Theories and Frameworks in Student Argumentation in Mathematics

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Comprehensive Exam Question

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Original Prompt: How have theories or frameworks been used by educational researchers to understand (and support) student argumentation in mathematics education settings? Please consider the origins (historical, social, or philosophical) of the theories or frameworks discussed and how they are satisfactory (or not satisfactory) for understanding student argumentation in your response.

Theories and Frameworks in Student Argumentation in Mathematics

Current international curricula and frameworks for mathematics education have advocated for an emphasis on proof and argumentation in K-12 mathematics. Taking a systematic approach to reviewing current research about conceptions, content, and supportive actions for proof and argumentation in K-12 mathematics, this literature review highlights how current mathematical education researchers view argumentation in classes today, particularly in regard to conceptions and implementation of argumentation techniques in K-12 mathematic classrooms. In looking closely at the empirical research that has been most recently published since the emphasis of argumentation via the National Council of Teachers of Mathematics publication of their national standards, firstly in 1989 then again in 2000, this paper highlights both major theories and frameworks that mathematics education researchers have adopted to deeply study the impact of argumentation on K-12 mathematics in the past two decades. This research remains important and relevant in the mathematics education field because the implementation of argumentation in K-12 mathematics is becoming a priority not only in the United States according to published national standards, but also internationally amongst policymakers across the globe (Nardi & Knuth, 2017). Further supported by the Common Core State Standards for Mathematics in 2010, the practice of justification and argumentation in K-12 mathematics warranted deeper empirical research that have transformed and improved argumentation classroom techniques in the classroom. By reviewing the literature systematically, this paper fulfills the purpose of understanding various theories and frameworks that underline current educational researchers’ understanding of K-12 mathematical argumentation, proof, and justification, as well as highlights future considerations of argumentation through a lens that supports all stakeholders in the investment of K-12 student advancement of mathematical understanding and learning.

Background of the Study

In taking a deep dive into the current published research on justification and argumentation in K-2 mathematics, it became evident that practitioners, researchers, and other stakeholders in the mathematics education community consider the concept of argumentation and the implementation of teaching practices that support argumentation in different manners and at different levels of priority (Campbell, Boyle, & King, 2019). Whereas argumentation has taken a priority in mathematics education in the past twenty decades, several different researchers emphasize various aspects of the process of argumentation, whether it be the social aspect via the discourse that students engage in (Berland & McNeill, 2010; Evagorou & Osborne; 2013, Knight & McNeill, 2015) or the advancement of the level of mathematical arguments (Harel & Sowder, 1998; Stylianides & Stylianides, 2009), as mere examples. No matter the emphasis of the various research studies that have previously been conducted concerning K-12 mathematics argumentation, a concentration of frameworks and theories have regularly served as the foundations of these research studies. As a result of this analysis of research studies, this paper will report on the current state of mathematics education research about the practice of teacher implementation of proof, justification, and argumentation techniques for K-12 mathematics. Further, it will expound on future directions that this research can take, carefully considering past research and areas that can be further explored and the theoretical and framework foundations supporting them. Finally, as a result of this research, research can begin to take shape to inspire best practices to most effectively impact practices for current mathematics teachers and practitioners in the classroom.

Framework

As one member of a team of three mathematics education researchers, I situate this review in the context of two recent literature reviews performed by Stylianides, Stylianides, and Weber (2017) and one working group session by Staples, Newton, Kosko, Cirillo, Weber, Vieda, and Conner (2016). Stylianides et al. (2017) reviewed proceedings of papers for the annual conference of the Psychology of Mathematics Education (PME) written between 2005-2015 related to proof and argumentation. They chose to focus on proof and argumentation because argumentation and proof are closely related, and the consideration of both argumentation and proof together emphasizes a wider range of mathematical processes than if considered individually. Aligning with this sentiment, our review similarly considers articles related to proof and argumentation. In their review, these authors noted themes related to student conceptions, classroom-based research, and teacher knowledge. Following this review, Stylianides et al. reviewed literature related to proof from three differing perspectives. The three perspectives included proving as a form of problem-solving, proving as convincing, and proving as a socially-embedded activity. Staples et al. formed a working group at the annual conference for the North American Chapter of the Psychology of Mathematics Education (PME-NA) to determine the perspectives of what the field calls argumentation, justification, and proof. Through their interactions, they noted there was some variability in how the field understood argumentation and proof. In noting the different conceptualizations of argumentation and proof in K-12 mathematics, this study reveals the various frameworks for research study in this field.

**Purpose**

The purpose of this review, then, is to summarize theoretical approaches and frameworks of published research on K-12 mathematics teachers’ implementation of argumentation in their classrooms, emphasizing various aspects of argumentation and highlighting their theoretical approaches to the field. The goal of this review is the begin to dissect the research base on K-12 mathematics that supports the conceptualization of arguments for K-12 mathematics as well as the student development of mathematical argumentation, with the hopes of making sense of the background and the foundational understanding of mathematics argumentation at the K-12 level. Therefore, implications for future areas of research will also be provided. Particular attention will be given to contexts and classroom settings in which empirical research studies about student argumentation and mathematical argumentation were conducted and the impact that this research has had on different stakeholders in the mathematics education community.

Methodology

In this review of literature, we used a systematic review methodology (Cooper, 2017; Hannes & Claes, 2007) to find and analyze research studies on K-12 proof and argumentation. The protocol for a systematic literature review entails creating a list of search terms and choosing databases in which the search will be run, running a search and gathering all articles which contain the search terms, and using a screening process to choose articles based on a predetermined set of criteria. After consulting with an education research librarian, we ran an initial search in three databases: ERIC EBSCOhost, PsycINFO, and Education Full Text (H.W. Wilson). Due to the release of the NCTM recommendations for reasoning and proof in 2000, we considered articles written after the year 2000. Therefore, the following limiters were applied to the initial search: published between January 1, 2000 and May 16, 2018 and peer-reviewed. We created a bank of search terms using words commonly found in nationally accepted curriculum documents (NCTM, 2000; CCSSI, 2010; Department of Education, 2013, literature reviews (Stylianides, 2016; Stylianides et al, 2017), working groups (Staples et al, 2016), or research studies which claim to examine proof and argumentation in mathematics. While it is likely impossible to consider all terms related to proof and argumentation, the chosen bank of terms is found often and well-represented in the literature, and the terms are used in conjunction with the words *proof* and *argumentation*. We retrieved articles that included the search terms appearing in any location of the article.

After retrieving an initial 8,094 articles, we transported each article from the research databases to Refworks, a computer tool to create bibliographies. Then, we removed all duplicates, leaving a remaining 5,263 articles. After deleting duplicates, we transported the citations to Google Sheets for purposes of screening. We screened the title and abstract of all 5,263 articles using the following inclusion criteria: content (the focus of the study must be on proof or argumentation), participants (research subjects must be K-12 students), empirical(the study must only report empirical finding), publication type (the publication must be a journal article), and language (the article must be written in English). Screening involved testing for overall inclusion through three phases: the article title met the inclusion criteria, the abstract of the article met the inclusion criteria, and the article met the inclusion criteria after a full scan of the article. After the first phase, 902 articles remained to go through the abstract screen. After the second phase, 119 articles remained to go through a full scan. Upon completion of the third phase, 73 articles remained for inclusion in the systematic review. I double coded 10% of the articles at phase 1 and phase 2 of the screening process to calculate inter-rater reliability (as calculated by dividing the number of agreements by the sum of agreements and disagreements and multiplying by 100) with a cutoff set for 85% agreement. For phase 1, the inter-rater reliability was 90.7%, and for phase 2 the inter-rater reliability was 91.1%. After screening articles for inclusion, we followed the same screening process to evaluate references of the 73 articles. The ancestral search yielded 3 more articles for a total of 76 articles included in this review.

**Results**

In completing this systematic review, I recognized that mathematics argumentation was generally categorized into three different theoretical perspectives. Within each of these perspectives, various research has found different frameworks to guide their research. As a result of this review, the following results summarize main theories behind these views of mathematics argumentation and the historical perspectives of these findings.

**Argumentation for Problem Solving**

Much literature is devoted to understanding K-12 mathematics students’ proving and argumentation knowledge and capabilities (e.g. Harel & Sowder, 1998; Healy & Hoyles, 2000; Flores, 2006; Lin, et al., 2004; Liu & Manouchehri, 2013; Martin & Harel, 1989; Weber, 2001). Scholars claim that students often rely on either the authority of an expert in the field or previously-proven or previously-accepted mathematical concepts (Flores, 2006; Healy & Hoyles, 2000; Lin, et al., 2004; Martin & Harel, 1989). As students learn to engage in argumentation that does not rely on other truths or other people, students are engaging in the act of problem-solving as they move through the process of solving a problem by defending a claim by backing it with evidence and critiquing the reasoning of others.

To take a deeper look at how students choose to create an argument and defend their thoughts with others, several seminal mathematics education research studies about proof and argumentation are widely cited for confirming deficiencies in K-16 students’ proving practices (Healy & Hoyles, 2000; Martin, 1989; Weber, 2001). Research is beginning to uncover that students and teachers alike often rely on their written final submissions as proof of student understanding through argumentation, whereas student oral argumentation is just as powerful at allowing students to solve problems through the process of argumentation in their discourse out loud with one another or in explanation to a teacher or other mathematical authority. Students’ written understandings are not always reflective of their cognitive knowledge (Evens & Houssart, 2004). Thus, the framework of argumentation as problem-solving gives credibility and importance to future studies of student engagement in oral argumentation, which also provides foundation to the second view of argumentation, which is argumentation as the act of convincing.

**Argumentation for Convincing**

The view of argumentation for the sake of convincing has led to several well-established and well-accepted frameworks for measuring K-12 mathematics arguments tangibly on a hierarchy of student understanding. These frameworks largely reflect on accuracy of the mathematical concept as well as what is socially and academically appropriate for the students at those levels, which guides many of the frameworks (Balacheff, 1988) used in mathematics education today.

One example of a commonly accepted mathematical argumentation framework is the proof scheme of Harel and Sowder (1998), who developed a holistic view of students’ proof and argumentation schemes by observing the proving practices of mostly college students, along with a case study of one junior high school student. In their analysis, they concluded that students’ proof schemes fell within three categories: external conviction, empirical, or analytical. The external conviction proof scheme occurs when a student relies on authority such as a teacher, a parent, or a textbook to create a mathematical argument. As an example, Harel and Sowder (1998) found that students often believed proofs should contain complicated language or symbols because they often saw sophisticated arguments in textbooks or other authoritative domains. They also found that students used authority figures as warrants for claims by using language such as ‘because my teacher said so’. Other scholars similarly found authority to be a proving scheme exhibited by students of all ages (Flores, 2006; Fried & Amit, 2008; Sen & Guler, 2015). The empirical proof scheme refers to student usage of examples, experiences, or perceptions to justify a claim (Harel & Sowder, 1998). Students use this proof scheme when they attempt to show that a conjecture is true by verifying a finite number of cases. Often a first step observed in some research studies (King & Campbell, 2019), some students rely on an empirical proof scheme to begin their argumentation reasoning processes before being able to advance into a more advanced proof scheme. Lastly, the analytical proof scheme is defined as arriving at a conclusion by using a sequence of logical deductions (Harel & Sowder, 1998). In current K-12 mathematics, particularly at the 9-12 level, students prove that a statement is true in mathematics by using axioms, definitions, and previously proved theorems to create a logical argument. According to the standards published by NCTM (2000), students should develop analytical proof schemes by the time they reach high school, although it is regularly encouraged in these standards across all levels of K-12 mathematics.

Though Harel and Sowder (1998) warned against treating their taxonomy as hierarchical structure, they acknowledge that “there is often at least a partially hierarchical nature implicit in the categories” (p. 277). In fact, several scholars have used Harel and Sowder’s (1998) taxonomy as a hierarchical rubric for determining how students’ arguments change over time (e.g. Ellis, 2007; Flores, 2006; Lee, Chen, & Chang, 2014; Liu & Manouchehri, 2013; Stylianou, 2013). This framework of convincing has been used in recent research concerning the role of interpersonal discourse among students on mathematics argument development (King & Campbell, in press). Descriptions of the levels and types of arguments are seen in Figure 1 below.

**Figure 1**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Level 1** | | **Level 2** | | **Level 3** |
| Empirical | Appeal to Authority | Unsuccessful Valid Argument | Valid Argument, Not a Proof | Proof |

Because the act of convincing has become a widely accepted goal of argumentation in K-12 mathematics, it is important to note that convincing refers back to the content of mathematics, as the engagement of students in discourse with one another for reasoning, critique, and justification leads students to a deeper understanding of the mathematics so that they can gain authority of their own learning of the mathematics (Stein, Engle, Smith, & Hughes, 2008). Students often take ownership or authority of their own mathematical learning when they take on the role of the one who does the convincing, either of himself or of another student.

Of course, convincing oneself or another of the validity of a mathematical claim in argumentation does requires students to engage with the content and often with one another according to the standard of mathematical practice set by NCTM (2000), where students not only create viable mathematical arguments, but they also critique the reasoning of others. Thus, argumentation as convincing does emphasize student ownership of the learning but also recognizes the need to do this in the social practice of argumentation with one another as students engage in discourse with one another.

**Argumentation as a Socially-Embedded Practice**

Most recently in mathematics education research, scholars have recognized argumentation as a socially-embedded practice, as research is highlighting the necessity of studying student discourse with one another. Primarily, this is in the rise of mathematics education research because of the recent emphasis on student voice because of the reformed mathematical practices that place the authority of learning on the student (Stein et al, 2008). Argumentation as a socially embedded practice looks at both the role of students individually as they engage in discussion as well as collectively as they make up a mathematical learning community.

Creating an environment that encourages student collaboration and communication, particularly in the context of developing mathematics arguments and proofs, opens up spaces for students to test their ideas publicly through a classroom discussion. When done carefully, Lampert (1990) argues that teachers can monitor student learning in the context of their discussion. However, this recommended consideration of argumentation is not regularly practiced in mathematics classrooms, as students struggle with reasoning and justification due to a lack of opportunities to engage in problem-solving activities that encourage development of reasoning skills that enhance mathematical arguments (Mueller & Yankelewitz, 2014). The idea of using collaboration to develop mathematical arguments and solve mathematical problems is a research-based student-centered approach to allow for mathematical argumentation development in the classroom, because it allows students to build knowledge, develop metacognition, and develop higher-order thinking skills (Cáceres, Nussbaum, Marroquín, Glesiner, & Marquínez, 2018). Fostering meaningful and purposeful collaboration that supports student growth in argumentation is a social practice that fosters student learning in the context of mathematical argumentation development. One primary example of student discussion as a social practice is the idea of using the classroom as a learning community, which can go deeper in the context of mathematical argumentation as they develop their own communal criteria by which they can judge arguments and proofs.

One way to balance authority in the classroom is to allow students to engage in the process of deciding what counts as proof in a specific setting. Mathematicians rely on the broader mathematical community to decide what does and does not count as proof (Harel & Sowder, 2007), and the criteria for what counts as proof is socially negotiated among the field of mathematics education. However, students rarely understand the criteria for what counts as proof in a given context (Harel & Sowder, 2007), so recently, scholars suggested that students should negotiate communal criteria for proof to engage them in practices similar to those of mathematicians (Boyle, Bleiler-Baxter, Yee, & Ko, 2015; Stylianides & Stylianides, 2009; Yee, Boyle, Ko, & Bleiler-Baxter, 2018).

Stylianides & Stylianides (2009) first implemented the idea of communal criteria with prospective elementary teachers in a mathematics education course on proof. The class socially agreed upon three criteria for what constitutes a proof, but it is important to note that the teacher acted as the representative of the mathematical community by helping to shape the criteria and making sure it met certain standards for proving. A major finding from this study is that teachers relied less on empirical arguments over time as a result of explicating the criteria for proof. Additionally, preservice mathematics teachers that used empirical arguments were aware of their limitations.

Looking at an example from collegiate mathematics instruction, Yee et al. (2018) designed an instructional sequence in undergraduate mathematics classes wherein the classroom community developed communal criteria and judged other classmates’ arguments based on the criteria. Similar to Stylianides and Stylianides (2009), they found that explicitly stating the criteria for proof aided students in creating increasingly sophisticated arguments over time. In another study, Boyle et al. (2015) found that the undergraduate mathematics students in his study changed their perceptions of proof as a result of engaging with communal criteria. Students learned that proof is socially negotiated, and they gained further clarity for what counted as proof.

**Looking Forward to the Future: Implications for Future Research and Practice**

Although much research has been completed on the theories and frameworks for argumentation in K-12 mathematics and the usage of mathematical arguments to support student learning, there are still more implications for future research and practice that remain. Recognizing argumentation as both a pedagogical practice, a student learning process, and an end product and goal for K-12 mathematics, future research can be broken down into different studies that highlight the student participation as well as their advancement of mathematical thinking through tools and teaching strategies developed as a result of current mathematics education research.

As a relevant example evident in the social approach to mathematics argumentation, the use of student discourse with one another can appropriately take on a more discursive theoretical approach in a social theoretical foundation to better study and understand the aspects of dialogue that students have with one another to better support student advancement of mathematical arguments. Even further, research can then utilize a framework that has been previously established, or one created with the inspiration and research-based evidence of other argumentation frameworks for mathematics education, to study how the social practice of student discourse can impact student understanding as arguments develop and increase in level through these argumentation frameworks, emphasizing argumentation more for problem-solving and for convincing. A hierarchical framework such as Healy and Hoyles (2000) helps emphasize how students go through the process of problem-solving to engage in convincing that truly defines mathematical understanding for the sake of student understanding of mathematics at the K-12 level.

Thus, overall argumentation by nature has historically taken on several different frameworks for research study, and different theories have served as foundations for these studies on mathematics argumentation. Because of the broad nature and the recent emphasis on argumentation in the context of K-12 mathematics, mathematics education research has viewed argumentation in several different lights, requiring various theories and frameworks as foundations to the recent research. More research can move forward with these theoretical and framework underpinnings to uncover more practical approaches to K-12 mathematics argumentation.

**Conclusion**

This research study served the purposes of understanding how educational researchers in mathematics education conceptualize argumentation for K-12 mathematics through the frameworks and theories in which they ground their studies. It is important to recognize that argumentation is recognized as both a content-specific practice that emphasizes the process of problem solving, the social activity of discourse within a learning community, and the end product of learning to convince oneself and another of a proof or claim to advance mathematical thinking and understanding. By studying mathematical argumentation through these theoretical frameworks, future research can utilize similar or perhaps different frameworks to unlock new studies that will positively impact classroom practices of argumentation in K-12 mathematics as research turns into and informs practitioners about best practices for the future of mathematics argumentation at the K-12 level.

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