AE 5222 – Homework

Due date: Tuesday, 30-Aril-2019 at 5:00 PM.

Instructions

- Solve and submit each problem separately, and as soon as possible. Do not wait until the end of the term to solve all problems. Each submission must include a signed cover page.
- No submissions will be accepted after the specified due date and time.
- Verbal collaboration between students is encouraged; however, the submitted work must reflect your individual efforts. *Do not share software code.*
- Refer to the syllabus for grading policies.
- The solutions to each problem must be presented, *without exception,* in the following format:
  - Method: e.g., "I used these equations [insert equations] from the notes/textbook/other source [insert reference if other than AE 5222 lecture notes/videos], then derived these equations [insert equations]. I developed/reused code to implement these equations [insert code snippets as appropriate], and obtained the following results."
  - Results: e.g. "The following plots/numbers indicate [what they indicate; insert plots / numbers etc]."
  - Discussion: e.g. "The [results above; details] match expected [behavior/values], possibly subject to minor numerical errors, because [reasons why you think the results make sense]" OR "The [results above; details] are not correct because [reasons why you think the results do not make sense]. This method appears to be correct, but I was not able to resolve [issue] despite trying [attempts]."

Problem 1. *Lift* and *drag* are aerodynamic forces acting on an aircraft that are often approximated as follows for a simplified analysis of the gross motion of an aircraft:

$$L = \tfrac{1}{2}\rho V^2 S C_L, \qquad D = \tfrac{1}{2}\rho V^2 S (C_{D,0} + K C_L^2).$$

Here, L and D are the lift and drag forces, respectively, ρ is the atmospheric density, S is the wing surface area, V is the speed of the aircraft (assuming no wind), and C_L is a dimensionless quantity called the *coefficient of lift.* $C_{D,0}$ and K are dimensionless constants.

To maintain steady level flight (i.e., constant speed, constant altitude), the lift must be equal to the weight W of the aircraft, i.e. $L = W$. It is easy to show that the propulsive thrust required to maintain steady level flight is minimized when the lift-to-drag ratio, i.e., the quantity (L/D) is maximum. Due to the relationship between lift and drag, the ratio (L/D) varies with speed.

Approximate data for the Boeing B-52 Stratofortress strategic bomber aircraft are as follows:

$$W = 430,000 \text{ lb}, \qquad S = 4,000 \text{ ft}^2, \qquad C_{D,0} = 0.02, \qquad K = 0.08.$$

We are interested in steady level flight at an altitude of 30,000 ft, where $\rho = 8.9 \times 10^{-4}$ slugs/ft^3.

a) Determine an analytical expression for the speed V^* at which (L/D) is maximized, and an analytical expression for the maximum value of (L/D).

b) Determine the numerical value of V^*, and the numerical value of the maximum (L/D) using the given B-52 aircraft data.

c) Using the given B-52 aircraft data, plot a curve of (L/D) for a range of values of speed. Indicate V^*, as previously computed, on this plot.

Problem 2. An open rectangular box is made by folding a thin rectangular sheet of metal. Find the width, length, and height of a box with volume V that minimizes the area of the required metal sheet.

Problem 3. Minimize the objective function $f(\mathbf{x}) := x_1^3 - 6x_1^2 + 11x_1 + x_3$ in the domain $\mathbf{x} \in \mathbb{R}^3_{\geqslant 0}$ (i.e., the nonnegative octant of $\mathbb{R}^3$) subject to the constraints

$$x_1^2 + x_2^2 - x_3^2 \leqslant 0, \qquad \|\mathbf{x}\| \geqslant 2, \qquad x_3 \leqslant 5.$$

Problem 4. Consider the problem of designing the cross-sectional areas of beams in a three-beam truss shown in Fig. 1. The objective is to minimize the total weight of the truss, which is given in normalized units by the formula

$$\text{Weight} = 2\sqrt{2}x_1 + x_2.$$

The minimum and maximum values for the cross section of any beam are 0.1 and 5.0 units, respectively. The constraints on the design are that the stresses induced in each beam must be lower than prespecified stress limits. Beams 1 and 2 (left and center) experience stresses in tension, given by the formulae

$$\sigma_1 = P\frac{x_2 + \sqrt{2}x_1}{\sqrt{2}x_1^2 + 2x_1x_2}, \qquad \sigma_2 = P\frac{1}{x_1 + \sqrt{2}x_2},$$

where $P = 20$ units. The maximum permissible stress in tension is 20 units. Beam 3 (right) experiences stress in compression, the magnitude of which is given by the formula

$$\sigma_3 = P\frac{x_2}{\sqrt{2}x_1^2 + 2x_1x_2}.$$

The magnitude of maximum permissible stress in compression is 15 units. Determine the optimal cross-sectional areas of the beams in this truss.

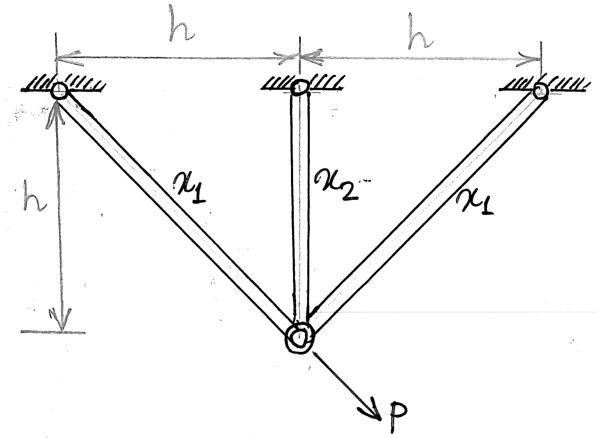

Figure 1

Problem 5. Consider the problem of designing a welded beam shown in Fig. 2. The design variables are the dimensions (all measured in inches) h, b, t, and ℓ indicated in Fig. 2. The design objective is to minimize cost of the total material used, which is computed as the sum of cost for the weld support (shaded region) at \$2.21 per cubic inch, and the cost of the beam at 4.8 cents per cubic inch. The design constraints are:

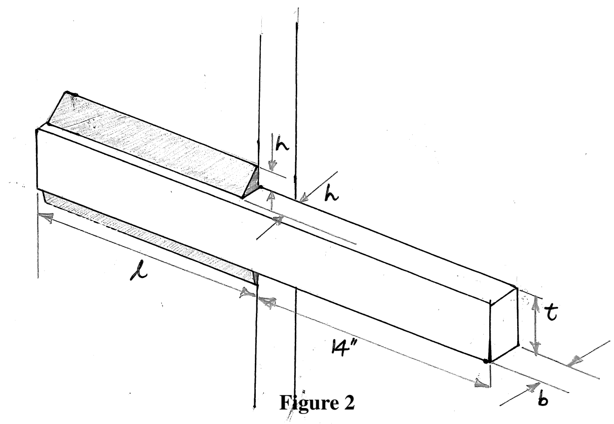

Figure 2

1. The beam should support a buckling load of at least $P = 6,000$ lb. The theoretical expression for buckling load supported is given by

$$\text{Buckling load supported} = \frac{4.013\sqrt{EGt^2b^6/36}}{196}\left(1 - \frac{t}{28}\sqrt{\frac{E}{4G}}\right) \text{ lb.}$$

   Here $E = 30 \times 10^6$ psi and $G = 12 \times 10^6$ psi are constants.
2. The thickness b of the beam must not exceed 2 in.
3. The base of the weld support (i.e., the dimension h) must be at least $\frac{1}{8}^{\text{th}}$ in, must not exceed 2 in, and must not exceed the thickness b of the beam.
4. The width t of the beam must be at least $\frac{1}{10}^{\text{th}}$ in, and must not exceed 10 in.
5. The supported length ℓ must be at least $\frac{1}{10}^{\text{th}}$ in, and must not exceed 10 in.
6. The shear stress in weld must not exceed $\tau_{\max} = 13,600$ psi. The theoretical shear stress in weld is given by

$$\text{Shear stress in weld } = \sqrt{\left(\frac{P}{\sqrt{2}h\ell}\right)^2 + \frac{P}{\sqrt{2}h}\frac{M}{J} + \left(\frac{MR}{J}\right)^2} \text{ psi,}$$

$$\text{where } M = P(14 + \ell/2), \quad R = \tfrac{1}{2}\sqrt{\ell^2 + (h+t)^2}, \quad J = \sqrt{2}h\ell\left(\frac{\ell^2}{12} + \frac{(h+t)^2}{4}\right).$$

7. The bending stress in the beam must not exceed $30,000$ psi. The theoretical bending stress is $84P/(bt^2)$ psi.
8. The end deflection of the beam must not exceed $\frac{1}{4}$ in. The theoretical end deflection of the beam is $\dfrac{10976P}{Ebt^3}$ in.

Find the beam design parameters (i.e., dimensions h, b, t, and ℓ) to minimize cost of total material.

Problem 6. An autonomous forklift in a warehouse is to perform the task of shelving 5 pallets of goods, marked $p_1, p_2, \ldots, p_5$ in the bins where they belong. These bins are marked $b_1, b_2, \ldots, b_5$. Unfortunately, a previous forklift mistakenly placed pallets in wrong bins as follows:

$$\begin{array}{ccccc} p_1 & p_2 & p_3 & p_4 & p_5 \\ b_2 & b_5 & b_1 & b_3 & b_4 \end{array}$$

The new forklift is required to correct the mistake by picking up pallets and placing them in the correct bins. At any given instant of time, the forklift can lift only one pallet, each bin can hold only one pallet, and each pallet must be either in a bin or on the forklift. There is an spare bin b^{T} where pallets can be placed temporarily.

The objective is to enable the forklift to complete this task while traveling a minimum distance. The distances of travel between bins are known.

Identify a graph (i.e., define vertices and edges) that can be used to model this problem.

For the given inter-bin distances, solve this problem.

	b_1	b_2	b_3	b_4	b_5	b^{T}
b_1	–	66	94	64	132	124
b_2	66	–	36	54	70	145
b_3	94	36	–	52	72	140
b_4	64	54	52	–	118	92
b_5	132	70	72	118	–	209
b^{T}	124	145	140	92	209	–

The next two problems are related to the problem of navigating a threat field with minimum threat exposure. A vehicle moves in the square region $\mathcal{W} = [-1, 1] \times [-1, 1]$. We denote by $\mathsf{x} = (x, y)$ the position coordinates of any point in this region. It is common to model threat fields by a series expansion of the form:

$$c(\mathsf{x}) = c_{\mathrm{offset}} + \sum_{n=1}^{N_{\mathrm{P}}} \theta_n \phi_n(\mathsf{x}) \tag{1}$$

where c_{offset} and N_{P} are given constants and $\phi_n(\mathsf{x})$ are prespecified spatial basis functions of the form

$$\phi_n(\mathsf{x}) := \frac{1}{\sqrt{2\pi\nu_n}} \exp(-\frac{1}{2\nu_n} \cdot (\mathsf{x} - \bar{\mathsf{x}}_n)^{\mathrm{T}}(\mathsf{x} - \bar{\mathsf{x}}_n)),$$

where $\nu_n \in \mathbb{R}_{>0}$ and $\bar{\mathsf{x}}_n \in \mathcal{W}$ are given constants for each $n = 1, \ldots, N_{\mathrm{P}}$. In a real-world application, where the "threat" could be, say, a weather system, the parameters θ_n are estimated using measurements of the field. For additional details, see for example the paper by Cooper & Cowlagi available on Canvas. For this assignment, consider θ_n as known constants.

Problem 7. Formulate a grid consisting of N_{G}^2 uniformly placed in N_{G} rows and N_{G} columns. Label these grid points starting from the bottom left, similar to Example 2.4, p. 21 in the lecture notes. The coordinates the i^{th} grid point are denoted by x^i, for each $i = 1, ..., N_{\mathrm{G}}^2$. The vehicle is assumed to traverse grid points according to a "4$-$connectivity rule." i.e., it can travel from a grid point to immediately adjacent grid points in the same row or the same column.

The attached MATLAB® files provides values of the threat field at any point in the region $\mathcal{W}$, and include values of the various constants introduced above. This is the same threat field used in Example 2.4 in the lecture notes.

Write a MATLAB® program that executes Dijkstra's algorithm, and apply this program to find a path in the given threat field from the bottom left corner ($\mathsf{x} = (-1, -1)$) to the top right corner ($\mathsf{x} = (1, 1)$.) Find such a path in three different cases: (1) $N_{\mathrm{G}} = 3$: This will replicate the results of Example 2.4, and help you validate the correctness of your code. (2) $N_{\mathrm{G}} = 15$. (3) $N_{\mathrm{G}} = 101$.

In each case, report (1) the optimal path found, (2) the cost of this path, and (3) the number of iterations for which the algorithm executed. One batch of execution of Lines 3–12 in Fig. 2.3 of the lecture notes is counted as one iteration. **Notes:**

- Your implementation of Dijkstra's algorithm does not necessarily have to be in MATLAB®. You are free use a different programming language.
- If you can find an open-source implementation of Dijkstra's algorithm online in any programming language, you are free to use that implementation. If you do so, you must cite the source from which you obtained this implementation. Do not share code with your classmates.

Problem 8. Consider the problem of selecting a sequence of heading directions of the vehicle to find a "piecewise straight" path. The start and goal locations are the same. We can formulate this problem as an N-stage optimal control problem; the in-class example used $N = 1$. For the following discretized vehicle kinematic model and cost function, write down the set of necessary conditions to find a sequence of headings that minimize the cost.

$$\text{Kinematics: } x(k+1) = x(k) + V\cos\psi(k), \quad y(k+1) = y(k) + V\sin\psi(k), \quad k = 0, \ldots, N-1.$$

$$\text{Cost: } \mathcal{J} = \tfrac{1}{2}(x(N)-1)^2 + \tfrac{1}{2}(y(N)-1)^2 + \sum_{k=1}^{N} c(x(k), y(k)).$$

For $N = 10$ and threat field as in the previous problem (details in MATLAB® file), solve the necessary conditions to find a sequence of headings. □

Problem 9. Use the following inter-city driving distances to solve the traveling salesman problem of visiting each city exactly once while minimizing the total driving distance. The starting city is not given, and any of the five cities is acceptable as the starting city. Determine the optimal sequence of cities and determine the total driving distance.

	Boston, MA	Providence, RI	New York, NY	Albany, NY	Buffalo, NY
Boston, MA	–	51	217	169	454
Providence, RI	51	–	182	163	449
New York, NY	217	182	–	151	373
Albany, NY	169	163	151	–	289
Buffalo, NY	454	449	373	289	–

Table 1: Inter-city distances in driving miles.

Problem 10. A certain process with discrete stages is modeled by the following linear system

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k),$$

where $\mathbf{x}$ is the state of the process, $\mathbf{u}$ is the control input, and A and B are constant matrices of appropriate dimensions. The initial condition $\mathbf{x}(0)$ is given, but the final condition $\mathbf{x}(N)$ is free. A typical control objective is to stabilize the state to zero, while conserving control effort. The trade-off between requiring state stabilization and conservation of control effort is typically captured by the quadratic cost function

$$J = \tfrac{1}{2}\sum_{k=0}^{N} \left(\mathbf{x}^{\mathrm{T}}(k)Q\mathbf{x}(k) + \mathbf{u}^{\mathrm{T}}(k)R\mathbf{u}(k)\right),$$

where Q, R are square positive symmetric definite matrices.

- Write down the necessary conditions for a candidate minimum sequence $\mathbf{x}^*(k), \mathbf{u}^*(k), \mathbf{p}^*(k)$.

- Prove that a solution to the necessary conditions is given by $\mathbf{p}^*(k) = S(k)\mathbf{x}^*(k)$, where $S(k)$ is a square symmetric matrix obtained by solving the difference equation

$$S(k) = A^{\mathrm{T}}\left(S^{-1}(k+1) + BR^{-1}B^{\mathrm{T}}\right)^{-1}A + Q, \qquad \text{for } k = 0, \dots, N-1, \text{ with } S(N) = Q.$$

- Write down numerical values for $\mathbf{x}^*(k), \mathbf{u}^*(k), \mathbf{p}^*(k)$ for the following problem data:

$$N = 5, \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.05 \end{bmatrix}, \qquad R = 1.$$

Problem 11. Consider the problem of navigating a threat field, similar to Problems 7 and 8. The threat field is the same as before. Here we consider a continuous-time kinematic model of the vehicle:

$$\dot{x}(t) = V\cos\psi(t), \qquad\qquad \dot{y}(t) = V\sin\psi(t),$$

where $V = 0.1$ is constant. The cost functional is defined by

$$J(\psi) = \int_0^{t_{\mathrm{f}}} c(\mathsf{x}(t))\mathrm{d}t.$$

The initial time is zero, and the initial position is specified $\mathsf{x}(0) = (-1, -1)$. The final time t_{f} is free, but the final state is specified $\mathsf{x}(t_{\mathrm{f}}) = (1, 1)$. Find

- A differential equation for a candidate optimal control input ψ^* (similar to the Zermelo problem).
- The value of $\psi^*(0)$ such that the corresponding state trajectory satisfies the specified boundary condition $\mathsf{x}^*(t_{\mathrm{f}}) = (1, 1)$.
- Plot ψ^* as a function of time, and plot the trajectory x^* superimposed on a contour plot or intensity map of the threat field.

Problem 12. In Problem 11, the boundary condition is changed. Instead of specifying $\mathsf{x}(t_{\mathrm{f}})$, it is required to lie on a surface, i.e, the following condition is to be satisfied:

$$m(\mathsf{x}(t_{\mathrm{f}})) \equiv (x(t_{\mathrm{f}}) - 1)^2 + (y(t_{\mathrm{f}}) - 1)^2 - 0.05^2 = 0$$

Find the value of $\psi^*(0)$ such that the corresponding *costate* trajectory satisfies the necessary boundary conditions of optimality.

- Plot ψ^* as a function of time, and plot the trajectory x^* superimposed on a contour plot or intensity map of the threat field.
- Plot also the costates as functions of time.

Problem 13. Consider *Goddard's problem* of finding a thrust profile to maximize the altitude of a single-stage rocket. The equations of motion are:

$$\dot{h} = v, \qquad\qquad \dot{v} = \frac{1}{m}\left(f(t) - D(v, h)\right) - g, \qquad\qquad \dot{m} = -\frac{f(t)}{c}.$$

Here, h is the altitude, v is the speed, m is the mass of the rocket, $D(v, h) = \frac{1}{2}\rho_0 e^{-\beta h} v^2 C_{\mathrm{D}} S$ is aerodynamic drag, and f is the thrust, which is the control variable. The constants ρ_0 (atmospheric density at sea level), C_{D} (drag coefficient), and S (surface area), and c (impulse per unit mass) are known. The objective is to maximize $h(t_{\mathrm{f}})$, where t_{f} is free. The quantity $m(t_{\mathrm{f}})$ is specified, whereas $v(t_{\mathrm{f}})$ is free. The rocket is launched from rest at $h(0) = 0$ with initial mass m_0. The minimum and maximum thrust values are, respectively, 0 and $f_{\max}$.

Find a candidate optimal control input, and a candidate optimal control on singular arcs. □

Problem 14. Consider the system

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = -ax_2(t) + u(t),$$

where $|u(t)| \leqslant 1$, and a is a constant. Find a candidate optimal control input that drives the system from the initial state (x_{10}, x_{20}) to the origin and minimizes the cost functional

$$J(u) = \int_0^{t_\mathrm{f}} (k + |u(t)|)\mathrm{d}t,$$

where k is a constant and t_f is free.