

# Factoradic Representation of Rational Numbers

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*From ‘A Course in Pure Mathematics’ by G. H. Hardy. Chapter 1, Miscellaneous Examples.*

Miscellaneous example\* #2 at the end of chapter 1 in Hardy’s ‘Pure Mathematics’ presents us with a fascinating result (which was new to me). It feels like a kind of basis-representation-theorem, but for rational numbers, ... beautiful!

Here it is, followed by my proof which starts out with some lemmas to get us rolling.

## Theorem

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \dots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \dots \cdot k},$$

where  $a_1, a_2, \dots, a_k$  are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \dots, \quad 0 < a_k < k$$

## Lemma-1

$$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-1}{k!} = \frac{k!-1}{k!}$$

## Proof of Lemma-1

This equality is fairly trivial to demonstrate by induction, since  $\frac{1}{2!} = \frac{2!-1}{2!}$  and,

$$\begin{aligned} & \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\ &= \frac{(k-1)!-1}{(k-1)!} + \frac{k-1}{k!} \\ &= \frac{k((k-1)!-1)}{k(k-1)!} + \frac{k-1}{k!} \\ &= \frac{k!-k+k-1}{k!} \\ &= \frac{k!-1}{k!} \end{aligned}$$

... thus establishing lemma-1 for all values of k.

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\*Hardy doesn’t call them ‘Exercises’ or ‘Questions’, but that’s what they are, math exercises like calculations to perform, theorems to prove etc.

## Lemma-2

For integers  $i, k$  where  $2 \leq i < k$  such that,

$$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{i-1}{i!} + \frac{i}{(i+1)!} + \dots + \frac{k-1}{k!},$$

then

$$\frac{1}{i!} - \frac{1}{k!} = \frac{i}{(i+1)!} + \dots + \frac{k-1}{k!}$$

## Proof of Lemma-2

$$\begin{aligned} & \frac{i}{(i+1)!} + \dots + \frac{k-1}{k!} \\ = & \left( \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-1}{k!} \right) - \left( \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{i-1}{i!} \right) \\ = & \frac{k!-1}{k!} - \frac{i!-1}{i!} \\ = & \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\ = & \frac{1}{i!} - \frac{1}{k!} \end{aligned}$$

## Lemma-3

The set of rational numbers,

$$\mathcal{S} = \left\{ \frac{a_2}{2!} + \frac{a_3}{3!} + \dots + \frac{a_k}{k!} \mid 0 \leq a_2 < 2, 0 \leq a_3 < 3, \dots, 0 \leq a_k < k \right\},$$

is identical to the set of rational numbers,

$$\mathcal{F} = \left\{ \frac{0}{k!}, \frac{1}{k!}, \frac{2}{k!}, \dots, \frac{k!-1}{k!} \right\}$$

## Proof of Lemma-3

Now we make note of the fact that the smallest member of the set  $\mathcal{S}$  occurs when all the coefficients of the sum are zero, i.e.;  $\frac{0}{k!}$ . Furthermore, the largest member of the set occurs when all the coefficients are set to their maximum value, which we have just seen gives us  $\frac{k!-1}{k!}$ .

We also note that every members of  $\mathcal{S}$  can be written as a rational number with  $k!$  as the denominator, like so:

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \dots + \frac{a_k}{k!} = \frac{k \cdot (k-1) \dots 3 \cdot a_2}{k!} + \frac{k \cdot (k-1) \dots 4 \cdot a_3}{k!} + \dots + \frac{a_k}{k!}$$

Also when  $i < k$  such that,

$$\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{i-1}{i!} + \frac{i}{(i+1)!} + \dots + \frac{k-1}{k!}$$

Then we can conclude that,

$$\frac{1}{i!} - \frac{1}{k!} = \frac{i}{(i+1)!} + \dots + \frac{k-1}{k!}$$

Because,

From here we can deduce that any assignment of values to the coefficients of a member of  $\mathcal{S}$  produces a unique member of the set, for if it didn't and  $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \dots + \frac{a_k}{k!}$  is not uniquely determined by the coefficients  $a_2, a_3, \dots, a_k$ . That is, suppose there is a second DIFFERENT sequence of coefficients  $b_2, b_3, \dots, b_k$  such that  $\frac{p}{q} = \frac{b_2}{2!} + \frac{b_3}{3!} + \dots + \frac{b_k}{k!}$ .

The number of values that the coefficient  $a_2$  can assume is 2,  $a_3$  can take on 3 values,  $\dots$ , up to  $a_k$  which can take on  $k$  values. So the total number of combinations of values that can be assigned to all the coefficients is  $2 \cdot 3 \cdot 4 \cdot \dots \cdot k = k!$ .

The size of  $\mathcal{F}$  is clearly  $k!$