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Homework 13

Method

In Goddard's Problem the goal is to maximize the final altitude reached by a rocket, given that t_f is free, the final velocity is free, and the final mass is given. The input in this problem is thrust f(t). Since this is a realistic scenario, naturally there will be a control constraint, defined in this case as $f(t) \in [0, f_{max}]$. This will be more of a concern later.

Since the goal is to maximize the final height, the cost function is defined below. Note that since this only aims to maximize a final value the integral part of the cost function is equal to zero, leaving just the following equation:

$$J(f) = -h(t_f)$$

This cost function along with the three differential equations describing the three states (h, v, m) which were included in the problem statement result in the following Hamiltonian equation. For the sake of reducing clutter I have omitted the dependencies on time for h, v, and m.

$$H(h, v, m, \overline{p, t}) = p_1 v + \frac{p_2}{m} f(t) - p_2 D(v, h) - p_2 g - p_3 \frac{f(t)}{c}$$

Using necessary conditions the following equations are true:

$$\begin{aligned} \frac{d}{dt} p_1^* &= \frac{p_1^*}{m} \frac{\partial D}{\partial h} \\ \frac{d}{dt} p_2^* &= -p_1^* + \left(\frac{p_2^*}{m}\right) \frac{\partial D}{\partial v} \\ \frac{d}{dt} p_3^* &= -\frac{p_2^*}{m^2} \left(f(t) - D(v, h)\right) \end{aligned}$$

In situations where the input u(t) is unconstrained we know that $\frac{\partial H}{\partial u(t)} = 0$. However this problem is different simply because the control input is now constrained. We cannot rely on that assumption since the value of u(t) that satisfies that condition could very well be outside of the control boundaries. Therefore that equation is useless in this case. We must now rely on Pontryagin's Minimum Principle (PMP) to determine optimal control inputs.

PMP concludes that the the optimal control of a system must minimize the Hamiltonian (using admissible control inputs). This means that we must now find all possible equations for thrust throughout the rocket's flight which will minimize the Hamiltonian. Doing so will successfully determine the optimal control input (thrust value) throughout the rocket's flight.

At this point we are concerned solely with the relationship between the thrust input and the Hamiltonian since we want to find control inputs which minimize the Hamiltonian. Therefore it is possible to determine what will minimize the Hamiltonian by focusing only on the variables which are multiplied by thrust. Doing so results in a “slope” term which directly relates the thrust input to the Hamiltonian. I have named this q*(t):

$$q^*(t) \;=\; \left(\frac{p_2^*}{m} + \frac{p_3^*}{c}\right)$$

q*(t) relates to the Hamiltonian as shown below:

$$H \;= q^*(t)\, f^*(t) \;+ (everything\ else)$$

Since we now have a relationship between the optimal control input and the Hamiltonian, we can define values of f*(t) in which the Hamiltonian will be minimized by examining three possible conditions for q*(t). The resulting piecewise function describes the optimal thrust input:

$$f(t) \;=\; \begin{cases} 0 & q^{\cdot}(t) \;>\; 0 \\ f_{\,max} & q^{\cdot}(t) \;<\; 0 \\ singular & q^{\cdot}(t) \;=\; 0\ (\,over\ an\ interval\ of\ \ time\,) \end{cases}$$

Because there could be a possibility that q*(t) is equal to zero over a sustained period of time the thrust input would therefore need to be determined by some function on a singular arc which would minimize the Hamiltonian.

<u>Results</u>

In the singular case we know that q*(t) is equal to zero over time. Therefore we know that all *n* number of time derivatives of this function must also equal zero. If we keep taking the time derivative of q*(t) until f(t) shows up we can therefore find an equation which defines f(t) on the singular arc. Unfortunately, the first time derivative ends up cancelling out f(t). Luckily however the second derivative did end up having an f(t) term. The following equation is the function I found which describes the optimal thrust input on the singular arc:

$$f^*(t) \;=\; D + mg + \frac{m}{\frac{2c^2}{v^2}+4\frac{c}{v}+1}\left[\beta c\,(c-v) - g\,\left(1-\frac{2c}{v}\right)\right]$$

The rocket should switch to this control once q*(t) reaches 0, and should continue until the rocket reaches its specified final mass. At this point the thrust will switch to zero (it doesn't really have any other choice since the propellant mass should be all gone), then the rocket will coast until apogee is reached. Using this optimal control structure the rocket's final height will be maximized.

<u>Discussion</u>

The Goddard problem is a good example of a "bang-singular-bang" optimal control structure. Intuitively one might assume that launching the rocket at maximum thrust might have been the way to maximize height, however once the effects of propellant mass and drag on the rocket's flight are taken into account it becomes clear that the solution is not that simple.

Since we went over the majority of this problem in class I am fairly confident in my approach and final answers to the problem, but of course am aware that I could have made errors while solving it. In all this problem really helped me to develop a practical understanding in Pontryagin's Minimum Principle.