Rational Approximations to $\sqrt{2}$

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From 'A Course in Pure Mathematics' by G. H. Hardy. Chapter 1, Section 5.

From "Example 2" in Hardy's "Pure Mathematics" (Ch. 1, Section 5, Examples III) we looked at one sequence of rational approximations to $\sqrt{2}$, that is:

$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \tfrac{99}{70}$$

So these numbers squared are:

$$\tfrac{1}{1},\ \tfrac{9}{4},\ \tfrac{49}{25},\ \tfrac{289}{144},\ \tfrac{1681}{841},\ \tfrac{9801}{4900}$$

And their differences from 2 are:

$$-1,\ \tfrac{1}{4},\ -\tfrac{1}{25},\ \tfrac{1}{144},\ -\tfrac{1}{841},\ \tfrac{1}{4900}$$

Theorem: Examples III, Number 3.

Show that if $\frac{m}{m}$ is a good approximation to $\sqrt{2}$, then $\frac{(m+2n)}{(m+n)}$ is a better one, and that the errors in the two cases are in opposite directions.

Lemma

Let m, n and a be positive integers

$$\frac{m^2}{n^2} + \frac{a}{n^2} = 2 \qquad \Leftrightarrow \qquad \frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} = 2$$

Proof of Lemma

$$
\begin{array}{lrcl}
 & \dfrac{(m+2n)^2}{(m+n)^2} - \dfrac{a}{(m+n)^2} & = & 2 \\
\Leftrightarrow & (m+2n)^2 - a & = & 2(m+n)^2 \\
\Leftrightarrow & m^2 + 4mn + 4n^2 & = & 2m^2 + 4mn + 2n^2 + a \\
\Leftrightarrow & 4n^2 - 2n^2 & = & 2m^2 - m^2 + a \\
\Leftrightarrow & 2n^2 & = & m^2 + a \\
\Leftrightarrow & 2 & = & \dfrac{m^2}{n^2} + \dfrac{a}{n^2}
\end{array}
$$

QED

Proof of Theorem

The error for any rational approximation to $\sqrt{2}$ is the difference between the approximation-squared and 2. So we learn from the Lemma that the error for the $\frac{m}{n}$ approximation of $\sqrt{2}$ is

$$\left|\tfrac{a}{n^2}\right| = \left|2 - \tfrac{m^2}{n^2}\right|,$$

and the error for the $\frac{(m+2n)}{(m+n)}$ approximation of $\sqrt{2}$ is

$$\left|\tfrac{-a}{(m+n)^2}\right| = \left|2 - \tfrac{(m+2n)^2}{(m+n)^2}\right|.$$

The denominator $(m+n)^2$ is always larger than n^2 therefore,

$$\left|\frac{-a}{(m+n)^2}\right| < \left|\frac{a}{n^2}\right|,$$

showing that the approximation $\frac{(m+2n)}{(m+n)}$ for $\sqrt{2}$ has a smaller error that that of $\frac{m}{n}$ and is therefore a better approximation to $\sqrt{2}$.

Furthermore, we also learn from the Lemma that the successive approximations flip to either side of 2 because:

If $\frac{m^2}{n^2} < 2$ then a is positive therefore $\frac{(m+2n)}{(m+n)} > 2$, similarly,

If $\frac{m^2}{n^2} > 2$ then a is negative therefore $\frac{(m+2n)}{(m+n)} < 2$.

QED