Counting

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July 24, 2017

There are 10 sorts of people in the world: those who understand binary and those who don't.

What does "10" mean?

We got it drilled into us watching Sesame Street that "10" is the symbol for the number "ten" which is this many apples "🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎" or the number of fingers on a typical person's two hands.

Once we are trained to automatically think of "10" as representing ten things, we quickly move past it to learn about 100 and 1000 and how to interpret a string of digits like 92507. Even at a young age we'd be able to accurately count out a pile of ninety-two-thousand-five-hundred-and-seven apples as time consuming and agonizing as it might be. Furthermore, learning how to add and multiply is easy once you can count in base-ten since the techniques are simple and straight-forward.

What about kids in ancient Rome, was it as easy for them? Try adding two numbers together in ancient Rome, or worse, multiplying or dividing them. What's XI times IX? Would you believe me if I told you it's XCIX?

Unless you convert those to Hindu-Arabic decimal or base-ten numerals to check, you're just gonna have to trust me. Truth is - I don't know how to multiply using Roman numerals - nor did most Romans. Not only that, but I'll bet that most modern eight-year-olds can count higher than any Roman could - as the Roman system only effectively allowed counting up to 4999.

Even though we use different symbols, the ancient Romans and us are talking about the same abstract set of numbers underneath, which we call integers*. Mathematics deals with numbers in this pure sort of way, divorced from the symbols used to represent each number. When we talk about positive integers in mathematics, it's best to remind yourself that we are really talking about the set of numbers that represent successively larger piles of apples, forgetting the symbols.

However we use numbers written out in base-ten all the time in mathematics, rarely thinking in terms of piles of apples. We take it for granted that we can use base-ten to represent the set of positive integers. *Caution*: the only thing modern mathematics takes for granted are axioms and the fact that we can use base-ten to represent the integers is NOT among the list of axioms.

Briefly; the axioms describe a few simple properites about addition and multiplication. These properites are *so simple* that they can't be expressed in yet other even-simpler ideas. The

*Integers are the set of all the postive whole numbers, as well as zero and all the negative counterparts to each positive number.

axioms are the minimal set of simple, obvious, irrefutable ideas from which everything else in mathematics is built*.

Since our ability to count in base-ten is not axiomatic, we need to define what it means to write out a number in base-ten, then state its properties in a theorem which we must prove to be true with a series of arguments that logically connects it directly† to the axioms. In doing so, the only way that the theorem could be false is if the axioms themselves are false.

Here's what that theorem looks like.

Basis Representation Theorem

Let b be a positive integer greater than 1.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

where $0 \leq d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-b by the string of base-b-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$

"That's nuts!" you might say, I don't even see a "ten" in there so how could that be about how we learned to count watching Sesame Street? If you let $b =$ ten in the above theorem, then we have the "Base-Ten Representation Theorem". We could let $b = 2$, then we'd have "Base-Two Representation Theorem" which states that we can count in binary.

Anyway, if you're not a "math-person" don't fear, by the end of this paper you will know how to read that theorem so that it makes sense plus you'll know how to prove it.

Let's take a big leap back and work up to the statement of that theorem step by step, using our familiar base-ten in our discussion.

*The axioms: For every integer a, b and c: Associativity: $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$; Commutativity: $a + b = b + a$ and $ab = ba$; Distributive: $a(b + c) = (b + c)a = ab + ac$; Identities: There are integers 0 and 1 such that, $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$ and Additive Inverse: $a + (-a) = 0$. Note: in general integers do NOT have multiplicative inverses that are also integers. (eg. $\frac{1}{2}$ is the multiplicative inverse of 2 because $\frac{1}{2} \cdot 2 = 1$ but $\frac{1}{2}$ is not an integer.)

†directly . . . or indirectly via other previously proven theorems.