Magic Numbers

James Rowell

February 12, 2017

In the delightful film "School of Rock" Jack Black's character Dewey Finn, pretending to be substitute teacher Ned Schneebly, is put on the spot when Miss. Mullins, the school principle, comes into class and demands that he show her his teaching methods which apparently involve an electric guitar. So "Mr. S" sings "The Math Song" - in the final line of his improvised tune he sings "...yes it's 9, and that's a magic-number..."

He's right - he's referring to the fact that if you add up the digits of any number then if that sum is divisible by 9, then the original number must be divisible by 9. It works the other way too, that is, if we know that a number is divisible by 9 then the sum of its digits must also be divisible by 9.

For example:

$9^5 = 59049$ and $5 + 9 + 0 + 4 + 9 = 27$ and 9 divides 27 because $27 = 3 \times 9$.

You may recall that 3 is also a magic-number, for example:

$$3 \times 67 = 201 \text{ and } 2 + 0 + 1 = 3$$

In this case, 9 does not divide 201* but 3 does (67 times). We can see that adding up the digits of 201 gives us 3, which of course is divisible by three but not by 9.

Anytime I want to check to see if a number is divisible by 3 (but don't feel like busting out a calculator) I use this little trick of checking to see if the digits add up to some multiple of 3. Along with the tricks to see if the number is divisible by 2 (last digit is even) or is divisible by 5 (last digit is 0 or 5) - you can decide if you can factor out 2, 3 or 5 from any number pretty quickly. 7 takes real work . . . but I digress.

3 and 9 are the only magic-numbers in base-10. However 7 is a magic-number in base-8; And 4 is a magic-number in base-5; 15 is a magic-number in base-16; . . . and 30 is a magic-number in base-31.

Here are some examples demonstrating† this fact about magic-numbers in alternate bases:

- $7 \times 7 = 49$ written in base-8 is $(61)_8$ and $6 + 1 = 7$.
- $7^3 \times 5 = 1715$ written in base-8 is $(3263)_8$ and $3 + 2 + 6 + 3 = 14$ (7 divides 14).
- $4 \times 67 = 268$ written in base-5 is $(2033)_5$ and $2 + 0 + 3 + 3 = 8$ (4 divides 8).

*Why is it immediately apparent that 9 does not divide 201? Hint: both 3 and 67 are prime numbers. Google the "Fundamental Theorem of Arithmetic" for further insight.

†We are going to use the conventions for specifying numbers in alternate bases as outlined in the paper "Sesame Street++".

- $15 \times 23 = 345$ written in base-16 is $(159)_{16}$ and $1+5+9=15$.
- $30 \times 35951 = 1078530$ written in base-31 is $(15699)_{31}$ and $1+5+6+9+9=30$

You may notice a pattern: the biggest "digit" in any base (which is always one less than the base) is a magic-number. Using our examples, 9 is the biggest digit in base-10; 7 is the biggest digit in base-8; and 4 is the biggest digit in base-5. To write this out in a general way we would say that if b is our base, then $(b-1)$ is a magic-number in base-b.

So what about 3 also being a magic-number in base-10? Let's take a look at the interesting (but probably never used outside this paper) base-31. It appears that 2, 3, 5, 6, 10, 15 (along with 30) are *all* magic-numbers in base-31, as we we can see with the following examples:

- $2 \times 13 \times 37 = 962$ written in base-31 is $(101)_{31}$ and $1+0+1=2$.
- $3 \times 641 = 1923$ written in base-31 is $(201)_{31}$ and $2+0+1=3$.
- $5 \times 139 \times 2659 = 1848005$ written in base-31 is $(20102)_{31}$ and $2+0+1+0+2=5$.
- $6 \times 10091 = 60546$ written in base-31 is $(2103)_{31}$ and $2+1+0+3=6$.
- $10 \times 197 \times 941 = 1853770$ written in base-31 is $(20701)_{31}$ and $2+0+7+0+1=10$.
- $15 \times 71503 = 1072545$ written in base-31 is $(15027)_{31}$ and $1+5+0+2+7=15$.

See the pattern? All the divisors of 30 are magic-numbers in base-31. It turns out that in base-b, then any divisor of $(b-1)$ is a magic-number in base-b.

I guess that begs the question: What about 1? Isn't 1 a magic-number in all bases? Since we already know that all numbers are divisible by 1, then we don't need to go to any trouble of adding up digits to find this out. So let's exclude 1 from being considered as a magic-number - it's not helpful.

Magic Number Theorem

Let $b, m, n \in \mathbb{Z}$ such that $b \geq 2$ and $m \big| (b-1)$.

If $n = d_k b^k + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0$, $d_i \in \mathbb{Z} : 0 \leq d_i \leq (b-1)\ \forall\ i \in \{0, 1, \ldots k\}$ then

$$m \big| n \Leftrightarrow m \big| (d_k + \cdots + d_2 + d_1 + d_0)$$

Definition: $m \neq 1$ is called a "magic-number" in base-b.

As an interesting side note - according to our definition, binary, or base-2, doesn't have a magic-number. No great loss, as we said above 1 isn't helpful to consider as a magic-number.

Let's look at the definition of a number represented in a given base.

...wip... break down what a number is (refer to baseTheorem paper) and explain motivation for mod

We assume the following properties of modular arithmetic where a,b,c ? Zsuch that c¿1.

(a+b) mod c=(a mod c+b mod c) mod c ab mod c=((a mod c)·(b mod c)) mod c

As a generalization of 2), we get:

3) a mod c=b mod c akmod c=bk mod c

Importantly, a mod c=0 c divides a.

In other words, there exists a number n such that nc=a. Lemma: if d¿1 is a divisor of m then: (m+1)k mod d=1 Proof of Lemma: Let d be a divisor of m. Soc such that cd=m. Hence, (cd + 1) mod d = cd mod d + 1 mod d = (c mod d) (d mod d) + 1 mod d = (c mod d) 0 + 1 mod d = 1 mod d By property 3) above, since (cd+1) mod d=1 mod d then: (m + 1)k mod d = (cd + 1)k mod d = (cd+1) mod d = 1 QED

Proof of "Magic Number" Theorem:

Let N=M+1.

That is; we are going to consider representations of numbers in base-M+1, so think of M as our magic-number.

Also let M=1, so our smallest possible base will be 2. Then every integer n is uniquely expressible in base-M+1 as follows: n=ak(M+1)k+ak-1(M+1)k-1+?+a2(M+1)2+a1(M+1)+a0 where 0=ai=M for all the "digits" ai where 0=i=k. n is divisible by M if and only if n mod M=0. Substitute the base-M+1 expression of n into the above relationship, therefor 0=(ak(M+1)k+ak-1(M+1)k-1+?+a2(M+1)2+a1(M+1)+a0)mod M

TEST

Let $n, k \in \mathbb{Z}_{\geq 0}$. Then every n can be uniquely expressed as follows:

$$n = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_2 10^2 + d_1 10^1 + d_0 10^0$$

for some k such that $0 \leq d_i \leq 9$ where $d_i, i \in \mathbb{Z}$ and $0 \leq i \leq k$.

Furthermore $d_k \neq 0$ except when $n = 0$.

Definition: n is represented in base-ten as $d_k d_{k-1} \ldots d_2 d_1 d_0$