Assignment 6

① $\mathbb{E}[U(X)] = \mathbb{E}[X] - \frac{\alpha}{2}\mathbb{E}[X^2]$

$$= \boxed{\mu - \frac{\alpha}{2}(\mu^2+\sigma^2)}$$

$$X_{CE} - \frac{\alpha}{2}X_{CE}^2 = \mu - \frac{\alpha}{2}(\mu^2+\sigma^2)$$

$$X_{CE} = \frac{1+\sqrt{1-2\alpha(\mu-\frac{\alpha}{2}(\mu^2+\sigma^2))}}{\alpha} = \frac{1+\sqrt{(1-\alpha\mu)^2-\alpha^2\sigma^2}}{\alpha}$$

$$= \boxed{\frac{1}{\alpha}+\sqrt{(\mu-1/\alpha)^2-\sigma^2}}$$

$$\pi_A = \mathbb{E}[X]-X_{CE} = \boxed{\mu - 1/\alpha - \sqrt{(\mu-1/\alpha)^2-\sigma^2}}$$

Investment: maximize $\mathbb{E}[U(zX+(W-z)r)]$ where $X\sim N(\mu,\sigma^2)$, $U(x)=x-\frac{\alpha}{2}x^2$

$$\mathbb{E}[U(zX+(W-z)r)] = z\mathbb{E}[X]+(W-z)r - \left(z^2\mathbb{E}[X^2]+2z(W-z)r\,\mathbb{E}[X]+(W-z)^2r^2\right)\frac{\alpha}{2}$$

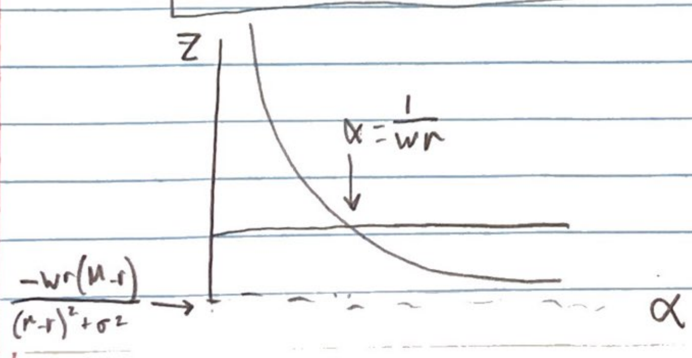
$$= z\mu+(W-z)r-\frac{\alpha}{2}\left(z^2(\mu^2+\sigma^2)+2z(W-z)r\mu+(W-z)^2r^2\right)$$

$$\frac{d}{dz}\mathbb{E}[\;''\;] = \mu - r - \alpha z(\mu^2+\sigma^2) - \alpha(W-z)r\mu + \alpha z r\mu - \alpha(z-W)r^2$$

$$= \mu - r + \alpha W r\mu + \alpha W r^2 - z(\sigma^2\alpha+\alpha\mu^2-\alpha r\mu-\alpha r\mu+\alpha r^2)$$

$$0 = (\mu-r)(1-\alpha W r) - \alpha z\left((\mu-r)^2+\sigma^2\right)$$

$$\Rightarrow \boxed{z = \frac{(\mu-r)(\frac{1}{\alpha}-Wr)}{((\mu-r)^2+\sigma^2)}}$$ ← where W is your total wealth/investment capital



③ Outcomes: $W_1 = \begin{cases} (1-f)W_0 + fW_0(1+\alpha) & \text{w/ prob } p \\ (1-f)W_0 + fW_0(1-\beta) & \text{w/ prob } 1-p \end{cases}$

$$W_1 = \begin{cases} W_0 + \alpha f W_0 & \text{w/ prob } p \\ W_0 - \beta f W_0 & \text{w/ prob } 1-p \end{cases}$$

$$\log W_1 = \log W_0 + \begin{cases} \log(1+\alpha f) & \text{w/ prob } p \\ \log(1-\beta f) & \text{w/ prob } 1-p \end{cases}$$

$$\mathbb{E}[\log W_1] = \log W_0 + p\log(1+\alpha f) + (1-p)\log(1-\beta f)$$

$$\frac{d}{df}\mathbb{E}[\log W_1] = \frac{p}{1+\alpha f}\cdot\alpha + \frac{1-p}{1-\beta f}\cdot -\beta$$

$$\frac{d^2}{df^2}(\,''\,) = -\alpha^2 p\left(\frac{1}{1+\alpha f}\right)^2 - \beta^2(1-p)\left(\frac{1}{1-\beta f}\right)^2 < 0$$

$$\frac{d}{df}\mathbb{E}[\log W_1] = 0 \Rightarrow \alpha p(1-\beta f^*) = \beta(1-p)(1+\alpha f^*)$$

$$\alpha p + \beta p - \beta = \alpha p\beta f^* + \alpha\cancel{\beta} f^* - \alpha\cancel{p\beta} f^*$$

$$\Rightarrow \boxed{f^* = \frac{p}{\beta} - \frac{1-p}{\alpha}}$$

This makes sense: • as β gets large, f^* gets smaller
• as α gets large, f^* gets larger up to some fixed quantity.
• as p increases, f^* increases