Factoradic Representation of Rational Numbers

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From 'A Course in Pure Mathematics' by G. H. Hardy. Chapter 1, Miscellaneous Examples.

Miscellaneous example* #2 at the end of chapter 1 in Hardy's 'Pure Mathematics' presents us with a fascinating result. I had never seen it before, but upon seeing it felt like I was looking at a kind of basis-representation-theorem but for rational numbers, ... beautiful!

Here it is, followed by my proof starting with the lemma.

Theorem

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Lemma

The set of rational numbers,

$$\mathcal{S} = \{ \tfrac{a_2}{2!} + \tfrac{a_3}{3!} + \ldots + \tfrac{a_k}{k!} \ \ | \ \ 0 \leq a_2 < 2,\ 0 \leq a_3 < 3,\ \ldots,\ 0 \leq a_k < k \},$$

is identical to the set of rational numbers,

$$\mathcal{F} = \{ \tfrac{0}{k!},\ \tfrac{1}{k!},\ \tfrac{2}{k!},\ \ldots,\ \tfrac{k!-1}{k!} \}$$

Proof of Lemma

The number of values that the coefficient a_2 can assume is 2, a_3 can take on 3 values, ..., up to a_k which can take on k values. So the total number of combinations of values that can be assigned to all the coefficients is $2 \cdot 3 \cdot 4 \cdot \ldots \cdot k = k!$.

*Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, math exercises like calculations to perform, theorems to prove etc.

Suppose that the rational number $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is not uniquely determined by the coefficients $a_2, a_3, \ldots, a_k$. That is, suppose there is a second DIFFERENT sequence of coefficients $b_2, b_3, \ldots, b_k$ such that $\frac{p}{q} = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$.

The size of $\mathcal{F}$ is clearly $k!$