Assignment 9

(2) $P_{t+1} = P_t \cdot e^{Z_t}$ $\quad R_{t+1} = R_t - N_t$ $\quad Z_t$ iid mean μ_Z Var σ_Z^2

$X_{t+1} = \rho X_t + \eta_t$ $\quad \eta_t$ iid mean 0

$Q_t = P_t \cdot (1 - \beta \cdot N_t - \theta X_t)$

Sales proceeds at time t: $N_t \cdot Q_t$

~~Objective: maximize~~ $\cancel{\mathbb{E}_\pi\left[\sum_{t=0}^{T-1} N_t (P_t (1-\beta N_t - \theta X_t)) \mid (P_t, R_t)\right]}$

We can denote the π-implied value function at step t as:

$$V_t^\pi((P_t, R_t, X_t)) = \mathbb{E}_\pi\left[\sum_{i=t}^{T-1} N_i P_i (1 - \beta N_i - \theta X_i) \mid (P_t, R_t, X_t)\right]$$

$$V_t^*((P_t, R_t, X_t)) = \max_\pi V_t^\pi((P_t, R_t, X_t))$$

V_t^* satisfies:

$$\begin{cases} V_t^*((P_t, R_t, X_t)) = \max_{N_t} \left\{ N_t P_t (1 - \beta N_t - \theta X_t) + \mathbb{E}\left[V_{t+1}^*((P_{t+1}, R_{t+1}, X_{t+1}))\right] \right\} \\ V_T^*(P_T, R_T, X_T) = 0 \ ; \ N_T = 0 \end{cases}$$

We can solve by backward induction:

$$V_{T-1}^* = \max_{N_{T-1}} \left\{ N_{T-1} P_{T-1} (1 - \beta N_{T-1} - \theta X_{T-1}) \right\} \text{ subject to } N_{T-1} = R_{T-1}$$

$$= R_{T-1} P_{T-1} (1 - \beta R_{T-1} - \theta X_{T-1})$$

$$\Rightarrow V_{T-2}^*(P_{T-2}, R_{T-2}, X_{T-2}) = \max_{N_{T-2}} \left\{ N_{T-2} P_{T-2} (1 - \beta N_{T-2} - \theta X_{T-2}) + \mathbb{E}\left[R_{T-1} P_{T-1} (1 - \beta R_{T-1} - \theta X_{T-1})\right] \right\}$$

$$= \max_{N_{T-2}} \Big\{ N_{T-2} P_{T-2} (1 - \theta X_{T-2}) - \beta P_{T-2} N_{T-2}^2$$

$$+ \mathbb{E}\left[(R_{T-2} - N_{T-2}) \cdot P_{T-2} e^{Z_{T-2}} \left(1 - \beta (R_{T-2} - N_{T-2}) - \theta(\rho X_{T-2} + \eta_{T-2})\right)\right] \Big\}$$

~~If we assume no higher order mom~~

$$M = M_{Z_t}(1) = \mathbb{E}\left[e^{Z_t}\right]$$

$$V^*_{T-2}(P_{T-2}, R_{T-2}, X_{T-2}) = \max_{N_{T-2}} \Big\{ N_{T-2}P_{T-2}(1-\theta X_{T-2}) - \beta P_{T-2} N_{T-2}^2$$

$$+ (R_{T-2}-N_{T-2})P_{T-2}\Big(1-\beta(R_{T-2}-N_{T-2}) - \theta\rho X_{T-2} - \theta\,\cancelto{0}{\mathbb{E}[Z_{T-1}]}\Big)\mathbb{E}\left[e^{Z_{T-1}}\right]\Big\}$$

$$= R_{T-2} M P_{T-2}(1-\beta R_{T-2} - \theta\rho X_{T-2}) + P_{T-2}\big((1-M-\theta X_{T-2} + M_\rho\theta X_{T-2} + 2M_\beta R_{T-2})N_{T-2}$$

$$- \cancel{M} P_{T-2}\beta(1+M)N_{T-2}^2$$

Deriving wrt N_{T-2} yields:

$$0 = 1 - M - \theta X_{T-2} + M_\rho\theta X_{T-2} + 2M_\beta R_{T-2} - 2M_\beta N_{T-2}$$

$$\Rightarrow N_{T-2} = \frac{1}{2\beta(1+M)}\left(1 - M - \theta X_{T-2} + M_\rho\theta X_{T-2} + 2M_\beta R_{T-2}\right)$$

$$= \frac{1}{2\beta(1+M)}\left((1-M) - (1-M_\rho)\theta X_{T-2} + 2M_\beta R_{T-2}\right)$$

$$= \delta_{x,1}X_{T-2} + \delta_{r,1}R_{T-2} + \delta_{1,1}$$

where $\delta_{x,1} = \frac{-(1-M_\rho)\theta}{2\beta(1+M)}$; $\delta_{r,1} = \frac{M}{1+M}$; $\delta_{1,1} = \frac{1-M}{2\beta(1+M)}$

Plugging into V^*:

$$V^*_{T-2}(P_{T-2}, R_{T-2}, X_{T-2}) = P_{T-2}\Big(\overbrace{(1-M)\delta_{1,1} - \beta(1+M)\delta_{1,1}^2}^{\frac{1}{2}(1-M)\delta_{1,1}} + X_{T-2}\big(\theta(M_\rho - 1)\delta_{1,1} + (1-M)\delta_{x,1} - 2\beta(1+M)\delta_{1,1}\delta_{x,1}\big)$$

$$+ X_{T-2}^2\left(\theta(M_\rho-1)\delta_{x,1} - \beta(1+M)\delta_{x,1}^2\right)$$

$$+ X_{T-2}R_{T-2}\left(-M_\rho\theta + \theta(M_\rho - 1)\delta_{r,1} + 2M_\beta\delta_{x,1} - 2\beta(1+M)\delta_{x,1}\delta_{r,1}\right)$$

$$+ R_{T-2}\left(M + 2M_\beta\delta_{1,1} + (1-M)\delta_{r,1} - 2\beta(1+M)\delta_{1,1}\delta_{r,1}\right)$$

$$+ R_{T-2}^2\left(-M_\beta + 2M_\beta\delta_{r,1} - \beta(1+M)\delta_{r,1}^2\right)\Big)$$

$$= P_{T-2}\left(a_1 + b_1 X_{T-2} + c_1 X_{T-2}^2 + d_1 X_{T-2}R_{T-2} + e_1 R_{T-2} + f_1 R_{T-2}^2\right.$$

where $a_1, b_1, \ldots, f_1$ come from here

This allows us to recursively find N^*_{T-k-1}:

$$V_{T-k-1}(P_{T-k-1}, R_{T-k-1}, X_{T-k-1}) =$$

$$= \max_{N_{T-k-1}} \left\{ N_{T-k-1} P_{T-k-1}(1-\beta N_{T-k-1} - \theta X_{T-k-1}) + \mathbb{E}\left[V^*_{T-k}(\cancel{X}P_{T-k}, R_{T-k}, X_{T-k})\right] \right\}$$

$$= \max_{N_{T-k-1}} \left\{ N_{T-k-1} P_{T-k-1}(1-\beta N_{T-k-1} - \theta X_{T-k-1}) + \right.$$

$$\left. \mathbb{E}\left[P_k\left(a_{k-1} + b_{k-1} X_{T-k} + c_{k-1} X^2_{T-k} + d_{k-1} X_{T-k} R_{T-k} + e_{k-1} R_{T-k} + f_{k-1} R^2_{T-k}\right)\right] \right\}$$

$$= \max_{N_{T-k-1}} \left\{ N_{T-k-1} P_{T-k-1}(1-\beta N_{T-k-1} - \theta X_{T-k-1}) + \left(a_{k-1} + b_{k-1} \mathbb{E}[X_{T-k} | X_{T-k-1}] \right.\right.$$

$$\left.\left. + c_{k-1} \mathbb{E}[X^2_{T-k} | X_{T-k-1}] + d_{k-1} \mathbb{E}[X_{T-k} R_{T-k} | X_{T-k-1}] + e_{k-1} R_{T-k} + f_{k-1} R^2_{T-k}\right) \mathbb{E}[P] \right\}$$

$$\mathbb{E}[X_{t+1} | X_t] = \rho X_t \ ; \ \mathbb{E}[X^2_{t+1} | X_t] = (\rho X_t)^2 + \sigma_\eta^2$$

$$\mathbb{E}[P_{t+1} | P_t] = P_t M$$

$$V_{T-k-1}(P_{T-k-1}, R_{T-k-1}, X_{T-k-1}) =$$

$$\max_{N_{T-k-1}} \left\{ N_{T-k-1} P_{T-k-1}(1-\beta N_{T-k-1} - \theta X_{T-k-1}) + M P_{T-k-1}\left(\quad \right) \right\}$$

We can take derivatives to get an expression for N^*_{T-k-1} & then plug in to get a recursive relation for $a_k, b_k, \dots f_k$ in terms of $a_{k-1}, \dots, f_{k-1}$.

This isn't super instructive so I'm not gonna do it. The results are given in Bertsimas-Lo.