2y\sqrt{1+x} &= x^2y\left[1 + \frac{x}{2} + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots\right] \\ &= y\left[x^2 + \frac{x^3}{2} - \frac{1}{2^2 2!}x^4 + \frac{1\cdot 3}{2^3 3!}x^5 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^6 + \cdots\right] \\ &= y\left[x^2 + \frac{x^3}{2} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1}[1\cdot 3\cdot 5\cdots(2n-7)]}{2^{n-2}(n-2)!}x^n\right], \end{aligned}$$

valid for all y and $-1 \le x \le 1$.

6. Since $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$,

$$\begin{aligned} \ln(1+x^2+y^2) &= x^2 + y^2 - \frac{(x^2+y^2)^2}{2} + \frac{(x^2+y^2)^3}{3} - \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x^2+y^2)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sum_{r=0}^{n}\binom{n}{r}x^{2r}y^{2(n-r)} \\ &= \sum_{n=1}^{\infty}\sum_{r=0}^{n} \frac{(-1)^{n+1}(n-1)!}{(n-r)!\,r!}x^{2r}y^{2(n-r)} \end{aligned}$$

7. Since the function is undefined at $(3,-4)$, there is no Taylor series for the function about this point.

8. $$\frac{xy^2}{1+y^2} = [(x+1)-1]\frac{y^2}{1+y^2} = [(x+1)-1]\sum_{n=0}^{\infty}(-1)^n y^{2n+2}$$

9. With $f(-1,1) = -\frac{1}{2}$,

$$\begin{aligned} f_x(-1,1) &= \frac{(x^2+y^2)(y) - xy(2x)}{(x^2+y^2)^2}\Big|_{(-1,1)} = \frac{y^3 - x^2y}{(x^2+y^2)^2}\Big|_{(-1,1)} = 0, \\ f_y(-1,1) &= \frac{(x^2+y^2)(x) - xy(2y)}{(x^2+y^2)^2}\Big|_{(-1,1)} = \frac{x^3 - xy^2}{(x^2+y^2)^2}\Big|_{(-1,1)} = 0, \\ f_{xx}(-1,1) &= \frac{(x^2+y^2)^2(-2xy) - (y^3-x^2y)(4x)(x^2+y^2)}{(x^2+y^2)^4}\Big|_{(-1,1)} = \frac{1}{2}, \\ f_{xy}(-1,1) &= \frac{(x^2+y^2)^2(3y^2-x^2) - (y^3-x^2y)(4y)(x^2+y^2)}{(x^2+y^2)^4}\Big|_{(-1,1)} = \frac{1}{2}, \\ f_{yy}(-1,1) &= \frac{(x^2+y^2)^2(-2xy) - (x^3-xy^2)(4y)(x^2+y^2)}{(x^2+y^2)^4}\Big|_{(-1,1)} = \frac{1}{2}, \end{aligned}$$

$$\frac{xy}{x^2+y^2} = -\frac{1}{2} + \frac{1}{2!}\left[\frac{1}{2}(x+1)^2 + (x+1)(y-1) + \frac{1}{2}(y-1)^2\right] + \cdots.$$

10. With $f(2,1)=\sqrt{1+2}=\sqrt{3}$,

$$f_x(2,1)=\frac{y}{2\sqrt{1+xy}}\Big|_{(2,1)}=\frac{1}{2\sqrt{3}},$$

$$f_y(2,1)=\frac{x}{2\sqrt{1+xy}}\Big|_{(2,1)}=\frac{1}{\sqrt{3}}, \qquad f_{xx}(2,1)=\frac{-y^2}{4(1+xy)^{3/2}}\Big|_{(2,1)}=-\frac{1}{12\sqrt{3}},$$

$$f_{xy}(2,1)=\left[\frac{1}{2\sqrt{1+xy}}-\frac{xy}{4(1+xy)^{3/2}}\right]_{|(2,1)}=\frac{1}{3\sqrt{3}}, \qquad f_{yy}(2,1)=\frac{-x^2}{4(1+xy)^{3/2}}\Big|_{(2,1)}=-\frac{1}{3\sqrt{3}},$$

$$\begin{aligned}\sqrt{1+xy}&=\sqrt{3}+\frac{1}{2\sqrt{3}}(x-2)+\frac{1}{\sqrt{3}}(y-1)\\&\quad+\frac{1}{2!}\left[-\frac{1}{12\sqrt{3}}(x-2)^2+\frac{2}{3\sqrt{3}}(x-2)(y-1)-\frac{1}{3\sqrt{3}}(y-1)^2\right]+\cdots\\&=\frac{1}{24\sqrt{3}}\left[72+12(x-2)+24(y-1)-(x-2)^2+8(x-2)(y-1)-4(y-1)^2\right]+\cdots.\end{aligned}$$

11. With $f(-1,0)=-e^{-1}\sin 3$,

$$\begin{aligned}f_x(-1,0)&=[e^x\sin(3x-y)+3e^x\cos(3x-y)]_{|(-1,0)}=e^{-1}(3\cos 3-\sin 3),\\f_y(-1,0)&=[-e^x\cos(3x-y)]_{|(-1,0)}=-e^{-1}\cos 3,\\f_{xx}(-1,0)&=[6e^x\cos(3x-y)-8e^x\sin(3x-y)]_{|(-1,0)}=e^{-1}(8\sin 3+6\cos 3),\\f_{xy}(-1,0)&=[3e^x\sin(3x-y)-3e^x\cos(3x-y)]_{|(-1,0)}=-3e^{-1}(\cos 3+\sin 3),\\f_{yy}(-1,0)&=[-e^x\sin(3x-y)]_{|(-1,0)}=e^{-1}\sin 3,\end{aligned}$$

$$\begin{aligned}e^x\sin(3x-y)&=-\frac{\sin 3}{e}+\frac{(3\cos 3-\sin 3)}{e}(x+1)-\frac{y\cos 3}{e}\\&\quad+\frac{1}{2!}\left[\frac{8\sin 3+6\cos 3}{e}(x+1)^2-\frac{6(\cos 3+\sin 3)}{e}(x+1)y+\frac{\sin 3}{e}y^2\right]+\cdots.\end{aligned}$$

12. With $f(0,1)=0$,

$$\begin{aligned}f_x(0,1)&=[2(x+y)\ln(x+y)+x+y]_{|(0,1)}=1,\\f_y(0,1)&=[2(x+y)\ln(x+y)+x+y]_{|(0,1)}=1,\\f_{xx}(0,1)&=[2\ln(x+y)+2+1]_{|(0,1)}=3,\\f_{xy}(0,1)&=[2\ln(x+y)+2+1]_{|(0,1)}=3,\\f_{yy}(0,1)&=[2\ln(x+y)+2+1]_{|(0,1)}=3,\end{aligned}$$

$$\begin{aligned}(x+y)^2\ln(x+y)&=x+(y-1)+\frac{1}{2!}[3x^2+6x(y-1)+3(y-1)^2]+\cdots\\&=\frac{1}{2}[2x+2(y-1)+3x^2+6x(y-1)+3(y-1)^2]+\cdots.\end{aligned}$$

13. With $f(1,-1)=\dfrac{\pi}{4}$,

$$\begin{aligned}f_x(1,-1)&=\frac{3}{1+(3x+2y)^2}\Big|_{(1,-1)}=\frac{3}{2},\\f_y(1,-1)&=\frac{2}{1+(3x+2y)^2}\Big|_{(1,-1)}=1,\\f_{xx}(1,-1)&=\frac{-18(3x+2y)}{[1+(3x+2y)^2]^2}\Big|_{(1,-1)}=-\frac{9}{2},\\f_{xy}(1,-1)&=\frac{-12(3x+2y)}{[1+(3x+2y)^2]^2}\Big|_{(1,-1)}=-3,\end{aligned}$$

$$f_{yy}(1,-1) = \frac{-8(3x+2y)}{[1+(3x+2y)^2]^2}_{|(1,-1)} = -2,$$

$$\mathrm{Tan}^{-1}(3x+2y) = \frac{\pi}{4} + \frac{3}{2}(x-1) + (y+1) + \frac{1}{2!}\left[-\frac{9}{2}(x-1)^2 - 6(x-1)(y+1) - 2(y+1)^2\right] + \cdots .$$

14. The first six terms vanish since the function is its own Taylor series about $(0,0)$.

15. $$f(x,y,z) = f(c,d,e) + [f_x(c,d,e)(x-c) + f_y(c,d,e)(y-d) + f_z(c,d,e)(z-e)]$$
$$+\frac{1}{2!}[f_{xx}(c,d,e)(x-c)^2 + f_{yy}(c,d,e)(y-d)^2 + f_{zz}(c,d,e)(z-e)^2$$
$$+2f_{xy}(c,d,e)(x-c)(y-d) + 2f_{xz}(c,d,e)(x-c)(z-e) + 2f_{yz}(c,d,e)(y-d)(z-e)] + \cdots .$$

16. If the operator is defined as

$$\left[(x-c)\frac{\partial}{\partial x} + (y-d)\frac{\partial}{\partial y}\right]^n f(c,d) = \sum_{r=0}^{n}\binom{n}{r}(x-c)^{n-r}(y-d)^r \frac{\partial^n f(x,y)}{\partial x^{n-r}\partial y^r}_{|(c,d)},$$

then equation 12.76 becomes

$$f(x,y) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[(x-c)\frac{\partial}{\partial x} + (y-d)\frac{\partial}{\partial y}\right]^n f(c,d).$$

REVIEW EXERCISES

1. $\dfrac{\partial f}{\partial x} = \dfrac{2x}{y^3} - \dfrac{y}{\sqrt{1-x^2y^2}}$

2. From $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2+z^2}$, we obtain $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{(x^2+y^2+z^2)(2) - 2y(2y)}{(x^2+y^2+z^2)^2} = \dfrac{2(x^2-y^2+z^2)}{(x^2+y^2+z^2)^2}$.

3. From $\partial f/\partial y = x^3e^y - \cos(x+y+z+t)$, we obtain $\partial^2 f/\partial x\partial y = 3x^2e^y + \sin(x+y+z+t)$, and $\partial^3 f/\partial x^2\partial y = 6xe^y + \cos(x+y+z+t)$.

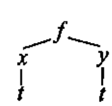
4. If we set $F(x,y,z) = z^2x + \mathrm{Tan}^{-1}z + y - 3x$, then

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} = -\frac{z^2-3}{2xz + \dfrac{1}{1+z^2}} = \frac{(3-z^2)(1+z^2)}{1+2xz+2xz^3}.$$

5. If we set $F(x,y,z,u) = u\cos y + y\cos(xu) + z^2 - 5x$, then

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(u)}} = -\frac{F_y}{F_u} = -\frac{-u\sin y + \cos(xu)}{\cos y - xy\sin(xu)} = \frac{u\sin y - \cos(xu)}{\cos y - xy\sin(xu)}.$$

6. $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= (2x - ye^{xy})(3t^2+3) + (2y - xe^{xy})(\ln t + 1)$$



7. If we set $F(x,y) = x - y^3 - 3y^2 + 2y - 4$, then

$$\frac{dy}{dx} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(y)}} = -\frac{F_x}{F_y} = -\frac{1}{-3y^2-6y+2} = \frac{1}{3y^2+6y-2}.$$

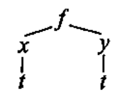
8. If we set $F(x,y,u,v) = u^2 + v^2 - xy - 5$ and $G(x,u,v) = 3u - 2v + x^2u - 2v^3$, then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -y & 2v \\ 2xu & -2-6v^2 \end{vmatrix}}{\begin{vmatrix} 2u & 2v \\ 3+x^2 & -2-6v^2 \end{vmatrix}}$$

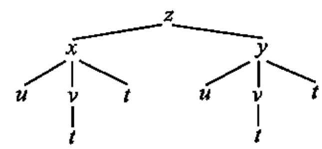
$$= -\frac{y(2+6v^2) - 4xuv}{-2u(2+6v^2) - 2v(3+x^2)} = \frac{y(1+3v^2) - 2xuv}{u(2+6v^2) + v(3+x^2)}.$$

9. From $\dfrac{\partial f}{\partial v} = -\dfrac{u^2}{2v^{3/2}} - \dfrac{1}{\sqrt{u}}$, we obtain $\dfrac{\partial^2 f}{\partial u \partial v} = -\dfrac{u}{v^{3/2}} + \dfrac{1}{2u^{3/2}}$.

10.
$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (y-2x)(te^t + e^t) + (x-2y)(-te^{-t} + e^{-t}) \\ &= e^t(t+1)(y-2x) + e^{-t}(1-t)(x-2y)\end{aligned}$$



11.
$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= 2x(-6v + 3ut)(2t-2) + 2x(3uv) + (-2y)[-ut\sin(vt)](2t-2) \\ &\quad + (-2y)[-uv\sin(vt)] \\ &= 6x[uv + 2(t-1)(ut-2v)] + 2uy\sin(vt)[2t(t-1) + v]\end{aligned}$$



12. If we set $F(x,r,\theta) = r\cos\theta - x$ and $G(y,r,\theta) = r\sin\theta - y$, then

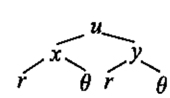
$$\frac{\partial r}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,\theta)}}{\dfrac{\partial(F,G)}{\partial(r,\theta)}} = -\frac{\begin{vmatrix} F_x & F_\theta \\ G_x & G_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_\theta \\ G_r & G_\theta \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & -r\sin\theta \\ 0 & r\cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}} = \frac{r\cos\theta}{r} = \cos\theta.$$

13. If we set $F(x,r,\phi,\theta) = r\sin\phi\cos\theta - x$, $G(y,r,\phi,\theta) = r\sin\phi\sin\theta - y$ and $H(z,r,\phi) = r\cos\phi - z$, then

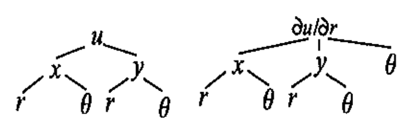
$$\frac{\partial \theta}{\partial x} = -\frac{\dfrac{\partial(F,G,H)}{\partial(r,\phi,x)}}{\dfrac{\partial(F,G,H)}{\partial(r,\phi,\theta)}} = -\frac{\begin{vmatrix} F_r & F_\phi & F_x \\ G_r & G_\phi & G_x \\ H_r & H_\phi & H_x \end{vmatrix}}{\begin{vmatrix} F_r & F_\phi & F_\theta \\ G_r & G_\phi & G_\theta \\ H_r & H_\phi & H_\theta \end{vmatrix}} = -\frac{\begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -1 \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}}{\begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}}$$

$$= -\frac{r\sin\theta}{\cos\phi(r^2\sin\phi\cos\phi) + r\sin\phi(r\sin^2\phi)} = -\frac{r\sin\theta}{r^2\sin\phi} = -\frac{\sin\theta}{r\sin\phi}.$$

14.
$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x - 3x^2y^2)\cos\theta + (-2yx^3)\sin\theta\end{aligned}$$

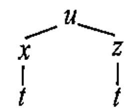


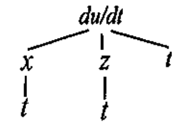
15.
$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x - 3x^2y^2)\cos\theta + (-2x^3y)\sin\theta\end{aligned}$$



$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right)\frac{\partial y}{\partial r} \\ &= [(2 - 6xy^2)\cos\theta - 6x^2y\sin\theta]\cos\theta + (-6x^2y\cos\theta - 2x^3\sin\theta)\sin\theta \\ &= 2(1 - 3xy^2)\cos^2\theta - 12x^2y\sin\theta\cos\theta - 2x^3\sin^2\theta\end{aligned}$$

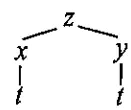
16. $$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}=\left(\frac{1}{z^2}+\frac{2z}{x^3}\right)(3t^2)+\left(-\frac{2x}{z^3}-\frac{1}{x^2}\right)\left(-\frac{3}{t^4}\right)$$





$$\frac{d^2u}{dt^2}=\frac{\partial}{\partial x}\left(\frac{du}{dt}\right)\frac{dx}{dt}+\frac{\partial}{\partial z}\left(\frac{du}{dt}\right)\frac{dz}{dt}+\frac{\partial}{\partial t}\left(\frac{du}{dt}\right)$$
$$=\left(-\frac{18zt^2}{x^4}+\frac{6}{z^3t^4}-\frac{6}{x^3t^4}\right)(3t^2)+\left(-\frac{6t^2}{z^3}+\frac{6t^2}{x^3}-\frac{18x}{z^4t^4}\right)\left(-\frac{3}{t^4}\right)$$
$$+\left(\frac{1}{z^2}+\frac{2z}{x^3}\right)(6t)+\left(\frac{2x}{z^3}+\frac{1}{x^2}\right)\left(-\frac{12}{t^5}\right)$$
$$=\frac{6}{x^4t^8z^4}(-9z^5t^{12}+6zt^6x^4-6xt^6z^4+9x^5+t^9z^2x^4+2z^5xt^9-4x^5zt^3-2x^2t^3z^4)$$

17. $$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$



$$=(1-y^2)\frac{dx}{dt}+(1-2xy)\frac{dy}{dt}$$

If we set $F(x,y,t)=x^2-y^2+xt-2t$ and $G(x,y,t)=xy-4t^2$, then

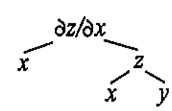
$$\frac{dx}{dt}=-\frac{\dfrac{\partial(F,G)}{\partial(t,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}}=-\frac{\begin{vmatrix}F_t & F_y\\ G_t & G_y\end{vmatrix}}{\begin{vmatrix}F_x & F_y\\ G_x & G_y\end{vmatrix}}=-\frac{\begin{vmatrix}x-2 & -2y\\ -8t & x\end{vmatrix}}{\begin{vmatrix}2x+t & -2y\\ y & x\end{vmatrix}}=\frac{2x+16ty-x^2}{2x^2+xt+2y^2},$$

$$\frac{dy}{dt}=-\frac{\dfrac{\partial(F,G)}{\partial(x,t)}}{\dfrac{\partial(F,G)}{\partial(x,y)}}=-\frac{\begin{vmatrix}F_x & F_t\\ G_x & G_t\end{vmatrix}}{2x^2+xt+2y^2}=-\frac{\begin{vmatrix}2x+t & x-2\\ y & -8t\end{vmatrix}}{2x^2+xt+2y^2}=\frac{16xt+8t^2+xy-2y}{2x^2+xt+2y^2},$$

Thus, $\dfrac{dz}{dt}=(1-y^2)\left(\dfrac{2x+16ty-x^2}{2x^2+xt+2y^2}\right)+(1-2xy)\left(\dfrac{16xt+8t^2+xy-2y}{2x^2+xt+2y^2}\right).$

18. If we set $F(x,y,z)=xz-x^2z^3+y^2-3$, then $\dfrac{\partial z}{\partial x}=-\dfrac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}}=-\dfrac{F_x}{F_z}=-\dfrac{z-2xz^3}{x-3x^2z^2}=\dfrac{2xz^3-z}{x-3x^2z^2}.$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial x}$$
$$=\left[\frac{(x-3x^2z^2)(2z^3)-(2xz^3-z)(1-6xz^2)}{(x-3x^2z^2)^2}\right]$$

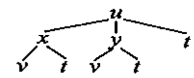


$$+\left[\frac{(x-3x^2z^2)(6xz^2-1)-(2xz^3-z)(-6x^2z)}{(x-3x^2z^2)^2}\right]\left(\frac{2xz^3-z}{x-3x^2z^2}\right)$$
$$=\frac{1}{(x-3x^2z^2)^3}[x(1-3xz^2)(2xz^3-6x^2z^5-2xz^3+z+12x^2z^5-6xz^3)$$
$$+z(2xz^2-1)(6x^2z^2-x-18x^3z^4+3x^2z^2+12x^3z^4-6x^2z^2)]$$
$$=\frac{xz[(1-3xz^2)(1+6x^2z^4-6xz^2)+(1-2xz^2)(1+6x^2z^4-3xz^2)]}{(x-3x^2z^2)^3}$$

19. If we set $F(x,y,z)=yx-x^2z^2+5x-3$ and $G(x,y,z)=2xz-3x^2y^2-4z^4$, then

$$\frac{dy}{dx}=-\frac{\dfrac{\partial(F,G)}{\partial(x,z)}}{\dfrac{\partial(F,G)}{\partial(y,z)}}=-\frac{\begin{vmatrix}F_x & F_z\\ G_x & G_z\end{vmatrix}}{\begin{vmatrix}F_y & F_z\\ G_y & G_z\end{vmatrix}}=-\frac{\begin{vmatrix}y-2xz^2+5 & -2x^2z\\ 2z-6xy^2 & 2x-16z^3\end{vmatrix}}{\begin{vmatrix}x & -2x^2z\\ -6x^2y & 2x-16z^3\end{vmatrix}}$$
$$=-\frac{(2x-16z^3)(y-2xz^2+5)+2x^2z(2z-6xy^2)}{x(2x-16z^3)-12x^4yz}=\frac{6x^3y^2z-xy-5x+8yz^3-16xz^5+40z^3}{x^2-8xz^3-6x^4yz}.$$

20. $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \left.\frac{\partial u}{\partial t}\right)_{x,y}$$



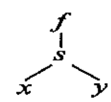
$$= \left(yt^2 - \frac{3y}{\sqrt{1-x^2y^2}}\right)(2v^2t - 2)$$
$$+ \left(xt^2 - \frac{3x}{\sqrt{1-x^2y^2}}\right)(v\sec^2 t) + 2xyt$$
$$= 2y(v^2t-1)\left(t^2 - \frac{3}{\sqrt{1-x^2y^2}}\right) + xv\sec^2 t\left(t^2 - \frac{3}{\sqrt{1-x^2y^2}}\right) + 2xyt$$

21. This follows from Theorem 12.3 since the function is homogeneous of degree 2.

22. With $\dfrac{\partial u}{\partial x} = 4x + y$, $\dfrac{\partial^2 u}{\partial x^2} = 4$, $\dfrac{\partial^2 u}{\partial x \partial y} = 1$, $\dfrac{\partial u}{\partial y} = -6y + x$, $\dfrac{\partial^2 u}{\partial y^2} = -6$,

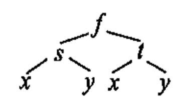
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 4x^2 + 2xy - 6y^2 = 2u.$$

23. If we set $s = 3x - 2y$, then
$$\frac{\partial f}{\partial x} = \frac{df}{ds}\frac{\partial s}{\partial x} = 3f'(s), \quad \frac{\partial f}{\partial y} = \frac{df}{ds}\frac{\partial s}{\partial y} = -2f'(s).$$



Thus, $2\dfrac{\partial f}{dx} + 3\dfrac{\partial f}{\partial y} = 6f'(s) - 6f'(s) = 0$.

24. If we set $s = x^2 - y^2$ and $t = y^2 - x^2$, then
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial f}{\partial s}(2x) + \frac{\partial f}{\partial t}(-2x)$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial f}{\partial s}(-2y) + \frac{\partial f}{\partial t}(2y).$$



Consequently,
$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = y\left(2x\frac{\partial f}{\partial s} - 2x\frac{\partial f}{\partial t}\right) + x\left(-2y\frac{\partial f}{\partial s} + 2y\frac{\partial f}{\partial t}\right) = 0.$$

25. $D_{\mathbf{v}}f = \nabla f_{|(3,-1)} \cdot \hat{\mathbf{v}} = (2x\sin y, x^2\cos y)_{|3,-1)} \cdot \dfrac{(2,4)}{2\sqrt{5}} = (-6\sin 1, 9\cos 1)\cdot\dfrac{(1,2)}{\sqrt{5}} = \dfrac{18\cos 1 - 6\sin 1}{\sqrt{5}}$.

26. With $\nabla f_{|(1,0,1)} = (2x, 2y, 2z)_{|(1,0,1)} = (2,0,2)$, and a unit vector $\hat{\mathbf{v}} = (1,-1,2)/\sqrt{1+1+4}$ in the direction from $(1,0,1)$ to $(2,-1,3)$, $D_{\hat{\mathbf{v}}}f = (2,0,2)\cdot\dfrac{(1,-1,2)}{\sqrt{6}} = \sqrt{6}$.

27. Since $\nabla f_{|(-1,2,5)} = \left(\dfrac{z}{1+(x+y)^2}, \dfrac{z}{1+(x+y)^2}, \mathrm{Tan}^{-1}(x+y)\right)_{|(-1,2,5)} = \left(\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{\pi}{4}\right)$, and a vector perpendicular to the surface is $\mathbf{n} = \nabla(z - x^2 - y^2)_{|(-1,2,5)} = (-2x, -2y, 1)_{|(-1,2,5)} = (2,-4,1)$, the directional derivative is

$$D_{\mathbf{n}}f = \left(\frac{5}{2}, \frac{5}{2}, \frac{\pi}{4}\right)\cdot\hat{\mathbf{n}} = \left(\frac{5}{2}, \frac{5}{2}, \frac{\pi}{4}\right)\cdot\frac{(2,-4,1)}{\sqrt{21}} = \frac{\pi/4 - 5}{\sqrt{21}}.$$

28. With parametric equations, $x = 2t - 1$, $y = -t$, $z = 5 - 3t$ for the line, a vector along the line is $\mathbf{T} = (2,-1,-3)$. Since $\nabla f_{|(1,-1,2)} = (2x, 1, -2)_{|(1,-1,2)} = (2,1,-2)$,

$$D_{\mathbf{T}}f = (2,1,-2)\cdot\frac{(2,-1,-3)}{\sqrt{4+1+9}} = \frac{9}{\sqrt{14}}.$$

29. Since the slope of the curve at $(3,10)$ is 6, a tangent vector to the curve at the point is $\mathbf{T} = (-1,-6)$. The directional derivative is

$$D_{\mathbf{T}}f = \nabla f_{|(3,10)} \cdot \hat{\mathbf{T}} = \left(\frac{1}{x+y}, \frac{1}{x+y}\right)_{|(3,10)} \cdot \frac{(-1,-6)}{\sqrt{37}} = \left(\frac{1}{13}, \frac{1}{13}\right) \cdot \frac{(-1,-6)}{\sqrt{37}} = -\frac{7}{13\sqrt{37}}.$$

30. A vector tangent to the curve at $(0,1,1)$ is

$$\mathbf{T} = \nabla(x^2+y^2+z^2-2)_{|(0,1,1)} \times \nabla(z-y)_{|(0,1,1)} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & 2 \\ 0 & -1 & 1 \end{vmatrix} = (4,0,0).$$

Since $\nabla f_{|(0,1,1)} = (2yz-2x, 2xz, 2xy-2z)_{|(0,1,1)} = (2,0,-2)$, $D_{\mathbf{T}}f = (2,0,-2)\cdot(1,0,0) = 2$.

31. Since a vector normal to the plane is $\nabla(x^2+y^2-z)_{|(1,3,10)} = (2x,2y,-1)_{|(1,3,10)} = (2,6,-1)$, its equation is $0 = (2,6,-1)\cdot(x-1,y-3,z-10) = 2x+6y-z-10$.

32. Since a vector normal to the plane is $\nabla(x^2-y^2+z^3)_{|(-1,3,2)} = (2x,-2y,3z^2)_{|(-1,3,2)} = (-2,-6,12)$, its equation is $0 = (1,3,-6)\cdot(x+1,y-3,z-2) = x+3y-6z+4$.

33. Since a vector normal to the plane is $\nabla(x^2+y^2-z^2-1)_{|(1,0,0)} = (2x,2y,-2z)_{|(1,0,0)} = (2,0,0)$, its equation is $0 = (1,0,0)\cdot(x-1,y-0,z-0) = x-1$.

34. Since a tangent vector to the curve at $(2,0,6)$ is $\mathbf{T} = \dfrac{d\mathbf{r}}{dt}_{|t=1} = (2t,2t,3t^2+5)_{|t=1} = (2,2,8)$, parametric equations for the tangent line are $x = 2+u$, $y = u$, $z = 6+4u$.

35. Since the curve is a straight line, the tangent line to the curve is the curve itself.

36. A tangent vector to the curve at $(1,1,1)$ is

$$\begin{aligned}&\nabla(xy-z)_{|(1,1,1)} \times \nabla(x^2+y^2-2)_{|(1,1,1)}\\ &\quad = (y,x,-1)_{|(1,1,1)} \times (2x,2y,0)_{|(1,1,1)} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -1 \\ 2 & 2 & 0 \end{vmatrix} = (2,-2,0).\end{aligned}$$

Parametric equations for the tangent line are $x = 1+t$, $y = 1-t$, $z = 1$.

37. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 3x^2-6$, $0 = \dfrac{\partial f}{\partial y} = 6y$. Solutions are $(\pm\sqrt{2},0)$. Since $f_{xx} = 6x$, $f_{xy} = 0$, and $f_{yy} = 6$, we obtain $(f_{xy})^2 - f_{xx}f_{yy} = -36x$. At $(\sqrt{2},0)$, $B^2-AC = -36\sqrt{2}$ and $A = 6\sqrt{2}$, so that this critical point gives a relative minimum. At $(-\sqrt{2},0)$, $B^2-AC = 36\sqrt{2}$, so that this critical point yields a saddle point.

38. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = ye^x$, $0 = \dfrac{\partial f}{\partial y} = e^x$. There are no solutions to these equations.

39. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 2x-y+1$, $0 = \dfrac{\partial f}{\partial y} = -x+2y-4$. The only solution is $(2/3, 7/3)$. Since $f_{xx} = 2$, $f_{xy} = -1$, and $f_{yy} = 2$, it follows that $B^2-AC = 1-4$. Since $A = 2$, the critical point gives a relative minimum.

40. For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 4x(x^2+y^2-1)$, $0 = \dfrac{\partial f}{\partial y} = 4y(x^2+y^2-1)$. The solutions are $(0,0)$ and every point on the circle $x^2+y^2 = 1$. Since $f(x,y) = 0$ for every point on $x^2+y^2=1$, but is otherwise positive, each of these critical points yields a relative minimum.

$$\frac{\partial^2 f}{\partial x^2} = 4(x^2+y^2-1)+8x^2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 8xy, \qquad \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2-1)+8y^2.$$

At $(0,0)$, $B^2-AC = 0-(-4)(-4) < 0$, and $A = -4$, and therefore $(0,0)$ gives a relative maximum.

41. Since $f(x,y) = (x^2+y^2)\left(\dfrac{x^3}{y} - \dfrac{y^3}{x}\right) = \dfrac{x^5}{y} - xy^3 + x^3y - \dfrac{y^5}{x}$,

$$\frac{\partial f}{\partial x} = \frac{5x^4}{y} - y^3 + 3x^2y + \frac{y^5}{x^2}, \qquad \frac{\partial f}{\partial y} = -\frac{x^5}{y^2} - 3xy^2 + x^3 - \frac{5y^4}{x},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{20x^3}{y} + 6xy - \frac{2y^5}{x^3}, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2x^5}{y^3} - 6xy - \frac{20y^3}{x}.$$

Thus, $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \dfrac{20x^3}{y} - \dfrac{2y^5}{x^3} + \dfrac{2x^5}{y^3} - \dfrac{20y^3}{x}$. On the other hand,

$$\frac{\partial F}{\partial x} = \frac{3x^2}{y} + \frac{y^3}{x^2}, \quad \frac{\partial F}{\partial y} = -\frac{x^3}{y^2} - \frac{3y^2}{x}, \quad \frac{\partial^2 F}{\partial x^2} = \frac{6x}{y} - \frac{2y^3}{x^3}, \quad \frac{\partial^2 F}{\partial y^2} = \frac{2x^3}{y^3} - \frac{6y}{x}.$$

Hence,

$$\begin{aligned}(x^2+y^2)\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) + 12F(x,y) &= (x^2+y^2)\left(\frac{6x}{y} - \frac{2y^3}{x^3} + \frac{2x^3}{y^3} - \frac{6y}{x}\right) + 12\left(\frac{x^3}{y} - \frac{y^3}{x}\right)\\ &= \frac{20x^3}{y} - \frac{2y^5}{x^3} + \frac{2x^5}{y^3} - \frac{20y^3}{x}.\end{aligned}$$

42. For critical points of $f(x,y)$ we solve $0 = \dfrac{\partial f}{\partial x} = y,\ 0 = \dfrac{\partial f}{\partial y} = x$. At the critical point $(0,0)$, $f(0,0) = \boxed{0}$. On the boundary of the region we set $x = \cos t$, $y = \sin t$, in which case $f(x,y)$ becomes

$$z(t) = \cos t \sin t = \frac{1}{2}\sin 2t, \quad 0 \le t \le 2\pi.$$

For critical points of $z(t)$, we set $0 = dz/dt = \cos 2t$. At the critical points $t = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$,

$$z(\pi/4) = z(5\pi/4) = \boxed{1/2}, \qquad z(3\pi/4) = z(7\pi/4) = \boxed{-1/2}.$$

Finally, $z(0) = z(2\pi) = \boxed{0}$. Maximum and minimum values are therefore $\pm 1/2$.

43. The function has no critical points inside the sphere so that its maximum and minimum values must occur on the surface of the sphere. To find them we define the Lagrangian $F(x,y,z,\lambda) = 2x + 3y - 4z + \lambda(x^2+y^2+z^2-2)$. For critical points, we solve

$$0 = \frac{\partial F}{\partial x} = 2 + 2\lambda x,\ 0 = \frac{\partial F}{\partial y} = 3 + 2\lambda y,\ 0 = \frac{\partial F}{\partial z} = -4 + 2\lambda z,\ 0 = \frac{\partial F}{\partial \lambda} = x^2+y^2+z^2-2.$$

Solutions are $(\pm\sqrt{8/29}, \pm\sqrt{18/29}, \mp\sqrt{32/29})$. Since the sphere is a closed surface, we need only evaluate $f(x,y,z)$ at these critical points, $f(\pm\sqrt{8/29}, \pm\sqrt{18/29}, \mp\sqrt{32/29}) = \pm\sqrt{58}$. These are maximum and minimum values of $f(x,y,z)$ for the sphere.

44. The distance D from the origin to any point $P(x,y)$ on the curve is given by

$$D^2 = x^2 + y^2 = x^2 + (1 - x^2 - x^4) = 1 - x^4.$$

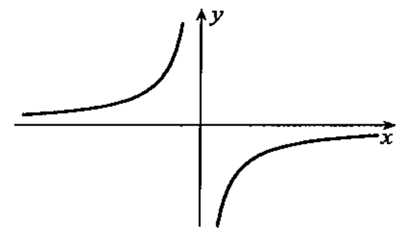
This function must be maximized and minimized on the interval $|x| \le \sqrt{(\sqrt{5}-1)/2}$. Since the only critical point of D^2 is $x = 0$, we evaluate

$$D^2(0) = 1, \qquad D^2\left(\pm\sqrt{(\sqrt{5}-1)/2}\right) = (\sqrt{5}-1)/2.$$

The closest and farthest points are therefore $(\pm\sqrt{(\sqrt{5}-1)/2}, 0)$ and $(0, \pm 1)$ respectively.

45. The distance d from the origin to a point (x, y, z) on the surface is given by $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + 1 + xy$. Since z^2 must be positive, we consider this function for points between the branches of the hyperbola $y = -1/x$. For critical points of d^2, we solve

$$0 = 2x + y, \qquad 0 = 2y + x.$$



The only solution is $(0,0)$ at which $d^2 = 1$. Along the right branch of the hyperbola, we can write that

$$d^2 = f(x) = x^2 + \frac{1}{x^2} + 1 - 1 = x^2 + \frac{1}{x^2}, \quad 0 < x < \infty.$$

Critical points of this function are defined by $0 = f'(x) = 2x - 2/x^3$ with solution $x = 1$, at which $f(1) = 2$. A similar result is found on the left branch of the hyperbola. Points on the surface closest to the origin are therefore $(0, 0, \pm 1)$.

46. If the farmer plants x hectares of corn, y hectares of potatoes, and z hectares of sunflowers, his losses are

$$L = pax^2 + qby^2 + rcz^2.$$

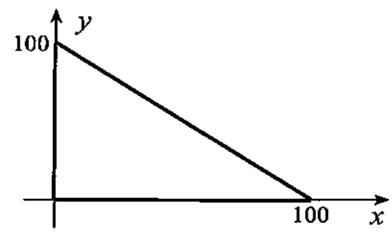
Since $x + y + z = 100$,

$$L = pax^2 + qby^2 + rc(100 - x - y)^2.$$

This function must be minimized for (x, y) in the triangle shown. For critical points of L, we solve

$$0 = \frac{\partial L}{\partial x} = 2apx - 2rc(100 - x - y),$$

$$0 = \frac{\partial L}{\partial y} = 2bqy - 2rc(100 - x - y).$$



The solution of these equations is $x_0 = \dfrac{100bcqr}{acpr + abpq + bcqr}$, $y_0 = \dfrac{100acpr}{acpr + abpq + bcqr}$.
When $x = 0$,

$$L = qby^2 + rc(100 - y)^2, \quad 0 \le y \le 100.$$

For critical points of this function, $0 = \dfrac{dL}{dy} = 2qby - 2rc(100 - y)$. The solution is $y_1 = 100rc/(cr + bq)$.

Similarly, when $y = 0$, we obtain the critical point $x_1 = 100rc/(cr + ap)$ of

$$L = pax^2 + rc(100 - x)^2, \quad 0 \le x \le 100.$$

When $x + y = 100$,

$$L = pax^2 + qb(100 - x)^2, \quad 0 \le x \le 100.$$

The critical point of this function is $x_2 = 100bq/(bq + ap)$.

We now evaluate L at each of these critical points and the corners of the triangle:

$$L(x_0, y_0) = \frac{10^4 abcpqr}{acpr + abpq + bcqr}, \quad L(y_1) = \frac{10^4 bcqr}{cr + bq}, \quad L(x_1) = \frac{10^4 acpr}{cr + ap}, \quad L(x_2) = \frac{10^4 abpq}{ap + bq},$$

$$L(0, 100) = 10^4 bq, \quad L(100, 0) = 10^4 ap, \quad L(0, 0) = 10^4 cr.$$

Notice now that:

$$\frac{1}{L(x_0, y_0)} = 10^{-4}\left(\frac{1}{bq} + \frac{1}{cr} + \frac{1}{ap}\right),$$

$$\frac{1}{L(x_1)} = 10^{-4}\left(\frac{1}{ap} + \frac{1}{cr}\right), \ \frac{1}{L(y_1)} = 10^{-4}\left(\frac{1}{bq} + \frac{1}{cr}\right), \ \frac{1}{L(x_2)} = 10^{-4}\left(\frac{1}{bq} + \frac{1}{ap}\right),$$

$$\frac{1}{L(0,100)} = \frac{10^{-4}}{bq}, \quad \frac{1}{L(100,0)} = \frac{10^{-4}}{ap}, \quad \frac{1}{L(0,0)} = \frac{10^{-4}}{cr}.$$

It follows that $1/L(x_0, y_0)$ is the largest of these numbers, and therefore $L(x_0, y_0)$ is the smallest value for L. Thus L is minimized when $x = x_0$, $y = y_0$, and

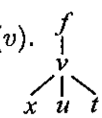
$$z = 100 - x_0 - y_0 = \frac{100abpq}{acpr + abpq + bcqr}.$$

47. If we set $F(x,t,u) = u - f(x - ut)$, then

$$\frac{\partial u}{\partial t} = -\frac{\dfrac{\partial(F)}{\partial(t)}}{\dfrac{\partial(F)}{\partial(u)}} = -\frac{\left.\dfrac{\partial F}{\partial t}\right)_{u,x}}{\left.\dfrac{\partial F}{\partial u}\right)_{x,t}} = -\frac{\left.\dfrac{\partial f}{\partial t}\right)_{u,x}}{1 - \left.\dfrac{\partial f}{\partial u}\right)_{x,t}}, \quad \frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(u)}} = -\frac{\left.\dfrac{\partial F}{\partial x}\right)_{u,t}}{\left.\dfrac{\partial F}{\partial u}\right)_{x,t}} = -\frac{\left.\dfrac{\partial f}{\partial x}\right)_{u,t}}{1 - \left.\dfrac{\partial f}{\partial u}\right)_{x,t}}.$$

If we set $v = x - ut$, then

$$\frac{\partial f}{\partial t} = \frac{df}{dv}\frac{\partial v}{\partial t} = -uf'(v), \quad \frac{\partial f}{\partial x} = \frac{df}{dv}\frac{\partial v}{\partial x} = f'(v), \quad \frac{\partial f}{\partial u} = \frac{df}{dv}\frac{\partial v}{\partial u} = -tf'(v).$$



Consequently, $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = -\dfrac{-uf'(v)}{1 + tf'(v)} + u\dfrac{-f'(v)}{1 + tf'(v)} = 0.$

48. With $\quad f(1, \pi/4) = 1/\sqrt{2}$,

$$f_x(1,\pi/4) = [3x^2 \sin(x^2y) + 2x^4 y \cos(x^2 y)]_{|(1,\pi/4)} = \frac{3}{\sqrt{2}} + \frac{2\pi}{4}\frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{4},$$

$$f_y(1,\pi/4) = x^5 \cos(x^2 y)_{|(1,\pi/4)} = \frac{1}{\sqrt{2}},$$

$$\begin{aligned} f_{xx}(1,\pi/4) &= [6x \sin(x^2y) + 14x^3 y \cos(x^2 y) - 4x^5 y^2 \sin(x^2 y)]_{|(1,\pi/4)} \\ &= \frac{6}{\sqrt{2}} + 14\left(\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - 4\left(\frac{\pi^2}{16}\right)\frac{1}{\sqrt{2}} = \frac{6\sqrt{2}}{2} + \frac{7\sqrt{2}\pi}{4} - \frac{\sqrt{2}\pi^2}{8}, \end{aligned}$$

$$\begin{aligned} f_{xy}(1,\pi/4) &= [5x^4 \cos(x^2 y) - 2x^6 y \sin(x^2 y)]_{|(1,\pi/4)} \\ &= \frac{5}{\sqrt{2}} - 2\left(\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}} - \frac{\sqrt{2}\pi}{4}, \end{aligned}$$

$$f_{yy}(1,\pi/4) = -x^7 \sin(x^2 y)_{|(1,\pi/4)} = -\frac{1}{\sqrt{2}},$$

$$\begin{aligned} x^3 \sin(x^2 y) &= \frac{1}{\sqrt{2}} + \left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{4}\right)(x-1) + \frac{1}{\sqrt{2}}(y - \pi/4) \\ &\quad + \frac{1}{2}\left[\left(\frac{6\sqrt{2}}{2} + \frac{7\sqrt{2}\pi}{4} - \frac{\sqrt{2}\pi^2}{8}\right)(x-1)^2 + 2\left(\frac{5\sqrt{2}}{2} - \frac{\sqrt{2}\pi}{4}\right)(x-1)(y-\pi/4) \right. \\ &\qquad \left. - \frac{1}{\sqrt{2}}(y - \pi/4)^2\right] + \cdots. \end{aligned}$$

49. Least squares estimates for parameters a and b in a linear function $y = ax + b$ must satisfy equations similar to 12.71. For the tabular values these equations are

$$204a + 36b = 656.8, \qquad 36a + 8b = 11.6.$$

Solutions are $a = 3.6810$ and $b = -2.6143$.

CHAPTER 13

EXERCISES 13.1

1. $$\int_{-1}^{2}\int_{y}^{y+2}(x^2-xy)\,dx\,dy = \int_{-1}^{2}\left\{\frac{x^3}{3}-\frac{x^2y}{2}\right\}_{y}^{y+2}dy = \frac{1}{6}\int_{-1}^{2}[2(y+2)^3-3y(y+2)^2-2y^3+3y^3]\,dy$$
$$= \frac{1}{6}\int_{-1}^{2}[2(y+2)^3-2y^3-12y^2-12y]\,dy$$
$$= \frac{1}{6}\left\{\frac{1}{2}(y+2)^4-\frac{y^4}{2}-4y^3-6y^2\right\}_{-1}^{2} = 11$$

2. $$\int_{-3}^{3}\int_{-\sqrt{18-2y^2}}^{\sqrt{18-2y^2}}x\,dx\,dy = \int_{-3}^{3}\left\{\frac{x^2}{2}\right\}_{-\sqrt{18-2y^2}}^{\sqrt{18-2y^2}}dy = \int_{-3}^{3}0\,dy = 0$$

3. $$\int_{0}^{1}\int_{x^2}^{x}(2xy+3y^2)\,dy\,dx = \int_{0}^{1}\left\{xy^2+y^3\right\}_{x^2}^{x}dx = \int_{0}^{1}(x^3+x^3-x^5-x^6)\,dx = \left\{\frac{x^4}{2}-\frac{x^6}{6}-\frac{x^7}{7}\right\}_{0}^{1} = \frac{4}{21}$$

4. $$\int_{-1}^{0}\int_{y}^{2}(1+y)^2\,dx\,dy = \int_{-1}^{0}\left\{x(1+y)^2\right\}_{y}^{2}dy = \int_{-1}^{0}(2+3y-y^3)\,dy = \left\{2y+\frac{3y^2}{2}-\frac{y^4}{4}\right\}_{-1}^{0} = \frac{3}{4}$$

5. $$\int_{3}^{4}\int_{0}^{\pi/2}x\,\sin y\,dy\,dx = \int_{3}^{4}\left\{-x\,\cos y\right\}_{0}^{\pi/2}dx = \int_{3}^{4}x\,dx = \left\{\frac{x^2}{2}\right\}_{3}^{4} = \frac{7}{2}$$

6. $$\int_{1}^{2}\int_{1}^{y}e^{x+y}\,dx\,dy = \int_{1}^{2}\left\{e^{x+y}\right\}_{1}^{y}dy = \int_{1}^{2}(e^{2y}-e^{y+1})\,dy = \left\{\frac{e^{2y}}{2}-e^{y+1}\right\}_{1}^{2} = \frac{e^2(1-e)^2}{2}$$

7. $$\int_{-1}^{1}\int_{-x}^{5}(x^2+y^2)\,dy\,dx = \int_{-1}^{1}\left\{x^2y+\frac{y^3}{3}\right\}_{-x}^{5}dx = \int_{-1}^{1}\left(5x^2+\frac{125}{3}+x^3+\frac{x^3}{3}\right)dx$$
$$= \left\{\frac{5x^3}{3}+\frac{125x}{3}+\frac{x^4}{3}\right\}_{-1}^{1} = \frac{260}{3}$$

8. $$\int_{-1}^{1}\int_{x}^{2x}(xy+x^3y^3)\,dy\,dx = \int_{-1}^{1}\left\{\frac{xy^2}{2}+\frac{x^3y^4}{4}\right\}_{x}^{2x}dx = \frac{1}{4}\int_{-1}^{1}(6x^3+15x^7)\,dx = \frac{1}{4}\left\{\frac{3x^4}{2}+\frac{15x^8}{8}\right\}_{-1}^{1} = 0$$

9. $$\int_{0}^{1}\int_{x}^{1}(x+y)^4\,dy\,dx = \int_{0}^{1}\left\{\frac{1}{5}(x+y)^5\right\}_{x}^{1}dx = \frac{1}{5}\int_{0}^{1}[(x+1)^5-(2x)^5]\,dx$$
$$= \frac{1}{5}\left\{\frac{1}{6}(x+1)^6-\frac{16x^6}{3}\right\}_{0}^{1} = \frac{31}{30}$$

10. $$\int_{1}^{2}\int_{x}^{2x}\frac{1}{(x+y)^3}\,dy\,dx = \int_{1}^{2}\left\{\frac{-1}{2(x+y)^2}\right\}_{x}^{2x}dx = \frac{5}{72}\int_{1}^{2}\frac{1}{x^2}dx = \frac{5}{72}\left\{-\frac{1}{x}\right\}_{1}^{2} = \frac{5}{144}$$

11. $$\int_{0}^{1}\int_{0}^{3x}\sqrt{x+y}\,dy\,dx = \int_{0}^{1}\left\{\frac{2}{3}(x+y)^{3/2}\right\}_{0}^{3x}dx = \frac{2}{3}\int_{0}^{1}(8x^{3/2}-x^{3/2})\,dx = \frac{14}{3}\left\{\frac{2x^{5/2}}{5}\right\}_{0}^{1} = \frac{28}{15}$$

12. $$\int_{-1}^{1}\int_{1}^{e}\frac{y}{x}\,dx\,dy = \int_{-1}^{1}\left\{y\ln|x|\right\}_{1}^{e}dy = \int_{-1}^{1}y\,dy = \left\{\frac{y^2}{2}\right\}_{-1}^{1} = 0$$

13. $$\int_{1}^{4}\int_{\sqrt{x}}^{x^2}(x^2+2xy-3y^2)\,dy\,dx = \int_{1}^{4}\left\{x^2y+xy^2-y^3\right\}_{\sqrt{x}}^{x^2}dx = \int_{1}^{4}(x^4+x^5-x^6-x^{5/2}-x^2+x^{3/2})\,dx$$
$$= \left\{\frac{x^5}{5}+\frac{x^6}{6}-\frac{x^7}{7}-\frac{2x^{7/2}}{7}-\frac{x^3}{3}+\frac{2x^{5/2}}{5}\right\}_{1}^{4} = -\frac{20\,975}{14}$$

14. $\displaystyle\int_0^2\int_{x^2}^{2x^2} x\cos y\,dy\,dx = \int_0^2 \left\{x\sin y\right\}_{x^2}^{2x^2} dx = \int_0^2 x[\sin 2x^2 - \sin x^2]\,dx$

$$= \left\{-\frac{1}{4}\cos 2x^2 + \frac{1}{2}\cos x^2\right\}_0^2 = -0.54$$

15. $\displaystyle\int_0^1\int_1^{\tan x} \frac{1}{1+y^2}\,dy\,dx = \int_0^1 \left\{\mathrm{Tan}^{-1} y\right\}_1^{\tan x} dx = \int_0^1 (x-\pi/4)\,dx = \left\{\frac{(x-\pi/4)^2}{2}\right\}_0^1 = \frac{2-\pi}{4}$

16. $\displaystyle\int_0^1\int_0^{y^3} \frac{1}{1+y^2}\,dx\,dy = \int_0^1 \left\{\frac{x}{1+y^2}\right\}_0^{y^3} dy = \int_0^1 \frac{y^3}{1+y^2}\,dy = \int_0^1 \left(y - \frac{y}{1+y^2}\right) dy$

$$= \left\{\frac{y^2}{2} - \frac{1}{2}\ln(y^2+1)\right\}_0^1 = \frac{1-\ln 2}{2}$$

17. $\displaystyle\int_2^3\int_0^1 \frac{x}{\sqrt{1-y^2}}\,dy\,dx = \int_2^3 \left\{x\,\mathrm{Sin}^{-1} y\right\}_0^1 dx = \int_2^3 \frac{\pi x}{2}\,dx = \frac{\pi}{2}\left\{\frac{x^2}{2}\right\}_2^3 = \frac{5\pi}{4}$

18. $\displaystyle\int_0^2\int_{-x}^{x} (8-2x^2)^{3/2}\,dy\,dx = \int_0^2 \left\{y(8-2x^2)^{3/2}\right\}_{-x}^{x} dx = 2\int_0^2 x(8-2x^2)^{3/2}\,dx$

$$= 2\left\{-\frac{(8-2x^2)^{5/2}}{10}\right\}_0^2 = \frac{128\sqrt{2}}{5}$$

19. $\displaystyle\int_0^1\int_0^x \frac{1}{\sqrt{1-y^2}}\,dy\,dx = \int_0^1 \left\{\mathrm{Sin}^{-1} y\right\}_0^x dx = \int_0^1 \mathrm{Sin}^{-1} x\,dx$

If we set $u = \mathrm{Sin}^{-1} x$, $dv = dx$, $du = \dfrac{1}{\sqrt{1-x^2}}dx$, $v = x$, and use integration by parts,

$$\int_0^1\int_0^x \frac{1}{\sqrt{1-y^2}}\,dy\,dx = \left\{x\,\mathrm{Sin}^{-1} x\right\}_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} + \left\{\sqrt{1-x^2}\right\}_0^1 = \frac{\pi}{2} - 1.$$

20. $\displaystyle\int_{-9}^0\int_0^{x^2\sqrt{9+x}} dy\,dx = \int_{-9}^0 \left\{y\right\}_0^{x^2\sqrt{9+x}} dx = \int_{-9}^0 x^2\sqrt{9+x}\,dx$ If we set $u = 9+x$, then $du = dx$, and

$$\int_{-9}^0\int_0^{x^2\sqrt{9+x}} dy\,dx = \int_0^9 (u-9)^2\sqrt{u}\,du = \int_0^9 (81\sqrt{u} - 18u^{3/2} + u^{5/2})\,du$$

$$= \left\{\frac{162u^{3/2}}{3} - \frac{36u^{5/2}}{5} + \frac{2u^{7/2}}{7}\right\}_0^9 = \frac{11\,664}{35}.$$

21. $\displaystyle\int_0^2\int_{\sqrt{4-x^2}}^2 y^2\,dy\,dx = \int_0^2 \left\{\frac{y^3}{3}\right\}_{\sqrt{4-x^2}}^2 dx = \frac{1}{3}\int_0^2 8 - (4-x^2)^{3/2}]\,dx$

If we set $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$, and

$$\int_0^2\int_{\sqrt{4-x^2}}^2 y^2\,dy\,dx = \frac{1}{3}\left\{8x\right\}_0^2 - \frac{1}{3}\int_0^{\pi/2} 8\cos^3\theta(2\cos\theta\,d\theta) = \frac{16}{3} - \frac{16}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{16}{3} - \frac{4}{3}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{16}{3} - \frac{4}{3}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{16}{3} - \pi.$$

22. $\displaystyle\int_{-1}^0\int_y^0 x\sqrt{x^2+y^2}\,dx\,dy = \int_{-1}^0 \left\{\frac{1}{3}(x^2+y^2)^{3/2}\right\}_y^0 dy$

$$= \frac{1}{3}\int_{-1}^0 (2\sqrt{2}-1)y^3\,dy = \frac{2\sqrt{2}-1}{3}\left\{\frac{y^4}{4}\right\}_{-1}^0 = \frac{1-2\sqrt{2}}{12}$$

23. $$\int_2^3\int_1^{2x}\frac{1}{(xy+x^2)^2}\,dy\,dx=\int_2^3\left\{\frac{-1}{x(xy+x^2)}\right\}_1^{2x}dx=\int_2^3\left[\frac{-1}{x(2x^2+x^2)}+\frac{1}{x(x+x^2)}\right]dx$$
$$=\int_2^3\left[\frac{-1}{3x^3}+\frac{1}{x^2(1+x)}\right]dx=\int_2^3\left(\frac{-1}{3x^3}-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx$$
$$=\left\{\frac{1}{6x^2}-\ln|x|-\frac{1}{x}+\ln|x+1|\right\}_2^3=0.0257$$

24. $$\int_0^1\int_0^{\mathrm{Cos}^{-1}x}x\cos y\,dy\,dx=\int_0^1\left\{x\sin y\right\}_0^{\mathrm{Cos}^{-1}x}dx=\int_0^1x\sqrt{1-x^2}\,dx=\left\{-\frac{1}{3}(1-x^2)^{3/2}\right\}_0^1=\frac{1}{3}$$

25. If we set $u=x^2-y^2$, then $du=2x\,dx$, and

$$\int_0^1\int_{\sqrt{y^2+y}}^{\sqrt{2}}x^3\sqrt{x^2-y^2}\,dx\,dy=\int_0^1\int_y^{y^2}(u+y^2)\sqrt{u}\frac{du}{2}dy=\frac{1}{2}\int_0^1\int_y^{y^2}(u^{3/2}+y^2\sqrt{u})\,du\,dy$$
$$=\frac{1}{2}\int_0^1\left\{\frac{2u^{5/2}}{5}+\frac{2y^2u^{3/2}}{3}\right\}_y^{y^2}dy=\frac{1}{15}\int_0^1(-3y^{5/2}-5y^{7/2}+8y^5)\,dy$$
$$=\frac{1}{15}\left\{-\frac{6y^{7/2}}{7}-\frac{10y^{9/2}}{9}+\frac{4y^6}{3}\right\}_0^1=-\frac{8}{189}.$$

26. If we set $u=x^2-y^2$, then $du=2x\,dx$, and

$$\int_0^1\int_{\sqrt{2}y}^{\sqrt{y^2+y}}x^3\sqrt{x^2-y^2}\,dx\,dy=\int_0^1\int_{y^2}^{y}(u+y^2)\sqrt{u}\frac{du}{2}dy=\frac{1}{2}\int_0^1\int_{y^2}^{y}(u^{3/2}+y^2\sqrt{u})\,du\,dy$$
$$=\frac{1}{2}\int_0^1\left\{\frac{2u^{5/2}}{5}+\frac{2y^2u^{3/2}}{3}\right\}_{y^2}^{y}dy=\frac{1}{15}\int_0^1(3y^{5/2}+5y^{7/2}-8y^5)\,dy$$
$$=\frac{1}{15}\left\{\frac{6y^{7/2}}{7}+\frac{10y^{9/2}}{9}-\frac{4y^6}{3}\right\}_0^1=\frac{8}{189}.$$

27. $$\int_{-2}^0\int_{x^4}^{4x^2}\sqrt{y-x^4}\,dy\,dx=\int_{-2}^0\left\{\frac{2}{3}(y-x^4)^{3/2}\right\}_{x^4}^{4x^2}dx=\frac{2}{3}\int_{-2}^0(4x^2-x^4)^{3/2}\,dx=\frac{2}{3}\int_{-2}^0-x^3(4-x^2)^{3/2}\,dx$$

If we set $u=\sqrt{4-x^2}$, then $du=\dfrac{-x}{\sqrt{4-x^2}}dx$, and

$$\int_{-2}^0\int_{x^4}^{4x^2}\sqrt{y-x^4}\,dy\,dx=-\frac{2}{3}\int_0^2(4-u^2)u^3(-u\,du)=\frac{2}{3}\left\{\frac{4u^5}{5}-\frac{u^7}{7}\right\}_0^2=\frac{512}{105}.$$

28. $$\int_{-2}^0\int_y^0\frac{x}{\sqrt{x^2+y^2}}\,dx\,dy=\int_{-2}^0\left\{\sqrt{x^2+y^2}\right\}_y^0dy=\int_{-2}^0(\sqrt{2}-1)y\,dy=(\sqrt{2}-1)\left\{\frac{y^2}{2}\right\}_{-2}^0=2(1-\sqrt{2})$$

29. $$\int_{-1}^2\int_{-1}^{y^3}\sqrt{1+y}\,dx\,dy=\int_{-1}^2\left\{x\sqrt{1+y}\right\}_{-1}^{y^3}dy=\int_{-1}^2(y^3\sqrt{1+y}+\sqrt{1+y})\,dy$$

If we set $u=\sqrt{1+y}$, then $du=\dfrac{1}{2\sqrt{1+y}}dy$, and

$$\int_{-1}^2\int_{-1}^{y^3}\sqrt{1+y}\,dx\,dy=\int_0^{\sqrt{3}}[(u^2-1)^3u+u](2u\,du)=2\int_0^{\sqrt{3}}(u^8-3u^6+3u^4)\,du$$
$$=2\left\{\frac{u^9}{9}-\frac{3u^7}{7}+\frac{3u^5}{5}\right\}_0^{\sqrt{3}}=\frac{198\sqrt{3}}{35}.$$

30. If we set $y = x\tan\theta$, then $dy = x\sec^2\theta\,d\theta$, and

$$\int_0^1\int_0^x \sqrt{x^2+y^2}\,dy\,dx = \int_0^1\int_0^{\pi/4} x\sec\theta\,x\sec^2\theta\,d\theta\,dx = \int_0^1\int_0^{\pi/4} x^2\sec^3\theta\,d\theta\,dx$$

$$= \int_0^1 \frac{x^2}{2}\Big\{\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\Big\}_0^{\pi/4} dx \quad \text{(see Example 8.9)}$$

$$= \frac{\sqrt{2}+\ln(\sqrt{2}+1)}{2}\int_0^1 x^2\,dx = \frac{\sqrt{2}+\ln(\sqrt{2}+1)}{6}.$$

31. From the continuity equation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -k.$$

Integration gives $v(x,y) = -ky + f(x)$, where $f(x)$ is any differentiable function of x.

32. From the continuity equation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2x.$$

Integration gives $v(x,y) = -2xy + f(x)$, where $f(x)$ is any differentiable function of x.

33. From the continuity equation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right) = \frac{y}{x^2+y^2}.$$

Integration gives $v(x,y) = (1/2)\ln(x^2+y^2) + f(x)$, where $f(x)$ is any differentiable function of x.

34. From the continuity equation,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = \frac{-xy}{\sqrt{x^2+y^2}}.$$

Integration gives $u(x,y) = -y\sqrt{x^2+y^2} + f(y)$, where $f(y)$ is any differentiable function of y.

35. From the continuity equation,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = \sin x\sin y.$$

Integration gives $u(x,y) = -\cos x\sin y + f(y)$, where $f(y)$ is any differentiable function of y.

36. Stream functions must satisfy

$$\frac{\partial\psi}{\partial x} = y, \qquad \frac{\partial\psi}{\partial y} = x.$$

Integration of the first gives $\psi(x,y) = xy + f(y)$, where $f(y)$ is any differentiable function of y. Substitution of this into the second equation requires $x + f'(y) = x \implies f(y) = C$, where C is a constant. Thus, $\psi(x,y) = xy + C$.

37. Stream functions must satisfy

$$\frac{\partial\psi}{\partial x} = 2xy, \qquad \frac{\partial\psi}{\partial y} = x^2+y^2.$$

Integration of the first gives $\psi(x,y) = x^2y + f(y)$, where $f(y)$ is any differentiable function of y. Substitution of this into the second equation requires $x^2 + f'(y) = x^2 + y^2 \implies f(y) = y^3/3 + C$, where C is a constant. Thus, $\psi(x,y) = x^2y + y^3/3 + C$.

38. Stream functions must satisfy

$$\frac{\partial \psi}{\partial x} = -x\sqrt{x^2+y^2}, \qquad \frac{\partial \psi}{\partial y} = -y\sqrt{x^2+y^2}.$$

Integration of the first gives $\psi(x,y) = -\frac{1}{3}(x^2+y^2)^{3/2} + f(y)$, where $f(y)$ is any differentiable function of y. Substitution of this into the second equation requires

$$-y\sqrt{x^2+y^2} + f'(y) = -y\sqrt{x^2+y^2} \quad \Longrightarrow \quad f(y) = C,$$

where C is a constant. Thus, $\psi(x,y) = -(1/3)(x^2+y^2)^{3/2} + C$.

39. Stream functions must satisfy

$$\frac{\partial \psi}{\partial x} = -\sin x \cos y - x, \qquad \frac{\partial \psi}{\partial y} = -\cos x \sin y.$$

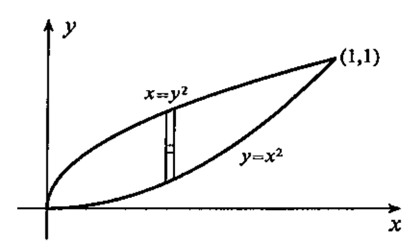
Integration of the second gives $\psi(x,y) = \cos x \cos y + f(x)$, where $f(x)$ is any differentiable function of x. Substitution of this into the first equation requires

$$-\sin x \cos y + f'(x) = -\sin x \cos y - x \quad \Longrightarrow \quad f(x) = -\frac{x^2}{2} + C,$$

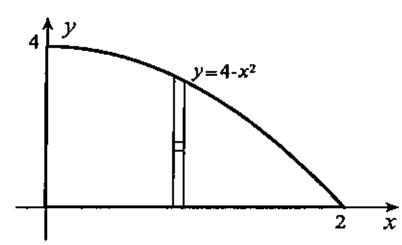
where C is a constant. Thus, $\psi(x,y) = \cos x \cos y - x^2/2 + C$.

EXERCISES 13.2

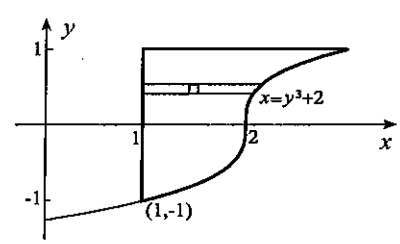
1.
$$\begin{aligned}\iint_R (x^2+y^2)\,dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2+y^2)\,dy\,dx \\ &= \int_0^1 \left\{x^2y + \frac{y^3}{3}\right\}_{x^2}^{\sqrt{x}} dx \\ &= \frac{1}{3}\int_0^1 (3x^{5/2} + x^{3/2} - 3x^4 - x^6)\,dx \\ &= \frac{1}{3}\left\{\frac{6x^{7/2}}{7} + \frac{2x^{5/2}}{5} - \frac{3x^5}{5} - \frac{x^7}{7}\right\}_0^1 = \frac{6}{35}\end{aligned}$$



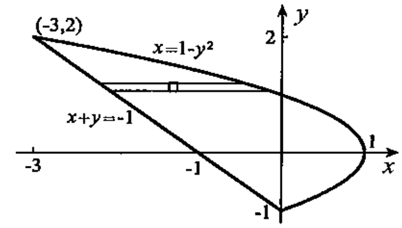
2.
$$\begin{aligned}\iint_R (4-x^2-y)\,dA &= \int_0^2 \int_0^{4-x^2} (4-x^2-y)\,dy\,dx \\ &= \int_0^2 \left\{4y - x^2y - \frac{y^2}{2}\right\}_0^{4-x^2} dx \\ &= \frac{1}{2}\int_0^2 (16 - 8x^2 + x^4)\,dx \\ &= \frac{1}{2}\left\{16x - \frac{8x^3}{3} + \frac{x^5}{5}\right\}_0^2 = \frac{128}{15}\end{aligned}$$



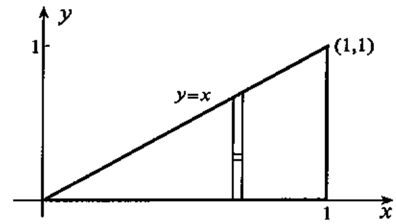
3.
$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{-1}^1 \int_1^{y^3+2} (x+y)\,dx\,dy \\ &= \int_{-1}^1 \left\{\frac{x^2}{2} + xy\right\}_1^{y^3+2} dy \\ &= \frac{1}{2}\int_{-1}^1 (y^6 + 2y^4 + 4y^3 + 2y + 3)\,dy \\ &= \frac{1}{2}\left\{\frac{y^7}{7} + \frac{2y^5}{5} + y^4 + y^2 + 3y\right\}_{-1}^1 = \frac{124}{35}\end{aligned}$$



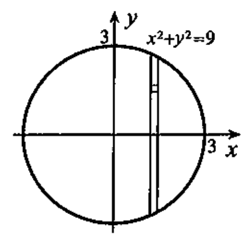
4. $$\iint_R xy^2\,dA = \int_{-1}^{2}\int_{-1-y}^{1-y^2} xy^2\,dx\,dy = \int_{-1}^{2}\left\{\frac{x^2y^2}{2}\right\}_{-1-y}^{1-y^2} dy$$
$$= \frac{1}{2}\int_{-1}^{2}(y^6 - 3y^4 - 2y^3)\,dy$$
$$= \frac{1}{2}\left\{\frac{y^7}{7} - \frac{3y^5}{5} - \frac{y^4}{2}\right\}_{-1}^{2} = -\frac{621}{140}$$



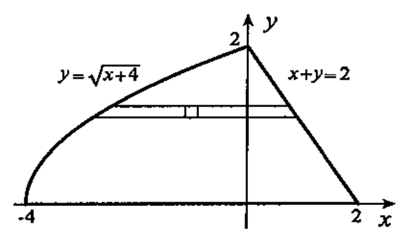
5. $$\iint_R xe^y\,dA = \int_0^1\int_0^x xe^y\,dy\,dx = \int_0^1\left\{xe^y\right\}_0^x dx$$
$$= \int_0^1 (xe^x - x)\,dx$$
$$= \left\{xe^x - e^x - \frac{x^2}{2}\right\}_0^1 = \frac{1}{2}$$



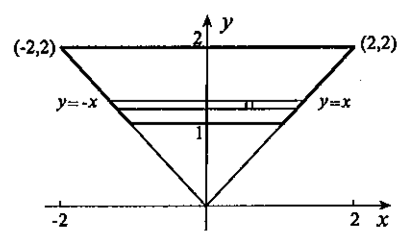
6. $$\iint_R (x+y)\,dA = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(x+y)\,dy\,dx$$
$$= \int_{-3}^{3}\left\{xy + \frac{y^2}{2}\right\}_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$
$$= 2\int_{-3}^{3} x\sqrt{9-x^2}\,dx = 2\left\{-\frac{1}{3}(9-x^2)^{3/2}\right\}_{-3}^{3} = 0$$



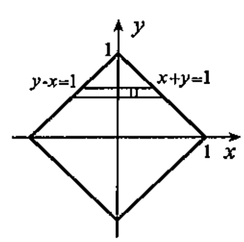
7. $$\iint_R x^2y\,dA = \int_0^2\int_{y^2-4}^{2-y} x^2y\,dx\,dy = \int_0^2\left\{\frac{x^3y}{3}\right\}_{y^2-4}^{2-y} dy$$
$$= \frac{1}{3}\int_0^2(-y^7 + 12y^5 - y^4 - 42y^3 - 12y^2 + 72y)\,dy$$
$$= \frac{1}{3}\left\{-\frac{y^8}{8} + 2y^6 - \frac{y^5}{5} - \frac{21y^4}{2} - 4y^3 + 36y^2\right\}_0^2 = \frac{56}{5}$$



8. $$\iint_R (xy + y^2 - 3x^2)\,dA = \int_1^2\int_{-y}^{y}(xy + y^2 - 3x^2)\,dx\,dy$$
$$= \int_1^2\left\{\frac{x^2y}{2} + xy^2 - x^3\right\}_{-y}^{y} dy$$
$$= \int_1^2 (0)\,dy = 0$$



9. Integration over the top half of the square is equal to that over the bottom half. Hence,
$$\iint_R (1-x)^2\,dA = 2\int_0^1\int_{y-1}^{1-y}(1-x)^2\,dx\,dy$$
$$= 2\int_0^1\left\{-\frac{(1-x)^3}{3}\right\}_{y-1}^{1-y} dy$$
$$= -\frac{2}{3}\int_0^1 [y^3 - (2-y)^3]\,dy$$
$$= -\frac{2}{3}\left\{\frac{y^4}{4} + \frac{(2-y)^4}{4}\right\}_0^1 = \frac{7}{3}$$

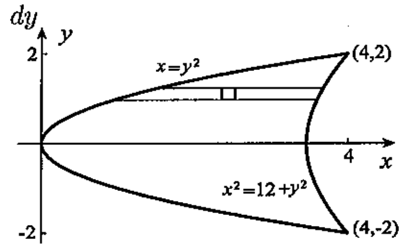


10. $\displaystyle\iint_R (x+y)\,dA = \int_{-2}^{2}\int_{y^2}^{\sqrt{12+y^2}} (x+y)\,dx\,dy = \int_{-2}^{2}\left\{\frac{x^2}{2}+xy\right\}_{y^2}^{\sqrt{12+y^2}} dy$

$\displaystyle = \frac{1}{2}\int_{-2}^{2}(12+y^2-2y^3-y^4+2y\sqrt{12+y^2})\,dy$

$\displaystyle = \frac{1}{2}\left\{12y+\frac{y^3}{3}-\frac{y^4}{2}-\frac{y^5}{5}+\frac{2}{3}(12+y^2)^{3/2}\right\}_{-2}^{2}$

$= 304/15$

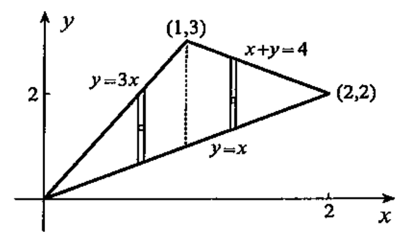


11. $\displaystyle\iint_R x\,dA = \int_0^1\int_x^{3x} x\,dy\,dx + \int_1^2\int_x^{4-x} x\,dy\,dx$

$\displaystyle = \int_0^1 \left\{xy\right\}_x^{3x} dx + \int_1^2 \left\{xy\right\}_x^{4-x} dx$

$\displaystyle = \int_0^1 2x^2\,dx + \int_1^2 (4x-2x^2)\,dx$

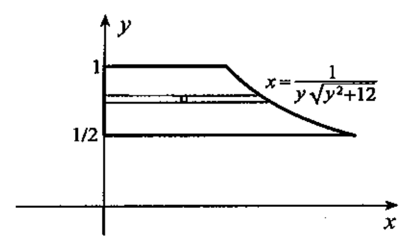
$\displaystyle = \left\{\frac{2x^3}{3}\right\}_0^1 + \left\{2x^2-\frac{2x^3}{3}\right\}_1^2 = 2$



12. $\displaystyle\iint_R y^2\,dA = \int_{1/2}^{1}\int_0^{1/(y\sqrt{y^2+12})} y^2\,dx\,dy$

$\displaystyle = \int_{1/2}^{1}\left\{xy^2\right\}_0^{1/(y\sqrt{y^2+12})} dy$

$\displaystyle = \int_{1/2}^{1}\frac{y}{\sqrt{y^2+12}}dy = \left\{\sqrt{y^2+12}\right\}_{1/2}^{1} = \sqrt{13}-\frac{7}{2}$

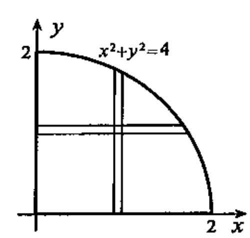


13. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} (4-y^2)^{3/2}\,dy\,dx = \int_0^2\int_0^{\sqrt{4-y^2}} (4-y^2)^{3/2}\,dx\,dy$

$\displaystyle = \int_0^2 \left\{x(4-y^2)^{3/2}\right\}_0^{\sqrt{4-y^2}} dy$

$\displaystyle = \int_0^2 (16-8y^2+y^4)\,dy$

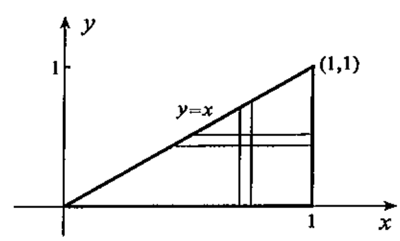
$\displaystyle = \left\{16y-\frac{8y^3}{3}+\frac{y^5}{5}\right\}_0^2 = \frac{256}{15}$



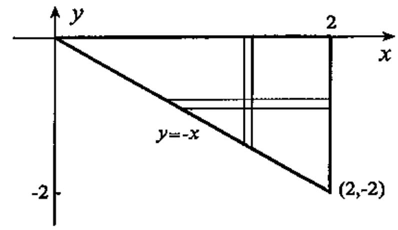
14. $\displaystyle\int_0^1\int_y^1 \sin x^2\,dx\,dy = \int_0^1\int_0^x \sin x^2\,dy\,dx = \int_0^1 \left\{y\sin x^2\right\}_0^x dx$

$\displaystyle = \int_0^1 x\sin x^2\,dx$

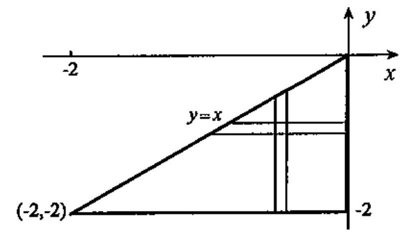
$\displaystyle = \left\{-\frac{\cos x^2}{2}\right\}_0^1 = \frac{1-\cos 1}{2}$



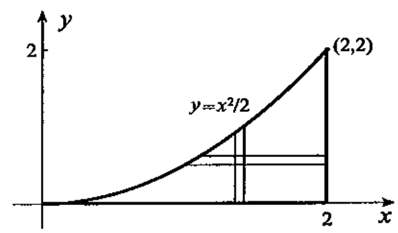
15. $$\int_{-2}^{0}\int_{-y}^{2} y(x^2+y^2)^8\,dx\,dy = \int_0^2\int_{-x}^{0} y(x^2+y^2)^8\,dy\,dx$$
$$= \int_0^2 \left\{\frac{1}{18}(x^2+y^2)^9\right\}_{-x}^{0} dx$$
$$= \frac{1}{18}\int_0^2 (x^{18} - 512x^{18})\,dx$$
$$= \frac{1}{18}\left\{-\frac{511x^{19}}{19}\right\}_0^2 = \frac{-511(2^{18})}{171}$$



16. $$\int_{-2}^{0}\int_{-2}^{x} \frac{x}{\sqrt{x^2+y^2}}dy\,dx = \int_{-2}^{0}\int_{y}^{0} \frac{x}{\sqrt{x^2+y^2}}dx\,dy$$
$$= \int_{-2}^{0}\left\{\sqrt{x^2+y^2}\right\}_y^0 dy = (\sqrt{2}-1)\int_{-2}^{0} y\,dy$$
$$= (\sqrt{2}-1)\left\{\frac{y^2}{2}\right\}_{-2}^{0} = 2(1-\sqrt{2})$$



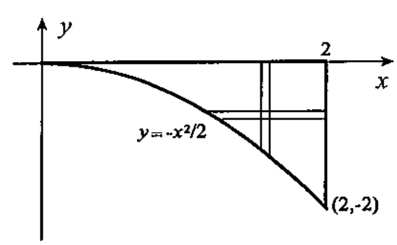
17. $$\int_0^2\int_0^{x^2/2} \frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_0^2\int_{\sqrt{2y}}^{2} \frac{x}{\sqrt{1+x^2+y^2}}dx\,dy$$
$$= \int_0^2\left\{\sqrt{1+x^2+y^2}\right\}_{\sqrt{2y}}^{2} dy$$
$$= \int_0^2 [\sqrt{5+y^2} - (1+y)]\,dy$$



If we set $y = \sqrt{5}\tan\theta$, then $dy = \sqrt{5}\sec^2\theta\,d\theta$, and

$$\int_0^2\int_0^{x^2/2} \frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} \sqrt{5}\sec\theta\,\sqrt{5}\sec^2\theta\,d\theta - \left\{y+\frac{y^2}{2}\right\}_0^2$$
$$= 5\int_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} \sec^3\theta\,d\theta - 4$$
$$= \frac{5}{2}\left\{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right\}_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} - 4 \quad \text{(see Example 8.9)}$$
$$= \frac{5}{4}\ln 5 - 1$$

18. $$\int_0^2\int_{-x^2/2}^{0} \frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_{-2}^{0}\int_{\sqrt{-2y}}^{2} \frac{x}{\sqrt{1+x^2+y^2}}dx\,dy$$
$$= \int_{-2}^{0}\left\{\sqrt{1+x^2+y^2}\right\}_{\sqrt{-2y}}^{2} dy$$
$$= \int_{-2}^{0}(\sqrt{5+y^2} - \sqrt{1-2y+y^2})\,dy$$



In the first term we set $y = \sqrt{5}\tan\theta$ and $dy = \sqrt{5}\sec^2\theta\,d\theta$,

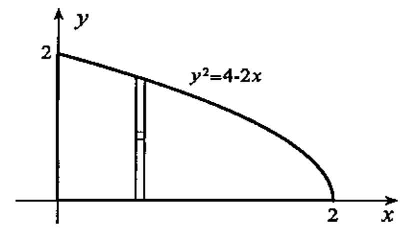
$$\int_0^2\int_{-x^2/2}^{0} \frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_{-\mathrm{Tan}^{-1}(2/\sqrt{5})}^{0} \sqrt{5}\sec\theta\,\sqrt{5}\sec^2\theta\,d\theta - \int_{-2}^{0}|y-1|\,dy$$
$$= \frac{5}{2}\left\{\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\right\}_{-\mathrm{Tan}^{-1}(2/\sqrt{5})}^{0}$$
$$- \int_{-2}^{0}(1-y)\,dy \quad \text{(see Example 8.9)}$$

$$= -\frac{5}{2}\left[\frac{3}{\sqrt{5}}\left(\frac{-2}{\sqrt{5}}\right) + \ln\left|\frac{3}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right|\right] - \left\{y - \frac{y^2}{2}\right\}_{-2}^{0}$$
$$= -\frac{5}{2}\left(-\frac{6}{5} - \ln\sqrt{5}\right) + (-2-2) = \frac{5}{4}\ln 5 - 1$$

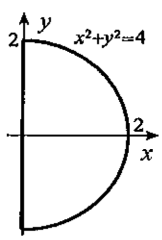
19. We verify the right inequality; the left is similar. Using equation 13.3,

$$\iint_R f(x,y)\,dA = \lim_{\|\Delta A_i\|\to 0}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i \le \lim_{\|\Delta A_i\|\to 0}\sum_{i=1}^{n} M\,\Delta A_i = M \lim_{\|\Delta A_i\|\to 0}\sum_{i=1}^{n}\Delta A_i = M(\text{Area of } R).$$

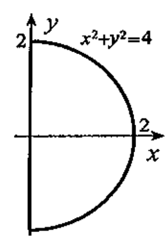
20. $$\iint_R \frac{1}{\sqrt{2x-x^2}}dA = \int_0^2\int_0^{\sqrt{4-2x}} \frac{1}{\sqrt{2x-x^2}}dy\,dx$$
$$= \int_0^2 \frac{\sqrt{4-2x}}{\sqrt{2x-x^2}}dx$$
$$= \int_0^2 \frac{\sqrt{2}\sqrt{2-x}}{\sqrt{x}\sqrt{2-x}}dx$$
$$= \sqrt{2}\left\{2\sqrt{x}\right\}_0^2 = 4$$



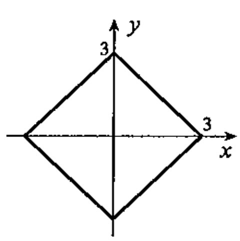
21. Since x^2y^3 is an odd function of y, and the region is symmetric about the x-axis, the value of the integral is zero.



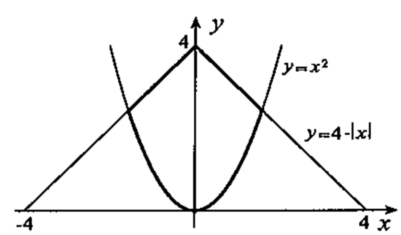
22. Since x^2y^2 is an even function of y, and the region is symmetric about the x-axis, we may double the value of the integral over that part of the region above the x-axis.



23. Since x is an odd function of x, and the area is symmetric about the y-axis, the double integal of x is equal to zero. Since y is an odd function of y, and the area is symmetric about the x-axis, the double integral of y is equal to zero also.

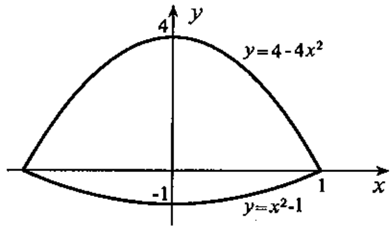


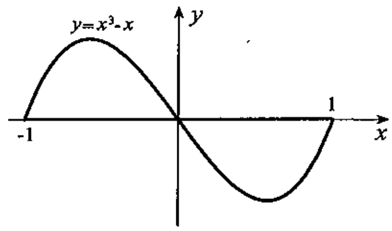
24. Since the integrand is an odd function of x, and the region is symmetric about the y-axis, the value of the integral is zero.



25. Since $e^{x^2+y^2}$ is an even function of x, and the region is symmetric about the y-axis, we could integrate over the right half, and double the result.

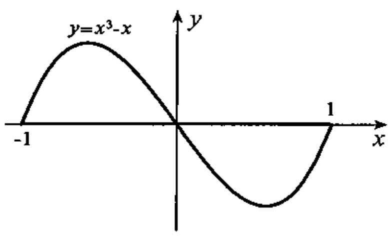
26. Since the integrand is an even function of x and y, and the region is symmetric about the origin, we may double the value of the integral over that part of the region to the right of the y-axis.

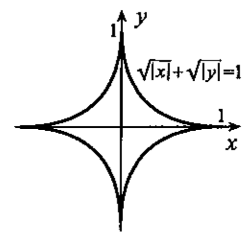


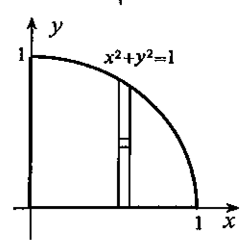


27. Since $\sin(x^2y)$ is an even function of x, and an odd function of y, and the region is symmetric about the origin, the value of the double integral is zero.

28. The first term of the integrand is an odd function of y and the second term is an odd function of x. Since the region is symmetric about the x- and y-axes, the value of the integral is zero.







29. $\overline{f} = \dfrac{1}{\pi/4}\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \dfrac{4}{\pi}\int_0^1 \left\{\dfrac{xy^2}{2}\right\}_0^{\sqrt{1-x^2}} dx$

$$= \frac{2}{\pi}\int_0^1 x(1-x^2)\,dx$$

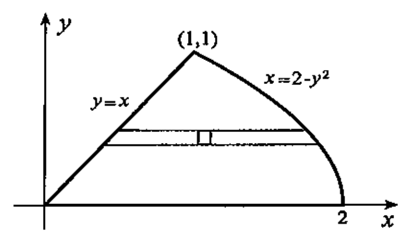
$$= \frac{2}{\pi}\left\{\frac{x^2}{2} - \frac{x^4}{4}\right\}_0^1 = \frac{1}{2\pi}$$

30. Since $\text{Area}(R) = \displaystyle\int_0^1 (2-y^2-y)\,dy = \left\{2y - \frac{y^3}{3} - \frac{y^2}{2}\right\}_0^1 = \frac{7}{6}$,

$$\overline{f} = \frac{6}{7}\int_0^1\int_y^{2-y^2} (x+y)\,dx\,dy = \frac{6}{7}\int_0^1 \left\{\frac{x^2}{2} + xy\right\}_y^{2-y^2} dy$$

$$= \frac{3}{7}\int_0^1 (4+4y-7y^2-2y^3+y^4)\,dy$$

$$= \frac{3}{7}\left\{4y + 2y^2 - \frac{7y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right\}_0^1 = \frac{101}{70}.$$



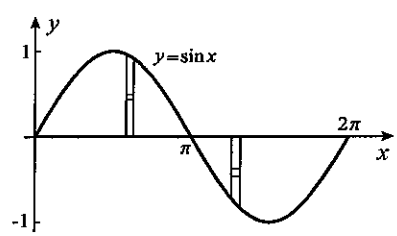
31. Since $\text{Area}(R) = 2\displaystyle\int_0^\pi \sin x\,dx = 2\Big\{-\cos x\Big\}_0^\pi = 4$,

$$\overline{f} = \frac{1}{4}\int_0^\pi\int_0^{\sin x} x\,dy\,dx + \frac{1}{4}\int_\pi^{2\pi}\int_{\sin x}^0 x\,dy\,dx$$

$$= \frac{1}{4}\int_0^\pi \Big\{xy\Big\}_0^{\sin x} dx + \frac{1}{4}\int_\pi^{2\pi}\Big\{xy\Big\}_{\sin x}^0 dx$$

$$= \frac{1}{4}\int_0^\pi x\sin x\,dx + \frac{1}{4}\int_\pi^{2\pi} -x\sin x\,dx$$

$$= \frac{1}{4}\Big\{-x\cos x + \sin x\Big\}_0^\pi - \frac{1}{4}\Big\{-x\cos x + \sin x\Big\}_\pi^{2\pi} = \pi$$



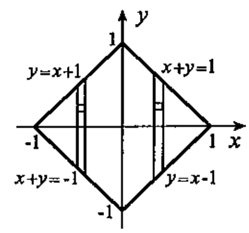
32. $\overline{f} = \dfrac{1}{2}\iint_R e^{x+y}\,dA$

$$= \frac{1}{2}\int_{-1}^{0}\int_{-1-x}^{x+1} e^{x+y}\,dy\,dx + \frac{1}{2}\int_0^1\int_{x-1}^{1-x} e^{x+y}\,dy\,dx$$

$$= \frac{1}{2}\int_{-1}^{0}\left\{e^{x+y}\right\}_{-1-x}^{x+1}dx + \frac{1}{2}\int_0^1\left\{e^{x+y}\right\}_{x-1}^{1-x}dx$$

$$= \frac{1}{2}\int_{-1}^{0}(e^{2x+1} - e^{-1})\,dx + \frac{1}{2}\int_0^1 (e - e^{2x-1})\,dx$$

$$= \frac{1}{2}\left\{\frac{1}{2}e^{2x+1} - \frac{x}{e}\right\}_{-1}^{0} + \frac{1}{2}\left\{ex - \frac{1}{2}e^{2x-1}\right\}_0^1 = \frac{e^2-1}{2e}$$

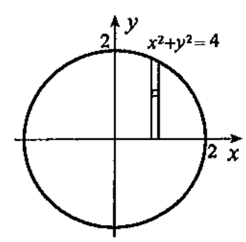


33. $\text{Average} = \dfrac{1}{(4)(10)}\displaystyle\int_{45}^{55}\int_8^{12} 10\,000x^{0.3}y^{0.7}\,dy\,dx = 250\int_{45}^{55}\left\{\frac{x^{0.3}y^{1.7}}{1.7}\right\}_8^{12}dx$

$$= \frac{250(12^{1.7} - 8^{1.7})}{1.7}\left\{\frac{x^{1.3}}{1.3}\right\}_{45}^{55} = 161\,781$$

34. $\text{Average} = \dfrac{1}{(4)(10)}\displaystyle\int_{45}^{55}\int_8^{12} 10\,000x^{0.7}y^{0.3}\,dy\,dx = 250\int_{45}^{55}\left\{\frac{x^{0.7}y^{1.3}}{1.3}\right\}_8^{12}dx$

$$= \frac{250(12^{1.3} - 8^{1.3})}{1.3}\left\{\frac{x^{1.7}}{1.7}\right\}_{45}^{55} = 307\,973$$



35. $\displaystyle\iint_R x^2\,dA = 4\int_0^2\int_0^{\sqrt{4-x^2}} x^2\,dy\,dx = 4\int_0^2\left\{x^2y\right\}_0^{\sqrt{4-x^2}}dx$

$$= 4\int_0^2 x^2\sqrt{4-x^2}\,dx$$

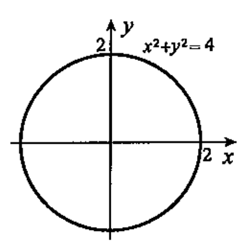
If we set $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$, and

$$\iint_R x^2\,dA = 4\int_0^{\pi/2} 4\sin^2\theta(2\cos\theta)2\cos\theta\,d\theta = 64\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= 64\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta = 16\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = 8\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = 4\pi.$$

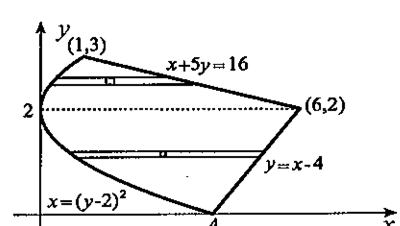
36. Since $f(x,y) = x$ is an odd function of x, and R is symmetric about the y-axis, the integral of the second term vanishes. Similarly, the integral of the third term vanishes. Thus,

$$\iint_R (6 - x - 2y)\,dA = 6\iint_R dA$$
$$= 6(\text{area of } R) = 6\pi(2)^2 = 24\pi.$$



37. $\displaystyle\iint_R 6x^5\,dA = \int_0^2\int_{(y-2)^2}^{y+4} 6x^5\,dx\,dy + \int_2^3\int_{(y-2)^2}^{16-5y} 6x^5\,dx\,dy = \int_0^2\left\{x^6\right\}_{(y-2)^2}^{y+4}dy + \int_2^3\left\{x^6\right\}_{(y-2)^2}^{16-5y}dy$

$$= \int_0^2 [(y+4)^6 - (y-2)^{12}]\,dy$$
$$+ \int_2^3 [(16-5y)^6 - (y-2)^{12}]\,dy$$
$$= \left\{\frac{(y+4)^7}{7} - \frac{(y-2)^{13}}{13}\right\}_0^2$$
$$+ \left\{-\frac{(16-5y)^7}{35} - \frac{(y-2)^{13}}{13}\right\}_2^3 = 4.50 \times 10^4$$

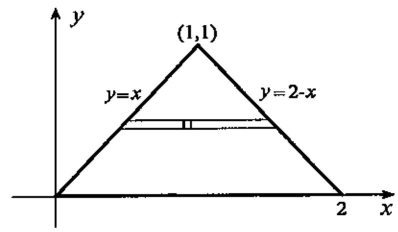


38. $\displaystyle \iint_R ye^x\,dA = \int_0^1 \int_y^{2-y} ye^x\,dx\,dy = \int_0^1 \left\{ye^x\right\}_y^{2-y} dy$

$\displaystyle = \int_0^1 (ye^{2-y} - ye^y)dy$

$\displaystyle = \left\{-ye^{2-y} - e^{2-y} - ye^y + e^y\right\}_0^1$

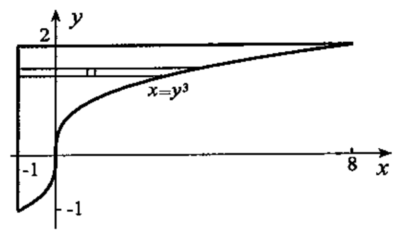
$= e^2 - 2e - 1$



39. $\displaystyle \iint_R \sqrt{1+y}\,dA = \int_{-1}^2 \int_{-1}^{y^3} \sqrt{1+y}\,dx\,dy$

$\displaystyle = \int_{-1}^2 \left\{x\sqrt{1+y}\right\}_{-1}^{y^3} dy$

$\displaystyle = \int_{-1}^2 (y^3+1)\sqrt{y+1}\,dy$



If we set $u = y+1$, then $du = dy$, and

$$\iint_R \sqrt{1+y}\,dA = \int_0^3 (u-1)^3\sqrt{u}\,du + \left\{\frac{2(y+1)^{3/2}}{3}\right\}_{-1}^2 = \int_0^3 (u^{7/2} - 3u^{5/2} + 3u^{3/2} - u^{1/2})\,du + 2\sqrt{3}$$

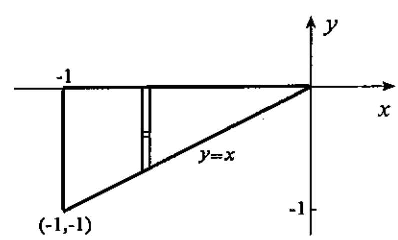
$$= \left\{\frac{2u^{9/2}}{9} - \frac{6u^{7/2}}{7} + \frac{6u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right\}_0^3 + 2\sqrt{3} = \frac{198\sqrt{3}}{35}.$$

40. $\displaystyle \iint_R y\sqrt{x^2+y^2}\,dA = \int_{-1}^0 \int_x^0 y\sqrt{x^2+y^2}\,dy\,dx$

$\displaystyle = \int_{-1}^0 \left\{\frac{1}{3}(x^2+y^2)^{3/2}\right\}_x^0 dx$

$\displaystyle = \frac{2\sqrt{2}-1}{3}\int_{-1}^0 x^3\,dx$

$\displaystyle = \frac{2\sqrt{2}-1}{3}\left\{\frac{x^4}{4}\right\}_{-1}^0 = \frac{1-2\sqrt{2}}{12}$

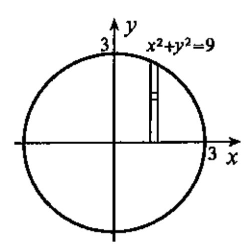


41. $\displaystyle \iint_R (x^2+y^2)\,dA = 4\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)\,dy\,dx$

$\displaystyle = 4\int_0^3 \left\{x^2y + \frac{y^3}{3}\right\}_0^{\sqrt{9-x^2}} dx$

$\displaystyle = \frac{4}{3}\int_0^3 [3x^2\sqrt{9-x^2} + (9-x^2)^{3/2}]\,dx$

$\displaystyle = \frac{4}{3}\int_0^3 (2x^2+9)\sqrt{9-x^2}\,dx$



If we set $x = 3\sin\theta$, then $dx = 3\cos\theta\,d\theta$, and

$$\iint_R (x^2+y^2)\,dA = \frac{4}{3}\int_0^{\pi/2} (18\sin^2\theta + 9)(3\cos\theta)3\cos\theta\,d\theta = 108\int_0^{\pi/2} (2\sin^2\theta\cos^2\theta + \cos^2\theta)\,d\theta$$

$$= 108\int_0^{\pi/2} \left[2\left(\frac{\sin 2\theta}{2}\right)^2 + \frac{1+\cos 2\theta}{2}\right] d\theta = 54\int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2} + 1 + \cos 2\theta\right) d\theta$$

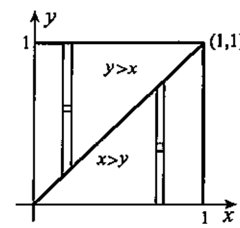
$$= 54\left\{\frac{3\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{81\pi}{2}.$$

42. $\displaystyle\int_0^1\int_0^1 |x-y|\,dy\,dx = \int_0^1\int_0^x (x-y)\,dy\,dx + \int_0^1\int_x^1 (y-x)\,dy\,dx$

$$= \int_0^1 \left\{-\frac{1}{2}(x-y)^2\right\}_0^x dx + \int_0^1 \left\{\frac{1}{2}(y-x)^2\right\}_x^1 dx$$

$$= \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= \frac{1}{2}\left\{\frac{x^3}{3}\right\}_0^1 + \frac{1}{2}\left\{-\frac{(1-x)^3}{3}\right\}_0^1 = \frac{1}{3}$$



43. $\displaystyle\iint_R |y-2x^2+1|\,dA = \iint_{R_1} (y-2x^2+1)\,dA + \iint_{R_2} (-y+2x^2-1)\,dA$

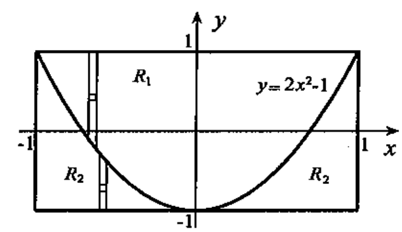
$$= \int_{-1}^1\int_{2x^2-1}^1 (y-2x^2+1)\,dy\,dx + \int_{-1}^1\int_{-1}^{2x^2-1} (-y+2x^2-1)\,dy\,dx$$

$$= \int_{-1}^1 \left\{\frac{y^2}{2} - 2x^2y + y\right\}_{2x^2-1}^1 dx$$

$$+ \int_{-1}^1 \left\{-\frac{y^2}{2} + 2x^2y - y\right\}_{-1}^{2x^2-1} dx$$

$$= 2\int_{-1}^1 (x^4-2x^2+1)\,dx + 2\int_{-1}^1 x^4\,dx$$

$$= 2\left\{\frac{x^5}{5} - \frac{2x^3}{3} + x\right\}_{-1}^1 + 2\left\{\frac{x^5}{5}\right\}_{-1}^1 = \frac{44}{15}$$



44. If we set $b = ay/c$, then

$$n = \frac{2n_cL}{\pi}\int_0^{2d} (1-b^2)\int_0^\infty \frac{x^2}{(1+x^2)(x^2+b^2)}\,dx\,dy,$$

and partial fractions gives

$$n = \frac{2n_cL}{\pi}\int_0^{2d} (1-b^2)\int_0^\infty \left[\frac{1/(1-b^2)}{1+x^2} - \frac{b^2/(1-b^2)}{b^2+x^2}\right] dx\,dy$$

$$= \frac{2n_cL}{\pi}\int_0^{2d} \left\{\mathrm{Tan}^{-1}x - b\,\mathrm{Tan}^{-1}\left(\frac{x}{b}\right)\right\}_0^\infty dy$$

$$= \frac{2n_cL}{\pi}\int_0^{2d} \left(\frac{\pi}{2} - \frac{b\pi}{2}\right) dy$$

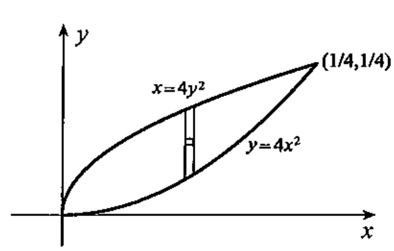
$$= n_cL\int_0^{2d} \left(1 - \frac{ay}{c}\right) dy = n_cL\left\{y - \frac{ay^2}{2c}\right\}_0^{2d}$$

$$= n_cL\left(2d - \frac{2ad^2}{c}\right) = 2n_cdL\left(1 - \frac{ad}{c}\right).$$

EXERCISES 13.3

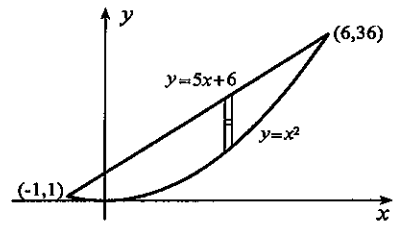
1. $\displaystyle A = \int_0^{1/4}\int_{4x^2}^{\sqrt{x}/2} dy\,dx = \int_0^{1/4} \left(\frac{\sqrt{x}}{2} - 4x^2\right) dx$

$$= \left\{\frac{x^{3/2}}{3} - \frac{4x^3}{3}\right\}_0^{1/4} = \frac{1}{48}$$



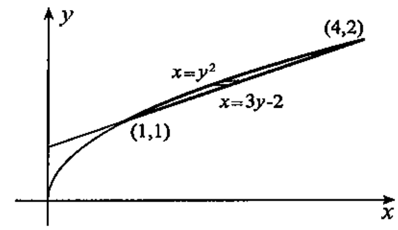
2. $A = \int_{-1}^{6} \int_{x^2}^{5x+6} dy\,dx = \int_{-1}^{6} (5x + 6 - x^2)\,dx$

$$= \left\{ \frac{5x^2}{2} + 6x - \frac{x^3}{3} \right\}_{-1}^{6} = \frac{343}{6}$$



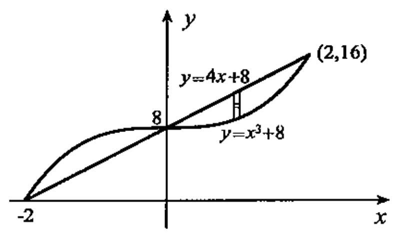
3. $A = \int_{1}^{2} \int_{y^2}^{3y-2} dx\,dy = \int_{1}^{2} (3y - 2 - y^2)\,dy$

$$= \left\{ \frac{3y^2}{2} - 2y - \frac{y^3}{3} \right\}_{1}^{2} = \frac{1}{6}$$



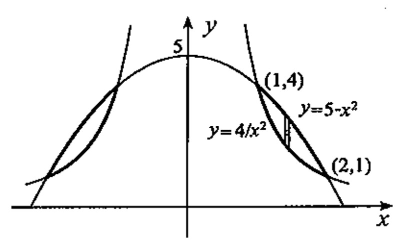
4. $A = 2\int_{0}^{2} \int_{x^3+8}^{4x+8} dy\,dx = 2\int_{0}^{2} (4x - x^3)\,dx$

$$= 2\left\{ 2x^2 - \frac{x^4}{4} \right\}_{0}^{2} = 8$$



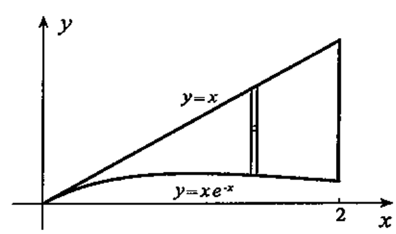
5. $A = 2\int_{1}^{2} \int_{4/x^2}^{5-x^2} dy\,dx = 2\int_{1}^{2} \left(5 - x^2 - \frac{4}{x^2} \right) dx$

$$= 2\left\{ 5x - \frac{x^3}{3} + \frac{4}{x} \right\}_{1}^{2} = \frac{4}{3}$$



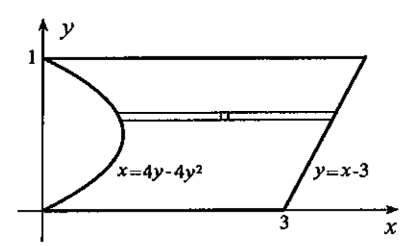
6. $A = \int_{0}^{2} \int_{xe^{-x}}^{x} dy\,dx = \int_{0}^{2} (x - xe^{-x})\,dx$

$$= \left\{ \frac{x^2}{2} + xe^{-x} + e^{-x} \right\}_{0}^{2} = 1 + 3e^{-2}$$



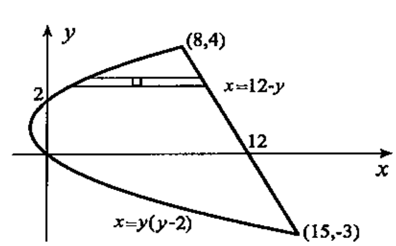
7. $A = \int_{0}^{1} \int_{4y-4y^2}^{y+3} dx\,dy = \int_{0}^{1} (3 - 3y + 4y^2)\,dy$

$$= \left\{ 3y - \frac{3y^2}{2} + \frac{4y^3}{3} \right\}_{0}^{1} = \frac{17}{6}$$



8. $A = \int_{-3}^{4} \int_{y(y-2)}^{12-y} dx\,dy = \int_{-3}^{4} (12 - y^2 + y)\,dy$

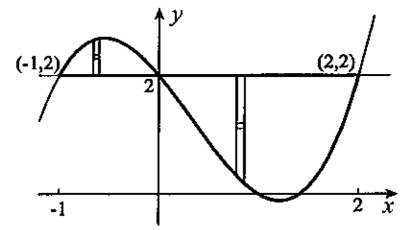
$$= \left\{ 12y - \frac{y^3}{3} + \frac{y^2}{2} \right\}_{-3}^{4} = \frac{343}{6}$$



9. $A = \int_{-1}^{0}\int_{2}^{x^3-x^2-2x+2} dy\,dx + \int_0^2\int_{x^3-x^2-2x+2}^{2} dy\,dx$

$= \int_{-1}^{0}(x^3 - x^2 - 2x)\,dx + \int_0^2(-x^3 + x^2 + 2x)\,dx$

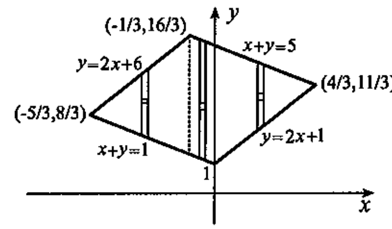
$= \left\{\frac{x^4}{4} - \frac{x^3}{3} - x^2\right\}_{-1}^{0} + \left\{-\frac{x^4}{4} + \frac{x^3}{3} + x^2\right\}_0^2 = \frac{37}{12}$



10. $A = \int_{-5/3}^{-1/3}\int_{1-x}^{2x+6} dy\,dx + \int_{-1/3}^{0}\int_{1-x}^{5-x} dy\,dx + \int_0^{4/3}\int_{2x+1}^{5-x} dy\,dx$

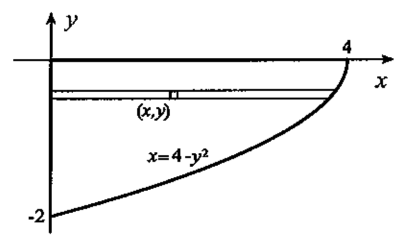
$= \int_{-5/3}^{-1/3}(3x+5)\,dx + \int_{-1/3}^{0}4\,dx + \int_0^{4/3}(4-3x)\,dx$

$= \left\{\frac{3x^2}{2} + 5x\right\}_{-5/3}^{-1/3} + \left\{4x\right\}_{-1/3}^{0} + \left\{4x - \frac{3x^2}{2}\right\}_0^{4/3} = \frac{20}{3}$



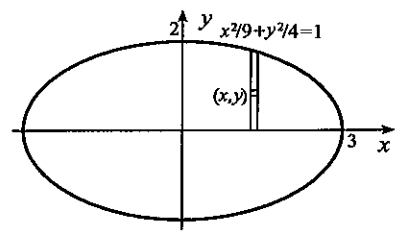
11. $V = \int_{-2}^{0}\int_0^{4-y^2} 2\pi(-y)\,dx\,dy = -2\pi\int_{-2}^{0}\left\{xy\right\}_0^{4-y^2} dy$

$= -2\pi\int_{-2}^{0} y(4-y^2)\,dy = -2\pi\left\{2y^2 - \frac{y^4}{4}\right\}_{-2}^{0} = 8\pi$



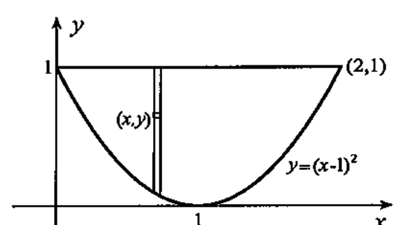
12. $V = 2\int_0^3\int_0^{2\sqrt{9-x^2}/3} 2\pi y\,dy\,dx = 2\pi\int_0^3\left\{y^2\right\}_0^{2\sqrt{9-x^2}/3} dx$

$= \frac{8\pi}{9}\int_0^3(9-x^2)\,dx = \frac{8\pi}{9}\left\{9x - \frac{x^3}{3}\right\}_0^3 = 16\pi$



13. $V = \int_0^2\int_{(x-1)^2}^{1} 2\pi x\,dy\,dx = 2\pi\int_0^2\left\{xy\right\}_{(x-1)^2}^{1} dx$

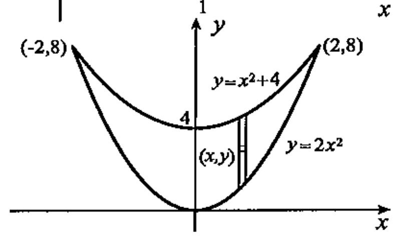
$= 2\pi\int_0^2(-x^3 + 2x^2)\,dx = 2\pi\left\{-\frac{x^4}{4} + \frac{2x^3}{3}\right\}_0^2 = \frac{8\pi}{3}$



14. $V = 2\int_0^2\int_{2x^2}^{x^2+4} 2\pi y\,dy\,dx = 2\pi\int_0^2\left\{y^2\right\}_{2x^2}^{x^2+4} dx$

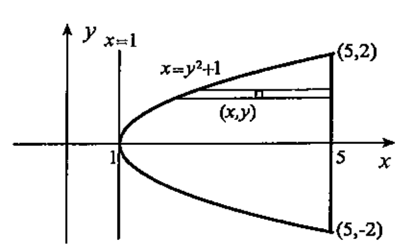
$= 2\pi\int_0^2(16 + 8x^2 - 3x^4)\,dx$

$= 2\pi\left\{16x + \frac{8x^3}{3} - \frac{3x^5}{5}\right\}_0^2 = \frac{1024\pi}{15}$



15. $V = 2\int_0^2\int_{y^2+1}^{5} 2\pi(x-1)\,dx\,dy = 4\pi\int_0^2\left\{\frac{x^2}{2} - x\right\}_{y^2+1}^{5} dy$

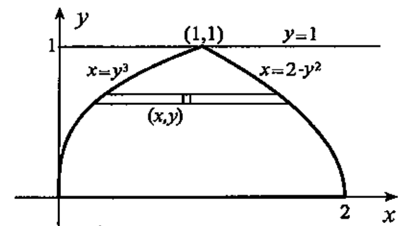
$= 2\pi\int_0^2(16 - y^4)\,dy = 2\pi\left\{16y - \frac{y^5}{5}\right\}_0^2 = \frac{256\pi}{5}$



16. $V = \int_0^1 \int_{y^3}^{2-y^2} 2\pi(1-y)\,dx\,dy = 2\pi \int_0^1 \left\{x(1-y)\right\}_{y^3}^{2-y^2} dy$

$= 2\pi \int_0^1 (2 - 2y - y^2 + y^4)\,dy$

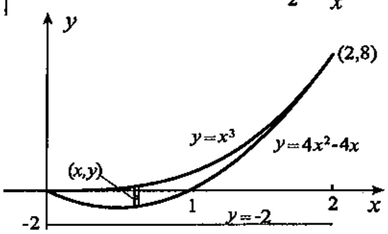
$= 2\pi \left\{2y - y^2 - \frac{y^3}{3} + \frac{y^5}{5}\right\}_0^1 = \frac{26\pi}{15}$



17. $V = \int_0^2 \int_{4x^2-4x}^{x^3} 2\pi(y+2)\,dy\,dx = 2\pi \int_0^2 \left\{\frac{y^2}{2} + 2y\right\}_{4x^2-4x}^{x^3} dx$

$= \pi \int_0^2 (x^6 - 16x^4 + 36x^3 - 32x^2 + 16x)\,dx$

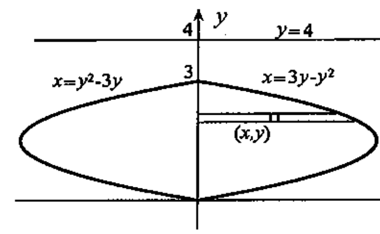
$= \pi \left\{\frac{x^7}{7} - \frac{16x^5}{5} + 9x^4 - \frac{32x^3}{3} + 8x^2\right\}_0^2 = \frac{668\pi}{105}$



18. $V = 2\int_0^3 \int_0^{3y-y^2} 2\pi(4-y)\,dx\,dy = 4\pi \int_0^3 \left\{x(4-y)\right\}_0^{3y-y^2} dy$

$= 4\pi \int_0^3 (12y - 7y^2 + y^3)\,dy$

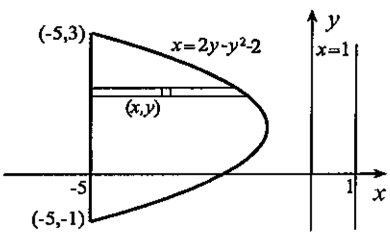
$= 4\pi \left\{6y^2 - \frac{7y^3}{3} + \frac{y^4}{4}\right\}_0^3 = 45\pi$



19. $V = \int_{-1}^3 \int_{-5}^{2y-y^2-2} 2\pi(1-x)\,dx\,dy = 2\pi \int_{-1}^3 \left\{x - \frac{x^2}{2}\right\}_{-5}^{2y-y^2-2} dy$

$= \pi \int_{-1}^3 (27 + 12y - 10y^2 + 4y^3 - y^4)\,dy$

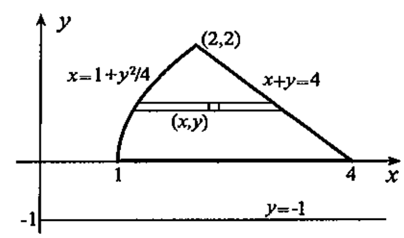
$= \pi \left\{27y + 6y^2 - \frac{10y^3}{3} + y^4 - \frac{y^5}{5}\right\}_{-1}^3 = \frac{1408\pi}{15}$



20. $V = \int_0^2 \int_{y^2/4+1}^{4-y} 2\pi(y+1)\,dx\,dy = 2\pi \int_0^2 \left\{x(y+1)\right\}_{y^2/4+1}^{4-y} dy$

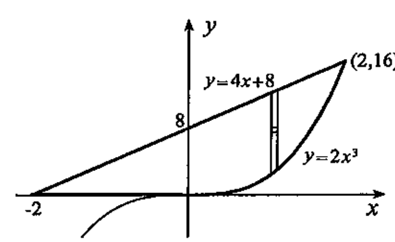
$= \frac{\pi}{2} \int_0^2 (12 + 8y - 5y^2 - y^3)\,dy$

$= \frac{\pi}{2} \left\{12y + 4y^2 - \frac{5y^3}{3} - \frac{y^4}{4}\right\}_0^2 = \frac{34\pi}{3}$



21. $A = \frac{1}{2}(2)(8) + \int_0^2 \int_{2x^3}^{4x+8} dy\,dx = 8 + \int_0^2 (8 + 4x - 2x^3)\,dx$

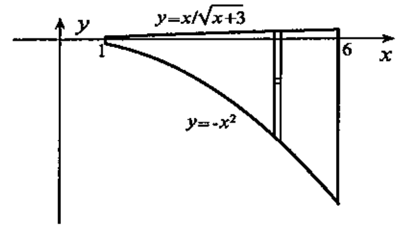
$= 8 + \left\{8x + 2x^2 - \frac{x^4}{2}\right\}_0^2 = 24$



22. $A = \int_1^6 \int_{-x^2}^{x/\sqrt{x+3}} dy\,dx = \int_1^6 \left(\frac{x}{\sqrt{x+3}} + x^2\right) dx$

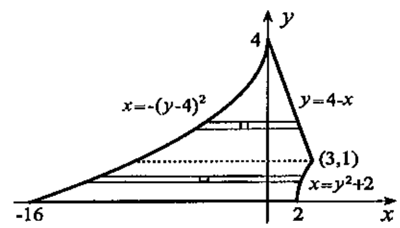
If we set $u = x + 3$ and $du = dx$,

$$A = \int_4^9 \left(\frac{u-3}{\sqrt{u}}\right) du + \left\{\frac{x^3}{3}\right\}_1^6 = \int_4^9 \left(\sqrt{u} - \frac{3}{\sqrt{u}}\right) du + \frac{215}{3}$$
$$= \left\{\frac{2}{3}u^{3/2} - 6\sqrt{u}\right\}_4^9 + \frac{215}{3} = \frac{235}{3}$$



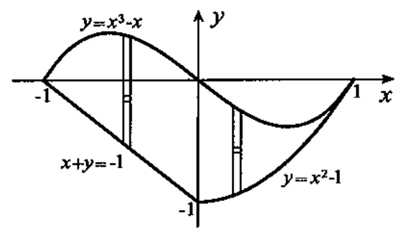
23. $A = \int_0^1 \int_{-(y-4)^2}^{y^2+2} dx\,dy + \int_1^4 \int_{-(y-4)^2}^{4-y} dx\,dy$

$$= \int_0^1 [y^2 + 2 + (y-4)^2]\,dy + \int_1^4 [4 - y + (y-4)^2]\,dy$$
$$= \left\{\frac{y^3}{3} + 2y + \frac{(y-4)^3}{3}\right\}_0^1 + \left\{4y - \frac{y^2}{2} + \frac{(y-4)^3}{3}\right\}_1^4$$
$$= \frac{169}{6}$$



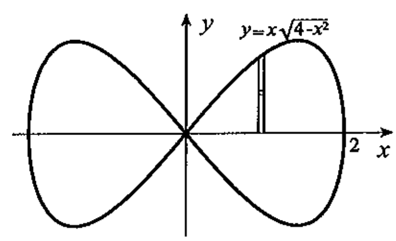
24. $A = \int_{-1}^0 \int_{-1-x}^{x^3-x} dy\,dx + \int_0^1 \int_{x^2-1}^{x^3-x} dy\,dx$

$$= \int_{-1}^0 (x^3 + 1)\,dx + \int_0^1 (x^3 - x - x^2 + 1)\,dx$$
$$= \left\{\frac{x^4}{4} + x\right\}_{-1}^0 + \left\{\frac{x^4}{4} - \frac{x^2}{2} - \frac{x^3}{3} + x\right\}_0^1 = \frac{7}{6}$$



25. $A = 4\int_0^2 \int_0^{x\sqrt{4-x^2}} dy\,dx = 4\int_0^2 x\sqrt{4-x^2}\,dx$

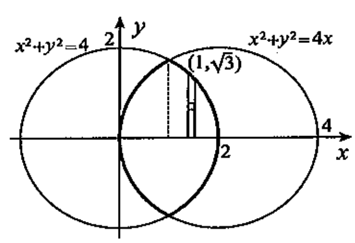
$$= 4\left\{-\frac{1}{3}(4-x^2)^{3/2}\right\}_0^2 = \frac{32}{3}$$



26. $A = 4\int_1^2 \int_0^{\sqrt{4-x^2}} dy\,dx = 4\int_1^2 \sqrt{4-x^2}\,dx$

If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

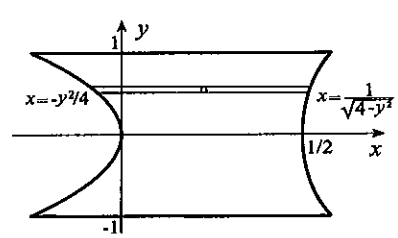
$$A = 4\int_{\pi/6}^{\pi/2} 2\cos\theta\, 2\cos\theta\,d\theta = 16\int_{\pi/6}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$
$$= 8\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{\pi/6}^{\pi/2} = \frac{8\pi}{3} - 2\sqrt{3}$$



27. $A = 2\int_0^1 \int_{-y^2/4}^{1/\sqrt{4-y^2}} dx\,dy = 2\int_0^1 \left(\frac{1}{\sqrt{4-y^2}} + \frac{y^2}{4}\right) dy$

If we set $y = 2\sin\theta$ and $dy = 2\cos\theta\,d\theta$, then

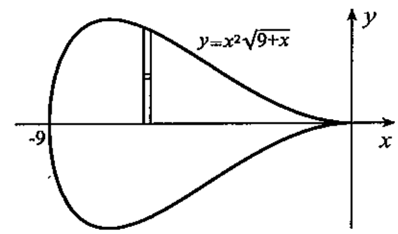
$$A = 2\int_0^{\pi/6} \frac{1}{2\cos\theta} 2\cos\theta\,d\theta + 2\left\{\frac{y^3}{12}\right\}_0^1$$
$$= 2\Big\{\theta\Big\}_0^{\pi/6} + \frac{1}{6} = \frac{\pi}{3} + \frac{1}{6}$$



28. $A = 2\int_{-9}^{0}\int_{0}^{x^2\sqrt{9+x}} dy\,dx = 2\int_{-9}^{0} x^2\sqrt{9+x}\,dx$

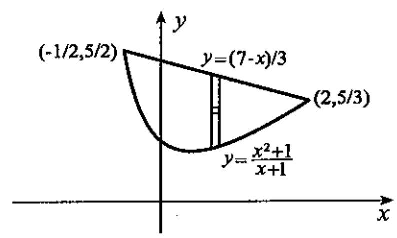
If we set $u = 9 + x$ and $du = dx$,

$$A = 2\int_0^9 (u-9)^2\sqrt{u}\,du$$
$$= 2\int_0^9 (u^{5/2} - 18u^{3/2} + 81u^{1/2})\,du$$
$$= 2\left\{\frac{2u^{7/2}}{7} - \frac{36u^{5/2}}{5} + \frac{162u^{3/2}}{3}\right\}_0^9 = \frac{23\,328}{35}$$



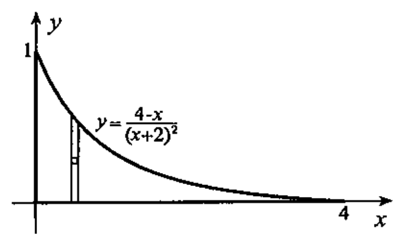
29. $A = \int_{-1/2}^{2}\int_{(x^2+1)/(x+1)}^{(7-x)/3} dy\,dx$

$$= \int_{-1/2}^{2}\left(\frac{7-x}{3} - \frac{x^2+1}{x+1}\right)dx$$
$$= \int_{-1/2}^{2}\left(\frac{10}{3} - \frac{4x}{3} - \frac{2}{x+1}\right)dx$$
$$= \left\{\frac{10x}{3} - \frac{2x^2}{3} - 2\ln|x+1|\right\}_{-1/2}^{2} = \frac{35}{6} - 2\ln 6$$



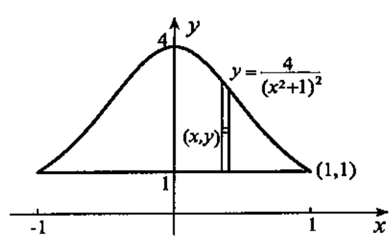
30. $A = \int_0^4\int_0^{(4-x)/(x+2)^2} dy\,dx = \int_0^4 \frac{4-x}{(x+2)^2}dx$

$$= \int_0^4\left[\frac{6}{(x+2)^2} - \frac{1}{x+2}\right]dx$$
$$= \left\{-\frac{6}{x+2} - \ln|x+2|\right\}_0^4 = 2 - \ln 3$$



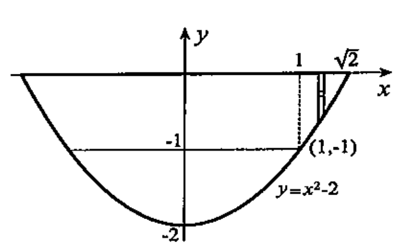
31. $V = \int_0^1\int_1^{4/(x^2+1)^2} 2\pi x\,dy\,dx = 2\pi\int_0^1\left\{xy\right\}_1^{4/(x^2+1)^2}dx$

$$= 2\pi\int_0^1\left[\frac{4x}{(x^2+1)^2} - x\right]dx = 2\pi\left\{\frac{-2}{x^2+1} - \frac{x^2}{2}\right\}_0^1 = \pi$$



32. We reject the area below $y = -1$ to obtain

$$V = 2\pi(1)^2(1) + 2\int_1^{\sqrt{2}}\int_{x^2-2}^{0} 2\pi(y+1)\,dy\,dx$$
$$= 2\pi + 4\pi\int_1^{\sqrt{2}}\left\{\frac{1}{2}(y+1)^2\right\}_{x^2-2}^{0}dx$$
$$= 2\pi + 2\pi\int_1^{\sqrt{2}}(-x^4 + 2x^2)\,dx$$
$$= 2\pi + 2\pi\left\{-\frac{x^5}{5} + \frac{2x^3}{3}\right\}_1^{\sqrt{2}} = \frac{16\pi(\sqrt{2}+1)}{15}$$

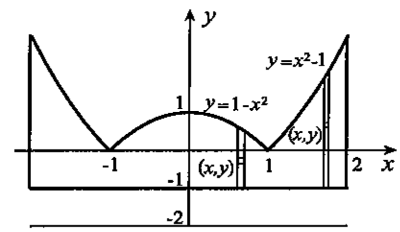


33. $V = 2\int_0^1 \int_{-1}^{1-x^2} 2\pi(y+2)\,dy\,dx + 2\int_1^2 \int_{-1}^{x^2-1} 2\pi(y+2)\,dy\,dx$

$$= 4\pi \int_0^1 \left\{\frac{(y+2)^2}{2}\right\}_{-1}^{1-x^2} dx + 4\pi \int_1^2 \left\{\frac{(y+2)^2}{2}\right\}_{-1}^{x^2-1} dx$$

$$= 2\pi \int_0^1 (8 - 6x^2 + x^4)\,dx + 2\pi \int_1^2 (2x^2 + x^4)\,dx$$

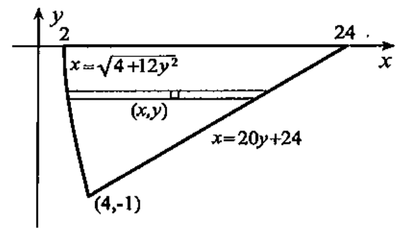
$$= 2\pi \left\{8x - 2x^3 + \frac{x^5}{5}\right\}_0^1 + 2\pi \left\{\frac{2x^3}{3} + \frac{x^5}{5}\right\}_1^2 = \frac{512\pi}{15}$$



34. $V = \int_{-1}^0 \int_{\sqrt{4+12y^2}}^{20y+24} 2\pi(-y)\,dx\,dy = 2\pi \int_{-1}^0 \left\{-xy\right\}_{\sqrt{4+12y^2}}^{20y+24} dy$

$$= 2\pi \int_{-1}^0 (y\sqrt{4+12y^2} - 20y^2 - 24y)\,dy$$

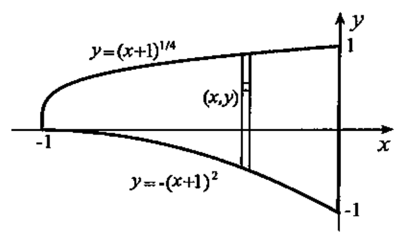
$$= 2\pi \left\{\frac{1}{36}(4+12y^2)^{3/2} - \frac{20y^3}{3} - 12y^2\right\}_{-1}^0 = \frac{68\pi}{9}$$



35. $V = \int_{-1}^0 \int_{-(x+1)^2}^{(x+1)^{1/4}} 2\pi(-x)\,dy\,dx = -2\pi \int_{-1}^0 \left\{xy\right\}_{-(x+1)^2}^{(x+1)^{1/4}} dx$

$$= -2\pi \int_{-1}^0 [x(x+1)^{1/4} + x(x+1)^2]\,dx$$

If we set $u = x+1$ and $du = dx$ in the first term,



$$V = -2\pi \int_0^1 (u-1)u^{1/4}\,du - 2\pi \int_{-1}^0 (x^3 + 2x^2 + x)\,dx$$

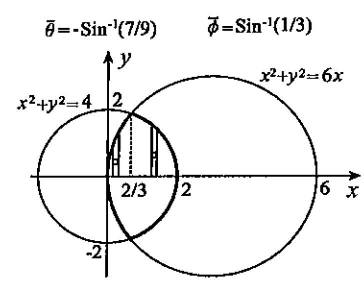
$$= -2\pi \left\{\frac{4u^{9/4}}{9} - \frac{4u^{5/4}}{5}\right\}_0^1 - 2\pi \left\{\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right\}_{-1}^0 = \frac{79\pi}{90}$$

36. $A = 2\int_0^{2/3} \int_0^{\sqrt{6x-x^2}} dy\,dx + 2\int_{2/3}^2 \int_0^{\sqrt{4-x^2}} dy\,dx$

$$= 2\int_0^{2/3} \sqrt{6x - x^2}\,dx + 2\int_{2/3}^2 \sqrt{4-x^2}\,dx$$

$$= 2\int_0^{2/3} \sqrt{9 - (x-3)^2}\,dx + 2\int_{2/3}^2 \sqrt{4-x^2}\,dx$$

If we set $x - 3 = 3\sin\theta$ in the first integral, and $x = 2\sin\phi$ in the second,



$$A = 2\int_{-\pi/2}^{\overline{\theta}} 3\cos\theta\, 3\cos\theta\,d\theta + 2\int_{\overline{\phi}}^{\pi/2} 2\cos\phi\, 2\cos\phi\,d\phi$$

$$= 18\int_{-\pi/2}^{\overline{\theta}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta + 8\int_{\overline{\phi}}^{\pi/2} \left(\frac{1+\cos 2\phi}{2}\right) d\phi$$

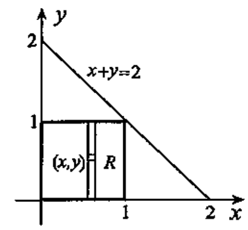
$$= 9\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/2}^{\overline{\theta}} + 4\left\{\phi + \frac{1}{2}\sin 2\phi\right\}_{\overline{\phi}}^{\pi/2}$$

$$= 9\left(\overline{\theta} + \sin\overline{\theta}\,\cos\overline{\theta} + \frac{\pi}{2}\right) + 4\left(\frac{\pi}{2} - \overline{\phi} - \sin\overline{\phi}\,\cos\overline{\phi}\right) = 5.38.$$

37. $V = \iint_R 2\pi \dfrac{|x+y-2|}{\sqrt{2}} dA = \sqrt{2}\pi \int_0^1 \int_0^1 (2-x-y)\,dy\,dx$

$= \sqrt{2}\pi \int_0^1 \left\{ 2y - xy - \dfrac{y^2}{2} \right\}_0^1 dx$

$= \sqrt{2}\pi \int_0^1 \left(2 - x - \dfrac{1}{2} \right) dx = \sqrt{2}\pi \left\{ \dfrac{3x}{2} - \dfrac{x^2}{2} \right\}_0^1 = \sqrt{2}\pi$

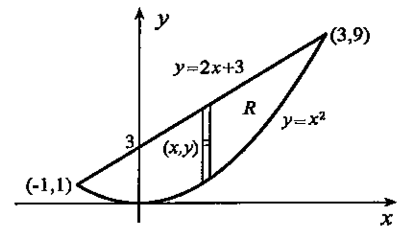


38. $V = \iint_R 2\pi \dfrac{|2x-y+3|}{\sqrt{5}} dA = \dfrac{2\pi}{\sqrt{5}} \int_{-1}^3 \int_{x^2}^{2x+3} (2x-y+3)\,dy\,dx$

$= \dfrac{2\pi}{\sqrt{5}} \int_{-1}^3 \left\{ (2x+3)y - \dfrac{y^2}{2} \right\}_{x^2}^{2x+3} dx$

$= \dfrac{2\pi}{\sqrt{5}} \int_{-1}^3 \left[\dfrac{1}{2}(2x+3)^2 + \dfrac{x^4}{2} - 2x^3 - 3x^2 \right] dx$

$= \dfrac{2\pi}{\sqrt{5}} \left\{ \dfrac{1}{12}(2x+3)^3 + \dfrac{x^5}{10} - \dfrac{x^4}{2} - x^3 \right\}_{-1}^3 = \dfrac{512\pi}{15\sqrt{5}}$

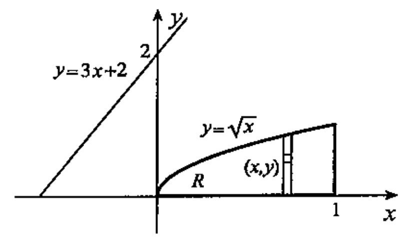


39. $V = \iint_R 2\pi \dfrac{|y-3x-2|}{\sqrt{10}} dA = \dfrac{2\pi}{\sqrt{10}} \int_0^1 \int_0^{\sqrt{x}} (3x+2-y)\,dy\,dx$

$= \dfrac{2\pi}{\sqrt{10}} \int_0^1 \left\{ (3x+2)y - \dfrac{y^2}{2} \right\}_0^{\sqrt{x}} dx$

$= \dfrac{\pi}{\sqrt{10}} \int_0^1 (6x^{3/2} + 4\sqrt{x} - x)\,dx$

$= \dfrac{\pi}{\sqrt{10}} \left\{ \dfrac{12x^{5/2}}{5} + \dfrac{8x^{3/2}}{3} - \dfrac{x^2}{2} \right\}_0^1 = \dfrac{137\pi}{30\sqrt{10}}$

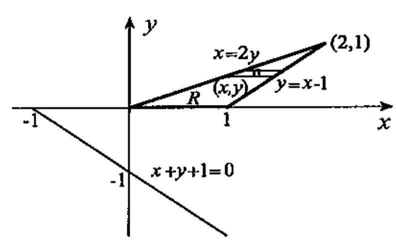


40. $V = \iint_R 2\pi \dfrac{|x+y+1|}{\sqrt{2}} dA = \sqrt{2}\pi \int_0^1 \int_{2y}^{y+1} (x+y+1)\,dx\,dy$

$= \sqrt{2}\pi \int_0^1 \left\{ \dfrac{1}{2}(x+y+1)^2 \right\}_{2y}^{y+1} dy$

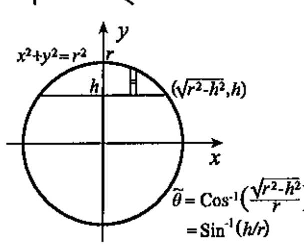
$= \dfrac{\pi}{\sqrt{2}} \int_0^1 [(2y+2)^2 - (3y+1)^2]\,dy$

$= \dfrac{\pi}{\sqrt{2}} \left\{ \dfrac{1}{6}(2y+2)^3 - \dfrac{1}{9}(3y+1)^3 \right\}_0^1 = \dfrac{7\sqrt{2}\pi}{6}$



41. $A = 2\int_0^{\sqrt{r^2-h^2}} \int_h^{\sqrt{r^2-x^2}} dy\,dx = 2\int_0^{\sqrt{r^2-h^2}} (\sqrt{r^2-x^2} - h)\,dx$

If we set $x = r\cos\theta$ and $dx = -r\sin\theta\,d\theta$,



$A = 2\int_{\pi/2}^{\tilde{\theta}} r\sin\theta(-r\sin\theta\,d\theta) - 2h\left\{ x \right\}_0^{\sqrt{r^2-h^2}}$

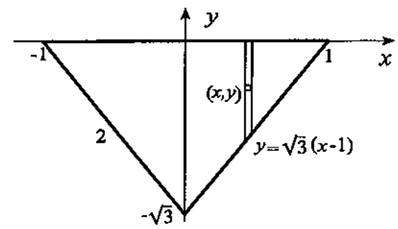
$= 2r^2 \int_{\pi/2}^{\tilde{\theta}} \left(\dfrac{\cos 2\theta - 1}{2} \right) d\theta - 2h\sqrt{r^2-h^2}$

$= r^2 \left\{ \dfrac{1}{2}\sin 2\theta - \theta \right\}_{\pi/2}^{\tilde{\theta}} - 2h\sqrt{r^2-h^2} = r^2(\sin\tilde{\theta}\cos\tilde{\theta} - \tilde{\theta} + \pi/2) - 2h\sqrt{r^2-h^2}$

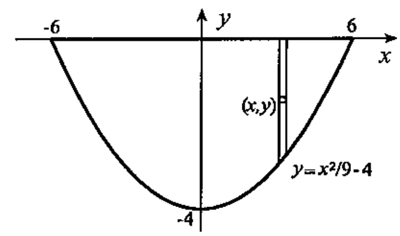
$= \dfrac{\pi r^2}{2} + r^2 \left[\dfrac{h}{r} \dfrac{\sqrt{r^2-h^2}}{r} - \mathrm{Sin}^{-1}(h/r) \right] - 2h\sqrt{r^2-h^2} = \dfrac{\pi r^2}{2} - h\sqrt{r^2-h^2} - r^2\mathrm{Sin}^{-1}(h/r).$

EXERCISES 13.4

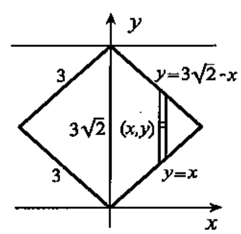
1. $$F = 2\int_0^1 \int_{\sqrt{3}(x-1)}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -2\rho g \int_0^1 \left\{\frac{y^2}{2}\right\}_{\sqrt{3}(x-1)}^0 dx$$
$$= \rho g \int_0^1 3(x-1)^2\,dx = \rho g \Big\{(x-1)^3\Big\}_0^1 = \rho g$$



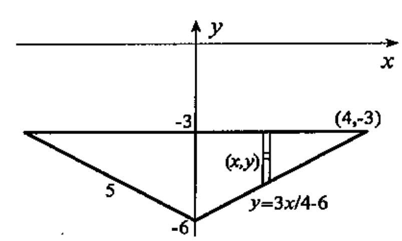
2. $$F = 2\int_0^6 \int_{x^2/9-4}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -2\rho g \int_0^6 \left\{\frac{y^2}{2}\right\}_{x^2/9-4}^0 dx = \rho g \int_0^6 \left(\frac{x^2}{9} - 4\right)^2 dx$$
$$= \frac{\rho g}{81}\int_0^6 (x^4 - 72x^2 + 1296)\,dx$$
$$= \frac{\rho g}{81}\left\{\frac{x^5}{5} - 24x^3 + 1296x\right\}_0^6 = \frac{256\rho g}{5}$$



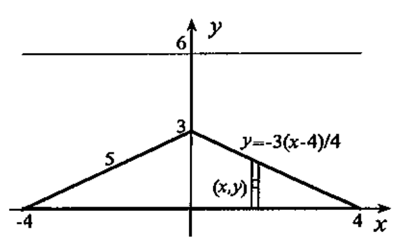
3. $$F = 2\int_0^{3/\sqrt{2}} \int_x^{3\sqrt{2}-x} \rho g(3\sqrt{2} - y)\,dy\,dx \quad (g = 9.81)$$
$$= 2\rho g \int_0^{3/\sqrt{2}} \left\{-\frac{1}{2}(3\sqrt{2} - y)^2\right\}_x^{3\sqrt{2}-x} dx$$
$$= \rho g \int_0^{3/\sqrt{2}} \left[(3\sqrt{2} - x)^2 - x^2\right] dx$$
$$= \rho g \left\{-\frac{1}{3}(3\sqrt{2} - x)^3 - \frac{x^3}{3}\right\}_0^{3/\sqrt{2}} = \frac{27\rho g}{\sqrt{2}}$$



4. $$F = 2\int_0^4 \int_{3x/4-6}^{-3} \rho g(-y)\,dy\,dx \quad (g = 9.81)$$
$$= -2\rho g \int_0^4 \left\{\frac{y^2}{2}\right\}_{3x/4-6}^{-3} dx = \rho g \int_0^4 \left[\left(\frac{3x}{4} - 6\right)^2 - 9\right] dx$$
$$= \rho g \left\{\frac{4}{9}\left(\frac{3x}{4} - 6\right)^3 - 9x\right\}_0^4 = 48\rho g$$



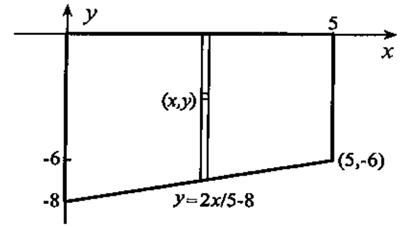
5. $$F = 2\int_0^4 \int_0^{-3(x-4)/4} \rho g(6 - y)\,dy\,dx \quad (g = 9.81)$$
$$= 2\rho g \int_0^4 \left\{-\frac{1}{2}(6 - y)^2\right\}_0^{-3(x-4)/4} dx$$
$$= \rho g \int_0^4 \left\{36 - \left[6 + \frac{3}{4}(x-4)\right]^2\right\} dx$$
$$= \rho g \left\{36x - \frac{4}{9}\left(3 + \frac{3x}{4}\right)^3\right\}_0^4 = 60\rho g$$



6. $F = \int_0^5 \int_{2x/5-8}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$

$$= -\rho g \int_0^5 \left\{ \frac{y^2}{2} \right\}_{2x/5-8}^0 dx = \frac{\rho g}{2} \int_0^5 \left(\frac{2x}{5} - 8 \right)^2 dx$$

$$= \frac{\rho g}{2} \left\{ \frac{5}{6} \left(\frac{2x}{5} - 8 \right)^3 \right\}_0^5 = \frac{370\rho g}{3}$$

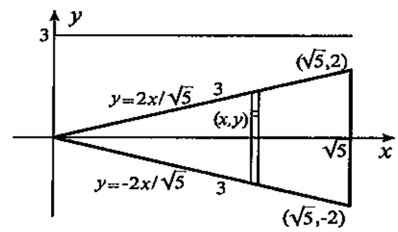


7. $F = \int_0^{\sqrt{5}} \int_{-2x/\sqrt{5}}^{2x/\sqrt{5}} \rho g(3-y)\,dy\,dx \quad (g = 9.81)$

$$= \rho g \int_0^{\sqrt{5}} \left\{ -\frac{1}{2}(3-y)^2 \right\}_{-2x/\sqrt{5}}^{2x/\sqrt{5}} dx$$

$$= \frac{\rho g}{2} \int_0^{\sqrt{5}} \left[\left(3 + \frac{2x}{\sqrt{5}}\right)^2 - \left(3 - \frac{2x}{\sqrt{5}}\right)^2 \right] dx$$

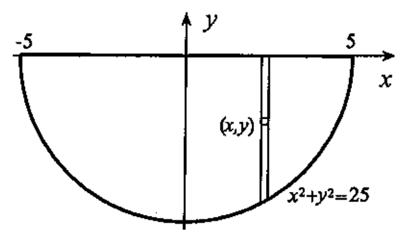
$$= \frac{\rho g}{2} \left\{ \frac{\sqrt{5}}{6} \left(3 + \frac{2x}{\sqrt{5}}\right)^3 + \frac{\sqrt{5}}{6} \left(3 - \frac{2x}{\sqrt{5}}\right)^3 \right\}_0^{\sqrt{5}} = 6\sqrt{5}\rho g$$



8. $F = 2\int_0^5 \int_{-\sqrt{25-x^2}}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$

$$= -2\rho g \int_0^5 \left\{ \frac{y^2}{2} \right\}_{-\sqrt{25-x^2}}^0 dx = \rho g \int_0^5 (25 - x^2)\,dx$$

$$= \rho g \left\{ 25x - \frac{x^3}{3} \right\}_0^5 = \frac{250\rho g}{3}$$

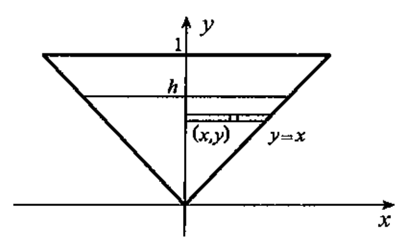


9. The trough is half-full by volume when the vertical end is half covered by area. If h is the depth of water when this happens, $h^2 = 1/2 \implies h = 1/\sqrt{2}$.

$$F = 2\int_0^{1/\sqrt{2}} \int_0^y \rho g \left(\frac{1}{\sqrt{2}} - y \right) dx\,dy \quad (g = 9.81)$$

$$= 2\rho g \int_0^{1/\sqrt{2}} \left\{ x \left(\frac{1}{\sqrt{2}} - y \right) \right\}_0^y dy$$

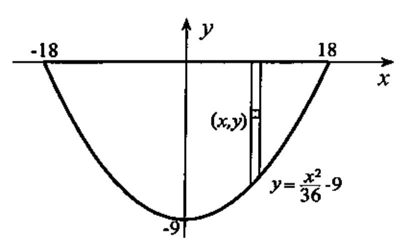
$$= 2\rho g \int_0^{1/\sqrt{2}} \left(\frac{y}{\sqrt{2}} - y^2 \right) dy = 2\rho g \left\{ \frac{y^2}{2\sqrt{2}} - \frac{y^3}{3} \right\}_0^{1/\sqrt{2}} = \frac{\rho g}{6\sqrt{2}}$$



10. $F = 2\int_0^{18} \int_{x^2/36-9}^0 \rho g(-y)\,dy\,dx \quad (g = 9.81)$

$$= -2\rho g \int_0^{18} \left\{ \frac{y^2}{2} \right\}_{x^2/36-9}^0 dx = \rho g \int_0^{18} \left(\frac{x^4}{1296} - \frac{x^2}{2} + 81 \right) dx$$

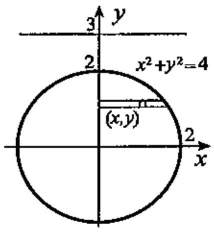
$$= 9810 \left\{ \frac{x^5}{1296 \cdot 5} - \frac{x^3}{6} + 81x \right\}_0^{18} = 7.63 \times 10^6 \text{ N}$$



11. $F = 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} \rho g(3-y)\,dx\,dy \quad (g = 9.81)$

$= 2\rho g \int_{-2}^{2} \left\{ x(3-y) \right\}_{0}^{\sqrt{4-y^2}} dy$

$= 2\rho g \int_{-2}^{2} (3-y)\sqrt{4-y^2}\,dy$



If we set $y = 2\sin\theta$ and $dy = 2\cos\theta\,d\theta$,

$$F = 6\rho g \int_{-\pi/2}^{\pi/2} 2\cos\theta(2\cos\theta\,d\theta) - 2\rho g \left\{ -\frac{1}{3}(4-y^2)^{3/2} \right\}_{-2}^{2}$$

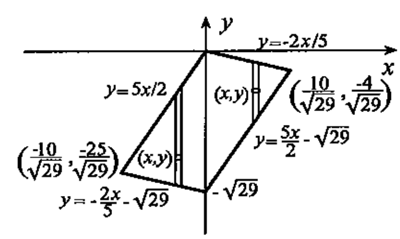
$$= 24\rho g \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right) d\theta = 12\rho g \left\{ \theta + \frac{1}{2}\sin 2\theta \right\}_{-\pi/2}^{\pi/2} = 12\rho g\pi$$

12. $F = \int_{-10/\sqrt{29}}^{0}\int_{-2x/5-\sqrt{29}}^{5x/2} \rho g(-y)\,dy\,dx \quad (g = 9.81)$

$+ \int_{0}^{10/\sqrt{29}}\int_{5x/2-\sqrt{29}}^{-2x/5} \rho g(-y)\,dy\,dx$

$= -\rho g \int_{-10/\sqrt{29}}^{0} \left\{ \frac{y^2}{2} \right\}_{-2x/5-\sqrt{29}}^{5x/2} dx$

$-\rho g \int_{0}^{10/\sqrt{29}} \left\{ \frac{y^2}{2} \right\}_{5x/2-\sqrt{29}}^{-2x/5} dx$



$= \frac{\rho g}{2} \int_{-10/\sqrt{29}}^{0} \left[\left(-\frac{2x}{5} - \sqrt{29} \right)^2 - \frac{25x^2}{4} \right] dx$

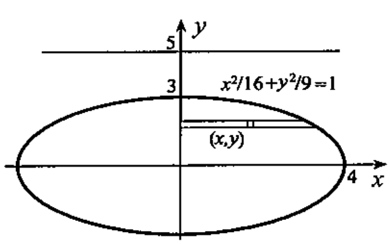
$+ \frac{\rho g}{2} \int_{0}^{10/\sqrt{29}} \left[\left(\frac{5x}{2} - \sqrt{29} \right)^2 - \frac{4x^2}{25} \right] dx$

$$= \frac{\rho g}{2} \left\{ \frac{5}{6}\left(\frac{2x}{5} + \sqrt{29} \right)^3 - \frac{25x^3}{12} \right\}_{-10/\sqrt{29}}^{0} + \frac{\rho g}{2} \left\{ \frac{2}{15}\left(\frac{5x}{2} - \sqrt{29} \right)^3 - \frac{4x^3}{75} \right\}_{0}^{10/\sqrt{29}} = 5\sqrt{29}\rho g$$

13. $F = 2\int_{-3}^{3}\int_{0}^{(4/3)\sqrt{9-y^2}} \rho g(5-y)\,dx\,dy \quad (g = 9.81)$

$= 2\rho g \int_{-3}^{3} \left\{ x(5-y) \right\}_{0}^{(4/3)\sqrt{9-y^2}} dy$

$= \frac{8\rho g}{3} \int_{-3}^{3} (5-y)\sqrt{9-y^2}\,dy$



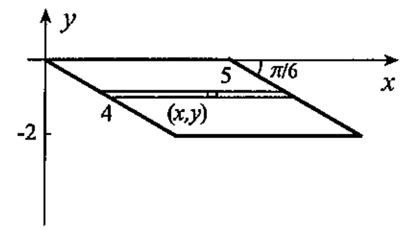
If we set $y = 3\sin\theta$ and $dy = 3\cos\theta\,d\theta$,

$$F = \frac{8\rho g}{3} \int_{-\pi/2}^{\pi/2} 5(3\cos\theta)(3\cos\theta\,d\theta) - \frac{8\rho g}{3} \left\{ -\frac{1}{3}(9-y^2)^{3/2} \right\}_{-3}^{3} = 120\rho g \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= 60\rho g \left\{ \theta + \frac{1}{2}\sin 2\theta \right\}_{-\pi/2}^{\pi/2} = 60\rho g\pi.$$

14. $F = \int_{-2}^{0} \rho g(-y) 5\, dy \quad (g = 9.81)$

$$= -5\rho g \left\{ \frac{y^2}{2} \right\}_{-2}^{0} = 10\rho g$$

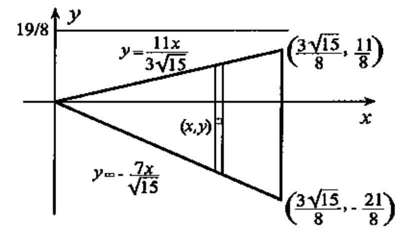


15. $F = \int_0^{3\sqrt{15}/8} \int_{-7x/\sqrt{15}}^{11x/(3\sqrt{15})} \rho g \left(\frac{19}{8} - y \right) dy\, dx \quad (g = 9.81)$

$$= \rho g \int_0^{3\sqrt{15}/8} \left\{ -\frac{1}{2} \left(\frac{19}{8} - y \right)^2 \right\}_{-7x/\sqrt{15}}^{11x/(3\sqrt{15})} dx$$

$$= \frac{\rho g}{2} \int_0^{3\sqrt{15}/8} \left[\left(\frac{19}{8} + \frac{7x}{\sqrt{15}} \right)^2 - \left(\frac{19}{8} - \frac{11x}{3\sqrt{15}} \right)^2 \right] dx$$

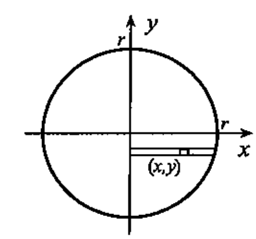
$$= \frac{\rho g}{2} \left\{ \frac{\sqrt{15}}{21} \left(\frac{19}{8} + \frac{7x}{\sqrt{15}} \right)^3 + \frac{\sqrt{15}}{11} \left(\frac{19}{8} - \frac{11x}{3\sqrt{15}} \right)^3 \right\}_0^{3\sqrt{15}/8} = \frac{67\sqrt{15}\rho g}{32}$$



16. $F = 2 \int_{-r}^{r} \int_0^{\sqrt{r^2-y^2}} \rho g(r - y)\, dx\, dy \quad (g = 9.81)$

$$= 2\rho g \int_{-r}^{r} \left\{ x(r - y) \right\}_0^{\sqrt{r^2-y^2}} dy$$

$$= 2\rho g \int_{-r}^{r} (r - y)\sqrt{r^2 - y^2}\, dy$$



If we set $y = r\sin\theta$ and $dy = r\cos\theta\, d\theta$ in the first term,

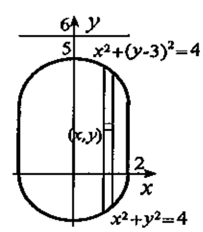
$$F = 2\rho g \int_{-\pi/2}^{\pi/2} r(r\cos\theta) r\cos\theta\, d\theta + 2\rho g \left\{ \frac{1}{3}(r^2 - y^2)^{3/2} \right\}_{-r}^{r}$$

$$= 2\rho g r^3 \int_{-\pi/2}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta = \rho g r^3 \left\{ \theta + \frac{\sin 2\theta}{2} \right\}_{-\pi/2}^{\pi/2} = \pi\rho g r^3$$

17. $F = 2 \int_0^2 \int_{-\sqrt{4-x^2}}^{3+\sqrt{4-x^2}} \rho g(6 - y)\, dy\, dx \quad (g = 9.81)$

$$= 2\rho g \int_0^2 \left\{ -\frac{1}{2}(6 - y)^2 \right\}_{-\sqrt{4-x^2}}^{3+\sqrt{4-x^2}} dx$$

$$= 9\rho g \int_0^2 (3 + 2\sqrt{4 - x^2})\, dx$$



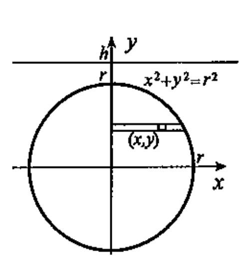
If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$,

$$F = 9\rho g \left\{ 3x \right\}_0^2 + 18\rho g \int_0^{\pi/2} 2\cos\theta 2\cos\theta\, d\theta = 54\rho g + 72\rho g \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= 54\rho g + 36\rho g \left\{ \theta + \frac{1}{2}\sin 2\theta \right\}_0^{\pi/2} = 1.08 \times 10^6 \text{ N}.$$

18. According to Exercise 40 in Section 7.7, the force on the plate is $F = \rho g h(\pi r^2) = \pi\rho g h r^2$. By symmetry, $x_c = 0$. If we integrate over the right-half of the circle and double the result,

$$\begin{aligned} Fy_c &= 2\int_{-r}^{r}\int_0^{\sqrt{r^2-y^2}} y\rho g(h-y)\,dx\,dy \\ &= 2\rho g\int_{-r}^{r}\left\{xy(h-y)\right\}_0^{\sqrt{r^2-y^2}} dy \\ &= 2\rho g\int_{-r}^{r}(hy\sqrt{r^2-y^2} - y^2\sqrt{r^2-y^2})\,dy. \end{aligned}$$

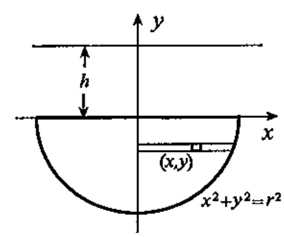


If we set $y = r\sin\theta$ and $dy = r\cos\theta\,d\theta$ in the last term,

$$\begin{aligned} y_c &= \frac{2\rho g h}{F}\left\{-\frac{1}{3}(r^2-y^2)^{3/2}\right\}_{-r}^{r} - \frac{2\rho g}{F}\int_{-\pi/2}^{\pi/2} r^2\sin^2\theta\, r\cos\theta\, r\cos\theta\,d\theta \\ &= -\frac{2\rho g r^4}{F}\int_{-\pi/2}^{\pi/2}\left(\frac{1-\cos 4\theta}{8}\right)d\theta = -\frac{\rho g r^4}{4F}\left\{\theta - \frac{\sin 4\theta}{4}\right\}_{-\pi/2}^{\pi/2} = -\frac{\rho g\pi r^4}{4(\pi\rho g h r^2)} = -\frac{r^2}{4h}. \end{aligned}$$

19. The fluid force on the surface is

$$\begin{aligned} F &= 2\int_{-r}^{0}\int_0^{\sqrt{r^2-y^2}}\rho g(h-y)\,dx\,dy \\ &= 2\rho g\int_{-r}^{0}\left\{x(h-y)\right\}_0^{\sqrt{r^2-y^2}}dy = 2\rho g\int_{-r}^{0}(h-y)\sqrt{r^2-y^2}\,dy \\ &= 2\rho g h\int_{-r}^{0}\sqrt{r^2-y^2}\,dy - 2\rho g\int_{-r}^{0} y\sqrt{r^2-y^2}\,dy \\ &= 2\rho g h\left(\frac{\pi r^2}{4}\right) - 2\rho g\left\{-\frac{(r^2-y^2)^{3/2}}{3}\right\}_{-r}^{0} = \frac{\rho g r^2(3\pi h + 4r)}{6}. \end{aligned}$$



By symmetry, $x_c = 0$, and

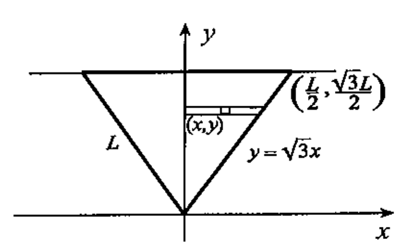
$$Fy_c = 2\int_{-r}^{0}\int_0^{\sqrt{r^2-y^2}}\rho g y(h-y)\,dx\,dy = 2\rho g\int_{-r}^{0}\left\{xy(h-y)\right\}_0^{\sqrt{r^2-y^2}}dy = 2\rho g\int_{-r}^{0} y(h-y)\sqrt{r^2-y^2}\,dy.$$

If we set $y = r\sin\theta$ and $dy = r\cos\theta\,d\theta$ in the second term,

$$\begin{aligned} y_c &= \frac{2\rho g h}{F}\left\{-\frac{(r^2-y^2)^{3/2}}{3}\right\}_{-r}^{0} - \frac{2\rho g}{F}\int_{-\pi/2}^{0} r^2\sin^2\theta\, r\cos\theta\, r\cos\theta\,d\theta \\ &= -\frac{2\rho g h r^3}{3F} - \frac{2\rho g r^4}{F}\int_{-\pi/2}^{0}\left(\frac{1-\cos 4\theta}{8}\right)d\theta = -\frac{2\rho g h r^3}{3F} - \frac{\rho g r^4}{4F}\left\{\theta - \frac{\sin 4\theta}{4}\right\}_{-\pi/2}^{0} \\ &= -\frac{2\rho g h r^3}{3F} - \frac{\rho g r^4\pi}{8F} = -\frac{\rho g r^3(3\pi r + 16h)}{24}\,\frac{6}{\rho g r^2(3\pi h + 4r)} = -\frac{r(3\pi r + 16h)}{4(3\pi h + 4r)}. \end{aligned}$$

20. The fluid force on the triangle is

$$\begin{aligned} F &= 2\int_0^{\sqrt{3}L/2}\int_0^{y/\sqrt{3}}\rho g\left(\frac{\sqrt{3}L}{2} - y\right)dx\,dy \\ &= \rho g\int_0^{\sqrt{3}L/2}\left\{x(\sqrt{3}L - 2y)\right\}_0^{y/\sqrt{3}}dy \\ &= \frac{\rho g}{\sqrt{3}}\int_0^{\sqrt{3}L/2}(\sqrt{3}Ly - 2y^2)\,dy \\ &= \frac{\rho g}{\sqrt{3}}\left\{\frac{\sqrt{3}Ly^2}{2} - \frac{2y^3}{3}\right\}_0^{\sqrt{3}L/2} = \frac{\rho g L^3}{8}. \end{aligned}$$

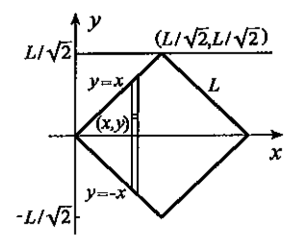


By symmetry, $x_c = 0$, and

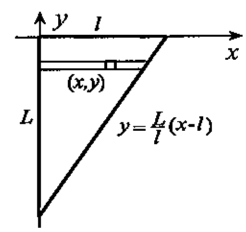
$$y_c = \frac{2}{F}\int_0^{\sqrt{3}L/2}\int_0^{y/\sqrt{3}} \rho g y\left(\frac{\sqrt{3}L}{2} - y\right) dx\,dy = \frac{\rho g}{F}\int_0^{\sqrt{3}L/2} \left\{xy(\sqrt{3}L - 2y)\right\}_0^{y/\sqrt{3}} dy$$
$$= \frac{\rho g}{\sqrt{3}F}\int_0^{\sqrt{3}L/2} (\sqrt{3}Ly^2 - 2y^3)\,dy = \frac{\rho g}{\sqrt{3}F}\left\{\frac{Ly^3}{\sqrt{3}} - \frac{y^4}{2}\right\}_0^{\sqrt{3}L/2}$$
$$= \frac{\sqrt{3}\rho g L^4}{32F} = \frac{\sqrt{3}\rho g L^4}{32}\frac{8}{\rho g L^3} = \frac{\sqrt{3}L}{4}.$$

21. According to Exercise 40 in Section 7.7, the force on the square is $F = \rho g(L/\sqrt{2})L^2 = \rho g L^3/\sqrt{2}$. By symmetry, $x_c = L/\sqrt{2}$, and

$$y_c = \frac{2}{F}\int_0^{L/\sqrt{2}}\int_{-x}^{x} \rho g y\left(\frac{L}{\sqrt{2}} - y\right) dy\,dx$$
$$= \frac{2\rho g}{F}\int_0^{L/\sqrt{2}}\left\{\frac{Ly^2}{2\sqrt{2}} - \frac{y^3}{3}\right\}_{-x}^{x} dx$$
$$= -\frac{4\rho g}{3F}\int_0^{L/\sqrt{2}} x^3\,dx = -\frac{4\rho g}{3F}\left\{\frac{x^4}{4}\right\}_0^{L/\sqrt{2}}$$
$$= -\frac{\rho g L^4}{12F} = -\frac{\rho g L^4}{12}\frac{\sqrt{2}}{\rho g L^3} = -\frac{L}{6\sqrt{2}}.$$



22. The force on the triangle is



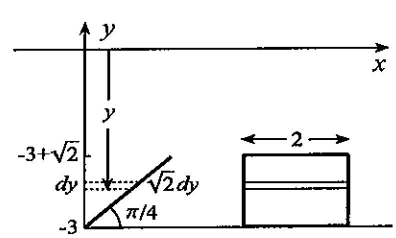
$$F = \int_{-L}^{0}\int_0^{ly/L+l} \rho g(-y)\,dx\,dy = -\rho g\int_{-L}^{0}\left\{xy\right\}_0^{ly/L+l} dy$$
$$= -\rho g\int_{-L}^{0} y\left(\frac{ly}{L} + l\right) dy = -\rho g\left\{\frac{ly^3}{3L} + \frac{ly^2}{2}\right\}_{-L}^{0} = \frac{\rho g l L^2}{6}.$$

According to equations 13.30,

$$x_c = \frac{1}{F}\int_{-L}^{0}\int_0^{ly/L+l} \rho g(-y)x\,dx\,dy = -\frac{\rho g}{F}\int_{-L}^{0}\left\{\frac{x^2 y}{2}\right\}_0^{ly/L+l} dy$$
$$= -\frac{\rho g}{2F}\int_{-L}^{0} y\left(\frac{l^2y^2}{L^2} + \frac{2l^2y}{L} + l^2\right) dy = -\frac{\rho g l^2}{2F}\left\{\frac{y^4}{4L^2} + \frac{2y^3}{3L} + \frac{y^2}{2}\right\}_{-L}^{0}$$
$$= \frac{\rho g l^2 L^2}{24F} = \frac{\rho g l^2 L^2}{24}\frac{6}{\rho g l L^2} = \frac{l}{4},$$
$$y_c = \frac{1}{F}\int_{-L}^{0}\int_0^{ly/L+l} \rho g(-y)y\,dx\,dy = -\frac{\rho g}{F}\int_{-L}^{0}\left\{xy^2\right\}_0^{ly/L+l} dy$$
$$= -\frac{\rho g}{F}\int_{-L}^{0} y^2\left(\frac{ly}{L} + l\right) dy = -\frac{\rho g l}{F}\left\{\frac{y^4}{4L} + \frac{y^3}{3}\right\}_{-L}^{0} = -\frac{\rho g l L^3}{12F} = -\frac{\rho g l L^3}{12}\frac{6}{\rho g l L^2} = -\frac{L}{2}.$$

23. When the geometric centre of a circle with radius r is at depth $h > r$ below the surface of a fluid, its centre of pressure is at depth $h + r^2/(4h)$ (see Exercise 18). This is not a fixed point in the plate. The centre of pressure is below the geometric centre and approaches the geometric centre as h increases.

24. $$F = \int_{-3}^{-3+\sqrt{2}} \rho g(-y)(2)(\sqrt{2}\,dy) \quad (g = 9.81)$$
$$= -\sqrt{2}\rho g\left\{y^2\right\}_{-3}^{-3+\sqrt{2}}$$
$$= 9.00 \times 10^4 \text{ N}$$

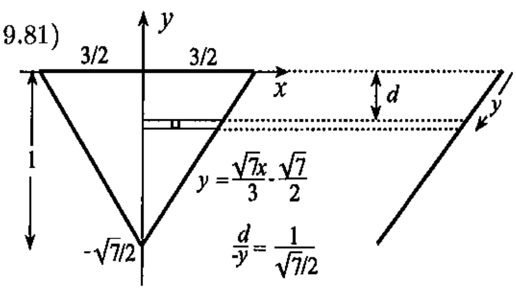


25. $F = 2\int_{-\sqrt{7}/2}^{0}\int_{0}^{3(2y+\sqrt{7})/(2\sqrt{7})} \rho g\left(-\frac{2y}{\sqrt{7}}\right) dx\, dy \quad (g = 9.81)$

$= -\frac{4\rho g}{\sqrt{7}}\int_{-\sqrt{7}/2}^{0}\left\{xy\right\}_{0}^{3(2y+\sqrt{7})/(2\sqrt{7})} dy$

$= -\frac{6\rho g}{7}\int_{-\sqrt{7}/2}^{0}(2y^2+\sqrt{7}y)\, dy$

$= -\frac{6\rho g}{7}\left\{\frac{2y^3}{3}+\frac{\sqrt{7}y^2}{2}\right\}_{-\sqrt{7}/2}^{0} = \frac{\sqrt{7}\rho g}{4}$



26. With the coordinate system in Figure 13.24,

$$y_c = \frac{1}{F}\iint_R yP\, dA = \frac{1}{\rho g(-\overline{y})A}\iint_R y\rho g(-y)\, dA = \frac{I_x}{\overline{y}A},$$

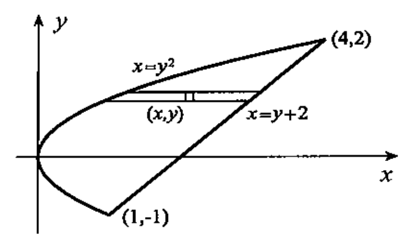
where A is the area of R and I_x is the second moment of area of R about the x-axis. But, according to the parallel axis theorem (see Exercise 20 in Section 7.8), $I_x = I_{CM} + \overline{y}^2 A$, where I_{CM} is the second moment of area of R about the line through the centroid and parallel to the x-axis. Thus,

$$y_c = \frac{I_{CM}+\overline{y}^2 A}{\overline{y}A} = \overline{y} + \frac{I_{CM}}{\overline{y}A}.$$

Since $\overline{y} < 0$, it follows that $y_c < \overline{y}$.

EXERCISES 13.5

1. $A = \int_{-1}^{2}\int_{y^2}^{y+2} dx\, dy = \int_{-1}^{2}(y+2-y^2)\, dy = \left\{\frac{y^2}{2}+2y-\frac{y^3}{3}\right\}_{-1}^{2} = \frac{9}{2}$



Since $A\overline{x} = \int_{-1}^{2}\int_{y^2}^{y+2} x\, dx\, dy = \int_{-1}^{2}\left\{\frac{x^2}{2}\right\}_{y^2}^{y+2} dy$

$= \frac{1}{2}\int_{-1}^{2}[(y+2)^2 - y^4]\, dy = \frac{1}{2}\left\{\frac{(y+2)^3}{3}-\frac{y^5}{5}\right\}_{-1}^{2} = \frac{36}{5},$

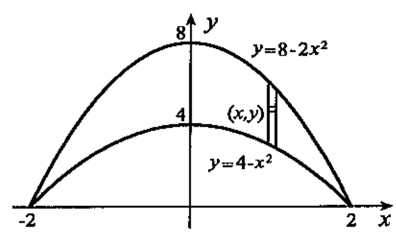
it follows that $\overline{x} = (36/5)(2/9) = 8/5$. Since

$$A\overline{y} = \int_{-1}^{2}\int_{y^2}^{y+2} y\, dx\, dy = \int_{-1}^{2}\left\{xy\right\}_{y^2}^{y+2} dy = \int_{-1}^{2}(y^2+2y-y^3)\, dy = \left\{\frac{y^3}{3}+y^2-\frac{y^4}{4}\right\}_{-1}^{2} = \frac{9}{4},$$

we obtain $\overline{y} = (9/4)(2/9) = 1/2$.

2. $A = 2\int_{0}^{2}\int_{4-x^2}^{8-2x^2} dy\, dx = 2\int_{0}^{2}(4-x^2)\, dx$

$= 2\left\{4x - \frac{x^3}{3}\right\}_{0}^{2} = \frac{32}{3}$



By symmetry, $\overline{x} = 0$. Since

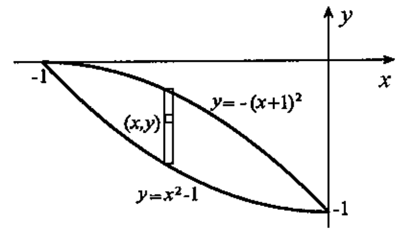
$A\overline{y} = 2\int_{0}^{2}\int_{4-x^2}^{8-2x^2} y\, dy\, dx = 2\int_{0}^{2}\left\{\frac{y^2}{2}\right\}_{4-x^2}^{8-2x^2} dx$

$= 3\int_{0}^{2}(16-8x^2+x^4)\, dx = 3\left\{16x - \frac{8x^3}{3}+\frac{x^5}{5}\right\}_{0}^{2} = \frac{256}{5},$

we obtain $\overline{y} = \frac{256}{5}\frac{3}{32} = \frac{24}{5}$.

3. $A = \int_{-1}^{0}\int_{x^2-1}^{-(x+1)^2} dy\,dx = \int_{-1}^{0} [-(x+1)^2 - x^2 + 1]\,dx$

$= \left\{-\frac{(x+1)^3}{3} - \frac{x^3}{3} + x\right\}_{-1}^{0} = \frac{1}{3}$



Since

$A\overline{x} = \int_{-1}^{0}\int_{x^2-1}^{-(x+1)^2} x\,dy\,dx = \int_{-1}^{0}\left\{xy\right\}_{x^2-1}^{-(x+1)^2} dx$

$= \int_{-1}^{0}(-2x^3 - 2x^2)\,dx = \left\{-\frac{x^4}{2} - \frac{2x^3}{3}\right\}_{-1}^{0} = -\frac{1}{6}$

it follows that $\overline{x} = -(1/6)3 = -1/2$. Since

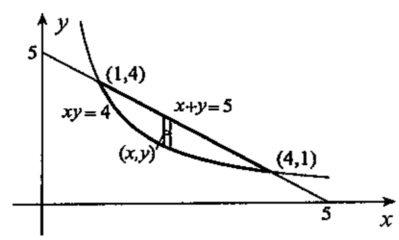
$$A\overline{y} = \int_{-1}^{0}\int_{x^2-1}^{-(x+1)^2} y\,dy\,dx = \int_{-1}^{0}\left\{\frac{y^2}{2}\right\}_{x^2-1}^{-(x+1)^2} dx = \frac{1}{2}\int_{-1}^{0}[(x+1)^4 - x^4 + 2x^2 - 1]\,dx$$

$$= \frac{1}{2}\left\{\frac{(x+1)^5}{5} - \frac{x^5}{5} + \frac{2x^3}{3} - x\right\}_{-1}^{0} = -\frac{1}{6},$$

$\overline{y}$ is also equal to $-1/2$.

4. $A = \int_{1}^{4}\int_{4/x}^{5-x} dy\,dx = \int_{1}^{4}(5 - x - 4/x)\,dx$

$= \left\{5x - \frac{x^2}{2} - 4\ln|x|\right\}_{1}^{4} = \frac{15}{2} - 4\ln 4$



From

$A\overline{x} = \int_{1}^{4}\int_{4/x}^{5-x} x\,dy\,dx = \int_{1}^{4}\left\{xy\right\}_{4/x}^{5-x} dx$

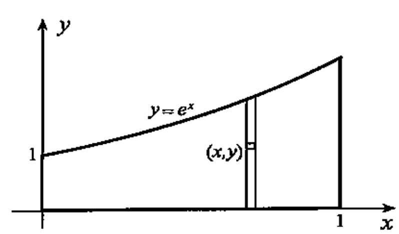
$= \int_{1}^{4}(5x - x^2 - 4)\,dx = \left\{\frac{5x^2}{2} - \frac{x^3}{3} - 4x\right\}_{1}^{4} = \frac{9}{2},$

we obtain $\overline{x} = \frac{9}{2}\frac{2}{15 - 8\ln 4} = \frac{9}{15 - 16\ln 2}$. Since

$$A\overline{y} = \int_{1}^{4}\int_{4/x}^{5-x} y\,dy\,dx = \int_{1}^{4}\left\{\frac{y^2}{2}\right\}_{4/x}^{5-x} dx = \frac{1}{2}\int_{1}^{4}\left[(5-x)^2 - \frac{16}{x^2}\right]dx = \frac{1}{2}\left\{-\frac{1}{3}(5-x)^3 + \frac{16}{x}\right\}_{1}^{4} = \frac{9}{2},$$

$\overline{y}$ is also equal to $9/(15 - 16\ln 2)$.

5. $A = \int_{0}^{1}\int_{0}^{e^x} dy\,dx = \int_{0}^{1} e^x\,dx = \left\{e^x\right\}_{0}^{1} = e - 1$



From

$A\overline{x} = \int_{0}^{1}\int_{0}^{e^x} x\,dy\,dx = \int_{0}^{1}\left\{xy\right\}_{0}^{e^x} dx$

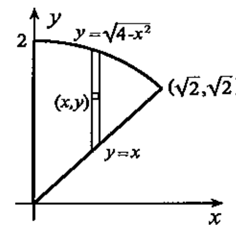
$= \int_{0}^{1} xe^x\,dx = \left\{xe^x - e^x\right\}_{0}^{1} = 1,$

we obtain $\overline{x} = \frac{1}{e-1}$. Since

$$A\overline{y} = \int_{0}^{1}\int_{0}^{e^x} y\,dy\,dx = \int_{0}^{1}\left\{\frac{y^2}{2}\right\}_{0}^{e^x} dx = \frac{1}{2}\int_{0}^{1} e^{2x}\,dx = \frac{1}{2}\left\{\frac{e^{2x}}{2}\right\}_{0}^{1} = \frac{e^2 - 1}{4},$$

we find $\overline{y} = \frac{e^2 - 1}{4}\frac{1}{e - 1} = \frac{e + 1}{4}$.

6. $A = \dfrac{1}{8}\pi(2)^2 = \dfrac{\pi}{2}$ Since



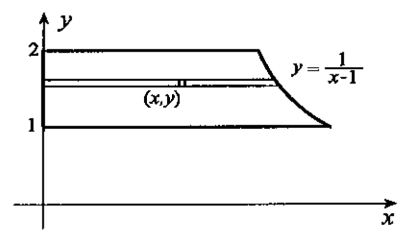
$$A\overline{x} = \int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} x\,dy\,dx$$
$$= \int_0^{\sqrt{2}} \left\{xy\right\}_x^{\sqrt{4-x^2}} dx = \int_0^{\sqrt{2}} (x\sqrt{4-x^2} - x^2)\,dx$$
$$= \left\{-\frac{1}{3}(4-x^2)^{3/2} - \frac{x^3}{3}\right\}_0^{\sqrt{2}} = \frac{4}{3}(2-\sqrt{2}),$$

it follows that $\overline{x} = \dfrac{4}{3}(2-\sqrt{2})\dfrac{2}{\pi} = \dfrac{8(2-\sqrt{2})}{3\pi}$. Since

$$A\overline{y} = \int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} y\,dy\,dx = \int_0^{\sqrt{2}} \left\{\frac{y^2}{2}\right\}_x^{\sqrt{4-x^2}} dx = \int_0^{\sqrt{2}} (2-x^2)\,dx = \left\{2x - \frac{x^3}{3}\right\}_0^{\sqrt{2}} = \frac{4\sqrt{2}}{3},$$

we obtain $\overline{y} = \dfrac{4\sqrt{2}}{3}\dfrac{2}{\pi} = \dfrac{8\sqrt{2}}{3\pi}$.

7. $A = \displaystyle\int_1^2\int_0^{(y+1)/y} dx\,dy = \int_1^2 \left(\frac{y+1}{y}\right) dy = \Big\{y + \ln|y|\Big\}_1^2 = 1 + \ln 2$



Since

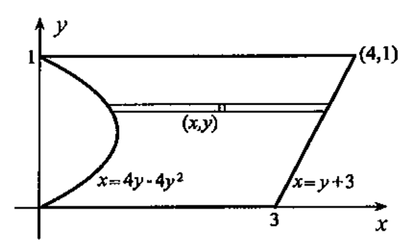
$$A\overline{x} = \int_1^2\int_0^{(y+1)/y} x\,dx\,dy = \int_1^2 \left\{\frac{x^2}{2}\right\}_0^{(y+1)/y} dy$$
$$= \frac{1}{2}\int_1^2 \left(1 + \frac{2}{y} + \frac{1}{y^2}\right) dy = \frac{1}{2}\left\{y + 2\ln|y| - \frac{1}{y}\right\}_1^2 = \frac{3+4\ln 2}{4},$$

it follows that $\overline{x} = \dfrac{3+4\ln 2}{4}\,\dfrac{1}{1+\ln 2} = \dfrac{3+4\ln 2}{4+4\ln 2}$. Since

$$A\overline{y} = \int_1^2\int_0^{(y+1)/y} y\,dx\,dy = \int_1^2 \Big\{xy\Big\}_0^{(y+1)/y} dy = \int_1^2 (y+1)\,dy = \left\{\frac{y^2}{2} + y\right\}_1^2 = \frac{5}{2},$$

we obtain $\overline{y} = \dfrac{5}{2}\,\dfrac{1}{1+\ln 2} = \dfrac{5}{2+2\ln 2}$.

8. $A = \displaystyle\int_0^1\int_{4y-4y^2}^{y+3} dx\,dy = \int_0^1 (3 - 3y + 4y^2)\,dy$



$$= \left\{3y - \frac{3y^2}{2} + \frac{4y^3}{3}\right\}_0^1 = \frac{17}{6}$$

Since

$$A\overline{x} = \int_0^1\int_{4y-4y^2}^{y+3} x\,dx\,dy = \int_0^1 \left\{\frac{x^2}{2}\right\}_{4y-4y^2}^{y+3} dy$$
$$= \frac{1}{2}\int_0^1 [(3+y)^2 - 16y^2 + 32y^3 - 16y^4]\,dy = \frac{1}{2}\left\{\frac{(3+y)^3}{3} - \frac{16y^3}{3} + 8y^4 - \frac{16y^5}{5}\right\}_0^1 = \frac{59}{10},$$

we find $\overline{x} = \dfrac{59}{10}\dfrac{6}{17} = \dfrac{177}{85}$. Since

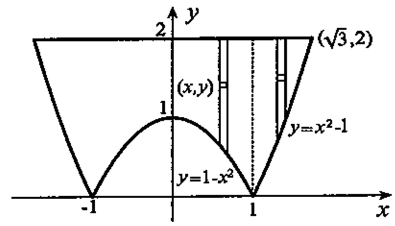
$$A\overline{y} = \int_0^1\int_{4y-4y^2}^{y+3} y\,dx\,dy = \int_0^1 \Big\{xy\Big\}_{4y-4y^2}^{y+3} dy = \int_0^1 (3y - 3y^2 + 4y^3)\,dy = \left\{\frac{3y^2}{2} - y^3 + y^4\right\}_0^1 = \frac{3}{2},$$

we obtain $\overline{y} = \dfrac{3}{2}\dfrac{6}{17} = \dfrac{9}{17}$.

9. $A = 2\int_0^1 \int_{1-x^2}^{2} dy\, dx + 2\int_1^{\sqrt{3}} \int_{x^2-1}^{2} dy\, dx$

$= 2\int_0^1 (1+x^2)\, dx + 2\int_1^{\sqrt{3}} (3-x^2)\, dx$

$= 2\left\{x + \frac{x^3}{3}\right\}_0^1 + 2\left\{3x - \frac{x^3}{3}\right\}_1^{\sqrt{3}} = \frac{12\sqrt{3}-8}{3}$



By symmetry, $\overline{x} = 0$. Since

$$A\overline{y} = 2\int_0^1 \int_{1-x^2}^{2} y\, dy\, dx + 2\int_1^{\sqrt{3}} \int_{x^2-1}^{2} y\, dy\, dx = 2\int_0^1 \left\{\frac{y^2}{2}\right\}_{1-x^2}^{2} dx + 2\int_1^{\sqrt{3}} \left\{\frac{y^2}{2}\right\}_{x^2-1}^{2} dx$$

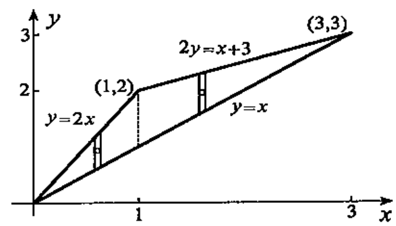
$$= \int_0^1 (3+2x^2-x^4)\, dx + \int_1^{\sqrt{3}} (3+2x^2-x^4)\, dx = \left\{3x + \frac{2x^3}{3} - \frac{x^5}{5}\right\}_0^{\sqrt{3}} = \frac{16\sqrt{3}}{5}$$

it follows that $\overline{y} = \frac{16\sqrt{3}}{5}\,\frac{3}{12\sqrt{3}-8} = \frac{12\sqrt{3}}{15\sqrt{3}-10}$.

10. $A = \int_0^1 \int_x^{2x} dy\, dx + \int_1^3 \int_x^{(x+3)/2} dy\, dx$

$= \int_0^1 x\, dx + \frac{1}{2}\int_1^3 (3-x)\, dx$

$= \left\{\frac{x^2}{2}\right\}_0^1 + \frac{1}{2}\left\{-\frac{1}{2}(3-x)^2\right\}_1^3 = \frac{3}{2}$



Since

$$A\overline{x} = \int_0^1 \int_x^{2x} x\, dy\, dx + \int_1^3 \int_x^{(x+3)/2} x\, dy\, dx$$

$$= \int_0^1 x^2\, dx + \frac{1}{2}\int_1^3 (3x - x^2)\, dx = \left\{\frac{x^3}{3}\right\}_0^1 + \frac{1}{2}\left\{\frac{3x^2}{2} - \frac{x^3}{3}\right\}_1^3 = 2,$$

it follows that $\overline{x} = 2 \cdot \frac{2}{3} = \frac{4}{3}$. With

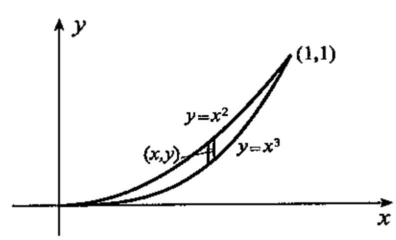
$$A\overline{y} = \int_0^1 \int_x^{2x} y\, dy\, dx + \int_1^3 \int_x^{(x+3)/2} y\, dy\, dx = \int_0^1 \left\{\frac{y^2}{2}\right\}_x^{2x} dx + \int_1^3 \left\{\frac{y^2}{2}\right\}_x^{(x+3)/2} dx$$

$$= \frac{3}{2}\int_0^1 x^2\, dx + \frac{1}{2}\int_1^3 \left[\frac{1}{4}(x+3)^2 - x^2\right] dx = \frac{3}{2}\left\{\frac{x^3}{3}\right\}_0^1 + \frac{1}{2}\left\{\frac{1}{12}(x+3)^3 - \frac{x^3}{3}\right\}_1^3 = \frac{5}{2},$$

we obtain $\overline{y} = (5/2)(2/3) = 5/3$.

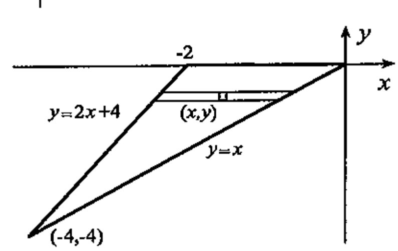
11. $I = \int_0^1 \int_{x^3}^{x^2} x^2\, dy\, dx = \int_0^1 \left\{x^2 y\right\}_{x^3}^{x^2} dx$

$= \int_0^1 (x^4 - x^5)\, dx = \left\{\frac{x^5}{5} - \frac{x^6}{6}\right\}_0^1 = \frac{1}{30}$



12. $I = \int_{-4}^0 \int_{(y-4)/2}^{y} y^2\, dx\, dy = \int_{-4}^0 \left\{xy^2\right\}_{(y-4)/2}^{y} dy$

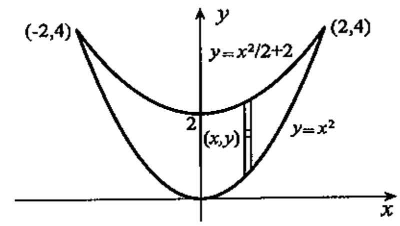
$= \frac{1}{2}\int_{-4}^0 (4y^2 + y^3)\, dy = \frac{1}{2}\left\{\frac{4y^3}{3} + \frac{y^4}{4}\right\}_{-4}^0 = \frac{32}{3}$



13. $I = 2\int_0^2 \int_{x^2}^{2+x^2/2} y^2\,dy\,dx = 2\int_0^2 \left\{\frac{y^3}{3}\right\}_{x^2}^{2+x^2/2} dx$

$$= \frac{2}{3}\int_0^2 \left(8 + 6x^2 + \frac{3x^4}{2} - \frac{7x^6}{8}\right) dx$$

$$= \frac{2}{3}\left\{8x + 2x^3 + \frac{3x^5}{10} - \frac{x^7}{8}\right\}_0^2 = \frac{256}{15}$$

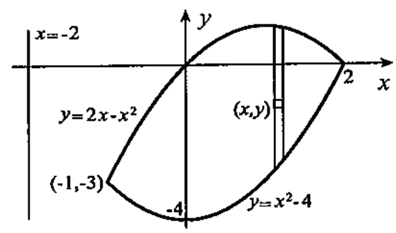


14. $I = \int_{-1}^2 \int_{x^2-4}^{2x-x^2} (x+2)^2\,dy\,dx$

$$= \int_{-1}^2 \left\{y(x+2)^2\right\}_{x^2-4}^{2x-x^2} dx$$

$$= 2\int_{-1}^2 (8 + 12x + 2x^2 - 3x^3 - x^4)\,dx$$

$$= 2\left\{8x + 6x^2 + \frac{2x^3}{3} - \frac{3x^4}{4} - \frac{x^5}{5}\right\}_{-1}^2 = \frac{603}{10}$$

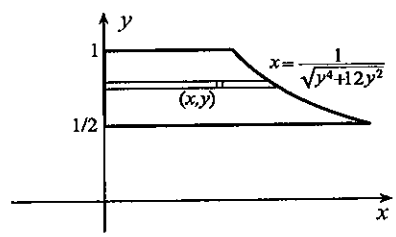


15. $I = \int_{1/2}^1 \int_0^{1/\sqrt{y^4+12y^2}} y^2\,dx\,dy$

$$= \int_{1/2}^1 \left\{xy^2\right\}_0^{1/\sqrt{y^4+12y^2}} dy$$

$$= \int_{1/2}^1 \frac{y^2}{\sqrt{y^4+12y^2}} dy = \int_{1/2}^1 \frac{y}{\sqrt{y^2+12}} dy$$

$$= \left\{\sqrt{y^2+12}\right\}_{1/2}^1 = \sqrt{13} - 7/2$$

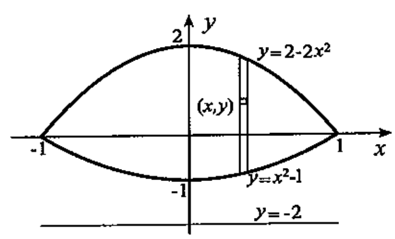


16. Moment $= 2\int_0^1 \int_{x^2-1}^{2-2x^2} \rho(y+2)\,dy\,dx$

$$= 2\rho\int_0^1 \left\{\frac{1}{2}(y+2)^2\right\}_{x^2-1}^{2-2x^2} dx$$

$$= 3\rho\int_0^1 (5 - 6x^2 + x^4)\,dx$$

$$= 3\rho\left\{5x - 2x^3 + \frac{x^5}{5}\right\}_0^1 = \frac{48\rho}{5}$$



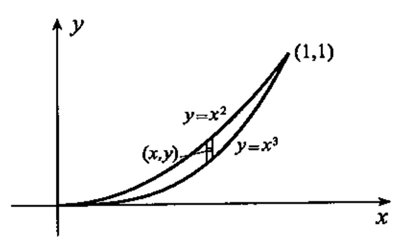
17. Due to the symmetry of the plate, the product moment of inertia about the axes is zero.

18. $I_{xy} = \int_0^1 \int_{x^3}^{x^2} xy\rho\,dy\,dx$

$$= \rho\int_0^1 \left\{\frac{xy^2}{2}\right\}_{x^3}^{x^2} dx$$

$$= \frac{\rho}{2}\int_0^1 (x^5 - x^7)\,dx$$

$$= \frac{\rho}{2}\left\{\frac{x^6}{6} - \frac{x^8}{8}\right\}_0^1 = \frac{\rho}{48}$$

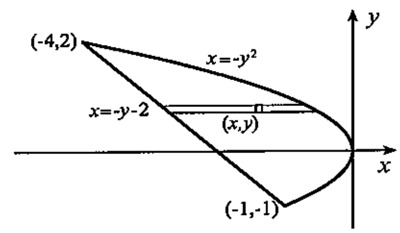


19. $I_{xy} = \int_{-1}^{2}\int_{-y-2}^{-y^2} xy\rho\,dx\,dy$

$= \rho \int_{-1}^{2} \left\{\frac{x^2 y}{2}\right\}_{-y-2}^{-y^2} dy$

$= \frac{\rho}{2}\int_{-1}^{2} (y^5 - y^3 - 4y^2 - 4y)\,dy$

$= \frac{\rho}{2}\left\{\frac{y^6}{6} - \frac{y^4}{4} - \frac{4y^3}{3} - 2y^2\right\}_{-1}^{2} = -\frac{45\rho}{8}$



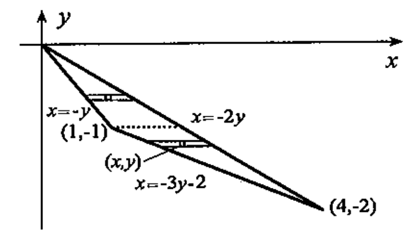
20. $I_{xy} = \int_{-2}^{-1}\int_{-3y-2}^{-2y} xy\rho\,dx\,dy + \int_{-1}^{0}\int_{-y}^{-2y} xy\rho\,dx\,dy$

$= \rho\int_{-2}^{-1}\left\{\frac{x^2y}{2}\right\}_{-3y-2}^{-2y} dy + \rho\int_{-1}^{0}\left\{\frac{x^2y}{2}\right\}_{-y}^{-2y} dy$

$= \frac{\rho}{2}\int_{-2}^{-1}(-5y^3 - 12y^2 - 4y)\,dy + \frac{3\rho}{2}\int_{-1}^{0} y^3\,dy$

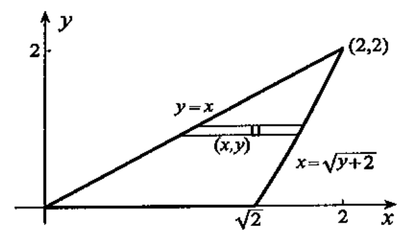
$= \frac{\rho}{2}\left\{-\frac{5y^4}{4} - 4y^3 - 2y^2\right\}_{-2}^{-1} + \frac{3\rho}{2}\left\{\frac{y^4}{4}\right\}_{-1}^{0}$

$= -2\rho$



21. $A = \int_0^2\int_y^{\sqrt{y+2}} dx\,dy = \int_0^2 (\sqrt{y+2} - y)\,dy$

$= \left\{\frac{2(y+2)^{3/2}}{3} - \frac{y^2}{2}\right\}_0^2 = \frac{10-4\sqrt{2}}{3}$



Since

$$A\overline{x} = \int_0^2\int_y^{\sqrt{y+2}} x\,dx\,dy = \int_0^2\left\{\frac{x^2}{2}\right\}_y^{\sqrt{y+2}} dy$$

$$= \frac{1}{2}\int_0^2 (y + 2 - y^2)\,dy = \frac{1}{2}\left\{\frac{y^2}{2} + 2y - \frac{y^3}{3}\right\}_0^2 = \frac{5}{3},$$

it follows that $\overline{x} = \frac{5}{3}\,\frac{3}{10-4\sqrt{2}} = \frac{5}{10-4\sqrt{2}}$. We now calculate

$$A\overline{y} = \int_0^2\int_y^{\sqrt{y+2}} y\,dx\,dy = \int_0^2 \left\{xy\right\}_y^{\sqrt{y+2}} dy = \int_0^2 (y\sqrt{y+2} - y^2)\,dy.$$

If we set $u = y + 2$ and $du = dy$ in the first term,

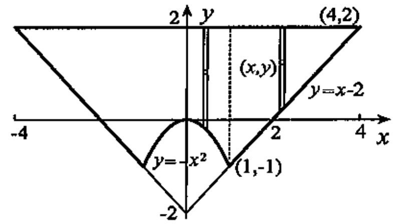
$$A\overline{y} = \int_2^4 (u-2)\sqrt{u}\,du - \left\{\frac{y^3}{3}\right\}_0^2 = \int_2^4 (u^{3/2} - 2\sqrt{u})\,du - \frac{8}{3} = \left\{\frac{2u^{5/2}}{5} - \frac{4u^{3/2}}{3}\right\}_2^4 - \frac{8}{3} = \frac{16\sqrt{2}-8}{15}.$$

Thus, $\overline{y} = \frac{16\sqrt{2}-8}{15}\,\frac{3}{10-4\sqrt{2}} = \frac{8\sqrt{2}-4}{25-10\sqrt{2}}$.

22. $A = 2\int_0^1 \int_{-x^2}^2 dy\,dx + 2\left(\frac{1}{2}\right)(3)(3)$

$= 2\int_0^1 (2+x^2)\,dx + 9$

$= 2\left\{2x + \frac{x^3}{3}\right\}_0^1 + 9 = \frac{41}{3}$



By symmetry, $\overline{x} = 0$. Since

$$A\overline{y} = 2\int_0^1 \int_{-x^2}^2 y\,dy\,dx + 2\int_1^4 \int_{x-2}^2 y\,dy\,dx$$

$$= 2\int_0^1 \left\{\frac{y^2}{2}\right\}_{-x^2}^2 dx + 2\int_1^4 \left\{\frac{y^2}{2}\right\}_{x-2}^2 dx$$

$$= \int_0^1 (4-x^4)\,dx + \int_1^4 [4-(x-2)^2]\,dx = \left\{4x - \frac{x^5}{5}\right\}_0^1 + \left\{4x - \frac{1}{3}(x-2)^3\right\}_1^4 = \frac{64}{5},$$

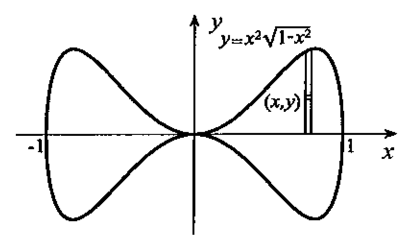
we obtain $\overline{y} = \frac{64}{5}\frac{3}{41} = \frac{192}{205}$.

23. $A = 2\int_0^1 \int_0^{x^2\sqrt{1-x^2}} dy\,dx = 2\int_0^1 x^2\sqrt{1-x^2}\,dx$

If we set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$, then

$A = 2\int_0^{\pi/2} \sin^2\theta(\cos\theta)\cos\theta\,d\theta = 2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = \frac{1}{4}\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{\pi}{8}.$



By symmetry, $\overline{y} = 0$. We now calculate

$$A\overline{x} = 2\int_0^1 \int_0^{x^2\sqrt{1-x^2}} x\,dy\,dx = 2\int_0^1 \left\{xy\right\}_0^{x^2\sqrt{1-x^2}} dx = 2\int_0^1 x^3\sqrt{1-x^2}\,dx.$$

If we set $u = 1-x^2$ and $du = -2x\,dx$, then

$$A\overline{x} = 2\int_1^0 (1-u)\sqrt{u}\left(\frac{du}{-2}\right) = \int_0^1 (\sqrt{u} - u^{3/2})\,du = \left\{\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right\}_0^1 = \frac{4}{15}.$$

Thus, $\overline{x} = (4/15)(8/\pi) = 32/(15\pi)$.

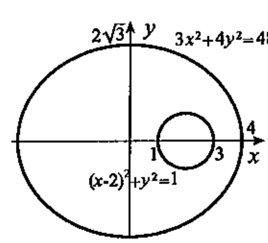
24. Since the area inside an ellipse is π multiplied by the product of half the major and minor axes,

$A = \pi(4)(2\sqrt{3}) - \pi(1)^2 = \pi(8\sqrt{3}-1).$

By symmetry, $\overline{y} = 0$. Since the first moment of the area about the y-axis is that of the ellipse less that of the circle,

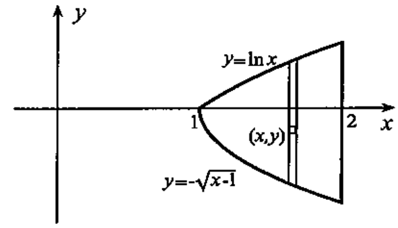
$A\overline{x} = 0 - 2\pi(1)^2 = -2\pi.$

Thus, $\overline{x} = \frac{-2\pi}{\pi(8\sqrt{3}-1)} = \frac{-2}{8\sqrt{3}-1}.$



25. $A = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} dy\,dx = \int_1^2 (\ln x + \sqrt{x-1})\,dx$

$= \left\{ x\ln x - x + \frac{2(x-1)^{3/2}}{3} \right\}_1^2 = 2\ln 2 - 1/3$



We now calculate that

$A\overline{x} = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} x\,dy\,dx = \int_1^2 \left\{ xy \right\}_{-\sqrt{x-1}}^{\ln x} dx$

$= \int_1^2 (x\ln x + x\sqrt{x-1})\,dx$

If we use integration by parts on the first term with $u = \ln x$, $dv = x\,dx$, $du = (1/x)dx$, and $v = x^2/2$, and set $u = x-1$ and $du = dx$ in the second,

$$A\overline{x} = \left\{ \frac{x^2}{2}\ln x \right\}_1^2 - \int_1^2 \frac{x}{2}dx + \int_0^1 (u+1)\sqrt{u}\,du = 2\ln 2 - \left\{ \frac{x^2}{4} \right\}_1^2 + \left\{ \frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3} \right\}_0^1 = \frac{120\ln 2 + 19}{60}.$$

Thus, $\overline{x} = \frac{120\ln 2 + 19}{60} \frac{3}{6\ln 2 - 1} = \frac{120\ln 2 + 19}{120\ln 2 - 20}$. Next, we find

$$A\overline{y} = \int_1^2 \int_{-\sqrt{x-1}}^{\ln x} y\,dy\,dx = \int_1^2 \left\{ \frac{y^2}{2} \right\}_{-\sqrt{x-1}}^{\ln x} dx = \frac{1}{2}\int_1^2 [(\ln x)^2 - (x-1)]\,dx.$$

If we set $u = (\ln x)^2$, $dv = dx$, $du = (2/x)\ln x\,dx$ and $v = x$ in the first term,

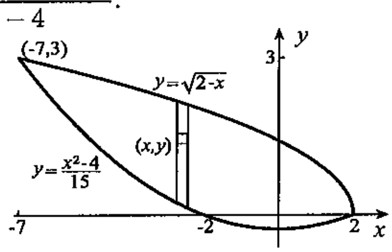
$$A\overline{y} = \frac{1}{2}\left\{ x(\ln x)^2 \right\}_1^2 - \frac{1}{2}\int_1^2 2\ln x\,dx - \frac{1}{2}\left\{ \frac{x^2}{2} - x \right\}_1^2 = (\ln 2)^2 - \left\{ x\ln x - x \right\}_1^2 - \frac{1}{4} = (\ln 2)^2 - 2\ln 2 + \frac{3}{4}.$$

Thus, $\overline{y} = \frac{4(\ln 2)^2 - 8\ln 2 + 3}{4} \frac{3}{6\ln 2 - 1} = \frac{12(\ln 2)^2 - 24\ln 2 + 9}{24\ln 2 - 4}$.

26. $A = \int_{-7}^2 \int_{(x^2-4)/15}^{\sqrt{2-x}} dy\,dx$

$= \int_{-7}^2 \left[\sqrt{2-x} - \frac{1}{15}(x^2 - 4) \right] dx$

$= \left\{ -\frac{2}{3}(2-x)^{3/2} - \frac{x^3}{45} + \frac{4x}{15} \right\}_{-7}^2 = \frac{63}{5}$



$A\overline{x} = \int_{-7}^2 \int_{(x^2-4)/15}^{\sqrt{2-x}} x\,dy\,dx = \int_{-7}^2 \left[x\sqrt{2-x} - \frac{1}{15}(x^3 - 4x) \right] dx$

If we set $u = 2 - x$ and $du = -dx$ in the first term,

$$A\overline{x} = \int_9^0 (2-u)\sqrt{u}(-du) - \frac{1}{15}\left\{ \frac{x^4}{4} - 2x^2 \right\}_{-7}^2 = \left\{ \frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right\}_0^9 - \frac{1}{15}\left\{ \frac{x^4}{4} - 2x^2 \right\}_{-7}^2 = -\frac{549}{20}.$$

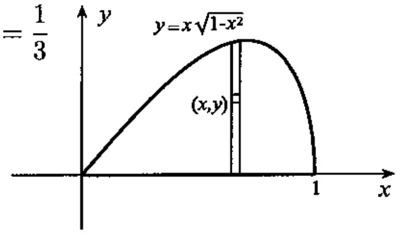
Thus, $\overline{x} = -(549/20)(5/63) = -61/28$. Since

$$A\overline{y} = \int_{-7}^2 \int_{(x^2-4)/15}^{\sqrt{2-x}} y\,dy\,dx = \int_{-7}^2 \left\{ \frac{y^2}{2} \right\}_{(x^2-4)/15}^{\sqrt{2-x}} dx = \frac{1}{2}\int_{-7}^2 \left[2 - x - \frac{1}{225}(x^4 - 8x^2 + 16) \right] dx$$

$$= \frac{1}{450}\left\{ 450x - \frac{225x^2}{2} - \frac{x^5}{5} + \frac{8x^3}{3} - 16x \right\}_{-7}^2 = \frac{7263}{500},$$

we find $\overline{y} = (7263/500)(5/63) = 807/700$.

27. $A = \int_0^1 \int_0^{x\sqrt{1-x^2}} dy\,dx = \int_0^1 x\sqrt{1-x^2}\,dx = \left\{-\frac{1}{3}(1-x^2)^{3/2}\right\}_0^1 = \frac{1}{3}$



We now calculate

$$A\overline{x} = \int_0^1 \int_0^{x\sqrt{1-x^2}} x\,dy\,dx = \int_0^1 \left\{xy\right\}_0^{x\sqrt{1-x^2}} dx$$
$$= \int_0^1 x^2\sqrt{1-x^2}\,dx$$

If we set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$,

$$A\overline{x} = \int_0^{\pi/2} \sin^2\theta(\cos\theta)\cos\theta\,d\theta = \int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$
$$= \frac{1}{8}\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{\pi}{16}.$$

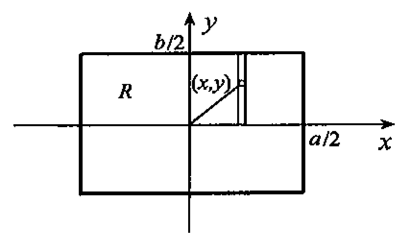
Thus, $\overline{x} = (\pi/16)3 = 3\pi/16$. Since

$$A\overline{y} = \int_0^1 \int_0^{x\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left\{\frac{y^2}{2}\right\}_0^{x\sqrt{1-x^2}} dx = \frac{1}{2}\int_0^1 x^2(1-x^2)\,dx = \frac{1}{2}\left\{\frac{x^3}{3} - \frac{x^5}{5}\right\}_0^1 = \frac{1}{15},$$

we obtain $\overline{y} = (1/15)3 = 1/5$.

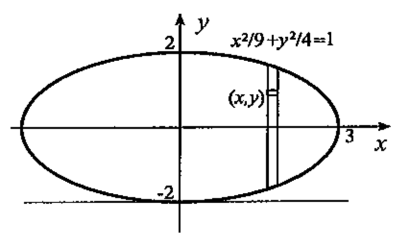
28. $I = \iint_R (x^2+y^2)\rho\,dA = 4\rho\int_0^{a/2}\int_0^{b/2}(x^2+y^2)\,dy\,dx$

$$= 4\rho\int_0^{a/2}\left\{x^2y + \frac{y^3}{3}\right\}_0^{b/2} dx = 4\rho\int_0^{a/2}\left(\frac{bx^2}{2} + \frac{b^3}{24}\right)dx$$
$$= 4\rho\left\{\frac{bx^3}{6} + \frac{b^3x}{24}\right\}_0^{a/2} = \frac{\rho ab}{12}(a^2+b^2)$$



29. $I = 2\int_0^3 \int_{-(2/3)\sqrt{9-x^2}}^{(2/3)\sqrt{9-x^2}} (y+2)^2\,dy\,dx$

$$= 2\int_0^3 \left\{\frac{(y+2)^3}{3}\right\}_{-(2/3)\sqrt{9-x^2}}^{(2/3)\sqrt{9-x^2}} dx$$
$$= \frac{32}{81}\int_0^3 (36-x^2)\sqrt{9-x^2}\,dx$$

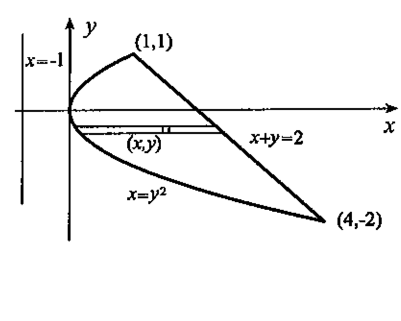


If we set $x = 3\sin\theta$ and $dx = 3\cos\theta\,d\theta$,

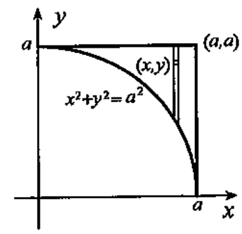
$$I = \frac{32}{81}\int_0^{\pi/2}(36 - 9\sin^2\theta)(3\cos\theta)3\cos\theta\,d\theta = 32\int_0^{\pi/2}(4\cos^2\theta - \sin^2\theta\cos^2\theta)\,d\theta$$
$$= 32\int_0^{\pi/2}\left[2 + 2\cos 2\theta - \left(\frac{\sin 2\theta}{2}\right)^2\right]d\theta = 32\int_0^{\pi/2}\left[2 + 2\cos 2\theta - \frac{1}{4}\left(\frac{1-\cos 4\theta}{2}\right)\right]d\theta$$
$$= 32\left\{\frac{15\theta}{8} + \sin 2\theta + \frac{1}{32}\sin 4\theta\right\}_0^{\pi/2} = 30\pi.$$

30. $I = \int_{-2}^1\int_{y^2}^{2-y}(x+1)^2\,dx\,dy = \int_{-2}^1\left\{\frac{1}{3}(x+1)^3\right\}_{y^2}^{2-y} dy$

$$= \frac{1}{3}\int_{-2}^1 [(3-y)^3 - y^6 - 3y^4 - 3y^2 - 1]\,dy$$
$$= \frac{1}{3}\left\{-\frac{1}{4}(3-y)^4 - \frac{y^7}{7} - \frac{3y^5}{5} - y^3 - y\right\}_{-2}^1 = \frac{4761}{140}$$



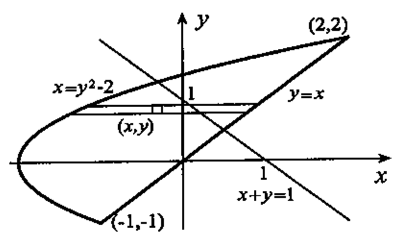
31. $I = \int_0^a \int_{\sqrt{a^2-x^2}}^a y^2\,dy\,dx = \int_0^a \left\{\frac{y^3}{3}\right\}_{\sqrt{a^2-x^2}}^a dx$
$= \frac{1}{3}\int_0^a [a^3 - (a^2-x^2)^{3/2}]\,dx$



If we set $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$,

$$\begin{aligned}I &= \frac{1}{3}\left\{a^3x\right\}_0^a - \frac{1}{3}\int_0^{\pi/2} a^3\cos^3\theta(a\cos\theta\,d\theta)\\ &= \frac{a^4}{3} - \frac{a^4}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{a^4}{3} - \frac{a^4}{12}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta\\ &= \frac{a^4}{3} - \frac{a^4}{12}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{a^4(16-3\pi)}{48}.\end{aligned}$$

32. If we take directed distances to the right of the line $x + y = 1$ as positive, then the directed distance from the line to the area $dy\,dx$ at point (x, y) is $(x + y - 1)/\sqrt{2}$. The first moment of the plate about the line is

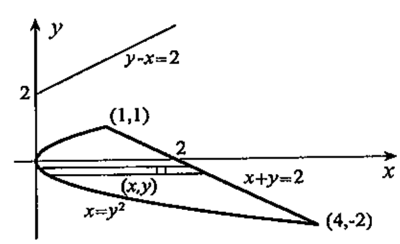


$$\begin{aligned}M &= \int_{-1}^2\int_{y^2-2}^y \frac{x+y-1}{\sqrt{2}}dx\,dy = \frac{1}{\sqrt{2}}\int_{-1}^2\left\{\frac{(x+y-1)^2}{2}\right\}_{y^2-2}^y dy\\ &= \frac{1}{2\sqrt{2}}\int_{-1}^2 [(2y-1)^2 - y^4 - 2y^3 + 5y^2 + 6y - 9]\,dy\\ &= \frac{1}{2\sqrt{2}}\left\{\frac{(2y-1)^3}{6} - \frac{y^5}{5} - \frac{y^4}{2} + \frac{5y^3}{3} + 3y^2 - 9y\right\}_{-1}^2 = -\frac{81\sqrt{2}}{40}.\end{aligned}$$

The second moment about the line is

$$\begin{aligned}I &= \int_{-1}^2\int_{y^2-2}^y \frac{(x+y-1)^2}{2}dx\,dy = \frac{1}{2}\int_{-1}^2\left\{\frac{(x+y-1)^3}{3}\right\}_{y^2-2}^y dy\\ &= \frac{1}{6}\int_{-1}^2 [(2y-1)^3 - y^6 - 3y^5 + 6y^4 + 17y^3 - 18y^2 - 27y + 27]\,dy\\ &= \frac{1}{6}\left\{\frac{(2y-1)^4}{8} - \frac{y^7}{7} - \frac{y^6}{2} + \frac{6y^5}{5} + \frac{17y^4}{4} - 6y^3 - \frac{27y^2}{2} + 27y\right\}_{-1}^2 = \frac{1863}{280}.\end{aligned}$$

33. If we take directed distances to the right of the line $y - x = 2$ as positive, then the directed distance from the line to the area $dy\,dx$ at point (x, y) is $(x - y + 2)/\sqrt{2}$. The first moment of the plate about the line is



$$\begin{aligned}M &= \int_{-2}^1\int_{y^2}^{2-y} \frac{x-y+2}{\sqrt{2}}dx\,dy = \frac{1}{\sqrt{2}}\int_{-2}^1\left\{\frac{(x-y+2)^2}{2}\right\}_{y^2}^{2-y} dy\\ &= \frac{1}{2\sqrt{2}}\int_{-2}^1 [(4-2y)^2 - y^4 + 2y^3 - 5y^2 + 4y - 4]\,dy\\ &= \frac{1}{2\sqrt{2}}\left\{\frac{(4-2y)^3}{-6} - \frac{y^5}{5} + \frac{y^4}{2} - \frac{5y^3}{3} + 2y^2 - 4y\right\}_{-2}^1 = \frac{369\sqrt{2}}{40}.\end{aligned}$$

The second moment about the line is

$$\begin{aligned}I &= \int_{-2}^1\int_{y^2}^{2-y} \frac{(x-y+2)^2}{2}dx\,dy = \frac{1}{2}\int_{-2}^1\left\{\frac{(x-y+2)^3}{3}\right\}_{y^2}^{2-y} dy\\ &= \frac{1}{6}\int_{-2}^1 [(4-2y)^3 - y^6 + 3y^5 - 9y^4 + 13y^3 - 18y^2 + 12y - 8]\,dy\\ &= \frac{1}{6}\left\{\frac{(4-2y)^4}{-8} - \frac{y^7}{7} + \frac{y^6}{2} - \frac{9y^5}{5} + \frac{13y^4}{4} - 6y^3 + 6y^2 - 8y\right\}_{-2}^1 = \frac{11\,943}{280}.\end{aligned}$$

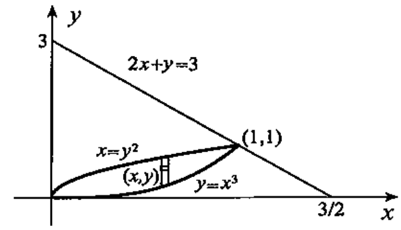
34. If we take directed distances to the right of the line $2x+y=3$ as positive, then the directed distance from the line to the area $dy\,dx$ at point (x,y) is $(2x+y-3)/\sqrt{5}$. The first moment of the plate about the line is

$$M = \int_0^1 \int_{x^3}^{\sqrt{x}} \frac{2x+y-3}{\sqrt{5}} dy\,dx$$

$$= \frac{1}{\sqrt{5}} \int_0^1 \left\{ \frac{(2x+y-3)^2}{2} \right\}_{x^3}^{\sqrt{x}} dx$$

$$= \frac{1}{2\sqrt{5}} \int_0^1 (-x^6 - 4x^4 + 6x^3 + 4x^{3/2} + x - 6\sqrt{x})\,dx$$

$$= \frac{1}{2\sqrt{5}} \left\{ -\frac{x^7}{7} - \frac{4x^5}{5} + \frac{3x^4}{2} + \frac{8x^{5/2}}{5} + \frac{x^2}{2} - 4x^{3/2} \right\}_0^1 = -\frac{47\sqrt{5}}{350}.$$



The second moment about the line is

$$I = \int_0^1 \int_{x^3}^{\sqrt{x}} \frac{(2x+y-3)^2}{5} dy\,dx = \frac{1}{5} \int_0^1 \left\{ \frac{(2x+y-3)^3}{3} \right\}_{x^3}^{\sqrt{x}} dx$$

$$= \frac{1}{15} \int_0^1 (-x^9 - 6x^7 + 9x^6 - 12x^5 + 36x^4 - 27x^3 + 6x^2 - 9x + 12x^{5/2} - 35x^{3/2} + 27\sqrt{x})\,dx$$

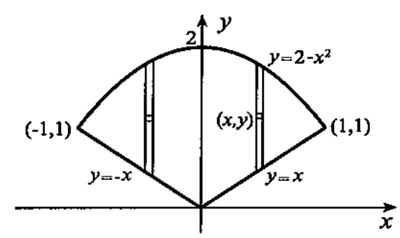
$$= \frac{1}{15} \left\{ -\frac{x^{10}}{10} - \frac{3x^8}{4} + \frac{9x^7}{7} - 2x^6 + \frac{36x^5}{5} - \frac{27x^4}{4} + 2x^3 - \frac{9x^2}{2} + \frac{24x^{7/2}}{7} - 14x^{5/2} + 18x^{3/2} \right\}_0^1 = \frac{89}{350}.$$

35. If we take directed distances to the left of the line $y=x$ as positive, then the directed distance from the line to the area $dy\,dx$ at point (x,y) is $(y-x)/\sqrt{2}$. The first moment of the plate about the line is

$$M = \int_{-1}^0 \int_{-x}^{2-x^2} \frac{y-x}{\sqrt{2}} dy\,dx + \int_0^1 \int_x^{2-x^2} \frac{y-x}{\sqrt{2}} dy\,dx$$

$$= \frac{1}{\sqrt{2}} \int_{-1}^0 \left\{ \frac{(y-x)^2}{2} \right\}_{-x}^{2-x^2} dx$$

$$+ \frac{1}{\sqrt{2}} \int_0^1 \left\{ \frac{(y-x)^2}{2} \right\}_x^{2-x^2} dx$$



$$= \frac{1}{2\sqrt{2}} \int_{-1}^0 (x^4 + 2x^3 - 7x^2 - 4x + 4)\,dx + \frac{1}{2\sqrt{2}} \int_0^1 (x^4 + 2x^3 - 3x^2 - 4x + 4)\,dx$$

$$= \frac{1}{2\sqrt{2}} \left\{ \frac{x^5}{5} + \frac{x^4}{2} - \frac{7x^3}{3} - 2x^2 + 4x \right\}_{-1}^0 + \frac{1}{2\sqrt{2}} \left\{ \frac{x^5}{5} + \frac{x^4}{4} - x^3 - 2x^2 + 4x \right\}_0^1 = \frac{19\sqrt{2}}{15}.$$

The second moment about the line is

$$I = \int_{-1}^0 \int_{-x}^{2-x^2} \frac{(y-x)^2}{2} dy\,dx + \int_0^1 \int_x^{2-x^2} \frac{(y-x)^2}{2} dy\,dx$$

$$= \frac{1}{2} \int_{-1}^0 \left\{ \frac{(y-x)^3}{3} \right\}_{-x}^{2-x^2} dx + \frac{1}{2} \int_0^1 \left\{ \frac{(y-x)^3}{3} \right\}_x^{2-x^2} dx$$

$$= \frac{1}{6} \int_{-1}^0 (8 - 12x - 6x^2 + 19x^3 + 3x^4 - 3x^5 - x^6)\,dx$$

$$+ \frac{1}{6} \int_0^1 (8 - 12x - 6x^2 + 11x^3 + 3x^4 - 3x^5 - x^6)\,dx$$

$$= \frac{1}{6} \left\{ 8x - 6x^2 - 2x^3 + \frac{19x^4}{4} + \frac{3x^5}{5} - \frac{x^6}{2} - \frac{x^7}{7} \right\}_{-1}^0 + \frac{1}{6} \left\{ 8x - 6x^2 - 2x^3 + \frac{11x^4}{4} + \frac{3x^5}{5} - \frac{x^6}{2} - \frac{x^7}{7} \right\}_0^1$$

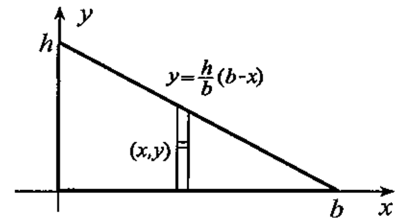
$$= \frac{191}{105}.$$

36. (a) $I_{xy} = \int_0^b \int_0^{h(b-x)/b} xy\rho\, dy\, dx$

$= \rho \int_0^b \left\{ \frac{xy^2}{2} \right\}_0^{h(b-x)/b} dx$

$= \frac{\rho h^2}{2b^2} \int_0^b (b^2x - 2bx^2 + x^3)\, dx$

$= \frac{\rho h^2}{2b^2} \left\{ \frac{b^2x^2}{2} - \frac{2bx^3}{3} + \frac{x^4}{4} \right\}_0^b = \frac{\rho b^2 h^2}{24}$



(b) The centre of mass is $(b/3, h/3)$. The product moment of inertia about horizontal and vertical lines through this point is

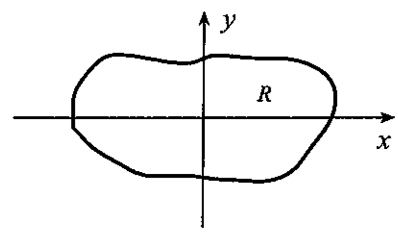
$$I = \int_0^b \int_0^{h(b-x)/b} \left(x - \frac{b}{3}\right)\left(y - \frac{h}{3}\right)\rho\, dy\, dx = \rho \int_0^b \left\{ \frac{1}{2}\left(x - \frac{b}{3}\right)\left(y - \frac{h}{3}\right)^2 \right\}_0^{h(b-x)/b} dx$$

$$= \frac{\rho h^2}{18b^2} \int_0^b (9x^3 - 15bx^2 + 7b^2x - b^3)\, dx = \frac{\rho h^2}{18b^2} \left\{ \frac{9x^4}{4} - 5bx^3 + \frac{7b^2x^2}{2} - b^3x \right\}_0^b = -\frac{\rho b^2 h^2}{72}.$$

37. Since $\iint_R (x-y)^2 \rho\, dA \ge 0$, it follows that

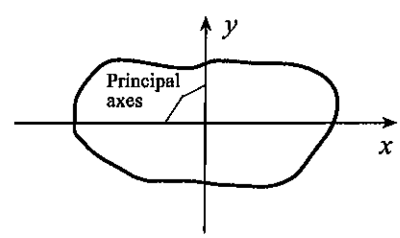
$0 \le \iint_R x^2\rho\, dA - 2\iint_R xy\rho\, dA + \iint_R y^2 \rho\, dA$
$= I_y - 2I_{xy} + I_x,$

and this implies that $I_{xy} \le (I_x + I_y)/2$. Similarly, by considering the double integral of $\rho(x+y)^2$ over R, we obtain $I_{xy} \ge -(I_x + I_y)/2$. Together these give



$$-\frac{I_x + I_y}{2} \le I_{xy} \le \frac{I_x + I_y}{2} \quad \Longrightarrow \quad |I_{xy}| \le \frac{I_x + I_y}{2}.$$

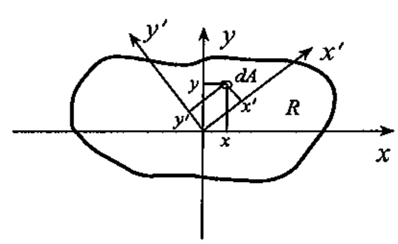
38. Suppose we choose the point as the origin and axes along the principal axes of the plate. Then one principal axis has slope zero while the slope of the other is undefined. According to equation 13.39, this occurs when $I_{xy} = 0$. This can also be seen by taking $\theta = 0$ or $\theta = \pi/2$ in Exercise 45.



39. $I_{x'} + I_{y'} = \iint_R (y')^2\rho\, dA + \iint_R (x')^2 \rho\, dA$

$= \iint_R [(x')^2 + (y')^2]\rho\, dA$

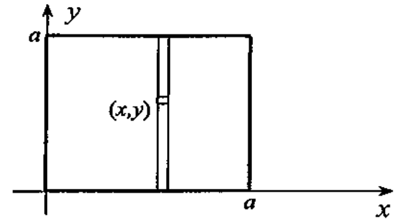
But $(x')^2 + (y')^2$ is the square of the distance from dA to the origin, and therefore $(x')^2 + (y')^2 = x^2 + y^2$.



Hence,

$$I_{x'} + I_{y'} = \iint_R (x^2 + y^2)\rho\, dA = \iint_R x^2\rho\, dA + \iint_R y^2 \rho\, dA = I_x + I_y.$$

40. Since



$$I_x = I_y = \int_0^a \int_0^a y^2 \rho\, dy\, dx = \rho \int_0^a \left\{\frac{y^3}{3}\right\}_0^a dx$$

$$= \frac{\rho a^3}{3}\left\{x\right\}_0^a = \frac{\rho a^4}{3},$$

and

$$I_{xy} = \int_0^a \int_0^a xy\rho\, dy\, dx = \rho \int_0^a \left\{\frac{xy^2}{2}\right\}_0^a dx = \frac{\rho a^2}{2}\left\{\frac{x^2}{2}\right\}_0^a = \frac{\rho a^4}{4},$$

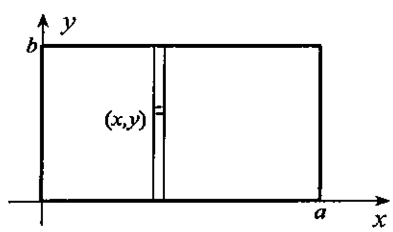
slopes of the principal axes are defined by equation 13.39,

$$m = \frac{I_x - I_y}{2I_{xy}} \pm \sqrt{1 + \left(\frac{I_x - I_y}{2I_{xy}}\right)^2} = \pm 1.$$

Principal axes are therefore the lines $y = \pm x$. According to equation 13.40, principal moments of inertia are

$$\frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} = I_x \pm I_{xy} = \frac{7\rho a^4}{12},\ \frac{\rho a^4}{12}.$$

41. For the rectangle,



$$I_x = \int_0^a \int_0^b y^2 \rho\, dy\, dx = \rho \int_0^a \left\{\frac{y^3}{3}\right\}_0^b dx = \frac{\rho b^3}{3}\left\{x\right\}_0^a = \frac{\rho a b^3}{3},$$

$$I_y = \int_0^a \int_0^b x^2 \rho\, dy\, dx = \rho \int_0^a \left\{x^2 y\right\}_0^b dx = \rho b \left\{\frac{x^3}{3}\right\}_0^a = \frac{\rho a^3 b}{3},$$

$$I_{xy} = \int_0^a \int_0^b xy\rho\, dy\, dx = \rho \int_0^a \left\{\frac{xy^2}{2}\right\}_0^b dx = \frac{\rho b^2}{2}\left\{\frac{x^2}{2}\right\}_0^a = \frac{\rho a^2 b^2}{4}.$$

With $\displaystyle \frac{I_x - I_y}{2I_{xy}} = \left(\frac{\rho a b^3}{3} - \frac{\rho a^3 b}{3}\right)\frac{2}{\rho a^2 b^2} = \frac{2}{3}\left(\frac{b}{a} - \frac{a}{b}\right),$ slopes of the principal axes are defined by equation 13.39,

$$m = \frac{2}{3}\left(\frac{b}{a} - \frac{a}{b}\right) \pm \sqrt{1 + \frac{4}{9}\left(\frac{b}{a} - \frac{a}{b}\right)^2}.$$

Principal moments of inertia in these directions are

$$\frac{1}{2}\left(\frac{\rho a b^3}{3} + \frac{\rho a^3 b}{3}\right) \mp \sqrt{\left(\frac{\rho a b^3}{6} - \frac{\rho a^3 b}{6}\right)^2 + \left(\frac{\rho a^2 b^2}{4}\right)^2} = \frac{\rho a b}{6}(a^2 + b^2) \mp \frac{\rho a b}{12}\sqrt{4(b^2 - a^2)^2 + 9a^2 b^2}.$$

42. Since $\displaystyle\lim_{m \to \pm\infty} I(m) = I_y$, we must show that

$$\frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \le I_y \le \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}.$$

But this is equivalent to

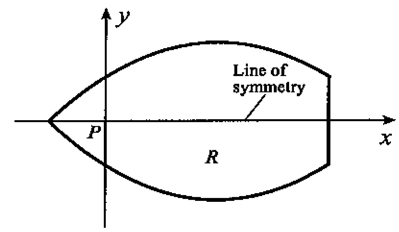
$$-\sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \le \frac{I_y - I_x}{2} \le \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2},$$

which is valid for any I_x and I_y.

43. Choose a coordinate system with the x-axis along ℓ and the y-axis through P. The product moment of inertia about ℓ (the x-axis) and the y-axis is

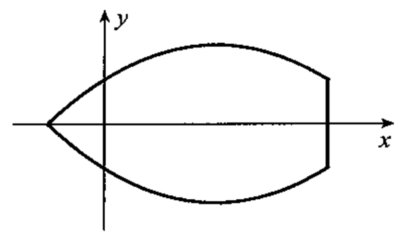
$$I_{xy} = \iint_R xy\, dA.$$

Because xy is an odd function of y and R is symmetric about the x-axis, this integral has value zero.



44. Suppose the x-axis is chosen as the axis of symmetry. Choose the y-axis through any point on the line. According to Exercise 43, the product moment of inertia about the origin for the coordinate axes is zero. Consequently the moment of inertia about any line through the origin with slope m is

$$I(m) = \frac{1}{m^2+1}(I_x + m^2 I_y) = I_y + \frac{I_x - I_y}{m^2+1},$$



(see equation 13.38). If $I_x > I_y$, then this is an even function of m, decreasing from $I(0) = I_x$ to $\lim_{m\to\infty} I(m) = I_y$; that is, principal axes are $x = 0$ and $y = 0$. If $I_x < I_y$, then this even function increases from $I(0) = I_x$ to $\lim_{m\to\infty} I(m) = I_y$, and once again I_x and I_y are principal moments of inertia. If $I_x = I_y$, then $I(m) = I_y$ for all m, in which case all pairs of perpendicular lines through the origin are principal axes.

45. We know that when θ is the angle of inclination of a line, then $\tan\theta = m$, and when the line is a principal axis about the origin, m is defined by equation 13.39. Using the double angle formula for $\tan 2\theta$ gives

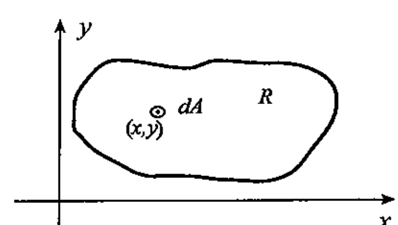
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2m}{1-m^2} = \frac{2\left[\dfrac{I_x - I_y}{2I_{xy}} \pm \sqrt{1 + \left(\dfrac{I_x - I_y}{2I_{xy}}\right)^2}\,\right]}{1 - \left[\dfrac{I_x - I_y}{2I_{xy}} \pm \sqrt{1 + \left(\dfrac{I_x - I_y}{2I_{xy}}\right)^2}\,\right]^2}$$

$$= \frac{\dfrac{I_x - I_y \pm \sqrt{(I_x - I_y)^2 + 4(I_{xy})^2}}{I_{xy}}}{\dfrac{4(I_{xy})^2 - (I_x - I_y)^2 \mp 2(I_x - I_y)\sqrt{(I_x - I_y)^2 + 4(I_{xy})^2} - (I_x - I_y)^2 - 4(I_{xy})^2}{4(I_{xy})^2}}$$

$$= \frac{4I_{xy}\left[I_x - I_y \pm \sqrt{(I_x - I_y)^2 + 4(I_{xy})^2}\right]}{-2(I_x - I_y)^2 \mp 2(I_x - I_y)\sqrt{(I_x - I_y)^2 + 4(I_{xy})^2}} = \frac{2I_{xy}}{-(I_x - I_y)} = \frac{2I_{xy}}{I_y - I_x}.$$

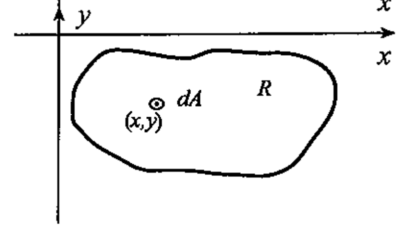
46. If we orient the area so that the axis of rotation is the y-axis, then

$$V = \iint_R 2\pi x\, dA = 2\pi \iint_R x\, dA$$
$$= 2\pi(A\overline{x}) = (2\pi\overline{x})A.$$

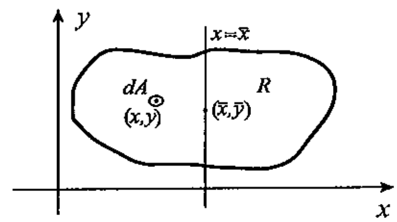


47. The fluid force on each side of the plate R is

$$F = \iint_R \rho g(-y)\, dA \quad (g = 9.81)$$
$$= -\rho g \iint_R y\, dA$$
$$= -\rho g(A\overline{y}) = \rho g(-\overline{y})A.$$



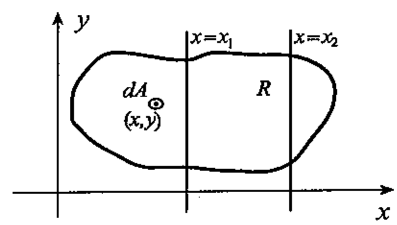
48. If we orient the area so that the line is the y-axis, then



$$\begin{aligned} I &= \iint_R x^2\rho\,dA = \rho\iint_R [(x-\overline{x})+\overline{x}]^2\,dA \\ &= \rho\iint_R [(x-\overline{x})^2 + 2\overline{x}(x-\overline{x})+\overline{x}^2]\,dA \\ &= \iint_R \rho(x-\overline{x})^2\,dA + 2\overline{x}\iint_R x\rho\,dA - \overline{x}^2\iint_R \rho\,dA \\ &= \iint_R \rho(x-\overline{x})^2\,dA + 2\overline{x}(M\overline{x}) - \overline{x}^2 M = \iint_R \rho(x-\overline{x})^2\,dA + M\overline{x}^2. \end{aligned}$$

49. Since $I_{x_2} = \iint_R (x-x_2)^2\rho\,dA$ and $I_{x_1} = \iint_R (x-x_1)^2\rho\,dA$,

$$\begin{aligned} I_{x_2} - I_{x_1} &= \iint_R (x-x_2)^2\rho\,dA - \iint_R (x-x_1)^2\rho\,dA = \iint_R (x^2 - 2xx_2 + x_2^2 - x^2 + 2xx_1 - x_1^2)\rho\,dA \\ &= (x_2^2 - x_1^2)\iint_R \rho\,dA + 2(x_1 - x_2)\iint_R x\rho\,dA \\ &= (x_2^2 - x_1^2)M + 2(x_1 - x_2)M\overline{x}. \end{aligned}$$



Thus,

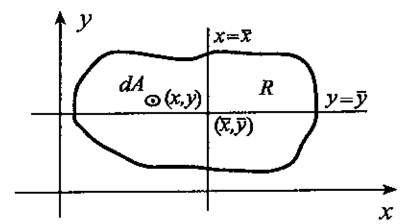
$$I_{x_2} = I_{x_1} + M[x_2^2 - x_1^2 + 2\overline{x}(x_1 - x_2)].$$

When $x_1 = \overline{x}$, this reduces to

$$I_{x_2} = I_{\overline{x}} + M[x_2^2 - \overline{x}^2 + 2\overline{x}(\overline{x} - x_2)] = I_{\overline{x}} + M(x_2^2 - 2\overline{x}x_2 + \overline{x}^2) = I_{\overline{x}} + M(x_2 - \overline{x})^2,$$

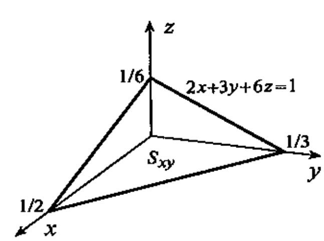
and this is the parallel axis theorem.

50.
$$\begin{aligned} I_{\overline{x}\,\overline{y}} &= \iint_R (x-\overline{x})(y-\overline{y})\rho\,dA \\ &= \iint_R xy\rho\,dA - \overline{x}\iint_R y\rho\,dA \\ &\quad -\overline{y}\iint_R x\rho\,dA + \iint_R \overline{x}\,\overline{y}\rho\,dA \\ &= I_{xy} - \overline{x}(M\overline{y}) - \overline{y}(M\overline{x}) + \overline{x}\,\overline{y}(M) = I_{xy} - M\overline{x}\,\overline{y} \end{aligned}$$

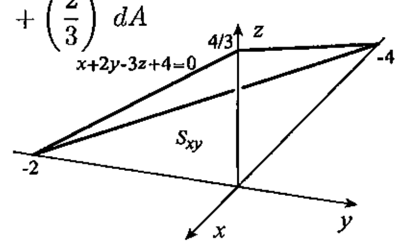


EXERCISES 13.6

1.
$$\begin{aligned} \text{Area} &= \iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= \iint_{S_{xy}} \sqrt{1 + (-1/3)^2 + (-1/2)^2}\,dA \\ &= \frac{7}{6}\iint_{S_{xy}} dA = \frac{7}{6}(\text{Area of } S_{xy}) = \frac{7}{6}\left(\frac{1}{12}\right) = \frac{7}{72} \end{aligned}$$

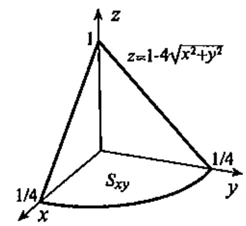


2.
$$\begin{aligned} \text{Area} &= \iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_{S_{xy}} \sqrt{1 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2}\,dA \\ &= \frac{\sqrt{14}}{3}\iint_{S_{xy}} dA = \frac{\sqrt{14}}{3}(\text{Area of } S_{xy}) \\ &= \frac{\sqrt{14}}{3}\frac{1}{2}(2)(4) = \frac{4\sqrt{14}}{3} \end{aligned}$$

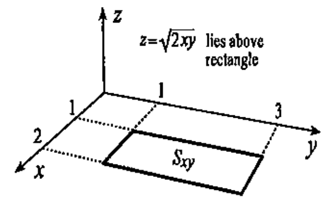


3. We quadruple the area in the first octant.

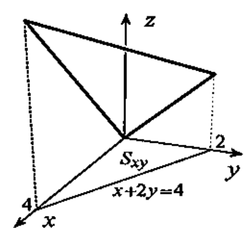
$$\begin{aligned}\text{Area } &= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{-4x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{-4y}{\sqrt{x^2+y^2}}\right)^2}\,dA\\ &= 4\sqrt{17}\iint_{S_{xy}} dA = 4\sqrt{17}(\text{Area of } S_{xy}) = 4\sqrt{17}\frac{\pi}{64} = \frac{\sqrt{17}\pi}{16}\end{aligned}$$



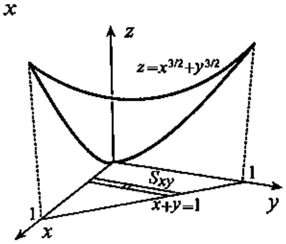
4.
$$\begin{aligned}\text{Area } &= \iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}} \sqrt{1+\left(\frac{y}{\sqrt{2xy}}\right)^2+\left(\frac{x}{\sqrt{2xy}}\right)^2}\,dA\\ &= \iint_{S_{xy}} \left(\frac{x+y}{\sqrt{2xy}}\right)dA = \frac{1}{\sqrt{2}}\int_1^2\int_1^3\left(\frac{\sqrt{x}}{\sqrt{y}}+\frac{\sqrt{y}}{\sqrt{x}}\right)dy\,dx\\ &= \frac{1}{\sqrt{2}}\int_1^2\left\{2\sqrt{xy}+\frac{2y^{3/2}}{3\sqrt{x}}\right\}_1^3 dx = \frac{1}{\sqrt{2}}\int_1^2\left[2(\sqrt{3}-1)\sqrt{x}+\frac{6\sqrt{3}-2}{3\sqrt{x}}\right]dx\\ &= \frac{1}{\sqrt{2}}\left\{\frac{4}{3}(\sqrt{3}-1)x^{3/2}+\frac{4(3\sqrt{3}-1)\sqrt{x}}{3}\right\}_1^2 = \frac{4}{3}(5\sqrt{3}-2\sqrt{6}-3+\sqrt{2})\end{aligned}$$



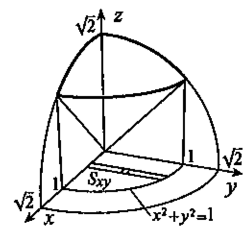
5.
$$\begin{aligned}\text{Area } &= \iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}} \sqrt{1+(1)^2+(1)^2}\,dA\\ &= \sqrt{3}\iint_{S_{xy}} dA = \sqrt{3}(\text{Area of } S_{xy})\\ &= \sqrt{3}\left(\frac{1}{2}\right)(4)(2) = 4\sqrt{3}\end{aligned}$$



6.
$$\begin{aligned}\text{Area } &= \iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\ &= \iint_{S_{xy}} \sqrt{1+\left(\frac{3\sqrt{x}}{2}\right)^2+\left(\frac{3\sqrt{y}}{2}\right)^2}\,dA\\ &= \frac{1}{2}\iint_{S_{xy}} \sqrt{4+9x+9y}\,dA\\ &= \frac{1}{2}\int_0^1\int_0^{1-x}\sqrt{4+9x+9y}\,dy\,dx = \frac{1}{2}\int_0^1\left\{\frac{2}{27}(4+9x+9y)^{3/2}\right\}_0^{1-x}dx\\ &= \frac{1}{27}\int_0^1[13\sqrt{13}-(4+9x)^{3/2}]\,dx = \frac{1}{27}\left\{13\sqrt{13}x-\frac{2(4+9x)^{5/2}}{45}\right\}_0^1 = \frac{247\sqrt{13}+64}{1215}\end{aligned}$$

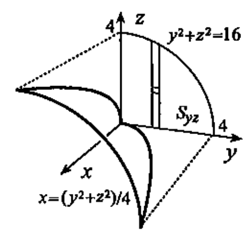


7. We quadruple the first octant area.



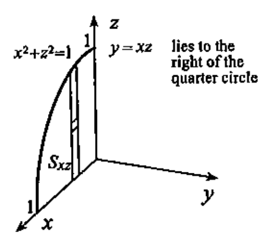
$$\begin{aligned}\text{Area } &= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\
&= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{-x}{\sqrt{2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{2-x^2-y^2}}\right)^2}\,dA\\
&= 4\iint_{S_{xy}} \frac{\sqrt{2}}{\sqrt{2-x^2-y^2}}dA = 4\sqrt{2}\int_0^1\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{2-x^2-y^2}}dy\,dx\end{aligned}$$

8. We quadruple the area in the first octant.



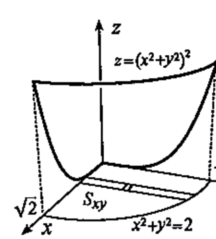
$$\begin{aligned}\text{Area } &= 4\iint_{S_{yz}} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dA\\
&= 4\iint_{S_{yz}} \sqrt{1+\left(\frac{y}{2}\right)^2+\left(\frac{z}{2}\right)^2}\,dA\\
&= 2\iint_{S_{yz}} \sqrt{4+y^2+z^2}\,dA\\
&= 2\int_0^4\int_0^{\sqrt{16-y^2}} \sqrt{4+y^2+z^2}\,dz\,dy\end{aligned}$$

9.



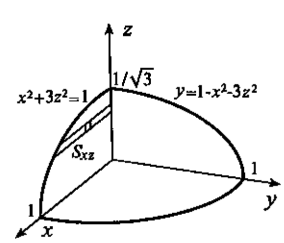
$$\begin{aligned}\text{Area } &= \iint_{S_{xz}} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA\\
&= \iint_{S_{xz}} \sqrt{1+(z)^2+(x)^2}dA\\
&= \int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1+x^2+z^2}\,dz\,dx\end{aligned}$$

10. We quadruple the area in the first octant.



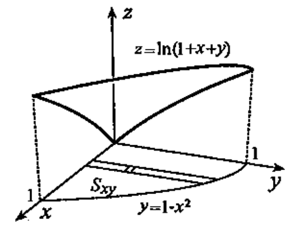
$$\begin{aligned}\text{Area } &= 4\iint_{S_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\
&= 4\iint_{S_{xy}} \sqrt{1+[4x(x^2+y^2)]^2+[4y(x^2+y^2)]^2}dA\\
&= 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} \sqrt{1+16(x^2+y^2)^3}\,dy\,dx\end{aligned}$$

11. We quadruple the area in the first octant.

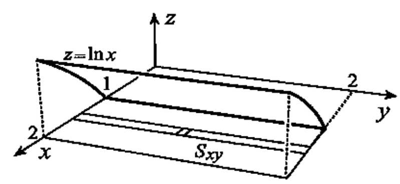


$$\begin{aligned}\text{Area } &= 4\iint_{S_{xz}} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA\\
&= 4\iint_{S_{xz}} \sqrt{1+(-2x)^2+(-6z)^2}\,dA\\
&= 4\int_0^{1/\sqrt{3}}\int_0^{\sqrt{1-3z^2}} \sqrt{1+4x^2+36z^2}\,dx\,dz\end{aligned}$$

12.
$$\text{Area} = \iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= \iint_{S_{xy}} \sqrt{1 + \left(\frac{1}{1+x+y}\right)^2 + \left(\frac{1}{1+x+y}\right)^2}\, dA$$
$$= \int_0^1 \int_0^{1-x^2} \sqrt{1 + \frac{2}{(1+x+y)^2}}\, dy\, dx$$



13.
$$\text{Area} = \iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= \iint_{S_{xy}} \sqrt{1 + (1/x)^2}\, dA = \int_1^2 \int_0^2 \frac{\sqrt{x^2+1}}{x} dy\, dx$$
$$= \int_1^2 \left\{ \frac{y\sqrt{x^2+1}}{x} \right\}_0^2 dx = 2\int_1^2 \frac{\sqrt{x^2+1}}{x} dx$$

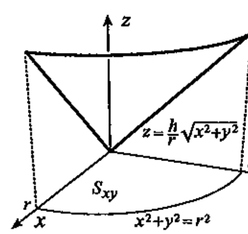


If we set $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$,

$$\text{Area} = 2\int_{\pi/4}^{\mathrm{Tan}^{-1}2} \frac{\sec\theta}{\tan\theta} \sec^2\theta\, d\theta = 2\int_{\pi/4}^{\mathrm{Tan}^{-1}2} \frac{(1+\tan^2\theta)\sec\theta}{\tan\theta} d\theta = 2\int_{\pi/4}^{\mathrm{Tan}^{-1}2} (\csc\theta + \sec\theta\tan\theta)\, d\theta$$
$$= 2\Big\{ \ln|\csc\theta - \cot\theta| + \sec\theta \Big\}_{\pi/4}^{\mathrm{Tan}^{-1}2} = 2[\ln(\sqrt{5}-1) - \ln 2 + \sqrt{5} - \ln(\sqrt{2}-1) - \sqrt{2}].$$

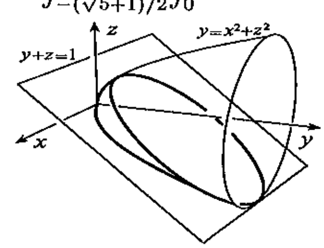
14. We quadruple the area in the first octant.

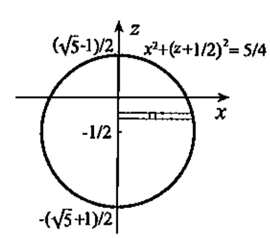
$$\text{Area} = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{hx}{r\sqrt{x^2+y^2}}\right)^2 + \left(\frac{hy}{r\sqrt{x^2+y^2}}\right)^2}\, dA$$
$$= \frac{4\sqrt{r^2+h^2}}{r} \iint_{S_{xy}} dA = \frac{4\sqrt{r^2+h^2}}{r} (\text{Area of } S_{xy})$$
$$= \frac{4\sqrt{r^2+h^2}}{r} \left(\frac{\pi r^2}{4}\right) = \pi r\sqrt{r^2+h^2}$$



15. The surface projects onto the area in the xz-plane bounded by the circle $1 - z = x^2 + z^2$, or, $x^2 + (z + 1/2)^2 = 5/4$. Thus,

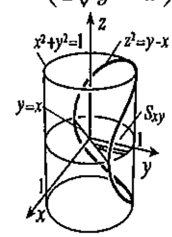
$$A = \iint_{S_{xz}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dA = \iint_{S_{xz}} \sqrt{1 + (2x)^2 + (2z)^2}\, dA$$
$$= 2\int_{-(\sqrt{5}+1)/2}^{(\sqrt{5}-1)/2} \int_0^{\sqrt{1-z-z^2}} \sqrt{1 + 4x^2 + 4z^2}\, dx\, dz$$





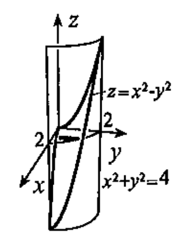
16. We double the area of the upper half.

$$\begin{aligned}\text{Area } &= 2\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 2\iint_{S_{xy}}\sqrt{1+\left(\frac{-1}{2\sqrt{y-x}}\right)^2+\left(\frac{1}{2\sqrt{y-x}}\right)^2}\,dA\\
&= \sqrt{2}\iint_{S_{xy}}\sqrt{2+\frac{1}{y-x}}\,dA\\
&= \sqrt{2}\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{x}^{\sqrt{1-x^2}}\sqrt{2+\frac{1}{y-x}}\,dy\,dx\\
&\quad +\sqrt{2}\int_{-1}^{-1/\sqrt{2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{2+\frac{1}{y-x}}\,dy\,dx\end{aligned}$$

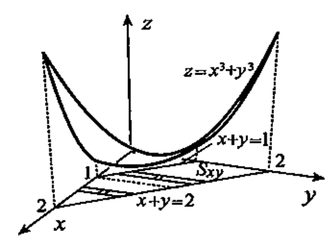


17. We quadruple the area in the first octant.

$$\begin{aligned}\text{Area } &= 4\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\
&= 4\iint_{S_{xy}}\sqrt{1+(-2x)^2+(2y)^2}\,dA\\
&= 4\int_0^2\int_0^{\sqrt{4-x^2}}\sqrt{1+4x^2+4y^2}\,dy\,dx\end{aligned}$$

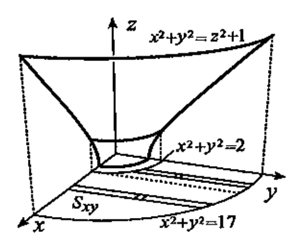


18.
$$\begin{aligned}\text{Area } &= \iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\
&= \iint_{S_{xy}}\sqrt{1+(3x^2)^2+(3y^2)^2}dA\\
&= \int_0^1\int_{1-x}^{2-x}\sqrt{1+9x^4+9y^4}\,dy\,dx\\
&\quad +\int_1^2\int_0^{2-x}\sqrt{1+9x^4+9y^4}\,dy\,dx\end{aligned}$$



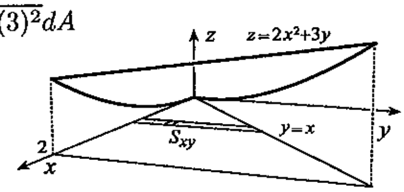
19. We quadruple the area in the first octant.

$$\begin{aligned}\text{Area } &= 4\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 4\iint_{S_{xy}}\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2-1}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2-1}}\right)^2}\,dA\\
&= 4\iint_{S_{xy}}\sqrt{\frac{2x^2+2y^2-1}{x^2+y^2-1}}dA\\
&= 4\int_0^{\sqrt{2}}\int_{\sqrt{2-x^2}}^{\sqrt{17-x^2}}\sqrt{\frac{2x^2+2y^2-1}{x^2+y^2-1}}dy\,dx\\
&\quad +4\int_{\sqrt{2}}^{\sqrt{17}}\int_0^{\sqrt{17-x^2}}\sqrt{\frac{2x^2+2y^2-1}{x^2+y^2-1}}dy\,dx\end{aligned}$$

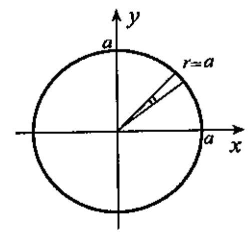


20. If S_{xy} is the region of the xy-plane bounded by the lines $x=2$, $y=0$, and $y=x$, then

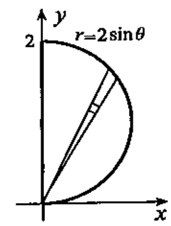
$$\begin{aligned}\text{Area } &= \iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_{S_{xy}}\sqrt{1+(4x)^2+(3)^2}dA\\
&= \int_0^2\int_0^x\sqrt{10+16x^2}\,dy\,dx = \int_0^2 x\sqrt{10+16x^2}\,dx\\
&= \left\{\frac{1}{48}(10+16x^2)^{3/2}\right\}_0^2 = \frac{1}{24}(37\sqrt{74}-5\sqrt{10}).\end{aligned}$$



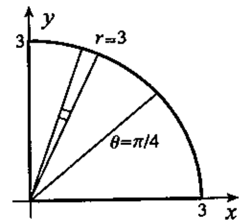
EXERCISES 13.7



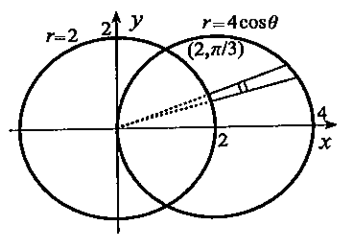
1. $$\iint_R e^{x^2+y^2}\,dA = \int_{-\pi}^{\pi}\int_0^a e^{r^2} r\,dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\left\{\frac{1}{2}e^{r^2}\right\}_0^a d\theta$$
$$= \frac{1}{2}(e^{a^2}-1)\left\{\theta\right\}_{-\pi}^{\pi} = \pi(e^{a^2}-1)$$



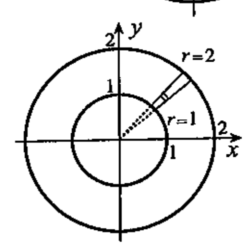
2. $$\iint_R x\,dA = \int_0^{\pi/2}\int_0^{2\sin\theta} r\cos\theta\, r\,dr\,d\theta = \int_0^{\pi/2}\left\{\frac{r^3}{3}\cos\theta\right\}_0^{2\sin\theta} d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\sin^3\theta\,\cos\theta\,d\theta$$
$$= \frac{8}{3}\left\{\frac{\sin^4\theta}{4}\right\}_0^{\pi/2} = \frac{2}{3}$$



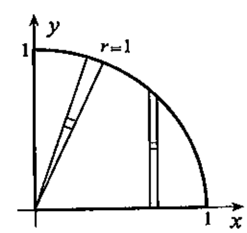
3. $$\iint_R \sqrt{x^2+y^2}\,dA = \int_{\pi/4}^{\pi/2}\int_0^3 (r)r\,dr\,d\theta$$
$$= \int_{\pi/4}^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^3 d\theta$$
$$= 9\left\{\theta\right\}_{\pi/4}^{\pi/2} = \frac{9\pi}{4}$$



4. $$\iint_R \frac{1}{\sqrt{x^2+y^2}}dA = 2\int_0^{\pi/3}\int_2^{4\cos\theta}\frac{1}{r}r\,dr\,d\theta$$
$$= 2\int_0^{\pi/3}(4\cos\theta-2)\,d\theta$$
$$= 4\left\{2\sin\theta-\theta\right\}_0^{\pi/3} = \frac{4}{3}(3\sqrt{3}-\pi)$$

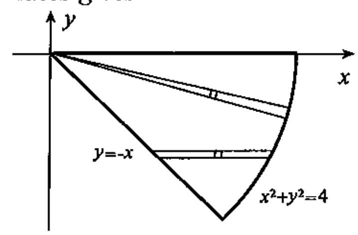


5. $$\iint_R \sqrt{1+2x^2+2y^2}\,dA = \int_{-\pi}^{\pi}\int_1^2 \sqrt{1+2r^2}\,r\,dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\left\{\frac{1}{6}(1+2r^2)^{3/2}\right\}_1^2 d\theta$$
$$= \frac{1}{6}(27-3\sqrt{3})\left\{\theta\right\}_{-\pi}^{\pi} = (9-\sqrt{3})\pi$$



6. This double iterated integral represents the double integral of $\sqrt{x^2+y^2}$ over the quarter circle shown. When we change to polar coordinates,
$$\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^1 r\,r\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^1 d\theta = \frac{1}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{\pi}{6}.$$

7. Limits define the quarter-circle shown. Changing to polar coordinates gives

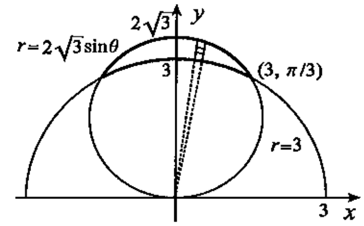


$$\int_{-\sqrt{2}}^{0}\int_{-y}^{\sqrt{4-y^2}} x^2\,dx\,dy = \int_{-\pi/4}^{0}\int_{0}^{2} r^2\cos^2\theta\, r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{0}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^2 d\theta = 4\int_{-\pi/4}^{0}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 2\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/4}^{0} = \frac{2+\pi}{2}$$

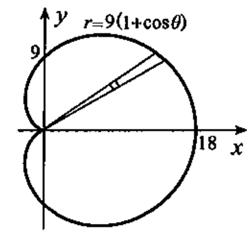
8. $$A = 2\int_{\pi/3}^{\pi/2}\int_{3}^{2\sqrt{3}\sin\theta} r\,dr\,d\theta = 2\int_{\pi/3}^{\pi/2}\left\{\frac{r^2}{2}\right\}_3^{2\sqrt{3}\sin\theta} d\theta$$



$$= 3\int_{\pi/3}^{\pi/2}(4\sin^2\theta - 3)\,d\theta = 3\int_{\pi/3}^{\pi/2}[2(1-\cos 2\theta) - 3]\,d\theta$$

$$= 3\Big\{-\theta - \sin 2\theta\Big\}_{\pi/3}^{\pi/2} = \frac{3\sqrt{3}-\pi}{2}$$

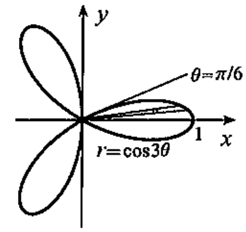
9. $$A = 2\int_{0}^{\pi}\int_{0}^{9(1+\cos\theta)} r\,dr\,d\theta = 2\int_{0}^{\pi}\left\{\frac{r^2}{2}\right\}_0^{9(1+\cos\theta)} d\theta$$



$$= \int_0^{\pi} 81(1+\cos\theta)^2\,d\theta = 81\int_0^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 81\left\{\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right\}_0^{\pi} = \frac{243\pi}{2}$$

10. $$A = 6\int_{0}^{\pi/6}\int_{0}^{\cos 3\theta} r\,dr\,d\theta = 6\int_{0}^{\pi/6}\left\{\frac{r^2}{2}\right\}_0^{\cos 3\theta} d\theta$$



$$= 3\int_0^{\pi/6}\cos^2 3\theta\,d\theta = 3\int_0^{\pi/6}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

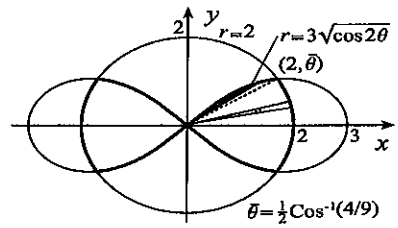
$$= \frac{3}{2}\left\{\theta + \frac{\sin 6\theta}{6}\right\}_0^{\pi/6} = \frac{\pi}{4}$$

11. $$A = 4\int_0^{\overline{\theta}}\int_0^2 r\,dr\,d\theta + 4\int_{\overline{\theta}}^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\overline{\theta}}\left\{\frac{r^2}{2}\right\}_0^2 d\theta + 4\int_{\overline{\theta}}^{\pi/4}\left\{\frac{r^2}{2}\right\}_0^{3\sqrt{\cos 2\theta}} d\theta$$

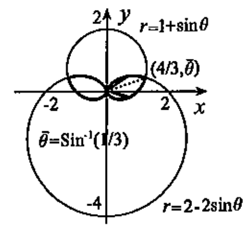
$$= 8\Big\{\theta\Big\}_0^{\overline{\theta}} + 2\int_{\overline{\theta}}^{\pi/4} 9\cos 2\theta\,d\theta = 8\overline{\theta} + 18\left\{\frac{1}{2}\sin 2\theta\right\}_{\overline{\theta}}^{\pi/4}$$

$$= 8\overline{\theta} + 9(1-\sin 2\overline{\theta}) = 4\mathrm{Cos}^{-1}(4/9) + 9 - \sqrt{65}$$



12. $A = 2\int_{-\pi/2}^{\overline{\theta}}\int_0^{1+\sin\theta} r\,dr\,d\theta + 2\int_{\overline{\theta}}^{\pi/2}\int_0^{2-2\sin\theta} r\,dr\,d\theta = 2\int_{-\pi/2}^{\overline{\theta}}\left\{\frac{r^2}{2}\right\}_0^{1+\sin\theta} d\theta + 2\int_{\overline{\theta}}^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{2-2\sin\theta} d\theta$

$$= \int_{-\pi/2}^{\overline{\theta}} (1+\sin\theta)^2\,d\theta + 4\int_{\overline{\theta}}^{\pi/2}(1-\sin\theta)^2\,d\theta$$



$$= \int_{-\pi/2}^{\overline{\theta}}\left(1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right) d\theta$$

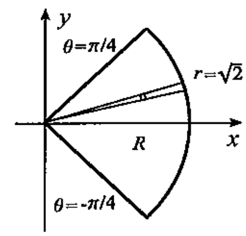
$$+4\int_{\overline{\theta}}^{\pi/2}\left(1 - 2\sin\theta + \frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \left\{\frac{3\theta}{2} - 2\cos\theta - \frac{\sin 2\theta}{4}\right\}_{-\pi/2}^{\overline{\theta}} + 4\left\{\frac{3\theta}{2} + 2\cos\theta - \frac{\sin 2\theta}{4}\right\}_{\overline{\theta}}^{\pi/2}$$

$$= \frac{15\pi}{4} - \frac{9\overline{\theta}}{2} - 10\cos\overline{\theta} + \frac{3}{4}\sin 2\overline{\theta} = \frac{15\pi}{4} - \frac{9}{2}\mathrm{Sin}^{-1}(1/3) - \frac{19\sqrt{2}}{3}$$

13. By symmetry, $\overline{y} = 0$. The area is $A = (1/4)\pi(2) = \pi/2$. Since

$$A\overline{x} = \iint_R x\,dA = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{2}} r\cos\theta\, r\,dr\,d\theta$$



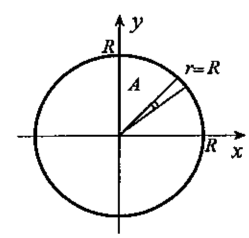
$$= \int_{-\pi/4}^{\pi/4}\left\{\frac{r^3}{3}\cos\theta\right\}_0^{\sqrt{2}} d\theta$$

$$= \frac{2\sqrt{2}}{3}\left\{\sin\theta\right\}_{-\pi/4}^{\pi/4} = \frac{4}{3},$$

it follows that $\overline{x} = (4/3)(2/\pi) = 8/(3\pi)$.

14. If we choose the x-axis as diameter,

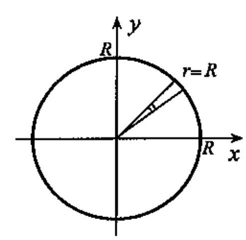
$$I = \iint_A y^2\,dA = \int_{-\pi}^{\pi}\int_0^R r^2\sin^2\theta\, r\,dr\,d\theta$$



$$= \int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\sin^2\theta\right\}_0^R d\theta = \frac{R^4}{4}\int_{-\pi}^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{R^4}{8}\left\{\theta - \frac{\sin 2\theta}{2}\right\}_{-\pi}^{\pi} = \frac{\pi R^4}{4}$$

15. $F = \int_{-\pi}^{\pi}\int_0^R 1000(9.81)(R - r\sin\theta)r\,dr\,d\theta$



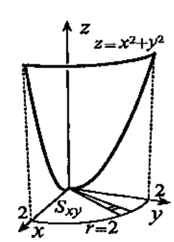
$$= 9810\int_{-\pi}^{\pi}\left\{\frac{Rr^2}{2} - \frac{r^3}{3}\sin\theta\right\}_0^R d\theta$$

$$= \frac{9810}{6}\int_{-\pi}^{\pi}(3R^3 - 2R^3\sin\theta)\,d\theta$$

$$= 1635R^3\left\{3\theta + 2\cos\theta\right\}_{-\pi}^{\pi} = 9810\pi R^3 \text{ N}$$

16. We quadruple the area in the first octant.

$$A = 4\iint_{S_{xy}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = 4\iint_{S_{xy}}\sqrt{1 + (2x)^2 + (2y)^2}\,dA$$



$$= 4\int_0^{\pi/2}\int_0^2\sqrt{1+4r^2}\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{(1+4r^2)^{3/2}}{12}\right\}_0^2 d\theta$$

$$= \frac{17\sqrt{17}-1}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{(17\sqrt{17}-1)\pi}{6}$$

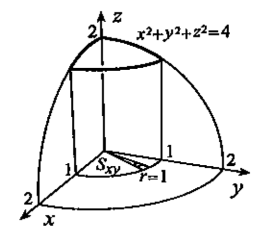
17. We multiply the area in the first octant by 8.

$$A = 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2}\, dA$$

$$= 8\iint_{S_{xy}} \frac{2}{\sqrt{4-x^2-y^2}} dA = 16\int_0^{\pi/2}\int_0^1 \frac{1}{\sqrt{4-r^2}} r\, dr\, d\theta$$

$$= 16\int_0^{\pi/2}\left\{-\sqrt{4-r^2}\right\}_0^1 d\theta = 16(2-\sqrt{3})\left\{\theta\right\}_0^{\pi/2} = 8\pi(2-\sqrt{3})$$

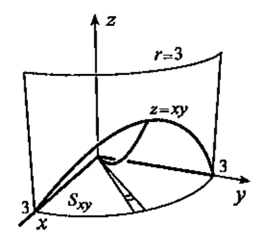


18. We quadruple the area in the first octant.

$$A = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = 4\iint_{S_{xy}} \sqrt{1+(y)^2+(x)^2}\, dA$$

$$= 4\int_0^{\pi/2}\int_0^3 \sqrt{1+r^2}\, r\, dr\, d\theta = 4\int_0^{\pi/2}\left\{\frac{(1+r^2)^{3/2}}{3}\right\}_0^3 d\theta$$

$$= 4\frac{10\sqrt{10}-1}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{2\pi}{3}(10\sqrt{10}-1)$$



19. We multiply the first octant area by 8.

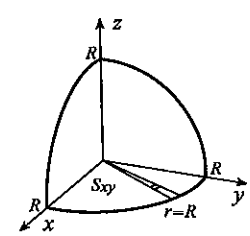
$$A = 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{-x}{\sqrt{R^2-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{R^2-x^2-y^2}}\right)^2}\, dA$$

$$= 8\iint_{S_{xy}} \frac{R}{\sqrt{R^2-x^2-y^2}} dA$$

$$= 8R\int_0^{\pi/2}\int_0^R \frac{1}{\sqrt{R^2-r^2}} r\, dr\, d\theta$$

$$= 8R\int_0^{\pi/2}\left\{-\sqrt{R^2-r^2}\right\}_0^R d\theta$$

$$= 8R^2\left\{\theta\right\}_0^{\pi/2} = 4\pi R^2$$



20. We quadruple the area in the first octant.

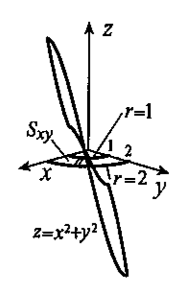
$$A = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= 4\iint_{S_{xy}} \sqrt{1+(2x)^2+(-2y)^2}\, dA$$

$$= 4\int_0^{\pi/2}\int_1^2 \sqrt{1+4r^2}\, r\, dr\, d\theta$$

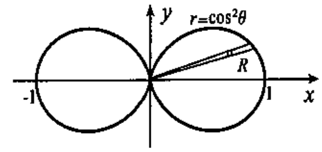
$$= 4\int_0^{\pi/2}\left\{\frac{(1+4r^2)^{3/2}}{12}\right\}_1^2 d\theta$$

$$= \frac{17\sqrt{17}-5\sqrt{5}}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{(17\sqrt{17}-5\sqrt{5})\pi}{6}.$$



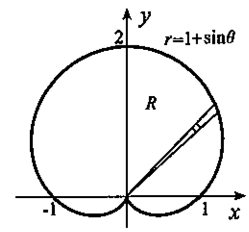
21. If R is the area in the first quadrant.

$$V = 2\iint_R 2\pi y\,dA = 4\pi\int_0^{\pi/2}\int_0^{\cos^2\theta} r\sin\theta\, r\,dr\,d\theta$$
$$= 4\pi\int_0^{\pi/2}\left\{\frac{r^3}{3}\sin\theta\right\}_0^{\cos^2\theta} d\theta = \frac{4\pi}{3}\int_0^{\pi/2}\cos^6\theta\sin\theta\,d\theta$$
$$= \frac{4\pi}{3}\left\{-\frac{1}{7}\cos^7\theta\right\}_0^{\pi/2} = \frac{4\pi}{21}$$



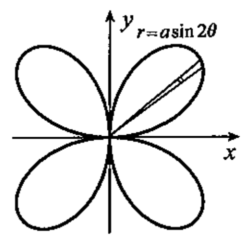
22. If R is that part of the cardioid to the right of the y-axis, then

$$V = \iint_R 2\pi x\,dA = 2\pi\int_{-\pi/2}^{\pi/2}\int_0^{1+\sin\theta} r\cos\theta\, r\,dr\,d\theta$$
$$= 2\pi\int_{-\pi/2}^{\pi/2}\left\{\frac{r^3}{3}\cos\theta\right\}_0^{1+\sin\theta} d\theta$$
$$= \frac{2\pi}{3}\int_{-\pi/2}^{\pi/2}(1+\sin\theta)^3\cos\theta\,d\theta = \frac{2\pi}{3}\left\{\frac{1}{4}(1+\sin\theta)^4\right\}_{-\pi/2}^{\pi/2} = \frac{8\pi}{3}$$



23. The equation of the curve in polar coordinates is
$r^6 = 4a^2(r^2\cos^2\theta)(r^2\sin^2\theta) \implies r^2 = a^2\sin^2 2\theta$.

$$A = 4\int_0^{\pi/2}\int_0^{a\sin 2\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{a\sin 2\theta} d\theta$$
$$= 2\int_0^{\pi/2} a^2\sin^2 2\theta\,d\theta = 2a^2\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$
$$= a^2\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{\pi a^2}{2}$$



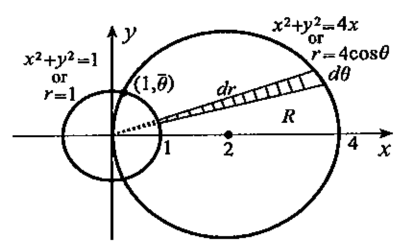
24. The equation of the inner surface of the shell is $r = 700$ in polar coordinates. The equation of the right-half of the outer surface of the shell is $r = 710 - 10\theta/\pi$. The volume of the shell is the product of its length 5000 cm and the cross-sectional area shown,

$$V = 5000(2)\int_0^{\pi/2}\int_{700}^{710-10\theta/\pi} r\,dr\,d\theta = 10\,000\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_{700}^{710-10\theta/\pi} d\theta$$
$$= 5000\int_0^{\pi/2}\left[\left(710 - \frac{10\theta}{\pi}\right)^2 - 490\,000\right]d\theta = 5000\left\{-\frac{\pi}{30}\left(710 - \frac{10\theta}{\pi}\right)^3 - 490\,000\theta\right\}_0^{\pi/2}$$
$$= 8.29\times 10^7 \text{ cc.}$$

25. If R is the region bounded by these circles and above the x-axis, then the required area is

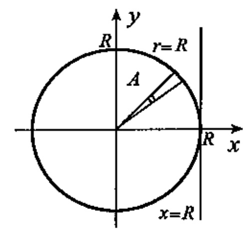
$$2\iint_R dA.$$

Since the curves intersect in the first quadrant at a point where $\theta = \overline{\theta} = \mathrm{Cos}^{-1}(\frac{1}{4})$, then



$$\text{area} = 2\int_0^{\overline{\theta}}\int_1^{4\cos\theta} r\,dr\,d\theta = 2\int_0^{\overline{\theta}}\left\{\frac{r^2}{2}\right\}_1^{4\cos\theta} d\theta = \int_0^{\overline{\theta}}(16\cos^2\theta - 1)d\theta$$
$$= \int_0^{\overline{\theta}}\left[16\left(\frac{1+\cos 2\theta}{2}\right) - 1\right]d\theta = \int_0^{\overline{\theta}}(7 + 8\cos 2\theta)d\theta = \{7\theta + 4\sin 2\theta\}_0^{\overline{\theta}}$$
$$= 7\overline{\theta} + 4\sin 2\overline{\theta} = 7\mathrm{Cos}^{-1}(\tfrac{1}{4}) + 8\cos\overline{\theta}\sin\overline{\theta}$$
$$= 7\,\mathrm{Cos}^{-1}(\tfrac{1}{4}) + 8(\tfrac{1}{4})\sqrt{1 - \tfrac{1}{16}} = 7\,\mathrm{Cos}^{-1}(\tfrac{1}{4}) + \sqrt{15}/2.$$

26. If we rotate $x^2+y^2 \le R^2$ about $x = R$,



$$V = \iint_A 2\pi(R-x)\,dA = 2\pi \int_{-\pi}^{\pi}\int_0^R (R - r\cos\theta)\,r\,dr\,d\theta$$
$$= 2\pi \int_{-\pi}^{\pi} \left\{ \frac{Rr^2}{2} - \frac{r^3}{3}\cos\theta \right\}_0^R d\theta = 2\pi R^3 \int_{-\pi}^{\pi} \left(\frac{1}{2} - \frac{1}{3}\cos\theta \right) d\theta$$
$$= 2\pi R^3 \left\{ \frac{\theta}{2} - \frac{1}{3}\sin\theta \right\}_{-\pi}^{\pi} = 2\pi^2 R^3.$$

27. (a) We set $s = \sqrt{r^2+d^2}$ where (r,θ) are the polar coordinates of dA, and integrate over the plate,

$$V = \int_{-\pi}^{\pi}\int_0^R \frac{\rho}{4\pi\epsilon_0\sqrt{r^2+d^2}}\,r\,dr\,d\theta = \frac{\rho}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^R \frac{r}{\sqrt{r^2+d^2}}\,dr\,d\theta.$$

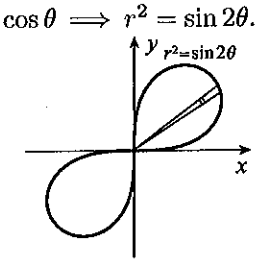
(b) $V = \dfrac{\rho}{4\pi\epsilon_0}\displaystyle\int_{-\pi}^{\pi}\left\{\sqrt{r^2+d^2}\right\}_0^R d\theta = \dfrac{\rho}{4\pi\epsilon_0}(\sqrt{R^2+d^2}-d)\Big\{\theta\Big\}_{-\pi}^{\pi} = \dfrac{\rho}{2\epsilon_0}(\sqrt{R^2+d^2}-d)$

28. The force on q due to the charge ρdA in dA has magnitude $\dfrac{q\rho dA}{4\pi\epsilon_0 s^2}$. Since x- and y-components of contributions from all parts of the plate cancel, only the z-components survive, and for the contribution from dA, the z-component is $\dfrac{q\rho dA}{4\pi\epsilon_0 s^2}\cos\psi$, where ψ is the angle between the z-axis and the line joining P and dA. The total force therefore has z-component

$$F_z = \iint_A \frac{q\rho\cos\psi}{4\pi\epsilon_0 s^2}dA = \frac{q\rho}{4\pi\epsilon_0}\iint_A \frac{d}{s^3}dA = \frac{q\rho d}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^R \frac{1}{(r^2+d^2)^{3/2}}\,r\,dr\,d\theta$$
$$= \frac{q\rho d}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\left\{\frac{-1}{\sqrt{r^2+d^2}}\right\}_0^R d\theta = \frac{q\rho d}{4\pi\epsilon_0}\left(\frac{1}{d} - \frac{1}{\sqrt{R^2+d^2}}\right)(2\pi) = \frac{q\rho}{2\epsilon_0}\left(1 - \frac{d}{\sqrt{R^2+d^2}}\right).$$

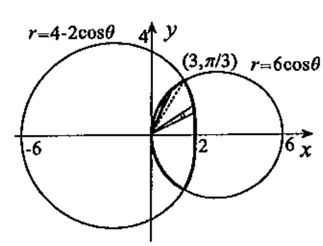
As the radius of the plate becomes very large, $\displaystyle\lim_{R\to\infty} F_z = \frac{q\rho}{2\epsilon_0}$.

29. The equation of the curve in polar coordinates is $r^4 = 2r^2\sin\theta\cos\theta \implies r^2 = \sin 2\theta$.



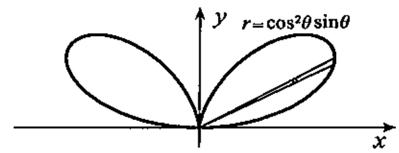
$$A = 2\int_0^{\pi/2}\int_0^{\sqrt{\sin 2\theta}} r\,dr\,d\theta$$
$$= 2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{\sqrt{\sin 2\theta}} d\theta$$
$$= \int_0^{\pi/2}\sin 2\theta\,d\theta = \left\{-\frac{1}{2}\cos 2\theta\right\}_0^{\pi/2} = 1$$

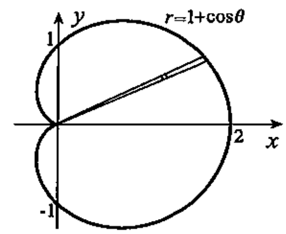
30. $$A = 2\int_0^{\pi/3}\int_0^{4-2\cos\theta} r\,dr\,d\theta + 2\int_{\pi/3}^{\pi/2}\int_0^{6\cos\theta} r\,dr\,d\theta$$
$$= 2\int_0^{\pi/3}\left\{\frac{r^2}{2}\right\}_0^{4-2\cos\theta} d\theta + 2\int_{\pi/3}^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{6\cos\theta} d\theta = 4\int_0^{\pi/3}(2-\cos\theta)^2\,d\theta + 36\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta$$
$$= 4\int_0^{\pi/3}\left(4 - 4\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$
$$+36\int_{\pi/3}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= 4\left\{\frac{9\theta}{2} - 4\sin\theta + \frac{\sin 2\theta}{4}\right\}_0^{\pi/3} + 18\left\{\theta + \frac{\sin 2\theta}{2}\right\}_{\pi/3}^{\pi/2}$$
$$= 9\pi - 12\sqrt{3}$$



31. $A = 2\int_0^{\pi/2}\int_0^{\cos^2\theta\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{\cos^2\theta\sin\theta} d\theta$

$$= \int_0^{\pi/2}\cos^4\theta\sin^2\theta\,d\theta$$
$$= \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$
$$= \frac{1}{8}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2} + \sin^2 2\theta\cos 2\theta\right) d\theta$$
$$= \frac{1}{8}\left\{\frac{\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{\pi}{32}$$





32. $A = 2\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi}(1+\cos\theta)^2\,d\theta$

$$= \int_0^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$
$$= \left\{\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right\}_0^{\pi} = \frac{3\pi}{2}$$

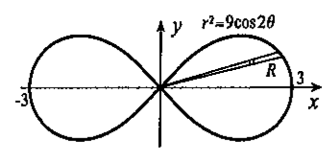
By symmetry, $\overline{y} = 0$. Since

$$A\overline{x} = 2\int_0^{\pi}\int_0^{1+\cos\theta} r\cos\theta\, r\,dr\,d\theta = 2\int_0^{\pi}\left\{\frac{r^3}{3}\cos\theta\right\}_0^{1+\cos\theta} d\theta = \frac{2}{3}\int_0^{\pi}(1+\cos\theta)^3\cos\theta\,d\theta$$
$$= \frac{2}{3}\int_0^{\pi}\left[\cos\theta + 3\left(\frac{1+\cos 2\theta}{2}\right) + 3\cos\theta(1-\sin^2\theta) + \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)\right] d\theta$$
$$= \frac{2}{3}\left\{4\sin\theta + \frac{15\theta}{8} + \sin 2\theta - \sin^3\theta + \frac{\sin 4\theta}{32}\right\}_0^{\pi} = \frac{5\pi}{4},$$

we find $\overline{x} = \frac{5\pi}{4}\frac{2}{3\pi} = \frac{5}{6}$.

33. $I = 4\iint_R y^2\,dA = 4\int_0^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} r^2\sin^2\theta\, r\,dr\,d\theta = 4\int_0^{\pi/4}\left\{\frac{r^4}{4}\sin^2\theta\right\}_0^{3\sqrt{\cos 2\theta}} d\theta$

$$= 81\int_0^{\pi/4}\cos^2 2\theta\sin^2\theta\,d\theta = 81\int_0^{\pi/4}\cos^2 2\theta\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$
$$= \frac{81}{2}\int_0^{\pi/4}\left[\frac{1+\cos 4\theta}{2} - (1-\sin^2 2\theta)\cos 2\theta\right] d\theta$$
$$= \frac{81}{2}\left\{\frac{\theta}{2} + \frac{1}{8}\sin 4\theta - \frac{1}{2}\sin 2\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/4} = \frac{27(3\pi-8)}{16}$$



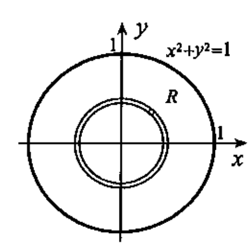
34. $I = \iint_R\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dA = \int_0^1\int_{-\pi}^{\pi}\sqrt{\frac{1-r^2}{1+r^2}}\,r\,d\theta\,dr = 2\pi\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr$

If we set $u = \sqrt{1+r^2}$, then $2u\,du = 2r\,dr$, and

$$I = 2\pi\int_1^{\sqrt{2}}\sqrt{\frac{1-(u^2-1)}{u^2}}\,u\,du = 2\pi\int_1^{\sqrt{2}}\sqrt{2-u^2}\,du.$$

If we now set $u = \sqrt{2}\sin\phi$ and $du = \sqrt{2}\cos\phi\,d\phi$, then

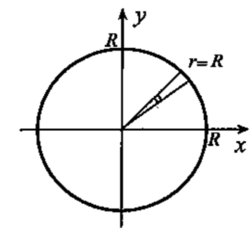
$$I = 2\pi\int_{\pi/4}^{\pi/2}\sqrt{2}\cos\phi\,\sqrt{2}\cos\phi\,d\phi = 4\pi\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos 2\phi}{2}\right) d\phi$$
$$= 2\pi\left\{\phi + \frac{\sin 2\phi}{2}\right\}_{\pi/4}^{\pi/2} = \frac{\pi(\pi-2)}{2}.$$



35. $B = \int_{-\pi}^{\pi}\int_0^R v\, r\, dr\, d\theta = \int_{-\pi}^{\pi}\int_0^R \frac{P}{4nL}(R^2 - r^2) r\, dr\, d\theta$

$= \frac{P}{4nL}\int_{-\pi}^{\pi}\left\{\frac{R^2r^2}{2} - \frac{r^4}{4}\right\}_0^R d\theta$

$= \frac{PR^4}{16nL}\left\{\theta\right\}_{-\pi}^{\pi} = \frac{\pi PR^4}{8nL}$



36. The volume of blood flowing through any cross-section of the larger blood vessel per unit time is

$$\int_{-\pi}^{\pi}\int_0^R V_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right] r\, dr\, d\theta = V_{\max}\int_{-\pi}^{\pi}\left\{\frac{r^2}{2} - \frac{r^4}{4R^2}\right\}_0^R d\theta = \frac{R^2V_{\max}}{4}\left\{\theta\right\}_{-\pi}^{\pi} = \frac{\pi R^2 V_{\max}}{2}.$$

Similarly, the volume of blood flowing through any cross-section of the smaller blood vessel is

$$\int_{-\pi}^{\pi}\int_0^{R_1} U_{\max}\left[1 - \left(\frac{r}{R_1}\right)^2\right] r\, dr\, d\theta = \frac{\pi R_1^2 U_{\max}}{2}.$$

When we equate these and set $R_1 = \alpha R$,

$$\frac{\pi R^2 V_{\max}}{2} = \frac{\pi\alpha^2 R^2 U_{\max}}{2} \quad\Longrightarrow\quad U_{\max} = \frac{V_{\max}}{\alpha^2}.$$

37. The volume of blood flowing through any cross-section of the larger blood vessel per unit time is

$$\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} V_{\max}\left(1 - \frac{4x^2}{L^2}\right)\left(1 - \frac{4y^2}{L^2}\right) dy\, dx = V_{\max}\int_{-L/2}^{L/2}\left(1 - \frac{4x^2}{L^2}\right)\left\{y - \frac{4y^3}{3L^2}\right\}_{-L/2}^{L/2} dx$$

$$= \frac{2LV_{\max}}{3}\left\{x - \frac{4x^3}{3L^2}\right\}_{-L/2}^{L/2} = \frac{4L^2V_{\max}}{9}.$$

A similar calculation for the flow through the smaller pipe gives $4(\alpha^2L^2)U_{\max}/9$, where $U_{\max}$ is the maximum velocity at the centre of the pipe. When we equate flows,

$$\frac{4L^2V_{\max}}{9} = \frac{4\alpha^2L^2U_{\max}}{9} \quad\Longrightarrow\quad U_{\max} = \frac{V_{\max}}{\alpha^2}.$$

38. We multiply the area in the first octant by 8,

$$A = 8\iint_{S_{xy}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = 8\iint_{S_{xy}}\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\, dA$$

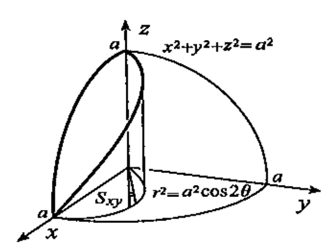
$$= 8\iint_{S_{xy}}\sqrt{\frac{a^2 - x^2 - y^2 + x^2 + y^2}{a^2 - x^2 - y^2}}\, dA = 8a\iint_{S_{xy}}\frac{1}{\sqrt{a^2 - x^2 - y^2}}\, dA$$

$$= 8a\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}}\frac{1}{\sqrt{a^2 - r^2}} r\, dr\, d\theta$$

$$= 8a\int_0^{\pi/4}\left\{-\sqrt{a^2 - r^2}\right\}_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 8a\int_0^{\pi/4}(a - \sqrt{a^2 - a^2\cos 2\theta})\, d\theta = 8a^2\int_0^{\pi/4}[1 - \sqrt{1 - (1 - 2\sin^2\theta)}]\, d\theta$$

$$= 8a^2\int_0^{\pi/4}(1 - \sqrt{2}\sin\theta)\, d\theta = 8a^2\left\{\theta + \sqrt{2}\cos\theta\right\}_0^{\pi/4} = 2a^2(\pi + 4 - 4\sqrt{2})$$



39. We multiply the area in the first octant by 8,

$$A = 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = 8\iint_{S_{xy}} \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\, dA = 8a\iint_{S_{xy}} \frac{1}{\sqrt{a^2 - x^2}} dA$$

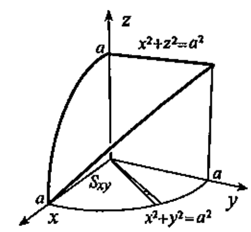
$$= 8a\int_0^{\pi/2}\int_0^a \frac{1}{\sqrt{a^2 - r^2\cos^2\theta}} r\, dr\, d\theta = 8a\int_0^{\pi/2} \left\{\frac{\sqrt{a^2 - r^2\cos^2\theta}}{-\cos^2\theta}\right\}_0^a d\theta$$

$$= 8a^2\int_0^{\pi/2} \frac{1 - \sin\theta}{\cos^2\theta} d\theta = 8a^2\int_0^{\pi/2} (\sec^2\theta - \tan\theta\sec\theta)\, d\theta$$

$$= 8a^2\left\{\tan\theta - \sec\theta\right\}_0^{\pi/2} = 8a^2\left[\lim_{\theta\to\pi/2^-} (\tan\theta - \sec\theta) + 1\right]$$

$$= 8a^2 + 8a^2 \lim_{\theta\to\pi/2^-} \frac{\sin\theta - 1}{\cos\theta} \qquad \text{(and now using L'hôpital's rule)}$$

$$= 8a^2 + 8a^2 \lim_{\theta\to\pi/2^-} \frac{\cos\theta}{-\sin\theta} = 8a^2$$



40. $I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\, dx$

We now interpret this double iterated integral as the double integral of $e^{-(x^2+y^2)}$ over the first quadrant of the xy-plane, and change to polar coordinates,

$$I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, dr\, d\theta = \int_0^{\pi/2} \left\{-\frac{1}{2}e^{-r^2}\right\}_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}.$$

Thus, $I = \sqrt{\pi}/2$.

41. When $n = 1/2$, $\quad \Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x}\, dx$. If we set $u = \sqrt{x} \Longrightarrow x = u^2$, and $dx = 2u\, du$,

$$\Gamma(1/2) = \int_0^\infty \frac{1}{u} e^{-u^2} (2u\, du) = 2\int_0^\infty e^{-u^2}\, du = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi} \quad \text{(from Exercise 40).}$$

EXERCISES 13.8

1. $\displaystyle\iiint_V (x^2 z + ye^x)\, dV = \int_0^1\int_1^2\int_0^1 (x^2 z + ye^x)\, dz\, dy\, dx = \int_0^1\int_1^2 \left\{\frac{x^2 z^2}{2} + zye^x\right\}_0^1 dy\, dx$

$$= \int_0^1\int_1^2 \left(\frac{x^2}{2} + ye^x\right) dy\, dx = \int_0^1 \left\{\frac{x^2 y}{2} + \frac{y^2 e^x}{2}\right\}_1^2 dx$$

$$= \frac{1}{2}\int_0^1 (x^2 + 3e^x)\, dx = \frac{1}{2}\left\{\frac{x^3}{3} + 3e^x\right\}_0^1 = \frac{9e - 8}{6}$$

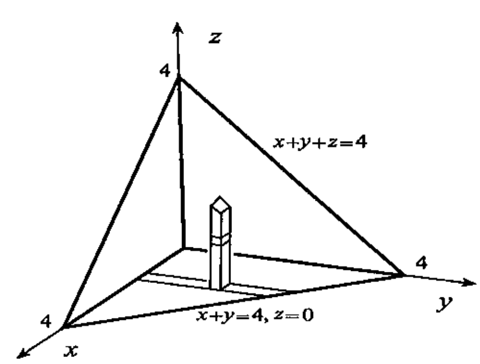
2. $\displaystyle\iiint_V x\, dV = \int_0^4\int_0^{4-x}\int_0^{4-x-y} x\, dz\, dy\, dx$

$$= \int_0^4\int_0^{4-x} x(4 - x - y)\, dy\, dx$$

$$= \int_0^4 \left\{x(4 - x)y - \frac{xy^2}{2}\right\}_0^{4-x} dx$$

$$= \frac{1}{2}\int_0^4 (16x - 8x^2 + x^3)\, dx$$

$$= \frac{1}{2}\left\{8x^2 - \frac{8x^3}{3} + \frac{x^4}{4}\right\}_0^4 = \frac{32}{3}$$



3. $$\iiint_V \sin(y+z)\,dV = \int_0^1 \int_0^{2x} \int_0^{x+2y} \sin(y+z)\,dz\,dy\,dx = \int_0^1 \int_0^{2x} \Big\{ -\cos(y+z) \Big\}_0^{x+2y} dy\,dx$$

$$= \int_0^1 \int_0^{2x} [\cos y - \cos(x+3y)]\,dy\,dx = \int_0^1 \left\{ \sin y - \frac{1}{3}\sin(x+3y) \right\}_0^{2x} dx$$

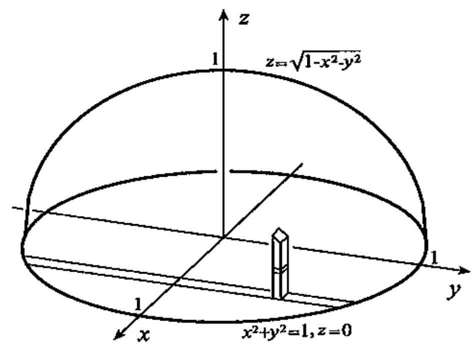
$$= \int_0^1 \left(\sin 2x - \frac{1}{3}\sin 7x + \frac{1}{3}\sin x \right) dx = \left\{ -\frac{1}{2}\cos 2x + \frac{1}{21}\cos 7x - \frac{1}{3}\cos x \right\}_0^1$$

$$= (2\cos 7 - 14\cos 1 - 21\cos 2 + 33)/42$$

4. $$\iiint_V xy\,dV = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xy\,dz\,dy\,dx$$

$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\sqrt{1-x^2-y^2}\,dy\,dx$$

$$= \int_{-1}^1 \left\{ -\frac{x}{3}(1-x^2-y^2)^{3/2} \right\}_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 0$$



5. $$\iiint_V dV = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\,dy\,dx = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx$$

$$= 8\int_0^1 \left\{ y\sqrt{1-x^2} \right\}_0^{\sqrt{1-x^2}} dx = 8\int_0^1 (1-x^2)\,dx = 8\left\{ x - \frac{x^3}{3} \right\}_0^1 = \frac{16}{3}$$

6. $$\iiint_V (x^2+2z)\,dV = \int_{-2}^2 \int_{x^2}^4 \int_0^{4-y} (x^2+2z)\,dz\,dy\,dx$$

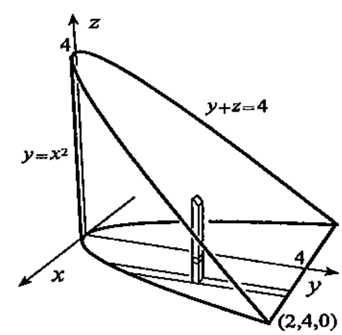
$$= \int_{-2}^2 \int_{x^2}^4 \left\{ x^2 z + z^2 \right\}_0^{4-y} dy\,dx$$

$$= \int_{-2}^2 \int_{x^2}^4 [x^2(4-y) + (4-y)^2]\,dy\,dx$$

$$= \int_{-2}^2 \left\{ -\frac{x^2}{2}(4-y)^2 - \frac{(4-y)^3}{3} \right\}_{x^2}^4 dx$$

$$= \frac{1}{6}\int_{-2}^2 (128 - 48x^2 + x^6)\,dx$$

$$= \frac{1}{6}\left\{ 128x - 16x^3 + \frac{x^7}{7} \right\}_{-2}^2 = \frac{1024}{21}$$

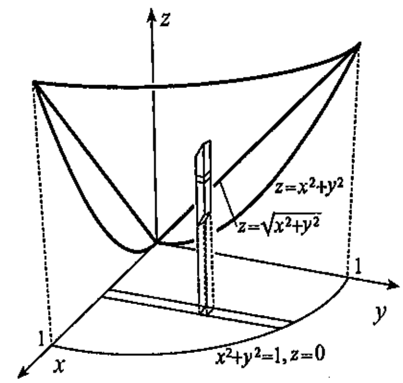


7. $$\iiint_V x^2y^2z^2\,dV = \int_0^1 \int_{z-1}^{1-z} \int_0^1 x^2y^2z^2\,dx\,dy\,dz = \int_0^1 \int_{z-1}^{1-z} \left\{ \frac{x^3y^2z^2}{3} \right\}_0^1 dy\,dz = \frac{1}{3}\int_0^1 \int_{z-1}^{1-z} y^2z^2\,dy\,dz$$

$$= \frac{1}{3}\int_0^1 \left\{ \frac{y^3z^2}{3} \right\}_{z-1}^{1-z} dz = \frac{2}{9}\int_0^1 (z^2 - 3z^3 + 3z^4 - z^5)\,dz$$

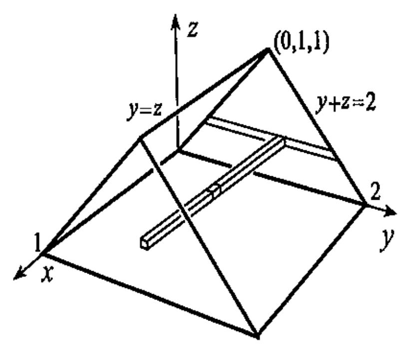
$$= \frac{2}{9}\left\{ \frac{z^3}{3} - \frac{3z^4}{4} + \frac{3z^5}{5} - \frac{z^6}{6} \right\}_0^1 = \frac{1}{270}$$

8. $$\iiint_V xyz\,dV = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \left\{ \frac{xyz^2}{2} \right\}_{x^2+y^2}^{\sqrt{x^2+y^2}} dy\,dx$$
$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (x^3y + xy^3 - x^5y - 2x^3y^3 - xy^5)\,dy\,dx$$
$$= \frac{1}{2} \int_0^1 \left\{ \frac{x^3y^2}{2} + \frac{xy^4}{4} - \frac{x^5y^2}{2} - \frac{x^3y^4}{2} - \frac{xy^6}{6} \right\}_0^{\sqrt{1-x^2}} dx$$
$$= \frac{1}{24} \int_0^1 [3x(1-x^2)^2 - 2x(1-x^2)^3]\,dx$$
$$= \frac{1}{24} \left\{ -\frac{1}{2}(1-x^2)^3 + \frac{1}{4}(1-x^2)^4 \right\}_0^1 = \frac{1}{96}$$



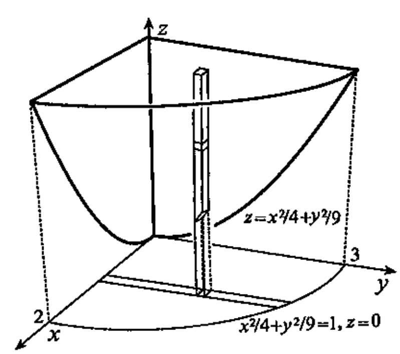
9. We double the integal over the first octant volume.
$$\iiint_V dV = 2\int_0^2 \int_{x^2}^4 \int_0^{4-z} dy\,dz\,dx = 2\int_0^2 \int_{x^2}^4 (4-z)\,dz\,dx = 2\int_0^2 \left\{ 4z - \frac{z^2}{2} \right\}_{x^2}^4 dx$$
$$= 2\int_0^2 \left(16 - 8 - 4x^2 + \frac{x^4}{2} \right) dx = 2\left\{ 8x - \frac{4x^3}{3} + \frac{x^5}{10} \right\}_0^2 = \frac{256}{15}$$

10. $$\iiint_V (x+y+z)\,dV = \int_0^1 \int_z^{2-z} \int_0^1 (x+y+z)\,dx\,dy\,dz$$
$$= \int_0^1 \int_z^{2-z} \left\{ \frac{1}{2}(x+y+z)^2 \right\}_0^1 dy\,dz$$
$$= \frac{1}{2} \int_0^1 \int_z^{2-z} [(1+y+z)^2 - (y+z)^2]\,dy\,dz$$
$$= \frac{1}{2} \int_0^1 \left\{ \frac{1}{3}(1+y+z)^3 - \frac{1}{3}(y+z)^3 \right\}_z^{2-z} dz$$
$$= \frac{1}{6} \int_0^1 [19 + 8z^3 - (1+2z)^3]\,dz$$
$$= \frac{1}{6} \left\{ 19z + 2z^4 - \frac{(1+2z)^4}{8} \right\}_0^1 = \frac{11}{6}$$



11. Because of the symmetry of the volume about the z-axis, and the fact that the integrand xyz is an odd function of x and y, the triple integral must have value zero.

12. $$\iiint_V x^2y\,dV = \int_0^2 \int_0^{3\sqrt{4-x^2}/2} \int_{x^2/4+y^2/9}^1 x^2y\,dz\,dy\,dx$$
$$= \int_0^2 \int_0^{3\sqrt{4-x^2}/2} x^2y \left(1 - \frac{x^2}{4} - \frac{y^2}{9} \right) dy\,dx$$
$$= \int_0^2 \left\{ x^2 \left(1 - \frac{x^2}{4} \right) \frac{y^2}{2} - \frac{x^2y^4}{36} \right\}_0^{3\sqrt{4-x^2}/2} dx$$
$$= \frac{9}{64} \int_0^2 (16x^2 - 8x^4 + x^6)\,dx$$
$$= \frac{9}{64} \left\{ \frac{16x^3}{3} - \frac{8x^5}{5} + \frac{x^7}{7} \right\}_0^2 = \frac{48}{35}$$



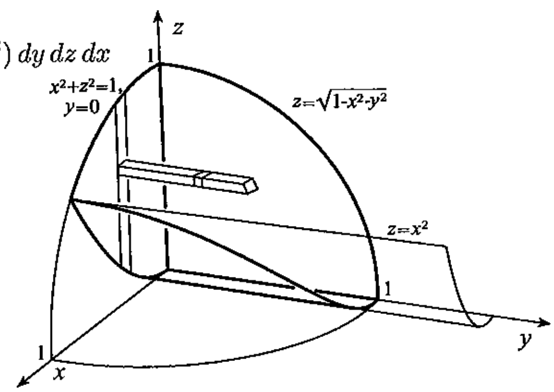
13. The six triple iterated integrals are:

$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} f(x,y,z)\,dz\,dy\,dx,\ \int_{0}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\int_{0}^{y} f(x,y,z)\,dz\,dx\,dy,\ \int_{-1}^{1}\int_{0}^{1-x^2}\int_{z}^{1-x^2} f(x,y,z)\,dy\,dz\,dx,$$

$$\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{z}^{1-x^2} f(x,y,z)\,dy\,dx\,dz,\ \int_{0}^{1}\int_{0}^{y}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dz\,dy,\ \int_{0}^{1}\int_{z}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dy\,dz.$$

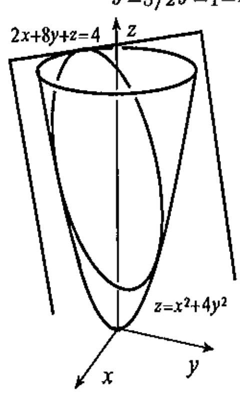
14. $\displaystyle\iiint_V (x^2+y^2+z^2)\,dV$

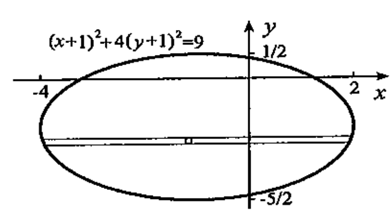
$$= 4\int_{0}^{\sqrt{(\sqrt{5}-1)/2}}\int_{x^2}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-z^2}} (x^2+y^2+z^2)\,dy\,dz\,dx$$



15. $\displaystyle\iiint_V xz\sin(x+y)\,dV = \int_{-1}^{1}\int_{-(1/2)\sqrt{3-3x^2}}^{(1/2)\sqrt{3-3x^2}}\int_{\sqrt{1+4x^2+4z^2}}^{\sqrt{4+x^2}} xz\sin(x+y)\,dy\,dz\,dx$

16. $\displaystyle\iiint_V xyz\,dV = \int_{-5/2}^{1/2}\int_{-1-\sqrt{9-4(y+1)^2}}^{-1+\sqrt{9-4(y+1)^2}}\int_{x^2+4y^2}^{4-2x-8y} xyz\,dz\,dx\,dy$





Area onto which vertical columns project

17. The surfaces intersect in a plane parallel to the yz-plane defined by $x+1=x^2$, from which $x=(1\pm\sqrt{1+4})/2=(1\pm\sqrt{5})/2$, only the positive result being acceptable. The equation of the projection of the curve in the yz-plane is $y^2+z^2=(1+\sqrt{5})/2$. Hence,

$$\iiint_V x^2y^2z^2\,dV = 4\int_{0}^{\sqrt{(1+\sqrt{5})/2}}\int_{0}^{\sqrt{(1+\sqrt{5})/2-y^2}}\int_{(y^2+z^2)^2-1}^{y^2+z^2} x^2y^2z^2\,dx\,dz\,dy.$$

18. $\iiint_V (y+x^2)\,dV = \int_{-2}^{1}\int_{z-1}^{1-z^2}\int_{-1}^{1}(y+x^2)\,dy\,dx\,dz$

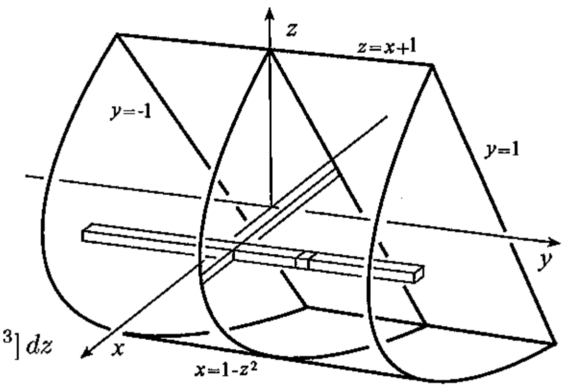
$$= \int_{-2}^{1}\int_{z-1}^{1-z^2}\left\{\frac{y^2}{2}+x^2y\right\}_{-1}^{1}dx\,dz$$

$$= 2\int_{-2}^{1}\int_{z-1}^{1-z^2}x^2\,dx\,dz$$

$$= 2\int_{-2}^{1}\left\{\frac{x^3}{3}\right\}_{z-1}^{1-z^2}dz$$

$$= \frac{2}{3}\int_{-2}^{1}[1-3z^2+3z^4-z^6-(z-1)^3]\,dz$$

$$= \frac{2}{3}\left\{z-z^3+\frac{3z^5}{5}-\frac{z^7}{7}-\frac{(z-1)^4}{4}\right\}_{-2}^{1} = \frac{729}{70}$$



19. $\iiint_V (xy+z)\,dV = \int_0^{1/3}\int_y^{2y}\int_0^3 (xy+z)\,dx\,dz\,dy + \int_{1/3}^{1/2}\int_y^{1-y}\int_0^3 (xy+z)\,dx\,dz\,dy$

$$= \int_0^{1/3}\int_y^{2y}\left\{\frac{x^2y}{2}+xz\right\}_0^3 dz\,dy + \int_{1/3}^{1/2}\int_y^{1-y}\left\{\frac{x^2y}{2}+xz\right\}_0^3 dz\,dy$$

$$= \frac{1}{2}\int_0^{1/3}\int_y^{2y}(9y+6z)\,dz\,dy + \frac{1}{2}\int_{1/3}^{1/2}\int_y^{1-y}(9y+6z)\,dz\,dy$$

$$= \frac{1}{2}\int_0^{1/3}\left\{9yz+3z^2\right\}_y^{2y}dy + \frac{1}{2}\int_{1/3}^{1/2}\left\{9yz+3z^2\right\}_y^{1-y}dy$$

$$= \frac{1}{2}\int_0^{1/3}18y^2\,dy + \frac{1}{2}\int_{1/3}^{1/2}[9y-21y^2+3(1-y)^2]\,dy$$

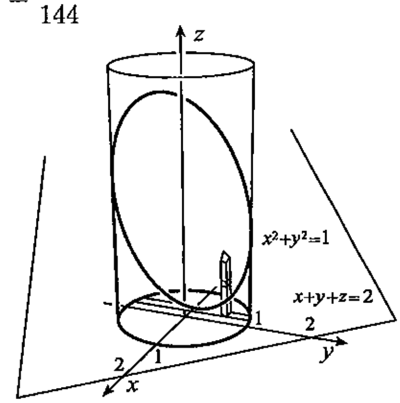
$$= \frac{1}{2}\left\{6y^3\right\}_0^{1/3} + \frac{1}{2}\left\{\frac{9y^2}{2}-7y^3-(1-y)^3\right\}_{1/3}^{1/2} = \frac{29}{144}$$

20. $\iiint_V dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{2-x-y}dz\,dy\,dx$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2-x-y)\,dy\,dx$$

$$= \int_{-1}^{1}\left\{(2-x)y-\frac{y^2}{2}\right\}_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx$$

$$= 2\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx$$



If we set $x=\sin\theta$, then $dx=\cos\theta\,d\theta$, and

$$\iiint_V dV = 2\int_{-\pi/2}^{\pi/2}(2-\sin\theta)\cos\theta\,\cos\theta\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta-\cos^2\theta\,\sin\theta)\,d\theta = 2\left\{\theta+\frac{\sin 2\theta}{2}+\frac{\cos^3\theta}{3}\right\}_{-\pi/2}^{\pi/2} = 2\pi.$$

21. We quadruple the integral over the first octant volume.

$$\iiint_V dV = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}(4-2x^2-2y^2)\,dy\,dx$$

$$= 4\int_0^{\sqrt{2}}\left\{(4-2x^2)y - \frac{2y^3}{3}\right\}_0^{\sqrt{2-x^2}} dx = \frac{16}{3}\int_0^{\sqrt{2}}(2-x^2)^{3/2}dx$$

If we set $x = \sqrt{2}\sin\theta$ and $dx = \sqrt{2}\cos\theta\,d\theta$,

$$\iiint_V dV = \frac{16}{3}\int_0^{\pi/2} 2\sqrt{2}\cos^3\theta(\sqrt{2}\cos\theta\,d\theta) = \frac{64}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{16}{3}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \frac{16}{3}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = 4\pi.$$

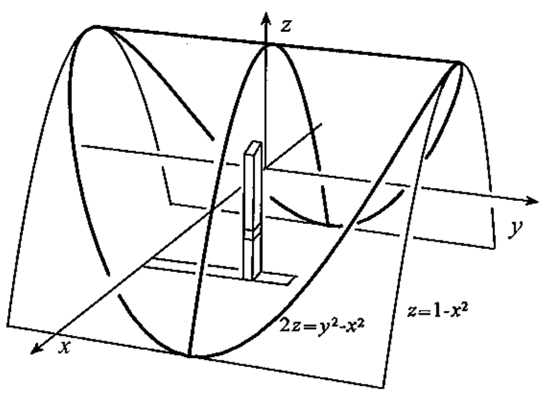
22. Because of the symmetry, integrals of x and y vanish. We multiply the integral of z over the first octant volume by 4,

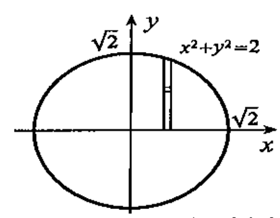
$$\iiint_V (x+y+z)\,dV = \iiint_V z\,dV = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{y^2/2-x^2/2}^{1-x^2} z\,dz\,dy\,dx$$

$$= 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\left\{\frac{z^2}{2}\right\}_{y^2/2-x^2/2}^{1-x^2} dy\,dx = \frac{1}{2}\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}(4-8x^2+3x^4-y^4+2x^2y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^{\sqrt{2}}\left\{4y - 8x^2y + 3x^4y - \frac{y^5}{5} + \frac{2x^2y^3}{3}\right\}_0^{\sqrt{2-x^2}} dx$$

$$= \frac{1}{2}\int_0^{\sqrt{2}}\left[4\sqrt{2-x^2} - 8x^2\sqrt{2-x^2} + 3x^4\sqrt{2-x^2} - \frac{1}{5}(2-x^2)^{5/2} + \frac{2x^2}{3}(2-x^2)^{3/2}\right]dx.$$





Columns project onto interior of circle

If we set $x = \sqrt{2}\sin\theta$, then $dx = \sqrt{2}\cos\theta\,d\theta$, and

$$\iiint_V (x+y+z)\,dV = \frac{1}{2}\int_0^{\pi/2}\left(4\sqrt{2}\cos\theta - 16\sqrt{2}\sin^2\theta\,\cos\theta + 12\sqrt{2}\sin^4\theta\,\cos\theta - \frac{4\sqrt{2}}{5}\cos^5\theta\right.$$

$$\left. + \frac{8\sqrt{2}}{3}\sin^2\theta\,\cos^3\theta\right)\sqrt{2}\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/2}\left[\cos^2\theta - \sin^2 2\theta + 3\left(\frac{\sin^2 2\theta}{4}\right)\left(\frac{1-\cos 2\theta}{2}\right)\right.$$

$$\left. - \frac{1}{5}\left(\frac{1+\cos 2\theta}{2}\right)^3 + \frac{2}{3}\left(\frac{\sin^2 2\theta}{4}\right)\left(\frac{1+\cos 2\theta}{2}\right)\right]d\theta$$

$$= 4\int_0^{\pi/2}\left\{\frac{1+\cos 2\theta}{2} - \left(\frac{1-\cos 4\theta}{2}\right) + \frac{3}{8}\left[\frac{1-\cos 4\theta}{2} - \sin^2 2\theta\,\cos 2\theta\right]\right.$$

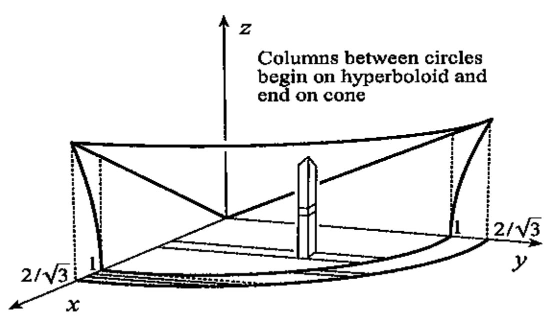
$$-\frac{1}{40}\left[1+3\cos 2\theta+\frac{3}{2}(1+\cos 4\theta)+\cos 2\theta(1-\sin^2 2\theta)\right]$$
$$+\frac{1}{12}\left(\frac{1-\cos 4\theta}{2}+\sin^2 2\theta\,\cos 2\theta\right)\bigg\}\,d\theta$$
$$=4\bigg\{\frac{\theta}{2}+\frac{\sin 2\theta}{4}-\frac{\theta}{2}+\frac{\sin 4\theta}{8}+\frac{3\theta}{16}-\frac{3\sin 4\theta}{64}-\frac{\sin^3 2\theta}{16}$$
$$-\frac{\theta}{40}-\frac{3\sin 2\theta}{80}-\frac{3\theta}{80}-\frac{3\sin 4\theta}{320}-\frac{\sin 2\theta}{80}$$
$$+\frac{\sin^3 2\theta}{240}+\frac{\theta}{24}-\frac{\sin 4\theta}{96}+\frac{\sin^3 2\theta}{72}\bigg\}_0^{\pi/2}$$
$$=\pi/3.$$

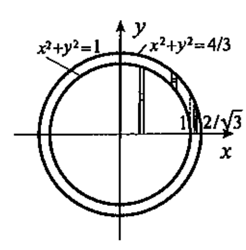
23. We quadruple the integral over the first octant volume.

$$\iiint_V |yz|\,dV=4\int_0^{\sqrt{3/2}}\int_0^{\sqrt{3/2-x^2}}\int_{\sqrt{1+x^2+y^2}}^{\sqrt{4-x^2-y^2}} yz\,dz\,dy\,dx=4\int_0^{\sqrt{3/2}}\int_0^{\sqrt{3/2-x^2}}\left\{\frac{yz^2}{2}\right\}_{\sqrt{1+x^2+y^2}}^{\sqrt{4-x^2-y^2}} dy\,dx$$
$$=2\int_0^{\sqrt{3/2}}\int_0^{\sqrt{3/2-x^2}}(3y-2x^2y-2y^3)\,dy\,dx=2\int_0^{\sqrt{3/2}}\left\{\frac{3y^2}{2}-x^2y^2-\frac{y^4}{2}\right\}_0^{\sqrt{3/2-x^2}} dx$$
$$=\int_0^{\sqrt{3/2}}[3(3/2-x^2)-2x^2(3/2-x^2)-(3/2-x^2)^2]\,dx=\frac{1}{4}\int_0^{\sqrt{3/2}}(9-12x^2+4x^4)\,dx$$
$$=\frac{1}{4}\left\{9x-4x^3+\frac{4x^5}{5}\right\}_0^{\sqrt{3/2}}=\frac{3\sqrt{6}}{5}$$

24. We quadruple the integral over the first octant volume.

$$\iiint_V (x^2+y^2+z^2)\,dV=4\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}/2}(x^2+y^2+z^2)\,dz\,dy\,dx$$
$$+4\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4/3-x^2}}\int_{\sqrt{x^2+y^2-1}}^{\sqrt{x^2+y^2}/2}(x^2+y^2+z^2)\,dz\,dy\,dx$$
$$+4\int_1^{2/\sqrt{3}}\int_0^{\sqrt{4/3-x^2}}\int_{\sqrt{x^2+y^2-1}}^{\sqrt{x^2+y^2}/2}(x^2+y^2+z^2)\,dz\,dy\,dx$$





EXERCISES 13.9

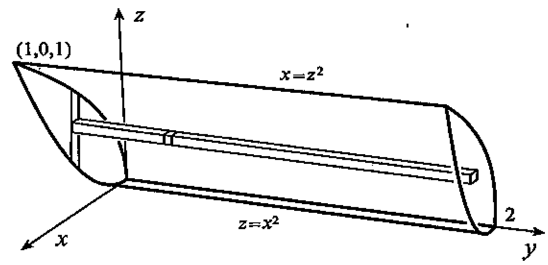
1. We double the first octant volume.

$$V=2\int_0^1\int_{x^2}^1\int_0^4 dz\,dy\,dx=2\int_0^1\int_{x^2}^1 4\,dy\,dx=8\int_0^1(1-x^2)\,dx=8\left\{x-\frac{x^3}{3}\right\}_0^1=\frac{16}{3}$$

2. $V = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^2 dy\,dz\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} 2\,dz\,dx$

$= 2\int_0^1 (\sqrt{x} - x^2)\,dx$

$= 2\left\{\frac{2x^{3/2}}{3} - \frac{x^3}{3}\right\}_0^1 = \frac{2}{3}$



3. $V = \int_0^2 \int_{x/3}^{3x} \int_0^1 dy\,dz\,dx = \int_0^2 \int_{x/3}^{3x} dz\,dx = \int_0^2 \left(3x - \frac{x}{3}\right) dx = \left\{\frac{4x^2}{3}\right\}_0^2 = \frac{16}{3}$

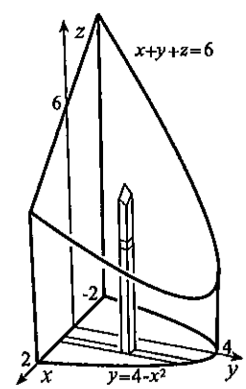
4. $V = \int_{-2}^2 \int_0^{4-x^2} \int_0^{6-x-y} dz\,dy\,dx$

$= \int_{-2}^2 \int_0^{4-x^2} (6 - x - y)\,dy\,dx$

$= \int_{-2}^2 \left\{(6-x)y - \frac{y^2}{2}\right\}_0^{4-x^2} dx$

$= \frac{1}{2}\int_{-2}^2 (32 - 8x - 4x^2 + 2x^3 - x^4)\,dx$

$= \frac{1}{2}\left\{32x - 4x^2 - \frac{4x^3}{3} + \frac{x^4}{2} - \frac{x^5}{5}\right\}_{-2}^2 = \frac{704}{15}$



5. We double the first octant volume.

$V = 2\int_0^2 \int_{x^2}^4 \int_0^{x^2+y^2} dz\,dy\,dx = 2\int_0^2 \int_{x^2}^4 (x^2 + y^2)\,dy\,dx = 2\int_0^2 \left\{x^2 y + \frac{y^3}{3}\right\}_{x^2}^4 dx$

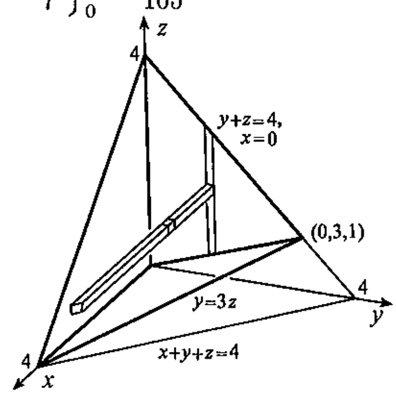
$= \frac{2}{3}\int_0^2 (12x^2 + 64 - 3x^4 - x^6)\,dx = \frac{2}{3}\left\{4x^3 + 64x - \frac{3x^5}{5} - \frac{x^7}{7}\right\}_0^2 = \frac{8576}{105}$

6. $V = \int_0^3 \int_{y/3}^{4-y} \int_0^{4-y-z} dx\,dz\,dy = \int_0^3 \int_{y/3}^{4-y} (4 - y - z)\,dz\,dy$

$= \int_0^3 \left\{(4-y)z - \frac{z^2}{2}\right\}_{y/3}^{4-y} dy$

$= \frac{1}{18}\int_0^3 [7y^2 - 24y + 9(4-y)^2]\,dy$

$= \frac{1}{18}\left\{\frac{7y^3}{3} - 12y^2 - 3(4-y)^3\right\}_0^3 = 8$

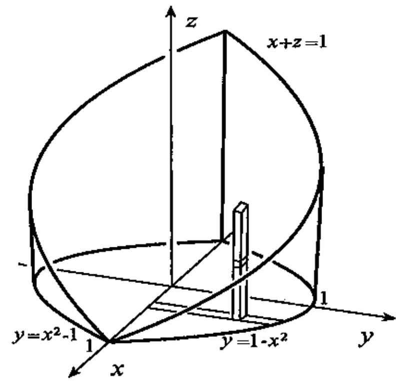


7. We multiply the first octant volume by 8.

$V = 8\int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-y^2}} dz\,dx\,dy = 8\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4-y^2}\,dx\,dy = 8\int_0^2 \left\{x\sqrt{4-y^2}\right\}_0^{\sqrt{4-y^2}} dy$

$= 8\int_0^2 (4 - y^2)\,dy = 8\left\{4y - \frac{y^3}{3}\right\}_0^2 = \frac{128}{3}$

8. We double the volume to the right of the xz-plane.

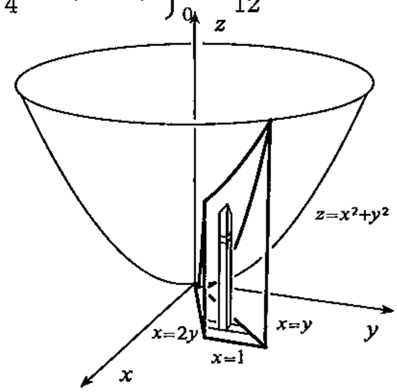


$$\begin{aligned} V &= 2\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{1-x} dz\,dy\,dx \\ &= 2\int_{-1}^{1}\int_{0}^{1-x^2}(1-x)\,dy\,dx \\ &= 2\int_{-1}^{1}(1-x)(1-x^2)\,dx \\ &= 2\left\{x-\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}\right\}_{-1}^{1}=\frac{8}{3} \end{aligned}$$

9. $$V=\int_0^1\int_0^{1-x}\int_{16-x^2-4y^2}^{16} dz\,dy\,dx=\int_0^1\int_0^{1-x}(16-16+x^2+4y^2)\,dy\,dx=\int_0^1\left\{x^2y+\frac{4y^3}{3}\right\}_0^{1-x}dx$$

$$=\frac{1}{3}\int_0^1[3x^2(1-x)+4(1-x)^3]\,dx=\frac{1}{3}\left\{x^3-\frac{3x^4}{4}-(1-x)^4\right\}_0^1=\frac{5}{12}$$

10.



$$\begin{aligned} V &= \int_0^1\int_{x/2}^{x}\int_0^{x^2+y^2} dz\,dy\,dx \\ &= \int_0^1\int_{x/2}^{x}(x^2+y^2)\,dy\,dx \\ &= \int_0^1\left\{x^2y+\frac{y^3}{3}\right\}_{x/2}^{x}dx \\ &= \frac{19}{24}\int_0^1 x^3\,dx=\frac{19}{24}\left\{\frac{x^4}{4}\right\}_0^1=\frac{19}{96} \end{aligned}$$

11. We quadruple the first octant volume.

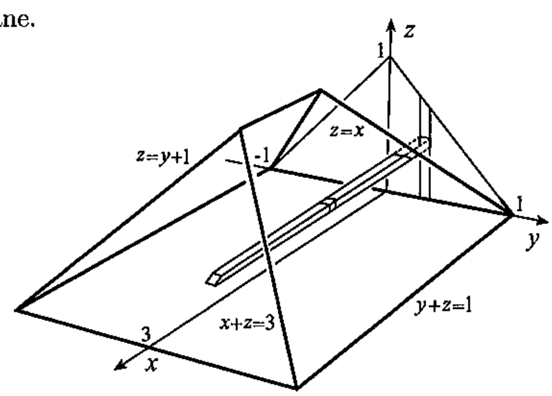
$$V=4\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2}dz\,dy\,dx=4\int_0^1\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx=4\int_0^1\left\{y-x^2y-\frac{y^3}{3}\right\}_0^{\sqrt{1-x^2}}dx$$

$$=\frac{8}{3}\int_0^1(1-x^2)^{3/2}\,dx$$

If we set $x=\sin\theta$ and $dx=\cos\theta\,d\theta$,

$$\begin{aligned} V &= \frac{8}{3}\int_0^{\pi/2}\cos^3\theta\cos\theta\,d\theta=\frac{8}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\ &= \frac{2}{3}\int_0^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta=\frac{2}{3}\left\{\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2}=\frac{\pi}{2}. \end{aligned}$$

12. We double the volume to the right of the xz-plane.



$$\begin{aligned} V &= 2\int_0^1\int_0^{1-y}\int_z^{3-z}dx\,dz\,dy \\ &= 2\int_0^1\int_0^{1-y}(3-2z)\,dz\,dy \\ &= 2\int_0^1\left\{-\frac{1}{4}(3-2z)^2\right\}_0^{1-y}dy \\ &= \frac{1}{2}\int_0^1[9-(2y+1)^2]\,dy \\ &= \frac{1}{2}\left\{9y-\frac{1}{6}(2y+1)^3\right\}_0^1=\frac{7}{3} \end{aligned}$$

13. $V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{2-x-y} dz\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2-x-y)\,dy\,dx = \int_{-1}^{1}\left\{(2-x)y - \frac{y^2}{2}\right\}_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$

$= 2\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx$

If we set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$ in the first term,

$$V = 4\int_{-\pi/2}^{\pi/2}\cos\theta\cos\theta\,d\theta - 2\left\{-\frac{1}{3}(1-x^2)^{3/2}\right\}_{-1}^{1} = 4\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 2\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/2}^{\pi/2} = 2\pi.$$

14. $V = \int_{0}^{1/3}\int_{y}^{2y}\int_{0}^{4-y-z} dx\,dz\,dy + \int_{1/3}^{1/2}\int_{y}^{1-y}\int_{0}^{4-y-z} dx\,dz\,dy$

$= \int_{0}^{1/3}\int_{y}^{2y}(4-y-z)\,dz\,dy + \int_{1/3}^{1/2}\int_{y}^{1-y}(4-y-z)\,dz\,dy$

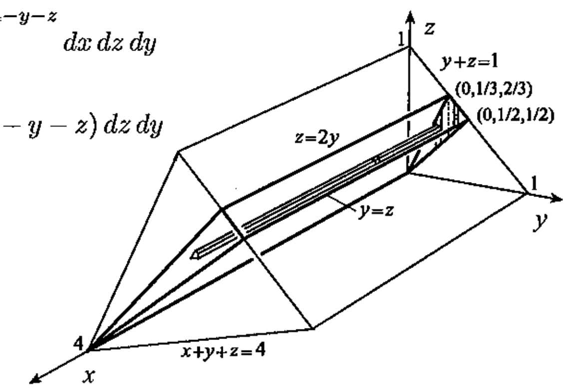
$= \int_{0}^{1/3}\left\{-\frac{1}{2}(4-y-z)^2\right\}_{y}^{2y} dy$

$+ \int_{1/3}^{1/2}\left\{-\frac{1}{2}(4-y-z)^2\right\}_{y}^{1-y} dy$

$= \frac{1}{2}\int_{0}^{1/3}(8y-5y^2)\,dy$

$+\frac{1}{2}\int_{1/3}^{1/2}(7-16y+4y^2)\,dy$

$= \frac{1}{2}\left\{4y^2 - \frac{5y^3}{3}\right\}_{0}^{1/3} + \frac{1}{2}\left\{7y - 8y^2 + \frac{4y^3}{3}\right\}_{1/3}^{1/2} = \frac{5}{18}$



15. We quadruple the first octant volume.

$V = 4\int_{0}^{1}\int_{0}^{2\sqrt{1-y^2}}\int_{x^2+4y^2}^{6-x^2/2-2y^2} dz\,dx\,dy = 4\int_{0}^{1}\int_{0}^{2\sqrt{1-y^2}}\left(6 - \frac{3x^2}{2} - 6y^2\right)dx\,dy$

$= 6\int_{0}^{1}\int_{0}^{2\sqrt{1-y^2}}(4-x^2-4y^2)\,dx\,dy = 6\int_{0}^{1}\left\{4x - \frac{x^3}{3} - 4xy^2\right\}_{0}^{2\sqrt{1-y^2}} dy = 32\int_{0}^{1}(1-y^2)^{3/2}\,dy$

If we set $y = \sin\theta$ and $dy = \cos\theta\,d\theta$,

$$V = 32\int_{0}^{\pi/2}\cos^3\theta\cos\theta\,d\theta = 32\int_{0}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 8\int_{0}^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = 8\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_{0}^{\pi/2} = 6\pi.$$

16. We multiply the volume in the first octant for which $x > y$ by 16.

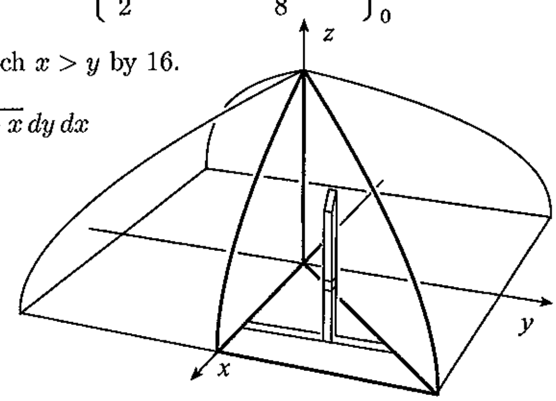
$V = 16\int_{0}^{1}\int_{0}^{x}\int_{0}^{\sqrt{1-x}} dz\,dy\,dx = 16\int_{0}^{1}\int_{0}^{x}\sqrt{1-x}\,dy\,dx$

$= 16\int_{0}^{1}x\sqrt{1-x}\,dx$

If we set $u = 1-x$, then $du = -dx$, and

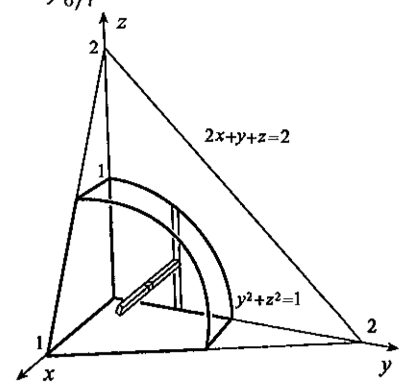
$V = 16\int_{1}^{0}(1-u)\sqrt{u}(-du)$

$= 16\left\{\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right\}_{0}^{1} = \frac{64}{15}.$



17. $$V = \int_0^{6/7}\int_x^{2x}\int_0^{3-x/2-3y/2} dz\,dy\,dx + \int_{6/7}^{3/2}\int_x^{2-x/3}\int_0^{3-x/2-3y/2} dz\,dy\,dx$$
$$= \frac{1}{2}\int_0^{6/7}\int_x^{2x}(6-x-3y)\,dy\,dx + \frac{1}{2}\int_{6/7}^{3/2}\int_x^{2-x/3}(6-x-3y)\,dy\,dx$$
$$= \frac{1}{2}\int_0^{6/7}\left\{-\frac{1}{6}(6-x-3y)^2\right\}_x^{2x} dx + \frac{1}{2}\int_{6/7}^{3/2}\left\{-\frac{1}{6}(6-x-3y)^2\right\}_x^{2-x/3} dx$$
$$= -\frac{1}{12}\int_0^{6/7}[(6-7x)^2-(6-4x)^2]\,dx - \frac{1}{12}\int_{6/7}^{3/2}-(6-4x)^2\,dx$$
$$= -\frac{1}{12}\left\{-\frac{1}{21}(6-7x)^3+\frac{1}{12}(6-4x)^3\right\}_0^{6/7} + \frac{1}{12}\left\{-\frac{1}{12}(6-4x)^3\right\}_{6/7}^{3/2} = \frac{9}{14}$$

18. $$V = \int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{1-y/2-z/2} dx\,dz\,dy$$
$$= \int_0^1\int_0^{\sqrt{1-y^2}}\left(1-\frac{y}{2}-\frac{z}{2}\right)dz\,dy$$
$$= \frac{1}{2}\int_0^1\left\{(2-y)z-\frac{z^2}{2}\right\}_0^{\sqrt{1-y^2}} dy$$
$$= \frac{1}{4}\int_0^1(4\sqrt{1-y^2}-2y\sqrt{1-y^2}-1+y^2)\,dy$$

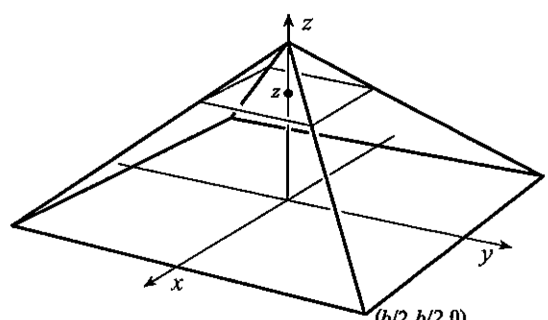


If we set $y=\sin\theta$ and $dy=\cos\theta\,d\theta$ in the first term,

$$V = \int_0^{\pi/2}\cos\theta\,\cos\theta\,d\theta + \frac{1}{4}\left\{\frac{2}{3}(1-y^2)^{3/2}-y+\frac{y^3}{3}\right\}_0^1$$
$$= \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - \frac{1}{3} = \frac{1}{2}\left\{\theta+\frac{\sin 2\theta}{2}\right\}_0^{\pi/2} - \frac{1}{3} = \frac{\pi}{4}-\frac{1}{3}$$

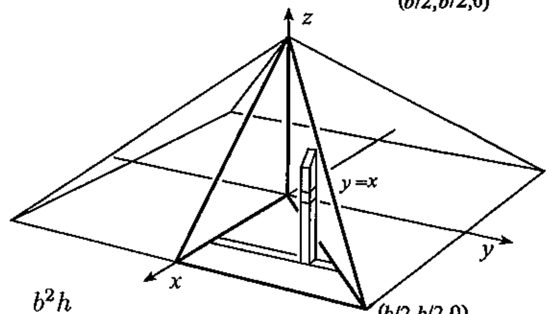
19. (a) The square cross section at height z has sides of length $b(h-z)/h$. Consequently, the area of the cross section is $b^2(h-z)^2/h^2$, and the volume of the pyramid is

$$V = \int_0^h \frac{b^2}{h^2}(h-z)^2\,dz = \frac{b^2}{h^2}\left\{-\frac{1}{3}(h-z)^3\right\}_0^h = \frac{b^2h}{3}.$$



(b) Since the equation of the face of the pyramid containing the point $(b/2,0,0)$ is $2x/b+z/h=1$,

$$V = 8\int_0^{b/2}\int_0^x\int_0^{h(1-2x/b)} dz\,dy\,dx$$
$$= 8\int_0^{b/2}\int_0^x h\left(1-\frac{2x}{b}\right)dy\,dx$$
$$= \frac{8h}{b}\int_0^{b/2}\left\{(b-2x)y\right\}_0^x dx$$
$$= \frac{8h}{b}\int_0^{b/2}(bx-2x^2)\,dx = \frac{8h}{b}\left\{\frac{bx^2}{2}-\frac{2x^3}{3}\right\}_0^{b/2} = \frac{b^2h}{3}.$$

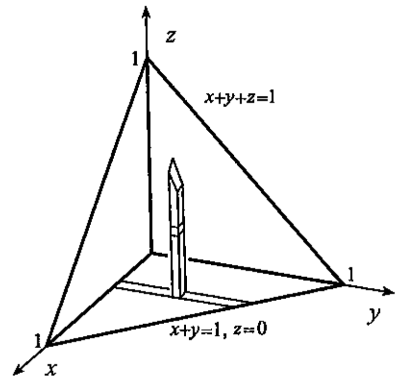


20. Since Volume $= \iiint_V dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 \left\{-\frac{1}{2}(1-x-y)^2\right\}_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = \frac{1}{2}\left\{-\frac{1}{3}(1-x)^3\right\}_0^1 = \frac{1}{6},$$



$$\overline{f} = 6\iiint_V xy\,dV = 6\int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = 6\int_0^1\int_0^{1-x} xy(1-x-y)\,dy\,dx$$

$$= 6\int_0^1 \left\{x(1-x)\frac{y^2}{2} - \frac{xy^3}{3}\right\}_0^{1-x} dx = \int_0^1 (x - 3x^2 + 3x^3 - x^4)\,dx$$

$$= \left\{\frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}\right\}_0^1 = \frac{1}{20}$$

21. Since $\mathrm{V} = \int_0^1\int_0^1\int_0^{9-x^2-y^2} dz\,dy\,dx = \int_0^1\int_0^1 (9-x^2-y^2)\,dy\,dx = \int_0^1 \left\{9y - x^2y - \frac{y^3}{3}\right\}_0^1 dx$

$$= \int_0^1 \left(9 - x^2 - \frac{1}{3}\right) dx = \left\{\frac{26x}{3} - \frac{x^3}{3}\right\}_0^1 = \frac{25}{3},$$

$$\overline{f} = \frac{3}{25}\int_0^1\int_0^1\int_0^{9-x^2-y^2} (x+y+z)\,dz\,dy\,dx = \frac{3}{25}\int_0^1\int_0^1 \left\{(x+y)z + \frac{z^2}{2}\right\}_0^{9-x^2-y^2} dy\,dx$$

$$= \frac{3}{50}\int_0^1\int_0^1 (81 + 18x + 18y - 18x^2 - 18y^2 - 2x^3 - 2y^3 - 2xy^2 - 2x^2y + x^4 + y^4 + 2x^2y^2)\,dy\,dx$$

$$= \frac{3}{50}\int_0^1 \left\{81y + 18xy + 9y^2 - 18x^2y - 6y^3 - 2x^3y - \frac{y^4}{2} - \frac{2xy^3}{3} - x^2y^2 + x^4y + \frac{y^5}{5} + \frac{2x^2y^3}{3}\right\}_0^1 dx$$

$$= \frac{3}{50}\int_0^1 \left(\frac{837}{10} + \frac{52x}{3} - \frac{55x^2}{3} - 2x^3 + x^4\right) dx = \frac{3}{50}\left\{\frac{837x}{10} + \frac{26x^2}{3} - \frac{55x^3}{9} - \frac{x^4}{2} + \frac{x^5}{5}\right\}_0^1 = \frac{1934}{375}.$$

22. Since Volume $= \frac{1}{2}(2)(2)(1) = 2$,

$$\overline{f} = \frac{1}{2}\iiint_V (x^2+y^2+z^2)\,dV$$

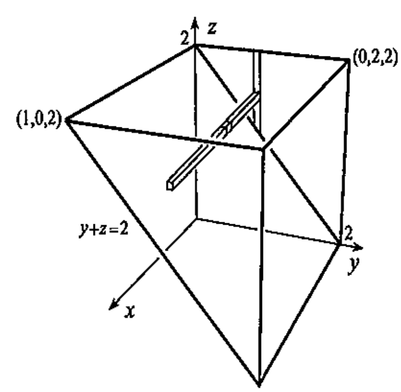
$$= \frac{1}{2}\int_0^2\int_{2-y}^2\int_0^1 (x^2+y^2+z^2)\,dx\,dz\,dy$$

$$= \frac{1}{2}\int_0^2\int_{2-y}^2 \left\{\frac{x^3}{3} + x(y^2+z^2)\right\}_0^1 dz\,dy$$

$$= \frac{1}{6}\int_0^2\int_{2-y}^2 [1 + 3(y^2+z^2)]\,dz\,dy$$

$$= \frac{1}{6}\int_0^2 \left\{z(1+3y^2) + z^3\right\}_{2-y}^2 dy$$

$$= \frac{1}{6}\int_0^2 [8 + y + 3y^3 - (2-y)^3]\,dy = \frac{1}{6}\left\{8y + \frac{y^2}{2} + \frac{3y^4}{4} + \frac{(2-y)^4}{4}\right\}_0^2 = \frac{13}{3}$$



23. The projection in the xy-plane of the curve of intersection of the surfaces has equation $x^2 - y^2 = 4 - 2(x^2 + y^2) \Longrightarrow 3x^2 + y^2 = 4$. We quadruple the first octant volume.

$$V = 4\int_0^{2/\sqrt{3}}\int_0^{\sqrt{4-3x^2}}\int_{x^2-y^2}^{4-2x^2-2y^2} dz\,dy\,dx = 4\int_0^{2/\sqrt{3}}\int_0^{\sqrt{4-3x^2}}(4-3x^2-y^2)\,dy\,dx$$
$$= 4\int_0^{2/\sqrt{3}}\left\{(4-3x^2)y - \frac{y^3}{3}\right\}_0^{\sqrt{4-3x^2}} dx = \frac{4}{3}\int_0^{2/\sqrt{3}}[3(4-3x^2)^{3/2} - (4-3x^2)^{3/2}]\,dx$$
$$= \frac{8}{3}\int_0^{2/\sqrt{3}}(4-3x^2)^{3/2}\,dx$$

If we set $x = (2/\sqrt{3})\sin\theta$ and $dx = (2/\sqrt{3})\cos\theta\,d\theta$,

$$V = \frac{8}{3}\int_0^{\pi/2} 8\cos^3\theta(2/\sqrt{3})\cos\theta\,d\theta = \frac{128}{3\sqrt{3}}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$
$$= \frac{32}{3\sqrt{3}}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \frac{32}{3\sqrt{3}}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{8\pi}{\sqrt{3}}.$$

24. If we set $x^2 - y^2 = 4 - x^2 - y^2$, then $2x^2 = 4$ or $x = \pm\sqrt{2}$. This implies that the curve of intersection of the surfaces divides into two parts, two parabolas $z = 2 - y^2$, $x = \pm\sqrt{2}$ in parallel planes. There is no bounded volume.

25. The projection in the yz-plane of the curve of intersection of the surfaces has equation $4y^2 = (2-z)^2 \Longrightarrow z = 2 \pm 2y$. We double the first octant volume.

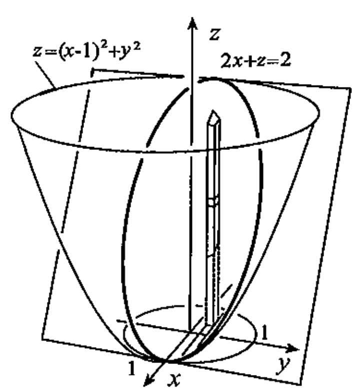
$$V = 2\int_0^1\int_0^{2-2y}\int_{4y^2/(2-z)}^{2-z} dx\,dz\,dy = 2\int_0^1\int_0^{2-2y}\left(2 - z - \frac{4y^2}{2-z}\right)dz\,dy$$
$$= 2\int_0^1\left\{2z - \frac{z^2}{2} + 4y^2\ln|2-z|\right\}_0^{2-2y} dy = \int_0^1(4 - 4y^2 + 8y^2\ln y)\,dy.$$

If we set $u = \ln y$, $dv = y^2\,dy$, $du = (1/y)\,dy$, and $v = y^3/3$, in the last term,

$$V = \left\{4y - \frac{4y^3}{3}\right\}_0^1 + 8\left\{\frac{y^3}{3}\ln y\right\}_0^1 - 8\int_0^1\frac{y^2}{3}dy = \frac{8}{3} - \frac{8}{3}\left\{\frac{y^3}{3}\right\}_0^1 = \frac{16}{9}.$$

26. We double the volume to the right of the xz-plane.

$$V = 2\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{(x-1)^2+y^2}^{2-2x} dz\,dx\,dy$$
$$= 2\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(1 - x^2 - y^2)\,dx\,dy$$
$$= 2\int_0^1\left\{x(1-y^2) - \frac{x^3}{3}\right\}_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$
$$= \frac{8}{3}\int_0^1(1-y^2)^{3/2}\,dy$$



If we set $y = \sin\theta$, then $dy = \cos\theta\,d\theta$, and

$$V = \frac{8}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$
$$= \frac{2}{3}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \frac{2}{3}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{\pi}{2}.$$

27. We multiple the first octant volume by eight.

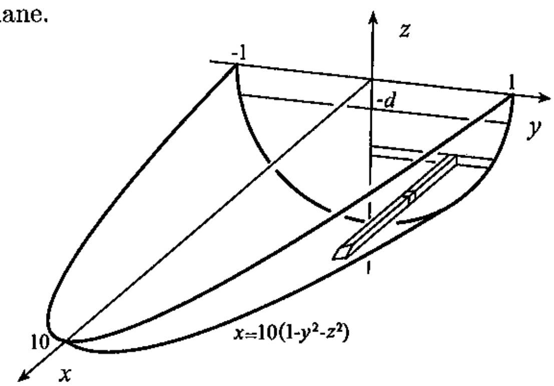
$$V = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-x^2/a^2-y^2/b^2}\,dy\,dx$$

If we set $y = b\sqrt{1-x^2/a^2}\sin\theta$ and $dy = b\sqrt{1-x^2/a^2}\cos\theta\,d\theta$, then

$$V = 8c\int_0^a \int_0^{\pi/2} \sqrt{1-x^2/a^2}\cos\theta\, b\sqrt{1-x^2/a^2}\cos\theta\,d\theta\,dx = 8c\int_0^a b(1-x^2/a^2)\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\,dx$$

$$= 4bc\int_0^a (1-x^2/a^2)\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} dx = 2\pi bc\int_0^a (1-x^2/a^2)\,dx = 2\pi bc\left\{x - \frac{x^3}{3a^2}\right\}_0^a = \frac{4\pi abc}{3}.$$

28. (a) We double the volume to the right of the xz-plane.



$$V = 2\int_{-1}^{-d}\int_0^{\sqrt{1-z^2}}\int_0^{10(1-y^2-z^2)} dx\,dy\,dz$$

$$= 20\int_{-1}^{-d}\int_0^{\sqrt{1-z^2}} (1-y^2-z^2)\,dy\,dz$$

$$= 20\int_{-1}^{-d}\left\{y(1-z^2) - \frac{y^3}{3}\right\}_0^{\sqrt{1-z^2}} dz$$

$$= \frac{40}{3}\int_{-1}^{-d}(1-z^2)^{3/2}\,dz$$

If we set $z = \sin\theta$, then $dz = \cos\theta\,d\theta$, and

$$V = \frac{40}{3}\int_{-\pi/2}^{\overline{\theta}} \cos^4\theta\,d\theta \qquad (\sin\overline{\theta} = -d)$$

$$= \frac{40}{3}\int_{-\pi/2}^{\overline{\theta}}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{10}{3}\int_{-\pi/2}^{\overline{\theta}}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta = \frac{10}{3}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_{-\pi/2}^{\overline{\theta}}$$

$$= \frac{5}{3}[3\overline{\theta} + 4\sin\overline{\theta}\,\cos\overline{\theta} + \sin\overline{\theta}\,\cos\overline{\theta}(1 - 2\sin^2\overline{\theta}) + 3\pi/2] = \frac{5}{3}[3\pi/2 - 3\mathrm{Sin}^{-1}d - d\sqrt{1-d^2}(5-2d^2)]$$

(b) The boat will sink when $d = 0$, at which point $V = 5\pi/2$. The buoyant force when $d = 0$ is $1000gV = 2500\pi g$, and this is the maximum weight.

29. $$V = 16\int_0^{a/\sqrt{2}}\int_0^x\int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx + 16\int_{a/\sqrt{2}}^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 16\int_0^{a/\sqrt{2}}\int_0^x \sqrt{a^2-x^2}\,dy\,dx + 16\int_{a/\sqrt{2}}^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy\,dx$$

$$= 16\int_0^{a/\sqrt{2}} x\sqrt{a^2-x^2}\,dx + 16\int_{a/\sqrt{2}}^a (a^2-x^2)\,dx = 16\left\{-\frac{1}{3}(a^2-x^2)^{3/2}\right\}_0^{a/\sqrt{2}} + 16\left\{a^2x - \frac{x^3}{3}\right\}_{a/\sqrt{2}}^a$$

$$= \frac{16a^3(\sqrt{2}-1)}{\sqrt{2}} = 8(2-\sqrt{2})a^3$$

EXERCISES 13.10

1. $$M = \int_0^1\int_0^1\int_0^{x^2+y^2}\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^1 (x^2+y^2)\,dy\,dx = \rho\int_0^1\left\{x^2y + \frac{y^3}{3}\right\}_0^1 dx$$

$$= \frac{\rho}{3}\int_0^1 (3x^2+1)\,dx = \frac{\rho}{3}\left\{x^3 + x\right\}_0^1 = \frac{2\rho}{3}$$

Since $M\overline{x} = \int_0^1\int_0^1\int_0^{x^2+y^2} x\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^1 x(x^2+y^2)\,dy\,dx = \rho\int_0^1\left\{x^3y + \frac{xy^3}{3}\right\}_0^1 dx$

$$= \frac{\rho}{3}\int_0^1 (3x^3+x)\,dx = \frac{\rho}{3}\left\{\frac{3x^4}{4}+\frac{x^2}{2}\right\}_0^1 = \frac{5\rho}{12},$$

it follows that $\overline{x} = \frac{5\rho}{12}\frac{3}{2\rho} = \frac{5}{8}$. By symmetry, $\overline{y} = 5/8$. Since

$$M\overline{z} = \int_0^1\int_0^1\int_0^{x^2+y^2} z\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^1\left\{\frac{z^2}{2}\right\}_0^{x^2+y^2} dy\,dx = \frac{\rho}{2}\int_0^1\int_0^1 (x^4+2x^2y^2+y^4)\,dy\,dx$$

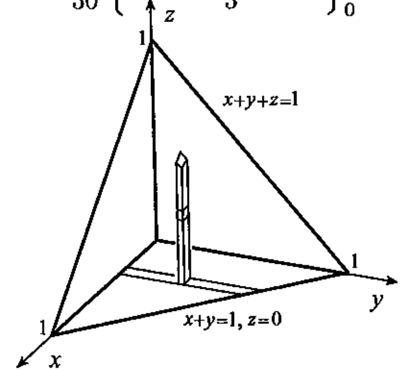
$$= \frac{\rho}{2}\int_0^1\left\{x^4y + \frac{2x^2y^3}{3}+\frac{y^5}{5}\right\}_0^1 dx = \frac{\rho}{30}\int_0^1 (15x^4+10x^2+3)\,dx = \frac{\rho}{30}\left\{3x^5+\frac{10x^3}{3}+3x\right\}_0^1 = \frac{14\rho}{45},$$

we obtain $\overline{z} = \frac{14\rho}{45}\frac{3}{2\rho} = \frac{7}{15}$.

2. $M = \int_0^1\int_0^{1-x}\int_0^{1-x-y}\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$

$$= \rho\int_0^1\left\{-\frac{1}{2}(1-x-y)^2\right\}_0^{1-x} dx$$

$$= \frac{\rho}{2}\int_0^1 (1-x)^2\,dx = \frac{\rho}{2}\left\{-\frac{1}{3}(1-x)^3\right\}_0^1 = \frac{\rho}{6}$$



Since $M\overline{x} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^{1-x} x(1-x-y)\,dy\,dx = \rho\int_0^1\left\{x(1-x)y - \frac{xy^2}{2}\right\}_0^{1-x} dx$

$$= \frac{\rho}{2}\int_0^1 (x-2x^2+x^3)\,dx = \frac{\rho}{2}\left\{\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\right\}_0^1 = \frac{\rho}{24},$$

it follows by symmetry that $\overline{x} = \overline{y} = \overline{z} = \frac{\rho}{24}\frac{6}{\rho} = \frac{1}{4}$.

3. $M = 2\int_0^2\int_{x^2}^4\int_0^{4-z}\rho\,dy\,dz\,dx = 2\rho\int_0^2\int_{x^2}^4 (4-z)\,dz\,dx = 2\rho\int_0^2\left\{-\frac{1}{2}(4-z)^2\right\}_{x^2}^4 dx$

$$= \rho\int_0^2 (16-8x^2+x^4)\,dx = \rho\left\{16x - \frac{8x^3}{3}+\frac{x^5}{5}\right\}_0^2 = \frac{256\rho}{15}$$

By symmetry, $\overline{x} = 0$. Since

$$M\overline{y} = 2\int_0^2\int_{x^2}^4\int_0^{4-z} y\rho\,dy\,dz\,dx = 2\rho\int_0^2\int_{x^2}^4\left\{\frac{y^2}{2}\right\}_0^{4-z} dz\,dx = \rho\int_0^2\int_{x^2}^4 (4-z)^2\,dz\,dx$$

$$= \rho\int_0^2\left\{\frac{-(4-z)^3}{3}\right\}_{x^2}^4 dx = \frac{\rho}{3}\int_0^2 (64-48x^2+12x^4-x^6)\,dx = \frac{\rho}{3}\left\{64x-16x^3+\frac{12x^5}{5}-\frac{x^7}{7}\right\}_0^2$$

$$= \frac{2048\rho}{105},$$

it follows that $\overline{y} = \frac{2048\rho}{105}\frac{15}{256\rho} = \frac{8}{7}$. Since

$$M\overline{z} = 2\int_0^2\int_{x^2}^4\int_0^{4-z} z\rho\,dy\,dz\,dx = 2\rho\int_0^2\int_{x^2}^4 z(4-z)\,dz\,dx = 2\rho\int_0^2\left\{2z^2-\frac{z^3}{3}\right\}_{x^2}^4 dx$$

$$= \frac{2\rho}{3}\int_0^2 (96-64-6x^4+x^6)\,dx = \frac{2\rho}{3}\left\{32x-\frac{6x^5}{5}+\frac{x^7}{7}\right\}_0^2 = \frac{1024\rho}{35},$$

we obtain $\overline{z} = \frac{1024\rho}{35}\frac{15}{256\rho} = \frac{12}{7}$.

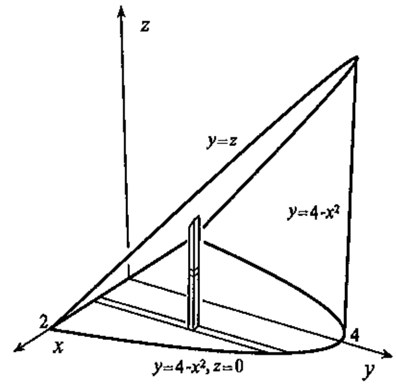
4. $M = 2\int_0^2\int_0^{4-x^2}\int_0^y \rho\,dz\,dy\,dx$

$= 2\rho\int_0^2\int_0^{4-x^2} y\,dy\,dx$

$= 2\rho\int_0^2 \left\{\frac{y^2}{2}\right\}_0^{4-x^2} dx$

$= \rho\int_0^2 (16-8x^2+x^4)\,dx$

$= \rho\left\{16x - \frac{8x^3}{3} + \frac{x^5}{5}\right\}_0^2 = \frac{256\rho}{15}$



Since $M\overline{y} = 2\int_0^2\int_0^{4-x^2}\int_0^y y\rho\,dz\,dy\,dx = 2\rho\int_0^2\int_0^{4-x^2} y^2\,dy\,dx = 2\rho\int_0^2\left\{\frac{y^3}{3}\right\}_0^{4-x^2} dx$

$$= \frac{2\rho}{3}\int_0^2 (64-48x^2+12x^4-x^6)\,dx = \frac{2\rho}{3}\left\{64x - 16x^3 + \frac{12x^5}{5} - \frac{x^7}{7}\right\}_0^2 = \frac{4096\rho}{105},$$

it follows that $\overline{y} = \frac{4096\rho}{105}\frac{15}{256\rho} = \frac{16}{7}$. By symmetry, $\overline{x} = 0$. We find that $\overline{z} = 8/7$ since

$$M\overline{z} = 2\int_0^2\int_0^{4-x^2}\int_0^y z\rho\,dz\,dy\,dx = 2\rho\int_0^2\int_0^{4-x^2}\left\{\frac{z^2}{2}\right\}_0^y dy\,dx = \rho\int_0^2\int_0^{4-x^2} y^2\,dy\,dx = \frac{1}{2}M\overline{y}.$$

5. $M = \int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z}\rho\,dx\,dz\,dy = \rho\int_0^3\int_{y/3}^{4-y}(4-y-z)\,dz\,dy = \rho\int_0^3\left\{-\frac{1}{2}(4-y-z)^2\right\}_{y/3}^{4-y} dy$

$= \frac{\rho}{2}\int_0^3\left(4-\frac{4y}{3}\right)^2 dy = \frac{8\rho}{9}\left\{-\frac{1}{3}(3-y)^3\right\}_0^3 = 8\rho$

Since $M\overline{x} = \int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z} x\rho\,dx\,dz\,dy = \rho\int_0^3\int_{y/3}^{4-y}\left\{\frac{x^2}{2}\right\}_0^{4-y-z} dz\,dy = \frac{\rho}{2}\int_0^3\int_{y/3}^{4-y}(4-y-z)^2\,dz\,dy$

$$= \frac{\rho}{2}\int_0^3\left\{-\frac{1}{3}(4-y-z)^3\right\}_{y/3}^{4-y} dy = \frac{\rho}{6}\int_0^3\left(4-\frac{4y}{3}\right)^3 dy = \frac{32\rho}{81}\left\{-\frac{1}{4}(3-y)^4\right\}_0^3 = 8\rho,$$

we find that $\overline{x} = 1$. Since

$$M\overline{y} = \int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z} y\rho\,dx\,dz\,dy = \rho\int_0^3\int_{y/3}^{4-y} y(4-y-z)\,dz\,dy = \rho\int_0^3\left\{-\frac{y}{2}(4-y-z)^2\right\}_{y/3}^{4-y} dy$$

$$= \frac{\rho}{2}\int_0^3 y\left(4-\frac{4y}{3}\right)^2 dy = \frac{8\rho}{9}\int_0^3 (9y-6y^2+y^3)\,dy = \frac{8\rho}{9}\left\{\frac{9y^2}{2} - 2y^3 + \frac{y^4}{4}\right\}_0^3 = 6\rho,$$

it follows that $\overline{y} = 6\rho/(8\rho) = 3/4$. Since

$$M\overline{z} = \int_0^3\int_{y/3}^{4-y}\int_0^{4-y-z} z\rho\,dx\,dz\,dy = \rho\int_0^3\int_{y/3}^{4-y} z(4-y-z)\,dz\,dy = \rho\int_0^3\left\{2z^2 - \frac{yz^2}{2} - \frac{z^3}{3}\right\}_{y/3}^{4-y} dy$$

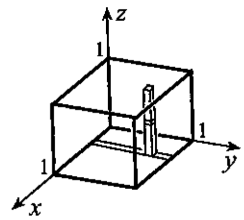
$$= \frac{\rho}{6}\int_0^3\left[12(4-y)^2 - 3y(4-y)^2 - 2(4-y)^3 - \frac{4y^2}{3} + \frac{y^3}{3} + \frac{2y^3}{27}\right] dy$$

$$= \frac{\rho}{6}\int_0^3\left[12(4-y)^2 - 48y + \frac{68y^2}{3} - \frac{70y^3}{27} - 2(4-y)^3\right] dy$$

$$= \frac{\rho}{6}\left\{-4(4-y)^3 - 24y^2 + \frac{68y^3}{9} - \frac{35y^4}{54} + \frac{(4-y)^4}{2}\right\}_0^3 = 10\rho,$$

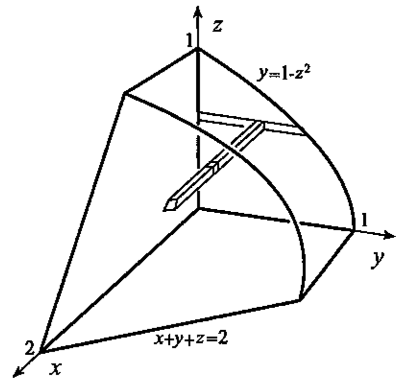
we find that $\overline{z} = 10\rho/(8\rho) = 5/4$.

6. $$I = \int_0^1\int_0^1\int_0^1 (y^2+z^2)\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^1 \left\{y^2z+\frac{z^3}{3}\right\}_0^1 dy\,dx$$
$$= \frac{\rho}{3}\int_0^1\int_0^1 (3y^2+1)\,dy\,dx$$
$$= \frac{\rho}{3}\int_0^1 \left\{y^3+y\right\}_0^1 dx = \frac{2\rho}{3}\left\{x\right\}_0^1 = \frac{2\rho}{3}$$



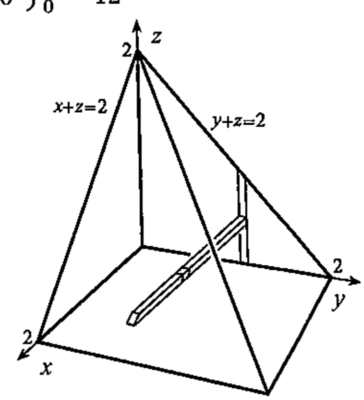
7. $$I = \int_0^3\int_0^2\int_0^{2x} (x^2+z^2)\rho\,dz\,dy\,dx = \rho\int_0^3\int_0^2 \left\{x^2z+\frac{z^3}{3}\right\}_0^{2x} dy\,dx = \frac{\rho}{3}\int_0^3\int_0^2 (6x^3+8x^3)\,dy\,dx$$
$$= \frac{14\rho}{3}\int_0^3 \left\{x^3y\right\}_0^2 dx = \frac{28\rho}{3}\left\{\frac{x^4}{4}\right\}_0^3 = 189\rho$$

8. $$I = \int_0^1\int_0^{1-z^2}\int_0^{2-y-z} (y^2+z^2)\rho\,dx\,dy\,dz$$
$$= \rho\int_0^1\int_0^{1-z^2} \left\{(y^2+z^2)x\right\}_0^{2-y-z} dy\,dz$$
$$= \rho\int_0^1\int_0^{1-z^2} (2y^2-y^3-zy^2+2z^2-yz^2-z^3)\,dy\,dz$$
$$= \rho\int_0^1 \left\{\frac{2y^3}{3}-\frac{y^4}{4}-\frac{zy^3}{3}+2z^2y-\frac{y^2z^2}{2}-yz^3\right\}_0^{1-z^2} dz$$
$$= \frac{\rho}{12}\int_0^1 [5+6z^2-12z^3-6z^4+12z^5-2z^6-3z^8-4z(1-z^2)^3]\,dz$$
$$= \frac{\rho}{12}\left\{5z+2z^3-3z^4-\frac{6z^5}{5}+2z^6-\frac{2z^7}{7}-\frac{z^9}{3}+\frac{1}{2}(1-z^2)^4\right\}_0^1 = \frac{773\rho}{2520}$$



9. $$I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{xy} (x^2+y^2)\rho\,dz\,dy\,dx = \rho\int_0^1\int_0^{\sqrt{1-x^2}} xy(x^2+y^2)\,dy\,dx$$
$$= \rho\int_0^1 \left\{\frac{x^3y^2}{2}+\frac{xy^4}{4}\right\}_0^{\sqrt{1-x^2}} dx = \frac{\rho}{4}\int_0^1 (x-x^5)\,dx = \frac{\rho}{4}\left\{\frac{x^2}{2}-\frac{x^6}{6}\right\}_0^1 = \frac{\rho}{12}$$

10. $$I = \int_0^2\int_0^{2-y}\int_0^{2-z} (x^2+y^2)\rho\,dx\,dz\,dy$$
$$= \rho\int_0^2\int_0^{2-y} \left\{\frac{x^3}{3}+xy^2\right\}_0^{2-z} dz\,dy$$
$$= \frac{\rho}{3}\int_0^2\int_0^{2-y} [(2-z)^3+3y^2(2-z)]\,dz\,dy$$
$$= \frac{\rho}{3}\int_0^2 \left\{-\frac{1}{4}(2-z)^4-\frac{3}{2}y^2(2-z)^2\right\}_0^{2-y} dy$$
$$= \frac{\rho}{12}\int_0^2 (16+24y^2-7y^4)\,dy = \frac{\rho}{12}\left\{16y+8y^3-\frac{7y^5}{5}\right\}_0^2 = \frac{64\rho}{15}$$



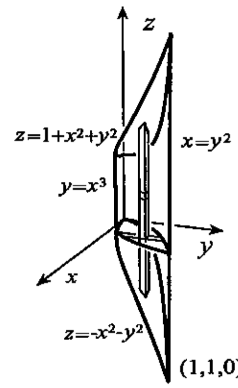
11. $$\text{Moment } = \int_0^2\int_{-z}^{z}\int_0^2 z\rho\,dy\,dx\,dz = \rho\int_0^2\int_{-z}^{z} 2z\,dx\,dz = 2\rho\int_0^2 \left\{xz\right\}_{-z}^{z} dz$$
$$= 4\rho\int_0^2 z^2\,dz = 4\rho\left\{\frac{z^3}{3}\right\}_0^2 = \frac{32\rho}{3}$$

12. $M = \int_0^1 \int_{x^3}^{\sqrt{x}} \int_{-x^2-y^2}^{1+x^2+y^2} \rho\, dz\, dy\, dx = \rho \int_0^1 \int_{x^3}^{\sqrt{x}} (1 + 2x^2 + 2y^2)\, dy\, dx$

$$= \rho \int_0^1 \left\{ y + 2x^2 y + \frac{2y^3}{3} \right\}_{x^3}^{\sqrt{x}} dx$$

$$= \frac{\rho}{3} \int_0^1 (3\sqrt{x} + 6x^{5/2} + 2x^{3/2} - 3x^3 - 6x^5 - 2x^9)\, dx$$

$$= \frac{\rho}{3} \left\{ 2x^{3/2} + \frac{12x^{7/2}}{7} + \frac{4x^{5/2}}{5} - \frac{3x^4}{4} - x^6 - \frac{x^{10}}{5} \right\}_0^1 = \frac{359\rho}{420}$$



Since

$$M\overline{x} = \int_0^1 \int_{x^3}^{\sqrt{x}} \int_{-x^2-y^2}^{1+x^2+y^2} x\rho\, dz\, dy\, dx = \rho \int_0^1 \int_{x^3}^{\sqrt{x}} x(1 + 2x^2 + 2y^2)\, dy\, dx$$

$$= \rho \int_0^1 \left\{ xy + 2x^3 y + \frac{2xy^3}{3} \right\}_{x^3}^{\sqrt{x}} dx = \frac{\rho}{3} \int_0^1 (3x^{3/2} + 6x^{7/2} + 2x^{5/2} - 3x^4 - 6x^6 - 2x^{10})\, dx$$

$$= \frac{\rho}{3} \left\{ \frac{6x^{5/2}}{5} + \frac{4x^{9/2}}{3} + \frac{4x^{7/2}}{7} - \frac{3x^5}{5} - \frac{6x^7}{7} - \frac{2x^{11}}{11} \right\}_0^1 = \frac{1693\rho}{3465},$$

we obtain $\overline{x} = \dfrac{1693\rho}{3465} \dfrac{420}{359\rho} = \dfrac{6772}{11\,847}$. Since

$$M\overline{y} = \int_0^1 \int_{x^3}^{\sqrt{x}} \int_{-x^2-y^2}^{1+x^2+y^2} y\rho\, dz\, dy\, dx = \rho \int_0^1 \int_{x^3}^{\sqrt{x}} y(1 + 2x^2 + 2y^2)\, dy\, dx$$

$$= \rho \int_0^1 \left\{ \frac{y^2}{2} + x^2 y^2 + \frac{y^4}{2} \right\}_{x^3}^{\sqrt{x}} dx = \frac{\rho}{2} \int_0^1 (x + x^2 + 2x^3 - x^6 - 2x^8 - x^{12})\, dx$$

$$= \frac{\rho}{2} \left\{ \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{2} - \frac{x^7}{7} - \frac{2x^9}{9} - \frac{x^{13}}{13} \right\}_0^1 = \frac{365\rho}{819},$$

we find $\overline{y} = \dfrac{365\rho}{819} \dfrac{420}{359\rho} = \dfrac{7300}{14\,001}$. Since the top and bottom surfaces have exactly the same shape, it follows that $\overline{z} = 1/2$.

13. $M = \int_{-2}^1 \int_{x^2}^{2-x} \int_0^z \rho\, dy\, dz\, dx = \rho \int_{-2}^1 \int_{x^2}^{2-x} z\, dz\, dx = \rho \int_{-2}^1 \left\{ \frac{z^2}{2} \right\}_{x^2}^{2-x} dx$

$$= \frac{\rho}{2} \int_{-2}^1 [(2-x)^2 - x^4]\, dx = \frac{\rho}{2} \left\{ -\frac{1}{3}(2-x)^3 - \frac{x^5}{5} \right\}_{-2}^1 = \frac{36\rho}{5}$$

Since

$$M\overline{x} = \int_{-2}^1 \int_{x^2}^{2-x} \int_0^z x\rho\, dy\, dz\, dx = \rho \int_{-2}^1 \int_{x^2}^{2-x} xz\, dz\, dx = \rho \int_{-2}^1 \left\{ \frac{xz^2}{2} \right\}_{x^2}^{2-x} dx$$

$$= \frac{\rho}{2} \int_{-2}^1 (4x - 4x^2 + x^3 - x^5)\, dx = \frac{\rho}{2} \left\{ 2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} - \frac{x^6}{6} \right\}_{-2}^1 = -\frac{45\rho}{8},$$

we obtain $\overline{x} = -\dfrac{45\rho}{8} \dfrac{5}{36\rho} = -\dfrac{25}{32}$. Since

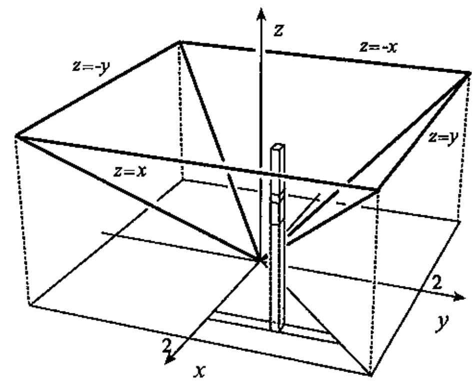
$$M\overline{y} = \int_{-2}^1 \int_{x^2}^{2-x} \int_0^z y\rho\, dy\, dz\, dx = \rho \int_{-2}^1 \int_{x^2}^{2-x} \left\{ \frac{y^2}{2} \right\}_0^z dz\, dx = \frac{\rho}{2} \int_{-2}^1 \int_{x^2}^{2-x} z^2\, dz\, dx$$

$$= \frac{\rho}{2} \int_{-2}^1 \left\{ \frac{z^3}{3} \right\}_{x^2}^{2-x} dx = \frac{\rho}{6} \int_{-2}^1 [(2-x)^3 - x^6]\, dx = \frac{\rho}{6} \left\{ -\frac{1}{4}(2-x)^4 - \frac{x^7}{7} \right\}_{-2}^1 = \frac{423\rho}{56},$$

we find that $\overline{y} = \dfrac{423\rho}{56}\,\dfrac{5}{36\rho} = \dfrac{235}{224}$. Since

$$M\overline{z} = \int_{-2}^{1}\int_{x^2}^{2-x}\int_0^z z\rho\,dy\,dz\,dx = \rho\int_{-2}^{1}\int_{x^2}^{2-x} z^2\,dz\,dx = \frac{423\rho}{28} \quad \text{(see } M\overline{y} \text{ integral)},$$

it follows that $\overline{z} = 235/112$.



14. $M = 8\displaystyle\int_0^2\int_0^x\int_x^2 \rho\,dz\,dy\,dx = 8\rho\int_0^2\int_0^x (2-x)\,dy\,dx$

$$= 8\rho\int_0^2 (2x - x^2)\,dx = 8\rho\left\{x^2 - \frac{x^3}{3}\right\}_0^2 = \frac{32\rho}{3}$$

By symmetry, $\overline{x} = \overline{y} = 0$. Since

$$M\overline{z} = 8\int_0^2\int_0^x\int_x^2 z\rho\,dz\,dy\,dx$$

$$= 8\rho\int_0^2\int_0^x\left\{\frac{z^2}{2}\right\}_x^2 dy\,dx = 4\rho\int_0^2\int_0^x (4 - x^2)\,dy\,dx$$

$$= 4\rho\int_0^2 (4x - x^3)\,dx = 4\rho\left\{2x^2 - \frac{x^4}{4}\right\}_0^2 = 16\rho,$$

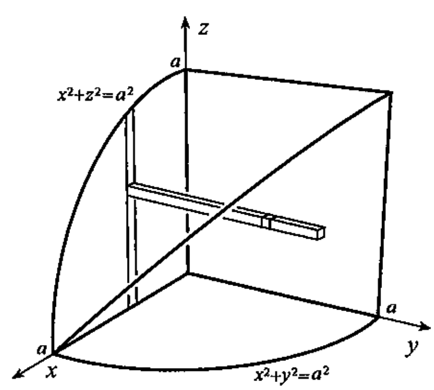
we find $\overline{z} = 16\rho\dfrac{3}{32\rho} = \dfrac{3}{2}$.

15. $I = \displaystyle\int_{-2}^{3}\int_{-2-x}^{4-x^2}\int_0^2 (x^2+z^2)\rho\,dy\,dz\,dx = \rho\int_{-2}^{3}\int_{-2-x}^{4-x^2} 2(x^2+z^2)\,dz\,dx = 2\rho\int_{-2}^{3}\left\{x^2z + \frac{z^3}{3}\right\}_{-2-x}^{4-x^2} dx$

$$= \frac{2\rho}{3}\int_{-2}^{3}[3x^2(4-x^2) + (4-x^2)^3 - 3x^2(-2-x) - (-2-x)^3]\,dx$$

$$= \frac{2\rho}{3}\int_{-2}^{3}(72 + 12x - 24x^2 + 4x^3 + 9x^4 - x^6)\,dx$$

$$= \frac{2\rho}{3}\left\{72x + 6x^2 - 8x^3 + x^4 + \frac{9x^5}{5} - \frac{x^7}{7}\right\}_{-2}^{3} = \frac{4750\rho}{21}$$



16. $I = 8\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} (y^2+z^2)\rho\,dy\,dz\,dx$

$$= 8\rho\int_0^a\int_0^{\sqrt{a^2-x^2}}\left\{\frac{y^3}{3} + yz^2\right\}_0^{\sqrt{a^2-x^2}} dz\,dx$$

$$= \frac{8\rho}{3}\int_0^a\int_0^{\sqrt{a^2-x^2}} [(a^2-x^2)^{3/2} + 3z^2\sqrt{a^2-x^2}]\,dz\,dx$$

$$= \frac{8\rho}{3}\int_0^a\left\{(a^2-x^2)^{3/2}z + z^3\sqrt{a^2-x^2}\right\}_0^{\sqrt{a^2-x^2}} dx$$

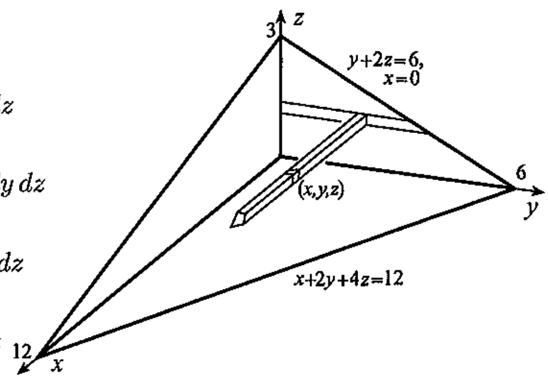
$$= \frac{16\rho}{3}\int_0^a (a^4 - 2a^2x^2 + x^4)\,dx = \frac{16\rho}{3}\left\{a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right\}_0^a = \frac{128\rho a^5}{45}$$

17. $I = \displaystyle\int_0^3\int_{x/3}^{2x/3}\int_0^{x+y} (x^2+y^2)\rho\,dz\,dy\,dx = \rho\int_0^3\int_{x/3}^{2x/3}(x^2+y^2)(x+y)\,dy\,dx$

$$= \rho\int_0^3\left\{x^3y + \frac{x^2y^2}{2} + \frac{xy^3}{3} + \frac{y^4}{4}\right\}_{x/3}^{2x/3} dx = \frac{205\rho}{324}\int_0^3 x^4\,dx = \frac{205\rho}{324}\left\{\frac{x^5}{5}\right\}_0^3 = \frac{123\rho}{4}$$

18. The distance from the volume $dz\,dy\,dx$ at point (x, y, z) to the plane $x+y+z=1$ is $|x+y+z-1|/\sqrt{3}$. If we take distances from those points on the origin side of the plane as negative, then the required first moment is

$$\begin{aligned}
&\int_0^3 \int_0^{6-2z} \int_0^{12-2y-4z} \frac{x+y+z-1}{\sqrt{3}} \rho\,dx\,dy\,dz \\
&= \frac{\rho}{\sqrt{3}} \int_0^3 \int_0^{6-2z} \left\{ \frac{(x+y+z-1)^2}{2} \right\}_0^{12-2y-4z} dy\,dz \\
&= \frac{\rho}{2\sqrt{3}} \int_0^3 \int_0^{6-2z} [(11-y-3z)^2 - (y+z-1)^2]\,dy\,dz \\
&= \frac{\rho}{2\sqrt{3}} \int_0^3 \left\{ \frac{(11-y-3z)^3}{-3} - \frac{(y+z-1)^3}{3} \right\}_0^{6-2z} dz \\
&= \frac{\rho}{6\sqrt{3}} \int_0^3 [-2(5-z)^3 + (11-3z)^3 + (z-1)^3]\,dz \\
&= \frac{\rho}{6\sqrt{3}} \left\{ \frac{(5-z)^4}{2} - \frac{(11-3z)^4}{12} + \frac{(z-1)^4}{4} \right\}_0^3 = 51\sqrt{3}\rho.
\end{aligned}$$



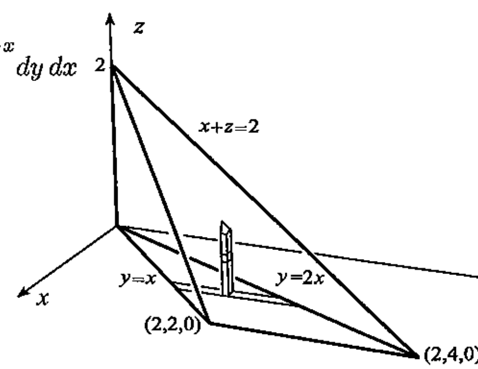
19. The product moment of inertia I_{xy} is

$$\begin{aligned}
I_{xy} &= \int_0^{1/a} \int_0^{(1-ax)/b} \int_0^{(1-ax-by)/c} xy\rho\,dz\,dy\,dx = \rho \int_0^{1/a} \int_0^{(1-ax)/b} \Big\{ xyz \Big\}_0^{(1-ax-by)/c} dy\,dx \\
&= \frac{\rho}{c} \int_0^{1/a} \int_0^{(1-ax)/b} xy(1-ax-by)\,dy\,dx = \frac{\rho}{c} \int_0^{1/a} \left\{ \frac{xy^2}{2} - \frac{ax^2y^2}{2} - \frac{bxy^3}{3} \right\}_0^{(1-ax)/b} dx \\
&= \frac{\rho}{6b^2c} \int_0^{1/a} (x - 3ax^2 + 3a^2x^3 - a^3x^4)\,dx = \frac{\rho}{6b^2c} \left\{ \frac{x^2}{2} - ax^3 + \frac{3a^2x^4}{4} - \frac{a^3x^5}{5} \right\}_0^{1/a} = \frac{\rho}{120a^2b^2c}.
\end{aligned}$$

Similarly, $I_{yz} = \rho/(120ab^2c^2)$ and $I_{xz} = \rho/(120a^2bc^2)$.

20. The product moment of inertia I_{xy} is

$$\begin{aligned}
I_{xy} &= \int_0^2 \int_x^{2x} \int_0^{2-x} xy\rho\,dz\,dy\,dx = \rho \int_0^2 \int_x^{2x} \Big\{ xyz \Big\}_0^{2-x} dy\,dx \\
&= \rho \int_0^2 \int_x^{2x} xy(2-x)\,dy\,dx \\
&= \rho \int_0^2 \left\{ \frac{x(2-x)y^2}{2} \right\}_x^{2x} dx \\
&= \frac{3\rho}{2} \int_0^2 (2x^3 - x^4)\,dx = \frac{3\rho}{2} \left\{ \frac{x^4}{2} - \frac{x^5}{5} \right\}_0^2 = \frac{12\rho}{5}.
\end{aligned}$$



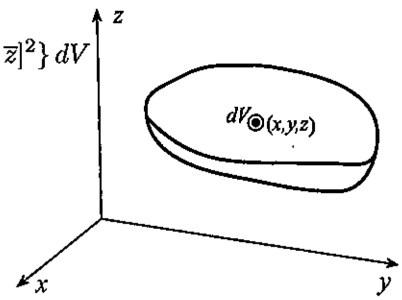
The other two are

$$\begin{aligned}
I_{yz} &= \int_0^2 \int_x^{2x} \int_0^{2-x} yz\rho\,dz\,dy\,dx = \rho \int_0^2 \int_x^{2x} \left\{ \frac{yz^2}{2} \right\}_0^{2-x} dy\,dx = \frac{\rho}{2} \int_0^2 \int_x^{2x} y(2-x)^2\,dy\,dx \\
&= \frac{\rho}{2} \int_0^2 \left\{ \frac{(2-x)^2y^2}{2} \right\}_x^{2x} dx = \frac{3\rho}{4} \int_0^2 (4x^2 - 4x^3 + x^4)\,dx = \frac{3\rho}{4} \left\{ \frac{4x^3}{3} - x^4 + \frac{x^5}{5} \right\}_0^2 = \frac{4\rho}{5}, \\
I_{xz} &= \int_0^2 \int_x^{2x} \int_0^{2-x} xz\rho\,dz\,dy\,dx = \rho \int_0^2 \int_x^{2x} \left\{ \frac{xz^2}{2} \right\}_0^{2-x} dy\,dx = \frac{\rho}{2} \int_0^2 \int_x^{2x} x(2-x)^2\,dy\,dx \\
&= \frac{\rho}{2} \int_0^2 \Big\{ x(2-x)^2y \Big\}_x^{2x} dx = \frac{\rho}{2} \int_0^2 (4x^2 - 4x^3 + x^4)\,dx = \frac{\rho}{2} \left\{ \frac{4x^3}{3} - x^4 + \frac{x^5}{5} \right\}_0^2 = \frac{8\rho}{15}.
\end{aligned}$$

21. We show that $I_x \le I_y + I_z$.

$$\begin{aligned} I_y + I_z &= \iiint_V (x^2+z^2)\rho\,dV + \iiint_V (x^2+y^2)\rho\,dV = \iiint_V (y^2+z^2)\rho\,dV + 2\iiint_V x^2\rho\,dV \\ &= I_x + 2\iiint_V x^2\rho\,dV \ge I_x, \qquad \text{(since the last integral is positive).} \end{aligned}$$

22. If we orient the volume so that the line is the x-axis, then



$$\begin{aligned} I_x &= \iiint_V (y^2+z^2)\rho\,dV = \iiint_V \{[(y-\overline{y})+\overline{y}]^2 + [(z-\overline{z})+\overline{z}]^2\}\,dV \\ &= \iiint_V [(y-\overline{y})^2 + 2\overline{y}(y-\overline{y}) + \overline{y}^2 \\ &\qquad + (z-\overline{z})^2 + 2\overline{z}(z-\overline{z}) + \overline{z}^2]\,dV \\ &= \iiint_V [(y-\overline{y})^2 + (z-\overline{z})^2]\rho\,dV + 2\overline{y}\iiint_V y\rho\,dV \\ &\qquad - \overline{y}^2\iiint_V \rho\,dV + 2\overline{z}\iiint_V z\rho\,dV - \overline{z}^2\iiint_V \rho\,dV \\ &= I_{\overline{x}} + 2\overline{y}(M\overline{y}) - \overline{y}^2(M) + 2\overline{z}(M\overline{z}) - \overline{z}^2(M) = I_{\overline{x}} + M(\overline{y}^2+\overline{z}^2). \end{aligned}$$

23. $M = \displaystyle\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} \rho\,dz\,dy\,dx = \rho\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$

If we set $y = b\sqrt{1-x^2/a^2}\sin\theta$, then $dy = b\sqrt{1-x^2/a^2}\cos\theta\,d\theta$, and

$$\begin{aligned} M &= \rho c\int_0^a\int_0^{\pi/2}\sqrt{\left(1-\frac{x^2}{a^2}\right) - \left(1-\frac{x^2}{a^2}\right)\sin^2\theta}\, b\sqrt{1-\frac{x^2}{a^2}}\cos\theta\,d\theta\,dx \\ &= \rho bc\int_0^a\int_0^{\pi/2}\left(1-\frac{x^2}{a^2}\right)\cos^2\theta\,d\theta\,dx = \frac{\rho bc}{a^2}\int_0^a\int_0^{\pi/2}(a^2-x^2)\left(\frac{1+\cos 2\theta}{2}\right)d\theta\,dx \\ &= \frac{\rho bc}{2a^2}\int_0^a\left\{(a^2-x^2)\left(\theta + \frac{\sin 2\theta}{2}\right)\right\}_0^{\pi/2} dx = \frac{\rho bc}{2a^2}\left(\frac{\pi}{2}\right)\left\{a^2x - \frac{x^3}{3}\right\}_0^a = \frac{\rho\pi abc}{6}. \end{aligned}$$

We now calculate

$$\begin{aligned} M\overline{x} &= \int_0^b\int_0^{a\sqrt{1-y^2/b^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} \rho x\,dz\,dx\,dy = \rho\int_0^b\int_0^{a\sqrt{1-y^2/b^2}} cx\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy \\ &= \rho c\int_0^b\left\{-\frac{a^2}{3}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2}\right\}_0^{a\sqrt{1-y^2/b^2}} dy = \frac{\rho a^2 c}{3b^3}\int_0^b (b^2-y^2)^{3/2}\,dy. \end{aligned}$$

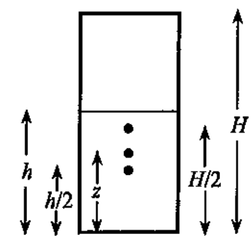
If we set $y = b\sin\theta$, then $dy = b\cos\theta\,d\theta$, and

$$\begin{aligned} M\overline{x} &= \frac{\rho a^2 c}{3b^3}\int_0^{\pi/2} b^3\cos^3\theta\, b\cos\theta\,d\theta = \frac{\rho a^2 bc}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\ &= \frac{\rho a^2 bc}{12}\int_0^{\pi/2}\left[1 + 2\cos 2\theta + \left(\frac{1+\cos 4\theta}{2}\right)\right]d\theta = \frac{\rho a^2 bc}{12}\left\{\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{8}\right\}_0^{\pi/2} = \frac{\rho\pi a^2 bc}{16}. \end{aligned}$$

Thus, $\overline{x} = \dfrac{\rho\pi a^2 bc}{16}\dfrac{6}{\rho\pi abc} = \dfrac{3a}{8}$. Similarly, $\overline{y} = 3b/8$ and $\overline{z} = 3c/8$.

24. Let H be the height of the can and h be the depth of pop. Let m and M be the mass of the pop and can, respectively. Let A be the cross-sectional area of the can and pop and ρ be the density of the pop. If z is the centre of mass of can plus pop, then $(m+M)z = m(h/2) + M(H/2)$. Hence,

$$z = \frac{mh+MH}{2(m+M)} = \frac{(\rho Ah)h+MH}{2(\rho Ah+M)},$$



where $0 < h < H$. For critical points of z as a function of h, we solve

$$0 = \frac{dz}{dh} = \frac{(\rho Ah+M)(2\rho Ah) - (\rho Ah^2+MH)(\rho A)}{2(\rho Ah+M)^2} = \frac{\rho A(\rho Ah^2 + 2Mh - MH)}{2(\rho Ah+M)^2}.$$

Solutions are $h = \dfrac{-2M \pm \sqrt{4M^2+4\rho AMH}}{2\rho A}$, only the positive root being acceptable. Since $z(0) = z(H) = H/2$, and there is only one critical point, it follows that this critical point must yield a minimum for z. We could substitute the critical value of h into the function $z(h)$ to find the minimum. Instead, notice that if we substitute $\rho Ah = M(H-2h)/h$, then the minimum value is

$$z = \frac{M(H-2h)+MH}{(2M/h)(H-2h)+2M} = \frac{2M(H-h)}{(2M/h)(H-2h+h)} = h;$$

that is, the centre of mass is in the surface of the pop.

25. Since the density of the sphere is half that of water, half the sphere will be above water and half under water. Suppose we take the xy-plane as the surface of the water and $z = -\sqrt{R^2-x^2-y^2}$ as the surface of the hemisphere beneath the surface. The mass of displaced water is $M = (2/3)\pi R^3(1000)$. If $\overline{z}$ is the z-coordinate of the centre of mass of this displaced water, then

$$\begin{aligned} M\overline{z} &= 4\int_0^R \int_0^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^0 z(1000)\,dz\,dy\,dx = 4000\int_0^R\int_0^{\sqrt{R^2-x^2}} \left\{\frac{z^2}{2}\right\}_{-\sqrt{R^2-x^2-y^2}}^0 dy\,dx \\ &= 2000\int_0^R\int_0^{\sqrt{R^2-x^2}} -(R^2-x^2-y^2)\,dy\,dx. \end{aligned}$$

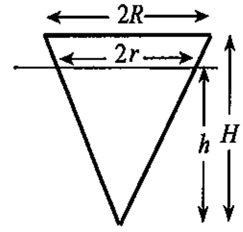
If we transform this double iterated integral to polar coordinates,

$$M\overline{z} = -2000\int_0^{\pi/2}\int_0^R (R^2-r^2)r\,dr\,d\theta = -2000\int_0^{\pi/2}\left\{\frac{R^2r^2}{2} - \frac{r^4}{4}\right\}_0^R d\theta = -500R^4\Big\{\theta\Big\}_0^{\pi/2} = -250\pi R^4.$$

Hence, $\overline{z} = -250\pi R^4\left(\dfrac{3}{2000\pi R^3}\right) = -\dfrac{3R}{8}$. The centre of buoyancy is $3R/8$ below the surface.

26. Suppose we let r and h be the radius and height of that part of the cone under water. For buoyancy, the weight of the water displaced must be equal to the weight of the cone,

$$\frac{1}{3}\pi r^2h(1000)g = \frac{1}{3}\pi R^2H(800)g \quad\Longrightarrow\quad r^2h = \frac{4}{5}R^2H.$$

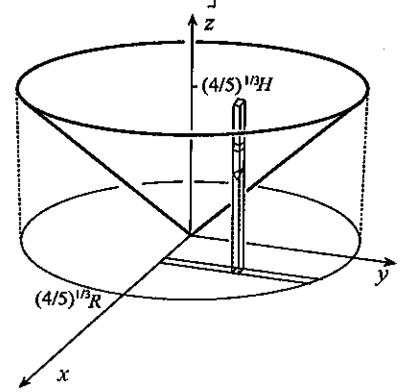


By similar triangles, $r/h = R/H \Longrightarrow r = Rh/H$, and therefore

$$\left(\frac{Rh}{H}\right)^2 h = \frac{4}{5}R^2H \quad\Longrightarrow\quad h = \left(\frac{4}{5}\right)^{1/3}H \quad\text{and}\quad r = \left(\frac{4}{5}\right)^{1/3}R.$$

To find the centre of buoyancy we require the centre of mass of a right-circular cone of water with radius $r = (4/5)^{1/3}R$ and height $h = (4/5)^{1/3}H$. Such a cone with apex at the origin has equation $z = (H/R)\sqrt{x^2+y^2}$. The mass of the displaced water is $M = (1000/3)\pi R^2 H$ kg. If $\overline{z}$ is the z-coordinate of its centre of mass, then

$$\begin{aligned}
M\overline{z} &= 4\int_0^{(4/5)^{1/3}R}\int_0^{\sqrt{(4/5)^{2/3}R^2-x^2}}\int_{(H/R)\sqrt{x^2+y^2}}^{(4/5)^{1/3}H} z(1000)\,dz\,dy\,dx\\
&= 4000\int_0^{(4/5)^{1/3}R}\int_0^{\sqrt{(4/5)^{2/3}R^2-x^2}}\left\{\frac{z^2}{2}\right\}_{(H/R)\sqrt{x^2+y^2}}^{(4/5)^{1/3}H} dy\,dx\\
&= 2000\int_0^{(4/5)^{1/3}R}\int_0^{\sqrt{(4/5)^{2/3}R^2-x^2}}\left[\left(\frac{4}{5}\right)^{2/3}H^2 - \frac{H^2}{R^2}(x^2+y^2)\right] dy\,dx.
\end{aligned}$$



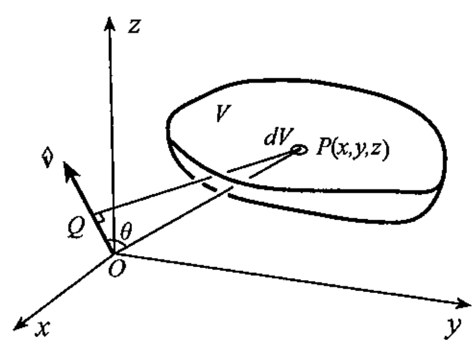
If we transform this double iterated integral to polar coordinates,

$$\begin{aligned}
M\overline{z} &= 2000\int_0^{\pi/2}\int_0^{(4/5)^{1/3}R}\left[\left(\frac{4}{5}\right)^{2/3}H^2 - \frac{H^2r^2}{R^2}\right] r\,dr\,d\theta\\
&= 2000\int_0^{\pi/2}\left\{\left(\frac{4}{5}\right)^{2/3}\frac{H^2r^2}{2} - \frac{H^2r^4}{4R^2}\right\}_0^{(4/5)^{1/3}R} d\theta\\
&= 500(4/5)^{4/3}H^2R^2\Big\{\theta\Big\}_0^{\pi/2} = 250(4/5)^{4/3}\pi H^2R^2.
\end{aligned}$$

Thus, $\overline{z} = 250(4/5)^{4/3}\pi H^2R^2\dfrac{3}{1000\pi R^2H} = \dfrac{3(4/5)^{4/3}H}{4}$. The centre of buoyancy of the floating cone is therefore $\left(\dfrac{4}{5}\right)^{1/3}H - \dfrac{3}{4}\left(\dfrac{4}{5}\right)^{4/3}H = \dfrac{2}{5}\left(\dfrac{4}{5}\right)^{1/3}H$ below the surface.

27. If $\mathbf{PQ}$ is the line from (x,y,z) perpendicular to $\hat{\mathbf{v}}$ at Q, then

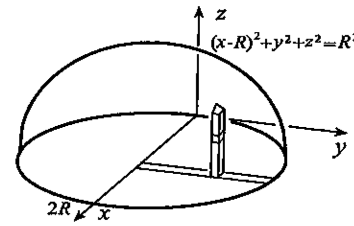
$$\begin{aligned}
|\mathbf{PQ}| &= |\mathbf{OP}|\sin\theta = |\mathbf{OP}||\hat{\mathbf{v}}|\sin\theta = |\mathbf{OP}\times\hat{\mathbf{v}}|\\
&= \left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ x & y & z\\ v_x & v_y & v_z\end{matrix}\right\|\\
&= \sqrt{(yv_z-zv_y)^2+(zv_x-xv_z)^2+(xv_y-yv_x)^2}.
\end{aligned}$$



The moment of inertia of the mass about the line containing $\hat{\mathbf{v}}$ is

$$\begin{aligned}
I &= \iiint_V [yv_z-zv_y)^2+(zv_x-xv_z)^2+(xv_y-yv_x)^2]\rho\,dV\\
&= \iiint_V (y^2v_z^2 - 2yzv_yv_z + z^2v_y^2 + z^2v_x^2 - 2xzv_xv_z + x^2v_z^2 + x^2v_y^2 - 2xyv_xv_y + y^2v_x^2)\rho\,dV\\
&= v_x^2\iiint_V (y^2+z^2)\rho\,dV + v_y^2\iiint_V (x^2+z^2)\rho\,dV + v_z^2\iiint_V (x^2+y^2)\rho\,dV\\
&\quad - 2v_xv_y\iiint_v xy\,\rho\,dV - 2v_yv_z\iiint_V yz\,\rho\,dV - 2v_zv_x\iiint_V xz\,\rho\,dV\\
&= v_x^2I_x + v_y^2I_y + v_z^2I_z - 2v_xy_yI_{xy} - 2v_yv_zI_{yz} - 2v_zv_xI_{xz}.
\end{aligned}$$

28. We choose the sphere $(x-R)^2+y^2+z^2=R^2$ and the z-axis. Then



$$I = 4\int_0^{2R}\int_0^{\sqrt{R^2-(x-R)^2}}\int_0^{\sqrt{R^2-(x-R)^2-y^2}} \rho(x^2+y^2)\,dz\,dy\,dx$$
$$= 4\rho\int_0^{2R}\int_0^{\sqrt{R^2-(x-R)^2}} (x^2+y^2)\sqrt{R^2-(x-R)^2-y^2}\,dy\,dx$$

In the inner integral, we set $a^2 = R^2-(x-R)^2$, in which case

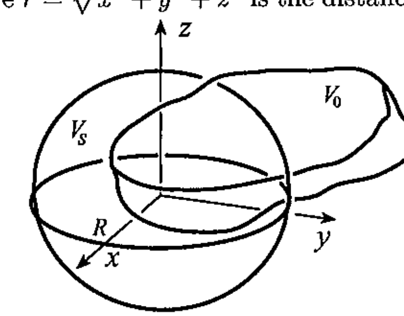
$$\int_0^{\sqrt{R^2-(x-R)^2}} (x^2+y^2)\sqrt{R^2-(x-R)^2-y^2}\,dy = \int_0^a (x^2+y^2)\sqrt{a^2-y^2}\,dy.$$

If we set $y = a\sin\theta$, then $dy = a\cos\theta\,d\theta$,

$$\int_0^{\sqrt{R^2-(x-R)^2}} (x^2+y^2)\sqrt{R^2-(x-R)^2-y^2}\,dy = \int_0^{\pi/2} (x^2+a^2\sin^2\theta)a\cos\theta\,a\cos\theta\,d\theta$$
$$= a^2\int_0^{\pi/2}\left[\frac{x^2(1+\cos 2\theta)}{2}+\frac{a^2}{4}\left(\frac{1-\cos 4\theta}{2}\right)\right]d\theta = a^2\left\{\frac{x^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+\frac{a^2}{8}\theta-\frac{a^2}{32}\sin 4\theta\right\}_0^{\pi/2}$$
$$= \frac{\pi a^2}{16}(a^2+4x^2) = \frac{\pi}{16}[R^2-(x-R)^2][R^2-(x-R)^2+4x^2].$$

Thus, $$I = \frac{\rho\pi}{4}\int_0^{2R}[2Rx-x^2)(2Rx+3x^2)]\,dx = \frac{\rho\pi}{4}\int_0^{2R}(4R^2x^2+4Rx^3-3x^4)\,dx$$
$$= \frac{\rho\pi}{4}\left\{\frac{4R^2x^3}{3}+Rx^4-\frac{3x^5}{5}\right\}_0^{2R} = \frac{28\rho\pi R^5}{15}.$$

29. $I_x+I_y+I_z = \iiint_V 2(x^2+y^2+z^2)\rho\,dV = 2\iiint_V r^2\rho\,dV$ where $r=\sqrt{x^2+y^2+z^2}$ is the distance from the origin to the element of mass ρdV at point (x,y,z). Let R be the radius of the sphere and $\overline{V}$ the region that it occupies. Let V_0 represent the region occupied by that part of the object outside the sphere and V_S represent the region occupied by that part of the sphere outside the object. The masses M_0 and M_S of these regions must be the same. Since $V = \overline{V}+V_0-V_S$, it follows that



$$\iiint_V r^2\rho\,dV = \iiint_{\overline{V}} r^2\rho\,dV + \iiint_{V_0} r^2\rho\,dV - \iiint_{V_S} r^2\rho\,dV.$$

In V_0, $r>R$, and in V_S, $r<R$, so that

$$\iiint_V r^2\rho\,dV \ge \iiint_{\overline{V}} r^2\rho\,dV + \iiint_{V_0} R^2\rho\,dV - \iiint_{V_S} R^2\rho\,dV$$
$$= \iiint_{\overline{V}} r^2\rho\,dV + R^2M_0 - R^2M_S = \iiint_{\overline{V}} r^2\rho\,dV.$$

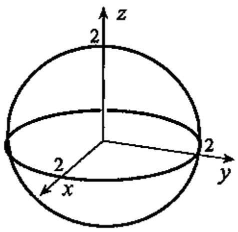
Thus,

$$I_x+I_y+I_z = 2\iiint_V r^2\rho\,dV \ge 2\iiint_{\overline{V}}(x^2+y^2+z^2)\rho\,dV,$$

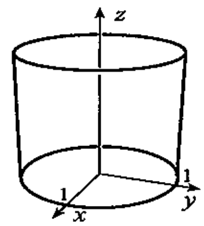
and the right side is the sum of the moments of inertia of the sphere about the axes.

EXERCISES 13.11

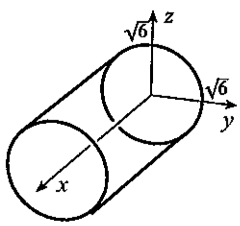
1. The equation is $r^2 + z^2 = 4$.
It is symmetric about the z-axis.



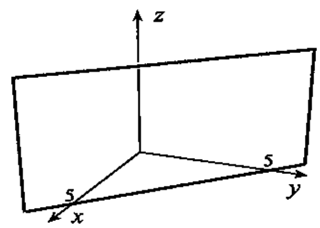
2. The equation is $r = 1$.
It is symmetric about the z-axis.



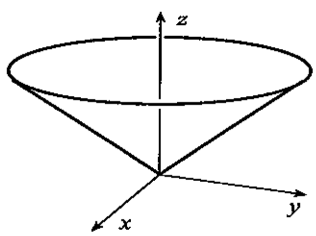
3. The equation is $r^2 \sin^2\theta + z^2 = 6$.
It is not symmetric about the z-axis.



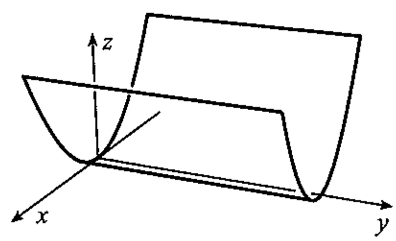
4. The equation is $r\cos\theta + r\sin\theta = 5$.
It is not symmetric about the z-axis.



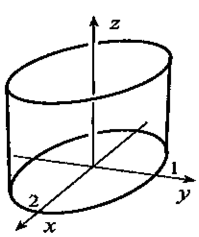
5. The equation is $z = 2r$.
It is symmetric about the z-axis.



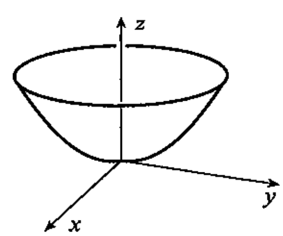
6. The equation is $z = r^2 \cos^2\theta$.
It is not symmetric about the z-axis.



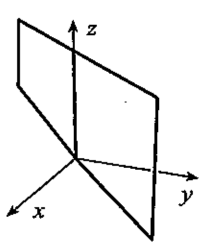
7. The equation is $r^2 = 4/(\cos^2\theta + 4\sin^2\theta)$.
It is not symmetric about the z-axis.



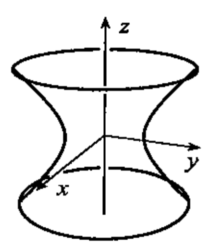
8. The equation is $4z = r^2$.
It is symmetric about the z-axis.



9. The equations are $\theta = \pi/4$ and $\theta = 5\pi/4$.
It is symmetric about the z-axis.

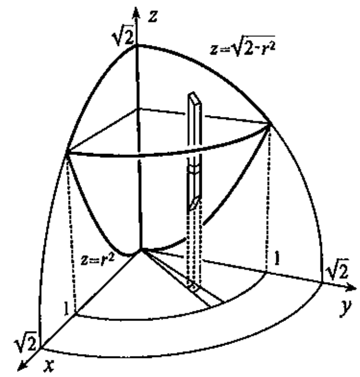


10. The equation is $r^2 = 1 + z^2$.
It is symmetric about the z-axis.



11. $$V = 4\int_0^{\pi/2}\int_0^2\int_0^r r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^2 \left\{rz\right\}_0^r dr\,d\theta = 4\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^2 d\theta = \frac{32}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{16\pi}{3}$$

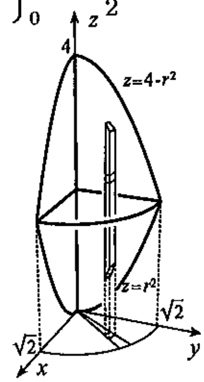
12. We quadruple the volume in the first octant.



$$V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^1 (r\sqrt{2-r^2} - r^3)\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\left\{-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^4}{4}\right\}_0^1 d\theta$$
$$= \frac{8\sqrt{2}-7}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{(8\sqrt{2}-7)\pi}{6}$$

13. $$V = 4\int_0^{\pi/2}\int_0^1\int_0^{r^2\sin\theta\cos\theta} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 \left\{rz\right\}_0^{r^2\sin\theta\cos\theta} dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 r^3\sin\theta\cos\theta\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\sin\theta\cos\theta\right\}_0^1 d\theta = \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \left\{\frac{1}{2}\sin^2\theta\right\}_0^{\pi/2} = \frac{1}{2}$$

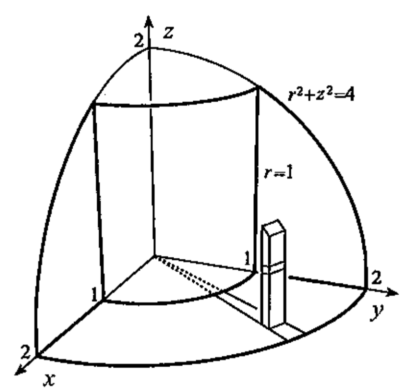
14. We quadruple the volume in the first octant.



$$V = 4\int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{r^2}^{4-r^2} r\,dz\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{\sqrt{2}} r(4-2r^2)\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\left\{2r^2 - \frac{r^4}{2}\right\}_0^{\sqrt{2}} d\theta = 8\left\{\theta\right\}_0^{\pi/2} = 4\pi$$

15. $$V = \int_{-\pi}^{\pi}\int_0^1\int_0^{2-r\cos\theta-r\sin\theta} r\,dz\,dr\,d\theta = \int_{-\pi}^{\pi}\int_0^1 r(2-r\cos\theta-r\sin\theta)\,dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\left\{r^2 - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right\}_0^1 d\theta = \frac{1}{3}\int_{-\pi}^{\pi}(3-\cos\theta-\sin\theta)\,d\theta$$
$$= \frac{1}{3}\left\{3\theta - \sin\theta + \cos\theta\right\}_{-\pi}^{\pi} = 2\pi$$

16. We multiply the first octant volume by eight.

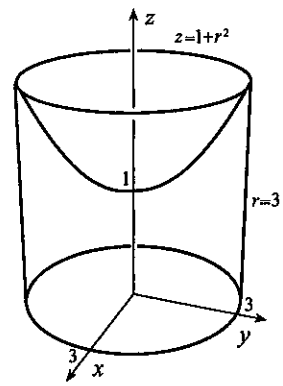


$$V = 8\int_0^{\pi/2}\int_1^2\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$
$$= 8\int_0^{\pi/2}\int_1^2 r\sqrt{4-r^2}\,dr\,d\theta$$
$$= 8\int_0^{\pi/2}\left\{-\frac{1}{3}(4-r^2)^{3/2}\right\}_1^2 d\theta$$
$$= 8\sqrt{3}\left\{\theta\right\}_0^{\pi/2} = 4\sqrt{3}\pi$$

17. By symmetry, $\overline{x} = \overline{y} = 0$ for the hemisphere bounded by $z = \sqrt{R^2 - x^2 - y^2}$ and $z = 0$. Since $M = (2/3)\pi R^3 \rho$, where ρ is the density, and

$$M\overline{z} = 4\int_0^{\pi/2}\int_0^R\int_0^{\sqrt{R^2-r^2}} z\rho r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^R \left\{\frac{rz^2}{2}\right\}_0^{\sqrt{R^2-r^2}} dr\,d\theta = 2\rho\int_0^{\pi/2}\int_0^R r(R^2-r^2)\,dr\,d\theta$$
$$= 2\rho\int_0^{\pi/2}\left\{\frac{R^2r^2}{2} - \frac{r^4}{4}\right\}_0^R d\theta = \frac{\rho R^4}{2}\left\{\theta\right\}_0^{\pi/2} = \frac{\rho\pi R^4}{4},$$

it follows that $\overline{z} = \dfrac{\rho\pi R^4}{4}\dfrac{3}{2\pi R^3\rho} = \dfrac{3R}{8}$.



18. The six triple iterated integrals are

$$\int_{-\pi}^{\pi}\int_0^3\int_0^{1+r^2} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta,$$
$$\int_0^3\int_{-\pi}^{\pi}\int_0^{1+r^2} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,d\theta\,dr,$$
$$\int_0^3\int_0^{1+r^2}\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta, z)\,r\,d\theta\,dz\,dr,$$
$$\int_0^1\int_0^3\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta, z)\,r\,d\theta\,dr\,dz + \int_1^{10}\int_{\sqrt{z-1}}^3\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta, z)\,r\,d\theta\,dr\,dz,$$
$$\int_{-\pi}^{\pi}\int_0^1\int_0^3 f(r\cos\theta, r\sin\theta, z)\,r\,dr\,dz\,d\theta + \int_{-\pi}^{\pi}\int_1^{10}\int_{\sqrt{z-1}}^3 f(r\cos\theta, r\sin\theta, z)\,r\,dr\,dz\,d\theta,$$
$$\int_0^1\int_{-\pi}^{\pi}\int_0^3 f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz + \int_1^{10}\int_{-\pi}^{\pi}\int_{\sqrt{z-1}}^3 f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

19. For the cylinder $x^2 + y^2 \le R^2$, $0 \le z \le h$,

(a) $$I = 4\int_0^{\pi/2}\int_0^R\int_0^h (x^2+y^2)\rho r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^R \left\{r^3 z\right\}_0^h dr\,d\theta$$
$$= 4\rho h\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^R d\theta = \rho h R^4\left\{\theta\right\}_0^{\pi/2} = \frac{\rho\pi R^4 h}{2}$$

(b) The moment of inertia about the x-axis is

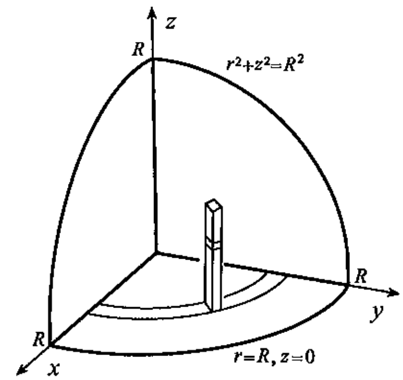
$$I = 4\int_0^{\pi/2}\int_0^R\int_0^h (y^2+z^2)\rho r\,dz\,dr\,d\theta = 4\rho\int_0^{\pi/2}\int_0^R \left\{r\left(zr^2\sin^2\theta + \frac{z^3}{3}\right)\right\}_0^h dr\,d\theta$$
$$= \frac{4\rho}{3}\int_0^{\pi/2}\int_0^R (3hr^3\sin^2\theta + h^3 r)\,dr\,d\theta = \frac{4\rho}{3}\int_0^{\pi/2}\left\{\frac{3hr^4}{4}\sin^2\theta + \frac{h^3r^2}{2}\right\}_0^R d\theta$$
$$= \frac{\rho}{3}\int_0^{\pi/2}(3hR^4\sin^2\theta + 2h^3R^2)\,d\theta = \frac{\rho}{3}\int_0^{\pi/2}\left[2h^3R^2 + 3hR^4\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$$
$$= \frac{\rho}{3}\left\{2h^3R^2\theta + 3hR^4\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\right\}_0^{\pi/2} = \frac{\rho\pi h R^2(4h^2+3R^2)}{12}.$$

20. We multiply the moment of inertia about the z-axis of that part in the first octant by eight.

$$I_z = 8\int_0^R \int_0^{\pi/2} \int_0^{\sqrt{R^2-r^2}} r^2 \rho\, r\, dz\, d\theta\, dr$$
$$= 8\rho \int_0^R \int_0^{\pi/2} r^3\sqrt{R^2-r^2}\, d\theta\, dr$$
$$= 4\pi\rho \int_0^R r^3\sqrt{R^2-r^2}\, dr$$

If we set $u = R^2 - r^2$, then $du = -2r\, dr$, and

$$I_z = 4\pi\rho \int_{R^2}^0 (R^2-u)\sqrt{u}\left(-\frac{du}{2}\right)$$
$$= 2\pi\rho\left\{\frac{2}{3}R^2u^{3/2} - \frac{2}{5}u^{5/2}\right\}_0^{R^2} = \frac{8\pi\rho R^5}{15}.$$



21. The limits define the first octant volume under the cone $z = \sqrt{x^2+y^2}$ and inside the cylinder $x^2+y^2 = 9$. The value of the triple integral is therefore

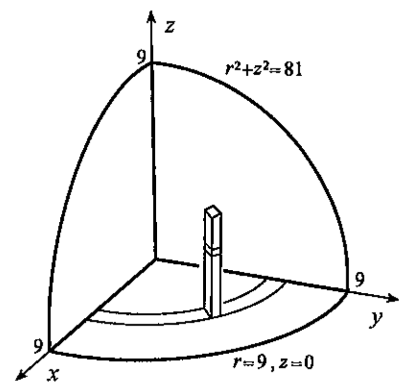
$$\int_0^{\pi/2}\int_0^3\int_0^r r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^3 r^2\, dr\, d\theta = \int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^3 d\theta = 9\Big\{\theta\Big\}_0^{\pi/2} = \frac{9\pi}{2}.$$

22. The limits define the first octant volume inside the sphere $x^2+y^2+z^2 = 81$. The value of the triple iterated integral is therefore given by

$$\int_0^9\int_0^{\pi/2}\int_0^{\sqrt{81-r^2}} \frac{1}{r} r\, dz\, d\theta\, dr$$
$$= \int_0^9\int_0^{\pi/2} \sqrt{81-r^2}\, d\theta\, dr = \frac{\pi}{2}\int_0^9 \sqrt{81-r^2}\, dr.$$

If we set $r = 9\sin\phi$, then $dr = 9\cos\phi\, d\phi$, and

$$\int_0^9\int_0^{\pi/2}\int_0^{\sqrt{81-r^2}} \frac{1}{r} r\, dz\, d\theta\, dr = \frac{\pi}{2}\int_0^{\pi/2} 9\cos\phi\, 9\cos\phi\, d\phi$$
$$= \frac{81\pi}{2}\int_0^{\pi/2}\left(\frac{1+\cos 2\phi}{2}\right) d\phi = \frac{81\pi}{4}\left\{\phi + \frac{\sin 2\phi}{2}\right\}_0^{\pi/2} = \frac{81\pi^2}{8}.$$

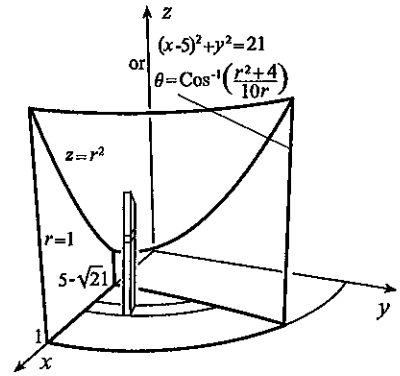


23. The limits define the first octant volume under $z = y + x^2$ and inside the cylinder $x^2+y^2 = 4y$. The value of the triple iterated integral is therefore given by

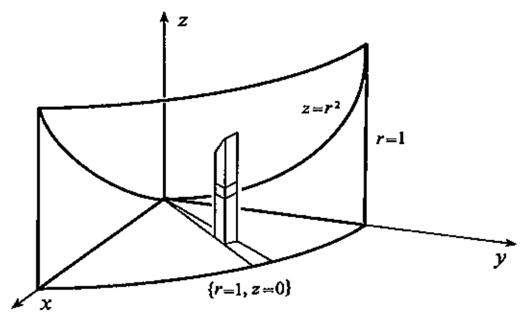
$$\int_0^{\pi/2}\int_0^{4\sin\theta}\int_0^{r\sin\theta + r^2\cos^2\theta} r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^{4\sin\theta} r(r\sin\theta + r^2\cos^2\theta)\, dr\, d\theta$$
$$= \int_0^{\pi/2}\left\{\frac{r^3}{3}\sin\theta + \frac{r^4}{4}\cos^2\theta\right\}_0^{4\sin\theta} d\theta = \frac{1}{12}\int_0^{\pi/2}[4\sin\theta(4\sin\theta)^3 + 3\cos^2\theta(4\sin\theta)^4]\, d\theta$$
$$= \frac{64}{3}\int_0^{\pi/2}(\sin^4\theta + 3\cos^2\theta\sin^4\theta)\, d\theta = \frac{64}{3}\int_0^{\pi/2}\left[\left(\frac{1-\cos 2\theta}{2}\right)^2 + 3\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)^2\right] d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}[2(1-2\cos 2\theta + \cos^2 2\theta) + 3(1-2\cos 2\theta+\cos^2 2\theta) + 3\cos 2\theta(1-2\cos 2\theta + \cos^2 2\theta)]\, d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\left[5 - 7\cos 2\theta - \left(\frac{1+\cos 4\theta}{2}\right) + 3\cos 2\theta(1-\sin^2 2\theta)\right] d\theta$$
$$= \frac{8}{3}\left\{\frac{9\theta}{2} - 2\sin 2\theta - \frac{1}{8}\sin 4\theta - \sin^3 2\theta\right\}_0^{\pi/2} = 6\pi$$

24. The limits define the volume under $z = x^2 + y^2$, above $z = 0$, and bounded on the sides by the cylinders $x^2+y^2 = 1$ and $(x-5)^2+y^2 = 21$, and the xz-plane. The value of the triple iterated integral is therefore given by

$$\begin{aligned}
&\int_{5-\sqrt{21}}^{1}\int_{0}^{\theta(r)}\int_{0}^{r^2} r\sin\theta\, r\, dz\, d\theta\, dr \\
&= \int_{5-\sqrt{21}}^{1}\int_{0}^{\theta(r)} r^4\sin\theta\, d\theta\, dr \\
&= \int_{5-\sqrt{21}}^{1}\left\{-r^4\cos\theta\right\}_{0}^{\theta(r)} dr \\
&= \int_{5-\sqrt{21}}^{1}\left(r^4 - \frac{r^5}{10} - \frac{2r^3}{5}\right) dr \\
&= \left\{\frac{r^5}{5} - \frac{r^6}{60} - \frac{r^4}{10}\right\}_{5-\sqrt{21}}^{1} = 0.084.
\end{aligned}$$



25. The limits determine the volume in the first octant bounded by the paraboloid $z = x^2 + y^2$ and the right circular cylinder $x^2 + y^2 = 1$. If we use a triple iterated integral with respect to z, r, and θ, then



$$\begin{aligned}
I &= \int_0^{\pi/2}\int_0^1\int_0^{r^2} r^2\sin^2\theta\; r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^1 r^5\sin^2\theta\, dr\, d\theta = \int_0^{\pi/2}\frac{1}{6}\sin^2\theta\, d\theta \\
&= \frac{1}{6}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right) d\theta = \frac{1}{12}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{\pi}{24}.
\end{aligned}$$

26. The moment of inertia of the upper leg about a line through its centre of mass G_U is

$$I_{G_U} = (0.137)(73)\left(\frac{0.07^2}{4} + \frac{0.45^2}{12}\right) = 0.181 \text{ kg}\cdot\text{m}^2.$$

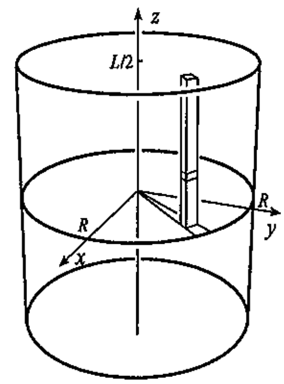
Similarly,

$$I_{G_L} = (0.06)(73)\left(\frac{0.05^2}{4} + \frac{0.5^2}{12}\right) = 0.094 \text{ kg}\cdot\text{m}^2.$$

Since $HG_U = 0.225$ and $HG_L = \sqrt{0.45^2 + 0.25^2 - 2(0.45)(0.25)\cos(\pi/3)} = 0.391$, the moment of inertia of the leg about the hip is

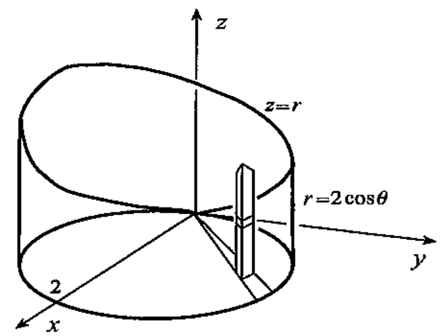
$$[0.181 + (0.137)(73)(0.225)^2] + [0.094 + (0.06)(73)(0.391)^2] = 1.45 \text{ kg}\cdot\text{m}^2.$$

27. The moment of inertia about the x-axis is eight times that in the first octant.



$$\begin{aligned}
I &= 8\int_0^{\pi/2}\int_0^R\int_0^{L/2}(y^2+z^2)\rho r\,dz\,dr\,d\theta \\
&= 8\rho\int_0^{\pi/2}\int_0^R\int_0^{L/2}(r^2\sin^2\theta+z^2)r\,dz\,dr\,d\theta \\
&= 8\rho\int_0^{\pi/2}\int_0^R\left\{r^3\sin^2\theta\,z+\frac{rz^3}{3}\right\}_0^{L/2}dr\,d\theta \\
&= \frac{\rho L}{3}\int_0^{\pi/2}\int_0^R(12r^3\sin^2\theta+L^2r)\,dr\,d\theta \\
&= \frac{\rho L}{3}\int_0^{\pi/2}\left\{3r^4\sin^2\theta+\frac{L^2r^2}{2}\right\}_0^R d\theta \\
&= \frac{\rho LR^2}{6}\int_0^{\pi/2}(6R^2\sin^2\theta+L^2)\,d\theta=\frac{\rho LR^2}{6}\int_0^{\pi/2}[3R^2(1-\cos 2\theta)+L^2]\,d\theta \\
&= \frac{\rho LR^2}{6}\left\{3R^2\left(\theta-\frac{\sin 2\theta}{2}\right)+L^2\theta\right\}_0^{\pi/2}=\frac{\rho\pi LR^2(3R^2+L^2)}{12}=m\left(\frac{R^2}{4}+\frac{L^2}{12}\right).
\end{aligned}$$

28.
$$\begin{aligned}
M &= 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^r \rho r\,dz\,dr\,d\theta = 2\rho\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta \\
&= 2\rho\int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^{2\cos\theta}d\theta=\frac{16\rho}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta \\
&= \frac{16\rho}{3}\int_0^{\pi/2}\cos\theta(1-\sin^2\theta)\,d\theta \\
&= \frac{16\rho}{3}\left\{\sin\theta-\frac{1}{3}\sin^3\theta\right\}_0^{\pi/2}=\frac{32\rho}{9}
\end{aligned}$$



By symmetry, $\overline{y}=0$. Since

$$\begin{aligned}
M\overline{x} &= 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^r r\cos\theta\rho r\,dz\,dr\,d\theta = 2\rho\int_0^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\,dr\,d\theta \\
&= 2\rho\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos\theta\right\}_0^{2\cos\theta}d\theta = 8\rho\int_0^{\pi/2}\cos^5\theta\,d\theta = 8\rho\int_0^{\pi/2}\cos\theta(1-\sin^2\theta)^2\,d\theta \\
&= 8\rho\int_0^{\pi/2}\cos\theta(1-2\sin^2\theta+\sin^4\theta)\,d\theta = 8\rho\left\{\sin\theta-\frac{2}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta\right\}_0^{\pi/2}=\frac{64\rho}{15},
\end{aligned}$$

it follows that $\overline{x}=\dfrac{64\rho}{15}\dfrac{9}{32\rho}=\dfrac{6}{5}$. Since

$$\begin{aligned}
M\overline{z} &= 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^r z\rho r\,dz\,dr\,d\theta = 2\rho\int_0^{\pi/2}\int_0^{2\cos\theta}\left\{\frac{rz^2}{2}\right\}_0^r dr\,d\theta \\
&= \rho\int_0^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta = \rho\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^{2\cos\theta}d\theta = 4\rho\int_0^{\pi/2}\cos^4\theta\,d\theta \\
&= 4\rho\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \rho\int_0^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta \\
&= \rho\left\{\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2}=\frac{3\pi\rho}{4},
\end{aligned}$$

we obtain $\overline{z}=\dfrac{3\pi\rho}{4}\dfrac{9}{32\rho}=\dfrac{27\pi}{128}$.

29. First we find the volume interior to the cylinder $x^2+y^2=a^2$ and the sphere $x^2+y^2+z^2=b^2$,

$$V_1 = 8\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{b^2-r^2}} r\,dz\,dr\,d\theta = 8\int_0^{\pi/2}\int_0^a r\sqrt{b^2-r^2}\,dr\,d\theta$$

$$= 8\int_0^{\pi/2}\left\{-\frac{1}{3}(b^2-r^2)^{3/2}\right\}_0^a d\theta = -\frac{8}{3}[(b^2-a^2)^{3/2}-b^3]\left\{\theta\right\}_0^{\pi/2} = \frac{4\pi[b^3-(b^2-a^2)^{3/2}]}{3}.$$

We now find the volume common to both cylinders $x^2+y^2=a^2$ and $y^2+z^2=a^2$,

$$V_2 = 8\int_0^a\int_0^{\sqrt{a^2-y^2}}\int_0^{\sqrt{a^2-y^2}} dz\,dx\,dy = 8\int_0^a\int_0^{\sqrt{a^2-y^2}}\sqrt{a^2-y^2}\,dx\,dy$$

$$= 8\int_0^a\left\{x\sqrt{a^2-y^2}\right\}_0^{\sqrt{a^2-y^2}} dy = 8\int_0^a (a^2-y^2)\,dy = 8\left\{a^2y-\frac{y^3}{3}\right\}_0^a = \frac{16a^3}{3}.$$

It now follows that the volume for the casting is

$$V = (\text{volume of sphere}) - 2V_1 + V_2 = \frac{4}{3}\pi b^3 - \frac{8\pi}{3}[b^3-(b^2-a^2)^{3/2}] + \frac{16a^3}{3} = \frac{16a^3}{3} + \frac{4\pi}{3}[2(b^2-a^2)^{3/2}-b^3].$$

30. The volume bounded by the planes and cylinder is

$$V = \int_0^{2\pi}\int_0^R\int_{my}^{my+h} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^R r(h)\,dr\,d\theta = h(\pi R^2).$$

31. We multiply the first octant volume by eight.

$$V = 8\int_0^{\pi/2}\int_0^{1/\sqrt{3}}\int_{2z}^{\sqrt{1+z^2}} r\,dr\,dz\,d\theta = 8\int_0^{\pi/2}\int_0^{1/\sqrt{3}}\left\{\frac{r^2}{2}\right\}_{2z}^{\sqrt{1+z^2}} dz\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{1/\sqrt{3}}(1-3z^2)\,dz\,d\theta = 4\int_0^{\pi/2}\left\{z-z^3\right\}_0^{1/\sqrt{3}} d\theta = \frac{8\sqrt{3}}{9}\left\{\theta\right\}_0^{\pi/2} = \frac{4\sqrt{3}\pi}{9}$$

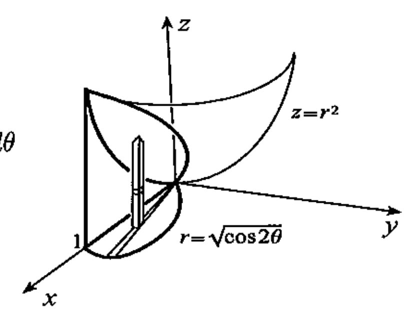
32. We quadruple the first octant volume.

$$\text{V} = 4\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}}\int_0^{r^2} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r^3\,dr\,d\theta = 4\int_0^{\pi/4}\left\{\frac{r^4}{4}\right\}_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= \int_0^{\pi/4}\cos^2 2\theta\,d\theta = \int_0^{\pi/4}\left(\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left\{\theta+\frac{1}{4}\sin 4\theta\right\}_0^{\pi/4} = \frac{\pi}{8}$$



33. We quadruple the first octant volume.

$$V = 4\int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{\sqrt{4-r^2}}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta + 4\int_0^{\pi/2}\int_{\sqrt{2}}^{2\sqrt{2}}\int_r^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{\sqrt{2}}(r\sqrt{16-r^2}-r\sqrt{4-r^2})\,dr\,d\theta + 4\int_0^{\pi/2}\int_{\sqrt{2}}^{2\sqrt{2}}(r\sqrt{16-r^2}-r^2)\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left\{-\frac{1}{3}(16-r^2)^{3/2}+\frac{1}{3}(4-r^2)^{3/2}\right\}_0^{\sqrt{2}} d\theta + 4\int_0^{\pi/2}\left\{-\frac{1}{3}(16-r^2)^{3/2}-\frac{r^3}{3}\right\}_{\sqrt{2}}^{2\sqrt{2}} d\theta$$

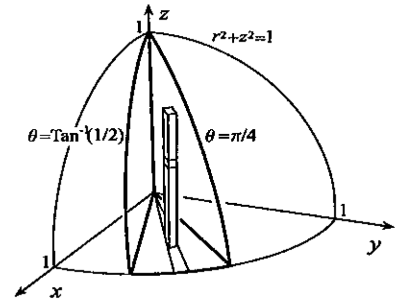
$$= \frac{112}{3}(2-\sqrt{2})\left\{\theta\right\}_0^{\pi/2} = \frac{56(2-\sqrt{2})\pi}{3}$$

34. $V = \int_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} \int_0^1 \int_0^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta$

$= \int_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} \int_0^1 r\sqrt{1-r^2}\, dr\, d\theta$

$= \int_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} \left\{-\frac{1}{3}(1-r^2)^{3/2}\right\}_0^1 d\theta$

$= \frac{1}{3}\left\{\theta\right\}_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} = \frac{1}{3}\left[\frac{\pi}{4} - \mathrm{Tan}^{-1}(1/2)\right]$



35. We multiply the first octant volume by eight.

$$V = 8\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{8-2r^2}} r\, dz\, dr\, d\theta = 8\int_0^{\pi/2}\int_0^1 r\sqrt{8-2r^2}\, dr\, d\theta$$

$$= 8\int_0^{\pi/2}\left\{-\frac{1}{6}(8-2r^2)^{3/2}\right\}_0^1 d\theta = \frac{4}{3}(16\sqrt{2}-6\sqrt{6})\left\{\theta\right\}_0^{\pi/2} = \frac{4(8\sqrt{2}-3\sqrt{6})\pi}{3}$$

36. We quadruple the first octant volume.

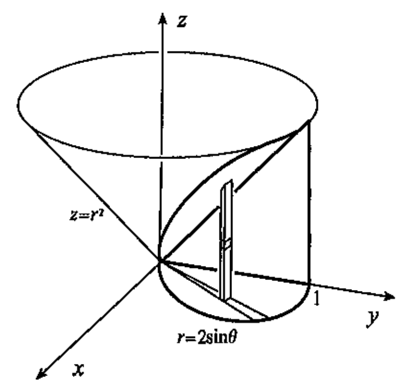
$V = 4\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{r^2} r\, dz\, dr\, d\theta$

$= 4\int_0^{\pi/2}\int_0^{2\sin\theta} r^3\, dr\, d\theta = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^{2\sin\theta} d\theta$

$= \int_0^{\pi/2} 16\sin^4\theta\, d\theta = 16\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$

$= 4\int_0^{\pi/2}\left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$

$= 4\left\{\frac{3\theta}{2} - \sin 2\theta + \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = 3\pi$



37. $V = \frac{4}{3}\pi a^3 - 4\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta = \frac{4}{3}\pi a^3 - 4\int_0^{\pi/2}\int_0^{a\sin\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$

$= \frac{4}{3}\pi a^3 - 4\int_0^{\pi/2}\left\{-\frac{1}{3}(a^2-r^2)^{3/2}\right\}_0^{a\sin\theta} d\theta = \frac{4}{3}\pi a^3 + \frac{4}{3}\int_0^{\pi/2}(a^3\cos^3\theta - a^3)\, d\theta$

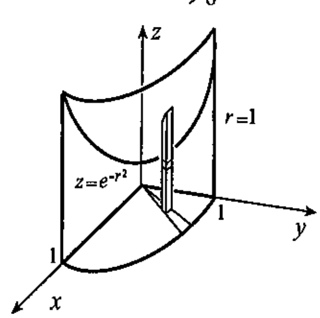
$= \frac{4}{3}\pi a^3 + \frac{4}{3}a^3\int_0^{\pi/2}[\cos\theta(1-\sin^2\theta) - 1]\, d\theta = \frac{4}{3}\pi a^3 + \frac{4}{3}a^3\left\{\sin\theta - \frac{1}{3}\sin^3\theta - \theta\right\}_0^{\pi/2} = \frac{2a^3(3\pi+4)}{9}$

38. We quadruple the first octant volume.

$V = 4\int_0^{\pi/2}\int_0^1\int_0^{e^{-r^2}} r\, dz\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 re^{-r^2}\, dr\, d\theta$

$= 4\int_0^{\pi/2}\left\{-\frac{1}{2}e^{-r^2}\right\}_0^1 d\theta$

$= -2(e^{-1}-1)\left\{\theta\right\}_0^{\pi/2} = \pi(1-1/e)$

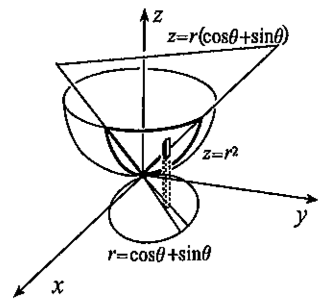


39. We multiply the first octant volume by eight.

$$V = 8\int_0^2\int_0^{\pi/2}\int_{\sqrt{1+z^2}}^{\sqrt{9-z^2}} r\, dr\, d\theta\, dz = 8\int_0^2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_{\sqrt{1+z^2}}^{\sqrt{9-z^2}} d\theta\, dz$$

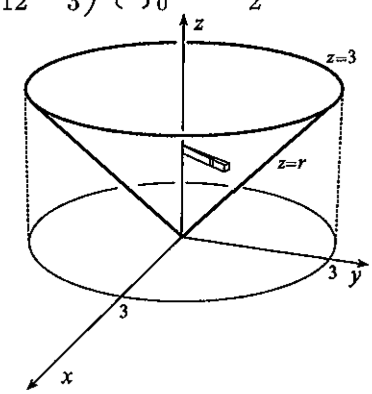
$$= 4\int_0^2\int_0^{\pi/2}(9-z^2-1-z^2)\, d\theta\, dz = 4\int_0^2\left\{(8-2z^2)\theta\right\}_0^{\pi/2} dz = 2\pi\left\{8z - \frac{2z^3}{3}\right\}_0^2 = \frac{64\pi}{3}$$

40. $$V = \int_{-\pi/4}^{3\pi/4}\int_0^{\cos\theta+\sin\theta}\int_{r^2}^{r\cos\theta+r\sin\theta} r\,dz\,dr\,d\theta$$
$$= \int_{-\pi/4}^{3\pi/4}\int_0^{\cos\theta+\sin\theta}(r^2\cos\theta + r^2\sin\theta - r^3)\,dr\,d\theta$$
$$= \int_{-\pi/4}^{3\pi/4}\left\{\frac{r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta - \frac{r^4}{4}\right\}_0^{\cos\theta+\sin\theta} d\theta$$
$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}[4(\cos\theta+\sin\theta)^4 - 3(\cos\theta+\sin\theta)^4]\,d\theta$$
$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}(\cos^4\theta + 4\cos^3\theta\sin\theta + 6\cos^2\theta\sin^2\theta + 4\cos\theta\sin^3\theta + \sin^4\theta)\,d\theta$$
$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}[(\cos^2\theta+\sin^2\theta)^2 + 4(\cos^3\theta\sin\theta + \cos^2\theta\sin^2\theta + \cos\theta\sin^3\theta)]\,d\theta$$
$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}[1 + 4(\cos^3\theta\sin\theta + \cos\theta\sin^3\theta) + (\sin 2\theta)^2]\,d\theta$$
$$= \frac{1}{12}\int_{-\pi/4}^{3\pi/4}\left[1 + 4(\cos^3\theta\sin\theta + \cos\theta\sin^3\theta) + \frac{1-\cos 4\theta}{2}\right]d\theta$$
$$= \frac{1}{12}\left\{\frac{3\theta}{2} - \cos^4\theta + \sin^4\theta - \frac{1}{8}\sin 4\theta\right\}_{-\pi/4}^{3\pi/4} = \frac{\pi}{8}$$



41. $$V = \frac{4}{3}\pi(2)^3 - 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_{r^2/3}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \frac{32\pi}{3} - 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\left(r\sqrt{4-r^2} - \frac{r^3}{3}\right)dr\,d\theta$$
$$= \frac{32\pi}{3} - 4\int_0^{\pi/2}\left\{-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^4}{12}\right\}_0^{\sqrt{3}} d\theta = \frac{32\pi}{3} + 4\left(\frac{1}{3} + \frac{9}{12} - \frac{8}{3}\right)\left\{\theta\right\}_0^{\pi/2} = \frac{15\pi}{2}$$

42. $$\iiint_V \sqrt{x^2+y^2+z^2}\,dV = 4\int_0^{\pi/2}\int_0^3\int_0^z \sqrt{r^2+z^2}\,r\,dr\,dz\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^3\left\{\frac{1}{3}(r^2+z^2)^{3/2}\right\}_0^z dz\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\int_0^3(2\sqrt{2}z^3 - z^3)\,dz\,d\theta$$
$$= \frac{4(2\sqrt{2}-1)}{3}\int_0^{\pi/2}\left\{\frac{z^4}{4}\right\}_0^3 d\theta$$
$$= 27(2\sqrt{2}-1)\left\{\theta\right\}_0^{\pi/2} = \frac{27\pi(2\sqrt{2}-1)}{2}$$

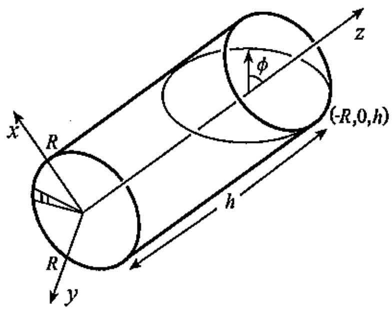


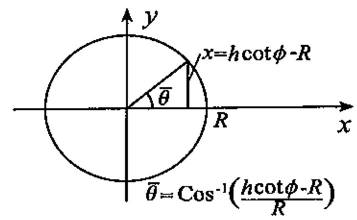
43. We quadruple the integral over the first octant volume.

$$\iiint_V |yz|\,dV = 4\int_0^{\pi/2}\int_0^{\sqrt{3/2}}\int_{\sqrt{1+r^2}}^{\sqrt{4-r^2}} r\sin\theta\,z\,r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{\sqrt{3/2}}\left\{\frac{r^2\sin\theta z^2}{2}\right\}_{\sqrt{1+r^2}}^{\sqrt{4-r^2}} dr\,d\theta$$
$$= 2\int_0^{\pi/2}\int_0^{\sqrt{3/2}}(3r^2 - 2r^4)\sin\theta\,dr\,d\theta = 2\int_0^{\pi/2}\left\{\left(r^3 - \frac{2r^5}{5}\right)\sin\theta\right\}_0^{\sqrt{3/2}} d\theta$$
$$= \frac{3\sqrt{6}}{5}\left\{-\cos\theta\right\}_0^{\pi/2} = \frac{3\sqrt{6}}{5}$$

44. If we choose a coordinate system as shown, the equation of the surface of the water is

$$0 = (\sin\phi, 0, \cos\phi) \cdot (x + R, 0, z - h) \quad \Longrightarrow \quad x\sin\phi + z\cos\phi = h\cos\phi - R\sin\phi.$$





CASE 1: Water touches only sides of the tumbler $(0 \le \phi \le \mathrm{Tan}^{-1}[h/(2R)])$.
In this case, the volume of water is

$$\begin{aligned}
V &= 2\int_0^{\pi}\int_0^{R}\int_0^{h-R\tan\phi-r\tan\phi\cos\theta} r\,dz\,dr\,d\theta = 2\int_0^{\pi}\int_0^{R} r(h - R\tan\phi - r\tan\phi\cos\theta)\,dr\,d\theta \\
&= 2\int_0^{\pi}\left(\frac{R^2h}{2} - \frac{R^3}{2}\tan\phi - \frac{R^3}{3}\tan\phi\cos\theta\right)d\theta = \pi R^2(h - R\tan\phi).
\end{aligned}$$

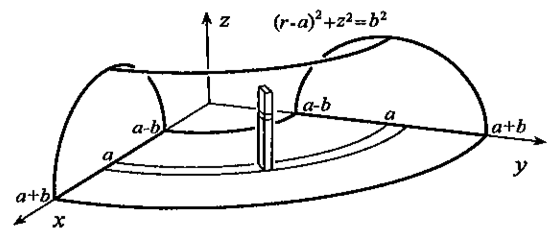
CASE 2: Water touches bottom of tumbler $(\mathrm{Tan}^{-1}[h/(2R)] < \phi < \pi/2)$
In this case, the volume of water is

$$\begin{aligned}
V &= 2\int_0^{\overline{\theta}}\int_0^{(h\cot\phi-R)\sec\theta}\int_0^{h-R\tan\phi-r\tan\phi\cos\theta} r\,dz\,dr\,d\theta \\
&\quad + 2\int_{\overline{\theta}}^{\pi}\int_0^{R}\int_0^{h-R\tan\phi-r\tan\phi\cos\theta} r\,dz\,dr\,d\theta \\
&= 2\int_0^{\overline{\theta}}\int_0^{(h\cot\phi-R)\sec\theta} r(h - R\tan\phi - r\tan\phi\cos\theta)\,dr\,d\theta \\
&\quad + 2\int_{\overline{\theta}}^{\pi}\int_0^{R} r(h - R\tan\phi - r\tan\phi\cos\theta)\,dr\,d\theta \\
&= 2\int_0^{\overline{\theta}}\left\{\frac{r^2}{2}(h - R\tan\phi) - \frac{r^3}{3}\tan\phi\cos\theta\right\}\Bigg|_0^{(h\cot\phi-R)\sec\theta} d\theta \\
&\quad + 2\int_{\overline{\theta}}^{\pi}\left\{\frac{r^2}{2}(h - R\tan\phi) - \frac{r^3}{3}\tan\phi\cos\theta\right\}\Bigg|_0^{R} d\theta \\
&= \int_0^{\overline{\theta}}\left[(h - R\tan\phi)(h\cot\phi - R)^2\sec^2\theta - \frac{2}{3}\tan\phi(h\cot\phi - R)^3\sec^2\theta\right]d\theta \\
&\quad + \int_{\overline{\theta}}^{\pi}\left[R^2(h - R\tan\phi) - \frac{2R^3}{3}\tan\phi\cos\theta\right]d\theta \\
&= \left[(h - R\tan\phi)(h\cot\phi - R)^2 - \frac{2}{3}\tan\phi(h\cot\phi - R)^3\right]\tan\overline{\theta} \\
&\quad + R^2(h - R\tan\phi)(\pi - \overline{\theta}) + \frac{2R^3}{3}\tan\phi\sin\overline{\theta} \\
&= \frac{1}{3}\tan\phi(h\cot\phi - R)^3\frac{\sqrt{2Rh\cot\phi - h^2\cot^2\phi}}{h\cot\phi - R} \\
&\quad + R^2\tan\phi(h\cot\phi - R)\left[\pi - \mathrm{Cos}^{-1}\left(\frac{h\cot\phi - R}{R}\right)\right]
\end{aligned}$$

$$+\frac{2R^3}{3}\tan\phi\frac{\sqrt{2Rh\cot\phi - h^2\cot^2\phi}}{R}$$
$$= \frac{1}{3}\tan\phi\sqrt{2Rh\cot\phi - h^2\cot^2\phi}[(h\cot\phi - R)^2 + 2R^2]$$
$$+ R^2\tan\phi(h\cot\phi - R)\left[\pi - \mathrm{Cos}^{-1}\left(\frac{h\cot\phi - R}{R}\right)\right].$$

45. We multiply the first octant volume by eight.

$$V = 8\int_{a-b}^{a+b}\int_0^{\pi/2}\int_0^{\sqrt{b^2-(r-a)^2}} r\,dz\,d\theta\,dr$$
$$= 8\int_{a-b}^{a+b}\int_0^{\pi/2} r\sqrt{b^2-(r-a)^2}\,d\theta\,dr$$
$$= 4\pi\int_{a-b}^{a+b} r\sqrt{b^2-(r-a)^2}\,dr$$



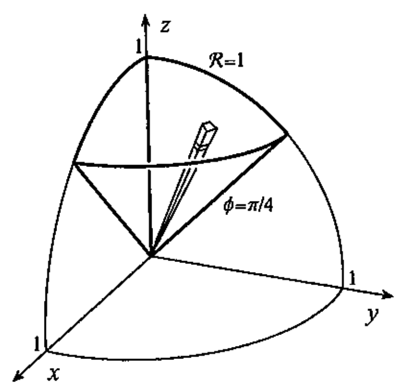
If we set $r - a = b\sin\phi$, then $dr = b\cos\phi\,d\phi$, and

$$V = 4\pi\int_{-\pi/2}^{\pi/2}(a + b\sin\phi)b\cos\phi\, b\cos\phi\,d\phi = 4\pi b^2\int_{-\pi/2}^{\pi/2}\left[a\left(\frac{1+\cos 2\phi}{2}\right) + b\cos^2\phi\sin\phi\right]d\phi$$
$$= 4\pi b^2\left\{\frac{a}{2}\left(\phi + \frac{\sin 2\phi}{2}\right) - \frac{b}{3}\cos^3\phi\right\}_{-\pi/2}^{\pi/2} = 2\pi^2 ab^2.$$

EXERCISES 13.12

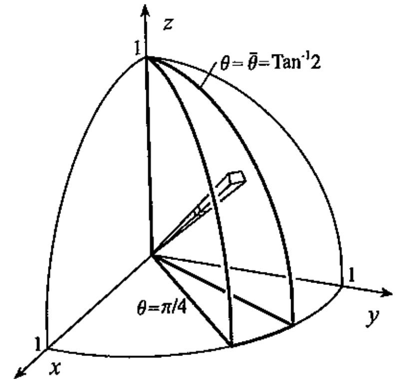
1. The equation is $\Re = 2$. See figure for Exercise 13.11-1.
2. The equation is $\Re\sin\phi = 1$. See figure for Exercise 13.11-2.
3. The equation is $\phi = \mathrm{Tan}^{-1}3$. The figure has the same shape as that in Exercise 13.11-5.
4. The equation is $\Re = 4\csc\phi\cot\phi$. See figure for Exercise 13.11-8.
5. The equations are $\theta = \pi/4$ and $\theta = 5\pi/4$. See figure for Exercise 13.11-9.
6. The equation is $\Re^2 = -\sec 2\phi$. See figure for Exercise 13.11-10.
7. The equation is $\phi = \pi - \mathrm{Tan}^{-1}(1/2)$. Turn the figure in Exercise 13.11-5 upside down.
8. We quadruple the volume in the first octant.

$$V = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^1 \Re^2\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\left\{-\cos\phi\right\}_0^{\pi/4}d\theta$$
$$= \frac{2(2-\sqrt{2})}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{(2-\sqrt{2})\pi}{3}$$



9. $$V = 4\int_0^{\pi/2}\int_0^{\pi/3}\int_{\sec\phi}^2 \Re^2\sin\phi\,d\Re\,d\phi\,d\theta = 4\int_0^{\pi/2}\int_0^{\pi/3}\left\{\frac{\Re^3}{3}\sin\phi\right\}_{\sec\phi}^2 d\phi\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/3}(8\sin\phi - \sec^3\phi\sin\phi)d\phi\,d\theta = \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/3}(8\sin\phi - \tan\phi\sec^2\phi)d\phi\,d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\left\{-8\cos\phi - \frac{1}{2}\tan^2\phi\right\}_0^{\pi/3}d\theta = \frac{10}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{5\pi}{3}$$

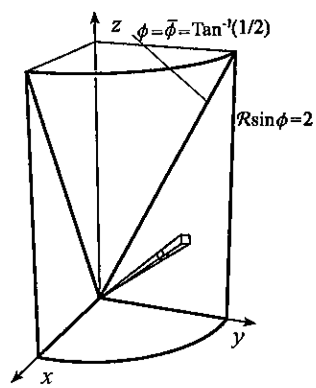
10. $$V = \int_{\pi/4}^{\overline{\theta}}\int_0^{\pi/2}\int_0^1 \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta$$
$$= \frac{1}{3}\int_{\pi/4}^{\overline{\theta}}\int_0^{\pi/2} \sin\phi \, d\phi \, d\theta$$
$$= \frac{1}{3}\int_{\pi/4}^{\overline{\theta}} \left\{-\cos\phi\right\}_0^{\pi/2} d\theta$$
$$= \frac{1}{3}\left\{\theta\right\}_{\pi/4}^{\overline{\theta}} = \frac{1}{3}(\mathrm{Tan}^{-1}2 - \pi/4)$$



11. $$V = 8\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_{\csc\phi}^{\sqrt{2}} \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta = 8\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\left\{\frac{\Re^3}{3}\sin\phi\right\}_{\csc\phi}^{\sqrt{2}} d\phi \, d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}(2\sqrt{2}\sin\phi - \csc^2\phi)\, d\phi \, d\theta = \frac{8}{3}\int_0^{\pi/2}\left\{-2\sqrt{2}\cos\phi + \cot\phi\right\}_{\pi/4}^{\pi/2} d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}(2-1)\, d\theta = \frac{8}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{4\pi}{3}$$

12. We quadruple the volume in the first octant.

$$V = 4\int_0^{\pi/2}\int_{\overline{\phi}}^{\pi/2}\int_0^{2\csc\phi} \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta$$
$$= 4\int_0^{\pi/2}\int_{\overline{\phi}}^{\pi/2}\left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{2\csc\phi} d\phi \, d\theta$$
$$= \frac{32}{3}\int_0^{\pi/2}\int_{\overline{\phi}}^{\pi/2}\csc^2\phi \, d\phi \, d\theta$$
$$= \frac{32}{3}\int_0^{\pi/2}\left\{-\cot\phi\right\}_{\overline{\phi}}^{\pi/2} d\theta$$
$$= \frac{64}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{32\pi}{3}$$

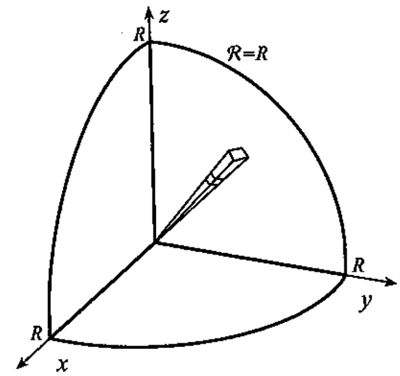


13. For the hemisphere bounded by $z = \sqrt{R^2 - x^2 - y^2}$ and $z = 0$, $\overline{x} = \overline{y} = 0$. Since $M = (2/3)\pi\rho R^3$, and

$$M\overline{z} = 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R (\Re\cos\phi)\rho\Re^2 \sin\phi \, d\Re \, d\phi \, d\theta = 4\rho\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^4}{4}\cos\phi\sin\phi\right\}_0^R d\phi \, d\theta$$
$$= \rho R^4\int_0^{\pi/2}\left\{\frac{1}{2}\sin^2\phi\right\}_0^{\pi/2} d\theta = \frac{\rho R^4}{2}\left\{\theta\right\}_0^{\pi/2} = \frac{\pi\rho R^4}{4},$$

it follows that $\overline{z} = \dfrac{\pi\rho R^4}{4}\dfrac{3}{2\pi\rho R^3} = \dfrac{3R}{8}$.

14. We multiply the moment of inertia of the first octant portion of the sphere about the z-axis by eight.



$$\begin{aligned}
I_z &= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R (\Re^2\sin^2\phi)\rho\Re^2\sin\phi\,d\Re\,d\phi\,d\theta \\
&= 8\rho\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^5}{5}\sin^3\phi\right\}_0^R d\phi\,d\theta \\
&= \frac{8\rho R^5}{5}\int_0^{\pi/2}\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi\,d\theta \\
&= \frac{8\rho R^5}{5}\int_0^{\pi/2}\left\{-\cos\phi+\frac{\cos^3\phi}{3}\right\}_0^{\pi/2} d\theta \\
&= \frac{16\rho R^5}{15}\left\{\theta\right\}_0^{\pi/2} = \frac{8\pi\rho R^5}{15}.
\end{aligned}$$

15.
$$\begin{aligned}
\text{Moment } &= \iiint_V x\rho\,dV = \int_0^{\pi/3}\int_0^{\pi/2}\int_2^3 (\Re\sin\phi\cos\theta)\Re^2\sin\phi\,d\Re\,d\phi\,d\theta \\
&= \int_0^{\pi/3}\int_0^{\pi/2}\left\{\frac{\Re^4}{4}\sin^2\phi\cos\theta\right\}_2^3 d\phi\,d\theta = \frac{65}{4}\int_0^{\pi/3}\int_0^{\pi/2}\cos\theta\left(\frac{1-\cos 2\phi}{2}\right)d\phi\,d\theta \\
&= \frac{65}{8}\int_0^{\pi/3}\left\{\cos\theta\left(\phi-\frac{1}{2}\sin 2\phi\right)\right\}_0^{\pi/2} d\theta = \frac{65\pi}{16}\int_0^{\pi/3}\cos\theta\,d\theta = \frac{65\pi}{16}\left\{\sin\theta\right\}_0^{\pi/3} = \frac{65\sqrt{3}\pi}{32}
\end{aligned}$$

16. Using the figure in Exercise 14,

$$\begin{aligned}
Q &= \int_0^{2\pi}\int_0^{\pi}\int_0^R k\Re\,\Re^2\sin\phi\,d\Re\,d\phi\,d\theta = \frac{kR^4}{4}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta \\
&= \frac{kR^4}{4}\int_0^{2\pi}\left\{-\cos\phi\right\}_0^{\pi} d\theta = \frac{kR^4}{2}\left\{\theta\right\}_0^{2\pi} = k\pi R^4 \text{ C.}
\end{aligned}$$

17. In each of the following integrals f stands for $f(\Re\sin\phi\cos\theta, \Re\sin\phi\sin\theta, \Re\cos\phi)$.

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} f\Re^2\sin\phi\,d\Re\,d\phi\,d\theta + \int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi} f\Re^2\sin\phi\,d\Re\,d\phi\,d\theta,$$

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} f\Re^2\sin\phi\,d\Re\,d\theta\,d\phi + \int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\csc\phi} f\Re^2\sin\phi\,d\Re\,d\theta\,d\phi,$$

$$\int_0^{\pi/4}\int_0^{\sqrt{2}}\int_0^{\pi/2} f\Re^2\sin\phi\,d\theta\,d\Re\,d\phi + \int_{\pi/4}^{\pi/2}\int_0^{\csc\phi}\int_0^{\pi/2} f\Re^2\sin\phi\,d\theta\,d\Re\,d\phi,$$

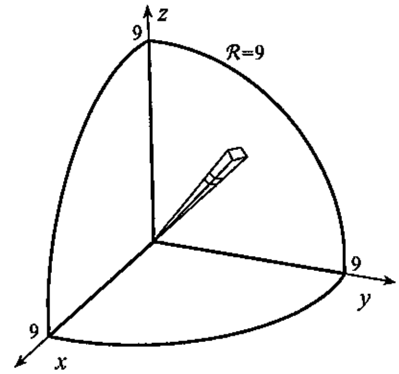
$$\int_0^{1}\int_0^{\pi/2}\int_0^{\pi/2} f\Re^2\sin\phi\,d\theta\,d\phi\,d\Re + \int_1^{\sqrt{2}}\int_0^{\mathrm{Csc}^{-1}\Re}\int_0^{\pi/2} f\Re^2\sin\phi\,d\theta\,d\phi\,d\Re,$$

$$\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{\mathrm{Csc}^{-1}\Re} f\Re^2\sin\phi\,d\phi\,d\Re\,d\theta + \int_0^{\pi/2}\int_0^{1}\int_0^{\pi/2} f\Re^2\sin\phi\,d\phi\,d\Re\,d\theta,$$

$$\int_1^{\sqrt{2}}\int_0^{\pi/2}\int_0^{\mathrm{Csc}^{-1}\Re} f\Re^2\sin\phi\,d\phi\,d\theta\,d\Re + \int_0^{1}\int_0^{\pi/2}\int_0^{\pi/2} f\Re^2\sin\phi\,d\phi\,d\theta\,d\Re.$$

18. The limits define the first octant volume inside the sphere $x^2 + y^2 + z^2 = 81$. The value of the triple iterated integral is therefore given by

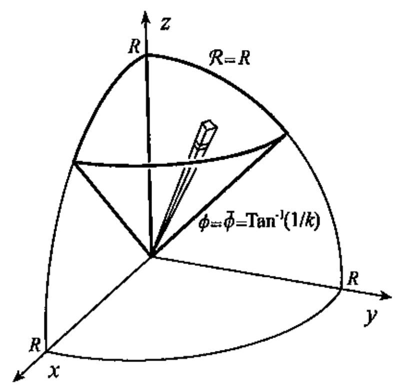
$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^9 \frac{1}{\Re^2}\Re^2 \sin\phi \, d\Re \, d\phi \, d\theta$$
$$= 9\int_0^{\pi/2}\int_0^{\pi/2} \sin\phi \, d\phi \, d\theta$$
$$= 9\int_0^{\pi/2} \Big\{-\cos\phi\Big\}_0^{\pi/2} d\theta$$
$$= 9\Big\{\theta\Big\}_0^{\pi/2} = \frac{9\pi}{2}$$



19. The limits define the first octant volume under the sphere $x^2 + y^2 + z^2 = 2$ and above the cone $z = \sqrt{x^2 + y^2}$. The value of the triple iterated integral is therefore given by

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta = \int_0^{\pi/2}\int_0^{\pi/4} \left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{\sqrt{2}} d\phi \, d\theta = \frac{2\sqrt{2}}{3}\int_0^{\pi/2}\Big\{-\cos\phi\Big\}_0^{\pi/4} d\theta$$
$$= \frac{2\sqrt{2}}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\Big\{\theta\Big\}_0^{\pi/2} = \frac{(\sqrt{2}-1)\pi}{3}.$$

20. $$V = 4\int_0^{\pi/2}\int_0^{\overline{\phi}}\int_0^R \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta$$
$$= \frac{4R^3}{3}\int_0^{\pi/2}\int_0^{\overline{\phi}} \sin\phi \, d\phi \, d\theta$$
$$= \frac{4R^3}{3}\int_0^{\pi/2}\Big\{-\cos\phi\Big\}_0^{\overline{\phi}} d\theta$$
$$= \frac{4R^3}{3}(1 - \cos\overline{\phi})\Big\{\theta\Big\}_0^{\pi/2}$$
$$= \frac{2\pi R^3}{3}\left(1 - \frac{k}{\sqrt{1+k^2}}\right)$$

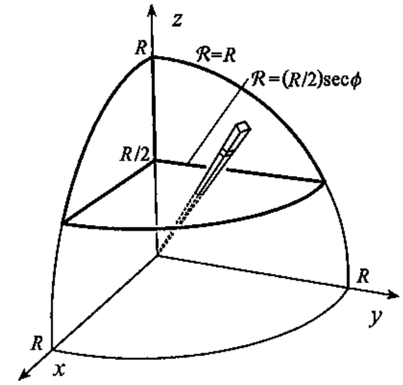


21. In spherical coordinates the equation of the surface is $\Re^4 = \Re\sin\phi\cos\theta \implies \Re^3 = \sin\phi\cos\theta$. As ϕ increases from 0 to π, values of $\sin\phi$ increase from 0 to 1, and then decrease from 1 to 0. Only for θ in the interval $-\pi/2 \le \theta \le \pi/2$ is $\Re > 0$. This leads to

$$V = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{(\sin\phi\cos\theta)^{1/3}} \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{(\sin\phi\cos\theta)^{1/3}} d\phi \, d\theta$$
$$= \frac{1}{3}\int_{-\pi/2}^{\pi/2}\int_0^{\pi} \sin^2\phi\cos\theta \, d\phi \, d\theta = \frac{1}{3}\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\left(\frac{1-\cos 2\phi}{2}\right)\cos\theta \, d\phi \, d\theta$$
$$= \frac{1}{6}\int_{-\pi/2}^{\pi/2}\left\{\left(\phi - \frac{1}{2}\sin 2\phi\right)\cos\theta\right\}_0^{\pi} d\theta = \frac{\pi}{6}\Big\{\sin\theta\Big\}_{-\pi/2}^{\pi/2} = \frac{\pi}{3}.$$

22. (a) Let ρ_b and ρ_w represent the densities of the ball and water. The magnitude of the force of gravity on the ball is $(4/3)\pi R^3\rho_b g$ where R is its radius, and $g > 0$ is the acceleration due to gravity. Since this must be equal to the weight of water displaced by the half-submerged ball, $\frac{4}{3}\pi R^3\rho_b g = \frac{2}{3}\pi R^3\rho_w g$. This equation implies that $\rho_b = \rho_w/2$.

(b) In the diagram, we let the plane $z = R/2$ represent the surface of the water. The volume of ball above water is given by



$$4\int_0^{\pi/2}\int_0^{\pi/3}\int_{(R/2)\sec\phi}^{R} \Re^2 \sin\phi \, d\Re \, d\phi \, d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{\pi/3}\left\{\frac{\Re^3}{3}\sin\phi\right\}_{(R/2)\sec\phi}^{R} d\phi \, d\theta$$
$$= \frac{R^3}{6}\int_0^{\pi/2}\int_0^{\pi/3}(8\sin\phi - \tan\phi\sec^2\phi)\, d\phi \, d\theta$$
$$= \frac{R^3}{6}\int_0^{\pi/2}\left\{-8\cos\phi - \frac{\tan^2\phi}{2}\right\}_0^{\pi/3} d\theta$$
$$= \frac{5R^3}{12}\left\{\theta\right\}_0^{\pi/2} = \frac{5\pi R^3}{24}.$$

The force required to keep the ball at this position is equal to the extra weight of water (above that in (a)) displaced; i.e., $\left(\frac{2}{3}\pi R^3 - \frac{5}{24}\pi R^3\right)\rho_w g = \frac{11}{24}\pi\rho_w g R^3$.

23. The equation of the surface in spherical coordinates is $\Re^4 = 2\Re\cos\phi(\Re^2\sin^2\phi) \implies \Re = 2\sin^2\phi\cos\phi$.

$$V = 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin^2\phi\cos\phi} \Re^2\sin\phi\, d\Re\, d\phi\, d\theta = 4\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{2\sin^2\phi\cos\phi} d\phi\, d\theta$$
$$= \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/2} 8\sin^7\phi\cos^3\phi\, d\phi\, d\theta = \frac{32}{3}\int_0^{\pi/2}\int_0^{\pi/2}\sin^7\phi(1-\sin^2\phi)\cos\phi\, d\phi\, d\theta$$
$$= \frac{32}{3}\int_0^{\pi/2}\left\{\frac{1}{8}\sin^8\phi - \frac{1}{10}\sin^{10}\phi\right\}_0^{\pi/2} d\theta = \frac{4}{15}\left\{\theta\right\}_0^{\pi/2} = \frac{2\pi}{15}$$

24. (a) Since $s^2 = \Re^2 + d^2 - 2\Re d\cos\phi$,

$$V = \int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{R}\frac{\rho}{4\pi\epsilon_0 s}\Re^2\sin\phi\, d\Re\, d\phi\, d\theta = \frac{\rho}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{R}\frac{\Re^2\sin\phi}{\sqrt{\Re^2+d^2-2\Re d\cos\phi}}d\Re\, d\phi\, d\theta.$$

(b) In order to change ϕ to s we first write $V = \frac{\rho}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^{R}\int_0^{\pi}\frac{\Re^2\sin\phi}{\sqrt{\Re^2+d^2-2\Re d\cos\phi}}d\phi\, d\Re\, d\theta$. If $s^2 = \Re^2 + d^2 - 2\Re d\cos\phi$, then $2s\, ds = 2\Re d\sin\phi\, d\phi$, and

$$V = \frac{\rho}{4\pi\epsilon_0}\int_{-\pi}^{\pi}\int_0^{R}\int_{d-\Re}^{d+\Re}\frac{\Re^2}{s}\left(\frac{s\, ds}{\Re d}\right)d\Re\, d\theta = \frac{\rho}{4\pi\epsilon_0 d}\int_{-\pi}^{\pi}\int_0^{R}\int_{d-\Re}^{d+\Re}\Re\, ds\, d\Re\, d\theta.$$

(c) $$V = \frac{\rho}{4\pi\epsilon_0 d}\int_{-\pi}^{\pi}\int_0^{R}\left\{\Re s\right\}_{d-\Re}^{d+\Re}d\Re\, d\theta = \frac{\rho}{2\pi\epsilon_0 d}\int_{-\pi}^{\pi}\int_0^{R}\Re^2\, d\Re\, d\theta$$
$$= \frac{\rho}{2\pi\epsilon_0 d}\int_{-\pi}^{\pi}\left\{\frac{\Re^3}{3}\right\}_0^{R}d\theta = \frac{\rho R^3}{6\pi\epsilon_0 d}\left\{\theta\right\}_{-\pi}^{\pi} = \frac{\rho R^3}{3\epsilon_0 d}$$

Since $Q = (4/3)\pi R^3\rho$, $\frac{1}{4\pi\epsilon_0}\frac{Q}{d} = \frac{1}{4\pi\epsilon_0 d}\left(\frac{4}{3}\pi R^3\rho\right) = \frac{\rho R^3}{3\epsilon_0 d}$, and therefore $V = \frac{1}{4\pi\epsilon_0}\frac{Q}{d}$.

25. (a) The cosine law for the triangle joining O, P, and dV gives $\Re^2 = s^2 + d^2 - 2sd\cos\psi$, and therefore

$$F_z = \iiint_V -\frac{Gm\rho}{s^2}\left(\frac{s^2+d^2-\Re^2}{2sd}\right)dV = -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^R\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\Re\,d\phi\,d\theta.$$

(b) In order to replace ϕ with s we first write

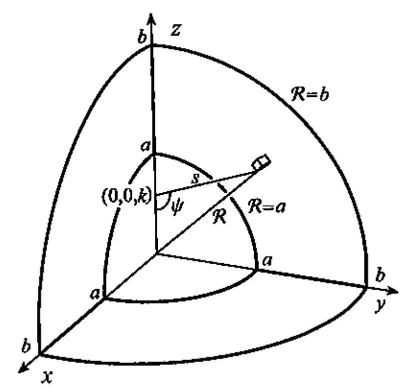
$$F_z = -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^R\int_0^{\pi}\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\phi\,d\Re\,d\theta.$$

If we set $s = \sqrt{\Re^2+d^2-2d\Re\cos\phi} \implies s^2 = \Re^2+d^2-2d\Re\cos\phi$, from which $2s\,ds = 2d\Re\sin\phi\,d\phi$, then

$$\begin{aligned}
F_z &= -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^R\int_{d-\Re}^{d+\Re}\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\left(\frac{s\,ds}{d\Re}\right)d\Re\,d\theta\\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^R\int_{d-\Re}^{d+\Re}\Re\left(\frac{s^2+d^2-\Re^2}{s^2}\right)ds\,d\Re\,d\theta\\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^R\left\{\Re\left(s-\frac{d^2-\Re^2}{s}\right)\right\}\Bigg|_{d-\Re}^{d+\Re}d\Re\,d\theta\\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^R\Re\left(d+\Re-\frac{d^2-\Re^2}{d+\Re}-d+\Re+\frac{d^2-\Re^2}{d-\Re}\right)d\Re\,d\theta\\
&= -\frac{2Gm\rho}{d^2}\int_{-\pi}^{\pi}\int_0^R\Re^2\,d\Re\,d\theta = -\frac{2Gm\rho}{d^2}\int_{-\pi}^{\pi}\left\{\frac{\Re^3}{3}\right\}\Bigg|_0^R d\theta\\
&= -\frac{2Gm\rho R^3}{3d^2}\left\{\theta\right\}\Bigg|_{-\pi}^{\pi} = -\frac{4\pi Gm\rho R^3}{3d^2} = -\frac{GmM}{d^2},
\end{aligned}$$

where M is the mass of the sphere.

26. We can always choose a coordinate system so that the point is on the z-axis. Symmetry makes it clear that x- and y-components of the force vanish.



The contribution to the z-component of the force due to the mass in dV is

$$-\frac{Gm\rho dV}{s^2}\cos\psi = -\frac{Gm\rho}{s^2}\left(\frac{k^2+s^2-\Re^2}{2ks}\right)dV.$$

Therefore

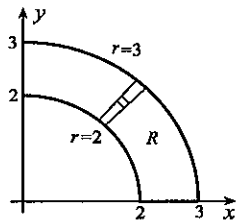
$$\begin{aligned}
F_z &= \int_0^{\pi}\int_a^b\int_{-\pi}^{\pi}-\frac{Gm\rho}{2ks^3}(k^2+s^2-\Re^2)\Re^2\sin\phi\,d\theta\,d\Re\,d\phi\\
&= -\frac{Gm\rho\pi}{k}\int_0^{\pi}\int_a^b\left(\frac{s^2+k^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\Re\,d\phi\\
&= -\frac{Gm\rho\pi}{k}\int_a^b\int_0^{\pi}\left(\frac{s^2+k^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\phi\,d\Re.
\end{aligned}$$

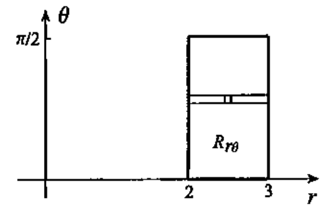
If we set $s = \sqrt{\Re^2+k^2-2k\Re\cos\phi}$ in the inner integral, then $2s\,ds = 2k\Re\sin\phi\,d\phi$, and

$$\begin{aligned}
F_z &= -\frac{Gm\rho\pi}{k}\int_a^b\int_{\Re-k}^{\Re+k}\left(\frac{s^2+k^2-\Re^2}{s^3}\right)\Re^2\sin\phi\left(\frac{s\,ds}{k\Re\sin\phi}\right)d\Re\\
&= -\frac{Gm\rho\pi}{k^2}\int_a^b\int_{\Re-k}^{\Re+k}\Re\left(\frac{s^2+k^2-\Re^2}{s^2}\right)ds\,d\Re = -\frac{Gm\rho\pi}{k^2}\int_a^b\Re\left\{s-\frac{k^2-\Re^2}{s}\right\}\Bigg|_{\Re-k}^{\Re+k}d\Re\\
&= -\frac{Gm\rho\pi}{k^2}\int_a^b\Re\left[\Re+k-\frac{k^2-\Re^2}{\Re+k}-(\Re-k)+\frac{k^2-\Re^2}{\Re-k}\right]d\Re = 0.
\end{aligned}$$

EXERCISES 13.13

1. (a) This is a change to polar coordinates, $\displaystyle\iint_R \sqrt{x^2+y^2}\,dA = \int_0^{\pi/2}\int_2^3 r\,r\,dr\,d\theta = \int_0^{\pi/2}\int_2^3 r^2\,dr\,d\theta.$



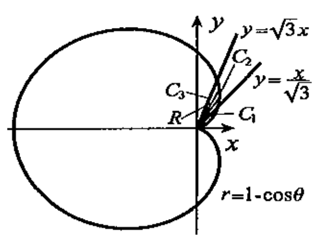


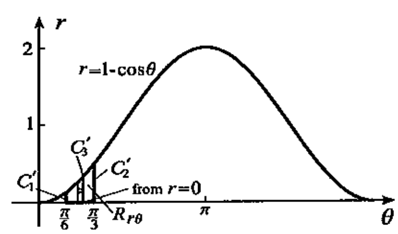
(b) Alternatively, region R in the xy-plane is mapped to the rectangle $R_{r\theta}$ in the $r\theta$-plane shown above. With $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$, equation 13.70 gives

$$\iint_R \sqrt{x^2+y^2}\,dA = \iint_{R_{r\theta}} r\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \int_0^{\pi/2}\int_2^3 r^2\,dr\,d\theta.$$

2. (a) This is a change to polar coordinates,

$$\iint_R xy\,dA = \int_{\pi/6}^{\pi/3}\int_0^{1-\cos\theta} r\cos\theta\, r\sin\theta\, r\,dr\,d\theta = \int_{\pi/6}^{\pi/3}\int_0^{1-\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta.$$

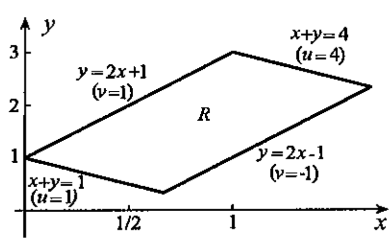


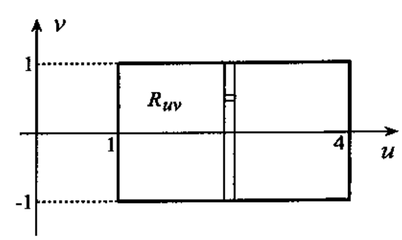


(b) Alternatively, region R in the xy-plane is mapped to the region $R_{r\theta}$ in the $r\theta$-plane shown above. With $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$, equation 13.70 gives

$$\iint_R xy\,dA = \iint_{R_{r\theta}} r\cos\theta\, r\sin\theta\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \int_{\pi/6}^{\pi/3}\int_0^{1-\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta.$$

3. The parallelogram R in the xy-plane is mapped to the rectangle R_{uv} in the uv-plane shown below.

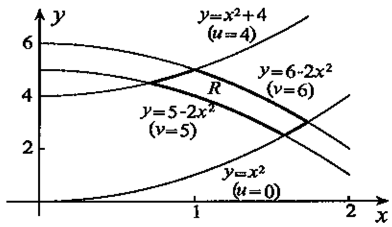


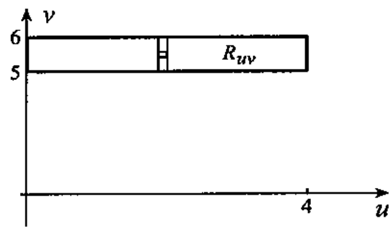


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \dfrac{1}{3}$, equation 13.70 gives

$$\iint_R x^2\cos y\,dA = \iint_{R_{uv}} \left(\frac{u-v}{3}\right)^2 \cos\left(\frac{2u+v}{3}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \frac{1}{27}\int_1^4\int_{-1}^1 (u-v)^2\cos\left(\frac{2u+v}{3}\right) dv\,du.$$

4. Region R in the xy-plane is mapped to the rectangle R_{uv} in the uv-plane shown below.



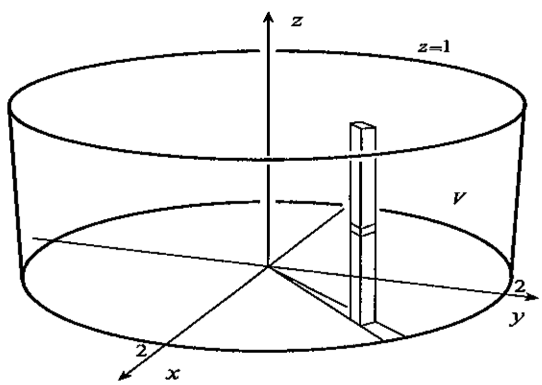


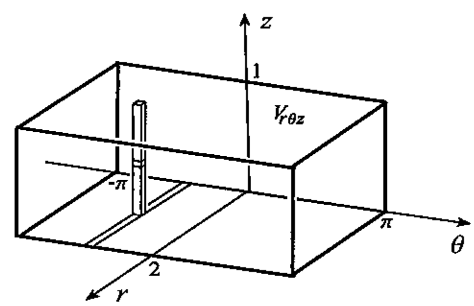
With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} -2x & 1 \\ 4x & 1 \end{vmatrix}} = -\dfrac{1}{6x}$, equation 13.70 gives

$$\iint_R (x^2+y)\,dA = \iint_{R_{uv}} (x^2+y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \frac{1}{6}\iint_{R_{uv}} \left(\frac{x^2+y}{x}\right) dv\,du$$

$$= \frac{1}{6}\int_0^4\int_5^6 \left(\sqrt{\frac{v-u}{3}} + \frac{(v+2u)/3}{\sqrt{(v-u)/3}}\right) dv\,du = \frac{1}{6\sqrt{3}}\int_0^4\int_5^6 \frac{2v+u}{\sqrt{v-u}}\,dv\,du.$$

5. (a) This is a change to cylindrical coordinates, $\displaystyle\iiint_V ze^{x^2+y^2}\,dV = \int_{-\pi}^{\pi}\int_0^2\int_0^1 ze^{r^2}\,r\,dz\,dr\,d\theta.$



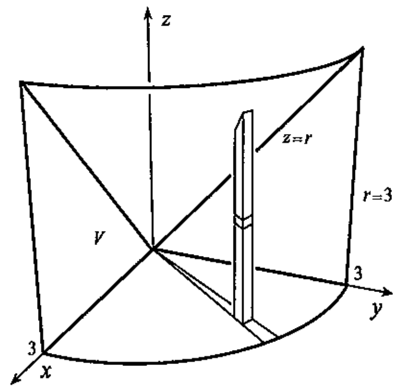


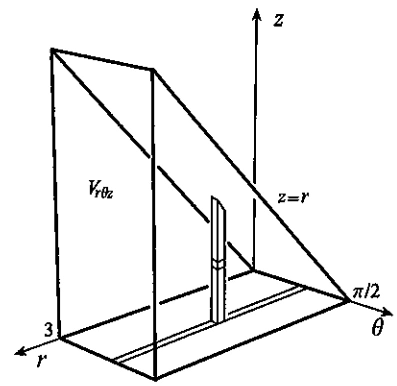
(b) Region V in xyz-space is mapped to the region $V_{r\theta z}$ in $r\theta z$-space shown above. With $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$, equation 13.73 gives

$$\iiint_V ze^{x^2+y^2}\,dV = \iiint_{V_{r\theta z}} ze^{x^2+y^2}\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dz\,dr\,d\theta = \int_{-\pi}^{\pi}\int_0^2\int_0^1 ze^{r^2}\,r\,dz\,dr\,d\theta.$$

6. (a) This is a change to cylindrical coordinates,

$$\iiint_V (x^2+y^2)\,dV = \int_0^{\pi/2}\int_0^3\int_0^r r^2\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3\int_0^r r^3\,dz\,dr\,d\theta.$$



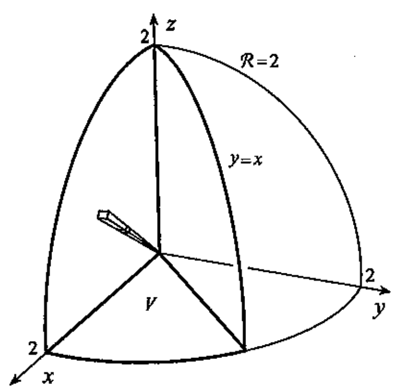


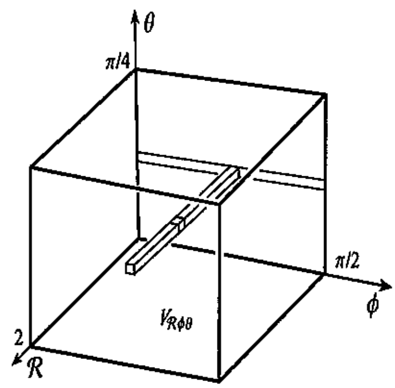
(b) Region V in xyz-space is mapped to the region $V_{r\theta z}$ in $r\theta z$-space shown above. With $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$, equation 13.73 gives

$$\iiint_V (x^2+y^2)\,dV = \iiint_{V_{r\theta z}} r^2 \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3\int_0^r r^3\,dz\,dr\,d\theta.$$

7. (a) This is a change to spherical coordinates,

$$\iiint_V \frac{1}{x^2+y^2}dV = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 \frac{1}{\Re^2\sin\phi}\Re^2\sin\phi\,d\Re\,d\phi\,d\theta = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 d\Re\,d\phi\,d\theta.$$





(b) Region V in xyz-space is mapped to the region $V_{\Re\phi\theta}$ in $\Re\phi\theta$-space shown above. With

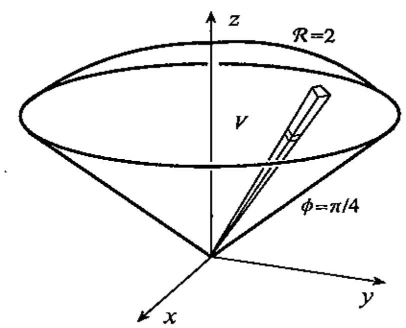
$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(\Re,\phi,\theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \Re\cos\phi\cos\theta & -\Re\sin\phi\sin\theta \\ \sin\phi\sin\theta & \Re\cos\phi\sin\theta & \Re\sin\phi\cos\theta \\ \cos\phi & -\Re\sin\phi & 0 \end{vmatrix} \\ &= \cos\phi(\Re^2\sin\phi\cos\phi\cos^2\theta + \Re^2\sin\phi\cos\phi\sin^2\theta) \\ &\quad + \Re\sin\phi(\Re\sin^2\phi\cos^2\theta + \Re\sin^2\phi\sin^2\theta) \\ &= \Re^2\cos^2\phi\sin\phi + \Re^2\sin^3\phi = \Re^2\sin\phi,\end{aligned}$$

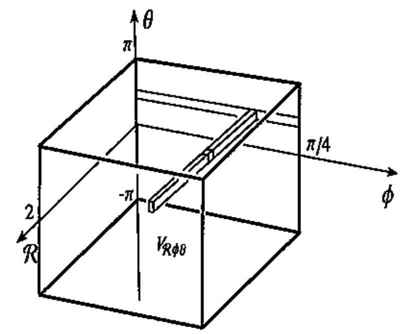
equation 13.73 gives

$$\iiint_V \frac{1}{x^2+y^2}dV = \iiint_{V_{\Re\phi\theta}} \frac{1}{\Re^2\sin\phi}\left|\frac{\partial(x,y,z)}{\partial(\Re,\phi,\theta)}\right| d\Re\,d\phi\,d\theta = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 d\Re\,d\phi\,d\theta.$$

8. (a) This is a change to spherical coordinates,

$$\iiint_V x^2y^2z\,dV = \int_{-\pi}^{\pi}\int_0^{\pi/4}\int_0^2 (\Re^2\sin^2\phi\cos^2\theta)(\Re^2\sin^2\phi\sin^2\theta)(\Re\cos\phi)\,\Re^2\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^{\pi/4}\int_0^2 \Re^7\sin^5\phi\cos\phi\sin^2\theta\cos^2\theta\,d\Re\,d\phi\,d\theta.$$





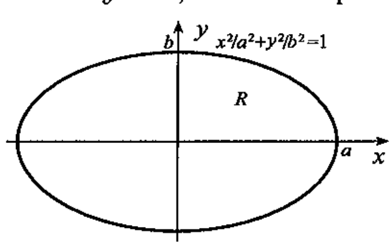
(b) Region V in xyz-space is mapped to the region $V_{\Re\phi\theta}$ in $\Re\phi\theta$-space shown above. With

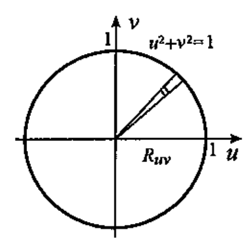
$$\frac{\partial(x,y,z)}{\partial(\Re,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \Re\cos\phi\cos\theta & -\Re\sin\phi\sin\theta \\ \sin\phi\sin\theta & \Re\cos\phi\sin\theta & \Re\sin\phi\cos\theta \\ \cos\phi & -\Re\sin\phi & 0 \end{vmatrix}$$
$$= \cos\phi(\Re^2\sin\phi\cos\phi\cos^2\theta + \Re^2\sin\phi\cos\phi\sin^2\theta)$$
$$+ \Re\sin\phi(\Re\sin^2\phi\cos^2\theta + \Re\sin^2\phi\sin^2\theta)$$
$$= \Re^2\cos^2\phi\sin\phi + \Re^2\sin^3\phi = \Re^2\sin\phi,$$

equation 13.73 gives

$$\iiint_V x^2y^2z\,dV = \iiint_{V_{\Re\phi\theta}} x^2y^2z\left|\frac{\partial(x,y,z)}{\partial(\Re,\phi,\theta)}\right| d\Re\,d\phi\,d\theta$$
$$= \iiint_{V_{\Re\phi\theta}} (\Re^2\sin^2\phi\cos^2\theta)(\Re^2\sin^2\phi\sin^2\theta)(\Re\cos\phi)\Re^2\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^{\pi/4}\int_0^2 \Re^7\sin^5\phi\cos\phi\sin^2\theta\cos^2\theta\,d\Re\,d\phi\,d\theta.$$

9. If we let $x = au$ and $y = bv$, then the ellipse $x^2/a^2 + y^2/b^2 = 1$ is mapped to the circle $u^2 + v^2 = 1$.





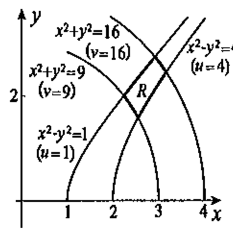
With $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$, equation 13.70 gives

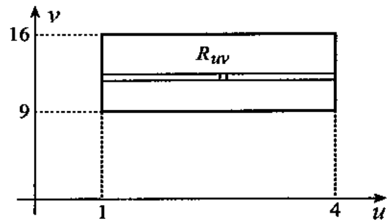
$$\iint_R \sqrt{x^2/a^2 + y^2/b^2}\,dA = \iint_{R_{uv}} \sqrt{u^2+v^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{R_{uv}} \sqrt{u^2+v^2}(ab)\,du\,dv.$$

If we now use polar coordinates in the R_{uv}-plane,

$$\iint_R \sqrt{x^2/a^2 + y^2/b^2}\,dA = ab\int_{-\pi}^{\pi}\int_0^1 r\,r\,dr\,d\theta = ab\int_{-\pi}^{\pi}\left\{\frac{r^3}{3}\right\}_0^1 d\theta = \frac{ab}{3}\Big\{\theta\Big\}_{-\pi}^{\pi} = \frac{2\pi ab}{3}.$$

10. If we let $u = x^2 - y^2$ and $v = x^2 + y^2$, then the region R in the xy-plane is mapped to the rectangle R_{uv} in the uv-plane shown below.

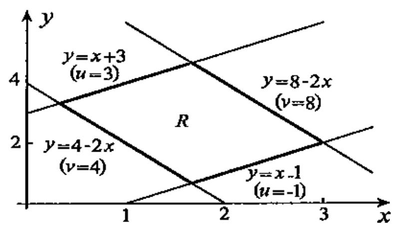


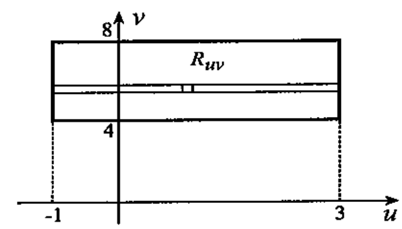


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix}} = \dfrac{1}{8xy}$, equation 13.70 gives

$$\iint_R xy\,dA = \iint_{R_{uv}} xy \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{R_{uv}} xy \left|\frac{1}{8xy}\right| du\,dv = \frac{1}{8}\iint_{R_{uv}} du\,dv = \frac{1}{8}(\text{Area of } R_{uv}) = \frac{21}{8}.$$

11. If we let $u = y - x$ and $v = y + 2x$, then the region R in the xy-plane is mapped to the rectangle R_{uv} in the uv-plane shown below.



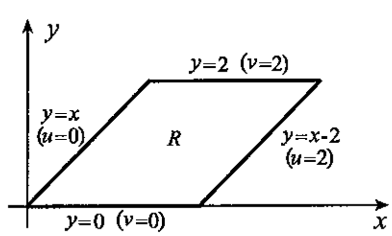


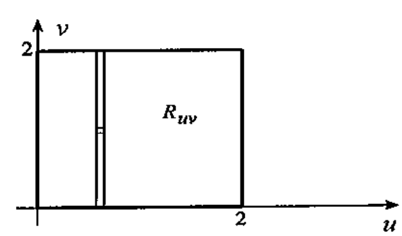
With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}} = -\dfrac{1}{3}$, and the fact that $2x^2 - xy - y^2 = (2x+y)(x-y) = -uv$, equation 13.70 gives

$$\iint_R (2x^2 - xy - y^2)\,dA = \iint_{R_{uv}} (-uv)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = -\iint_{R_{uv}} uv\left|-\frac{1}{3}\right| du\,dv = -\frac{1}{3}\int_4^8\int_{-1}^3 uv\,du\,dv$$

$$= -\frac{1}{3}\int_4^8 \left\{\frac{u^2v}{2}\right\}_{-1}^{3} dv = -\frac{4}{3}\left\{\frac{v^2}{2}\right\}_4^8 = -32.$$

12. The transformation $u = x - y$ and $v = y$, maps the parallelogram R in the xy-plane to the square R_{uv} in the uv-plane shown below.



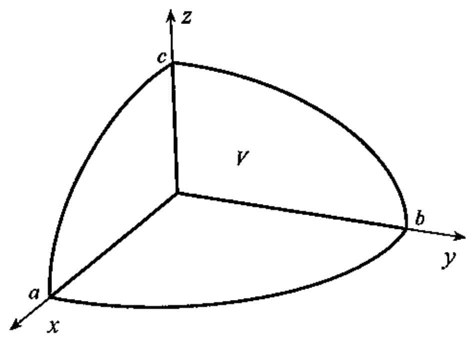


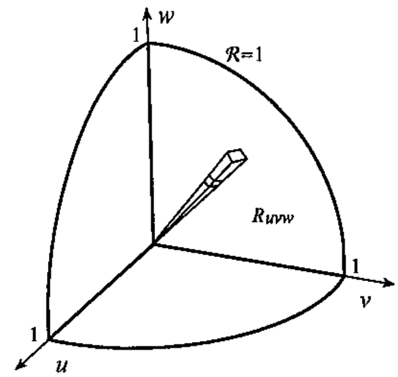
With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}} = 1$, equation 13.70 gives

$$\iint_R (x+y)\,dA = \iint_{R_{uv}} (u+2v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \iint_{R_{uv}} (u+2v)\,dv\,du = \int_0^2\int_0^2 (u+2v)\,dv\,du$$

$$= \int_0^2 \left\{uv + v^2\right\}_0^2 du = \int_0^2 (2u+4)\,du = \left\{u^2 + 4u\right\}_0^2 = 12.$$

13. If we let $u = x/a$, $v = y/b$, and $w = z/c$, then the region V in the first octant of xyz-space bounded by the ellipsoid is mapped to the first octant part of the sphere $u^2 + v^2 + w^2 = 1$ in uvw-space.





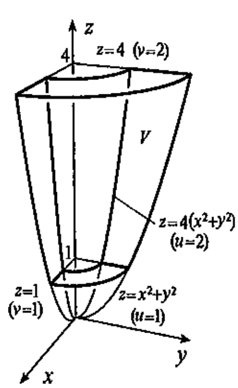
Since $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$, equation 13.73 gives

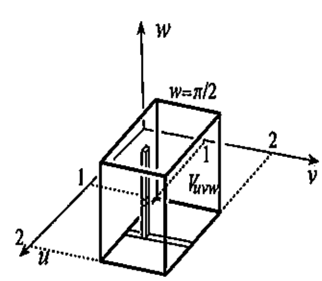
$$8\iiint_V x^2y^2z^2\,dV = 8\iiint_{R_{uvw}} (au)^2(bv)^2(cw)^2 \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = 8a^3b^3c^3 \iiint_{R_{uvw}} u^2v^2w^2\,du\,dv\,dw.$$

If we now change to spherical coordinates in uvw-space,

$$\begin{aligned} 8\iiint_V x^2y^2z^2\,dV &= 8a^3b^3c^3 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \Re^2\sin^2\phi\cos^2\theta\,\Re^2\sin^2\phi\sin^2\theta\,\Re^2\cos^2\phi\,\Re^2\sin\phi\,d\Re\,d\phi\,d\theta \\ &= 8a^3b^3c^3 \int_0^{\pi/2}\int_0^{\pi/2} \left\{\frac{\Re^9}{9}\sin^5\phi\cos^2\phi\sin^2\theta\cos^2\theta\right\}_0^1 d\phi\,d\theta \\ &= \frac{8a^3b^3c^3}{9}\int_0^{\pi/2}\int_0^{\pi/2} \cos^2\phi(1 - 2\cos^2\phi + \cos^4\phi)\sin\phi\sin^2\theta\cos^2\theta\,d\phi\,d\theta \\ &= \frac{8a^3b^3c^3}{9}\int_0^{\pi/2} \left\{\left(-\frac{1}{3}\cos^3\phi + \frac{2}{5}\cos^5\phi - \frac{1}{7}\cos^7\phi\right)\left(\frac{1}{4}\right)\sin^2 2\theta\right\}_0^{\pi/2} d\theta \\ &= \frac{16a^3b^3c^3}{945}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = \frac{8a^3b^3c^3}{945}\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{4\pi a^3b^3c^3}{945}. \end{aligned}$$

14. The transformation maps the first octant volume V bounded by the surfaces to the box V_{uvw} in uvw-space shown below.



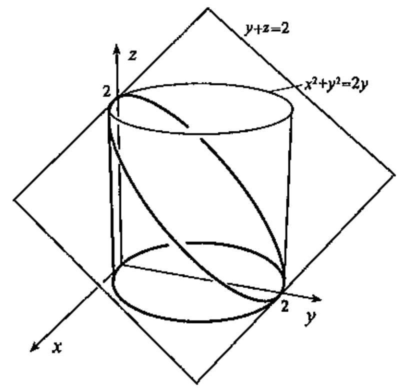


With $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} -(v/u^2)\cos w & (1/u)\cos w & -(v/u)\sin w \\ -(v/u^2)\sin w & (1/u)\sin w & (v/u)\cos w \\ 0 & 2v & 0 \end{vmatrix} = \dfrac{2v^3}{u^3}$, equation 13.73 gives

$$\begin{aligned}4\iiint_V (x^2+y^2)\,dV &= 4\iiint_{V_{uvw}} \left(\frac{v^2}{u^2}\cos^2 w + \frac{v^2}{u^2}\sin^2 w\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\,dv\,du\\
&= 4\iiint_{V_{uvw}} \left(\frac{v^2}{u^2}\right)\left(\frac{2v^3}{u^3}\right) dw\,dv\,du = 8\int_1^2\int_1^2\int_0^{\pi/2} \frac{v^5}{u^5} dw\,dv\,du\\
&= 8\int_1^2\int_1^2 \left\{\frac{v^5 w}{u^5}\right\}_0^{\pi/2} dv\,du = 4\pi\int_1^2 \left\{\frac{v^6}{6u^5}\right\}_1^2 du = 42\pi\left\{-\frac{1}{4u^4}\right\}_1^2 = \frac{315\pi}{32}.\end{aligned}$$

15. (a) With the usual cylindrical coordinates,

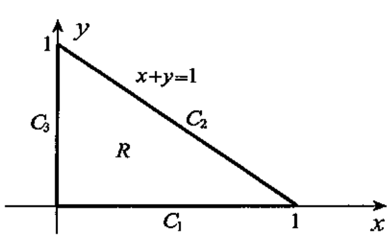
$$\begin{aligned}\iiint_V y\,dV &= 2\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{2-r\sin\theta} r\sin\theta\, r\,dz\,dr\,d\theta\\
&= 2\int_0^{\pi/2}\int_0^{2\sin\theta} r^2\sin\theta(2-r\sin\theta)\,dr\,d\theta\\
&= 2\int_0^{\pi/2}\left\{\frac{2r^3\sin\theta}{3} - \frac{r^4\sin^2\theta}{4}\right\}_0^{2\sin\theta} d\theta\\
&= \frac{8}{3}\int_0^{\pi/2}(4\sin^4\theta - 3\sin^6\theta)\,d\theta\\
&= \frac{8}{3}\int_0^{\pi/2}\left[4\left(\frac{1-\cos 2\theta}{2}\right)^2 - 3\left(\frac{1-\cos 2\theta}{2}\right)^3\right] d\theta\\
&= \frac{8}{3}\int_0^{\pi/2}\left[1 - 2\cos 2\theta + \cos^2 2\theta - \frac{3}{8}(1 - 3\cos 2\theta + 3\cos^2 2\theta - \cos^3 2\theta)\right] d\theta\\
&= \frac{8}{3}\int_0^{\pi/2}\left[\frac{5}{8} - \frac{7}{8}\cos 2\theta - \frac{1}{16}(1+\cos 4\theta) + \frac{3}{8}\cos 2\theta(1-\sin^2 2\theta)\right] d\theta\\
&= \frac{8}{3}\left\{\frac{9\theta}{16} - \frac{1}{4}\sin 2\theta - \frac{1}{64}\sin 4\theta + \frac{1}{16}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{3\pi}{4}.\end{aligned}$$

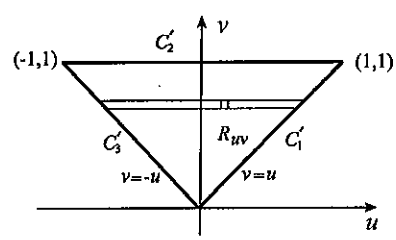


(b) With cylindrical coordinates based at $(0,1)$, $x = r\cos\theta$, $y = 1 + r\sin\theta$, and

$$\begin{aligned}\iiint_V y\,dV &= 2\int_0^{\pi}\int_0^1\int_0^{1-r\sin\theta}(1+r\sin\theta)r\,dz\,dr\,d\theta = 2\int_0^{\pi}\int_0^1 (1-r\sin\theta)(1+r\sin\theta)r\,dr\,d\theta\\
&= 2\int_0^{\pi}\int_0^1 (r - r^3\sin^2\theta)\,dr\,d\theta = 2\int_0^{\pi}\left\{\frac{r^2}{2} - \frac{r^4\sin^2\theta}{4}\right\}_0^1 d\theta\\
&= \int_0^{\pi}\left[1 - \frac{1}{4}(1-\cos 2\theta)\right] d\theta = \left\{\frac{3\theta}{4} + \frac{\sin 2\theta}{8}\right\}_0^{\pi} = \frac{3\pi}{4}.\end{aligned}$$

16. The transformation maps the triangle R in the xy-plane to the triangle R_{uv} in the uv-plane shown below.

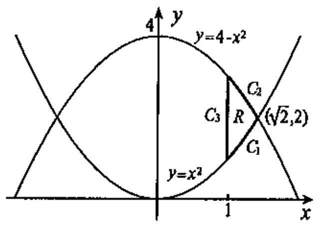


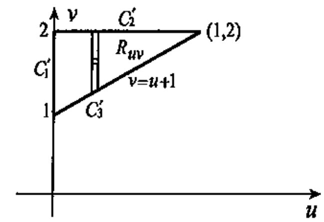


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}} = \dfrac{1}{2}$, equation 13.70 gives

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dA = \iint_{R_{uv}} \cos\left(\frac{u}{v}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{2}\int_0^1\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du\,dv$$
$$= \frac{1}{2}\int_0^1 \left\{v\sin\left(\frac{u}{v}\right)\right\}_{-v}^{v} dv = \sin 1\int_0^1 v\,dv = \sin 1\left\{\frac{v^2}{2}\right\}_0^1 = \frac{\sin 1}{2}.$$

17. The transformation maps the region R in the xy-plane to the triangle R_{uv} in the uv-plane shown below.

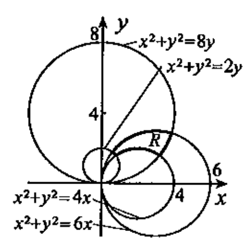


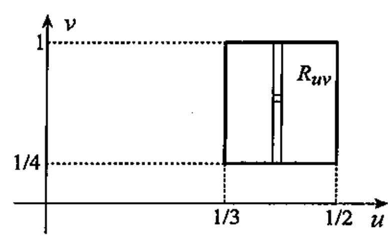


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\dfrac{1}{2\sqrt{v-u}} & \dfrac{1}{2\sqrt{v-u}} \\ 1 & 1 \end{vmatrix} = -\dfrac{1}{\sqrt{v-u}}$, equation 13.70 gives

$$\iint_R \frac{x}{x^2+y} dA = \iint_{R_{uv}} \frac{\sqrt{v-u}}{v-u+u+v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \iint_{R_{uv}} \frac{\sqrt{v-u}}{2v}\left(\frac{1}{\sqrt{v-u}}\right) dv\,du$$
$$= \frac{1}{2}\int_0^1\int_{u+1}^{2} \frac{1}{v} dv\,du = \frac{1}{2}\int_0^1 \Big\{\ln|v|\Big\}_{u+1}^{2} du = \frac{1}{2}\int_0^1 [\ln 2 - \ln(u+1)]\,du$$
$$= \frac{1}{2}\Big\{u\ln 2 - (u+1)\ln|u+1| + u\Big\}_0^1 = \frac{1}{2}(1-\ln 2).$$

18. The transformation maps the region R in the xy-plane to the rectangle R_{uv} in the uv-plane shown below.

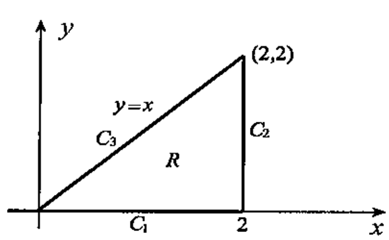


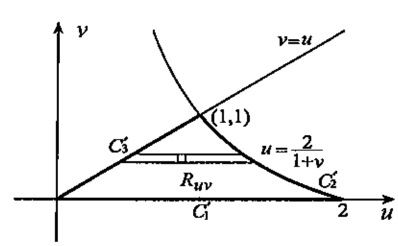


With $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix} \dfrac{2(y^2-x^2)}{(x^2+y^2)^2} & \dfrac{-4xy}{(x^2+y^2)^2} \\ \dfrac{-4xy}{(x^2+y^2)^2} & \dfrac{2(x^2-y^2)}{(x^2+y^2)^2} \end{vmatrix}} = -\dfrac{(x^2+y^2)^2}{4}$, equation 13.70 gives

$$\iint_R \frac{1}{(x^2+y^2)^2} dA = \iint_{R_{uv}} \frac{1}{(x^2+y^2)^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \frac{1}{4}\int_{1/3}^{1/2}\int_{1/4}^{1} dv\,du = \frac{1}{4}\left(\frac{3}{4}\right)\left(\frac{1}{6}\right) = \frac{1}{32}.$$

19. The transformation maps the triangle R in the xy-plane to the region R_{uv} in the uv-plane shown below.





The Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v$, and

$$\frac{1}{\sqrt{(x-y)^2+2(x+y)+1}} = \frac{1}{\sqrt{(u-v)^2+2(u+v+2uv)+1}} = \frac{1}{\sqrt{(u+v+1)^2}} = \frac{1}{u+v+1}.$$

Equation 13.70 gives

$$\iint_R \frac{1}{\sqrt{(x-y)^2+2(x+y)+1}}dA = \iint_{R_{uv}} \frac{1}{u+v+1}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{R_{uv}} du\,dv = \int_0^1\int_v^{2/(1+v)} du\,dv$$

$$= \int_0^1 \left(\frac{2}{1+v} - v\right) dv = \left\{2\ln|1+v| - \frac{v^2}{2}\right\}_0^1 = 2\ln 2 - \frac{1}{2}.$$

20. (a) We require two iterated integrals for direct evaluation,

$$\iiint_V (x+y+z)\,dV = \int_0^1\int_{1-x}^{1+x}\int_{2x+y}^{2x+y+1} (x+y+z)\,dz\,dy\,dx + \int_1^2\int_{x-1}^{3-x}\int_{2x+y}^{2x+y+1} (x+y+z)\,dz\,dy\,dx$$

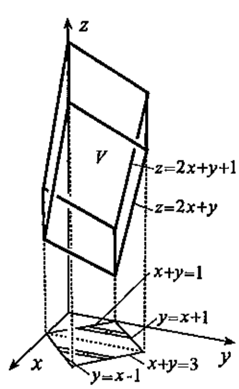
$$= \int_0^1\int_{1-x}^{1+x} \left\{\frac{(x+y+z)^2}{2}\right\}_{2x+y}^{2x+y+1} dy\,dx + \int_1^2\int_{x-1}^{3-x} \left\{\frac{(x+y+z)^2}{2}\right\}_{2x+y}^{2x+y+1} dy\,dx$$

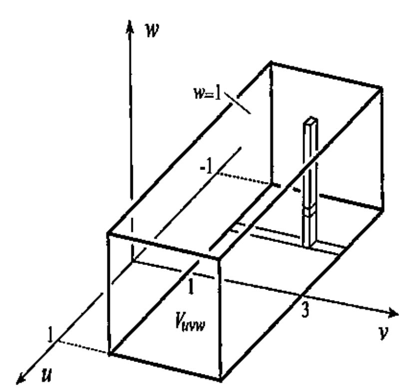
$$= \frac{1}{2}\int_0^1\int_{1-x}^{1+x} (6x+4y+1)\,dy\,dx + \frac{1}{2}\int_1^2\int_{x-1}^{3-x} (6x+4y+1)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left\{\frac{(6x+4y+1)^2}{8}\right\}_{1-x}^{1+x} dx + \frac{1}{2}\int_1^2 \left\{\frac{(6x+4y+1)^2}{8}\right\}_{x-1}^{3-x} dx$$

$$= \frac{1}{16}\int_0^1 [(10x+5)^2 - (2x+5)^2]\,dx + \frac{1}{16}\int_1^2 [(2x+13)^2 - (10x-3)^2]\,dx$$

$$= \frac{1}{16}\left\{\frac{(10x+5)^3}{30} - \frac{(2x+5)^3}{6}\right\}_0^1 + \frac{1}{16}\left\{\frac{(2x+13)^3}{6} - \frac{(10x-3)^3}{30}\right\}_1^2 = 11.$$





(b) The transformation maps the region V in xyz-space to the box V_{uvw} in uvw-space shown above. The Jacobian of the transformation is $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{\dfrac{\partial(u,v,w)}{\partial(x,y,z)}} = \dfrac{1}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix}} = \dfrac{1}{2}$. Equation 13.73 gives

$$\iiint_V (x+y+z)\,dV = \iiint_{V_{uvw}} (x+y+z)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\,dv\,du$$

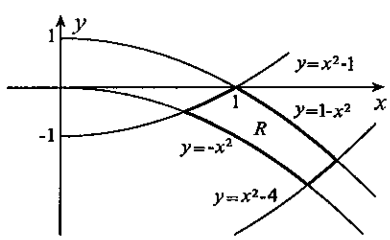
$$= \frac{1}{2}\iiint_{V_{uvw}} \left(\frac{u+v}{2} + \frac{v-u}{2} + w + \frac{u}{2} + \frac{3v}{2}\right) dw\,dv\,du$$

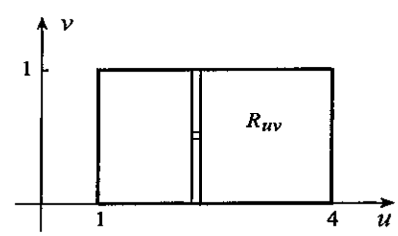
$$= \frac{1}{4}\int_{-1}^1\int_1^3\int_0^1 (2w+u+5v)\,dw\,dv\,du = \frac{1}{4}\int_{-1}^1\int_1^3 \left\{w^2+uw+5vw\right\}_0^1 dv\,du$$

$$= \frac{1}{4}\int_{-1}^{1}\int_{1}^{3}(1+u+5v)\,dv\,du = \frac{1}{4}\int_{-1}^{1}\left\{\frac{(1+u+5v)^2}{10}\right\}_1^3 du$$
$$= \frac{1}{40}\int_{-1}^{1}[(u+16)^2-(u+6)^2]\,du = \frac{1}{40}\left\{\frac{(u+16)^3}{3}-\frac{(u+6)^3}{3}\right\}_{-1}^{1} = 11.$$

21. (a) We require three iterated integrals for direct evaluation,

$$\iint_R (x+y)\,dA = \int_{1/\sqrt{2}}^{1}\int_{-x^2}^{x^2-1}(x+y)\,dy\,dx + \int_{1}^{\sqrt{2}}\int_{-x^2}^{1-x^2}(x+y)\,dy\,dx + \int_{\sqrt{2}}^{\sqrt{5/2}}\int_{x^2-4}^{1-x^2}(x+y)\,dy\,dx$$
$$= \int_{1/\sqrt{2}}^{1}\left\{\frac{(x+y)^2}{2}\right\}_{-x^2}^{x^2-1}dx + \int_{1}^{\sqrt{2}}\left\{\frac{(x+y)^2}{2}\right\}_{-x^2}^{1-x^2}dx + \int_{\sqrt{2}}^{\sqrt{5/2}}\left\{\frac{(x+y)^2}{2}\right\}_{x^2-4}^{1-x^2}dx$$
$$= \frac{1}{2}\int_{1/\sqrt{2}}^{1}(4x^3-2x^2-2x+1)\,dx + \frac{1}{2}\int_{1}^{\sqrt{2}}(1+2x-2x^2)\,dx + \frac{1}{2}\int_{\sqrt{2}}^{\sqrt{5/2}}(-4x^3+6x^2+10x-15)\,dx$$
$$= \frac{1}{2}\left\{x^4-\frac{2x^3}{3}-x^2+x\right\}_{1/\sqrt{2}}^{1} + \frac{1}{2}\left\{x+x^2-\frac{2x^3}{3}\right\}_{1}^{\sqrt{2}} + \frac{1}{2}\left\{-x^4+2x^3+5x^2-15x\right\}_{\sqrt{2}}^{\sqrt{5/2}}$$
$$= \frac{1}{12}(9+62\sqrt{2}-30\sqrt{10}).$$





(b) The transformation maps the region R in the xy-plane to the rectangle R_{uv} in the uv-plane shown above. The Jacobian of the transformation is $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \dfrac{1}{\begin{vmatrix}2x & -1\\ 2x & 1\end{vmatrix}} = \dfrac{1}{4x}$. Equation 13.70 gives

$$\iint_R (x+y)\,dA = \iint_{R_{uv}}(x+y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dv\,du = \iint_{R_{uv}}(x+y)\left(\frac{1}{4x}\right)dv\,du$$
$$= \frac{1}{4}\int_{1}^{4}\int_{0}^{1}\left(\sqrt{\frac{u+v}{2}}+\frac{v-u}{2}\right)\sqrt{\frac{2}{u+v}}\,dv\,du = \frac{1}{4}\int_{1}^{4}\int_{0}^{1}\left(1+\frac{v-u}{\sqrt{2}\sqrt{u+v}}\right)dv\,du$$
$$= \frac{1}{4}\int_{1}^{4}\left\{v\right\}_0^1 du - \frac{\sqrt{2}}{8}\int_{1}^{4}\left\{2u\sqrt{u+v}\right\}_0^1 du + \frac{\sqrt{2}}{8}\int_{1}^{4}\int_{0}^{1}\frac{v}{\sqrt{u+v}}\,dv\,du.$$

We set $z=u+v$ and $dz=dv$ in the last integral,

$$\iint_R (x+y)\,dA = \frac{1}{4}\left\{u\right\}_1^4 - \frac{\sqrt{2}}{4}\int_{1}^{4}(u\sqrt{u+1}-u^{3/2})\,du + \frac{\sqrt{2}}{8}\int_{1}^{4}\int_{u}^{u+1}\frac{z-u}{\sqrt{z}}\,dz\,du$$
$$= \frac{3}{4} - \frac{\sqrt{2}}{4}\int_{1}^{4}u\sqrt{u+1}\,du + \frac{\sqrt{2}}{4}\left\{\frac{2u^{5/2}}{5}\right\}_1^4 + \frac{\sqrt{2}}{8}\int_{1}^{4}\left\{\frac{2z^{3/2}}{3}-2u\sqrt{z}\right\}_u^{u+1}du$$
$$= \frac{3}{4} - \frac{\sqrt{2}}{4}\int_{1}^{4}u\sqrt{u+1}\,du + \frac{31\sqrt{2}}{10} + \frac{\sqrt{2}}{8}\int_{1}^{4}\left[\frac{2(u+1)^{3/2}}{3}-2u\sqrt{u+1}+\frac{4u^{3/2}}{3}\right]du.$$

We now combine the first integral and the second term in the last integral and set $z=u+1$ and $dz=du$,

$$\iint_R (x+y)\,dA = \frac{3}{4} - \frac{\sqrt{2}}{2}\int_2^5 (z-1)\sqrt{z}\,dz + \frac{31\sqrt{2}}{10} + \frac{\sqrt{2}}{8}\left\{\frac{4(u+1)^{5/2}}{15} + \frac{8u^{5/2}}{15}\right\}_1^4$$
$$= \frac{3}{4} - \frac{\sqrt{2}}{2}\left\{\frac{2z^{5/2}}{5} - \frac{2z^{3/2}}{3}\right\}_2^5 + \frac{31\sqrt{2}}{10} + \frac{\sqrt{2}}{30}(25\sqrt{5}+62-4\sqrt{2})$$
$$= \frac{1}{12}(9+62\sqrt{2}-30\sqrt{10}).$$

EXERCISES 13.14

1. With Leibnitz's rule, $F'(x) = \int_0^3 (2xy^2+3y)\,dy = \left\{\frac{2xy^3}{3} + \frac{3y^2}{2}\right\}_0^3 = 18x + \frac{27}{2}$. If we evaluate the integral, $F(x) = \int_0^3 (x^2y^2+3xy)\,dy = \left\{\frac{x^2y^3}{3} + \frac{3xy^2}{2}\right\}_0^3 = 9x^2 + \frac{27x}{2}$, and therefore $F'(x) = 18x + 27/2$.

2. With Leibnitz's rule, $F'(x) = \int_1^x \left(\frac{2x}{y^2}\right)dy + \left(\frac{x^2}{x^2} + e^x\right)(1) = \left\{-\frac{2x}{y}\right\}_1^x + 1 + e^x = e^x + 2x - 1$. If we evaluate the integral, $F(x) = \left\{-\frac{x^2}{y} + e^y\right\}_1^x = -x + e^x + x^2 - e$, in which case $F'(x) = -1 + e^x + 2x$.

3. With Leibnitz's rule,
$$F'(x) = \int_{x-1}^{x^2} 3x^2y\,dy + (x^5+x^4+1)(2x) - [x^3(x-1)+(x-1)^2+1](1)$$
$$= \left\{\frac{3x^2y^2}{2}\right\}_{x-1}^{x^2} + 2x^6 + 2x^5 - x^4 + x^3 - x^2 + 4x - 2$$
$$= \frac{3x^6}{2} - \frac{3x^2(x-1)^2}{2} + 2x^6 + 2x^5 - x^4 + x^3 - x^2 + 4x - 2$$
$$= \frac{7x^6}{2} + 2x^5 - \frac{5x^4}{2} + 4x^3 - \frac{5x^2}{2} + 4x - 2.$$

If we evaluate the integral,
$$F(x) = \int_{x-1}^{x^2} (x^3y+y^2+1)\,dy = \left\{\frac{x^3y^2}{2} + \frac{y^3}{3} + y\right\}_{x-1}^{x^2}$$
$$= \frac{x^7}{2} + \frac{x^6}{3} + x^2 - \frac{x^3(x-1)^2}{2} - \frac{(x-1)^3}{3} - (x-1),$$

and therefore
$$F'(x) = \frac{7x^6}{2} + 2x^5 + 2x - \frac{3x^2(x-1)^2}{2} - x^3(x-1) - (x-1)^2 - 1$$
$$= \frac{7x^6}{2} + 2x^5 - \frac{5x^4}{2} + 4x^3 - \frac{5x^2}{2} + 4x - 2.$$

4. With Leibnitz's rule,
$$F'(x) = \int_{x^2}^{x^3-1} dy + [x + (x^3-1)\ln(x^3-1)](3x^2) - [x + x^2\ln(x^2)](2x)$$
$$= x^3 - 1 - x^2 + 3x^2[x + (x^3-1)\ln(x^3-1)] - 2x[x + x^2\ln(x^2)]$$
$$= 4x^3 - 3x^2 - 1 + 3x^2(x^3-1)\ln(x^3-1) - 2x^3\ln(x^2).$$

If we evaluate the integral,

$$F(x) = \left\{xy + \frac{y^2}{2}\ln y - \frac{y^2}{4}\right\}_{x^2}^{x^3-1} = x(x^3-1) + \frac{1}{2}(x^3-1)^2\ln(x^3-1) - \frac{1}{4}(x^3-1)^2 - x^3 - \frac{x^4}{2}\ln(x^2) + \frac{x^4}{4},$$

in which case
$$F'(x) = 4x^3 - 1 + 3x^2(x^3-1)\ln(x^3-1) + \frac{1}{2}(x^3-1)(3x^2) - \frac{1}{2}(x^3-1)(3x^2)$$

$$-3x^2 - 2x^3 \ln(x^2) - x^3 + x^3$$
$$= 4x^3 - 3x^2 - 1 + 3x^2(x^3 - 1)\ln(x^3 - 1) - 2x^3 \ln(x^2).$$

5. With Leibnitz's rule,

$$F'(x) = \int_0^x \left[\frac{(y+x)(-1) - (y-x)(1)}{(y+x)^2}\right] dy = \int_0^x \frac{-2y}{(y+x)^2} dy = -2\int_0^x \left[\frac{1}{y+x} - \frac{x}{(y+x)^2}\right] dy$$
$$= -2\left\{\ln|y+x| + \frac{x}{y+x}\right\}_0^x = -2\left(\ln|2x| + \frac{1}{2} - \ln|x| - 1\right) = 1 - 2\ln 2.$$

If we evaluate the integral, $F(x) = \int_0^x \frac{y-x}{y+x} dy = \int_0^x \left(1 - \frac{2x}{y+x}\right) dy = \left\{y - 2x\ln|y+x|\right\}_0^x$
$$= x - 2x\ln|2x| + 2x\ln|x| = x - (2\ln 2)x,$$

and therefore, $F'(x) = 1 - 2\ln 2$.

6. $F(x) = \left\{\frac{y^4}{4}\ln y - \frac{y^4}{16} + x^3 e^y\right\}_x^{2x} = (4\ln 2)x^4 + \frac{15x^4}{4}\ln x - \frac{15x^4}{16} + x^3(e^{2x} - e^x)$, and therefore

$$F'(x) = (16\ln 2)x^3 + 15x^3\ln x + \frac{15x^3}{4} - \frac{15x^3}{4} + 3x^2(e^{2x} - e^x) + x^3(2e^{2x} - e^x)$$
$$= x^3(16\ln 2 + 15\ln x + 2e^{2x} - e^x) + 3x^2(e^{2x} - e^x).$$

7. Using Example 13.39,

$$\int_0^1 \frac{x^p - x^q}{\ln x} dx = \int_0^1 \frac{x^p - 1}{\ln x} dx - \int_0^1 \frac{x^q - 1}{\ln x} dx = \ln(p+1) - \ln(q+1) = \ln\left(\frac{p+1}{q+1}\right).$$

8. With Leibnitz's rule,

$$F'(x) = \int_{\sin x}^{e^x} 0\, dy + \sqrt{1 + e^{3x}}(e^x) - \sqrt{1 + \sin^3 x}(\cos x) = e^x\sqrt{1 + e^{3x}} - \cos x\sqrt{1 + \sin^3 x}.$$

9. $x\frac{dy}{dx} + 2y = x\left[-\frac{2}{x^3}\int_0^x t^2 f(t)\, dt + \frac{1}{x^2}x^2 f(x)\right] + \frac{2}{x^2}\int_0^x t f(t)\, dt = x f(x)$

10. Since $\frac{dy}{dx} = \frac{1}{2}\int_0^x f(t)(e^{x-t} + e^{t-x})\, dt$, it follows that $\frac{d^2y}{dx^2} = \frac{1}{2}\int_0^x f(t)(e^{x-t} - e^{t-x})\, dt + \frac{1}{2}f(x)(2)(1)$, and therefore

$$\frac{d^2y}{dx^2} - y = \frac{1}{2}\int_0^x f(t)(e^{x-t} - e^{t-x})\, dt + f(x) - \frac{1}{2}\int_0^x f(t)(e^{x-t} - e^{t-x})\, dt = f(x).$$

11. Since $\frac{dy}{dx} = \frac{1}{\sqrt{2}}\int_0^x \{-2e^{2(t-x)}\sin[\sqrt{2}(x-t)] + \sqrt{2}e^{2(t-x)}\cos[\sqrt{2}(x-t)]\}f(t)\, dt$, and

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{2}}\int_0^x \{4e^{2(t-x)}\sin[\sqrt{2}(x-t)] - 4\sqrt{2}e^{2(t-x)}\cos[\sqrt{2}(x-t)] - 2e^{2(t-x)}\sin[\sqrt{2}(x-t)]\}f(t)\, dt + f(x),$$

it follows that

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = \frac{1}{\sqrt{2}}\int_0^x \{2e^{2(t-x)}\sin[\sqrt{2}(x-t)] - 4\sqrt{2}e^{2(t-x)}\cos[\sqrt{2}(x-t)]\}f(t)\, dt + f(x)$$
$$+ \frac{4}{\sqrt{2}}\int_0^x \{-2e^{2(t-x)}\sin[\sqrt{2}(x-t)] + \sqrt{2}e^{2(t-x)}\cos[\sqrt{2}(x-t)]\}f(t)\, dt$$
$$+ \frac{6}{\sqrt{2}}\int_0^x e^{2(t-x)}\sin[\sqrt{2}(x-t)]f(t)\, dt = f(x).$$

12. Differentiation of $\displaystyle\int_0^b \frac{1}{1+ax}dx = \frac{1}{a}\ln(1+ab)$ with respect to a gives

$$\int_0^b \frac{-x}{(1+ax)^2}dx = -\frac{1}{a^2}\ln(1+ab) + \frac{1}{a}\frac{b}{1+ab} \implies \int_0^b \frac{x}{(1+ax)^2}dx = \frac{1}{a^2}\ln(1+ab) - \frac{b}{a(1+ab)}.$$

13. If we write $\displaystyle\int_0^b \frac{1}{\sqrt{a^2-x^2}}dx = \mathrm{Sin}^{-1}\left(\frac{b}{a}\right)$, then differentiation with respect to a gives

$$\int_0^b \frac{-a}{(a^2-x^2)^{3/2}}dx = \frac{1}{\sqrt{1-b^2/a^2}}\left(\frac{-b}{a^2}\right),$$

and therefore, $\displaystyle\int_0^b \frac{1}{(a^2-x^2)^{3/2}}dx = -\frac{1}{a}\left[\frac{a}{\sqrt{a^2-b^2}}\left(\frac{-b}{a^2}\right)\right] = \frac{b}{a^2\sqrt{a^2-b^2}}.$
Thus, $\displaystyle\int \frac{1}{(a^2-x^2)^{3/2}}dx = \frac{x}{a^2\sqrt{a^2-x^2}} + C.$

14. If we write $\displaystyle\int_0^b \frac{1}{a^2+x^2}dx = \frac{1}{a}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right)$, then differentiation with respect to a gives

$$\int_0^b \frac{-2a}{(a^2+x^2)^2}dx = -\frac{1}{a^2}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) + \frac{1}{a}\frac{1}{1+b^2/a^2}\left(\frac{-b}{a^2}\right),$$

or,

$$\int_0^b \frac{1}{(a^2+x^2)^2}dx = \frac{1}{2a^3}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) + \frac{b}{2a^2(a^2+b^2)}.$$

Another differentiation gives

$$\int_0^b \frac{-4a}{(a^2+x^2)^3}dx = -\frac{3}{2a^4}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) + \frac{1}{2a^3}\frac{1}{1+b^2/a^2}\left(\frac{-b}{a^2}\right) - \frac{b(8a^3+4ab^2)}{4a^4(a^2+b^2)^2},$$

or,

$$\begin{aligned}\int_0^b \frac{1}{(a^2+x^2)^3}dx &= -\frac{1}{4a}\left[-\frac{3}{2a^4}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) - \frac{b}{2a^3(a^2+b^2)} - \frac{b(2a^2+b^2)}{a^3(a^2+b^2)^2}\right]\\ &= \frac{3}{8a^5}\mathrm{Tan}^{-1}\left(\frac{b}{a}\right) + \frac{b(3b^2+5a^2)}{8a^4(a^2+b^2)^2}.\end{aligned}$$

Thus, $\displaystyle\int \frac{1}{(a^2+x^2)^3}dx = \frac{3}{8a^5}\mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + \frac{x(3x^2+5a^2)}{8a^4(a^2+x^2)^2} + C.$

15. Differentiation of the given formula with respect to a and b gives

$$\int_0^{\pi/2} \frac{-2a\cos^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2}dx = \frac{-\pi}{2a|ab|}, \qquad \int_0^{\pi/2} \frac{-2b\sin^2 x}{(a^2\cos^2 x + b^2\sin^2 x)^2}dx = \frac{-\pi}{2b|ab|}.$$

If we divide the first by $-2a$, the second by $-2b$, and add, the result is

$$\int_0^{\pi/2} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}dx = \frac{\pi(a^2+b^2)}{4|ab|^3}.$$

16. If we set $\displaystyle F(a) = \int_0^\pi \frac{\ln(1+a\cos x)}{\cos x}dx$, then $\displaystyle F'(a) = \int_0^\pi \frac{1}{1+a\cos x}dx.$

To evaluate this integral we let $t = \tan\dfrac{x}{2}$. Then $\cos x = \dfrac{1-t^2}{1+t^2}$, $dx = \dfrac{2}{1+t^2}dt$ (see Exercise 35 in Section 8.6), and

$$F'(a) = \int_0^\infty \frac{1}{1 + a\left(\dfrac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2} dt = 2\int_0^\infty \frac{1}{1+t^2+a(1-t^2)} dt = 2\int_0^\infty \frac{1}{(a+1)+(1-a)t^2} dt.$$

We now set $t = \sqrt{(1+a)/(1-a)}\tan\theta$ and $dt = \sqrt{(1+a)/(1-a)}\sec^2\theta\, d\theta$,

$$F'(a) = 2\int_0^{\pi/2} \frac{1}{(a+1)+(a+1)\tan^2\theta}\sqrt{\frac{1+a}{1-a}}\sec^2\theta\, d\theta = 2\left\{\frac{\theta}{\sqrt{1-a^2}}\right\}_0^{\pi/2} = \frac{\pi}{\sqrt{1-a^2}}.$$

Hence, $F(a) = \pi\mathrm{Sin}^{-1}a + C$. Since $F(0) = 0$, it follows that $0 = C$, and $F(a) = \pi\mathrm{Sin}^{-1}a$.

17. If we set $F(a) = \displaystyle\int_0^\infty \frac{\mathrm{Tan}^{-1}(ax)}{x(1+x^2)}dx$, then differentiation with respect to a gives

$$\begin{aligned} F'(a) &= \int_0^\infty \frac{x}{x(1+x^2)(1+a^2x^2)}\,dx = \int_0^\infty \left(\frac{1/(1-a^2)}{1+x^2} - \frac{a^2/(1-a^2)}{1+a^2x^2}\right)dx \\ &= \left\{\frac{1}{1-a^2}\mathrm{Tan}^{-1}x - \frac{a}{1-a^2}\mathrm{Tan}^{-1}(ax)\right\}_0^\infty = \frac{\pi/2}{1+a}. \end{aligned}$$

Integration gives $F(a) = \dfrac{\pi}{2}\ln(1+a) + C$. Since $F(0) = 0$, it follows that $C = 0$ and therefore $F(a) = (\pi/2)\ln(1+a)$.

18. We calculate:

$$\begin{aligned} \frac{\partial T}{\partial t} &= e^{-(1-x)^2/(4t)}\left(\frac{x-1}{4t^{3/2}}\right) + e^{-(1+x)^2/(4t)}\left(\frac{-x-1}{4t^{3/2}}\right), \\ \frac{\partial T}{\partial x} &= e^{-(1-x)^2/(4t)}\left(\frac{-1}{2\sqrt{t}}\right) + e^{-(1+x)^2/(4t)}\left(\frac{1}{2\sqrt{t}}\right), \\ \frac{\partial^2 T}{\partial x^2} &= e^{-(1-x)^2/(4t)}\left(\frac{x-1}{4t^{3/2}}\right) + e^{-(1+x)^2/(4t)}\left(\frac{-1-x}{4t^{3/2}}\right). \end{aligned}$$

Thus, $\partial T/\partial t = \partial^2 T/\partial x^2$.

19. (a) Since $1 - x^2y^2$ must be positive, it follows that $x^2 < 1/y^2$. Since y ranges from 0 to 9, it follows that x must be restricted to $-1/9 < x < 1/9$. Clearly, $F(0) = \displaystyle\int_0^9 \ln(1)\,dy = 0$.

(b)
$$\begin{aligned} F'(x) &= \int_0^9 \frac{-2xy^2}{1-x^2y^2}dy = \int_0^9 \left(\frac{2}{x} + \frac{1/x}{xy-1} - \frac{1/x}{xy+1}\right)dy \\ &= \left\{\frac{2y}{x} + \frac{1}{x^2}\ln|xy-1| - \frac{1}{x^2}\ln|xy+1|\right\}_0^9 = \frac{18}{x} + \frac{1}{x^2}\ln\left(\frac{1-9x}{1+9x}\right), \quad x \neq 0. \end{aligned}$$

From $F'(x) = \displaystyle\int_0^9 \frac{-2xy^2}{1-x^2y^2}dy$, we obtain $F'(0) = 0$.

(c) If we use Leibnitz's rule,

$$F''(x) = \int_0^9 \left[\frac{(1-x^2y^2)(-2y^2) - (-2xy^2)(-2xy^2)}{(1-x^2y^2)^2}\right]dy = \int_0^9 \frac{-2y^2(1+x^2y^2)}{(1-x^2y^2)^2}dy.$$

Since $F''(x)$ is clearly negative, the graph of $F(x)$ is always concave downward.

20. We calculate that

$$\begin{aligned} \frac{\partial u}{\partial r} &= -\frac{2r}{2\pi}\int_{-\pi}^{\pi} \frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi + \frac{R^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{-u(R,\phi)[2r-2R\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi \\ &= -\frac{r}{\pi}\int_{-\pi}^{\pi} \frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi + \frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi} \frac{-u(R,\phi)[r-R\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi \\ \frac{\partial u}{\partial \theta} &= \frac{R^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{-u(R,\phi)[2rR\sin(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi = \frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi} \frac{-u(R,\phi)rR\sin(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi \end{aligned}$$

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi - \frac{r}{\pi}\int_{-\pi}^{\pi}\frac{-u(R,\phi)[2r-2R\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$-\frac{2r}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)[R\cos(\theta-\phi)-r]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi + \frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}u(R,\phi)\left\{\frac{-1}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}\right.$$
$$\left.+\frac{4[R\cos(\theta-\phi)-r]^2}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi$$
$$= -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi - \frac{4r}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)[R\cos(\theta-\phi)-r]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$+\frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}u(R,\phi)\left\{\frac{-1}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}+\frac{4[R\cos(\theta-\phi)-r]^2}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi$$
$$\frac{\partial^2 u}{\partial \theta^2} = \frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}-u(R,\phi)\left\{\frac{rR\cos(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}-\frac{4r^2R^2\sin^2(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi.$$

With these,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial r^2} = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi - \frac{4r}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)[R\cos(\theta-\phi)-r]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$+\frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}u(R,\phi)\left\{\frac{-1}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}+\frac{4[R\cos(\theta-\phi)-r]^2}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi$$
$$-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi + \frac{R^2-r^2}{\pi r}\int_{-\pi}^{\pi}\frac{-u(R,\phi)[r-R\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$\frac{R^2-r^2}{\pi r^2}\int_{-\pi}^{\pi}-u(R,\phi)\left\{\frac{rR\cos(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}-\frac{4r^2R^2\sin^2(\theta-\phi)}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}\right\}d\phi$$
$$= -\frac{2}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)\{-4r[R\cos(\theta-\phi)-r]-2R^2+2r^2\}}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$+\frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}\frac{4u(R,\phi)\{[R\cos(\theta-\phi)-r]^2+R^2\sin^2(\theta-\phi)\}}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}d\phi$$
$$= -\frac{2}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)}{R^2+r^2-2rR\cos(\theta-\phi)}d\phi - \frac{2}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)[R^2-3r^2+2rR\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi$$
$$+\frac{R^2-r^2}{\pi}\int_{-\pi}^{\pi}\frac{4u(R,\phi)[R^2+r^2-2rR\cos(\theta-\phi)]}{[R^2+r^2-2rR\cos(\theta-\phi)]^3}d\phi$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(R,\phi)\{-2[R^2+r^2-2rR\cos(\theta-\phi)]-2[R^2-3r^2+2rR\cos(\theta-\phi)]+4R^2-4r^2\}}{[R^2+r^2-2rR\cos(\theta-\phi)]^2}d\phi = 0.$$

21. Using Leibnitz's rule:

$$\frac{\partial u}{\partial r} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)[2r-2R\cos(\theta-u)]}{R^2+r^2-2rR\cos(\theta-u)}du;$$
$$\frac{\partial u}{\partial \theta} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)[2rR\sin(\theta-u)]}{R^2+r^2-2rR\cos(\theta-u)}du;$$
$$\frac{\partial^2 u}{\partial r^2} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}f(u)\left\{\frac{2}{R^2+r^2-2rR\cos(\theta-u)}-\frac{[2r-2R\cos(\theta-u)]^2}{[R^2+r^2-2rR\cos(\theta-u)]^2}\right\}du;$$
$$\frac{\partial^2 u}{\partial \theta^2} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}f(u)\left\{\frac{2rR\cos(\theta-u)}{R^2+r^2-2rR\cos(\theta-u)}-\frac{[2rR\sin(\theta-u)]^2}{[R^2+r^2-2rR\cos(\theta-u)]^2}\right\}du.$$

Thus,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{R}{2\pi}\int_{-\pi}^{\pi}f(u)\left\{\frac{2[R^2+r^2-2rR\cos(\theta-u)]-[2r-2R\cos(\theta-u)]^2}{[R^2+r^2-2rR\cos(\theta-u)]^2}du\right\}du$$

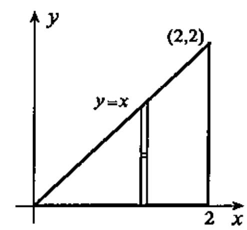
$$-\frac{R}{2\pi r}\int_{-\pi}^{\pi}\frac{f(u)[2r-2R\cos(\theta-u)]}{R^2+r^2-2rR\cos(\theta-u)}du$$
$$-\frac{R}{2\pi r^2}\int_{-\pi}^{\pi}f(u)\left\{\frac{2rR\cos(\theta-u)[R^2+r^2-2rR\cos(\theta-u)]-[2rR\sin(\theta-u)]^2}{[R^2+r^2-2rR\cos(\theta-u)]^2}\right\}du.$$

If we bring all three integrals together, and factor $-Rf(u)/\{2\pi r^2[R^2+r^2-2rR\cos(\theta-u)]^2$ from each term, the remaining factor in the integrand is

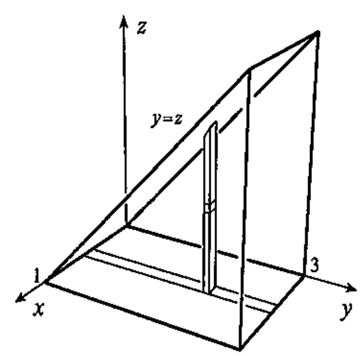
$$\begin{aligned}&2r^2[R^2+r^2-2rR\cos(\theta-u)]-r^2[2r-2R\cos(\theta-u)]^2+r[2r-2R\cos(\theta-u)][R^2+r^2-2rR\cos(\theta-u)]\\&\qquad+2rR\cos(\theta-u)[R^2+r^2-2rR\cos(\theta-u)]-[2rR\sin(\theta-u)]^2\\&=2r^2[R^2+r^2-2rR\cos(\theta-u)]-r^2[4r^2-8rR\cos(\theta-u)+4R^2\cos^2(\theta-u)]\\&\qquad+2r^2[R^2+r^2-2rR\cos(\theta-u)]-2rR\cos(\theta-u)[R^2+r^2-2rR\cos(\theta-u)]\\&\qquad+2rR\cos(\theta-u)[R^2+r^2-2rR\cos(\theta-u)]-4r^2R^2\sin^2(\theta-u)\\&=4r^2R^2-4r^2R^2\cos^2(\theta-u)-4r^2R^2\sin^2(\theta-u)=0.\end{aligned}$$

REVIEW EXERCISES

1. $$\iint_R(2x+y)\,dA=\int_0^2\int_0^x(2x+y)\,dy\,dx=\int_0^2\left\{2xy+\frac{y^2}{2}\right\}_0^x dx$$
$$=\frac{1}{2}\int_0^2(4x^2+x^2)\,dx$$
$$=\frac{5}{2}\left\{\frac{x^3}{3}\right\}_0^2=\frac{20}{3}$$

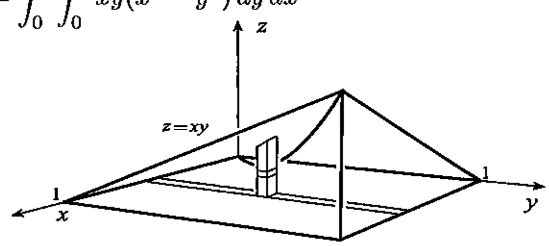


2. $$\iiint_V xyz\,dV=\int_0^1\int_0^3\int_0^y xyz\,dz\,dy\,dx$$
$$=\int_0^1\int_0^3\left\{\frac{xyz^2}{2}\right\}_0^y dy\,dx$$
$$=\frac{1}{2}\int_0^1\int_0^3 xy^3\,dy\,dx$$
$$=\frac{1}{2}\int_0^1\left\{\frac{xy^4}{4}\right\}_0^3 dx$$
$$=\frac{81}{8}\int_0^1 x\,dx=\frac{81}{8}\left\{\frac{x^2}{2}\right\}_0^1=\frac{81}{16}$$



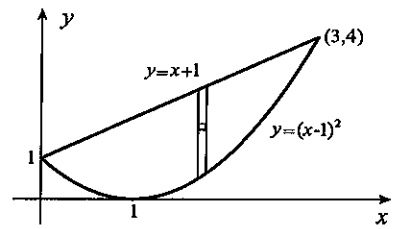
3. Since R is symmetric about the y-axis, and x^3y^2 is an odd function of x, the value of the double integral is 0.

4. $$\iiint_V(x^2-y^3)\,dV=\int_0^1\int_0^1\int_0^{xy}(x^2-y^3)\,dz\,dy\,dx=\int_0^1\int_0^1 xy(x^2-y^3)\,dy\,dx$$
$$=\int_0^1\left\{\frac{x^3y^2}{2}-\frac{xy^5}{5}\right\}_0^1 dx$$
$$=\frac{1}{10}\int_0^1(5x^3-2x)\,dx$$
$$=\frac{1}{10}\left\{\frac{5x^4}{4}-x^2\right\}_0^1=\frac{1}{40}$$

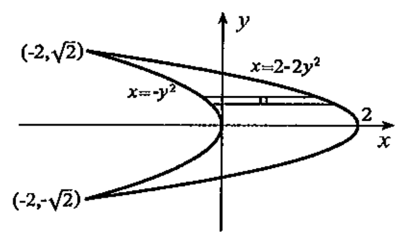


5. $$\iiint_V(x^2-y^2)\,dV=\int_0^1\int_0^1\int_0^{xy}(x^2-y^2)\,dz\,dy\,dx=\int_0^1\int_0^1(x^3y-xy^3)\,dy\,dx$$
$$=\int_0^1\left\{\frac{x^3y^2}{2}-\frac{xy^4}{4}\right\}_0^1 dx=\frac{1}{4}\int_0^1(2x^3-x)\,dx=\frac{1}{4}\left\{\frac{x^4}{2}-\frac{x^2}{2}\right\}_0^1=0$$

6. $$\iint_R y\,dA = \int_0^3 \int_{(x-1)^2}^{x+1} y\,dy\,dx$$
$$= \int_0^3 \left\{ \frac{y^2}{2} \right\}_{(x-1)^2}^{x+1} dx$$
$$= \frac{1}{2}\int_0^3 [(x+1)^2 - (x-1)^4]\,dx$$
$$= \frac{1}{2}\left\{ \frac{1}{3}(x+1)^3 - \frac{1}{5}(x-1)^5 \right\}_0^3 = \frac{36}{5}$$



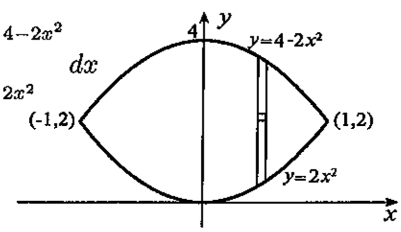
7. $$\iint_R xy^2\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-y^2}^{2-2y^2} xy^2\,dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left\{ \frac{x^2y^2}{2} \right\}_{-y^2}^{2-2y^2} dy$$
$$= \frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} [y^2(2-2y^2)^2 - y^2(-y^2)^2]\,dy$$
$$= \frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} (4y^2 - 8y^4 + 3y^6)\,dy$$
$$= \frac{1}{2}\left\{ \frac{4y^3}{3} - \frac{8y^5}{5} + \frac{3y^7}{7} \right\}_{-\sqrt{2}}^{\sqrt{2}} = -\frac{32\sqrt{2}}{105}$$



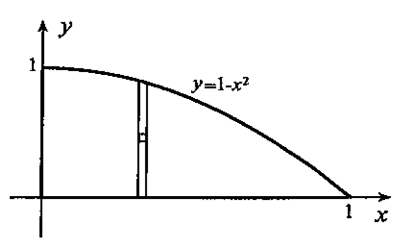
8. Since R is symmetric about the x-axis and x^2y is an odd function of y, $\iint_R x^2y\,dA = 0$.

9. $$\iiint_V (x^2+y^2+z^2)\,dV = \int_0^2 \int_{-z}^{z} \int_0^1 (x^2+y^2+z^2)\,dy\,dx\,dz = \int_0^2 \int_{-z}^{z} \left\{ x^2y + \frac{y^3}{3} + z^2y \right\}_0^1 dx\,dz$$
$$= \frac{1}{3}\int_0^2 \int_{-z}^{z} (3x^2 + 1 + 3z^2)\,dx\,dz = \frac{1}{3}\int_0^2 \left\{ x^3 + x + 3z^2x \right\}_{-z}^{z} dz$$
$$= \frac{1}{3}\int_0^2 (8z^3 + 2z)\,dz = \frac{1}{3}\left\{ 2z^4 + z^2 \right\}_0^2 = 12$$

10. $$\iint_R (xy - x^2y^2)\,dA = 2\int_0^1 \int_{2x^2}^{4-2x^2} -x^2y^2\,dy\,dx = -2\int_0^1 \left\{ \frac{x^2y^3}{3} \right\}_{2x^2}^{4-2x^2} dx$$
$$= -\frac{32}{3}\int_0^1 (4x^2 - 6x^4 + 3x^6 - x^8)\,dx$$
$$= -\frac{32}{3}\left\{ \frac{4x^3}{3} - \frac{6x^5}{5} + \frac{3x^7}{7} - \frac{x^9}{9} \right\}_0^1 = -\frac{4544}{945}$$

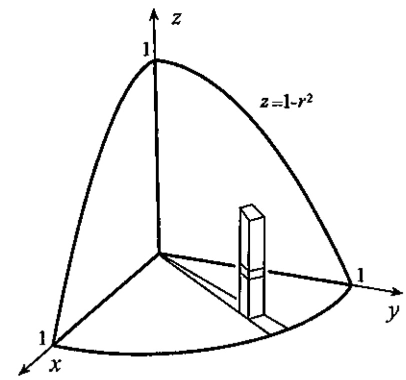


11. $$\iint_R x\sin y\,dA = \int_0^1 \int_0^{1-x^2} x\sin y\,dy\,dx$$
$$= \int_0^1 \left\{ -x\cos y \right\}_0^{1-x^2} dx$$
$$= \int_0^1 [-x\cos(1-x^2) + x]\,dx$$
$$= \left\{ \frac{1}{2}\sin(1-x^2) + \frac{x^2}{2} \right\}_0^1 = \frac{1-\sin 1}{2}$$

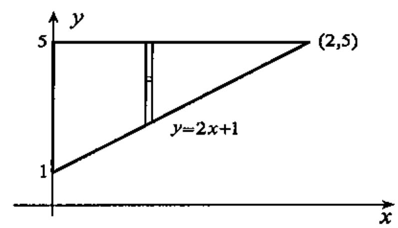


12. Integrals of x and y yield zero. We quadruple the integral of z over the first octant volume.

$$\begin{aligned}\iiint_V (x+y+z)\,dV &= \iiint_V z\,dV\\ &= 4\int_0^{\pi/2}\int_0^1\int_0^{1-r^2} z\,r\,dz\,dr\,d\theta\\ &= 4\int_0^{\pi/2}\int_0^1 \left\{\frac{rz^2}{2}\right\}_0^{1-r^2} dr\,d\theta\\ &= 2\int_0^{\pi/2}\int_0^1 r(1-r^2)^2\,dr\,d\theta\\ &= 2\int_0^{\pi/2}\left\{-\frac{1}{6}(1-r^2)^3\right\}_0^1 d\theta\\ &= \frac{1}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{\pi}{6}\end{aligned}$$



13.
$$\begin{aligned}\iint_R xe^y\,dA &= \int_0^2\int_{2x+1}^5 xe^y\,dy\,dx\\ &= \int_0^2 \Big\{xe^y\Big\}_{2x+1}^5 dx\\ &= \int_0^2 (e^5x - xe^{2x+1})\,dx\end{aligned}$$

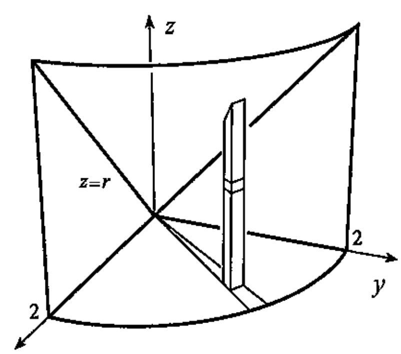


If we set $u = x$, $dv = e^{2x+1}\,dx$, $du = dx$, and $v = (1/2)e^{2x+1}$ in the second term,

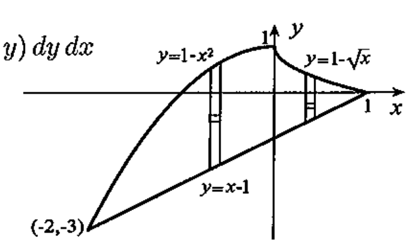
$$\iint_R xe^y\,dA = \left\{\frac{e^5x^2}{2}\right\}_0^2 - \left\{\frac{x}{2}e^{2x+1}\right\}_0^2 + \int_0^2 \frac{1}{2}e^{2x+1}\,dx = e^5 + \frac{1}{2}\left\{\frac{e^{2x+1}}{2}\right\}_0^2 = \frac{5}{4}e^5 - \frac{e}{4}.$$

14. We multiply the first octant volume by 8.

$$\begin{aligned}\iiint_V dV &= 8\int_0^{\pi/2}\int_0^2\int_0^r r\,dz\,dr\,d\theta\\ &= 8\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta\\ &= 8\int_0^{\pi/2}\left\{\frac{r^3}{3}\right\}_0^2 d\theta\\ &= \frac{64}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{32\pi}{3}\end{aligned}$$

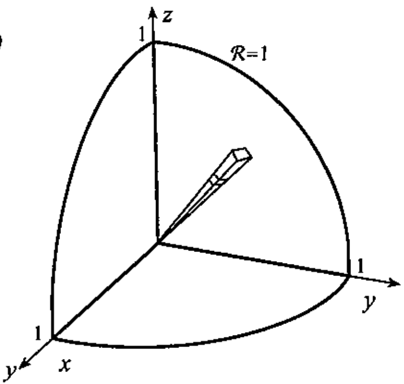


15.
$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{-2}^0\int_{x-1}^{1-x^2}(x+y)\,dy\,dx + \int_0^1\int_{x-1}^{1-\sqrt{x}}(x+y)\,dy\,dx\\ &= \int_{-2}^0\left\{\frac{(x+y)^2}{2}\right\}_{x-1}^{1-x^2} dx\\ &\quad + \int_0^1\left\{\frac{(x+y)^2}{2}\right\}_{x-1}^{1-\sqrt{x}} dx\\ &= \frac{1}{2}\int_{-2}^0 (x^4 - 2x^3 - 5x^2 + 6x)\,dx + \frac{1}{2}\int_0^1 (7x - 3x^2 - 2\sqrt{x} - 2x^{3/2})\,dx\\ &= \frac{1}{2}\left\{\frac{x^5}{5} - \frac{x^4}{2} - \frac{5x^3}{3} + 3x^2\right\}_{-2}^0 + \frac{1}{2}\left\{\frac{7x^2}{2} - x^3 - \frac{4x^{3/2}}{3} - \frac{4x^{5/2}}{5}\right\}_0^1 = -\frac{317}{60}\end{aligned}$$

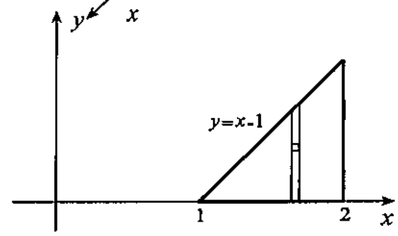


16. We quadruple the integral over the volume in the first octant.

$$\iiint_V (x^2+y^2+z^2)\,dV = 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \Re^2\,\Re^2\,\sin\phi\,d\Re\,d\phi\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^5}{5}\right\}_0^1 \sin\phi\,d\phi\,d\theta$$
$$= \frac{4}{5}\int_0^{\pi/2}\left\{-\cos\phi\right\}_0^{\pi/2} d\theta$$
$$= \frac{4}{5}\left\{\theta\right\}_0^{\pi/2} = \frac{2\pi}{5}$$



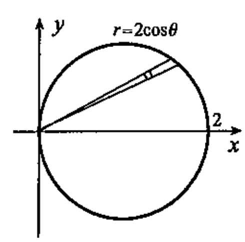
17. $$\iint_R \frac{x}{x+y}\,dA = \int_1^2\int_0^{x-1}\frac{x}{x+y}\,dy\,dx$$
$$= \int_1^2\left\{x\ln|x+y|\right\}_0^{x-1} dx$$
$$= \int_1^2 [x\ln(2x-1) - x\ln x]\,dx$$



We use integration by parts on each of these. In the first we set $u=\ln(2x-1)$, $dv = x\,dx$, $du = 2dx/(2x-1)$, $v = x^2/2$; in the second $u=\ln x$, $dv = x\,dx$, $du = (1/x)dx$, $v = x^2/2$;

$$\iint_R \frac{x}{x+y}\,dA = \left\{\frac{x^2}{2}\ln(2x-1)\right\}_1^2 - \int_1^2 \frac{x^2}{2x-1}dx - \left\{\frac{x^2}{2}\ln x\right\}_1^2 + \int_1^2 \frac{x}{2}dx$$
$$= 2\ln 3 - \int_1^2\left(\frac{x}{2}+\frac{1}{4}+\frac{1/4}{2x-1}\right)dx - 2\ln 2 + \left\{\frac{x^2}{4}\right\}_1^2$$
$$= 2\ln 3 - 2\ln 2 + \frac{3}{4} - \left\{\frac{x^2}{4}+\frac{x}{4}+\frac{1}{8}\ln(2x-1)\right\}_1^2 = \frac{15}{8}\ln 3 - 2\ln 2 - \frac{1}{4}.$$

18. $$\iint_R (x^2+y^2)\,dA = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,r\,dr\,d\theta = 2\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^{2\cos\theta} d\theta$$
$$= 8\int_0^{\pi/2}\cos^4\theta\,d\theta$$
$$= 2\int_0^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$
$$= 2\left\{\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = \frac{3\pi}{2}$$



19. $$\iiint_V \frac{x^2}{z^2}dV = 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}}\frac{r^2\cos^2\theta}{z^2}r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\left\{\frac{-r^3\cos^2\theta}{z}\right\}_1^{\sqrt{4-r^2}} dr\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\left(r^3 - \frac{r^3}{\sqrt{4-r^2}}\right)\cos^2\theta\,dr\,d\theta$$

If we set $u = 4-r^2$ and $du = -2r\,dr$ in the second term,

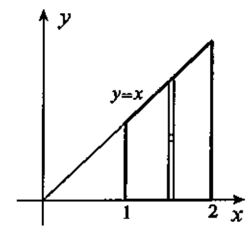
$$\iiint_V \frac{x^2}{z^2}dV = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^{\sqrt{3}} d\theta - 4\int_0^{\pi/2}\int_4^1\left(\frac{4-u}{\sqrt{u}}\right)\cos^2\theta\left(\frac{du}{-2}\right)d\theta$$
$$= 9\int_0^{\pi/2}\cos^2\theta\,d\theta + 2\int_0^{\pi/2}\left\{\left(8\sqrt{u}-\frac{2u^{3/2}}{3}\right)\cos^2\theta\right\}_4^1 d\theta$$
$$= \frac{7}{3}\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{7}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{7}{6}\left\{\theta+\frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{7\pi}{12}.$$

20. $\displaystyle\iint_R \frac{1}{x^2+y^2}\,dA = \int_1^2\int_0^x \frac{1}{x^2+y^2}\,dy\,dx$

If we set $y = x\tan\theta$, then $dy = x\sec^2\theta\,d\theta$, and

$$\iint_R \frac{1}{x^2+y^2}\,dA = \int_1^2\int_0^{\pi/4}\frac{1}{x^2\sec^2\theta}x\sec^2\theta\,d\theta\,dx$$

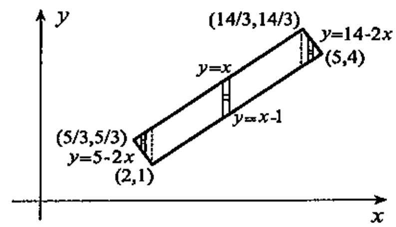
$$= \int_1^2\left\{\frac{\theta}{x}\right\}_0^{\pi/4}dx = \frac{\pi}{4}\int_1^2\frac{1}{x}dx = \frac{\pi}{4}\Big\{\ln|x|\Big\}_1^2 = \frac{\pi}{4}\ln 2.$$



21. $\displaystyle\iint_R (x^2-y^2)\,dA = \int_{5/3}^2\int_{5-2x}^x (x^2-y^2)\,dy\,dx + \int_2^{14/3}\int_{x-1}^x (x^2-y^2)\,dy\,dx + \int_{14/3}^5\int_{x-1}^{14-2x}(x^2-y^2)\,dy\,dx$

$$= \int_{5/3}^2\left\{x^2y - \frac{y^3}{3}\right\}_{5-2x}^x dx + \int_2^{14/3}\left\{x^2y-\frac{y^3}{3}\right\}_{x-1}^x dx$$

$$+ \int_{14/3}^5\left\{x^2y-\frac{y^3}{3}\right\}_{x-1}^{14-2x}dx$$



$$= \frac{1}{3}\int_{5/3}^2[3x^3 - x^3 - 3x^2(5-2x) + (5-2x)^3]\,dx$$

$$+\frac{1}{3}\int_2^{14/3}[3x^3 - x^3 - 3x^2(x-1) + (x-1)^3]\,dx$$

$$+\frac{1}{3}\int_{14/3}^5[3x^2(14-2x) - (14-2x)^3 - 3x^2(x-1) + (x-1)^3]\,dx$$

$$= \frac{1}{3}\left\{2x^4 - 5x^3 - \frac{(5-2x)^4}{8}\right\}_{5/3}^2 + \frac{1}{3}\left\{-\frac{x^4}{4} + x^3 + \frac{(x-1)^4}{4}\right\}_2^{14/3}$$

$$+\frac{1}{3}\left\{15x^3 - \frac{9x^4}{4} + \frac{(14-2x)^4}{8} + \frac{(x-1)^4}{4}\right\}_{14/3}^5 = \frac{55}{6}$$

22. See answer in text.

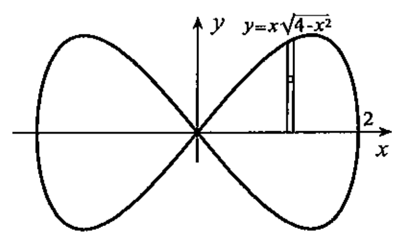
23. $\displaystyle V = 4\int_0^{\pi/2}\int_0^{\sqrt{\ln 2}}\int_{1-2e^{-r^2}}^0 r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{\sqrt{\ln 2}} r(-1+2e^{-r^2})\,dr\,d\theta = 4\int_0^{\pi/2}\left\{-\frac{r^2}{2} - e^{-r^2}\right\}_0^{\sqrt{\ln 2}}d\theta$

$$= 2(-\ln 2 - 2e^{-\ln 2} + 2)\Big\{\theta\Big\}_0^{\pi/2} = \pi(1-\ln 2)$$

24. $\displaystyle A = 4\int_0^2\int_0^{x\sqrt{4-x^2}}dy\,dx$

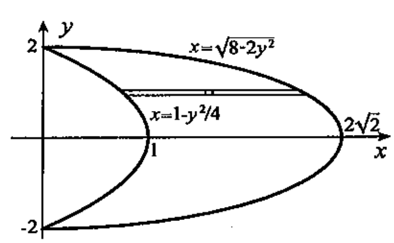
$$= 4\int_0^2 x\sqrt{4-x^2}\,dx$$

$$= 4\left\{-\frac{1}{3}(4-x^2)^{3/2}\right\}_0^2 = \frac{32}{3}$$



25. $\displaystyle A = 2\int_0^2\int_{1-y^2/4}^{\sqrt{8-2y^2}}dx\,dy$

$$= 2\int_0^2\left(\sqrt{8-2y^2} - 1 + \frac{y^2}{4}\right)dy$$



If we set $y = 2\sin\theta$ and $dy = 2\cos\theta\,d\theta$,

$$A = 2\int_0^{\pi/2}2\sqrt{2}\cos\theta(2\cos\theta\,d\theta) + 2\left\{-y+\frac{y^3}{12}\right\}_0^2 = 8\sqrt{2}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta - \frac{8}{3}$$

$$= 4\sqrt{2}\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} - \frac{8}{3} = \frac{6\sqrt{2}\pi - 8}{3}.$$

By symmetry, $\overline{y} = 0$. Since

$$A\overline{x} = 2\int_0^2\int_{1-y^2/4}^{\sqrt{8-2y^2}} x\,dx\,dy = 2\int_0^2\left\{\frac{x^2}{2}\right\}_{1-y^2/4}^{\sqrt{8-2y^2}} dy = \int_0^2 [(8-2y^2) - (1-y^2/4)^2]\,dy$$

$$= \int_0^2\left(7 - \frac{3y^2}{2} - \frac{y^4}{16}\right)dy = \left\{7y - \frac{y^3}{2} - \frac{y^5}{80}\right\}_0^2 = \frac{48}{5},$$

it follows that $\overline{x} = \dfrac{48}{5}\dfrac{3}{6\sqrt{2}\pi - 8} = \dfrac{72}{15\sqrt{2}\pi - 20}$.

26. We quadruple the volume in the first octant.

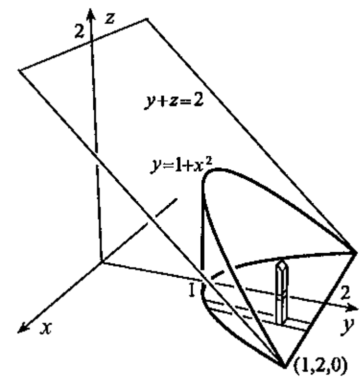
$$V = 4\int_0^1\int_{1+x^2}^2\int_0^{2-y} dz\,dy\,dx$$

$$= 4\int_0^1\int_{1+x^2}^2 (2-y)\,dy\,dx$$

$$= 4\int_0^1\left\{-\frac{1}{2}(2-y)^2\right\}_{1+x^2}^2 dx$$

$$= 2\int_0^1 (1 - 2x^2 + x^4)\,dx$$

$$= 2\left\{x - \frac{2x^3}{3} + \frac{x^5}{5}\right\}_0^1 = \frac{16}{15}$$



27. $F = 2\displaystyle\int_{-50.05}^{-49.95}\int_0^{\sqrt{1/400-(y+50)^2}} 1000(9.81)(-y)\,dx\,dy = -19\,620\int_{-50.05}^{-49.95} y\sqrt{\frac{1}{400} - (y+50)^2}\,dy$

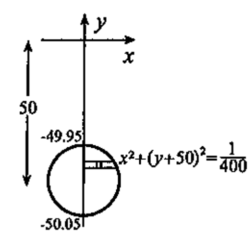
If we set $y + 50 = (1/20)\sin\theta$ and $dy = (1/20)\cos\theta\,d\theta$,

$$F = -19\,620\int_{-\pi/2}^{\pi/2}\left(-50 + \frac{1}{20}\sin\theta\right)\left(\frac{1}{20}\cos\theta\right)\frac{1}{20}\cos\theta\,d\theta$$

$$= -\frac{981}{400}\int_{-\pi/2}^{\pi/2}(-1000\cos^2\theta + \cos^2\theta\sin\theta)\,d\theta$$

$$= -\frac{981}{400}\int_{-\pi/2}^{\pi/2}[\cos^2\theta\sin\theta - 500(1 + \cos 2\theta)]\,d\theta$$

$$= -\frac{981}{400}\left\{-\frac{1}{3}\cos^3\theta - 500\theta - 250\sin 2\theta\right\}_{-\pi/2}^{\pi/2} = \frac{4905\pi}{4}\text{ N}.$$



28. We quadruple the area in the first octant.

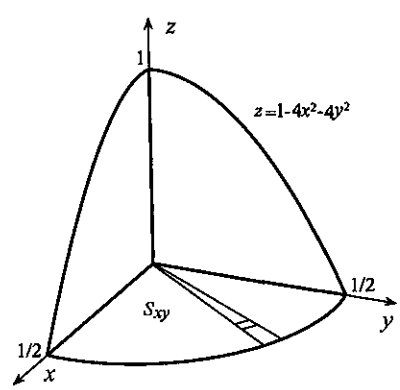
$$A = 4\iint_{S_{xy}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= 4\iint_{S_{xy}}\sqrt{1 + (-8x)^2 + (-8y)^2}\,dA$$

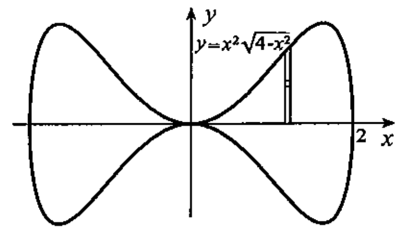
$$= 4\int_0^{\pi/2}\int_0^{1/2}\sqrt{1 + 64r^2}\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left\{\frac{1}{192}(1 + 64r^2)^{3/2}\right\}_0^{1/2} d\theta$$

$$= \frac{17\sqrt{17} - 1}{48}\left\{\theta\right\}_0^{\pi/2} = \frac{(17\sqrt{17} - 1)\pi}{96}$$



29. $A = 4\int_0^2 \int_0^{x^2\sqrt{4-x^2}} dy\, dx = 4\int_0^2 x^2\sqrt{4-x^2}\, dx$

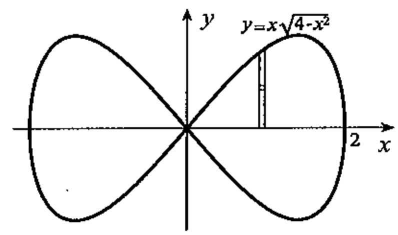


If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$,

$$A = 4\int_0^{\pi/2} 4\sin^2\theta(2\cos\theta)(2\cos\theta\, d\theta) = 64\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= 16\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = 8\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = 4\pi.$$

30. $V_x = 2\int_0^2 \int_0^{x\sqrt{4-x^2}} 2\pi y\, dy\, dx = 2\pi\int_0^2 \left\{y^2\right\}_0^{x\sqrt{4-x^2}} dx$



$$= 2\pi\int_0^2 x^2(4-x^2)\, dx = 2\pi\left\{\frac{4x^3}{3} - \frac{x^5}{5}\right\}_0^2 = \frac{128\pi}{15}$$

$$V_y = 2\int_0^2 \int_0^{x\sqrt{4-x^2}} 2\pi x\, dy\, dx = 4\pi\int_0^2 x^2\sqrt{4-x^2}\, dx$$

If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$, then

$$V_y = 4\pi\int_0^{\pi/2} 4\sin^2\theta\, 2\cos\theta\, 2\cos\theta\, d\theta = 64\pi\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= 16\pi\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta = 8\pi\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = 4\pi^2$$

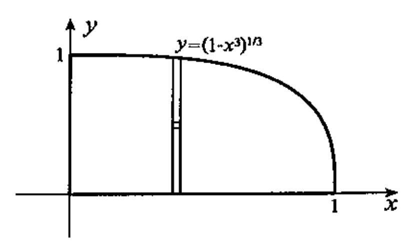
31. $I = 4\int_0^{\pi/2}\int_0^1\int_r^{2-r} \rho r^2\, r\, dz\, dr\, d\theta = 4\rho\int_0^{\pi/2}\int_0^1 \left\{r^3 z\right\}_r^{2-r} dr\, d\theta = 8\rho\int_0^{\pi/2}\int_0^1 (r^3 - r^4)\, dr\, d\theta$

$$= 8\rho\int_0^{\pi/2}\left\{\frac{r^4}{4} - \frac{r^5}{5}\right\}_0^1 d\theta = \frac{2\rho}{5}\left\{\theta\right\}_0^{\pi/2} = \frac{\rho\pi}{5}$$

32. $I = \int_0^1 \int_0^{(1-x^3)^{1/3}} x^2\, dy\, dx$

$$= \int_0^1 x^2(1-x^3)^{1/3}\, dx$$

$$= \left\{-\frac{1}{4}(1-x^3)^{4/3}\right\}_0^1 = \frac{1}{4}$$



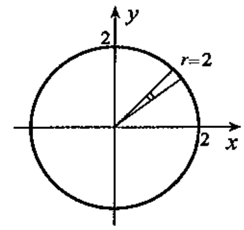
33. By symmetry, $\overline{x} = \overline{y} = 0$.

$$M = 4\int_0^{\pi/2}\int_0^1\int_0^{1+r^2} \rho r\, dz\, dr\, d\theta = 4\rho\int_0^{\pi/2}\int_0^1 r(1+r^2)\, dr\, d\theta = 4\rho\int_0^{\pi/2}\left\{\frac{r^2}{2} + \frac{r^4}{4}\right\}_0^1 d\theta = 3\rho\left\{\theta\right\}_0^{\pi/2} = \frac{3\pi\rho}{2}$$

Since $M\overline{z} = 4\int_0^{\pi/2}\int_0^1\int_0^{1+r^2} z\rho r\, dz\, dr\, d\theta = 4\rho\int_0^{\pi/2}\int_0^1 \left\{\frac{rz^2}{2}\right\}_0^{1+r^2} dr\, d\theta$

$$= 2\rho\int_0^{\pi/2}\int_0^1 r(1+r^2)^2\, dr\, d\theta = 2\rho\int_0^{\pi/2}\left\{\frac{1}{6}(1+r^2)^3\right\}_0^1 d\theta = \frac{7\rho}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{7\pi\rho}{6},$$

we find that $\overline{z} = \dfrac{7\pi\rho}{6}\dfrac{2}{3\pi\rho} = \dfrac{7}{9}$.



34. $\overline{f} = \dfrac{1}{\pi(2)^2}\displaystyle\int_{-\pi}^{\pi}\int_0^2 r^2\, r\, dr\, d\theta$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\right\}_0^2 d\theta$$

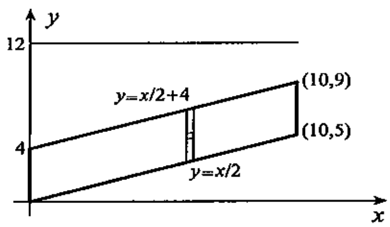
$$= \frac{1}{\pi}\left\{\theta\right\}_{-\pi}^{\pi} = 2$$

35. $F = \displaystyle\int_0^{10}\int_{x/2}^{4+x/2} 1000(9.81)(12-y)\, dy\, dx$

$$= 9810\int_0^{10}\left\{-\frac{1}{2}(12-y)^2\right\}_{x/2}^{4+x/2} dx$$

$$= -4905\int_0^{10}[(8-x/2)^2-(12-x/2)^2]\, dx$$

$$= -4905\left\{-\frac{2}{3}\left(8-\frac{x}{2}\right)^3+\frac{2}{3}\left(12-\frac{x}{2}\right)^3\right\}_0^{10} = 2943\ \text{kN}$$



36. We quadruple the moment of inertia of the first octant volume.

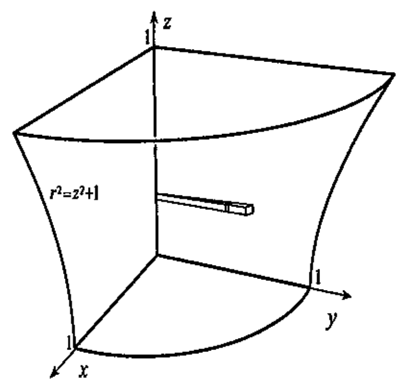
$$I = 4\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1+z^2}} r^2\,\rho\, r\, dr\, dz\, d\theta$$

$$= 4\rho\int_0^{\pi/2}\int_0^1\left\{\frac{r^4}{4}\right\}_0^{\sqrt{1+z^2}} dz\, d\theta$$

$$= \rho\int_0^{\pi/2}\int_0^1(1+2z^2+z^4)\, dz\, d\theta$$

$$= \rho\int_0^{\pi/2}\left\{z+\frac{2z^3}{3}+\frac{z^5}{5}\right\}_0^1 d\theta$$

$$= \frac{28\rho}{15}\left\{\theta\right\}_0^{\pi/2} = \frac{14\pi\rho}{15}$$



37. Since $V = \displaystyle\int_0^2\int_0^{2-y}\int_{2y+2z-4}^{2-y-z} dx\, dz\, dy = \int_0^2\int_0^{2-y}(6-3y-3z)\, dz\, dy$

$$= 3\int_0^2\left\{-\frac{1}{2}(2-y-z)^2\right\}_0^{2-y} dy = \frac{3}{2}\int_0^2(2-y)^2\, dy$$

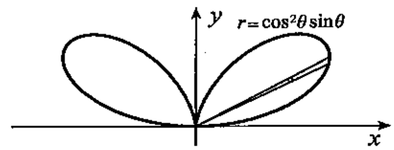
$$= \frac{3}{2}\left\{-\frac{1}{3}(2-y)^3\right\}_0^2 = 4,$$

$$\overline{f} = \frac{1}{4}\int_0^2\int_0^{2-y}\int_{2y+2z-4}^{2-y-z}(x+y+z)\, dx\, dz\, dy = \frac{1}{4}\int_0^2\int_0^{2-y}\left\{\frac{1}{2}(x+y+z)^2\right\}_{2y+2z-4}^{2-y-z} dz\, dy$$

$$= \frac{1}{8}\int_0^2\int_0^{2-y}[4-(3y+3z-4)^2]\, dz\, dy = \frac{1}{8}\int_0^2\left\{4z-\frac{1}{9}(3y+3z-4)^3\right\}_0^{2-y} dy$$

$$= \frac{1}{72}\int_0^2[36(2-y)-8+(3y-4)^3]\, dy = \frac{1}{72}\left\{-18(2-y)^2-8y+\frac{1}{12}(3y-4)^4\right\}_0^2 = \frac{1}{2}.$$

38. $$A = 2\int_0^{\pi/2}\int_0^{\sin\theta\cos^2\theta} r\,dr\,d\theta$$
$$= 2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{\sin\theta\cos^2\theta} d\theta$$
$$= \int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$$
$$= \int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{8}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2} + \sin^2 2\theta\cos 2\theta\right)d\theta = \frac{1}{8}\left\{\frac{\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{\pi}{32}$$



By symmetry, $\overline{x} = 0$. Since $$A\overline{y} = 2\int_0^{\pi/2}\int_0^{\sin\theta\cos^2\theta} r\sin\theta\, r\,dr\,d\theta = 2\int_0^{\pi/2}\left\{\frac{r^3}{3}\sin\theta\right\}_0^{\sin\theta\cos^2\theta} d\theta$$
$$= \frac{2}{3}\int_0^{\pi/2}\sin^4\theta\cos^6\theta\,d\theta = \frac{2}{3}\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^4\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{48}\int_0^{\pi/2}\left[\left(\frac{1-\cos 2\theta}{2}\right)^2 + \sin^4 2\theta\cos 2\theta\right]d\theta$$
$$= \frac{1}{48}\int_0^{\pi/2}\left[\frac{1}{4}\left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) + \sin^4 2\theta\cos 2\theta\right]d\theta$$
$$= \frac{1}{48}\left\{\frac{3\theta}{8} - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + \frac{1}{10}\sin^5 2\theta\right\}_0^{\pi/2} = \frac{\pi}{256},$$

it follows that $\overline{y} = \dfrac{\pi}{256}\dfrac{32}{\pi} = \dfrac{1}{8}$.

39. By Leibnitz's rule, $$\frac{dy}{dx} = -3e^{-3x}(C_1\cos x + C_2\sin x) + e^{-3x}(-C_1\sin x + C_2\cos x)$$
$$+ \int_0^x f(t)[-3e^{3(t-x)}\sin(x-t) + e^{3(t-x)}\cos(x-t)]\,dt$$
$$= (-3C_1 + C_2)e^{-3x}\cos x + (-3C_2 - C_1)e^{-3x}\sin x$$
$$+ \int_0^x f(t)e^{3(t-x)}[\cos(x-t) - 3\sin(x-t)]\,dt,$$

and

$$\frac{d^2y}{dx^2} = (-3C_1 + C_2)(-3e^{-3x}\cos x - e^{-3x}\sin x) + (-3C_2 - C_1)(-3e^{-3x}\sin x + e^{-3x}\cos x)$$
$$+ \int_0^x f(t)\{-3e^{3(t-x)}[\cos(x-t) - 3\sin(x-t)] + e^{3(t-x)}[-\sin(x-t) - 3\cos(x-t)]\}dt + f(x).$$

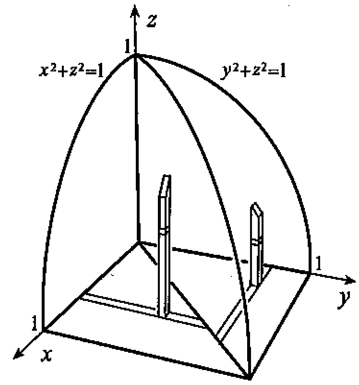
Thus, $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = (8C_1 - 6C_2)e^{-3x}\cos x + (6C_1 + 8C_2)e^{-3x}\sin x$$
$$+ \int_0^x f(t)e^{3(t-x)}[-6\cos(x-t) + 8\sin(x-t)]\,dt + f(x)$$
$$+6(-3C_1 + C_2)e^{-3x}\cos x + 6(-3C_2 - C_1)e^{-3x}\sin x$$
$$+6\int_0^x f(t)e^{3(t-x)}[\cos(x-t) - 3\sin(x-t)]\,dt$$
$$+10e^{-3x}(C_1\cos x + C_2\sin x) + 10\int_0^x f(t)e^{3(t-x)}\sin(x-t)\,dt$$
$$= f(x).$$

40. $M = 2\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} \rho\, dz\, dy\, dx$

$$= 2\rho \int_0^1 \int_0^x \sqrt{1-x^2}\, dy\, dx$$

$$= 2\rho \int_0^1 x\sqrt{1-x^2}\, dx$$

$$= 2\rho \left\{-\frac{1}{3}(1-x^2)^{3/2}\right\}_0^1 = \frac{2\rho}{3}$$



$$M\overline{x} = \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} x\rho\, dz\, dy\, dx + \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} x\rho\, dz\, dx\, dy$$

$$= \rho \int_0^1 \int_0^x x\sqrt{1-x^2}\, dy\, dx + \rho \int_0^1 \int_0^y x\sqrt{1-y^2}\, dx\, dy$$

$$= \rho \int_0^1 x^2\sqrt{1-x^2}\, dx + \frac{\rho}{2}\int_0^1 y^2\sqrt{1-y^2}\, dy = \frac{3\rho}{2}\int_0^1 x^2\sqrt{1-x^2}\, dx$$

If we now set $x = \sin\theta$ and $dx = \cos\theta\, d\theta$, then

$$M\overline{x} = \frac{3\rho}{2}\int_0^{\pi/2} \sin^2\theta\, \cos\theta\, \cos\theta\, d\theta = \frac{3\rho}{2}\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{3\rho}{8}\int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2}\right) d\theta = \frac{3\rho}{16}\left\{\theta - \frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{3\pi\rho}{32},$$

we obtain $\overline{x} = \dfrac{3\pi\rho}{32}\dfrac{3}{2\rho} = \dfrac{9\pi}{64}$. By symmetry, $\overline{y} = \overline{x} = 9\pi/64$. Since

$$M\overline{z} = 2\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} z\rho\, dz\, dy\, dx = 2\rho \int_0^1 \int_0^x \left\{\frac{z^2}{2}\right\}_0^{\sqrt{1-x^2}} dy\, dx$$

$$= \rho \int_0^1 \int_0^x (1-x^2)\, dy\, dx = \rho \int_0^1 (x - x^3)\, dx = \rho\left\{\frac{x^2}{2} - \frac{x^4}{4}\right\}_0^1 = \frac{\rho}{4},$$

we find $\overline{z} = \dfrac{\rho}{4}\dfrac{3}{2\rho} = \dfrac{3}{8}$.

41. We quadruple the first octant area.

$$A = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{2x}{x^2+y^2}\right)^2 + \left(\frac{2y}{x^2+y^2}\right)^2}\, dA = 4\iint_{S_{xy}} \sqrt{\frac{(x^2+y^2)^2 + 4(x^2+y^2)}{(x^2+y^2)^2}}\, dA$$

$$= 4\int_0^{\pi/2} \int_1^2 \sqrt{\frac{r^2+4}{r^2}}\, r\, dr\, d\theta = 4\int_0^{\pi/2} \int_1^2 \sqrt{4+r^2}\, dr\, d\theta = 4\int_1^2 \left\{\sqrt{4+r^2}\,\theta\right\}_0^{\pi/2} dr = 2\pi \int_1^2 \sqrt{4+r^2}\, dr$$

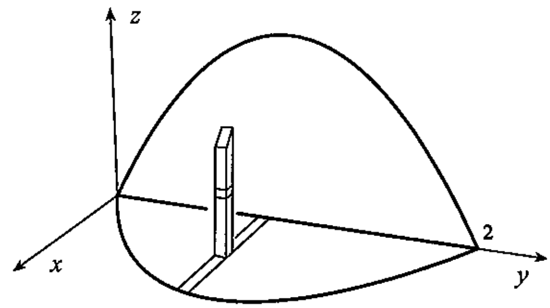
If we set $r = 2\tan\theta$ and $dr = 2\sec^2\theta\, d\theta$,

$$A = 2\pi \int_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} (2\sec\theta) 2\sec^2\theta\, d\theta = 8\pi \int_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} \sec^3\theta\, d\theta$$

$$= \frac{8\pi}{2}\left\{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right\}_{\mathrm{Tan}^{-1}(1/2)}^{\pi/4} \quad \text{(see Example 8.9)}$$

$$= 4\pi\left[\sqrt{2} - \frac{\sqrt{5}}{4} + \ln(\sqrt{2}+1) - \ln(\sqrt{5}+1) + \ln 2\right].$$

42. We quadruple the first octant volume.



$$V = 4\int_0^2 \int_0^{y(2-y)} \int_0^{\sqrt{y^2(2-y)^2 - x^2}} dz\, dx\, dy$$
$$= 4\int_0^2 \int_0^{y(2-y)} \sqrt{y^2(2-y)^2 - x^2}\, dx\, dy$$

If we set $x = y(2-y)\sin\theta$ and $dx = y(2-y)\cos\theta\, d\theta$,

$$V = 4\int_0^2 \int_0^{\pi/2} y(2-y)\cos\theta\, y(2-y)\cos\theta\, d\theta\, dy$$
$$= 4\int_0^2 \int_0^{\pi/2} y^2(2-y)^2 \left(\frac{1+\cos 2\theta}{2}\right) d\theta\, dy = 2\int_0^2 \left\{ y^2(2-y)^2 \left(\theta + \frac{\sin 2\theta}{2}\right)\right\}_0^{\pi/2} dy$$
$$= \pi \int_0^2 (4y^2 - 4y^3 + y^4)\, dy = \pi \left\{\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right\}_0^2 = \frac{16\pi}{15}.$$

CHAPTER 14

EXERCISES 14.1

1. Open, connected, simply-connected domain

2. Closed, connected

3. Open, connected, domain, but not a simply-connected domain

4. Connected

5. Open, connected, simply-connected domain

6. Closed, connected

7. Open, connected, simply-connected domain

8. Open

9. Open, connected, domain, but not a simply-connected domain

10. For interior, exterior, and boundary points replace circle with sphere in planar definitions. Open, closed, connected, and domain definitions are identical. A domain is simply-connected if every closed curve in the domain is the boundary of a surface that contains only points of the domain.

11. Open, connected, simply-connected domain

12. Closed, connected

13. Open, connected, simply-connected domain

14. Connected

15. Open, connected, domain, but not a simply connected domain

16. Open

17. Open

18. Open, connected, simply-connected domain

19. Open, connected, domain, but not a simply-connected domain

20. To be open a set must not contain any of its boundary points. To be closed it must contain all of its boundary points. The only way to satisfy both conditions is for the set to have no boundary points. The whole plane is the only nonempty set that has no boundary points.

21. $\nabla f = 6xy\hat{\mathbf{i}} + (3x^2 - 3y^2z^2)\hat{\mathbf{j}} - 2y^3z\hat{\mathbf{k}}$

22. $\nabla f = -\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}) = -(x^2+y^2+z^2)^{-3/2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$

23. $\nabla f = \dfrac{1}{1+y^2/x^2}\left(-\dfrac{y}{x^2}\hat{\mathbf{i}} + \dfrac{1}{x}\hat{\mathbf{j}}\right) = \dfrac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{x^2+y^2}$

24. $\nabla f_{|(1,2)} = (3x^2y - 2\cos y)\hat{\mathbf{i}} + (x^3 + 2x\sin y)\hat{\mathbf{j}}_{|(1,2)} = (6 - 2\cos 2)\hat{\mathbf{i}} + (1 + 2\sin 2)\hat{\mathbf{j}}$

25. $\nabla f_{|(1,-1,4)} = e^{xyz}(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})_{|(1,-1,4)} = e^{-4}(-4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}})$

26. $\nabla \cdot \mathbf{F} = 2e^y + 0 - 2x^2y = 2(e^y - x^2y)$

27. $\nabla \cdot \mathbf{F} = \ln y - 3y^2e^x$

28. $\nabla \cdot \mathbf{F} = 2x\cos(x^2+y^2+z^2) - \sin(y+z)$

29. $\nabla \cdot \mathbf{F} = e^x + e^y$

30. $\nabla \cdot \mathbf{F}_{(1,1,1)} = (2xy^3 - 3x + 2z)_{(1,1,1)} = 2 - 3 + 2 = 1.$

31. $\nabla \cdot \mathbf{F}_{|(-1,3)} = [2(x+y) + 2(x+y)]_{|(-1,3)} = 8$

32. $$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$
$$= \left[\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}}\right] + \left[\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{y^2}{(x^2+y^2+z^2)^{3/2}}\right]$$
$$+ \left[\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{z^2}{(x^2+y^2+z^2)^{3/2}}\right]$$
$$= \frac{3(x^2+y^2+z^2)-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

33. $$\nabla \cdot \mathbf{F} = \frac{-y}{1+x^2y^2} + \frac{x}{1+x^2y^2} = \frac{x-y}{1+x^2y^2}$$

34. $$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z & 12xyz & 32y^2z^4 \end{vmatrix} = (64yz^4 - 12xy)\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 12yz\hat{\mathbf{k}}$$

35. $$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^y & -2xy^2 & 0 \end{vmatrix} = (-2y^2 - xe^y)\hat{\mathbf{k}}$$

36. $$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix} = \mathbf{0}$$

37. $$\nabla \times \mathbf{F}_{|(1,-1,1)} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}_{|(1,-1,1)} = [(2z^4 + 2x^2y)\hat{\mathbf{i}} + 3xz^2\hat{\mathbf{j}} - 4xyz\hat{\mathbf{k}}]_{|(1,-1,1)} = 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

38. $$\nabla \times \mathbf{F}_{|(2,0)} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix}_{|(2,0)} = (-2)\hat{\mathbf{k}}$$

39. $$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \ln(x+y+z) & \ln(x+y+z) & \ln(x+y+z) \end{vmatrix} = \mathbf{0}$$

40. $$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \mathrm{Sec}^{-1}(x+y) & \mathrm{Csc}^{-1}(y+x) & 0 \end{vmatrix}$$
$$= \left(\frac{-1}{(y+x)\sqrt{(y+x)^2-1}} - \frac{1}{(x+y)\sqrt{(x+y)^2-1}}\right)\hat{\mathbf{k}} = \frac{-2}{(x+y)\sqrt{(x+y)^2-1}}\hat{\mathbf{k}}$$

41. Since $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix} = (2z+2x)\hat{\mathbf{i}} + (-2z - x^2)\hat{\mathbf{k}}$,

$$\nabla \times \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z+2x & 0 & -2z-x^2 \end{vmatrix} = (2+2x)\hat{\mathbf{j}}.$$

42. (a) For $f(x,y,z)$, the equation $\nabla \cdot \nabla f = 0$ can be written in the form

$$0 = \nabla \cdot \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

which is equation 12.12.

(b) Since $\dfrac{\partial f}{\partial x} = \dfrac{-x}{(x^2+y^2+z^2)^{3/2}}$, it follows that

$$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}.$$

Similarly, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}}$. When these second partial derivatives are added together, the result is zero.

43. 14.7:
$$\begin{aligned}\nabla(f+g) &= (f_x+g_x)\hat{\mathbf{i}} + (f_y+g_y)\hat{\mathbf{j}} + (f_z+g_z)\hat{\mathbf{k}}\\ &= (f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}} + f_z\hat{\mathbf{k}}) + (g_x\hat{\mathbf{i}} + g_y\hat{\mathbf{j}} + g_z\hat{\mathbf{k}}) = \nabla f + \nabla g\end{aligned}$$

14.8:
$$\begin{aligned}\nabla\cdot(\mathbf{F}+\mathbf{G}) &= \frac{\partial}{\partial x}(F_x+G_x) + \frac{\partial}{\partial y}(F_y+G_y) + \frac{\partial}{\partial z}(F_z+G_z)\\ &= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}\end{aligned}$$

14.9:
$$\begin{aligned}\nabla\times(\mathbf{F}+\mathbf{G}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_x+G_x & F_y+G_y & F_z+G_z\end{vmatrix}\\ &= \left[\frac{\partial}{\partial y}(F_z+G_z) - \frac{\partial}{\partial z}(F_y+G_y)\right]\hat{\mathbf{i}} + \left[\frac{\partial}{\partial z}(F_x+G_x) - \frac{\partial}{\partial x}(F_z+G_z)\right]\hat{\mathbf{j}}\\ &\quad + \left[\frac{\partial}{\partial x}(F_y+G_y) - \frac{\partial}{\partial y}(F_x+G_x)\right]\hat{\mathbf{k}}\\ &= \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\right]\\ &\quad + \left[\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)\hat{\mathbf{k}}\right]\\ &= \nabla\times\mathbf{F} + \nabla\times\mathbf{G}\end{aligned}$$

14.10:
$$\begin{aligned}\nabla(fg) &= \frac{\partial}{\partial x}(fg)\hat{\mathbf{i}} + \frac{\partial}{\partial y}(fg)\hat{\mathbf{j}} + \frac{\partial}{\partial z}(fg)\hat{\mathbf{k}}\\ &= \left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}\right)\hat{\mathbf{i}} + \left(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\right)\hat{\mathbf{j}} + \left(g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}\right)\hat{\mathbf{k}}\\ &= g\left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) + f\left(\frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}\right) = g\nabla f + f\nabla g\end{aligned}$$

14.11:
$$\begin{aligned}\nabla\cdot(f\mathbf{F}) &= \frac{\partial}{\partial x}(fF_x) + \frac{\partial}{\partial y}(fF_y) + \frac{\partial}{\partial z}(fF_z)\\ &= \left(\frac{\partial f}{\partial x}F_x + f\frac{\partial F_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}F_y + f\frac{\partial F_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}F_z + f\frac{\partial F_z}{\partial z}\right)\\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot(F_x, F_y, F_z) + f\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = \nabla f\cdot\mathbf{F} + f\,\nabla\cdot\mathbf{F}\end{aligned}$$

14.12:
$$\begin{aligned}\nabla\times(f\mathbf{F}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ fF_x & fF_y & fF_z\end{vmatrix}\\ &= \left(\frac{\partial f}{\partial y}F_z + f\frac{\partial F_z}{\partial y} - \frac{\partial f}{\partial z}F_y - f\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z}F_x + f\frac{\partial F_x}{\partial z} - \frac{\partial f}{\partial x}F_z - f\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}}\\ &\quad + \left(\frac{\partial f}{\partial x}F_y + f\frac{\partial F_y}{\partial x} - \frac{\partial f}{\partial y}F_x - f\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\\ &= \left[\left(\frac{\partial f}{\partial y}F_z - \frac{\partial f}{\partial z}F_y\right)\hat{\mathbf{i}} + \left(\frac{\partial f}{\partial z}F_x - \frac{\partial f}{\partial x}F_z\right)\hat{\mathbf{j}} + \left(\frac{\partial f}{\partial x}F_y - \frac{\partial f}{\partial y}F_x\right)\hat{\mathbf{k}}\right]\end{aligned}$$

$$+f\left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\right]$$

$$=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z\\ F_x & F_y & F_z\end{vmatrix}+f\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_x & F_y & F_z\end{vmatrix}$$

$$=\nabla f\times\mathbf{F}+f(\nabla\times\mathbf{F})$$

14.14: $$\nabla\times(\nabla f)=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z\end{vmatrix}$$

$$=\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)\hat{\mathbf{i}}+\left(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\right)\hat{\mathbf{j}}+\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)\hat{\mathbf{k}}=\mathbf{0}$$

14.15: $$\nabla\cdot(\nabla\times\mathbf{F})=\nabla\cdot\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_x & F_y & F_z\end{vmatrix}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$$

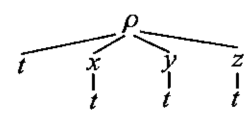
$$=\left(\frac{\partial^2 F_z}{\partial x\partial y}-\frac{\partial^2 F_z}{\partial y\partial x}\right)+\left(\frac{\partial^2 F_y}{\partial z\partial x}-\frac{\partial^2 F_y}{\partial x\partial z}\right)+\left(\frac{\partial^2 F_x}{\partial y\partial z}-\frac{\partial^2 F_x}{\partial z\partial y}\right)=0$$

44. From the schematic,

$$\frac{d\rho}{dt}=\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}\frac{dx}{dt}+\frac{\partial\rho}{\partial y}\frac{dy}{dt}+\frac{\partial\rho}{\partial z}\frac{dz}{dt}$$

$$=\frac{\partial\rho}{\partial t}+\left(\frac{\partial\rho}{\partial x}\hat{\mathbf{i}}+\frac{\partial\rho}{\partial y}\hat{\mathbf{j}}+\frac{\partial\rho}{\partial z}\hat{\mathbf{k}}\right)\cdot\left(\frac{dx}{dt}\hat{\mathbf{i}}+\frac{dy}{dt}\hat{\mathbf{j}}+\frac{dz}{dt}\hat{\mathbf{k}}\right)$$

$$=\frac{\partial\rho}{\partial t}+\nabla\rho\cdot\frac{d\mathbf{r}}{dt}.$$



45. If $\nabla f=2xy\hat{\mathbf{i}}+x^2\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x}=2xy$, $\dfrac{\partial f}{\partial y}=x^2$. Integrating the first gives $f(x,y)=x^2y+v(y)$. Substitution into the second requires $x^2+v'(y)=x^2\implies v(y)=C=\text{constant}$. Thus, $f(x,y)=x^2y+C$.

46. If $\nabla f=\mathbf{F}$, then $\dfrac{\partial f}{\partial x}=3x^2y^2+3$, $\dfrac{\partial f}{\partial y}=2x^3y+2$. From the first of these, $f(x,y)=x^3y^2+3x+v(y)$, which substituted into the second requires $2x^3y+v'(y)=2x^3y+2$. Thus, $v(y)=2y+C$, where C is a constant, and $f(x,y)=x^3y^2+3x+2y+C$.

47. If $\nabla f=e^y\hat{\mathbf{i}}+(xe^y+4y^2)\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x}=e^y$, $\dfrac{\partial f}{\partial y}=xe^y+4y^2$. Integrating the first gives $f(x,y)=xe^y+v(y)$. Substitution into the second requires $xe^y+v'(y)=xe^y+4y^2\implies v(y)=4y^3/3+C$, where C is a constant. Thus, $f(x,y)=xe^y+4y^3/3+C$.

48. If $\nabla f=\mathbf{F}$, then $\dfrac{\partial f}{\partial x}=\dfrac{1}{x+y}$, $\dfrac{\partial f}{\partial y}=\dfrac{1}{x+y}$. From the first, $f(x,y)=\ln|x+y|+v(y)$, which substituted into the second requires $1/(x+y)+v'(y)=1/(x+y)$. Thus, $v(y)=C$, where C is a constant, and $f(x,y)=\ln|x+y|+C$.

49. If $\nabla f=-xy(1-x^2y^2)^{-1/2}(y\hat{\mathbf{i}}+x\hat{\mathbf{j}})$, then $\dfrac{\partial f}{\partial x}=\dfrac{-xy^2}{\sqrt{1-x^2y^2}}$, $\dfrac{\partial f}{\partial y}=\dfrac{-x^2y}{\sqrt{1-x^2y^2}}$. Integration of the first gives $f(x,y)=\sqrt{1-x^2y^2}+v(y)$. Substitution into the second requires

$$\frac{-x^2y}{\sqrt{1-x^2y^2}}+v'(y)=\frac{-x^2y}{\sqrt{1-x^2y^2}}\quad\implies v(y)=C=\text{constant}.$$

Thus, $f(x,y)=\sqrt{1-x^2y^2}+C$.

50. If $\nabla f=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x}=x$, $\dfrac{\partial f}{\partial y}=y$, $\dfrac{\partial f}{\partial z}=z$. From the first, $f(x,y,z)=x^2/2+v(y,z)$, which substituted into the second requires $\partial v/\partial y = y$. Thus, $v(y,z) = y^2/2 + w(z)$, and $f(x,y,z) = x^2/2 + y^2/2 + w(z)$. Substitution into the third equation gives $w'(z) = z$. Thus, $w(z) = z^2/2 + C$, where C is a constant, and $f(x,y,z) = x^2/2 + y^2/2 + z^2/2 + C$.

51. If $\nabla f = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + (yx-3)\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x} = yz$, $\dfrac{\partial f}{\partial y} = xz$, $\dfrac{\partial f}{\partial z} = yx - 3$. From the first, $f(x,y,z) = xyz + v(y,z)$, which substituted into the second requires $xz + \partial v/\partial y = xz$. Thus, $v(y,z) = w(z)$, and $f(x,y,z) = xyz + w(z)$. Substitution into the third equation gives $xy + w'(z) = yx - 3$. Thus, $w(z) = -3z + C$, where C is a constant, and $f(x,y,z) = xyz - 3z + C$.

52. If $\nabla f = (1+x+y+z)^{-1}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$, then

$$\frac{\partial f}{\partial x} = \frac{1}{1+x+y+z}, \quad \frac{\partial f}{\partial y} = \frac{1}{1+x+y+z}, \quad \frac{\partial f}{\partial z} = \frac{1}{1+x+y+z}.$$

From the first, $f(x,y,z) = \ln|1+x+y+z| + v(y,z)$, which substituted into the second requires

$$\frac{1}{1+x+y+z} + \frac{\partial v}{\partial y} = \frac{1}{1+x+y+z}.$$

Thus, $v(y,z) = w(z)$, and $f(x,y,z) = \ln|1+x+y+z| + w(z)$. Substitution into the third equation gives $\dfrac{1}{1+x+y+z} + w'(z) = \dfrac{1}{1+x+y+z}$. Thus, $w(z) = C$, and $f(x,y,z) = \ln|1+x+y+z| + C$.

53. If $\nabla f = (2x/y^2 + 1)\hat{\mathbf{i}} - (2x^2/y^3)\hat{\mathbf{j}} - 2z\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x} = \dfrac{2x}{y^2} + 1$, $\dfrac{\partial f}{\partial y} = -\dfrac{2x^2}{y^3}$, $\dfrac{\partial f}{\partial z} = -2z$. From the first, $f(x,y,z) = x^2/y^2 + x + v(y,z)$, which substituted into the second requires $-\dfrac{2x^2}{y^3} + \dfrac{\partial v}{\partial y} = -\dfrac{2x^2}{y^3}$. Thus, $v(y,z) = w(z)$, and $f(x,y,z) = x^2/y^2 + x + w(z)$. Substitution into the third equation gives $w'(z) = -2z$, from which $w(z) = -z^2 + C$, where C is a constant. Thus, $f(x,y,z) = x^2/y^2 + x - z^2 + C$.

54. If $\nabla f = (1+x^2y^2)^{-1}(y\hat{\mathbf{i}} + x\hat{\mathbf{j}}) + z\hat{\mathbf{k}}$, then $\dfrac{\partial f}{\partial x} = \dfrac{y}{1+x^2y^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{x}{1+x^2y^2}$, $\dfrac{\partial f}{\partial z} = z$. From the first, $f(x,y,z) = \mathrm{Tan}^{-1}(xy) + v(y,z)$, which substituted into the second requires

$$\frac{x}{1+x^2y^2} + \frac{\partial v}{\partial y} = \frac{x}{1+x^2y^2}.$$

Thus, $v(y,z) = w(z)$, and $f(x,y,z) = \mathrm{Tan}^{-1}(xy) + w(z)$. Substitution into the third equation gives $w'(z) = z$. Thus, $w(z) = z^2/2 + C$, where C is a constant, and $f(x,y,z) = \mathrm{Tan}^{-1}(xy) + z^2/2 + C$.

55. If $\nabla f = \mathbf{F}$, then $\dfrac{\partial f}{\partial x} = 3x^2y + yz + 2xz^2$, $\dfrac{\partial f}{\partial y} = xz + x^3 + 3z^2 - 6y^2z$, $\dfrac{\partial f}{\partial z} = 2x^2z + 6yz - 2y^3 + xy$. Integrating the first gives $f(x,y,z) = x^3y + xyz + x^2z^2 + v(y,z)$. Substitution into the second requires

$$x^3 + xz + \frac{\partial v}{\partial y} = xz + x^3 + 3z^2 - 6y^2z \quad \Longrightarrow \quad v(y,z) = 3yz^2 - 2y^3z + w(z).$$

Thus, $f(x,y,z) = x^3y + xyz + x^2z^2 + 3yz^2 - 2y^3z + w(z)$. Substitution into the third equation gives

$$xy + 2x^2z + 6yz - 2y^3 + w'(z) = 2x^2z + 6yz - 2y^3 + xy \quad \Longrightarrow \quad w(z) = C, \text{ a constant.}$$

Thus, $f(x,y,z) = x^3y + xyz + x^2z^2 + 3yz^2 - 2y^3z + C$.

56. (a) $\mathbf{F}$ is irrotational if

$$\mathbf{0} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = (c+1)\hat{\mathbf{i}} + (a-4)\hat{\mathbf{j}} + (b-2)\hat{\mathbf{k}}.$$

Thus, $a = 4$, $b = 2$, $c = -1$.

(b) If $\nabla f = \mathbf{F}$, then $\dfrac{\partial f}{\partial x} = x^2 + 2y + 4z$, $\dfrac{\partial f}{\partial y} = 2x - 3y - z$, $\dfrac{\partial f}{\partial z} = 4x - y + 2z$. From the first, $f(x, y, z) = x^3/3 + 2xy + 4xz + v(y, z)$, which substituted into the second requires $2x + \dfrac{\partial v}{\partial y} = 2x - 3y - z$. Thus, $v(y, z) = -3y^2/2 - yz + w(z)$, and $f(x, y, z) = x^3/3 + 2xy + 4xz - 3y^2/2 - yz + w(z)$. Substitution into the third equation gives $4x - y + w'(z) = 4x - y + 2z$. Thus, $w(z) = z^2 + C$, where C is a constant, and $f(x, y, z) = x^3/3 + 2xy + 4xz - 3y^2/2 - yz + z^2 + C$.

57. (a) Since $\nabla \cdot \mathbf{F} = (4x + 8y^2 z) + (3x^3 - 3x) - (8y^2 z + 2x^3) = x + x^3 \neq 0$, this vector field is not solenoidal. For the second field,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \nabla(xyz^2) \cdot \mathbf{F} + xyz^2(\nabla \cdot \mathbf{F}) \\ &= (yz^2, xz^2, 2xyz) \cdot [(2x^2 + 8xy^2 z)\hat{\mathbf{i}} + (3x^3 y - 3xy)\hat{\mathbf{j}} - (4y^2 z^2 + 2x^3 z)\hat{\mathbf{k}}] + xyz^2(x + x^3) = 0.\end{aligned}$$

This vector field is solenoidal.

$$\begin{aligned}\text{(b)}\ \nabla \cdot (\nabla f \times \nabla g) &= \nabla \cdot \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} \\ &= \nabla \cdot [(f_y g_z - f_z g_y)\hat{\mathbf{i}} + (f_z g_x - f_x g_z)\hat{\mathbf{j}} + (f_x g_y - f_y g_x)\hat{\mathbf{k}}] \\ &= (f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx}) + (f_{zy} g_x + f_z g_{xy} - f_{xy} g_z - f_x g_{zy}) \\ &\quad + (f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz}) = 0\end{aligned}$$

58. (a) $\nabla V = -\mathbf{E} = -\dfrac{\mathbf{F}}{Q} = -\dfrac{q}{4\pi\epsilon_0 |\mathbf{r}|^3}\mathbf{r} = \dfrac{-q}{4\pi\epsilon_0}\dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{3/2}}$. This implies that

$$\frac{\partial V}{\partial x} = \frac{-q}{4\pi\epsilon_0}\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{\partial V}{\partial y} = \frac{-q}{4\pi\epsilon_0}\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{\partial V}{\partial z} = \frac{-q}{4\pi\epsilon_0}\frac{z}{(x^2 + y^2 + z^2)^{3/2}}.$$

From the first, $V(x, y, z) = \dfrac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + z^2}} + w(y, z)$, which substituted into the second requires

$$\frac{-q}{4\pi\epsilon_0}\frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial w}{\partial y} = \frac{-q}{4\pi\epsilon_0}\frac{y}{(x^2 + y^2 + z^2)^{3/2}}.$$

Thus, $w(y, z) = k(z)$, and $V(x, y, z) = \dfrac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + z^2}} + k(z)$. Substitution into the third equation gives $\dfrac{-q}{4\pi\epsilon_0}\dfrac{z}{(x^2 + y^2 + z^2)^{3/2}} + k'(z) = \dfrac{-q}{4\pi\epsilon_0}\dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$. Hence, $k(z) = C$, where C is a constant, and $V(x, y, z) = \dfrac{q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + z^2}} + C$.

(b) $\nabla V = -\mathbf{E} = -\dfrac{\mathbf{F}}{Q} = -\dfrac{\sigma}{2\epsilon_0}\hat{\mathbf{k}}$. This implies that $\dfrac{\partial V}{\partial x} = 0$, $\dfrac{\partial V}{\partial y} = 0$, $\dfrac{\partial V}{\partial z} = -\dfrac{\sigma}{2\epsilon_0}$. These require $V(x, y, z) = -\dfrac{\sigma z}{2\epsilon_0} + C$.

59. If $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (z\omega_y - y\omega_z)\hat{\mathbf{i}} + (x\omega_z - z\omega_x)\hat{\mathbf{j}} + (y\omega_x - x\omega_y)\hat{\mathbf{k}}$, then

$$\frac{1}{2}(\nabla \times \mathbf{v}) = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z\omega_y - y\omega_z & x\omega_z - z\omega_x & y\omega_x - x\omega_y \end{vmatrix} = \frac{1}{2}[(\omega_x + \omega_x)\hat{\mathbf{i}} + (\omega_y + \omega_y)\hat{\mathbf{j}} + (\omega_z + \omega_z)\hat{\mathbf{k}}] = \boldsymbol{\omega}.$$

60. Exercise 42(a) indicates that if $f(x, y, z)$ satisfies Laplace's equation, then $\nabla \cdot (\nabla f) = 0$. In other words, ∇f is solenoidal. On the other hand, for any function whatsoever, equation 14.14 indicates that $\nabla \times (\nabla f) = \mathbf{0}$; i.e., ∇f is irrotational.

61. (a) If $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}} = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ L & M & N \end{vmatrix}$, then

$$P = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \qquad Q = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \qquad R = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}.$$

(b) Since $\mathbf{F} \times (x,y,z) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ P & Q & R \\ x & y & z \end{vmatrix} = (zQ - yR)\hat{\mathbf{i}} + (xR - zP)\hat{\mathbf{j}} + (yP - xQ)\hat{\mathbf{k}}$,

$$\mathbf{v} = \int_0^1 [t(zQ - yR)\hat{\mathbf{i}} + t(xR - zP)\hat{\mathbf{j}} + t(yP - xQ)\hat{\mathbf{k}}]\,dt,$$

where the arguments of P, Q and R are tx, ty, and tz. Consequently,

$$L = \int_0^1 t[zQ(tx,ty,tz) - yR(tx,ty,tz)]\,dt, \quad M = \int_0^1 t[xR(tx,ty,tz) - zP(tx,ty,tz)]\,dt,$$

$$N = \int_0^1 t[yP(tx,ty,tz) - xQ(tx,ty,tz)]\,dt.$$

We now show that $P = \partial N/\partial y - \partial M/\partial z$. To do this we set $a = tx$, $b = ty$, and $c = tz$. Then

$$\begin{aligned}\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} &= \int_0^1 t\left[y\frac{\partial P(a,b,c)}{\partial b}t + P(a,b,c) - x\frac{\partial Q(a,b,c)}{\partial b}t\right]dt \\ &\quad - \int_0^1 t\left[x\frac{\partial R(a,b,c)}{\partial c}t - z\frac{\partial P(a,b,c)}{\partial c}t - P(a,b,c)\right]dt \\ &= \int_0^1 t\left[2P + t\left(y\frac{\partial P}{\partial b} + z\frac{\partial P}{\partial c}\right) - xt\left(\frac{\partial Q}{\partial b} + \frac{\partial R}{\partial c}\right)\right]dt,\end{aligned}$$

where the arguments of P, Q, and R are a, b, and c. Now $\mathbf{F}$ is solenoidal so that $\dfrac{\partial P}{\partial a} + \dfrac{\partial Q}{\partial b} + \dfrac{\partial R}{\partial c} = 0$. Hence

$$\begin{aligned}\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} &= \int_0^1 t\left[2P + t\left(y\frac{\partial P}{\partial b} + z\frac{\partial P}{\partial c}\right) - xt\left(-\frac{\partial P}{\partial a}\right)\right]dt \\ &= \int_0^1 t\left[2P + t\left(x\frac{\partial P}{\partial a} + y\frac{\partial P}{\partial b} + z\frac{\partial P}{\partial c}\right)\right]dt.\end{aligned}$$

But $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial a}\dfrac{da}{dt} + \dfrac{\partial P}{\partial b}\dfrac{db}{dt} + \dfrac{\partial P}{\partial c}\dfrac{dc}{dt} = x\dfrac{\partial P}{\partial a} + y\dfrac{\partial P}{\partial b} + z\dfrac{\partial P}{\partial c}$, and therefore

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = \int_0^1 t\left(2P + t\frac{dP}{dt}\right)dt = \int_0^1 \frac{d}{dt}(t^2P)\,dt = \left\{t^2P\right\}_0^1 = P(x,y,z).$$

A similar proof shows that $Q = \dfrac{\partial L}{\partial z} - \dfrac{\partial N}{\partial x}$ and $R = \dfrac{\partial M}{\partial x} - \dfrac{\partial L}{\partial y}$.

The vector $\mathbf{v}$ defined by this equation is unique, but the vector $\mathbf{v}$ satisfying $\mathbf{F} = \nabla \times \mathbf{v}$ is not. Equation 14.14 indicates that for any function u, $\nabla \times (\nabla u) = \mathbf{0}$. This means that if to $\mathbf{v}$ we add any gradient ∇u, then

$$\nabla \times (\mathbf{v} + \nabla u) = \nabla \times \mathbf{v} + \nabla \times (\nabla u) = \nabla \times \mathbf{v} = \mathbf{F}.$$

Thus, we can add to $\mathbf{v}$ any gradient and still have a vector whose curl is $\mathbf{F}$.

(c) If we use part (b) to define $\mathbf{v}$,

$$\mathbf{v} = \int_0^1 t(t^n)\mathbf{F}(x,y,z) \times (x,y,z)\,dt = \mathbf{F}(x,y,z) \times (x,y,z)\left\{\frac{t^{n+2}}{n+2}\right\}_0^1 = \frac{1}{n+2}\mathbf{F} \times \mathbf{r}.$$

62. Since $\nabla \cdot \mathbf{F} = 1 + 1 - 2 = 0$, $\mathbf{F}$ is solenoidal. Because $\mathbf{F}(tx, ty, tz) = t\mathbf{F}(x, y, z)$, we use the formula in Exercise 61(c) to obtain

$$\mathbf{v} = \frac{1}{3}\mathbf{F} \times \mathbf{r} = \frac{1}{3}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & -2z \\ x & y & z \end{vmatrix} = \frac{1}{3}(3yz\hat{\mathbf{i}} - 3xz\hat{\mathbf{j}}) = yz\hat{\mathbf{i}} - xz\hat{\mathbf{j}}.$$

63. Since $\nabla \cdot \mathbf{F} = 1 - 1 = 0$, the vector field is solenoidal. We use the formula in Exercise 61(b) to find $\mathbf{v}$. Since

$$\begin{aligned}\mathbf{F}(tx, ty, tz) \times (x, y, z) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 + tx & 0 & -tx - tz \\ x & y & z \end{vmatrix} \\ &= y(tx + tz)\hat{\mathbf{i}} + (-tx^2 - txz - z - txz)\hat{\mathbf{j}} + y(1 + tx)\hat{\mathbf{k}},\end{aligned}$$

$$\begin{aligned}\mathbf{v} &= \int_0^1 [ty(tx + tz)\hat{\mathbf{i}} - t(tx^2 + 2txz + z)\hat{\mathbf{j}} + ty(1 + tx)\hat{\mathbf{k}}]\, dt \\ &= \left\{\frac{t^3}{3}(xy + yz)\hat{\mathbf{i}} - \left(\frac{t^3x^2}{3} + \frac{2t^3xz}{3} + \frac{t^2z}{2}\right)\hat{\mathbf{j}} + \left(\frac{t^2y}{2} + \frac{t^3xy}{3}\right)\hat{\mathbf{k}}\right\}_0^1 \\ &= \frac{1}{3}(xy + yz)\hat{\mathbf{i}} - \left(\frac{x^2}{3} + \frac{2xz}{3} + \frac{z}{2}\right)\hat{\mathbf{j}} + \left(\frac{y}{2} + \frac{xy}{3}\right)\hat{\mathbf{k}}.\end{aligned}$$

64. Since $\nabla \cdot \mathbf{F} = 4x - 2y + 2y - 4x = 0$, $\mathbf{F}$ is solenoidal. Because $\mathbf{F}(tx, ty, tz) = t^2\mathbf{F}(x, y, z)$, we use the formula in Exercise 61(c) to obtain

$$\begin{aligned}\mathbf{v} = \frac{1}{4}\mathbf{F} \times \mathbf{r} &= \frac{1}{4}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2x^2 & -y^2 & 2yz - 4xz \\ x & y & z \end{vmatrix} \\ &= \frac{1}{4}\left[(-y^2z - 2y^2z + 4xyz)\hat{\mathbf{i}} + (2xyz - 4x^2z - 2x^2z)\hat{\mathbf{j}} + (2x^2y + xy^2)\hat{\mathbf{k}}\right] \\ &= \frac{1}{4}\left[(4xyz - 3y^2z)\hat{\mathbf{i}} + (2xyz - 6x^2z)\hat{\mathbf{j}} + (2x^2y + xy^2)\hat{\mathbf{k}}\right].\end{aligned}$$

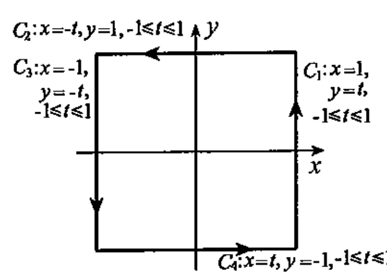
EXERCISES 14.2

1. Using x as the parameter along the curve,

$$\int_C x\, ds = \int_0^1 x\sqrt{1 + (2x)^2}\, dx = \left\{\frac{1}{12}(1 + 4x^2)^{3/2}\right\}_0^1 = \frac{1}{12}(5\sqrt{5} - 1).$$

2.

$$\begin{aligned}\oint_C (x^2 + y^2)\, ds &= \int_{C_1} (x^2 + y^2)\, ds + \int_{C_2} (x^2 + y^2)\, ds \\ &\quad + \int_{C_3} (x^2 + y^2)\, ds + \int_{C_4} (x^2 + y^2)\, ds \\ &= \int_{-1}^1 (1 + t^2)\, dt + \int_{-1}^1 (t^2 + 1)\, dt \\ &\quad + \int_{-1}^1 (1 + t^2)\, dt + \int_{-1}^1 (t^2 + 1)\, dt \\ &= 4\int_{-1}^1 (1 + t^2)\, dt = 4\left\{t + \frac{t^3}{3}\right\}_{-1}^1 = \frac{32}{3}\end{aligned}$$



3. With parametric equations $C: \ x = 2\cos t, \ y = -2\sin t, \ -\pi < t \le \pi$,

$$\oint_C (2 + x - 2xy)\, ds = \int_{-\pi}^{\pi} (2 + 2\cos t + 8\cos t \sin t)\sqrt{(-2\sin t)^2 + (-2\cos t)^2}\, dt$$
$$= 4\int_{-\pi}^{\pi} (1 + \cos t + 4\cos t \sin t)\, dt = 4\Big\{t + \sin t + 2\sin^2 t\Big\}_{-\pi}^{\pi} = 8\pi$$

4. With parametric equations $C: \ x = 1 + 2t, \ y = 2, \ z = -1 + 6t, \ 0 \le t \le 1$,

$$\int_C (x^2 + yz)\, ds = \int_0^1 [(1 + 2t)^2 + 2(-1 + 6t)]\sqrt{2^2 + 0 + 6^2}\, dt$$
$$= 2\sqrt{10}\left\{\frac{1}{6}(1 + 2t)^3 + \frac{1}{6}(-1 + 6t)^2\right\}_0^1 = \frac{50\sqrt{10}}{3}$$

5. With parametric equations $x = \cos t$, $y = \sin t$, $z = \sin t$, $0 \le t \le \pi/2$,

$$\int_C xy\, ds = \int_0^{\pi/2} \cos t \sin t\sqrt{(-\sin t)^2 + (\cos t)^2 + (\cos t)^2}\, dt = \int_0^{\pi/2} \cos t \sin t\sqrt{1 + \cos^2 t}\, dt$$
$$= \left\{-\frac{1}{3}(1 + \cos^2 t)^{3/2}\right\}_0^{\pi/2} = \frac{2\sqrt{2} - 1}{3}.$$

6. With parametric equations $C: \ x = 1 - 4t, \ y = -1/2 + 4t, \ z = 1/2, \ 0 \le t \le 1$,

$$\int_C x^2yz\, ds = \int_0^1 (1 - 4t)^2\left(-\frac{1}{2} + 4t\right)\left(\frac{1}{2}\right)\sqrt{(-4)^2 + (4)^2}\, dt = \sqrt{2}\int_0^1 (1 - 4t)^2(8t - 1)\, dt$$
$$= \sqrt{2}\int_0^1 (128t^3 - 80t^2 + 16t - 1)\, dt = \sqrt{2}\left\{32t^4 - \frac{80t^3}{3} + 8t^2 - t\right\}_0^1 = \frac{37\sqrt{2}}{3}$$

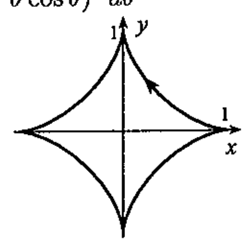
7. Parametric equations for $x^2 + y^2 = r^2$ are $x = r\cos t, \ y = r\sin t, \ -\pi < t \le \pi$. Its length is

$$L = \int_C ds = \int_{-\pi}^{\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = r\int_{-\pi}^{\pi} dt = 2\pi r.$$

8. $$L = \int_C ds = \int_0^{12\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + [3/(4\pi)]^2}\, dt$$
$$= \int_0^{12\pi} \sqrt{9 + \frac{9}{16\pi^2}}dt = \frac{3}{4\pi}\sqrt{16\pi^2 + 1}\Big\{t\Big\}_0^{12\pi} = 9\sqrt{1 + 16\pi^2}\ \text{cm}$$

9. If C is the first quadrant part of the astroid,

$$A = 4\int_C (x^2 + y^2)\, ds = 4\int_0^{\pi/2} (\cos^6\theta + \sin^6\theta)\sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2}\, d\theta$$
$$= 12\int_0^{\pi/2} (\cos^6\theta + \sin^6\theta)\cos\theta\sin\theta\, d\theta$$
$$= 12\left\{-\frac{1}{8}\cos^8\theta + \frac{1}{8}\sin^8\theta\right\}_0^{\pi/2} = 3$$



10. With parametric equations $C: \ y = x^2, \ z = 1 - x^2, \ 0 \le x \le 1$,

$$\int_C xz\, ds = \int_0^1 x(1 - x^2)\sqrt{1^2 + (2x)^2 + (-2x)^2}\, dx = \int_0^1 x(1 - x^2)\sqrt{1 + 8x^2}\, dx.$$

If we set $u = 1 + 8x^2$ and $du = 16x\, dx$, then

$$\int_C xz\, ds = \int_1^9 \left[1 - \left(\frac{u - 1}{8}\right)\right]\sqrt{u}\left(\frac{du}{16}\right) = \frac{1}{128}\int_1^9 (9\sqrt{u} - u^{3/2})\, du = \frac{1}{128}\left\{6u^{3/2} - \frac{2}{5}u^{5/2}\right\}_1^9 = \frac{37}{80}.$$

11. $$\int_C (x+y)^5\,ds = \int_1^4 (2t)^5 \sqrt{\left(1-\frac{1}{t^2}\right)^2 + \left(1+\frac{1}{t^2}\right)^2}\,dt = 32\sqrt{2}\int_1^4 t^3\sqrt{1+t^4}\,dt$$
$$= 32\sqrt{2}\left\{\frac{1}{6}(1+t^4)^{3/2}\right\}_1^4 = \frac{16\sqrt{2}(257\sqrt{257}-2\sqrt{2})}{3}$$

12. With parametric equations $C:\ x=1-t,\ y=9t/14,\ z=1+4t/7,\ 0\le t\le 1$,
$$\int_C x\sqrt{y+z}\,ds = \int_0^1 (1-t)\sqrt{\frac{9t}{14}+1+\frac{4t}{7}}\sqrt{(-1)^2+\left(\frac{9}{14}\right)^2+\left(\frac{4}{7}\right)^2}\,dt = \frac{\sqrt{341}}{14\sqrt{14}}\int_0^1 (1-t)\sqrt{14+17t}\,dt.$$
If we set $u=14+17t$ and $du=17\,dt$, then
$$\int_C x\sqrt{y+z}\,ds = \frac{\sqrt{341}}{14\sqrt{14}}\int_{14}^{31}\left[1-\left(\frac{u-14}{17}\right)\right]\sqrt{u}\left(\frac{du}{17}\right) = \frac{\sqrt{341}}{4046\sqrt{14}}\int_{14}^{31}(31\sqrt{u}-u^{3/2})\,du$$
$$= \frac{\sqrt{341}}{4046\sqrt{14}}\left\{\frac{62u^{3/2}}{3}-\frac{2u^{5/2}}{5}\right\}_{14}^{31} = 0.78.$$

13. Using y as the parameter along the curve,
$$\int_C xy\,ds = \int_0^1 (1-y^2)y\sqrt{1+(-2y)^2}\,dy.$$
If we set $u=1+4y^2$ and $du=8y\,dy$, then
$$\int_C xy\,ds = \int_1^5\left(1-\frac{u-1}{4}\right)\sqrt{u}\left(\frac{du}{8}\right) = \frac{1}{32}\int_1^5 (5\sqrt{u}-u^{3/2})\,du$$
$$= \frac{1}{32}\left\{\frac{10u^{3/2}}{3}-\frac{2u^{5/2}}{5}\right\}_1^5 = \frac{25\sqrt{5}-11}{120}$$

14. With parametric equations $C:\ y=x,\ z=1+x^4,\ -1\le x\le 1$,
$$\int_C (x+y)z\,ds = \int_{-1}^1 (x+x)(1+x^4)\sqrt{1+1+(4x^3)^2}\,dx = 2\int_{-1}^1 (x+x^5)\sqrt{2+16x^6}\,dx = 0,$$
since the integrand is an odd function of x.

15. Using x as the parameter along the curve,
$$\int_C \frac{1}{y+z}ds = \int_1^2 \frac{1}{x^2+x^2}\sqrt{1+(2x)^2+(2x)^2}\,dx = \frac{1}{2}\int_1^2\frac{\sqrt{1+8x^2}}{x^2}dx.$$
If we set $x=[1/(2\sqrt{2})]\tan\theta$ and $dx=[1/(2\sqrt{2})/\sec^2\theta\,d\theta$, then for $\theta_1=\mathrm{Tan}^{-1}(2\sqrt{2})$ and $\theta_2=\mathrm{Tan}^{-1}(4\sqrt{2})$,
$$\int_C\frac{1}{y+z}ds = \frac{1}{2}\int_{\theta_1}^{\theta_2}\frac{8\sec\theta}{\tan^2\theta}\frac{1}{2\sqrt{2}}\sec^2\theta\,d\theta = \sqrt{2}\int_{\theta_1}^{\theta_2}\frac{\sec\theta(1+\tan^2\theta)}{\tan^2\theta}d\theta$$
$$= \sqrt{2}\int_{\theta_1}^{\theta_2}(\csc\theta\cot\theta+\sec\theta)\,d\theta = \sqrt{2}\Big\{-\csc\theta+\ln|\sec\theta+\tan\theta|\Big\}_{\theta_1}^{\theta_2} = 1.013.$$

16. With parametric equations $C:\ x=t^2,\ y=-t,\ z=-t^3,\ 0\le t\le 2$,
$$\int_C (2y+9z)\,ds = \int_0^2(-2t-9t^3)\sqrt{(2t)^2+1+(-3t^2)^2}\,dt = -\int_0^2(2t+9t^3)\sqrt{1+4t^2+9t^4}\,dt.$$
If we set $u=1+4t^2+9t^4$ and $du=(8t+36t^3)\,dt=4(2t+9t^3)\,dt$, then
$$\int_C(2y+9z)\,ds = -\int_1^{161}\sqrt{u}\left(\frac{du}{4}\right) = -\left\{\frac{1}{6}u^{3/2}\right\}_1^{161} = \frac{1-161\sqrt{161}}{6}.$$

17. (a) When a small length ds at position (x, y) on C is rotated around the y-axis, it traces a ribbon with approximate area $2\pi x\, ds$. The total surface area is therefore the limit of the summation of all such ribbons, namely, $\int_C 2\pi x\, ds$.

(b) The surface area in this case is $\int_C 2\pi y\, ds$.

18. When we use x as parameter along the curve,

$$A = \int_C 2\pi y\, ds = 2\pi \int_1^2 x^3\sqrt{1 + (3x^2)^2}\, dx = 2\pi \int_1^2 x^3\sqrt{1 + 9x^4}\, dx = 2\pi \left\{\frac{(1 + 9x^4)^{3/2}}{54}\right\}_1^2$$
$$= \frac{(145\sqrt{145} - 10\sqrt{10})\pi}{27}.$$

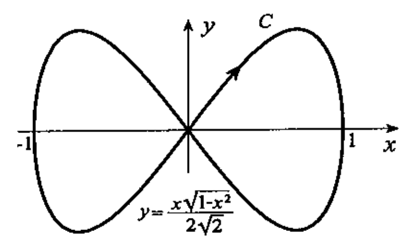
19. Since $y = x^3/24 + 2/x$, $(ds)^2 = \left[1 + \left(\frac{x^2}{8} - \frac{2}{x^2}\right)^2\right](dx)^2 = \left[1 + \frac{x^4}{64} - \frac{1}{2} + \frac{4}{x^4}\right](dx)^2$
$= \left(\frac{x^2}{8} + \frac{2}{x^2}\right)^2 (dx)^2$. The surface area is

$$A = \int_C 2\pi x\, ds = 2\pi \int_1^2 x\left(\frac{x^2}{8} + \frac{2}{x^2}\right) dx = 2\pi \left\{\frac{x^4}{32} + 2\ln x\right\}_1^2 = \frac{\pi(15 + 64\ln 2)}{16}.$$

20. We double the area obtained by rotating the first quadrant part of the curve.

$A = 2\int_C 2\pi y\, ds$ where

$$ds = \sqrt{1 + \left[\frac{1}{2\sqrt{2}}\left(\sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}}\right)\right]^2} dx$$
$$= \sqrt{1 + \frac{1}{8}\left(\frac{1 - 2x^2}{\sqrt{1 - x^2}}\right)^2} dx$$
$$= \sqrt{\frac{8(1 - x^2) + (1 - 2x^2)^2}{8(1 - x^2)}} dx = \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}} dx$$



Thus, $A = 4\pi \int_0^1 \frac{x\sqrt{1 - x^2}}{2\sqrt{2}} \frac{3 - 2x^2}{2\sqrt{2}\sqrt{1 - x^2}} dx = \frac{\pi}{2}\int_0^1 (3x - 2x^3)\, dx = \frac{\pi}{2}\left\{\frac{3x^2}{2} - \frac{x^4}{2}\right\}_0^1 = \frac{\pi}{2}.$

21. $L = \int_0^1 \sqrt{1 + (2x)^2}\, dx = \int_0^1 \sqrt{1 + 4x^2}\, dx$ If we set $x = (1/2)\tan\theta$ and $dx = (1/2)\sec^2\theta\, d\theta$,

$$L = \int_0^{\mathrm{Tan}^{-1}2} \sec\theta(1/2)\sec^2\theta\, d\theta = \frac{1}{2}\int_0^{\mathrm{Tan}^{-1}2} \sec^3\theta\, d\theta$$
$$= \frac{1}{4}\Big\{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big\}_0^{\mathrm{Tan}^{-1}2} \quad \text{(see Example 8.9)}$$
$$= \frac{1}{4}[2\sqrt{5} + \ln(\sqrt{5} + 2)].$$

22. Using x as the parameter along the curve,

$$\begin{aligned}\int_C xy\,ds &= \int_0^{1/2} x^4\sqrt{1+(3x^2)^2}\,dx = \int_0^{1/2} x^4\sqrt{1+9x^4}\,dx\\ &= \int_0^{1/2} x^4\left[1+\frac{1}{2}(9x^4)+\frac{(1/2)(-1/2)}{2}(9x^4)^2+\cdots\right]dx\\ &= \int_0^{1/2}\left(x^4+\frac{9}{2}x^8-\frac{9^2}{2^2 2!}x^{12}+\frac{9^3 3}{2^3 3!}x^{16}-\frac{9^4 3\cdot 5}{2^4 4!}x^{20}+\cdots\right]dx\\ &= \left\{\frac{x^5}{5}+\frac{x^9}{2}-\frac{9^2x^{13}}{2^2 2!13}+\frac{9^3 3x^{17}}{2^3 3!17}-\frac{9^4 3\cdot 5x^{21}}{2^4 4!21}+\cdots\right\}_0^{1/2}\\ &= \frac{1}{5\cdot 2^5}+\frac{1}{2^{10}}-\frac{9^2}{2^{15}2!13}+\frac{9^3 3}{2^{20}3!17}-\frac{9^4 3\cdot 5}{2^{25}4!21}+\frac{9^5 3\cdot 5\cdot 7}{2^{30}5!25}+\cdots.\end{aligned}$$

Because this series is alternating (after the first term), and absolute values of terms decrease and approach zero, the sum is between any two consecutive partial sums. Since the sum of the first two terms is 0.007 22 and the sum of the first three terms is 0.007 13, we can say that to three decimals, the value of the integral is 0.007.

23. Using parametric equations $x=-1+t$, $y=3-2t$, $z=t$, $0\le t\le 1$,

$$\begin{aligned}\int_C e^{-(x+y-2)^2}ds &= \int_0^1 e^{-t^2}\sqrt{(1)^2+(-2)^2+(1)^2}\,dt = \sqrt{6}\int_0^1 e^{-t^2}dt\\ &= \sqrt{6}\int_0^1\left(1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}+\cdots\right)dt = \sqrt{6}\left\{t-\frac{t^3}{3}+\frac{t^5}{5\cdot 2!}-\frac{t^7}{7\cdot 3!}+\cdots\right\}_0^1\\ &= \sqrt{6}\left(1-\frac{1}{3}+\frac{1}{5\cdot 2!}-\frac{1}{7\cdot 3!}+\cdots\right).\end{aligned}$$

Because this series is alternating and absolute values of terms decrease and approach zero, the sum is between any two consecutive partial sums. Since the sum of the first six terms is 1.829 11 and the sum of the first seven terms is 1.829 38, we can say that to three decimals, the value of the integral is 1.829.

24. With parametric equations $C:\ x=2\cos t,\ y=2\sin t,\ -\pi<t\le\pi$,

$$\begin{aligned}\overline{f} &= \frac{1}{2\pi(2)}\int_C x^2y^2\,ds = \frac{1}{4\pi}\int_{-\pi}^{\pi}4\cos^2 t\,4\sin^2 t\sqrt{4\sin^2 t+4\cos^2 t}\,dt = \frac{8}{\pi}\int_{-\pi}^{\pi}\cos^2 t\,\sin^2 t\,dt\\ &= \frac{8}{\pi}\int_{-\pi}^{\pi}\left(\frac{\sin 2t}{2}\right)^2dt = \frac{2}{\pi}\int_{-\pi}^{\pi}\left(\frac{1-\cos 4t}{2}\right)dt = \frac{1}{\pi}\left\{t-\frac{\sin 4t}{4}\right\}_{-\pi}^{\pi} = 2.\end{aligned}$$

25. Since $L=\displaystyle\int_0^{\pi}\sqrt{(-\sin t)^2+(\cos t)^2+1}\,dt=\int_0^{\pi}\sqrt{2}\,dt=\sqrt{2}\pi$,

$$\begin{aligned}\overline{f} &= \frac{1}{\sqrt{2}\pi}\int_C(x^2+y^2+z^2)\,ds = \frac{1}{\sqrt{2}\pi}\int_0^{\pi}(\cos^2 t+\sin^2 t+t^2)\sqrt{2}\,dt\\ &= \frac{1}{\pi}\int_0^{\pi}(1+t^2)\,dt = \frac{1}{\pi}\left\{t+\frac{t^3}{3}\right\}_0^{\pi} = \frac{3+\pi^2}{3}.\end{aligned}$$

26. With parametric equations $C:\ y=x^2,\ z=x^2,\ 0\le x\le 1$, the length of the curve is $L=\displaystyle\int_0^1\sqrt{1+(2x)^2+(2x)^2}\,dx=\int_0^1\sqrt{1+8x^2}\,dx$. If we set $x=[1/(2\sqrt{2})]\tan\theta$, then

$$\begin{aligned}L &= \int_0^{\overline{\theta}}\sec\theta\frac{1}{2\sqrt{2}}\sec^2\theta\,d\theta \quad (\overline{\theta}=\mathrm{Tan}^{-1}(2\sqrt{2}))\\ &= \frac{1}{4\sqrt{2}}\Big\{\sec\theta\,\tan\theta+\ln|\sec\theta+\tan\theta|\Big\}_0^{\overline{\theta}} \quad \text{(see Example 8.9)}\\ &= \frac{1}{4\sqrt{2}}[6\sqrt{2}+\ln(3+2\sqrt{2})].\end{aligned}$$

The average value of the function is $\overline{f} = \frac{1}{L}\int_0^1 xyz\,ds = \frac{1}{L}\int_0^1 x^5\sqrt{1+8x^2}\,dx$. If we set $u = 1+8x^2$, then $du = 16x\,dx$, and

$$\overline{f} = \frac{1}{L}\int_1^9 \left(\frac{u-1}{8}\right)^2 \sqrt{u}\left(\frac{du}{16}\right) = \frac{1}{1024L}\int_1^9 (u^{5/2} - 2u^{3/2} + \sqrt{u})\,du$$
$$= \frac{1}{1024L}\left\{\frac{2u^{7/2}}{7} - \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right\}_1^9 = 0.242.$$

27. Since $ds^2 = \left[1 + \left(\frac{3x^2}{4} - \frac{1}{3x^2}\right)^2\right] dx^2 = \left(1 + \frac{9x^4}{16} - \frac{1}{2} + \frac{1}{9x^4}\right) dx^2 = \left(\frac{3x^2}{4} + \frac{1}{3x^2}\right)^2 dx^2$, the length of the curve is $L = \int_1^2 \left(\frac{3x^2}{4} + \frac{1}{3x^2}\right) dx = \left\{\frac{x^3}{4} - \frac{1}{3x}\right\}_1^2 = \frac{23}{12}$. The average value is

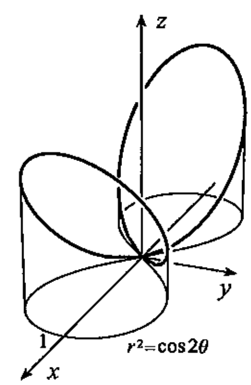
$$\overline{f} = \frac{12}{23}\int_C y\,ds = \frac{12}{23}\int_1^2 \left(\frac{x^3}{4} + \frac{1}{3x}\right)\left(\frac{3x^2}{4} + \frac{1}{3x^2}\right) dx$$
$$= \frac{12}{23}\int_1^2 \left(\frac{3x^5}{16} + \frac{x}{3} + \frac{1}{9x^3}\right) dx = \frac{12}{23}\left\{\frac{x^6}{32} + \frac{x^2}{6} - \frac{1}{18x^2}\right\}_1^2 = \frac{241}{184}.$$

28. $A = 4\int_C \sqrt{x^2+y^2}\,ds$ where C is that part of the lemniscate in the first quadrant. According to formula 9.14,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

Hence,

$$A = 4\int_0^{\pi/4} r\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$= 4\int_0^{\pi/4} \sqrt{\cos 2\theta}\sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\,d\theta = 4\int_0^{\pi/4} d\theta = \pi.$$



29. $M = \int_C \left(1 - \frac{|x|}{40}\right) ds = 2\int_0^{20}\left(1 - \frac{x}{40}\right)\sqrt{1 + \sinh^2\left(\frac{x}{40}\right)}\,dx = 2\int_0^{20}\left(1 - \frac{x}{40}\right)\cosh\left(\frac{x}{40}\right) dx$

If we set $u = 1 - x/40$, $dv = \cosh(x/40)\,dx$, $du = -dx/40$, and $v = 40\sinh(x/40)$,

$$M = 2\left\{\left(1 - \frac{x}{40}\right) 40\sinh\left(\frac{x}{40}\right)\right\}_0^{20} - 2\int_0^{20} 40\sinh\left(\frac{x}{40}\right)\left(\frac{-dx}{40}\right)$$
$$= 40\sinh\left(\frac{1}{2}\right) + 2\left\{40\cosh\left(\frac{x}{40}\right)\right\}_0^{20} = 31.05 \text{ kg}.$$

30. With $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \sqrt{(2-\sin\theta)^2 + (-\cos\theta)^2}\,d\theta = \sqrt{5 - 4\sin\theta}\,d\theta$, we obtain

$$\int_C \frac{x}{\sqrt{x^2+y^2}}\,ds = \int_0^{\pi/2} \frac{r\cos\theta}{r}\sqrt{5-4\sin\theta}\,d\theta = \left\{-\frac{1}{6}(5 - 4\sin\theta)^{3/2}\right\}_0^{\pi/2} = \frac{5\sqrt{5}-1}{6}.$$

31. $$\oint_C (x^2+y^2)\,ds = \int_{-\pi}^{\pi} r^2\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{-\pi}^{\pi}(1+\cos\theta)^2\sqrt{(1+\cos\theta)^2+(-\sin\theta)^2}\,d\theta$$
$$= \int_{-\pi}^{\pi}(1+\cos\theta)^2\sqrt{2+2\cos\theta}\,d\theta = \sqrt{2}\int_{-\pi}^{\pi}(1+\cos\theta)^{5/2}\,d\theta$$
$$= \sqrt{2}\int_{-\pi}^{\pi}[1+2\sin^2(\theta/2)-1]^{5/2}\,d\theta = 8\int_{-\pi}^{\pi}\sin^5(\theta/2)\,d\theta$$
$$= 8\int_{-\pi}^{\pi}\sin(\theta/2)[1-\cos^2(\theta/2)]^2\,d\theta = 8\int_{-\pi}^{\pi}\sin(\theta/2)[1-2\cos^2(\theta/2)+\cos^4(\theta/2)]\,d\theta$$
$$= 8\left\{-2\cos(\theta/2)+\frac{4}{3}\cos^3(\theta/2)-\frac{2}{5}\cos^5(\theta/2)\right\}_{-\pi}^{\pi} = \frac{256}{15}$$

32. With $ds = \sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\,d\theta = \sqrt{e^{2\theta}+e^{2\theta}}\,d\theta = \sqrt{2}e^{\theta}\,d\theta$,
$$\int_C xy\,ds = \int_0^{2\pi} r\cos\theta\, r\sin\theta\,\sqrt{2}e^{\theta}\,d\theta = \sqrt{2}\int_0^{2\pi}\sin\theta\,\cos\theta\,e^{3\theta}d\theta = \frac{1}{\sqrt{2}}\int_0^{2\pi}e^{3\theta}\sin 2\theta\,d\theta$$
$$= \frac{1}{\sqrt{2}}\left\{\frac{e^{3\theta}}{13}(3\sin 2\theta - 2\cos 2\theta)\right\}_0^{2\pi} = \frac{\sqrt{2}(1-e^{6\pi})}{13}.$$

33. $$\oint_C \cos^3 2\theta\,ds = \int_0^{\pi/2}\cos^3 2\theta\sqrt{\sin 2\theta+\left(\frac{\cos 2\theta}{\sqrt{\sin 2\theta}}\right)^2}\,d\theta = \int_0^{\pi/2}\frac{\cos^3 2\theta}{\sqrt{\sin 2\theta}}d\theta$$
$$= \int_0^{\pi/2}\frac{(1-\sin^2 2\theta)\cos 2\theta}{\sqrt{\sin 2\theta}}d\theta = \left\{\sqrt{\sin 2\theta}-\frac{1}{5}\sin^{5/2}2\theta\right\}_0^{\pi/2} = 0$$

34. With parametric equations $C:\ y=1-x,\ z=2x^2-2x+1,\ -1\le x\le 1$,
$$I = \int_{-1}^{1}[x^2(1-x)+2x^2-2x+1]\sqrt{1+(-1)^2+(4x-2)^2}\,dx$$
$$= \int_{-1}^{1}(1-2x+3x^2-x^3)\sqrt{6-16x+16x^2}\,dx = \sqrt{2}\int_{-1}^{1}(1-2x+3x^2-x^3)\sqrt{3-8x+8x^2}\,dx.$$

If we denote the integrand by $f(x)$, then Simpson's rule with 10 equal subdivisions gives
$$I \approx \sqrt{2}\left(\frac{1/5}{3}\right)[f(-1)+4f(-0.8)+2f(-0.6)+\cdots+2f(0.6)+4f(0.8)+f(1)] = 17.08.$$

35. With parametric equations $C:\ x=3\cos t,\ y=2\sin t,\ -\pi<t\le\pi$,
$$\oint_C x^2y^2\,ds = 4\int_0^{\pi/2}(9\cos^2 t)(4\sin^2 t)\sqrt{(-3\sin t)^2+(2\cos t)^2}\,dt$$
$$= 144\int_0^{\pi/2}\cos^2 t\sin^2 t\sqrt{9\sin^2 t+4\cos^2 t}\,dt.$$

If we denote the integrand by $f(t)$, then Simpson's rule with ten equal subdivisions gives
$$\oint_C x^2y^2\,ds \approx \frac{144(\pi/2)}{30}[f(0)+4f(\pi/20)+2f(\pi/10)+\cdots+2f(2\pi/5)+4f(9\pi/20)+f(\pi/2)] = 71.74.$$

36. Using the result of Exercise 17, and parametric equations $C:\ x=a+b\cos t,\ y=b\sin t,\ -\pi<t\le\pi$,
$$A = \int_C 2\pi x\,ds = 2\pi\int_{-\pi}^{\pi}(a+b\cos t)\sqrt{(-b\sin t)^2+(b\cos t)^2}\,dt$$
$$= 2\pi b\int_{-\pi}^{\pi}(a+b\cos t)\,dt = 2\pi b\Big\{at+b\sin t\Big\}_{-\pi}^{\pi} = 4\pi^2 ab.$$

37. Suppose that a curve C has parametrization

$$C: \; x = x(t), \quad y = y(t), \quad z = z(t), \quad \alpha \leq \; t \leq \beta,$$

and we change parameters by setting $t = t(u)$ (or $u = u(t)$). Then

$$C: \; x = x[t(u)], \quad y = y[t(u)], \quad z = z[t(u)], \quad a = u(\alpha) \leq u \leq u(\beta) = b.$$

According to equation 14.20,

$$\int_C f(x,y,z)\,ds = \int_\alpha^\beta f[x(t),y(t),z(t)]\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$$

and if we change the variable of integration by $t = t(u)$, then $dt = t'(u)\,du$, and

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_a^b f\{x[t(u)],y[t(u)],z[t(u)]\}\sqrt{\left(\frac{dx}{du}\frac{du}{dt}\right)^2 + \left(\frac{dy}{du}\frac{du}{dt}\right)^2 + \left(\frac{dz}{du}\frac{du}{dt}\right)^2}\,dt \\ &= \int_a^b f\{x[t(u)],y[t(u)],z[t(u)]\}\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du.\end{aligned}$$

But this is equation 14.20 with $f(x,y,z)$ and ds expressed in term of parameter u instead of t. In other words, the value of the line integral is the same for any parametrization of the curve.

EXERCISES 14.3

1. Using x as parameter along the curve,

$$\int_C x\,dx + x^2y\,dy = \int_{-1}^2 x\,dx + x^2(x^3)(3x^2\,dx) = \int_{-1}^2 (x + 3x^7)\,dx = \left\{\frac{x^2}{2} + \frac{3x^8}{8}\right\}_{-1}^2 = \frac{777}{8}.$$

2. Using x as parameter along the curve,

$$\int_C x\,dx + yz\,dy + x^2\,dz = \int_{-1}^2 [x\,dx + xx^2\,dx + x^2(2x\,dx)] = \int_{-1}^2 (x + 3x^3)\,dx = \left\{\frac{x^2}{2} + \frac{3x^4}{4}\right\}_{-1}^2 = \frac{51}{4}.$$

3. Using parametric equations $x = 1 + t^2$, $y = -t$, $-1 \leq t \leq 1$,

$$\begin{aligned}\int_C x\,dx + (x+y)\,dy &= \int_{-1}^1 (1+t^2)(2t\,dt) + (1+t^2-t)(-dt) = \int_{-1}^1 (-1 + 3t - t^2 + 2t^3)\,dt \\ &= \left\{-t + \frac{3t^2}{2} - \frac{t^3}{3} + \frac{t^4}{2}\right\}_{-1}^1 = -\frac{8}{3}.\end{aligned}$$

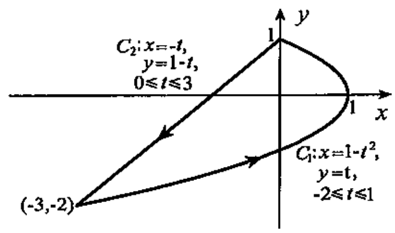
4. With parametric equations $C: \; x = -2 + 3t, \; y = 3 - 3t, \; z = 3 - 3t, \; 0 \leq t \leq 1$, for the straight line,

$$\begin{aligned}\int_C x^2\,dx + y^2\,dy + z^2\,dz &= \int_0^1 [(-2+3t)^2(3\,dt) + (3-3t)^2(-3\,dt) + (3-3t)^2(-3\,dt)] \\ &= 3\int_0^1 [(-2+3t)^2 - 2(3-3t)^2]\,dt = 3\left\{\frac{1}{9}(-2+3t)^3 + \frac{2}{9}(3-3t)^3\right\}_0^1 = -15.\end{aligned}$$

5. Using y as parameter along the curve,

$$\int_C (y + 2x^2z)\,dx = \int_{-2}^1 [y + 2(y^2)^2(y^4)](2y\,dy) = 2\int_{-2}^1 (y^2 + 2y^9)\,dy = 2\left\{\frac{y^3}{3} + \frac{y^{10}}{5}\right\}_{-2}^1 = -\frac{2016}{5}.$$

6. $$\oint_C x^2y\,dx + (x-y)\,dy = \int_{C_1} x^2y\,dx + (x-y)\,dy + \int_{C_2} x^2y\,dx + (x-y)\,dy$$
$$= \int_{-2}^{1} [(1-t^2)^2 t(-2t\,dt) + (1-t^2-t)\,dt]$$
$$+ \int_0^3 [t^2(1-t)(-dt) + (-t-1+t)(-dt)]$$
$$= \int_{-2}^{1} (-2t^6 + 4t^4 - 3t^2 - t + 1)\,dt + \int_0^3 (t^3 - t^2 + 1)\,dt$$
$$= \left\{-\frac{2t^7}{7} + \frac{4t^5}{5} - t^3 - \frac{t^2}{2} + t\right\}_{-2}^{1} + \left\{\frac{t^4}{4} - \frac{t^3}{3} + t\right\}_0^3 = -\frac{99}{140}$$

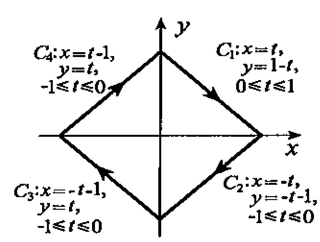


7. With parametric equations $C:\ x = \cos t,\ y = -\sin t,\ -\pi/2 \le t \le \pi/2$,
$$\int_C y^2\,dx + x^2\,dy = \int_{-\pi/2}^{\pi/2} \sin^2 t(-\sin t\,dt) + \cos^2 t(-\cos t\,dt) = \int_{-\pi/2}^{\pi/2} [-(1-\cos^2 t)\sin t - (1-\sin^2 t)\cos t]\,dt$$
$$= \left\{\cos t - \frac{1}{3}\cos^3 t - \sin t + \frac{1}{3}\sin^3 t\right\}_{-\pi/2}^{\pi/2} = -\frac{4}{3}.$$

8. With parametric equations $C:\ x = 1-y,\ z = 2y^2 - 2y + 1,\ 0 \le y \le 2$,
$$\int_C y\,dx + x\,dy + z\,dz = \int_0^2 [y(-dy) + (1-y)\,dy + (2y^2-2y+1)(4y-2)\,dy]$$
$$= \int_0^2 [-2y + 1 + (2y^2-2y+1)(4y-2)]\,dy = \left\{-y^2 + y + \frac{(2y^2-2y+1)^2}{2}\right\}_0^2 = 10.$$

9. With parametric equations $C:\ x = \cos t,\ y = \sin t,\ z = 1 - \cos t - \sin t,\ -\pi < t \le \pi$,
$$\oint_C x^2y\,dy + z\,dx = \int_{-\pi}^{\pi} \cos^2 t(\sin t)(\cos t\,dt) + (1 - \cos t - \sin t)(-\sin t\,dt)$$
$$= \int_{-\pi}^{\pi} \left(\cos^3 t \sin t - \sin t + \cos t \sin t + \frac{1-\cos 2t}{2}\right) dt$$
$$= \left\{-\frac{1}{4}\cos^4 t + \cos t + \frac{1}{2}\sin^2 t + \frac{t}{2} - \frac{1}{4}\sin 2t\right\}_{-\pi}^{\pi} = \pi.$$

10. $$\oint_C y^2\,dx + x^2\,dy = \int_{C_1} y^2\,dx + x^2\,dy + \int_{C_2} y^2\,dx + x^2\,dy + \int_{C_3} y^2\,dx + x^2\,dy + \int_{C_4} y^2\,dx + x^2\,dy$$
$$= \int_0^1 [(1-t)^2\,dt + t^2(-dt)] + \int_{-1}^0 [(t+1)^2(-dt) + t^2(-dt)]$$
$$+ \int_{-1}^0 [t^2(-dt) + (t+1)^2\,dt] + \int_0^1 [t^2\,dt + (t-1)^2\,dt]$$
$$= \int_0^1 2(t-1)^2\,dt + \int_{-1}^0 (-2t^2)\,dt$$
$$= \left\{\frac{2(t-1)^3}{3}\right\}_0^1 + \left\{-\frac{2t^3}{3}\right\}_{-1}^0 = 0$$



11. With parametric equations $C: \ x = 1 + 5t, \ y = 5t, \ 0 \le t \le 1$,

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x^2 y\, dx + x\, dy = \int_0^1 (1+5t)^2(5t)(5\,dt) + (1+5t)(5\,dt)$$
$$= 5\int_0^1 (1 + 10t + 50t^2 + 125t^3)\, dt = 5\left\{t + 5t^2 + \frac{50t^3}{3} + \frac{125t^4}{4}\right\}_0^1 = \frac{3235}{12}.$$

12. (a) $$\int_C xy\, dx + x^2\, dy = \int_0^{\pi/2} [3\cos t(3\sin t)(-3\sin t\, dt) + 9\cos^2 t(3\cos t\, dt)]$$
$$= 27\int_0^{\pi/2} [-\sin^2 t\, \cos t + \cos t(1 - \sin^2 t)]\, dt = 27\left\{-\frac{2}{3}\sin^3 t + \sin t\right\}_0^{\pi/2} = 9$$

(b) $$\int_C xy\, dx + x^2\, dy = \int_0^3 \left[\sqrt{9-y^2}(y)\left(\frac{-y}{\sqrt{9-y^2}}dy\right) + (9-y^2)\, dy\right] = \int_0^3 (9 - 2y^2)\, dy$$
$$= \left\{9y - \frac{2y^3}{3}\right\}_0^3 = 9$$

13. (a) Along the straight line with parametric equations $C_1: \ x = -5 + 9t, \ y = 3 - 3t, \ 0 \le t \le 1$,

$$\int_{C_1} xy\, dx + x\, dy = \int_0^1 (-5+9t)(3-3t)(9\,dt) + (-5+9t)(-3\,dt)$$
$$= 3\int_0^1 (-40 + 117t - 81t^2)\, dt = 3\left\{-40t + \frac{117t^2}{2} - 27t^3\right\}_0^1 = -\frac{51}{2}.$$

(b) Along the parabola with parametric equations $C_2: \ x = 4 - t^2, \ y = -t, \ -3 \le t \le 0$,

$$\int_{C_2} xy\, dx + x\, dy = \int_{-3}^0 (4-t^2)(-t)(-2t\, dt) + (4-t^2)(-dt) = \int_{-3}^0 (-4 + 9t^2 - 2t^4)\, dt$$
$$= \left\{-4t + 3t^3 - \frac{2t^5}{5}\right\}_{-3}^0 = -\frac{141}{5}.$$

(c) Along the parabola with equation $C_3: \ y = (x^2 - 16)/3, \ -5 \le x \le 4$,

$$\int_{C_3} xy\, dx + x\, dy = \int_{-5}^4 x\left(\frac{x^2-16}{3}\right) dx + x\left(\frac{2x\, dx}{3}\right) = \frac{1}{3}\int_{-5}^4 (x^3 + 2x^2 - 16x)\, dx$$
$$= \frac{1}{3}\left\{\frac{x^4}{4} + \frac{2x^3}{3} - 8x^2\right\}_{-5}^4 = \frac{141}{4}.$$

14. With parametric equations $C: \ x = a\cos t, \ y = b\sin t, \ -\pi \le t < \pi$,

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C x\, dx + y\, dy = \int_{-\pi}^{\pi} [a\cos t(-a\sin t\, dt) + b\sin t(b\cos t\, dt)]$$
$$= (b^2 - a^2)\int_{-\pi}^{\pi} \sin t\, \cos t\, dt = (b^2 - a^2)\left\{\frac{\sin^2 t}{2}\right\}_{-\pi}^{\pi} = 0.$$

15. With parametric equations $C: \ x = \cos t, \ y = \sin t, \ z = \sin t, \ \pi/4 \le t \le 3\pi/4$,

$$\int_C \frac{1}{yz}dx = \int_{\pi/4}^{3\pi/4} \frac{1}{\sin^2 t}(-\sin t\, dt) = \int_{\pi/4}^{3\pi/4} -\csc t\, dt = -\Big\{\ln|\csc t - \cot t|\Big\}_{\pi/4}^{3\pi/4} = 0.$$

16. With parametric equations $C:\ x = 2+\cos t,\ y = -\sin t,\ 0 \le t < 4\pi$,

$$\begin{aligned}\oint_C (x^2+2y^2)\,dy &= \int_0^{4\pi} [(2+\cos t)^2 + 2(-\sin t)^2](-\cos t\,dt)\\ &= -\int_0^{4\pi} (4+4\cos t+\cos^2 t+2\sin^2 t)\cos t\,dt = -\int_0^{4\pi} (5+4\cos t+\sin^2 t)\cos t\,dt\\ &= -\int_0^{4\pi} [5\cos t + 2(1+\cos 2t)+\sin^2 t\,\cos t]\,dt = -\left\{5\sin t + 2t + \sin 2t + \frac{1}{3}\sin^3 t\right\}_0^{4\pi}\\ &= -8\pi.\end{aligned}$$

17. With parametric equations $C:\ x = 1+\cos t,\ y = \sin t,\ z = \sqrt{2-2\cos t},\ 0 \le t \le \pi$,

$$\begin{aligned}\int_C y\,dx - y(x-1)\,dy + y^2 z\,dz &= \int_0^{\pi} \sin t(-\sin t\,dt) - \sin t(\cos t)(\cos\,dt) + \sin^2 t\sqrt{2-2\cos t}\left(\frac{\sin t\,dt}{\sqrt{2-2\cos t}}\right)\\ &= \int_0^{\pi}\left[\frac{\cos 2t-1}{2} - \cos^2 t\sin t + (1-\cos^2 t)\sin t\right]dt\\ &= \left\{\frac{1}{4}\sin 2t - \frac{t}{2} + \frac{2}{3}\cos^3 t - \cos t\right\}_0^{\pi} = \frac{2}{3} - \frac{\pi}{2}.\end{aligned}$$

18. With parametric equations $C:\ x = t-1,\ y = 1+2t^2,\ z = t,\ 1 \le t \le 2$,

$$\begin{aligned}\int_C x^2 y\,dx + y\,dy + \sqrt{1-x^2}\,dz &= \int_1^2 [(t-1)^2(1+2t^2)\,dt + (1+2t^2)(4t\,dt) + \sqrt{1-(t-1)^2}\,dt]\\ &= \int_1^2 [2t^4+4t^3+3t^2+2t+1+\sqrt{1-(t-1)^2}]\,dt.\end{aligned}$$

In the last term we set $t-1 = \sin\theta$ and $dt = \cos\theta\,d\theta$,

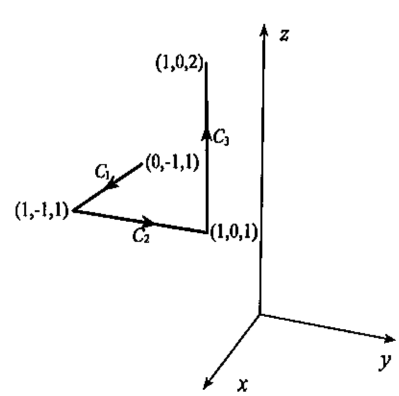
$$\begin{aligned}\int_C x^2 y\,dx + y\,dy + \sqrt{1-x^2}\,dz &= \left\{\frac{2t^5}{5} + t^4 + t^3 + t^2 + t\right\}_1^2 + \int_0^{\pi/2}\cos^2\theta\,d\theta\\ &= \frac{192}{5} + \int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{192}{5} + \left\{\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right\}_0^{\pi/2} = \frac{192}{5} + \frac{\pi}{4}.\end{aligned}$$

19. With parametric equations $C:\ x = 2+t,\ y = -1+2t,\ z = 4-5t,\ 0 \le t \le 1$,

$$\begin{aligned}\int_C x\,dx + xy\,dy + 2\,dz &= \int_0^1 (2+t)\,dt + (2+t)(-1+2t)(2\,dt) + 2(-5\,dt)\\ &= \int_0^1 (-12+7t+4t^2)\,dt = \left\{-12t + \frac{7t^2}{2} + \frac{4t^3}{3}\right\}_0^1 = -\frac{43}{6}.\end{aligned}$$

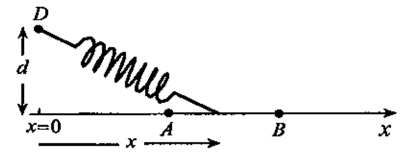
20. The line integral along C is equal to the sum of the line integrals along C_1, C_2, and C_3; that is,

$$\begin{aligned}&\int_C \frac{x^3}{(1+x^4)^3}dx + y^2 e^y\,dy + \frac{z}{\sqrt{1+z^2}}dz\\ &= \int_0^1 \frac{x^3}{(1+x^4)^3}dx + \int_{-1}^0 y^2 e^y\,dy\\ &\quad + \int_1^2 \frac{z}{\sqrt{1+z^2}}dz\\ &= \left\{\frac{-1}{8(1+x^4)^2}\right\}_0^1 + \left\{y^2e^y - 2ye^y + 2e^y\right\}_{-1}^0\\ &\quad + \left\{\sqrt{1+z^2}\right\}_1^2\\ &= \frac{67}{32} + \sqrt{5} - \sqrt{2} - \frac{5}{e}.\end{aligned}$$



21. Since the spring is stretched an amount ℓ at A, its natural length is $L = \sqrt{a^2 + d^2} - \ell$.
Its stretch at x is therefore $\sqrt{x^2 + d^2} - L$.
The force necessary to counteract the spring at this position is

$$k(\sqrt{x^2+d^2} - L)\frac{(x\hat{\mathbf{i}} - d\hat{\mathbf{j}})}{\sqrt{x^2+d^2}}.$$



The work done by this force along that part C of the x-axis from A to B is

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b k(\sqrt{x^2+d^2} - L)\left(\frac{x\hat{\mathbf{i}} - d\hat{\mathbf{j}}}{\sqrt{x^2+d^2}}\right)\cdot(dx\,\hat{\mathbf{i}}) = k\int_a^b(\sqrt{x^2+d^2} - L)\frac{x}{\sqrt{x^2+d^2}}dx \\ &= k\int_a^b\left(x - \frac{Lx}{\sqrt{x^2+d^2}}\right)dx = k\left\{\frac{x^2}{2} - L\sqrt{x^2+d^2}\right\}_a^b \\ &= k\left[\left(\frac{b^2}{2} - L\sqrt{b^2+d^2}\right) - \left(\frac{a^2}{2} - L\sqrt{a^2+d^2}\right)\right]\end{aligned}$$

22. At position x, the magnitude of the force $\mathbf{F}_1$ of q_1 on q_3 is $|\mathbf{F}_1| = \dfrac{q_1q_3}{4\pi\epsilon_0[(x-5)^2+25]}$. Since a unit vector in the direction of $\mathbf{F}_1$ is

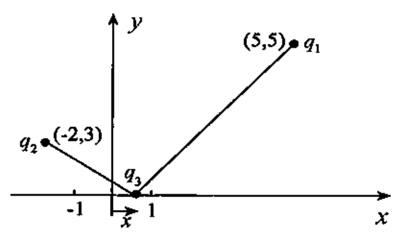
$$\hat{\mathbf{F}}_1 = \frac{(x-5,-5)}{\sqrt{(x-5)^2+25}},$$

it follows that

$$\mathbf{F}_1 = \frac{q_1q_3}{4\pi\epsilon_0[(x-5)^2+25]^{3/2}}(x-5,-5).$$

Similarly, the force $\mathbf{F}_2$ of q_2 on q_3 is

$$\mathbf{F}_2 = \frac{q_2q_3}{4\pi\epsilon_0[(x+2)^2+9]^{3/2}}(x+2,-3).$$



The work done by these forces is

$$\begin{aligned}W &= \int_C(\mathbf{F}_1+\mathbf{F}_2)\cdot d\mathbf{r} = \int_C(\mathbf{F}_1+\mathbf{F}_2)\cdot(dx\,\hat{\mathbf{i}}) \\ &= \int_C\left\{\frac{q_1q_3}{4\pi\epsilon_0[(x-5)^2+25]^{3/2}}(x-5) + \frac{q_2q_3}{4\pi\epsilon_0[(x+2)^2+9]^{3/2}}(x+2)\right\}dx \\ &= \int_{-1}^1\left\{\frac{q_1q_3}{4\pi\epsilon_0[(-t-5)^2+25]^{3/2}}(-t-5) + \frac{q_2q_3}{4\pi\epsilon_0[(2-t)^2+9]^{3/2}}(2-t)\right\}(-dt) \\ &= \left\{\frac{-q_1q_3}{4\pi\epsilon_0\sqrt{(t+5)^2+25}} + \frac{-q_2q_3}{4\pi\epsilon_0\sqrt{(2-t)^2+9}}\right\}_{-1}^1 \\ &= \frac{1}{4\pi\epsilon_0}\left[q_1q_3\left(\frac{1}{\sqrt{41}} - \frac{1}{\sqrt{61}}\right) + q_2q_3\left(\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{10}}\right)\right].\end{aligned}$$

23. Using points A and D we obtain $k_2 = 12\,000$ and $k_1 = 20\,000$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

$$\begin{aligned}W &= \int_{C_1}P\,dV + \int_{C_2}P\,dV + \int_{C_3}P\,dV + \int_{C_4}P\,dV \\ &= \int_{3/5}^{1/5}\frac{k_2}{V}dV + 0 + \int_{1/5}^{3/5}\frac{k_1}{V}dV + 0 = (k_1-k_2)\int_{1/5}^{3/5}\frac{1}{V}dV = (k_1-k_2)\Big\{\ln V\Big\}_{1/5}^{3/5} = 8.8\times10^3\text{ J}.\end{aligned}$$

24. Using points A and D we obtain $k_2 = 1000$ and $k_1 = 20\,000$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV = \int_{1/10}^{1/100} \frac{k_2}{V} dV + \int_{1/100}^{1/5} 10^5\,dV + \int_{1/5}^{2} \frac{k_1}{V} dV + \int_{2}^{1/10} 10^4\,dV$$
$$= k_2 \Big\{\ln V\Big\}_{1/10}^{1/100} + 10^5 \left(\frac{1}{5} - \frac{1}{100}\right) + k_1 \Big\{\ln V\Big\}_{1/5}^{2} + 10^4 \left(\frac{1}{10} - 2\right) = 4.4 \times 10^4 \text{ J}.$$

25. Using points B and C we obtain $k_2 = 4.64$ and $k_1 = 6.89$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV = \int_{8\times 10^{-4}}^{2\times 10^{-4}} \frac{k_2}{V^{1.4}} dV + 0 + \int_{2\times 10^{-4}}^{8\times 10^{-4}} \frac{k_1}{V^{1.4}} dV + 0$$
$$= (k_1 - k_2) \int_{2\times 10^{-4}}^{8\times 10^{-4}} \frac{1}{V^{1.4}} dV = (k_1 - k_2) \left\{ \frac{-1}{0.4V^{0.4}} \right\}_{2\times 10^{-4}}^{8\times 10^{-4}} = 72 \text{ J}.$$

26. Using points A and D we obtain $k_2 = 15.0$ and $k_1 = 66.6$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

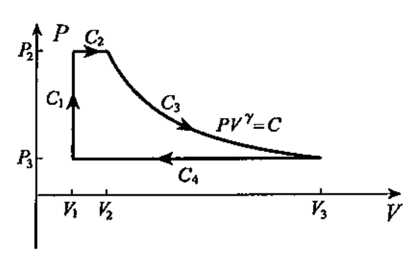
$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV$$
$$= \int_{20\times 10^{-4}}^{2\times 10^{-4}} \frac{k_2}{V^{1.4}} dV + \int_{2\times 10^{-4}}^{5.75\times 10^{-4}} (23 \times 10^5)\,dV + \int_{5.75\times 10^{-4}}^{20\times 10^{-4}} \frac{k_1}{V^{1.4}} dV + 0$$
$$= k_2 \left\{ \frac{-1}{0.4V^{0.4}} \right\}_{20\times 10^{-4}}^{2\times 10^{-4}} + 23 \times 10^5 (5.75 \times 10^{-4} - 2 \times 10^{-4}) + k_1 \left\{ \frac{-1}{04.V^{0.4}} \right\}_{5.75\times 10^{-4}}^{20\times 10^{-4}} = 1.5 \times 10^3 \text{ J}.$$

27. Using points A and D we obtain $k_2 = 2.62$ and $k_1 = 32.2$. Designating the four parts of the cycle starting at A by C_1, C_2, C_3, and C_4,

$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV$$
$$= \int_{4\times 10^{-4}}^{10^{-4}} \frac{k_2}{V^{1.4}} dV + \int_{10^{-4}}^{6\times 10^{-4}} (10.4 \times 10^5)\,dV + \int_{6\times 10^{-4}}^{24\times 10^{-4}} \frac{k_1}{V^{1.4}} dV + \int_{24\times 10^{-4}}^{4\times 10^{-4}} (1.5 \times 10^5)\,dV$$
$$= k_2 \left\{ \frac{-1}{0.4V^{0.4}} \right\}_{4\times 10^{-4}}^{10^{-4}} + 10.4 \times 10^5 (5 \times 10^{-4}) + k_1 \left\{ \frac{-1}{0.4V^{0.4}} \right\}_{6\times 10^{-4}}^{24\times 10^{-4}} + 1.5 \times 10^5 (-20 \times 10^{-4})$$
$$= 7.8 \times 10^2 \text{ J}.$$

28. Designating the four parts of the cycle by C_1, C_2, C_3, and C_4,

$$W = \int_{C_1} P\,dV + \int_{C_2} P\,dV + \int_{C_3} P\,dV + \int_{C_4} P\,dV$$
$$= 0 + \int_{V_1}^{V_2} P_2\,dV + \int_{V_2}^{V_3} CV^{-\gamma}\,dV + \int_{V_3}^{V_2} P_3\,dV$$
$$= P_2(V_2 - V_1) + C \left\{ \frac{V^{1-\gamma}}{1-\gamma} \right\}_{V_2}^{V_3} + P_3(V_2 - V_3)$$
$$= P_2(V_2 - V_1) + \frac{C}{1-\gamma}(V_3^{1-\gamma} - V_2^{1-\gamma}) + P_3(V_2 - V_3).$$



Since $P_2V_2^\gamma = C = P_3V_3^\gamma$, it follows that $P_3 = P_2(V_2/V_3)^\gamma$, and

$$W = P_2(V_2 - V_1) + \frac{P_2V_2^\gamma}{1-\gamma}(V_3^{1-\gamma} - V_2^{1-\gamma}) + P_2 \left(\frac{V_2}{V_3}\right)^\gamma (V_2 - V_3).$$

29. $\displaystyle\int_C xy\,dx + xy^2\,dy = \int_0^2 \frac{x}{\sqrt{1+x^3}}dx + \frac{x}{1+x^3}\left[\frac{-3x^2}{2(1+x^3)^{3/2}}dx\right] = \int_0^2 \frac{2x(1+x^3)^2 - 3x^3}{2(1+x^3)^{5/2}}dx$

If we set $f(x) = [2x(1+x^3)^2 - 3x^3]/(1+x^3)^{5/2}$, and use Simpson's rule with 10 equal subdivisions,

$$\int_C xy\,dx + xy^2\,dy \approx \frac{1/5}{2(3)}[f(0) + 4f(1/5) + 2f(2/5) + \cdots + 2f(8/5) + 4f(9/5) + f(2)] = 0.8584.$$

30. When we use y as the parameter along the curve, the value of the line integral is

$$\begin{aligned} I &= \int_{-1}^1 y^2y^3(2y\,dy) + \tan y^2\,dy + e^{y^3}(3y^2\,dy) = \int_{-1}^1 (2y^6 + 3y^2e^{y^3} + \tan y^2)\,dy \\ &= \left\{\frac{2y^7}{7} + e^{y^3}\right\}_{-1}^1 + 2\int_0^1 \tan y^2\,dy. \end{aligned}$$

If we use Simpson's rule with 10 equal subdivisions on the remaining integral

$$\begin{aligned} I \approx \left(\frac{2}{7} + e\right) - \left(-\frac{2}{7} + e^{-1}\right) + \frac{1/5}{3}\big[&\tan(0) + 4\tan(0.01) + 2\tan(0.04) \\ &+ \cdots + 2\tan(0.64) + 4\tan(0.81) + \tan(1)\big] = 3.719. \end{aligned}$$

31. $$\begin{aligned} \int_C \sqrt{1+y^2}\,dz + zy\,dy &= \int_0^{\pi/2} \sqrt{1+\cos^6 t}(3\sin^2 t\cos t\,dt) + \cos^3 t\sin^3 t(-3\cos^2 t\sin t\,dt) \\ &= 3\int_0^{\pi/2} (\sin^2 t\cos t\sqrt{1+\cos^6 t} - \cos^5 t\sin^4 t)\,dt \end{aligned}$$

If we set $f(t) = \sin^2 t\cos t\sqrt{1+\cos^6 t} - \cos^5 t\sin^4 t$, and use Simpson's rule with 10 equal subdivisions,

$$\begin{aligned} \int_C \sqrt{1+y^2}\,dz + zy\,dy &\approx \frac{3(\pi/20)}{3}[f(0) + 4f(\pi/20) + 2f(2\pi/20) + \cdots + 2f(8\pi/20) + 4f(9\pi/20) + f(\pi/2)] \\ &= 0.9934. \end{aligned}$$

32. $$\begin{aligned} \int_C xyz\,dy &= \int_{-1}^1 \left(\frac{1-t^2}{1+t^2}\right)\left[\frac{t(1-t^2)}{1+t^2}\right]t\left[\frac{(1+t^2)(1-3t^2) - (t-t^3)(2t)}{(1+t^2)^2}\right]dt \\ &= \int_{-1}^1 \frac{t^2(1-t^2)^2(1-4t^2-t^4)}{(1+t^2)^4}dt \end{aligned}$$

If we denote the integrand by $f(t)$, then Simpson's rule with 10 equal subdivisions gives

$$\begin{aligned} \int_C xyz\,dy &\approx \frac{2/10}{3}[f(-1) + 4f(-0.8) + 2f(-0.6) + \cdots + 2f(0.6) + 4f(0.8) + f(1)] \\ &= -4.26 \times 10^{-4}. \end{aligned}$$

33. With $x = r\cos\theta = (1-\cos\theta)\cos\theta$ and $y = r\sin\theta = (1-\cos\theta)\sin\theta$,

$$\begin{aligned} \oint_C y\,dx &= \int_{-\pi}^{\pi} (1-\cos\theta)\sin\theta(-\sin\theta + 2\cos\theta\sin\theta)\,d\theta \\ &= \int_{-\pi}^{\pi} (-\sin^2\theta + 3\sin^2\theta\cos\theta - 2\sin^2\theta\cos^2\theta)\,d\theta \\ &= \int_{-\pi}^{\pi} \left(\frac{\cos 2\theta - 1}{2} + 3\sin^2\theta\cos\theta - \frac{\sin^2 2\theta}{2}\right)d\theta \\ &= \int_{-\pi}^{\pi} \left(\frac{\cos 2\theta - 1}{2} + 3\sin^2\theta\cos\theta - \frac{1-\cos 4\theta}{4}\right)d\theta \\ &= \left\{-\frac{3\theta}{4} + \frac{1}{4}\sin 2\theta + \sin^3\theta + \frac{1}{16}\sin 4\theta\right\}_{-\pi}^{\pi} = -\frac{3\pi}{2} \end{aligned}$$

34. Since $x = r\cos\theta$, $y = r\sin\theta$, and $r = \theta$, it follows that $x = \theta\cos\theta$ and $y = \theta\sin\theta$. Hence,

$$\begin{aligned}\int_C y\,dx + x\,dy &= \int_0^\pi \theta\sin\theta(\cos\theta - \theta\sin\theta)\,d\theta + \theta\cos\theta(\sin\theta + \theta\cos\theta)\,d\theta\\ &= \int_0^\pi [2\theta\,\sin\theta\,\cos\theta + \theta^2(\cos^2\theta - \sin^2\theta)]\,d\theta\\ &= \int_0^\pi [\theta\sin 2\theta + \theta^2\cos 2\theta]\,d\theta = \left\{\frac{\theta^2}{2}\sin 2\theta\right\}_0^\pi = 0.\end{aligned}$$

35. With parametric equations $x = r\cos t$, $y = r\sin t$, $z = 1$, $-\pi < t \le \pi$,

$$\text{(a) } \Gamma = \int_C \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}} = \int_{-\pi}^{\pi} \frac{r\cos t(-r\sin t\,dt) + r\sin t(r\cos t\,dt)}{(r^2+1)^{3/2}} = 0$$

$$\text{(b) } \Gamma = \int_C -y\,dx + x\,dy = \int_{-\pi}^{\pi} -r\sin t(-r\sin t\,dt) + r\cos t(r\cos t\,dt) = r^2\int_{-\pi}^{\pi} dt = 2\pi r^2$$

36. When C: $x = x(t)$, $y = y(t)$, $z = z(t)$, $\alpha \le t \le \beta$ are parametric equations for C, then parametric equations for $-C$ are $x = x(t)$, $y = y(t)$, $z = z(t)$, $\beta \ge t \ge \alpha$. To obtain an increasing parameter along $-C$, we set $u = -t$, in which case

$$-C:\quad x = x(-u),\quad y = y(-u),\quad z = z(-u),\quad -\beta \le u \le -\alpha.$$

If $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$, then the value of the line integral along $-C$ can be expressed as a definite integral with respect to u:

$$\begin{aligned}\int_{-C}\mathbf{F}\cdot d\mathbf{r} &= \int_{-C} P\,dx + Q\,dy + R\,dz\\ &= \int_{-\beta}^{-\alpha}\left\{P[x(-u),y(-u),z(-u)]\frac{dx}{du} + Q[x(-u),y(-u),z(-u)]\frac{dy}{du}\right.\\ &\qquad\left. + R[x(-u),y(-u),z(-u)]\frac{dz}{du}\right\}du.\end{aligned}$$

If we now change variables of integration by setting $t = -u$,

$$\frac{dx}{du} = \frac{dx}{dt}\frac{dt}{du} = -\frac{dx}{dt},$$

and similarly for dy/du and dz/du. Consequently,

$$\begin{aligned}\int_{-C}\mathbf{F}\cdot d\mathbf{r} &= \int_\beta^\alpha \left\{P[x(t),y(t),z(t)]\left(-\frac{dx}{dt}\right) + Q[x(t),y(t),z(t)]\left(-\frac{dy}{dt}\right)\right.\\ &\qquad\left. + R[x(t),y(t),z(t)]\left(-\frac{dz}{dt}\right)\right\}(-dt)\\ &= -\int_\alpha^\beta \left\{P[x(t),y(t),z(t)]\frac{dx}{dt} + Q[x(t),y(t),z(t)]\frac{dy}{dt}\right.\\ &\qquad\left. + R[x(t),y(t),z(t)]\frac{dz}{dt}\right\}dt\\ &= -\int_C \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

37. For $\mathbf{F}$ to satisfy $f = \mathbf{F}\cdot\hat{\mathbf{T}}$, we must have $f = |\mathbf{F}|\cos\theta$. This leaves us the freedom to choose the magnitude $|\mathbf{F}|$ and the angle θ so that $f = |\mathbf{F}|\cos\theta$. This can be done in many ways so that $\mathbf{F}$ is not unique. For example, suppose C is that part of the x-axis from $x = a$ to $x = b$, and $f(x,y,z) > 0$ is given. Then for $f = |\mathbf{F}|\cos\theta$, one possible choice is $|\mathbf{F}| = f$ and $\theta = 0$. Another is $|\mathbf{F}| = \sqrt{2}f$ and $\theta = \pi/4$.

38. (a) Since the centre of the circle has coordinates $(R\theta, R)$, the direction of the unit force is

$$(R\theta - R\theta + R\sin\theta, R - R + R\cos\theta) = R(\sin\theta, \cos\theta).$$

Hence, the unit force has components $\mathbf{F} = (\sin\theta, \cos\theta)$. The work done during a half revolution is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \sin\theta\, dx + \cos\theta\, dy = \int_0^\pi [\sin\theta(R - R\cos\theta)\, d\theta + \cos\theta(R\sin\theta)\, d\theta \\ &= R\int_0^\pi \sin\theta\, d\theta = R\left\{-\cos\theta\right\}_0^\pi = 2R. \end{aligned}$$

(b) The work done by the vertical component is

$$W = \int_C \cos\theta\hat{\mathbf{j}} \cdot d\mathbf{r} = \int_C \cos\theta\, dy = \int_0^\pi \cos\theta(R\sin\theta)\, d\theta = \left\{\frac{R}{2}\sin^2\theta\right\}_0^\pi = 0.$$

39. Suppose we divide the integral into three separate integrals

$$\oint_C f(x)\, dx + g(y)\, dy + h(z)\, dz = \oint_C f(x)\, dx + \oint_C g(y)\, dy + \oint_C h(z)\, dz.$$

To evaluate the first integral, we could evaluate an antiderivative $F(x)$ of $f(x)$ at initial and final points on the curve. But because the curve is closed, these are the same point, and the value is zero. The same is true for the second and third terms.

EXERCISES 14.4

1. Since $\nabla(x^2y^2/2) = xy^2\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}}$, the line integral is independent of path in the xy-plane, and

$$\int_C xy^2\, dx + x^2y\, dy = \left\{\frac{x^2y^2}{2}\right\}_{(0,0,0)}^{(1,1,0)} = \frac{1}{2}.$$

2. Since $\nabla(x^3 + xy) = (3x^2 + y)\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, the line integral is independent of path in space, and

$$\int_C (3x^2 + y)\, dx + x\, dy = \left\{x^3 + xy\right\}_{(2,1,5)}^{(-3,2,4)} = -43.$$

3. Since $\nabla(x^2e^y + 3y) = 2xe^y\hat{\mathbf{i}} + (x^2e^y + 3)\hat{\mathbf{j}}$, the line integral is independent of path in the xy-plane, and

$$\int_C 2xe^y\, dx + (x^2e^y + 3)\, dy = \left\{x^2e^y + 3y\right\}_{(1,0,0)}^{(-1,0,0)} = 1 - 1 = 0.$$

4. Since $\nabla(x^3yz - 2z^2) = 3x^2yz\hat{\mathbf{i}} + x^3z\hat{\mathbf{j}} + (x^3y - 4z)\hat{\mathbf{k}}$, the line integral is independent of path in space, and

$$\int_C 3x^2yz\, dx + x^3z\, dy + (x^3y - 4z)\, dz = \left\{x^3yz - 2z^2\right\}_{(-1,-1,1)}^{(1,1,-1)} = -2.$$

5. Since $\nabla\left(\frac{y}{z}\cos x\right) = \left(-\frac{y}{z}\sin x\right)\hat{\mathbf{i}} + \left(\frac{1}{z}\cos x\right)\hat{\mathbf{j}} - \left(\frac{y}{z^2}\cos x\right)\hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the xy-plane. Since C does not pass through the xy-plane,

$$\int_C -\frac{y}{z}\sin x\, dx + \frac{1}{z}\cos x\, dy - \frac{y}{z^2}\cos x\, dz = \left\{\frac{y}{z}\cos x\right\}_{(2,0,2\pi)}^{(2,0,4\pi)} = 0.$$

6. Since $\nabla(y\sin x) = y\cos x\,\hat{\mathbf{i}} + \sin x\,\hat{\mathbf{j}}$, the line integral is independent of path in the xy-plane, and

$$\oint_C y\cos x\, dx + \sin x\, dy = 0.$$

7. Since $\nabla(x^3/3+y^3/3+z^3/3)=x^2\hat{\mathbf{i}}+y^2\hat{\mathbf{j}}+z^2\hat{\mathbf{k}}$, the line integral is independent of path in space, and

$$\int_C x^2\,dx+y^2\,dy+z^2\,dz=\left\{\frac{x^3}{3}+\frac{y^3}{3}+\frac{z^3}{3}\right\}\Bigg|_{(-2,3,3)}^{(1,0,0)}=\left(\frac{1}{3}\right)-\left(-\frac{8}{3}+9+9\right)=-15.$$

8. Since $\nabla(xy+z^2/2)=y\hat{\mathbf{i}}+x\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, the line integral is independent of path in space, and

$$\int_C y\,dx+x\,dy+z\,dz=\left\{xy+\frac{z^2}{2}\right\}\Bigg|_{(1,0,1)}^{(-1,2,5)}=10.$$

9. Since $\nabla(x/y+z)=(1/y)\hat{\mathbf{i}}-(x/y^2)\hat{\mathbf{j}}+\hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the xz-plane. Since C does not pass through the xz-plane,

$$\int_C \frac{1}{y}dx-\frac{x}{y^2}dy+dz=\left\{\frac{x}{y}+z\right\}\Bigg|_{(0,1,1)}^{(3,10,-11)}=\left(\frac{3}{10}-11\right)-1=-\frac{117}{10}.$$

10. Since $\nabla(x^3y^3)=3x^2y^3\hat{\mathbf{i}}+3x^3y^2\hat{\mathbf{j}}$, the line integral is independent of path in the xy-plane, and

$$\int_C 3x^2y^3\,dx+3x^3y^2\,dy=\left\{x^3y^3\right\}\Bigg|_{(0,1)}^{(1,e)}=e^3.$$

11. Since $\nabla\times[f(x)\hat{\mathbf{i}}+g(y)\hat{\mathbf{j}}+h(z)\hat{\mathbf{k}}]=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ f(x) & g(y) & h(z)\end{vmatrix}=\mathbf{0}$, the line integral is independent of path.

12. No. It may be independent of path. All we know is that Theorem 14.4 fails to imply independence of path.

13. Since $\nabla(ze^{xy}-z)=yze^{xy}\hat{\mathbf{i}}+xze^{xy}\hat{\mathbf{j}}+(e^{xy}-1)\hat{\mathbf{k}}$, the line integral is independent of path in space, and

$$\int_C yze^{xy}\,dx+xze^{xy}\,dy+(e^{xy}-1)\,dz=\left\{ze^{xy}-z\right\}\Bigg|_{(1,1,1)}^{(2,4,8)}=(8e^8-8)-(e-1)=8e^8-e-7.$$

14. Since $\nabla(xy\tan x+z)=y(\tan x+x\sec^2 x)\hat{\mathbf{i}}+x\tan x\,\hat{\mathbf{j}}+\hat{\mathbf{k}}$, the line integral is certainly independent of path in the domain $-1.1<x<1.1$ containing the curve $x^2+y^2=1$. Hence

$$\oint_C y(\tan x+x\sec^2 x)\,dx+x\tan x\,dy+dz=0.$$

15. Since $\nabla\left(\frac{1+y^2}{-2x^2}-\frac{y^2}{2}+\frac{z^2}{2}\right)=\left(\frac{1+y^2}{x^3}\right)\hat{\mathbf{i}}+\left(-\frac{y}{x^2}-y\right)\hat{\mathbf{j}}+z\hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the yz-plane. Since C does not pass through the yz-plane,

$$\int_C\left(\frac{1+y^2}{x^3}\right)dx-\left(\frac{y+x^2y}{x^2}\right)dy+z\,dz=\left\{\frac{1+y^2}{-2x^2}-\frac{y^2}{2}+\frac{z^2}{2}\right\}\Bigg|_{(1,0,0)}^{(5,2,1)}=-\frac{11}{10}.$$

16. Since $\nabla(xz/y)=(z/y)\hat{\mathbf{i}}-(xz/y^2)\hat{\mathbf{j}}+(x/y)\hat{\mathbf{k}}$, the line integral is independent of path in any domain which not containing points in the xz-plane. Hence

$$\oint_C\frac{zy\,dx-xz\,dy+xy\,dz}{y^2}=0.$$

17. Since $\nabla\left(\frac{1}{x}\mathrm{Tan}^{-1}y\right)=\left(-\frac{1}{x^2}\mathrm{Tan}^{-1}y\right)\hat{\mathbf{i}}+\frac{1}{x(1+y^2)}\hat{\mathbf{j}}$, the line integral is independent of path in any domain not containing points on the y-axis. Since C does not pass through this axis,

$$\int_C -\frac{1}{x^2}\mathrm{Tan}^{-1}y\,dx+\frac{1}{x+xy^2}dy=\left\{\frac{1}{x}\mathrm{Tan}^{-1}y\right\}\Bigg|_{(2,-1)}^{(10,3)}=\frac{1}{10}\mathrm{Tan}^{-1}3+\frac{\pi}{8}.$$

18. Since $\nabla\left(\dfrac{-1}{(x-3)(y+5)}+\ln|z+4|\right)=\dfrac{1}{(x-3)^2(y+5)}\hat{\mathbf{i}}+\dfrac{1}{(x-3)(y+5)^2}\hat{\mathbf{j}}+\dfrac{1}{z+4}\hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the planes $x=3$, $y=-5$, and $z=-4$. Thus,

$$\int_C \frac{1}{(x-3)^2(y+5)}dx+\frac{1}{(x-3)(y+5)^2}dy+\frac{1}{z+4}dz=\left\{\frac{-1}{(x-3)(y+5)}+\ln|z+4|\right\}_{0,0,0)}^{(2,2,2)}$$
$$=\frac{8}{105}+\ln(3/2).$$

19. (a) With parametric equations $x=\cos t$, $y=\sin t$, $-\pi\le t\le\pi$,

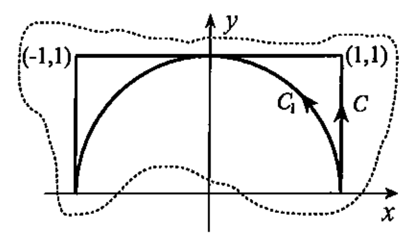
$$\oint_C\frac{-y\,dx+x\,dy}{x^2+y^2}=\int_{-\pi}^{\pi}-\sin t(-\sin t\,dt)+\cos t(\cos t\,dt)=\int_{-\pi}^{\pi}dt=2\pi.$$

(b) Since $\nabla\times\left(\dfrac{-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}}{x^2+y^2}\right)=\mathbf{0}$ at every point except $(0,0)$, the line integral is independent of path in a domain that contains the circle (the circle does not contain $(0,0)$). The value of the line integral is therefore zero.

20. Since

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

in the simply-connected domain shown, the line integral is independent of path therein. We may therefore replace C with the semicircle C': $x=\cos t$, $y=\sin t$, $0\le t\le\pi$. Hence,



$$\int_C\frac{y}{x^2+y^2}dx-\frac{x}{x^2+y^2}dy=\int_{C'}\frac{y}{x^2+y^2}dx-\frac{x}{x^2+y^2}dy$$
$$=\int_0^{\pi}\frac{\sin t(-\sin t\,dt)-\cos t(\cos t\,dt)}{1}=-\int_0^{\pi}dt=-\pi.$$

21. Since $\nabla(\sqrt{x^2+y^2})=(x\hat{\mathbf{i}}+y\hat{\mathbf{j}})/\sqrt{x^2+y^2}$, the line integral is independent of path in any domain not containing the origin. A similar argument holds for the 3-space line integral.

22. A quick calculation shows that each of the partial derivatives $\dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right)$ and $\dfrac{\partial}{\partial x}\left(\dfrac{-x}{x^2+y^2}\right)$ is equal to $\dfrac{x^2-y^2}{(x^2+y^2)^2}$. These derivatives are equal in each of the domains specified. Since domains $x>0$, $x<0$, $y>0$, and $y<0$ are simply-connected, the line integral is independent of path therein. The domain $x^2+y^2>0$ is not simply-connected. The line integral around C: $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$ is

$$\oint_C\frac{y\,dx-x\,dy}{x^2+y^2}=\int_0^{2\pi}\sin t(-\sin t\,dt)-\cos t(\cos t\,dt)=-2\pi.$$

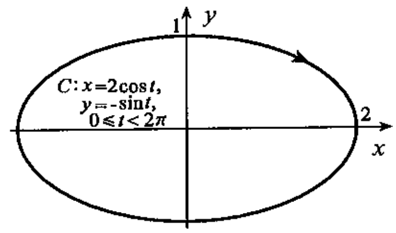
By Corollary 2 to Theorem 14.3, the line integral is not independent of path in $x^2+y^2>0$.

23. (a) With $dU=f'(T)\,dT$ and $P=nRT/V$, $\quad I=\displaystyle\int T^{-1}(dU+P\,dV)=\int\frac{f'(T)}{T}dT+\frac{nR}{V}dV.$

(b) When $f'(T)=k$, a constant, then in the TV-plane, $\quad\nabla(k\ln T+nR\ln V)=\dfrac{k}{T}\hat{\mathbf{T}}+\dfrac{nR}{V}\hat{\mathbf{V}}$, except when $T=0$ or $V=0$. Thus, $S=k\ln T+nR\ln V+S_0$, where S_0 is a constant.

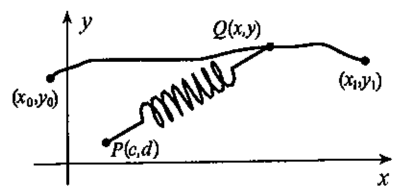
24. Since $\nabla(e^{x^2y}) = (2xye^{x^2y})\hat{\mathbf{i}} + x^2e^{x^2y}\hat{\mathbf{j}}$, the line integral $\int 2xye^{x^2y}\,dx + x^2e^{x^2y}dy$ is independent of path in space, and its value around the given curve is zero. The given line integral therefore reduces to

$$\begin{aligned}\oint_C x^2y\,dx &= \int_0^{2\pi} 4\cos^2 t(-\sin t)(-2\sin t\,dt)\\ &= 8\int_0^{2\pi}\left(\frac{\sin 2t}{2}\right)^2 dt\\ &= 2\int_0^{2\pi}\left(\frac{1-\cos 4t}{2}\right)dt = \left\{t - \frac{\sin 4t}{4}\right\}_0^{2\pi} = 2\pi.\end{aligned}$$



25. Suppose P has coordinates (c,d) and the unstretched length of the spring is L. At $Q(x,y)$, the force to counteract the spring is

$$\begin{aligned}\mathbf{F} &= k[\sqrt{(x-c)^2+(y-d)^2} - L]\widehat{\mathbf{PQ}}\\ &= k[\sqrt{(x-c)^2+(y-d)^2} - L]\left[\frac{(x-c)\hat{\mathbf{i}} + (y-d)\hat{\mathbf{j}}}{\sqrt{(x-c)^2+(y-d)^2}}\right].\end{aligned}$$



The work done by this force along C is

$$\begin{aligned}W &= \int_C \mathbf{F}\cdot d\mathbf{r} = k\int_C [\sqrt{(x-c)^2+(y-d)^2} - L]\frac{(x-c)\,dx + (y-d)\,dy}{\sqrt{(x-c)^2+(y-d)^2}}\\ &= k\int_C \left[1 - \frac{L}{\sqrt{(x-c)^2+(y-d)^2}}\right][(x-c)\,dx + (y-d)\,dy].\end{aligned}$$

Since

$$\begin{aligned}\nabla\left[\frac{1}{2}(x-c)^2 + \frac{1}{2}(y-d)^2 - L\sqrt{(x-c)^2+(y-d)^2}\right] &= \left[1 - \frac{L}{\sqrt{(x-c)^2+(y-d)^2}}\right](x-c)\hat{\mathbf{i}}\\ &\quad + \left[1 - \frac{L}{\sqrt{(x-c)^2+(y-d)^2}}\right](y-d)\hat{\mathbf{j}},\end{aligned}$$

the line integral is independent of path, and its value is

$$\begin{aligned}W &= k\left\{\frac{1}{2}(x-c)^2 + \frac{1}{2}(y-d)^2 - L\sqrt{(x-c)^2+(y-d)^2}\right\}_{(x_0,y_0)}^{(x_1,y_1)}\\ &= \frac{k}{2}\Big[(x_1-c)^2 + (y_1-d)^2 - 2L\sqrt{(x_1-c)^2+(y_1-d)^2} - (x_0-c)^2\\ &\qquad - (y_0-d)^2 + 2L\sqrt{(x_0-c)^2+(y_0-d)^2}\Big]\\ &= \frac{k}{2}\left[(x_1-c)^2 + (y_1-d)^2 - 2L\sqrt{(x_1-c)^2+(y_1-d)^2} + L^2\right]\\ &\qquad - \frac{k}{2}\left[(x_0-c)^2 + (y_0-d)^2 - 2L\sqrt{(x_0-c)^2+(y_0-d)^2} + L^2\right]\\ &= \frac{k}{2}[\sqrt{(x_1-c)^2+(y_1-d)^2} - L]^2 - \frac{k}{2}[(x_0-c)^2 + (y_0-d)^2 - L]^2\\ &= \frac{k}{2}(b^2 - a^2).\end{aligned}$$

26. (a) The curl of $\mathbf{F}$ is $\nabla \times \mathbf{F} = k \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{(x^2+y^2+z^2)^{3/2}} & \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$.

The x-component is k times $\dfrac{\partial}{\partial y}\left[\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right] - \dfrac{\partial}{\partial z}\left[\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right] = 0$. Similarly, the y- and z-components vanish. Hence, in any simply-connected domain that does not contain the origin, the line integral representing work done by $\mathbf{F}$ is independent of path.

(b) Since any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ can be enclosed in a simply-connected domain not containing the origin, the line integral $\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$ is independent of path, and its value is $\phi(P_2) - \phi(P_1)$ where $\phi(x, y, z)$ is any function satisfying $\nabla\phi = \mathbf{F}$. Since

$$\nabla\left(\frac{k}{|\mathbf{r}|}\right) = \frac{-k}{|\mathbf{r}|^2}\nabla|\mathbf{r}| = \frac{-k}{|\mathbf{r}|^2}\nabla\left(\sqrt{x^2+y^2+z^2}\right) = \frac{-k}{|\mathbf{r}|^2}\frac{1}{2\sqrt{x^2+y^2+z^2}}(2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}) = \frac{-k\hat{\mathbf{r}}}{|\mathbf{r}|^2},$$

it follows that $\displaystyle\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left\{\frac{-k}{|\mathbf{r}|}\right\}_{P_1}^{P_2} = \frac{k}{d_1} - \frac{k}{d_2}$ where d_1 and d_2 are distances from P_1 and P_2 to the origin.

EXERCISES 14.5

1. Since $\nabla\left(\dfrac{-q_1 q_2}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}}\right) = \dfrac{q_1 q_2}{4\pi\epsilon_0(x^2+y^2+z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, the force field is conservative in any domain not containing the origin. It is the electrostatic force between charges q_1 and q_2. A potential energy function is $V = \dfrac{q_1 q_2}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}}$.

2. Since $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ mx & xy & 0 \end{vmatrix} = y\hat{\mathbf{k}}$, $\mathbf{F}$ is not conservative.

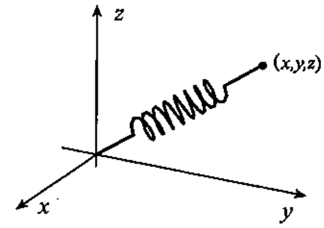
3. Since $\nabla(-kx^2/2) = -kx\hat{\mathbf{i}}$, the force field is conservative. It is that due to a spring with potential energy function $kx^2/2$.

4. Since $\nabla(-mgz) = \mathbf{F}$, $\mathbf{F}$ is conservative with potential energy function $U(z) = mgz$. $\mathbf{F}$ is the force of gravity on a mass m.

5. $\nabla\left(\dfrac{-GMm}{\sqrt{x^2+y^2+z^2}}\right) = \dfrac{GMm}{(x^2+y^2+z^2)^{3/2}}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, the force field is conservative in any domain not containing the origin. It is the gravitational force between masses M and m. A potential energy function is $V = \dfrac{GMm}{\sqrt{x^2+y^2+z^2}}$.

6. A normal vector to the equipotential surface $U(x, y, z) = C$ at any point $P(x, y, z)$ on the surface is $\nabla(U - C) = \nabla U$. But $\mathbf{F} = -\nabla U$, and therefore $\mathbf{F}$ is normal to the surface at P.

7. For Exercises 1 and 5, they are spheres centred at the origin. In Exercise 4 they are planes parallel to the xy-plane.



8. The magnitude of the force is
$$|\mathbf{F}| = k\left(\sqrt{x^2+y^2+z^2} - L\right),$$
where k is the spring constant. Since $\mathbf{F}$ is directed toward the origin, it follows that
$$\mathbf{F} = -k\left(\sqrt{x^2+y^2+z^2} - L\right)\frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{\sqrt{x^2+y^2+z^2}}.$$
Since $\nabla\left[-\frac{k}{2}\left(\sqrt{x^2+y^2+z^2} - L\right)^2\right] = \mathbf{F}$ in the domain $x^2+y^2+z^2 > 0$, it follows that the force is conservative.

9. Friction is not conservative because work done by (or against) friction depends on the path followed.

10. (a) Suppose we take the school at the origin. When a student is at position (x, y, z), the magnitude of the force is $|\mathbf{F}| = \dfrac{d}{x^2+y^2+z^2}$, where d a constant. Since the force is toward the origin,
$$\mathbf{F} = \frac{-d}{x^2+y^2+z^2}\frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{\sqrt{x^2+y^2+z^2}} = \frac{-d(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{(x^2+y^2+z^2)^{3/2}}.$$
In the domain $x^2+y^2+z^2 > 10\,000$, $\nabla\left(\dfrac{d}{\sqrt{x^2+y^2+z^2}}\right) = \mathbf{F}$, and therefore $\mathbf{F}$ is conservative.

(b) Suppose the donut shop is at position (a, b, c). Then the force is
$$\mathbf{F} = \frac{-d}{[(x-a)^2+(y-b)^2+(z-c)^2]^{3/2}}[(x-a)\hat{\mathbf{i}} + (y-b)\hat{\mathbf{j}} + (z-c)\hat{\mathbf{k}}].$$
Since $\nabla\left(\dfrac{d}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\right) = \mathbf{F}$, the force is still conservative.

11. (a) When a mass moves under the influence of a conservative force field, the sum of its potential energy $U(x)$ and its kinetic energy $K(x)$ is constant, $U + K = E$. Since $K = mv^2/2 = (1/2)m(dx/dt)^2$,
$$U + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = E \quad\Longrightarrow\quad \frac{dx}{dt} = \sqrt{\frac{2}{m}(E-U)} \quad\Longrightarrow\quad \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}dt.$$
This is a separated differential equation with solutions defined implicitly by
$$\int \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}t + C.$$
In order to incorporate the initial condition $x(0) = x_0$, we rewrite the indefinite integral as a definite integral with a variable upper limit,
$$\int_0^x \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}t + C.$$
The initial condition now implies that
$$\int_0^{x_0} \frac{1}{\sqrt{E-U(x)}}dx = C.$$
Thus,
$$\int_0^x \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}t + \int_0^{x_0} \frac{1}{\sqrt{E-U(x)}}dx \quad\Longrightarrow\quad \int_{x_0}^x \frac{1}{\sqrt{E-U(x)}}dx = \sqrt{\frac{2}{m}}t.$$
(b) When $U(x) = kx^2/2$,

$$\sqrt{\frac{2}{m}}t = \int_{x_0}^{x} \frac{1}{\sqrt{E - kx^2/2}}dx = \sqrt{2}\int_{x_0}^{x} \frac{1}{\sqrt{2E - kx^2}}dx.$$

If we set $x = \sqrt{2E/k}\sin\theta$, and $dx = \sqrt{2E/k}\cos\theta\, d\theta$, then

$$\sqrt{2}\int \frac{1}{\sqrt{2E - kx^2}}dx = \sqrt{2}\int \frac{\sqrt{2E/k}\cos\theta}{\sqrt{2E}\cos\theta}d\theta = \sqrt{\frac{2}{k}}\theta + C = \sqrt{\frac{2}{k}}\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x\right) + C.$$

Thus,

$$\sqrt{\frac{2}{m}}t = \left\{\sqrt{\frac{2}{k}}\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x\right)\right\}_{x_0}^{x} = \sqrt{\frac{2}{k}}\left[\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x\right) - \mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x_0\right)\right].$$

This implies that

$$\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x\right) = \mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x_0\right) + \sqrt{\frac{k}{m}}t,$$

and if we take sines of both sides, we obtain

$$\begin{aligned}
x &= \sqrt{\frac{2E}{k}}\sin\left[\mathrm{Sin}^{-1}\left(\sqrt{\frac{k}{2E}}x_0\right) + \sqrt{\frac{k}{m}}t\right] \\
&= \sqrt{\frac{2E}{k}}\left[\sqrt{\frac{k}{2E}}x_0\cos\sqrt{\frac{k}{m}}t + \sqrt{1 - \frac{kx_0^2}{2E}}\sin\sqrt{\frac{k}{m}}t\right] \\
&= x_0\cos\sqrt{\frac{k}{m}}t + \sqrt{\frac{2E}{k} - x_0^2}\sin\sqrt{\frac{k}{m}}t.
\end{aligned}$$

This describes simple harmonic motion for the mass, as we would expect. When $v(0) = 0$, total energy is $E = kx_0^2/k$, and $x = x_0\cos\sqrt{\frac{k}{m}}t$.

12. (a) The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(r)x & f(r)y & f(r)z \end{vmatrix} = \left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right)\hat{\mathbf{i}} + \left(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right)\hat{\mathbf{j}} + \left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)\hat{\mathbf{k}}.$$

The x-component is

$$z\left(\frac{df}{dr}\frac{\partial r}{\partial y}\right) - y\left(\frac{df}{dr}\frac{\partial r}{\partial z}\right) = \frac{df}{dr}\left(\frac{zy}{\sqrt{x^2 + y^2 + z^2}} - \frac{yz}{\sqrt{x^2 + y^2 + z^2}}\right) = 0.$$

Similarly, the y- and z-components of $\nabla \times \mathbf{F}$ vanish, and $\nabla \times \mathbf{F} = \mathbf{0}$. According to Theorem 14.4, $\mathbf{F}$ is conservative in any simply-connected domain that does not contain the origin.

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f(r)\mathbf{r}\cdot d\mathbf{r} = \int_C f(r)(x\,dx + y\,dy + z\,dz)$

Since $r = \sqrt{x^2 + y^2 + z^2}$, it follows that

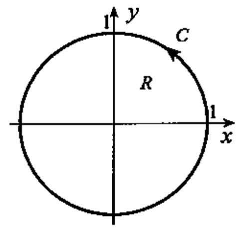
$$dr = \frac{x\,dx}{\sqrt{x^2 + y^2 + z^2}} + \frac{y\,dy}{\sqrt{x^2 + y^2 + z^2}} + \frac{z\,dz}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\,dx + y\,dy + z\,dz}{r},$$

and therefore $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b f(r)r\,dr.$

(c) The electrostatic force in Example 14.14 and the gravitational force in Exercise 5 are radially symmetric.

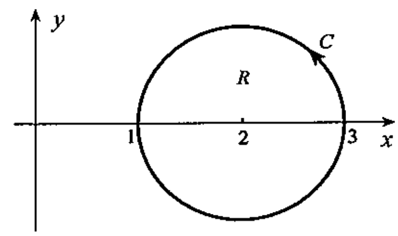
EXERCISES 14.6

1. With Green's theorem, $\oint_C y^2\,dx + x^2\,dy = \iint_R (2x - 2y)\,dA = 0$ because x and y are odd functions, and the circle is symmetric about the x- and y-axes.



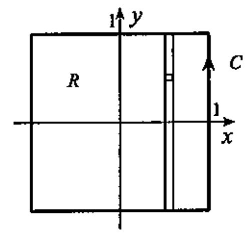
2. With Green's theorem,

$$\begin{aligned}\oint_C (x^2 + 2y^2)\,dy &= \iint_R 2x\,dA \\ &= 2(\text{First moment of } R \text{ about } y\text{-axis}) \\ &= 2(\text{Area of R})(\overline{x}) = 2(\pi)(2) = 4\pi.\end{aligned}$$



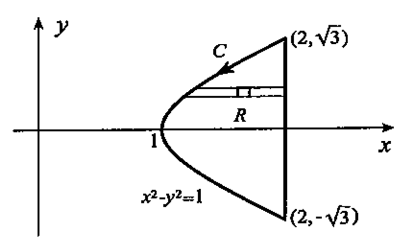
3. With Green's theorem,

$$\begin{aligned}\oint_C x^2e^y\,dx + (x+y)\,dy &= -\iint_R (1 - x^2e^y)\,dA \\ &= -\int_{-1}^{1}\int_{-1}^{1}(1 - x^2e^y)\,dy\,dx = -\int_{-1}^{1}\left\{y - x^2e^y\right\}_{-1}^{1}dx \\ &= -\int_{-1}^{1}[2 - (e - e^{-1})x^2]\,dx \\ &= -\left\{2x - (e - e^{-1})\frac{x^3}{3}\right\}_{-1}^{1} = -4 + \frac{2}{3}(e - e^{-1}).\end{aligned}$$



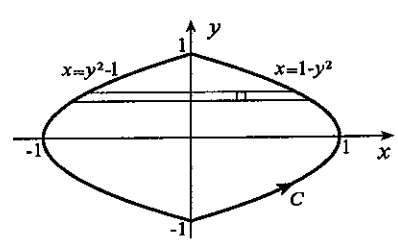
4. By Green's theorem,

$$\begin{aligned}\oint_C xy^3\,dx + x^2\,dy &= \iint_R (2x - 3xy^2)\,dA = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{\sqrt{1+y^2}}^{2}(2x - 3xy^2)\,dx\,dy \\ &= \int_{-\sqrt{3}}^{\sqrt{3}}\left\{x^2 - \frac{3x^2y^2}{2}\right\}_{\sqrt{1+y^2}}^{2}dy \\ &= \int_{-\sqrt{3}}^{\sqrt{3}}\left(3 - \frac{11y^2}{2} + \frac{3y^4}{2}\right)dy \\ &= \left\{3y - \frac{11y^3}{6} + \frac{3y^5}{10}\right\}_{-\sqrt{3}}^{\sqrt{3}} = \frac{2\sqrt{3}}{5}.\end{aligned}$$

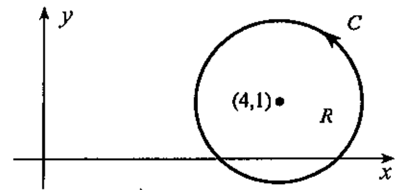


5. By Green's theorem,

$$\begin{aligned}\oint_C (x^3 + y^3)\,dx + (x^3 - y^3)\,dy &= \iint_R (3x^2 - 3y^2)\,dA \\ &= 6\int_0^1\int_{y^2-1}^{1-y^2}(x^2 - y^2)\,dx\,dy \\ &= 6\int_0^1\left\{\frac{x^3}{3} - xy^2\right\}_{y^2-1}^{1-y^2}dy \\ &= 2\int_0^1[(1-y^2)^3 - 3y^2(1-y^2) - (y^2-1)^3 + 3y^2(y^2-1)]\,dy \\ &= 2\int_0^1\{(2 - 12y^2 + 12y^4 - 2y^6)\,dy = 2\left\{2y - 4y^3 + \frac{12y^5}{5} - \frac{2y^7}{7}\right\}_0^1 = \frac{8}{35}\end{aligned}$$

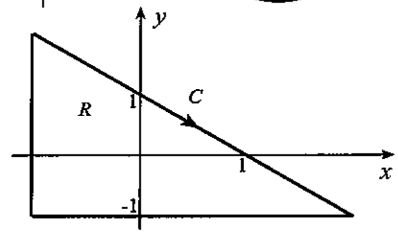


6. By Green's theorem,



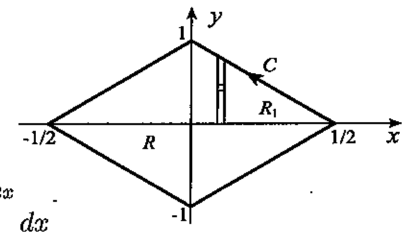
$$\oint_C 2\operatorname{Tan}^{-1}\left(\frac{y}{x}\right) dx + \ln(x^2+y^2)\, dy$$
$$= \iint_R \left[\frac{2x}{x^2+y^2} - \frac{2}{1+y^2/x^2}\left(\frac{1}{x}\right)\right] dA$$
$$= 0.$$

7. By Green's theorem,



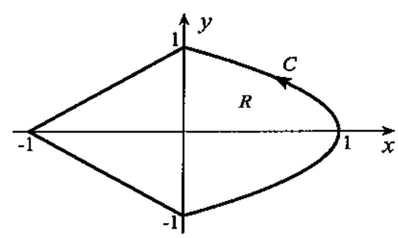
$$\oint_C (3x^2y^3+y)\,dx + (3x^3y^2+2x)\,dy$$
$$= -\iint_R (9x^2y^2+2-9x^2y^2-1)\,dA$$
$$= -\iint_R dA = -\frac{1}{2}(3)(3) = -\frac{9}{2}$$

8. By Green's theorem,



$$\oint_C (x^3+y^3)\,dx + (x^3-y^3)\,dy = \iint_R (3x^2-3y^2)\,dA$$
$$= 12\iint_{R_1}(x^2-y^2)\,dA \quad (R_1 = \text{first quadrant part of } R)$$
$$= 12\int_0^{1/2}\int_0^{1-2x}(x^2-y^2)\,dy\,dx = 12\int_0^{1/2}\left\{x^2y-\frac{y^3}{3}\right\}_0^{1-2x} dx$$
$$= 4\int_0^{1/2}[3x^2-6x^3-(1-2x)^3]\,dx = 4\left\{x^3-\frac{3x^4}{2}+\frac{(1-2x)^4}{8}\right\}_0^{1/2} = -\frac{3}{8}.$$

9. By Green's theorem,

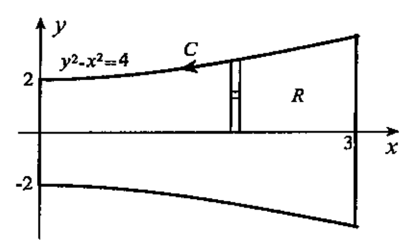


$$\oint_C (x^2y^2+3x)\,dx + (2xy-y)\,dy = \iint_R (2y-2x^2y)\,dA$$

The integral has value zero because $2y-2x^2y$ is an odd function of y and R is symmetric about the x-axis.

10. By Green's theorem,

$$\oint_C (xy^2+2x)\,dx + (x^2y+y+x^2)\,dy$$
$$= \iint_R (2xy+2x-2xy)\,dA$$
$$= 2(2)\int_0^3\int_0^{\sqrt{4+x^2}} x\,dy\,dx = 4\int_0^3\left\{xy\right\}_0^{\sqrt{4+x^2}} dx$$
$$= 4\int_0^3 x\sqrt{4+x^2}\,dx = 4\left\{\frac{(4+x^2)^{3/2}}{3}\right\}_0^3 = \frac{4}{3}(13\sqrt{13}-8).$$



11. Green's theorem cannot be used since $x/(x^2+y^2)$ and $y/(x^2+y^2)$ are not continuous at $(0,0)$. With the parametric equations, $x=\cos t$, $y=\sin t$, $-\pi \le t \le \pi$,

$$\oint_C \frac{-y\,dx+x\,dy}{x^2+y^2} = \int_{-\pi}^{\pi} -\sin t(-\sin t\,dt)+\cos t(\cos t\,dt) = \int_{-\pi}^{\pi} dt = 2\pi.$$

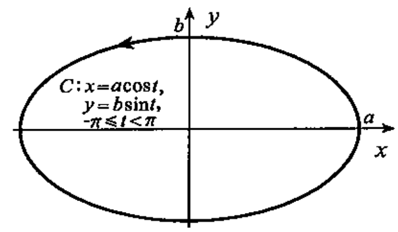
12. Since $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\hat{\mathbf{k}},$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dA = \iint_R (\nabla\times\mathbf{F})\cdot\hat{\mathbf{k}}\,dA.$$

13. If we set $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ and $\hat{\mathbf{n}} = (dy, -dx)/ds$, then

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C (P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}) \cdot \frac{(dy, -dx)}{ds} ds = \oint_C -Q\, dx + P\, dy = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_R \nabla \cdot \mathbf{F}\, dA.$$

14.
$$\begin{aligned} A &= \frac{1}{2}\oint_C x\, dy - y\, dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [a\cos t(b\cos t\, dt) - b\sin t(-a\sin t\, dt)] \\ &= \frac{ab}{2}\int_{-\pi}^{\pi} dt = \pi ab \end{aligned}$$

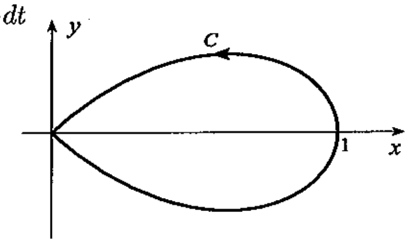


15. Since $\dfrac{dx}{dt} = \dfrac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} = \dfrac{-4t}{(1+t^2)^2}$, the area enclosed by the strophoid is

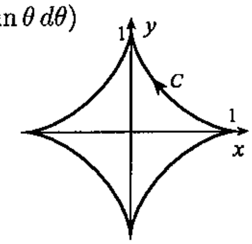
$$A = -\oint_C y\, dx = -\int_{-1}^{1} \left[\frac{t(1-t^2)}{1+t^2}\right]\left[\frac{-4t}{(1+t^2)^2}\right] dt = 4\int_{-1}^{1} \frac{t^2(1-t^2)}{(1+t^2)^3} dt$$

If we set $t = \tan\theta$ and $dt = \sec^2\theta\, d\theta$,

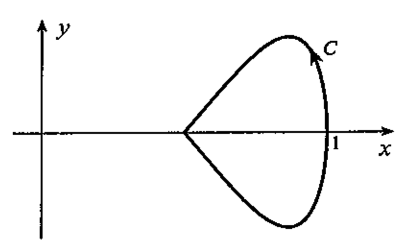
$$\begin{aligned} A &= 4\int_{-\pi/4}^{\pi/4} \frac{\tan^2\theta(1-\tan^2\theta)}{\sec^6\theta}(\sec^2\theta\, d\theta) \\ &= 4\int_{-\pi/4}^{\pi/4} (\sin^2\theta\cos^2\theta - \sin^4\theta)\, d\theta \\ &= 4\int_{-\pi/4}^{\pi/4} \sin^2\theta(\cos^2\theta - \sin^2\theta)\, d\theta = 4\int_{-\pi/4}^{\pi/4} \left(\frac{1-\cos 2\theta}{2}\right)\cos 2\theta\, d\theta \\ &= 2\int_{-\pi/4}^{\pi/4} \left(\cos 2\theta - \frac{1+\cos 4\theta}{2}\right) d\theta = 2\left\{\frac{1}{2}\sin 2\theta - \frac{\theta}{2} - \frac{1}{8}\sin 4\theta\right\}_{-\pi/4}^{\pi/4} = \frac{4-\pi}{2}. \end{aligned}$$



16.
$$\begin{aligned} A &= \frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2}\int_0^{2\pi} \cos^3\theta(3\sin^2\theta\ \cos\theta\, d\theta) - \sin^3\theta(-3\cos^2\theta\ \sin\theta\, d\theta) \\ &= \frac{3}{2}\int_0^{2\pi} (\cos^4\theta\ \sin^2\theta + \sin^4\theta\ \cos^2\theta)\, d\theta \\ &= \frac{3}{2}\int_0^{2\pi} \cos^2\theta\ \sin^2\theta\, d\theta = \frac{3}{2}\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta\, d\theta \\ &= \frac{3}{8}\int_0^{2\pi} \left(\frac{1-\cos 4\theta}{2}\right) d\theta = \frac{3}{16}\left\{\theta - \frac{\sin 4\theta}{4}\right\}_0^{2\pi} = \frac{3\pi}{8} \end{aligned}$$

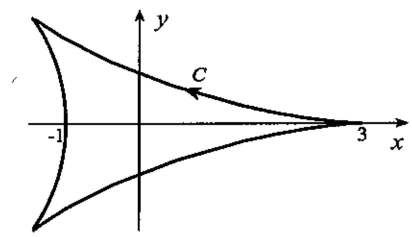


17.
$$\begin{aligned} A &= \oint_C x\, dy = \int_{-\pi/3}^{\pi/3} \cos\theta(3\cos 3\theta\, d\theta) \\ &= 3\int_{-\pi/3}^{\pi/3} \frac{1}{2}(\cos 4\theta + \cos 2\theta)\, d\theta \\ &= \frac{3}{2}\left\{\frac{1}{4}\sin 4\theta + \frac{1}{2}\sin 2\theta\right\}_{-\pi/3}^{\pi/3} = \frac{3\sqrt{3}}{8} \end{aligned}$$



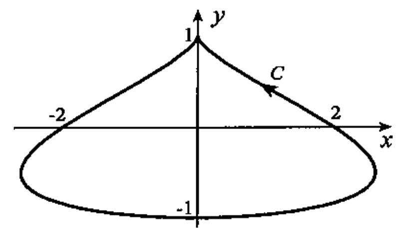
18. $A = \dfrac{1}{2}\oint_C x\,dy - y\,dx$

$$= \frac{1}{2}\int_0^{2\pi} (2\cos t + \cos 2t)(2\cos t - 2\cos 2t)\,dt$$
$$-(2\sin t - \sin 2t)(-2\sin t - 2\sin 2t)\,dt$$
$$= \int_0^{2\pi} (2\cos^2 t - \cos t\,\cos 2t - \cos^2 2t + 2\sin^2 t$$
$$+ \sin t\,\sin 2t - \sin^2 2t)\,dt$$
$$= \int_0^{2\pi} (1 - \cos 3t)\,dt = \left\{t - \frac{\sin 3t}{3}\right\}_0^{2\pi} = 2\pi$$



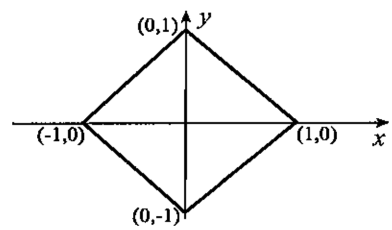
19. $A = \oint_C x\,dy = \int_{-\pi}^{\pi} (2\cos t - \sin 2t)(\cos t\,dt)$

$$= \int_{-\pi}^{\pi} (1 + \cos 2t - 2\sin t\cos^2 t)\,dt$$
$$= \left\{t + \frac{1}{2}\sin 2t + \frac{2}{3}\cos^3 t\right\}_{-\pi}^{\pi} = 2\pi$$



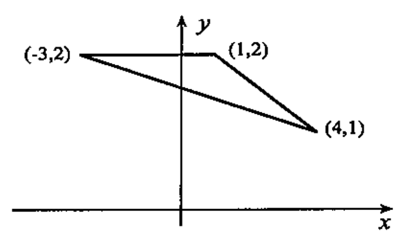
20. $A = \dfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$= \frac{1}{2}[(1+1) - (-1-1)] = 2$$



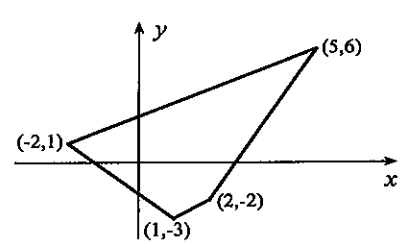
21. $A = \dfrac{1}{2}\begin{bmatrix} 1 & 2 \\ -3 & 2 \\ 4 & 1 \\ 1 & 2 \end{bmatrix}$

$$= \frac{1}{2}[(2 - 3 + 8) - (-6 + 8 + 1)] = 2$$



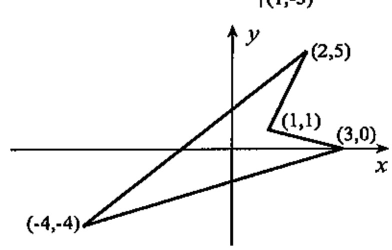
22. $A = \dfrac{1}{2}\begin{bmatrix} 2 & -2 \\ 5 & 6 \\ -2 & 1 \\ 1 & -3 \\ 2 & -2 \end{bmatrix}$

$$= \frac{1}{2}[(12 + 5 + 6 - 2) - (-10 - 12 + 1 - 6)] = 24$$

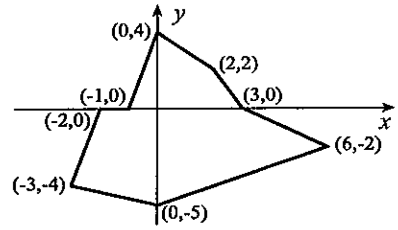


23. $A = \dfrac{1}{2}\begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 2 & 5 \\ -4 & -4 \\ 3 & 0 \end{bmatrix}$

$$= \frac{1}{2}[(3 + 5 - 8) - (2 - 20 - 12)] = 15$$

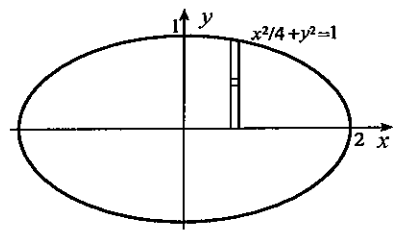


24. $A = \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -1 & 0 \\ -2 & 0 \\ -3 & -4 \\ 0 & -5 \\ 6 & -2 \\ 3 & 0 \\ 2 & 2 \\ 0 & 4 \end{bmatrix} = \frac{1}{2}[(8+15+6+8)-(-4-30-6)] = 77/2$



25. By Green's theorem,

$$\oint_C (2xye^{x^2y} + 3x^2y)\,dx + x^2e^{x^2y}\,dy$$
$$= \iint_R (2xe^{x^2y} + 2x^3ye^{x^2y} - 2xe^{x^2y} - 2x^3ye^{x^2y} - 3x^2)\,dA$$
$$= -12\int_0^2\int_0^{(1/2)\sqrt{4-x^2}} x^2\,dy\,dx = -12\int_0^2 \left\{x^2y\right\}_0^{(1/2)\sqrt{4-x^2}} dx$$
$$= -6\int_0^2 x^2\sqrt{4-x^2}\,dx$$

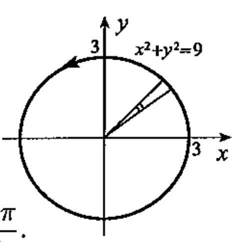


If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

$$\oint_C (2xye^{x^2y} + 3x^2y)\,dx + x^2e^{x^2y}\,dy = -6\int_0^{\pi/2} 4\sin^2\theta(2\cos\theta)(2\cos\theta\,d\theta) = -96\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$
$$= -24\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = -12\left\{\theta - \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = -6\pi.$$

26. By Green's theorem,

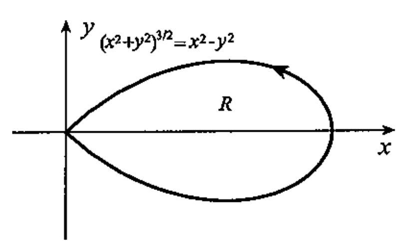
$$\oint_C (3x^2y^3 - x^2y)\,dx + (xy^2 + 3x^3y^2)\,dy$$
$$= \iint_R (y^2 + 9x^2y^2 - 9x^2y^2 + x^2)\,dA = \iint_R (x^2+y^2)\,dA$$
$$= 4\int_0^{\pi/2}\int_0^3 r^2\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\right\}_0^3 d\theta = 81\left\{\theta\right\}_0^{\pi/2} = \frac{81\pi}{2}.$$



27. By Green's theorem,

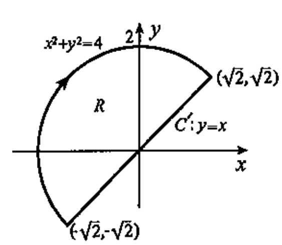
$$\oint_C -x^3y^2\,dx + x^2y^3\,dy = \iint_R (2xy^3 + 2x^3y)\,dA$$

Since the integrand is an odd function of y and R is symmetric about the x-axis, the integral has value zero.



28. Since $\int_{C'} (x-y)(dx+dy) = 0$, we may write

$$\int_C (x-y)(dx+dy) = \oint_{C+C'} (x-y)(dx+dy)$$
$$= -\iint_R (1+1)\,dA$$
$$= -2(\text{Area of } R) = -2(2\pi) = -4\pi.$$



29. If C' is the y-axis from $(0,1)$ to $(0,-1)$, then

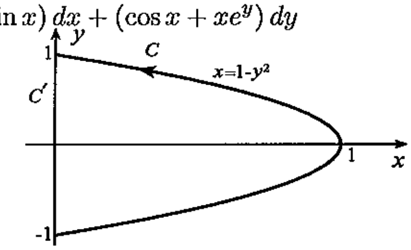
$$\int_C (e^y - y\sin x)\,dx + (\cos x + xe^y)\,dy$$

$$= \oint_{C'+C} (e^y - y\sin x)\,dx + (\cos x + xe^y)\,dy - \int_{C'} (e^y - y\sin x)\,dx + (\cos x + xe^y)\,dy$$

$$= \iint_R (-\sin x + e^y - e^y + \sin x)\,dA$$

$$+ \int_{-C'} (e^y - y\sin x)\,dx + (\cos x + xe^y)\,dy$$

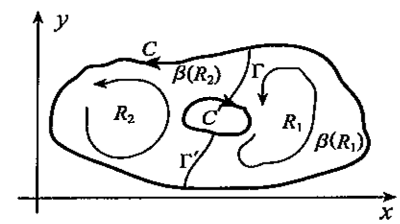
$$= \int_{-1}^{1} dy = 2.$$



30. (a) If we draw curves Γ and Γ' as shown, then P and Q have continuous first partial derivatives in a domain that contains R_1 and its boundary, and also in a domain that contains R_2 and its boundary. If we apply Green's theorem to these regions, denoting their boundaries by $\beta(R_1)$ and $\beta(R_2)$,

$$\oint_{\beta(R_1)} P\,dx + Q\,dy = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

$$\oint_{\beta(R_2)} P\,dx + Q\,dy = \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$



When these results are added,

$$\oint_{\beta(R_1)} P\,dx + Q\,dy + \oint_{\beta(R_2)} P\,dx + Q\,dy = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Now $R_1 + R_2 = R$. Furthermore, tracing $\beta(R_1)$ and $\beta(R_2)$ in the directions indicated is equivalent to tracing C and C' in the directions indicated, plus Γ and Γ' each traversed once in one direction and then in the reverse direction. Consequently,

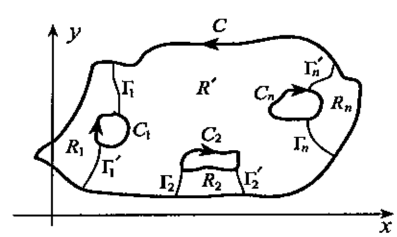
$$\oint_C P\,dx + Q\,dy + \oint_{C'} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

(b) In this case we draw curves Γ_i and Γ_i' from C_i to C as shown. This divides R into regions R_i and R' to which we apply Green's theorem,

$$\oint_{\beta(R_i)} P\,dx + Q\,dy = \iint_{R_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

for $i = 1, \ldots, n$, and

$$\oint_{\beta(R')} P\,dx + Q\,dy = \iint_{R'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$



Since $R_1 + R_2 + \cdots + R_n + R' = R$, and traversing the $\beta(R_i)$ and $\beta(R')$ is equivalent to traversing C and the C_i in the directions shown, when we add these equations we obtain

$$\oint_C P\,dx + Q\,dy + \sum_{i=1}^{n} \oint_{C_i} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

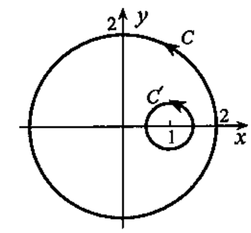
(c) If $\partial Q/\partial x = \partial P/\partial y$ in R, then in part (a), $\displaystyle\oint_C P\,dx + Q\,dy = \oint_{-C'} P\,dx + Q\,dy,$

and in part (b), $\displaystyle\oint_C P\,dx + Q\,dy = \sum_{i=1}^{n} \oint_{-C_i} P\,dx + Q\,dy.$

31. The circle C': $(x-1)^2+y^2=1/4$ is interior to C, and everywhere except at $(1,0)$,

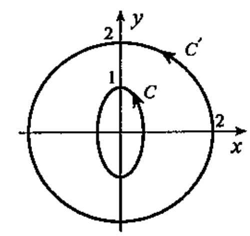
$$\frac{\partial}{\partial x}\left[\frac{-(x-1)}{(x-1)^2+y^2}\right]=\frac{\partial}{\partial y}\left[\frac{y}{(x-1)^2+y^2}\right].$$

Consequently, from Exercise 30(c), (and using the parametric equations $C':\ x=1+(1/2)\cos t,\ y=(1/2)\sin t,\ -\pi<t\le\pi$),



$$\begin{aligned}
\oint_C\frac{y\,dx-(x-1)\,dy}{(x-1)^2+y^2}&=\oint_{C'}\frac{y\,dx-(x-1)\,dy}{(x-1)^2+y^2}\\
&=\int_{-\pi}^{\pi}\frac{(1/2)\sin t(-(1/2)\sin t\,dt)-(1/2)\cos t(1/2)\cos t\,dt}{1/4}\\
&=\int_{-\pi}^{\pi}-dt=-2\pi.
\end{aligned}$$

32. The circle C': $x^2+y^2=4$ encloses C, and everywhere except at $(0,0)$,



$$\frac{\partial}{\partial x}\left[\frac{x^3}{(x^2+y^2)^2}\right]=\frac{\partial}{\partial y}\left[\frac{-x^2y}{(x^2+y^2)^2}\right].$$

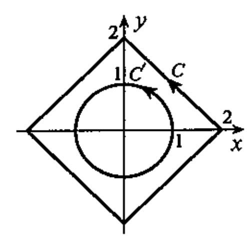
Hence, by Exercise 30(c), (and using the parametric equations $C':\ x=2\cos t,\ y=2\sin t,\ -\pi<t\le\pi$),

$$\begin{aligned}
\oint_C\frac{-x^2y\,dx+x^3\,dy}{(x^2+y^2)^2}&=\oint_{C'}\frac{-x^2y\,dx+x^3\,dy}{(x^2+y^2)^2}\\
&=\int_{-\pi}^{\pi}\frac{-4\cos^2 t\,2\sin t(-2\sin t\,dt)+(2\cos t)^3(2\cos t\,dt)}{16}\\
&=\int_{-\pi}^{\pi}(\cos^2 t\,\sin^2 t+\cos^4 t)\,dt=\int_{-\pi}^{\pi}\cos^2 t\,dt\\
&=\int_{-\pi}^{\pi}\left(\frac{1+\cos 2t}{2}\right)dt=\frac{1}{2}\left\{t+\frac{\sin 2t}{2}\right\}_{-\pi}^{\pi}=\pi.
\end{aligned}$$

33. The circle C': $x^2+y^2=1$ is interior to C, and everywhere except at $(0,0)$,

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right).$$

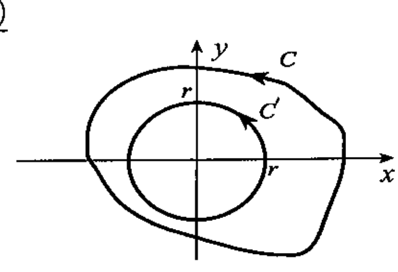
Hence, by Exercise 30(c), (and using the parametric equations $C':\ x=\cos t,\ y=\sin t,\ -\pi<t\le\pi$),



$$\begin{aligned}
\oint_C\frac{-y\,dx+x\,dy}{x^2+y^2}&=\oint_{C'}\frac{-y\,dx+x\,dy}{x^2+y^2}\\
&=\int_{-\pi}^{\pi}-\sin t(-\sin t\,dt)+\cos t(\cos t\,dt)=\int_{-\pi}^{\pi}dt=2\pi.
\end{aligned}$$

34. Suppose C is a curve enclosing the origin in the counterclockwise sense. It is always possible to find a circle C' of radius $r>0$ centred at the origin which is interior to C. Since $\partial Q/\partial x=\partial P/\partial y$ in a domain containing C and C' and the area between them, it follows by Exercise 30(c) that

$$\begin{aligned}
\oint_C\frac{-y\,dx+x\,dy}{x^2+y^2}&=\oint_{C'}\frac{-y\,dx+x\,dy}{x^2+y^2}\\
&=\int_{-\pi}^{\pi}\frac{(-r\sin t)(-r\sin t\,dt)+r\cos t(r\cos t\,dt)}{r^2}\\
&=\int_{-\pi}^{\pi}dt=2\pi,
\end{aligned}$$



where we have used the parametric equations $C':x=r\cos t,\ y=r\sin t,\ -\pi<t\le\pi$. If C encloses the origin in the opposite direction, then the value of the line integral is -2π.

35. (a) Since $\nabla[(1/2)\ln(x^2+y^2)] = (x\hat{\mathbf{i}}+y\hat{\mathbf{j}})/(x^2+y^2)$, the line integral is independent of path in any domain not containing $(0,0)$.

(b) Since the line integral is independent of path in the domain consisting of the xy-plane with $(0,0)$ removed, the value of the line integral is zero.

36. According to Exercise 13,

$$\oint_C \frac{\partial P}{\partial n}ds = \oint_C \nabla P\cdot\hat{\mathbf{n}}\,ds = \iint_R \nabla\cdot\nabla P\,dA = \iint_R\left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial y}\right)\right]dA = \iint_R \nabla^2 P\,dA.$$

If P satisfies Laplace's equation in R, then $\nabla^2 P = 0$, and $\displaystyle\oint_C \frac{\partial P}{\partial n}ds = 0$.

37. Using Green's theorem in the form of Exercise 13, we may write that

$$\oint_C (P\nabla Q)\cdot\hat{\mathbf{n}}\,ds = \iint_R \nabla\cdot(P\nabla Q)\,dA.$$

Using identity 14.11 on the right side gives

$$\oint_C P\frac{\partial Q}{\partial n}ds = \iint_R(\nabla P\cdot\nabla Q + P\nabla\cdot\nabla Q)\,dA = \iint_R \nabla P\cdot\nabla Q\,dA + \iint_R P\,\nabla^2 Q\,dA.$$

38. If we reverse the roles of P and Q in Exercise 37,

$$\oint_C \frac{Q\partial P}{\partial n}ds = \iint_R Q\nabla^2 P\,dA + \iint_R \nabla Q\cdot\nabla P\,dA.$$

When we subtract this result from that in Exercise 37, we obtain

$$\oint_C\left(P\frac{\partial Q}{\partial n} - Q\frac{\partial P}{\partial n}\right)ds = \iint_R (P\nabla^2 Q - Q\nabla^2 P)\,dA.$$

39. According to Exercise 19(a) in Section 14.4, if C is the unit circle $x^2+y^2=1$ directed counterclockwise, the value of the line integral is 2π. Notice that

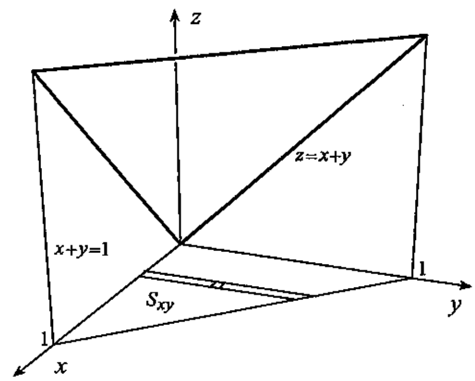
$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2},\qquad \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2},$$

and this is valid everywhere except at $(0,0)$. According to Exercise 30(c), the value of the line integral around every other curve (besides $x^2+y^2=1$) that encloses the origin in the same direction is also equal to 2π. For a curve encircling the origin n times, the value is $\pm 2\pi n$, the $\pm$ depending on the direction of the curve. If a curve does not encircle the origin, then Green's theorem can be involked to yield a value of zero. The only possible values are therefore $2\pi n$ (n an integer).

EXERCISES 14.7

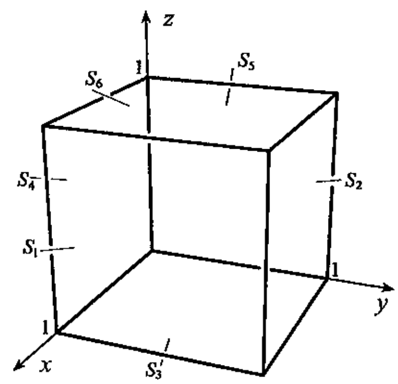
1.
$$\begin{aligned}\iint_S (x^2y+z)\,dS &= \iint_{S_{xy}}(x^2y+z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\
&= \iint_{S_{xy}}(x^2y+6-2x-3y)\sqrt{1+(-2)^2+(-3)^2}\,dA\\
&= \sqrt{14}\int_0^3\int_0^{2-2x/3}(x^2y+6-2x-3y)\,dy\,dx = \sqrt{14}\int_0^3\left\{\frac{x^2y^2}{2}+6y-2xy-\frac{3y^2}{2}\right\}_0^{2-2x/3}dx\\
&= \frac{2\sqrt{14}}{9}\int_0^3(27-18x+12x^2-6x^3+x^4)\,dx = \frac{2\sqrt{14}}{9}\left\{27x-9x^2+4x^3-\frac{3x^4}{2}+\frac{x^5}{5}\right\}_0^3 = \frac{39\sqrt{14}}{5}\end{aligned}$$

2. $\displaystyle\iint_S (x^2+y^2)z\,dS$



$$= \iint_{S_{xy}} (x^2+y^2)(x+y)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \iint_{S_{xy}} (x^2+y^2)(x+y)\sqrt{1+(1)^2+(1)^2}\,dA$$
$$= \sqrt{3}\int_0^1\int_0^{1-x}(x^3+xy^2+x^2y+y^3)\,dy\,dx$$
$$= \sqrt{3}\int_0^1\left\{x^3y+\frac{xy^3}{3}+\frac{x^2y^2}{2}+\frac{y^4}{4}\right\}_0^{1-x}dx$$
$$= \frac{\sqrt{3}}{12}\int_0^1 [4x-6x^2+12x^3-10x^4+3(1-x)^4]\,dx$$
$$= \frac{\sqrt{3}}{12}\left\{2x^2-2x^3+3x^4-2x^5-\frac{3}{5}(1-x)^5\right\}_0^1 = \frac{2\sqrt{3}}{15}$$

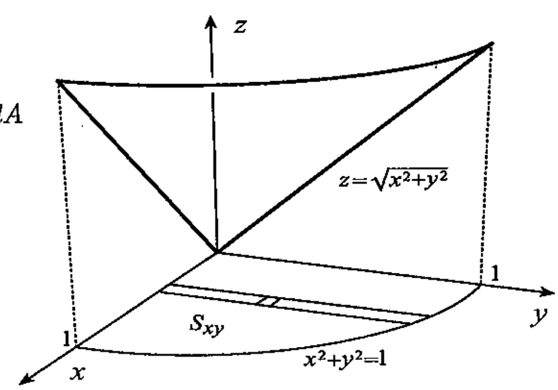
3. Integrals over S_3, S_4, and S_5 in the coordinate planes vanish.



$$\iint_{S_1} xyz\,dS = \iint_{S_{yz}} yz\,dA$$
$$= \int_0^1\int_0^1 yz\,dz\,dy = \int_0^1\left\{\frac{yz^2}{2}\right\}_0^1 dy$$
$$= \frac{1}{2}\int_0^1 y\,dy = \frac{1}{2}\left\{\frac{y^2}{2}\right\}_0^1 = \frac{1}{4}$$

Integrals over S_2 and S_6 are the same. Hence, the integral over S is $3(1/4) = 3/4$.

4. $$\iint_S xy\,dS = \iint_{S_{xy}} xy\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$



$$= \iint_{S_{xy}} xy\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dA$$
$$= \sqrt{2}\iint_{S_{xy}} xy\,dA = \sqrt{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$
$$= \sqrt{2}\int_0^1\left\{\frac{xy^2}{2}\right\}_0^{\sqrt{1-x^2}}dx = \frac{1}{\sqrt{2}}\int_0^1 (x-x^3)\,dx$$
$$= \frac{1}{\sqrt{2}}\left\{\frac{x^2}{2}-\frac{x^4}{4}\right\}_0^1 = \frac{1}{4\sqrt{2}}$$

5. $$\iint_S \frac{1}{\sqrt{z-y+1}}dS = \iint_{S_{xy}} \frac{1}{\sqrt{z-y+1}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \iint_{S_{xy}} \frac{1}{\sqrt{x^2/2+y-y+1}}\sqrt{1+(x)^2+(1)^2}\,dA = \sqrt{2}\iint_{S_{xy}} dA = \sqrt{2}(1) = \sqrt{2}$$

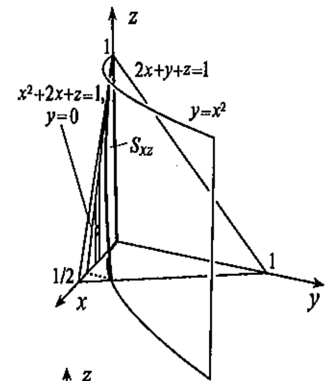
6. $$\iint_S \sqrt{4y+1}\,dS = \iint_{S_{xz}} \sqrt{4x^2+1}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA = \iint_{S_{xz}} \sqrt{4x^2+1}\sqrt{1+(2x)^2}\,dA$$

$$= \iint_{S_{xz}} (1+4x^2)\,dA = \int_0^{\sqrt{2}-1}\int_0^{1-2x-x^2} (1+4x^2)\,dz\,dx$$

$$= \int_0^{\sqrt{2}-1} (1+4x^2)(1-2x-x^2)\,dx$$

$$= \int_0^{\sqrt{2}-1} (1-2x+3x^2-8x^3-4x^4)\,dx$$

$$= \left\{x - x^2 + x^3 - 2x^4 - \frac{4x^5}{5}\right\}_0^{\sqrt{2}-1} = \frac{44\sqrt{2}-61}{5}$$

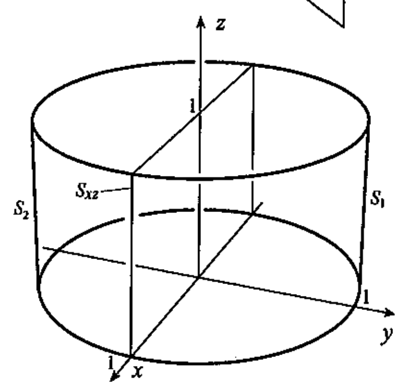


7. $$\iint_S x^2 z\,dS = \iint_{S_1} x^2 z\,dS + \iint_{S_2} x^2 z\,dS \qquad \text{On } S_1 \text{ and } S_2,$$

$$dS = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA$$

$$= \sqrt{1+\left(\frac{\mp x}{\sqrt{1-x^2}}\right)^2}\,dA = \frac{1}{\sqrt{1-x^2}}dA.$$

Hence,



$$\iint_S x^2 z\,dS = \iint_{S_{xz}} \frac{x^2 z}{\sqrt{1-x^2}}dA + \iint_{S_{xz}} \frac{x^2 z}{\sqrt{1-x^2}}dA$$

$$= 4\int_0^1\int_0^1 \frac{x^2 z}{\sqrt{1-x^2}}dz\,dx = 4\int_0^1 \left\{\frac{x^2 z^2}{2\sqrt{1-x^2}}\right\}_0^1 dx = 2\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx.$$

If we set $x = \sin\theta$ and $dx = \cos\theta\,d\theta$,

$$\iint_S x^2 z\,dS = 2\int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta}(\cos\theta\,d\theta) = 2\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{\pi}{2}.$$

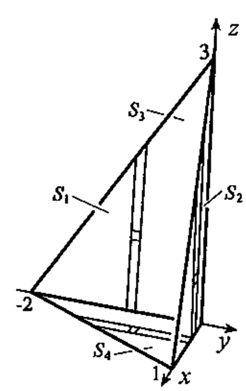
8. $$\iint_S (x+y)\,dS = \iint_{S_1}(x+y)\,dS + \iint_{S_2}(x+y)\,dS$$

$$+ \iint_{S_3}(x+y)\,dS + \iint_{S_4}(x+y)\,dS$$

$$= \iint_{S_{1xy}} (x+y)\sqrt{1+(-3)^2+(3/2)^2}\,dA$$

$$+ \iint_{S_{2xz}} x\,dA + \iint_{S_{3yz}} y\,dA + \iint_{S_{4xy}} (x+y)\,dA.$$

Since $S_{1xy} = S_{4xy}$,



$$\iint_S (x+y)\,dS = \frac{9}{2}\iint_{S_{1xy}} (x+y)\,dA + \iint_{S_{2xz}} x\,dA + \iint_{S_{3yz}} y\,dA$$

$$= \frac{9}{2}\int_0^1\int_{2x-2}^0 (x+y)\,dy\,dx + \int_0^1\int_0^{3-3x} x\,dz\,dx + \int_{-2}^0\int_0^{3+3y/2} y\,dz\,dy$$

$$= \frac{9}{2}\int_0^1 \left\{xy + \frac{y^2}{2}\right\}_{2x-2}^{0} dx + \int_0^1 \left\{xz\right\}_0^{3-3x} dx + \int_{-2}^{0}\left\{yz\right\}_0^{3+3y/2} dy$$

$$= \frac{9}{4}\int_0^1 [-4x^2 + 4x - 4(x-1)^2]\,dx + 3\int_0^1 (x - x^2)\,dx + \frac{3}{2}\int_{-2}^{0}(2y + y^2)\,dy$$

$$= \frac{9}{4}\left\{\frac{-4x^3}{3} + 2x^2 - \frac{4(x-1)^3}{3}\right\}_0^1 + 3\left\{\frac{x^2}{2} - \frac{x^3}{3}\right\}_0^1 + \frac{3}{2}\left\{y^2 + \frac{y^3}{3}\right\}_{-2}^{0} = -3.$$

9. For projection in the xy-plane, $dS = \sqrt{1 + (-2x)^2 + (-8y)^2}\,dA = \sqrt{1 + 4x^2 + 64y^2}\,dA$. Thus,

$$\iint_S f(x,y,z)\,dS = \int_0^1 \int_0^{\sqrt{4-4y^2}} f(x, y, 4 - x^2 - 4y^2)\sqrt{1 + 4x^2 + 64y^2}\,dx\,dy$$

For projection in the xz-plane,

$$dS = \sqrt{1 + \left(\frac{-x}{2\sqrt{4 - x^2 - z}}\right)^2 + \left(\frac{-1}{4\sqrt{4 - x^2 - z}}\right)^2}\,dA = \frac{1}{4}\sqrt{\frac{65 - 12x^2 - 16z}{4 - x^2 - z}}\,dA.$$

Thus,

$$\iint_S f(x,y,z)\,dS = \frac{1}{4}\int_0^2 \int_0^{4-x^2} f(x, \sqrt{4 - x^2 - z}/2, z)\sqrt{\frac{65 - 12x^2 - 16z}{4 - x^2 - z}}\,dz\,dx.$$

For projection in the yz-plane,

$$dS = \sqrt{1 + \left(\frac{-4y}{\sqrt{4 - 4y^2 - z}}\right)^2 + \left(\frac{-1}{2\sqrt{4 - 4y^2 - z}}\right)^2}\,dA = \frac{1}{2}\sqrt{\frac{17 + 48y^2 - 4z}{4 - 4y^2 - z}}\,dA.$$

Thus,

$$\iint_S f(x,y,z)\,dS = \frac{1}{2}\int_0^1 \int_0^{4-4y^2} f(\sqrt{4 - 4y^2 - z}, y, z)\sqrt{\frac{17 + 48y^2 - 4z}{4 - 4y^2 - z}}\,dz\,dy.$$

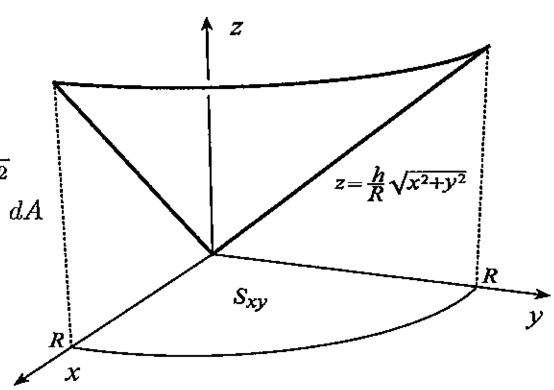
10. If S is that portion of the cone in the first octant,

$$\text{Area} = 4\iint_S dS = 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= 4\iint_{S_{xy}} \sqrt{1 + \left(\frac{hx}{R\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{hy}{R\sqrt{x^2 + y^2}}\right)^2}\,dA$$

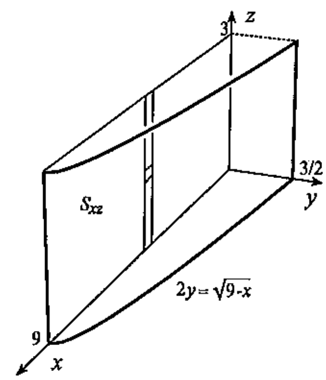
$$= \frac{4\sqrt{R^2 + h^2}}{R}\iint_{S_{xy}} dA = \frac{4\sqrt{R^2 + h^2}}{R}(\text{Area of } S_{xy})$$

$$= \frac{4\sqrt{R^2 + h^2}}{R}\left(\frac{1}{4}\pi R^2\right) = \pi R\sqrt{R^2 + h^2}.$$



11. Since xyz^3 is an odd function of y and the surface is symmetric about the xz-plane, the integral has value zero.

12. $$\iint_S xyz\,dS = \iint_{S_{xz}} xyz\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$= \iint_{S_{xz}} xyz\sqrt{1+\left(\frac{-1}{4\sqrt{9-x}}\right)^2}\,dA$$
$$= \iint_{S_{xz}} xyz\sqrt{\frac{145-16x}{16(9-x)}}\,dA$$
$$= \frac{1}{4}\int_0^9\int_0^3 xz\frac{\sqrt{9-x}}{2}\sqrt{\frac{145-16x}{9-x}}\,dz\,dx$$
$$= \frac{1}{8}\int_0^9\left\{x\sqrt{145-16x}\,\frac{z^2}{2}\right\}_0^3 dx = \frac{9}{16}\int_0^9 x\sqrt{145-16x}\,dx$$



If we now set $u = 145-16x$, then $du = -16\,dx$, and

$$\iint_S xyz\,dS = \frac{9}{16}\int_{145}^1\left(\frac{145-u}{16}\right)\sqrt{u}\left(\frac{du}{-16}\right) = \frac{-9}{4096}\left\{\frac{290u^{3/2}}{3}-\frac{2u^{5/2}}{5}\right\}_{145}^1 = \frac{3(145^{5/2}-361)}{5120}.$$

13. If S_{xy} is the projection of the surface in the xy-plane,

$$dS = \sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{a^2-x^2-y^2}}\right)^2}\,dA = \frac{a}{\sqrt{a^2-x^2-y^2}}dA.$$

Thus,

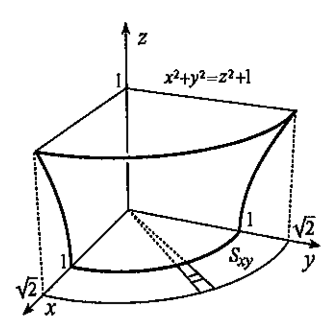
$$\iint_S \frac{1}{\sqrt{2az-z^2}}dS = \iint_{S_{xy}}\frac{1}{\sqrt{x^2+y^2}}\frac{a}{\sqrt{a^2-x^2-y^2}}dA = a\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{1}{r\sqrt{a^2-r^2}}r\,dr\,d\theta$$
$$= a\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{1}{\sqrt{a^2-r^2}}dr\,d\theta.$$

If we set $r = a\sin\phi$ and $dr = a\cos\phi\,d\phi$,

$$\iint_S \frac{1}{\sqrt{2az-z^2}}dS = a\int_0^{\pi/2}\int_0^{\theta}\frac{1}{a\cos\phi}(a\cos\phi\,d\phi) = a\int_0^{\pi/2}\left\{\phi\right\}_0^{\theta}d\theta = a\int_0^{\pi/2}\theta\,d\theta = a\left\{\frac{\theta^2}{2}\right\}_0^{\pi/2} = \frac{a\pi^2}{8}.$$

14. We quadruple the integral over that part of the surface in the first octant.

$$\iint_S z\,dS = 4\iint_{S_{xy}} z\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} z\sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} z\sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dA$$
$$= 4\iint_{S_{xy}} \sqrt{2x^2+2y^2-1}\,dA$$
$$= 4\int_0^{\pi/2}\int_1^{\sqrt{2}}\sqrt{2r^2-1}\,r\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\left\{\frac{(2r^2-1)^{3/2}}{6}\right\}_1^{\sqrt{2}}d\theta = \frac{2(3\sqrt{3}-1)}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{\pi(3\sqrt{3}-1)}{3}$$



15. We quadruple the integral over that part of the surface in the first octant.

$$\iint_S x^2y^2\,dS = 4\iint_{S_{xy}} x^2y^2\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = 4\iint_{S_{xy}} x^2y^2\sqrt{1+(2x)^2+(2y)^2}\,dA$$

$$= 4\int_0^{\pi/2}\int_0^1 r^4\cos^2\theta\sin^2\theta\sqrt{1+4r^2}\,r\,dr\,d\theta$$

If we set $u = 1+4r^2$ and $du = 8r\,dr$,

$$\iint_S x^2y^2\,dS = 4\int_0^{\pi/2}\int_1^5 \cos^2\theta\sin^2\theta\left(\frac{u-1}{4}\right)^2\sqrt{u}\left(\frac{du}{8}\right)d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2}\int_1^5 \cos^2\theta\sin^2\theta(u^{5/2}-2u^{3/2}+\sqrt{u})\,du\,d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2}\left\{\cos^2\theta\sin^2\theta\left(\frac{2u^{7/2}}{7}-\frac{4u^{5/2}}{5}+\frac{2u^{3/2}}{3}\right)\right\}_1^5 d\theta = \frac{125\sqrt{5}-1}{210}\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{125\sqrt{5}-1}{840}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = \frac{125\sqrt{5}-1}{1680}\left\{\theta-\frac{1}{4}\sin 4\theta\right\}_0^{\pi/2} = \frac{(125\sqrt{5}-1)\pi}{3360}.$$

16. We quadruple the integral over that part of the surface in the first octant.

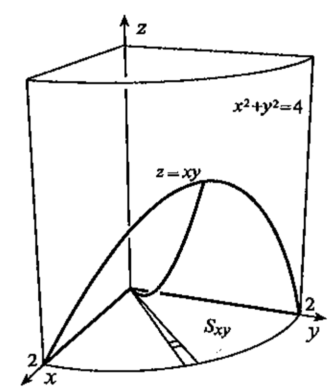
$$\iint_S x^2\,dS = 4\iint_{S_{xy}} x^2\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= 4\iint_{S_{xy}} x^2\sqrt{1+y^2+x^2}\,dA$$

$$= 4\int_0^2\int_0^{\pi/2} r^2\cos^2\theta\sqrt{1+r^2}\,r\,d\theta\,dr$$

$$= 4\int_0^2\int_0^{\pi/2} r^3\sqrt{1+r^2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta\,dr$$

$$= 2\int_0^2\left\{r^3\sqrt{1+r^2}\left(\theta+\frac{\sin 2\theta}{2}\right)\right\}_0^{\pi/2} dr = \pi\int_0^2 r^3\sqrt{1+r^2}\,dr$$



If we set $u = 1+r^2$, then $du = 2r\,dr$, and

$$\iint_S x^2\,dS = \pi\int_1^5 (u-1)\sqrt{u}\left(\frac{du}{2}\right) = \frac{\pi}{2}\int_1^5 (u^{3/2}-\sqrt{u})\,du = \frac{\pi}{2}\left\{\frac{2u^{5/2}}{5}-\frac{2u^{3/2}}{3}\right\}_1^5 = \frac{(50\sqrt{5}+2)\pi}{15}.$$

17. $$\iint_S z(y+x^2)\,dS = \iint_{S_{xz}} z(y+x^2)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA = \iint_{S_{xz}} z(1-x^2+x^2)\sqrt{1+(-2x)^2}\,dA$$

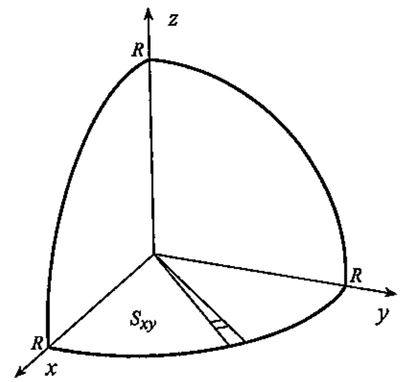
$$= 2\int_0^1\int_0^2 z\sqrt{1+4x^2}\,dz\,dx = 2\int_0^1\left\{\frac{z^2}{2}\sqrt{1+4x^2}\right\}_0^2 dx = 4\int_0^1 \sqrt{1+4x^2}\,dx$$

If we set $x = (1/2)\tan\theta$ and $dx = (1/2)\sec^2\theta\,d\theta$, and use Example 8.9,

$$\iint_S z(y+x^2)\,dS = 4\int_0^{\mathrm{Tan}^{-1}2}\sec\theta(1/2)\sec^2\theta\,d\theta = 2\int_0^{\mathrm{Tan}^{-1}2}\sec^3\theta\,d\theta$$

$$= \Big\{\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big\}_0^{\mathrm{Tan}^{-1}2} = 2\sqrt{5}+\ln(\sqrt{5}+2).$$

18. The surface integral over S is eight times that over that part of the the upper hemisphere $z = \sqrt{R^2 - x^2 - y^2}$ in the first octant. Since



$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\, dA$$
$$= \frac{R}{\sqrt{R^2 - x^2 - y^2}}\, dA,$$

it follows that

$$\iint_S dS = 8 \iint_{S_{xy}} \frac{R}{\sqrt{R^2 - x^2 - y^2}}\, dA = 8R \int_0^{\pi/2} \int_0^R \frac{1}{\sqrt{R^2 - r^2}}\, r\, dr\, d\theta$$
$$= 8R \int_0^{\pi/2} \left\{-\sqrt{R^2 - r^2}\right\}_0^R d\theta = 8R^2 \left\{\theta\right\}_0^{\pi/2} = 4\pi R^2.$$

Alternatively, using area element 14.56,

$$\iint_S dS = 8 \int_0^{\pi/2} \int_0^{\pi/2} R^2 \sin\phi\, d\phi\, d\theta = 8R^2 \int_0^{\pi/2} \left\{-\cos\phi\right\}_0^{\pi/2} d\theta = 8R^2 \left\{\theta\right\}_0^{\pi/2} = 4\pi R^2.$$

19. If S_{xy} is the projection of the first octant part of the sphere in the xy-plane,

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{1 - x^2 - y^2}}\right)^2}\, dA = \frac{1}{\sqrt{1 - x^2 - y^2}}\, dA.$$

Thus,

$$\iint_S x^2 z^2\, dS = 8 \iint_{S_{xz}} x^2 (1 - x^2 - y^2) \frac{1}{\sqrt{1 - x^2 - y^2}}\, dA = 8 \int_0^{\pi/2} \int_0^1 r^2 \cos^2\theta \sqrt{1 - r^2}\, r\, dr\, d\theta.$$

If we set $u = 1 - r^2$ and $du = -2r\, dr$,

$$\iint_S x^2 z^2\, dS = 8 \int_0^{\pi/2} \int_1^0 (1 - u)\sqrt{u} \cos^2\theta \left(\frac{du}{-2}\right) d\theta = 4 \int_0^{\pi/2} \int_0^1 (\sqrt{u} - u^{3/2}) \cos^2\theta\, du\, d\theta$$
$$= 4 \int_0^{\pi/2} \left\{\left(\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right) \cos^2\theta\right\}_0^1 d\theta = \frac{16}{15} \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= \frac{8}{15} \left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{4\pi}{15}.$$

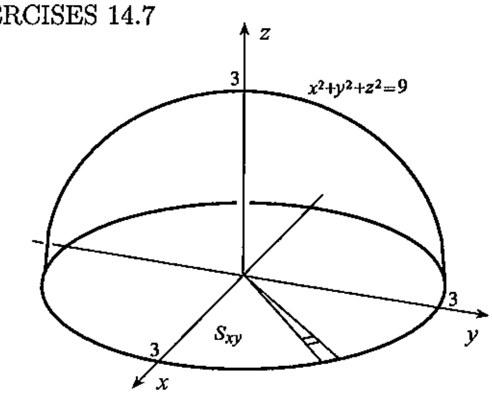
Alternatively, using area element 14.56 with $R = 1$,

$$\iint_S x^2 z^2\, dS = 8 \int_0^{\pi/2} \int_0^{\pi/2} (\sin^2\phi \cos^2\theta) \cos^2\phi \sin\phi\, d\phi\, d\theta$$
$$= 8 \int_0^{\pi/2} \int_0^{\pi/2} \cos^2\theta (1 - \cos^2\phi) \cos^2\phi \sin\phi\, d\phi\, d\theta$$
$$= 8 \int_0^{\pi/2} \left\{\cos^2\theta \left(-\frac{1}{3}\cos^3\phi + \frac{1}{5}\cos^5\phi\right)\right\}_0^{\pi/2} d\theta = \frac{16}{15} \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= \frac{8}{15} \left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{4\pi}{15}.$$

20. The hemisphere projects one-to-one onto the circle $S_{xy} : x^2 + y^2 \le 9$, $z = 0$. Since

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \sqrt{1 + \frac{x^2}{9 - x^2 - y^2} + \frac{y^2}{9 - x^2 - y^2}}\,dA = \frac{3}{\sqrt{9 - x^2 - y^2}}\,dA,$$

it follows that



$$\iint_S (x^2 - y^2)\,dS = \iint_{S_{xy}} \frac{3(x^2 - y^2)}{\sqrt{9 - x^2 - y^2}}\,dA = 3\int_0^3 \int_{-\pi}^{\pi} \frac{(r^2\cos^2\theta - r^2\sin^2\theta)}{\sqrt{9 - r^2}}\,r\,d\theta\,dr$$

$$= 3\int_0^3 \int_{-\pi}^{\pi} \frac{r^3}{\sqrt{9 - r^2}}\cos 2\theta\,d\theta\,dr = 3\int_0^3 \left\{\frac{r^3}{\sqrt{9 - r^2}}\frac{\sin 2\theta}{2}\right\}_{-\pi}^{\pi} dr = 0.$$

Alternatively, using $dS = R^2 \sin\phi\,d\phi\,d\theta$, with $R = 3$,

$$\iint_S (x^2 - y^2)\,dS = \int_{-\pi}^{\pi}\int_0^{\pi/2} (9\sin^2\phi\,\cos^2\theta - 9\sin^2\phi\,\sin^2\theta)\,9\sin\phi\,d\phi\,d\theta$$

$$= 81\int_{-\pi}^{\pi}\int_0^{\pi/2} \sin^3\phi\,\cos 2\theta\,d\phi\,d\theta = 81\int_{-\pi}^{\pi}\int_0^{\pi/2} \sin\phi(1 - \cos^2\phi)\cos 2\theta\,d\phi\,d\theta$$

$$= 81\int_{-\pi}^{\pi} \left\{\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\cos 2\theta\right\}_0^{\pi/2} d\theta = 54\left\{\frac{\sin 2\theta}{2}\right\}_{-\pi}^{\pi} = 0.$$

21. If S_{xy} is the projection of the sphere in the xy-plane,

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{R^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{R^2 - x^2 - y^2}}\right)^2}\,dA = \frac{R}{\sqrt{R^2 - x^2 - y^2}}\,dA.$$

Thus, $\displaystyle\iint_S (x^2 + y^2)\,dS = 2\iint_{S_{xy}} (x^2 + y^2)\frac{R}{\sqrt{R^2 - x^2 - y^2}}\,dA = 8R\int_0^{\pi/2}\int_0^R \frac{r^2}{\sqrt{R^2 - r^2}}\,r\,dr\,d\theta.$

If we set $u = R^2 - r^2$ and $du = -2r\,dr$,

$$\iint_S (x^2 + y^2)\,dS = 8R\int_0^{\pi/2}\int_{R^2}^0 \frac{R^2 - u}{\sqrt{u}}\left(\frac{du}{-2}\right)d\theta = 4R\int_0^{\pi/2}\int_0^{R^2} \left(\frac{R^2}{\sqrt{u}} - \sqrt{u}\right)du\,d\theta$$

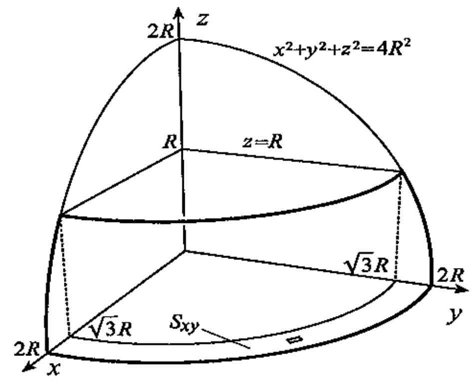
$$= 4R\int_0^{\pi/2} \left\{2R^2\sqrt{u} - \frac{2u^{3/2}}{3}\right\}_0^{R^2} d\theta = \frac{16R^4}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{8\pi R^4}{3}.$$

Alternatively, if we use area element 14.56,

$$\iint_S (x^2 + y^2)\,dS = 8\int_0^{\pi/2}\int_0^{\pi/2} R^2\sin^2\phi R^2\sin\phi\,d\phi\,d\theta = 8R^4\int_0^{\pi/2}\int_0^{\pi/2} (1 - \cos^2\phi)\sin\phi\,d\phi\,d\theta$$

$$= 8R^4\int_0^{\pi/2} \left\{-\cos\phi + \frac{1}{3}\cos^3\phi\right\}_0^{\pi/2} d\theta = \frac{16R^4}{3}\Big\{\theta\Big\}_0^{\pi/2} = \frac{8\pi R^4}{3}.$$

22. The first octant part of the surface projects one-to-one onto the area $S_{xy}: 3R^2 \le x^2 + y^2 \le 4R^2$ in the first quadrant of the xy-plane. Since



$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= \sqrt{1 + \frac{x^2}{4R^2 - x^2 - y^2} + \frac{y^2}{4R^2 - x^2 - y^2}}\, dA$$
$$= \frac{2R}{\sqrt{4R^2 - x^2 - y^2}} dA,$$

it follows that

$$\iint_S \frac{1}{x^2 + y^2} dS = 4 \iint_{S_{xy}} \frac{1}{x^2 + y^2} \frac{2R}{\sqrt{4R^2 - x^2 - y^2}}\, dA$$
$$= 8R \int_{\sqrt{3}R}^{2R} \int_0^{\pi/2} \frac{1}{r^2\sqrt{4R^2 - r^2}} r\, d\theta\, dr = 4\pi R \int_{\sqrt{3}R}^{2R} \frac{1}{r\sqrt{4R^2 - r^2}}\, dr.$$

If we set $r = 2R\sin\phi$ and $dr = 2R\cos\phi\, d\phi$, then

$$\iint_S \frac{1}{x^2 + y^2} dS = 4\pi R \int_{\pi/3}^{\pi/2} \frac{1}{2R\sin\phi\, 2R\cos\phi} 2R\cos\phi\, d\phi = 2\pi \int_{\pi/3}^{\pi/2} \csc\phi\, d\phi$$
$$= 2\pi \Big\{ \ln|\csc\phi - \cot\phi| \Big\}_{\pi/3}^{\pi/2} = \pi \ln 3.$$

Alternatively, using area element 14.56,

$$\iint_S \frac{1}{x^2 + y^2} dS = 4 \int_0^{\pi/2} \int_{\pi/3}^{\pi/2} \frac{1}{4R^2\sin^2\phi\, \cos^2\theta + 4R^2\sin^2\phi\, \sin^2\theta} 4R^2 \sin\phi\, d\phi\, d\theta$$
$$= 4 \int_0^{\pi/2} \int_{\pi/3}^{\pi/2} \csc\phi\, d\phi\, d\theta = 4 \int_0^{\pi/2} \Big\{ \ln|\csc\phi - \cot\phi| \Big\}_{\pi/3}^{\pi/2} d\theta = \pi \ln 3.$$

23. The thickness of material as a function of ϕ is $\rho(\phi) = \dfrac{0.004\phi}{\pi} + 0.001 = \dfrac{4\phi + \pi}{1000\pi}$. Using area element 14.56 with $R = 1$,

$$V = \iint_S \rho\, dS = \int_{-\pi}^{\pi} \int_0^{\pi} \frac{4\phi + \pi}{1000\pi} \sin\phi\, d\theta\, d\phi = \frac{1}{1000\pi} \int_0^{\pi} \Big\{ (4\phi + \pi)\sin\phi\, \theta \Big\}_{-\pi}^{\pi} d\phi = \frac{1}{500} \int_0^{\pi} (4\phi + \pi)\sin\phi\, d\phi.$$

If we set $u = 4\phi + \pi$, $dv = \sin\phi\, d\phi$, $du = 4\, d\phi$, and $v = -\cos\phi$,

$$V = \frac{1}{500} \left[\Big\{ -(4\phi + \pi)\cos\phi \Big\}_0^{\pi} - \int_0^{\pi} -4\cos\phi\, d\phi \right] = \frac{1}{500} \left(6\pi + 4\Big\{\sin\phi\Big\}_0^{\pi} \right) = \frac{3\pi}{250}\ \text{m}^3.$$

24. If S projects one-to-one onto S_{xy}, then an element of area dS on S is related to its projection dA in the xy-plane by $dS = \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\, dA$, where

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(z)}} = -\frac{F_y}{F_z}.$$

Thus, $dS = \sqrt{1 + \left(-\dfrac{F_x}{F_z}\right)^2 + \left(-\dfrac{F_y}{F_z}\right)^2}\, dA = \dfrac{\sqrt{(F_x)^2 + (F_y)^2 + (F_z)^2}}{|F_z|}\, dA = \dfrac{|\nabla F|}{|F_z|}\, dA$, and

$$\iint_S f(x,y,z)\,dS = \iint_{S_{xy}} f[x,y,g(x,y)]\frac{|\nabla F|}{|F_z|}\,dA,$$

where $z = g(x,y)$ on S.

25. (a) If S_{xy} is the projection of the first octant part of $z^2 = x^2 + y^2$ in the xy-plane,

$$dS = \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dA = \sqrt{2}\,dA.$$

Thus, $A = 4\displaystyle\iint_{S_{xy}} \sqrt{2}\,dA = 4\sqrt{2}\left(\frac{\pi}{2}\right) = 2\sqrt{2}\pi$.

(b) The projection in the xz-plane of the curve of intersection of the cone and the cylinder in the first octant has equation $z = \sqrt{2x}$. Since $dS = \sqrt{1 + \left(\dfrac{1-x}{\sqrt{2x-x^2}}\right)^2}\,dA = \dfrac{1}{\sqrt{2x-x^2}}dA$,

$$\begin{aligned} A &= 4\iint_{S_{xz}} \frac{1}{\sqrt{2x-x^2}}dA = 4\int_0^2\int_0^{\sqrt{2x}} \frac{1}{\sqrt{2x-x^2}}dz\,dx = 4\int_0^2 \frac{\sqrt{2x}}{\sqrt{2x-x^2}}dx \\ &= 4\sqrt{2}\int_0^2 \frac{1}{\sqrt{2-x}}dx = 4\sqrt{2}\left\{-2\sqrt{2-x}\right\}_0^2 = 16. \end{aligned}$$

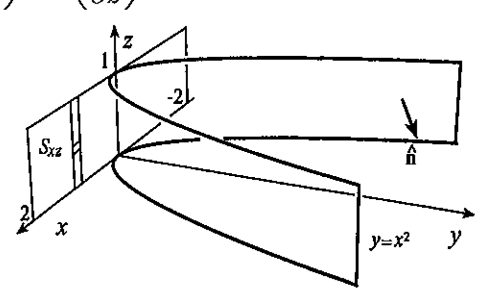
EXERCISES 14.8

1. Since $\hat{\mathbf{n}} = (1,1,1)/\sqrt{3}$,

$$\begin{aligned} \iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS &= \iint_{S_{xy}} \frac{x+y}{\sqrt{3}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \frac{1}{\sqrt{3}}\iint_{S_{xy}} (x+y)\sqrt{1+(-1)^2+(-1)^2}\,dA \\ &= \int_0^3\int_0^{3-x}(x+y)\,dy\,dx = \int_0^3\left\{\frac{1}{2}(x+y)^2\right\}_0^{3-x}dx = \frac{1}{2}\int_0^3(9-x^2)\,dx = \frac{1}{2}\left\{9x - \frac{x^3}{3}\right\}_0^3 = 9. \end{aligned}$$

2. Since $\hat{\mathbf{n}} = \dfrac{\nabla(y-x^2)}{|\nabla(y-x^2)|} = \dfrac{(-2x,1,0)}{\sqrt{4x^2+1}}$,

$$\begin{aligned} \iint_S (yz^2\hat{\mathbf{i}} + ye^x\hat{\mathbf{j}} + x\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS &= \iint_{S_{xz}} \frac{-2xyz^2 + ye^x}{\sqrt{4x^2+1}}\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA \\ &= \iint_{S_{xz}} \frac{-2x^3z^2 + x^2e^x}{\sqrt{4x^2+1}}\sqrt{1+(2x)^2}\,dA \\ &= \int_{-2}^2\int_0^1 (x^2e^x - 2x^3z^2)\,dz\,dx = \int_{-2}^2\left\{x^2ze^x - \frac{2x^3z^3}{3}\right\}_0^1 dx \\ &= \int_{-2}^2\left(x^2e^x - \frac{2x^3}{3}\right)dx = \left\{x^2e^x - 2xe^x + 2e^x - \frac{x^4}{6}\right\}_{-2}^2 \\ &= 2e^2 - 10e^{-2}. \end{aligned}$$

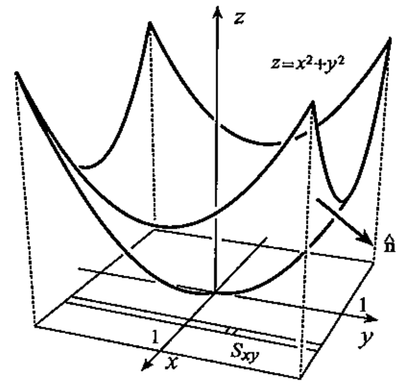


3. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-1)}{|\nabla(x^2+y^2+z^2-1)|} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = (x,y,z)$,

$$\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iint_S (x^2+y^2+z^2)\,dS = \iint_S dS = \frac{1}{2}(4\pi) = 2\pi.$$

4. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-z)}{|\nabla(x^2+y^2-z)|} = \dfrac{(2x, 2y, -1)}{\sqrt{1+4x^2+4y^2}}$,

$$\begin{aligned}
&\iint_S (yz\hat{\mathbf{i}} + zx\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS \\
&= \iint_{S_{xy}} \frac{2xyz + 2xyz - xy}{\sqrt{1+4x^2+4y^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \iint_{S_{xy}} \frac{4xy(x^2+y^2) - xy}{\sqrt{1+4x^2+4y^2}} \sqrt{1 + (2x)^2 + (2y)^2}\, dA \\
&= \int_{-1}^{1}\int_{-1}^{1} (4x^3y + 4xy^3 - xy)\, dy\, dx \\
&= \int_{-1}^{1} \left\{2x^3y^2 + xy^4 - \frac{xy^2}{2}\right\}_{-1}^{1} dx = 0.
\end{aligned}$$



5. Let S_1 be the hemisphere and S_2 be that part of the xy-plane bounded by $x^2+y^2=4$. On S_1, $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-1)}{|\nabla(x^2+y^2+z^2-1)|} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(x,y,z)}{2}$, and on S_2, $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$. Thus,

$$\oiint_S (z\hat{\mathbf{i}} - x\hat{\mathbf{j}} + y\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \frac{1}{2}\iint_{S_1} (xz - xy + yz)\, dS + \iint_{S_2} -y\, dS.$$

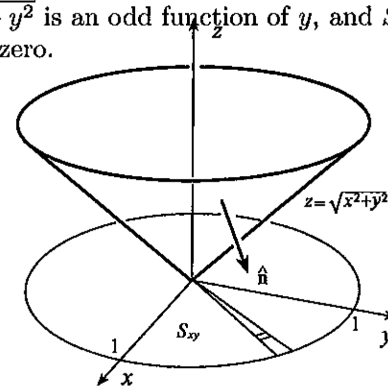
The integral over S_2 is zero since y is an odd function of y and S_2 is symmetric about the xz-plane. If S_{xy} is the projection of S_1 in the xy-plane,

$$\begin{aligned}
\oiint_S (z\hat{\mathbf{i}} - x\hat{\mathbf{j}} + y\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= \frac{1}{2}\iint_{S_{xy}} (xz - xy + yz)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \frac{1}{2}\iint_{S_{xy}} (xz - xy + yz)\sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2}\, dA \\
&= \frac{1}{2}\iint_{S_{xy}} [(x+y)\sqrt{4-x^2-y^2} - xy]\frac{2}{\sqrt{4-x^2-y^2}} dA.
\end{aligned}$$

Since $x\sqrt{4-x^2-y^2} - xy$ is an odd function of x, and $y\sqrt{4-x^2-y^2}$ is an odd function of y, and S_{xy} is symmetric about the x- and y-axes, this integral is also equal to zero.

6. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-z^2)}{|\nabla(x^2+y^2-z^2)|} = \dfrac{(2x, 2y, -2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x, y, -z)}{\sqrt{2}z}$,

$$\begin{aligned}
\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \cdot \hat{\mathbf{n}}\, dS &= \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{2}z}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \frac{1}{\sqrt{2}}\iint_{S_{xy}} \frac{x^2+y^2}{z}\sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\, dA \\
&= \frac{1}{\sqrt{2}}\iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{x^2+y^2}}\sqrt{2}\, dA = 4\int_0^{\pi/2}\int_0^1 r\, r\, dr\, d\theta = 4\int_0^{\pi/2} \left\{\frac{r^3}{3}\right\}_0^1 d\theta = \frac{4}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{2\pi}{3}.
\end{aligned}$$

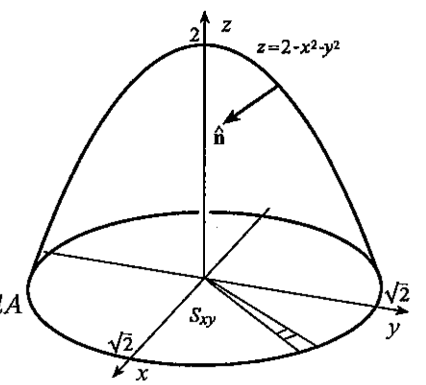


7. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-9)}{|\nabla(x^2+y^2-9)|} = \dfrac{(2x,2y,0)}{\sqrt{4x^2+4y^2}} = \dfrac{(x,y,0)}{3}$,

$$\begin{aligned}\iint_S (xyz\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS &= \frac{1}{3}\iint_{S_{xz}} (x^2yz - xy)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA\\
&= \frac{1}{3}\iint_{S_{xz}} (x^2z - x)y\sqrt{1+\left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\,dA\\
&= \frac{1}{3}\iint_{S_{xz}} (x^2z - x)\sqrt{9-x^2}\,\frac{3}{\sqrt{9-x^2}}\,dA\\
&= \int_{-3/\sqrt{2}}^{3/\sqrt{2}}\int_0^2 (x^2z-x)\,dz\,dx = \int_{-3/\sqrt{2}}^{3/\sqrt{2}}\left\{\frac{x^2z^2}{2} - xz\right\}_0^2 dx\\
&= \int_{-3/\sqrt{2}}^{3/\sqrt{2}}(2x^2-2x)\,dx = \left\{\frac{2x^3}{3} - x^2\right\}_{-3/\sqrt{2}}^{3/\sqrt{2}} = 9\sqrt{2}.\end{aligned}$$

8. Since $\hat{\mathbf{n}} = \dfrac{\nabla(2-x^2-y^2-z)}{|\nabla(2-x^2-y^2-z)|} = \dfrac{(-2x,-2y,-1)}{\sqrt{1+4x^2+4y^2}}$,

$$\begin{aligned}&\iint_S (x^2y\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS\\
&= \iint_{S_{xy}} \frac{-2x^3y - 2xy^2 - z}{\sqrt{1+4x^2+4y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\
&= \iint_{S_{xy}} \frac{-(2x^3y+2xy^2+2-x^2-y^2)}{\sqrt{1+4x^2+4y^2}}\sqrt{1+(-2x)^2+(-2y)^2}\,dA\\
&= -\iint_{S_{xy}} [2x^3y + 2xy^2 + 2 - (x^2+y^2)]\,dA.\end{aligned}$$

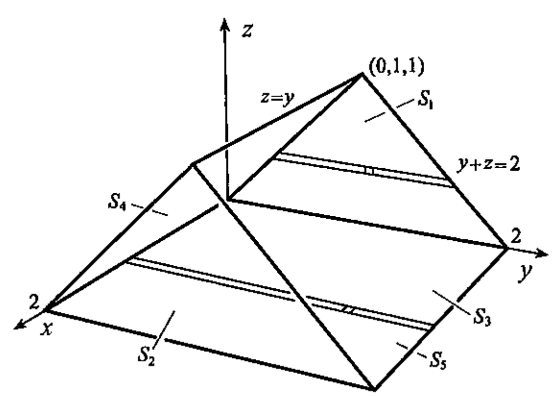


Because the first two terms are odd functions of x and S_{xy} is symmetric about the y-axis, their double integral vanishes, and

$$\iint_S (x^2y\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = -\int_{-\pi}^{\pi}\int_0^{\sqrt{2}}(2-r^2)\,r\,dr\,d\theta = -\int_{-\pi}^{\pi}\left\{r^2 - \frac{r^4}{4}\right\}_0^{\sqrt{2}} d\theta = -\Big\{\theta\Big\}_{-\pi}^{\pi} = -2\pi.$$

9. On S_1, $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$, and therefore

$$\begin{aligned}&\iint_{S_1}(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS\\
&= \iint_{S_1} -yz\,dA = \int_0^1\int_z^{2-z} -yz\,dy\,dz\\
&= -\int_0^1\left\{\frac{y^2z}{2}\right\}_z^{2-z} dz = -\frac{1}{2}\int_0^1 (4z-4z^2)\,dz\\
&= -\frac{1}{2}\left\{2z^2 - \frac{4z^3}{3}\right\}_0^1 = -\frac{1}{3}.\end{aligned}$$



On S_2, $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, and therefore

$$\iint_{S_2}(yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_2} yz\,dS = \iint_{S_1} yz\,dA = \frac{1}{3}.$$

On S_3, $\mathbf{n} = (0,1,1)/\sqrt{2}$, and therefore

$$\iint_{S_3} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_3} \left(\frac{xz+xy}{\sqrt{2}}\right) dS = \frac{1}{\sqrt{2}} \iint_{S_{xy}} x(2) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \sqrt{2} \iint_{S_{xy}} x\sqrt{1+(-1)^2}\, dA = 2\int_0^2 \int_1^2 x\, dy\, dx$$

$$= 2\int_0^2 \Big\{xy\Big\}_1^2 dx = 2\int_0^2 x\, dx = 2\left\{\frac{x^2}{2}\right\}_0^2 = 4.$$

On S_4, $\mathbf{n} = (0,-1,1)/\sqrt{2}$, and therefore

$$\iint_{S_4} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_4} \left(\frac{-xz+xy}{\sqrt{2}}\right) dS = \frac{1}{\sqrt{2}} \iint_S x(0)\, dS = 0.$$

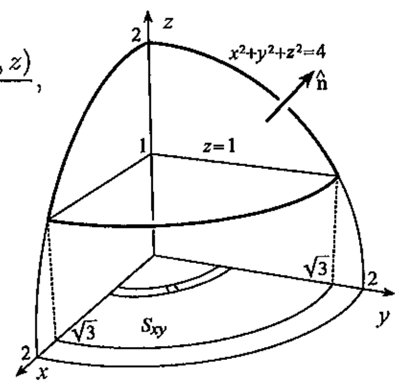
On S_5, $\mathbf{n} = -\hat{\mathbf{k}}$, and therefore

$$\iint_{S_5} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_5} -xy\, dS = \int_0^2 \int_0^2 -xy\, dy\, dx = -\int_0^2 \left\{\frac{xy^2}{2}\right\}_0^2 dx = -\frac{1}{2}\int_0^2 4x\, dx$$

$$= -2\left\{\frac{x^2}{2}\right\}_0^2 = -4.$$

Thus, $\displaystyle\oiint_S (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = -\frac{1}{3} + \frac{1}{3} + 4 + 0 - 4 = 0.$

10. Since $\displaystyle\hat{\mathbf{n}} = \frac{\nabla(x^2+y^2+z^2-4)}{|\nabla(x^2+y^2+z^2-4)|} = \frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \frac{(x,y,z)}{2}$,



$$\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \cdot \hat{\mathbf{n}}\, dS$$

$$= \iint_{S_{xy}} \frac{x^2+y^2}{2} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \frac{1}{2} \iint_{S_{xy}} (x^2+y^2) \sqrt{1 + \left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2}\, dA$$

$$= \frac{1}{2} \iint_{S_{xy}} (x^2+y^2) \left(\frac{2}{z}\right) dA$$

$$= \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{4-x^2-y^2}}\, dA = \int_0^{\sqrt{3}} \int_{-\pi}^{\pi} \frac{r^2}{\sqrt{4-r^2}}\, r\, d\theta\, dr = 2\pi \int_0^{\sqrt{3}} \frac{r^3}{\sqrt{4-r^2}}\, dr.$$

If we set $u = 4 - r^2$ and $du = -2r\, dr$, then

$$\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \cdot \hat{\mathbf{n}}\, dS = 2\pi \int_4^1 \frac{(4-u)}{\sqrt{u}} \left(-\frac{du}{2}\right) = \pi \int_1^4 \left(\frac{4}{\sqrt{u}} - \sqrt{u}\right) du = \pi \left\{8\sqrt{u} - \frac{2u^{3/2}}{3}\right\}_1^4 = \frac{10\pi}{3}.$$

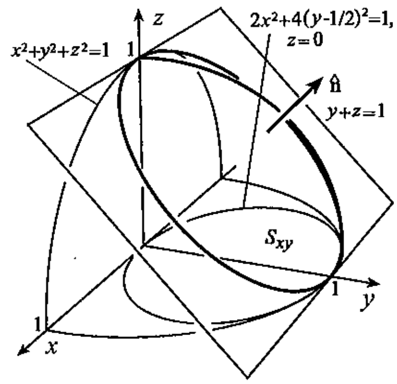
11. Since $\displaystyle\hat{\mathbf{n}} = \frac{\nabla(x^2+y^2+z^2-a^2)}{|\nabla(x^2+y^2+z^2-a^2)|} = \frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \frac{(x,y,z)}{a}$,

$$\oiint_S (x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \oiint_S (x^3+y^3+z^3)\, dS.$$

Since each of x^3, y^3, and z^3 is an odd function, and S is symmetric about the coordinate planes, the surface integral must be equal to zero.

12. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-1)}{|\nabla(x^2+y^2+z^2-1)|}$

$$= \frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = (x,y,z),$$



$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\,dS$$

$$= \iint_{S_{xy}} (yx - xy + z)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

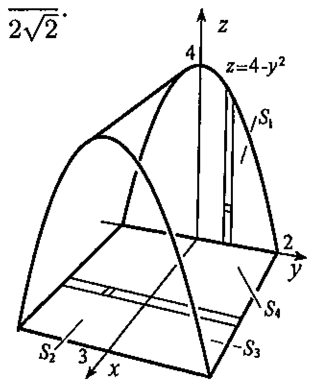
$$= \iint_{S_{xy}} z\sqrt{1 + \left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2}\,dA$$

$$= \iint_{S_{xy}} dA = \text{Area of } S_{xy}.$$

Since S_{xy} is the region inside the ellipse $2x^2 + 4(y-1/2)^2 = 1$,

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\,dS = \pi\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\pi}{2\sqrt{2}}.$$

13. On S_1, $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$, and therefore



$$\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iint_{S_1} (x - z^2)\,dS = \iint_{S_1} -z^2\,dA$$

$$= -2\int_0^2 \int_0^{4-y^2} z^2\,dz\,dy = -2\int_0^2 \left\{\frac{z^3}{3}\right\}_0^{4-y^2} dy$$

$$= -\frac{2}{3}\int_0^2 (64 - 48y^2 + 12y^4 - y^6)\,dy$$

$$= -\frac{2}{3}\left\{64y - 16y^3 + \frac{12y^5}{5} - \frac{y^7}{7}\right\}_0^2 = -\frac{4096}{105}.$$

On S_2, $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, and therefore

$$\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iint_{S_2} (z^2 - x)\,dS = \iint_{S_2} z^2\,dS - \iint_{S_2} x\,dS = -\iint_{S_1} z^2\,dS - \iint_{S_1} 3\,dS$$

$$= \frac{4096}{105} - 6\int_0^2 \int_0^{4-y^2} dz\,dy = \frac{4096}{105} - 6\int_0^2 (4 - y^2)\,dy = \frac{4096}{105} - 6\left\{4y - \frac{y^3}{3}\right\}_0^2 = \frac{4096}{105} - 32.$$

On S_3, $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, and therefore $\displaystyle\iint_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iint_{S_3} -3z\,dS = 0.$

On S_4, $\hat{\mathbf{n}} = \dfrac{\nabla(y^2 + z - 4)}{|\nabla(y^2 + z - 4)|} = \dfrac{(0, 2y, 1)}{\sqrt{4y^2 + 1}}$, and therefore

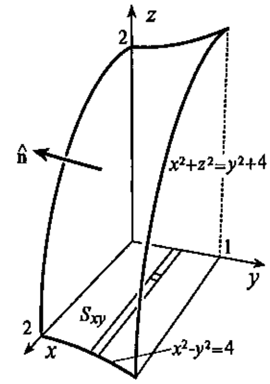
$$\iint_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iint_{S_4} \frac{(-2xy^2 + 3z)}{\sqrt{1 + 4y^2}}dS = \iint_{S_{xy}} \frac{(-2xy^2 + 12 - 3y^2)}{\sqrt{1 + 4y^2}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$= \int_0^3 \int_{-2}^2 (12 - 2xy^2 - 3y^2)\,dy\,dx = \int_0^3 \left\{12y - \frac{2xy^3}{3} - y^3\right\}_{-2}^2 dx$$

$$= \int_0^3 \left(24 - \frac{16x}{3} - 8 + 24 - \frac{16x}{3} - 8\right) dx = \frac{32}{3}\int_0^3 (3 - x)\,dx = \frac{32}{3}\left\{3x - \frac{x^2}{2}\right\}_0^3 = 48.$$

Thus, $\displaystyle\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = -\frac{4096}{105} + \frac{4096}{105} - 32 + 48 = 16.$

14. The surface projects one-to-one onto the area S_{xy} in the xy-plane shown. Since



$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= \sqrt{1 + \frac{x^2}{4+y^2-x^2} + \frac{y^2}{4+y^2-x^2}}\, dA$$
$$= \sqrt{\frac{4+2y^2}{4+y^2-x^2}}\, dA,$$

and $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+z^2-y^2-4)}{|\nabla(x^2+z^2-y^2-4)|} = \dfrac{(2x,-2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x,-y,z)}{\sqrt{x^2+y^2+4+y^2-x^2}} = \dfrac{(x,-y,z)}{\sqrt{4+2y^2}}$,
it follows that

$$\iint_S (x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + xz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_{xy}} \frac{x^3 - xy^2 + xz^2}{\sqrt{4+2y^2}} \sqrt{\frac{4+2y^2}{4+y^2-x^2}}\, dA$$
$$= \iint_{S_{xy}} \frac{4x}{\sqrt{4+y^2-x^2}}\, dA = 4\int_0^1 \int_0^{\sqrt{4+y^2}} \frac{x}{\sqrt{4+y^2-x^2}}\, dx\, dy$$
$$= 4\int_0^1 \left\{-\sqrt{4+y^2-x^2}\right\}_0^{\sqrt{4+y^2}} dy = 4\int_0^1 \sqrt{4+y^2}\, dy.$$

If we set $y = 2\tan\theta$ and $dy = 2\sec^2\theta\, d\theta$, then

$$\iint_S (x^2\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + xz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 4\int_0^{\overline{\theta}} 2\sec\theta\, 2\sec^2\theta\, d\theta \quad (\overline{\theta} = \mathrm{Tan}^{-1}(1/2))$$
$$= 8\Big\{\sec\theta\, \tan\theta + \ln|\sec\theta + \tan\theta|\Big\}_0^{\overline{\theta}} \quad \text{(see Example 8.9)}$$
$$= 2\sqrt{5} + 8\ln[(\sqrt{5}+1)/2].$$

15. If S_{yz} is the projection of the surface in the yz-plane, then $dS = \sqrt{1+z^2+y^2}\,dA$ and $\hat{\mathbf{n}} = \dfrac{\nabla(x-yz)}{|\nabla(x-yz)|} = \dfrac{(1,-z,-y)}{\sqrt{1+y^2+z^2}}$. Thus,

$$\iint_S (x^2\hat{\mathbf{i}} + yz\hat{\mathbf{j}} - x\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_S \frac{(x^2 - yz^2 + xy)}{\sqrt{1+y^2+z^2}} dS = \iint_{S_{yz}} \frac{(y^2z^2 - yz^2 + y^2z)}{\sqrt{1+y^2+z^2}} \sqrt{1+y^2+z^2}\, dA$$
$$= \int_0^1 \int_0^{\sqrt{1-y^2}} (y^2z^2 - yz^2 + y^2z)\, dz\, dy = \int_0^1 \left\{\frac{y^2z^3}{3} - \frac{yz^3}{3} + \frac{y^2z^2}{2}\right\}_0^{\sqrt{1-y^2}} dy$$
$$= \frac{1}{6}\int_0^1 [2y^2(1-y^2)^{3/2} - 2y(1-y^2)^{3/2} + 3y^2(1-y^2)]\, dy.$$

If we set $y = \sin\theta$ and $dy = \cos\theta\, d\theta$ in the first term,

$$\iint_S (x^2\hat{\mathbf{i}} + yz\hat{\mathbf{j}} - x\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \frac{1}{6}\int_0^{\pi/2} 2\sin^2\theta\cos^3\theta\cos\theta\, d\theta + \frac{1}{6}\left\{\frac{2}{5}(1-y^2)^{5/2} + y^3 - \frac{3y^5}{5}\right\}_0^1$$
$$= \frac{1}{3}\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{1}{24}\int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2} + \sin^2 2\theta\cos 2\theta\right) d\theta$$
$$= \frac{1}{24}\left\{\frac{\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{6}\sin^3 2\theta\right\}_0^{\pi/2} = \frac{\pi}{96}.$$

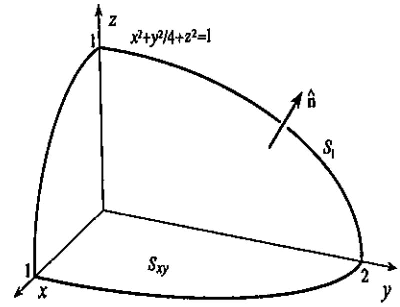
16. We divide S into two parts

$S_1 : \ x^2 + y^2/4 + z^2 = 1, \ z \ge 0,$

$S_2 : \ x^2 + y^2/4 + z^2 = 1, \ z \le 0,$

both of which project onto the ellipse

$$S_{xy} : \ x^2 + \frac{y^2}{4} \le 1, \ z = 0.$$



We have shown one-quarter of S_1 and S_{xy} in the figure. On S_1, $\partial z/\partial x = -x/z$ and $\partial z/\partial y = -y/(4z)$, so that

$$dS = \sqrt{1 + \left(\frac{x^2}{z^2}\right) + \left(\frac{y^2}{16z^2}\right)}\, dA = \frac{\sqrt{16z^2 + 16x^2 + y^2}}{4z}\, dA.$$

Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2 + y^2/4 + z^2 - 1)}{|\nabla(x^2 + y^2/4 + z^2 - 1)|} = \dfrac{(2x, y/2, 2z)}{\sqrt{4x^2 + y^2/4 + 4z^2}} = \dfrac{(4x, y, 4z)}{\sqrt{16x^2 + y^2 + 16z^2}},$

$$\iint_{S_1} (yx\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_{xy}} \frac{(4x^2y + y^3 + 4yz^2)}{\sqrt{16x^2 + y^2 + 16z^2}} \frac{\sqrt{16z^2 + 16x^2 + y^2}}{4z}\, dA$$

$$= \iint_{S_{xy}} \frac{y}{z}\, dA = \iint_{S_{xy}} \frac{y}{\sqrt{1 - x^2 - y^2/4}}\, dA.$$

On S_2, $dS = \dfrac{\sqrt{16z^2 + 16x^2 + y^2}}{-4z}\, dA$ and $\hat{\mathbf{n}} = \dfrac{(4x, y, 4z)}{\sqrt{16x^2 + y^2 + 16z^2}}$, and therefore

$$\iint_{S_2} (yx\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_{xy}} \frac{y}{-z}\, dA = \iint_{S_{xy}} \frac{y}{\sqrt{1 - x^2 - y^2/4}}\, dA.$$

Hence, $\displaystyle\iint_S (yx\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 2\iint_{S_{xy}} \frac{y}{\sqrt{1 - x^2 - y^2/4}}\, dA = 0$, because the integrand is an odd function of y and S_{xy} is symmetric about the x-axis.

17. If the surface projects one-to-one onto area S_{xy} in the xy-plane, we can take its equation in the form $z = f(x, y)$. Then

$$\hat{\mathbf{n}} = \pm \frac{\left(-\dfrac{\partial z}{\partial x}, -\dfrac{\partial z}{\partial y}, 1\right)}{\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}} \quad \text{and} \quad dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

Thus, $\displaystyle\iint_S (P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \pm \iint_{S_{xy}} \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA.$

Corresponding formulas when S projects onto S_{yz} and S_{xz} are

$$\pm \iint_{S_{yz}} \left(P - Q\frac{\partial x}{\partial y} - R\frac{\partial x}{\partial z}\right) dA \quad \text{and} \quad \pm \iint_{S_{xz}} \left(Q - P\frac{\partial y}{\partial x} - R\frac{\partial y}{\partial z}\right) dA.$$

18. (a) Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z-9)}{|\nabla(x^2+y^2+z-9)|}$

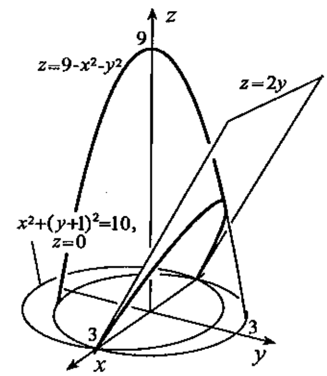
$$= \frac{(2x, 2y, 1)}{\sqrt{1+4x^2+4y^2}},$$

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS$$

$$= \iint_{S_{xy}} \frac{(2yx - 2xy + z)}{\sqrt{1+4x^2+4y^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_{S_{xy}} \frac{z}{\sqrt{1+4x^2+4y^2}} \sqrt{1+(-2x)^2+(-2y)^2}\, dA$$

$$= \iint_{S_{xy}} (9 - x^2 - y^2)\, dA.$$



If we set up polar coordinates with the pole at $(0, -1)$ and polar axis parallel to the positive x-axis, then $x = r\cos\theta$ and $y = -1 + r\sin\theta$, and

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \int_{-\pi}^{\pi}\int_0^{\sqrt{10}} [9 - r^2\cos^2\theta - (-1 + r\sin\theta)^2]\, r\, dr\, d\theta$$

$$= \int_{-\pi}^{\pi}\int_0^{\sqrt{10}} (8 + 2r\sin\theta - r^2)\, r\, dr\, d\theta = \int_{-\pi}^{\pi} \left\{4r^2 + \frac{2r^3\sin\theta}{3} - \frac{r^4}{4}\right\}_0^{\sqrt{10}} d\theta$$

$$= \int_{-\pi}^{\pi} \left(15 + \frac{20\sqrt{10}}{3}\sin\theta\right) d\theta = \left\{15\theta - \frac{20\sqrt{10}}{3}\cos\theta\right\}_{-\pi}^{\pi} = 30\pi.$$

(b) With $\hat{\mathbf{n}} = \dfrac{\nabla(z-2y)}{|\nabla(z-2y)|} = \dfrac{(0,-2,1)}{\sqrt{5}}$,

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_{xy}} \frac{(2x+z)}{\sqrt{5}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \frac{1}{\sqrt{5}} \iint_{S_{xy}} (2x + 2y)\sqrt{1+(2)^2}\, dA = 2\int_{-\pi}^{\pi}\int_0^{\sqrt{10}} (r\cos\theta - 1 + r\sin\theta)\, r\, dr\, d\theta$$

$$= 2\int_{-\pi}^{\pi} \left\{\frac{r^3}{3}(\cos\theta + \sin\theta) - \frac{r^2}{2}\right\}_0^{\sqrt{10}} d\theta = 2\int_{-\pi}^{\pi} \left[\frac{10\sqrt{10}}{3}(\cos\theta + \sin\theta) - 5\right] d\theta$$

$$= 2\left\{\frac{10\sqrt{10}}{3}(\sin\theta - \cos\theta) - 5\theta\right\}_{-\pi}^{\pi} = -20\pi.$$

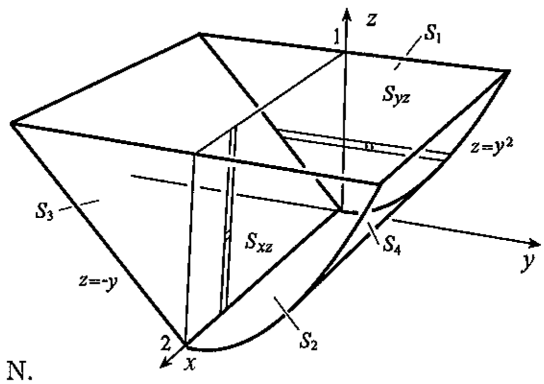
19. Normals to the surface are $\hat{\mathbf{n}} = \pm\nabla G/|\nabla G|$, and $dS = \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\, dA$. Since

$$\frac{\partial z}{\partial x} = -\frac{G_x}{G_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{G_y}{G_z},$$

it follows that $dS = \sqrt{1 + \left(-\dfrac{G_x}{G_z}\right)^2 + \left(-\dfrac{G_y}{G_z}\right)^2}\, dA = \dfrac{|\nabla G|}{|G_z|} dA.$

Thus, $\displaystyle\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \pm \iint_{S_{xy}} \mathbf{F} \cdot \left(\frac{\nabla G}{|\nabla G|}\right) \frac{|\nabla G|}{|G_z|} dA = \pm \iint_{S_{xy}} \frac{\mathbf{F} \cdot \nabla G}{|G_z|} dA.$

20. (a) The force on the end $S_1: x=0$ is in the negative x-direction and has magnitude



$$F_1 = \int_0^1 \int_{-z}^{\sqrt{z}} 9810(1-z)\,dy\,dz$$
$$= 9810 \int_0^1 \Big\{y(1-z)\Big\}_{-z}^{\sqrt{z}} dz$$
$$= 9810 \int_0^1 (\sqrt{z} - z^{3/2} + z - z^2)\,dz$$
$$= 9810 \left\{\frac{2z^{3/2}}{3} - \frac{2z^{5/2}}{5} + \frac{z^2}{2} - \frac{z^3}{3}\right\}_0^1 = 4251 \text{ N.}$$

Thus, $\mathbf{F}_1 = -4251\hat{\mathbf{i}}$ N. The force on $S_2: x=2$ is $\mathbf{F}_2 = 4251\hat{\mathbf{i}}$ N. Forces on all parts of $S_3: y=-z$ are in the same direction, namely $-\hat{\mathbf{j}} - \hat{\mathbf{k}}$. The magnitude of the force is

$$F_3 = \iint_{S_3} P\,dS = \iint_{S_3} 9810(1-z)\,dS = 9810 \iint_{S_{xz}} (1-z)\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$= 9810 \iint_{S_{xz}} (1-z)\sqrt{1+1}\,dA = 9810\sqrt{2}\int_0^2\int_0^1 (1-z)\,dz\,dx$$
$$= 9810\sqrt{2}\int_0^2 \left\{z - \frac{z^2}{2}\right\}_0^1 dx = 4905\sqrt{2}\Big\{x\Big\}_0^2 = 9810\sqrt{2} \text{ N.}$$

Thus, $\mathbf{F}_3 = 9810\sqrt{2}\left(\dfrac{-\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{2}}\right) = -9810(\hat{\mathbf{j}} + \hat{\mathbf{k}})$ N. The force on an element dS on $S_4: z=y^2$ points in the direction normal to S_4. At a point (x,y,z), the unit downward normal is $\hat{\mathbf{n}} = (0, 2y, -1)/\sqrt{1+4y^2}$. The magnitude of the force on dS is $P\,dS = 9810(1-z)\,dS$. The force on dS at (x,y,z) is therefore $\dfrac{9810(1-z)dS(0,2y,-1)}{\sqrt{1+4y^2}}$. The y-component of the total force on S_4 is

$$F_{4_y} = \iint_{S_4} \frac{9810(1-z)(2y)}{\sqrt{1+4y^2}}dS = 19\,620 \iint_{S_{xz}} \frac{y(1-z)}{\sqrt{1+4y^2}}\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$= 19\,620 \iint_{S_{xz}} \frac{y(1-z)}{\sqrt{1+4y^2}}\sqrt{1 + \left(\frac{1}{2\sqrt{z}}\right)^2}\,dA = 19\,620 \int_0^2\int_0^1 \left(\frac{1-z}{2}\right) dz\,dx$$
$$= 9810 \int_0^2 \left\{z - \frac{z^2}{2}\right\}_0^1 dx = 4905\Big\{x\Big\}_0^2 = 9810 \text{ N.}$$

The z-component of the total force on S_4 is

$$F_{4_z} = \iint_{S_4} \frac{9810(1-z)(-1)}{\sqrt{1+4y^2}}dS = 9810 \iint_{S_{xz}} \frac{(z-1)}{\sqrt{1+4y^2}}\sqrt{1 + \left(\frac{1}{2\sqrt{z}}\right)^2}\,dA$$
$$= 9810 \int_0^2\int_0^1 \left(\frac{z-1}{2\sqrt{z}}\right) dz\,dx = 4905 \int_0^2 \left\{\frac{2z^{3/2}}{3} - 2\sqrt{z}\right\}_0^1 dx = -6540\Big\{x\Big\}_0^2 = -13080 \text{ N.}$$

Thus, $\mathbf{F}_4 = 9810\hat{\mathbf{j}} - 13080\hat{\mathbf{k}}$ N.

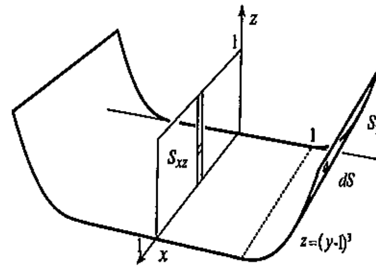
(b) The sum of the four forces is $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = -22\,890\,\hat{\mathbf{k}}$ N. The magnitude of the weight of the water in the trough is

$$\int_0^1 \int_{-z}^{\sqrt{z}} \int_0^2 9810\,dx\,dy\,dz = 9810 \int_0^1 \int_{-z}^{\sqrt{z}} \Big\{x\Big\}_0^2 dy\,dx = 19\,620 \int_0^1 \Big\{y\Big\}_{-z}^{\sqrt{z}} dz$$
$$= 19\,620 \int_0^1 (\sqrt{z} + z)\,dz = 19\,620 \left\{\frac{2z^{3/2}}{3} + \frac{z^2}{2}\right\}_0^1 = 22\,890 \text{ N.}$$

21. (a) The force on the bottom of the channel is equal to the weight of the water above it, namely, $\mathbf{F}_1 = -9810(2)(1)(1)\hat{\mathbf{k}} = -19\,620\hat{\mathbf{k}}$ N.

(b) The force on an elemental dS on $S_1 : z = (y-1)^3$ points in the direction normal to S_1. At a point (x, y, z), the unit downward normal to S_1 is $\hat{\mathbf{n}} = (0, 3(y-1)^2, -1)/\sqrt{1+9(y-1)^4}$. The magnitude of the force on dS is $P\,dS = 9810(1-z)\,dS$. The force on dS at (x, y, z) is therefore

$$\frac{9810(1-z)dS(0, 3(y-1)^2, -1)}{\sqrt{1+9(y-1)^4}}.$$



The y-component of the total force on S_1 is

$$\begin{aligned}
F_{2_y} &= \iint_{S_1} \frac{9810(1-z)(3)(y-1)^2}{\sqrt{1+9(y-1)^4}}dS = 29\,430 \iint_{S_{xz}} \frac{(1-z)(y-1)^2}{\sqrt{1+9(y-1)^4}}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA \\
&= 29\,430 \iint_{S_{xz}} \frac{(1-z)(y-1)^2}{\sqrt{1+9(y-1)^4}}\sqrt{1+\frac{1}{9(y-1)^4}}\,dA = 9810 \iint_{S_{xz}} (1-z)\,dA \\
&= 9810 \int_0^1\int_0^1 (1-z)\,dz\,dx = 9810\int_0^1 \left\{z - \frac{z^2}{2}\right\}_0^1 dx = 4905\Big\{x\Big\}_0^1 = 4905 \text{ N.}
\end{aligned}$$

The z-component of the total force on S_1 is

$$\begin{aligned}
F_{2_z} &= \iint_{S_1} \frac{-9810(1-z)}{\sqrt{1+9(y-1)^4}}dS = -9810 \iint_{S_{xz}} \frac{1-z}{\sqrt{1+9(y-1)^4}}\sqrt{1+\frac{1}{9(y-1)^4}}\,dA \\
&= -9810 \iint_{S_{xz}} \frac{1-z}{3(y-1)^2}\,dA = -3270 \iint_{S_{xz}} \frac{1-z}{z^{2/3}}dA = -3270\int_0^1\int_0^1 \left(\frac{1}{z^{2/3}} - z^{1/3}\right)dz\,dx \\
&= -3270\int_0^1 \left\{3z^{1/3} - \frac{3z^{4/3}}{4}\right\}_0^1 dx = -3270\left(\frac{9}{4}\right)\Big\{x\Big\}_0^1 = -\frac{14\,715}{2} \text{ N.}
\end{aligned}$$

Thus, $\mathbf{F}_2 = 4905\hat{\mathbf{j}} - (14\,715/2)\hat{\mathbf{k}}$ N.

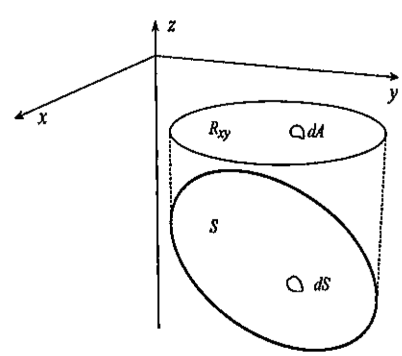
(c) By symmetry, the force on the left wall is $\mathbf{F}_3 = -4905\hat{\mathbf{j}} - (14\,715/2)\hat{\mathbf{k}}$ N.

The sum of all three forces is $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = -34\,335\hat{\mathbf{k}}$ N. The magnitude of the weight of the water in 1 metre of length is

$$\begin{aligned}
19\,620 + 2\int_0^1\int_1^2\int_{(y-1)^3}^1 9810\,dz\,dy\,dx &= 19\,620 + 19\,620\int_0^1\int_1^2 [1-(y-1)^3]\,dy\,dx \\
&= 19\,620 + 19\,620\int_0^1 \left\{y - \frac{(y-1)^4}{4}\right\}_1^2 dx = 19\,620 + 19\,620\left(\frac{3}{4}\right)\Big\{x\Big\}_0^1 = 34\,335 \text{ N.}
\end{aligned}$$

22. The magnitude of the fluid force on S is

$$\begin{aligned}
\iint_S P\,dS &= \iint_{R_{xy}} -\rho g z\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \\
&= -\rho g \iint_{R_{xy}} z\sqrt{1+\left(\frac{-A}{C}\right)^2+\left(\frac{-B}{C}\right)^2}\,dA \\
&= -\frac{\rho g\sqrt{A^2+B^2+C^2}}{|C|}\iint_{R_{xy}} \left(\frac{-D-Ax-By}{C}\right)dA \\
&= \frac{\rho g\sqrt{A^2+B^2+C^2}}{C|C|}\iint_{R_{xy}} (D+Ax+By)\,dA.
\end{aligned}$$



A unit normal vector to S is $\pm(A,B,C)/\sqrt{A^2+B^2+C^2}$, where the plus or minus is chosen depending on the sign of C and whether we consider the top or bottom of S. The force on S is

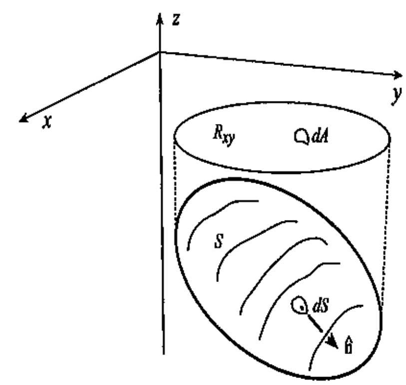
$$\left[\frac{\rho g\sqrt{A^2+B^2+C^2}}{C|C|}\iint_{R_{xy}}(D+Ax+By)\,dA\right]\frac{\pm(A,B,C)}{\sqrt{A^2+B^2+C^2}}.$$

The magnitude of the z-component of this force is $F_z=\dfrac{\rho g}{C}\displaystyle\iint_{R_{xy}}(D+Ax+By)\,dA$. On the other hand, the weight of the column of fluid above S is

$$W=\iint_{R_{xy}}\int_{(-D-Ax-By)/C}^{0}\rho g\,dz\,dA=\iint_{R_{xy}}\rho g\left(\frac{D+Ax+By}{C}\right)dA=\frac{\rho g}{C}\iint_{R_{xy}}(D+Ax+By)\,dA.$$

Consequently, the magnitude of F_z is equal to W.

23. The force due to fluid pressure on a small area dS on (the top of) S is $(P\,dS)\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit downward pointing normal to S at dS. The z-component of this force is $(P\,dS)\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}$, and the total z-component of the fluid force on S is



$$F_z=\iint_S P\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,dS=\iint_S -\rho gz\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,dS.$$

If we take the equation for S in the form $z=f(x,y)$, then $\hat{\mathbf{n}}=\dfrac{(f_x,f_y,-1)}{\sqrt{1+(f_x)^2+(f_y)^2}}$, and $dS=\sqrt{1+(f_x)^2+(f_y)^2}\,dA$. Hence,

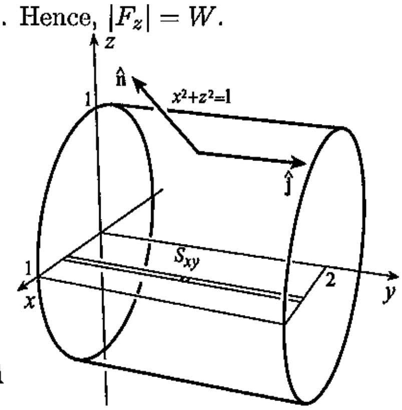
$$F_z=\iint_{R_{xy}}-\rho gf(x,y)\left[\frac{-1}{\sqrt{1+(f_x)^2+(f_y)^2}}\right]\sqrt{1+(f_x)^2+(f_y)^2}dA=\rho g\iint_{R_{xy}}f(x,y)\,dA.$$

This is a negative quantity since $f(x,y)<0$ for all (x,y) in R_{xy}. On the other hand, the weight of the column of fluid above S is

$$W=\iint_{R_{xy}}\int_{f(x,y)}^{0}\rho g\,dz\,dA=\rho g\iint_{R_{xy}}-f(x,y)\,dA=-\rho g\iint_{R_{xy}}f(x,y)\,dA.$$

This is a positive quantity since $f(x,y)<0$ for all (x,y) in R_{xy}. Hence, $|F_z|=W$.

24. The total amount of blood per second is



$$\iint_S\mathbf{F}\cdot\hat{\mathbf{n}}\,dS=\iint_S e^{-y}\,dS.$$

The same amount of diffusion occurs in each of the four octants. If S_{xy} is the projection of the first octant part of S in the xy-plane, then

$$\begin{aligned}\iint_S\mathbf{F}\cdot\hat{\mathbf{n}}\,dS&=4\iint_{S_{xy}}e^{-y}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA\\&=4\iint_{S_{xy}}e^{-y}\sqrt{1+\left(-\frac{x}{z}\right)^2}dA=4\iint_{S_{xy}}\frac{e^{-y}}{\sqrt{1-x^2}}\,dA\\&=4\int_0^1\int_0^2\frac{e^{-y}}{\sqrt{1-x^2}}\,dy\,dx=4\int_0^1\left\{\frac{-e^{-y}}{\sqrt{1-x^2}}\right\}_0^2dx=4(1-e^{-2})\int_0^1\frac{1}{\sqrt{1-x^2}}\,dx\\&=4(1-e^{-2})\left\{\mathrm{Sin}^{-1}x\right\}_0^1=2\pi(1-e^{-2}).\end{aligned}$$

25. (a) The normal to S is $\hat{\mathbf{n}} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(x,y,z)}{a}$, and when S is projected into the xz-plane,

$$dS = \sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2-z^2}}\right)^2+\left(\frac{-z}{\sqrt{a^2-x^2-z^2}}\right)^2}\,dA = \frac{a}{\sqrt{a^2-x^2-z^2}}dA.$$

Thus, absorption in time dt is

$$\begin{aligned}\iint_S \mathbf{I}\cdot\hat{\mathbf{n}}\,dS\,dt &= dt\iint_S \frac{ye^{-t}}{ay^2}dS = \frac{e^{-t}\,dt}{a}\iint_{S_{xz}}\frac{a}{a^2-x^2-z^2}dA\\ &= 4e^{-t}\,dt\int_0^{\pi/2}\int_0^{\sqrt{3}a/2}\frac{1}{a^2-r^2}r\,dr\,d\theta = 4e^{-t}\,dt\int_0^{\pi/2}\left\{-\frac{1}{2}\ln|a^2-r^2|\right\}_0^{\sqrt{3}a/2}d\theta\\ &= 4\ln 2e^{-t}\,dt\Big\{\theta\Big\}_0^{\pi/2} = 2\pi\ln 2e^{-t}\,dt.\end{aligned}$$

(b) Absorption from $t=0$ to $t=5$ is $\displaystyle\int_0^5 2\pi\ln 2e^{-t}\,dt = 2\pi\ln 2\Big\{e^{-t}\Big\}_0^5 = 2\pi\ln 2(1-e^{-5})$.

EXERCISES 14.9

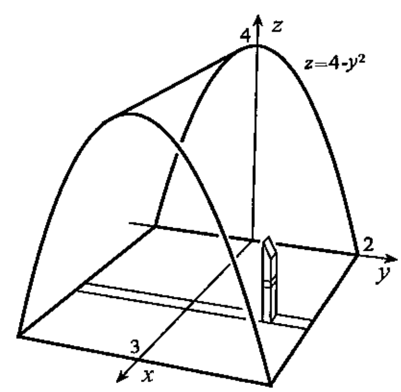
1. By the divergence theorem, $\displaystyle\oiint_S (x\hat{\mathbf{i}}+y\hat{\mathbf{j}}-2z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (1+1-2)\,dV = 0.$

2. By the divergence theorem, $\displaystyle\oiint_S (x^2\hat{\mathbf{i}}+y^2\hat{\mathbf{j}}+z^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (2x+2y+2z)\,dV$. Since the triple integral is twice the sum of the first moments of the sphere about the coordinate planes, it must have value zero.

3. By the divergence theorem, $\displaystyle\oiint_S (yz\hat{\mathbf{i}}+xz\hat{\mathbf{j}}+xy\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (0+0+0)\,dV = 0.$

4. By the divergence theorem,

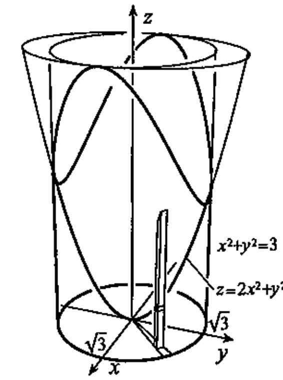
$$\begin{aligned}\oiint_S [(z^2-x)\hat{\mathbf{i}}-xy\hat{\mathbf{j}}+3z\hat{\mathbf{k}}]\cdot\hat{\mathbf{n}}\,dS &= \iiint_V (-1-x+3)\,dV\\ &= \int_0^3\int_{-2}^2\int_0^{4-y^2}(2-x)\,dz\,dy\,dx\\ &= \int_0^3\int_{-2}^2(2-x)(4-y^2)\,dy\,dx\\ &= \int_0^3\left\{(2-x)\left(4y-\frac{y^3}{3}\right)\right\}_{-2}^2dx\\ &= \frac{32}{3}\left\{2x-\frac{x^2}{2}\right\}_0^3 = 16.\end{aligned}$$



5. By the divergence theorem,

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= -\iiint_V (xy+yz+xz)\,dV = -\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^1(xy+yz+xz)\,dz\,dy\,dx\\ &= -\int_0^1\int_0^{\sqrt{1-x^2}}\left\{xyz+\frac{yz^2}{2}+\frac{xz^2}{2}\right\}_0^1dy\,dx = -\frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}}(2xy+y+x)\,dy\,dx\\ &= -\frac{1}{2}\int_0^1\left\{xy^2+\frac{y^2}{2}+xy\right\}_0^{\sqrt{1-x^2}}dx = -\frac{1}{4}\int_0^1[2x(1-x^2)+(1-x^2)+2x\sqrt{1-x^2}]\,dx\\ &= -\frac{1}{4}\left\{x+x^2-\frac{x^3}{3}-\frac{x^4}{2}-\frac{2}{3}(1-x^2)^{3/2}\right\}_0^1 = -\frac{11}{24}.\end{aligned}$$

6. By the divergence theorem,

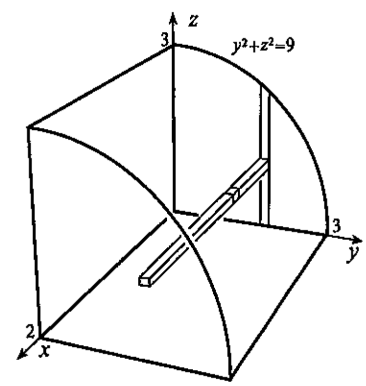


$$\oiint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (1+1+2)\,dV$$
$$= 4\int_{-\pi}^{\pi}\int_0^{\sqrt{3}}\int_0^{2r^2\cos^2\theta + r^2\sin^2\theta} r\,dz\,dr\,d\theta$$
$$= 4\int_{-\pi}^{\pi}\int_0^{\sqrt{3}} r(2r^2\cos^2\theta + r^2\sin^2\theta)\,dr\,d\theta$$
$$= 4\int_{-\pi}^{\pi}\left\{\frac{r^4}{2}\cos^2\theta + \frac{r^4}{4}\sin^2\theta\right\}_0^{\sqrt{3}} d\theta = 9\int_{-\pi}^{\pi}(2\cos^2\theta + \sin^2\theta)\,d\theta$$
$$= 9\int_{-\pi}^{\pi}\left(1 + \frac{1+\cos 2\theta}{2}\right)d\theta = 9\left\{\frac{3\theta}{2} + \frac{\sin 2\theta}{4}\right\}_{-\pi}^{\pi} = 27\pi.$$

7. By the divergence theorem, $\oiint_S (z\hat{\mathbf{i}} - x\hat{\mathbf{j}} + y\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (0+0+0)\,dV = 0.$

8. By the divergence theorem,

$$\oiint_S (2x^2y\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} + 4xz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (4xy - 2y + 8xz)\,dV$$

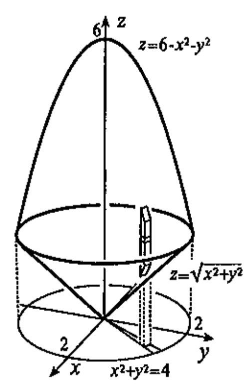


$$= 2\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^2 (2xy - y + 4xz)\,dx\,dz\,dy$$
$$= 2\int_0^3\int_0^{\sqrt{9-y^2}}\left\{x^2y - xy + 2x^2z\right\}_0^2 dz\,dy$$
$$= 2\int_0^3\int_0^{\sqrt{9-y^2}}(2y + 8z)\,dz\,dy$$
$$= 4\int_0^3\left\{yz + 2z^2\right\}_0^{\sqrt{9-y^2}} dy = 4\int_0^3\left(y\sqrt{9-y^2} + 18 - 2y^2\right)dy$$
$$= 4\left\{-\frac{1}{3}(9-y^2)^{3/2} + 18y - \frac{2y^3}{3}\right\}_0^3 = 180.$$

9. By the divergence theorem, $\oiint_S (yx\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (y + 2y + y)\,dV = 4\iiint_V y\,dV$. Since the integrand is an odd function of y, and V is symmetric about the xz-plane, the value of the triple integral is zero.

10. By the divergence theorem,

$$\oiint_S (x^3\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} - z^3\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (3x^2 + 3y^2 - 3z^2)\,dV$$

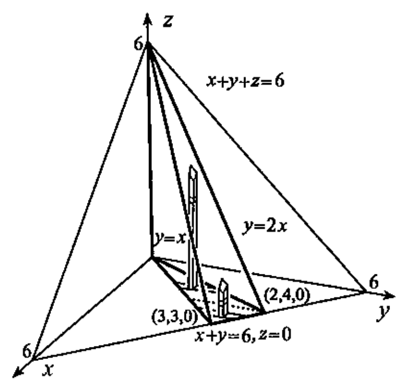


$$= 3\int_{-\pi}^{\pi}\int_0^2\int_r^{6-r^2}(r^2 - z^2)\,r\,dz\,dr\,d\theta$$
$$= 3\int_{-\pi}^{\pi}\int_0^2\left\{r^3z - \frac{rz^3}{3}\right\}_r^{6-r^2} dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^2 (r^7 - 21r^5 - 2r^4 + 126r^3 - 216r)\,dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\left\{\frac{r^8}{8} - \frac{7r^6}{2} - \frac{2r^5}{5} + \frac{63r^4}{2} - 108r^2\right\}_0^2 d\theta$$
$$= \frac{-664}{5}\left\{\theta\right\}_{-\pi}^{\pi} = -\frac{1328\pi}{5}.$$

11. By the divergence theorem, $\oiint_S (y\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + zy^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = -\iiint_V (-x + y^2)\, dV$. Since x is an odd function of x and V is symmetric about the yz-plane, this term contributes nothing to the integral. If we introduce polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$ in the xz-plane, then

$$\begin{aligned}\oiint_S (y\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + zy^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= -4\int_0^{\pi/2}\int_0^{2\sqrt{3}}\int_{\sqrt{4+r^2}}^4 y^2 r\, dy\, dr\, d\theta \\ &= -4\int_0^{\pi/2}\int_0^{2\sqrt{3}} \left\{\frac{ry^3}{3}\right\}_{\sqrt{4+r^2}}^4 dr\, d\theta \\ &= -\frac{4}{3}\int_0^{\pi/2}\int_0^{2\sqrt{3}} [64r - r(4+r^2)^{3/2}]\, dr\, d\theta \\ &= -\frac{4}{3}\int_0^{\pi/2}\left\{32r^2 - \frac{1}{5}(4+r^2)^{5/2}\right\}_0^{2\sqrt{3}} d\theta = -\frac{3712}{15}\left\{\theta\right\}_0^{\pi/2} = -\frac{1856\pi}{15}.\end{aligned}$$

12. By the divergence theorem,



$$\begin{aligned}\oiint_S (xy\hat{\mathbf{i}} + z^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= \iiint_V (y + 2z)\, dV \\ &= \int_0^2\int_x^{2x}\int_0^{6-x-y} (y+2z)\, dz\, dy\, dx \\ &\quad + \int_2^3\int_x^{6-x}\int_0^{6-x-y} (y+2z)\, dz\, dy\, dx \\ &= \int_0^2\int_x^{2x}\left\{yz + z^2\right\}_0^{6-x-y} dy\, dx \\ &\quad + \int_2^3\int_x^{6-x}\left\{yz + z^2\right\}_0^{6-x-y} dy\, dx \\ &= \int_0^2\int_x^{2x} [y(6-x-y) + (6-x-y)^2]\, dy\, dx + \int_2^3\int_x^{6-x} [y(6-x-y) + (6-x-y)^2]\, dy\, dx \\ &= \int_0^2\left\{3y^2 - \frac{xy^2}{2} - \frac{y^3}{3} - \frac{(6-x-y)^3}{3}\right\}_x^{2x} dx + \int_2^3\left\{3y^2 - \frac{xy^2}{2} - \frac{y^3}{3} - \frac{(6-x-y)^3}{3}\right\}_x^{6-x} dx \\ &= \int_0^2\left[9x^2 - \frac{23x^3}{6} - \frac{(6-3x)^3}{3} + \frac{(6-2x)^3}{3}\right] dx \\ &\quad + \int_2^3\left[3(6-x)^2 - \frac{(6-x)^3}{3} + \frac{(6-2x)^3}{3} - 18x + 3x^2 + \frac{x^3}{3}\right] dx \\ &= \left\{3x^3 - \frac{23x^4}{24} + \frac{(6-3x)^4}{36} - \frac{(6-2x)^4}{24}\right\}_0^2 \\ &\quad + \left\{-(6-x)^3 + \frac{(6-x)^4}{12} - \frac{(6-2x)^4}{24} - 9x^2 + x^3 + \frac{x^4}{12}\right\}_2^3 = \frac{57}{2}.\end{aligned}$$

13. If we let S' be that part of the xy-plane bounded by $x^2 + 4y^2 = 36$, then

$$\oiint_{S+S'} (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (1+1+1)\, dV = 3\iiint_V dV.$$

Therefore, $\oiint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 3\iiint_V dV - \iint_{S'} (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS$.

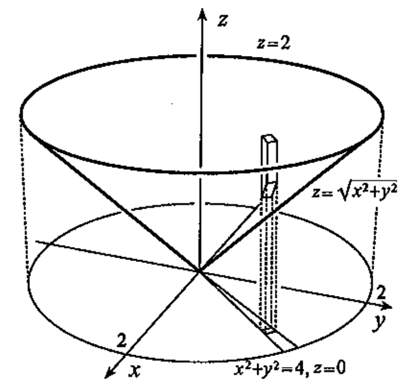
The volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $4\pi abc/3$ (see Exercise 27 in Section 13.9). Since $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ on S', $\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 3\left(\frac{2\pi}{3}\right)(6)(3)(2) - \iint_{S'} -z\, dS = 72\pi$.

14. If we create a closed surface by including with S the surface S' : $z = 2,\ x^2 + y^2 \le 4$, then

$$\iint_S (xy\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS + \iint_{S'} (xy\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = -\iiint_V (y - z + x^2)\,dV$$

provided $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ on S'. Now,

$$\iiint_V (y - z + x^2)\,dV = \int_{-\pi}^{\pi}\int_0^2\int_r^2 (-z + r^2\cos^2\theta)\,r\,dz\,dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^2 r\left\{-\frac{z^2}{2} + zr^2\cos^2\theta\right\}_r^2 dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\int_0^2 \left(-2r + 2r^3\cos^2\theta + \frac{r^3}{2} - r^4\cos^2\theta\right) dr\,d\theta$$
$$= \int_{-\pi}^{\pi}\left\{-r^2 + \frac{r^4\cos^2\theta}{2} + \frac{r^4}{8} - \frac{r^5\cos^2\theta}{5}\right\}_0^2 d\theta$$
$$= \int_{-\pi}^{\pi}\left(-4 + 8\cos^2\theta + 2 - \frac{32}{5}\cos^2\theta\right) d\theta$$
$$= \int_{-\pi}^{\pi}\left[-2 + \frac{4(1+\cos 2\theta)}{5}\right] d\theta = \left\{-\frac{6\theta}{5} + \frac{2\sin 2\theta}{5}\right\}_{-\pi}^{\pi} = -\frac{12\pi}{5}.$$



We now calculate that

$$\iint_{S'} (xy\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iint_{S'} -x^2z\,dS = -2\iint_{S'} x^2\,dS = -2\iint_{S'_{xy}} x^2\,dA$$
$$= -2\int_{-\pi}^{\pi}\int_0^2 r^2\cos^2\theta\,r\,dr\,d\theta = -2\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^2 d\theta = -8\int_{-\pi}^{\pi}\cos^2\theta\,d\theta$$
$$= -8\int_{-\pi}^{\pi}\left(\frac{1+\cos 2\theta}{2}\right) d\theta = -4\left\{\theta + \frac{\sin 2\theta}{2}\right\}_{-\pi}^{\pi} = -8\pi.$$

Thus, $\displaystyle\iint_S (xy\hat{\mathbf{i}} - yz\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \frac{12\pi}{5} + 8\pi = \frac{52\pi}{5}.$

15. If S' is that part of the plane $z = 2y$ cut out by $z = 4 - x^2 - y^2$, then

$$\oiint_{S+S'} (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (-x + 1)\,dV.$$

The integral of x over V is zero because V is symmetric about the yz-plane. Therefore,

$$\iint_S (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V dV - \iint_{S'} (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS.$$

On S', $\hat{\mathbf{n}} = (0, 2, -1)/\sqrt{5}$, and if S' projects onto S_{xy} in the xy-plane, then $dS = \sqrt{1 + (2)^2}\,dA = \sqrt{5}\,dA$. Thus,

$$\iint_S (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = 2\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}}\int_{2y}^{4-x^2-y^2} dz\,dy\,dx - \iint_{S'} \frac{(-2xy - z)}{\sqrt{5}}dS$$
$$= 2\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} (4 - x^2 - y^2 - 2y)\,dy\,dx + \iint_{S_{xy}} \frac{(2xy + 2y)}{\sqrt{5}}\sqrt{5}\,dA$$
$$= 2\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} (4 - x^2 - y^2 - 2y)\,dy\,dx + 4\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} y\,dy\,dx$$
$$= 2\int_0^{\sqrt{5}}\int_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} (4 - x^2 - y^2)\,dy\,dx$$

$$= 2\int_0^{\sqrt{5}} \left\{(4-x^2)y - \frac{y^3}{3}\right\}_{-1-\sqrt{5-x^2}}^{-1+\sqrt{5-x^2}} dx = \frac{8}{3}\int_0^{\sqrt{5}} (2-x^2)\sqrt{5-x^2}\, dx.$$

If we set $x = \sqrt{5}\sin\theta$, and $dx = \sqrt{5}\cos\theta\, d\theta$, then

$$\begin{aligned}\iint_S (y^2e^z\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= \frac{8}{3}\int_0^{\pi/2} (2 - 5\sin^2\theta)\sqrt{5}\cos\theta\sqrt{5}\cos\theta\, d\theta \\ &= \frac{40}{3}\int_0^{\pi/2} \left[1 + \cos 2\theta - \frac{5}{4}\left(\frac{1-\cos 4\theta}{2}\right)\right] d\theta \\ &= \frac{40}{3}\left\{\frac{3\theta}{8} + \frac{1}{2}\sin 2\theta + \frac{5}{32}\sin 4\theta\right\}_0^{\pi/2} = \frac{5\pi}{2}.\end{aligned}$$

16. By the divergence theorem, $\frac{1}{3}\oiint_S \mathbf{r}\cdot\hat{\mathbf{n}}\, dS = \frac{1}{3}\iiint_V \nabla\cdot\mathbf{r}\, dV = \frac{1}{3}\iiint_V (1+1+1)\, dV = \iiint_V dV = V.$

17. If we set $\mathbf{F} = \hat{\mathbf{n}}$ in the divergence theorem, then $\iiint_V \nabla\cdot\hat{\mathbf{n}}\, dV = \oiint_S \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\, dS = \oiint_S dS = \text{area}(S).$

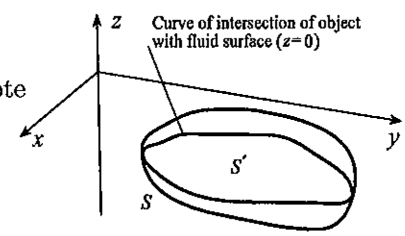
18. Using the discussion in Example 14.26, we can state that the total buoyant force must be

$$\iint_S (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS$$

where S is the submerged portion of the surface. Suppose we remove that part of the object above the fluid surface and denote by S' the remaining part of the solid in the fluid surface. Since

$$\iint_{S'} (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS = 0,$$

we may write that



$$\iint_S (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS + \iint_{S'} (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS = \iiint_V \nabla\cdot(9.81\rho z\hat{\mathbf{k}})\, dV,$$

or, $$\begin{aligned}\iint_S (9.81\rho z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{n}})\, dS = \iiint_V 9.81\rho\, dV &= 9.81\rho(\text{Volume of object below fluid surface}) \\ &= \text{Weight of fluid displaced by object.}\end{aligned}$$

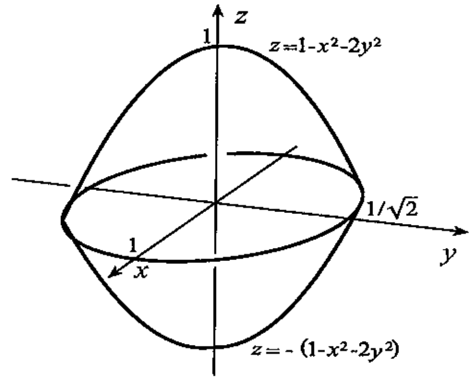
19. By the divergence theorem,

$$\begin{aligned}&\oiint_S [(x+y)\hat{\mathbf{i}} + y^3\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS \\ &= \iiint_V (1 + 3y^2 + x^2)\, dV = 8\int_0^{\pi/2}\int_0^1\int_{\sqrt{2}z}^{\sqrt{z^2+1}} (1 + 3r^2\sin^2\theta + r^2\cos^2\theta)\, r\, dr\, dz\, d\theta \\ &= 8\int_0^{\pi/2}\int_0^1 \left\{\frac{r^2}{2} + \frac{3r^4}{4}\sin^2\theta + \frac{r^4}{4}\cos^2\theta\right\}_{\sqrt{2}z}^{\sqrt{z^2+1}} dr\, d\theta \\ &= 2\int_0^{\pi/2}\int_0^1 [2(z^2+1) + 3(z^2+1)^2\sin^2\theta + (z^2+1)^2\cos^2\theta - 4z^2 - 12z^4\sin^2\theta - 4z^4\cos^2\theta]\, dz\, d\theta \\ &= 2\int_0^{\pi/2}\int_0^1 [2 - 2z^2 + (-9z^4 + 6z^2 + 3)\sin^2\theta + (-3z^4 + 2z^2 + 1)\cos^2\theta]\, dz\, d\theta \\ &= 2\int_0^{\pi/2} \left\{2z - \frac{2z^3}{3} + \left(-\frac{9z^5}{5} + 2z^3 + 3z\right)\sin^2\theta + \left(-\frac{3z^5}{5} + \frac{2z^3}{3} + z\right)\cos^2\theta\right\}_0^1 d\theta \\ &= 2\int_0^{\pi/2} \left[\frac{4}{3} + \frac{8}{5}(1 - \cos 2\theta) + \frac{8}{15}(1 + \cos 2\theta)\right] d\theta \\ &= 2\left\{\frac{4\theta}{3} + \frac{8}{5}\left(\theta - \frac{1}{2}\sin 2\theta\right) + \frac{8}{15}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right\}_0^{\pi/2} = \frac{52\pi}{15}.\end{aligned}$$

20. By the divergence theorem,

$$\oiint_S [(x+y)^2\hat{\mathbf{i}} + x^2y\hat{\mathbf{j}} - x^2z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS = -\iiint_V [2(x+y) + x^2 - x^2]\, dV = -2\iiint_V (x+y)\, dV.$$

Since the triple integral is the sum of the first moments of V about the yz- and xz-coordinate planes, its value must be zero.



21. If S' is that part of the xy-plane bounded by $x^2+y^2=1$, then

$$\oiint_{S+S'} [(y^3+x^2y)\hat{\mathbf{i}} + (x^3-xy^2)\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS = \iiint_V (2xy - 2xy + 1)\, dV = \iiint_V dV.$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$ on S',

$$\iint_S [(y^3+x^2y)\hat{\mathbf{i}} + (x^3-xy^2)\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS = \iiint_V dV - \oiint_{S'} [(y^3+x^2y)\hat{\mathbf{i}} + (x^3-xy^2)\hat{\mathbf{j}} + z\hat{\mathbf{k}}] \cdot (-\hat{\mathbf{k}})\, dS$$
$$= \frac{2\pi}{3} - \iint_{S'} -z\, dS = \frac{2\pi}{3}.$$

22. By the divergence theorem, $\oiint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{B}\, dV = \iiint_V \nabla \cdot (\nabla \times \mathbf{A})\, dV$. But according to equation 14.15, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, and hence the required result follows immediately.

23. The divergence theorem is the three-dimensional analogue of Green's theorem when it is written in the form in Exercise 13 of Section 14.6.

24. By the divergence theorem, $\oiint_S \nabla P \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \nabla P\, dV = \iiint_V \nabla^2 P\, dV$. If $\nabla^2 P = 0$ in V, then $\oiint_S \nabla P \cdot \hat{\mathbf{n}}\, dS = 0$.

25. If we set $\mathbf{F} = P\,\nabla Q$ in the divergence theorem, $\oiint_S (P\,\nabla Q) \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot (P\,\nabla Q)\, dV$. Using identity 14.11 on the right gives

$$\oiint_S (P\,\nabla Q) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (\nabla P \cdot \nabla Q + P\nabla \cdot \nabla Q)\, dV = \iiint_V (\nabla P \cdot \nabla Q + P\nabla^2 Q)\, dV.$$

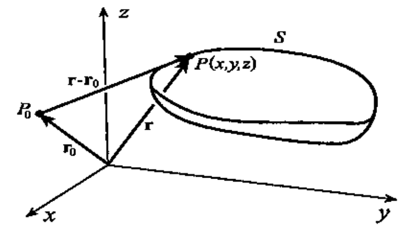
26. If we reverse the roles of P and Q in Exercise 25, $\oiint_S Q\nabla P \cdot \hat{\mathbf{n}}\, dS = \iiint_V (Q\nabla^2 P + \nabla Q \cdot \nabla P)\, dV$. When we subtract this result from that in Exercise 25, we obtain

$$\oiint_S (P\nabla Q - Q\nabla P) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (P\nabla^2 Q - Q\nabla^2 P)\, dV.$$

27. Exercises 24–26 in this section are the three-dimensional analogues of Exercises 36–38 in Section 14.6.

28. If P_0 is outside V, then $\mathbf{r}-\mathbf{r}_0 \neq \mathbf{0}$ on or inside S, and by the divergence theorem,

$\oiint_S \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = \iiint_V \nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\right)dV.$



Now,

$$\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\right) = \nabla\cdot\left(\frac{(x-x_0)\hat{\mathbf{i}}+(y-y_0)\hat{\mathbf{j}}+(z-z_0)\hat{\mathbf{k}}}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\right),$$

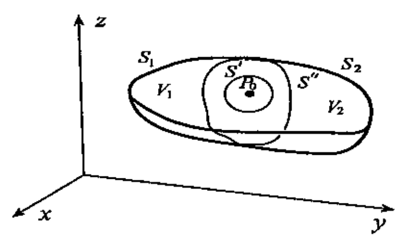
and it is a straightforward calculation to show that

$$\frac{\partial}{\partial x}\left(\frac{x-x_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\right) = \frac{(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{5/2}}.$$

With similar calculations for the partial derivatives of the y- and z-components with respect to y and z, we obtain

$$\begin{aligned}\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\right) &= \frac{1}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{5/2}}\{[(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2]\\ &\quad+[(x-x_0)^2+(z-z_0)^2-2(y-y_0)^2]+[(x-x_0)^2+(y-y_0)^2-2(z-z_0)^2]\}\\ &= 0.\end{aligned}$$

Thus, when P_0 is outside S, $\oiint_S \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = 0$.
Now suppose that P_0 is inside S. We construct a sphere S' of radius R and centre P_0 which is entirely within V. We now join S and S' by a surface S'' so that S'' divides V into two parts V_1 and V_2, with bounding surfaces S_1 and S_2. Since P_0 is outside both V_1 and V_2, we have



$\oiint_{S_1} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = 0, \quad \oiint_{S_2} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = 0.$

When these results are added together,

$$0 = \oiint_{S_1} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS + \oiint_{S_2} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS.$$

But in evaluating integrals over S_1 and S_2, integration is performed over S'' twice, once with $\hat{\mathbf{n}}$ in one direction, and once with $\hat{\mathbf{n}}$ in the opposite direction. As a result, the contributions over S'' vanish, leaving only the surface integrals over S and S', where $\hat{\mathbf{n}}$ is the outer normal on S and the inner normal on S',

$$0 = \oiint_{S} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS + \oiint_{S'} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS.$$

On S' : $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = R^2$, $\quad \hat{\mathbf{n}} = -\frac{1}{R}(x-x_0, y-y_0, z-z_0) = -\frac{\mathbf{r}-\mathbf{r}_0}{R}$, and $|\mathbf{r}-\mathbf{r}_0| = R$, so that

$$\oiint_{S'} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = \oiint_{S'} \frac{(-R\hat{\mathbf{n}})}{R^3}\cdot\hat{\mathbf{n}}\,dS = -\frac{1}{R^2}\oiint_{S'} dS = -\frac{1}{R^2}(\text{Area of } S') = -\frac{1}{R^2}(4\pi R^2) = -4\pi.$$

Consequently, when P_0 is enclosed by S, $\oiint_{S} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = -\oiint_{S'} \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\,dS = 4\pi.$

EXERCISES 14.10

1. According to Stokes's theorem, $\oint_C x^2y\,dx + y^2z\,dy + z^2x\,dz = \iint_S \nabla \times (x^2y, y^2z, z^2x) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla \times (x^2y, y^2z, z^2x) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2x \end{vmatrix} = (-y^2, -z^2, -x^2).$$

If we choose S as that part of the plane $z = 4$ inside C, then $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, and

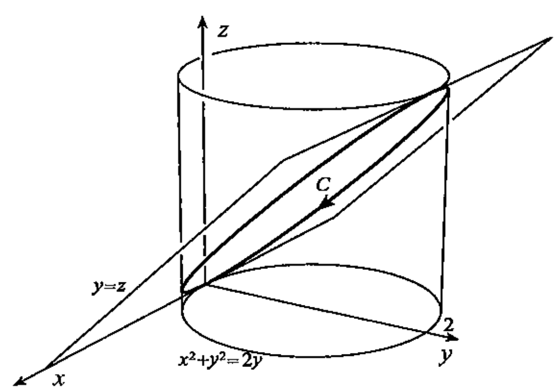
$$\begin{aligned} \oint_C x^2y\,dx + y^2z\,dy + z^2x\,dz &= \iint_S (-y^2, -z^2 - x^2) \cdot (-\hat{\mathbf{k}})\,dS = \iint_S x^2\,dS \\ &= \iint_{S_{xy}} x^2\,dA = 4\int_0^{\pi/2}\int_0^2 r^2\cos^2\theta\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^2 d\theta \\ &= 16\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 8\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = 4\pi. \end{aligned}$$

2. According to Stokes's theorem,

$$\oint_C y^2\,dx + xy\,dy + xz\,dz = \iint_S \nabla \times (y^2, xy, xz) \cdot \hat{\mathbf{n}}\,dS$$

where S is any surface with C as boundary. Now,

$$\nabla \times (y^2, xy, xz) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & xy & xz \end{vmatrix} = (0, -z, -y).$$



If we choose S as that part of $z = y$ bounded by C, then $\hat{\mathbf{n}} = (0, -1, 1)/\sqrt{2}$, and

$$\oint_C y^2\,dx + xy\,dy + xz\,dz = \iint_S (0, -z, -y) \cdot \frac{(0,-1,1)}{\sqrt{2}}dS = \frac{1}{\sqrt{2}}\iint_S (z - y)\,dS = 0.$$

3. According to Stokes's theorem, $\oint_C (xyz + 2yz)\,dx + xz\,dy + 2xy\,dz = \iint_S \nabla \times (xyz + 2yz, xz, 2xy) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla \times (xyz + 2yz, xz, 2xy) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz + 2yz & xz & 2xy \end{vmatrix} = (x, xy, -z - xz).$$

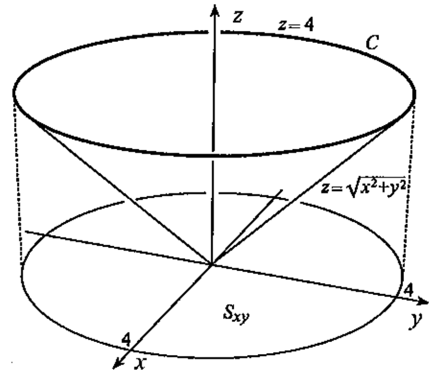
If we choose S as that part of the plane $z = 1$ inside C, then $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, and

$$\begin{aligned} \oint_C (xyz + 2yz)\,dx + xz\,dy + 2xy\,dz &= \iint_S (x, xy, -z - xz) \cdot \hat{\mathbf{k}}\,dS = \iint_S -(z + xz)\,dS \\ &= -\iint_S z\,dS - \iint_S xz\,dS = -\iint_S dS - 0 = -3\pi. \end{aligned}$$

4. By Stokes's theorem, $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary and $\mathbf{F} = (2xy+y)\hat{\mathbf{i}} + (x^2+xy-3y)\hat{\mathbf{j}} + 2xz\hat{\mathbf{k}}$. Now

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+y & x^2+xy-3y & 2xz \end{vmatrix}$$
$$= -2z\hat{\mathbf{j}} + (y-1)\hat{\mathbf{k}}.$$

If we choose S as that part of the plane $z=4$ inside C, then $\hat{\mathbf{n}} = \pm\hat{\mathbf{k}}$, depending on the direction along C, and $\nabla\times\mathbf{F}\cdot\hat{\mathbf{n}} = \pm(y-1)$. Hence



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \pm\iint_S (y-1)\,dS = \pm\left[\iint_{S_{xy}} y\,dA - \iint_{S_{xy}} dA\right] = \pm(0-16\pi) = \pm 16\pi.$$

5. According to Stokes's theorem, $\oint_C x^2\,dx + y^2\,dy + (x^2+y^2)\,dz = \iint_S \nabla\times(x^2,y^2,x^2+y^2)\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla\times(x^2,y^2,x^2+y^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & x^2+y^2 \end{vmatrix} = (2y,-2x,0).$$

If we choose S as that part of the plane $x+y+z=1$ bounded by C, then $\hat{\mathbf{n}} = (-1,-1,-1)/\sqrt{3}$, and

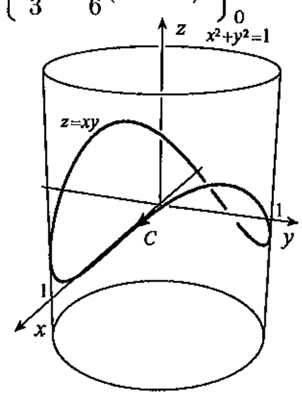
$$\oint_C x^2\,dx + y^2\,dy + (x^2+y^2)\,dz = \iint_S \frac{-2y+2x}{\sqrt{3}}dS = \frac{2}{\sqrt{3}}\iint_{S_{xy}} (x-y)\sqrt{1+(-1)^2+(-1)^2}\,dA$$
$$= 2\int_0^1\int_0^{1-x} (x-y)\,dy\,dx = 2\int_0^1 \left\{-\frac{1}{2}(x-y)^2\right\}_0^{1-x} dx$$
$$= \int_0^1 [x^2-(2x-1)^2]\,dx = \left\{\frac{x^3}{3} - \frac{1}{6}(2x-1)^3\right\}_0^1 = 0.$$

6. By Stokes's theorem,

$$\oint_C y\,dx + x\,dy + (x^2+y^2+z^2)\,dz$$
$$= \iint_S \nabla\times(y,x,x^2+y^2+z^2)\cdot\hat{\mathbf{n}}\,dS$$

where S is any surface with C as boundary. Now

$$\nabla\times(y,x,x^2+y^2+z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & x^2+y^2+z^2 \end{vmatrix}$$
$$= (2y,-2x,0).$$



If we choose S as that part of $z=xy$ inside C, then $\hat{\mathbf{n}} = (y,x,-1)/\sqrt{1+x^2+y^2}$. Thus,

$$\oint_C y\,dx + x\,dy + (x^2+y^2+z^2)\,dz = \iint_S \frac{2y^2-2x^2}{\sqrt{1+x^2+y^2}}\,dS$$
$$= 2\iint_{S_{xy}} \frac{y^2-x^2}{\sqrt{1+x^2+y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= 2\iint_{S_{xy}} \frac{y^2-x^2}{\sqrt{1+y^2+x^2}}\sqrt{1+y^2+x^2}\,dA = 2\iint_{S_{xy}} y^2\,dA - 2\iint_{S_{xy}} x^2\,dA = 0$$

since these integrals are equal.

7. According to Stokes's theorem, $\oint_C zy^2\,dx + xy\,dy + (y^2+z^2)\,dz = \iint_S \nabla \times (zy^2, xy, y^2+z^2)\cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla \times (zy^2, xy, y^2+z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ zy^2 & xy & y^2+z^2 \end{vmatrix} = (2y, y^2, y-2yz).$$

If we choose S as that part of the plane $y=3$ inside C, then $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$, and

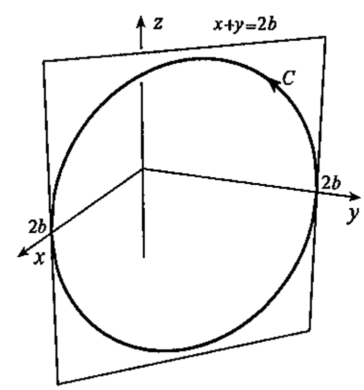
$$\begin{aligned}\oint_C zy^2\,dx + xy\,dy + (y^2+z^2)\,dz &= \iint_S (2y, y^2, y-2yz)\cdot(-\hat{\mathbf{j}})\,dS \\ &= \iint_S -y^2\,dS = -9\iint_S dS = -9(9\pi) = -81\pi.\end{aligned}$$

8. By Stokes's theorem, $\oint_C y\,dx + z\,dy + x\,dz = \iint_S \nabla\times(y,z,x)\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now

$$\nabla\times(y,z,x) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = (-1,-1,-1).$$

If we choose S as that part of $x+y=2b$ inside C, then $\hat{\mathbf{n}} = (1,1,0)/\sqrt{2}$, and,

$$\begin{aligned}&\oint_C y\,dx + z\,dy + x\,dz \\ &= \iint_S \frac{-2}{\sqrt{2}}\,dS = -\sqrt{2}\iint_S dS \\ &= -\sqrt{2}(\text{Area of } S) = -2\sqrt{2}\pi b^2.\end{aligned}$$



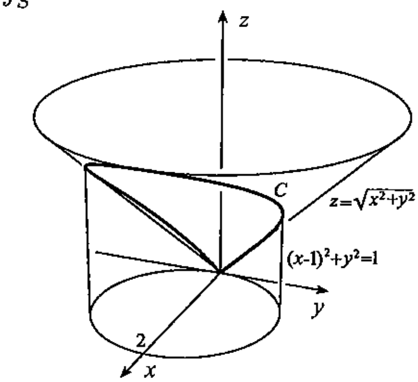
9. According to Stokes's theorem, $\oint_C y^2\,dx + (x+y)\,dy + yz\,dz = \iint_S \nabla\times(y^2, x+y, yz)\cdot\hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla\times(y^2, x+y, yz) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x+y & yz \end{vmatrix} = (z, 0, 1-2y).$$

If we choose S as that part of the plane $x+y+z=2$ bounded by C, then $\hat{\mathbf{n}} = (1,1,1)/\sqrt{3}$, and

$$\begin{aligned}\oint_C y^2\,dx + (x+y)\,dy + yz\,dz &= \iint_S (z,0,1-2y)\cdot\frac{(1,1,1)}{\sqrt{3}}dS = \frac{1}{\sqrt{3}}\iint_S (z+1-2y)\,dS \\ &= \frac{1}{\sqrt{3}}\iint_{S_{xy}} (2-x-y+1-2y)\sqrt{1+(-1)^2+(-1)^2}\,dA \\ &= \iint_{S_{xy}} 3\,dA - \iint_{S_{xy}} x\,dA - 3\iint_{S_{xy}} y\,dA = 3(2\pi) - 0 - 0 = 6\pi.\end{aligned}$$

10. By Stokes's theorem, $\oint_C (x+y)^2\,dx + (x+y)^2\,dy + yz^3\,dz = \iint_S \nabla \times \big((x+y)^2, (x+y)^2, yz^3\big) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now



$$\nabla \times \big((x+y)^2, (x+y)^2, yz^3\big) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y)^2 & (x+y)^2 & yz^3 \end{vmatrix} = (z^3, 0, 0).$$

If we choose S as that part of $z = \sqrt{x^2+y^2}$ inside C, then

$$\hat{\mathbf{n}} = \frac{\pm\nabla(x^2+y^2-z^2)}{|\nabla(x^2+y^2-z^2)|} = \frac{\pm(2x, 2y, -2z)}{\sqrt{4x^2+4y^2+4z^2}} = \frac{\pm(x, y, -z)}{\sqrt{2}z},$$

the sign depending on the direction along C. Hence,

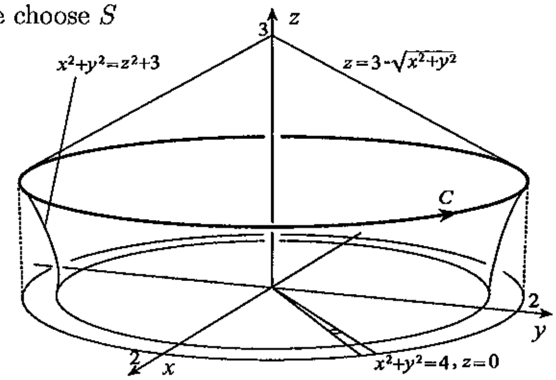
$$\begin{aligned}
\oint_C (x+y)^2\,dx + (x+y)^2\,dy + yz^3\,dz &= \pm \iint_S \frac{xz^3}{\sqrt{2}z}\,dS = \frac{\pm 1}{\sqrt{2}} \iint_S xz^2\,dS \\
&= \frac{\pm 1}{\sqrt{2}} \iint_{S_{xy}} x(x^2+y^2)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \\
&= \frac{\pm 1}{\sqrt{2}} \iint_{S_{xy}} x(x^2+y^2)\sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dA \\
&= \pm \iint_{S_{xy}} x(x^2+y^2)\,dA = \pm 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^4\cos\theta\,dr\,d\theta = \pm 2\int_0^{\pi/2}\left\{\frac{r^5}{5}\cos\theta\right\}_0^{2\cos\theta} d\theta \\
&= \frac{\pm 64}{5}\int_0^{\pi/2}\cos^6\theta\,d\theta = \frac{\pm 64}{5}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^3 d\theta \\
&= \frac{\pm 8}{5}\int_0^{\pi/2}\left[1 + 3\cos 2\theta + \frac{3}{2}(1+\cos 4\theta) + \cos 2\theta(1-\sin^2 2\theta)\right] d\theta \\
&= \frac{\pm 8}{5}\left\{\frac{5\theta}{2} + 2\sin 2\theta + \frac{3\sin 4\theta}{8} - \frac{\sin^3 2\theta}{6}\right\}_0^{\pi/2} = \pm 2\pi.
\end{aligned}$$

11. According to Stokes's theorem, $\oint_C xy\,dx - zx\,dy + yz\,dz = \iint_S \nabla \times (xy, -zx, yz) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now, $\nabla \times (xy, -zx, yz) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -zx & yz \end{vmatrix} = (z+x, 0, -z-x)$. If we choose S as that part of $z = x+y$ inside C, then $\hat{\mathbf{n}} = (-1,-1,1)/\sqrt{3}$, and

$$\begin{aligned}
\oint_C xy\,dx - zx\,dy + yz\,dz &= \iint_S (z+x, 0, -z-x) \cdot \frac{(-1,-1,1)}{\sqrt{3}} dS = -\frac{2}{\sqrt{3}} \iint_S (z+x)\,dS \\
&= -\frac{2}{\sqrt{3}} \iint_{S_{xy}} (x+y+x)\sqrt{1+(1)^2+(1)^2}\,dA = -2\int_0^1\int_0^{1-x}(2x+y)\,dy\,dx \\
&= -2\int_0^1\left\{\frac{1}{2}(2x+y)^2\right\}_0^{1-x} dx = \int_0^1 [4x^2 - (x+1)^2]\,dx \\
&= \left\{\frac{4x^3}{3} - \frac{1}{3}(x+1)^3\right\}_0^1 = -1.
\end{aligned}$$

12. The curve of intersection lies in the plane $z = 1$. If we choose S as that part of the plane interior to C, then

$$\oint_C y^3\,dx - x^3\,dy + xyz\,dz$$
$$= \iint_S \nabla \times (y^3, -x^3, xyz) \cdot \hat{\mathbf{n}}\,dS.$$



Now, $\nabla \times (y^3, -x^3, xyz) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3 & -x^3 & xyz \end{vmatrix}$
$= (xz, -yz, -3x^2 - 3y^2).$

Since $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ on S,

$$\oint_C y^3\,dx - x^3\,dy + xyz\,dz = \iint_S (-3x^2 - 3y^2)\,dS = -3\iint_{S_{xy}} (x^2 + y^2)\,dA$$
$$= -3\int_{-\pi}^{\pi}\int_0^2 r^2\,r\,dr\,d\theta = -3\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\right\}_0^2 d\theta = -12\Big\{\theta\Big\}_{-\pi}^{\pi} = -24\pi.$$

13. If S is that part of the plane $y = x$ inside C, then according to Stokes's theorem,

$$I = \oint_C z(x+y)^2\,dx + (y-x)^2\,dy + z^2\,dz = \iint_S \nabla \times \big(z(x+y)^2, (y-x)^2, z^2\big) \cdot \hat{\mathbf{n}}\,dS.$$

Now, $\nabla \times \big(z(x+y)^2, (y-x)^2, z^2\big) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z(x+y)^2 & (y-x)^2 & z^2 \end{vmatrix} = \big(0, (x+y)^2, 2(x - y - xz - yz)\big).$

Since $\hat{\mathbf{n}} = (-1, 1, 0)/\sqrt{2}$,

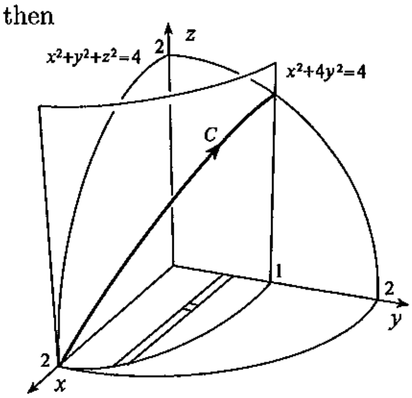
$$I = \iint_S \big(0, (x+y)^2, 2(x - y - xz - yz)\big) \cdot \frac{(-1,1,0)}{\sqrt{2}}dS = \frac{1}{\sqrt{2}}\iint_S (x+y)^2\,dS$$
$$= \frac{1}{\sqrt{2}}\iint_{S_{xz}} (x+x)^2\sqrt{1 + (1)^2}\,dA = 4\iint_{S_{xz}} x^2\,dA.$$

If we set up polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$ in the xz-plane,

$$I = 16\int_0^{\pi/2}\int_0^a r^2\cos^2\theta\,r\,dr\,d\theta = 16\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^a d\theta$$
$$= 4a^4\int_0^{\pi/2}\left(\frac{1 + \cos 2\theta}{2}\right)d\theta = 2a^4\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \pi a^4.$$

14. If we choose S as the upper part of the sphere bounded by C, then

$$\oint_C -2y^3x^2\,dx + x^3y^2\,dy + z\,dz$$
$$= \iint_S \nabla \times (-2y^3x^2, x^3y^2, z) \cdot \hat{\mathbf{n}}\,dS.$$



Now,

$$\nabla \times (-2y^3x^2, x^3y^2, z) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2y^3x^2 & x^3y^2 & z \end{vmatrix}$$
$$= (0, 0, 9x^2y^2),$$

and $\hat{\mathbf{n}} = \dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(x,y,z)}{2}$. Thus,

$$\oint_C -2y^3x^2\,dx + x^3y^2\,dy + z\,dz = \iint_S \frac{z}{2}(9x^2y^2)\,dS = \frac{9}{2}\iint_S x^2y^2\sqrt{4-x^2-y^2}\,dS$$
$$= \frac{9}{2}\iint_{S_{xy}} x^2y^2\sqrt{4-x^2-y^2}\sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}\,dA$$
$$= 9\iint_{S_{xy}} x^2y^2\,dA = 36\int_0^1\int_0^{\sqrt{4-4y^2}} x^2y^2\,dx\,dy$$
$$= 36\int_0^1 \left\{\frac{x^3y^2}{3}\right\}_0^{\sqrt{4-4y^2}} dy = 96\int_0^1 y^2(1-y^2)^{3/2}\,dy.$$

If we set $y=\sin\theta$ and $dy=\cos\theta\,d\theta$, then

$$\oint_C -2y^3x^2\,dx + x^3y^2\,dy + z\,dz = 96\int_0^{\pi/2}\sin^2\theta\,\cos^3\theta\,\cos\theta\,d\theta = 96\int_0^{\pi/2}\sin^2\theta\,\cos^4\theta\,d\theta$$
$$= 96\int_0^{\pi/2}\frac{\sin^2 2\theta}{4}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= 12\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}+\sin^2 2\theta\,\cos 2\theta\right)d\theta$$
$$= 12\left\{\frac{\theta}{2}-\frac{\sin 4\theta}{8}+\frac{\sin^3 2\theta}{6}\right\}_0^{\pi/2} = 3\pi.$$

15. (a) With parametric equations $x=\cos t$, $y=\sin t$, $z=\sqrt{3}$, $-\pi\le t\le\pi$,

$$I = \oint_C 2x^2y\,dx - yz\,dy + xz\,dz = \int_{-\pi}^{\pi} 2\cos^2 t\sin t(-\sin t\,dt) - \sqrt{3}\sin t(\cos t\,dt)$$
$$= -\int_{-\pi}^{\pi}\left(\frac{1}{2}\sin^2 2t + \sqrt{3}\sin t\cos t\right)dt = -\frac{1}{2}\int_{-\pi}^{\pi}\left[\left(\frac{1-\cos 4t}{2}\right)+2\sqrt{3}\sin t\cos t\right]dt$$
$$= -\frac{1}{2}\left\{\frac{t}{2}-\frac{1}{8}\sin 4t+\sqrt{3}\sin^2 t\right\}_{-\pi}^{\pi} = -\frac{\pi}{2}.$$

(b) Since $\nabla\times(2x^2y,-yz,xz) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2x^2y & -yz & xz\end{vmatrix} = (y,-z,-2x^2)$,

and $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-4)}{|\nabla(x^2+y^2+z^2-4)|} = \dfrac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x,y,z)}{2}$,

$$I = \iint_S (y,-z,-2x^2)\cdot\frac{(x,y,z)}{2}dS = \frac{1}{2}\iint_S (xy-yz-2x^2z)\,dS$$
$$= \frac{1}{2}\iint_S (xy - y\sqrt{4-x^2-y^2} - 2x^2\sqrt{4-x^2-y^2})\,dS$$
$$= \frac{1}{2}\iint_{S_{xy}} [xy-(y+2x^2)\sqrt{4-x^2-y^2}]\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2}\,dA$$
$$= \frac{1}{2}\iint_{S_{xy}} [xy-(y+2x^2)\sqrt{4-x^2-y^2}]\frac{2}{\sqrt{4-x^2-y^2}}\,dA$$
$$= \iint_{S_{xy}}\left(\frac{xy}{\sqrt{4-x^2-y^2}} - y - 2x^2\right)dA.$$

Integrals of the first two terms give zero, and therefore

$$I = -2\iint_{S_{xy}} x^2\,dA = -8\int_0^{\pi/2}\int_0^1 r^2\cos^2\theta\, r\,dr\,d\theta = -8\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^1 d\theta$$
$$= -2\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = -\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = -\frac{\pi}{2}.$$

(c) Since $\hat{\mathbf{n}} = \dfrac{\nabla(z-\sqrt{3}(x^2+y^2))}{|\nabla(z-\sqrt{3}(x^2+y^2))|} = \dfrac{(-2\sqrt{3}x, -2\sqrt{3}y, 1)}{\sqrt{1+12x^2+12y^2}}$,

$$I = \iint_S (y,-z,-2x^2)\cdot\frac{(-2\sqrt{3}x,-2\sqrt{3}y,1)}{\sqrt{1+12x^2+12y^2}}dS = \iint_S \frac{-2\sqrt{3}xy+2\sqrt{3}yz-2x^2}{\sqrt{1+12x^2+12y^2}}dS$$
$$= \iint_{S_{xy}} \frac{2\sqrt{3}y(\sqrt{3}x^2+\sqrt{3}y^2)-2\sqrt{3}xy-2x^2}{\sqrt{1+12x^2+12y^2}}\sqrt{1+(2\sqrt{3}x)^2+(2\sqrt{3}y)^2}\,dA$$
$$= \iint_{S_{xy}} [2\sqrt{3}y(\sqrt{3}x^2+\sqrt{3}y^2)-2\sqrt{3}xy-2x^2]\,dA.$$

Integrals of the first two terms give zero, leaving $I = \iint_{S_{xy}} -2x^2\,dA$, the same integral as in part (b).

(d) Since $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, $I = \iint_S (y,-z,-2x^2)\cdot\hat{\mathbf{k}}\,dS = \iint_S -2x^2\,dS = \iint_{S_{xy}} -2x^2\,dA = -\dfrac{\pi}{2}$, the same integral as in part (b).

16. Both surfaces have the curve $C: x^2+y^2=1,\ z=0$ as boundary. Consequently, by Stokes's theorem, both surface integrals are equal to $\oint_C \mathbf{F}\cdot d\mathbf{r}$ and are therefore equal to each other.

REVIEW EXERCISES

1. $\nabla f = (2xy^3 - y)\hat{\mathbf{i}} + (3x^2y^2 - x)\hat{\mathbf{j}} + \hat{\mathbf{k}}$

2. $\nabla\cdot\mathbf{F} = 3x^2y + x^2/y^2$

3. $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin(xy) & \cos(xy) & xy \end{vmatrix} = x\hat{\mathbf{i}} - y\hat{\mathbf{j}} + [-y\sin(xy) - x\cos(xy)]\hat{\mathbf{k}}$

4. $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} = (1-1)\hat{\mathbf{i}} + (1-1)\hat{\mathbf{j}} + (1-1)\hat{\mathbf{k}} = \mathbf{0}$

5. $\nabla f = \left(\dfrac{2x}{x^2+y^2+z^2}\right)\hat{\mathbf{i}} + \left(\dfrac{2y}{x^2+y^2+z^2}\right)\hat{\mathbf{j}} + \left(\dfrac{2z}{x^2+y^2+z^2}\right)\hat{\mathbf{k}}$

6. $\nabla\cdot\mathbf{F} = ye^x + ze^y + xe^z$

7. $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xyz & 0 \end{vmatrix} = -xy\hat{\mathbf{i}} + yz\hat{\mathbf{k}}$

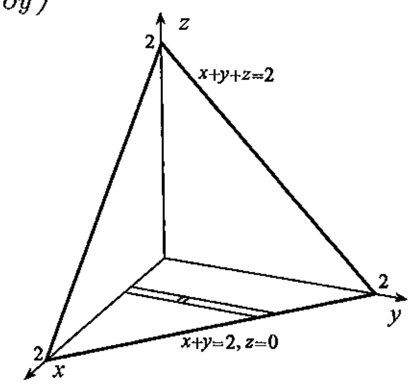
8. $\nabla f = \dfrac{1}{\sqrt{1-(x+y)^2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

9. $\nabla\cdot\mathbf{F} = -2y + 2y^2z$

10. $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \mathrm{Cot}^{-1}(xyz) & 0 & 0 \end{vmatrix} = \dfrac{-xy}{1+x^2y^2z^2}\hat{\mathbf{j}} + \dfrac{xz}{1+x^2y^2z^2}\hat{\mathbf{k}}$

11. $\displaystyle\int_C y\,ds = \int_{-1}^2 x^3\sqrt{1+(3x^2)^2}\,dx = \int_{-1}^2 x^3\sqrt{1+9x^4}\,dx = \left\{\frac{(1+9x^4)^{3/2}}{54}\right\}_{-1}^2 = \frac{145\sqrt{145}-10\sqrt{10}}{54}$

12. $$\iint_S (x^2+yz)\,dS = \iint_{S_{xy}} [x^2+y(2-x-y)]\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \iint_{S_{xy}} (x^2+2y-xy-y^2)\sqrt{1+(-1)^2+(-1)^2}\,dA$$
$$= \sqrt{3}\int_0^2\int_0^{2-x}(x^2+2y-xy-y^2)\,dy\,dx$$
$$= \sqrt{3}\int_0^2\left\{x^2y+y^2-\frac{xy^2}{2}-\frac{y^3}{3}\right\}_0^{2-x}dx$$
$$= \frac{\sqrt{3}}{6}\int_0^2[24x^2-9x^3-12x+6(2-x)^2-2(2-x)^3]\,dx$$
$$= \frac{\sqrt{3}}{6}\left\{8x^3-\frac{9x^4}{4}-6x^2-2(2-x)^3+\frac{(2-x)^4}{2}\right\}_0^2 = 2\sqrt{3}$$

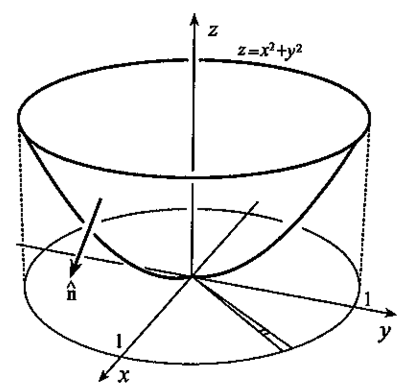


13. Since $\hat{\mathbf{n}} = \dfrac{(2x,2y,-1)}{\sqrt{1+4x^2+4y^2}}$,

$$\iint_S (x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS = \iint_{S_{xy}}\frac{2x^2+2y^2}{\sqrt{1+4x^2+4y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \iint_{S_{xy}}\frac{2x^2+2y^2}{\sqrt{1+4x^2+4y^2}}\sqrt{1+(2x)^2+(2y)^2}\,dA = 8\int_0^1\int_0^1(x^2+y^2)\,dy\,dx$$
$$= 8\int_0^1\left\{x^2y+\frac{y^3}{3}\right\}_0^1dx = \frac{8}{3}\int_0^1(3x^2+1)\,dx = \frac{8}{3}\left\{x^3+x\right\}_0^1 = \frac{16}{3}$$

14. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-z)}{|\nabla(x^2+y^2-z)|} = \dfrac{(2x,2y,-1)}{\sqrt{1+4x^2+4y^2}}$,

$$\iint_S (x\hat{\mathbf{i}}+y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS$$
$$= \iint_{S_{xy}}\frac{2x^2+2y^2}{\sqrt{1+4x^2+4y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \iint_{S_{xy}}\frac{2(x^2+y^2)}{\sqrt{1+4x^2+4y^2}}\sqrt{1+(2x)^2+(2y)^2}\,dA$$
$$= 2\iint_{S_{xy}}(x^2+y^2)\,dA = 2\int_{-\pi}^{\pi}\int_0^1 r^2\,r\,dr\,d\theta$$
$$= 2\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}\right\}_0^1d\theta = \frac{1}{2}\left\{\theta\right\}_{-\pi}^{\pi} = \pi.$$



15. Since $\nabla(x^2/2+y^2/2-z^3/3) = x\hat{\mathbf{i}}+y\hat{\mathbf{j}}-z^2\hat{\mathbf{k}}$, the line integral is independent of path in space. Because C is a closed curve, the line integral has value zero.

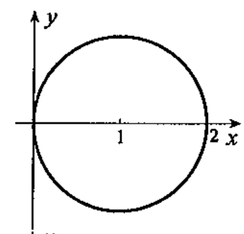
16. With parametric equations $C:\ x=-t,\ y=\sqrt{1+t^2},\ z=\sqrt{1-2t^2},\ -1/\sqrt{2}\le t\le 1/\sqrt{2}$,

$$\int_C xy\,dx+xz\,dz = \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left[-t\sqrt{1+t^2}(-dt)-t\sqrt{1-2t^2}\left(\frac{-2t}{\sqrt{1-2t^2}}\right)dt\right]$$
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}(t\sqrt{1+t^2}+2t^2)\,dt = \left\{\frac{1}{3}(1+t^2)^{3/2}+\frac{2t^3}{3}\right\}_{-1/\sqrt{2}}^{1/\sqrt{2}} = \frac{\sqrt{2}}{3}.$$

17. By Green's theorem,

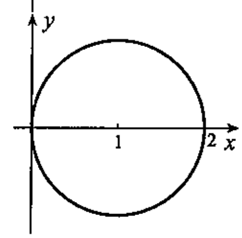
$$\oint_C 2xy^3\,dx + (3x^2y^2 + 2xy)\,dy$$
$$= \iint_R (6xy^2 + 2y - 6xy^2)\,dA = 2\iint_R y\,dA.$$

This integral has value zero since y is an odd function of y and R is symmetric about the x-axis.



18. By Green's theorem,

$$\oint_C 2xy^3\,dx + (3x^2y^2 + x^2)\,dy$$
$$= \iint_R (6xy^2 + 2x - 6xy^2)\,dA$$
$$= 2\iint_R x\,dA = 2\overline{x}(\text{Area of } R) = 2(1)\pi(1)^2 = 2\pi.$$

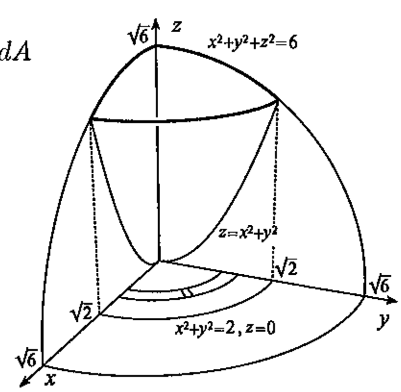


19. By the divergence theorem,

$$\oiint_S (x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS = \iiint_V (2x + 2y + 2z)\,dV = 2\int_0^1\int_z^{2-z}\int_0^1 (x + y + z)\,dx\,dy\,dz$$
$$= 2\int_0^1\int_z^{2-z}\left\{\frac{x^2}{2} + xy + xz\right\}_0^1 dy\,dz = \int_0^1\int_z^{2-z}(1 + 2y + 2z)\,dy\,dz$$
$$= \int_0^1\left\{\frac{1}{4}(1 + 2y + 2z)^2\right\}_z^{2-z} dz = \frac{1}{4}\int_0^1 [25 - (1 + 4z)^2]\,dz$$
$$= \frac{1}{4}\left\{25z - \frac{1}{12}(1 + 4z)^3\right\}_0^1 = \frac{11}{3}$$

20. We quadruple the integral over that part of the surface in the first octant.

$$\iint_S (x^2 + y^2)\,dS = 4\iint_{S_{xy}} (x^2 + y^2)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} (x^2 + y^2)\sqrt{1 + \left(-\frac{x}{z}\right)^2 + \left(-\frac{y}{z}\right)^2}\,dA$$
$$= 4\iint_{S_{xy}} (x^2 + y^2)\frac{\sqrt{6}}{z}\,dA$$
$$= 4\sqrt{6}\iint_{S_{xy}} \frac{x^2 + y^2}{\sqrt{6 - x^2 - y^2}}\,dA$$
$$= 4\sqrt{6}\int_0^{\sqrt{2}}\int_0^{\pi/2} \frac{r^2}{\sqrt{6 - r^2}}\,r\,d\theta\,dr$$
$$= 2\sqrt{6}\pi\int_0^{\sqrt{2}} \frac{r^3}{\sqrt{6 - r^2}}\,dr$$



If we set $u = 6 - r^2$ and $du = -2r\,dr$, then

$$\iint_S (x^2 + y^2)\,dS = 2\sqrt{6}\pi\int_6^4 \frac{6 - u}{\sqrt{u}}\left(-\frac{du}{2}\right) = \sqrt{6}\pi\int_6^4\left(-\frac{6}{\sqrt{u}} + \sqrt{u}\right)du$$
$$= \sqrt{6}\pi\left\{-12\sqrt{u} + \frac{2u^{3/2}}{3}\right\}_6^4 = \frac{8\pi(18 - 7\sqrt{6})}{3}.$$

21. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2+z^2-6)}{|\nabla(x^2+y^2+z^2-6)|} = \dfrac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x,y,z)}{\sqrt{6}}$,

$$\iint_S (x^2+y^2)\hat{\mathbf{i}}\cdot\hat{\mathbf{n}}\,dS = \iint_S \frac{x(x^2+y^2)}{\sqrt{6}}dS = \frac{1}{\sqrt{6}}\iint_{S_{xy}} x(x^2+y^2)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \frac{1}{\sqrt{6}}\iint_{S_{xy}} x(x^2+y^2)\sqrt{1+\left(-\frac{x}{\sqrt{6-x^2-y^2}}\right)^2+\left(-\frac{y}{\sqrt{6-x^2-y^2}}\right)^2}\,dA$$
$$= \frac{1}{\sqrt{6}}\iint_{S_{xy}} x(x^2+y^2)\frac{\sqrt{6}}{\sqrt{6-x^2-y^2}}dA = \iint_{S_{xy}}\frac{x(x^2+y^2)}{\sqrt{6-x^2-y^2}}dA = 0.$$

The integrand is an odd function of x and S_{xy} is symmetric about the y-axis.

22. By Stokes's theorem,

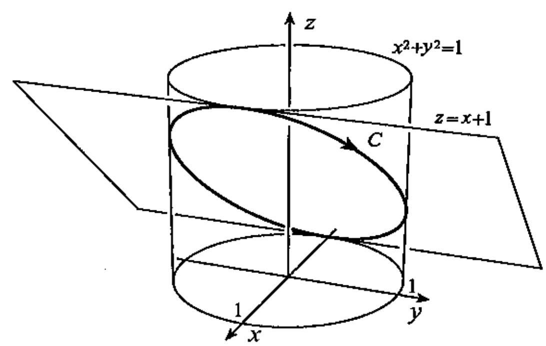
$$\oint_C (x^2\hat{\mathbf{i}}+y\hat{\mathbf{j}}-xz\hat{\mathbf{k}})\cdot d\mathbf{r} = \iint_S \nabla\times(x^2,y,-xz)\cdot\hat{\mathbf{n}}\,dS$$

where S is any surface with C as boundary. Now

$$\nabla\times(x^2,y,-xz) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2 & y & -xz\end{vmatrix} = (0,z,0).$$

If we choose S as that part of $z = x+1$ inside C, then $\hat{\mathbf{n}} = (-1,0,1)/\sqrt{2}$, and

$$\oint_C (x^2\hat{\mathbf{i}}+y\hat{\mathbf{j}}-xz\hat{\mathbf{k}})\cdot d\mathbf{r} = \iint_S 0\,dS = 0.$$



23. If S is that part of $z = x+1$ inside C, then by Stokes's theorem,

$$\oint_C (xy\hat{\mathbf{i}}+z\hat{\mathbf{j}}-x^2\hat{\mathbf{k}})\cdot d\mathbf{r} = \iint_S \nabla\times(xy,z,-x^2)\cdot\hat{\mathbf{n}}\,dS.$$

Since $\nabla\times(xy,z,-x^2) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xy & z & -x^2\end{vmatrix} = (-1,2x,-x)$, and $\hat{\mathbf{n}} = (-1,0,1)/\sqrt{2}$,

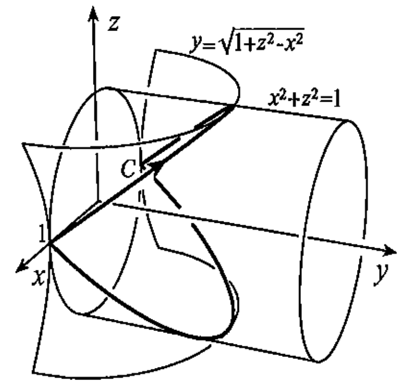
$$\oint_C (xy\hat{\mathbf{i}}+z\hat{\mathbf{j}}-x^2\hat{\mathbf{k}})\cdot d\mathbf{r} = \iint_S (-1,2x,-x)\cdot\frac{(-1,0,1)}{\sqrt{2}}dS = \frac{1}{\sqrt{2}}\iint_S (1-x)\,dS$$
$$= \frac{1}{\sqrt{2}}\iint_S (1-x)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA = \frac{1}{\sqrt{2}}\iint_{S_{xy}}(1-x)\sqrt{1+(1)^2}\,dA$$
$$= \iint_{S_{xy}} dA = \pi$$

24. By Stokes's theorem, $\oint_C y\,dx + 2x\,dy - 3z^2\,dz = \iint_S \nabla \times (y, 2x, -3z^2) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with C as boundary. Now,

$$\nabla \times (y, 2x, -3z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 2x & -3z^2 \end{vmatrix} = (0, 0, 1).$$

If we choose S as that part of $y = \sqrt{1 + z^2 - x^2}$ inside C, then

$$\hat{\mathbf{n}} = \frac{\nabla(z^2 - x^2 - y^2 + 1)}{|\nabla(z^2 - x^2 - y^2 + 1)|} = \frac{(-2x, -2y, 2z)}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$
$$= \frac{(-x, -y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$



Hence,

$$\oint_C y\,dx + 2x\,dy - 3z^2\,dz = \iint_S \frac{z}{\sqrt{x^2 + y^2 + z^2}}\,dS$$
$$= \iint_{S_{xz}} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$= \iint_{S_{xz}} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \sqrt{1 + \left(-\frac{x}{y}\right)^2 + \left(\frac{z}{y}\right)^2}\,dA$$
$$= \iint_{S_{xz}} \frac{z}{y}\,dA = \iint_{S_{xz}} \frac{z}{\sqrt{1 + z^2 - x^2}}\,dA = 0,$$

since the integrand is an odd function of z and S_{xz} is symmetric about the x-axis.

25. If S is that part of the plane $y = z$ inside C, then by Stokes's theorem,

$$I = \oint_C (xy + 4x^3y^2)\,dx + (z + 2x^4y)\,dy + (z^5 + x^2z^2)\,dz = \iint_S \nabla \times (xy + 4x^3y^2, z + 2x^4y, z^5 + x^2z^2) \cdot \hat{\mathbf{n}}\,dS.$$

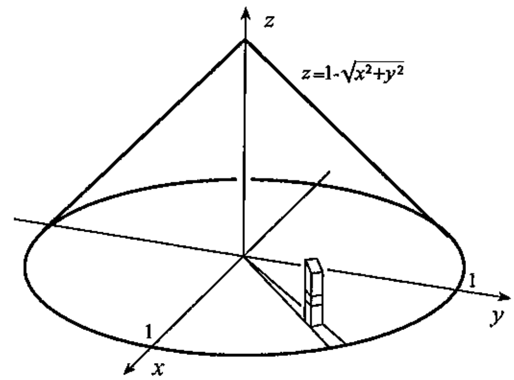
Since $\nabla \times (xy + 4x^3y, z + 2x^4y, z^5 + x^2z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy + 4x^3y^2 & z + 2x^4y & z^5 + x^2z^2 \end{vmatrix} = (-1, -2xz^2, -x),$

and $\hat{\mathbf{n}} = (0, -1, 1)/\sqrt{2}$,

$$I = \iint_S (-1, -2xz^2, -x) \cdot \frac{(0, -1, 1)}{\sqrt{2}}\,dS = \frac{1}{\sqrt{2}} \iint_S (2xz^2 - x)\,dS$$
$$= \frac{1}{\sqrt{2}} \iint_{S_{xy}} (2xz^2 - x) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \frac{1}{\sqrt{2}} \iint_{S_{xy}} (2xy^2 - x)\sqrt{1 + (1)^2}\,dA$$
$$= \iint_{S_{xy}} (2xy^2 - x)\,dA = 0$$

(since $2xy^2 - x$ is an odd function of x and S_{xy} is symmetric about the y-axis).

26. If S' is that part of the xy-plane bounded by $x^2+y^2=1$, $z=0$, then
$$\iint_{S'}(x^2yz\hat{\mathbf{i}}-x^2yz\hat{\mathbf{j}}-xyz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS=0.$$
By the divergence theorem,
$$\oiint_{S+S'}(x^2yz\hat{\mathbf{i}}-x^2yz\hat{\mathbf{j}}-xyz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS$$
$$=\iiint_V(2xyz-x^2z-2xyz)\,dV$$

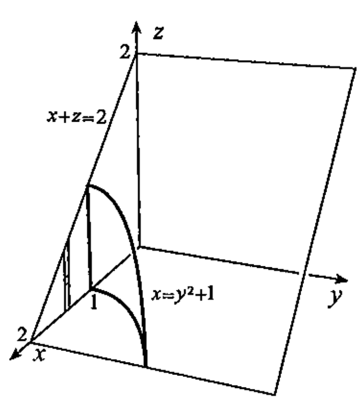


Thus,

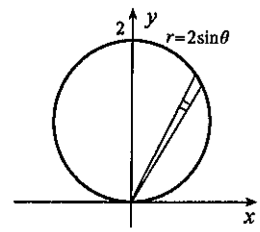
$$\begin{aligned}\iint_S(x^2yz\hat{\mathbf{i}}-x^2yz\hat{\mathbf{j}}-xyz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS&=-\iiint_V x^2z\,dV-\iint_{S'}(x^2yz\hat{\mathbf{i}}-x^2yz\hat{\mathbf{j}}-xyz^2\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}\,dS\\&=-\int_{-\pi}^{\pi}\int_0^1\int_0^{1-r}zr^2\cos^2\theta\,r\,dz\,dr\,d\theta=-\int_{-\pi}^{\pi}\int_0^1\left\{\frac{z^2r^3\cos^2\theta}{2}\right\}_0^{1-r}dr\,d\theta\\&=-\frac{1}{2}\int_{-\pi}^{\pi}\int_0^1(r^3-2r^4+r^5)\cos^2\theta\,dr\,d\theta\\&=-\frac{1}{2}\int_{-\pi}^{\pi}\left\{\frac{r^4}{4}-\frac{2r^5}{5}+\frac{r^6}{6}\right\}_0^1\cos^2\theta\,d\theta\\&=\frac{-1}{120}\int_{-\pi}^{\pi}\left(\frac{1+\cos2\theta}{2}\right)d\theta=\frac{-1}{240}\left\{\theta+\frac{\sin2\theta}{2}\right\}_{-\pi}^{\pi}=\frac{-\pi}{120}.\end{aligned}$$

27. $$\iint_S dS=\iint_{S_{xy}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA=\iint_{S_{xy}}\sqrt{1+(2x)^2+(-2y)^2}\,dA$$
$$=4\int_0^{\pi/2}\int_0^2\sqrt{1+4r^2}\,r\,dr\,d\theta=4\int_0^{\pi/2}\left\{\frac{1}{12}(1+4r^2)^{3/2}\right\}_0^2d\theta$$
$$=\frac{1}{3}(17\sqrt{17}-1)\Big\{\theta\Big\}_0^{\pi/2}=\frac{(17\sqrt{17}-1)\pi}{6}$$

28. $$\iint_S y\,dS=\iint_{S_{xz}}y\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dA$$
$$=\iint_{S_{xz}}y\sqrt{1+[1/(2y)]^2}\,dA=\frac{1}{2}\iint_{S_{xz}}\sqrt{4y^2+1}\,dA$$
$$=\frac{1}{2}\iint_{S_{xz}}\sqrt{4(x-1)+1}\,dA=\frac{1}{2}\iint_{S_{xz}}\sqrt{4x-3}\,dA$$
$$=\frac{1}{2}\int_1^2\int_0^{2-x}\sqrt{4x-3}\,dz\,dx=\frac{1}{2}\int_1^2(2-x)\sqrt{4x-3}\,dx$$
If we set $u=4x-3$ and $du=4\,dx$ in the second term,
$$\iint_S y\,dS=\left\{\frac{1}{6}(4x-3)^{3/2}\right\}_1^2-\frac{1}{2}\int_1^5\left(\frac{u+3}{4}\right)\sqrt{u}\left(\frac{du}{4}\right)$$
$$=\frac{1}{6}(5\sqrt{5}-1)-\frac{1}{32}\left\{\frac{2u^{5/2}}{5}+2u^{3/2}\right\}_1^5=\frac{25\sqrt{5}-11}{120}.$$

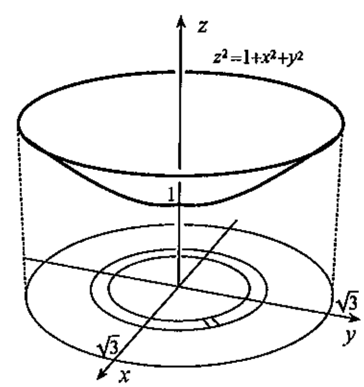


29. By Green's theorem,



$$\begin{aligned}
&\oint_C (ye^{xy} + xy^2e^{xy})\,dx + (xe^{xy} + x^2ye^{xy} + x^3y)\,dy \\
&= \iint_R (e^{xy} + xye^{xy} + 2xye^{xy} + x^2y^2e^{xy} + 3x^2y \\
&\qquad - e^{xy} - xye^{xy} - 2xye^{xy} - x^2y^2e^{xy})\,dA \\
&= 6\int_0^{\pi/2}\int_0^{2\sin\theta} (r^2\cos^2\theta)(r\sin\theta)r\,dr\,d\theta \\
&= 6\int_0^{\pi/2}\left\{\frac{r^5}{5}\cos^2\theta\sin\theta\right\}_0^{2\sin\theta} d\theta \\
&= \frac{192}{5}\int_0^{\pi/2}\cos^2\theta\sin^6\theta\,d\theta = \frac{192}{5}\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta \\
&= \frac{12}{5}\int_0^{\pi/2}\sin^2 2\theta(1 - 2\cos 2\theta + \cos^2 2\theta)\,d\theta = \frac{12}{5}\int_0^{\pi/2}\left[\frac{1-\cos 4\theta}{2} - 2\sin^2 2\theta\cos 2\theta + \left(\frac{\sin 4\theta}{2}\right)^2\right] d\theta \\
&= \frac{12}{5}\int_0^{\pi/2}\left[\frac{1-\cos 4\theta}{2} - 2\sin^2 2\theta\cos 2\theta + \frac{1}{4}\left(\frac{1-\cos 8\theta}{2}\right)\right] d\theta \\
&= \frac{12}{5}\left\{\frac{5\theta}{8} - \frac{1}{8}\sin 4\theta - \frac{1}{3}\sin^3 2\theta - \frac{1}{64}\sin 8\theta\right\}_0^{\pi/2} = \frac{3\pi}{4}
\end{aligned}$$

30. Since $\hat{\mathbf{n}} = \dfrac{\nabla(x^2+y^2-z^2+1)}{|\nabla(x^2+y^2-z^2+1)|} = \dfrac{(2x, 2y, -2z)}{\sqrt{4x^2+4y^2+4z^2}} = \dfrac{(x, y, -z)}{\sqrt{x^2+y^2+z^2}}$,



$$\begin{aligned}
\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS &= \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \\
&= \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}\sqrt{1 + \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2}\,dA \\
&= \iint_{S_{xy}} \frac{x^2+y^2}{z}\,dA \\
&= \iint_{S_{xy}} \frac{x^2+y^2}{\sqrt{1+x^2+y^2}}\,dA \\
&= \int_0^{\sqrt{3}}\int_{-\pi}^{\pi} \frac{r^2}{\sqrt{1+r^2}} r\,d\theta\,dr = 2\pi\int_0^{\sqrt{3}} \frac{r^3}{\sqrt{1+r^2}}\,dr.
\end{aligned}$$

If we set $u = 1 + r^2$ and $du = 2r\,dr$, then

$$\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}})\cdot\hat{\mathbf{n}}\,dS = 2\pi\int_1^4 \frac{u-1}{\sqrt{u}}\left(\frac{du}{2}\right) = \pi\int_1^4\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du = \pi\left\{\frac{2u^{3/2}}{3} - 2\sqrt{u}\right\}_1^4 = \frac{8\pi}{3}.$$

31. (a) With $dS = \sqrt{1 + \left(\dfrac{-x}{\sqrt{1-x^2-y^2}}\right)^2 + \left(\dfrac{-y}{\sqrt{1-x^2-y^2}}\right)^2}\,dA = \dfrac{1}{\sqrt{1-x^2-y^2}}dA$,

$$\begin{aligned}
\iint_S x^2y^2z^2\,dS &= \iint_{S_{xy}} x^2y^2(1-x^2-y^2)\frac{1}{\sqrt{1-x^2-y^2}}dA \\
&= 4\int_0^{\sqrt{2\sqrt{2}-2}}\int_0^{\sqrt{1-x^2-x^4/4}} x^2y^2\sqrt{1-x^2-y^2}\,dy\,dx
\end{aligned}$$

(b) With $dS = \sqrt{1 + \left(\frac{-y}{\sqrt{1-y^2-z^2}}\right)^2 + \left(\frac{-z}{\sqrt{1-y^2-z^2}}\right)^2}\,dA = \frac{1}{\sqrt{1-y^2-z^2}}dA,$

$$\iint_S x^2y^2z^2\,dS = \iint_{S_{yz}} y^2z^2(1-y^2-z^2)\frac{1}{\sqrt{1-y^2-z^2}}dA = 4\int_0^1\int_{-1+\sqrt{2-y^2}}^{\sqrt{1-y^2}} y^2z^2\sqrt{1-y^2-z^2}\,dz\,dy.$$

(c) With $dS = \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2-z^2}}\right)^2 + \left(\frac{-z}{\sqrt{1-x^2-z^2}}\right)^2}\,dA = \frac{1}{\sqrt{1-x^2-z^2}}dA,$

$$\iint_S x^2y^2z^2\,dS = 4\iint_{S_{xz}} x^2z^2(1-x^2-z^2)\frac{1}{\sqrt{1-x^2-z^2}}dA = 4\int_0^{\sqrt{2\sqrt{2}-2}}\int_{x^2/2}^{\sqrt{1-x^2}} x^2z^2\sqrt{1-x^2-z^2}\,dz\,dx.$$

32.
$$\begin{aligned}\nabla(|\mathbf{r}|^n) &= \nabla[(x^2+y^2+z^2)^{n/2}] = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x\hat{\mathbf{i}}+2y\hat{\mathbf{j}}+2z\hat{\mathbf{k}})\\ &= n(x^2+y^2+z^2)^{(n-2)/2}(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}) = n|\mathbf{r}|^{n-2}\mathbf{r}\end{aligned}$$

33. If $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$, then

$$\nabla\times\mathbf{F} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ P & Q & R\end{vmatrix} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\hat{\mathbf{k}},$$

and
$$\begin{aligned}\nabla\times(\nabla\times\mathbf{F}) &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z} & \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x} & \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\end{vmatrix}\\ &= \left(\frac{\partial^2 Q}{\partial y\partial x}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}+\frac{\partial^2 R}{\partial z\partial x}\right)\hat{\mathbf{i}} + \left(\frac{\partial^2 R}{\partial z\partial y}-\frac{\partial^2 Q}{\partial z^2}-\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 P}{\partial x\partial y}\right)\hat{\mathbf{j}}\\ &\quad + \left(\frac{\partial^2 P}{\partial x\partial z}-\frac{\partial^2 R}{\partial x^2}-\frac{\partial^2 R}{\partial y^2}+\frac{\partial^2 Q}{\partial y\partial z}\right)\hat{\mathbf{k}}.\end{aligned}$$

On the other hand,
$$\begin{aligned}\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F} &= \nabla\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right) - \nabla^2(P\hat{\mathbf{i}}+Q\hat{\mathbf{j}}+R\hat{\mathbf{k}})\\ &= \left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 Q}{\partial x\partial y}+\frac{\partial^2 R}{\partial z\partial x}-\frac{\partial^2 P}{\partial x^2}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}\right)\hat{\mathbf{i}}\\ &\quad + \left(\frac{\partial^2 P}{\partial y\partial x}+\frac{\partial^2 Q}{\partial y^2}+\frac{\partial^2 R}{\partial y\partial z}-\frac{\partial^2 Q}{\partial x^2}-\frac{\partial^2 Q}{\partial y^2}-\frac{\partial^2 Q}{\partial z^2}\right)\hat{\mathbf{j}}\\ &\quad + \left(\frac{\partial^2 P}{\partial z\partial x}+\frac{\partial^2 Q}{\partial z\partial y}+\frac{\partial^2 R}{\partial z^2}-\frac{\partial^2 R}{\partial x^2}-\frac{\partial^2 R}{\partial y^2}-\frac{\partial^2 R}{\partial z^2}\right)\hat{\mathbf{k}}\\ &= \left(\frac{\partial^2 Q}{\partial y\partial x}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}+\frac{\partial^2 R}{\partial z\partial x}\right)\hat{\mathbf{i}}\\ &\quad + \left(\frac{\partial^2 R}{\partial z\partial y}-\frac{\partial^2 Q}{\partial z^2}-\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 P}{\partial x\partial y}\right)\hat{\mathbf{j}}\\ &\quad + \left(\frac{\partial^2 P}{\partial x\partial z}-\frac{\partial^2 R}{\partial x^2}-\frac{\partial^2 R}{\partial y^2}+\frac{\partial^2 Q}{\partial y\partial z}\right)\hat{\mathbf{k}}.\end{aligned}$$

CHAPTER 15

EXERCISES 15.1

1. For the given function, $\dfrac{dy}{dx} + 2xy = -2Cxe^{-x^2} + 2x(2 + Ce^{-x^2}) = 4x$.

2. For the given function, $\dfrac{dy}{dx} - \dfrac{y^2}{x^2} = \dfrac{(1+Cx)(1) - x(C)}{(1+Cx)^2} - \dfrac{x^2}{x^2(1+Cx)^2} = 0$.

3. For the given function,

$$x^3\frac{dy}{dx} + (2-3x^2)y = x^3\left[\frac{3x^2}{2} + 3Cx^2e^{1/x^2} + Cx^3e^{1/x^2}\left(-\frac{2}{x^3}\right)\right] + (2-3x^2)\left(\frac{x^3}{2} + Cx^3e^{1/x^2}\right) = x^3.$$

4. For the given function, $\dfrac{d^2y}{dx^2} + 9y = (-9C_1\sin 3x - 9C_2\cos 3x) + 9(C_1\sin 3x + C_2\cos 3x) = 0$.

5. If we write the function in the form $y = \dfrac{1}{2C_1}(C_1^2e^x + e^{-x})$, then

$$\left(\frac{d^2y}{dx^2}\right)^2 - 1 - \left(\frac{dy}{dx}\right)^2 = \left[\frac{1}{2C_1}(C_1^2e^x + e^{-x})\right]^2 - 1 - \left[\frac{1}{2C_1}(C_1^2e^x - e^{-x})\right]^2$$
$$= \frac{1}{4C_1^2}(C_1^4e^{2x} + 2C_1^2 + e^{-2x}) - 1 - \frac{1}{4C_1^2}(C_1^4e^{2x} - 2C_1^2 + e^{-2x}) = 0.$$

6. For the given function,

$$\begin{aligned}2\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 9y &= 2[4C_1e^{2x}\cos(x/\sqrt{2}) - (4/\sqrt{2})C_1e^{2x}\sin(x/\sqrt{2}) - (C_1/2)e^{2x}\cos(x/\sqrt{2})\\ &\quad + 4C_2e^{2x}\sin(x/\sqrt{2}) + (4/\sqrt{2})C_2e^{2x}\cos(x/\sqrt{2}) - (C_2/2)e^{2x}\sin(x/\sqrt{2})]\\ &\quad - 8[2C_1e^{2x}\cos(x/\sqrt{2}) - (C_1/\sqrt{2})e^{2x}\sin(x/\sqrt{2}) + 2C_2e^{2x}\sin(x/\sqrt{2})\\ &\quad + (C_2/\sqrt{2})e^{2x}\cos(x/\sqrt{2})] + 9[C_1e^{2x}\cos(x/\sqrt{2}) + C_2e^{2x}\sin(x/\sqrt{2})] = 0.\end{aligned}$$

7. For the given function,

$$\begin{aligned}\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} + 4y &= (16C_1\cos 2x + 16C_2\sin 2x + C_3\cos x + C_4\sin x)\\ &\quad + 5(-4C_1\cos 2x - 4C_2\sin 2x - C_3\cos x - C_4\sin x)\\ &\quad + 4(C_1\cos 2x + C_2\sin 2x + C_3\cos x + C_4\sin x) = 0.\end{aligned}$$

8. For the given function,

$$\begin{aligned}2\frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 32y &= 2[-(1/2)e^{4x} + 2(C_2 - x/2)4e^{4x} + 16(C_1 + C_2x - x^2/4)e^{4x}]\\ &\quad - 16[(C_2 - x/2)e^{4x} + 4(C_1 + C_2x - x^2/4)e^{4x}] + 32(C_1 + C_2x - x^2/4)e^{4x} = -e^{4x}.\end{aligned}$$

9. For the given function,

$$\begin{aligned}x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y &= x^2\left[\frac{2C_1}{x^2}\sin(2\ln x) - \frac{4C_1}{x^2}\cos(2\ln x) - \frac{2C_2}{x^2}\cos(2\ln x) - \frac{4C_2}{x^2}\sin(2\ln x)\right]\\ &\quad + x\left[-\frac{2C_1}{x}\sin(2\ln x) + \frac{2C_2}{x}\cos(2\ln x)\right] + 4[C_1\cos(2\ln x) + C_2\sin(2\ln x) + 1/4]\\ &= 1.\end{aligned}$$

10. For the given function,

$$\begin{aligned}x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1/4)y &= x^2\left(-\frac{C_1\sin x}{\sqrt{x}} - \frac{C_1\cos x}{x^{3/2}} + \frac{3C_1\sin x}{4x^{5/2}} - \frac{C_2\cos x}{\sqrt{x}}\right.\\ &\quad\left.+\frac{C_2\sin x}{x^{3/2}} + \frac{3C_2\cos x}{4x^{5/2}}\right) + x\left(\frac{C_1\cos x}{\sqrt{x}} - \frac{C_1\sin x}{2x^{3/2}} - \frac{C_2\sin x}{\sqrt{x}} - \frac{C_2\cos x}{2x^{3/2}}\right)\\ &\quad + x^2\left(\frac{C_1\sin x}{\sqrt{x}} + \frac{C_2\cos x}{\sqrt{x}}\right) - \frac{1}{4}\left(\frac{C_1\sin x}{\sqrt{x}} + \frac{C_2\cos x}{\sqrt{x}}\right) = 0.\end{aligned}$$

11. For $y(0) = 1$ and $y'(0) = 6$, constants C_1 and C_2 must satisfy $1 = C_2$ and $6 = 3C_1$. Thus, $y(x) = 2\sin 3x + \cos 3x$.

12. For $y(0) = 2$ and $y(\pi/2) = 3$, constants C_1 and C_2 must satisfy $2 = C_2$ and $3 = -C_1$. Thus, $y(x) = -3\sin 3x + 2\cos 3x$.

13. For $y(\pi/12) = 0$ and $y'(\pi/12) = 1$, constants C_1 and C_2 must satisfy $0 = C_1/\sqrt{2} + C_2/\sqrt{2}$ and $1 = 3C_1/\sqrt{2} - 3C_2/\sqrt{2}$. These give $C_1 = -C_2 = \sqrt{2}/6$, and $y(x) = (\sqrt{2}/6)(\sin 3x - \cos 3x)$.

14. For $y(1) = 1$ and $y(2) = 2$, constants C_1 and C_2 must satisfy

$$1 = C_1\sin 3 + C_2\cos 3 \qquad \text{and} \qquad 2 = C_1\sin 6 + C_2\cos 6.$$

The solution of these equations is $C_1 = (2\cos 3 - \cos 6)/\sin 3$ and $C_2 = (\sin 6 - 2\sin 3)/\sin 3$, and therefore $y(x) = [(2\cos 3 - \cos 6)\sin 3x + (\sin 6 - 2\sin 3)\cos 3x]/\sin 3$.

15. Integration with respect to x gives a general solution $y(x) = 2x^3 + x^2 + C$.

16. Integration with respect to x gives $y(x) = \int \frac{1}{9 + x^2}dx$, and if we set $x = 3\tan\theta$ and $dx = 3\sec^2\theta\, d\theta$, then, $y(x) = \int \frac{1}{9\sec^2\theta}3\sec^2\theta\, d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\text{Tan}^{-1}(x/3) + C$.

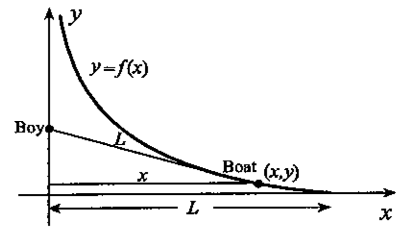
17. Integration with respect to x gives $\frac{dy}{dx} = x^2 + e^x + C$. A second integration gives a general solution $y(x) = \frac{x^3}{3} + e^x + Cx + D$.

18. Integration with respect to x gives $\frac{dy}{dx} = \int x\ln x\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$. A further integration now yields $y(x) = \int\left(\frac{x^2}{2}\ln x - \frac{x^2}{4} + C\right)dx = \frac{x^3}{6}\ln x - \frac{5x^3}{36} + Cx + D$.

19. Integration with respect to x gives $\frac{d^2y}{dx^2} = -\frac{1}{12x^4} + C$. From a second integration, $\frac{dy}{dx} = \frac{1}{36x^3} + Cx + D$. Finally, one more integration gives a general solution $y(x) = -\frac{1}{72x^2} + \frac{Cx^2}{2} + Dx + E$.

20. (a) Since the slope of the curve is also the slope of the string,

$$\frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}.$$



(b) If we integrate with respect to x,

$$y = -\int \frac{\sqrt{L^2 - x^2}}{x} dx.$$

We now set $x = L \sin\theta$ and $dx = L \cos\theta\, d\theta$,

$$y = -\int \frac{L \cos\theta}{L \sin\theta} L \cos\theta\, d\theta = -L \int \frac{1 - \sin^2\theta}{\sin\theta} d\theta = L \int (\sin\theta - \csc\theta)\, d\theta$$

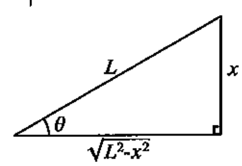
$$= L[-\cos\theta - \ln|\csc\theta - \cot\theta|] + C = -L\left(\frac{\sqrt{L^2 - x^2}}{L} + \ln\left|\frac{L}{x} - \frac{\sqrt{L^2 - x^2}}{x}\right|\right) + C$$

$$= -\sqrt{L^2 - x^2} - L \ln\left|\frac{L - \sqrt{L^2 - x^2}}{x}\right| + C = -\sqrt{L^2 - x^2} - L \ln\left|\frac{L - \sqrt{L^2 - x^2}}{x} \frac{L + \sqrt{L^2 - x^2}}{L + \sqrt{L^2 - x^2}}\right| + C$$

$$= -\sqrt{L^2 - x^2} - L \ln\left|\frac{x}{L + \sqrt{L^2 - x^2}}\right| + C = L \ln\left|\frac{L + \sqrt{L^2 - x^2}}{x}\right| - \sqrt{L^2 - x^2} + C.$$

Since $y = 0$ when $x = L$, it follows that $C = 0$, and

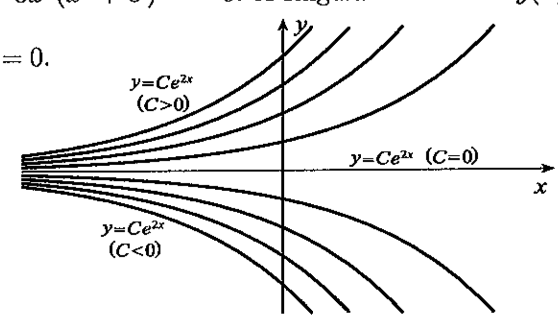
$$y = L \ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right) - \sqrt{L^2 - x^2}.$$



21. For this function, $\dfrac{dy}{dx} - 2xy^2 = \dfrac{2x}{(x^2 + C)^2} - 2x\left(\dfrac{-1}{x^2 + C}\right)^2 = 0$. A singular solution is $y(x) \equiv 0$.

22. For this function, $\dfrac{dy}{dx} - 3x^2(y - 1)^2 = \dfrac{3x^2}{(x^3 + C)^2} - 3x^2(x^3 + C)^{-2} = 0$. A singular solution is $y(x) \equiv 1$.

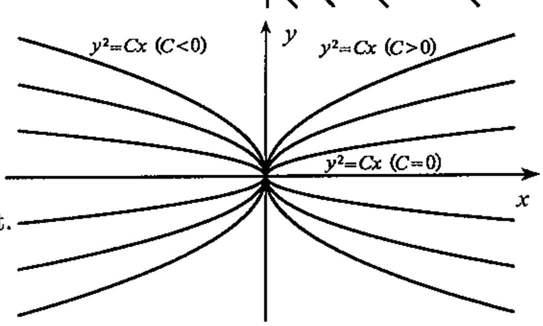
23. (a) For this function, $\dfrac{dy}{dx} - 2y = 2Ce^{2x} - 2(Ce^{2x}) = 0$.
(b) Graphs of curves in the family are shown to the right.
(c) For that solution passing through (x_0, y_0), C must satisfy $y_0 = Ce^{2x_0}$. Thus, $C = y_0 e^{-2x_0}$, and the required solution is $y(x) = y_0 e^{2(x - x_0)}$.



24. (a) Differentiation of $y^2 = Cx$ (implicitly) with respect to x gives $2y\dfrac{dy}{dx} = C$. If we substitute $C = y^2/x$, we obtain

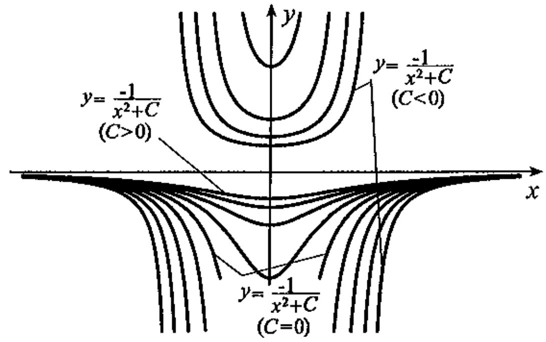
$$2y\frac{dy}{dx} = \frac{y^2}{x} \implies 2x\frac{dy}{dx} = y.$$

(b) The curves are the parabolas shown to the right.
(c) For that solution passing through (x_0, y_0), C must satisfy $y_0^2 = Cx_0$. Thus, $C = y_0^2/x_0$, except when $x_0 = 0$.



25. (a) Graphs are shown to the right.

(b) For the solution to pass through a point (x_0, y_0), C must satisfy $y_0 = -1/(x_0^2 + C)$. When we solve for C, we obtain $C = -1/y_0 - x_0^2$, provided, of course, that $y_0 \neq 0$.



26. If we integrate both sides of the differential equation with respect to x, we obtain the general solution

$$y(x) = \begin{cases} -1/x + C, & x < 0 \\ -1/x + D, & x > 0 \end{cases}.$$

(a) For the solution to satisfy $y(1) = 1$, it is necessary for $D = 2$. Constant C is undetermined.
(b) For the solution to satisfy $y(-1) = 2$, it is necessary for $C = 1$. Constant D is undetermined.
(c) For the solution to satisfy $y(1) = 1$ and $y(-1) = 2$, we choose $C = 1$ and $D = 2$.

27. (a) Since arc length along a circle is given by $s = r\theta$, it follows that $ds/dt = r d\theta/dt + \theta dr/dt$. Now dr/dt is very small compared to $d\theta/dt$ so that we set $ds/dt = r d\theta/dt$. Since $dn/dt = (2\pi)^{-1} d\theta/dt$, and $v = ds/dt$, it follows that $dn/dt = v/(2\pi r)$.
(b) If we set $\pi(r^2 - r_0^2) = wvt$, then $r = \sqrt{wvt/\pi + r_0^2}$. Hence,

$$\frac{dn}{dt} = \frac{v}{2\pi\sqrt{wvt/\pi + r_0^2}} = \frac{v/(2\pi r_0)}{\sqrt{wvt/(\pi r_0^2) + 1}}.$$

(c) Integration gives

$$n(t) = \frac{v}{2\pi r_0}\left(\frac{2\pi r_0^2}{wv}\right)\sqrt{\frac{wvt}{\pi r_0^2} + 1} + C = \frac{r_0}{w}\sqrt{\frac{wvt}{\pi r_0^2} + 1} + C.$$

Since $n(0) = 0$, we obtain $0 = r_0/w + C$, and therefore

$$n(t) = \frac{r_0}{w}\sqrt{\frac{wvt}{\pi r_0^2} + 1} - \frac{r_0}{w} = \frac{r_0}{w}\left(\sqrt{\frac{wvt}{\pi r_0^2} + 1} - 1\right).$$

EXERCISES 15.2

1. This equation can be separated, $\dfrac{dy}{y^2} = \dfrac{dx}{x^2}$, (provided $y \neq 0$), and therefore a one-parameter family of solutions is defined implicitly by $-1/y = -1/x + C$. This equation can be solved for $y(x) = x/(1 - Cx)$. The function $y = 0$ is a solution of the differential equation, and because it is not contained in the one-parameter family of solutions, it is a singular solution.

2. This equation can be separated, $\dfrac{1}{2-y}\,dy = 2x\,dx$, (provided $y \neq 2$), and therefore a one-parameter family of solutions is defined implicitly by $-\ln|2 - y| = x^2 + C$. This equation can be solved for $y(x) = 2 + De^{-x^2}$, where $D = \pm e^{-C}$. The function $y = 2$ is a solution of the differential equation. If we allow D to be equal to zero, this solution is contained in the family, and it is not, therefore, a singular solution.

3. When the equation is separated, $\dfrac{dy}{y} = \dfrac{-2x\,dx}{x^2+1}$, (provided $y \neq 0$). A one-parameter family of solutions is defined implicitly by $\ln|y| = -\ln|x^2 + 1| + C$. When we solve for x, the result is $y = D/(x^2 + 1)$, where $D = \pm e^C$. The function $y = 0$ is a solution of the differential equation. If we allow D to be equal to zero, this solution is contained in the family, and it is not, therefore, a singular solution.

4. When this equation is separated, $\dfrac{1}{3y+2}\,dy = dx$, (provided $y \neq -2/3$). A one-parameter family of solutions is defined implicitly by $(1/3)\ln|3y+2| = x + C$. This equation can be solved for $y(x) = De^{3x} - 2/3$, where $D = \pm(1/3)e^{3C}$. The function $y = -2/3$ is a solution of the differential equation. If we allow D to be equal to zero, this solution is contained in the family, and it is not, therefore, a singular solution.

5. Separation gives $\dfrac{4y\,dy}{y^2+2} = \dfrac{3\,dx}{x-1}$, and therefore a one-parameter family of solutions is defined implicitly by $2\ln|y^2+2| = 3\ln|x-1| + C$. When we solve for the explicit solution, we obtain $y(x) = \pm\sqrt{D|x-1|^{3/2} - 2}$ where $D = \pm e^{C/2}$.

6. This equation can be separated, $\dfrac{x^2\,dx}{1-x} = \dfrac{y^2\,dy}{1+y}$, (provided $y \neq -1$), and a one-parameter family of solutions is therefore defined implicitly by $\displaystyle\int \frac{x^2}{1-x}\,dx = \int \frac{y^2}{1+y}\,dy$. To integrate, we write

$$\int\left(-x - 1 + \frac{1}{1-x}\right)dx = \int\left(y - 1 + \frac{1}{y+1}\right)dy,$$

and hence, $-\dfrac{x^2}{2} - x - \ln|x-1| + C = \dfrac{y^2}{2} - y + \ln|y+1|$. The function $y = -1$ is a solution of the differential equation, and because it is not contained in the one-parameter family, it is a singular solution.

7. Separation gives $\sec y\,dy = -\csc x\,dx$, (provided $y \neq (2n+1)\pi/2$, where n is an integer). A one-parameter family of solutions is defined implicitly by $\ln|\sec y + \tan y| = -\ln|\csc x - \cot x| + C$. When we exponentiate, $(\csc x - \cot x)(\sec y + \tan y) = D$, where $D = \pm e^C$. The functions $y = (2n+1)\pi/2$ are solutions of the differential equation, and because they are not contained in the one-parameter family, they are singular solutions.

8. When we separate this equation, $y^2\,dy = \left(\dfrac{1 - x^2e^x}{x}\right)dx$, and a one-parameter family of solutions is defined implicitly by $\dfrac{y^3}{3} = \ln|x| - \displaystyle\int xe^x\,dx = \ln|x| - xe^x + e^x + C$. Explicitly we obtain $y(x) = [D + 3\ln|x| - 3xe^x + 3e^x]^{1/3}$, where $D = 3C$.

9. When we separate, $y\,dy = -(x\sec x\tan x + \sec x)\,dx$, and a one-parameter family of solutions is therefore defined implicitly by $y^2/2 = -x\sec x + C$. When we solve, $y(x) = \pm\sqrt{D - 2x\sec x}$, where $D = 2C$.

10. When we separate this equation, $\dfrac{dy}{1+y^2} = \dfrac{dx}{1+x^2}$, and a one-parameter family of solutions is defined implicitly by $\mathrm{Tan}^{-1}y = \mathrm{Tan}^{-1}x + C$. If we apply the tangent function to both sides of $C = \mathrm{Tan}^{-1}y - \mathrm{Tan}^{-1}x$, we find

$$\tan C = D = \tan(\mathrm{Tan}^{-1}y - \mathrm{Tan}^{-1}x) = \frac{\tan(\mathrm{Tan}^{-1}y) - \tan(\mathrm{Tan}^{-1}x)}{1 + \tan(\mathrm{Tan}^{-1}y)\tan(\mathrm{Tan}^{-1}x)} = \frac{y-x}{1+xy}.$$

When this equation is solved for y the result is $y(x) = \dfrac{x+D}{1-Dx}$.

11. When we separate the equation, $\dfrac{dy}{y} = \dfrac{-2\,dx}{x+1}$, (provided $y \neq 0$). A one-parameter family of solutions is defined implicitly by $\ln|y| = -2\ln|x+1| + C$. When we solve for y, the result is $y(x) = D/(x+1)^2$, where $D = \pm e^C$. For the solution to satisfy $y(1) = 2$, we must have $2 = D/4$. Thus, $D = 8$, and $y(x) = 8/(x+1)^2$.

12. We separate this equation, $\left(\dfrac{y-1}{y}\right)dy = \left(\dfrac{x+1}{x}\right)dx$, (provided $y \neq 0$). A one-parameter family of solutions is defined implicitly by $y - \ln|y| = x + \ln|x| + C$. When we take exponentials, $xy = De^{y-x}$, where $D = \pm e^{-C}$. To satisfy $y(1) = 2$, we must have $(1)(2) = De^{2-1}$. Thus, $D = 2/e$, and $xy = 2e^{y-x-1}$.

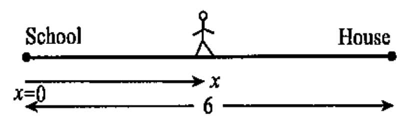
13. When we separate the equation, $e^{-y}dy = e^x dx$. A one-parameter family of solutions is defined implicitly by $-e^{-y} = e^x + C$. For the solution to satisfy $y(0) = 0$, we must have $-1 = 1 + C$. Thus, $C = -2$, and $-e^{-y} = e^x - 2 \Longrightarrow y = -\ln(2 - e^x)$.

14. When we separate variables, $\dfrac{1}{1+y^2}\,dy = 2x\,dx$. A one-parameter family of solutions is defined implicitly by $\operatorname{Tan}^{-1}y = x^2 + C$. Thus, $y(x) = \tan(x^2 + C)$. To satisfy $y(2) = 4$, we must have $4 = \tan(4 + C)$. This implies that $C = \operatorname{Tan}^{-1}4 - 4 + n\pi$, where n is an integer, and hence

$$y(x) = \tan(x^2 + \operatorname{Tan}^{-1}4 - 4 + n\pi) = \frac{\tan(x^2-4) + \tan(\operatorname{Tan}^{-1}4 + n\pi)}{1 - \tan(x^2-4)\,\tan(\operatorname{Tan}^{-1}4 + n\pi)} = \frac{4 + \tan(x^2-4)}{1 - 4\tan(x^2-4)}.$$

15. When we separate the equation, $\csc^2 y\,dy = \sec^2 x\,dx$. A one-parameter family of solutions is defined implicitly by $-\cot y = \tan x + C$. For the solution to satisfy $y(0) = \pi/2$, we must have $C = 0$. Thus, $\cot y = -\tan x$. This can be simplified by writing $\cot y = -\cot(\pi/2 - x) = \cot(x - \pi/2)$. This implies that $y = x - \pi/2 + n\pi$ for some integer n. The condition $y(0) = \pi/2$ requires $n = 1$, and the solution of the differential equation is $y = x + \pi/2$.

16. (a) Since the girl's speed is proportional to x^2, $dx/dt = kx^2$. Separation of this equation gives



$$\frac{1}{x^2}\,dx = k\,dt,$$

a one-parameter family of solutions of which is defined implicitly by $-1/x = kt + C$. Thus, $x = -1/(kt + C)$. If we choose time $t = 0$ when $x = 6$, then $6 = -1/C$ and $x = 6/(1 - 6kt)$ km.

(b) The girl reaches school when $x = 0$, but this only happens after an infinitely long time.

17. When we separate the equation, $\dfrac{y^2\,dy}{y^3+1} = \dfrac{-dx}{x(x^2+1)} = \left(\dfrac{x}{x^2+1} - \dfrac{1}{x}\right)dx$. A one-parameter family of solutions is defined implicitly by $(1/3)\ln|y^3+1| = (1/2)\ln|x^2+1| - \ln|x| + C$. When we solve this equation for y, we obtain an explicit definition of the solution $y(x) = \left[\dfrac{D(x^2+1)^{3/2}}{|x|^3} - 1\right]^{1/3}$, where $D = \pm e^{3C}$.

18. If $T(t)$ represents the temperature of the water as a function of time t, then according to Newton's law of cooling, $\dfrac{dT}{dt} = k(T - 20)$, where $k < 0$ is a constant. Separation of variables leads to $\dfrac{1}{T-20}\,dT = k\,dt$, and therefore $\ln|T - 20| = kt + C$. When we solve for T, the result is $T(t) = 20 + De^{kt}$. If we choose time $t = 0$ when $T = 80$, then $80 = 20 + D$. Thus, $D = 60$, and $T(t) = 20 + 60e^{kt}$. Because $T(2) = 60$, it follows that $60 = 20 + 60e^{2k}$, from which $k = (1/2)\ln(2/3) = -0.203$. Finally then, $T(t) = 20 + 60e^{-0.203t}$, or, $T(t) = 20 + 60e^{(t/2)\ln(2/3)} = 20 + 60(2/3)^{t/2}$.

19. If $T(t)$ represents the temperature of the mercury in the thermometer as a function of time t, then according to Newton's law of cooling, $\dfrac{dT}{dt} = k(T + 20)$, where $k < 0$ is a constant. Separation of variables leads to $\dfrac{1}{T+20}\,dT = k\,dt$, and therefore $\ln|T + 20| = kt + C$. When we solve for T, the result is $T(t) = -20 + De^{kt}$. If we choose time $t = 0$ when $T = 23$, then $23 = -20 + D$. Thus, $D = 43$, and $T(t) = -20 + 43e^{kt}$. Because $T(4) = 0$, it follows that $0 = -20 + 43e^{4k}$, from which $k = (1/4)\ln(20/43)$. The temperature is $-19°$ C when $-19 = -20 + 43e^{kt} \Longrightarrow t = (1/k)\ln(1/43) = 19.7$ minutes.

20. If $A(t)$ represents the amount of drug in the body at time t (in hours), then $\dfrac{dA}{dt} = kA$, where $k < 0$ is a constant. Separation of variables gives $\dfrac{1}{A}\,dA = k\,dt$, and therefore $\ln|A| = kt + C$, or, $A = De^{kt}$. If A_0 is the size of the original dose injected at time $t = 0$, then $A_0 = D$, and $A = A_0e^{kt}$. Since $A(1) = 0.95A_0$, it follows that $0.95A_0 = A_0e^k$. Thus, $k = \ln(0.95)$, and $A = A_0e^{t\ln(0.95)}$. The dose decreases to $A_0/2$ when $A_0/2 = A_0e^{t\ln(0.95)}$, the solution of which is $t = -\ln 2/\ln(0.95) = 13.51$ h.

21. Since the rate of change of the amount of nitrogen is proportional to $\overline{N} - N$, we can write that $\dfrac{dN}{dt} = k(\overline{N} - N)$, where $k > 0$ is a constant. If we separate this equation $\dfrac{1}{\overline{N} - N}dN = k\,dt$, a one-parameter family of solutions is defined implicitly by

$$-\ln(\overline{N} - N) = kt + C \quad \Longrightarrow \quad \overline{N} - N = e^{-kt-C} = De^{-kt} \quad \Longrightarrow \quad N = \overline{N} - De^{-kt},$$

where $D = e^{-C}$ is a constant. The initial value $N(0) = N_0$ requires $N_0 = \overline{N} - D$, and therefore $N = \overline{N} - (\overline{N} - N_0)e^{-kt} = N_0e^{-kt} + \overline{N}(1 - e^{-kt})$.

22. Because the total rate of change dA/dt of the amount of glucose in the bloodstream is the rate at which it is added less the rate at which it is used up, $\dfrac{dA}{dt} = R - kA$, where $k > 0$ is a constant. This equation can be separated, $\dfrac{1}{R - kA}\,dA = dt$, and therefore a one-parameter family of solutions is defined implicitly by $-(1/k)\ln|R - kA| = t + C$. When this equation is solved for A, the result is $A(t) = R/k + De^{-kt}$. If we choose time $t = 0$ when $A = A_0$, then $A_0 = R/k + D$, and

$$A(t) = \frac{R}{k} + \left(A_0 - \frac{R}{k}\right)e^{-kt} = \frac{R}{k}(1 - e^{-kt}) + A_0e^{-kt}.$$

23. The equation of the normal line to the curve $y = f(x)$ at any point $P(x_0, y_0)$ is

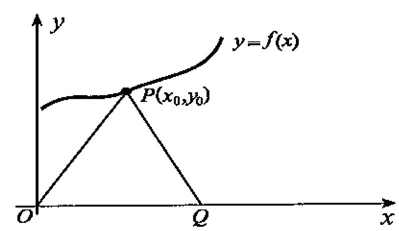
$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0).$$

The x-intercept of this line is

$$x = x_0 + f'(x_0)y_0.$$

Triangle OPQ is isosceles if $\|OP\|^2 = \|PQ\|^2$, and this can be expressed in the form

$${x_0}^2 + {y_0}^2 = [f'(x_0)y_0]^2 + {y_0}^2.$$



Since this must be true at each point on the curve, we drop subscripts and replace $f'(x)$ with dy/dx, the result being $x^2 = (dy/dx)^2y^2$. This equation can be separated, $y\,dy = \pm x\,dx$, and a one-parameter family of solutions is defined implicitly by $y^2/2 = \pm x^2/2 + C$. In the case of the circles $x^2 + y^2 = 2C$, the normal always passes through the origin, and the triangle OPQ degenerates to a straight line. Hence the only nondegenerate case is the hyperbolas $y^2 - x^2 = 2C$.

24. If x is the number of grams of C at time t, then $x/2$ came from each of A and B. This means that there is $10 - x/2$ grams of A and $15 - x/2$ grams of B remaining. Hence, the rate dx/dt at which C is formed is related to x by

$$\frac{dx}{dt} = K\left(10 - \frac{x}{2}\right)\left(15 - \frac{x}{2}\right) = \frac{K}{4}(20 - x)(30 - x) = k(20 - x)(30 - x),$$

where we have set $k = K/4$. If we choose time $t = 0$ when A and B are brought together, then $x(t)$ must also satisfy $x(0) = 0$. We separate the equation and use partial fractions to write it in the form

$$k\,dt = \frac{1}{(20 - x)(30 - x)}dx = \left(\frac{1/10}{20 - x} - \frac{1/10}{30 - x}\right)dx.$$

A one-parameter family of solutions is defined implicitly by

$$kt + C = \frac{1}{10}\left[-\ln(20 - x) + \ln(30 - x)\right].$$

Absolute values are unnecessary because x cannot exceed 20. We now solve this equation for x by writing

$$10(kt + C) = \ln\left(\frac{30 - x}{20 - x}\right), \quad \text{and exponentiating,} \quad \frac{30 - x}{20 - x} = De^{10kt},$$

where $D = e^{10C}$. Cross multiplying gives $(20-x)De^{10kt} = 30 - x$, and therefore $x = \dfrac{30-20De^{10kt}}{1-De^{10kt}}$. The initial condition $x(0) = 0$ requires $D = 3/2$, in which case

$$x = \frac{30-30e^{10kt}}{1-(3/2)e^{10kt}} = \frac{60(1-e^{10kt})}{2-3e^{10kt}} \text{ g}.$$

25. If $x(t)$ represents the amount of C in the mixture at time t, then

$$\frac{dx}{dt} = k\left(10-\frac{x}{2}\right)\left(10-\frac{x}{2}\right) = \frac{k}{4}(20-x)^2,$$

where factors $10-x/2$ represent the amounts of A and B in the mixture at time t, and k is a constant. We can separate this equation, $\dfrac{dx}{(20-x)^2} = \dfrac{k\,dt}{4}$, and a one-parameter family of solutions is defined implicitly by $1/(20-x) = kt/4 + D$. When this equation is solved for x, the result is $x(t) = 20 - 4/(kt+4D)$. If we choose time $t = 0$ when the reaction begins, then $0 = 20 - 1/D$. Thus, $D = 1/20$, and

$$x(t) = 20 - \frac{4}{kt+1/5} = \frac{100kt}{5kt+1} \text{ g}.$$

26. In Section 5.5 we showed that if V is the volume of water in the left sphere, then

$$\frac{dV}{dt} = A(y)\frac{dy}{dt}$$

where $A(y)$ is the surface area of the water. Since water leaves this container at rate

$$\frac{a}{3}\sqrt{2gh} = \frac{a}{3}\sqrt{2g(2y)} = \frac{2a}{3}\sqrt{gy},$$

it follows that

$$A(y)\frac{dy}{dt} = -\frac{2a}{3}\sqrt{gy}.$$

Because $A(y) = \pi x^2 = \pi(R^2-y^2)$,

$$\pi(R^2-y^2)\frac{dy}{dt} = -\frac{2a}{3}\sqrt{gy} \quad \Longrightarrow \quad \frac{R^2-y^2}{\sqrt{y}}\,dy = -\frac{2a\sqrt{g}}{3\pi}\,dt,$$

a separated equation. A one-parameter family of solutions is defined implicitly by

$$2R^2\sqrt{y} - \frac{2}{5}y^{5/2} = -\frac{2a\sqrt{g}t}{3\pi} + C.$$

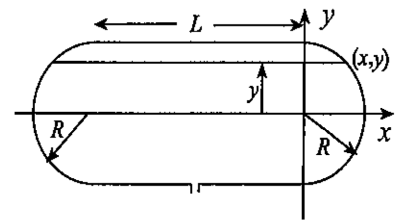
If we choose $t = 0$ when $y = R$, then $2R^2\sqrt{R} - \dfrac{2}{5}R^{5/2} = C$, and

$$2R^2\sqrt{y} - \frac{2}{5}y^{5/2} = -\frac{2a\sqrt{g}t}{3\pi} + \frac{8}{5}R^{5/2}.$$

The water levels are the same in both spheres when $y = 0$, in which case

$$0 = -\frac{2a\sqrt{g}t}{3\pi} + \frac{8}{5}R^{5/2} \qquad \Longrightarrow \qquad t = \frac{12\pi R^{5/2}}{5a\sqrt{g}}.$$

27. The differential equation describing the height y of the surface is
$A(y)\frac{dy}{dt} = -0.6a\sqrt{2g(R+y)}$,
where $A(y)$, the surface area of the water is
$A(y) = 2xL + \pi x^2 = 2L\sqrt{R^2-y^2} + \pi(R^2-y^2)$.
Thus,
$[2L\sqrt{R^2-y^2} + \pi(R^2-y^2)]\frac{dy}{dt} = -0.6a\sqrt{2g(R+y)}$.
This is separable,



$$\left[\frac{2L\sqrt{(R+y)(R-y)}}{\sqrt{R+y}} + \pi\frac{(R+y)(R-y)}{\sqrt{R+y}}\right] dy = -0.6a\sqrt{2g}\,dt,$$

or,

$$\left[2L\sqrt{R-y} + \pi(R-y)\sqrt{R+y}\right] dy = -0.6a\sqrt{2g}\,dt.$$

Integration leads to the following equation that implicitly defines a one-parameter family of solutions

$$-\frac{4L}{3}(R-y)^{3/2} + \pi\int(R-y)\sqrt{R+y}\,dy = -0.6a\sqrt{2g}t + C.$$

If we set $u = R+y$ and $du = dy$ in the integral, then

$$\int(R-y)\sqrt{R+y}\,dy = \int(2R-u)\sqrt{u}\,du = \frac{4R}{3}u^{3/2} - \frac{2}{5}u^{5/2} = \frac{4R}{3}(R+y)^{3/2} - \frac{2}{5}(R+y)^{5/2}.$$

Hence, solutions of the differential equation are defined implicitly by

$$-\frac{4L}{3}(R-y)^{3/2} + \frac{4\pi R}{3}(R+y)^{3/2} - \frac{2\pi}{5}(R+y)^{5/2} = -0.6a\sqrt{2g}t + C.$$

If we choose $t = 0$ when $y = R$, then

$$C = \frac{4\pi R}{3}(2R)^{3/2} - \frac{2\pi}{5}(2R)^{5/2} = \frac{16\sqrt{2}\pi R^{5/2}}{15}.$$

The tank empties when $y = -R$ in which case $-0.6a\sqrt{2g}t + C = -\frac{4L}{3}(2R)^{3/2}$. The emptying time is therefore

$$t = \frac{1}{0.6a\sqrt{2g}}\left(C + \frac{8\sqrt{2}L}{3}R^{3/2}\right) = \frac{1}{0.6a\sqrt{2g}}\left(\frac{16\sqrt{2}\pi R^{5/2}}{15} + \frac{8\sqrt{2}LR^{3/2}}{3}\right) = \frac{8R^{3/2}}{9a\sqrt{g}}(2\pi R + 5L).$$

28. Volume of water in the lock above the downstream level is $V = (8)(16)(2-h)$. Consequently,

$$\frac{dV}{dt} = -128\frac{dh}{dt}.$$

Since water enters the lock at $0.04\sqrt{2gh}$ m^3/s, it follows that

$$-128\frac{dh}{dt} = 0.04\sqrt{2gh} \quad\Longrightarrow\quad \frac{1}{\sqrt{h}}\,dh = -\frac{\sqrt{2g}}{3200}\,dt,$$

a separated equation. A one-parameter family of solutions is defined implicitly by

$$2\sqrt{h} = -\frac{\sqrt{2g}t}{3200} + C.$$

If valve A is opened at $t = 0$ (when $h = 2$), then $2\sqrt{2} = C$, and

$$2\sqrt{h} = -\frac{\sqrt{2g}t}{3200} + 2\sqrt{2}.$$

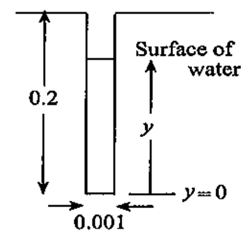
The upstream gate is opened when $h = 0.02$ in which case

$$2\sqrt{0.02} = -\frac{\sqrt{2g}t}{3200} + 2\sqrt{2} \quad \Longrightarrow \quad t = -\frac{3200}{\sqrt{2g}}(2\sqrt{0.02} - 2\sqrt{2}) = 1839.$$

Operation of the lock takes a little over 30 minutes.

29. When the height of the water above the bottom of the slit is y, Exercise 30 in Section 7.9 with $H = 0$ indicates that the volume of water per unit time through the slit is

$$\frac{2\sqrt{2g}c(10^{-3})}{3}y^{3/2} = \frac{\sqrt{2g}cy^{3/2}}{1500}.$$



But the rate of change of the volume of the water in the container is $A\,dy/dt$ where $A = 1$ is the cross-sectional area of the container. Hence

$$-\frac{dy}{dt} = \frac{\sqrt{2g}cy^{3/2}}{1500} \quad \Longrightarrow \quad \frac{1}{y^{3/2}}\,dy = -\frac{\sqrt{2g}c}{1500}\,dt,$$

a separate differential equation. A one-parameter family of solutions is defined implicitly by

$$-\frac{2}{\sqrt{y}} = -\frac{\sqrt{2g}ct}{1500} + C.$$

The initial condition $y(0) = 0.2$ requires $C = -2/\sqrt{0.2}$, and therefore

$$-\frac{2}{\sqrt{y}} = -\frac{\sqrt{2g}ct}{1500} - \frac{2}{\sqrt{0.2}}.$$

The water level has dropped 10 cm when

$$-\frac{2}{\sqrt{0.1}} = -\frac{\sqrt{2g}ct}{1500} - \frac{2}{\sqrt{0.2}} \quad \Longrightarrow \quad t = \frac{1500}{\sqrt{2g}(0.6)}\left(\frac{2}{\sqrt{0.1}} - \frac{2}{\sqrt{0.2}}\right) = 1046 \text{ s}.$$

30. The differential equation is separable, $\dfrac{1}{2gh - v^2}dv = \dfrac{1}{2L}dt$, so that a one-parameter of solutions is defined implicitly by

$$\frac{t}{2L} + C = \int \frac{1}{2gh - v^2}dv = \int \left(\frac{\frac{1}{2\sqrt{2gh}}}{\sqrt{2gh} + v} + \frac{\frac{1}{2\sqrt{2gh}}}{\sqrt{2gh} - v} \right) dv$$

$$= \frac{1}{2\sqrt{2gh}}\left(\ln|\sqrt{2gh} + v| - \ln|\sqrt{2gh} - v|\right) = \frac{1}{2\sqrt{2gh}}\ln\left|\frac{\sqrt{2gh} + v}{\sqrt{2gh} - v}\right|.$$

Exponentiation gives

$$\frac{\sqrt{2gh} + v}{\sqrt{2gh} - v} = De^{\sqrt{2gh}t/L} \quad \Longrightarrow \quad \sqrt{2gh} + v = De^{\sqrt{2gh}t/L}(\sqrt{2gh} - v) \quad \Longrightarrow \quad v = \frac{D\sqrt{2gh}e^{\sqrt{2gh}t/L} - \sqrt{2gh}}{1 + De^{\sqrt{2gh}t/L}}.$$

Since $v(0) = 0$,

$$0 = \frac{D\sqrt{2gh} - \sqrt{2gh}}{1 + D} \quad \Longrightarrow \quad D = 1.$$

Thus, $v(t) = \dfrac{\sqrt{2gh}(e^{\sqrt{2gh}t/L} - 1)}{e^{\sqrt{2gh}t/L} + 1}$.

31. (a) When $a = b = c$,

$$\frac{dx}{dt} = k(a-x)^3 \qquad \Longrightarrow \qquad \frac{1}{(a-x)^3}\,dx = k\,dt,$$

a separated equation. A one-parameter family of solutions is defined implicitly by

$$\frac{1}{2(a-x)^2} = kt + C \quad \Longrightarrow \quad (a-x)^2 = \frac{1}{2(kt+C)}.$$

Square roots give $\quad x = a \pm \dfrac{1}{\sqrt{2(kt+C)}}$. If, as is normal, the initial condition is $x(0) = 0$, we would choose the negative sign, in which case $x(t) = a - 1/\sqrt{2(kt+C)}$.

(b) When $a = b$,

$$\frac{dx}{dt} = k(a-x)^2(c-x) \quad \Longrightarrow \quad \frac{1}{(a-x)^2(c-x)}\,dx = k\,dt.$$

A one-parameter family of solutions is defined implicitly by

$$\begin{aligned} kt + C &= \int \left[\frac{-1/(a-c)^2}{a-x} + \frac{1/(c-a)}{(a-x)^2} + \frac{1/(a-c)^2}{c-x}\right] dx \\ &= \frac{1}{(a-c)^2}\ln|a-x| + \frac{1}{(c-a)(a-x)} - \frac{1}{(a-c)^2}\ln|c-x|. \end{aligned}$$

(c) When $a \neq b \neq c$,

$$\frac{1}{(a-x)(b-x)(c-x)}\,dx = k\,dt.$$

A one-parameter family of solutions is defined implicitly by

$$\begin{aligned} kt + C &= \int \left[\frac{\dfrac{1}{(b-a)(c-a)}}{a-x} + \frac{\dfrac{1}{(a-b)(c-b)}}{b-x} + \frac{\dfrac{1}{(a-c)(b-c)}}{c-x}\right] dx \\ &= \frac{-1}{(b-a)(c-a)}\ln|a-x| - \frac{1}{(a-b)(c-b)}\ln|b-x| - \frac{1}{(a-c)(b-c)}\ln|c-x|. \end{aligned}$$

32. The differential equation can be separated, and partial fractions leads to

$$\frac{1}{A}\,dA = \frac{M^2-1}{M\left[\left(\dfrac{k-1}{2}\right)M^2+1\right]}\,dM = \left[\frac{(k+1)M/2}{\left(\dfrac{k-1}{2}\right)M^2+1} - \frac{1}{M}\right] dM.$$

A one-parameter family of solutions is defined implicitly by

$$\ln A = \left(\frac{k+1}{2}\right)\left(\frac{1}{k-1}\right)\ln\left[\left(\frac{k-1}{2}\right)M^2+1\right] - \ln M + C,$$

from which

$$A = \frac{D\left[\left(\dfrac{k-1}{2}\right)M^2+1\right]^{(k+1)/(2k-2)}}{M}.$$

The condition $A(1) = A_0$ gives

$$A_0 = D\left[\left(\frac{k-1}{2}\right)+1\right]^{(k+1)/(2k-2)} = D\left(\frac{k+1}{2}\right)^{(k+1)/(2k-2)},$$

from which

$$A = \frac{A_0}{M} \frac{\left[\left(\frac{k-1}{2}\right) M^2 + 1\right]^{(k+1)/(2k-2)}}{\left(\frac{k+1}{2}\right)^{(k+1)/(2k-2)}} = \frac{A_0}{M} \left[\frac{(k-1)M^2 + 2}{k+1}\right]^{(k+1)/(2k-2)}.$$

33. (a) When H is a constant, we can separate the differential equation

$$\frac{v}{v+H} dv = \frac{K}{n} dt.$$

A one-parameter family of solutions is defined implicitly by

$$\int \left(1 + \frac{-H}{v+H}\right) dv = \frac{K}{n}t + C \qquad \Longrightarrow \qquad v - H\ln(v+H) = \frac{K}{n}t + C.$$

For $v(0) = 0$, we find $0 - H\ln H = C$, and therefore the solution is defined implicitly by

$$v - H\ln\left(\frac{v+H}{H}\right) = \frac{Kt}{n}.$$

(b) When $H(t) = qt - nv$, the differential equation becomes

$$\frac{dv}{dt} = \frac{K}{n}\left(\frac{v + qt - nv}{v}\right) \qquad \Longrightarrow \qquad \frac{dv}{dt} = \frac{K}{n}\left[(1-n) + \frac{qt}{v}\right].$$

If we try a solution of the form $v(t) = At$, and substitute into the differential equation,

$$A = \frac{K}{n}\left[(1-n) + \frac{qt}{At}\right] \qquad \Longrightarrow \qquad A^2 - \frac{K}{n}(1-n)A - \frac{qK}{n} = 0.$$

Solutions of this quadratic are

$$A = \frac{\frac{K}{n}(1-n) \pm \sqrt{\frac{K^2}{n^2}(1-n)^2 + \frac{4qK}{n}}}{2}.$$

Since the differential equation requires dv/dt to be positive, we much choose the positive radical, in which case

$$v(t) = \left[\frac{K}{n}(1-n) + \sqrt{\frac{K^2}{n^2}(1-n)^2 + \frac{4qK}{n}}\right] t.$$

34. If we set $v = y/x$ or $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substitution into the differential equation gives

$$v + x\frac{dv}{dx} = f(v) \qquad \Longrightarrow \qquad \frac{1}{f(v) - v} dv = \frac{1}{x} dx,$$

which is separated.

35. The functions should be positively homogeneous of the same degree (see Section 12.6 for a definition of homogeneous functions).

36. We write $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{xy} = \dfrac{1 - (y/x)^2}{y/x}$, a homogeneous differential equation. If we set $v = y/x$, or $y = vx$, then $dy/dx = v + xdv/dx$, and $v + x\dfrac{dv}{dx} = \dfrac{1 - v^2}{v}$. This can be separated in the form

$$\frac{1}{x} dx = \frac{1}{\frac{1-v^2}{v} - v} dv = \frac{v}{1 - 2v^2} dv.$$

A one-parameter family of solutions is defined implicitly by $-(1/4)\ln|1-2v^2| = \ln|x| + C$. Exponentiation of both sides leads to $2v^2 = (x^4 - D)/x^4$, and when we substitute $v = y/x$, the solution reduces to $x^2(x^2 - 2y^2) = D$.

37. We write $\dfrac{dy}{dx} = \dfrac{2y + \sqrt{x^2+4y^2}}{2x} = \dfrac{2(y/x) + \sqrt{1+4(y/x)^2}}{2}$, a homogeneous differential equation. If we set $v = y/x$, or $y = vx$, then $dy/dx = v + x\,dv/dx$, and

$$v + x\frac{dv}{dx} = \frac{2v + \sqrt{1+4v^2}}{2} \quad \Longrightarrow \quad \frac{dv}{\sqrt{1+4v^2}} = \frac{dx}{2x},$$

a separated differential equation, with one-parameter family of solutions defined implicitly by

$$\int \frac{1}{\sqrt{1+4v^2}}dv = \int \frac{1}{2x}dx = \frac{1}{2}\ln|x| + C.$$

If we set $v = (1/2)\tan\theta$ and $dv = (1/2)\sec^2\theta\, d\theta$,

$$\frac{1}{2}\ln|x| + C = \int \frac{1}{\sec\theta}\frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| = \frac{1}{2}\ln|\sqrt{1+4v^2} + 2v|.$$

When this equation is solved for v in terms of x, the result is

$$v = \frac{D^2x^2 - 1}{4Dx} \qquad \Longrightarrow \qquad y(x) = \frac{D^2x^2-1}{4D}.$$

38. When we write $\dfrac{dy}{dx} = \dfrac{y/x + 1}{y/x - 1}$, the differential equation is clearly homogeneous. We therefore set $v = y/x$, or, $y = vx$, in which case $dy/dx = v + x\,dv/dx$, and $v + x\dfrac{dv}{dx} = \dfrac{v+1}{v-1}$. This can be separated in the form $\dfrac{v-1}{-v^2+2v+1}\,dv = \dfrac{1}{x}\,dx$. A one-parameter family of solutions of this equation is defined implicitly by $-(1/2)\ln|-v^2+2v+1| = \ln|x| + C$. When this equation is exponentiated, $-v^2 + 2v + 1 = D/x^2$, and substitution of $v = y/x$ gives $x^2 + 2xy - y^2 = D$.

39. We write $\dfrac{dy}{dx} = \dfrac{y + x\cos(y/x)}{x} = \dfrac{y}{x} + \cos(y/x)$, a homogeneous differential equation. If we set $v = y/x$, or $y = vx$, then $dy/dx = v + x\,dv/dx$, and $v + x\dfrac{dv}{dx} = v + \cos v \implies \sec v\, dv = \dfrac{dx}{x}$, a separated differential equation, with one-parameter family of solutions defined implicitly by $\ln|\sec v + \tan v| = \ln|x| + C$. Exponentiation gives $\sec v + \tan v = Dx$, and therefore $\sec(y/x) + \tan(y/x) = Dx$.

40. We write $\dfrac{dy}{dx} = \dfrac{e^{-y/x} + (y/x)^2}{y/x}$, a homogeneous differential equation. When we set $v = y/x$, or, $y = vx$, then $dy/dx = v + x\,dv/dx$, and $v + x\dfrac{dv}{dx} = \dfrac{e^{-v}+v^2}{v} \implies ve^v\, dv = \dfrac{1}{x}\,dx$, a separated differential equation with one-parameter family of solutions defined implicitly by $ve^v - e^v = \ln|x| + C$. Substitution of $v = y/x$ leads to $e^{y/x}(y - x) = x\ln|x| + Cx$.

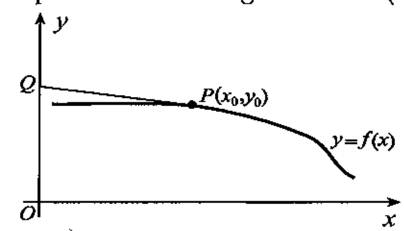
41. We write $\dfrac{dy}{dx} = -\dfrac{x^2y + y^3}{x^3} = -\dfrac{y}{x} - \left(\dfrac{y}{x}\right)^3$, a homogeneous differential equation. If we set $v = y/x$, or $y = vx$, then $dy/dx = v + x\,dv/dx$, and

$$v + x\frac{dv}{dx} = -v - v^3 \quad \Longrightarrow \quad -\frac{dx}{x} = \frac{dv}{v(2+v^2)} = \frac{1}{2}\left(\frac{1}{v} - \frac{v}{2+v^2}\right)dv.$$

A one-parameter family of solutions of this separated equation is defined implicitly by $-\ln|x| + C = (1/2)\ln|v| - (1/4)\ln|2+v^2|$. When we exponentiate and solve for v^2, the result is $v^2 = 2D/(x^4 - D)$. Setting $v = y/x$ leads to $x^4y^2 = D(2x^2 + y^2)$.

42. Let $P(x_0, y_0)$ be any point on the required curve $y = f(x)$. The equation of the tangent line at (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0).$$



The y-intercept of this line is $y_0 - f'(x_0)x_0$.
Since $\|OQ\|^2 = \|PQ\|^2$,

$$[y_0 - f'(x_0)x_0]^2 = x_0^2 + [f'(x_0)x_0]^2, \text{ or,}$$

$$y_0^2 - 2f'(x_0)x_0y_0 + [f'(x_0)]^2x_0^2 = x_0^2 + [f'(x_0)x_0]^2.$$

Thus, $y_0^2 - x_0^2 = 2x_0y_0f'(x_0)$. Since this must be valid at every point on the curve, we drop the subscripts and set $f'(x) = dy/dx$,

$$y^2 - x^2 = 2xy\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

In this homogeneous differential equation, we set $y = vx$ and $dy/dx = v + xdv/dx$,

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}.$$

Thus, $x\dfrac{dv}{dx} = \dfrac{v^2 - 1}{2v} - v = -\dfrac{v^2 + 1}{2v} \implies \dfrac{2v}{v^2 + 1}\,dv = -\dfrac{1}{x}\,dx$. A one-parameter family of solutions is defined implicitly by $\ln(v^2 + 1) = -\ln|x| + C$, or, $v^2 + 1 = D/x$. Substitution of $v = y/x$ now gives $x^2 + y^2 = Dx$. Since $(1, 2)$ is on the curve, it follows that $1 + 4 = D$, and the required curve is $y = \sqrt{5x - x^2}$.

43. The differential equation is separable $\sin y\,dy = dx$ with a one-parameter family of solutions defined implicitly by $-\cos y = x + C$.
(a) For $y(0) = \pi/4$, we must set $C = -1/\sqrt{2}$. The solution is then defined implicitly by $\cos y = 1/\sqrt{2} - x$, or explicitly by $y = \mathrm{Cos}^{-1}(1/\sqrt{2} - x)$.
(b) For $y(0) = 7\pi/4$, we again find $C = -1/\sqrt{2}$, and $\cos y = 1/\sqrt{2} - x$. Because $y = 7\pi/4$ is not in the principal value range for the inverse cosine function, we write that $\cos(2\pi - y) = 1/\sqrt{2} - x$, and now take inverse cosines, $2\pi - y = \mathrm{Cos}^{-1}(1/\sqrt{2} - x)$, from which $y = 2\pi - \mathrm{Cos}^{-1}(1/\sqrt{2} - x)$.

44. If $y(t)$ represents the amount of trypsin at any given time, then

$$\frac{dy}{dt} = ky[A - (y - y_0)] = ky(A + y_0 - y),$$

where k is a constant. This equation can be separated,

$$k\,dt = \frac{1}{y(A + y_0 - y)}\,dy = \left(\frac{\dfrac{1}{A + y_0}}{y} + \frac{\dfrac{1}{A + y_0}}{A + y_0 - y}\right)dy.$$

Thus, $\left(\dfrac{1}{y} + \dfrac{1}{A + y_0 - y}\right)dy = k(A + y_0)\,dt$, a one-parameter family of solutions of which is defined implicitly by $\ln y - \ln(A + y_0 - y) = k(A + y_0)t + C$. When this equation is exponentiated and solved for y, the result is

$$y(t) = \frac{D(A + y_0)e^{k(A+y_0)t}}{1 + De^{k(A+y_0)t}}.$$

Since $y(0) = y_0$, it follows that $y_0 = \dfrac{D(A + y_0)}{1 + D}$, from which $D = y_0/A$. Hence,

$$y(t) = \frac{\dfrac{y_0}{A}(A + y_0)e^{k(A+y_0)t}}{1 + \dfrac{y_0}{A}e^{k(A+y_0)t}} = \frac{y_0(A + y_0)}{y_0 + Ae^{-k(A+y_0)t}}.$$

45. If we set $v = ax + by$, then $\dfrac{dv}{dx} = a + b\dfrac{dy}{dx}$. Substitution of these into the differential equation gives

$$\frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = f(v) \qquad \Longrightarrow \qquad \frac{dv}{bf(v)+a} = dx,$$

a separated differential equation.

46. If we set $v = x + y$, then $dv/dx = 1 + dy/dx$, and $\dfrac{dv}{dx} - 1 = v$. This equation can be separated, $\dfrac{1}{v+1}\,dv = dx$, a one-parameter family of solutions of which is defined implicitly by $\ln|v+1| = x + C$. Exponentiation and substitution of $v = x + y$ leads to $y = De^x - x - 1$.

47. If we set $v = x + y$, then $dv/dx = 1 + dy/dx$, and $\dfrac{dv}{dx} - 1 = v^2$. This equation can be separated, $\dfrac{1}{1+v^2}\,dv = dx$, a one-parameter family of solutions of which is defined implicitly by $\mathrm{Tan}^{-1}v = x + C$. If we take tangents and substitute $v = x + y$, we obtain $x + y = \tan(x + C)$. Thus, $y = -x + \tan(x + C)$.

48. If we set $v = 2x + 3y$, then $dv/dx = 2 + 3dy/dx$, and $\dfrac{1}{3}\dfrac{dv}{dx} - \dfrac{2}{3} = \dfrac{1}{v}$. This equation can be separated, $\dfrac{v}{3+2v}\,dv = dx$, a one-parameter family of solutions of which is defined implicitly by

$$x + C = \int\left(\frac{1}{2} - \frac{3/2}{2v+3}\right)dv = \frac{v}{2} - \frac{3}{4}\ln|2v+3|.$$

Substitution of $v = 2x + 3y$ gives $6y - 3\ln|4x + 6y + 3| = D$.

49. If we set $v = x - y$, then $dv/dx = 1 - dy/dx$, and $1 - \dfrac{dv}{dx} = \sin^2 v$. This equation can be separated, $\sec^2 v\,dv = dx$, a one-parameter family of solutions of which is defined implicitly by $\tan v = x + C$. When we take inverse tangents and set $v = x - y$, the result is $x - y = \mathrm{Tan}^{-1}(x + C) + n\pi$, where n is an integer. Thus, $y = x - \mathrm{Tan}^{-1}(x + C) - n\pi$.

50. When we substitute for k and C, and separate variables,

$$\frac{dN}{N(1 - N/10^6)} = dt \quad \Longrightarrow \quad \frac{10^6 dN}{N(10^6 - N)} = dt \quad \Longrightarrow \quad \left(\frac{1}{N} + \frac{1}{10^6 - N}\right)dN = dt.$$

A one-parameter family of solutions is defined implicitly by

$$\ln|N| - \ln|10^6 - N| = t + D \quad \Longrightarrow \quad \ln\left|\frac{N}{10^6 - N}\right| = t + D \quad \Longrightarrow \quad \frac{N}{10^6 - N} = \pm e^{t+D} = Ee^t,$$

where $E = \pm e^D$. Consequently, $N = (10^6 - N)Ee^t \Longrightarrow N = \dfrac{10^6 Ee^t}{1 + Ee^t} = \dfrac{10^6}{1 + Fe^{-t}}$, where $F = 1/E$. From $N(0) = 100$, we obtain $100 = 10^6/(1 + F) \Longrightarrow F = 9999$. The number of bacteria is therefore $N(t) = 10^6/(1 + 9999e^{-t})$.

51. When we separate variables,

$$\frac{dN}{N(1 - N/C)} = k\,dt \quad \Longrightarrow \quad \frac{C\,dN}{N(C - N)} = k\,dt \quad \Longrightarrow \quad \left(\frac{1}{N} + \frac{1}{C - N}\right)dN = k\,dt.$$

A one-parameter family of solutions is defined implicitly by

$$\ln|N| - \ln|C - N| = kt + D \quad \Longrightarrow \quad \ln\left|\frac{N}{C - N}\right| = kt + D \quad \Longrightarrow \quad \frac{N}{C - N} = \pm e^{kt+D} = Ee^{kt},$$

where $E = \pm e^D$. Consequently, $N = (C - N)Ee^{kt} \Longrightarrow N = \dfrac{CEe^{kt}}{1 + Ee^{kt}} = \dfrac{C}{1 + Fe^{-kt}}$, where $F = 1/E$. From $N(0) = N_0$, we obtain $N_0 = C/(1 + F) \Longrightarrow F = (C - N_0)/N_0$. The population is therefore $N(t) = C/\{1 + [(C - N_0)/N_0]e^{-kt}\}$.

52. The differential equation is separable, $\dfrac{dw}{aw^{2/3} - bw} = dt$, in which case a one-parameter family of solutions is defined implicitly by

$$\int \frac{1}{aw^{2/3} - bw} dw = t + C.$$

If we set $u = w^{1/3} \Longrightarrow w = u^3$, and $dw = 3u^2\,du$, then

$$t + C = \int \frac{3u^2}{au^2 - bu^3} du = 3\int \frac{1}{a - bu} du = -\frac{3}{b} \ln|a - bu|.$$

Consequently,

$$\ln|a - bu| = -\frac{b}{3}(t + C) \quad \Longrightarrow \quad |a - bu| = e^{-b(t+C)/3} \quad \Longrightarrow \quad u = \frac{a}{b} + De^{-bt/3},$$

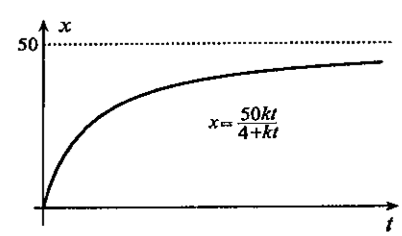
where $D = \pm(1/b)e^{-bC/3}$. Since $u = w^{1/3}$, we find $w = (a/b + De^{-bt/3})^3$. From $w(0) = w_0$, we obtain $w_0 = (a/b + D)^3 \Longrightarrow D = w_0^{1/3} - a/b$. Finally, then

$$w(t) = \left[\frac{a}{b} + \left(w_0^{1/3} - \frac{a}{b}\right) e^{-bt/3}\right]^3 = \left[\frac{a}{b}\left(1 - e^{-bt/3}\right) + w_0^{1/3} e^{-bt/3}\right]^3.$$

53. If $x(t)$ represents the number of grams of dissolved chemical at time t, then

$$\frac{dx}{dt} = k(50 - x)\left(\frac{25}{100} - \frac{x}{200}\right) = \frac{k}{200}(50 - x)^2,$$

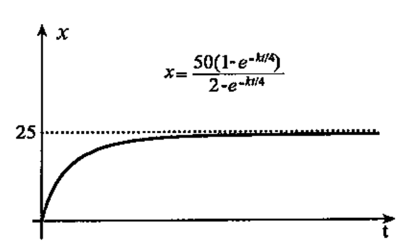
where k is a constant. This equation can be separated, $\dfrac{1}{(50 - x)^2}\,dx = \dfrac{k}{200}\,dt$, and a one-parameter family of solutions is defined implicitly by $\dfrac{1}{50 - x} = \dfrac{kt}{200} + C$. Since $x(0) = 0$, it follows that $1/50 = C$, and $\dfrac{1}{50 - x} = \dfrac{kt}{200} + \dfrac{1}{50}$. When we solve this equation for x, we obtain $x(t) = \dfrac{50kt}{4 + kt}$ g.



(b) In this case

$$\frac{dx}{dt} = k(50 - x)\left(\frac{25}{100} - \frac{x}{100}\right) = \frac{k}{100}(50 - x)(25 - x).$$

Once again we separate variables,
$\dfrac{k}{100}\,dt = \dfrac{1}{(50 - x)(25 - x)}\,dx = \left(\dfrac{-1/25}{50 - x} + \dfrac{1/25}{25 - x}\right) dx.$
Integration gives $\dfrac{kt}{4} + C = \ln(50 - x) - \ln(25 - x)$.
Since $x(0) = 0$, it follows that $C = \ln 50 - \ln 25 = \ln 2$, and $\dfrac{kt}{4} + \ln 2 = \ln\left(\dfrac{50 - x}{25 - x}\right)$. When we solve this equation for x, the result is $x(t) = \dfrac{50(1 - e^{-kt/4})}{2 - e^{-kt/4}}$ g.



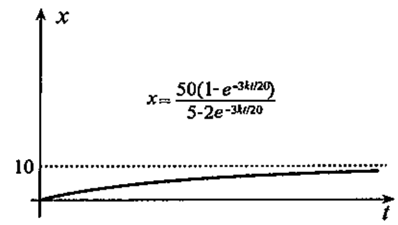
(c) In this case

$$\frac{dx}{dt} = k(10 - x)\left(\frac{25}{100} - \frac{x}{100}\right) = \frac{k}{100}(10 - x)(25 - x).$$

We separate variables again,

$$\frac{k}{100}\,dt = \frac{1}{(10-x)(25-x)}\,dx = \left(\frac{1/15}{10-x} + \frac{-1/15}{25-x}\right)dx,$$

and integrate for $\frac{3kt}{20} + C = -\ln(10-x) + \ln(25-x)$.
Since $x(0)=0$, it follows that $C = -\ln 10 + \ln 25$
$= \ln(5/2)$, and $\frac{3kt}{20} + \ln\left(\frac{5}{2}\right) = \ln\left(\frac{25-x}{10-x}\right)$.
When this equation is solved for x, the result is
$x(t) = \dfrac{50(1-e^{-3kt/20})}{5-2e^{-3kt/20}}$ g.



54. When H is a constant, the differential becomes

$$\frac{dv}{dt} = \frac{K}{n}\left[\frac{v^2-(D-\gamma-H)v-DH}{v(v-D)}\right] \qquad \Longrightarrow \qquad \frac{v(v-D)}{v^2-(D-\gamma-H)v-DH}dv = \frac{K}{n}dt.$$

a separated differential equation. A one-parameter family of solutions is defined implicitly by

$$\int\left[1+\frac{-(H+\gamma)v+DH}{v^2-(D-\gamma-H)v-DH}\right]dv = \frac{K}{n}t + C.$$

The roots of the quadratic $v^2-(D-\gamma-H)v-DH=0$ are

$$v = \frac{(D-\gamma-H)\pm\sqrt{(D-\gamma-H)^2+4DH}}{2}.$$

Let us denote them by r_1 and r_2 where r_1 uses the positive radical and r_2 the negative one. Then

$$\frac{Kt}{n}+C = \int\left[1+\frac{DH-(H+\gamma)v}{(v-r_1)(v-r_2)}\right]dv = v + \int\left[\frac{\dfrac{DH-(H+\gamma)r_1}{r_1-r_2}}{v-r_1} + \frac{\dfrac{DH-(H+\gamma)r_2}{r_2-r_1}}{v-r_2}\right]dv$$

$$= v + \frac{1}{r_1-r_2}\left\{[DH-(H+\gamma)r_1]\ln|v-r_1| - [DH-(H+\gamma)r_2]\ln|v-r_2|\right\}.$$

For $v(0)=0$,

$$\frac{1}{r_1-r_2}\left\{[DH-(H+\gamma)r_1]\ln r_1 - [DH-(H+\gamma)r_2]\ln r_2\right] = C.$$

Thus, $v(t)$ is defined implicitly by

$$\begin{aligned}\frac{Kt}{n} &= v + \frac{1}{r_1-r_2}\left\{[DH-(H+\gamma)r_1]\ln|v-r_1| - [DH-(H+\gamma)r_2]\ln|v-r_2|\right\}\\ &\quad - \frac{1}{r_1-r_2}\left\{[DH-(H+\gamma)r_1]\ln r_1 - [DH-(H+\gamma)r_2]\ln r_2\right]\\ &= v + \frac{1}{r_1-r_2}\left\{[DH-(H+\gamma)r_1]\ln\left|\frac{v-r_1}{r_1}\right| - [DH-(H+\gamma)r_2]\ln\left|\frac{v-r_2}{r_2}\right|\right\}.\end{aligned}$$

55. Suppose that the snow started falling T hours before 12:00. We assume first that snow falls at a constant rate of r metres per hour. The depth of snow on the ground at time t (if we choose $t=0$ at 12:00) is $d = r(t+T)$. Next we assume that the speed v of the plow (in kilometres per hour) is inversely proportional to the depth of snow (in metres),

$$v = \frac{k}{d} = \frac{k}{r(t+T)} = \frac{R}{t+T}, \quad R = \frac{k}{r}.$$

Since $v = dx/dt$ (where x measures distance travelled), integration of this equation gives $x(t) = R\ln(t+T)+C$. We now use the conditions $x(0)=0$, $x(1)=2$, and $x(2)=3$:

$$0 = R\ln T + C, \quad 2 = R\ln(1+T) + C, \quad 3 = R\ln(2+T) + C.$$

When each of these is solved for C and results equated,

$$-R\ln T = 2 - R\ln(1+T), \quad -R\ln T = 3 - R\ln(2+T).$$

If these are solved for R and results equated, the equation $T^2 + T - 1 = 0$ is obtained. Solutions are $T = (-1 \pm \sqrt{5})/2$. Since T must be positive, $T = (\sqrt{5} - 1)/2$ hours, or 37 minutes. Snow therefore started falling at 11:23.

56. If $S(t)$ represents the amount of drug in the dog as a function of time t, then $\dfrac{dS}{dt} = kS$, where $k < 0$ is a constant. Separation of this equation gives $\dfrac{1}{S}\,dS = k\,dt$, a one-parameter family of solutions of which is defined by $\ln S = kt + C$. Exponentiation gives $S = De^{kt}$. If S_0 represents the amount injected at time $t = 0$, then $S_0 = D$, and $S = S_0e^{kt}$. Since $S = S_0/2$ when $t = 5$, it follows that $S_0/2 = S_0e^{5k}$, and this implies that $k = -(1/5)\ln 2$. At the end of the one hour operation, the amount of drug in the body must be 400 mg, and therefore $400 = S_0e^k$. Thus, $S_0 = 400e^{-k} = 459.5$ mg.

57. It is a change of variable of integration $y = y(x)$.

58. We can separate this equation, $\left(\dfrac{y^6-1}{y^4}\right)dy = (x^2+2)\,dx$, and therefore a one-parameter family of solutions is defined implicitly by $\dfrac{y^3}{3} + \dfrac{1}{3y^3} = \dfrac{x^3}{3} + 2x + C$. For $y(1) = 1$, we must have $1/3 + 1/3 = 1/3 + 2 + C$. Thus, $C = -5/3$, and $\dfrac{y^3}{3} + \dfrac{1}{3y^3} = \dfrac{x^3}{3} + 2x - \dfrac{5}{3}$. Multiplication by $3y^3$ gives $y^3(x^3+6x-5) = 1 + y^6$.

59. (a) If $x(t)$ represents the amount of C in the mixture at time t, then

$$\frac{dx}{dt} = k\left(20 - \frac{2x}{3}\right)\left(10 - \frac{x}{3}\right) = \frac{2k}{9}(30-x)^2.$$

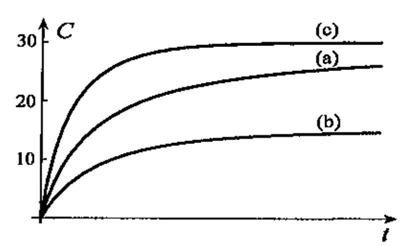
We can separate this equation, $\dfrac{dx}{(30-x)^2} = \dfrac{2k\,dt}{9} \implies \dfrac{1}{30-x} = \dfrac{2kt}{9} + C$. For $x(0) = 0$, we must have $1/30 = C$, and therefore $\dfrac{1}{30-x} = \dfrac{2kt}{9} + \dfrac{1}{30} \implies x(t) = \dfrac{600kt}{20kt+3}$ g.

(b) In this case the differential equation is $\dfrac{dx}{dt} = k\left(20 - \dfrac{2x}{3}\right)\left(5 - \dfrac{x}{3}\right) = \dfrac{2k}{9}(30-x)(15-x)$. It is separable, $\dfrac{2k\,dt}{9} = \dfrac{dx}{(30-x)(15-x)} = \dfrac{1}{15}\left(\dfrac{1}{15-x} - \dfrac{1}{30-x}\right)dx$, and a one-parameter family of solutions is defined implicitly by $10kt/3 + C = \ln|30-x| - \ln|15-x|$. Since x cannot exceed 15, we may drop absolute values. The condition $x(0) = 0$ implies that $C = \ln 30 - \ln 15 = \ln 2$. Hence, $10kt/3 + \ln 2 = \ln(30-x) - \ln(15-x)$. When we exponentiate and solve for x, the result is $x(t) = \dfrac{30(1-e^{-10kt/3})}{2-e^{-10kt/3}}$ g.

(c) In this case the differential equation is

$$\frac{dx}{dt} = k\left(20 - \frac{2x}{3}\right)\left(20 - \frac{x}{3}\right) = \frac{2k}{9}(30-x)(60-x).$$

The equation is separable and the solution is similar to that in part (b). The result is $x(t) = \dfrac{60(1-e^{-20kt/3})}{2-e^{-20kt/3}}$ g. Graphs of all three functions are shown to the right.



60. If $x(t)$ and $y(t)$ are x and y coordinates of the bird as functions of time t, then

$$\frac{dx}{dt} = -V\cos\theta, \quad \frac{dy}{dt} = v - V\sin\theta.$$

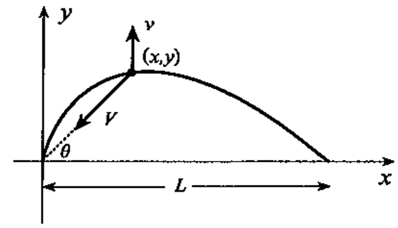
Division of these gives

$$\frac{dy}{dx} = \frac{v - V\sin\theta}{-V\cos\theta}.$$

Since $\tan\theta = y/x$ it follows that

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}, \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}},$$

and therefore



$$\frac{dy}{dx} = \frac{v - \dfrac{Vy}{\sqrt{x^2+y^2}}}{\dfrac{-Vx}{\sqrt{x^2+y^2}}} = \frac{Vy - v\sqrt{x^2+y^2}}{Vx}.$$

This is a homogeneous differential equation so that we set $y = px$ and $dy/dx = p + x\,dp/dx$,

$$p + x\frac{dp}{dx} = \frac{Vpx - v\sqrt{x^2+p^2x^2}}{Vx} = p - \frac{v\sqrt{1+p^2}}{V}.$$

We can now separate, $\dfrac{1}{\sqrt{1+p^2}}\,dp = -\dfrac{v}{Vx}\,dx$, and a one-parameter family of solutions is defined implicitly by $\displaystyle\int \frac{1}{\sqrt{1+p^2}}\,dp = -\frac{v}{V}\ln x + C$. If we set $p = \tan\phi$ and $dp = \sec^2\phi\,d\phi$, then

$$C - \frac{v}{V}\ln x = \int \frac{\sec^2\phi}{\sec\phi}d\phi = \ln|\sec\phi + \tan\phi| = \ln|\sqrt{1+p^2} + p|.$$

When we exponentiate, $p + \sqrt{1+p^2} = Dx^{-v/V} \implies \sqrt{1+p^2} = Dx^{-v/V} - p$. Squaring gives $1 + p^2 = D^2x^{-2v/V} - 2pDx^{-v/V} + p^2$. This equation can be solved for p, and therefore

$$y = px = \frac{x}{2}\left(Dx^{-v/V} - \frac{1}{D}x^{v/V}\right).$$

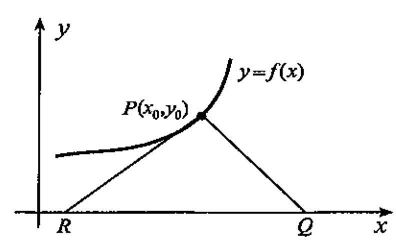
Since $y(L) = 0$, it follows that $0 = \dfrac{L}{2}\left(DL^{-v/V} - \dfrac{1}{D}L^{v/V}\right)$, and therefore $D = L^{v/V}$. The curve followed by the bird is

$$y = \frac{x}{2}\left(L^{v/V}x^{-v/V} - L^{-v/V}x^{v/V}\right) = \frac{x}{2}\left[\left(\frac{L}{x}\right)^{v/V} - \left(\frac{x}{L}\right)^{v/V}\right] = \frac{L}{2}\left[\left(\frac{x}{L}\right)^{1-v/V} - \left(\frac{x}{L}\right)^{1+v/V}\right].$$

61. Equations of the tangent and normal lines at any point $P(x_0, y_0)$ on the required curve $y = f(x)$ are $y - y_0 = f'(x_0)(x - x_0)$ and $y - y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$. The x-coordinates of Q and R are

$$x_Q = x_0 + f'(x_0)y_0, \quad x_R = x_0 - \frac{y_0}{f'(x_0)}.$$

Since the area of ΔPQR is equal to the slope of the curve,



$$f'(x_0) = \frac{1}{2}\|PQ\|\|PR\| = \frac{1}{2}\sqrt{[f'(x_0)y_0]^2 + y_0{}^2}\sqrt{\left[\frac{y_0}{f'(x_0)}\right]^2 + y_0{}^2}.$$

Since $P(x_0, y_0)$ is any point on the curve, we may drop subscripts,

$$2f'(x) = y^2\sqrt{[f'(x)]^2+1}\sqrt{\frac{1}{[f'(x)]^2}+1} = \frac{y^2\{[f'(x)]^2+1\}}{|f'(x)|}.$$

Since $f'(x)$ must be nonnegative, we may write that

$$2[f'(x)]^2 = y^2\{[f'(x)]^2+1\} \quad\Longrightarrow\quad \frac{dy}{dx} = \sqrt{\frac{y^2}{2-y^2}} = \frac{\pm y}{\sqrt{2-y^2}}.$$

If we choose the positive sign so that dy/dx will be positive at $(1,1)$, then

$$\frac{\sqrt{2-y^2}}{y}dy = dx \quad\Longrightarrow\quad x + C = \int\frac{\sqrt{2-y^2}}{y}dy.$$

We set $y = \sqrt{2}\sin\theta$ and $dy = \sqrt{2}\cos\theta\,d\theta$,

$$\begin{aligned} x + C &= \int\frac{\sqrt{2}\cos\theta}{\sqrt{2}\sin\theta}\sqrt{2}\cos\theta\,d\theta = \sqrt{2}\int(\csc\theta - \sin\theta)\,d\theta \\ &= \sqrt{2}[\ln|\csc\theta - \cot\theta| + \cos\theta] = \sqrt{2}\ln\left|\frac{\sqrt{2}}{y} - \frac{\sqrt{2-y^2}}{y}\right| + \sqrt{2-y^2}. \end{aligned}$$

For $y(1) = 1$, we must have $1 + C = \sqrt{2}\ln(\sqrt{2}-1) + 1$, and therefore the curve is defined implicitly by

$$x = \sqrt{2}\ln\left|\frac{\sqrt{2}-\sqrt{2-y^2}}{y}\right| + \sqrt{2-y^2} - \sqrt{2}\ln(\sqrt{2}-1).$$

EXERCISES 15.3

1. An integrating factor is $e^{\int 2x\,dx} = e^{x^2}$. When the differential equation is multiplied by e^{x^2},

$$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = 4xe^{x^2} \qquad\Longrightarrow\qquad \frac{d}{dx}(ye^{x^2}) = 4xe^{x^2}.$$

Integration now gives $ye^{x^2} = 2e^{x^2} + C \implies y = 2 + Ce^{-x^2}$.

2. An integrating factor is $e^{\int 2/x\,dx} = e^{2\ln|x|} = x^2$. When the differential equation is multiplied by x^2,

$$x^2\frac{dy}{dx} + 2xy = 6x^5 \qquad\Longrightarrow\qquad \frac{d}{dx}(yx^2) = 6x^5.$$

Integration now gives $yx^2 = x^6 + C \implies y = x^4 + C/x^2$.

3. If we write $\dfrac{dy}{dx} + 2y = x$, an integrating factor is $e^{\int 2\,dx} = e^{2x}$. When we multiply the differential equation by e^{2x},

$$e^{2x}\frac{dy}{dx} + 2ye^{2x} = xe^{2x} \qquad\Longrightarrow\qquad \frac{d}{dx}(ye^{2x}) = xe^{2x}.$$

Integration now gives $ye^{2x} = \dfrac{x}{2}e^{2x} - \dfrac{1}{4}e^{2x} + C \implies y = x/2 - 1/4 + Ce^{-2x}$.

4. An integrating factor is $e^{\int\cot x\,dx} = e^{\ln|\sin x|} = |\sin x|$. For either $\sin x < 0$ or $\sin x > 0$, multiplication of the differential equation by $|\sin x|$ gives

$$\sin x\frac{dy}{dx} + y\cos x = 5\sin x e^{\cos x} \qquad\Longrightarrow\qquad \frac{d}{dx}(y\sin x) = 5\sin x\,e^{\cos x}.$$

Integration gives $y\sin x = -5e^{\cos x} + C \implies y = \csc x(C - 5e^{\cos x})$.

5. If we write $\dfrac{dy}{dx}+\dfrac{2xy}{x^2+1}=\dfrac{-x^2}{x^2+1}$, an integrating factor is $e^{\int 2x/(x^2+1)\,dx}=e^{\ln(x^2+1)}=x^2+1$. When we multiply the differential equation by x^2+1,

$$(x^2+1)\frac{dy}{dx}+2xy=-x^2 \qquad \Longrightarrow \qquad \frac{d}{dx}[y(x^2+1)]=-x^2.$$

Integration now gives $y(x^2+1)=-x^3/3+C \implies y=(3C-x^3)/(3x^2+3)$.

6. If we write $\dfrac{dy}{dx}-\dfrac{2}{x+1}y=2$, an integrating factor is $e^{\int -2/(x+1)\,dx}=e^{-2\ln|x+1|}=1/(x+1)^2$. When we multiply the differential equation by $1/(x+1)^2$,

$$\frac{1}{(x+1)^2}\frac{dy}{dx}-\frac{2}{(x+1)^3}y=\frac{2}{(x+1)^2} \qquad \Longrightarrow \qquad \frac{d}{dx}\left[\frac{y}{(x+1)^2}\right]=\frac{2}{(x+1)^2}.$$

Integration now gives $\dfrac{y}{(x+1)^2}=\dfrac{-2}{x+1}+C \implies y=-2(x+1)+C(x+1)^2$.

7. The differential equation can be expressed in the form $\dfrac{d}{dx}\left(\dfrac{y}{x}\right)=\dfrac{1}{x^3}$, so that $\dfrac{y}{x}=-\dfrac{1}{2x^2}+C$, and $y=\dfrac{-1}{2x}+Cx$.

8. Since $dy/dx-y=e^{2x}$, an integrating factor is $e^{\int -dx}=e^{-x}$. When we multiply the differential equation by this integrating factor,

$$e^{-x}\frac{dy}{dx}-ye^{-x}=e^x \qquad \Longrightarrow \qquad \frac{d}{dx}(ye^{-x})=e^x.$$

Integration gives $ye^{-x}=e^x+C \implies y=e^{2x}+Ce^x$.

9. An integrating factor is $e^{\int dx}=e^x$. When we multiply the differential equation be e^x,

$$e^x\frac{dy}{dx}+ye^x=2e^x\cos x \qquad \Longrightarrow \qquad \frac{d}{dx}(ye^x)=2e^x\cos x.$$

Integration now gives

$$ye^x=2\int e^x\cos x\,dx=e^x(\cos x+\sin x)+C \qquad \Longrightarrow \qquad y=\cos x+\sin x+Ce^{-x}.$$

10. Since $\dfrac{dy}{dx}+\left(\dfrac{2-3x^2}{x^3}\right)y=1$, an integrating factor is $e^{\int \frac{2-3x^2}{x^3}dx}=e^{-1/x^2-3\ln|x|}=\dfrac{e^{-1/x^2}}{|x|^3}$. For either $x<0$ or $x>0$, multiplication of the differential equation by this factor gives

$$\frac{1}{x^3}e^{-1/x^2}\frac{dy}{dx}+\left(\frac{2-3x^2}{x^6}\right)e^{-1/x^2}y=\frac{1}{x^3}e^{-1/x^2} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{y}{x^3}e^{-1/x^2}\right)=\frac{1}{x^3}e^{-1/x^2}.$$

Integration gives $\dfrac{y}{x^3}e^{-1/x^2}=\dfrac{1}{2}e^{-1/x^2}+C \implies y=x^3/2+Cx^3e^{1/x^2}$.

11. An integrating factor is $e^{\int 1/(x\ln x)dx}=e^{\ln(\ln x)}=\ln x$. When we multiply the differential equation by $\ln x$,

$$\ln x\frac{dy}{dx}+\frac{1}{x}y=x^2\ln x \qquad \Longrightarrow \qquad \frac{d}{dx}(y\ln x)=x^2\ln x.$$

Integration now gives

$$y\ln x=\int x^2\ln x\,dx=\frac{x^3}{3}\ln x-\frac{x^3}{9}+C \qquad \Longrightarrow \qquad y=\frac{x^3}{3}+\frac{9C-x^3}{9\ln x}.$$

12. If we write $\dfrac{dy}{dx} - (2\cot 2x)y = 1 - 2x\cot 2x - 2\csc 2x$, an integrating factor is $e^{\int -2\cot 2x\,dx} = e^{-\ln|\sin 2x|}$ $= |\csc 2x|$. For either $\csc 2x < 0$ or $\csc 2x > 0$, multiplication of the differential equation by $|\csc 2x|$ leads to

$$\csc 2x\frac{dy}{dx} - 2y\cot 2x\csc 2x = \csc 2x - 2x\cot 2x\ \csc 2x - 2\csc^2 2x,$$

or,

$$\frac{d}{dx}(y\csc 2x) = \csc 2x - 2x\cot 2x\ \csc 2x - 2\csc^2 2x.$$

Integration gives $y\csc 2x = x\csc 2x + \cot 2x + C \implies y = x + \cos 2x + C\sin 2x$.

13. An integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$. When we multiply the differential equation by e^{x^3},

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = x^2e^{x^3} \qquad \implies \qquad \frac{d}{dx}(ye^{x^3}) = x^2e^{x^3}.$$

Integration now gives $ye^{x^3} = \dfrac{1}{3}e^{x^3} + C \implies y = \dfrac{1}{3} + Ce^{-x^3}$. For $y(1) = 2$, we must have $2 = 1/3 + Ce^{-1}$. Hence, $C = 5e/3$, and $y = (1 + 5e^{1-x^3})/3$.

14. Since $dy/dx + y = e^x\sin x$, an integrating factor is $e^{\int dx} = e^x$. Multiplication of the differential equation by this factor gives

$$e^x\frac{dy}{dx} + ye^x = e^{2x}\sin x \qquad \implies \qquad \frac{d}{dx}(ye^x) = e^{2x}\sin x.$$

Integration now yields $ye^x = \dfrac{1}{5}(2e^{2x}\sin x - e^{2x}\cos x) + C \implies y = e^x(2\sin x - \cos x)/5 + Ce^{-x}$. For $y(0) = -1$, we must have $-1 = -1/5 + C$. Thus, $y = e^x(2\sin x - \cos x)/5 - (4/5)e^{-x}$.

15. An integrating factor is $e^{\int x^3/(x^4+1)\,dx} = e^{(1/4)\ln(x^4+1)} = (x^4+1)^{1/4}$. When we multiply the differential equation by $(x^4+1)^{1/4}$,

$$(x^4+1)^{1/4}\frac{dy}{dx} + \frac{x^3y}{(x^4+1)^{3/4}} = x^7(x^4+1)^{1/4} \qquad \implies \qquad \frac{d}{dx}[y(x^4+1)^{1/4}] = x^7(x^4+1)^{1/4}.$$

Integration now gives $y(x^4+1)^{1/4} = \displaystyle\int x^7(x^4+1)^{1/4}\,dx$. If we set $u = x^4 + 1$ and $du = 4x^3\,dx$, then

$$\begin{aligned} y(x^4+1)^{1/4} &= \int (u-1)u^{1/4}\left(\frac{du}{4}\right) = \frac{1}{4}\left(\frac{4}{9}u^{9/4} - \frac{4}{5}u^{5/4}\right) + C \\ &= \frac{1}{9}(x^4+1)^{9/4} - \frac{1}{5}(x^4+1)^{5/4} + C. \end{aligned}$$

For $y(0) = 1$, we must have $1 = 1/9 - 1/5 + C \implies C = 49/45$. Hence, $y = (x^4+1)^2/9 - (x^4+1)/5 + (49/45)(x^4+1)^{-1/4}$.

16. If we write $dx/dy + (1/y)x = y^2$, the differential equation is linear in $x = x(y)$. An integrating factor is $e^{\int (1/y)\,dy} = e^{\ln|y|} = |y|$. For either $y < 0$ or $y > 0$, multiplication by $|y|$ yields

$$y\frac{dx}{dy} + x = y^3 \qquad \implies \qquad \frac{d}{dy}(yx) = y^3.$$

Integration gives $yx = \dfrac{y^4}{4} + C$, an implicit definition for solutions.

17. If we set $z = y^{1-n}$, then $dz/dx = (1-n)y^{-n}dy/dx$. If we divide all terms in the differential equation by y^n, and substitute

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \qquad \implies \qquad \frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x).$$

Multiplication by $1 - n$ gives a linear first order equation.

18. If we set $z = 1/y$, then $dz/dx = (-1/y^2)dy/dx$, and $-y^2\dfrac{dz}{dx} + y = y^2e^x$. Division by $-y^2$ gives $\dfrac{dz}{dx} - \dfrac{1}{y} = -e^x \Longrightarrow \dfrac{dz}{dx} - z = -e^x$. An integrating factor for this equation is $e^{\int -dx} = e^{-x}$. When we multiply the differential equation by this factor,

$$e^{-x}\frac{dz}{dx} - ze^{-x} = -1 \qquad \Longrightarrow \qquad \frac{d}{dx}(ze^{-x}) = -1.$$

Integration now yields

$$ze^{-x} = -x + C \quad \Longrightarrow \quad z = (C-x)e^x \quad \Longrightarrow \quad \frac{1}{y} = (C-x)e^x \quad \Longrightarrow \quad y = \frac{e^{-x}}{C-x}.$$

19. If we set $z = 1/y$, then $dz/dx = (-1/y^2)dy/dx$, and $-y^2\dfrac{dz}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$. Division by $-y^2$ gives $\dfrac{dz}{dx} - \dfrac{1}{xy} = -\dfrac{1}{x^2} \Longrightarrow \dfrac{dz}{dx} - \dfrac{z}{x} = -\dfrac{1}{x^2}$. An integrating factor for this equation is $e^{\int -(1/x)\,dx} = 1/|x|$. For $x > 0$ or $x < 0$, multiplication of the differential equation by this factor gives

$$\frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -\frac{1}{x^3} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{z}{x}\right) = -\frac{1}{x^3}.$$

Integration now yields

$$\frac{z}{x} = \frac{1}{2x^2} + C \Longrightarrow z = \frac{1}{2x} + Cx \quad \Longrightarrow \quad \frac{1}{y} = \frac{1}{2x} + Cx \Longrightarrow y = \frac{2x}{2Cx^2+1}.$$

20. If we set $z = 1/y$, then $dz/dx = -(1/y^2)dy/dx$, and $-y^2\dfrac{dz}{dx} - y = -(x^2+2x)y^2$. Division by $-y^2$ gives $\dfrac{dz}{dx} + \dfrac{1}{y} = x^2 + 2x \Longrightarrow \dfrac{dz}{dx} + z = x^2 + 2x$. An integrating factor for this equation is $e^{\int dx} = e^x$. Multiplication by this factor now yields

$$e^x\frac{dz}{dx} + ze^x = (x^2+2x)e^x \qquad \Longrightarrow \qquad \frac{d}{dx}(ze^x) = (x^2+2x)e^x.$$

This can be integrated to give

$$ze^x = \int (x^2+2x)e^x\,dx = x^2e^x + C \quad \Longrightarrow \quad z = \frac{1}{y} = x^2 + Ce^{-x} \quad \Longrightarrow \quad y = \frac{1}{x^2+Ce^{-x}}.$$

21. In the form $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^5$, we see that the differential equation is Bernoulli. We set $z = y^{-4}$ and $dz/dx = -4y^{-5}dy/dx$. Substitution gives $-\dfrac{1}{4}y^5\dfrac{dz}{dx} + \dfrac{y}{x} = x^2y^5$. Multiplication by $-4y^{-5}$ leads to

$$\frac{dz}{dx} - \frac{4}{xy^4} = -4x^2 \quad \Longrightarrow \quad \frac{dz}{dx} - \frac{4}{x}z = -4x^2.$$

An integrating factor for this linear equation is $e^{\int -4/x\,dx} = e^{-4\ln|x|} = 1/x^4$. Multiplication by this factor now yields

$$\frac{1}{x^4}\frac{dz}{dx} - \frac{4}{x^5}z = -\frac{4}{x^2} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{z}{x^4}\right) = -\frac{4}{x^2}.$$

Integration gives

$$\frac{z}{x^4} = \frac{4}{x} + C \quad \Longrightarrow \quad z = 4x^3 + Cx^4.$$

When we set $z = 1/y^4$, we obtain $\dfrac{1}{y^4} = 4x^3 + Cx^4 \Longrightarrow y = \dfrac{\pm 1}{(4x^3+Cx^4)^{1/4}}$.

22. If we set $z = y^{-3}$, then $dz/dx = -(3/y^4)dy/dx$, and $-\dfrac{y^4}{3}\dfrac{dz}{dx} + y\tan x = y^4 \sin x$. Multiplication by $-3y^{-4}$ gives

$$\frac{dz}{dx} - \frac{3\tan x}{y^3} = -3\sin x \qquad \Longrightarrow \qquad \frac{dz}{dx} - (3\tan x)z = -3\sin x.$$

An integrating factor for this equation is $e^{\int -3\tan x\,dx} = e^{3\ln|\cos x|} = |\cos x|^3$. For either $\cos x < 0$ or $\cos x > 0$, multiplication by $|\cos x|^3$ gives

$$\cos^3 x\frac{dz}{dx} - 3z\cos^2 x \sin x = -3\cos^3 x \sin x \qquad \Longrightarrow \qquad \frac{d}{dx}(z\cos^3 x) = -3\sin x\,\cos^3 x.$$

Integration now yields

$$z\cos^3 x = \frac{3}{4}\cos^4 x + C \quad \Longrightarrow \quad z = \frac{3}{4}\cos x + C\sec^3 x.$$

When we replace z with $1/y^3$,

$$\frac{1}{y^3} = \frac{3}{4}\cos x + C\sec^3 x \quad \Longrightarrow \quad y = \left(\frac{3}{4}\cos x + C\sec^3 x\right)^{-1/3}.$$

23. If glucose is added at rate $R(t)$, the differential equation becomes

$$\frac{dA}{dt} = R(t) - kA \qquad \Longrightarrow \qquad \frac{dA}{dt} + kA = R(t).$$

An integrating factor is $e^{\int k\,dt} = e^{kt}$. When we multiply the differential equation by e^{kt},

$$e^{kt}\frac{dA}{dt} + ke^{kt}A = R(t)e^{kt} \qquad \Longrightarrow \qquad \frac{d}{dt}(Ae^{kt}) = R(t)e^{kt}.$$

Integration now gives

$$Ae^{kt} = \int R(t)e^{kt}\,dt + C \quad \Longrightarrow \quad A(t) = e^{-kt}\int R(t)e^{kt}\,dt + Ce^{-kt}.$$

In order to incorporate the initial condition $A(0) = A_0$, we rewrite the antiderivative as a definite integral with a variable upper limit, $A(t) = e^{-kt}\displaystyle\int_0^t R(u)e^{ku}\,du + Ce^{-kt}$. The initial condition now implies that $A_0 = C$, and therefore $A(t) = \displaystyle\int_0^t R(u)e^{k(u-t)}\,du + A_0e^{-kt}$.

24. If $S(t)$ represents the number of grams of sugar in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which sugar is added less the rate at which it is removed. The rate at which sugar is added is 2 g/min. Since the concentration of sugar at time t is $S/(10^5 + 100t)$ g/mL, the rate at which sugar is removed at time t is $100S/(10^5 + 100t)$ g/min. Thus,

$$\frac{dS}{dt} = 2 - \frac{100S}{10^5 + 100t}, \qquad \Longrightarrow \qquad \frac{dS}{dt} + \frac{S}{1000 + t} = 2.$$

An integrating factor for this equation is $e^{\int 1/(1000+t)\,dt} = e^{\ln|1000+t|} = 1000 + t$. Multiplication of the differential equation by $1000 + t$ gives

$$(1000+t)\frac{dS}{dt} + S = 2(1000+t) \quad \Longrightarrow \quad \frac{d}{dt}[(1000+t)S] = 2(1000+t) \quad \Longrightarrow \quad (1000+t)S = (1000+t)^2 + C.$$

Since $S(0) = 4000$, it follows that $(1000)(4000) = (1000)^2 + C$. Thus, $C = 3 \times 10^6$, and

$$S = 1000 + t + \frac{3 \times 10^6}{1000 + t}\text{ g}.$$

25. If $S(t)$ represents the number of grams of sugar in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which sugar is added less the rate at which it is removed. The rate at which sugar is added is 2 g/min. Since the concentration of sugar at time t is $S/(10^5 - 100t)$ g/mL, the rate at which sugar is removed at time t is $300S/(10^5 - 100t)$ g/min. Thus,

$$\frac{dS}{dt} = 2 - \frac{300S}{10^5 - 100t}, \qquad \Longrightarrow \qquad \frac{dS}{dt} + \frac{3S}{1000 - t} = 2.$$

An integrating factor for this equation is $e^{\int 3/(1000-t)\,dt} = e^{-3\ln|1000-t|} = 1/(1000-t)^3$. Multiplication by $1/(1000-t)^3$ gives

$$\frac{1}{(1000-t)^3}\frac{dS}{dt} + \frac{3S}{(1000-t)^4} = \frac{2}{(1000-t)^3} \quad \Longrightarrow \quad \frac{d}{dt}\left[\frac{S}{(1000-t)^3}\right] = \frac{2}{(1000-t)^3}.$$

Integration now yields

$$\frac{S}{(1000-t)^3} = \frac{1}{(1000-t)^2} + C \quad \Longrightarrow \quad S(t) = 1000 - t + C(1000-t)^3.$$

Since $S(0) = 4000$, it follows that $4000 = 1000 + C(1000)^3$. Thus, $C = 3 \times 10^{-6}$, and $S = 1000 - t + 3 \times 10^{-6}(1000-t)^3$ g. This solution is valid only until the tank empties $0 \le t \le 1000$.

26. If $S(t)$ represents the number of grams of salt in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which salt is added less the rate at which it is removed. The rate at which salt is added is 0.2 g/s. Since the concentration of salt at time t is $S/(10^6 + 5t)$ g/mL, the rate at which salt is removed at time t is $5S/(10^6 + 5t)$ g/s. Thus,

$$\frac{dS}{dt} = \frac{1}{5} - \frac{5S}{10^6 + 5t} \qquad \Longrightarrow \qquad \frac{dS}{dt} + \frac{5S}{10^6 + 5t} = \frac{1}{5}.$$

An integrating factor for this equation is $e^{\int 5/(10^6+5t)\,dt} = e^{\ln|10^6+5t|} = 10^6 + 5t$. Multiplication by $10^6 + 5t$ gives

$$(10^6 + 5t)\frac{dS}{dt} + 5S = \frac{1}{5}(10^6 + 5t) \quad \Longrightarrow \quad \frac{d}{dt}\left[(10^6 + 5t)S\right] = \frac{1}{5}(10^6 + 5t).$$

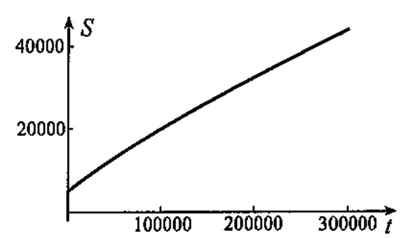
Integration now yields

$$(10^6 + 5t)S = \frac{1}{50}(10^6 + 5t)^2 + C \quad \Longrightarrow \quad S(t) = \frac{1}{50}(10^6 + 5t) + \frac{C}{10^6 + 5t}.$$

Since $S(0) = 5000$, it follows that $5000 = \dfrac{10^6}{50} + \dfrac{C}{10^6}$. This gives $C = -15 \times 10^9$, and therefore

$$S(t) = \frac{1}{50}(10^6 + 5t) - \frac{15 \times 10^9}{10^6 + 5t} \text{ g}.$$

A graph is shown to the right; it is asymptotic to the line $S = (10^6 + 5t)/50$.



27. If $S(t)$ represents the number of grams of salt in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which salt is added less the rate at which it is removed. The rate at which salt is added is 0.2 g/s. Since the concentration of salt at time t is $S/10^6$ g/mL, the rate at which salt is removed at time t is $10S/10^6$ g/s. Thus,

$$\frac{dS}{dt} = \frac{1}{5} - \frac{10S}{10^6} \qquad \Longrightarrow \qquad \frac{dS}{dt} + \frac{S}{10^5} = \frac{1}{5}.$$

An integrating factor for this equation is $e^{\int 1/10^5\,dt} = e^{t/10^5}$. Multiplication by $e^{t/10^5}$ gives

$$e^{t/10^5}\frac{dS}{dt}+\frac{Se^{t/10^5}}{10^5}=\frac{1}{5}e^{t/10^5} \quad\Longrightarrow\quad \frac{d}{dt}(Se^{t/10^5})=\frac{1}{5}e^{t/10^5}.$$

Integration now yields

$$Se^{t/10^5}=20\,000e^{t/10^5}+C \quad\Longrightarrow\quad S(t)=20\,000+Ce^{-t/10^5}.$$

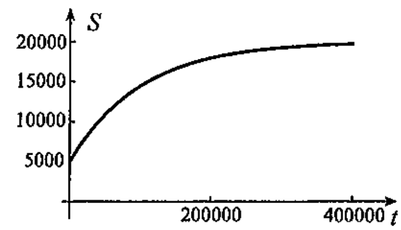
Since $S(0)=5000$, it follows that

$$5000=20\,000+C.$$

Thus, $C=-15\,000$, and $S(t)$ is

$$S(t)=20\,000-15\,000e^{-t/10^5} \text{ g}.$$

A graph of this function is shown to the right. The limit as $t\to\infty$ is 20 000.



28. If $S(t)$ represents the number of grams of salt in the tank as a function of time t, then dS/dt, the rate of change of S with respect to t, must be the rate at which salt is added less the rate at which it is removed. The rate at which salt is added is 0.2 g/s. Since the concentration of salt at time t is $S/(10^6-10t)$ g/mL, the rate at which salt is removed at time t is $20S/(10^6-10t)$ g/s. Thus,

$$\frac{dS}{dt}=\frac{1}{5}-\frac{20S}{10^6-10t} \quad\Longrightarrow\quad \frac{dS}{dt}+\frac{2S}{10^5-t}=\frac{1}{5},$$

valid for $0<t<10^5$. An integrating factor is $e^{\int 2/(10^5-t)\,dt}=e^{-2\ln(10^5-t)}=\dfrac{1}{(10^5-t)^2}$. When we multiply by this factor, the differential equation becomes

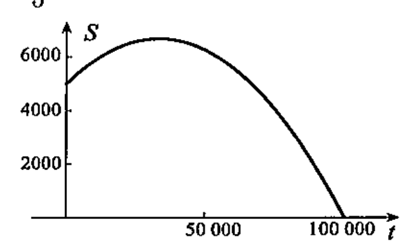
$$\frac{1}{(10^5-t)^2}\frac{dS}{dt}+\frac{2S}{(10^5-t)^3}=\frac{1}{5(10^5-t)^2} \quad\Longrightarrow\quad \frac{d}{dt}\left[\frac{S}{(10^5-t)^2}\right]=\frac{1}{5(10^5-t)^2}.$$

Integration now gives

$$\frac{S}{(10^5-t)^2}=\frac{1}{5(10^5-t)}+C \quad\Longrightarrow\quad S(t)=\frac{10^5-t}{5}+C(10^5-t)^2.$$

Since $S(0)=5000$, it follows that $5000=\dfrac{10^5}{5}+C(10^{10})$, and therefore $C=-15/10^7$. Thus,

$$S(t)=20\,000-\frac{t}{5}-\frac{15}{10^7}(10^5-t)^2.$$



29. If $C(t)$ represents the number of cubic metres of carbon dioxide in the room as a function of time t, then dC/dt, the rate of change of C with respect to t, must be the rate at which carbon dioxide is added less the rate at which it is removed. The rate at which carbon dioxide is added is $1/400$ m^3/min. Since the concentration of carbon dioxide at time t is $C/100$, the rate at which carbon dioxide is removed is $C/20$ m^3/min. Thus,

$$\frac{dC}{dt}=\frac{1}{400}-\frac{C}{20}, \quad\Longrightarrow\quad \frac{dC}{dt}+\frac{C}{20}=\frac{1}{400}.$$

An integrating factor for this equation is $e^{\int 1/20\,dt}=e^{t/20}$. Multiplication by $e^{t/20}$ gives

$$e^{t/20}\frac{dC}{dt}+\frac{Ce^{t/20}}{20}=\frac{e^{t/20}}{400} \quad\Longrightarrow\quad \frac{d}{dt}(Ce^{t/20})=\frac{1}{400}e^{t/20}.$$

Integration now yields

$$Ce^{t/20}=\frac{1}{20}e^{t/20}+D \quad\Longrightarrow\quad C=\frac{1}{20}+De^{-t/20}.$$

Since $C(0) = 1/10$, it follows that $1/10 = 1/20 + D \implies D = 1/20$, and $C(t) = 1/20 + (1/20)e^{-t/20}$. The limit of this function as $t \to \infty$ is $1/20$.

30. If $t = 0$ when the oven is turned on, its temperature is $T_o(t) = 20 + 36t$ for $0 \le t \le 5$, and 200 for $t > 5$. If $T(t)$ is the temperature of the potato, then

$$\frac{dT}{dt} = k(T_o - T), \quad T(0) = 20.$$

For $0 \le t \le 5$, this becomes $dT/dt = -kT + k(20 + 36t)$. An integrating factor for this linear first-order differential equation is e^{kt} so that

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = k(20+36t)e^{kt} \quad \implies \quad \frac{d}{dt}(Te^{kt}) = k(20+36t)e^{kt} \quad \implies \quad Te^{kt} = k\int(20+36t)e^{kt}dt + C.$$

Integration by parts leads to

$$T = ke^{-kt}\left(\frac{20e^{kt}}{k} - \frac{36e^{kt}}{k^2} + \frac{36te^{kt}}{k}\right) + Ce^{-kt} = 20 - \frac{36}{k} + 36t + Ce^{-kt}.$$

For $T(0) = 20$, we have $20 = 20 - 36/k + C \implies C = 36/k$, and

$$T(t) = 20 - \frac{36}{k} + 36t + \frac{36}{k}e^{-kt} = 20 + 36t + \frac{36}{k}(e^{-kt} - 1).$$

For $t > 5$,

$$\frac{dT}{dt} = k(200 - T), \quad T(5) = 200 + \frac{36}{k}(e^{-5k} - 1).$$

This equation is separable,

$$\frac{dT}{200 - T} = k\,dt \quad \implies \quad -\ln|200 - T| = kt + C \quad \implies \quad T = 200 + De^{-kt},$$

where $D = \pm e^{-C}$. The temperature at $t = 5$ requires

$$200 + De^{-5k} = 200 + \frac{36}{k}(e^{-5k} - 1) \quad \implies \quad D = \frac{36}{k}(1 - e^{5k}).$$

Hence, for $t > 5$, temperature is $T(t) = 200 + (36/k)(1 - e^{5k})e^{-kt}$.

31. The energy balance equation is

$$(0.03)(4190)(10) + 2000 = (0.03)(4190)T + 100(4190)\frac{dT}{dt} \quad \implies \quad \frac{dT}{dt} + \frac{3T}{10\,000} = \frac{3257}{419\,000}.$$

An integrating factor is $e^{\int(3/10\,000)\,dt} = e^{3t/10\,000}$, so that the differential equation can be expressed in the form

$$e^{3t/10\,000}\frac{dT}{dt} + \frac{3Te^{3t/10\,000}}{10\,000} = \frac{3257}{419\,000}e^{3t/10\,000} \quad \implies \quad \frac{d}{dt}(Te^{3t/10\,000}) = \frac{3257}{419\,000}e^{3t/10\,000}.$$

Integration now gives

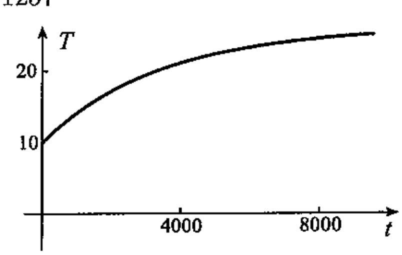
$$Te^{3t/10\,000} = \frac{32\,570}{1257}e^{3t/10\,000} + C \implies T = \frac{32\,570}{1257} + Ce^{-3t/10\,000}.$$

Since $T(0) = 10$, we find

$$10 = \frac{32\,570}{1257} + C \implies C = -\frac{20\,000}{1257}.$$

Temperature of the water is therefore

$$T(t) = \frac{32\,570}{1257} - \frac{20\,000}{1257}e^{-3t/10\,000}.$$



32. The energy balance equation is

$$\left(\frac{100}{t+1}\right)(4190)(10)+2000=\left(\frac{100}{t+1}\right)(4190)T+100(4190)\frac{dT}{dt} \implies \frac{dT}{dt}+\frac{T}{t+1}=\frac{2}{419}+\frac{10}{t+1}.$$

An integrating factor is $e^{\int [1/(t+1)]\,dt}=e^{\ln(t+1)}=t+1$, so that the differential equation can be expressed in the form

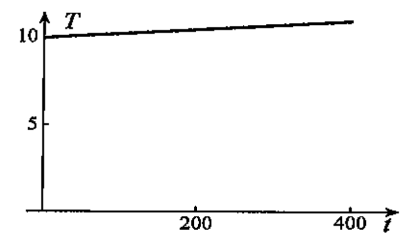
$$(t+1)\frac{dT}{dt}+T=\frac{2}{419}(t+1)+10 \implies \frac{d}{dt}[(t+1)T]=\frac{2}{419}(t+1)+10.$$

Integration now gives

$$(t+1)T=10t+\frac{1}{419}(t+1)^2+C \implies T(t)=\frac{10t}{t+1}+\frac{1}{419}(t+1)+\frac{C}{t+1}.$$

Since $T(0)=10$, we find $10=\dfrac{1}{419}+C \implies C=\dfrac{4189}{419}$.
Temperature of the water is therefore

$$T(t)=\frac{10t}{t+1}+\frac{t+1}{419}+\frac{4189}{419(t+1)}=\frac{4190t+4189}{419(t+1)}+\frac{t+1}{419}.$$



33. The energy balance equation is

$$(0.03)(4190)(10)+20t=(0.03)(4190)T+100(4190)\frac{dT}{dt} \implies \frac{dT}{dt}+\frac{3T}{10\,000}=\frac{t}{20\,950}+\frac{3}{1000}.$$

An integrating factor is $e^{\int (3/10\,000)\,dt}=e^{3t/10\,000}$, so that the differential equation can be expressed in the form

$$e^{3t/10\,000}\frac{dT}{dt}+\frac{3e^{3t/10\,000}T}{10\,000}=\left(\frac{t}{20\,950}+\frac{3}{1000}\right)e^{3t/10\,000} \implies \frac{d}{dt}(Te^{3t/10\,000})=\left(\frac{t}{20\,950}+\frac{3}{1000}\right)e^{3t/10\,000}.$$

Integration by parts on the right leads to

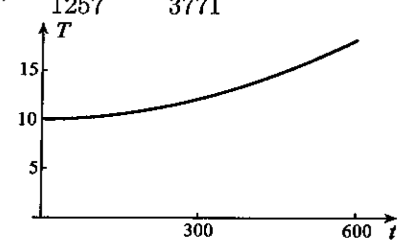
$$Te^{3t/10\,000}=\left(\frac{200t}{1257}-\frac{1\,962\,290}{3771}\right)e^{3t/10\,000}+C \implies T(t)=\frac{200t}{1257}-\frac{1\,962\,290}{3771}+Ce^{-3t/10\,000}.$$

Since $T(0)=10$, we find

$$10=-\frac{1\,962\,290}{3771}+C \implies C=\frac{2\times 10^6}{3771}.$$

Temperature of the water is therefore

$$T(t)=\frac{200t}{1257}-\frac{1\,962\,290}{3771}+\frac{2\times 10^6}{3771}e^{-3t/10\,000}.$$



34. The energy balance equation is

$$(0.03)(4190)(10e^{-t})+2000=(0.03)(4190)T+100(4190)\frac{dT}{dt} \implies \frac{dT}{dt}+\frac{3T}{10\,000}=\frac{2}{419}+\frac{3e^{-t}}{1000}.$$

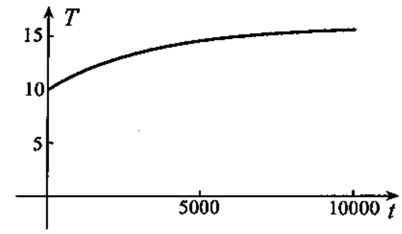
An integrating factor is $e^{\int (3/10\,000)\,dt}=e^{3t/10\,000}$, so that the differential equation can be expressed in the form

$$e^{3t/10\,000}\frac{dT}{dt}+\frac{3Te^{3t/10\,000}}{10\,000}=\left(\frac{2}{419}+\frac{3e^{-t}}{1000}\right)e^{3t/10\,000} \implies \frac{d}{dt}(Te^{3t/10\,000})=\left(\frac{2}{419}+\frac{3e^{-t}}{1000}\right)e^{3t/10\,000}.$$

Integration gives

$$Te^{3t/10\,000} = \frac{20\,000}{1257}e^{3t/10\,000} - \frac{30}{9997}e^{-9997t/10\,000} + C \quad \Longrightarrow \quad T(t) = \frac{20\,000}{1257} - \frac{30}{9997}e^{-t} + Ce^{-3t/10\,000}.$$

Since $T(0) = 10$, we find $10 = \dfrac{20\,000}{1257} - \dfrac{30}{9997} + C$,
and this implies that $C = -\dfrac{74\,240\,000}{12\,566\,229}$.
Temperature of the water is therefore
$T(t) = \dfrac{20\,000}{1257} - \dfrac{30}{9997}e^{-t} - \dfrac{74\,240\,000}{12\,566\,229}e^{-3t/10\,000}$.



35. The differential equation is linear first-order,

$$\frac{dV}{dt} + \frac{V}{RC} = \frac{1}{RC}h(t),$$

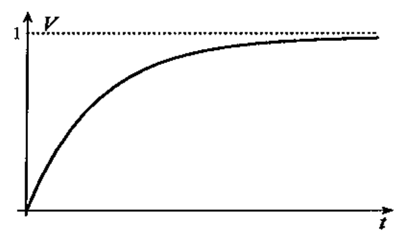
with integrating factor $e^{\int [1/(RC)]dt} = e^{t/(RC)}$. Multiplying the differential equation by this factor leads to

$$\frac{d}{dt}\left[Ve^{t/(RC)}\right] = \frac{1}{RC}h(t)e^{t/(RC)}.$$

Integration for $t > 0$ yields

$$Ve^{t/(RC)} = \frac{1}{RC}\int h(t)e^{t/(RC)}dt = \frac{1}{RC}\int e^{t/(RC)}dt = e^{t/(RC)} + D.$$

Division by $e^{t/(RC)}$ gives
$V(t) = 1 + De^{-t/(RC)}$.
The initial condition $V(0) = 0$ requires $0 = 1 + D$, so that the step response is
$V(t) = 1 - e^{-t/(RC)}$.
Its graph is shown to the right.



36. The differential equation is linear first-order,

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{R}{L}[h(t) - h(t-1)],$$

with integrating factor $e^{\int (R/L)dt} = e^{Rt/L}$. Multiplying the differential equation by this factor leads to

$$\frac{d}{dt}\left[ie^{Rt/L}\right] = \frac{R}{L}[h(t) - h(t-1)]e^{Rt/L}.$$

Integration for $0 < t < 1$ yields

$$ie^{Rt/L} = \frac{R}{L}\int e^{Rt/L}dt = e^{Rt/L} + D.$$

Division by $e^{Rt/L}$ gives $i(t) = 1 + De^{-Rt/L}$. The initial condition $i(0) = 0$ gives $0 = 1 + D$, so that for $0 < t < 1$,

$$i(t) = 1 - e^{-Rt/L}.$$

When $t > 1$, integration gives

$$ie^{Rt/L} = \frac{R}{L}\int (1-1)e^{Rt/L}dt = E \qquad \Longrightarrow \qquad i = Ee^{-Rt/L}.$$

To evaluate E we demaind that the current be continuous at $t = 1$. This requires

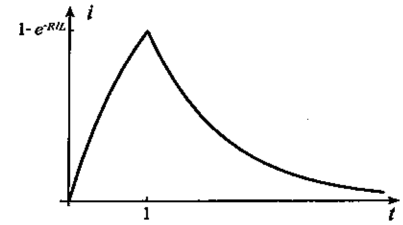
$\lim_{t\to 1^-} i(t) = \lim_{t\to 1^+} i(t)$, or

$1 - e^{-R/L} = Ee^{-R/L} \implies E = e^{R/L} - 1.$

The pulse response function is

$$i(t) = \begin{cases} 1 - e^{-Rt/L}, & 0 \le t \le 1 \\ (e^{R/L} - 1)e^{-Rt/L}, & t > 1 \end{cases}.$$

Its graph is shown to the right.



37. (a) Since $\dfrac{dI}{dt} + \dfrac{R}{L}I = \dfrac{E}{L}$ an integrating factor is $e^{\int (R/L)\,dt} = e^{Rt/L}$. Multiplication of the diferential equation by this factor leads to

$$\frac{d}{dt}\left[Ie^{Rt/L}\right] = \frac{E_0}{L}e^{Rt/L}\sin(\omega t).$$

Integration now yields

$$Ie^{Rt/L} = \frac{E_0 e^{Rt/L}}{L(R^2/L^2 + \omega^2)}\left[\frac{R}{L}\sin(\omega t) - \omega\cos(\omega t)\right] + A$$

or,

$$I(t) = Ae^{-Rt/L} + \frac{E_0}{R^2 + \omega^2 L^2}[R\sin(\omega t) - \omega L\cos(\omega t)].$$

If we set

$$R\sin(\omega t) - \omega L\cos(\omega t) = B\sin(\omega t - \phi) = B\sin(\omega t)\cos\phi - B\cos(\omega t)\sin\phi,$$

then $B\cos\phi = R$ and $B\sin\phi = \omega L$. These equations imply that

$$B = \sqrt{R^2 + \omega^2 L^2} \qquad \text{and} \qquad \tan\phi = \frac{\omega L}{R} \implies \phi = \mathrm{Tan}^{-1}\left(\frac{\omega L}{R}\right).$$

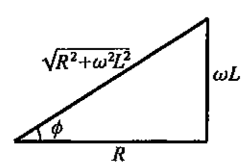
Hence, $I(t) = Ae^{-Rt/L} + \dfrac{E_0}{R^2 + \omega^2 L^2}\sqrt{R^2 + \omega^2 L^2}\sin(\omega t - \phi) = Ae^{-Rt/L} + \dfrac{E_0}{Z}\sin(\omega t - \phi)$,

where $Z = \sqrt{R^2 + \omega^2 L^2}$.

(b) If $I(0) = I_0$, then

$$I_0 = A + \frac{E_0}{Z}\sin(-\phi) = A - \frac{E_0}{Z}\frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} = A - \frac{E_0 \omega L}{Z^2},$$

and therefore $A = I_0 + E_0\omega L/Z^2$.



38. (a) Since $\dfrac{dI}{dt} + \dfrac{1}{RC}I = \dfrac{1}{R}\dfrac{dE}{dt} = \dfrac{\omega}{R}E_0\cos(\omega t)$, an integrating factor is $e^{\int [1/(RC)]\,dt} = e^{t/(RC)}$. Multiplication by this factor gives

$$e^{t/(RC)}\frac{dI}{dt} + \frac{1}{RC}Ie^{t/(RC)} = \frac{\omega}{R}E_0 e^{t/(RC)}\cos(\omega t) \implies \frac{d}{dt}\left[Ie^{t/(RC)}\right] = \frac{\omega}{R}E_0 e^{t/(RC)}\cos(\omega t).$$

Integration now yields

$$Ie^{t/(RC)} = \frac{\omega E_0}{R}\left\{\frac{e^{t/(RC)}}{\omega^2 + 1/(RC)^2}[(1/(RC))\cos(\omega t) + \omega\sin(\omega t)]\right\} + A.$$

Thus, $I(t) = Ae^{-t/(RC)} + \dfrac{\omega E_0}{R[\omega^2 + 1/(RC)^2]}\{[1/(RC)]\cos(\omega t) + \omega\sin(\omega t)\}$. If we set

$$\frac{1}{RC}\cos(\omega t) + \omega\sin(\omega t) = B\sin(\omega t - \phi) = B\sin(\omega t)\cos\phi - B\cos(\omega t)\sin\phi,$$

then $B\cos\phi = \omega$ and $-B\sin\phi = 1/(RC)$. These equations imply that

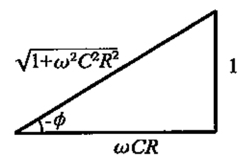
$$B = \sqrt{\omega^2 + \frac{1}{(RC)^2}} \qquad \text{and} \qquad \tan\phi = \frac{-1}{\omega CR} \quad \Longrightarrow \quad \phi = \operatorname{Tan}^{-1}\left(\frac{-1}{\omega CR}\right).$$

Hence,
$$I(t) = Ae^{-t/(RC)} + \frac{\omega E_0}{R[\omega^2 + 1/(RC)^2]}\sqrt{\omega^2 + 1/(RC)^2}\sin(\omega t - \phi)$$
$$= Ae^{-t/(RC)} + \frac{E_0}{Z}\sin(\omega t - \phi),$$

where $Z = \dfrac{R\sqrt{\omega^2 + 1/(RC)^2}}{\omega} = \sqrt{R^2 + \dfrac{1}{\omega^2 C^2}}$.

(b) If $I(0) = I_0$, then

$$I_0 = A + \frac{E_0}{Z}\sin(-\phi) = A + \frac{E_0}{Z}\frac{1}{\sqrt{1 + \omega^2 C^2 R^2}}$$
$$= A + \frac{E_0}{\omega CZ\sqrt{R^2 + 1/(\omega C)^2}} = A + \frac{E_0}{\omega CZ^2},$$



and therefore $A = I_0 - E_0/(\omega CZ^2)$.

39. For $0 < t < 5$, the solution is as before, $T(t) = 20 + 36t + (36/k)(e^{-kt} - 1)$. For $t > 5$, the temperature of the oven is $T_o(t) = 200 + 10\sin[\pi(t-5)/5] = 200 - 10\sin(\pi t/5)$, in which case $T(t)$ must satisfy

$$\frac{dT}{dt} = k[200 - 10\sin(\pi t/5) - T], \quad T(5) = 200 + \frac{36}{k}(e^{-5k} - 1).$$

An integrating factor is e^{kt} so that

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = k[200 - 10\sin(\pi t/5)]e^{kt} \quad \Longrightarrow \quad \frac{d}{dt}(Te^{kt}) = k[200 - 10\sin(\pi t/5)]e^{kt}.$$

Integration by parts on the right leads to

$$T(t) = 200 - \frac{10k}{k^2 + \pi^2/25}[k\sin(\pi t/5) - (\pi/5)\cos(\pi t/5)] + Ce^{-kt}.$$

For $T(5) = 200 + (36/k)(e^{-5k} - 1)$,

$$200 + \frac{36}{k}(e^{-5k} - 1) = 200 - \frac{10k}{k^2 + \pi^2/25}(\pi/5) + Ce^{-5k} \quad \Longrightarrow \quad C = \frac{36}{k}(1 - e^{5k}) + \frac{2k\pi e^{5k}}{k^2 + \pi^2/25}.$$

40. If we substitute $Q = \dfrac{S}{K(1-x)} - \dfrac{xI}{1-x}$ into the differential equation,

$$\frac{dS}{dt} = I - \frac{S}{K(1-x)} + \frac{xI}{1-x} \quad \Longrightarrow \quad \frac{dS}{dt} + \frac{S}{K(1-x)} = \frac{I}{1-x} = \frac{I_0 e^{-\lambda t}}{1-x}.$$

An integrating factor is $e^{t/b}$, where $b = K(1-x)$. Multiplication of the differential equation by $e^{t/b}$ gives

$$e^{t/b}\frac{dS}{dt} + \frac{Se^{t/b}}{b} = \frac{I_0 e^{t/b}e^{-\lambda t}}{1-x} \quad \Longrightarrow \quad \frac{d}{dt}[Se^{t/b}] = \frac{I_0}{1-x}e^{(-\lambda + 1/b)t}.$$

Integration gives

$$Se^{t/b} = \frac{I_0}{(1-x)(-\lambda + 1/b)}e^{(-\lambda+1/b)t} + C \quad \Longrightarrow \quad S(t) = Ce^{-t/b} + \frac{I_0}{(1-x)(-\lambda + 1/b)}e^{-\lambda t}.$$

Thus,

$$Q(t) = \frac{C}{b}e^{-t/b} + \frac{I_0}{b(1-x)(-\lambda + 1/b)}e^{-\lambda t} - \frac{xI_0 e^{-\lambda t}}{1-x}.$$

Since $Q(0) = 0$,

$$0 = \frac{C}{b} + \frac{I_0}{b(1-x)(-\lambda + 1/b)} - \frac{xI_0}{1-x}.$$

This gives C, and

$$\begin{aligned} Q(t) &= \left[\frac{xI_0}{1-x} - \frac{I_0}{b(1-x)(-\lambda+1/b)}\right] e^{-t/b} + \left[\frac{-xI_0}{1-x} + \frac{I_0}{b(1-x)(-\lambda+1/b)}\right] e^{-\lambda t} \\ &= I_0 \left[\frac{1}{(1-x)(1-\lambda b)} - \frac{x}{1-x}\right] (e^{-\lambda t} - e^{-t/b}). \end{aligned}$$

41. Since $\overline{N}$ is proportional to depth y below the surface, $\overline{N} = \overline{N}(y) = ky$, $0 \le y \le Y$, where $k > 0$ is a constant and Y is the depth of the water. Since dN/dt is proportional to $\overline{N} - N$, $dN/dt = l(\overline{N} - N) = l(ky - N)$, where $l > 0$ is a constant. Because the diver drops to the bottom in time T, $y = Yt/T$, where $t = 0$ on entry, and therefore $dN/dt = l(kYt/T - N)$, $0 \le t \le T$. When we write the differential equation in the form $\dfrac{dN}{dt} + lN = \dfrac{klYt}{T}$, an integrating factor is e^{lt}. Multiplication by e^{lt} yields

$$e^{lt}\frac{dN}{dt} + le^{lt}N = \frac{klYt}{T}e^{lt} \quad \Longrightarrow \quad \frac{d}{dt}(Ne^{lt}) = \frac{klY}{T}te^{lt} \quad \Longrightarrow \quad Ne^{lt} = \frac{klY}{T}\left(\frac{t}{l}e^{lt} - \frac{1}{l^2}e^{lt}\right) + C.$$

Thus, $N(t) = \dfrac{klY}{T}\left(\dfrac{t}{l} - \dfrac{1}{l^2}\right) + Ce^{-lt} = \dfrac{kY}{lT}(lt-1) + Ce^{-lt}$. Since $N(0) = 0$, we obtain $0 = -\dfrac{kY}{lT} + C$, and therefore

$$N(t) = \frac{kY}{lT}(lt-1) + \frac{kY}{lT}e^{-lt}, \quad 0 \le t \le T.$$

When the diver reaches the bottom, the nitrogen pressure in his body is $N(T) = \dfrac{kY}{lT}(lT-1) + \dfrac{kY}{lT}e^{-lT}$. For time $t > T$, nitrogen pressure must satisfy the differential equation

$$\frac{dN}{dt} = l(\tilde{N} - N), \quad \text{where } \tilde{N} = \overline{N}(Y),$$

a differential equation that can be separated, $\dfrac{dN}{\tilde{N} - N} = l\,dt$. A one-parameter family of solutions is defined implicitly by $-\ln|\tilde{N} - N| = lt + C \Longrightarrow N = \tilde{N} + De^{-lt}$. Since $N(T) = \dfrac{kY}{lT}(lT-1) + \dfrac{kY}{lT}e^{-lT}$,

$$\frac{kY}{lT}(lT-1) + \frac{kY}{lT}e^{-lT} = \tilde{N} + De^{-lT} \quad \Longrightarrow \quad D = \left[\frac{kY}{lT}(lT-1) - \tilde{N}\right]e^{lT} + \frac{kY}{lT}.$$

Thus, $N(t) = \tilde{N} + \left[\dfrac{kY}{lT}(lT-1) - \tilde{N}\right]e^{l(T-t)} + \dfrac{kY}{lT}e^{-lt}, \quad t > T.$

EXERCISES 15.4

1. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$x\frac{dv}{dx} + v = 4x \quad \Longrightarrow \quad \frac{d}{dx}(vx) = 4x.$$

Integration gives $vx = 2x^2 + C$. When we solve for v and set $v = dy/dx$,

$$v = \frac{dy}{dx} = 2x + \frac{C}{x} \quad \Longrightarrow \quad y = x^2 + C\ln|x| + D.$$

2. Since x is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\dfrac{dy}{dx} = v\dfrac{dv}{dy}$,

$$2yv\,\frac{dv}{dy} = 1 + v^2.$$

This equation can be separated, $\dfrac{2v}{1+v^2}\,dv = \dfrac{1}{y}\,dy$, and a one-parameter family of solutions is defined implicitly by $\ln(1+v^2) = \ln|y| + C$. When this equation is solved for v, $v = \dfrac{dy}{dx} = \pm\sqrt{Dy-1}$. This equation can also be separated, $\dfrac{1}{\sqrt{Dy-1}}\,dy = \pm dx$, and a two-parameter family of solutions is defined implicitly by $2\sqrt{Dy-1} = \pm Dx + F$.

3. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$\frac{dv}{dx} = v + 2x \qquad \Longrightarrow \qquad \frac{dv}{dx} - v = 2x.$$

An integrating factor for this linear first-order equation is e^{-x}, so that the differential equation can be expressed in the form

$$e^{-x}\frac{dv}{dx} - ve^{-x} = 2xe^{-x} \quad \Longrightarrow \quad \frac{d}{dx}(ve^{-x}) = 2xe^{-x} \quad \Longrightarrow \quad ve^{-x} = \int 2xe^{-x}\,dx = -2(x+1)e^{-x} + C.$$

When we solve for v and set it equal to dy/dx,

$$v = \frac{dy}{dx} = -2(x+1) + Ce^{x} \quad \Longrightarrow \quad y = -x^2 - 2x + Ce^{x} + D.$$

4. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$, giving $x^2\dfrac{dv}{dx} = v^2$. This equation can be separated, $\dfrac{1}{v^2}\,dv = \dfrac{1}{x^2}\,dx$, and a one-parameter family of solutions is defined implicitly by $-1/v = -1/x - C$. We now solve for v, $v = \dfrac{dy}{dx} = \dfrac{x}{Cx+1}$, and integrate to obtain

$$\begin{aligned} y &= \int \frac{x}{Cx+1}\,dx = \int \left(\frac{1}{C} + \frac{-1/C}{Cx+1}\right) dx = \frac{x}{C} - \frac{1}{C^2}\ln|Cx+1| + D \\ &= Ex - E^2\ln\left|\frac{x}{E} + 1\right| + D = Ex - E^2\ln|x+E| + F. \end{aligned}$$

5. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$\sin x\frac{dv}{dx} + v\cos x = \sin x \quad \Longrightarrow \quad \frac{d}{dx}(v\sin x) = \sin x.$$

Integration gives $v\sin x = -\cos x + C$. When we solve for v and set $v = dy/dx$,

$$v = \frac{dy}{dx} = -\cot x + C\csc x \quad \Longrightarrow \quad y = \ln|\csc x| + C\ln|\csc x - \cot x| + D.$$

6. When we substitute $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\dfrac{dy}{dx} = v\dfrac{dv}{dy}$, we obtain $v\dfrac{dv}{dy} = (1+v^2)^{3/2}$. This equation can be separated, $\dfrac{v}{(1+v^2)^{3/2}}\,dv = dy$, a one-parameter family of solutions of which is defined implicitly by $-1/\sqrt{1+v^2} = y + C$. When this is solved for v,

$$v = \frac{dy}{dx} = \pm\frac{\sqrt{1-(y+C)^2}}{y+C} \quad \Longrightarrow \quad \frac{y+C}{\sqrt{1-(y+C)^2}}\,dy = \pm dx.$$

Integration gives $-\sqrt{1-(y+C)^2} = \pm x + D$. By squaring this equation, the solution can be rewritten in the form $(x+E)^2 + (y+C)^2 = 1$.

7. Since x is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\dfrac{dy}{dx} = v\,\dfrac{dv}{dy}$, giving $v\dfrac{dv}{dy} + 4y = 0$. When we separate, $v\,dv = -4y\,dy$, a one-parameter family of solutions then being defined by $v^2/2 = -2y^2 + C$. We now solve for v and set it equal to dy/dx,

$$v = \frac{dy}{dx} = \pm\sqrt{2C - 4y^2} \qquad \Longrightarrow \qquad \frac{dy}{\sqrt{D - 4y^2}} = \pm dx,$$

where we have replaced $2C$ with D. When we substitute $y = (\sqrt{D}/2)\sin\theta$ and $dy = (\sqrt{D}/2)\cos\theta\,d\theta$,

$$\pm x + E = \int \frac{1}{\sqrt{D - 4y^2}}dy = \int \frac{1}{\sqrt{D}\cos\theta}\frac{\sqrt{D}}{2}\cos\theta\,d\theta = \frac{\theta}{2} = \frac{1}{2}\mathrm{Sin}^{-1}\left(\frac{2y}{\sqrt{D}}\right).$$

Thus, $\mathrm{Sin}^{-1}\left(\dfrac{2y}{\sqrt{D}}\right) = \pm 2x + 2E \Longrightarrow \dfrac{2y}{\sqrt{D}} = \sin(2E \pm 2x)$ and this implies that

$$y = \frac{\sqrt{D}}{2}(\sin 2E\cos 2x \pm \cos 2E\sin 2x) = F\cos 2x + G\sin 2x,$$

where $F = (1/2)\sqrt{D}\sin 2E$ and $G = \pm(1/2)\sqrt{D}\cos 2E$.

8. Since x is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} = \dfrac{dv}{dy}\dfrac{dy}{dx} = v\,\dfrac{dv}{dy}$, giving $v\,\dfrac{dv}{dy} = vy$. Division by v and integration gives

$$v = \frac{y^2}{2} + C \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{y^2}{2} + \frac{D}{2} \qquad (D = 2C).$$

Separation now yields $\dfrac{1}{D+y^2}\,dy = \dfrac{1}{2}\,dx \Longrightarrow \dfrac{x}{2} + E = \displaystyle\int \frac{1}{D+y^2}\,dy$. If $D = 0$, then $x/2 + E = -1/y$. If $D > 0$, we set $y = \sqrt{D}\tan\theta$ and $dy = \sqrt{D}\sec^2\theta\,d\theta$,

$$\frac{x}{2} + E = \int \frac{\sqrt{D}\sec^2\theta}{D\sec^2\theta}\,d\theta = \frac{1}{\sqrt{D}}\theta = \frac{1}{\sqrt{D}}\mathrm{Tan}^{-1}\left(\frac{y}{\sqrt{D}}\right) = F\,\mathrm{Tan}^{-1}(Fy).$$

If $D < 0$, we set $D = -F^2$ and use partial fractions to write

$$\begin{aligned}\frac{x}{2} + E &= \int \frac{1}{y^2 - F^2}\,dy = \int\left(\frac{\frac{-1}{2F}}{y+F} + \frac{\frac{1}{2F}}{y-F}\right)dy\\ &= -\frac{1}{2F}\ln|y+F| + \frac{1}{2F}\ln|y-F| = \frac{1}{2F}\ln\left|\frac{y-F}{y+F}\right|.\end{aligned}$$

9. Since y is explicitly missing, we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$,

$$\frac{dv}{dx} + v^2 = 1 \quad \Longrightarrow \quad \frac{dv}{1-v^2} = dx \quad \Longrightarrow \quad \frac{1}{2}\left(\frac{1}{1-v} + \frac{1}{1+v}\right)dv = dx.$$

Integration gives $-\ln|1-v| + \ln|1+v| = 2x + C$. When we solve for v we obtain

$$v = \frac{dy}{dx} = \frac{De^{2x}-1}{De^{2x}+1} \quad \Longrightarrow \quad y = \int \frac{De^{2x}-1}{De^{2x}+1}dx = \frac{1}{2}\ln(De^{2x}+1) - \int \frac{1}{De^{2x}+1}dx.$$

If we set $e^x = (1/\sqrt{D})\tan\theta$ and $e^x\,dx = (1/\sqrt{D})\sec^2\theta\,d\theta$, then

$$y = \frac{1}{2}\ln(De^{2x}+1) - \int \frac{1}{\sec^2\theta}\frac{\sec^2\theta}{\tan\theta}d\theta = \frac{1}{2}\ln(De^{2x}+1) + \ln|\csc\theta| + E$$

$$= \frac{1}{2}\ln(De^{2x}+1) + \ln\left|\frac{\sqrt{De^{2x}+1}}{\sqrt{D}e^x}\right| + E$$
$$= \ln(De^{2x}+1) - x + F, \quad F = E - \frac{1}{2}\ln D.$$

10. If we set $\dfrac{dy}{dx} = v$ and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$, then $\left(\dfrac{dv}{dx}\right)^2 = 1 + v^2$. This equation can be separated, $\dfrac{1}{\sqrt{1+v^2}}\,dv = \pm dx$, and a one-parameter family of solutions is defined implicitly by

$$\pm x + C = \int \frac{1}{\sqrt{1+v^2}}\,dv \quad \text{(and if we set } v = \tan\theta,\ dv = \sec^2\theta\,d\theta\text{)},$$
$$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1+v^2} + v|.$$

We now exponentiate, $\sqrt{1+v^2}+v = De^{\pm x}$, transpose the v and square, $1+v^2 = D^2e^{\pm 2x} - 2Dve^{\pm x} + v^2$. We can now solve for v,

$$v = \frac{dy}{dx} = \frac{D^2e^{\pm 2x} - 1}{2De^{\pm x}} = \frac{1}{2}\left(De^{\pm x} - \frac{1}{D}e^{\mp x}\right).$$

In either case dy/dx is of the form $\dfrac{dy}{dx} = \dfrac{1}{2}\left(De^x - \dfrac{1}{D}e^{-x}\right)$, and hence, $y = \dfrac{1}{2}\left(De^x + \dfrac{1}{D}e^{-x}\right) + E$.

11. We could use substitution 15.28 but it simpler to write

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \quad\Longrightarrow\quad r\frac{dT}{dr} = C \quad\Longrightarrow\quad \frac{dT}{dr} = \frac{C}{r} \quad\Longrightarrow\quad T = C\ln r + D.$$

For $T(a) = T_a$ and $T(b) = T_b$, C and D must satisfy $T_a = C\ln a + D$ and $T_b = C\ln b + D$. These can be solved for $C = (T_b - T_a)/\ln(b/a)$ and $D = (T_a\ln b - T_b\ln a)/\ln(b/a)$.

12. We could use substitution 15.28 but it simpler to write

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = k \quad\Longrightarrow\quad r\frac{dT}{dr} = kr + C \quad\Longrightarrow\quad \frac{dT}{dr} = k + \frac{C}{r} \quad\Longrightarrow\quad T = kr + C\ln r + D.$$

For $T(a) = T_a$ and $T(b) = T_b$, C and D must satisfy $T_a = ka + C\ln a + D$ and $T_b = kb + C\ln b + D$. These can be solved for $C = [T_b - T_a - k(b-a)]/\ln(b/a)$ and $D = [T_a\ln b - T_b\ln a + k(b\ln a - a\ln b)]/\ln(b/a)$.

13. We could use substitution 15.28 but it simpler to write

$$\frac{d}{dr}\left(r\frac{dh}{dr}\right) = 0 \quad\Longrightarrow\quad r\frac{dh}{dr} = C \quad\Longrightarrow\quad \frac{dh}{dr} = \frac{C}{r} \quad\Longrightarrow\quad h = C\ln r + D.$$

The boundary conditions require

$$C\ln r_w + D = h_w, \quad C\ln r_i + D = h_i.$$

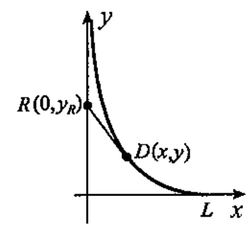
These imply that $C = \dfrac{h_i - h_w}{\ln(r_i/r_w)}$ and $D = h_w - \left[\dfrac{h_i - h_w}{\ln(r_i/r_w)}\right]\ln r_w$, and therefore

$$h = \frac{h_i - h_w}{\ln(r_i/r_w)}\ln r + h_w - \left[\frac{h_i - h_w}{\ln(r_i/r_w)}\right]\ln r_w = h_w + \left[\frac{h_i - h_w}{\ln(r_i/r_w)}\right]\ln\left(\frac{r}{r_w}\right).$$

14. We let $y = y(x)$ be the equation of the required curve, and use two principles. First, if $D(x, y)$ is any position of the dog, then the line DR joining the positions of the rabbit and dog is always tangent to $y = y(x)$. This can be expressed in the form

$$(y - y_R)/(x - 0) = y'(x) \implies y_R = y - xy'(x).$$

Secondly, distance run by the dog and rabbit are always the same. Distance run by the rabbit is y_R. Distance



run by the dog in the same time interval is represented by the integral

$$\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

When we equate these distances, we obtain

$$y - xy'(x) = \int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

We differentiate this equation to eliminate the integral,

$$\frac{dy}{dx} - x\frac{d^2y}{dx^2} - \frac{dy}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \implies \quad x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Since y is explicitly missing, we set $v = dy/dx$ and $dv/dx = d^2y/dx^2$,

$$x\frac{dv}{dx} = \sqrt{1 + v^2} \quad \implies \quad \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} = \ln x + C.$$

We set $v = \tan\theta$ and $dv = \sec^2\theta\, d\theta$,

$$\ln x + C = \int \frac{\sec^2\theta}{\sec\theta} d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1 + v^2} + v|.$$

Exponentiation gives

$$Dx = \sqrt{1 + v^2} + v \quad \implies \quad Dx - v = \sqrt{1 + v^2} \quad \implies \quad 1 + v^2 = D^2x^2 - 2Dxv + v^2,$$

where $D = e^C$. When we solve for v,

$$v = \frac{dy}{dx} = \frac{D^2x^2 - 1}{2Dx} = \frac{Dx}{2} - \frac{1}{2Dx}.$$

Since $y'(L) = 0$, we obtain $D = 1/L$, and therefore

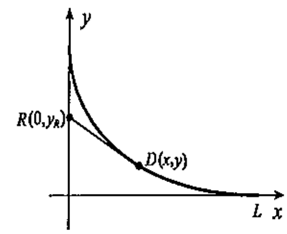
$$\frac{dy}{dx} = \frac{x}{2L} - \frac{L}{2x} \quad \implies \quad y = \frac{x^2}{4L} - \frac{L}{2}\ln x + C.$$

The condition $y(L) = 0$ implies that $0 = L/4 - (L/2)\ln L + C$. Consequently,

$$y = \frac{x^2}{4L} - \frac{L}{2}\ln x + \frac{L}{2}\ln L - \frac{L}{4} = \frac{1}{4L}(x^2 - L^2) + \frac{L}{2}\ln\left(\frac{L}{x}\right).$$

15. We let $y = y(x)$ be the equation of the required curve, and use two principles. First, if $D(x, y)$ is any position of the dog, then the line DR joining the positions of the rabbit and dog is always tangent to $y = y(x)$. This can be expressed in the form

$$(y - y_R)/(x - 0) = y'(x) \implies y_R = y - xy'(x).$$



Secondly, distance run by the dog is twice that run by the rabbit. Distance run by the rabbit is y_R. Distance run by the dog in the same time is represented by the integral

$$\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

When we equate this to $2y_R$, we obtain

$$2y - 2xy'(x) = \int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

We differentiate this equation to eliminate the integral,

$$2\frac{dy}{dx} - 2x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \implies 2x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Since y is explicitly missing, we set $v = dy/dx$ and $dv/dx = d^2y/dx^2$,

$$2x\frac{dv}{dx} = \sqrt{1 + v^2} \implies \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{2x} = \frac{1}{2}\ln x + C.$$

We set $v = \tan\theta$ and $dv = \sec^2\theta\, d\theta$,

$$\frac{1}{2}\ln x + C = \int \frac{\sec^2\theta}{\sec\theta} d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1 + v^2} + v|.$$

Exponentiation gives

$$D\sqrt{x} = \sqrt{1 + v^2} + v \implies D\sqrt{x} - v = \sqrt{1 + v^2} \implies 1 + v^2 = D^2x - 2Dv\sqrt{x} + v^2,$$

where $D = e^C$. When we solve for v,

$$v = \frac{dy}{dx} = \frac{D^2x - 1}{2D\sqrt{x}} = \frac{D\sqrt{x}}{2} - \frac{1}{2D\sqrt{x}}.$$

Since $y'(L) = 0$, we obtain $D = 1/\sqrt{L}$, and therefore

$$\frac{dy}{dx} = \frac{\sqrt{x}}{2\sqrt{L}} - \frac{\sqrt{L}}{2\sqrt{x}} \implies y = \frac{x^{3/2}}{3\sqrt{L}} - \sqrt{Lx} + C.$$

The condition $y(L) = 0$ implies that $0 = L/3 - L + C \implies C = 2L/3$. Consequently,

$$y = \frac{x^{3/2}}{3\sqrt{L}} - \sqrt{Lx} + \frac{2L}{3}.$$

The dog catches the rabbit if, and when, $x = 0$, and this occurs when $y = 2L/3$.

16. (a) We let $y = y(x)$ be the equation of the required curve, and use two principles. First, if $H(x, y)$ is any position of the hawk, then the line PH joining the positions of the pigeon and hawk is always tangent to $y = y(x)$. If $P(0, y_p)$ is the position of the pigeon, then this requirement can be expressed in the form

$$\frac{y - y_p}{x - 0} = y'(x) \quad \Longrightarrow \quad y_p = y - xy'(x).$$

Secondly, distance flown by the hawk is V/v times that by the pigeon. Distance flown by the pigeon is y_p. Distance flown by the hawk in the same time is represented by the integral

$$\int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

When we equate this to $(V/v)y_p$, we obtain

$$\frac{V}{v}[y - xy'(x)] = \int_x^L \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \Longrightarrow \quad x\frac{dy}{dx} - y = \frac{v}{V}\int_L^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

(b) Differentiation of this equation gives

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{v}{V}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad \Longrightarrow \quad x\frac{d^2y}{dx^2} = \frac{v}{V}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

(c) If we set $p = dy/dx$ and $dp/dx = d^2y/dx^2$, then $x\dfrac{dp}{dx} = \dfrac{v}{V}\sqrt{1 + p^2}$, a separable equation, $\dfrac{1}{\sqrt{1 + p^2}}\, dp = \dfrac{v}{Vx}\, dx$. A one-parameter family of solutions is defined implicitly by

$$\begin{aligned} \frac{v}{V}\ln x + C &= \int \frac{1}{\sqrt{1 + p^2}} dp \qquad \text{(and if we set } p = \tan\theta,\ dp = \sec^2\theta\, d\theta\text{)} \\ &= \int \frac{\sec^2\theta}{\sec\theta} d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1 + p^2} + p|. \end{aligned}$$

Since $p(L) = y'(L) = 0$, it follows that $\dfrac{v}{V}\ln L + C = \ln 1 = 0$. Thus,

$$\frac{v}{V}\ln x - \frac{v}{V}\ln L = \ln|\sqrt{1 + p^2} + p|.$$

Exponentiation gives $\left(\dfrac{x}{L}\right)^{v/V} = \sqrt{1 + p^2} + p$. When this equation is solved for p, the result is

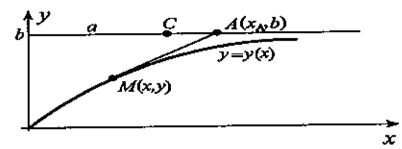
$$p = \frac{dy}{dx} = \frac{1}{2}\left[\left(\frac{x}{L}\right)^{v/V} - \left(\frac{x}{L}\right)^{-v/V}\right].$$

Integration yields $y = \dfrac{1}{2}\left[\dfrac{L}{v/V + 1}\left(\dfrac{x}{L}\right)^{v/V+1} - \dfrac{L}{-v/V + 1}\left(\dfrac{x}{L}\right)^{-v/V+1}\right] + D$. Because $y(L) = 0$ we obtain $0 = \dfrac{1}{2}\left[\dfrac{L}{v/V + 1} - \dfrac{L}{-v/V + 1}\right] + D$. This equation can be solved for $D = LVv/(V^2 - v^2)$, and therefore

$$y(x) = \frac{LV}{2}\left[\frac{1}{V + v}\left(\frac{x}{L}\right)^{v/V+1} - \frac{1}{V - v}\left(\frac{x}{L}\right)^{-v/V+1}\right] + \frac{LVv}{V^2 - v^2}.$$

(d) The hawk catches the pigeon when $x = 0$, in which case $y(0) = \dfrac{LVv}{V^2 - v^2}$.

17. (a) We let $y = y(x)$ be the equation of the required curve, and use two principles. First, if $M(x, y)$ is any position of the missile, then the line MA joining the positions of the missile and aircraft is always tangent to $y = y(x)$. If $A(x_A, b)$ is the position of the aircraft, then this requirement can be expressed in the form



$$\frac{y-b}{x-x_A} = y'(x) \quad \Longrightarrow \quad x_A = x - \frac{y-b}{y'(x)}.$$

Secondly, distance flown by the missile is V/v times that by the aircraft. Distance flown by the aircraft after the missile takes off is $x_A - a = x - a - (y-b)/y'(x)$. Distance flown by the missile in the same time is represented by the integral

$$\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

When we equate this to $(V/v)[x - a - (y-b)/y'(x)]$, we obtain

$$\frac{V}{v}\left[x - a - \frac{y-b}{y'(x)}\right] = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \Longrightarrow \quad a + \frac{v}{V}\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \frac{b-y}{y'(x)} + x.$$

If we differentiate with respect to x to eliminate the integral,

$$\frac{v}{V}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{y'(-y') - (b-y)y''}{(y')^2} + 1 = \frac{y''(y-b)}{(y')^2} \quad \Longrightarrow \quad \frac{v}{V}(y')^2\sqrt{1 + (y')^2} = y''(y-b).$$

If we set $u = y'$ and $y'' = u\, du/dy$, then

$$\frac{v}{V}u^2\sqrt{1+u^2} = u\frac{du}{dy}(y-b) \quad \Longrightarrow \quad \frac{v/V}{y-b}dy = \frac{1}{u\sqrt{1+u^2}}du,$$

a separated differential equation. A one-parameter family of solutions is defined implicitly by

$$\frac{v}{V}\ln|y-b| = \int \frac{1}{u\sqrt{1+u^2}}du.$$

The trigonometric substitution $u = \tan\theta$ leads to

$$\frac{v}{V}\ln|y-b| = \ln\left(\frac{\sqrt{1+u^2}-1}{u}\right) + C.$$

Exponentiation gives $D(b-y)^{v/V} = \dfrac{\sqrt{1+u^2}-1}{u}$. Since $u(0) = b/a$, we obtain

$$Db^{v/V} = \frac{\sqrt{1+b^2/a^2}-1}{b/a} \quad \Longrightarrow \quad D = \frac{\sqrt{a^2+b^2}-a}{b^{1+v/V}}.$$

If we square $1 + Du(b-y)^{v/V} = \sqrt{1+u^2}$, we obtain

$$1 + 2Du(b-y)^{v/V} + D^2u^2(b-y)^{2v/V} = 1 + u^2 \quad \Longrightarrow \quad \frac{dy}{dx} = u = \frac{2D(b-y)^{v/V}}{1 - D^2(b-y)^{2v/V}}.$$

This equation can be separated,

$$dx = \frac{1 - D^2(b-y)^{2v/V}}{2D(b-y)^{v/V}}dy = \left[\frac{1}{2D}(b-y)^{-v/V} - \frac{D}{2}(b-y)^{v/V}\right]dy,$$

a one-parameter family of solutions being defined by

$$-\frac{(b-y)^{1-v/V}}{2D(1-v/V)} + \frac{D(b-y)^{1+v/V}}{2(1+v/V)} = x + C.$$

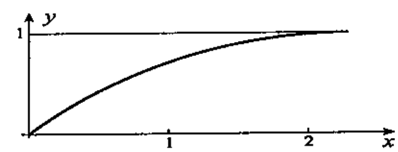
Since $y(0) = 0$, we obtain $C = -\dfrac{b^{1-v/V}}{2D(1-v/V)} + \dfrac{Db^{1+v/V}}{2(1+v/V)}$. When these values of C and D are substituted into the implicit definition of the curve, it simplifies to

$$x = \frac{V}{2}\left\{\frac{\sqrt{a^2+b^2}+a}{V-v}\left[1-\left(\frac{b-y}{b}\right)^{1-v/V}\right] - \frac{\sqrt{a^2+b^2}-a}{V+v}\left[1-\left(\frac{b-y}{b}\right)^{1+v/V}\right]\right\}.$$

(b) When $a = b = 1$ and $V = 2v$, the curve simplifies to

$$x = \frac{2}{3}(2+\sqrt{2}) - (\sqrt{2}+1)\sqrt{1-y} + \frac{\sqrt{2}-1}{3}(1-y)^{3/2}.$$

A plot is shown to the right. The misile catches the aircraft when
$y = 1 \Longrightarrow x = -C = 2(2+\sqrt{2})/3.$



18. (a) If we set $v = dy/dT$ and $d^2y/dT^2 = v\,dv/dy$,

$$v\frac{dv}{dy} = \frac{5v^2}{4y} - ay + \frac{b}{y} - \frac{c}{y^3} \quad\Longrightarrow\quad \frac{d}{dy}(v^2) - \frac{5v^2}{2y} = -2ay + \frac{2b}{y} - \frac{2c}{y^3}.$$

This is linear in v^2 with integrating factor $e^{\int -5/(2y)dy} = e^{-(5/2)\ln|y|} = 1/y^{5/2}$. Hence,

$$\frac{d}{dy}\left(\frac{v^2}{y^{5/2}}\right) = -\frac{2a}{y^{3/2}} + \frac{2b}{y^{7/2}} - \frac{2c}{y^{11/2}} \quad\Longrightarrow\quad \frac{v^2}{y^{5/2}} = \frac{4a}{\sqrt{y}} - \frac{4b}{5y^{5/2}} + \frac{4c}{9y^{9/2}} + D.$$

Solving for $v = dy/dT$ gives

$$\frac{dy}{dT} = \pm\sqrt{4ay^2 - \frac{4b}{5} + \frac{4c}{9y^2} + Dy^{5/2}}.$$

Since dy/dT must be negative, and when we set $D = 0$, $\dfrac{dy}{dT} = -2\sqrt{ay^2 - \dfrac{b}{5} + \dfrac{c}{9y^2}}$.

(b) The above differential equation is separable,

$$\frac{1}{\sqrt{ay^2 - \dfrac{b}{5} + \dfrac{c}{9y^2}}} = -2\,dT \quad\Longrightarrow\quad \frac{y}{\sqrt{\left(y^2 - \dfrac{b}{10a}\right)^2 + \left(\dfrac{c}{9a} - \dfrac{b^2}{100a^2}\right)}}dy = -2\sqrt{a}\,dT.$$

If we let $y^2 - \dfrac{b}{10a} = \sqrt{\dfrac{c}{9a} - \dfrac{b^2}{100a^2}}\tan\theta$, then $2y\,dy = \sqrt{\dfrac{c}{9a} - \dfrac{b^2}{100a^2}}\sec^2\theta\,d\theta$, and

$$\begin{aligned}
-2\sqrt{a}T + E &= \int \frac{\dfrac{1}{2}\sqrt{\dfrac{c}{9a} - \dfrac{b^2}{100a^2}}\sec^2\theta}{\sqrt{\dfrac{c}{9a} - \dfrac{b^2}{100a^2}}\sec\theta}d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| \\
&= \frac{1}{2}\ln\left|\frac{\sqrt{y^4 - \dfrac{by^2}{5a} + \dfrac{c}{9a}}}{\sqrt{\dfrac{c}{9a} - \dfrac{b^2}{100a^2}}} + \frac{y^2 - \dfrac{b}{10a}}{\sqrt{\dfrac{c}{9a} - \dfrac{b^2}{100a^2}}}\right|.
\end{aligned}$$

This can be expressed in the form $\ln\left|\sqrt{y^4 - \frac{by^2}{5a} + \frac{c}{9a}} + y^2 - \frac{b}{10a}\right| = -4\sqrt{a}T + F$. Since $y(0) = 1$, we obtain $F = \ln\left|\sqrt{1 - \frac{b}{5a} + \frac{c}{9a}} + 1 - \frac{b}{10a}\right|$, so that $y(T)$ is defined implicitly by

$$\ln\left|\sqrt{y^4 - \frac{by^2}{5a} + \frac{c}{9a}} + y^2 - \frac{b}{10a}\right| = -4\sqrt{a}T + \ln\left|\sqrt{1 - \frac{b}{5a} + \frac{c}{9a}} + 1 - \frac{b}{10a}\right|.$$

EXERCISES 15.5

1. The resistive force is of the form $F_R = -\beta\sqrt{v}$, where β is a constant and v is speed. Since $-F^* = -\beta\sqrt{v^*}$, it follows that $\beta = F^*/\sqrt{v^*}$. The differential equation describing motion is

$$m\frac{dv}{dt} = F + F_R = F - \beta\sqrt{v}.$$

Terminal velocity is attained when acceleration is zero, and this occurs when $F - \beta\sqrt{v} = 0$. This gives $v = F^2/\beta^2 = F^2 v^{*2}/F^{*2}$.

2. The condition $v(0) = v_0$ implies that $C = (1/k)\ln|kv_0 - mg|$. Hence

$$\frac{1}{k}\ln|kv - mg| = -\frac{t}{m} + \frac{1}{k}\ln|kv_0 - mg| \implies \ln\left|\frac{kv - mg}{kv_0 - mg}\right| = -\frac{kt}{m} \implies \left|\frac{kv - mg}{kv_0 - mg}\right| = e^{-kt/m}.$$

Since terminal velocity occurs when $dv/dt = 0$, it follows from the differential equation that $mg - kv = 0$. Terminal velocity is therefore $v = mg/k$. If the initial velocity v_0 is less than terminal velocity, then v will always be less than terminal velocity. In this case, both $kv - mg < 0$ and $kv_0 - mg < 0$, the quotient being positive. If v_0 is greater than terminal velocity, v will always be greater than terminal velocity. In this case, both $kv - mg > 0$ and $kv_0 - mg > 0$, as is their quotient. In both cases then, we may write

$$\frac{kv - mg}{kv_0 - mg} = e^{-kt/m} \implies kv - mg = (kv_0 - mg)e^{-kt/m} \implies v = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}.$$

3. (a) Let us take x as positive in the direction of motion of the boat with $x = 0$ and $t = 0$ when motion commences. Since the resistive force is proportional to velocity and is 200 N when the speed is 30 km/hr or 25/2 m/s, its magnitude is 16 times the velocity. Newton's second law for the acceleration of the boat is

$$250\frac{dv}{dt} = 250 - 16v \implies \frac{dv}{125 - 8v} = \frac{dt}{125}.$$

A one-parameter family of solutions of this separated equation is defined implicitly by $-(1/8)\ln|125 - 8v| = t/125 + C$. When we solve for v, we get $v = 125/8 - De^{-8t/125}$. Since $v(0) = 0$, we find $0 = 125/8 - D$, and therefore $v(t) = (125/8)(1 - e^{-8t/125})$ m/s.

(b) $\lim_{t\to\infty} v(t) = 125/8$ m/s

4. Let us measure x as positive in the direction of motion taking $x = 0$ and $t = 0$ at the instant the brakes are applied. Because the coefficient of kinetic friction is less than one, we can say that the x-component of the force of friction has magnitude less than $9.81m$, where m is the mass of the car. If we use this as the magnitude of the frictional force, then because this is the maximum possible, we will be finding the maximum possible speed before the brakes were applied. In other words, we are testifying for the defence. Newton's second law for the x-component of the acceleration dv/dt of the car gives

$$m\frac{dv}{dt} = -9.81m,$$

and this can be integrated for $v(t) = -9.81t + C$. If we set $v = v_0$ at time $t = 0$, then $v_0 = C$, and

$$v = \frac{dx}{dt} = -9.81t + v_0.$$

Integration gives $x(t) = -4.905t^2 + v_0t + D$. Because $x(0) = 0$, it follows that $D = 0$, and

$$x(t) = -4.905t^2 + v_0t.$$

The car comes to a stop when $0 = v = -9.81t + v_0$, and this implies that $t = v_0/9.81$. Since $x = 9$ at this instant,

$$9 = -4.905\left(\frac{v_0}{9.81}\right)^2 + v_0\left(\frac{v_0}{9.81}\right).$$

The solution of this equation is $v_0 = 13.29$ m/s or $v_0 = 47.8$ km/hr. Thus, the maximum possible speed of the car was 47.8 km/hr.

5. Let us take x as positive in the direction of motion of the car with $x = 0$ and $t = 0$ when motion commences. Newton's second law for the acceleration of the car is

$$1500\frac{dv}{dt} = 2500 - v^2 \quad \Longrightarrow \quad \frac{dv}{v^2 - 2500} = -\frac{dt}{1500}.$$

A one-parameter family of solutions of this separated differential equation is defined implicitly by

$$\begin{aligned} -\frac{t}{1500} + C &= \int \frac{1}{v^2 - 2500}dv = \frac{1}{100}\int\left(\frac{-1}{v+50} + \frac{1}{v-50}\right)dv \\ &= \frac{1}{100}(\ln|v-50| - \ln|v+50|) = \frac{1}{100}\ln\left|\frac{v-50}{v+50}\right|. \end{aligned}$$

Exponentiation gives

$$\frac{v-50}{v+50} = De^{-t/15} \quad \Longrightarrow \quad v - 50 = D(v+50)e^{-t/15} \quad \Longrightarrow \quad v(t) = \frac{50(1+De^{-t/15})}{1-De^{-t/15}}.$$

The initial condition $v(0) = 0$ gives $0 = 50(1+D)/(1-D) \Longrightarrow D = -1$. Hence,

$$v(t) = \frac{50(1-e^{-t/15})}{1+e^{-t/15}} \qquad \text{and} \qquad v(10) = \frac{50(1-e^{-10/15})}{1+e^{-10/15}} = 16.1 \text{ m/s}.$$

Integration of the velocity gives

$$x(t) = 50\int \frac{1-e^{-t/15}}{1+e^{-t/15}}dt = 50\int\left(1 - \frac{2e^{-t/15}}{1+e^{-t/15}}\right)dt = 50[t + 30\ln(1+e^{-t/15})] + C.$$

Since $x(0) = 0$, we find $0 = 50[30\ln 2] + C \Longrightarrow C = -1500\ln 2$. Thus,

$$x(t) = 50t + 1500[\ln(1+e^{-t/15}) - \ln 2] = 50t + 1500\ln\left[\frac{1}{2}(1+e^{-t/15})\right].$$

For $t = 10$, we find $x(10) = 500 + 1500\ln\left[\frac{1}{2}(1+e^{-10/15})\right] = 81.8$ m.

6. (a) Let us take y as positive downward with $y = 0$ and time $t = 0$ when motion begins. Newton's second gives

$$m\frac{dv}{dt} = mg - kv \qquad \Longrightarrow \qquad \frac{1}{mg-kv}dv = \frac{1}{m}dt,$$

a separated differential equation. A one-parameter family of solutions is defined implicitly by

$$-\frac{1}{k}\ln|mg-kv| = \frac{t}{m} + C \quad \Longrightarrow \quad \ln|mg-kv| = -\frac{kt}{m} - kC \quad \Longrightarrow \quad mg - kv = De^{-kt/m}.$$

For $v(0) = 0$, we find $mg = D$, and therefore

$$mg - kv = mge^{-kt/m} \qquad \Longrightarrow \qquad v(t) = \frac{mg}{k}(1-e^{-kt/m}).$$

Terminal velocity is mg/k. Velocity is 95% of this when

$$0.95\frac{mg}{k} = \frac{mg}{k}(1 - e^{-kt/m}) \quad \Longrightarrow \quad e^{-kt/m} = 1 - 0.95 \quad \Longrightarrow \quad t = \frac{m}{k}\ln 20.$$

(b) Integration of velocity gives

$$y(t) = \frac{mg}{k}\left(t + \frac{m}{k}e^{-kt/m}\right) + D.$$

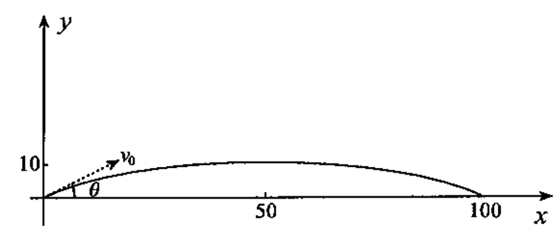
Since $y(0) = 0$, it follows that $0 = (mg/k)(m/k) + D$. Hence, $D = -m^2g/k^2$ and

$$y(t) = \frac{mg}{k}\left(t + \frac{m}{k}e^{-kt/m}\right) - \frac{m^2g}{k^2} = \frac{mgt}{k} - \frac{m^2g}{k^2}(1 - e^{-kt/m}).$$

If we substitute $t = (m/k)\ln 20$, we obtain distance fallen as

$$y = \frac{m^2g}{k^2}\ln 20 - \frac{m^2g}{k^2}(1 - e^{-\ln 20}) = \frac{m^2g}{k^2}\left(\ln 20 - \frac{19}{20}\right).$$

7. We can work with separate equations for x- and y-components of motion or in vectors. We choose the latter. The acceleration of the arrow is $\mathbf{a} = -g\hat{\mathbf{j}}$, so that $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$. If we take $t = 0$ to be the time when the arrow leaves the bow, then when the bow is held at angle θ, and the initial speed of the arrow is v_0,



$\mathbf{v}(0) = v_0(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})$. This gives $v_0(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = \mathbf{C}$. Integration of $\mathbf{v} = -gt\hat{\mathbf{j}} + \mathbf{C}$ gives $\mathbf{r} = -gt^2\hat{\mathbf{j}}/2 + \mathbf{C}t + \mathbf{D}$. If the arrow starts from the origin, then $\mathbf{r}(0) = \mathbf{0}$, from which $\mathbf{D} = \mathbf{0}$, and therefore

$$\mathbf{r} = -\frac{1}{2}gt^2\hat{\mathbf{j}} + v_0t(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (v_0t\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gt^2 + v_0t\sin\theta\right)\hat{\mathbf{j}}.$$

If T is the time for the arrow to reach maximum height, we can say that $\mathbf{R}(T) = 50\hat{\mathbf{i}} + 10\hat{\mathbf{j}}$ and $\mathbf{r}(2T) = 100\hat{\mathbf{i}}$. These imply that

$$50\hat{\mathbf{i}} + 10\hat{\mathbf{j}} = (v_0T\cos\theta)\hat{\mathbf{i}} + \left(-\frac{1}{2}gT^2 + v_0T\sin\theta\right)\hat{\mathbf{j}}, \quad 100\hat{\mathbf{i}} = [v_0(2T)\cos\theta]\hat{\mathbf{i}} + \left[-\frac{1}{2}g(2T)^2 + v_0(2T)\sin\theta\right]\hat{\mathbf{j}}.$$

When we equate components, we obtain four equations, three of which are independent,

$$50 = v_0T\cos\theta, \quad 10 = -\frac{1}{2}gT^2 + v_0T\sin\theta, \quad 0 = -2gT^2 + 2v_0T\sin\theta.$$

We eliminate T and solve for v_0 and θ. The result is $v_0 = 37.7$ m/s and $\theta = 0.381$ radians.

8. (a) When M is at position x, and is moving to the left, the spring force is $-kx$ and the frictional force is μMg. Newton's second law therefore gives

$$M\frac{d^2x}{dt^2} = -kx + \mu Mg, \quad x(0) = x_0, \quad v(0) = -v_0,$$

and this equation is valid until M comes to an instantaneous stop for the first time.

(b) If we set $\dfrac{dx}{dt} = v$ and $\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = v\dfrac{dv}{dx}$, then $Mv\dfrac{dv}{dx} = -kx + \mu Mg$, subject to $v(x_0) = -v_0$. This equation can be separated, $Mv\,dv = (-kx + \mu Mg)\,dx$, and a one-parameter family of solutions is defined implicitly by $\dfrac{Mv^2}{2} = -\dfrac{kx^2}{2} + \mu Mgx + C$. The initial condition requires $\dfrac{Mv_0^2}{2} = -\dfrac{kx_0^2}{2} + \mu Mgx_0 + C$. Thus,

$$\frac{Mv^2}{2} = -\frac{kx^2}{2} + \mu Mgx + \frac{Mv_0^2}{2} + \frac{kx_0^2}{2} - \mu Mgx_0 \implies \frac{k}{2}(x_0^2 - x^2) = \frac{M}{2}(v^2 - v_0^2) + \mu Mg(x_0 - x).$$

The left side is the loss of stored energy in the spring at x relative to that initially; $M(v^2 - v_0^2)/2$ is the gain in kinetic energy at x; and $\mu Mg(x_0 - x)$ is the work done against friction when m moves from x_0 to x.

(c) If we set $x = x^*$ when $v = 0$, then x^* is defined implicitly by

$$\frac{k}{2}(x_0^2 - x^{*2}) = -\frac{Mv_0^2}{2} + \mu Mg(x_0 - x^*).$$

This is a quadratic equation in x^*, $kx^{*2} - 2\mu Mgx^* + (2\mu Mgx_0 - kx_0^2 - Mv_0^2) = 0$, with solutions

$$\begin{aligned} x^* &= \frac{2\mu Mg \pm \sqrt{4\mu^2M^2g^2 - 4k(2\mu Mgx_0 - kx_0^2 - Mv_0^2)}}{2k} \\ &= \frac{\mu Mg \pm \sqrt{\mu^2M^2g^2 - k(2\mu Mgx_0 - kx_0^2 - Mv_0^2)}}{k}. \end{aligned}$$

Whether the mass stops to the left of, to the right of, or on the origin depends on the quantity $2\mu Mgx_0 - kx_0^2 - Mv_0^2$. When it is negative, the sum $(1/2)kx_0^2 + (1/2)Mv_0^2$ of the initial energy stored in the spring $(1/2)kx_0^2$ and the initial kinetic energy of the mass $(1/2)Mv_0^2$ is greater than the work done against friction μMgx_0 as the mass travels from $x = x_0$ to $x = 0$. The mass therefore stops at position

$$x^* = \frac{\mu Mg - \sqrt{\mu^2M^2g^2 + k(kx_0^2 + Mv_0^2 - 2\mu Mgx_0)}}{k}$$

to the left of the origin. When $2\mu Mgx_0 - kx_0^2 - Mv_0^2 = 0$, initial spring energy and kinetic energy are just sufficient to bring the mass back to the origin ($x^* = 0$). Finally, when $2\mu Mgx_0 - kx_0^2 - Mv_0^2 > 0$, there is not sufficient initial energy to return the mass to the origin; it stops at position

$$x^* = \frac{\mu Mg - \sqrt{\mu^2M^2g^2 - k(2\mu Mgx_0 - kx_0^2 - Mv_0^2)}}{k} > 0.$$

9. Let us choose y as positive downward taking $y = 0$ and $t = 0$ when the mass is released. Newton's second law for the acceleration of the mass is

$$(1)\frac{dv}{dt} = g - \frac{v^2}{500},$$

where $g = 9.81$. This equation is separable, $\dfrac{1}{v^2 - 500g}\,dv = -\dfrac{1}{500}\,dt$, and a one-parameter family of solutions is defined implicitly by

$$\begin{aligned} -\frac{t}{500} + C &= \int \frac{1}{v^2 - 500g}\,dv = \int \left(\frac{\frac{1}{2\sqrt{500g}}}{v - \sqrt{500g}} + \frac{\frac{-1}{2\sqrt{500g}}}{v + \sqrt{500g}} \right) dv \\ &= \frac{1}{2\sqrt{500g}}\left(\ln|v - \sqrt{500g}| - \ln|v + \sqrt{500g}|\right). \end{aligned}$$

When this equation is solved for v, the result is $v = \dfrac{\sqrt{500g}[1 + De^{-2\sqrt{g/500t}}]}{1 - De^{-2\sqrt{g/500t}}}$. Since $v(0) = 20$, it follows that $20 = \dfrac{\sqrt{500g}[1 + D]}{1 - D}$, and therefore $D = (20 - \sqrt{500g})/(20 + \sqrt{500g}) = -0.556$. Thus, $v(t) = 70.0\left(\dfrac{1 - 0.556e^{-0.280t}}{1 + 0.556e^{-0.280t}}\right)$ m/s.

10. Let us choose y as positive downward taking $y = 0$ and $t = 0$ when the mass is released. Newton's second law for the acceleration of the mass is

$$(1)\frac{dv}{dt} = g - \frac{v^2}{500},$$

where $g = 9.81$. This equation is separable, $\frac{1}{v^2 - 500g}\,dv = -\frac{1}{500}\,dt$, and a one-parameter family of solutions is defined implicitly by

$$-\frac{t}{500} + C = \int \frac{1}{v^2 - 500g}\,dv = \int \left(\frac{\frac{1}{2\sqrt{500g}}}{v - \sqrt{500g}} + \frac{\frac{-1}{2\sqrt{500g}}}{v + \sqrt{500g}} \right) dv$$
$$= \frac{1}{2\sqrt{500g}} \left(\ln|v - \sqrt{500g}| - \ln|v + \sqrt{500g}| \right).$$

When this equation is solved for v, the result is $v = \dfrac{\sqrt{500g}[1 + De^{-2\sqrt{g/500}t}]}{1 - De^{-2\sqrt{g/500}t}}$. Since $v(0) = 100$, it follows that $100 = \dfrac{\sqrt{500g}[1+D]}{1-D}$, and therefore $D = (100 - \sqrt{500g})/(100 + \sqrt{500g}) = 0.176$. Thus, $v(t) = 70.0\left(\dfrac{1 + 0.176e^{-0.280t}}{1 - 0.176e^{-0.280t}}\right)$.

11. Let us choose y as positive upward taking $y = 0$ and $t = 0$ when the mass is released. Newton's second law for the acceleration is

$$(1)\frac{dv}{dt} = -g - \frac{v^2}{500},$$

where $g = 9.81$. We can separate this equation, $\frac{1}{v^2 + 500g}\,dv = -\frac{1}{500}\,dt$, and a one-parameter family of solutions is defined implicitly by

$$\frac{1}{\sqrt{500g}}\,\mathrm{Tan}^{-1}\left(\frac{v}{\sqrt{500g}}\right) = -\frac{t}{500} + C.$$

When we solve this for v, $v(t) = \sqrt{500g}\tan\left[\sqrt{500g}\left(C - \dfrac{t}{500}\right)\right]$. Since $v(0) = 20$, it follows that $20 = \sqrt{500g}\tan[C\sqrt{500g}]$, and this equation can be solved for $C = (1/\sqrt{500g})\mathrm{Tan}^{-1}(20/\sqrt{500g}) = 0.00397$. The velocity is

$$v(t) = 70.0\tan\left[70.0\left(0.00397 - \frac{t}{500}\right)\right].$$

Maximum height is atained when $v = 0$ and this occurs when $t = 1.99$ s.

12. If we choose y as positive downward, then integration of the differential equation $m\,dv/dt = mg - kv^2$ as in Example 15.9 leads to

$$v(t) = \frac{V(1 + De^{-2kVt/m})}{1 - De^{-2kVt/m}},$$

where $V = \sqrt{mg/k}$ is the terminal velocity of the body. The initial velocity $v(0) = v_0$ requires

$$v_0 = \frac{V(1+D)}{1-D} \implies v_0(1-D) = V(1+D) \implies D = \frac{v_0 - V}{v_0 + V}.$$

Hence, $$v(t) = \frac{V\left[1 + \left(\dfrac{v_0 - V}{v_0 + V}\right)e^{-2kVt/m}\right]}{1 - \left(\dfrac{v_0 - V}{v_0 + V}\right)e^{-2kVt/m}} = \frac{V\left[1 - \left(\dfrac{V - v_0}{V + v_0}\right)e^{-2kVt/m}\right]}{1 + \left(\dfrac{V - v_0}{V + v_0}\right)e^{-2kVt/m}}.$$

13. If we choose y as positive downward, then integration of the differential equation $mdv/dt = mg - kv^2$ as in Example 15.9 leads to

$$v(t) = \frac{V(1 + De^{-2kVt/m})}{1 - De^{-2kVt/m}},$$

where $V = \sqrt{mg/k}$ is the terminal velocity of the body. The initial velocity $v(0) = v_0$ requires

$$v_0 = \frac{V(1+D)}{1-D} \implies v_0(1-D) = V(1+D) \implies D = \frac{v_0 - V}{v_0 + V}.$$

Hence, $$v(t) = \frac{V\left[1 + \left(\dfrac{v_0 - V}{v_0 + V}\right)e^{-2kVt/m}\right]}{1 - \left(\dfrac{v_0 - V}{v_0 + V}\right)e^{-2kVt/m}}.$$

14. If we choose y as positive upward, the differential equation describing the velocity of the body is

$$m\frac{dv}{dt} = -mg - kv^2 \implies \frac{dv}{v^2 + mg/k} = -\frac{k}{m}dt \implies \sqrt{\frac{k}{mg}}\mathrm{Tan}^{-1}\left(\frac{v}{\sqrt{mg/k}}\right) = -\frac{kt}{m} + C.$$

Hence, $v(t) = \sqrt{\dfrac{mg}{k}}\tan\left(D - \sqrt{\dfrac{kg}{m}}t\right)$, where $D = \sqrt{mg/k}C$. The initial condition $v(0) = v_0$ requires $v_0 = \sqrt{mg/k}\tan D \Longrightarrow D = \mathrm{Tan}^{-1}[\sqrt{k/(mg)}v_0]$. Thus,

$$v(t) = \sqrt{\frac{mg}{k}}\tan\left[\mathrm{Tan}^{-1}\left(\sqrt{\frac{k}{mg}}v_0\right) - \sqrt{\frac{kg}{m}}t\right].$$

Maximum height is attained when

$$0 = v(t) = \sqrt{\frac{mg}{k}}\tan\left[\mathrm{Tan}^{-1}\left(\sqrt{\frac{k}{mg}}v_0\right) - \sqrt{\frac{kg}{m}}t\right] \implies \mathrm{Tan}^{-1}\left(\sqrt{\frac{k}{mg}}v_0\right) - \sqrt{\frac{kg}{m}}t = n\pi,$$

where n is an integer. Solving for t gives $t = \sqrt{\dfrac{m}{kg}}\left[\mathrm{Tan}^{-1}\left(\sqrt{\dfrac{k}{mg}}v_0\right) - n\pi\right]$. When we choose $n = 0$ to obtain the smallest positive solution, $t = \sqrt{\dfrac{m}{kg}}\mathrm{Tan}^{-1}\left(\sqrt{\dfrac{k}{mg}}v_0\right)$.

15. If we choose y as positive upward, the differential equation describing the velocity of the rock is

$$\frac{dv}{dt} = -9.81 - \frac{v^2}{10} \implies \frac{dv}{v^2 + 98.1} = -\frac{dt}{10} \implies \frac{1}{\sqrt{98.1}}\mathrm{Tan}^{-1}\left(\frac{v}{\sqrt{98.1}}\right) = -\frac{t}{10} + C.$$

Hence, $v(t) = \sqrt{98.1}\tan\left(\dfrac{-\sqrt{98.1}t}{10} + D\right)$. The initial velocity $v(0) = 20$ requires $20 = \sqrt{98.1}\tan D \Longrightarrow$ $D = \mathrm{Tan}^{-1}(20/\sqrt{98.1})$. Thus, $v(t) = \sqrt{98.1}\tan\left[\mathrm{Tan}^{-1}\left(\dfrac{20}{\sqrt{98.1}}\right) - \dfrac{\sqrt{98.1}t}{10}\right]$. Integration gives $y(t) = 10\ln\left\{\cos\left[\mathrm{Tan}^{-1}\left(\dfrac{20}{\sqrt{98.1}}\right) - \dfrac{\sqrt{98.1}t}{10}\right]\right\} + C$. If we take $y(0) = 0$, then $0 = 10\ln\left\{\cos\left[\mathrm{Tan}^{-1}\left(\dfrac{20}{\sqrt{98.1}}\right)\right]\right\} + C$, which defines C. Hence,

$$y(t) = 10\ln\left\{\cos\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{98.1}}\right) - \frac{\sqrt{98.1}t}{10}\right]\right\} - 10\ln\left\{\cos\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{98.1}}\right)\right]\right\}.$$

For maximum height we set $v = 0$, and this implies that $\text{Tan}^{-1}\left(\dfrac{20}{\sqrt{98.1}}\right) - \dfrac{\sqrt{98.1}t}{10} = n\pi$, where n is an integer. When we choose $n = 0$ for the smallest positive solution, $t = (10/\sqrt{98.1})\text{Tan}^{-1}(20/\sqrt{98.1})$. For this t, the height of the rock is $-10\ln\{\cos[\text{Tan}^{-1}(20/\sqrt{98.1})]\} = 8.1$ m.

16. (a) Let us choose y as positive upward taking $y = 0$ and $t = 0$ when the mass is released. Newton's second law for acceleration during ascent is $m\dfrac{dv}{dt} = -mg - kv^2$. This is valid only during ascent since air resistance during descent is kv^2. If we set $V = \sqrt{mg/k}$, then

$$-\frac{m}{k}\frac{dv}{dt} = V^2 + v^2 \quad\Longrightarrow\quad \frac{1}{v^2 + V^2}\,dv = -\frac{k}{m}\,dt \quad\Longrightarrow\quad \frac{1}{V}\text{Tan}^{-1}\left(\frac{v}{V}\right) = -\frac{kt}{m} + C.$$

Since $v(0) = v_0$, we obtain $\dfrac{1}{V}\,\text{Tan}^{-1}\left(\dfrac{v_0}{V}\right) = C$. Thus,

$$\frac{1}{V}\,\text{Tan}^{-1}\left(\frac{v}{V}\right) = -\frac{kt}{m} + \frac{1}{V}\,\text{Tan}^{-1}\left(\frac{v_0}{V}\right).$$

When we solve this equation for v, the result is $v(t) = V\tan\left[\text{Tan}^{-1}\left(\dfrac{v_0}{V}\right) - \dfrac{kVt}{m}\right]$. Once again this is only valid during ascent since air resistance is kv^2 (rather than $-kv^2$) on descent.

(b) Integration of the velocity gives $y = \dfrac{m}{k}\ln\left|\cos\left[\text{Tan}^{-1}\left(\dfrac{v_0}{V}\right) - \dfrac{kVt}{m}\right]\right| + D$. Since $y(0) = 0$, we find that $0 = \dfrac{m}{k}\ln\left|\cos\left[\text{Tan}^{-1}\left(\dfrac{v_0}{V}\right)\right]\right| + D$, and therefore $D = (m/k)\ln(\sqrt{v_0^2 + V^2}/V)$. The height of the mass is

$$y = \frac{m}{k}\ln\left|\cos\left[\text{Tan}^{-1}\left(\frac{v_0}{V}\right) - \frac{kVt}{m}\right]\right| + \frac{m}{k}\ln\left(\frac{\sqrt{v_0^2 + V^2}}{V}\right).$$

Maximum height occurs when $0 = v = V\tan\left[\text{Tan}^{-1}\left(\dfrac{v_0}{V}\right) - \dfrac{kVt}{m}\right]$, and the solution of this equation is $t = \dfrac{m}{kV}\,\text{Tan}^{-1}\left(\dfrac{v_0}{V}\right)$. Maximum height is therefore

$$\frac{m}{k}\ln\left|\cos\left[\text{Tan}^{-1}\left(\frac{v_0}{V}\right) - \text{Tan}^{-1}\left(\frac{v_0}{V}\right)\right]\right| + \frac{m}{k}\ln\left(\frac{\sqrt{v_0^2 + V^2}}{V}\right) = \frac{m}{k}\ln\left(\frac{\sqrt{v_0^2 + V^2}}{V}\right).$$

17. If we take x as positive in the direction of motion, the differential equation describing motion is

$$m\frac{dv}{dt} = -F - \beta v \quad\Longrightarrow\quad \frac{dv}{F + \beta v} = -\frac{dt}{m} \quad\Longrightarrow\quad \frac{1}{\beta}\ln|F + \beta v| = -\frac{t}{m} + C.$$

Exponentiation gives $F + \beta v = De^{-\beta t/m} \Longrightarrow v = (1/\beta)(De^{-\beta t/m} - F)$. The initial velocity $v(0) = v_0$ requires $v_0 = (1/\beta)(D - F) \Longrightarrow D = F + \beta v_0$, and $v(t) = (1/\beta)(F + \beta v_0)e^{-\beta t/m} - F/\beta$. Integration of this gives

$$x(t) = -\frac{m}{\beta^2}(F + \beta v_0)e^{-\beta t/m} - \frac{Ft}{\beta} + C.$$

If we take $x(0) = 0$, then $0 = -(m/\beta^2)(F + \beta v_0) + C \Longrightarrow C = (m/\beta^2)(F + \beta v_0)$, and

$$x(t) = -\frac{m}{\beta^2}(F + \beta v_0)e^{-\beta t/m} - \frac{Ft}{\beta} + \frac{m}{\beta^2}(F + \beta v_0) = -\frac{Ft}{\beta} + \frac{m}{\beta^2}(F + \beta v_0)(1 - e^{-\beta t/m}).$$

The object comes to rest when $0 = v(t) = \dfrac{1}{\beta}(F + \beta v_0)e^{-\beta t/m} - \dfrac{F}{\beta}$. This can be solved for t,

$$e^{-\beta t/m} = \frac{F}{F+\beta v_0} \quad \Longrightarrow \quad t = -\frac{m}{\beta}\ln\left(\frac{F}{F+\beta v_0}\right) = \frac{m}{\beta}\ln\left(1+\frac{\beta v_0}{F}\right).$$

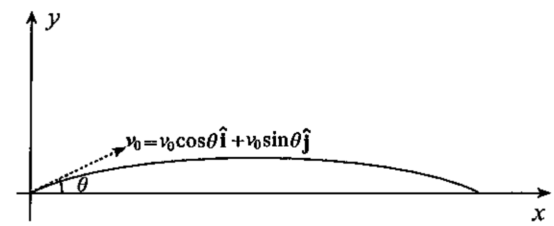
The position of the object at this time is

$$x = -\frac{F}{\beta}\left(\frac{m}{\beta}\right)\ln\left(1+\frac{\beta v_0}{F}\right) + \frac{m}{\beta^2}(F+\beta v_0)\left(1-\frac{F}{F+\beta v_0}\right) = -\frac{Fm}{\beta^2}\ln\left(1+\frac{\beta v_0}{F}\right) + \frac{mv_0}{\beta}.$$

18. The initial-value problem for the motion of the projectile (figure to the right) is

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\hat{\mathbf{j}} - \beta\mathbf{v},$$

subject to initial displacement $\mathbf{r}(0) = \mathbf{0}$ and initial velocity $\mathbf{r}'(0) = \mathbf{v}_0 = v_0\cos\theta\,\hat{\mathbf{i}} + v_0\sin\theta\,\hat{\mathbf{j}}$. We can solve this differential equation in vector form, or in component form. The component scalar differential equations are identical, so let us save space by integrating vectorially. When we write



$$\frac{d\mathbf{v}}{dt} + \frac{\beta}{m}\mathbf{v} = -g\hat{\mathbf{j}},$$

we have a linear first-order differential equation with integrating factor $e^{\int(\beta/m)dt} = e^{\beta t/m}$. When we multiply the differential equation by $e^{\beta t/m}$, it can be expressed in the form

$$\frac{d}{dt}\left(\mathbf{v}e^{\beta t/m}\right) = -ge^{\beta t/m}\hat{\mathbf{j}} \quad \Longrightarrow \quad \mathbf{v}e^{\beta t/m} = -\frac{mg}{\beta}e^{\beta t/m}\hat{\mathbf{j}} + \mathbf{C} \quad \Longrightarrow \quad \mathbf{v} = -\frac{mg}{\beta}\hat{\mathbf{j}} + \mathbf{C}e^{-\beta t/m}.$$

The initial velocity condition requires $\mathbf{v_0} = -\frac{mg}{\beta}\hat{\mathbf{j}} + \mathbf{C}$, and therefore

$$\mathbf{v}(t) = -\frac{mg}{\beta}\hat{\mathbf{j}} + \left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right)e^{-\beta t/m}.$$

Integrating once again gives

$$\mathbf{r}(t) = -\frac{mgt}{\beta}\hat{\mathbf{j}} - \frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right)e^{-\beta t/m} + \mathbf{D}.$$

For $\mathbf{r}(0) = \mathbf{0}$, we must have $\mathbf{0} = -\frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right) + \mathbf{D}$. Thus,

$$\mathbf{r}(t) = -\frac{mgt}{\beta}\hat{\mathbf{j}} - \frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right)e^{-\beta t/m} + \frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\right)\hat{\mathbf{j}} = -\frac{mgt}{\beta}\hat{\mathbf{j}} + \frac{m}{\beta}\left(\mathbf{v}_0 + \frac{mg}{\beta}\hat{\mathbf{j}}\right)(1 - e^{-\beta t/m}).$$

19. When resistance is proportional to the square of velocity, Newton's second law can be expressed in the form

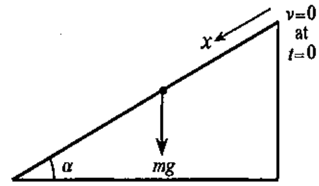
$$m\frac{d\mathbf{v}}{dt} = -mg\hat{\mathbf{j}} - \beta|\mathbf{v}|\mathbf{v}.$$

In vector form it is perhaps not clear that we cannot integrate this equation. If we separate it into x- and y-components, however, we obtain

$$m\frac{dv_x}{dt} = -\beta v_x\sqrt{v_x^2 + y_y^2}, \qquad m\frac{dv_y}{dt} = -mg - \beta v_y\sqrt{v_x^2 + v_y^2}.$$

Both equations contain both unknowns v_x and v_y, unlike the previous exercise where component equations would be uncoupled.

20. Newton's second law for acceleration gives



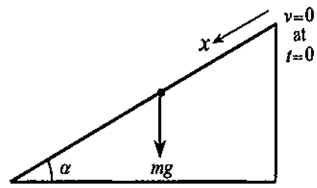
$$m\frac{dv}{dt} = mg\sin\alpha.$$

Integration gives $v = (g\sin\alpha)t + C$. Since $v(0) = 0$, it follows that $C = 0$, and

$$v = \frac{dx}{dt} = (g\sin\alpha)t.$$

Integration now gives $x = (g\sin\alpha)t^2/2 + E$. If we choose $x = 0$ when motion begins, then $E = 0$, and $x = (g\sin\alpha)t^2/2$. The time for the mass to travel distance D is given by $D = (g\sin\alpha)t^2/2 \implies t = \sqrt{2D/(g\sin\alpha)}$. The speed of the mass at this time is $g\sin\alpha\sqrt{\dfrac{2D}{g\sin\alpha}} = \sqrt{2gD\sin\alpha}$.

21. When air resistance proportional to velocity acts on the mass, Newton's second law gives



$$m\frac{dv}{dt} = mg\sin\alpha - kv.$$

This equation can be separated,

$$\frac{1}{v - \dfrac{mg}{k}\sin\alpha}\,dv = -\frac{k}{m}\,dt,$$

and a one-parameter family of solutions is defined implicitly by $\ln\left|v - \dfrac{mg}{k}\sin\alpha\right| = -\dfrac{kt}{m} + C$. When we solve for v, the result is $v = (mg/k)\sin\alpha + De^{-kt/m}$. Since $v(0) = 0$, it follows that $0 = (mg/k)\sin\alpha + D$. Hence, $v = (mg/k)\sin\alpha[1 - e^{-kt/m}]$. Integration now gives $x = \dfrac{mg}{k}\sin\alpha\left[t + \dfrac{m}{k}e^{-kt/m}\right] + E$. Since $x(0) = 0$, we obtain $0 = \dfrac{m^2g}{k^2}\sin\alpha + E$. Thus,

$$x = \frac{m^2g}{k^2}\sin\alpha\left[\frac{kt}{m} + e^{-kt/m}\right] - \frac{m^2g}{k^2}\sin\alpha = \frac{m^2g}{k^2}\sin\alpha\left[\frac{kt}{m} + e^{-kt/m} - 1\right].$$

22. Since $dy/dt = 2x + 4$ and $dx/dt = (3 - y)/2$, it follows that

$$\frac{dy}{dx} = \frac{2x+4}{(3-y)/2} \quad\implies\quad (3-y)\,dy = 4(x+2)\,dx,$$

a separated equation. A one-parameter family of solutions is defined implicitly by

$$3y - \frac{y^2}{2} = 4\left(\frac{x^2}{2} + 2x\right) + C.$$

Since the electron passes through $(0, 3)$, we must have $9 - 9/2 = C$, and therefore

$$3y - \frac{y^2}{2} = 2x^2 + 8x + \frac{9}{2} \quad\implies\quad 4x^2 + y^2 + 16x - 6y + 9 = 0.$$

This is an ellipse.

23. If we take the positive direction away from the earth's surface, then Newton's second law for the acceleration of a projectile gives $m\dfrac{dv}{dt} = -\dfrac{GMm}{r^2}$. If we set $\dfrac{dv}{dt} = \dfrac{dv}{dr}\dfrac{dr}{dt} = v\dfrac{dv}{dr}$, then

$$v\frac{dv}{dr} = -\frac{GM}{r^2} \quad\implies\quad v\,dv = -\frac{GM\,dr}{r^2}.$$

A one-parameter family of solutions of this separated differential equation is defined implicitly by $v^2/2 = GM/r + C$. If R is the radius of the earth, and v_0 is the initial velocity of the projectile, then $v(R) = v_0$, and this implies that ${v_0}^2/2 = GM/R + C$. Thus, $v^2/2 = GM/r + {v_0}^2/2 - GM/R$. The projectile escapes the gravitational pull of the earth if its velocity approaches 0 as r becomes infinite. This requires ${v_0}^2/2 - GM/R = 0$, and therefore the initial velocity of the projectile must be $v_0 = \sqrt{2GM/R}$. For an alternative expression, we note that on the earth's surface where $r = R$, the force of gravity on the mass is $-mg = -GMm/R^2 \Longrightarrow GM/R = gR$. Thus, $v_0 = \sqrt{2gR}$.

24. (a) At the surface of the earth, the magnitude of the force of gravity on m is

$$9.81m = \frac{GmM}{(6.37\times 10^6)^2} \quad \Longrightarrow \quad GM = 9.81(6.37\times 10^6)^2 = 3.98\times 10^{14}.$$

(b) If we set $\dfrac{dv}{dt} = \dfrac{dv}{dr}\dfrac{dr}{dt} = v\dfrac{dv}{dr}$, then $v\dfrac{dv}{dr} = -\dfrac{GM}{r^2}$. Variables are separable, $v\,dv = -(GM/r^2)dr$, with solutions defined implicitly by $v^2/2 = GM/r + C$. Since the object is dropped from height $r = 11\,370$, $0 = GM/(11.37\times 10^6) + C \Longrightarrow C = -GM/(11.37\times 10^6)$. Thus,

$$\frac{v^2}{2} = \frac{GM}{r} - \frac{GM}{11.37\times 10^6} \Longrightarrow v(r) = -\sqrt{\frac{2GM}{11.37\times 10^6}}\sqrt{\frac{11.37\times 10^6 - r}{r}} = -8.37\times 10^3\sqrt{\frac{11.37\times 10^6 - r}{r}}.$$

(c)If we substitute $v = dr/dt$,

$$\frac{dr}{dt} = -8.37\times 10^3\sqrt{\frac{11.37\times 10^6 - r}{r}} \quad \Longrightarrow \quad \sqrt{\frac{r}{11.37\times 10^6 - r}}dr = -8.37\times 10^3 dt,$$

from which

$$-8.37\times 10^3 t + C = \int \sqrt{\frac{r}{11.37\times 10^6 - r}}dr.$$

If we set $r = 11.37\times 10^6 \sin^2\theta$ and $dr = 22.74\times 10^6 \sin\theta\cos\theta\,d\theta$, then

$$\begin{aligned}
-8.37\times 10^3 t + C &= \int \frac{\sqrt{11.37\times 10^6}\sin\theta}{\sqrt{11.37\times 10^6}\cos\theta} 22.74\times 10^6 \sin\theta\cos\theta\,d\theta = 22.74\times 10^6 \int \sin^2\theta\,d\theta \\
&= 22.74\times 10^6 \int \left(\frac{1-\cos 2\theta}{2}\right) d\theta = 11.37\times 10^6\left(\theta - \frac{1}{2}\sin 2\theta\right) \\
&= 11.37\times 10^6(\theta - \sin\theta\cos\theta) \\
&= 11.37\times 10^6\left(\mathrm{Sin}^{-1}\sqrt{\frac{r}{11.37\times 10^6}} - \sqrt{\frac{r}{11.37\times 10^6}}\sqrt{1 - \frac{r}{11.37\times 10^6}}\right) \\
&= 11.37\times 10^6\ \mathrm{Sin}^{-1}\sqrt{\frac{r}{11.37\times 10^6}} - \sqrt{11.37\times 10^6 r - r^2}.
\end{aligned}$$

If the object is dropped at time $t = 0$, then $C = 11.37\times 10^6\ \mathrm{Sin}^{-1}\sqrt{\dfrac{11.37\times 10^6}{11.37\times 10^6}} = 5.685\times 10^6\pi$, and the following equation defines r as a function of t,

$$-8.37\times 10^3 t + 5.685\times 10^6\pi = 11.37\times 10^6\ \mathrm{Sin}^{-1}\sqrt{\frac{r}{11.37\times 10^6}} - \sqrt{11.37\times 10^6 r - r^2}.$$

(d) The object hits the earth when

$$-8.37\times 10^3 t + 5.685\times 10^6\pi = 11.37\times 10^6\ \mathrm{Sin}^{-1}\sqrt{\frac{6.37\times 10^6}{11.37\times 10^6}} - \sqrt{11.37\times 10^6(6.37\times 10^6) - (6.37\times 10^6)^2}.$$

When this is solved for t, the result is approximately 2000 s.

25. We take y positive upward with $y = 0$ and $t = 0$ at the point of release. The differential equation describing the motion of the stone during ascent is

$$\frac{1}{10}\frac{dv}{dt} = -\frac{9.81}{10} - \frac{v^2}{1000} \quad \Longrightarrow \quad \frac{dv}{v^2 + 981} = -\frac{dt}{100} \quad \Longrightarrow \quad \frac{1}{\sqrt{981}}\mathrm{Tan}^{-1}\left(\frac{v}{\sqrt{981}}\right) = -\frac{t}{100} + C.$$

The initial condition $v(0) = 20$ implies that $C = (1/\sqrt{981})\mathrm{Tan}^{-1}(20/\sqrt{981})$, and therefore

$$\frac{1}{\sqrt{981}}\mathrm{Tan}^{-1}\left(\frac{v}{\sqrt{981}}\right) = \frac{-t}{100} + \frac{1}{\sqrt{981}}\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) \implies v = \sqrt{981}\tan\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \frac{\sqrt{981}t}{100}\right].$$

Integration of this gives $y(t) = 100\ln\left|\cos\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \frac{\sqrt{981}t}{100}\right]\right| + D.$

Since $y(0) = 0$, we find $0 = 100\ln|\cos[\mathrm{Tan}^{-1}(20/\sqrt{981})]| + D$, from which $D = -100\ln(\sqrt{981}/\sqrt{1381}) = 50\ln(1381/981)$. The height of the stone during ascent is therefore

$$y(t) = 100\ln\left|\cos\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \frac{\sqrt{981}t}{100}\right]\right| + 50\ln\left(\frac{1381}{981}\right).$$

Maximum height occurs when $0 = v = \sqrt{981}\tan\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \frac{\sqrt{981}t}{100}\right]$, and the solution of this equation is $t = (100/\sqrt{981})\mathrm{Tan}^{-1}(20/\sqrt{981}) = 1.81$ s. The height of the stone at this time is

$$100\ln\left|\cos\left[\mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right) - \mathrm{Tan}^{-1}\left(\frac{20}{\sqrt{981}}\right)\right]\right| + 50\ln\left(\frac{1381}{981}\right) = 50\ln\left(\frac{1381}{981}\right).$$

Maintaining the same coordinate system, the differential equation describing the descent of the stone is

$$\frac{1}{10}\frac{dv}{dt} = -\frac{9.81}{10} + \frac{v^2}{1000} \implies \frac{dv}{v^2 - 981} = \frac{dt}{100} \implies \left(\frac{1}{v - \sqrt{981}} - \frac{1}{v + \sqrt{981}}\right)dv = \frac{\sqrt{981}}{50}dt.$$

Integration gives

$$\ln|v - \sqrt{981}| - \ln|v + \sqrt{981}| = \frac{\sqrt{981}t}{50} + E \implies \ln\left|\frac{v - \sqrt{981}}{v + \sqrt{981}}\right| = \frac{\sqrt{981}t}{50} + E.$$

Exponentiation gives

$$\frac{v - \sqrt{981}}{v + \sqrt{981}} = Fe^{\sqrt{981}t/50} \implies v - \sqrt{981} = (v + \sqrt{981})Fe^{\sqrt{981}t/50} \implies v = \frac{\sqrt{981}(1 + Fe^{\sqrt{981}t/50})}{1 - Fe^{\sqrt{981}t/50}}.$$

Let us reset the time to $t = 0$ when the stone begins its downward motion. Then $v(0) = 0$, and this implies that $0 = \sqrt{981}(1 + F)/(1 - F) \Longrightarrow F = -1$. Thus $v(t) = \dfrac{\sqrt{981}(1 - e^{\sqrt{981}t/50})}{1 + e^{\sqrt{981}t/50}}$. Integration gives the position of the stone on its descent,

$$y(t) = \sqrt{981}\int\left(1 - \frac{2e^{\sqrt{981}t/50}}{1 + e^{\sqrt{981}t/50}}\right)dt = \sqrt{981}\left[t - \frac{100}{\sqrt{981}}\ln|1 + e^{\sqrt{981}t/50}|\right] + G.$$

Since $y(0) = 50\ln(1381/981)$,

$$50\ln(1381/981) = \sqrt{981}\left(-\frac{100}{\sqrt{981}}\ln 2\right) + G \implies G = 50\ln\left(\frac{1381}{981}\right) + 100\ln 2 = 50\ln\left(\frac{5524}{981}\right).$$

The stone hits the ground when $0 = y(t) = \sqrt{981}\left[t - \frac{100}{\sqrt{981}}\ln(1 + e^{\sqrt{981}t/50})\right] + 50\ln\left(\frac{5524}{981}\right)$. This equation can be solved numerically for $t = 1.92$ s. When we add this to the ascent time, total time in the air is $1.81 + 1.92 = 3.73$ s.

If air resistance is ignored, acceleration of the stone is given by $dv/dt = -9.81 \Longrightarrow v = -9.81t + C$. With $v(0) = 20$, we find $C = 20$. A second integration gives $y = -4.905t^2 + 20t + D$. Since $y(0) = 0$, it follows that $D = 0$, and $y(t) = -4.905t^2 + 20t$. The stone returns to the ground when

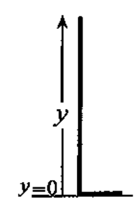
$$0 = y = -4.905t^2 + 20t = t(-4.905t + 20) \implies t = 20/4.905 = 4.08 \text{ s}.$$

26. (a) According to Newton's second law $m\dfrac{d^2r}{dt^2} = -\dfrac{GMm}{(r+R)^2} + \dfrac{GM^*m}{(a+R^*-r)^2}$. On the earth's surface we know that the magnitude of the force of gravity of the earth on m is mg, and therefore $mg = \dfrac{GMm}{R^2} \Longrightarrow GM = gR^2$. Similarly, on the moon's surface, we obtain $mg^* = \dfrac{GM^*m}{R^{*2}} \Longrightarrow GM^* = g^*R^{*2}$. Thus,

$$\frac{d^2r}{dt^2} = \frac{-gR^2}{(r+R)^2} + \frac{g^*R^{*2}}{(a+R^*-r)^2}.$$

(b) If we set $\dfrac{dr}{dt} = v$ and $\dfrac{d^2r}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dr}\dfrac{dr}{dt} = v\dfrac{dv}{dr}$, then $v\dfrac{dv}{dr} = \dfrac{-gR^2}{(r+R)^2} + \dfrac{g^*R^{*2}}{(a+R^*-r)^2}$. This equation can be separated, $v\,dv = \left[\dfrac{-gR^2}{(r+R)^2} + \dfrac{g^*R^{*2}}{(a+R^*-r)^2}\right]dr$, and a one-parameter family of solutions is defined implicitly by $\dfrac{v^2}{2} = \dfrac{gR^2}{r+R} + \dfrac{g^*R^{*2}}{a+R^*-r} + C$. Since $v(0) = v_0$, it follows that $\dfrac{v_0^2}{2} = \dfrac{gR^2}{R} + \dfrac{g^*R^{*2}}{a+R^*} + C$, and therefore $v^2 = \dfrac{2gR^2}{r+R} + \dfrac{2g^*R^{*2}}{a+R^*-r} + v_0^2 - 2gR - \dfrac{2g^*R^{*2}}{a+R^*}$.

27. If y is the length of chain still falling at any given time t, then the mass of falling chain at this time is $m = 2y$, and the force of gravity on this chain has y-component $-2gy$. Newton's second law requires



$$\frac{d}{dt}\left(2y\frac{dy}{dt}\right) = -2gy \quad \Longrightarrow \quad y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = -gy.$$

Because t is explicitly missing, we set $\dfrac{dy}{dt} = v$ and $\dfrac{d^2y}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$,

$$yv\frac{dv}{dy} + v^2 = -gy \quad \Longrightarrow \quad \frac{dv}{dy} + \frac{v}{y} = -\frac{g}{v}.$$

This is a Bernoulli equation (see Exercise 15.3-17) so that we set $w = v^2$ and $dw/dy = 2v\,dv/dy$,

$$\frac{1}{2v}\frac{dw}{dy} + \frac{v}{y} = -\frac{g}{v} \quad \Longrightarrow \quad \frac{dw}{dy} + \frac{2}{y}w = -2g.$$

An integrating factor is $e^{\int (2/y)\,dy} = y^2$, and therefore

$$\frac{d}{dy}(wy^2) = -2gy^2 \quad \Longrightarrow \quad wy^2 = -\frac{2gy^3}{3} + C \quad \Longrightarrow \quad v^2 = -\frac{2gy}{3} + \frac{C}{y^2}.$$

Since $v(3) = 0$, we find $0 = -2g + C/9$, and therefore $v^2 = -\dfrac{2gy}{3} + \dfrac{18g}{y^2}$. Since v must be negative,

$$v = -\sqrt{\left(-\frac{2y}{3} + \frac{18}{y^2}\right)g} = -\frac{1.81}{y}\sqrt{54 - 2y^3} \quad \text{m/s}.$$

The velocity of the end when it hits the floor is $\lim_{y\to 0^+} v(y) = -\infty$.

EXERCISES 15.6

1. Since $L(y_1 + y_2) = 5(y_1 + y_2) = 5y_1 + 5y_2 = L(y_1) + L(y_2)$, and
$L(cy_1) = 5(cy_1) = c(5y_1) = cL(y_1)$,
the operator L is linear.

2. Since $L(y_1 + y_2) = 15x(y_1 + y_2) = 15xy_1 + 15xy_2 = L(y_1) + L(y_2)$, and
$L(cy_1) = 15x(cy_1) = c(15xy_1) = cL(y_1)$,

the operator L is linear.

3. Since $\quad L(y_1+y_2)=y_1+y_2+z(x),$ but

$$L(y_1)+L(y_2)=[y_1+z(x)]+[y_2+z(x)],$$

the operator L is not linear.

4. Since $L(y_1+y_2)=\lim\limits_{x\to 3}(y_1+y_2)=\lim\limits_{x\to 3}y_1+\lim\limits_{x\to 3}y_2=L(y_1)+L(y_2),$ and

$$L(cy_1)=\lim_{x\to 3}cy_1=c\lim_{x\to 3}y_1=cL(y_1),$$

the operator L is linear.

5. Since $L(y_1+y_2)=\lim\limits_{x\to\infty}(y_1+y_2)=\lim\limits_{x\to\infty}y_1+\lim\limits_{x\to\infty}y_2=L(y_1)+L(y_2),$ and

$$L(cy_1)=\lim_{x\to\infty}cy_1=c\lim_{x\to\infty}y_1=cL(y_1),$$

the operator L is linear.

6. Since $L(y_1+y_2)=\dfrac{d}{dx}(y_1+y_2)=\dfrac{dy_1}{dx}+\dfrac{dy_2}{dx}=L(y_1)+L(y_2),$ and

$$L(cy_1)=\frac{d}{dx}(cy_1)=c\,\frac{dy_1}{dx}=cL(y_1),$$

the operator L is linear.

7. Since $L(y_1+y_2)=\dfrac{d^3}{dx^3}(y_1+y_2)=\dfrac{d^3y_1}{dx^3}+\dfrac{d^3y_2}{dx^3}=L(y_1)+L(y_2),$ and

$$L(cy_1)=\frac{d^3}{dx^3}(cy_1)=c\,\frac{d^3y_1}{dx^3}=cL(y_1),$$

the operator L is linear.

8. Since $L(y_1+y_2)=\displaystyle\int(y_1+y_2)\,dx=\int y_1\,dx+\int y_2\,dx=L(y_1)+L(y_2),$ and

$$L(cy_1)=\int cy_1\,dx=c\int y_1\,dx=cL(y_1),$$

the operator L is linear.

9. Since $L(y_1+y_2)=\displaystyle\int_{-1}^{4}(y_1+y_2)\,dx=\int_{-1}^{4}y_1\,dx+\int_{-1}^{4}y_2\,dx=L(y_1)+L(y_2),$ and

$$L(cy_1)=\int_{-1}^{4}cy_1\,dx=c\int_{-1}^{4}y_1\,dx=cL(y_1),$$

the operator L is linear.

10. Since $L(y_1+y_2)=(y_1+y_2)^{1/3}$, but $L(y_1)+L(y_2)=y_1^{1/3}+y_2^{1/3}$, the operator L is not linear.

11. This equation is linear, and in operator notation

$$\phi(x,D)y=x^2+5 \qquad \text{where } \phi(x,D)=2xD^2+x^3.$$

12. This equation is linear, and in operator notation

$$\phi(x,D)y=x^2 \qquad \text{where } \phi(x,D)=2xD^2+(x^3-5).$$

13. Because of the term $5y^2$, the equation is not linear.

14. This equation is linear, and in operator notation

$$\phi(x,D)y=10\sin x \qquad \text{where } \phi(x,D)=xD^3+3xD^2-2D+1.$$

15. Because of the term y^2, the equation is not linear.

16. Because of the term $y\,d^3y/dx^3$, the equation is not linear.

17. This equation is linear, and in operator notation

$$\phi(D)y=9\sec^2 x \qquad \text{where } \phi(D)=D^2-3D-2.$$

18. Because of the yy'' term, this equation is not linear.

19. Since the equation can be expressed in the form $\dfrac{dy}{dx} = (4-x^2)^2 - 1$, the equation is linear. In operator notation, $Dy = (4-x^2)^2 - 1$.

20. This equation is linear, and in operator notation

$$\phi(D)y = \ln x \qquad \text{where } \phi(D) = D^4 + D^2 - 1.$$

21. If $y_1(t)$ and $y_2(t)$ are any two functions in S, and c is a constant,

$$\begin{aligned} L(y_1+y_2) &= \int_0^\infty [y_1(t)+y_2(t)]e^{-st}\,dt = \int_0^\infty y_1(t)e^{-st}\,dt + \int_0^\infty y_2(t)e^{-st}\,dt \\ &= L(y_1) + L(y_2), \\ L(cy_1) &= \int_0^\infty cy_1(t)e^{-st}\,dt = c\int_0^\infty y_1(t)e^{-st}\,dt = cL(y_1). \end{aligned}$$

Thus, L is a linear operator.

22. If $y_1(x)$ and $y_2(x)$ are any two functions in S, and c is a constant,

$$\begin{aligned} L(y_1+y_2) &= \int_0^{2\pi} [y_1(x)+y_2(x)]\cos nx\,dx = \int_0^{2\pi} y_1(x)\,\cos nx\,dx + \int_0^{2\pi} y_2(x)\,\cos nx\,dx \\ &= L(y_1) + L(y_2), \\ L(cy_1) &= \int_0^{2\pi} cy_1(x)\,\cos nx\,dx = c\int_0^{2\pi} y_1(x)\,\cos nx\,dx = cL(y_1). \end{aligned}$$

Thus, L is a linear operator.

EXERCISES 15.7

1. Since $y_1'' + y_1' - 6y_1 = 4e^{2x} + 2e^{2x} - 6e^{2x} = 0$, and similarly, $y_2'' + y_2' - 6y_2 = 0$, $y_1(x)$ and $y_2(x)$ are solutions of the equation. Because the equation is linear and homogeneous, a general solution is $y(x) = C_1e^{2x} + C_2e^{-3x}$.

2. Since $y_1' + y_1\,\tan x = -\sin x + \cos x\,\tan x = 0$, $y_1(x)$ is a solution. Because the equation is linear and homogeneous, a general solution is $y(x) = C\cos x$.

3. Since $y_1'''' + 5y_1'' + 4y_1 = 16\cos 2x - 20\cos 2x + 4\cos 2x = 0$, $y_1(x)$ is a solution of the differential equation. Similarly, $y_2(x)$, $y_3(x)$, and $y_4(x)$ are solutions. Because the equation is linear and homogeneous, a general solution is $y(x) = C_1\cos 2x + C_2\sin 2x + C_3\cos x + C_4\sin x$.

4. Since $2y_1'' - 16y_1' + 32y_1 = 96e^{4x} - 192e^{4x} + 96e^{4x} = 0$, and

$$2y_2'' - 16y_2' + 32y_2 = -4(16xe^{4x} + 8e^{4x}) + 32(4xe^{4x} + e^{4x}) - 64xe^{4x} = 0,$$

$y_1(x)$ and $y_2(x)$ are solutions of the equation. Because the equation is linear and homogeneous, a general solution is $y(x) = D_1(3e^{4x}) + D_2(-2xe^{4x}) = (C_1 + C_2x)e^{4x}$.

5. The equation is clearly satisfied by $y_1(x) = 10$. Since $y_2''' - 3y_2'' + 2y_2' = 3e^x - 9e^x + 6e^x = 0$, $y_2(x)$ is also a solution. Similarly, $y_3(x)$ is a solution. Because the equation is linear and homogeneous, a general solution is $y(x) = D_1(1) + D_2(3e^x) + D_3(4e^{2x}) = C_1 + C_2e^x + C_3e^{2x}$.

6. Since

$$\begin{aligned} 2y_1'' - 8y_1' + 9y_1 &= 2[4e^{2x}\cos(x/\sqrt{2}) - (2\sqrt{2})e^{2x}\sin(x/\sqrt{2}) - (1/2)e^{2x}\cos(x/\sqrt{2})] \\ &\quad -8[2e^{2x}\cos(x/\sqrt{2}) - (1/\sqrt{2})e^{2x}\sin(x/\sqrt{2})] + 9e^{2x}\cos(x/\sqrt{2}) \\ &= 0, \end{aligned}$$

and similarly, $2y_2'' - 8y_2' + 9y_2 = 0$, $y_1(x)$ and $y_2(x)$ are solutions of the equation. Because the equation is linear and homogeneous, a general solution is $y(x) = C_1e^{2x}\cos(x/\sqrt{2}) + C_2e^{2x}\sin(x/\sqrt{2})$.

7. Since
$$x^2y_1''+xy_1'+(x^2-1/4)y_1=x^2\left(-\frac{\sin x}{\sqrt{x}}-\frac{\cos x}{x^{3/2}}+\frac{3\sin x}{4x^{5/2}}\right)$$
$$+x\left(\frac{\cos x}{\sqrt{x}}-\frac{\sin x}{2x^{3/2}}\right)+\left(x^2-\frac{1}{4}\right)\left(\frac{\sin x}{\sqrt{x}}\right)$$
$$=0,$$
$y_1(x)$ is a solution of the differential equation. Similarly, $y_2(x)$ is a solution. Because the equation is linear and homogeneous, a general solution is $y(x)=(C_1/\sqrt{x})\sin x+(C_2/\sqrt{x})\cos x$.

8. Since
$$4y_1+xy_1'+x^2y_1''=4\cos(2\ln x)+x[(-2/x)\sin(2\ln x)]$$
$$+x^2[(2/x^2)\sin(2\ln x)-(4/x^2)\cos(2\ln x)]$$
$$=0,$$
and similarly, $4y_2+xy_2'+x^2y_2''=0$, $y_1(x)$ and $y_2(x)$ are solution of the equation. Because the equation is linear and homogeneous, a general solution is $y(x)=C_1\cos(2\ln x)+C_2\sin(2\ln x)$.

9. Since $y_1''-y_1y_1'=\dfrac{-4}{(x+1)^3}-\left(\dfrac{-2}{x+1}\right)\left[\dfrac{2}{(x+1)^2}\right]=0$, $y_1(x)$ is a solution of the differential equation. Similarly, $y_2(x)$ is a solution. Because
$$(y_1+y_2)''-(y_1+y_2)(y_1'+y_2')=(y_1''-y_1y_1')+(y_2''-y_2y_2')-(y_1y_2'+y_1'y_2)$$
$$=-(y_1y_2'+y_1'y_2)=-\left(\frac{-2}{x+1}\right)\left[\frac{2}{(x+2)^2}\right]-\left[\frac{2}{(x+1)^2}\right]\left(\frac{-2}{x+2}\right)\neq 0,$$
y_1+y_2 is not a solution. We would not expect y_1+y_2 to be a solution because the equation is not linear.

10. If the $y_i(x)$ are linearly dependent, there exists constants C_i, not all zero, such that
$$C_1y_1(x)+C_2y_2(x)+\cdots+C_ny_n(x)=0.$$
When we differentiate this equation $n-1$ times, we obtain $n-1$ more equations:
$$\begin{array}{ccccccccc} C_1y_1'(x) & + & C_2y_2'(x) & + & \cdots & + & C_ny_n'(x) & = & 0,\\ C_1y_1''(x) & + & C_2y_2''(x) & + & \cdots & + & C_ny_n''(x) & = & 0,\\ \vdots & & \vdots & & & & \vdots & & \\ C_1y_1^{(n-1)}(x) & + & C_2y_2^{(n-1)}(x) & + & \cdots & + & C_ny_n^{(n-1)}(x) & = & 0. \end{array}$$
Because this system of n linear equations in $C_1,\ldots,C_n$ has a nontrivial solution, it follows that the determinant of its coefficients must vanish; that is,
$$W(y_1,\ldots,y_n)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n\\ y_1' & y_2' & \cdots & y_n'\\ \vdots & \vdots & & \vdots\\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}=0 \quad \text{on } I.$$

11. Since $W(1,x,x^2)=\begin{vmatrix}1 & x & x^2\\ 0 & 1 & 2x\\ 0 & 0 & 2\end{vmatrix}=2$, the functions are linearly independent.

12. Since $W(x,2x-3x^2,x^2)=\begin{vmatrix}x & 2x-3x^2 & x^2\\ 1 & 2-6x & 2x\\ 0 & -6 & 2\end{vmatrix}=x[2(2-6x)+12x]-[2(2x-3x^2)+6x^2]=0$, the functions are linearly dependent.

13. Since $W(\sin x,\cos x)=\begin{vmatrix}\sin x & \cos x\\ \cos x & -\sin x\end{vmatrix}=1$, the functions are linearly independent.

14. Since $W(x,xe^x,x^2e^x)=\begin{vmatrix}x & xe^x & x^2e^x\\ 1 & (x+1)e^x & (x^2+2x)e^x\\ 0 & (x+2)e^x & (x^2+4x+2)e^x\end{vmatrix}$, which at $x=1$ reduces to $\begin{vmatrix}1 & e & e\\ 1 & 2e & 3e\\ 0 & 3e & 7e\end{vmatrix}=e^2$, the functions are therefore linearly independent.

15. Since $W(x\sin x, e^{2x}) = \begin{vmatrix} x\sin x & e^{2x} \\ \sin x + x\cos x & 2e^{2x} \end{vmatrix} = e^{2x}[(2x-1)\sin x - x\cos x] \not\equiv 0$, the functions are linearly independent.

EXERCISES 15.8

1. The auxiliary equation is $0 = m^2 + m - 6 = (m+3)(m-2)$ with solutions $m = -3,\ 2$. A general solution of the differential equation is therefore $y(x) = C_1e^{-3x} + C_2e^{2x}$.
2. The auxiliary equation is $0 = 2m^2 - 16m + 32 = 2(m-4)^2$ with solutions $m = 4,\ 4$. A general solution of the differential equation is therefore $y(x) = (C_1 + C_2x)e^{4x}$.
3. The auxiliary equation is $0 = 2m^2 + 16m + 82$ with solutions $m = -4 \pm 5i$. A general solution of the differential equation is therefore $y(x) = e^{-4x}(C_1\cos 5x + C_2\sin 5x)$.
4. The auxiliary equation is $0 = m^2 + 2m - 2$ with solutions $m = -1 \pm \sqrt{3}$. A general solution of the differential equation is therefore $y(x) = C_1e^{-(1+\sqrt{3})x} + C_2e^{(-1+\sqrt{3})x}$.
5. The auxiliary equation is $0 = m^2 - 4m + 5$ with solutions $m = 2 \pm i$. A general solution of the differential equation is therefore $y(x) = e^{2x}(C_1\cos x + C_2\sin x)$.
6. The auxiliary equation is $0 = m^3 - 3m^2 + m - 3 = (m-3)(m^2+1)$ with solutions $m = 3,\ \pm i$. A general solution of the differential equation is therefore $y(x) = C_1e^{3x} + C_2\cos x + C_3\sin x$.
7. The auxiliary equation is $0 = m^4 + 2m^2 + 1 = (m^2+1)^2$ with solutions $m = \pm i, \pm i$. A general solution of the differential equation is therefore $y(x) = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x$.
8. The auxiliary equation is $0 = m^3 - 6m^2 + 12m - 8 = (m-2)^3$ with solutions $m = 2,\ 2,\ 2$. A general solution of the differential equation is therefore $y(x) = (C_1 + C_2x + C_3x^2)e^{2x}$.
9. The auxiliary equation is $0 = 3m^3 - 12m^2 + 18m - 12 = 3(m-2)(m^2-2m+2)$ with solutions $m = 2,\ 1 \pm i$. A general solution of the differential equation is therefore $y(x) = C_1e^{2x} + e^x(C_2\cos x + C_3\sin x)$.
10. The auxiliary equation is $0 = m^4 + 5m^2 + 4 = (m^2+1)(m^2+4)$ with solutions $m = \pm i,\ \pm 2i$. A general solution of the differential equation is therefore $y(x) = C_1\cos x + C_2\sin x + C_3\cos 2x + C_4\sin 2x$.
11. The auxiliary equation is $0 = m^3 - 3m^2 + 2m = m(m-1)(m-2)$ with solutions $m = 0,\ 1,\ 2$. A general solution of the differential equation is therefore $y(x) = C_1 + C_2e^x + C_3e^{2x}$.
12. The auxiliary equation is $0 = m^4 + 16 = (m^2+4i)(m^2-4i)$. To solve $m^2 = 4i$, we set $m = a + bi$, so that $4i = (a+bi)^2 = (a^2 - b^2) + 2abi$. When we equate real and imaginary parts, $a^2 - b^2 = 0$ and $2ab = 4$, These imply that $a = b = \pm\sqrt{2}$. Thus, $m = \pm\sqrt{2}(1+i)$. From $m^2 = -4i$, we obtain $m = \pm\sqrt{2}(1-i)$. A general solution of the differential equation is therefore
$$y(x) = e^{\sqrt{2}x}[C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x)] + e^{-\sqrt{2}x}[C_3\cos(\sqrt{2}x) + C_4\sin(\sqrt{2}x)].$$
13. For this general solution, roots of the auxiliary equation had to be $m = 1,\ -4,\ -4$, and therefore
$$\phi(m) = (m-1)(m+4)^2 = m^3 + 7m^2 + 8m - 16.$$
A possible differential equation is therefore $y''' + 7y'' + 8y' - 16y = 0$.
14. For this general solution, roots of the auxiliary equation had to be $m = -2 \pm 4i$, and therefore
$$\phi(m) = (m+2+4i)(m+2-4i) = m^2 + 4m + 20.$$
A possible differential equation is therefore $y'' + 4y' + 20y = 0$.
15. For this general solution, roots of the auxiliary equation had to be $m = 0,\ \pm\sqrt{3}$, and therefore
$$\phi(m) = m(m - \sqrt{3})(m + \sqrt{3}) = m^3 - 3m.$$
A possible differential equation is therefore $y''' - 3y' = 0$.

16. For this general solution, the roots of the auxiliary equation had to be $m = 1 \pm \sqrt{2}i,\ 1 \pm \sqrt{2}i$, and therefore

$$\phi(m) = (m - 1 + \sqrt{2}i)^2(m - 1 - \sqrt{2}i)^2 = (m^2 - 2m + 3)^2 = m^4 - 4m^3 + 10m^2 - 12m + 9.$$

A possible differential equation is therefore $y'''' - 4y''' + 10y'' - 12y' + 9y = 0$.

17. When we substitute $e^{ax}\sin bx$ into the differential equation,

$$\begin{aligned}0 &= 10e^{ax}\sin bx + 2(ae^{ax}\sin bx + be^{ax}\cos bx) + (a^2e^{ax}\sin bx + 2abe^{ax}\cos bx - b^2e^{ax}\sin bx)\\ &= e^{ax}[(10 + 2a + a^2 - b^2)\sin bx + (2b + 2ab)\cos bx].\end{aligned}$$

Since $\cos bx$ and $\sin bx$ are linearly independent functions, it follows that $10 + 2a + a^2 - b^2 = 0$ and $2b + 2ab = 0$. These equations imply that $a = -1$ and $b = \pm 3$. When we substitute $e^{-x}\cos 3x$ into the differential equation,

$$\begin{aligned}y'' + 2y' + 10y &= (e^{-x}\cos 3x + 6e^{-x}\sin 3x - 9e^{-x}\cos 3x)\\ &\qquad + 2(-e^{-x}\cos 3x - 3e^{-x}\sin 3x) + 10e^{-x}\cos 3x\\ &= 0.\end{aligned}$$

A general solution of the differential equation is therefore $e^{-x}(C_1\cos 3x + C_2\sin 3x)$.

18. For this $y(x)$ to be a solution, the roots of the auxiliary equation must be $m = -1,\ -2\pm 4i$, and therefore $\phi(m) = (m + 1)(m + 2 + 4i)(m + 2 - 4i) = m^3 + 5m^2 + 24m + 20$. It follows that

$$m^3 + 5m^2 + 24m + 20 = m^3 + am^2 + bm + c,$$

and we conclude that $a = 5$, $b = 24$, and $c = 20$.

19. The auxiliary equation is $m^2 + \lambda = 0$. Solutions depend on whether λ is positive, negative, or zero. We consider all three cases. When $\lambda < 0$, solutions are $m = \pm\sqrt{-\lambda}$, in which case $y(x) = C_1e^{\sqrt{-\lambda}x} + C_2e^{-\sqrt{-\lambda}x}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1 + C_2, \qquad 0 = y(3) = C_1e^{3\sqrt{-\lambda}} + C_2e^{-3\sqrt{-\lambda}}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 0$, the auxiliary equation has a double root $m = 0$, in which case $y(x) = C_1 + C_2x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y(3) = C_1 + 3C_2.$$

Once again, the only solution of these is $C_1 = C_2 = 0$, from which $y(x) = 0$.

When $\lambda > 0$, roots of the auxiliary equation are $m = \pm\sqrt{\lambda}\,i$, in which case $y(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y(3) = C_1\cos 3\sqrt{\lambda} + C_2\sin 3\sqrt{\lambda}.$$

With $C_1 = 0$, the second of these implies that $C_2\sin 3\sqrt{\lambda} = 0$. Since we cannot set $C_2 = 0$, else $y(x)$ would again be zero, we must set $\sin 3\sqrt{\lambda} = 0 \Longrightarrow 3\sqrt{\lambda} = n\pi$, where $n \neq 0$ is an integer. Thus, eigenvalues of the Sturm-Liouville system are $\lambda_n = n^2\pi^2/9$, with corresponding eigenfunctions $y_n(x) = C_2\sin(n\pi x/3)$.

20. The auxiliary equation is $m^2 + \lambda = 0$. Solutions depend on whether λ is positive, negative, or zero. We consider all three cases. When $\lambda < 0$, solutions are $m = \pm\sqrt{-\lambda}$, in which case $y(x) = C_1e^{\sqrt{-\lambda}x} + C_2e^{-\sqrt{-\lambda}x}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = \sqrt{-\lambda}\,C_1 - \sqrt{-\lambda}\,C_2, \qquad 0 = y'(4) = \sqrt{-\lambda}\,C_1e^{4\sqrt{-\lambda}} - \sqrt{-\lambda}\,C_2e^{-4\sqrt{-\lambda}}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 0$, the auxiliary equation has a double root $m = 0$, in which case $y(x) = C_1 + C_2x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = C_2, \qquad 0 = y'(4) = C_2.$$

Thus, $\lambda_0 = 0$ is an eigenvalue with eigenfunction $y_0(x) = C_1$.

When $\lambda > 0$, roots of the auxiliary equation are $m = \pm\sqrt{\lambda}\,i$, in which case $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = \sqrt{\lambda}\,C_2, \qquad 0 = y'(4) = -\sqrt{\lambda}\,C_1 \sin 4\sqrt{\lambda} + \sqrt{\lambda}\,C_2 \cos 4\sqrt{\lambda}.$$

With $C_2 = 0$, the second of these implies that $C_1 \sin 4\sqrt{\lambda} = 0$. Since we cannot set $C_1 = 0$, else $y(x)$ would again be zero, we must set $\sin 4\sqrt{\lambda} = 0 \Longrightarrow 4\sqrt{\lambda} = n\pi$, where $n \neq 0$ is an integer. Thus, additional eigenvalues of the Sturm-Liouville system are $\lambda_n = n^2\pi^2/16$, with corresponding eigenfunctions $y_n(x) = C_1 \cos(n\pi x/4)$.

21. The auxiliary equation is $m^2 + \lambda = 0$. Solutions depend on whether λ is positive, negative, or zero. We consider all three cases. When $\lambda < 0$, solutions are $m = \pm\sqrt{-\lambda}$, in which case $y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1 + C_2, \qquad 0 = y'(2) = \sqrt{-\lambda}\,C_1 e^{2\sqrt{-\lambda}} - \sqrt{-\lambda}\,C_2 e^{-2\sqrt{-\lambda}}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 0$, the auxiliary equation has a double root $m = 0$, in which case $y(x) = C_1 + C_2 x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y'(2) = C_2.$$

Once again, the only solution is $y(x) = 0$.

When $\lambda > 0$, roots of the auxiliary equation are $m = \pm\sqrt{\lambda}\,i$, in which case $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y'(2) = -\sqrt{\lambda}\,C_1 \sin 2\sqrt{\lambda} + \sqrt{\lambda}\,C_2 \cos 2\sqrt{\lambda}.$$

With $C_1 = 0$, the second of these implies that $C_2 \cos 2\sqrt{\lambda} = 0$. Since we cannot set $C_2 = 0$, else $y(x)$ would again be zero, we must set $\cos 2\sqrt{\lambda} = 0 \Longrightarrow 2\sqrt{\lambda} = (2n-1)\pi/2$, where n is an integer. Thus, eigenvalues of the Sturm-Liouville system are $\lambda_n = (2n-1)^2\pi^2/16$, with corresponding eigenfunctions $y_n(x) = C_2 \sin[(2n-1)\pi x/4)]$.

22. The auxiliary equation is $m^2 + \lambda = 0$. Solutions depend on whether λ is positive, negative, or zero. We consider all three cases. When $\lambda < 0$, solutions are $m = \pm\sqrt{-\lambda}$, in which case $y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = \sqrt{-\lambda}\,C_1 - \sqrt{-\lambda}\,C_2, \qquad 0 = y(5) = C_1 e^{5\sqrt{-\lambda}} + C_2 e^{-5\sqrt{-\lambda}}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 0$, the auxiliary equation has a double root $m = 0$, in which case $y(x) = C_1 + C_2 x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = C_2, \qquad 0 = y(5) = C_1 + 5C_2.$$

Once again, the only solution is $C_1 = C_2 = 0$, from which $y(x) = 0$.

When $\lambda > 0$, roots of the auxiliary equation are $m = \pm\sqrt{\lambda}\,i$, in which case $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y'(0) = \sqrt{\lambda}\,C_2, \qquad 0 = y(5) = C_1 \cos 5\sqrt{\lambda} + C_2 \sin 5\sqrt{\lambda}.$$

With $C_2 = 0$, the second of these implies that $C_1 \cos 5\sqrt{\lambda} = 0$. Since we cannot set $C_1 = 0$, else $y(x)$ would again be zero, we must set $\cos 5\sqrt{\lambda} = 0 \Longrightarrow 5\sqrt{\lambda} = (2n-1)\pi/2$, where n is an integer. Thus, eigenvalues of the Sturm-Liouville system are $\lambda_n = (2n-1)^2\pi^2/100$, with corresponding eigenfunctions $y_n(x) = C_1 \cos[(2n-1)\pi x/10)]$.

23. The auxiliary equation is $m^2 - m + \lambda = 0$ with solutions $m = (1 \pm \sqrt{1-4\lambda})/2$. The form of the solutions depends on whether $\lambda < 1/4$, $\lambda = 1/4$, or $\lambda > 1/4$. We consider all three cases. When $\lambda < 1/4$, we set $\omega = \sqrt{1-4\lambda}$, in which case roots of the auxiliary equation are $m = (1 \pm \omega)/2$, and $y(x) = C_1 e^{(1+\omega)x/2} + C_2 e^{(1-\omega)x/2}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1 + C_2, \qquad 0 = y(1) = C_1 e^{(1+\omega)/2} + C_2 e^{(1-\omega)/2}.$$

The only solution of these is $C_1 = C_2 = 0$, and therefore $y(x) = 0$.

When $\lambda = 1/4$, the auxiliary equation has a double root $m = 1/2$, in which case $y(x) = (C_1 + C_2 x)e^{x/2}$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y(1) = (C_1 + C_2)e^{1/2}.$$

Once again, the only solution of these is $C_1 = C_2 = 0$, from which $y(x) = 0$.

When $\lambda > 1/4$, we set $\omega = \sqrt{4\lambda - 1}$, in which case roots of the auxiliary equation are $m = (1 \pm \omega i)/2$, and $y(x) = e^{x/2}[C_1 \cos(\omega x/2) + C_2 \sin(\omega x/2)$. The boundary conditions require C_1 and C_2 to satisfy

$$0 = y(0) = C_1, \qquad 0 = y(1) = e^{1/2}[C_1 \cos(\omega/2) + C_2 \sin(\omega/2)].$$

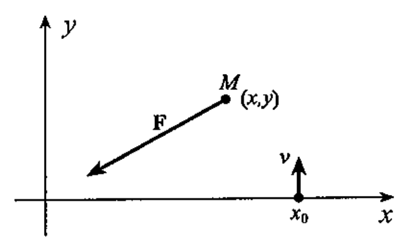
With $C_1 = 0$, the second of these implies that $C_2 \sin(\omega/2) = 0$. Since we cannot set $C_2 = 0$, else $y(x)$ would again be zero, we must set $\sin(\omega/2) = 0 \Longrightarrow \omega/2 = n\pi$, where $n \neq 0$ is an integer. Thus,

$$\omega = 2n\pi \quad \Longrightarrow \quad \sqrt{4\lambda - 1} = 2n\pi \quad \Longrightarrow \quad \lambda = n^2\pi^2 + \frac{1}{4}.$$

Eigenvalues of the Sturm-Liouville system are $\lambda_n = n^2\pi^2 + 1/4$ with corresponding eigenfunctions $y_n(x) = C_2 e^{x/2} \sin(n\pi x)$.

24. (a) When the mass is at position (x, y), the force acting on it is

$$\mathbf{F} = k\sqrt{x^2+y^2}\left(\frac{-x\hat{\mathbf{i}} - y\hat{\mathbf{j}}}{\sqrt{x^2+y^2}}\right) = -k(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}).$$



According to Newton's second law, the acceleration is given by

$$M\frac{d^2\mathbf{r}}{dt^2} = -k(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}).$$

When we equate components,

$$M\frac{d^2x}{dt^2} = -kx, \quad M\frac{d^2y}{dt^2} = -ky.$$

The auxiliary equation for each of these differential equations is $Mm^2 + k = 0$ with solutions $m = \pm\sqrt{k/M}\,i$. If we set $\omega = \sqrt{k/M}$, then

$$x(t) = A\cos\omega t + B\sin\omega t, \qquad y(t) = C\cos\omega t + D\sin\omega t.$$

With the initial conditions $x(0) = x_0$, $x'(0) = 0$, $y(0) = 0$, and $y'(0) = v$,

$$x_0 = A, \quad 0 = \omega B, \quad 0 = C, \quad v = \omega D.$$

Thus, $x = x_0 \cos\omega t$ and $y = (v/\omega)\sin\omega t$ define the path of the mass parametrically. Eliminating t gives

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{\omega y}{v}\right)^2 = 1 \quad \Longrightarrow \quad \frac{x^2}{x_0^2} + \frac{ky^2}{Mv^2} = 1,$$

an ellipse.

(b) In this case, the differential equations are

$$M\frac{d^2x}{dt^2} = kx, \quad M\frac{d^2y}{dt^2} = ky.$$

The auxiliary equation for each of these differential equations is $Mm^2 - k = 0$ with solutions $m = \pm\sqrt{k/M}$. If we set $\omega = \sqrt{k/M}$, then

$$x(t) = Ae^{\omega t} + Be^{-\omega t}, \qquad y(t) = Ce^{\omega t} + De^{-\omega t}.$$

The initial conditions require

$$x_0 = A + B, \quad 0 = \omega A - \omega B, \quad 0 = C + D, \quad v = \omega C - \omega D.$$

These give $A = B = x_0/2$ and $C = -D = v/(2\omega)$. Thus, parametric equations for the path of the mass are

$$x = \frac{x_0}{2}e^{\omega t} + \frac{x_0}{2}e^{-\omega t} = \frac{x_0}{2}(e^{\omega t} + e^{-\omega t}), \quad y = \frac{v}{2\omega}e^{\omega t} - \frac{v}{2\omega}e^{-\omega t} = \frac{v}{2\omega}(e^{\omega t} - e^{-\omega t}).$$

When t is eliminated, we obtain

$$\left(\frac{2x}{x_0}\right)^2 - \left(-\frac{2\omega y}{v}\right)^2 = (e^{\omega t} + e^{-\omega t})^2 - (e^{\omega t} - e^{-\omega t})^2 = 4 \quad \Longrightarrow \quad \frac{x^2}{x_0^2} - \frac{ky^2}{Mv^2} = 1,$$

a hyperbola. The mass moves along the right half of this hyperbola.

25. $D\{e^{px}f(x)\} = e^{px}f'(x) + pe^{px}f(x) = e^{px}\{f'(x) + pf(x)\} = e^{px}\{(D+p)f(x)\}$
The result $D^k\{e^{px}f(x)\} = e^{px}\{(D+p)^k f(x)\}$ has just been proven for $k = 1$. Suppose it is valid for some integer r; that is, suppose that $D^r\{e^{px}f(x)\} = e^{px}\{(D+p)^r f(x)\}$. Then,

$$\begin{aligned} D^{r+1}\{e^{px}f(x)\} &= D[D^r\{e^{px}f(x)\}] = D[e^{px}\{(D+p)^r f(x)\}] \\ &= e^{px}\{(D+p)^r f'(x)\} + pe^{px}\{(D+p)^r f(x)\} \\ &= e^{px}(D+p)^r[f'(x) + pf(x)] = e^{px}(D+p)^r(D+p)f(x) \\ &= e^{px}\{(D+p)^{r+1}f(x)\}. \end{aligned}$$

Consequently, the result is valid for $r+1$, and by mathematical induction it is valid for all $k \geq 1$.
If $\phi(D) = \sum_{i=0}^{n} a_{n-i}D^i$, (see 15.45), then

$$\begin{aligned} \phi(D)\{e^{px}f(x)\} &= \left(\sum_{i=0}^{n} a_{n-i}D^i\right)\{e^{px}f(x)\} = \sum_{i=0}^{n} a_{n-i}D^i\{e^{px}f(x)\} \\ &= \sum_{i=0}^{n} a_{n-i}e^{px}\{(D+p)^i f(x)\} = e^{px}\left[\sum_{i=0}^{n} a_{n-i}(D+p)^i\right] f(x) \\ &= e^{px}\{\phi(D+p)f(x)\}. \end{aligned}$$

26. (a) If $m = m_0$ is a root of multiplicity k of $\phi(m) = 0$, then $\phi(m) = (m - m_0)^k\psi(m)$, and therefore $\phi(D) = (D - m_0)^k\psi(D)$. We now calculate that

$$\begin{aligned} &\phi(D)[(C_1 + C_2x + \cdots + C_kx^{k-1})e^{m_0x}] \\ &\quad = e^{m_0x}[\phi(D + m_0)(C_1 + C_2x + \cdots + C_kx^{k-1})] \\ &\quad = e^{m_0x}[(D + m_0 - m_0)^k\psi(D + m_0)(C_1 + C_2x + \cdots + C_kx^{k-1})] \\ &\quad = e^{m_0x}[\psi(D + m_0)D^k(C_1 + C_2x + \cdots + C_kx^{k-1})] \\ &\quad = 0. \end{aligned}$$

Thus, $(C_1 + C_2x + \cdots + C_kx^{k-1})e^{m_0x}$ is a solution of $\phi(D)y = 0$.
(b) If $a \pm bi$ are complex conjugate roots of multiplicity k of $\phi(m) = 0$, then

$$\phi(m) = (m - a - bi)^k(m - a + bi)^k\psi(m) \quad \Longrightarrow \quad \phi(D) = (D - a - bi)^k(D - a + bi)^k\psi(D).$$

We now calculate that

$$\begin{aligned} &\phi(D)\{e^{ax}[(C_1 + C_2x + \cdots + C_kx^{k-1})\cos bx + (D_1 + D_2x + \cdots + D_kx^{k-1})\sin bx]\} \\ &\quad = \phi(D)\{\mathrm{Re}[e^{(a+bi)x}](C_1 + C_2x + \cdots + C_kx^{k-1}) + \mathrm{Im}[e^{(a+bi)x}](D_1 + D_2x + \cdots + D_kx^{k-1})\} \\ &\quad = \mathrm{Re}\{\phi(D)[e^{(a+bi)x}(C_1 + C_2x + \cdots + C_kx^{k-1})]\} \\ &\qquad + \mathrm{Im}\{\phi(D)[e^{(a+bi)x}(D_1 + D_2x + \cdots + D_kx^{k-1})]\} \\ &\quad = \mathrm{Re}\{e^{(a+bi)x}\phi(D + a + bi)(C_1 + C_2x + \cdots + C_kx^{k-1})\} \\ &\qquad + \mathrm{Im}\{e^{(a+bi)x}\phi(D + a + bi)(D_1 + D_2x + \cdots + D_kx^{k-1})\} \end{aligned}$$

$$\begin{aligned}
&= \mathrm{Re}\{e^{(a+bi)x}D^k(D+2bi)^k\psi(D+a+bi)(C_1+C_2x+\cdots+C_kx^{k-1})\} \\
&\quad + \mathrm{Im}\{e^{(a+bi)x}D^k(D+2bi)^k\psi(D+a+bi)(D_1+D_2x+\cdots+D_kx^{k-1})\} \\
&= \mathrm{Re}\{e^{(a+bi)x}\psi(D+a+bi)(D+2bi)^k[D^k(C_1+C_2x+\cdots+C_kx^{k-1})]\} \\
&\quad + \mathrm{Im}\{e^{(a+bi)x}\psi(D+a+bi)(D+2bi)^k[D^k(D_1+D_2x+\cdots+D_kx^{k-1})]\} \\
&= 0.
\end{aligned}$$

It follows that $e^{ax}[(C_1+C_2x+\cdots+C_kx^{k-1})\cos bx+(D_1+D_2x+\cdots+D_kx^{k-1})\sin bx]$ is a solution of $\phi(D)y=0$.

27. The auxiliary equation $Mm^2+\beta m+k=0$ has solutions $m=\dfrac{-\beta\pm\sqrt{\beta^2-4kM}}{2M}$. We consider three cases:

Case 1: $\beta^2-4kM=0$

In this case, the auxiliary equation has equal roots $m=-\beta/(2M)$, and a general solution of the differential equation is $x(t)=(C_1+C_2t)e^{-\beta t/(2M)}$.

Case 2: $\beta^2-4kM>0$

In this case, the auxiliary equation has real and distinct roots, and a general solution of the differential equation is

$$x(t)=C_1e^{(-\beta+\sqrt{\beta^2-4kM})t/(2M)}+C_2e^{(-\beta-\sqrt{\beta^2-4kM})t/(2M)}.$$

Case 3: $\beta^2-4kM<0$

In this case, the auxiliary equation has complex conjugate roots $-\dfrac{\beta}{2M}\pm\dfrac{\sqrt{4kM-\beta^2}}{2M}i$. A general solution of the differential equation is

$$x(t)=e^{-\beta t/(2M)}\left[C_1\cos\frac{\sqrt{4kM-\beta^2}}{2M}t+C_2\sin\frac{\sqrt{4kM-\beta^2}}{2M}t\right].$$

EXERCISES 15.9

1. The auxiliary equation is $0=2m^2-16m+32=2(m-4)^2$ with solutions $m=4, 4$. A general solution of the associated homogeneous equation is $y_h(x)=(C_1+C_2x)e^{4x}$. By operators,

$$y_p=\frac{1}{2D^2-16D+32}(-e^{4x})=-e^{4x}\frac{1}{2(D+4)^2-16(D+4)+32}(1)=-e^{4x}\frac{1}{2D^2}(1)=-\frac{x^2}{4}e^{4x}.$$

By undetermined coefficients, $y_p=Ax^2e^{4x}$. Subsititution into the differential equation gives

$$2(16Ax^2e^{4x}+16Axe^{4x}+2Ae^{4x})-16(4Ax^2e^{4x}+2Axe^{4x})+32Ax^2e^{4x}=-e^{4x},$$

and this simplifies to $4Ae^{4x}=-e^{4x}$. Thus, $A=-1/4$, and $y_p=-(x^2/4)e^{4x}$. A general solution of the differential equation is therefore $y(x)=(C_1+C_2x)e^{4x}-(x^2/4)e^{4x}$.

2. The auxiliary equation is $0=m^2+2m-2$ with solutions $m=-1\pm\sqrt{3}$. A general solution of the associated homogeneous equation is $y_h(x)=C_1e^{-(1+\sqrt{3})x}+C_2e^{(-1+\sqrt{3})x}$. By operators,

$$\begin{aligned}
y_p&=\frac{1}{D^2+2D-2}(x^2e^{-x})=e^{-x}\frac{1}{(D-1)^2+2(D-1)-2}(x^2)=e^{-x}\frac{1}{D^2-3}(x^2)\\
&=\frac{e^{-x}}{-3}\frac{1}{1-D^2/3}(x^2)=\frac{e^{-x}}{-3}\left(1+\frac{D^2}{3}+\cdots\right)x^2=\frac{e^{-x}}{-3}\left(x^2+\frac{2}{3}\right).
\end{aligned}$$

By undetermined coefficients, $y_p=Ax^2e^{-x}+Bxe^{-x}+Ce^{-x}$. Subsititution into the differential equation gives

$$\begin{aligned}
(2Ae^{-x}-4Axe^{-x}+Ax^2e^{-x}-2Be^{-x}+Bxe^{-x}+Ce^{-x})+2(2Axe^{-x}-Ax^2e^{-x}+Be^{-x}\\
-Bxe^{-x}-Ce^{-x})-2(Ax^2e^{-x}+Bxe^{-x}+Ce^{-x})=x^2e^{-x}.
\end{aligned}$$

When we equate coefficients of x^2e^{-x}, xe^{-x}, and e^{-x}:

$$-3A = 1, \qquad -3B = 0, \qquad 2A - 3C = 0.$$

Thus, $A = -1/3$, $B = 0$, and $C = -2/9$ and once again $y_p = (-1/3)x^2e^{-x} - (2/9)e^{-x}$. A general solution of the differential equation is therefore $y(x) = C_1e^{-(1+\sqrt{3})x} + C_2e^{(-1+\sqrt{3})x} - e^{-x}(3x^2+2)/9$.

3. The auxiliary equation is $0 = m^3 - 3m^2 + m - 3 = (m-3)(m^2+1)$ with solutions $m = 3, \pm i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{3x} + C_2\cos x + C_3\sin x$. By operators,

$$\begin{aligned} y_p &= \frac{1}{D^3 - 3D^2 + D - 3}(3xe^x + 2) = 3e^x\frac{1}{(D+1)^3 - 3(D+1)^2 + (D+1) - 3}(x) + \frac{1}{D^3 - 3D^2 + D - 3}(2) \\ &= 3e^x\frac{1}{D^3 - 2D - 4}(x) + \frac{1}{-3[1 - (D^3 - 3D^2 + D)/3]}(2) \\ &= -\frac{3e^x}{4}\frac{1}{1 - (D^3 - 2D)/4}(x) - \frac{1}{3}[1 + \cdots](2) \\ &= -\frac{3e^x}{4}\left[1 + \left(\frac{D^3 - 2D}{4}\right) + \cdots\right]x - \frac{2}{3} = -\frac{3e^x}{4}\left(x - \frac{1}{2}\right) - \frac{2}{3}. \end{aligned}$$

By undetermined coefficients, $y_p = (Ax + B)e^x + C$. Subsititution into the differential equation gives

$$\begin{aligned} (3Ae^x + Axe^x + Be^x) - 3(2Ae^x + Axe^x + Be^x) + (Ae^x + Axe^x + Be^x)& \\ - 3(Axe^x + Be^x + C) = 3xe^x + 2.& \end{aligned}$$

When we equate coefficients of xe^x, e^x, and 1:

$$-4A = 3, \qquad -2A - 4B = 0, \qquad -3C = 2.$$

Thus, $A = -3/4$, $B = 3/8$, and $C = -2/3$, and once again $y_p = 3e^x(1-2x)/8 - 2/3$. A general solution of the differential equation is therefore $y(x) = C_1e^{3x} + C_2\cos x + C_3\sin x + 3e^x(1-2x)/8 - 2/3$.

4. The auxiliary equation is $0 = m^4 + 2m^2 + 1 = (m^2+1)^2$ with solutions $m = \pm i$, $\pm i$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x$. By operators,

$$\begin{aligned} y_p &= \frac{1}{D^4 + 2D^2 + 1}\cos 2x = \text{Re}\left(\frac{1}{D^4 + 2D^2 + 1}e^{2ix}\right) \\ &= \text{Re}\left[e^{2ix}\frac{1}{(D+2i)^4 + 2(D+2i)^2 + 1}(1)\right] = \text{Re}\left[e^{2ix}\frac{1}{9 + \cdots}(1)\right] = \text{Re}\left[\frac{e^{2ix}}{9}\right] = \frac{1}{9}\cos 2x. \end{aligned}$$

By undetermined coefficients, $y_p = A\cos 2x + B\sin 2x$. Substitution into the differential equation gives

$$(16A\cos 2x + 16B\sin 2x) + 2(-4A\cos 2x - 4B\sin 2x) + (A\cos 2x + B\sin 2x) = \cos 2x.$$

When we equate coefficients of $\cos 2x$ and $\sin 2x$, we obtain $9A = 1$, and $9B = 0$. Thus, $A = 1/9$ and $B = 0$, and $y_p = (1/9)\cos 2x$. A general solution of the differential equation is

$$y(x) = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x + (1/9)\cos 2x.$$

5. The auxiliary equation is $0 = m^3 - 6m^2 + 12m - 8 = (m-2)^3$ with solutions $m = 2, 2, 2$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2x + C_3x^2)e^{2x}$. By operators,

$$y_p = \frac{1}{D^3 - 6D^2 + 12D - 8}2e^{2x} = 2e^{2x}\frac{1}{(D+2)^3 - 6(D+2)^2 + 12(D+2) - 8}(1) = 2e^{2x}\frac{1}{D^3}(1) = \frac{x^3}{3}e^{2x}.$$

By undetermined coefficients, $y_p = Ax^3e^{2x}$. Substitution into the differential equation gives

$$\begin{aligned} (6Ae^{2x} + 36Axe^{2x} + 36Ax^2e^{2x} + 8Ax^3e^{2x}) - 6(6Axe^{2x} + 12Ax^2e^{2x} + 4Ax^3e^{2x})& \\ + 12(3Ax^2e^{2x} + 2Ax^3e^{2x}) - 8Ax^3e^{2x} = 2e^{2x}.& \end{aligned}$$

This simplifies to $6Ae^{2x} = 2e^{2x}$, so that $A = 1/3$. Once again $y_p = (x^3/3)e^{2x}$. A general solution of the differential equation is $y(x) = (C_1 + C_2x + C_3x^2)e^{2x} + (x^3/3)e^{2x}$.

6. The auxiliary equation is $0 = m^4 + 5m^2 + 4 = (m^2+4)(m^2+1)$ with solutions $m = \pm i$, $\pm 2i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 \cos x + C_2 \sin x + C_3 \cos 2x + C_4 \sin 2x$. By operators,

$$y_p = \frac{1}{D^4+5D^2+4}e^{-2x} = e^{-2x}\frac{1}{(D-2)^4+5(D-2)^2+4}(1) = e^{-2x}\frac{1}{40+\cdots}(1) = \frac{1}{40}e^{-2x}.$$

By undetermined coefficients, $y_p = Ae^{-2x}$. Substitution into the differential equation gives

$$(16Ae^{-2x}) + 5(4Ae^{-2x}) + 4(Ae^{-2x}) = e^{-2x}.$$

This equation implies that $A = 1/40$, and a general solution of the differential equation is

$$y(x) = C_1\cos x + C_2 \sin x + C_3\cos 2x + C_4 \sin 2x + (1/40)e^{-2x}.$$

7. The auxiliary equation is $0 = m^3 - 3m^2 + 2m = m(m-1)(m-2)$ with solutions $m = 0, 1, 2$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 + C_2e^x + C_3e^{2x}$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{D^3-3D^2+2D}(x^2+e^{-x}) = \frac{1}{D^3-3D^2+2D}x^2 + e^{-x}\frac{1}{(D-1)^3-3(D-1)^2+2(D-1)}(1)\\
&= \frac{1}{D(D^2-3D+2)}(x^2) + e^{-x}\frac{1}{D^3-6D^2+11D-6}(1)\\
&= \frac{1}{2D[1+(D^2-3D)/2]}(x^2) + e^{-x}\frac{1}{-6[1-(D^3-6D^2+11D)/6]}(1)\\
&= \frac{1}{2D}\left[1 - \left(\frac{D^2-3D}{2}\right) + \left(\frac{D^2-3D}{2}\right)^2 + \cdots\right]x^2 - \frac{e^{-x}}{6}[1+\cdots](1)\\
&= \frac{1}{2D}\left(x^2+3x+\frac{7}{2}\right) - \frac{e^{-x}}{6} = \frac{1}{2}\left(\frac{x^3}{3}+\frac{3x^2}{2}+\frac{7x}{2}\right) - \frac{e^{-x}}{6}.
\end{aligned}$$

By undetermined coefficients, $y_p = Ae^{-x} + Bx^3 + Cx^2 + Dx$. Substitution into the differential equation gives

$$(-Ae^{-x}+6B) - 3(Ae^{-x}+6Bx+2C) + 2(-Ae^{-x}+3Bx^2+2Cx+D) = x^2 + e^{-x}.$$

When we equate coefficients of e^{-x}, x^2, x, and 1:

$$-6A = 1, \quad 6B = 1, \quad -18B + 4C = 0, \quad 6B - 6C + 2D = 0.$$

Thus, $A = -1/6$, $B = 1/6$, $C = 3/4$, and $D = 7/4$, and once again $y_p = (2x^3+9x^2+21x)/12 - (1/6)e^{-x}$. A general solution of the differential equation is

$$y(x) = C_1 + C_2e^x + C_3e^{2x} + (2x^3+9x^2+21x)/12 - (1/6)e^{-x}.$$

8. The auxiliary equation is $0 = 2m^2 + 16m + 82$ with solutions $m = -4 \pm 5i$. A general solution of the associated homogeneous equation is $y_h(x) = e^{-4x}(C_1\cos 5x + C_2 \sin 5x)$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{2D^2+16D+82}(-2e^{2x}\sin x) = -\frac{1}{D^2+8D+41}\mathrm{Im}[e^{(2+i)x}] = -\mathrm{Im}\left[\frac{1}{D^2+8D+41}e^{(2+i)x}\right]\\
&= -\mathrm{Im}\left[e^{(2+i)x}\frac{1}{(D+2+i)^2+8(D+2+i)+41}(1)\right] = -\mathrm{Im}\left[e^{(2+i)x}\frac{1}{60+12i+\cdots}(1)\right]\\
&= -\frac{1}{12}\mathrm{Im}\left[e^{(2+i)x}\frac{1}{5+i}\frac{5-i}{5-i}\right] = \frac{-1}{12}\mathrm{Im}\left[e^{(2+i)x}\frac{5-i}{26}\right]\\
&= \frac{-1}{312}\mathrm{Im}\left[e^{2x}(\cos x + i \sin x)(5-i)\right] = -\frac{e^{2x}}{312}(-\cos x + 5\sin x).
\end{aligned}$$

By undetermined coefficients, $y_p = Ae^{2x}\sin x + Be^{2x}\cos x$. Substitution into the differential equation gives

$$\begin{aligned}&2(4Ae^{2x}\sin x + 4Ae^{2x}\cos x - Ae^{2x}\sin x + 4Be^{2x}\cos x - 4Be^{2x}\sin x - Be^{2x}\cos x)\\&\quad + 16(2Ae^{2x}\sin x + Ae^{2x}\cos x + 2Be^{2x}\cos x - Be^{2x}\sin x)\\&\quad + 82(Ae^{2x}\sin x + Be^{2x}\cos x) = -2e^{2x}\sin x.\end{aligned}$$

When we equate coefficients of $e^{2x}\sin x$ and $e^{2x}\cos x$:

$$120A - 24B = -2, \qquad 120B + 24A = 0.$$

These imply that $A = -5/312$ and $B = 1/312$, and once again $y_p = e^{2x}(\cos x - 5\sin x)/312$. A general solution of the differential equation is therefore

$$y(x) = e^{-4x}(C_1\cos 5x + C_2\sin 5x) + e^{2x}(\cos x - 5\sin x)/312.$$

9. The auxiliary equation is $0 = m^2 + m - 6 = (m-2)(m+3)$ with solutions $m = -3, 2$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{-3x} + C_2e^{2x}$. By operators,

$$\begin{aligned}y_p &= \frac{1}{D^2+D-6}(x+\cos x) = \frac{1}{D^2+D-6}(x) + \mathrm{Re}\left(\frac{1}{D^2+D-6}e^{ix}\right)\\&= \frac{1}{-6[1-(D^2+D)/6]}(x) + \mathrm{Re}\left[e^{ix}\frac{1}{(D+i)^2+(D+i)-6}(1)\right]\\&= -\frac{1}{6}\left[1+\left(\frac{D^2+D}{6}\right)+\cdots\right]x + \mathrm{Re}\left[e^{ix}\frac{1}{D^2+(1+2i)D-7+i}(1)\right]\\&= -\frac{1}{6}\left(x+\frac{1}{6}\right) + \mathrm{Re}\left[e^{ix}\left(\frac{1}{-7+i}\right)\right]\\&= -\frac{1}{36}(6x+1) + \mathrm{Re}\left[e^{ix}\left(\frac{-7-i}{50}\right)\right] = -\frac{1}{36}(6x+1) - \frac{7}{50}\cos x + \frac{1}{50}\sin x.\end{aligned}$$

By undetermined coefficients, $y_p = Ax + B + C\cos x + D\sin x$. Substitution into the differential equation gives

$$(-C\cos x - D\sin x) + (A - C\sin x + D\cos x) - 6(Ax + B + C\cos x + D\sin x) = x + \cos x.$$

When we equate coefficients of x, 1, $\cos x$, and $\sin x$, we obtain

$$-6A = 1, \quad A - 6B = 0, \quad -7C + D = 1, \quad -C - 7D = 0.$$

Thus, $A = -1/6$, $B = -1/36$, $C = -7/50$, and $D = 1/50$. Once again $y_p = -(6x+1)/36 + (\sin x - 7\cos x)/50$. A general solution of the differential equation is

$$y(x) = C_1e^{-3x} + C_2e^{2x} - (6x+1)/36 + (\sin x - 7\cos x)/50.$$

10. The auxiliary equation is $0 = m^2 - 4m + 5$ with solutions $m = 2 \pm i$. A general solution of the associated homogeneous equation is $y_h(x) = e^{2x}(C_1\cos x + C_2\sin x)$. By operators,

$$\begin{aligned}y_p &= \frac{1}{D^2-4D+5}(x\cos x) = \mathrm{Re}\left[\frac{1}{D^2-4D+5}xe^{ix}\right]\\&= \mathrm{Re}\left[e^{ix}\frac{1}{(D+i)^2-4(D+i)+5}(x)\right] = \mathrm{Re}\left[e^{ix}\frac{1}{D^2+(-4+2i)D+(4-4i)}(x)\right]\\&= \mathrm{Re}\left[\frac{e^{ix}}{4-4i}\frac{1}{1+\dfrac{(-4+2i)D+D^2}{4-4i}}(x)\right] = \mathrm{Re}\left\{\frac{e^{ix}}{4-4i}\left[1-\left(\frac{(-4+2i)D+D^2}{4-4i}\right)+\cdots\right]x\right\}\\&= \mathrm{Re}\left[\frac{e^{ix}}{4(1-i)}\left(x+\frac{4-2i}{4-4i}\right)\right] = \mathrm{Re}\left[\frac{e^{ix}}{4}\left(\frac{x}{1-i}\frac{1+i}{1+i}+\frac{4-2i}{-8i}\right)\right]\\&= \mathrm{Re}\left[\frac{\cos x + i\sin x}{4}\left(\frac{x(1+i)}{2}+\frac{1+2i}{4}\right)\right] = \frac{x}{8}(\cos x - \sin x) + \frac{1}{16}(\cos x - 2\sin x).\end{aligned}$$

By undetermined coefficients, $y_p = Ax\cos x + Bx\sin x + C\cos x + D\sin x$. Substitution into the differential equation gives

$$\begin{aligned}&(-2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x - C\cos x - D\sin x)\\ &\quad - 4(A\cos x - Ax\sin x + B\sin x + Bx\cos x - C\sin x + D\cos x)\\ &\quad + 5(Ax\cos x + Bx\sin x + C\cos x + D\sin x) = x\cos x.\end{aligned}$$

When we equate coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$:

$$4A - 4B = 1, \quad 4A + 4B = 0, \quad -4A + 2B + 4C - 4D = 0, \quad -2A - 4B + 4C + 4D = 0.$$

These imply that $A = 1/8$, $B = -1/8$, $C = 1/16$, and $D = -1/8$, giving the same y_p as above. A general solution of the differential equation is therefore

$$y(x) = e^{2x}(C_1\cos x + C_2\sin x) + x(\cos x - \sin x)/8 + (\cos x - 2\sin x)/16.$$

11. The auxiliary equation is $0 = 3m^3 - 12m^2 + 18m - 12 = 3(m-2)(m^2 - 2m + 2)$ with solutions $m = 2$, $1 \pm i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{2x} + e^x(C_2\cos x + C_3\sin x)$. By operators,

$$\begin{aligned}y_p &= \frac{1}{3D^3 - 12D^2 + 18D - 12}(x^2 + 3x - 4) = \frac{1}{-12[1 - (D^3 - 4D^2 + 6D)/4]}(x^2 + 3x - 4)\\ &= -\frac{1}{12}\left[1 + \left(\frac{D^3 - 4D^2 + 6D}{4}\right) + \left(\frac{D^3 - 4D^2 + 6D}{4}\right)^2 + \cdots\right](x^2 + 3x - 4)\\ &= -\frac{1}{12}\left[(x^2 + 3x - 4) + \frac{3}{2}(2x + 3) + \frac{5}{4}(2)\right] = -\frac{1}{12}(x^2 + 6x + 3).\end{aligned}$$

By undetermined coefficients, $y_p = Ax^2 + Bx + C$. Subsititution into the differential equation gives

$$3(0) - 12(2A) + 18(2Ax + B) - 12(Ax^2 + Bx + C) = x^2 + 3x - 4.$$

When we equate coefficients of x^2, x, and 1:

$$-12A = 1, \qquad 36A - 12B = 3, \qquad -24A + 18B - 12C = -4.$$

Thus, $A = -1/12$, $B = -1/2$, and $C = -1/4$. Once again $y_p = -(x^2 + 6x + 3)/12$. A general solution of the differential equation is therefore $y(x) = C_1e^{2x} + e^x(C_2\cos x + C_3\sin x) - (x^2 + 6x + 3)/12$.

12. The auxiliary equation is $0 = m^3 + 9m^2 + 27m + 27 = (m + 3)^2$ with solutions $m = -3, -3, -3$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2x + C_3x^2)e^{-3x}$. Undetermined coefficients suggests $y_p(x) = Axe^{3x} + Be^{3x} + Cx\cos x + Dx\sin x + E\cos x + F\sin x$.

13. The auxiliary equation is $0 = m^3 + 4m^2 + m + 4 = (m+4)(m^2+1)$ with solutions $m = -4$, $\pm i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{-4x} + C_2\cos x + C_3\sin x$. Undetermined coefficients suggests $y_p(x) = Axe^x\sin x + Bxe^x\cos x + Ce^x\sin x + De^x\cos x$.

14. The auxiliary equation is $0 = 2m^3 - 6m^2 - 12m + 16 = 2(m - 1)(m - 4)(m + 2)$ with solutions $m = 1, -2, 4$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^x + C_2e^{-2x} + C_3e^{4x}$. Undetermined coefficients suggests $y_p(x) = Ax^2e^x + Bxe^x + Cx^3 + Dx^2 + Ex + F + G\cos x + H\sin x$.

15. The auxiliary equation is $0 = 2m^2 - 4m + 10$ with solutions $m = 1 \pm 2i$. A general solution of the associated homogeneous equation is $y_h(x) = e^x(C_1\cos 2x + C_2\sin 2x)$. Undetermined coefficients suggests

$$y_p(x) = Axe^x\sin 2x + Bxe^x\cos 2x.$$

16. According to the operator shift theorem, $\phi(D)\{e^{px}g(x)\} = e^{px}[\phi(D+p)g(x)]$, and therefore $e^{px}g(x) = \frac{1}{\phi(D)}\{e^{px}[\phi(D+p)g(x)]\}$. If we set $f(x) = \phi(D+p)g(x)$, in which case $g(x) = \dfrac{1}{\phi(D+p)}f(x)$, then

$$e^{px}\frac{1}{\phi(D+p)}f(x) = \frac{1}{\phi(D)}\{e^{px}f(x)\}.$$

17. The auxiliary equation is $0 = m^2 + 2m - 4$ with solutions $m = -1 \pm \sqrt{5}$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 e^{-(1+\sqrt{5})x} + C_2 e^{(-1+\sqrt{5})x}$. Since $\cos^2 x = (1 + \cos 2x)/2$, undetermined coefficients suggests $y_p = A + B\cos 2x + C\sin 2x$. Substitution into the differential equation gives

$$(-4B\cos 2x - 4C\sin 2x) + 2(-2B\sin 2x + 2C\cos 2x) - 4(A + B\cos 2x + C\sin 2x) = \frac{1}{2} + \frac{1}{2}\cos 2x.$$

When we equate coefficients of $\cos 2x$, $\sin 2x$, and 1:

$$-8B + 4C = 1/2, \qquad -8C - 4B = 0, \qquad -4A = 1/2.$$

Thus, $A = -1/8$, $B = -1/20$, and $C = 1/40$. A particular solution is $y_p = -1/8 + (\sin 2x - 2\cos 2x)/40$, and a general solution of the differential equation is

$$y(x) = C_1 e^{-(1+\sqrt{5})x} + C_2 e^{(-1+\sqrt{5})x} - 1/8 + (\sin 2x - 2\cos 2x)/40.$$

18. The auxiliary equation is $0 = 2m^2 - 4m + 3$ with solutions $m = 1 \pm (1/\sqrt{2})i$. A general solution of the associated homogeneous equation is $y_h(x) = e^x[C_1\cos(x/\sqrt{2}) + C_2\sin(x/\sqrt{2})]$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{2D^2 - 4D + 3}(\cos x \, \sin 2x) = \frac{1}{2D^2 - 4D + 3}\left[\frac{1}{2}(\sin 3x + \sin x)\right] \\
&= \frac{1}{2}\mathrm{Im}\left[\frac{1}{2D^2 - 4D + 3}e^{3ix}\right] + \frac{1}{2}\mathrm{Im}\left[\frac{1}{2D^2 - 4D + 3}e^{ix}\right] \\
&= \frac{1}{2}\mathrm{Im}\left[e^{3ix}\frac{1}{2(D+3i)^2 - 4(D+3i) + 3}(1)\right] + \frac{1}{2}\mathrm{Im}\left[e^{ix}\frac{1}{2(D+i)^2 - 4(D+i) + 3}(1)\right] \\
&= \frac{1}{2}\mathrm{Im}\left[\frac{e^{3ix}}{-15 - 12i}\right] + \frac{1}{2}\mathrm{Im}\left[\frac{e^{ix}}{1 - 4i}\right] = -\frac{1}{6}\mathrm{Im}\left[\frac{e^{3ix}}{5 + 4i}\frac{5 - 4i}{5 - 4i}\right] + \frac{1}{2}\mathrm{Im}\left[\frac{e^{ix}}{1 - 4i}\frac{1 + 4i}{1 + 4i}\right] \\
&= \frac{-1}{6}\mathrm{Im}\left[\frac{(\cos 3x + i\sin 3x)(5 - 4i)}{41}\right] + \frac{1}{2}\mathrm{Im}\left[\frac{(\cos x + i\sin x)(1 + 4i)}{17}\right] \\
&= -\frac{1}{246}(-4\cos 3x + 5\sin 3x) + \frac{1}{34}(4\cos x + \sin x).
\end{aligned}$$

A general solution of the differential equation is therefore

$$y(x) = e^x[C_1\cos(x/\sqrt{2}) + C_2\sin(x/\sqrt{2})] + (4\cos 3x - 5\sin 3x)/246 + (4\cos x + \sin x)/34.$$

19. The auxiliary equation $0 = m^2 - 3m + 2 = (m-1)(m-2)$ has solutions $m = 1, 2$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 e^x + C_2 e^{2x}$. By operators,

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 3D + 2}(8x^2 + 12e^{-x}) = \frac{8}{2[1 + (D^2 - 3D)/2]}(x^2) + 12e^{-x}\frac{1}{(D-1)^2 - 3(D-1) + 2}(1) \\
&= 4\left[1 - \left(\frac{D^2 - 3D}{2}\right) + \left(\frac{D^2 - 3D}{2}\right)^2 + \cdots\right]x^2 + 12e^{-x}\frac{1}{D^2 - 5D + 6}(1) \\
&= 4\left[x^2 + \frac{3}{2}(2x) + \frac{7}{4}(2)\right] + \frac{12e^{-x}}{6} = 4x^2 + 12x + 14 + 2e^{-x}.
\end{aligned}$$

A general solution of the differential equation is $y(x) = C_1 e^x + C_2 e^{2x} + 4x^2 + 12x + 14 + 2e^{-x}$. To satisfy the conditions $y(0) = 0$ and $y'(0) = 2$, we must have $0 = C_1 + C_2 + 14 + 2$ and $2 = C_1 + 2C_2 + 12 - 2$. These imply that $C_1 = -24$ and $C_2 = 8$.

20. The auxiliary equation $m^2 + 9 = 0$ has solutions $m = \pm 3i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 \cos 3x + C_2 \sin 3x$. By operators,

$$y_p = \frac{1}{D^2+9}\left\{x[\mathrm{Im}(e^{3ix}) + \mathrm{Re}(e^{3ix})]\right\} = \mathrm{Im}\left[\frac{1}{D^2+9}(xe^{3ix})\right] + \mathrm{Re}\left[\frac{1}{D^2+9}(xe^{3ix})\right].$$

Consider then
$$\begin{aligned}\frac{1}{D^2+9}(xe^{3ix}) &= e^{3ix}\frac{1}{(D+3i)^2+9}(x) = e^{3ix}\frac{1}{D^2+6iD}(x)\\ &= e^{3ix}\frac{1}{6iD[1+D/(6i)]}(x) = e^{3ix}\frac{1}{6iD}\left(1 - \frac{D}{6i} + \cdots\right)x\\ &= e^{3ix}\frac{1}{6iD}\left(x - \frac{1}{6i}\right) = -\frac{i}{6}e^{3ix}\left(\frac{x^2}{2} + \frac{ix}{6}\right) = \frac{1}{36}e^{3ix}(x - 3ix^2).\end{aligned}$$

Thus, $y_p(x) = \dfrac{1}{36}(x\sin 3x - 3x^2\cos 3x) + \dfrac{1}{36}(x\cos 3x + 3x^2\sin 3x)$, and

$$y(x) = C_1\cos 3x + C_2\sin 3x + \frac{x}{36}(\cos 3x + \sin 3x) + \frac{x^2}{12}(\sin 3x - \cos 3x).$$

For $y(0) = 0$ and $y'(0) = 0$, we must have $0 = C_1$ and $0 = 3C_2 + 1/36$. Hence,
$y(x) = -\dfrac{1}{108}\sin 3x + \dfrac{x}{36}(\cos 3x + \sin 3x) + \dfrac{x^2}{12}(\sin 3x - \cos 3x)$.

21. The auxiliary equation is $Jm^4 + k = 0 \implies m^2 = \pm\sqrt{k/J}i$. If we set $\lambda = (1/\sqrt{2})(k/J)^{1/4}$, then $m^2 = \pm 2\lambda^2 i$. If we now set $m = a + bi$ in $m^2 = 2\lambda^2 i$, then $a^2 - b^2 + 2abi = 2\lambda^2 i$. When we equate real and imaginary parts, $a^2 - b^2 = 0$ and $2ab = 2\lambda^2$. These give $a = b = \pm\lambda$; that is, $m = \pm\lambda(1+i)$. From $m^2 = -2\lambda^2 i$, we obtain $m = \pm\lambda(1-i)$. A general solution of the associated homogeneous equation is $y_h(x) = e^{\lambda x}(C_1\cos\lambda x + C_2\sin\lambda x) + e^{-\lambda x}(C_3\cos\lambda x + C_4\sin\lambda x)$. Since a particular solution is $y_p(x) = w/k$, a general solution of the differential equation is $y(x) = e^{\lambda x}(C_1\cos\lambda x + C_2\sin\lambda x) + e^{-\lambda x}(C_3\cos\lambda x + C_4\sin\lambda x) + w/k$.

22. Since $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx} = \dfrac{dy}{dz}\Big/\dfrac{dx}{dz} = \dfrac{1}{x}\dfrac{dy}{dz}$, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}.$$

Thus, $x\dfrac{dy}{dx} = \dfrac{dy}{dz}$ and $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$. When we substitute these into the differential equation,

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + a\frac{dy}{dz} + by = F(e^z) \quad\implies\quad \frac{d^2y}{dz^2} + (a-1)\frac{dy}{dz} + by = F(e^z),$$

a linear differential equation with constant coefficients.

23. If we set $r = e^z$ and use the results of Exercise 22 on $r^2\dfrac{d^2u}{dr^2} + r\dfrac{du}{dr} - u = 0$,

$$0 = \frac{d^2u}{dz^2} - \frac{du}{dz} + \frac{du}{dz} - u = \frac{d^2u}{dz^2} - u.$$

The auxiliary equation is $m^2 - 1 = 0$ with solutions $m = \pm 1$. A general solution of the differential equation is therefore $u(z) = C_1e^z + C_2e^{-z}$, and therefore $u(r) = C_1r + C_2/r$.

24. If we set $x = e^z$ and use the results of Exercise 22, $1 = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz} + \dfrac{dy}{dz} + 4y = \dfrac{d^2y}{dz^2} + 4y$. The auxiliary equation is $m^2 + 4 = 0$ with solutions $m = \pm 2i$. A general solution of the associated homogeneous equation is $y_h(z) = C_1\cos 2z + C_2\sin 2z$. Since $y_p(z) = 1/4$,

$$y(z) = C_1\cos 2z + C_2\sin 2z + 1/4 \quad\implies\quad y(x) = C_1\cos(2\ln x) + C_2\sin(2\ln x) + 1/4.$$

25. The solution of the associated homogeneous equation was discussed in Exercise 15.8-27. Substituting a particular solution of the form $x_p(t) = B\sin\omega t + C\cos\omega t$ into the differential equation gives

$$M(-\omega^2 B\sin\omega t - \omega^2 C\cos\omega t) + \beta(\omega B\cos\omega t - \omega C\sin\omega t) + k(B\sin\omega t + C\cos\omega t) = A\sin\omega t.$$

When we equate coefficients of $\sin\omega t$ and $\cos\omega t$,

$$(k - M\omega^2)B - \beta\omega C = A, \qquad \beta\omega B + (k - M\omega^2)C = 0.$$

Solutions of these are $B = \dfrac{A(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2}$, $C = \dfrac{-A\beta\omega}{(k - M\omega^2)^2 + \beta^2\omega^2}$. The particular solution is therefore $x_p(t) = \dfrac{A}{(k - M\omega^2)^2 + \beta^2\omega^2}[(k - M\omega^2)\sin\omega t - \beta\omega\cos\omega t]$.

This solution is unacceptable when $\beta = 0$ and $\omega^2 = k/M$. In this case, the general solution of the associated homogeneous equation is $C_1\cos\omega t + C_2\sin\omega t$. A particular solution is of the form $x_p(t) = Bt\sin\omega t + Ct\cos\omega t$. Substitution into the differential equation gives

$$\begin{aligned}A\sin\omega t &= M(2\omega B\cos\omega t - \omega^2 Bt\sin\omega t - 2\omega C\sin\omega t - \omega^2 Ct\cos\omega t) + k(Bt\sin\omega t + Ct\cos\omega t)\\ &= B(k - M\omega^2)t\sin\omega t + C(k - M\omega^2)t\cos\omega t + 2M\omega B\cos\omega t - 2M\omega C\sin\omega t\\ &= 2M\omega B\cos\omega t - 2M\omega C\sin\omega t.\end{aligned}$$

Thus, $B = 0$ and $C = -A/(2M\omega)$. The particular solution is $x_p(t) = -At/(2M\omega)\cos\omega t$.

26. (a) Since the auxiliary equation $0 = m^2 - 3m + 2 = (m-1)(m-2)$ has solutions $m = 1, 2$, a general solution of the associated homogeneous equation is $y_h(x) = C_1e^x + C_2e^{2x}$. When we assume a particular solution of the form $y_p = Ax + B$ and substitute into the differential equation, $-3A + 2(Ax + B) = x \implies A = 1/2$, $B = 3/4$. Thus, $y(x) = C_1e^x + C_2e^{2x} + x/2 + 3/4$. To satisfy the initial conditions, we must have $2 = C_1 + C_2 + 3/4$ and $-1/2 = C_1 + 2C_2 + 1/2$. These imply that $C_1 = 7/2$ and $C_2 = -9/4$. The solution for $0 \le x \le 1$ is $y(x) = (7/2)e^x - (9/4)e^{2x} + x/2 + 3/4$.
(b) Since the differential equation is homogeneous for $x > 1$, a general solution on this interval is $y(x) = D_1e^x + D_2e^{2x}$.
(c) For $y(x)$ and $y'(x)$ to be continuous at $x = 1$, we require

$$\frac{7e}{2} - \frac{9e^2}{4} + \frac{5}{4} = D_1e + D_2e^2, \quad \frac{7e}{2} - \frac{9e^2}{2} + \frac{1}{2} = D_1e + 2D_2e^2.$$

These can be solved for $D_1 = \dfrac{7e+4}{2e}$ and $D_2 = -\dfrac{9e^2+3}{4e^2}$.
(d) The function will not satisfy the differential equation at $x = 1$ since its second derivative does not exist there.

27. Since roots of the auxiliary equation $m^2 + 1 = 0$ are $m = \pm i$, a general solution of the associated homogeneous equation is $y_h(x) = C_1\cos x + C_2\sin x$. A particular solution on the interval $0 \le x \le \pi$ is $x - 1$, so that on this interval $y(x) = C_1\cos x + C_2\sin x + x - 1$. To satisfy the initial conditions, we must have $0 = C_1 - 1$ and $0 = C_2 + 1$. Thus, $y(x) = \cos x - \sin x + x - 1$. On the interval $x > \pi$, we substitute a particular solution of the form $y_p = Ae^{-x}$ into the differential equation, obtaining $e^{-x} = Ae^{-x} + Ae^{-x} \implies A = 1/2$. Thus, on $x > \pi$, a general solution is $y(x) = D_1\cos x + D_2\sin x + (1/2)e^{-x}$. For the solution to be continuous and have a continuous first derivative at $x = \pi$, we require

$$-1 + \pi - 1 = -D_1 + \frac{e^{-\pi}}{2}, \qquad 1 + 1 = -D_2 - \frac{e^{-\pi}}{2}.$$

Solutions of these are $D_1 = (1/2)e^{-\pi} + 2 - \pi$ and $D_2 = -(1/2)e^{-\pi} - 2$.

28. If we change dependent variables according to $y = d\Phi/dr$, then $r^3y''' + 2r^2y'' - ry' + y = 0$. If we now change independent variables with $r = e^z$, then as in Exercise 22,

$$r\frac{dy}{dr} = \frac{dy}{dz} \qquad \text{and} \qquad r^2\frac{d^2y}{dr^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}.$$

Furthermore,

$$\begin{aligned}\frac{d^3y}{dr^3} &= \frac{d}{dr}\left(\frac{d^2y}{dr^2}\right) = \frac{d}{dr}\left[\frac{1}{r^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right] = -\frac{2}{r^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{r^2}\frac{d}{dz}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\frac{dz}{dr}\\ &= -\frac{2}{r^3}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \frac{1}{r^2}\left(\frac{d^3y}{dz^3} - \frac{d^2y}{dz^2}\right)\frac{1}{r}.\end{aligned}$$

Hence, $r^3\dfrac{d^3y}{dr^3} = \dfrac{d^3y}{dz^3} - 3\dfrac{d^2y}{dz^2} + 2\dfrac{dy}{dz}$. Substitution of these into the differential equation for $y(r)$ gives

$$0 = \left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + 2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - \frac{dy}{dz} + y = \frac{d^3y}{dz^3} - \frac{d^2y}{dz^2} - \frac{dy}{dz} + y.$$

The auxiliary equation is $0 = m^3 - m^2 - m + 1 = (m-1)^2(m+1)$ with solutions $m = -1, 1, 1$. Thus, $y(z) = (C_1 + C_2z)e^z + C_3e^{-z}$, from which $y(r) = (C_1 + C_2\ln r)r + C_3/r$.
Integration now gives

$$\begin{aligned}\Phi(r) &= \int\left[r(C_1 + C_2\ln r) + \frac{C_3}{r}\right]dr + C_4 = \frac{C_1r^2}{2} + C_2\int r\ln r\,dr + C_3\ln r + C_4\\ &= \frac{C_1r^2}{2} + C_2\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + C_3\ln r + C_4 = C_5r^2 + C_6r^2\ln r + C_3\ln r + C_4.\end{aligned}$$

29. (a) If we substitute $\Phi(r,\theta) = f(r)\cos n\theta$ into the biharmonic partial differential equation,

$$\begin{aligned}0 &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[f''(r)\cos n\theta + \frac{f'(r)}{r}\cos n\theta - \frac{n^2f(r)}{r^2}\cos n\theta\right]\\ &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left\{\left[f''(r) + \frac{f'(r)}{r} - \frac{n^2f(r)}{r^2}\right]\cos n\theta\right\}\\ &= \left[f''''(r) + \frac{f'''(r)}{r} - \frac{2f''(r)}{r^2} + \frac{2f'(r)}{r^3} - \frac{n^2f''(r)}{r^2} + \frac{4n^2f'(r)}{r^3} - \frac{6n^2f(r)}{r^4}\right]\cos n\theta\\ &\quad + \frac{1}{r}\left[f'''(r) + \frac{f''(r)}{r} - \frac{f'(r)}{r^2} - \frac{n^2f'(r)}{r^2} + \frac{2n^2f(r)}{r^3}\right]\cos n\theta\\ &\quad - \frac{n^2}{r^2}\left[f''(r) + \frac{f'(r)}{r} - \frac{n^2f(r)}{r^2}\right]\cos n\theta.\end{aligned}$$

Thus, $f(r)$ must satisfy

$$\begin{aligned}0 &= f''''(r) + \frac{2f'''(r)}{r} + \frac{f''(r)}{r^2}(-2 - n^2 + 1 - n^2) + \frac{f'(r)}{r^3}(2 + 4n^2 - 1 - n^2 - n^2)\\ &\quad + \frac{f(r)}{r^4}(-6n^2 + 2n^2 + n^4)\\ &= f''''(r) + \frac{2f'''(r)}{r} - \frac{(1+2n^2)f''(r)}{r^2} + \frac{(1+2n^2)f'(r)}{r^3} + \frac{(n^4 - 4n^2)f(r)}{r^4}.\end{aligned}$$

If we set $r = e^z$ and $y = f(r)$, then as in Exercise 22 and 28,

$$r\frac{dy}{dr} = \frac{dy}{dz}, \qquad r^2\frac{d^2y}{dr^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}, \qquad r^3\frac{d^3y}{dr^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}.$$

Furthermore,

$$\begin{aligned}\frac{d^4y}{dr^4} &= \frac{d}{dr}\left(\frac{d^3y}{dr^3}\right) = \frac{d}{dr}\left[\frac{1}{r^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)\right]\\ &= -\frac{3}{r^4}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + \frac{1}{r^3}\frac{d}{dz}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)\frac{dz}{dr}\\ &= -\frac{3}{r^4}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + \frac{1}{r^3}\left(\frac{d^4y}{dz^4} - 3\frac{d^3y}{dz^3} + 2\frac{d^2y}{dz^2}\right)\frac{1}{r}.\end{aligned}$$

Hence, $r^4\dfrac{d^4y}{dr^4} = \dfrac{d^4y}{dz^4} - 6\dfrac{d^3y}{dz^3} + 11\dfrac{d^2y}{dz^2} - 6\dfrac{dy}{dz}$. Substitution of these into the differential equation for $y = f(r)$ gives

$$\begin{aligned}0 &= \left(\frac{d^4y}{dz^4} - 6\frac{d^3y}{dz^3} + 11\frac{d^2y}{dz^2} - 6\frac{dy}{dz}\right) + 2\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) - (1+2n^2)\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\\ &\quad + (1+2n^2)\frac{dy}{dz} + (n^4 - 4n^2)y\\ &= \frac{d^4y}{dz^4} - 4\frac{d^3y}{dz^3} + (4-2n^2)\frac{d^2y}{dz^2} + 4n^2\frac{dy}{dz} + (n^4-4n^2)y.\end{aligned}$$

The auxiliary equation is $0 = m^4 - 4m^3 + (4-2n^2)m^2 + 4n^2m + (n^4 - 4n^2)$. When $n = 1$, this becomes $0 = m^4 - 4m^3 + 2m^2 + 4m - 3 = (m-1)^2(m+1)(m-3)$, in which case $f(z) = (C_1 + C_2z)e^z + C_3e^{-z} + C_4e^{3z}$, and $f(r) = C_1r + C_2r\ln r + C_3/r + C_4r^3$. When $n > 1$, $0 = (m-n)(m+n)(m-n-2)(m+n-2)$, in which case

$$f(z) = C_1e^{nz} + C_2e^{-nz} + C_3e^{(n+2)z} + C_4e^{(2-n)z} \Longrightarrow f(r) = C_1r^n + C_2r^{-n} + C_3r^{n+2} + C_4r^{2-n}.$$

EXERCISES 15.10

1. (a) With the coordinate system of Figure 15.11, the differential equation describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + 16x = 0, \qquad x(0) = -1/10, \quad x'(0) = 0.$$

The auxiliary equation is $m^2 + 16 = 0$ with solutions $m = \pm 4i$. A general solution of the differential equation is $x(t) = C_1\cos 4t + C_2\sin 4t$. To satisfy the initial conditions, we must have $-1/10 = C_1$ and $0 = 4C_2$. Thus, $x(t) = -(1/10)\cos 4t$ m.

(b) In this case the differential equation describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + \frac{1}{10}\frac{dx}{dt} + 16x = 0 \quad \Longrightarrow \quad 10x'' + x' + 160x = 0.$$

The auxiliary equation is $10m^2 + m + 160 = 0$ with solutions $m = (-1 \pm 9\sqrt{79}i)/20$. A general solution of the differential equation is $x(t) = e^{-t/20}[C_1\cos(9\sqrt{79}t/20) + C_2\sin(9\sqrt{79}t/20)$. To satisfy the initial conditions, we must have $-1/10 = C_1$ and $0 = -C_1/20 + 9\sqrt{79}C_2/20$. These give

$$x(t) = e^{-t/20}\left(-\frac{1}{10}\cos\frac{9\sqrt{79}t}{20} - \frac{\sqrt{79}}{7110}\sin\frac{9\sqrt{79}t}{20}\right) \text{ m.}$$

(c) In this case the differential equation describing the position $x(t)$ of the mass is

$$(1)\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 16x = 0.$$

The auxiliary equation is $m^2 + 10m + 16 = 0$ with solutions $m = -2, -8$. A general solution of the differential equation is $x(t) = C_1e^{-2t} + C_2e^{-8t}$. The initial conditions require $-1/10 = C_1 + C_2$ and $0 = -2C_1 - 8C_2$. These give $C_1 = -2/15$ and $C_2 = 1/30$. Thus, $x(t) = (e^{-8t} - 4e^{-2t})/30$ m.

2. With the coordinate system of Figure 15.11, the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{5}\frac{d^2x}{dt^2}+\frac{3}{2}\frac{dx}{dt}+10x=4\sin 10t \quad \Longrightarrow \quad 2x''+15x'+100x=40\sin 10t,$$

subject to $x(0)=0$, $x'(0)=0$. The auxiliary equation is $0=2m^2+15m+100$ with solutions $m=(-15\pm 5\sqrt{23}i)/4$. A general solution of the associated homogeneous equation is

$$x_h(t)=e^{-15t/4}[C_1\cos(5\sqrt{23}t/4)+C_2\sin(5\sqrt{23}t/4)].$$

A particular solution of the differential equation is

$$\begin{aligned}
x_p(t)&=\frac{1}{2D^2+15D+100}[40\,\mathrm{Im}(e^{10it})]=40\,\mathrm{Im}\left[\frac{1}{2D^2+15D+100}(e^{10it})\right]\\
&=40\,\mathrm{Im}\left[e^{10it}\frac{1}{2(D+10i)^2+15(D+10i)+100}(1)\right]=40\,\mathrm{Im}\left(\frac{e^{10it}}{-100+150i}\right)\\
&=\frac{4}{5}\mathrm{Im}\left(\frac{e^{10it}}{-2+3i}\frac{-2-3i}{-2-3i}\right)=-\frac{4}{5}\mathrm{Im}\left[\frac{(2+3i)(\cos 10t+i\sin 10t)}{13}\right]\\
&=-\frac{4}{65}(3\cos 10t+2\sin 10t).
\end{aligned}$$

Thus, $x(t)=e^{-15t/4}[C_1\cos(5\sqrt{23}t/4)+C_2\sin(5\sqrt{23}t/4)]-(4/65)(3\cos 10t+2\sin 10t)$. To satisfy the initial conditions, we must have $0=C_1-12/65$ and $0=-15C_1/4+5\sqrt{23}C_2/4-16/13$. These imply that $C_1=12/65$ and $C_2=20/(13\sqrt{23})$, and therefore

$$\begin{aligned}
x(t)=e^{-15t/4}\{&(12/65)\cos(5\sqrt{23}t/4)+[20/(13\sqrt{23})]\sin(5\sqrt{23}t/4)\}\\
&-(4/65)(3\cos 10t+2\sin 10t)\ \mathrm{m}.
\end{aligned}$$

3. The differential equation describing charge $Q(t)$ on the capacitor is

$$2\frac{d^2Q}{dt^2}+\frac{1}{0.001}Q=20 \quad \Longrightarrow \quad Q''+500Q=10,$$

subject to $Q(0)=0$ and $Q'(0)=0$. The auxiliary equation is $0=m^2+500$ with solutions $m=\pm 10\sqrt{5}i$. A general solution of the differential equation is therefore $Q(t)=C_1\cos 10\sqrt{5}t+C_2\sin 10\sqrt{5}t+1/50$. To satisfy the initial conditions, we must have $0=C_1+1/50$ and $0=10\sqrt{5}C_2$. Thus, $Q(t)=-(1/50)\cos 10\sqrt{5}t+1/50$, and the current in the circuit is $I(t)=(1/\sqrt{5})\sin 10\sqrt{5}t$ A.

4. The differential equation describing charge $Q(t)$ on the capacitor is

$$(1)\frac{d^2Q}{dt^2}+100\frac{dQ}{dt}+\frac{1}{0.02}Q=0 \quad \Longrightarrow \quad Q''+100Q'+50Q=0,$$

subject to $Q(0)=5$ and $Q'(0)=0$. The auxiliary equation is $m^2+100m+50=0$ with solutions $m=-0.50, -99.50$. A general solution of the differential equation is therefore $Q(t)=C_1e^{-0.50t}+C_2e^{-99.50t}$. To satisfy the initial conditions, we must have $5=C_1+C_2$ and $0=-0.50C_1-99.50C_2$. These imply that $C_1=5.03$ and $C_2=-0.0253$, and therefore $Q(t)=5.03e^{-0.50t}-0.0253e^{-99.50t}$ C.

5. The differential equation describing the current $I(t)$ in the circuit is

$$5\frac{d^2I}{dt^2}+20\frac{dI}{dt}=20\cos 2t, \qquad I(0)=0, \quad I'(0)=0.$$

The auxiliary equation is $5m^2+20m=0$ with solutions $m=0, -4$. A general solution of the associated homogeneous differential equation is therefore $I(t)=C_1+C_2e^{-4t}$. Substituting a particular solution of the form $I_p=A\cos 2t+B\sin 2t$,

$$5(-4A\cos 2t-4B\sin 2t)+20(-2A\sin 2t+2B\cos 2t)=20\cos 2t.$$

This implies that $-20A + 40B = 20$ and $-20B - 40A = 0$, from which $A = -1/5$ and $B = 2/5$. The current is therefore $I(t) = C_1 + C_2e^{-4t} + (2\sin 2t - \cos 2t)/5$. The initial conditions require $0 = C_1 + C_2 - 1/5$ and $0 = -4C_2 + 4/5$, from which $C_1 = 0$ and $C_2 = 1/5$. The transient part of the current is $(1/5)e^{-4t}$ A, and the steady-state part is $(2\sin 2t - \cos 2t)/5$ A.

6. With the coordinate system of Figure 15.11, the differential equation describing the position of M is $M\dfrac{d^2x}{dt^2} + kx = 0$. According to equation 15.63, the stretch in the spring at equilibrium is Mg/k, and therefore the initial conditions are $x(0) = Mg/k$ and $x'(0) = 0$. The auxiliary equation is $0 = Mm^2 + k$ with solutions $m = \pm\sqrt{k/M}i$. Thus, $x(t) = C_1\cos(\sqrt{k/M}t) + C_2\sin(\sqrt{k/M}t)$. To satisfy the initial conditions, $Mg/k = C_1$ and $0 = \sqrt{k/M}C_2$. Thus, $x(t) = (Mg/k)\cos(\sqrt{k/M}t)$.

7. (a) Since the x-component of the force of friction when the mass is moving to the left is 1/2 N, the differential equation describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 18x = \frac{1}{2} \quad \Longrightarrow \quad x'' + 36x = 1,$$

subject to $x(0) = 0.05$ and $x'(0) = 0$.
(b) The auxiliary equation is $m^2 + 36 = 0$ with solutions $m = \pm 6i$, and therefore $x(t) = C_1\cos 6t + C_2\sin 6t + 1/36$. To satisfy the initial conditions, we must have $1/20 = C_1 + 1/36$ and $0 = 6C_2$. Thus, $x(t) = (1/45)\cos 6t + 1/36$. Since $v(t) = (-2/15)\sin 6t$, the mass comes to rest for the first time when $6t = \pi$, and at this time, its position is $x = (1/45)\cos\pi + 1/36 = 1/180$ m. Since this is to the right of the equilibrium position, further motion will not occur.

8. (a) Since the x-component of the force of friction when the mass is moving to the left is 1/2 N, the differential equation describing the position $x(t)$ of the mass from the time it starts until it comes to a stop for the first time is

$$\frac{1}{2}\frac{d^2x}{dt^2} + 18x = \frac{1}{2} \quad \Longrightarrow \quad x'' + 36x = 1,$$

subject to $x(0) = 1/4$ and $x'(0) = 0$.
(b) The auxiliary equation is $m^2 + 36 = 0$ with solutions $m = \pm 6i$, and therefore $x(t) = C_1\cos 6t + C_2\sin 6t + 1/36$. To satisfy the initial conditions, we must have $1/4 = C_1 + 1/36$ and $0 = 6C_2$. Thus, $x(t) = (2/9)\cos 6t + 1/36$. Since $v(t) = (-4/3)\sin 6t$, the mass comes to rest for the first time when $6t = \pi$, and at this time, its position is $x = (2/9)\cos\pi + 1/36 = -7/36$ m. At this position, the spring force is $18(7/36) = 7/2$ N. Because this is greater than the 1/2 N friction force, further motion will occur.

9. The differential equation describing charge $Q(t)$ on the capacitor is

$$\frac{1}{2}\frac{d^2Q}{dt^2} + 3\frac{dQ}{dt} + \frac{1}{0.1}Q = 0 \quad \Longrightarrow \quad Q'' + 6Q + 20Q = 0,$$

subject to $Q(0) = 0$ and $Q'(0) = 1$. The auxiliary equation $m^2 + +6m + 20 = 0$ has solutions $m = -3 \pm \sqrt{11}i$. A general solution of the differential equation is therefore $Q(t) = e^{-3t}(C_1\cos\sqrt{11}t + C_2\sin\sqrt{11}t)$. To satisfy the initial conditions, we must have $0 = C_1$ and $1 = -3C_1 + \sqrt{11}C_2$. Thus, $Q(t) = (1/\sqrt{11})e^{-3t}\sin\sqrt{11}t$. To find the maximum charge on the capacitor, we find critical points for $Q(t)$,

$$0 = Q'(t) = \frac{1}{\sqrt{11}}(-3e^{-3t}\sin\sqrt{11}t + \sqrt{11}e^{-3t}\cos\sqrt{11}t).$$

The smallest positive solution of this equation is $t = (1/\sqrt{11})\mathrm{Tan}^{-1}(\sqrt{11}/3)$, and the charge on the capacitor at this time is 0.105 C.

10. (a) If $\beta = 0$, then $M\dfrac{d^2x}{dt^2} + kx = 0$. The auxiliary equation is $Mm^2 + k = 0$ with solutions $m = \pm\sqrt{k/M}i$. Thus, $x(t) = C_1 \cos\left(\sqrt{k/M}t\right) + C_2 \sin\left(\sqrt{k/M}t\right)$.

(b) If $\beta \neq 0$ and $\beta^2 - 4kM < 0$, the auxiliary equation $Mm^2 + \beta m + k = 0$ has solutions

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M} = -\frac{\beta}{2M} \pm \frac{\sqrt{4kM - \beta^2}}{2M}i.$$

Thus, $x(t) = e^{-\beta t/(2M)}(C_1 \cos\omega t + C_2 \sin\omega t)$, where $\omega = \sqrt{4kM - \beta^2}/(2M)$.

(c) If $\beta \neq 0$ and $\beta^2 - 4kM > 0$, the auxiliary equation has solutions $m = \dfrac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M}$, and therefore

$$x(t) = C_1 e^{(-\beta + \sqrt{\beta^2 - 4kM})t/(2M)} + C_2 e^{(-\beta - \sqrt{\beta^2 - 4kM})t/(2M)} = e^{-\beta t/(2M)}(C_1 e^{\omega t} + C_2 e^{-\omega t}),$$

where $\omega = \sqrt{\beta^2 - 4kM}/(2M)$.

(d) If $\beta \neq 0$ and $\beta^2 - 4kM = 0$, the auxiliary equation has solutions $m = -\beta/(2M),\ -\beta/(2M)$, and therefore $x(t) = (C_1 + C_2 t)e^{-\beta t/(2M)}$.

11. The steady-state solution is the particular solution of the form $x_p = B\sin\omega t + C\cos\omega t$. When we substitute into the differential equation,

$$m(-B\omega^2 \sin\omega t - C\omega^2 \cos\omega t) + \beta(B\omega\cos\omega t - C\omega\sin\omega t) + k(B\sin\omega t + C\cos\omega t) = AP_0 \sin\omega t.$$

Equating coefficients of $\sin\omega t$ and $\cos\omega t$ gives

$$-m\omega^2 B - \beta\omega C + kB = AP_0, \qquad -m\omega^2 C + \beta B\omega + kC = 0,$$

the solution of which is $B = \dfrac{AP_0(k - m\omega^2)}{(k - m\omega^2)^2 + \beta^2\omega^2}, \qquad C = \dfrac{-AP_0\beta\omega}{(k - m\omega^2)^2 + \beta^2\omega^2}$. The steady-state solution is therefore $x_p(t) = \dfrac{AP_0[(k - m\omega^2)\sin\omega t - \beta\omega\cos\omega t]}{(k - m\omega^2)^2 + \beta^2\omega^2}$.

12. With the coordinate system of Figure 15.11, the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 4000x = 3\cos 200t \quad\Longrightarrow\quad x'' + 40\,000x = 30\cos 200t,$$

subject to $x(0) = 0,\ x'(0) = 10$. The auxiliary equation is $m^2 + 40\,000 = 0$ with solutions $m = \pm 200i$. A general solution of the associated homogeneous equation is $x_h(t) = C_1 \cos 200t + C_2 \sin 200t$. Substituting a particular solution of the form $x_p = At\cos 200t + Bt\sin 200t$ into the differential equation gives

$$\begin{aligned}(-400A\sin 200t - 40\,000At\cos 200t + 400B\cos 200t - 40\,000Bt\sin 200t)&\\ + 40\,000(At\cos 200t + Bt\sin 200t) &= 30\cos 200t.\end{aligned}$$

This implies that $A = 0$ and $B = 3/40$, so that $x(t) = C_1 \cos 200t + C_2 \sin 200t + (3t/40)\sin 200t$. The initial conditions require $0 = C_1$ and $10 = 200C_2$. Thus, $x(t) = (1/20 + 3t/40)\sin 200t$ m. Displacements become unbounded as t gets large.

13. With the coordinate system of Figure 15.11, the differential equation describing the position of M is $(1)\dfrac{d^2x}{dt^2} + 64x = 2\sin 8t$, subject to $x(0) = 0$ and $x'(0) = 0$. The auxiliary equation is $0 = m^2 + 64$ with solutions $m = \pm 8i$. The general solution of the associated homogeneous differential equation is $x(t) = C_1 \cos 8t + C_2 \sin 8t$. A particular solution is

$$\begin{aligned}x_p(t) &= \frac{1}{D^2 + 64}(2\sin 8t) = 2\frac{1}{D^2 + 64}\mathrm{Im}(e^{8it}) = 2\,\mathrm{Im}\left[\frac{1}{D^2 + 64}(e^{8it})\right]\\ &= 2\,\mathrm{Im}\left[e^{8it}\frac{1}{(D + 8i)^2 + 64}(1)\right] = 2\,\mathrm{Im}\left[e^{8it}\frac{1}{D^2 + 16iD}(1)\right] = 2\,\mathrm{Im}\left[e^{8it}\frac{1}{D}\frac{1}{D + 16i}(1)\right]\\ &= 2\,\mathrm{Im}\left[e^{8it}\frac{1}{D}\left(\frac{-i}{16}\right)\right] = -\frac{1}{8}\mathrm{Im}\left(ite^{8it}\right) = -\frac{t}{8}\cos 8t.\end{aligned}$$

Thus, $x(t) = C_1 \cos 8t + C_2 \sin 8t - (t/8)\cos 8t$. To satisfy the initial conditions, we must have $0 = C_1$ and $0 = 8C_2 - 1/8$. Thus, $x(t) = (1/64)\sin 8t - (t/8)\cos 8t$ m. For large t, the oscillations become unbounded.

14. The differential equation describing the position of the mass is $M\dfrac{d^2x}{dt^2} + kx = A\cos\omega t$. Solutions of the auxiliary equation $Mm^2 + k = 0$ are $m = \pm\sqrt{k/M}i$. Hence the general solution of the associated homogeneous equation is $x(t) = C_1 \cos\sqrt{k/M}t + C_2 \sin\sqrt{k/M}t$. Resonance occurs when $\sqrt{k/M} = \omega$.

15. The differential equation describing the current in the circuit is

$$\frac{25}{9}\frac{d^2I}{dt^2} + \frac{1}{0.04}I = -45\sin 3t \quad\Longrightarrow\quad 5I'' + 45I = -81\sin 3t,$$

subject to $I(0) = I'(0) = 0$. The auxiliary equation $5m^2 + 45 = 0$ has solutions $m = \pm 3i$, and therefore $I_h(t) = C_1\cos 3t + C_2 \sin 3t$. A particular solution is

$$\begin{aligned} I_p(t) &= \frac{1}{5D^2+45}[-81\,\mathrm{Im}(e^{3ti})] = -\frac{81}{5}\mathrm{Im}\left[\frac{1}{D^2+9}(e^{3ti})\right] = -\frac{81}{5}\mathrm{Im}\left[e^{3ti}\frac{1}{(D+3i)^2+9}(1)\right] \\ &= -\frac{81}{5}\mathrm{Im}\left[e^{3ti}\frac{1}{D(D+6i)}(1)\right] = -\frac{81}{5}\mathrm{Im}\left(\frac{e^{3ti}}{6i}t\right) = \frac{27t}{10}\mathrm{Im}(ie^{3ti}) = \frac{27}{10}t\cos 3t. \end{aligned}$$

Thus, $I(t) = C_1\cos 3t + C_2\sin 3t + (27t/10)\cos 3t$. To satisfy the initial conditions, we must have $0 = C_1$ and $0 = 3C_2 + 27/10$, and the solution becomes $I(t) = -(9/10)\sin 3t + (27/10)t\cos 3t$ A. Resonance does indeed occur.

16. (a) Substituting a particular solution of the form $x_p(t) = B\cos\omega t + C\sin\omega t$ into the differential equation gives

$$M(-\omega^2 B\cos\omega t - \omega^2 C\sin\omega t) + \beta(-\omega B\sin\omega t + \omega C\cos\omega t) + k(B\cos\omega t + C\sin\omega t) = A\cos\omega t.$$

When we equate coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain

$$(k - M\omega^2)B + \beta\omega C = A, \qquad -\beta\omega B + (k - M\omega^2)C = 0.$$

Solutions of these are $B = \dfrac{A(k - M\omega^2)}{(k-M\omega^2)^2 + \beta^2\omega^2}$, $\quad C = \dfrac{A\beta\omega}{(k-M\omega^2)^2+\beta^2\omega^2}$. The particular solution is therefore

$$x_p(t) = \frac{A}{(k-M\omega^2)^2+\beta^2\omega^2}[(k-M\omega^2)\cos\omega t + \beta\omega\sin\omega t].$$

(b) If we set $(k - M\omega^2)\cos\omega t + \beta\omega\sin\omega t = R\sin(\omega t + \phi) = R(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$, and equate coefficients of $\sin\omega t$ and $\cos\omega t$,

$$k - M\omega^2 = R\sin\phi, \quad \beta\omega = R\cos\phi.$$

These imply that $R^2 = (k - M\omega^2)^2 + \beta^2\omega^2$, and therefore

$$\sin\phi = \frac{k - M\omega^2}{\sqrt{(k-M\omega^2)^2+\beta^2\omega^2}}, \quad \cos\phi = \frac{\beta\omega}{\sqrt{(k-M\omega^2)^2+\beta^2\omega^2}}.$$

(c) The amplitude $x(t)$ is a maximum when $(k - M\omega^2)^2 + \beta^2\omega^2$ is smallest. To determine the value of ω that yields the minimum, we solve

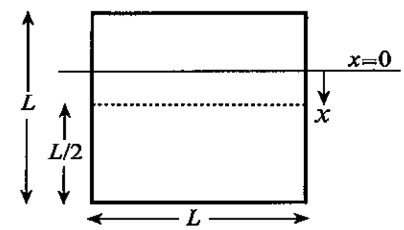
$$0 = 2(k - M\omega^2)(-2M\omega) + 2\beta^2\omega = 2\omega[-2M(k - M\omega^2) + \beta^2].$$

The nonzero solution is $\omega = \sqrt{k/M - \beta^2/(2M^2)}$. The amplitude at this value of ω is

$$\frac{A}{\sqrt{\left[k - M\left(\dfrac{k}{M} - \dfrac{\beta^2}{2M^2}\right)\right]^2 + \beta^2\left(\dfrac{k}{M} - \dfrac{\beta^2}{2M^2}\right)}} = \frac{2AM}{\beta\sqrt{4kM - \beta^2}}.$$

17. (a) Since the cube floats half submerged, its density is one-half that of water, namely 500 kg/m^3. Suppose we let x denote the distance of the midpoint of the cube below the surface of the water. When the midpoint is x m below the surface, the force on the cube is the buoyant force due to Archimedes' principle less the force of gravity,

$$-9810L^2\left(\frac{L}{2}+x\right)+4905L^3=-9810L^2x.$$



The differential equation describing oscillations of the cube is therefore

$$500L^3\frac{d^2x}{dt^2}=-9810L^2x \quad\Longrightarrow\quad x''+\frac{981}{50L}x=0.$$

(b) The auxiliary equation $m^2+981/(50L)=0$ has solutions $m=\pm\sqrt{981/(50L)}i$, and therefore

$$x(t)=C_1\cos\sqrt{\frac{981}{50L}}t+C_2\sin\sqrt{\frac{981}{50L}}t.$$

The frequency of the oscillations is $\dfrac{\sqrt{981/(50L)}}{2\pi}=\dfrac{0.705}{\sqrt{L}}$.

18. If x measures displacement of the platform from its equilibrium position, then the differential equation for the combined motion is

$$\left(\frac{W+w}{g}\right)\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=0.$$

The auxiliary equation is $\left(\dfrac{W+w}{g}\right)m^2+\beta m+k=0$ with solutions

$$m=\frac{-\beta\pm\sqrt{\beta^2-4k(W+w)/g}}{2(W+w)/g}.$$

Oscillations occur for large w, and for small values of w no oscillations occur. The largest value of w for no oscillations occurs when

$$\beta^2-\frac{4k(W+w)}{g}=0 \quad\Longrightarrow\quad w=\frac{\beta^2 g}{4k}-W.$$

19. The differential equation for vertical motion of the trailer is

$$400\frac{d^2y}{dt^2}+40\,000y=40\,000A\cos\frac{\pi vt}{5}.$$

Resonance occurs when the forcing frequency is equal to the natural frequency of the trailer; that is, when

$$\frac{\pi v}{5}=\sqrt{\frac{40\,000}{400}} \quad\Longrightarrow\quad v=\frac{50}{\pi}\ \text{m/s}.$$

This is equivalent to 57.3 km/hr.

20. (a) The differential equation is

$$200\frac{d^2y}{dt^2} + 3000\frac{dy}{dt} + 50\,000y = 5000\sin\frac{\pi vt}{40} \implies \frac{d^2y}{dt^2} + 15\frac{dy}{dt} + 250y = 25\sin\frac{\pi vt}{40}.$$

The auxiliary equation is $m^2 + 15m + 250 = 0$ with solutions

$$m = \frac{-15 \pm \sqrt{15^2 - 4(250)}}{2} = \frac{-15 \pm 5\sqrt{31}i}{2}.$$

Hence, $y_h(t) = e^{-15t/2}\left(C_1\cos\frac{5\sqrt{31}t}{2} + C_2\sin\frac{5\sqrt{31}t}{2}\right)$. A particular solution is of the form $y_p(t) = A\cos\frac{\pi vt}{40} + B\sin\frac{\pi vt}{40}$. Substituting into the differential equation,

$$\left(-\frac{\pi^2v^2A}{1600}\cos\frac{\pi vt}{40} - \frac{\pi^2v^2B}{1600}\sin\frac{\pi vt}{40}\right) + 15\left(\frac{-\pi vA}{40}\sin\frac{\pi vt}{40} + \frac{\pi vB}{40}\cos\frac{\pi vt}{40}\right)$$
$$+ 250\left(A\cos\frac{\pi vt}{40} + B\sin\frac{\pi vt}{40}\right) = 25\sin\frac{\pi vt}{40}.$$

Equating coefficients gives

$$-\frac{\pi^2v^2A}{1600} + \frac{3\pi vB}{8} + 250A = 0, \quad -\frac{\pi^2v^2B}{1600} - \frac{3\pi vA}{8} + 250B = 25.$$

The solution is

$$A = \frac{-24\,000\,000\pi v}{(400\,000 - \pi^2v^2)^2 + 360\,000\pi^2v^2}, \quad B = \frac{40\,000(400\,000 - \pi^2v^2)}{(400\,000 - \pi^2v^2)^2 + 360\,000\pi^2v^2}.$$

Thus, $y(t) = e^{-15t/2}\left(C_1\cos\frac{5\sqrt{31}t}{2} + C_2\sin\frac{5\sqrt{31}t}{2}\right) + A\cos\frac{\pi vt}{40} + B\sin\frac{\pi vt}{40}$. The initial conditions $y(0) = 0$ and $y'(0) = 0$ require

$$0 = C_1 + A, \qquad 0 = -\frac{15}{2}C_1 + \frac{5\sqrt{31}}{2}C_2 + \frac{\pi vB}{40}.$$

These give $C_1 = -A$ and $C_2 = -(300A + \pi vB)/(100\sqrt{31})$. When $v = 10$, the solution is

$$y(t) = e^{-15t/2}\left(0.00473\cos\frac{5\sqrt{31}t}{2} - 0.00310\sin\frac{5\sqrt{31}t}{2}\right) - 0.00473\cos\frac{\pi t}{4} + 0.100\sin\frac{\pi t}{4}.$$

The amplitude of the steady-state part is

$$\sqrt{(0.00473)^2 + (0.100)^2} = 0.100.$$

(b) When $v = 20$, the solution is

$$y(t) = e^{-15t/2}\left(0.00953\cos\frac{5\sqrt{31}t}{2} - 0.00616\sin\frac{5\sqrt{31}t}{2}\right) - 0.00953\cos\frac{\pi t}{2} + 0.100\sin\frac{\pi t}{2}.$$

The amplitude of the steady-state part is

$$\sqrt{(0.00953)^2 + (0.100)^2} = 0.100.$$

21. The auxiliary equation is

$$0 = m^2 + 3m + 2 = (m+1)(m+2) \implies m = -1, -2.$$

A general solution of the differential equation is $x(t) = C_1e^{-t} + C_2e^{-2t} + a/2$. The initial conditions require

$$0 = C_1 + C_2 + \frac{a}{2}, \qquad \frac{2a}{3} = -C_1 - 2C_2.$$

The solution is $C_1 = -a/3$ and $C_2 = -a/6$, so that

$$x(t) = -\frac{a}{3}e^{-t} - \frac{a}{6}e^{-2t} + \frac{a}{2} = \frac{a}{6}(3 - 2e^{-t} - e^{-2t}).$$

22. (a) The force field associated with the given potential is $\mathbf{F} = -\nabla V = (-36y + 96x)\hat{\mathbf{i}} - 36x\hat{\mathbf{j}}$. By Newton's second law,

$$3\left(\frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}}\right) = (-36y + 96x)\hat{\mathbf{i}} - 36x\hat{\mathbf{j}}.$$

When we equate components,

$$\frac{d^2x}{dt^2} = -12y + 32x, \qquad \frac{d^2y}{dt^2} = -12x.$$

(b) If we substitute from the second of the differential equations in part (a) into the first, we obtain

$$-\frac{1}{12}\frac{d^4y}{dt^4} = -12y + 32\left(-\frac{1}{12}\frac{d^2y}{dt^2}\right) \qquad \Longrightarrow \qquad \frac{d^4y}{dt^4} - 32\frac{d^2y}{dt^2} - 144y = 0.$$

The auxiliary equation is $0 = m^4 - 32m^2 - 144 = (m^2 - 36)(m^2 + 4)$ with solutions $m = \pm 6,\ \pm 2i$. Hence,

$$y(t) = C_1e^{6t} + C_2e^{-6t} + C_3\cos 2t + C_4\sin 2t.$$

The second equation gives

$$x(t) = -\frac{1}{12}\frac{d^2y}{dt^2} = -3C_1e^{6t} - 3C_2e^{-6t} + \frac{1}{3}C_3\cos 2t + \frac{1}{3}C_4\sin 2t.$$

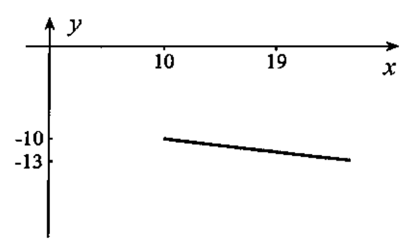
The initial conditions $x(0) = 10$, $x'(0) = 0$, $y(0) = -10$, and $y'(0) = 0$ require

$$\begin{aligned} 10 &= -3C_1 - 3C_2 + \frac{C_3}{3}, & 0 &= -18C_1 + 18C_2 + \frac{2C_4}{3}, \\ -10 &= C_1 + C_2 + C_3, & 0 &= 6C_1 - 6C_2 + 2C_4. \end{aligned}$$

These give $C_1 = -2$, $C_2 = -2$, $C_3 = -6$, and $C_4 = 0$. Thus,

$$x(t) = 6e^{6t} + 6e^{-6t} - 2\cos 2t, \qquad y(t) = -2e^{6t} - 2e^{-6t} - 6\cos 2t.$$

(c) A plot of the curve is shown to the right. It appears to be a straight line with slope -3. This is not the case, however. It looks this way only because the exponential function e^{6t} is so dominant, the oscillations of the cosine terms are obliterated.



23. If we take $\mathbf{E} = E\hat{\mathbf{j}}$ and $\mathbf{B} = -B\hat{\mathbf{i}}$, then

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = E\hat{\mathbf{j}} + \mathbf{v} \times (-B\hat{\mathbf{i}}) = E\hat{\mathbf{j}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ -B & 0 & 0 \end{vmatrix} = E\hat{\mathbf{j}} - Bv_z\hat{\mathbf{j}} + Bv_y\hat{\mathbf{k}}.$$

Newton's second law gives

$$m\frac{d\mathbf{v}}{dt} = q[(E - Bv_z)\hat{\mathbf{j}} + Bv_y\hat{\mathbf{k}}].$$

When we equate components, we obtain

$$m\frac{dv_x}{dt} = 0, \qquad m\frac{dv_y}{dt} = q(E - Bv_z), \qquad m\frac{dv_z}{dt} = qBv_y.$$

With $x(0) = x'(0) = 0$, the first of these gives $x(t) = 0$ for all t. If we substitute from the third differential equation into the second, we find

$$m\left(\frac{m}{qB}\frac{d^2v_z}{dt^2}\right) = q(E - Bv_z) \qquad \Longrightarrow \qquad m^2\frac{d^2v_z}{dt^2} + q^2B^2v_z = q^2BE.$$

The auxiliary equation is $m^2 + q^2B^2 = 0$ with solutions $m = \pm qBi$. Since a particular solution is E/B, the general solution is

$$v_z(t) = C_1\cos\left(\frac{qBt}{m}\right) + C_2\sin\left(\frac{qBt}{m}\right) + \frac{E}{B}.$$

The initial condition $v_z(0) = 0$ requires $0 = C_1 + E/B$, and therefore

$$v_z(t) = \frac{E}{B}\left[1 - \cos\left(\frac{qBt}{m}\right)\right] + C_2\sin\left(\frac{qBt}{m}\right).$$

Since $v_y = [m/(qB)]dv_z/dt$, we find that

$$v_y = \frac{m}{qB}\left[\frac{Eq}{m}\sin\left(\frac{qBt}{m}\right) + \frac{qBC_2}{m}\cos\left(\frac{qBt}{m}\right)\right].$$

The initial condition $v_y(0) = 0$ requires $C_2 = 0$. Thus,

$$v_y(t) = \frac{E}{B}\sin\left(\frac{qBt}{m}\right), \qquad v_z = \frac{E}{B}\left[1 - \cos\left(\frac{qBt}{m}\right)\right].$$

Integration of these gives

$$y = -\frac{Em}{qB^2}\cos\left(\frac{qBt}{m}\right) + D_1, \qquad z = \frac{E}{B}\left[t - \frac{m}{qB}\sin\left(\frac{qBt}{m}\right)\right] + D_2.$$

The initial conditions $y(0) = 0$ and $z(0) = 0$, yield $0 = -Em/(qB^2) + D_1$ and $0 = D_2$. Thus,

$$y(t) = \frac{Em}{qB^2}\left[1 - \cos\left(\frac{qBt}{m}\right)\right], \qquad z(t) = \frac{Em}{qB^2}\left[\frac{qBt}{m} - \sin\left(\frac{qBt}{m}\right)\right].$$

24. (a) We write $\dfrac{d^2h}{dx^2} - \dfrac{K_v}{KbB}h = -\dfrac{K_vH_0}{KbB}$. The auxiliary equation $m^2 - K_v/(KbB) = 0$ has real roots $\pm\sqrt{K_v/(KbB)}$. If we let $m = \sqrt{K_v/(KbB)}$, then a general solution of the differential equation is

$$h(x) = C_1e^{mx} + C_2e^{-mx} + H_0.$$

The boundary conditions require

$$H_0 = C_1 + C_2 + H_0, \qquad b = C_1e^{mL} + C_2e^{-mL} + H_0,$$

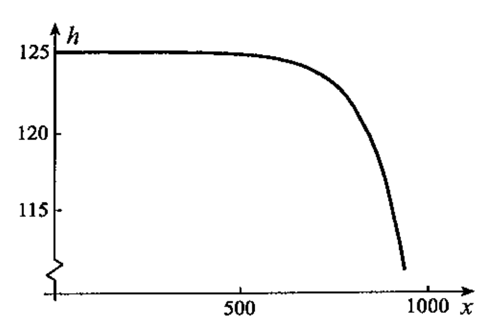
from which $C_1 = -C_2 = \dfrac{b - H_0}{e^{mL} - e^{-mL}}$. Thus,

$$h(x) = \frac{b - H_0}{e^{mL} - e^{-mL}}(e^{mx} - e^{-mx}) + H_0.$$

(b) With $K_v = 10^{-8}$, $K = 10^{-6}$, $b = 100$, $B = 1$, $H_0 = 125$, and $L = 1000$,

$$h(x) = 125 - \frac{25}{e^{10} - e^{-10}}(e^{x/100} - e^{-x/100}).$$

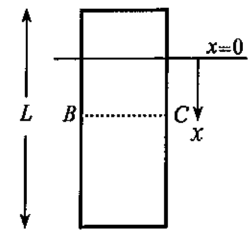
The plot is indeed relatively flat for $0 \le x \le 600$.



25. Let BC be the line on the cylinder that resides in the surface of the water when the cylinder is at equilibrium. If x represents the depth of BC below the surface when the cylinder is in motion, then Newton's second law for the acceleration of the cylinder is

$$M\frac{d^2x}{dt^2} = -9.81(1000)\rho(Ax),$$

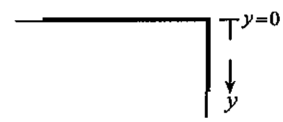
where M is the mass of the cylinder, A is its cross-sectional area, and ρ is its density. Since $M = \rho AL$, where L is the length of the cylinder,



$$\rho AL\frac{d^2x}{dt^2} = -9810\rho Ax \implies L\frac{d^2x}{dt^2} + 9810x = 0.$$

The auxiliary equation $Lm^2 + 9810 = 0$ has roots $m = \pm\sqrt{9810/L}i$, so that $x(t) = C_1\cos\sqrt{9810/L}t + C_2\sin\sqrt{9810/L}t$. Since the period of the oscillations is 4 s, it follows that $2\pi\sqrt{L/9810} = 4 \implies L = 39\,240/\pi^2$. The mass of the cylinder is therefore $\rho AL = \rho(\pi/100)(39\,240/\pi^2) = 124.9\rho$ kg.

26. Suppose the mass of the chain is M so that its mass per unit length is M/a. When the length of chain hanging from the edge of the table is y, then



$$M\frac{d^2y}{dt^2} = \frac{Mgy}{a}.$$

This differential equation is subject to the initial conditions $y(0) = b$ and $y'(0) = 0$, provided $t = 0$ is taken at the instant motion begins. The differential equation is linear with auxiliary equation $m^2 - g/a = 0 \Longrightarrow m = \pm\sqrt{g/a}$. A general solution is therefore $y(t) = C_1e^{\sqrt{g/a}t} + C_2e^{-\sqrt{g/a}t}$. The initial conditions require

$$b = C_1 + C_2, \quad 0 = \sqrt{\frac{g}{a}}C_1 - \sqrt{\frac{g}{a}}C_2 \quad \Longrightarrow \quad C_1 = C_2 = b/2.$$

Thus, $y(t) = \dfrac{b}{2}(e^{\sqrt{g/a}t} + e^{-\sqrt{g/a}t})$. The chain slides off the table when $y = a$ in which case

$$a = \frac{b}{2}(e^{\sqrt{g/a}t} + e^{-\sqrt{g/a}t}) \quad \Longrightarrow \quad e^{2\sqrt{g/a}t} - \frac{2a}{b}e^{\sqrt{g/a}t} + 1 = 0.$$

This is a quadratic in $e^{\sqrt{g/a}t}$ with solutions

$$e^{\sqrt{g/a}t} = \frac{2a/b \pm \sqrt{4a^2/b^2 - 4}}{2} = \frac{1}{b}(a \pm \sqrt{a^2 - b^2}) \quad \Longrightarrow t = \sqrt{\frac{a}{g}}\ln\left(\frac{a \pm \sqrt{a^2 - b^2}}{b}\right).$$

It is straightforward to verify that $(a - \sqrt{a^2 - b^2})/b < 1$ in which case t would be negative, an unacceptable value. Hence, $t = \sqrt{\dfrac{a}{g}}\ln\left(\dfrac{a + \sqrt{a^2 - b^2}}{b}\right)$.

27. (a) The auxiliary equation $m^4 + P/(EI)m^2 = 0$ has solutions $m = 0, 0, \pm\sqrt{P/(EI)}i$. A general solution of the differential equation is

$$y(x) = C_1\cos\sqrt{\frac{P}{EI}}x + C_2\sin\sqrt{\frac{P}{EI}}x + C_3x + C_4.$$

(b) The boundary conditions require

$$0 = y(0) = C_1 + C_4, \quad 0 = y'(0) = \sqrt{\frac{P}{EI}}C_2 + C_3, \quad 0 = y''(L) = -\frac{C_1P}{EI}\cos\sqrt{\frac{P}{EI}}L - \frac{C_2P}{EI}\sin\sqrt{\frac{P}{EI}}L,$$

$$0 = y'''(L) + \frac{P}{EI}y'(L) = \left(\frac{P}{EI}\right)^{3/2}C_1\sin\sqrt{\frac{P}{EI}}L - \left(\frac{P}{EI}\right)^{3/2}C_2\cos\sqrt{\frac{P}{EI}}L$$

$$+\frac{P}{EI}\left(-\sqrt{\frac{P}{EI}}C_1\sin\sqrt{\frac{P}{EI}}L+\sqrt{\frac{P}{EI}}C_2\cos\sqrt{\frac{P}{EI}}L+C_3\right).$$

These imply that $C_2 = C_3 = 0$, $0 = C_1 + C_4$, and $0 = C_1\cos\sqrt{\frac{P}{EI}}L$. Since $C_1 \neq 0$, else $y(x) \equiv 0$, we must set

$$0 = \cos\sqrt{\frac{P}{EI}}L = 0 \qquad \Longrightarrow \qquad \sqrt{\frac{P}{EI}}L = \frac{(2n+1)\pi}{2},$$

where n is an integer. Consequently, $P = (2n+1)^2\pi^2 EI/(4L^2)$, and

$$y(x) = -C_4\cos\frac{(2n+1)\pi x}{2L} + C_4 = C_4\left[1-\cos\frac{(2n+1)\pi x}{2L}\right].$$

(c) Euler's buckling load is the smallest buckling load $\pi^2 EI/(4L^2)$ occurring when $n = 0$.

28. (a) The auxiliary equation $m^4 + P/(EI)m^2 = 0$ has solutions $m = 0, 0 \pm \sqrt{P/(EI)}i$. A general solution of the differential equation is

$$y(x) = C_1\cos\sqrt{\frac{P}{EI}}x + C_2\sin\sqrt{\frac{P}{EI}}x + C_3 x + C_4.$$

(b) If we set $\mu = \sqrt{P/(EI)}$ for simplicity in notation, the boundary conditions require

$$0 = y(0) = C_1 + C_4, \quad 0 = y'(0) = \mu C_2 + C_3, \quad 0 = y(L) = C_1\cos\mu L + C_2\sin\mu L + C_3 L + C_4,$$

$$0 = y'(L) = -\mu C_1\sin\mu L + \mu C_2\cos\mu L + C_3.$$

If we solve the first two for C_4 and C_3 and substitute into the last two, the result is

$$0 = C_1\cos\mu L + C_2\sin\mu L - \mu L C_2 - C_1, \quad 0 = -\mu C_1\sin\mu L + \mu C_2\cos\mu L - \mu C_2,$$

or,

$$C_1(\cos\mu L - 1) + C_2(\sin\mu L - \mu L) = 0, \quad C_1(\sin\mu L) + C_2(1-\cos\mu L) = 0.$$

If we solve each for C_1 and equate results,

$$-\frac{C_2(\sin\mu L - \mu L)}{\cos\mu L - 1} = -\frac{C_2(1-\cos\mu L)}{\sin\mu L},$$

or,

$$\begin{aligned}0 &= C_2(\sin^2\mu L - \mu L\sin\mu L + \cos^2\mu L - 2\cos\mu L + 1) = C_2(2 - 2\cos\mu L - \mu L\sin\mu L)\\ &= C_2\left[2 - 2\left(1 - 2\sin^2\frac{\mu L}{2}\right) - 2\mu L\sin\frac{\mu L}{2}\cos\frac{\mu L}{2}\right] = 2C_2\sin\frac{\mu L}{2}\left(2\sin\frac{\mu L}{2} - \mu L\cos\frac{\mu L}{2}\right).\end{aligned}$$

There are three possibilities:
Case 1: $C_2 = 0$ — In this case, $C_3 = 0$, $C_4 = -C_1$, and $C_1(\cos\mu L - 1) = 0 = C_1\sin\mu L$. We cannot set $C_1 = 0$, else $C_4 = 0$ also, and $y(x) \equiv 0$. Hence we must set $\cos\mu L = 1$ and $\sin\mu L = 0$. These imply that $\mu L = 2n\pi$, where n is an integer, and therefore $\sqrt{P/(EI)}L = 2n\pi \Longrightarrow P = 4n^2\pi^2 EI/L^2$. The smallest positive P is $P = 4\pi^2 EI/L^2$.
Case 2: $\sin\frac{\mu L}{2} = 0$ — In this case, $\mu L/2 = n\pi$, where n is an integer, and $\sqrt{P/(EI)}L/2 = n\pi \Longrightarrow P = 4n^2\pi^2 EI/L^2$. The smallest positive P is once again $P = 4\pi^2 EI/L^2$.
Case 3: $2\sin\frac{\mu L}{2} - \mu L\cos\frac{\mu L}{2} = 0$ — In this case, $\mu L/2$ must satisfy the equation

$$\tan\frac{\mu L}{2} = \frac{\mu L}{2}.$$

This equation must be solved numerically. The smallest positive solution is $\mu L/2 = 4.49$. Hence, $\sqrt{P/(EI)}L/2 = 4.49 \Longrightarrow P = 4(4.49)^2 EI/L^2 = 80.6EI/L^2$. This value is larger than that in Cases 1 and 2. Thus, the Euler buckling load is $P = 4\pi^2 EI/L^2$.

29. (a)

$$\begin{aligned}\frac{d^2N_A}{dt^2} &= -k_1\frac{dN_A}{dt} + k_2\frac{dN_B}{dt} = -k_1\frac{dN_A}{dt} + k_2[-(k_2+k_3)N_B + k_1N_A + k_4N_C]\\ &= -k_1\frac{dN_A}{dt} + k_1k_2N_A - k_2(k_2+k_3)N_B + k_2k_4(1-N_A-N_B)\\ &= -k_1\frac{dN_A}{dt} + (k_1k_2 - k_2k_4)N_A - k_2(k_2+k_3+k_4)N_B + k_2k_4\\ &= -k_1\frac{dN_A}{dt} + (k_1k_2 - k_2k_4)N_A - (k_2+k_3+k_4)\left(\frac{dN_A}{dt} + k_1N_A\right) + k_2k_4\\ &= -(k_1+k_2+k_3+k_4)\frac{dN_A}{dt} + (k_1k_2 - k_2k_4 - k_1k_2 - k_1k_3 - k_1k_4)N_A + k_2k_4.\end{aligned}$$

Thus,

$$\frac{d^2N_A}{dt^2} + (k_1+k_2+k_3+k_4)\frac{dN_A}{dt} + (k_1k_3 + k_2k_4 + k_1k_4)N_A = k_2k_4.$$

(b) When $k_1 = 2$, $k_2 = 1$, $k_3 = 4$, and $k_4 = 3$,

$$\frac{d^2N_A}{dt^2} + 10\frac{dN_A}{dt} + 17N_A = 3.$$

The auxiliary equation $m^2 + 10m + 17 = 0$ has roots $m = \dfrac{-10 \pm \sqrt{100-68}}{2} = -5 \pm 2\sqrt{2}$. The general solution for $N_A(t)$ is therefore

$$N_A(t) = C_1e^{(-5+2\sqrt{2})t} + C_2e^{-(5+2\sqrt{2})t} + \frac{3}{17}.$$

The initial conditions require

$$C_1 + C_2 + \frac{3}{17} = 1, \qquad (-5+2\sqrt{2})C_1 - (5+2\sqrt{2})C_2 = 0.$$

These give

$$N_A(t) = \frac{7}{68}(4+5\sqrt{2})e^{(-5+2\sqrt{2})t} + \frac{7}{68}(4-5\sqrt{2})e^{-(5+2\sqrt{2})t} + \frac{3}{17}.$$

Its limit is $\displaystyle\lim_{t\to\infty} N_A(t) = \frac{3}{17}$.

(c) The differential equation for $N_B(t)$ is

$$\frac{dN_B}{dt} = -(k_2+k_3)N_B + k_1N_A + k_4(1-N_A-N_B)$$

or,

$$\frac{dN_B}{dt} + (k_2+k_3+k_4)N_B = (k_1-k_4)N_A + k_4.$$

With the given values for k_1, k_2, k_3, and k_4,

$$\frac{dN_B}{dt} + 8N_B = -N_A + 3.$$

This is a first-order linear differential equation with integating factor e^{8t}. We write therefore that

$$\frac{d}{dt}[e^{8t}N_B] = 3e^{8t} - \frac{7}{68}(4+5\sqrt{2})e^{(3+2\sqrt{2})t} + \frac{7}{68}(4-5\sqrt{2})e^{(3-2\sqrt{2})t} - \frac{3}{17}e^{8t}.$$

Integration gives

$$e^{8t}N_B = \frac{6}{17}e^{8t} - \frac{7(4+5\sqrt{2})}{68(3+2\sqrt{2})}e^{(3+2\sqrt{2})t} + \frac{7(4-5\sqrt{2})}{68(3-2\sqrt{2})}e^{(3-2\sqrt{2})t} + D,$$

from which

$$N_B(t) = \frac{6}{17} - \frac{7(7\sqrt{2}-8)}{68}e^{(-5+2\sqrt{2})t} - \frac{7(7\sqrt{2}+8)}{68}e^{-(5+2\sqrt{2})t} + De^{-8t}.$$

The initial condition $N_B(0) = 0$ requires

$$0 = \frac{6}{17} - \frac{7(7\sqrt{2}-8)}{68} - \frac{7(7\sqrt{2}+8)}{68} + D \quad \Longrightarrow \quad D = \frac{49\sqrt{2}-12}{34}.$$

Thus,

$$N_B(t) = \frac{6}{17} - \frac{7(7\sqrt{2}-8)}{68}e^{(-5+2\sqrt{2})t} - \frac{7(7\sqrt{2}+8)}{68}e^{-(5+2\sqrt{2})t} + \frac{49\sqrt{2}-12}{34}e^{-8t}.$$

Its limit is $\lim_{t\to\infty} N_B(t) = \frac{6}{17}$.

(d) $N_C(t) = 1 - N_A - N_B = \frac{8}{17} - \frac{7(6-\sqrt{2})}{34}e^{(-5+2\sqrt{2})t} + \frac{7(6+\sqrt{2})}{34}e^{-(5+2\sqrt{2})t} - \frac{49\sqrt{2}-12}{34}e^{-8t}$.

Its limit is $\lim_{t\to\infty} N_C(t) = \frac{8}{17}$.

30. (b) The auxiliary equation $m^2 + (k_1+k_2+k_3+k_4)m + (k_1k_3+k_2k_4+k_1k_4) = 0$ has solutions

$$m = \frac{-(k_1+k_2+k_3+k_4) \pm \sqrt{(k_1+k_2+k_3+k_4)^2 - 4(k_1k_3+k_2k_4+k_1k_4)}}{2}.$$

We denote these roots by ω_1 and ω_2 ($\omega_1 > \omega_2$), where it is important to note that both ω_1 and ω_2 are negative. The general solution of the differential equation is

$$N_A(t) = C_1e^{\omega_1 t} + C_2e^{\omega_2 t} + K,$$

where $K = k_2k_4/(k_1k_3+k_2k_4+k_1k_4)$. The initial conditions require

$$1 = C_1 + C_2 + K, \qquad 0 = \omega_1C_1 + \omega_2C_2,$$

solutions of which are

$$C_1 = \frac{\omega_2(1-K)}{\omega_2-\omega_1}, \qquad C_2 = \frac{\omega_1(1-K)}{\omega_1-\omega_2}.$$

Thus,

$$N_A(t) = \frac{\omega_2(1-K)}{\omega_2-\omega_1}e^{\omega_1 t} + \frac{\omega_1(1-K)}{\omega_1-\omega_2}e^{\omega_2 t} + K.$$

Since ω_1 and ω_2 are negative, $\lim_{t\to\infty} N_A(t) = K$.

(c) From Exercise 29, the differential equation for $N_B(t)$ is

$$\frac{dN_B}{dt} + (k_2+k_3+k_4)N_B = (k_1-k_4)N_A + k_4.$$

This is a first-order linear equation with integrating factor $e^{(k_2+k_3+k_4)t}$. We write therefore that

$$\begin{aligned}\frac{d}{dt}[e^{(k_2+k_3+k_4)t}N_B] &= k_4e^{(k_2+k_3+k_4)t} + (k_1-k_4)e^{(k_2+k_3+k_4)t}N_A(t)\\ &= k_4e^{(k_2+k_3+k_4)t} + (k_1-k_4)Ke^{(k_2+k_3+k_4)t}\\ &\quad + \frac{(k_1-k_4)(1-K)}{\omega_2-\omega_1}\left[\omega_2e^{(\omega_1+k_2+k_3+k_4)t} - \omega_1e^{(\omega_2+k_2+k_3+k_4)t}\right].\end{aligned}$$

Integration gives

$$e^{(k_2+k_3+k_4)t}N_B = \frac{k_4}{k_2+k_3+k_4}e^{(k_2+k_3+k_4)t} + \left[\frac{(k_1-k_4)K}{k_2+k_3+k_4}\right]e^{(k_2+k_3+k_4)t}$$
$$+\frac{(k_1-k_4)(1-K)}{\omega_2-\omega_1}\left[\frac{\omega_2 e^{(\omega_1+k_2+k_3+k_4)t}}{\omega_1+k_2+k_3+k_4} - \frac{\omega_1 e^{(\omega_2+k_2+k_3+k_4)t}}{\omega_2+k_2+k_3+k_4}\right] + D.$$

Thus,

$$N_B(t) = \frac{k_4}{k_2+k_3+k_4} + \frac{(k_1-k_4)K}{k_2+k_3+k_4} + \frac{(k_1-k_4)(1-K)}{\omega_2-\omega_1}\left[\frac{\omega_2 e^{\omega_1 t}}{\omega_1+k_2+k_3+k_4} - \frac{\omega_1 e^{\omega_2 t}}{\omega_2+k_2+k_3+k_4}\right]$$
$$+ De^{-(k_2+k_3+k_4)t}.$$

The initial condition $N_B(0) = 0$ requires

$$0 = \frac{k_4}{k_2+k_3+k_4} + \frac{(k_1-k_4)K}{k_2+k_3+k_4} + \frac{(k_1-k_4)(1-K)}{\omega_2-\omega_1}\left[\frac{\omega_2}{\omega_1+k_2+k_3+k_4} - \frac{\omega_1}{\omega_2+k_2+k_3+k_4}\right] + D,$$

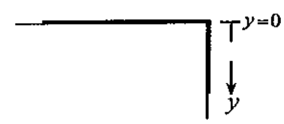
and this defines D. Since again ω_1 and ω_2 are negative,

$$\lim_{t\to\infty} N_B(t) = \frac{k_4}{k_2+k_3+k_4} + \frac{(k_1-k_4)K}{k_2+k_3+k_4}.$$

(d) $N_C(t) = 1 - N_A(t) - N_B(t)$ Its limit as $t \to \infty$ is

$$\lim_{t\to\infty} N_C(t) = 1 - \lim_{t\to\infty} N_A(t) - \lim_{t\to\infty} N_B(t) = 1 - K - \frac{k_4}{k_2+k_3+k_4} + \frac{(k_1-k_4)K}{k_2+k_3+k_4}.$$

31. Suppose the mass of the chain is M so that its mass per unit length is M/a. When the length of chain hanging from the edge of the table is y, then



$$M\frac{d^2y}{dt^2} = \frac{Mgy}{a} - \frac{\mu Mg}{a}(a-y) \quad \Longrightarrow \quad \frac{d^2y}{dt^2} - \frac{g}{a}(1+\mu)y = -\mu g.$$

This differential equation is subject to the initial conditions $y(0) = b$ and $y'(0) = 0$, provided $t = 0$ is taken at the instant motion begins. The differential equation is linear with auxiliary equation $m^2 - (g/a)(1+\mu) = 0 \Longrightarrow m = \pm\sqrt{g(1+\mu)/a}$. A general solution is therefore $y(t) = C_1 e^{\sqrt{g(1+\mu)/a}t} + C_2 e^{-\sqrt{g(1+\mu)/a}t} + a\mu/(1+\mu)$. The initial conditions require

$$b = C_1 + C_2 + \frac{a\mu}{1+\mu}, \quad 0 = \sqrt{\frac{g(1+\mu)}{a}}C_1 - \sqrt{\frac{g(1+\mu)}{a}}C_2 \qquad \Longrightarrow \qquad C_1 = C_2 = \frac{1}{2}\left(b - \frac{a\mu}{1+\mu}\right).$$

Thus, $y(t) = \dfrac{1}{2}\left(b - \dfrac{a\mu}{1+\mu}\right)(e^{\sqrt{g(1+\mu)/a}t} + e^{-\sqrt{g(1+\mu)/a}t}) + \dfrac{a\mu}{1+\mu}$. The chain slides off the table when $y = a$,

$$a = \frac{1}{2}\left(b - \frac{a\mu}{1+\mu}\right)(e^{\sqrt{g(1+\mu)/a}t} + e^{-\sqrt{g(1+\mu)/a}t}) + \frac{a\mu}{1+\mu},$$

which can be expressed in the form

$$e^{2\sqrt{g(1+\mu)/a}t} - \frac{2a}{b(1+\mu)-a\mu}e^{\sqrt{g(1+\mu)/a}t} + 1 = 0.$$

This is a quadratic in $e^{\sqrt{g(1+\mu)/a}t}$ with solutions

$$e^{\sqrt{g(1+\mu)/a}t} = \frac{1}{2}\left[\frac{2a}{b(1+\mu)-a\mu} \pm \sqrt{\frac{4a^2}{[b(1+\mu)-a\mu]^2} - 4}\right] = \frac{a \pm \sqrt{a^2 - [b(1+\mu)-a\mu]^2}}{b(1+\mu)-a\mu},$$

and

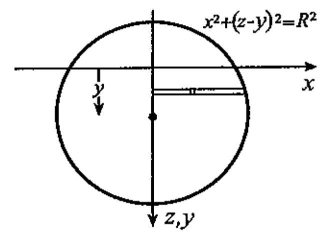
$$t = \sqrt{\frac{a}{g(1+\mu)}} \ln\left\{\frac{a \pm \sqrt{a^2 - [b(1+\mu) - a\mu]^2}}{b(1+\mu) - a\mu}\right\}.$$

It can be shown that the negative root is less than 1, in which case $t < 0$. Hence,

$$t = \sqrt{\frac{a}{g(1+\mu)}} \ln\left\{\frac{a + \sqrt{a^2 - [b(1+\mu) - a\mu]^2}}{b(1+\mu) - a\mu}\right\}.$$

32. Because the sphere floats half submerged, its density is one-half that of water, namely 500 kg/m^3. The resultant vertical force on the sphere when its centre is y m below the surface is the buoyant force due to the water displaced by the sphere less the force of gravity on the sphere,

$$-9810V + 4905\left(\frac{4}{3}\right)\pi R^3,$$



where V is the volume of water displaced by the sphere when its centre is y m below the surface. We can calculate V with the following double iterated integral,

$$\begin{aligned} V &= \int_0^{R+y}\int_0^{\sqrt{R^2-(z-y)^2}} 2\pi x\,dx\,dz = 2\pi\int_0^{R+y}\left\{\frac{x^2}{2}\right\}_0^{\sqrt{R^2-(z-y)^2}} dz \\ &= \pi\int_0^{R+y}[R^2 - (z-y)^2]\,dz = \pi\left\{R^2z - \frac{(z-y)^3}{3}\right\}_0^{R+y} = \frac{\pi}{3}(2R^3 + 3R^2y - y^3). \end{aligned}$$

The resultant force on the sphere when its centre is at depth y is therefore

$$\frac{-9810\pi}{3}(2R^3 + 3R^2y - y^3) + \frac{19\,620}{3}\pi R^3 = \frac{9810\pi}{3}(y^3 - 3R^2y).$$

Newton's second law now gives

$$\frac{4}{3}\pi R^3(500)\frac{d^2y}{dt^2} = \frac{9810\pi}{3}(y^3 - 3R^2y) \quad\Longrightarrow\quad \frac{d^2y}{dt^2} = -\frac{3(9.81)}{2R^3}\left(R^2y - \frac{y^3}{3}\right).$$

33. (a) If x is the length of the longer piece of cable, then Newton's second law for acceleration of the cable is

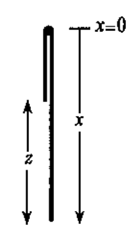
$$25\rho\frac{d^2x}{dt^2} = 9.81\rho z,$$

where ρ is the mass per unit length of the cable. Since $x + (x - z) = 25$, it follows that $z = 2x - 25$ and

$$25\frac{d^2x}{dt^2} = 9.81(2x - 25),$$

or,

$$25\frac{d^2x}{dt^2} - 19.62x = -245.25.$$



The auxiliary equation $25m^2 - 19.62 = 0$ has roots $\pm\sqrt{19.62/25}$. If we denote the positive root by m, then $x(t) = C_1e^{mt} + C_2e^{-mt} + 245.25/19.62$. The initial conditions $x(0) = 15$ and $x'(0) = 0$ require $15 = C_1 + C_2 + 245.25/19.62$ and $0 = mC_1 - mC_2$. These imply that $C_1 = C_2 = 1.25$. The cable slides off the peg when $25 = 1.25(e^{mt} + e^{-mt}) + 245.25/19.62$ and the solution of this equation is 2.59 s.

(b) In this case Newton's second is

$$25\rho\frac{d^2x}{dt^2} = 9.81\rho z - 9.81\rho \quad\Longrightarrow\quad 25\frac{d^2x}{dt^2} - 19.62x = -255.06.$$

The solution of this differential equation is $x(t) = C_1e^{mt} + C_2e^{-mt} + 255.06/19.62$, where m is as in part (a). The initial conditions require $15 = C_1 + C_2 + 255.06/19.62$ and $0 = mC_1 - mC_2$, and these gives $C_1 = C_2 = 1$. The cable slides off the peg when $25 = e^{mt} + e^{-mt} + 255.06/19.62$ and the solution of this equation is 2.80 s.

REVIEW EXERCISES

1. This equation can be separated $\dfrac{dy}{y} = \dfrac{dx}{x^2}$ (provided $y \neq 0$), and a one-parameter family of solutions is therefore defined implicitly by $\ln|y| = -\dfrac{1}{x} + C \implies y = De^{-1/x}$. By letting $D = 0$, we include the solution $y = 0$.

2. This equation can be separated $y\,dy = \left(\dfrac{x+1}{x}\right)dx$, and a one-parameter family of solutions is therefore defined implicitly by $\dfrac{y^2}{2} = x + \ln|x| + C \implies y = \pm\sqrt{2(x + \ln|x|) + D}$.

3. An integrating factor for this linear first-order equation is $e^{\int 3x\,dx} = e^{3x^2/2}$. When we multiply the differential equation by this factor,

$$e^{3x^2/2}\frac{dy}{dx} + 3xe^{3x^2/2}y = 2xe^{3x^2/2} \quad \implies \quad \frac{d}{dx}(ye^{3x^2/2}) = 2xe^{3x^2/2}.$$

Integration now gives

$$ye^{3x^2/2} = \int 2xe^{3x^2/2}\,dx = \frac{2}{3}e^{3x^2/2} + C \quad \implies \quad y(x) = Ce^{-3x^2/2} + 2/3.$$

4. An integrating factor for this linear first-order equation is $e^{\int 4\,dx} = e^{4x}$. When we multiply the differential equation by this factor,

$$e^{4x}\frac{dy}{dx} + 4ye^{4x} = x^2e^{4x} \quad \implies \quad \frac{d}{dx}(ye^{4x}) = x^2e^{4x}.$$

Integration now gives

$$ye^{4x} = \int x^2e^{4x}\,dx = \frac{1}{4}x^2e^{4x} - \frac{x}{8}e^{4x} + \frac{1}{32}e^{4x} + C \quad \implies \quad y(x) = Ce^{-4x} + x^2/4 - x/8 + 1/32.$$

5. The auxiliary equation is $0 = m^2 + 4m + 3$ with solutions $m = -1, -3$, and therefore $y(x) = C_1e^{-x} + C_2e^{-3x} + 2/3$.

6. The auxiliary equation is $0 = m^2 + 3m + 4$ with solutions $m = (-3 \pm \sqrt{7}i)/2$, and therefore $y(x) = e^{-3x/2}[C_1\cos(\sqrt{7}x/2) + C_2\sin(\sqrt{7}x/2)] + 1/2$.

7. This equation can be separated $\dfrac{y\,dy}{\sqrt{1+y^2}} = dx$, and a one-parameter family of solutions is therefore defined implicitly by $\sqrt{1+y^2} = x + C \implies y = \pm\sqrt{(x+C)^2 - 1}$.

8. If we set $dy/dx = v$ and $d^2y/dx^2 = dv/dx$, then $\dfrac{dv}{dx} + \dfrac{1}{x}v = x$, and multiplication by x results in $x^2 = x\dfrac{dv}{dx} + v = \dfrac{d}{dx}(xv)$. Integration now gives $xv = x^3/3 + C$, and hence $\dfrac{dy}{dx} = v = \dfrac{x^2}{3} + \dfrac{C}{x}$. Integration with respect to x now gives $y(x) = \dfrac{x^3}{9} + C\ln|x| + D$.

9. The auxiliary equation $m^2 + 6m + 3 = 0$ has solutions $m = -3 \pm \sqrt{6}$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{(-3+\sqrt{6})x} + C_2e^{-(3+\sqrt{6})x}$. A particular solution is

$$y_p(x) = \frac{1}{D^2 + 6D + 3}xe^x = e^x\frac{1}{(D+1)^2 + 6(D+1) + 3}(x) = e^x\frac{1}{D^2 + 8D + 10}(x)$$

$$= e^x \frac{1}{10[1 + (D^2 + 8D)/10]}(x) = \frac{1}{10}e^x \left[1 - \left(\frac{D^2 + 8D}{10}\right) + \cdots\right] x = \frac{1}{10}e^x(x - 4/5).$$

A general solution is therefore $y(x) = C_1 e^{(-3+\sqrt{6})x} + C_2 e^{-(3+\sqrt{6})x} + e^x(5x - 4)/50$.

10. An integrating factor for this linear first-order equation is $e^{\int 2x\,dx} = e^{x^2}$. When we multiply the differential equation by this factor,

$$e^{x^2/2}\frac{dy}{dx} + 2xe^{x^2/2}y = 2x^3 e^{x^2/2} \quad \Longrightarrow \quad \frac{d}{dx}(ye^{x^2}) = 2x^3 e^{x^2}.$$

Integration now gives

$$ye^{x^2} = \int 2x^3 e^{x^2}\,dx = x^2 e^{x^2} - e^{x^2} + C \quad \Longrightarrow \quad y(x) = x^2 - 1 + Ce^{-x^2}.$$

11. Since the dependent variable x is missing from the differential equation, we use substitutions 15.28 to write $y^2 v\dfrac{dv}{dy} = v \Longrightarrow dv = \dfrac{dy}{y^2}$. A one-parameter family of solutions of this separated differential equation is $v = -1/y + C$, from which $dy/dx = C - 1/y$. This equation can also be separated,

$$dx = \frac{y\,dy}{Cy - 1} = \frac{1}{C}\left(1 + \frac{1}{Cy - 1}\right) dy \quad \Longrightarrow \quad Cx + D = y + \frac{1}{C}\ln|Cy - 1|.$$

12. The auxiliary equation $0 = m^2 - 4m + 4 = (m - 2)^2$ has solutions $m = 2, 2$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2 x)e^{2x}$. A particular solution is

$$y_p(x) = \frac{1}{(D-2)^2}\mathrm{Im}(e^{ix}) = \mathrm{Im}\left[\frac{1}{(D-2)^2}e^{ix}\right] = \mathrm{Im}\left[e^{ix}\frac{1}{(D-2+i)^2}(1)\right]$$
$$= \mathrm{Im}\left(e^{ix}\frac{1}{3-4i}\right) = \mathrm{Im}\left(\frac{e^{ix}}{3-4i}\frac{3+4i}{3+4i}\right) = \frac{1}{25}(4\cos x + 3\sin x).$$

Thus, $y(x) = (C_1 + C_2 x)e^{2x} + (4\cos x + 3\sin x)/25$.

13. The auxiliary equation $0 = m^2 - 4m + 4 = (m - 2)^2$ has solutions $m = 2, 2$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1 + C_2 x)e^{2x}$. A particular solution is

$$y_p(x) = \frac{1}{(D-2)^2}x^2 e^{2x} = e^{2x}\frac{1}{D^2}(x^2) = \frac{x^4}{12}e^{2x}.$$

Thus, $y(x) = (C_1 + C_2 x + x^4/12)e^{2x}$.

14. The auxiliary equation $0 = m^2 + 4$ has solutions $m = \pm 2i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1\cos 2x + C_2\sin 2x$. A particular solution is

$$y_p(x) = \frac{1}{D^2+4}\mathrm{Im}(e^{2ix}) = \mathrm{Im}\left[\frac{1}{D^2+4}e^{2ix}\right] = \mathrm{Im}\left[e^{2ix}\frac{1}{(D+2i)^2+4}(1)\right]$$
$$= \mathrm{Im}\left[e^{2ix}\frac{1}{D(D+4i)}(1)\right] = \mathrm{Im}\left(\frac{e^{2ix}}{4i}x\right) = -\frac{x}{4}\mathrm{Im}(ie^{2ix}) = -\frac{x}{4}\cos 2x.$$

Thus, $y(x) = (C_1 - x/4)\cos 2x + C_2\sin 2x$.

15. The auxiliary equation $0 = m^2 + 4m = m(m + 4)$ has solutions $m = 0, -4$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 + C_2 e^{-4x}$. A particular solution is

$$y_p(x) = \frac{1}{D^2+4D}(x^2) = \frac{1}{4D(1+D/4)}(x^2) = \frac{1}{4D}\left(1 - \frac{D}{4} + \frac{D^2}{16} + \cdots\right)x^2$$
$$= \frac{1}{4D}\left(x^2 - \frac{x}{2} + \frac{1}{8}\right) = \frac{1}{4}\left(\frac{x^3}{3} - \frac{x^2}{4} + \frac{x}{8}\right).$$

Thus, $y(x) = C_1 + C_2 e^{-4x} + x^3/12 - x^2/16 + x/32$.

16. This equation can be separated $\frac{2}{y}\,dy = -\frac{(x+1)^2}{x}\,dx$, (provided $y \neq 0$), and a one-parameter family of solutions is defined implicitly by

$$2\ln|y| = -\int\left(x+2+\frac{1}{x}\right)dx = -\left(\frac{x^2}{2}+2x+\ln|x|\right)+C.$$

Exponentiation leads to $xy^2 = De^{-2x-x^2/2}$. The function $y=0$ is a solution of the differential equation, but because it can be obtained if we permit $D=0$, it is not a singular solution.

17. The auxiliary equation $0 = m^3+3m^2+3m+1 = (m+1)^3$ has solutions $m=-1,\,-1,\,-1$. A general solution of the associated homogeneous equation is $y_h(x) = (C_1+C_2x+C_3x^2)e^{-x}$. A particular solution is

$$y_p(x) = \frac{1}{(D+1)^3}(2e^{-x}) = 2e^{-x}\frac{1}{D^3}(1) = \frac{x^3}{3}e^{-x}.$$

Thus, $y(x) = (C_1+C_2x+C_3x^2+x^3/3)e^{-x}$.

18. The auxiliary equation $0 = m^2+2m+4$ has solutions $m = -1\pm\sqrt{3}i$. A general solution of the associated homogeneous equation is $y_h(x) = e^{-x}[C_1\cos(\sqrt{3}x)+C_2\sin(\sqrt{3}x)]$. A particular solution is

$$\begin{aligned}
y_p(x) &= \frac{1}{D^2+2D+4}\mathrm{Re}(e^{-x}e^{\sqrt{3}ix}) = \mathrm{Re}\left[\frac{1}{D^2+2D+4}e^{(-1+\sqrt{3}i)x}\right]\\
&= \mathrm{Re}\left[e^{(-1+\sqrt{3}i)x}\frac{1}{(D-1+\sqrt{3}i)^2+2(D-1+\sqrt{3}i)+4}(1)\right]\\
&= \mathrm{Re}\left[e^{(-1+\sqrt{3}i)x}\frac{1}{D(D+2\sqrt{3}i)}(1)\right] = \mathrm{Re}\left[\frac{e^{(-1+\sqrt{3}i)x}}{2\sqrt{3}i}x\right]\\
&= -\frac{xe^{-x}}{2\sqrt{3}}\mathrm{Re}\big(ie^{\sqrt{3}ix}\big) = \frac{x}{2\sqrt{3}}e^{-x}\sin(\sqrt{3}x).
\end{aligned}$$

Thus, $y(x) = e^{-x}[C_1\cos(\sqrt{3}x)+C_2\sin(\sqrt{3}x)] + (\sqrt{3}/6)xe^{-x}\sin(\sqrt{3}x)$.

19. An integrating factor for this linear first-order equation is $e^{\int -\tan x\,dx} = e^{\ln|\cos x|} = |\cos x|$. For $\cos x > 0$ or $\cos x < 0$, multiplication by $\cos x$ leads to

$$\cos x\frac{dy}{dx} - y\sin x = \cos^2 x \quad\Longrightarrow\quad \frac{d}{dx}(y\cos x) = \cos^2 x.$$

Integration now gives

$$y\cos x = \int\left(\frac{1+\cos 2x}{2}\right)dx = \frac{x}{2}+\frac{1}{4}\sin 2x + C \quad\Longrightarrow\quad y(x) = (C+x/2)\sec x + (1/2)\sin x.$$

20. An integrating factor for this linear first-order equation in $x(y)$, $\frac{dx}{dy}+3x = -2y^2$, is $e^{\int 3\,dy} = e^{3y}$. Multiplication by e^{3y} gives

$$e^{3y}\frac{dx}{dy} + 3xe^{3y} = -2y^2e^{3y} \quad\Longrightarrow\quad \frac{d}{dy}(xe^{3y}) = -2y^2e^{3y}.$$

Integrate with respect to y now yields

$$xe^{3y} = \int -2y^2e^{3y}\,dy = -\frac{2y^2}{3}e^{3y} + \frac{4y}{9}e^{3y} - \frac{4}{27}e^{3y} + C \quad\Longrightarrow\quad x = -2y^2/3 + 4y/9 - 4/27 + Ce^{-3y}.$$

21. A one-parameter family of solutions of the separated equation $\frac{dy}{y^2} = -\frac{dx}{x+1}$ is given by $-1/y = -\ln|x+1| + C$. For $y(0)=3$, we must have $-1/3 = C$, and therefore the required solution is $y = 3/(3\ln|x+1|+1)$.

22. The auxiliary equation $0 = m^2 - 8m - 9 = (m-9)(m+1)$ has solutions $m = -1, 9$. A general solution of the associated homogeneous equation is $y_h(x) = C_1e^{-x} + C_2e^{9x}$. A particular solution is

$$y_p(x) = \frac{1}{D^2 - 8D - 9}(2x+4) = \frac{1}{-9\left(1 + \dfrac{8D - D^2}{9}\right)}(2x+4)$$
$$= -\frac{1}{9}\left[1 - \left(\frac{8D - D^2}{9}\right) + \cdots\right](2x+4) = -\frac{1}{9}\left[(2x+4) - \frac{16}{9}\right].$$

Thus, $y(x) = C_1e^{-x} + C_2e^{9x} - 2x/9 - 20/81$. For $y(0) = 3$ and $y'(0) = 7$, we must have $3 = C_1 + C_2 - 20/81$ and $7 = -C_1 + 9C_2 - 2/9$. These imply that $C_1 = 11/5$ and $C_2 = 424/405$, and therefore $y(x) = \dfrac{11}{5}e^{-x} + \dfrac{424}{405}e^{9x} - \dfrac{2x}{9} - \dfrac{20}{81}$.

23. The auxiliary equation $m^2 + 9 = 0$ has solutions $m = \pm 3i$. A general solution of the associated homogeneous equation is $y_h(x) = C_1 \cos 3x + C_2 \sin 3x$. A particular solution is

$$y_p(x) = \frac{1}{D^2+9}e^x = e^x\frac{1}{(D+1)^2+9}(1) = e^x\frac{1}{D^2+2D+10}(1) = \frac{1}{10}e^x.$$

Thus, $y(x) = C_1 \cos 3x + C_2 \sin 3x + (1/10)e^x$. For $y(0) = 0$ and $y(\pi/2) = 4$, we must have $0 = C_1 + 1/10$ and $4 = -C_2 + (1/10)e^{\pi/2}$. These imply that $C_1 = -1/10$ and $C_2 = -4 + (1/10)e^{\pi/2}$, and therefore $y(x) = -(1/10)\cos 3x - (4 - e^{\pi/2}/10)\sin 3x + (1/10)e^x$.

24. An integrating factor for this linear first-order equation is $e^{\int 2/x\,dx} = e^{2\ln|x|} = x^2$. Multiplication of the differential equation by this factor gives

$$x^2\frac{dy}{dx} + 2xy = x^2 \sin x \quad \Longrightarrow \quad \frac{d}{dx}(x^2y) = x^2 \sin x.$$

We now integrate to obtain

$$x^2y = \int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

For $y(1) = 1$, we must have $1 = -\cos 1 + 2\sin 1 + 2\cos 1 + C$. Thus,

$$y(x) = -\cos x + \frac{2}{x}\sin x + \frac{2}{x^2}\cos x + \frac{1}{x^2}(1 - \cos 1 - 2\sin 1).$$

25. If $A(t)$ represents the amount of radioactive material in the sample, then $dA/dt = kA$, where k is a constant. This equation is separable, $dA/A = k\,dt$, and a one-parameter family of solutions is defined implicitly by $\ln A = kt + C \Longrightarrow A = De^{kt}$. If A_0 is the original size of the sample (at time $t = 0$), then $D = A_0$, and $A = A_0e^{kt}$. Since $A(5) = 3A_0/4$, it follows that $3A_0/4 = A_0e^{5k} \Longrightarrow k = (1/5)\ln(3/4)$. The sample is reduced to $A_0/10$ when $A_0/10 = A_0e^{kt}$. When we solve for t, the result is $t = -k^{-1}\ln 10 = 40$ years.

26. (a) According to Archimedes' principle, the buoyant force due to fluid pressure is the weight of fluid displaced by the wood, $(1.0 \times 10^{-6})(900)(9.81) = 8.829 \times 10^{-3}$ N. The total force due to fluid and gravity has magnitude $8.829 \times 10^{-3} - (1.0 \times 10^{-6})(500)(9.81) = 3.924 \times 10^{-3}$ N.

(b) If y measures distance from the bottom, then $\dfrac{1}{2000}\dfrac{d^2y}{dt^2} = 3.924 \times 10^{-3} - 2\dfrac{dy}{dt}$. We may integrate this equation, $\dfrac{1}{2000}\dfrac{dy}{dt} + 2y = 3.924 \times 10^{-3}t + C$. Since $y'(0) = 0 = y(0)$, it follows that C must also be zero, and $\dfrac{dy}{dt} + 4000y = 7.848t$. An integrating factor for this equation is $e^{\int 4000\,dt} = e^{4000t}$. Multiplication by this factor gives

$$e^{4000t}\frac{dy}{dt} + 4000e^{4000t}y = 7.848te^{4000t} \quad \Longrightarrow \quad \frac{d}{dt}(ye^{4000t}) = 7.848te^{4000t}.$$

Integration yields

$$ye^{4000t} = 7.848\left(\frac{t}{4000}e^{4000t} - \frac{1}{16\times 10^6}e^{4000t}\right) + D.$$

Consequently, $y(t) = 1.962\times 10^{-3}t - 4.905\times 10^{-7} + De^{-4000t}$. Because $y(0) = 0$, it follows that $0 = -4.905\times 10^{-7} + D$, and therefore $y(t) = 1.962\times 10^{-3}t + 4.905\times 10^{-7}(e^{-4000t} - 1)$ m.

27. (a) With the coordinate system of Figure 15.11, the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + x = 0 \quad\Longrightarrow\quad \frac{d^2x}{dt^2} + 10x = 0,$$

subject to $x(0) = 1/25$ and $x'(0) = 0$. The auxiliary equation is $m^2 + 10 = 0$ with solutions $m = \pm\sqrt{10}i$. A general solution of the differential equation is $x(t) = C_1\cos\sqrt{10}t + C_2\sin\sqrt{10}t$. To satisfy the initial conditions, we must have $1/25 = C_1$ and $0 = \sqrt{10}C_2$. Thus, $x(t) = (1/25)\cos\sqrt{10}t$ m.

(b) In this case the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{5}\frac{dx}{dt} + x = 0 \quad\Longrightarrow\quad \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0.$$

The auxiliary equation is $m^2 + 2m + 10 = 0$ with solutions $m = -1\pm 3i$. A general solution of the differential equation is $x(t) = e^{-t}(C_1\cos 3t + C_2\sin 3t)$. To satisfy the initial conditions, we must have $1/25 = C_1$ and $0 = -C_1 + 3C_2$. These give $x(t) = e^{-t}(3\cos 3t + \sin 3t)/75$ m.

(c) In this case the differential equation describing the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \sqrt{2/5}\frac{dx}{dt} + x = 0 \quad\Longrightarrow\quad \frac{d^2x}{dt^2} + 2\sqrt{10}\frac{dx}{dt} + 10x = 0.$$

The auxiliary equation is $0 = m^2 + 2\sqrt{10}m + 10 = (m+\sqrt{10})^2$ with solutions $m = -\sqrt{10}, -\sqrt{10}$. A general solution of the differential equation is $x(t) = (C_1 + C_2t)e^{-\sqrt{10}t}$. To satisfy the initial conditions, we must have $1/25 = C_1$ and $0 = -\sqrt{10}C_1 + C_2$. Thus, $x(t) = (1/25)(1+\sqrt{10}t)e^{-\sqrt{10}t}$ m.

28. (a),(b) Let us take $y = 0$ on the bridge and y positive downward. During free fall, $\dfrac{1}{100}\dfrac{dv}{dt} = \dfrac{1}{100}(9.81)$. Thus, $v(t) = 9.81t + C$, and the condition $v(0) = 0$ requires $C = 0$. Integration now gives $y(t) = 4.905t^2 + D$. Because $y(0) = 0$, we obtain $D = 0$, and $y(t) = 4.905t^2$. The stone strikes the water when $50 = 4.905t^2$, and this equation implies that $t = \sqrt{50/4.905}$ s. At this instant, its velocity is $v = 9.81\sqrt{50/4.905} = \sqrt{50(19.62)}$. When the stone is in the water,

$$\frac{1}{100}\frac{dv}{dt} = \frac{1}{100}(9.81) - \frac{v}{5} \quad\Longrightarrow\quad \frac{dv}{dt} + 20v = 9.81.$$

An integrating factor for this equation is $e^{\int 20\,dt} = e^{20t}$, so that

$$e^{20t}\frac{dv}{dt} + 20ve^{20t} = 9.81e^{20t} \quad\Longrightarrow\quad \frac{d}{dt}(ve^{20t}) = 9.81e^{20t}.$$

Integration gives

$$ve^{20t} = 0.4905e^{20t} + C \quad\Longrightarrow\quad v(t) = 0.4905 + Ce^{-20t}.$$

Since $v(\sqrt{50/4.905}) = 0.9\sqrt{50(19.62)}$,

$$0.9\sqrt{50(19.62)} = 0.4905 + Ce^{-20\sqrt{50/4.905}},$$

and this equation implies that $C = 1.494\times 10^{29}$. Integration of $v(t)$ yields $y(t) = 0.4905t - \dfrac{C}{20}e^{-20t} + D$. Because $y(\sqrt{50/4.905}) = 50$, we have $50 = 0.4905\sqrt{50/4.905} - \dfrac{1.494\times 10^{29}}{20}e^{-20\sqrt{50/4.905}} + D$, and this requires $D = 49.82$. Thus, $y(t) = 0.4905t - 7.47\times 10^{27}e^{-20t} + 49.82$.

(c) The stone reaches the bottom when $60 = 0.4905t - 7.47\times 10^{27}e^{-20t} + 49.82$. The solution of this equation is 20.75 seconds.

29. (a),(b) The velocity and position functions in Exercise 28 remain valid when the stone is falling in the air. When the stone is in the water,

$$\frac{1}{100}\frac{dv}{dt} = \frac{1}{100}(9.81) - \frac{v}{5} - 1000(9.81)\left(\frac{3}{10^6}\right) \quad \Longrightarrow \quad \frac{dv}{dt} + 20v = 6.867.$$

An integrating factor for this equation is $e^{\int 20\,dt} = e^{20t}$, so that

$$e^{20t}\frac{dv}{dt} + 20ve^{20t} = 6.867e^{20t} \quad \Longrightarrow \quad \frac{d}{dt}(ve^{20t}) = 6.867e^{20t}.$$

Integration gives

$$ve^{20t} = 0.343\,35e^{20t} + C \quad \Longrightarrow \quad v(t) = 0.343\,35 + Ce^{-20t}.$$

Since $v(\sqrt{50/4.905}) = 0.9\sqrt{50(19.62)}$,

$$0.9\sqrt{50(19.62)} = 0.343\,35 + Ce^{-20\sqrt{50/4.905}},$$

and this implies that $C = 1.502 \times 10^{29}$. Integration of $v(t)$ yields $y(t) = 0.343\,35t - \dfrac{C}{20}e^{-20t} + D$. Because $y(\sqrt{50/4.905}) = 50$, we have $50 = 0.343\,35\sqrt{50/4.905} - \dfrac{1.502 \times 10^{29}}{20}e^{-20\sqrt{50/4.905}} + D$, and this requires $D = 50.30$. Thus, $y(t) = 0.343\,35t - 7.51 \times 10^{27}e^{-20t} + 50.30$.
(c) The stone reaches the bottom when $60 = 0.343\,35t - 7.51 \times 10^{27}e^{-20t} + 50.30$. The solution of this equation is 28.25 seconds.