Assignment 6

① $E[U(x)] = E[X] - \frac{\alpha}{2} E[X^2]$

$= \boxed{\mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)}$

$X_{CE} - \frac{\alpha}{2} X_{CE}^2 = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)$

$X_{CE} = \frac{1 + \sqrt{1 - 2\alpha(\mu - \frac{\alpha}{2}(\mu^2 + \sigma^2))}}{\alpha} = \frac{1 + \sqrt{(1-\alpha\mu)^2 - \alpha^2\sigma^2}}{\alpha}$

$= \boxed{\frac{1}{\alpha} + \sqrt{(\mu - \frac{1}{\alpha})^2 - \sigma^2}}$

$\pi_A = E[X] - X_{CE} = \boxed{\mu - \frac{1}{\alpha} - \sqrt{(\mu - \frac{1}{\alpha})^2 - \sigma^2}}$

Investment: maximize $E[U(zX + (W-z)r)]$ where $X \sim N(\mu, \sigma^2)$,

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad U(x) = x - \frac{\alpha}{2}x^2$

$E[U(zX + (w-z)r)] = zE[X] + (w-z)r - (z^2 E[X^2] + 2z(w-z)r E[X] + (w-z)^2 r^2)\frac{\alpha}{2}$

$= z\mu + (w-z)r - \frac{\alpha}{2}(z^2(\mu^2 + \sigma^2) + 2z(w-z)r\mu + (w-z)^2 r^2)$

$\frac{d}{dz} E[\,\,||\,\,] = \mu - r - \alpha z(\mu^2 + \sigma^2) - \alpha(w-z)r\mu + \alpha z r\mu - \alpha(z-w)r^2$

$= \mu - r + \alpha w r\mu + \alpha w^2 r^2 - z(\sigma^2\alpha + \alpha\mu^2 - \alpha r\mu - \alpha r\mu + \alpha r^2)$

$0 = (\mu - r)(1 - \alpha w r) - \alpha z((\mu - r)^2 + \sigma^2)$

$\Rightarrow \boxed{z = \frac{(\mu - r)(\frac{1}{\alpha} - wr)}{((\mu - r)^2 + \sigma^2)}}$ ← where W is your total wealth / investment capital

③ Outcomes: $W_1 = \begin{cases} (1-f)W_0 + fW_0(1+\alpha) & \text{w/ prob } p \\ (1-f)W_0 + fW_0(1-\beta) & \text{w/ prob } 1-p \end{cases}$

$W_1 = \begin{cases} W_0 + \alpha f W_0 & \text{w/ prob } p \\ W_0 - \beta f W_0 & \text{w/ prob } 1-p \end{cases}$

$\log W_1 = \log W_0 + \begin{cases} \log(1+\alpha f) & \text{w/ prob } p \\ \log(1-\beta f) & \text{w/ prob } 1-p \end{cases}$

$\mathbb{E}[\log W_1] = \log W_0 + p\log(1+\alpha f) + (1-p)\log(1-\beta f)$

$\frac{d}{df}\mathbb{E}[\log W_1] = \frac{p}{1+\alpha f}\cdot\alpha + \frac{1-p}{1-\beta f}\cdot-\beta$

$\frac{d^2}{df^2}(||) = -\alpha^2 p\left(\frac{1}{1+\alpha f}\right)^2 - \beta^2(1-p)\left(\frac{1}{1-\beta f}\right)^2 < 0$

$\frac{d}{df}\mathbb{E}[\log W_1] = 0 \Rightarrow \alpha p(1-\beta f^*) = \beta(1-p)(1+\alpha f^*)$

$\alpha p + \beta p - \beta = \alpha p \beta f^* + \alpha \beta f^* - \alpha p \beta f^*$

$\Rightarrow \boxed{f^* = \frac{p}{\beta} - \frac{1-p}{\alpha}}$

This makes sense: • as β gets large, f^* ~~gets~~ gets smaller
• as α gets large, f^* gets larger up to some fixed quantity
• as p increases, f^* increases