Optional Stopping Theorems

This document reviews several famous theorems from the theory of martingales, with a focus on (sub-)martingale convergence as well as optional sampling and optional stopping. Building on the literature, in particular A. Klenke's *Probability Theory*, [1], and J.F. Le Gall's *Brownian motion, martingales and stochastic calculus*, [2], we consider (sub-, super-)martingales in both discrete and continuous time and obtain and prove somewhat more general forms of the theorems presented therein. For the continuous-time case, we consider not just stopping times, but also optional times.

1 *Introduction & Definitions*

Throughout this document, we work with a nonempty totally ordered index set I of times and filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$. Usually, we will have $I = \mathbb{N}_0$, $I = \mathbb{R}_+$ or $I \subseteq \mathbb{R}_+$ an interval. In all cases, we will denote by t^* an upper bound[1] of I and let $\overline{I} = I \cup \{t^*\}$.

[1] if $I \subseteq \mathbb{R}$, $t^* = \sup I \in \overline{\mathbb{R}}_+$ will do.

1.1 *The Doob decomposition in discrete time*

Theorem 1.1 (Doob decomposition). *Let $X = (X_n)_{n\in\mathbb{N}_0}$ be an adapted integrable process.*

(a) There exists a decomposition $X = M + A$, unique up to equality almost surely, where A is predictable[2] with $A_0 = 0$ and M is a martingale. This representation of X is called the Doob decomposition.
(b) X is a submartingale if and only if A is monotone increasing.
(c) Let X a submartingale, and suppose that there exists[3] an $\mathcal{F}_\infty$-measurable random variable X_∞ s.t. $X_n \to X_\infty$ as $n \to \infty$ almost surely and in L^1. Then $A_\infty := \lim_{n\to\infty} A_n$ is integrable and both M and A are uniformly integrable.

[2] A stochastic process $A = (A_n)_{n\in\mathbb{N}_0}$ is called $\mathbb{F}$-predictable, if A_0 is constant and for each $n \in \mathbb{N}$, A_n is $\mathcal{F}_{n-1}$ measurable.

[3] By theorem 2.12, this is the case iff X is uniformly integrable.

Proof. (a) To see uniqueness, suppose that $X = M + A = M' + A'$ are two decompositions. Then $W_n := M_n - M'_n = A'_n - A_n$, $n \in \mathbb{N}_0$ defines a predictable martingale, and thus $W_n \overset{\text{a.s.}}{=} W_0 = 0$ by the fairness property of martingales. For existence, let $\Delta X_n := X_n - X_{n-1}$, $n \in \mathbb{N}$. To motivate the proof, suppose that M and A exist and let $\Delta M_n = M_n - M_{n-1}$ and $\Delta A_n = A_n - A_{n-}$. Since M is a martingale and A is predictable, we have $\mathbb{E}[\Delta X_n|\mathcal{F}_{n-1}] = \mathbb{E}[\Delta M_n|\mathcal{F}_{n-1}] + \mathbb{E}[\Delta A_n|\mathcal{F}_{n-1}] = \mathbb{E}[\Delta A_n|\mathcal{F}_{n-1}] = \Delta A_n$, and thus necessarily $A_n = A_0 + \sum_{k=1}^n \Delta A_k = \sum_{k=1}^n \mathbb{E}[\Delta X_k|\mathcal{F}_{k-1}]$. Consequently, we define

$$A_n := \sum_{k=1}^n \mathbb{E}[\Delta X_k|\mathcal{F}_{k-1}],$$

$$\begin{aligned} M_n := X_n - A_n = X_0 + \sum_{k=1}^n \Delta X_k - \mathbb{E}[\Delta X_k|\mathcal{F}_{k-1}] \\ = X_0 + \sum_{k=1}^n X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}], \end{aligned}$$

and clearly A is predictable and M is adapted and integrable with $\mathbb{E}[\Delta M_n|\mathcal{F}_{n-1}] = 0$ for all $n \in \mathbb{N}$, i.e. M is a martingale.

(b) This follows from the equality $\mathbb{E}[\Delta X_n|\mathcal{F}_{n-1}] = \Delta A_n$ established in part (a) of the proof.

(c) Since $X_n \to X_\infty$ in L^1, it holds that $\mathbb{E}[X_n] \to \mathbb{E}[X_\infty]$. Since A_n increases to A_∞ as $n \to \infty$, monotone convergence yields

$$\mathbb{E}[A_\infty] = \lim_{n\to\infty} \mathbb{E}[A_n] = \lim_{n\to\infty} \mathbb{E}[X_n - M_n] = \mathbb{E}[X_\infty] - \mathbb{E}[M_0] < \infty,$$

i.e. A_∞ is integrable, implying that A is uniformly integrable. Since $|M_n| = |X_n - A_n| \leq |X_n| + |A_\infty|$ and X is uniformly integrable, it follows that M is uniformly integrable.

□

1.2 Backwards Martingales

This section briefly reviews the concept of backwards martingales, i.e. martingales defined on the index set $I = -\mathbb{N}_0$ or $I = \mathbb{R}_- \coloneqq (-\infty, 0]$.

Definition 1.2 (Backwards Sub/Super/Martingale). Let $I = -\mathbb{N}_0$ or $I = \mathbb{R}_-$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t \in I}$. Then an $\mathbb{F}$-(sub)martingale $X = (X_t)_{t \in I}$ is called an $\mathbb{F}$-backwards (sub)martingale, and similarly for supermartingales.

Definition 1.2 makes clear that backwards sub- and supermartingales are just ordinary martingales with a specific index set, which allows to carry over many results (e.g. the upcrossing inequality) that hold for usual martingales. However, for applications, this definition is suboptimal, and the following characterisation is frequently used.

Lemma 1.3. *Let $I = \mathbb{N}_0$ or $I = \mathbb{R}_+$, $X = (X_t)_{t \in -I}$ a real-valued stochastic process and $\mathbb{F} = (\mathcal{F}_t)_{t \in -I}$ a sequence of sub-σ-algebras. For $t \in I$, let $Y_t = X_{-t}$, $\mathcal{G}_t = \mathcal{F}_{-t}$ and let $Y = (Y_t)_{t \in I}$ and $\mathbb{G} = (\mathcal{G}_t)_{t \in I}$. Then X is an $\mathbb{F}$-backwards submartingale if and only if $\mathbb{G}$ is a decreasing sequence of sub-σ-algebras, i.e. $\mathcal{G}_s \supseteq \mathcal{G}_t$ for all $s, t \in I$ with $s \leq t$, and in addition the following hold:*

(i) *Y is $\mathbb{G}$-adapted, i.e. Y_t is $\mathcal{G}_t$-measurable for every $t \in I$.*
(ii) *Y is an integrable process.*
(iii) *It holds that for every $s, t \in I$ with $s \leq t$,*

$$\mathbb{E}[Y_s | \mathcal{G}_t] \geq Y_t. \tag{1}$$

The analogous statement holds for supermartingales.

To remember (iii) from lemma 1.3, note that eq. (1) is the usual defining inequality for submartingales with s and t reversed.

Remark 1.4. Let $M = (M_t)_{t \in -I}$ a backwards martingale. Then for all $t \in -I$, $M_t = \mathbb{E}[M_0 | \mathcal{F}_t]$, i.e. M is closed by M_0 and thus uniformly integrable, see [1, Cor. 8.22]. The next theorem shows that uniform integrability holds also for backwards sub- and supermartingales in discrete time under the mild added assumption of L^1-boundedness.

Theorem 1.5 (Doob decomposition for discrete time backwards submartingales). *Let $X = (X_n)_{n \in -\mathbb{N}_0}$ a backwards submartingale adapted to the filtration $\mathbb{F} = (\mathcal{F}_n)_{n \in -\mathbb{N}_0}$ and suppose that $L \coloneqq \lim_{n \to -\infty} \mathbb{E}[X_n] > -\infty$.*

(a) *There exist a real-valued, $\mathbb{F}$-predictable,[4] increasing process $A = (A_n)_{n \in -\mathbb{N}_0}$ with $A_0 \in \mathcal{L}^1(\mathbb{P})$ and $A_{-\infty} \coloneqq \lim_{n \to -\infty} A_n = 0$, and a backwards martingale $M = (M_n)_{n \in -\mathbb{N}_0}$ such that $X = M + A$.*
(b) *M, A and thus X are uniformly integrable.*

[4] This means that for each $n \in -\mathbb{N}_0$, A_n is $\mathcal{F}_{n-1}$-measurable.

Proof. (a) For $n \in -\mathbb{N}_0$, let $\Delta X_n = X_n - X_{n-1}$. If the decomposition exists, then necessarily $\Delta A_n \coloneqq A_n - A_{n-1} = \mathbb{E}[\Delta X_n | \mathcal{F}_{n-1}] = \mathbb{E}[X_n | \mathcal{F}_{n-1}] - X_{n-1} \geq 0$ for all $n \in -\mathbb{N}_0$. So let

$$A_n \coloneqq \sum_{k=-\infty}^{n} \mathbb{E}[\Delta X_k | \mathcal{F}_{k-1}], \qquad n \in -\mathbb{N}_0,$$

which defines an $\overline{\mathbb{R}}_+$-valued $\mathbb{F}$-predictable increasing process. By monotone convergence, $\mathbb{E}[A_0] = \sum_{k=-\infty}^{0} \mathbb{E}[\Delta X_k] = \lim_{N\to-\infty} \mathbb{E}[X_0 - X_N] = \mathbb{E}[X_0] - L < \infty$. Hence A_0 is integrable and in particular finite a.s., so by passing[5] to $(A_n \mathbb{1}_{\{A_n<\infty\}})_{n\in-\mathbb{N}_0}$, we may suppose that A is $\mathbb{R}_+$-valued. This ensures that $\Delta A_n \coloneqq A_n - A_{n-1}$ is well defined with $\Delta A_n \overset{\text{a.s.}}{=} \mathbb{E}[\Delta X_n|\mathcal{F}_{n-1}]$ and gives also that $A_{-\infty} = 0$. Now define $M = (M_n)_{n\in-\mathbb{N}_0}$ by $M_n = X_n - A_n$, which is $\mathbb{F}$-adapted and integrable with $\Delta M_n \coloneqq M_n - M_{n-1} = \Delta X_n - \Delta A_n$ and thus

$$\mathbb{E}[\Delta M_n|\mathcal{F}_{n-1}] = \mathbb{E}[\Delta X_n|\mathcal{F}_{n-1}] - \Delta A_n \overset{\text{a.s.}}{=} 0, \qquad n \in -\mathbb{N}_0,$$

i.e. M is an $\mathbb{F}$-martingale, proving the claim.

(b) For $n \in -\mathbb{N}_0$, $0 = A_{-\infty} \le A_n \le A_0 \in \mathcal{L}^1(\mathbb{P})$, hence A is uniformly integrable. Moreover the backwards martingale M is closed by M_0 and thus also uniformly integrable. It follows that $X = M + A$ is uniformly integrable.

□

[5] $\mathbb{E}[\Delta X_k|\mathcal{F}_{k-1}]$ can be assumed to be $\mathbb{R}_+$-valued, so the events $\{A_n < \infty\}$, $n \in -\mathbb{N}_0$ all equal $\{A_0 < \infty\}$, which has probability 1. This modification of A on a null set preserves predictability and leaves conditional expectations unchanged.

1.3 Facts about stopping times

This section leans heavily on [3], which is an invaluable resource on stochastic calculus. Let $I \neq \varnothing$ a totally ordered set index, t^* an upper bound of I and $\overline{I} \coloneqq I \cup \{t^*\}$. In the case $I \subseteq \overline{\mathbb{R}}$, take $t^* = \sup I$ and equip I and $\overline{I}$ with their respective subspace topologies and σ-algebras. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtered probability space and define $\mathcal{F}_{t^*} = \sigma(\bigcup_{t\in I} \mathcal{F}_t)$.

Definition 1.6. Let $I \subseteq \mathbb{R}_+$.

(a) A map $\tau\colon \Omega \to \overline{I}$ is called a random time, if τ is $\mathcal{F}$-$\mathcal{B}_{\overline{I}}$-measurable

(b) A map $\tau\colon \Omega \to \overline{I}$ is called an $\mathbb{F}$-stopping time, if $\{\tau \le t\} \in \mathcal{F}_t$ for every $t \in I$.

(c) With every $\mathbb{F}$-stopping time τ, we associated the σ-algebra

$$\mathcal{F}_\tau \coloneqq \{\, F \in \mathcal{F}_{t^*} : F \cap \{\, \tau \le t \,\} \in \mathcal{F}_t \text{ for all } t \in I \,\}.$$

Note that $\{\tau \le t^*\} = \Omega$, hence the definition can omit the case $t = t^*$. Moreover, every stopping time is a random time.[6]

[6] To see this, it suffices to show that $\{\tau \le s\} \in \mathcal{F}$ for every $s \in \mathbb{R}$. For $s \in I$ and $s \ge t^*$, this is clear, and for $s < t^*$ we have $\{\tau \le s\} = \{\tau \in I \cap [-\infty, s]\} = \{\tau \le t(s)\}$, where $t(s) \coloneqq \sup(I \cap [-\infty, s])$. Now either $t(s) \in I$ and we are done, or $t(s) \notin I$, in which case there exists $(t_n) \subseteq I \cap [-\infty, s]$, with $t_n \nearrow t(s)$. It follows that $\{\tau \le s\} = \{\tau \le t(s)\} = \{\tau < t(s)\} = \bigcap_{n\ge1}\{\tau \le t_n\} \in \mathcal{F}$.

Lemma 1.7 (Properties of stopping times). *Let σ and τ $\mathbb{F}$-stopping times.*

(a) τ is $\mathcal{F}_\tau$-measurable.

(b) If $\sigma \le \tau$, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

(c) $\mathcal{F}_\sigma \cap \mathcal{F}_\tau = \mathcal{F}_{\sigma\wedge\tau}$.

(d) Let σ and τ denote two $\mathbb{F}$-stopping times, and let $F \in \mathcal{F}_\sigma$. Then the events $\{\, \sigma \le \tau \,\} \cap F$, $\{\, \sigma = \tau \,\} \cap F$ and $\{\, \tau \le \sigma \,\}$ are all elements of $\mathcal{F}_{\sigma\wedge\tau}$.

Proof. (a) Exercise.

(b) Exercise.

(c) For every $t \in I$,

$$\{\, \sigma \le \tau \,\} \cap F \cap \{\, \tau \le t \,\} = \{\, \sigma \le t \,\} \cap F \cap \{\, \tau \le t \,\} \cap \{\, \sigma \wedge t \le \tau \wedge t \,\} \in \mathcal{F}_t,$$
$$\{\, \sigma \le \tau \,\} \cap F \cap \{\, \sigma \le t \,\} = \{\, \sigma \wedge t \le \tau \wedge t \,\} \cap F \cap \{\, \sigma \le t \,\} \in \mathcal{F}_t,$$

because $\sigma \wedge t$ and $\tau \wedge t$ are $\mathcal{F}_{\sigma\wedge t}$ resp. $\mathcal{F}_{\tau\wedge t}$-measurable and thus both $\mathcal{F}_t$-measurable. Hence $\{\, \sigma \le \tau \,\} \cap F \in \mathcal{F}_{\sigma\wedge\tau}$ for any $F \in \mathcal{F}_\sigma$. Taking $F = \Omega$ shows by symmetry that $\{\, \tau \le \sigma \,\} \in \mathcal{F}_{\sigma\wedge\tau}$. It follows that $\{\, \sigma = \tau \,\} \in \mathcal{F}_{\sigma\wedge\tau}$ and thus $\{\, \sigma = \tau \,\} \cap F \in \mathcal{F}_{\sigma\wedge\tau}$.

□

Definition 1.8. (a) Let $I \subseteq \overline{\mathbb{R}}$ an interval with upper bound t^*, and $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtration. Then the filtration $\mathbb{F}^+ = (\mathcal{F}_{t+})_{t\in I}$ is defined by

$$\mathcal{F}_{t+} = \bigcap_{\substack{u\in I\\ u>t}} \mathcal{F}_u, \quad t \in I \setminus \{t^*\},$$

and $\mathcal{F}_{t^*+} = \mathcal{F}_{t^*}$ if $t^* \in I$.

(b) A stopping time of the filtration $\mathbb{F}^+$ is called a weak ($\mathbb{F}$-)stopping time or ($\mathbb{F}$-)optional time.

(c) With every $\mathbb{F}^+$-stopping time τ, we associate the σ-algebra

$$\mathcal{F}_{\tau+} = \{F \in \mathcal{F}_{t^*} : F \cap \{\tau \le t\} \in \mathcal{F}_{t+} \text{ for all } t \in I\}.$$

Remark 1.9. Note that $\mathcal{F}_{\tau+} = \mathcal{G}_\tau$, where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in I}$ is defined by $\mathcal{G}_t = \mathcal{F}_{t+}$. Hence definition 1.8 (c) is a special case of definition 1.6 (c), and the results from lemma 1.7 carry over with the appropriate modifications. In particular, every $\mathbb{F}^+$-stopping time τ is $\mathcal{F}_{\tau+}$-measurable.

Lemma 1.10 (Properties of $\mathbb{F}^+$-stopping times). *Let $I \subseteq \overline{\mathbb{R}}$ an interval and $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtration.*

(a) Every $\mathbb{F}$-stopping time is an $\mathbb{F}^+$-stopping time.

(b) A map $\tau\colon \Omega \to \overline{I}$ is an $\mathbb{F}^+$-stopping time iff $\{\tau < t\} \in \mathcal{F}_t$ for every $t \in I\setminus\{\inf I\}$.

(c) $\mathcal{F}_{\tau+} = \{F \in \mathcal{F}_{t^*} : F \cap \{\tau < t\} \in \mathcal{F}_t$ for all $t \in I\}$.

(d) If σ is an $\mathbb{F}$-stopping time and τ is an $\mathbb{F}^+$-stopping time, then $\sigma\wedge\tau$ is $(\mathcal{F}_\sigma\cap\mathcal{F}_{\tau+})$-measurable and therefore $\mathcal{F}_\sigma$-measurable.

(e) If $(\tau_n)_{n\in\mathbb{N}_0}$ is a sequence of optional times and $\tau = \inf_{n\ge1}\tau_n$, then $\mathcal{F}_{\tau+} = \bigcap_{n\ge1}\mathcal{F}_{\tau_n+}$. In addition, if each τ_n is a stopping time and $\tau < \tau_n$ on $\{\tau < \infty\}$ for each $n \in \mathbb{N}$, then $\mathcal{F}_{\tau+} = \bigcap_{n\ge1}\mathcal{F}_{\tau_n}$.

Proof. (a) This follows, since for every $t \in I$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+}$.

(b) Let τ be an $\mathbb{F}^+$-stopping time, $t \in I \setminus \{\inf I\}$, and $(t_n)_{n\in\mathbb{N}}$ a sequence in I strictly increasing to t. Then

$$\{\tau < t\} = \bigcup_{n\in\mathbb{N}} \underbrace{\{\tau \le t_n\}}_{\in\mathcal{F}_{t_n+}\subseteq\mathcal{F}_t} \in \mathcal{F}_t.$$

To prove the reverse implication, fix $t \in I \setminus \{t^*\}$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence in I strictly decreasing to t as $n \to \infty$. Then, for every $u \in I$ with $u > t$,

$$\{\tau \le t\} = \bigcap_{\substack{n\in\mathbb{N}\\ t_n\le u}} \underbrace{\{\tau < t_n\}}_{\in\mathcal{F}_{t_n}\subseteq\mathcal{F}_u} \in \mathcal{F}_u,$$

hence $\{\tau \le t\} \in \bigcap_{u\in I, u>t}\mathcal{F}_u = \mathcal{F}_{t+}$.

(c) Suppose $F \in \mathcal{F}_{\tau+}$. Then $F \cap \{\tau \le t\} \in \mathcal{F}_{t+}$ for every $t \in I$, hence

$$F \cap \{\tau < t\} = \bigcup_{\substack{q\in\mathbb{Q}\cap I\\ q<t}} F \cap \underbrace{\{\tau \le q\}}_{\in\mathcal{F}_{u+}\subseteq\mathcal{F}_t} \in \mathcal{F}_t.$$

Conversely, assume that $F \cap \{\tau < q\} \in \mathcal{F}_q$ for every $q \in I$. Then for every $t \in I$ and every $s > t$,

$$F \cap \{\tau \le t\} = \bigcap_{\substack{q\in\mathbb{Q}\cap I\\ t<q<s}} \underbrace{F \cap \{\tau < q\}}_{\in\mathcal{F}_q\in\mathcal{F}_s} \in \mathcal{F}_s.$$

Hence $F \cap \{\tau \le t\} \in \mathcal{F}_{t+}$ and thus $F \in \mathcal{F}_{\tau+}$.

(d) By part (a), σ is an $\mathbb{F}^+$-stopping time, hence $\sigma \wedge \tau$ is an $\mathbb{F}^+$-stopping time. By lemma 1.7 (a) applied to $\mathbb{F}^+$, $\sigma \wedge \tau$ is $\mathcal{F}_{(\sigma\wedge\tau)+}$-measurable. By lemma 1.7 (b) we have $\mathcal{F}_{(\sigma\wedge\tau)+} \subseteq \mathcal{F}_{\tau+}$. Hence $\sigma \wedge \tau$ is $\mathcal{F}_{\tau+}$-measurable. We verify next that $\sigma \wedge \tau$ is $\mathcal{F}_\sigma$-measurable by checking that for all $t \in I$, $\{\sigma \wedge \tau \leq t\} \in \mathcal{F}_\sigma$. It holds that $\{\sigma \wedge \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t < \sigma\}$, hence for all $s \in I$,

$$\{\sigma \wedge \tau \leq t\} \cap \{\sigma \leq s\} = \{\sigma \leq s \wedge t\} \cup \{\tau \leq t < \sigma \leq s\} \in \mathcal{F}_s,$$

since $\{\sigma \leq s \wedge t\} \in \mathcal{F}_s$ and similarly

$$\{\tau \leq t < \sigma \leq s\} = \overbrace{\{\tau < s\} \cap \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_{\tau+}}}^{\in \mathcal{F}_s} \cap \overbrace{\{t < \sigma \leq s\}}^{\in \mathcal{F}_s} \in \mathcal{F}_s,$$

where we have used that τ is $\mathcal{F}_{\tau+}$-measurable by remark 1.9 and applied part (c) of the present lemma. Hence $\sigma \wedge \tau$ is both $\mathcal{F}_\sigma$ and $\mathcal{F}_{\tau+}$-measurable, and thus $\mathcal{F}_\sigma \cap \mathcal{F}_{\tau+}$-measurable.

(e) We first show $\mathcal{F}_{\tau+} = \bigcap_{n\geq 1} \mathcal{F}_{\tau_n+}$. Since $\tau \leq \tau_n$ for each $n \in \mathbb{N}$, the inclusion "$\subseteq$" follows from lemma 1.7 (b) applied to the filtration $\mathbb{F}^+$. For "$\supseteq$", let $A \in \bigcap_{n\geq 1} \mathcal{F}_{\tau_n+}$. Then for every $t \geq 0$,

$$A \cap \{\tau < t\} = \bigcup_{n\geq 1} A \cap \{\tau_n < t\} \in \mathcal{F}_t,$$

hence $A \in \mathcal{F}_{\tau+}$. For the addition, "$\supseteq$" follows from the first part, since $\mathcal{F}_{\tau_n} \subseteq \mathcal{F}_{\tau_n+}$ for each $n \in \mathbb{N}$ and thus $\bigcap_{n\geq 1} \mathcal{F}_{\tau_n} \subseteq \bigcap_{n\geq 1} \mathcal{F}_{\tau_n+} = \mathcal{F}_{\tau+}$. Next, "$\subseteq$" follows from the fact that under the stronger assumption, we have for any $A \in \mathcal{F}_{\tau+}$ and for all $t \geq 0$ that

$$A \cap \{\tau_n \leq t\} = A \cap \{\tau < t\} \cap \{\tau_n \leq t\} \in \mathcal{F}_t.$$

□

The next lemma shows that superlinear[7] functions of $\mathbb{F}$-stopping times and strictly superlinear functions of $\mathbb{F}^+$-stopping times are $\mathbb{F}$-stopping times.

[7] We call a function $f\colon \overline{I} \to \overline{I}$ superlinear, if $f(x) \geq x$ for all $x \in \overline{I}$. We call f strictly superlinear, if $f(x) > x$ for all $x \in I \setminus \{t^*\}$ and $f(t^*) = t^*$.

Lemma 1.11 (Superlinear functions of stopping times are stopping times). *Suppose that the (extended) index set of times $\overline{I}$ is a measurable subset of $\overline{\mathbb{R}}$.*

(a) If τ is an $\mathbb{F}$-stopping time and $\sigma\colon \Omega \to \overline{I}$ is $\mathcal{F}_\tau$-measurable with $\sigma \geq \tau$, then σ is an $\mathbb{F}$-stopping time. In particular for every measurable function $f\colon \overline{I} \to \overline{I}$ with $f(t) \geq t$ for every $t \in \overline{I}$, $f(\tau)$ is an $\mathbb{F}$-stopping time.

(b) If $\overline{I}$ is an interval, τ is an $\mathbb{F}^+$-stopping time and $\sigma\colon \Omega \to \overline{I}$ is $\mathcal{F}_{\tau+}$-measurable with $\sigma > \tau$ on $\{\tau < t^*\}$ and $\sigma = \tau$ on $\{\tau = t^*\}$, then σ is an $\mathbb{F}$-stopping time. In particular for every measurable function $f\colon \overline{I} \to \overline{I}$ with $f(t) > t$ for every $t \in I \setminus \{t^*\}$ and $f(t^*) = t^*$, $f(\tau)$ is an $\mathbb{F}$-stopping time.

Proof. (a) Since $\tau \leq \sigma$ and σ is $\mathcal{F}_\tau$-measurable, it holds that for every $t \in I$

$$\{\sigma \leq t\} = \underbrace{\{\sigma \leq t\}}_{\in \mathcal{F}_\tau} \cap \{\tau \leq t\} \in \mathcal{F}_t,$$

hence σ is an $\mathbb{F}$-stopping time. The addition follows, since the $\mathbb{F}$-stopping time τ is $\mathcal{F}_\tau$-measurable, see item (a).

(b) Let $t \in I \setminus \{t^*\}$. Then $\{\sigma \leq t\} \cap \{\tau = t^*\} = \{\tau \leq t\} \cap \{\tau = t^*\} = \varnothing$, hence $\{\sigma \leq t\} \subseteq \{\tau < t^*\}$. By assumption, $\{\tau < t^*\} \subseteq \{\tau < \sigma\}$, hence $\{\sigma \leq t\} \subseteq \{\tau < \sigma\}$. It follows that

$$\{\sigma \leq t\} = \underbrace{\{\sigma \leq t\}}_{\in \mathcal{F}_{\tau+}} \cap \{\tau < t\} \in \mathcal{F}_t,$$

where we have used $\mathcal{F}_{\tau+}$-measurability of σ and lemma 1.10 (b). The addition follows, since the $\mathbb{F}^+$-stopping time τ is $\mathcal{F}_{\tau+}$-measurable, see remark 1.9.

□

Let $\lceil \cdot \rceil \colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ the ceiling function, extended to $\{-\infty, +\infty\}$ by $\lceil \pm\infty \rceil := \pm\infty$. Similarly, denote by $\lfloor \cdot \rfloor \colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ the extended floor function. Lemma 1.11 then yields the following corollary.

Corollary 1.12. *Let $I \subseteq \overline{\mathbb{R}}$ an interval and $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtration. Let $\tau \colon \Omega \to \overline{I}$ an $\mathbb{F}^+$-stopping time, then*

$$\tau_n := 2^{-n}(\lfloor 2^n \tau \rfloor + 1) \wedge t^*$$

defines a sequence of $\mathbb{F}$-stopping times taking countably many values such that $\tau_n \searrow \tau$ pointwise. For a sequence of $\mathbb{F}$-stopping times, we may take alternatively $\tau_n := 2^{-n}\lceil 2^n \tau \rceil \wedge t^*$.

2 Martingale Convergence

2.1 Convergence Theorems for Discrete Time Martingales

Definition 2.1 (Upcrossing numbers). Let $I \subseteq \mathbb{R}_+$ and $f \colon I \to \mathbb{R}$ be a function. For any $a, b \in \mathbb{R}$ with $a < b$, the upcrossing number of f along $[a, b]$, denoted by $U^f_{ab}(I)$, is the maximal integer $k \geq 1$ such that there exists a finite increasing sequence $s_1 < t_1 < \cdots < s_k < t_k$ of elements of I such that $f(s_i) \leq a$ and $f(t_i) \geq b$ for every $i \in \{1, \ldots, k\}$. If, even for $k = 1$, there is no such subsequence, we take $U^f_{ab}(I) = 0$, and if such a subsequence exists for every $k \geq 1$, we take $U^f_{ab}(I) = \infty$.

Remark 2.2 (Measurability of upcrossing numbers). (a) If $I \subseteq \mathbb{R}_+$ is a finite set, then the set $\mathcal{S}$ of strictly increasing sequences of even length with values in I is finite. For every such sequence $S = (s_1, t_1, \ldots, s_k, t_k) \in \mathcal{S}$, the map

$$\mathbb{R}^{\mathbb{R}_+} \ni f \mapsto k \cdot \prod_{i=1}^{k} \mathbb{1}_{[-\infty,a]}(f(s_i))\, \mathbb{1}_{[b,\infty]}(f(t_i)) \in \mathbb{N}_0 \tag{2}$$

is $\mathcal{B}_{\mathbb{R}}^{\otimes \mathbb{R}_+}$-measurable by measurability of the projections $\pi_t \colon \mathbb{R}^{\mathbb{R}_+} \ni f \mapsto f(t) \in \mathbb{R}$. Hence $\mathbb{R}^{\mathbb{R}_+} \ni f \mapsto U^f_{ab}(I) \in \mathbb{N}_0$ is $\mathcal{B}_{\mathbb{R}}^{\otimes \mathbb{R}_+}$-measurable as a maximum of finitely many measurable maps of the form (2).

(b) For countable I, let $(I_n)_{n\in\mathbb{N}}$ a sequence of finite subsets of I with $I_n \nearrow I$, and observe that in this case, the $\overline{\mathbb{N}}_0$-valued map $f \mapsto U^f_{ab}(I) = \sup_{n\in\mathbb{N}} U^f_{ab}(I_n)$ is also $\mathcal{B}_{\mathbb{R}}^{\otimes \mathbb{R}_+}$-measurable.

(c) For any stochastic process $X = (X_t)_{t\in I}$, the map $\Omega \ni \omega \mapsto X(\omega) := (t \mapsto X_t(\omega)) \in \mathbb{R}^{\mathbb{R}_+}$ is $\mathcal{F}$-$\mathcal{B}_{\mathbb{R}}^{\otimes \mathbb{R}_+}$-measurable, hence $\omega \mapsto U^{X(\omega)}_{ab}(I)$ is an $\overline{\mathbb{N}}_0$-valued random variable whenever I is countable.

(d) With $I = [n] := \{0, \ldots, n\}$, the following equivalent definition[8] of the upcrossing number is useful in the discrete time setting. Let $X = (X_n)_{n\in\mathbb{N}_0}$ an $\mathbb{R}$-valued stochastic process adapted to the filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$, and define $\overline{\mathbb{R}}_+$-valued stopping times $\sigma_0 \equiv 0$ and, for all $k \in \mathbb{N}$, $\tau_k := \inf\{n \geq \sigma_{k-1} : X_n \leq a\}$ and $\sigma_k := \inf\{n \geq \tau_k : X_n \geq b\}$. Then $U^X_{ab}([n]) = \sup\{k \in \mathbb{N}_0 : \sigma_k \leq n\}$, and we say that X has its k th upcrossing over $[a, b]$ between τ_k and σ_k if $\sigma_k < \infty$.

[8] See [1, §11.2] for details.

Lemma 2.3 (Doob's upcrossing inequality). *Let $(X_n)_{n\in\mathbb{N}_0}$ be a submartingale. Then*

$$\mathbb{E}\left[U^X_{ab}([n])\right] \leq \frac{\mathbb{E}\left[(X_n - a)^+\right] - \mathbb{E}\left[(X_0 - a)^+\right]}{b - a}$$

Proof. An argument involving the discrete stochastic integral, see [1, Lemma 11.3]. □

Theorem 2.4 (Convergence theorem for discrete time submartingales). *Let $(X_n)_{n\in\mathbb{N}_0}$ be a submartingale bounded above in expectation, meaning that* $\sup\{\mathbb{E}[X_n^+] : n \in \mathbb{N}_0\} < \infty$*. Then there exists an $\mathcal{F}_\infty$-measurable random variable $X_\infty \in \mathcal{L}^1(\mathbb{P})$ with $X_n \overset{n\to\infty}{\longrightarrow} X_\infty$ almost surely.*

Proof sketch. This follows from the upcrossing inequality for discrete time submartingales, which guarantees that X has almost surely only finitely many upcrossings over any interval $[a, b]$ with rational endpoints, see [1, §11.2] for details. □

Convention 1. Let $X = (X_n)_{n\in\mathbb{N}_0}$ an $\mathbb{F}$-adapted stochastic process which converges almost surely as $n \to \infty$. For definiteness, we always denote by X_∞ the $\mathcal{F}_\infty$-measurable random variable defined by

$$X_\infty(\omega) = \begin{cases} \lim_{n\to\infty} X_n(\omega) & \omega \in C, \\ 0 & \omega \in C^c, \end{cases}$$

where $C \coloneqq \{\underline{\lim}_{n\to\infty} X_n = \overline{\lim}_{n\to\infty} X_n\}$ denotes the (measurable) set where (X_n) converges as $n \to \infty$.

Theorem 2.5 (Convergence theorem for nonnegative discrete time supermartingales). *If X is a nonnegative supermartingale, then there is an $\mathcal{F}_\infty$-measurable, integrable random variable $X_\infty \geq 0$ with $X_n \overset{n\to\infty}{\longrightarrow} X_\infty$ almost surely and $\mathbb{E}[X_\infty|\mathcal{F}_n] \leq X_n$ for all $n \in \mathbb{N}$.*

Proof. The preceding theorem with $(-X)$ establishes the almost sure limit. The inequality $\mathbb{E}[X_\infty|\mathcal{F}_n] \leq X_n$ follows by nonnegativity from Fatou's lemma for the conditional expectation, which also establishes the bound $\mathbb{E}[X_\infty] \leq \mathbb{E}[X_0]$ and thereby integrability. □

Corollary 2.6. *Let $X \geq 0$ be a martingale and let $X_\infty = \lim_{n\to\infty} X_n$. Then $\mathbb{E}[X_\infty] = \mathbb{E}[X_0]$ if and only if X is uniformly integrable.*

Proof sketch. The direction $\Leftarrow$ is clear, and $\Rightarrow$ follows from Scheffé's theorem. □

We next give the convergence theorem for discrete time backwards submartingales, see definition 1.2.

Theorem 2.7 (Convergence theorem for backwards submartingales). *Let $X = (X_n)_{n\in-\mathbb{N}_0}$ a discrete time backwards submartingale w.r.t. to the filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in-\mathbb{N}_0}$. Let $\mathcal{F}_{-\infty} \coloneqq \bigcap_{n\in-\mathbb{N}_0} \mathcal{F}_n$.*

(a) *There exists an $\mathcal{F}_{-\infty}$-measurable random variable $X_{-\infty}$ with*

$$X_n \xrightarrow[n\to-\infty]{a.s.} X_{-\infty}.$$

(b) *If* $\lim_{n\to-\infty} \mathbb{E}[X_n] > -\infty$*, in particular if X is a backwards martingale, then in addition*

$$X_n \xrightarrow[n\to-\infty]{L^1} X_{-\infty}.$$

Proof sketch. (a) By Doob's upcrossing inequality, the expectation of the number $U^X_{ab}(\{-n, \ldots, 0\})$ of upcrossings of X over $[a, b]$ between the times $-n$ and 0 satisfies the bound

$$\mathbb{E}[U^X_{a,b}(\{-n, \ldots, 0\})] \leq \frac{\mathbb{E}[(X_0 - a)^+]}{b - a}.$$

Taking $n \to -\infty$ and using monotone convergence, it follows that X has a.s. only finitely many upcrossings over $[a, b]$ and all times $n \in -\mathbb{N}_0$. Hence almost sure convergence of (X_n) to a random variable $X_{-\infty}$ follows as in the proof of the convergence theorem for discrete time submartingales, and clearly $X_{-\infty}$ is $\mathcal{F}_{-\infty}$-measurable.

(b) By theorem 1.5, the assumption $\lim_{n\to-\infty} \mathbb{E}[X_n] > -\infty$ ensures that X is uniformly integrable, which together with part (a) shows L^1-convergence.

□

Example 2.8. Let $X = (X_t)_{t\geq 0}$ a continuous time $(\mathcal{F}_t)_{t\geq 0}$-submartingale with right-continuous sample paths.[9] Then the map $t \mapsto \mathbb{E}[X_t]$ is right-continuous.[10] To see this, fix $t \geq 0$ and let $(t_n)_{n\in\mathbb{N}}$ a sequence with $t_n \searrow t$. Then $(X_{t_n})_{n\in\mathbb{N}}$ is a backwards submartingale w.r.t. the filtration $(\mathcal{F}_{t_n})_{n\in\mathbb{N}}$ by lemma 1.3. Indeed, $(\mathcal{F}_{t_n})_{n\in\mathbb{N}}$ is a decreasing sequence of sub-σ-algebras w.r.t. which $(X_{t_n})_{n\in\mathbb{N}}$ is adapted to. Moreover $(X_{t_n})_{n\in\mathbb{N}}$ is integrable and for every $0 \leq n \leq m$, $\mathbb{E}[X_{t_n}|\mathcal{F}_{t_m}] \geq X_{t_m}$ by the submartingale property of X. Since $\lim_{n\to\infty} \mathbb{E}[X_{t_n}] \geq \mathbb{E}[X_t] > -\infty$, theorem 2.7 implies that there is some random variable X_∞ with $X_{t_n} \to X_\infty$ a.s. and in L^1 as $n \to \infty$. By right-continuity, $X_\infty \overset{\text{a.s.}}{=} X_t$, and by L^1 convergence, it follows that $\mathbb{E}[X_{t_n}] \to \mathbb{E}[X_\infty] = \mathbb{E}[X_t]$ as $n \to \infty$.

[9] We use the subscript $t \geq 0$ as synonymous with $t \in \mathbb{R}_+$.

[10] In fact, if the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions, then right-continuity of $t \mapsto \mathbb{E}[X_t]$ suffices for the existence of a càdlàg modification of X, even when X itself is not right-continuous, see [2, §3].

2.2 Convergence Theorems for Continuous Time Martingales

In this section, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtered probability space and let $X = (X_t)_{t\geq 0}$ an $\mathbb{R}$-valued $\mathbb{F}$-adapted stochastic process.

Theorem 2.9 (Martingale convergence theorem for continuous time submartingales). *Let $I \subseteq \mathbb{R}_+$ an interval, $t^\circ := \inf I$, $t^* := \sup I \in \overline{\mathbb{R}}_+$ and $X = (X_t)_{t\in I}$ a submartingale with right-continuous sample paths which is bounded*[11] *in L^1. Then there exists an $\mathcal{F}_{t^*}$-measurable random variable $X_{t^*} \in \mathcal{L}^1(\mathbb{P})$ such that*

$$\lim_{t\to t^*} X_t = X_{t^*}, \qquad \mathbb{P}\text{-a.s.}$$

[11] Since X is a submartingale, we have $\mathbb{E}[X_t^-] = \mathbb{E}[X_t^+] - \mathbb{E}[X_t] \leq \mathbb{E}[X_t^+] - \mathbb{E}[X_0]$, hence this is equivalent to $\sup_{t\in I} \mathbb{E}[X_t^+] < \infty$.

Proof. Since we are interested in convergence of X_t for $t \to t^*$, we can suppose w.l.o.g. that $t^\circ \in I$. Let D a countable dense subset of I, $T \in D$ and $(D_n)_{n\in\mathbb{N}}$ a sequence of finite subsets of D increasing to $D \cap [t^\circ, T]$ with $\{t^\circ, T\} \subseteq D_n$ for each $n \in \mathbb{N}$.[12] By Doob's upcrossing inequality for discrete time submartingales (lemma 2.3), it holds that for every $a, b \in I$ with $a < b$ and every and $m \geq 1$,

$$\mathbb{E}\left[U_{ab}^X(D_m)\right] \leq \frac{1}{b-a}\mathbb{E}[(X_T - a)^+].$$

Taking $m \to \infty$, monotone converges implies that

$$\mathbb{E}\left[U_{ab}^X(D \cap [0, T])\right] \leq \frac{1}{b-a}\mathbb{E}[(X_T - a)^+],$$

and letting $T \to t^*$, another application of monotone convergence gives

$$\mathbb{E}\left[U_{ab}^X(D)\right] \leq \frac{1}{b-a}\sup_{t\in I}\mathbb{E}[(X_t - a)^+] < \infty,$$

since $X = (X_t)_{t\in I}$ is bounded in L^1. Hence almost surely, for all rationals $a < b$, we have $U_{ab}^X(D) < \infty$, which implies that the limit

$$X_{t^*} := \lim_{D\ni t\to t^*} X_t \tag{3}$$

exists almost surely in $\overline{\mathbb{R}}$. By Fatou's lemma and L^1-boundedness, $\mathbb{E}[|X_{t^*}|] \leq \underline{\lim}_{D\ni t\to t^*} \mathbb{E}[|X_t|] < \infty$, hence $X_{t^*} \in \mathcal{L}^1(\mathbb{P})$ is finite almost surely. The right-continuity of sample paths[13] now allows to remove the restriction $t \in D$ in the limit (3). □

[12] For example, $D = \bigcup_{n\geq 1} 2^{-n}\mathbb{N}_0$ where $D_n = \{t^\circ, T\} \cup ([t^\circ, T] \cap 2^{-n}\mathbb{N}_0)$ works.

[13] Note that right-continuity was not used up until this point.

2.3 *Interlude: Closeability of Sub- and Supermartingales*

This subsection draws from [4, §1.4] to discuss closeability of sub- and supermartingales, which will be useful to formulate the optional sampling theorems of the next section.

Definition 2.10 (Closeability of (sub-, super-)martingales). Let $I \neq \varnothing$ a totally ordered index set and $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtration.

(a) An $\mathbb{F}$-martingale $X = (X_t)_{t\in I}$ is called closable, if there exists $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_t = \mathbb{E}[\xi|\mathcal{F}_t]$ holds for all $t \in I$.
(b) An $\mathbb{F}$-submartingale $X = (X_t)_{t\in I}$ is called closable, if there exists $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_t \leq \mathbb{E}[\xi|\mathcal{F}_t]$ holds for all $t \in I$.
(c) An $\mathbb{F}$-supermartingale $X = (X_t)_{t\in I}$ is called closable, if there exists $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_t \geq \mathbb{E}[\xi|\mathcal{F}_t]$ holds for all $t \in I$.

In all cases, ξ is said to close X.

Remark 2.11. (a) Closed martingales are sometimes also referred to as Doob- or Lévy-martingales.
(b) Let t^* an upper bound of I. If ξ closes the submartingale $X = (X_t)_{t\in I}$ and $t^* \notin I$, then X can be extended to a submartingale indexed by $\overline{I} = I \cup \{t^*\}$ by setting $X_{t^*} = \mathbb{E}[\xi|\mathcal{F}_{t^*}]$, where $\mathcal{F}_{t^*} = \sigma(\bigcup_{t\in I} \mathcal{F}_t)$. In particular any closable (sub)martingale is always closable by an $\mathcal{F}_{t^*}$-measurable integrable random variable, and for this reason, some authors[14] call a closed (sub)martingale instead a (sub)martingale with a last element.
(c) Every closed martingale (with an arbitrary index set of times) is uniformly integrable, see [1, Cor. 8.22].
(d) If $X = (X_t)_{t\in I}$ is a uniformly integrable (sub-)martingale with $I = \mathbb{N}_0$ or $I = \mathbb{R}_+$ and in addition X is right-continuous in the latter case, then X is closed by its almost sure limit X_∞ by uniform integrability and martingale convergence.
(e) If $X = (X_t)_{t\in I}$ is either a nonnegative supermartingale or a nonpositive submartingale, then X is closed by $\xi \equiv 0$.

[14] e.g. Karatzas & Shreve.

As a consequence of the martingale convergence theorems 2.4 and 2.9, we obtain that every uniformly integrable (sub-, super-)martingale (with right-continuous sample paths in the continuous time case) converges almost surely and in L^1, and is closable by its almost sure limit.

Theorem 2.12 (Convergence theorem for discrete time uniformly integrable submartingales). *Let $I = \mathbb{N}_0$ or $I = \mathbb{R}_+$ and Let $X = (X_t)_{t\in I}$ a uniformly integrable $\mathbb{F}$-(sub-, super-)martingale. If $I = \mathbb{R}_+$, suppose in addition that X has right-continuous sample paths. Then there exists an $\mathcal{F}_\infty$-measurable integrable random variable X_∞ with $X_t \overset{t\to\infty}{\longrightarrow} X_\infty$ a.s. and in L^1. In addition, X_∞ closes X.*

Proof. It suffices to consider the (sub-)martingale case. Uniform integrability implies that $\sup\left\{\mathbb{E}\left[X_t^+\right] : t \in I\right\} < \infty$. Let $D \subseteq I$ a countable dense subset (if $I = \mathbb{N}_0$, take $D = \mathbb{N}_0$). By theorem 2.4 in the discrete time case theorem 2.9 in the continuous time case, the almost sure limit $\lim_{t\to\infty} X_t \overset{\text{a.s.}}{=} X_\infty$ exists, and L^1-convergence follows by uniform integrability. For the addition, note that $X_t \overset{t\to\infty}{\longrightarrow} X_\infty$ in L^1 and $\|\cdot\|_{L^1(\mathbb{P})}$-continuity of the conditional expectation imply that for any $s \in I$, $\mathbb{E}[X_t|\mathcal{F}_s] \overset{t\to\infty}{\longrightarrow} \mathbb{E}[X_\infty|\mathcal{F}_s]$ in L^1, and by passing to a subsequence $(t_n) \subseteq \mathbb{R}_+$ of times increasing to infinity, we can suppose that $\mathbb{E}[X_{t_n}|\mathcal{F}_s] \overset{n\to\infty}{\longrightarrow} \mathbb{E}[X_\infty|\mathcal{F}_s]$ almost surely. Since $\mathbb{E}[X_{t_n}|\mathcal{F}_s] \geq X_s$ for all large enough n, we find

$$\mathbb{E}[X_\infty|\mathcal{F}_s] \overset{\text{a.s.}}{=} \lim_{t\to\infty} \mathbb{E}[X_t|\mathcal{F}_s] \overset{\text{a.s.}}{\geq} X_s,$$

with equality in the martingale case, proving the claim. □

With remark 2.11 (c) and theorem 2.12, we obtain the following proposition.

Proposition 2.13. *Let $I = \mathbb{N}_0$ or $I = \mathbb{R}_+$ and $X = (X_t)_{t\in I}$ a martingale. If $I = \mathbb{R}_+$, suppose in addition that X has right-continuous sample paths.*[15] *Then the following are equivalent:*

(i) X is closed,
(ii) X is uniformly integrable,
(iii) X_t converges a.s. and in L^1 to a limit X_∞ as $t \to \infty$.

Moreover, if these properties hold, X is closed by X_∞, i.e. we have $X_t = \mathbb{E}[X_\infty|\mathcal{F}_t]$ for every $t \in I$.

[15] Right-continuity is only needed for the implication (ii) ⇒ (iii).

Remark 2.14. Closed sub- and supermartingales are not necessarily uniformly integrable. This is already suggested by remark 2.11 (e), by which any not uniformly integrable nonpositive submartingale would be a counterexample. Indeed, such supermartingales exist. However, for a closeable submartingale X, $(X_t^+)_{t\in I}$ necessarily is uniformly integrable, see proposition 2.16.

Proposition 2.15. *Let $I \neq \varnothing$ a totally ordered index set, t^* an upper bound of I, $\overline{I} = I \cup \{t^*\}$, and let $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtration with $\mathcal{F}_{t^*} = \sigma(\bigcup_{t\in I}\mathcal{F}_t)$.*

(a) A supermartingale $X = (X_t)_{t\in I}$ is closable iff X is a sum of a closable martingale and a nonnegative supermartingale.
(b) A submartingale $X = (X_t)_{t\in I}$ is closable iff X is a sum of a closable martingale and a nonpositive submartingale.

Proof. It suffices to consider the supermartingale case.
$\boxed{\Rightarrow}$ Suppose that X is closed, w.l.o.g. by a $\mathcal{F}_{t^*}$-measurable random variable X_{t^*}, see remark 2.11. Since $\mathbb{E}[X_{t^*}|\mathcal{F}_t] \le X_t$ for all $t \in I$,

$$X_t = \mathbb{E}[X_{t^*}|\mathcal{F}_t] + (X_t - \mathbb{E}[X_{t^*}|\mathcal{F}_t]),$$

gives a decomposition of X into a martingale closed by X_{t^*} and a nonnegative supermartingale.
$\boxed{\Leftarrow}$ Let $X = M + Z$ a decomposition of X as the sum of a martingale $M = (M_t)_{t\in I}$ closed by some $\mathcal{F}_{t^*}$-measurable random variable M_{t^*} and a nonnegative supermartingale $Z = (Z_t)_{t\in I}$. Since Z is nonnegative, it holds that for each $t \in I$,

$$\mathbb{E}[M_{t^*}|\mathcal{F}_t] = M_t \le M_t + Z_t = X_t,$$

i.e. M_{t^*} also closes X. □

Proposition 2.16 (Characterization of closable submartingales). *Let $I \subseteq \overline{\mathbb{R}}_+$ either countable or an interval, $t^* = \sup I \in \overline{\mathbb{R}}_+$ and $\overline{I} = I \cup \{t^*\}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtration and $X = (X_t)_{t\in I}$ an $\mathbb{F}$-submartingale. Then X is closable iff $X^+ := (X_t^+)_{t\in I}$ is uniformly integrable.*

Proof. $\boxed{\Rightarrow}$ Let X_{t^*} an $\mathcal{F}_{t^*}$-measurable random variable which closes X, i.e. it holds that $X_t \le \mathbb{E}[X_{t^*}|\mathcal{F}_t]$ for all $t \in I$. Then $M_t = \mathbb{E}[X_{t^*}|\mathcal{F}_t]$ defines a closed martingale which is uniformly integrable by proposition 2.13. It follows that $(M_t^+)_{t\in I}$ is uniformly integrable, and $X_t \le M_t$ implies that $0 \le X_t^+ \le M_t^+$ for all $t \in I$, i.e. $(X_t^+)_{t\in I}$ is uniformly integrable.
$\boxed{\Leftarrow}$ Suppose that X^+ is uniformly integrable and let $D \subseteq I$ a countable dense subset (if I is countable, we can take $D = I$). Since X^+ is also a submartingale, by the proof[16] of theorem 2.9, there exists an integrable random variable $\xi \ge 0$ with

[16] We are being slightly imprecise here: If I is an interval, this follows exactly as in the proof of theorem 2.9, while in the general case, similar arguments as in that proof are applicable. If $I = \mathbb{N}_0$, this is also clear from theorem 2.4. In the sequel, we will use the direction "⇐" of proposition 2.16 only in the discrete time case.

$$\lim_{\substack{t\to t^* \\ t\in D}} X_t^+ \overset{\text{a.s.}}{=} \xi.$$

By uniform integrability, the convergence is also in L^1, hence for any $s \in I \setminus \{t^*\}$,

$$\mathbb{E}[\xi|\mathcal{F}_s] = \lim_{\substack{t\to t^* \\ t\in D}} \mathbb{E}[X_t^+|\mathcal{F}_s] \geq X_s^+,$$

where the limit is in the sense of $\|\cdot\|_{L^1}$, the first equality is due to $\|\cdot\|_{L^1}$-continuity of the conditional expectation and the second follows from the submartingale property of X^+ and by passing to an almost surely convergent subsequence. Hence ξ closes X^+ and since for all $s \in I$,

$$X_s \leq X_s^+ \leq \mathbb{E}[\xi|\mathcal{F}_s],$$

ξ also closes X. □

3 Optional Sampling

3.1 Optional Sampling Theorems for Discrete Time Martingales

This section gives several optional sampling theorems for discrete time (sub-)martingales. Theorem 3.4 establishes that $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_{\sigma\wedge\tau}$ for an arbitrary submartingale X (with equality in the martingale case) and $\mathbb{N}_0$-valued stopping times σ and τ when τ is bounded, while theorem 3.8 extends this inequality to the case of $\overline{\mathbb{N}}_0$-valued stopping times when the positive part of the submartingale X is uniformly integrable.

We begin with some preliminaries. Let $I \neq \varnothing$ a totally ordered set index and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtered probability space. Furthermore let t^* an upper bound of I, $\mathcal{F}_{t^*} := \sigma(\bigcup_{t\in I} \mathcal{F}_t)$ and denote $\overline{I} = I \cup \{t^*\}$.

Definition 3.1. Let $X = (X_t)_{t\in I}$ an $\mathbb{F}$-adapted stochastic process taking values in a measurable space $(E, \mathcal{E})$.

(a) For any $\overline{I}$-valued random time τ, we define $X_\tau \colon \{\tau \in I\} \to E$ by $X_\tau(\omega) := X_{\tau(\omega)}(\omega)$.

(b) If X_{t^*} is a given $\mathcal{F}_{t^*}$-measurable[17] random variable, X_τ can be defined on all of Ω by

$$X_\tau(\omega) := \begin{cases} X_{\tau(\omega)}(\omega) & \omega \in \{\tau \in I\}, \\ X_{t^*} & \omega \in \{\tau = t^*\}. \end{cases}$$

We will always use this extension whenever X has an almost sure limit X_{t^*} as $t \to t^*$.

[17] $\mathcal{F}_{t^*}$-measurability is only necessary when $\mathcal{F}_\tau$ is by definition a sub-σ-algebra of $\mathcal{F}_{t^*}$ (this depends on the author).

Lemma 3.2. *If I is countable and $X = (X_t)_{t\in I}$ is an $\mathbb{F}$-adapted stochastic process taking values in a measurable space $(E, \mathcal{E})$, then X_τ is $\mathcal{F}_\tau$-measurable.*

Proof. For any $A \in \mathcal{E}$ and any $t \geq 0$, we have $\{X_\tau \in A\} \cap \{\tau \leq t\} = \bigcup_{s\leq t}\{\tau = s\} \cap \underbrace{\{X_s \in A\}}_{\in\mathcal{F}_s\subseteq\mathcal{F}_t} \in \mathcal{F}_t$. □

Lemma 3.3 (Discrete time optional sampling lemma). *Let I countable and $X = (X_t)_{t\in I}$ a martingale. Let τ an I-valued stopping time and suppose that $T \in I$ is such that $\tau \leq T$. Then X_τ is integrable and it holds that*

$$\mathbb{E}[X_T|\mathcal{F}_\tau] = X_\tau.$$

Proof. (i) We first show that X_τ is integrable. Note that $(|X_t|)_{t\in I}$ is a submartingale, hence $|X_t| \le \mathbb{E}[|X_T||\mathcal{F}_t]$ for all $t \le T$. Since $\{\tau = t\} \in \mathcal{F}_t$ due to countability of I, it follows that

$$\begin{aligned}\mathbb{E}[|X_\tau|] &= \sum_{t\le T} \mathbb{E}[|X_t|\,\mathbb{1}_{\{\tau=t\}}] \le \sum_{t\le T} \mathbb{E}[\mathbb{E}[|X_T||\mathcal{F}_t]\,\mathbb{1}_{\{\tau=t\}}] \\ &= \sum_{t\le T} \mathbb{E}[|X_T|\,\mathbb{1}_{\{\tau=t\}}] \le \mathbb{E}[|X_T|] < \infty,\end{aligned}$$

where we have used the monotone convergence theorem in the first and last equality.

(ii) By lemma 3.2, X_τ is $\mathcal{F}_\tau$-measurable. Moreover, using that $\{\tau = t\} \cap A \in \mathcal{F}_t$ for all $t \in I$, a similar calculation as above shows that $\mathbb{E}[X_\tau \mathbb{1}_A] = \mathbb{E}[X_T \mathbb{1}_A]$ holds for all $A \in \mathcal{F}_\tau$, which proves the claim.

□

Theorem 3.4 (Optional sampling for discrete time submartingales and bounded stopping times). *Let $X = (X_n)_{n\in\mathbb{N}_0}$ a submartingale and let σ and τ $\overline{\mathbb{N}}_0$-valued stopping times. Suppose that $N \in \mathbb{N}$ is such that $\tau \le N$. Then it holds that*

$$\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \ge X_{\sigma\wedge\tau},$$

with equality in the martingale case.

Proof. The idea is to treat the martingale case first and assume $\sigma \le \tau$, in which case the optional sampling lemma is helpful. The restriction $\sigma \le \tau$ is removed using properties of stopping times, and the extension to submartingales works by Doob-decomposition. By lemma 3.2, $X_{\sigma\wedge\tau}$ is $\mathcal{F}_{\sigma\wedge\tau}$ measurable, hence also $\mathcal{F}_\sigma$-measurable by lemma 1.7. Integrability of X_τ follows from $|X_\tau| \le \sum_{n=0}^N \mathbb{1}_{\{\tau=n\}} |X_n|$, and similarly for $X_{\sigma\wedge\tau}$.

(a) Suppose first that X is a martingale and that $\sigma \le \tau$. Then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$, so lemma 3.3 and the tower property of conditional expectation imply

$$\mathbb{E}[X_\tau|\mathcal{F}_\sigma] = \mathbb{E}[\mathbb{E}[X_T|\mathcal{F}_\tau]|\mathcal{F}_\sigma] = X_\sigma.$$

(b) Now let X a martingale but remove the assumption $\sigma \le \tau$. We verify that $\mathbb{E}[X_\tau \mathbb{1}_A] = \mathbb{E}[X_{\sigma\wedge\tau} \mathbb{1}_A]$ holds for every $A \in \mathcal{F}_\sigma$. Clearly,

$$\mathbb{E}\left[X_\tau \mathbb{1}_{A\cap\{\tau<\sigma\}}\right] = \mathbb{E}\left[X_{\sigma\wedge\tau} \mathbb{1}_{A\cap\{\tau<\sigma\}}\right],$$

and by lemma 1.7, it holds that $A \cap \{\sigma \le \tau\} \in \mathcal{F}_{\sigma\wedge\tau}$. Hence by part (a),

$$\mathbb{E}[X_\tau \mathbb{1}_{A\cap\{\sigma\le\tau\}}] = \mathbb{E}\left[\mathbb{E}[X_\tau|\mathcal{F}_{\sigma\wedge\tau}]\,\mathbb{1}_{A\cap\{\sigma\le\tau\}}\right] = \mathbb{E}[X_{\sigma\wedge\tau} \mathbb{1}_{A\cap\{\sigma\le\tau\}}],$$

proving the claim.

(c) If X is a submartingale, let $X = M + A$ the Doob decomposition of X from theorem 1.1, i.e. $M = (M_n)_{n\in\mathbb{N}_0}$ is a martingale and $A = (A_n)_{n\in\mathbb{N}_0}$ is predictable and increasing with $A_0 = 0$. It holds that $A_\tau = X_\tau - M_\tau$ is integrable, $A_\tau \ge A_{\sigma\wedge\tau}$, and $A_{\sigma\wedge\tau}$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable, and thus also $\mathcal{F}_\sigma$-measurable. With part (b) applied to the martingale M and monotonicity and linearity of the conditional expectation, we obtain

$$\begin{aligned}\mathbb{E}[X_\tau|\mathcal{F}_\sigma] &= \mathbb{E}[M_\tau|\mathcal{F}_\sigma] + \mathbb{E}[A_\tau|\mathcal{F}_\sigma] \\ &= M_{\sigma\wedge\tau} + \mathbb{E}[A_\tau|\mathcal{F}_\sigma] \ge M_{\sigma\wedge\tau} + A_{\sigma\wedge\tau} = X_{\sigma\wedge\tau},\end{aligned}$$

proving the claim.

□

Remark 3.5. (a) Theorem 3.4 does not assume boundedness of σ.

(b) It is not hard to check that the results in lemma 3.3 and theorem 3.4 remain valid when τ is only bounded above by some constant almost surely, if we define $X_\tau = 0$ on $\{\tau = \infty\}$ and similarly for $X_{\sigma\wedge\tau}$.

(c) For the steps (a) and (b) of the proof above, we used only the optional sampling lemma 3.3,[18] so for martingales, theorem 3.4 carries over to general countable index sets I without difficulties.

(d) For submartingales, we restricted to the case $I = \mathbb{N}_0$, since for a submartingale X and a stopping time τ with a general countable index set of times, integrability of X_τ requires an added assumption. In addition, we can in this case no longer rely on Doob's decomposition theorem theorem 1.1. However, the optional sampling theorem remains valid in this setting.[19]

[18] We showed integrability of X_τ using that $I = \mathbb{N}_0$, but integrability is entailed in lemma 3.3 in the martingale case

[19] See [3, §4.5] for a very general version of the optional sampling theorem treating several cases (discrete & continuous time, the uniformly integrable cases) in a uniform manner.

For nonnegative supermartingales, the optional sampling theorem holds even for $\overline{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$-valued stopping times.

Theorem 3.6 (Optional sampling for nonnegative supermartingales and nonpositive submartingales in discrete time). *Let $X = (X_n)_{n\in\mathbb{N}_0}$ a nonnegative supermartingale, and let σ and τ $\overline{\mathbb{N}}_0$-valued stopping times. Then it holds that*

$$\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \leq X_{\sigma\wedge\tau}.$$

Similarly, if $X = (X_n)_{n\in\mathbb{N}_0}$ is a nonpositive submartingale, then

$$\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_{\sigma\wedge\tau}.$$

Proof. It suffices to consider the supermartingale case. We have $X_{\tau\wedge n} \overset{n\to\infty}{\longrightarrow} X_\tau$ and similarly $X_{\sigma\wedge\tau\wedge n} \overset{n\to\infty}{\longrightarrow} X_{\sigma\wedge\tau}$. By theorem 3.4, it holds that $\mathbb{E}[X_{\tau\wedge n}|\mathcal{F}_\sigma] \leq X_{\sigma\wedge\tau\wedge n}$, and nonnegativity of X allows to apply Fatou's lemma for the conditional expectation, yielding

$$\mathbb{E}[X_\tau|\mathcal{F}_\sigma] = \mathbb{E}\left[\varliminf_{n\to\infty} X_{\tau\wedge n}|\mathcal{F}_\sigma\right] \leq \varliminf_{n\to\infty} \mathbb{E}\left[X_{\tau\wedge n}|\mathcal{F}_\sigma\right] \leq \varliminf_{n\to\infty} X_{\sigma\wedge\tau\wedge n} = X_{\sigma\wedge\tau}.$$

□

The next lemma will allow to obtain an optional sampling theorem for unbounded and even possibly infinite stopping times, provided the considered (sub-)martingale is uniformly integrable.

Lemma 3.7. *Let $X = (X_n)_{n\in\mathbb{N}_0}$ a uniformly integrable submartingale. Then the family $\mathcal{T} = \{X_\tau : \tau$ is an $\overline{\mathbb{N}}_0$-valued stopping time$\}$ is uniformly integrable.*

Proof. (a) Suppose that X is a uniformly integrable martingale and Let X_∞ the A.s. And L^1-limit of X_n as $n \to \infty$. Since X is uniformly integrable and X_∞ is integrable, the family $\mathcal{X} = \{X_n : n \in \overline{\mathbb{N}}_0\}$ is uniformly integrable. By [1, Theorem 6.19], there exists a test function of uniform integrability for $\mathcal{X}$, i.e. a monotone increasing, convex function $H\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\varliminf_{x\to\infty} H(x)/x = \infty$ and $L := \sup_{n\in\overline{\mathbb{N}}_0} \mathbb{E}[H(|X_n|)] < \infty$.
If τ is an $\overline{\mathbb{N}}_0$-valued stopping time, then by the optional sampling theorem for bounded stopping times (theorem 3.4 with $\tau = n$ and $\sigma = \tau \wedge n$), we have $\mathbb{E}[X_n \mid \mathcal{F}_{\tau\wedge n}] = X_{\tau\wedge n}$ and thus $|X_{\tau\wedge n}| \leq \mathbb{E}[|X_n||\mathcal{F}_{\tau\wedge n}]$ by the triangle inequality for the conditional expectation. Since $\{\tau \leq n\} \in \mathcal{F}_{\tau\wedge n}$ by lemma 1.7, Jensen's inequality for the conditional expectation yields[20]

[20] This argument works also with martingales and nonnegative submartingales in continuous time, as long as they have right-continuous sample paths in order to allow the application of theorem 3.14.
In general, the argument fails for submartingales, since here we have

$$\mathbb{E}[H(|X_\tau|)\,\mathbb{1}_{\{\tau\le n\}}] = \mathbb{E}[H(|X_{\tau\wedge n}|)\,\mathbb{1}_{\{\tau\le n\}}] \le \mathbb{E}[H(\mathbb{E}[|X_n|\,|\,\mathcal{F}_{\tau\wedge n}])\,\mathbb{1}_{\{\tau\le n\}}]$$
$$\le \mathbb{E}[\mathbb{E}[H(|X_n|)\,|\,\mathcal{F}_{\tau\wedge n}]\,\mathbb{1}_{\{\tau\le n\}}] = \mathbb{E}[H(|X_n|)\,\mathbb{1}_{\{\tau\le n\}}] \le L,$$

and thus $\mathbb{E}[H(|X_\tau|)\,\mathbb{1}_{\{\tau<\infty\}}] \le L$. In addition, $X_\infty \in \mathcal{X}$ by martingale convergence, which implies that $\mathbb{E}[H(|X_\infty|)\,\mathbb{1}_{\{\tau=\infty\}}] \le \mathbb{E}[H(|X_\infty|)] \le L$, hence $\mathbb{E}[H(|X_\tau|)] \le 2L$. It follows[21] $\mathcal{T} \subseteq \mathcal{L}^1(\mathbb{P})$ and that H is also a test function of uniform integrability for $\mathcal{T}$, which is therefore uniformly integrable by [1, Theorem 6.19].

(b) If X is a uniformly integrable submartingale, let $X = M + A$ the Doob decomposition of X, i.e. $M = (M_n)_{n\in\mathbb{N}_0}$ is a uniformly integrable martingale and $A = (A_n)_{n\in\mathbb{N}_0}$ is predictable and increasing with $A_0 = 0$ and $A_\infty \in \mathcal{L}^1(\mathbb{P})$, see theorem 1.1. By part (a) above, $\{M_\tau : \tau \text{ is an } \overline{\mathbb{N}}_0\text{-valued stopping time}\}$ is uniformly integrable, and since $0 \le A_\tau \le A_\infty$ for any $\overline{\mathbb{N}}_0$-valued stopping time τ, we also have that $\{A_\tau : \tau \text{ is an } \overline{\mathbb{N}}_0\text{-valued stopping time}\}$ is uniformly integrable, proving the claim.

□

Using lemma 3.7, we can extend the optional sampling theorem to $\overline{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$-valued stopping times and martingales, and by propositions 2.15 and 2.16, to sub- and supermartingales.

Theorem 3.8 (Optional sampling theorem for closable submartingales in discrete time). *Let $X = (X_n)_{n\in\mathbb{N}_0}$ a submartingale such that $(X_n^+)_{n\in\mathbb{N}_0}$ is uniformly integrable, and let σ and τ $\overline{\mathbb{N}}_0$-valued stopping times. Then X_τ is integrable and it holds that*

$$\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \ge X_{\sigma\wedge\tau}$$

with equality if X is a martingale.

Proof. The idea is to use lemma 3.7 to show the claim for uniformly integrable martingales, then decompose a general closeable submartingale into the sum of a closable martingale and a nonpositive submartingale, the latter of which theorem 3.6 applies to.

(a) Suppose first that X is a uniformly integrable martingale. By theorem 3.4, we have that for each $n \in \mathbb{N}_0$,

$$\mathbb{E}[X_{\tau\wedge n}|\mathcal{F}_\sigma] = X_{\sigma\wedge\tau\wedge n}.$$

As $n \to \infty$, it holds that $X_{\tau\wedge n} \to X_\tau$, and similarly $X_{\sigma\wedge\tau\wedge n} \to X_{\sigma\wedge n}$. By uniform integrability of X and lemma 3.7, the family $\{X_{\tau\wedge n} : n \in \mathbb{N}_0\}$ is uniformly integrable, implying that the convergence $X_{\tau\wedge n} \to X_\tau$ happens also in L^1, see [1, thm. 6.25].[22] Similarly, we obtain $X_{\sigma\wedge\tau\wedge n} \to X_{\sigma\wedge\tau}$ in L^1. Due to $\|\cdot\|_{L^1(\mathbb{P})}$-continuity (or the triangle inequality) of conditional expectation, it follows that

$$\mathbb{E}[X_{\tau\wedge n}|\mathcal{F}_\sigma] \xrightarrow{L^1} \mathbb{E}[X_\tau|\mathcal{F}_\sigma],$$

hence $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] = X_{\sigma\wedge\tau}$ by a.s. uniqueness of $\|\cdot\|_{L^1(\mathbb{P})}$ limits.[23]

(b) Let $X = (X_n)_{n\in\mathbb{N}_0}$ a submartingale such that $(X_n^+)_{n\in\mathbb{N}_0}$ is uniformly integrable.[24] By proposition 2.16, X is closable, and by proposition 2.15, there is a decomposition $X = M + Z$ with $M = (M_n)_{n\in\mathbb{N}_0}$ closable martingale and $Z = (Z_n)_{n\in\mathbb{N}_0}$ a nonpositive submartingale. By part (a), M_τ is integrable with $\mathbb{E}[M_\tau|\mathcal{F}_\sigma] = M_{\sigma\wedge\tau}$, and theorem 3.6 shows integrability of Z_τ as well as $\mathbb{E}[Z_\tau|\mathcal{F}_\sigma] \ge Z_{\sigma\wedge\tau}$, proving the claim.

□

[21] Since $\underline{\lim}_{x\to\infty} H(x)/x = \infty$, there exists $x_0 \in \mathbb{R}_+$ s.t. $H(x)/x \ge 1$ for all $x \ge x_0$. Hence for all $x \in \mathbb{R}_+$, it holds that $x \le \max(x_0, H(x))$.

[22] Integrability of X_τ follows here either from [1, thm. 6.25], or Fatou's lemma and L^1-boundedness of the UI family $\{X_{\tau\wedge n} : n \in \mathbb{N}_0\}$, or directly from lemma 3.7.

[23] Alternatively, we can invoke again the triangle inequality for $\|\cdot\|_{L^1(\mathbb{P})}$.

[24] If X is uniformly integrable, part (b) can be shown exactly as in part (a) (or inferred from part (a) via Doob-decomposition), i.e. use of propositions 2.15 and 2.16 can be avoided in that case.

Remark 3.9 (Summary: discrete time optional sampling). For a discrete time submartingale $X = (X_n)_{n\in\mathbb{N}_0}$ and σ, τ stopping times with values in $\overline{\mathbb{N}}_0$, theorems 3.4 and 3.8 show that X_τ is integrable and it holds that

$$\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_{\sigma\wedge\tau}$$

with equality in the martingale case if either

(a) there is a $T \in \mathbb{N}$ with $\tau \leq T$, or
(b) $(X_n^+)_{n\in\mathbb{N}_0}$ is uniformly integrable.

Corollary 3.10. *Let X a uniformly integrable martingale (resp. submartingale) and let $(\tau_n)_{n\in\mathbb{N}_0}$ an increasing sequence of $\overline{\mathbb{N}}_0$-valued stopping times. Then $(X_{\tau_n})_{n\in\mathbb{N}_0}$ is an $(\mathcal{F}_{\tau_n})_{n\in\mathbb{N}_0}$-martingale (resp. submartingale).*

3.2 Optional Sampling Theorems for Continuous Time Martingales

Lemma 3.11. *Let $X = (X_t)_{t\geq 0}$ an $\mathbb{F}$-progressive stochastic process taking values in a metric space (E, d) with Borel-σ-algebra $\mathcal{B}_E$.*

(a) Let τ an $\overline{\mathbb{R}}_+$-valued $\mathbb{F}$-stopping time. Then the function $X_\tau \colon \{\tau < \infty\} \to E$, $\omega \mapsto X_{\tau(\omega)}(\omega)$ is $\mathcal{F}_\tau$-measurable.
(b) Let τ an $\overline{\mathbb{R}}_+$-valued $\mathbb{F}^+$-stopping time. Then X_τ as defined in (a) is $\mathcal{F}_{\tau+}$-measurable.
(c) Suppose that $(X_t)_{t\geq 0}$ converges almost surely to some $\mathcal{F}_\infty$-measurable random variable X_∞ as $t \to \infty$.[25] In this case, the definition from part (a) is extended to all of Ω by[26]

$$X_\tau(\omega) = \begin{cases} X_{\tau(\omega)}(\omega) & \omega \in \{\tau < \infty\}, \\ X_\infty(\omega) & \omega \in \{\tau = \infty\}, \end{cases} \tag{4}$$

and this version of X_τ is also $\mathcal{F}_\tau$-measurable. Similarly, if τ is an $\mathbb{F}^+$-stopping time, then X_τ is $\mathcal{F}_{\tau+}$-measurable.
(d) Let σ an $\mathbb{F}$-stopping time and τ an $\mathbb{F}^+$-stopping time. Then $X_{\sigma\wedge\tau}$ (defined in this case analogously to (a) resp. (c)) is $(\mathcal{F}_\sigma \cap \mathcal{F}_{\tau+})$-measurable.

[25] We do not fix a choice of limit as in convention 1, since in the continuous time case it is less clear what the canonical choice would be.

[26] If $E = \mathbb{R}$, this is sometimes written slightly informally as $X_\tau(\omega) := \mathbb{1}_{\tau(\omega)<\infty} X_{\tau(\omega)}(\omega) + \mathbb{1}_{\tau(\omega)=\infty} X_\infty(\omega)$.

Proof. (a) It suffices to show that every restriction $X_\tau|_{\{\tau\leq t\}} \colon \{\tau \leq t\} \ni \omega \mapsto X_\tau(\omega)$, $t \geq 0$ is $\mathcal{F}_t$-measurable. For this, note that $X_\tau|_{\{\tau\leq t\}} = X|_{[0,t]\times\Omega} \circ \varphi$, where $\varphi \colon \{\tau \leq t\} \ni \omega \mapsto (\tau(\omega), \omega) \in \Omega \times [0, t]$ is $\mathcal{F}_t$-$(\mathcal{B}_{[0,t]\otimes\mathcal{F}_t})$-measurable. Indeed, $\{\tau \leq t\} \ni \omega \mapsto \omega \in \Omega$ is $\mathcal{F}_t$-$\mathcal{F}_t$-measurable, and similarly $\tau|_{\{\tau\leq t\}} \colon \{\tau \leq t\} \ni \omega \mapsto \tau(\omega) \in [0, t]$ is $\mathcal{F}_t$-$\mathcal{B}_{[0,t]}$-measurable, since for all $s \in [0, t]$, we have $\{\tau|_{\{\tau\leq t\}} \leq s\} = \{\tau \leq s \wedge t\} \in \mathcal{F}_t$. Since $X|_{[0,t]\times\Omega}$ is $(\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t)$-measurable by progressive measurability, the claim follows.
(b) This follows from part (a), since $\mathbb{F}$-progressiveness implies $\mathbb{F}^+$-progressiveness.
(c) Since X_∞ and $\{\tau = \infty\} = \bigcap_{n\geq 1}\{\tau > n\}$ are $\mathcal{F}_\infty$-measurable, it holds that X_τ as defined in (4) is $\mathcal{F}_\infty$-measurable.[27] For any $A \in \mathcal{B}_E$, we have

$$\{X_\tau \in A\} = \{\tau < \infty\} \cap \{X_\tau|_{\{\tau<\infty\}} \in A\} \sqcup \{\tau = \infty\} \cap \{X_\infty \in A\} \in \mathcal{F}_\infty,$$

hence by part (a), $\{X_\tau \in A\} \cap \{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$, which proves the claim in the case in which τ is a stopping time. The same argument with $\{\tau < t\}$ instead of $\{\tau \leq t\}$ proves the claim for $\mathbb{F}^+$-stopping times.
(d) Since $\sigma \wedge \tau$ is an optional time, $X_{\sigma\wedge\tau}$ is $\mathcal{F}_{(\sigma\wedge\tau)+}$-measurable and thus $\mathcal{F}_{\tau+}$-measurable by part (b) resp. part (c) of the present lemma, and it remains to show $\mathcal{F}_\sigma$-measurability. As in part (a) of the proof, it suffices that $X_{\sigma\wedge\tau}|_{\{\sigma\leq t\}}$

[27] $\mathcal{F}_\infty$-measurability of X_∞ is only required if $\mathcal{F}_\tau$ is defined as a sub-σ-algebra of $\mathcal{F}_\infty$.

is $\mathcal{F}_t$-measurable for every $t \geq 0$, which follows as in part (a) from $\mathcal{F}_t$-$\mathcal{B}_{[0,t]}$-measurability of $\omega \mapsto \sigma(\omega) \wedge \tau(\omega) \wedge t$ (see also lemma 1.10 (d), which with $\sigma = t$ and $\tau = \sigma$ gives that $\tau \wedge t$ is $\mathcal{F}_t$-measurable).

□

Theorem 3.12 (Optional sampling theorem for right-continuous closeable submartingales). *Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ a filtration and $(X_t)_{t\geq 0}$ a right-continuous $\mathbb{F}$-submartingale such that $(X_t^+)_{t\geq 0}$ is uniformly integrable. Let σ and τ two $\mathbb{F}^+$-stopping times. Then X_τ is integrable and it holds that*

$$\mathbb{E}[X_\tau | \mathcal{F}_{\sigma+}] \geq X_{\sigma\wedge\tau}.$$

If σ is an $\mathbb{F}$-stopping time, then $\mathcal{F}_\sigma$ can replace $\mathcal{F}_{\sigma+}$ above.

Proof. For $n \in \mathbb{N}$, let $D_n = 2^{-n}\mathbb{N}_0 = \{k \cdot 2^{-n} : k \in \mathbb{N}_0\}$, and let $\sigma_n = 2^{-n}(\lfloor 2^n\sigma \rfloor + 1)$ and $\tau_n = 2^{-n}(\lfloor 2^n\tau\rfloor + 1)$, which are D_n-valued $\mathbb{F}$-stopping times by corollary 1.12. In fact, σ_n and τ_n are stopping times of the "discretized" filtrations $(\mathcal{F}_t)_{t\in D_m}$ for all $m \geq n$.[28] Thus we may apply the discrete time optional sampling theorem 3.8 to the closed $(\mathcal{F}_t)_{t\in D_n}$-submartingale $(X_t)_{t\in D_n}$ to see that

$$\mathbb{E}[X_{\tau_n} | \mathcal{F}_{\sigma_n}] \geq X_{\sigma_n \wedge \tau_n}. \tag{5}$$

Since $\sigma_n \searrow \sigma$ and $\sigma < \sigma_n$ for all $n \in \mathbb{N}$ on $\{\sigma < \infty\}$, we have by lemma 1.10 (e) that $\mathcal{F}_{\sigma+} = \bigcap_{n\geq 1} \mathcal{F}_{\sigma_n}$, hence eq. (5) implies that

$$\mathbb{E}[X_{\tau_n} \mathbb{1}_A] \geq \mathbb{E}[X_{\sigma_n\wedge\tau_n} \mathbb{1}_A], \qquad A \in \mathcal{F}_{\sigma+}. \tag{6}$$

By theorem 3.8, $(X_{\tau_n})_{n\in\mathbb{N}}$ is a backwards submartingale w.r.t. the (backwards) filtration $(\mathcal{F}_{\tau_n})_{n\in\mathbb{N}}$.[29] Since $\inf_{n\in\mathbb{N}} \mathbb{E}[X_{\tau_n}] \geq \mathbb{E}[X_0]$, $(X_{\tau_n})_{n\in\mathbb{N}}$ is uniformly integrable by theorem 1.5, and the same is true for $(X_{\sigma_n\wedge\tau_n})_{n\in\mathbb{N}}$. By right-continuity, $X_\tau = \lim_{n\to\infty} X_{\tau_n}$ and $X_{\sigma\wedge\tau} = \lim_{n\to\infty} X_{\sigma_n\wedge\tau_n}$,[30] so uniform integrability of (X_{τ_n}) and $(X_{\sigma_n\wedge\tau_n})$ implies that X_τ and $X_{\sigma\wedge\tau}$ are integrable and allows to pass to the limit in eq. (6), which yields

$$\mathbb{E}[X_\tau \mathbb{1}_A] \geq \mathbb{E}[X_{\sigma\wedge\tau} \mathbb{1}_A], \qquad A \in \mathcal{F}_{\sigma+}.$$

If σ is a stopping time, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_{\sigma+} = \bigcap_{n\geq 1} \mathcal{F}_{\sigma_n}$, moreover in this case $X_{\sigma\wedge\tau}$ is $\mathcal{F}_\sigma$-measurable by lemma 3.11 (d), hence $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_\sigma$ follows. Similarly, if σ is an optional time, $X_{\sigma\wedge\tau}$ is $\mathcal{F}_{\sigma+}$-measurable by lemma 3.11 (c),[31] hence $\mathbb{E}[X_\tau|\mathcal{F}_{\sigma+}] \geq X_\sigma$, proving the claim.

□

[28] Indeed, $\{\sigma_n \leq 2^{-n}k\} = \{\lfloor 2^n\sigma\rfloor \leq k-1\} = \{2^n\sigma < k\} = \{\sigma < 2^{-n}k\} \in \mathcal{F}_{2^{-n}k}$, which takes care of the case $m = n$, and the case $m \geq n$ is similar.

[29] Adaptedness follows from lemma 3.11 and integrability from theorem 3.8 applied to the submartingale $(X_t)_{t\in D_m}$ and the $(\mathcal{F}_t)_{t\in D_m}$-stopping times τ_n and τ_m, which also yields that for any $0 \leq n \leq m$, $\mathbb{E}[X_{\tau_n}|\mathcal{F}_{\tau_m}] \geq X_{\tau_m}$. This shows that $(X_{\tau_n})_{n\in\mathbb{N}}$ is a backwards martingale by lemma 1.3.

[30] Its easy to see that $\sigma_n \wedge \tau_n \searrow \sigma\wedge\tau$ by considering the cases $\sigma \leq \tau$ and $\sigma > \tau$ separately.

[31] This follows also directly from $X_{\sigma\wedge\tau} = \lim_{n\to\infty} X_{\sigma_n\wedge\tau_n}$ and $\mathcal{F}_{(\sigma\wedge\tau)+} = \bigcap_{n\geq 1} \mathcal{F}_{\sigma_n\wedge\tau_n}$

Remark 3.13. In the proof of theorem 3.12, if we consider only uniformly integrable martingales instead of submartingales, the uniform integrability of (X_{τ_n}) and $(X_{\sigma_n\wedge\tau_n})$ is clear from closeability, so we can omit the backwards martingale argument in its entirety.

Theorem 3.14 (Optional sampling theorem for right-continuous submartingales and bounded stopping times). *Let $(X_t)_{t\geq 0}$ a right-continuous $\mathbb{F}$-submartingale, and let σ and τ two $\mathbb{F}^+$-stopping times. Suppose $T \in \mathbb{R}_+$ is such that $\tau \leq T$. Then X_τ is integrable and it holds that*

$$\mathbb{E}[X_\tau | \mathcal{F}_{\sigma+}] \geq X_{\sigma\wedge\tau}.$$

If σ is an $\mathbb{F}$-stopping time, then $\mathcal{F}_\sigma$ can replace $\mathcal{F}_{\sigma+}$ above.

Proof. Consider the $\mathbb{F}$-submartingale $X^T := (X_{t\wedge T})_{t\geq 0}$, which is closed by X_T.[32] By proposition 2.16, $(X^+_{t\wedge T})_{t\geq 0}$ is uniformly integrable, hence theorem 3.12 implies that $X_{\tau\wedge T} = X_\tau$ is integrable with $\mathbb{E}[X_{\tau\wedge T}|\mathcal{F}_{\sigma+}] \geq X_{\sigma\wedge\tau\wedge T}$. It follows that

$$\mathbb{E}[X_\tau|\mathcal{F}_{\sigma+}] \geq X_{\sigma\wedge\tau},$$

and the same argument shows that if σ is an $\mathbb{F}$-stopping time, then $\mathcal{F}_\sigma$ can replace $\mathcal{F}_{\sigma+}$. □

[32] Indeed, X^T is adapted and integrable, and for $0 \leq s \leq t$, $\mathbb{E}[X_{t\wedge T}|\mathcal{F}_s] \geq X_{s\wedge T}$ follows by considering separately the cases $s > T$, $s \leq t \leq T$ and $s \leq T < t$. Similarly, one verifies $\mathbb{E}[X_T|\mathcal{F}_s] \geq X_{s\wedge T}$ for all $s \geq 0$.

Corollary 3.15. *Every right-continuous $\mathbb{F}$-(sub)martingale $X = (X_t)_{t\geq 0}$ is also an $\mathbb{F}^+$-(sub)martingale.*

Proof. Since X is $\mathbb{F}$-adapted and $\mathbb{F}$ is a subfiltration of $\mathbb{F}^+$, X is $\mathbb{F}^+$-adapted. Integrability is clear, and for $0 \leq s \leq t$, the optional sampling theorem 3.14 with the bounded optional times $\tau \equiv t$ and $\sigma \equiv s$ yields $\mathbb{E}[X_t|\mathcal{F}_{s+}] \geq X_s$, proving the claim. □

4 Optional Stopping

Let $\varnothing \neq I \subseteq \mathbb{R}_+$ an index set, $t^* = \sup I$, $\overline{I} = I \cup \{t^*\}$ and $\mathbb{F} = (\mathcal{F}_t)_{t\in I}$ a filtration.

Definition 4.1 (Stopped Process). (a) Let $(X_t)_{t\in I}$ a stochastic process taking values in a measurable space $(E, \mathcal{E})$ and τ an $\overline{I}$-valued random time. Then the stopped process $X^\tau = (X^\tau_t)_{t\in I}$ is defined by

$$X^\tau_t = X_{\tau\wedge t}, \qquad \text{for all } t \in I.$$

(b) For an $\overline{I}$-valued stopping time τ, the filtration $\mathbb{F}^\tau$ is defined by $\mathbb{F}^\tau := (\mathcal{F}_{\tau\wedge t})_{t\in I}$.

Theorem 4.2 (Optional Stopping Theorem for Discrete Time Submartingales). *Let $X = (X_n)_{n\in\mathbb{N}_0}$ an $\mathbb{F}$-(sub-,super-)martingale and τ an $\overline{\mathbb{N}}_0$-valued stopping time. Then it holds that:*

(a) X^τ is a (sub-,super-)martingale w.r.t. the filtrations $\mathbb{F}$ and $\mathbb{F}^\tau$.
(b) If X is uniformly integrable, then X^τ is a uniformly integrable (sub-,super-)martingale closed by X_τ.

Proof. It suffices to consider the (sub-)martingale case.

(a) By lemma 1.10 (d), it holds that for each $n \geq 0$, $\tau \wedge n$ is $\mathcal{F}_n$-measurable. By lemma 3.2, X^τ is $\mathbb{F}^\tau$ adapted, and since $\mathbb{F}^\tau$ is a subfiltration of $\mathbb{F}$, X^τ is also $\mathbb{F}$-adapted. Thus, by the tower property of the conditional expectation, it suffices to show that X^τ is an $\mathbb{F}$-(sub-)martingale. By the optional sampling theorem 3.4 for bounded stopping times, X^τ is an integrable process, and we have for all $k, n \in \mathbb{N}_0$ with $k \leq n$ that

$$\mathbb{E}[X^\tau_n|\mathcal{F}_k] = \mathbb{E}[X_{\tau\wedge n}|\mathcal{F}_k] \geq X_{\tau\wedge n\wedge k} = X_{\tau\wedge k} = X^\tau_k,$$

with equality in the martingale case. Hence X^τ is an $\mathbb{F}$-(sub-)martingale.

(b) In addition, if X is uniformly integrable, uniform integrability of the family $\{X_{\tau\wedge n} : n \in \overline{\mathbb{N}}_0\} \subseteq \{X_\sigma : \sigma \text{ is an } \overline{\mathbb{N}}_0\text{-valued stopping time}\}$ follows from lemma 3.7. In this case, $\lim_{n\to\infty} X^\tau_n = X_\tau$ closes X^τ by theorem 2.12 or alternatively theorem 3.8.

□

For the optional stopping theorem in continuous time, we require the following lemma.

Lemma 4.3 (Adaptedness of the stopped process in continuous time). *Let $X = (X_t)_{t\geq 0}$ a progressively measurable stochastic process taking values in a measurable space $(E, \mathcal{E})$.*

(a) If τ is an $\mathbb{F}$-stopping time, then then X^τ is is adapted to $\mathbb{F}$ and $\mathbb{F}^\tau$.
(b) If τ is an $\mathbb{F}^+$-stopping time, then X^τ is adapted to $\mathbb{F}$ and $\mathbb{F}^{\tau+} := (\mathcal{F}_{(\tau\wedge t)+})_{t\in I}$.

Proof. (a) This follows from lemma 3.11 and $\mathcal{F}_{\tau\wedge t} \subseteq \mathcal{F}_t$, see lemma 1.7 (b).[33]
(b) Adaptedness to $\mathbb{F}^{\tau+}$ follows from lemma 3.11 (b). $\mathbb{F}$-adaptedness follows from lemma 3.11 (d).

□

[33] Implicitly, we use here the easily verifiable fact that for a deterministic stopping time $\sigma \equiv t$, $\mathcal{F}_\sigma = \mathcal{F}_t$.

Theorem 4.4 (Optional stopping theorem for right-continuous subpartingales). *Let $X = (X_t)_{t\geq 0}$ a right-continuous $\mathbb{F}$-(sub-,super-)martingale and τ an $\overline{\mathbb{R}}_+$-valued $\mathbb{F}^+$-stopping time. Then it holds that:*

(a) X^τ is an $\mathbb{F}$-(sub-,super-)martingale.
(b) If X is uniformly integrable, then X^τ is closed by X_τ. In particular, if X is a uniformly integrable martingale, then so is X^τ.

It is an interesting question whether (when) uniform integrability of X implies that of X^τ for sub- and supermartingales, see also this discussion.

Proof. It suffices to consider the (sub-)martingale case.

(a) $\mathbb{F}$-adaptedness follows from lemma 4.3, using that right-continuity and adaptedness of X imply that X is progressive, see [2, Prop. 3.4]. By the continuous time optional sampling theorem 3.14 for bounded stopping times, X^τ is an integrable process, and for all $0 \leq s \leq t$ it holds that

$$\mathbb{E}[X_t^\tau|\mathcal{F}_s] = \mathbb{E}[X_{\tau\wedge t}|\mathcal{F}_s] \geq X_{\tau\wedge t\wedge s} = X_{\tau\wedge s} = X_s^\tau,$$

with equality in the martingale case.
(b) If X is uniformly integrable, X_τ is well defined and integrable by theorem 3.12, which also yields that for all $t \geq 0$,

$$\mathbb{E}[X_\tau|\mathcal{F}_t] \geq X_{\tau\wedge t}$$

with equality in the martingale case.

□

Remark 4.5. Lemma 4.3 shows that for a right-continuous $\mathbb{F}$-(sub-)martingale X and any $\mathbb{F}$-stopping time resp. optional time the stopped process X^τ is adapted to $\mathbb{F}^\tau$ resp. $\mathbb{F}^{\tau+}$. Consequently:

(a) If τ is a stopping time, $\mathbb{F}^\tau$ being a subfiltration of $\mathbb{F}$ implies that X^τ is also an $\mathbb{F}^\tau$-(sub-)martingale, and
(b) If τ is an $\mathbb{F}$-optional time, then X^τ as a right-continuous $\mathbb{F}$-(sub-)martingale is also an $\mathbb{F}^+$-(sub-)martingale. Then since X^τ is $\mathbb{F}^{\tau+}$-adapted and $\mathbb{F}^{\tau+}$ is a subfiltration of $\mathbb{F}^+$, it follows that X^τ is also an $\mathbb{F}^{\tau+}$-martingale.

5 *Further Results*

Corollary 5.1 (Lévy's upward theorem, Lévy's zero–one law). *Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ a filtration, $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_\infty = \sigma(\bigcup_{n\in\mathbb{N}_0} \mathcal{F}_n)$. Then it holds that*

$$\mathbb{E}[\xi|\mathcal{F}_n] \overset{n\to\infty}{\longrightarrow} \mathbb{E}[\xi|\mathcal{F}_\infty], \qquad \textit{a.s. and in } L^1.$$

Proof. The closed martingale $X_n := \mathbb{E}[\xi|\mathcal{F}_n]$ is uniformly integrable and hence by martingale convergence converges to some X_∞, measurable w.r.t. $\mathcal{F}_\infty$, almost surely and in L^1. It remains to show that $\mathbb{E}[X_\infty \mathbb{1}_A] = \mathbb{E}[\xi \mathbb{1}_A]$ for all $A \in \mathcal{F}_\infty$. By L^1-convergence of X_n to X_∞, this is satisfied for all $A \in \mathcal{E} := \bigcup_{n\geq 1} \mathcal{F}_n$. Since $\mathcal{E}$ is a π-system generating $\mathcal{F}_\infty$, we can conclude by the generator criterion for the conditional expectation or by Dynkin's lemma. $\square$

Corollary 5.2 (Kolmogorov's 0-1 law). *Let $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ an independent family of sub-σ-algebras of $\mathcal{F}$. Then the tail-σ-algebra $\mathcal{T} := \bigcap_{n\in\mathbb{N}_0} \sigma(\bigcup_{k=n}^\infty \mathcal{A}_k)$ is $\mathbb{P}$-trivial, i.e. $\mathbb{P}[A] \in \{0,1\}$ for all $A \in \mathcal{T}$.*

Proof. Define the filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ by $\mathcal{F}_n := \sigma(\bigcup_{k=0}^n \mathcal{A}_k)$. Any event $A \in \mathcal{T}$ is independent of each $\mathcal{F}_n$, hence $\mathbb{E}[\mathbb{1}_A|\mathcal{F}_n] = \mathbb{P}[A]$. By Lévy's upward theorem and since $A \in \mathcal{F}_\infty$,

$$\mathbb{P}[A] = \mathbb{E}[\mathbb{1}_A|\mathcal{F}_n] \xrightarrow{\text{a.s.}} \mathbb{E}[\mathbb{1}_A|\mathcal{F}_\infty] = \mathbb{1}_A .$$

hence $\mathbb{1}_A = \mathbb{P}[A]$ almost surely. Thus there is at least one $\omega \in \Omega$ with $\mathbb{P}[A] = \mathbb{1}_A(\omega)$ implying $\mathbb{P}[A] \in \{0,1\}$. $\square$

Corollary 5.3 (Lévy's downward theorem). *Let $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be a decreasing sequence of sub-sigma algebras of $\mathcal{F}$, $\mathcal{F}_\infty := \bigcap_{n\geq 0} \mathcal{F}_n$ and $\xi \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then it holds that*

$$\mathbb{E}[\xi|\mathcal{F}_n] \xrightarrow{n\to\infty} \mathbb{E}[\xi|\mathcal{F}_\infty] \quad \textit{a.s. and in } L^1.$$

Proof. For $n \in -\mathbb{N}$, let $\mathcal{G}_n = \mathcal{F}_{-n}$ and $Y_n = \mathbb{E}[\xi|\mathcal{G}_n] = \mathbb{E}[\xi|\mathcal{F}_{-n}]$. Then $Y = (Y_n)_{n\in-\mathbb{N}}$ is a backwards martingale w.r.t. the filtration $(\mathcal{G}_n)_{n\in-\mathbb{N}}$. Indeed,

$$\mathbb{E}[Y_n|\mathcal{G}_{n-1}] = \mathbb{E}[\mathbb{E}[\xi|\mathcal{F}_{-n}]|\mathcal{F}_{-n+1}]] = \mathbb{E}[\xi|\mathcal{F}_{-n+1}] = Y_{n-1}.$$

Let $\mathcal{G}_{-\infty} = \bigcap_{n\geq 0} \mathcal{G}_n$. By the backwards martingale convergence theorem 2.7, it holds that Y_n converges a.s. and in L^1 to $Y_{-\infty} = \mathbb{E}[Y_0|\mathcal{G}_{-\infty}] = \mathbb{E}[\xi|\mathcal{F}_\infty]$. $\square$

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