Adjacent Block Interchange Problem

November 2, 2022

1 Intro

Motivation: You have a PDF but it's reversed! You can only move the pages by designating a range of pages and moving them to a specified location.

Operation: Given a list, you may designate a range and a point in that range (between values) and the blocks on either side are (rigidly) swapped. Notation:

$$|_0\ 1\ |_1\ 2\ |_2\ 3\ |_3\ \dots\ |_{n-1}\ n\ |_n$$

Denote a swap of the range a, c about point b (s.t. $a < b < c$) by $[a, b, c]$.

Problem: If the list is reversed, what is the optimal sequence of swaps to sort it?

A clear upper bound is $n - 1$: the sequence of swaps $s_i = [1, i-1, i]$ (for $1 \leq i < n$) reverses the list. Is this the best we can do?

Question: If we permute the swaps above by some permutation $\sigma \in S_{n-1}$ and associate to it the permutation defined on the reversal (so that $id \in S_{n-1}$ is sent to $id \in S_n$), what can we say about this map? It is certainly a map of sets $S_{n-1} \to S_n$ preserving the identity, but is it a group homomorphism? If so, what is the image?

Question: In general, what is the best sorting algorithm using this operation?

1.1 Duality

Observe that if σ is the reversing permutation $(n\ 1)(n-1\ 2)\ldots$ then it is its own inverse. It is fixed under conjugation by itself and, importantly, an operation of the type we are considering is sent to another under conjugation by σ, so conjugation by σ defines an involution on the solution set. Call orbits of solutions under conjugation by σ equivalence classes. Visually, this corresponds to flipping the entire permutation. Algebraically an opertation $[a, b, c]$ on n elements is sent to $[n-c, n-b, n-a]$ (note the order).

2 Optimal sequences

	length	equivalence classes
2	1	1
3	2	2
4	3	10
5	3	1
6	4	21
7	4	4
8	5	57
9	5	5
10	6	77

(Lexicographically earliest)

2. $[[0, 1, 2]]$
3. $[[0, 1, 3], [0, 1, 2]]$
4. $[[0, 1, 4], [0, 1, 3], [0, 1, 2]]$
5. $[[0, 2, 4], [1, 3, 5], [0, 2, 4]]$
6. $[[0, 1, 2], [0, 3, 5], [1, 4, 6], [0, 2, 4]]$
7. $[[0, 2, 5], [1, 4, 7], [0, 3, 5], [1, 4, 6]]$
8. $[[0, 1, 2], [0, 3, 6], [1, 5, 8], [0, 3, 5], [1, 4, 6]]$
9. $[[0, 2, 6], [1, 5, 9], [0, 4, 6], [1, 5, 7], [2, 6, 8]]$
10. $[[0, 1, 2], [0, 3, 7], [1, 6, 10], [0, 4, 6], [1, 5, 7], [2, 6, 8]]$

2.1 Almost symmetric optimal sequences

No self-dual solutions have been found yet. However, for $n = 6$, there are two nicely symmetric solutions.

$$[[0,3,6],[0,2,4],[1,3,5],[2,4,6]]$$
$$[[0,3,6],[2,4,6],[1,3,5],[0,2,4]]$$

$$[[1,3,5],[2,4,6],[1,3,5],[0,1,6]]$$
$$[[1,3,5],[0,2,4],[1,3,5],[0,5,6]]$$