# Adjacent Block Interchange Problem

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## 1 Intro

**Motivation:** You have a PDF but it's reversed! You can only move the pages by designating a range of pages and moving them to a specified location.

**Operation:** Given a list, you may designate a range and a point in that range (between values) and the blocks on either side are (rigidly) swapped. Notation:

$$|_0\ 1\ |_1\ 2\ |_2\ 3\ |_3\ \dots\ |_{n-1}\ n\ |_n$$

Denote a swap of the range $a, c$ about point $b$ (s.t. $a < b < c$) by $[a, b, c]$.

**Problem:** If the list is reversed, what is the optimal sequence of swaps to sort it?

A clear upper bound is $n - 1$: the sequence of swaps $s_i = [1, i - 1, i]$ (for $1 \leq i < n$) reverses the list. Is this the best we can do?

**Question:** If we permute the swaps above by some permutation $\sigma \in S_{n-1}$ and associate to it the permutation defined on the reversal (so that $id \in S_{n-1}$ is sent to $id \in S_n$), what can we say about this map? It is certainly a map of sets $S_{n-1} \to S_n$ preserving the identity, but is it a group homomorphism? If so, what is the image?

**Question:** In general, what is the best sorting algorithm using this operation?

### 1.1 Duality

Observe that if $\sigma$ is the reversing permutation $(n\ 1)(n-1\ 2)\ldots$ then it is its own inverse. It is fixed under conjugation by itself and, importantly, an operation of the type we are considering is sent to another under conjugation by $\sigma$, so conjugation by $\sigma$ defines an involution on the solution set. Call orbits of solutions under conjugation by $\sigma$ equivalence classes. Visually, this corresponds to flipping the entire permutation. Algebraically an opertation $[a, b, c]$ on $n$ elements is sent to $[n-c, n-b, n-a]$ (note the order).

## 2 Optimal sequences

| | length | equivalence classes |
|---|---|---|
| 2 | 1 | 1 |
| 3 | 2 | 2 |
| 4 | 3 | 10 |
| 5 | 3 | 1 |
| 6 | 4 | 21 |
| 7 | 4 | 4 |
| 8 | 5 | 57 |
| 9 | 5 | 5 |

(Lexicographically earliest)

2. $[[0, 1, 2]]$
3. $[[0, 1, 3], [0, 1, 2]]$
4. $[[0, 1, 4], [0, 1, 3], [0, 1, 2]]$
5. $[[0, 2, 4], [1, 3, 5], [0, 2, 4]]$
6. $[[0, 1, 2], [0, 3, 5], [1, 4, 6], [0, 2, 4]]$
7. $[[0, 2, 5], [1, 4, 7], [0, 3, 5], [1, 4, 6]]$
8. $[[0, 1, 2], [0, 3, 6], [1, 5, 8], [0, 3, 5], [1, 4, 6]]$
9. $[[0, 2, 6], [1, 5, 9], [0, 4, 6], [1, 5, 7], [2, 6, 8]]$
10. $[[0, 1, 2], [0, 3, 7], [1, 6, 10], [0, 4, 6], [1, 5, 7], [2, 6, 8]]$

## 2.1 Almost symmetric optimal sequences

No self-dual solutions have been found yet. However, for $n = 6$, there are two nicely symmetric solutions.

$$[[0,3,6],[0,2,4],[1,3,5],[2,4,6]]$$
$$[[0,3,6],[2,4,6],[1,3,5],[0,2,4]]$$

$$[[1,3,5],[2,4,6],[1,3,5],[0,1,6]]$$
$$[[1,3,5],[0,2,4],[1,3,5],[0,5,6]]$$