1. Louis and Harvey are the only two lawyers working in New York City where they engage in Cournot competition. Weekly demand for lawyers is given by $P = 45 - Q$. Louis takes on x amount of cases per week where he faces costs $C_x(x) = 0.5x^2 + 40$. Harvey takes on y cases per week and faces costs $C_y(y) = 20y$.

   (a) What are Louis and Harvey's profit functions? Label them π_x and π_y, respectively.

Louis: π_x

$Q = x + y$

$\pi_x = P \cdot x - C_x(x)$

$= (45 - x - y)x - 0.5x^2 - 40$

$= 45x - x^2 - yx - 0.5x^2 - 40$

$\pi_x = -1.5x^2 - xy + 45x - 40$

Harvey: π_y

$\pi_y = P \cdot y - C_y(y)$

$= (45 - x - y)y - 20y$

$= 45y - xy - y^2 - 20y$

$\pi_y = -y^2 - xy + 25y$

(b) What is Louis's best response function? What is Harvey's best response function?

$$\pi_x = -1.5x^2 - xy + 45x - 40$$

Set $\frac{\partial \pi_x}{\partial x} = 0$ & solve for x^*

$$\frac{\partial \pi_x}{\partial x} = -3x - y + 45 = 0$$

$$\rightarrow 3x = 45 - y$$

$$BR_x = x^* = 15 - y/3$$

$$\pi_y = -y^2 - xy + 25y$$

$$\frac{\partial \pi_y}{\partial y} = -2y - x + 25 = 0$$

$$\rightarrow 2y = -x + 25$$

$$BR_y = y^* = \frac{25}{2} - x/2$$

(c) What is the Nash Equilibrium of this model? How much does each lawyer earn in weekly profit?

$$NE: x^* = BR_x(y^*) \ \& \ y^* = BR_y(x^*)$$

$$x^* = 15 - \frac{1}{3} BR_y$$

$$= 15 - \frac{1}{3}\left(\frac{25}{2} - \frac{1}{2}x\right)$$

$$x^* = 15 - \frac{25}{6} + \frac{1}{6}x^*$$

$$\frac{5}{6}x^* = \frac{65}{6}$$

$$x^* = 13$$

$$y^* = \frac{25}{2} - \frac{1}{2}(13)$$

$$y^* = \frac{25}{2} - \frac{13}{2}$$

$$y^* = \frac{12}{2} = 6$$

$$\pi_x = Px - C_x(x) = 26 \cdot 13 - \frac{1}{2}(13^2) - 40 = 213.5$$

$$\pi_y = Py - C_y(y) = 26 \cdot 6 - 20(6) = 36$$

$$P = 45 - (x+y) = 45 - 13 - 6 = 26$$

2. Two firms engage in Cournot competition. They each face cost curves $C_x(x) = 8x^2$ and $C_y(y) = 6y^2 + 200$. Demand is given by $P = 261 - 4Q$ where $Q = x + y$.

   (a) What is firm X's best response function?

Need Π_x to find BR_x

$$\Pi_x = Px - C_x(x) = (261 - 4(x+y))x - 8x^2$$
$$= (261 - 4x - 4y)x - 8x^2$$
$$= 261x - 4x^2 - 4yx - 8x^2$$
$$= -12x^2 + 261x - 4yx$$

$$\frac{\partial \Pi_x}{\partial x} = 0 \rightarrow -24x + 261 - 4y = 0$$
$$24x = 261 - 4y$$
$$x^* = 10.875 - \frac{1}{6}y = BR_x$$

   (b) What is firm Y's best response function?

Need Π_y to find BR_y

$$\Pi_y = Py - C_y(y) = (261 - 4x - 4y)y - 6y^2 - 200$$
$$= 261y - 4xy - 4y^2 - 6y^2 - 200$$
$$= -10y^2 - 4xy + 261y - 200$$

$$\frac{\partial \Pi_y}{\partial y} = 0 \rightarrow -20y - 4x + 261 = 0$$
$$20y = -4x + 261$$
$$y^* = -\frac{1}{5}x + 13.05 = BR_y$$

(c) What is the Nash Equilibrium of this model? How much profit does each firm earn?

$x^* = 10.875 - \frac{1}{6}y = BR_x$

$x^* = 10.875 - \frac{1}{6}\left(\frac{-1}{5}x + 13.05\right)$

$x^* = 10.875 + \frac{1}{30}x - \frac{13.05}{6}$

$\frac{29}{30}x^* = 10.875 - 2.175$

$\boxed{x^* = 9}$

$y^* = -\frac{1}{5}x + 13.05 = BR_y$

$y^* = 13.05 - \frac{1}{5}(9)$

$y^* = 13.05 - 1.8$

$\boxed{y^* = 11.25}$

Profits ; $P = 261 - 4(x+y) = 261 - 4(20.25) = 180$

$\pi_x = Px - C_x(x) = (180)(9) - 8(9^2) = 1620 - 648 = \underline{972}$

$\pi_y = Py - C_y(y) = (180)(11.25) - (6(11.25^2) + 200) = 2025 - 959.375 = \underline{1065.625}$

(d) Suppose firm X and Y are considering forming a Cartel and splitting the profits. Would either/both of them be better off?
(Hint: $\pi_M = \pi_x + \pi_y = -12x^2 - 10y^2 - 8xy + 261x + 261y - 200$)

Need to find $\frac{\partial \pi_m}{\partial x^*}$ & $\frac{\partial \pi_m}{\partial y^*}$

$\frac{\partial \pi_m}{\partial x^*} = 0 \rightarrow -24x - 8y + 261 = 0 \rightarrow 24x = -8y + 261 \rightarrow x^* = 10.875 - \frac{1}{3}y^*$

$\frac{\partial \pi_m}{\partial y^*} = 0 \rightarrow -20y - 8x + 261 = 0 \rightarrow 20y = 261 - 8x \rightarrow y^* = 13.05 - \frac{2}{5}x$

$x^* = 10.875 - \frac{1}{3}\left(13.05 - \frac{2}{5}x^*\right)$

$x^* = 10.875 - 4.35 + \frac{2}{15}x^*$

$\frac{13}{15}x^* = 6.525 \rightarrow \boxed{x^* = 7.53}$

$y^* = 13.05 - \frac{2}{5}(7.53)$

$\boxed{y^* = 10.04}$

Find $\pi_m/2$:

$\pi_m = -12(7.53)^2 - 10(10.04)^2 - 8(7.53)(10.04) + 261(7.53) + 261(10.04) - 200$

$\pi_m = 2092.53$

$\frac{\pi_m}{2} = 1046.27$; Each firm earns 1046.27.

Firm x is better off

Firm y is worse off

3. Two firms are engaged in Stackelberg Competition. Firm X has the following cost curve $C_x(x) = 3x^2 + 4$ and firm Y faces the following cost curve $C_y(y) = y^2 + 4$. Market demand is given by $P = 60 - Q$.

   (a) What are each firm's best response functions?

$$\pi_x = P \cdot x - C_x(x) = (60 - x - y)x - 3x^2 - 4 = 60x - x^2 - xy - 3x^2 - 4$$
$$= -4x^2 + 60x - xy - 4$$

$$BR_x \Rightarrow \frac{\partial \pi_x}{\partial x} = 0 \rightarrow -8x + 60 - y = 0 \rightarrow 8x = 60 - y$$

$$\boxed{BR_x = x^* = 7.5 - 1/8\, y}$$

$$\pi_y = P \cdot y - C_y(y) = (60 - x - y)y - y^2 - 4 = 60y - xy - y^2 - y^2 - 4$$
$$= -2y^2 + 60y - xy - 4$$

$$BR_y \Rightarrow \frac{\partial \pi_y}{\partial y} = 0 \rightarrow -4y + 60 - x = 0 \rightarrow 4y = 60 - x$$

$$\boxed{BR_y = y^* = 15 - \frac{1}{4}x}$$

   (b) Assume firm X is the leader and firm Y is the follower. What is the Nash Equilibrium of this game? Firm X leading $\rightarrow$ Find x^* to max π_x when $y = BR_y$

$$\pi_x = -4x^2 + 60x - xy - 4 = -4x^2 + 60x - x\left(15 - \frac{1}{4}x\right) - 4$$
$$= -4x^2 + 60x - 15x + \frac{1}{4}x^2 - 4$$
$$\pi_x = -3.75x^2 + 45x - 4$$

$$\frac{\partial \pi_x}{\partial x} = 0 \rightarrow -7.5x + 45 = 0 \rightarrow 7.5x = 45 \rightarrow \boxed{x^* = 6}$$

$$y^* = 15 - \frac{1}{4}x^* = 15 - \frac{1}{4}(6) = 15 - 1.5$$

$$\boxed{y^* = 13.5}$$

(c) Now assume firm Y is the leader and firm X is the follower. What is the Nash Equilibrium of this game? Firm Y leadin $\rightarrow$ Find y^* to max π_y when $x = BR_x$

$$\pi_y = -2y^2+60y-xy-4 = -2y^2+60y-(7.5-\tfrac{1}{8}y)y-4$$

$$= -2y^2+60y-7.5y+\tfrac{1}{8}y^2-4$$

$$\pi_y = -1.875y^2+52.5y-4$$

$$\frac{\partial \pi_y}{\partial y}=0 \rightarrow -3.75y+52.5=0 \rightarrow 3.75y=52.5 \rightarrow \boxed{y^*=14}$$

$$x^* = 7.5-\tfrac{1}{8}y^* = 7.5-\tfrac{1}{8}(14) = 7.5-1.75$$

$$\boxed{y^* = 5.75}$$

(d) In terms of profit, how much does first mover's advantage help firm X? What about firm Y?

NE in Part b $\longrightarrow \pi_x = 131$

NE in Part c $\longrightarrow \pi_x = 128.25$

Firm X was leader in Part b, so first movers advantage is 131 - 128.25 = 2.75

NE in Part b $\longrightarrow \pi_y = 360.5$

NE in Part d $\longrightarrow \pi_y = 363.5$

Firm y led in part c, so their first movers advantage is 363.5 - 360.5 = 3

4. Two firms engage in Bertrand competition where they each face the following cost curve $C(Q_i) = 3Q_i + 3$ where $i = 1, 2$. Market demand is represented by $Q_D = 50 - P$.

   (a) What is the Bertrand Nash Equilibrium?

Both firms have the same costs so we will have P=MC

$$MC = \frac{\partial C(Q)}{\partial Q} = 3 \quad , \text{ so } \quad P^* = 3$$

At $P=3$, $Q_D = 50 - 3 = 47$ & since both firms will be in the market so each produces $\frac{47}{2} = 23.5$

NE is at $P^* = 3$ & $Q_i^* = 23.5$

   (b) ~~What is the Bertrand Nash Equilibrium~~? Why is this a Nash Equilibrium?

Because no firm has an incentive to deviate.

Any price under 3 will lead to neg. profits