

Rational Approximation to $\sqrt{2}$

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From 'A Course in Pure Mathematics' by G. H. Hardy. Chapter 1, Section 5.

Examples III, Number 3.

Show that if $\frac{m}{n}$ is a good approximation to $\sqrt{2}$, then $\frac{(m+2n)}{(m+n)}$ is a better one, and that the errors in the two cases are in opposite directions.

Lemma

Let m , n and a be positive integers

$$\frac{m^2}{n^2} + \frac{a}{n^2} = 2 \quad \Leftrightarrow \quad \frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} = 2$$

Proof of Lemma

$$\begin{aligned} & \frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} = 2 \\ \Leftrightarrow & \quad (m+2n)^2 - a = 2(m+n)^2 \\ \Leftrightarrow & \quad m^2 + 4mn + 4n^2 = 2m^2 + 4mn + 2n^2 + a \\ \Leftrightarrow & \quad 4n^2 - 2n^2 = 2m^2 - m^2 + a \\ \Leftrightarrow & \quad 2n^2 = m^2 + a \\ \Leftrightarrow & \quad 2 = \frac{m^2}{n^2} + \frac{a}{n^2} \quad \text{QED} \end{aligned}$$

Proof of Examples III, Number 3.

The error in any rational approximation to $\sqrt{2}$ is the difference between the approximation-squared and 2. Let e_1 represent the error for the $\frac{m}{n}$ approximation of $\sqrt{2}$, and let e_2 represent the error for the $\frac{(m+2n)}{(m+n)}$ approximation of $\sqrt{2}$, in other words:

$$\begin{aligned} e_1 &= \left| \frac{a}{n^2} \right| = \left| 2 - \frac{m^2}{n^2} \right|, \text{ and} \\ e_2 &= \left| \frac{-a}{(m+n)^2} \right| = \left| 2 - \frac{(m+2n)^2}{(m+n)^2} \right| \end{aligned}$$