The Role of Interpersonal Discourse on K-12 Mathematical Argumentation

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Comprehensive Exam Question

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Original Prompt: How does student discourse with one another impact K-12 student ability to create mathematical arguments?

Supporting K-12 Student Development of Mathematical Argumentation

Creating an environment that encourages student collaboration has long been considered an important practice in secondary mathematics learning. The National Council for Teachers of Mathematics (2000) and the Common Core State Standards for Mathematics (2010) recommended principles and standards for grades preK-12, which include the ability to communicate mathematical thinking coherently and clearly to peers, teachers, and others. Particularly, in the context of developing mathematics arguments and proofs, NCTM (2000) recommends students test their ideas publicly through a classroom discussion. In these public discussions, the teacher can monitor student learning (Lampert, 1990). However, this recommended consideration of argumentation is not regularly well-practiced in mathematics classrooms. In the context of K-12 mathematics, mathematical argumentation is largely accepted as students participating in discursive communication to develop and make sense of mathematics as they inquire about the reasoning of others and justify their own (Ball & Bass, 2003; Rojas-Drummond & Zapata, 2004). Students often are weak in their argumentation skills in this context, as they struggle with reasoning and justification due to a lack of opportunities to engage in problem-solving activities that encourage development of reasoning skills that enhance mathematical arguments (Mueller & Yankelewitz 2014). Teachers often use collaboration as a regular tool in the K-12 classroom, and the standards obviously support the idea of argumentation, but the use of collaboration for true argumentation is an underutilized tool in promoting student problem solving and mathematical reasoning via argumentation. Further, it still remains a challenge to effectively utilize collaboration in any K-12 setting because of the dynamics of working with various student opinions and mathematical abilities, but collaboration can be a tool in K-12 education to unlock students’ abilities to deeply analyze the mathematics. The idea of using collaboration to develop problem-solving skills is a research-based, student-centered approach to allow for mathematical argumentation development in the classroom, because it allows students to build knowledge, develop metacognition, and develop higher-order thinking skills (Cáceres, Nussbaum, Marroquín, Glesiner, & Marquínez 2018). Fostering meaningful and purposeful collaboration that supports student growth in these skills requires certain criteria for successful lessons at the secondary level.

Purpose of the Study

Although the power of student discourse is recognized and the importance of mathematical argumentation is emphasized in both national standards and recommended practices of practitioners and researchers alike across several national mathematics education organizations, there still remains extensive research to be completed to support the role that discourse plays in promoting the advancement of K-12 student mathematical arguments (Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017). This study analyzes the roles of collaboration and discourse K-12 mathematics classrooms, and the impact and conceptualization of discourse on students’ development of mathematical arguments. Further, the project will study the act of argumentation as a social practice and recognize the specifics of collaborative discussion that impact students to advance in their abilities to create their own mathematical arguments. Finally, this study will conclude with recommendations for future research and practice to support K-12 teacher understanding of how best to encourage meaningful discussion among their students in the form of interpersonal discourse to promote mathematical argumentation.

Review of the Literature

The concept of student collaboration through discursive activities has been a regular practice in K-12 mathematics classrooms. As students work with one another, they are given the chance to share their thinking with one another, which provides them the opportunity to individually and collectively approach problem-solving in a way that both challenges and supports all types of students and different learning styles (Hodge & Walther, 2017). To emphasize student thinking and improvement of their own mathematical processes and thinking, collaborative efforts among students emphasize their discourse with one another and the impact that student conversations have on student learning and progression of mathematical thinking. According to both researchers and practitioners in K-12 mathematics, mathematical discourse lies at the intersection of social, emotional, and mathematical intelligence (Bertolone-Smith & Gillette-Koyen, 2019). Thus, viewing mathematical discourse, as a social practice highlights the value of student discourse in the promotion of national standards and mathematical practices, particularly the creation of mathematical arguments. A primary focus in K-12 mathematics classrooms by national mathematics education organizations such as the National Council of Teachers of Mathematics and the Common Core State Standards, both NCTM (1989) and CCSS (2010) recommend that reasoning and proof should regularly be incorporated in K-12 classrooms. Although argumentation has been established as a recommendation for K-12 mathematics, classroom teachers still find it difficult to promote student discourse so that students successfully engage in argumentation through justification and generalization (Melhuish, Thanheiser, & Fagan, 2019). Thus, this review of literature delineates the role of student collaboration through the lens of student discourse and the current state of mathematics education in utilizing discourse for argumentation in K-12 mathematics, while also providing implications for the K-12 mathematics education community and recommendations for future practice and research.

**The Concept of Student Collaboration**

The concept of student collaboration establishes classrooms as communities of inquiry where students are able to build knowledge through collaborative group work and the dialogue with one another (Kendrick, 2010). In fact, a classroom that successfully facilitates student collaboration maintains that the teacher facilitates student thinking through orchestrating student-to-student interactions in a climate that promotes such collaboration (MacDougall, 2013). Within the mathematics education community, it has become a common teaching practice to facilitate classrooms that promote an environment that not only encourages students to discuss with one another in collaborative efforts, but in fact requires it through the design of the classroom activities or experiences. In this design, students share their ideas about the content concepts with each other, particularly in STEM fields such as science and mathematics (MacDougall, 2013), especially as students are required to justify their thinking and reach a consensus.

The idea that student collaboration leads to justification via evidence and the necessity of critiquing the reasoning of others’ arguments or ideas directs back to the standards of mathematical practice encouraged and published by the National Council of Teachers of Mathematics (NCTM, 2000) and the Common Core State Standards for Mathematics (CCSSM, 2010) that establish the standards and widely-accepted practices of K-12 classrooms today. As students work with one another to justify their own arguments and critique each other’s ideas in alignment with these recommended practices, K-12 STEM teachers highlight the necessity of empowering students to be able to use their own strengths via student collaboration by appropriately implementing these practices that keep students accountable and hold them to the expectations not only of behavior in the classroom, but also of high standards for academic learning in the context of the content of the classroom lesson (McGlynn & Kozlowski, 2016). Because both creating and critiquing an argument in a K-12 STEM classroom is both academically challenging and socially engaging, standards for accountability for the classroom task hold students accountable to their individual work and to the group as they work together towards this mathematical practice of mathematical argumentation. McGlynn and Kozlowski (2016) recognize that collaboration among students in their groupwork with one another requires students to hold each other accountable to be involved, to have a valid role or job with a set standard of completion, to be invested in the task or learning goal, and to maintain accountability both individually and collectively. In the context of mathematical argumentation, the creation of an individual argument and the critique of another’s argument provides students the valid role with a set standard of completion (both creation of an argument and a critique to challenge the justification of one’s rationale) while also holding students accountable to themselves and to each other as they create and critique each other. By the design of this task, students must be invested in the task because the claim of a mathematical argument sets the mathematical standard and the learning objective for the mathematics lesson. Overall, collaboration is seen as an action that promotes students to work with one another and maintains high standards of investment and academic work from all students both individually and collectively as they work in groups together.

**The Concept of Student Discourse**

The concept of student discourse is prioritized by the National Council for Teachers of Mathematics (2000) in the establishment of the communication standard that states: “Instructional programs from pre-kindergarten through grade 12 should enable all students to organize and consolidate their mathematical thinking through communication, and communicate their mathematical thinking coherently clearly to peers, teachers, and others.” As researchers have studied the power of student discourse in mathematical learning, they have come to realize that the social activity of student collaboration and communication creates mathematical understanding from mathematical context. More specifically, the power in student discourse lies in the transformation of language of/for mathematics into knowledge of mathematics that is gained socially, known as “sociomathematical knowledge,” through the classroom activity that is implemented by the classroom teacher (Ochs, 1998). This concept of transformation of mathematics language into mathematical understanding through carefully-facilitated instruction by the K-12 teacher has guided several specific practices that emphasize a greater understanding and an end goal of acquisition of mathematical content.

However, research is more recently emphasizing the importance of student discourse as the end product of mathematics-rich student conversations in themselves. The recognition of discourse as a social practice ties together the social aspect of student discourse with the mathematical content, which remains the goal in K-12 mathematics education research. A closer look at student discourse as students work in small groups with one another unpacks the role that discourse has on student development of mathematical understanding according to the national standards and mathematical practices that are regularly expected of K-12 mathematics classrooms.

**Using Collaborative Student Discourse in Mathematical Argumentation**

In looking at student discourse as a vehicle to the development of mathematical arguments, Kuhn and Udell (2003) suggest that argumentation is a skill that is not only essential to the critical thinking required in formal education, but also as a means to developing thinking skills in general. The basis in this concept is that the ability to propose, critique, and defend a position is universal across all domains of knowledge, mathematics-related or not. Thus, it is necessary that students learn to engage in topics of discussion with one another where they are required to not only pose their own thoughts, but also then critique the reasoning of others, as recommended by the NCTM Standards (200) and the CCSS Standards (2010). The call for collaborative student discourse for the sake of argumentation is recognized and heavily emphasized in current mathematics education.

To expound on the necessity of mathematics collaboration for argumentation, discourse analysis reveals that students are capable of using, learning, and expounding their thoughts about mathematical concepts in a variety of different ways, particularly as they discuss mathematics with one another in classroom activities (Huang, Normandia, & Greer, 2005). More specific, students are able to describe, sequence, and choose their thoughts about mathematical processes as they engage in discourse with one another initially, and as teachers intervene with guided practices for promote advancement in mathematical thought and argument development, students are able to reference other truths about mathematical content, explain a method, and/or justify a decision made for a method or solution in their mathematics (Huang et al, 2005). Thus, appropriate understanding and conceptualization of discourse for the development of argumentation to promote student mathematical learning continues to be an area of research that can impact K-12 mathematics instruction.

**Argumentation in the Development of Student Mathematical Learning**

Though argumentation has consistently been a recommended practice of K-12 mathematics, students are generally unable to produce valid arguments (Stylianou, Blanton, & Knuth, 2010), and current methods of teaching proofs and arguments are largely inaccessible to K-12 students (e.g. Karunakaran, Freeburn, Konuk, & Arbaugh, 2014). In several K-12 mathematics classrooms, students learn to create arguments simply by watching the teacher’s approach and attempting to recreate the same process. This type of instruction leads students to believe that the teacher is the supreme authority of an argument in terms of what is acceptable and what is not. Thus, students develop an authoritarian proof scheme (Harel & Sowder, 1998), wherein the process of developing a proof becomes a computational exercise to find the specific solution for which the teacher is searching.

In contrast to the traditional teaching approach to teaching mathematical arguments, encouraging collaboration and allowing students to engage in the process of proving can improve students’ proof development (e.g. Brown, 2017; Byrne, 2013; Yee, Boyle, Ko, & Bleiler-Baxter, 2017). Collective argumentation, as one example, allows students to engage in a process where they collaborate to create arguments and come to agreement on the arguments that can be accepted within a community. Students create arguments within a group and present their work to the class, helping students gain authority (Stein, Engle, Smith, & Hughes, 2008) in their work and participate in authentic mathematical community.

**Using Collaboration in Argumentation in the Mathematics Classroom**

Research has shown that mathematical communication within a classroom community is crucial for the development of students’ reasoning and mathematical understanding (Alrø & Skovsmose, 2003; Forman, 2003). Lampert and Cobb (2003) argue that by providing students the opportunity to discuss their ideas with others can develop their mathematic reasoning more readily. According to Howe and her colleagues, the most successful instances of collaboration occur when students propose and defend their ideas and when they explain their reasoning to each other (Howe et al, 2007). Further, Howe et al. (2007) discovered that collaboration was most productive when the teacher offered little intervention and allowed students to exercise their own authority in solving the mathematical tasks proposed to them.

**Teacher Use of Mathematical Arguments**

To discover more about the role of the teacher in fostering a community that encourage collaboration to develop student learning via mathematical arguments, Mercer (2008) builds upon the research of Howe et al. to explain that the teacher’s role should be one where he or she guides the students in creating mathematical arguments. In this capacity, the teacher assists the students as they learn to collaborate effectively and utilize “exploratory talk” as a cultural and psychological tool to contribute to their development of reasoning (Mercer, 2008; Mercer, Wegerif, Dawes, & Sams, 2004). Exploratory talk is the idea where partners engage critically but constructively with each other’s ideas. Their statements and suggestions are considered jointly, as they challenge and counter-challenge, requiring justification and alternative hypotheses (Mercer et al., 2004). Exploratory talk holds students accountable to reasoning.

To create an atmosphere where exploratory talk is commonplace in the mathematics classroom, Brown (2017) encourages teacher participation in a way that guides and pushes student thinking, as they listen and observe the activities of students in their small groups, as in the aforementioned collective argumentation model. The observation of activities can then inspire students by then challenging them to engage in different types of representations, explanations, and justifications (Brown, 2017) than what they had previously created. Brown (2017) continues that the teacher can do this by asking questions about representations, adding to the representations, or even by providing his or her own personal representation. The active role of the teacher can create an environment where students are not only accountable to developing viable mathematical arguments, but they also are inspired to actively engage in them as they are challenged to create new representations of the mathematics.

**Discourse in Collaboration of Mathematical Arguments in a Mathematical Learning Community**

Discourse in the classroom is dependent upon the social setting of the classroom and can have multiple meanings, involving more than language (Gee, 1996; Moschkovich, 2007). Moschkovic (2007) contends that discourse also involves representations and behaviors, which involves collaboration about arguments and proofs in the context of mathematics classrooms. The discourse of a mathematics classroom is important to note, then, because the language, representations, and behaviors in a class because the teacher and the students may have different interpretations to meanings and focus of attention.

During the act of collaboration in a mathematics class, collective mathematical understanding may take place when students work together on one mathematical task (Martin, Towers, & Pirie, 2006). This collective understanding requires the social context of the learning environment, and it cannot be described by looking at the actions of the individual learners. Through the process of working jointly on a problem, problem-solving leads students to share ideas and their ways of solving, so individual understanding becomes shared. Teachers should establish a classroom environment that encourages this type of discourse where students will jointly partake in a discourse that transforms individual student thinking about mathematics due to the collective understanding that takes place via student language, patterns, behavior, and interactions with the mathematics as well as each other. This study pursues the nature of the small group discussions and the class-wide, whole-group consensus in collective argumentation, hoping to clarify the elements of these discussions that most encourage student learning through the development of mathematical arguments.

**Teacher Support of Student Discourse for the Purpose of Mathematical Argumentation**

As recognized in previous research, the power of student discourse in promoting mathematical argumentation is recognized and encouraged in various classroom practices across all levels of K-12 mathematics. However, an important factor to note for the success of implementation of social practices to enhance mathematical argumentation is the assessment of student engagement and understanding in social discourse by the teacher. Thus, it is necessary to understand how teachers conceptualize and implement discussion as a social practice to enhance mathematical argumentation.

As a first step to promote student justification and generalization in mathematical argumentation, Melhuish et al. (2019) establish that teachers should be able to recognize when students are truly engaging in the activities that teachers use to implement successful discourse for mathematical argumentation. Even deeper, Melhuish et al. (2019) recognize that as a mathematics teacher community, educators must commonly agree on the definition of argumentation through justification and generalization, which they define as showing why an idea/solution is true, refusing the validity of an idea, and giving mathematical defense of an idea that was challenged. This brings the practice of argumentation back to the standards set by NCTM (2000) and CCSS (2010), while also setting forth concrete criteria for teachers to recognize.

Some research has combined with practical steps and recommendations for K-12 mathematics teachers to enhance purposeful student discourse in developing mathematical arguments. As one example that emphasizes the classroom as a community that engages in the social practice of student discourse, Ko, Yee, Bleiler-Baxter, and Boyle (2016) have recognized that a before-during-after instructional sequence would best give students the chance to use each other as a resource as a classroom community as they work via their social discourse towards a common goal of developing advanced mathematical arguments. In this framework (Ko et al., 2016), before engaging wholeheartedly in group discussion with one another, students individually create their own argument to a mathematical claim that is the center of the mathematical learning activity. This can often be done as homework or in a previous class. Then, during the activity, students evaluate the validity of arguments based on classroom criteria in small and whole groups, emphasizing discourse with one another, and then after the class they critique reasoning by creating a rubric and using that rubric to critique each other’s arguments (Ko et al., 2016). By engaging students in the process of creating a rubric for critique of arguments, students now have a standard by which they can make generalizations and justifications for their arguments, and they know how to properly engage in discourse with one another that is focused on the appropriate K-12 mathematics activity.

Melhuish et al. (2019) recently developed the SDOT, which stands for Student Discourse Observation Tool, as an effective measure to promote teacher noticing in student engagement in mathematical practices that advance their thinking and, particularly, their argumentation skills in K-12 mathematics. This tool serves as just one example of a practical support to effectively implement mathematical discourse to promote K-12 student development of argumentation through justification and generalization. As research pushes forward in mathematical discourse and argumentation, more tools and instruments for practical implication of mathematical arguments for practitioners will be developed to enhance student discussions and social practices in K-12 mathematics classrooms.

**Looking Forward to the Future: Implications for Future Research and Practice**

Research has consistently uncovered successful mathematical practices that promote student collaboration and discourse as a key to promote student mathematical learning. Particularly, taking a sociomathematical approach to discourse as a social practice has emphasized the transformation of mathematical content into student learning, the end goal of K-12 mathematics education (Ochs, 1998). However, a deeper dive into the practice of mathematical discourse for the development of arguments for K-12 students could benefit still the greater community of K-12 mathematics teachers and students. Since argumentation has recently become a standard in mathematical practices according to the National Council of Teachers of Mathematics (2000), mathematics education research still has room to explore the role of student discourse as students work with one another to promote mathematical argumentation advancement. Mathematics education researchers have recently begun to study and published study results of ways to change classroom cultures to support mathematical proofs and arguments that produce long-lasting effects in student understanding (Nardi & Knuth, 2017; Melhuish et al., 2019). For the purposes of future research, longitudinal studies and studies of student discourse to create and critique viable mathematical arguments could contribute to practitioners’ and researchers’ understanding of what makes mathematical argumentation successful in K-12 classrooms.

**Conclusion**

This literature reviewed served the purpose of understanding the current state of mathematics education research about student discourse and the role it plays on K-12 mathematics student development of mathematical arguments. It is important to lay a foundation to the understanding of how mathematical discourse is recognized as a social practice that connects mathematical content to student understanding, and how these practices achieve the goals and standards set forth by the National Council of Teachers of Mathematics (2000) and the Common Core State Standards Initiative (2010). In studying the role of student discourse as a social practice, research can now look into the impact of student discourse with one another to advance mathematical argumentation. Future research will help better understand how students utilize social discourse with one another to create a better mathematical argument and how teachers can support their teachers in creating and critiquing more viable mathematical arguments.

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