Counting

James Rowell

July 26, 2017

There are 10 sorts of people in the world: those who understand binary and those who don't.

What does "10" mean?

We got it drilled into us watching Sesame Street that "10" is the symbol for the number "ten" which is this many apples "" or the number of fingers on a typical person's two hands.

Once we are trained to automatically think of "10" as representing ten things, we quickly move past it to learn about 100 and 1000 and how to interpret a string of digits like 92507. Even at a young age we'd be able to accurately count out a pile of ninety-two-thousand-five-hundred-and-seven apples as time consuming and agonizing as it might be. Furthermore, learning how to add and multiply is easy once you can count in base-ten since the techniques are simple and straight-forward.

What about kids in ancient Rome, was it as easy for them? Try adding two numbers together in ancient Rome, or worse, multiplying or dividing them. What's XI times IX? Would you believe me if I told you it's XCIX?

Unless you convert those to Hindu-Arabic decimal or base-ten numbers to check, you're just gonna have to trust me. Truth is - I don't know how to multiply using Roman numerals - nor did most Romans. Not only that, but I'll bet that most modern eight-year-olds can count higher than any Roman could - as the Roman system only effectively allowed counting up to 4999.

Even though we use different symbols, the ancient Romans and us are talking about the same abstract set of numbers underneath, which we call integers[*]. Mathematics deals with numbers in this pure sort of way, divorced from the symbols used to represent each number. When we talk about positive integers in mathematics, it's best to remind yourself that we are really talking about the set of numbers that represent successively larger piles of apples, forgetting the symbols.

However we use numbers written out in base-ten all the time in mathematics, rarely thinking in terms of piles of apples. We take it for granted that we can use base-ten to represent the set of positive integers. *Caution*: the only thing modern mathematics takes for granted are axioms and the fact that we can use base-ten to represent the integers is NOT among the list of axioms.

Briefly; the axioms describe a few simple properites about addition and multiplication. These properites are *so simple* that they can't be expressed in yet other even-simpler ideas. The

[*] Integers are the set of all the postive whole numbers, as well as zero and all the negative counterparts to each positive number.

axioms are the minimal set of simple, obvious, irrefutable ideas from which everything else in mathematics is built*.

Since our ability to count in base-ten is not axiomatic, then to properly ground it in modern mathematics we should define what it means to write out a number in decimal, state its properties in a theorem, then provide a proof of that theorem - The proof being a series of arguments that logically connects it directly† to the axioms. In doing so, the only way that the theorem could be false is if the axioms themselves are false.

Here's what that theorem looks like.

Basis Representation Theorem

Let b be a positive integer greater than 1.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

where $0 \leq d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-b by the string of base-b-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$

"That's nuts!" you might say, I don't even see a "ten" in there so how could that describe how we learned to count watching Sesame Street? More likely if you're unfamiliar with mathematical notation then that stuff above likely looks like goblety-gook.

Try this: imagine that the above was written such that we replace the b with "ten". Does it make any more sense? At least then we'd have the "Base-Ten Representation Theorem". We could also let $b = 2$, which would give us the "Base-Two Representation Theorem" stating how we count in binary.

Anyway, don't worry if you can't read the theorem, we'll get to how to do that shortly, but this theorem is a good example of the kind of thing mathematicians like to do - generalize ideas.

Why restrict ourselves to ten when the idea applies equally well to two, three, four, five, ... etc.? The heptapods in "Arrival" have seven limbs with seven fingers each, perhaps they use base-fourty-nine, so our theorem should cover that case too. By generalizing the idea to a base b, where b is any number two or higher, we gain deeper understanding about the subject in question.

Even though doing arithmetic in base-ten has been going on for almost two-thousand years, formalizing it and generalizing it into a theorem is fairly recent. The earliest reference I've found to our theorem is in "Elementary Number Theory" by E. Landau in 1958. We probably don't need to look further back than Leibniz time when he introduced the idea of binary arithmetic in 1679. So our theorem is fairly recent on the world stage.

*The axioms: For every integer a, b and c: Associativity: $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$; Commutativity: $a + b = b + a$ and $ab = ba$; Distributive: $a(b + c) = (b + c)a = ab + ac$; Identities: There are integers 0 and 1 such that, $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$ and Additive Inverse: $a + (-a) = 0$. Note: in general integers do NOT have multiplicative inverses that are also integers. (eg. $\frac{1}{2}$ is the multiplicative inverse of 2 because $\frac{1}{2} \cdot 2 = 1$ but $\frac{1}{2}$ is not an integer.)

†directly ...or indirectly via other previously proven theorems.

Let's take a big leap back and work up to the statement of that theorem step by step, using our familiar base-ten for discussion.

When using some arbitrary base-b to count with, it's useful to have simple symbols to represent each of the integers from zero to up to $(b-1)$. So in our base-ten system we use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Base-ten strings a series of these digits together one after the other to be able to represent each positive integer. Let's look at the first two-digit number, that is, ten, which as you well know looks like this: "10". This extra digit on the left tells us how many tens we have and the last, or rightmost digit says how many additional units to add to it.

So our very first two-digit number 10 means "one lot of ten - plus zero units". When we see "11" - we interpret it to mean "one lot of ten - plus one unit", and "12" is "one lot of ten - plus two units", etc.

Continuing on; "20" - we interpret to mean "two lots of ten, plus zero units", etc. up to "90" meaning "nine lots of ten, plus zero units".

Following this line of reasoning since "10" now means the integer ten, then "100" must mean "ten lots of ten, plus zero units"- which is exactly what it means. We have a special word for this number we call it "one hundred" or "one lot of a hundred, plus zero lots of tens, plus zero units". Similarly "200" means "two lots of a hundred, plus zero lots of ten, plus zero units", etc.

We can keep going by one-hundred until we similarly get to "1000" or "ten lots of a hundred, plus zero lots of ten, plus zero units" otherwise known as "a thousand" or more specifically "one lot of a thousand, plus zero lots of a hundred, plus zero lots of ten, plus zero units".

It gets a little tedious to be so specific when reading out a number so our language has developed quite a few verbal shortcuts. Furthermore it doesn't take long before we run out of fancy names for these "powers-of-ten" like, million, billion, trillion, zillion etc. So let's introduce some nice clean mathematical notation to describe these powers-of-ten and let's forget the fancy words.

$$\begin{aligned}
100 &= 10 \times 10 = 10^2, \\
1000 &= 10 \times 10 \times 10 = 10^3, \\
10000 &= 10 \times 10 \times 10 \times 10 = 10^4, \\
&\dots \\
\underbrace{10\dots000}_{\text{k zeros}} &= \underbrace{10 \times 10 \times 10 \times 10 \times \dots \times 10}_{\text{k 10s}} = 10^k
\end{aligned}$$

10^k means there are k tens multiplied together - also written as a 1 followed by k zeros*. The above list explicitly shows the cases for $k = 2, 3$ and 4. Using the k like that is just a way to show that we can pick ANY whole number, i.e., there is no limit on how big k can be.

The notation of 10^k is handy and extends to the case when $k = 0$ and $k = 1$.

*k is called the "exponent" and you should read the symbol 10^k as "ten-raised-to-the-k^{th}-power" or "ten-to-the-k", so 10^2 is "ten raised to the second power" or 10^4 is "ten-to-the-fourth". You may also see 10^2 referred to as "ten squared", similarly 10^3 as "ten cubed" - but since we don't live in 4 dimensional hyperspace, we don't have a way of saying 10^4 that has geometric meaning.

So 10^1 means* that there is only one ten multiplied together, or one "0" following the "1", in other words just the number ten itself.

How about when $k = 0$? Examining the pattern of how the power k relates to how many zeros follow the "1" (eg, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc.) then it must be the case that $10^0 = 1$, i.e., no zeros follow the "1", which is exactly right. Furthermore every number raised to the 0^{th} power is 1.†

Let's look at an example. Reading the number 92507 out according to our technique we can see that it's "nine lots of ten-thousand, plus two lots of a thousand, plus five lots of a hundred, plus zero lots of ten, plus seven units":

$$\begin{array}{rrcr} & 9 & \times & 10000 \\ + & 2 & \times & 1000 \\ + & 5 & \times & 100 \\ + & 0 & \times & 10 \\ + & 7 & \times & 1 \end{array} \quad = \quad \begin{array}{rr} & 90000 \\ + & 2000 \\ + & 500 \\ + & 00 \\ + & 7 \\ \hline = & 92507 \end{array}$$

Written‡ in terms of powers-of-ten: $92507 = 9\times10^4 + 2\times10^3 + 5\times10^2 + 0\times10^1 + 7\times10^0$.

This way of breaking down the base-ten representation of a number into an algebraic expression can be done for EVERY string of decimal digits. It's the key to understanding what a string of decimal digits means.

Recalling that $10^0 = 1$ you might wonder why we bother to multiply $7\times10^0 = 7\times1 = 7$ since there is no actual effect when multiplying by one. Even though it's not necessary, including the 10^0 in the expression reveals a kind of mathematical symmetry. Each successive digit is multiplied by an ever decreasing power-of-ten, including the units digit, which is just some number from 0 to 9 times a power-of-ten like any of the other digits.

Our example number 92507 only has five digits and it's biggest power-of-ten is 10^4, but there's no limit on how big a power-of-ten could be involved in our expression. Look at "a trillion and one", i.e.; 1,000,000,000,001 which can be expressed as:

$$1\times10^{12} + 0\times10^{11} + 0\times10^{10} + \cdots + 0\times10^2 + 0\times10^1 + 1\times10^0$$

Or pushing that limit to silly heights we can also describe this next ludicrous number. It's twenty-thousand-and-one digits long§, a "7" followed by 9999 zeros, then a "3" followed by 9999 more zeros, then a "5", which means this:

$$7\times10^{20000} + 0\times10^{19999} + \cdots + 0\times10^{10001} + 3\times10^{10000} + 0\times10^{9999} + \cdots + 0\times10^1 + 5\times10^0$$

Clearly we can keep going as high as we like.

Let's use our understanding of counting in base-ten to build up to that a statement of the theorem above.

*Don't forget to read 10^1 as "ten-to-the-one".

†Proof: Since $a^{b+c} = a^b a^c$ consider when $c = 0$; that is, $a^b = a^{b+0} = a^b a^0$ so because of the uniqueness of the multiplicative identity "1", then a^0 *must* be 1 since it's behaving like a "1" in the expression $a^b = a^b a^0$.

‡Recall the mnemonic "bedmas" for the "Order of Operations" in evaluating an expression, which is no different from what we did in our table above the expression.

§A twenty-thousand-and-one digit long number is *ridiculously* large, consider that our estimate of the number of molecules in the entire universe would only need a base-ten number with the k set to somewhere between 78 and 82 to count them all.

We intuitively know that counting with base-ten covers all the positive integers. For example, the odometer in your car that keeps churning out new numbers for each mile you drive, starting from zero when it rolls off the production line. If your odometer was long enough that it stretched off past the horizon on your left, there's no limit on how many miles you could count.

Our intuition is good - let's write it down in our theorem. We might say:

Base-Ten Representation Theorem (initial draft)

> Every integer has a representation in base-ten.

Something else we know intuitively is that each number written in base-ten represents only ONE integer. It almost feels silly to spell it out, but if we were to count out 4 piles of 100-apples-each-pile, then 9 piles of 10-apples-each-pile, then count out 9 additional apples, *then* scoop them all into a big pile that we'd always get the exact same size big-pile-of-apples.

It goes the other way too. If we were handed the aforementioned big-pile-of-apples we could start counting out piles of 100. We'd try to make as many piles of 100 as we could, and we'd find that we'd have 4 piles of 100 before we couldn't make another such pile. Then we would start counting out piles of 10 with the remaining apples. After we made as many piles of ten as we could out of those remaining apples, we would discover that we'd have 9 such piles of ten with 9 apples left over, in other words 499 apples! There is NO other way to divvy up this big-pile-of-apples if we follow this procedure. In other words, each integer is represented by only ONE base-ten number.

Let's strengthen our theorem based on the last two observations.

Base-Ten Representation Theorem (second draft)

> Every integer has a *unique* representation in base-ten.

Let's not even worry about negative integers for now, they're easy to represent once you have a way to represent positive integers, just slap a minus sign on the front to get the negatives. Also, moving forward it would be helpful to have a name for our positive integer so that we can refer to it directly - how about n for "number":

Base-Ten Representation Theorem (third draft)

> Every *positive* integer n has a unique representation in base-ten.

At the moment it's not very helpful to have named n (the theorem as it stands doesn't say anything more about n so why did we bother naming it?) but as we flush out the remaining details of the theorem we can refer to n which carries the important information that it could be ANY positive integer.

Earlier we looked at the number 92507 by adding up each digit times a power-of-ten*:

$$92507 = 9{\cdot}10^4 + 2{\cdot}10^3 + 5{\cdot}10^2 + 0{\cdot}10^1 + 7{\cdot}10^0$$

Every base-ten number implicitly describes an algebraic expression like this, so let's come up with a general expression of this form that can describe ANY positive integer n.

*It's time to replace our "$\times$" symbol for multiplication, with "$\cdot$" because "$\times$" might get confused for an "x" in an expression, whereas "$\cdot$" never will be. Eg. $x \times 2$ vs. $x \cdot 2$, additionally ending up with something that is more aesthetically pleasing. You may also see the "$\cdot$" omitted entirely as in ab - which means $a \cdot b$ as you have seen in earlier footnotes.

Let's replace one of the digits in our example number 92507 with d, how about the 5 like this $92d07$. What I mean becomes clear if I write it out:

$$n = 92d07 = 9{\cdot}10^4 + 2{\cdot}10^3 + d{\cdot}10^2 + 0{\cdot}10^1 + 7{\cdot}10^0$$

So n is one of the following numbers: 92007, 92107, 92207, 92307, 92407, 92507, 92607, 92707, 92807 or 92907.

As you can see, the digit d must be an integer between 0 and 9 inclusive which we can write as "$0 \le d \le 9$" however I suggest that "$0 \le d < 10$" is better*! It's logically equivalent to "$0 \le d \le 9$" but conveys more important information to the reader. Why even talk about nine when the theorem is about base TEN?

$$\begin{gathered} n = 9{\cdot}10^4 + 2{\cdot}10^3 + d{\cdot}10^2 + 0{\cdot}10^1 + 7{\cdot}10^0, \text{ where} \\ d \text{ is an integer such that } 0 \le d < 10 \end{gathered}$$

This statement for n only represents the integers 92007, 92107, ... or 92907, so let's come up with a statement for n that will allow us to generate ANY five-digit number from 10000 the way up to 99999 (which is a complete list of all the five-digit numbers).

In order to describe this general five-digit number, we need five different 'd's, one for each of the five digits. In other words, we need to associate a different term d with each of the powers 10^4, 10^3, 10^2, 10^1 and 10^0.

Mathematics has a convention for coming up with a list of terms for situations just like this - we tack a subscript onto the name like so: d_2 which you read as "dee-two"†. d_2 is a term to represent a digit just like the d we used above. But now we can use that little subscript as a way to associate it to a specific power-of-ten. Naturally we'll associate d_2 with 10^2 (ten squared) as follows:

$$\begin{gathered} n = 9{\cdot}10^4 + 2{\cdot}10^3 + d_2{\cdot}10^2 + 0{\cdot}10^1 + 7{\cdot}10^0, \text{ where} \\ d_2 \text{ is an integer such that } 0 \le d_2 < 10 \end{gathered}$$

If we define d_2 like this, then we know that when we refer to the digit d_2 that we are talking about the digit that is multiplied with 10^2.

---

*Read $0 \le d \le 9$ as "zero is less-than-or-equal-to dee which is less-than-or-equal-to nine" and $0 \le d < 10$ as "zero is less-than-or-equal-to dee which is (strictly) less-than ten".

†...yes like artoo-detoo, which perhaps George should have written as "R_2D_2" and not "R2-D2"!

Let our five-digit-number n use d_0, d_1, d_2, d_3 and d_4 for its digits. Then the general expression for n looks like this*:

$$\begin{aligned} n = {} & d_4{\cdot}10^4 + d_3{\cdot}10^3 + d_2{\cdot}10^2 + d_1{\cdot}10^1 + d_0{\cdot}10^0, \text{ where} \\ & d_0 \text{ is an integer such that } 0 \le d_0 < 10, \text{ and} \\ & d_1 \text{ is an integer such that } 0 \le d_1 < 10, \text{ and} \\ & d_2 \text{ is an integer such that } 0 \le d_2 < 10, \text{ and} \\ & d_3 \text{ is an integer such that } 0 \le d_3 < 10, \text{ and} \\ & d_4 \text{ is an integer such that } 1 \le d_4 < 10. \end{aligned}$$

Ok, hold on a minute - that's getting a little cumbersome. it's clunky and hard to read - plus did you notice how we slipped in that different range for d_4?

First let's deal with the different range on that d_4. To make sure n is a legitimate five-digit number we have to call out the exception that d_4 can NOT be zero - it has to be at least 1. Why? Because if d_4 were zero then n would only be a four-digit number, or perhaps a three-digit number, or only two-digits etc.

Secondly, to clean up the presentation a common convention is to let another term like i, for perhaps "index", stand in for the subscript when you want to talk about all your 'd's at once:

Let d_0, d_1, d_2, d_3 and d_4 be integers such that:

$$n = d_4{\cdot}10^4 + d_3{\cdot}10^3 + d_2{\cdot}10^2 + d_1{\cdot}10^1 + d_0{\cdot}10^0$$

where $0 \le d_i < 10$ for all i in $\{0, 1, 2, 3, 4\}$ and $d_4 \ne 0$.

That's it! Those statements, and the expression for n describe EVERY five-digit number.

Now let's extend our five-digit expression for n to an arbitrary number of digits. Consider the following progression:

expression for n	number-of-digits
$d_0{\cdot}10^0$	1
$d_1{\cdot}10^1 + d_0{\cdot}10^0$	2
$d_2{\cdot}10^2 + d_1{\cdot}10^1 + d_0{\cdot}10^0$	3
$d_3{\cdot}10^3 + d_2{\cdot}10^2 + d_1{\cdot}10^1 + d_0{\cdot}10^0$	4
$d_4{\cdot}10^4 + d_3{\cdot}10^3 + d_2{\cdot}10^2 + d_1{\cdot}10^1 + d_0{\cdot}10^0$	5
$\dots$	$\dots$
$d_k{\cdot}10^k + \dots + d_4{\cdot}10^4 + d_3{\cdot}10^3 + d_2{\cdot}10^2 + d_1{\cdot}10^1 + d_0{\cdot}10^0$	k+1

Using k like this let's us specify any number of digits we want. If we let $k = 0$ we get the first "single digit" item on the list. $k = 4$ gives us our five-digit number above, or we could let k be a twenty-thousand, which would allow us to specify an integer that has a twenty-thousand-and-one digits in it.

*That expression looks like hard-core math, so let's take a moment to read it out loud, as a Math-Professor might do in a lecture. She might say: "n is equal to dee-four times ten-to-the-fourth, ... plus dee-three times ten-cubed, ... plus dee-two times ten-squared, ... plus dee-one times ten, ... plus dee-zero times one."

So there we have it, we found our expression for being able to express each positive integer, let's use it in a revised draft of our theorem:

Base-Ten Representation Theorem (close to final draft)

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k{\cdot}10^k + d_{k-1}{\cdot}10^{k-1} + \cdots + d_2{\cdot}10^2 + d_1{\cdot}10^1 + d_0{\cdot}10^0$$

where $0 \leq d_i < 10$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-ten by the string of digits $d_k d_{k-1} \cdots d_2 d_1 d_0$

Our newly added "Definition" introduces exactly what it means to write the number out in base-ten; that is, we toss out all the extraneous stuff from our expression and string all the digits one after another. Starting at the most-significant digit d_k on the left, down to the next digit to its right which is d_{k-1} (read as "dee-kay-minus-one"*) all the way down to the least-significant units-digit d_0 on the right.

We are so darn close, but there is one super-picky detail that we should be concerned about. Our theorem establishes what it means to represent a number in base-ten, so until we've proven it, how can we actually use the first two-digit number "10" to represent the integer ten!? We need a symbol for ten in the theorem, so what can we do?

Apart from "10" we don't have a symbol for the integer ten so we have to make one up, how about T for "Ten":

Let T represent the integer ten.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k{\cdot}T^k + d_{k-1}{\cdot}T^{k-1} + \cdots + d_2{\cdot}T^2 + d_1{\cdot}T^1 + d_0{\cdot}T^0$$

... etc.

That's a little confusing, so let's solve our problem by defining the two digit number "10" (i.e., a one followed by a zero) to be the integer ten. It's ok - we're not violating any rules by doing this. At this point we're just defining a symbol to stand in for the integer ten. Here's our FINAL draft of the theorem:

Base-Ten Representation Theorem

Let the two digit number "10" represent the integer ten.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k{\cdot}10^k + d_{k-1}{\cdot}10^{k-1} + \cdots + d_2{\cdot}10^2 + d_1{\cdot}10^1 + d_0{\cdot}10^0$$

where $0 \leq d_i < 10$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-ten by the string of digits $d_k d_{k-1} \cdots d_2 d_1 d_0$

* ... and d_{k-1} is multiplied by "ten-to-the-power-of-(kay-minus-one)".

Now let's begin to generalize our theorem to any base.

Since the introduction of the EDVAC* computer, around 1950, there have been many orders of magnitude more calculations done in base-two (otherwise known as binary) by computers than have EVER been done by people in base-ten for the entirety of human history. (This might even be true if we only count one-day's worth of binary computer calculations - someone needs to check this.)

Binary-computer logic gates (the building blocks of the modern computer) can only take one of two states, that is; "off" or "on". We interpret these two states to represent these two numbers: 0 and 1. By doing so, in the same way that base-ten uses ten numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 for its digits; we can represent integers in base-two with just the digits 0 and 1. How is this possible? Let's find out with an imaginary trip into space.

Consider distant Planet-Nova on which the emergent intelligent species only have nine fingers on their hands. They have three hands with three fingers each - anyway, that's why they use base-nine, so they only need the numbers 0, 1, 2, 3, 4, 5, 6, 7 and 8 for their digits†. So like we Earthlings do for the integer ten, instead of making up a new symbol for nine, they use "10" to represent the integer nine - which for them means "One lot of nine, plus zero units".

Similarly on Planet-Ocho, since they only have eight fingers, then they use base-eight and only use numbers 0, 1, 2, 3, 4, 5, 6 and 7 for their digits. For them "10" means "One lot of eight, plus zero units".

On and on past Planet-Gary-Seven, and Planet-Secks, Planet-Penta, . . .

Finally we come upon Planet-Claire (well someone has to come from Planet-Claire, I know she came from there), where the poor bastards only have two fingers so they only use the digits 0 and 1 and base-two, so for them "10" means "one lot of two and zero units". So on Planet-Claire "10" means two. Recall above how we arrived at our 100 in base-ten, being "ten lots of ten, plus zero units" - similarly on Planet-Claire "100" in base-two for them means "Two lots of two plus zero units" in other words, four! What is "11" in base-two? Using our technique to describe the digits we see that it's "One lot of two, plus one unit", in other words three.

Here's how they count on Planet-Claire using base-two:

base-two	base-ten	base-two	base-ten
0	0	(. . . cont)	
1	1	1101	13
10	2	1110	14
11	3	1111	15
100	4	10000	16
101	5	10001	17
110	6	. . .	
111	7	11111	31
1000	8	100000	32
1001	9	. . .	
1010	10	1000000	64
1011	11	10000000	128
1100	12 (cont. . .)	100000000	256

*You might be thinking, don't you mean ENIAC which was earlier? Actually no - the ENIAC used base-ten accumulators, not binary!

†Digit is another word for finger! Of course that's where the math term got its start.

Note something interesting in the list above - the powers of two, written in base-two, resemble our powers of 10 in base-ten! That is:

$$
\begin{aligned}
1 &= 2^0 = 1, & 32 &= 2^5 = 100000_{\text{(base-2)}},\\
2 &= 2^1 = 10_{\text{(base-2)}}, & 64 &= 2^6 = 1000000_{\text{(base-2)}},\\
4 &= 2^2 = 100_{\text{(base-2)}}, & 128 &= 2^7 = 10000000_{\text{(base-2)}},\\
8 &= 2^3 = 1000_{\text{(base-2)}}, & 256 &= 2^8 = 100000000_{\text{(base-2)}},\\
16 &= 2^4 = 10000_{\text{(base-2)}}, & & \dots
\end{aligned}
$$

Let's look at the binary number 11010 for example. Using our wordy technique to describe the number we can see that it's "One lot of sixteen, plus one lot of eight, plus zero lots of four, plus one lot of two, plus zero units":

$$
\begin{array}{rrcr}
 & 1 & \times & 10000 \\
+ & 1 & \times & 1000 \\
+ & 0 & \times & 100 \\
+ & 1 & \times & 10 \\
+ & 0 & \times & 1
\end{array}
\quad = \quad
\begin{array}{crl}
 & 10000 & (16) \\
+ & 1000 & (8) \\
+ & 000 & \\
+ & 10 & (2) \\
+ & 0 & \\
\hline
= & 11010 & (26)
\end{array}
$$

Written in terms of powers of two: $11010_{\text{(base-2)}} = 26 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$.

Does that expression look familiar? It has exactly the same form as the expression for our five-digit base-ten number 92507. All the reasoning we used to come up with the statement of the "Base-Ten Representation Theorem" can be used again, but swapping powers-of-two for powers-of-ten, and limiting the values for the digits to be zero or one. Following our line of reasoning this is what the Planet-Claire mathematicians would have come up with:

Base-Two Representation Theorem

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \dots, d_k$ such that:

$$n = d_k 2^k + d_{k-1} 2^{k-1} + \cdots + d_2 2^2 + d_1 2^1 + d_0 2^0,$$

where $0 \le d_i < 2$ for all i in $\{0, 1, 2, \dots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-two by the string of binary-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_2$

Our new Base-Two Representation Theorem introduced some helpful new notation. How do you know what I'm talking about if I just write "1000"? Do I mean 10^3 or 2^3? If there is any possibility for confusion we write the number like this $(1000)_{10}$ for the base-ten version meaning one-thousand and $(1000)_2$ for the binary version meaning eight. That's what the "Definition" is spelling out with the "$(\dots)_2$" extra notation.

As is hinted by the habits of our various alien friends above it seems that we can use ANY integer greater than or equal to 2 as a base (base-one doesn't really make sense - think about it for a while). In fact computer graphics artists are known to stumble upon numbers written in hexadecimal (usually relating to specifying a color-channel), which is base-sixteen.

Base-sixteen introduces some new single-character symbols to the usual numbers 0, 1, 2, thru 9, to represent the numbers 10, 11, 12, 13, 14 and 15. Base-sixteen adds the digits A, B, C, D, E and F where $A_{16}=(10)_{10}$, $B_{16}=(11)_{10}$, $C_{16}=(12)_{10}$, $D_{16}=(13)_{10}$, $E_{16}=(14)_{10}$, $F_{16}=(15)_{10}$. So $(80FB)_{16}$ is a four digit number in base-sixteen. (As we'll see shortly it means $(33019)_{10}$ in base-ten).

Note that if we omit the parentheses and subscript from a number, it means we're talking about it in base-ten; our "default" base. Case in point: the subscripts that we use to denote the base (like the "16" in $(80FB)_{16}$) are written in base-ten!

We could go ahead and prove our "Base-Ten" and "Base-Two" theorems above, but what about proving the "Base-Nine" version of the theorem for the aliens on Planet-Nova, or the "Base-Eight" version for the inhabitants of Planet-Ocho?

To cover all bases (pun intended) let's restate our theorem for the general case, call it "base-b", where b is some number greater than or equal to two. If we can prove that theorem, then we'll automatically get all the cases of specific bases for free.

Here is our hero-theorem again, but this time, armed with your new mathematical vocabulary, I expect that this theorem will make much more sense to you.

Basis Representation Theorem

> Let b be a positive integer greater than 1.
>
> For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:
>
> $$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$
>
> where $0 \le d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.
>
> Definition: n is represented in base-b by the string of base-b-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$

So to get the "Base-Ten Representation Theorem" let b equal ten. To get the "Base-Two Representation Theorem" let $b = 2$; or the "Base-Nine Representation Theorem" let $b = 9$; etc.

The Basis Representation Theorem implies that we can safely convert between different bases. Why? (Exercise left for student).

Recall how we defined $(A)_{16} = 10$ and $(F)_{16} = 15$ as base-sixteen digits, then:

$$(97A3F2)_{16} = 9 \cdot 16^5 + 7 \cdot 16^4 + 10 \cdot 16^3 + 3 \cdot 16^2 + 15 \cdot 16^1 + 2 \cdot 16^0 = 9,937,906$$

... which gives you an idea of how you can convert from an alternate base into base-ten.