Assignment 7

(1) $dS_t = \mu S_t dt + \sigma S_t dz_t$; $dR_t = r R_t dt$

I know I'm repeating a lot of work from the textbook but I wanted to work through this on my own

Suppose we allocate a fraction π_t of wealth at any time into the risky asset, and we have a consumption c_t. Then our change in wealth is:

$$dW_t = \mu \pi_t W_t dt + \sigma \pi_t W_t dz_t + r(1-\pi_t) W_t dt - c_t dt$$

$$= (((\mu - r)\pi_t + r) W_t - c_t) dt + \sigma \pi_t W_t dz_t$$

Our utility is increasing like

$$dU_t = \log(c_t) dt$$

Our total utility summed & discounted w/ rate ρ from t to T is:

$$U_{tot} = \int_t^T \log(c_s) e^{-\rho(s-t)} ds + B(T) e^{-\rho(T-t)} \log(W_T)$$

We want to maximize the following by choosing π_t, c_t intelligently:

$$\mathbb{E}[U_{tot} | W_t] = \mathbb{E}\left[\int_t^T \log(c_s) e^{-\rho(s-t)} ds + B(T) \log(W_T) e^{-\rho(T-t)} \Big| W_t\right]$$

This is modeled nicely as a continuous-time MDP, where our state is (W_t, t), our actions are pairs (π_t, c_t), and the discount rate is ρ, and rewards are $U(c_t)dt$. The discrete-time Bellman optimality equation in the short time step limit is:

$$V(W_t, t) = \max_{\pi_t, c_t} \{\log(c_t) dt + (1 - \rho dt) \cdot \mathbb{E}[V(W_{t+dt}, t+dt)]\}$$

We can expand:

$$V^*(W_t,t) = \max_{\pi_t, c_t} \left\{ \log(c_t)dt + (1-\rho dt)\left(V^*(W_t,t) + \mathbb{E}\left[\Delta V^*(W_t,t,\pi_t,c_t)\right] \right) \right\}$$

$$= V^*(W_t,t) - \rho dt\, V^*(W_t,t) + \max_{\pi_t, c_t} \left\{ \log(c_t)dt + \mathbb{E}\left[\Delta V^*(W_t,t,\pi_t,c_t)\right] \right\}$$

$$\Rightarrow \rho V^*(W_t,t)dt = \max_{\pi_t, c_t} \left\{ \log(c_t)dt + \mathbb{E}_t\left[dV^*(W_t,t \mid \pi_t,c_t) \right] \right\}$$

Ito's lemma lets us expand:

$$\rho V^*(W_t,t)dt = \max_{\pi_t, c_t} \left\{ \log(c_t)dt + \mathbb{E}_t\left[\frac{\partial V^*}{\partial t} dt + \frac{\partial V^*}{\partial W_t} dW_t + \frac{1}{2}\frac{\partial^2 V^*}{\partial W_t^2}(dW_t)^2 \right] \right\}$$

$$= \max_{\pi_t, c_t} \left\{ \log(c_t)dt + \frac{\partial V^*}{\partial t} dt + \frac{\partial V^*}{\partial W_t} \mathbb{E}[dW_t] + \frac{1}{2}\frac{\partial^2 V^*}{\partial W_t^2}\mathbb{E}\left[(dW_t)^2\right] \right\}$$

Notice:

$$\mathbb{E}[dW_t] = \underbrace{\left(\left((\mu-r)\pi_t + r\right)W_t - c_t\right)}_{A(W_t,t)}dt + \underbrace{\sigma \pi_t W_t}_{B(W_t,t)} \underbrace{\mathbb{E}[z_t]}_{\to 0} \qquad dt = dz_t^2$$

$$\mathbb{E}[dW_t^2] = \underbrace{A(W_t,t)^2 dt^2}_{\to 0} + \underbrace{2A(W_t,t)B(W_t,t)\,dt\,\mathbb{E}[dz_t]}_{\to 0} + B(W_t,t)^2 dz_t^2$$

$$= \sigma^2 \pi_t^2 W_t^2 dt$$

So:

$$\rho V^*(W_t,t) = \max_{\pi_t, c_t} \left\{ \log(c_t) + \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t}\cdot\left((r+(\mu-r)\pi_t)W_t - c_t\right) + \frac{1}{2}\frac{\partial^2 V^*}{\partial W_t^2}\sigma^2\pi_t^2 W_t^2 \right\}$$

Let's solve this maximum by taking derivatives w.r.t. c_t & π_t. Notice that we require $\frac{\partial^2 V^*}{\partial W_t^2} < 0$ for this to work.

$$\frac{1}{c_t^*} - \frac{\partial V^*}{\partial W_t} = 0 \Rightarrow c_t^* = \left(\frac{\partial V^*}{\partial W_t}\right)^{-1}$$

$$(\mu - r)W_t \frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2}\sigma^2 V_t^2 \pi_t^* = 0 \Rightarrow \pi_t^* = \frac{\mu - r}{\sigma^2 W_t}\cdot\frac{\frac{\partial V^*}{\partial W_t}}{\left(-\frac{\partial^2 V^*}{\partial W_t^2}\right)}$$

$\frac{1+(\rho\epsilon-1)e^{-\rho(T-t)}}{\rho}$ $\quad 1+c^{\rho t}$

$$\rho V^*(W_t,t) = -\log\left(\frac{\partial V^*}{\partial W_t}\right) + \frac{\partial V^*}{\partial t} - 1 + \overset{rW_t}{\downarrow}\frac{\partial V^*}{\partial W_t} + \left(\frac{\mu-r}{\sigma}\right)^2 \cdot \frac{(\partial V^*/\partial W_t)^2}{-(\partial^2 V^*/\partial W_t^2)}$$

$$+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 \frac{(\partial V^*/\partial W_t)^2}{(\partial^2 V^*/\partial W_t^2)}$$

This is a PDE we can solve

$$\rho V^*(W_t,t) = -\log\left(\frac{\partial V^*}{\partial W_t}\right) + \frac{\partial V^*}{\partial t} + rW_t\frac{\partial V^*}{\partial W_t} + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 \cdot \frac{(\partial V^*/\partial W_t)^2}{-\partial^2 V^*/\partial W_t^2} - 1$$

subject to $V^*(W_T,T) = \beta(T)\cdot\log(W_T) = \epsilon\log(W_T)$

Suppose $V^*(W_t,t) = \beta(t)\log(W_t) + f(t)$. We can plug in this guess:

$$\frac{\partial V^*}{\partial t} = f'(t), \quad \frac{\partial V^*}{\partial W_t} = \frac{\epsilon}{W_t}, \quad \frac{\partial^2 V^*}{\partial W_t} = -\frac{\epsilon}{W_t^2}$$

$$\epsilon\rho\log W_t + \rho f(t) = -\log\epsilon + \log W_t + f'(t) + rW_t\cdot\frac{\epsilon}{W_t} - 1$$

$$+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 \cdot \frac{(\epsilon/W_t)^2}{\epsilon/W_t^2}$$

$t\log t$

$$= -\log\epsilon + \log W_t + r\epsilon + \frac{\epsilon}{2}\left(\frac{\mu-r}{\sigma}\right)^2 - 1$$

$e^{\alpha t}$

$$(\epsilon\rho - 1)\log W_t + \log\epsilon + r\epsilon - \frac{\epsilon}{2}\left(\frac{\mu-r}{\sigma}\right)^2 + 1 = -\rho f(t) + f'(t)$$

$\alpha t - \log W_t$

$f'(t) =$

$1^{-1}(t)$

$$V^* = f(t)\log(W_t)$$

$\frac{d}{dx}\,1^{-1}(t)$

$= \frac{1}{\log(1^{-1}(t))}$

$$\frac{\rho W_t}{1+(\rho\epsilon-1)e^{-\rho(T-t)}} = \frac{1+(\rho\epsilon-1)e^{-\rho(T-t)}}{\rho W_t} = \partial$$

Let me copy this PDE to this page for reference:

$$\begin{cases} \frac{\partial V^*}{\partial t} + \frac{(\mu-r)^2}{2\sigma^2} \cdot \frac{\left(\frac{\partial V^*}{\partial W_t}\right)^2}{\frac{\partial^2 V^*}{\partial W_t}} + \frac{\partial V^*}{\partial W_t} \cdot r \cdot W_t - 1 - \log\left(\frac{\partial V^*}{\partial W_t}\right) = \rho V^* \\ \text{Subject to } V^*(W_T, T) = \epsilon \log W_T \end{cases}$$

where $\epsilon \equiv B(T)$

Suppose V^* has the form $V^*(W_t, t) = f(t) \log W_t + g(t)$. Then:

$$\frac{\partial V^*}{\partial t} = f'(t) \log W_t + g'(t) \; ; \; \frac{\partial V^*}{\partial W_t} = \frac{f(t)}{W_t} \; ; \; \frac{\partial^2 V^*}{\partial W_t^2} = \frac{-f(t)}{W_t^2}$$

Substituting in, we get:

simplify

$$f'(t) \log W_t + g'(t) + \frac{(\mu-r)^2}{2\sigma^2} f(t) + r f(t) - 1 - \log f(t) + \log W_t = \rho f(t) \log W_t + g(t) \rho$$

$$\left(f'(t) - \rho f(t) + 1\right) \log W_t + g'(t) - \rho g(t) + h(t) = 0$$

where we define $h(t) \equiv \left(r + \frac{(\mu-r)^2}{2\sigma^2}\right) f(t) - \log f(t) - 1$

This diffeq is solved by $f(t)$, $g(t)$ which are solutions to the ODEs:

$$\begin{cases} f'(t) - \rho f(t) + 1 = 0 \\ g'(t) - \rho g(t) + h(t) = 0 \end{cases}$$

This ODE system has a general solution:

$$\begin{cases} f(t) = C_1 e^{\rho t} + 1/\rho \\ g(t) = C_2 e^{\rho t} - e^{\rho t} \int h(t) e^{-\rho t} dt \end{cases}$$

subject to B.C.s

$$\begin{cases} f(T) = \epsilon \\ g(T) = 0 \end{cases}$$

Applying the boundary conditions, we get the solution:

$$\boxed{\begin{aligned} f(t) &= \epsilon + \frac{1}{\rho}\left(1 - e^{-\rho(T-t)}\right) \\ g(t) &= e^{\rho t}\int_t^T e^{-\rho s} h(s)\, ds \end{aligned}}$$

This gives us a value function:

$$\boxed{\begin{aligned} V^*(W_t,t) &= \left(\epsilon + \frac{1-e^{-\rho(T-t)}}{\rho}\right)\log W_t + e^{\rho t}\int_t^T e^{-\rho s} h(s)\, ds \\ \text{where } h(s) &\equiv \left(r + \frac{(\mu-r)^2}{2\sigma^2}\right)\left(\epsilon + \frac{1-e^{-\rho(T-t)}}{\rho}\right) - \log\left(\epsilon + \frac{1-e^{-\rho(T-t)}}{\rho}\right) - 1 \end{aligned}}$$

From the value function, we can recover the optimal policy:

$$\pi_t^* = \frac{\mu - r}{\sigma^2 W_t} \cdot \frac{\partial V^*/\partial W_t}{-\partial^2 V^*/\partial W_t^2} = \boxed{\frac{\mu - r}{\sigma^2}}$$

$$c_t^* = \left(\frac{\partial V^*}{\partial W_T}\right)^{-1} = \boxed{\frac{W_t}{\epsilon + \frac{1-e^{-\rho(T-t)}}{\rho}}}$$

③ States:

Your state is a tuple (s, J) where $s \in \mathbb{R}^+$ is your current skill level, and $J \in \{0,1\}$ is 1 if you are currently employed and zero otherwise

Actions:

Your action space is continuous: $\alpha \in [0,1]$ if $J=1$, but is null otherwise (you have no decisions if unemployed). We could also allow for consistency $A = \{\{\alpha \in [0,1]\}\}$ even if $J=0$ for consistency, as long as it is not included in the state transition probability calculation.

Rewards:

$\mathcal{R}(s, J) = J \cdot \alpha L f(s)$, where L is the length of the day in min, & $f(s)$ is wage per min

Transitions:

s varies deterministically: your next day's skill level s', given your current skill level s, your job status J, and your action α, is:

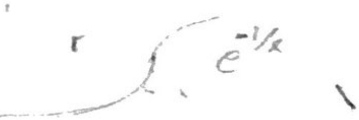
$$s' = \left(J \cdot g(s)^{\alpha L} + (1-J) \cdot \frac{1}{2^{1/\lambda}}\right) s$$

J varies stochastically: your next days job status J' is a random variable which is a function of both your current job status J & current skill s:

$$J' = J \cdot (1 - \mathbb{I}_p) + (1-J) \cdot \mathbb{I}_{h(s)}$$

where $\mathbb{I}_\alpha$ is a bernoulli which is 1 w/ prob α

1 2 3 4 5
25 20 10 5 2



~~Let's plug in some realistic-sounding numbers here. Let $L = 60 \cdot 9 = 540$, let $f(s) = 100{,}000/250 / L \cdot s$. Let $\lambda = 100$. Let $p = \frac{1}{250 \cdot 3}$. Let $h(s) = s^2 / (s^2 + 25)$. Let $g(s) = 1 + \frac{1}{5} \cdot \frac{1}{250} \cdot \frac{1}{2/3}$~~

The limiting factors here are primarily the size of the action space/state space, the long time horizon compared to the frequency of decisions. W/ a clever choice of $f(s), g(s), h(s)$, I'm sure this problem can be analytically tractable. If we wanted to do it numerically, we could restrict α to $\{0, 0.1, \ldots, 1\}$, and we could do some kind of linear function approx