Factoradic Representation of Rational Numbers

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From 'A Course in Pure Mathematics' by G. H. Hardy. Chapter 1, Miscellaneous Examples.

Miscellaneous example* #2 at the end of chapter 1 in Hardy's 'Pure Mathematics' presents us with a fascinating result (which was new to me). It feels like a kind of basis-representation-theorem, but for rational numbers, ... beautiful!

Here it is, followed by my proof which starts out with some lemmas to get us rolling.

Theorem

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Lemma-1

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}$$

Proof of Lemma-1

This equality is fairly trivial to demonstrate by induction, since $\frac{1}{2!} = \frac{2!-1}{2!}$ and,

$$\begin{aligned} & \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\ &= \frac{(k-1)!-1}{(k-1)!} + \frac{k-1}{k!} \\ &= \frac{k((k-1)!-1)}{k(k-1)!} + \frac{k-1}{k!} \\ &= \frac{k!-k+k-1}{k!} \\ &= \frac{k!-1}{k!} \end{aligned}$$

...thus establishing lemma-1 for all values of k. QED

*Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, math exercises like calculations to perform, theorems to prove etc.

Lemma-2

For integers i, k where $2 \leq i < k$ such that,

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!} + \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!},$$

then

$$\frac{1}{i!} - \frac{1}{k!} = \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!}$$

Proof of Lemma-2

$$\begin{aligned}
&\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \\
&= (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!}) - (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!}) \\
&= \frac{k!-1}{k!} - \frac{i!-1}{i!} \\
&= \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\
&= \frac{1}{i!} - \frac{1}{k!}
\end{aligned}$$

Lemma-3

The set of rational numbers,

$$\mathcal{S} = \{\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \quad | \quad 0 \leq a_2 < 2, \;\; 0 \leq a_3 < 3, \;\; \ldots, \;\; 0 \leq a_k < k\}$$

is identical to the set of rational numbers,

$$\mathcal{F} = \{\frac{0}{k!}, \;\; \frac{1}{k!}, \;\; \frac{2}{k!}, \;\; \ldots, \;\; \frac{k!-1}{k!}\}$$

Proof of Lemma-3

The smallest member of the set $\mathcal{S}$ is $\frac{0}{k!}$ and occurs when all the coefficients of the sum are zero. Furthermore, the largest member of the set occurs when all the coefficients of the sum are set to their maximum value, which adds up to $\frac{k!-1}{k!}$ as shown in lemma-1.

We also note that every member of $\mathcal{S}$ can be written as a rational number with $k!$ as the denominator, like so,

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_{k-1}}{(k-1)!} + \frac{a_k}{k!} = \frac{k \cdot (k-1) \cdot \; \cdot \; \cdot 3 \cdot a_2}{k!} + \frac{k \cdot (k-1) \cdot \; \cdot \; \cdot 4 \cdot a_2}{k!} + \ldots + \frac{k \cdot a_{k-1}}{k!} + \frac{a_k}{k!}$$

Furthermore, each possible assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produce a unique member of the set $\mathcal{S}$.

For this weren't true and both $\frac{p}{k!} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ and $\frac{p}{k!} = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$ for different coefficients $a_2, a_3, \ldots, a_k$ and $b_2, b_3, \ldots, b_k$, then we can arrive at a contradiction as follows.

First suppose that $a_i \neq b_i$ is the first such pair of coefficients that differ from each other. In other words, $a_2 = b_2, a_3 = b_3, \ldots, a_{i-1} = b_{i-1}$. Also, without loss of generality we can assume that $a_i > b_i$ and state the following equality:

$$\frac{a_i}{i!} + \frac{a_{i+1}}{(i+1)!} + \ldots + \frac{a_k}{k!} = \frac{b_i}{i!} + \frac{b_{i+1}}{(i+1)!} + \ldots + \frac{b_k}{k!}$$

$$\Leftrightarrow \qquad \frac{a_i - b_i}{i!} = \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!}$$

The number of values that the coefficient a_2 can assume is 2, a_3 can take on 3 values, ..., up to a_k which can take on k values. So the total number of combinations of values that can be assigned to all the coefficients is $2 \cdot 3 \cdot 4 \cdot \ldots \cdot k = k!$.

The size of $\mathcal{F}$ is clearly $k!$