Factorial Basis Representation of Rational Numbers

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From 'A Course of Pure Mathematics' by G. H. Hardy. Chapter 1, Miscellaneous Examples.

Miscellaneous example* #2 at the end of chapter 1 in Hardy's 'Pure Mathematics' presents us with a fascinating result. The theorem feels like what the 'basis-representation-theorem' is for integers, but for rational numbers ... beautiful!

Factorial Representation Theorem†

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Observations that led to the proof.

We know that any rational number‡, say $\frac{m}{q}$, can be written as an integer part, i, PLUS a fractional part, $\frac{p}{q}$, such that $\frac{m}{q} = i + \frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$ (note that i can be zero).

So if we're trying to represent any positive rational number $\frac{m}{q}$ in the form of the theorem then the integer a_1 wants to play the role of the integer part, i, and the remainder of the expression $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ looks to be playing the role of the rational part, $\frac{p}{q}$, where,

$$0 \leq \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} < 1$$

It seemed to me a good idea to forget about the integer a_1 and just focus on the integers $a_2, a_3, \ldots, a_k$. In other words, prove the theorem for rational numbers $\frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$, then it should be trivial to extend it to ALL rational numbers by tacking the a_1 back on at the end of the proof. Also, it started to become clear that including zero (that is, not JUST positive rational numbers) was going to simplify the task§.

*Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, math exercises for the student.

†...it's not named in the text, so I named the theorem.

‡Every variable, or constant (eg. a_1, a_k, m, n, i, p, q) in this paper is going to represent a non-negative integer. We aren't dealing with 'real numbers' here, just non-negative rational numbers which we will always discuss in terms of one integer divided by another integer, like $\frac{p}{q}$.

§Did you notice how the theorem restricts the last integer, a_k, to be strictly greater than zero, unlike all the other variables? We loosen up that restriction by allowing a_k to be equal to zero so that all the variables are treated the same. At the very end of the proof it's trivial to reintroduce that restriction on the integer a_k.

At first glance it wasn't remotely obvious to me how I'd go about calculating the values of the integers $a_2, a_3, \ldots, a_k$ for a given rational number $\frac{p}{q}$, where $0 \le \frac{p}{q} < 1$, let alone that it would be unique.

After playing around for a while, and finally figuring out a way to calculate the variables $a_2, a_3, \ldots, a_k$ for a given rational number $\frac{p}{q}$, (it's kinda like doing long-division) a few thing started to jump out at me. For example, look at these rational numbers,

$$\frac{1}{2} = \frac{1}{1 \cdot 2} = \frac{2! - 1}{2!}$$
$$\frac{5}{6} = \frac{1}{1 \cdot 2} + \frac{2}{1 \cdot 2 \cdot 3} = \frac{1 \cdot 3}{1 \cdot 2 \cdot 3} + \frac{2}{1 \cdot 2 \cdot 3} = \frac{3+2}{6} = \frac{3! - 1}{3!}$$
$$\frac{23}{24} = \frac{1}{1 \cdot 2} + \frac{2}{1 \cdot 2 \cdot 3} + \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{1 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{2 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{12+8+3}{24} = \frac{4! - 1}{4!}$$

An obvious pattern has emerged! It seems to be the case that if we assign the largest possible values to the variables, from a_2 up to say a_k (with all subsequent variables being zero) we get this rational number, $\frac{k!-1}{k!}$. This number is as close to 1 as you can get with a denominator of $k!$ without actually hitting 1. (What happens if you add $\frac{1}{k!}$ to $\frac{k!-1}{k!}$?) This turned out to be a pretty useful observation, and it became my 'Lemma 1' in the proof below.

Also, if we assign zeros to all the variables then naturally we get $\frac{0}{k!}$, plus it's pretty simple to figure out how to make the smallest non-zero such rational number $\frac{1}{k!}$. Then thinking about continually adding $\frac{1}{k!}$ to the result gives us an idea about how the a_i variables change as you keep incrementing by $\frac{1}{k!}$.

So if we restrict ourselves to using only $a_2, a_3, \ldots, a_k$, then we can generate the smallest rational number $(\frac{0}{k!})$ and the largest $(\frac{k!-1}{k!})$ where $0 \le \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} < 1$.

One further observation to help understand the motivation behind this proof is that by using combinatorics we can count how many possible combinations of a_i's are possible. So, we have two choices for the a_2 variable (0, 1), combined with three choices for the a_3 variable (0, 1, 2), combined with four choices for the a_4 variable (0, 1, 2, 3), ... combined with k choices for the a_k variable (0, 1, 2, ..., $k-1$), which gives us $2 \cdot 3 \cdot 4 \cdots k = k!$ possible different sums.

Hmmmmm, the following set has $k!$ members, $\{\frac{0}{k!}, \ \frac{1}{k!}, \ \frac{2}{k!}, \ \ldots, \ \frac{k!-1}{k!}\}$. This set contains each of the rational numbers between zero and one with denominators from 2 up to k, so if we let k grow without bound then we should get a set that contains all the rational numbers between zero and one.

So that, plus one or two other thoughts is what led me to the proof below. I won't spoil the rest of it; to find out, go ahead and read the rest of the paper!

Lemma 1

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}, \quad \text{for integer } k \ge 2$$

Proof

This equality is straightforward to demonstrate by induction, since $\frac{1}{2!} = \frac{2!-1}{2!}$ and,

$$\begin{aligned}
&\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\
=& \frac{(k-1)!-1}{(k-1)!} + \frac{k-1}{k!} \\
=& \frac{k((k-1)!-1)}{k(k-1)!} + \frac{k-1}{k!} \\
=& \frac{k!-k+k-1}{k!} \\
=& \frac{k!-1}{k!}
\end{aligned}$$

...thus establishing Lemma 1 for all values of $k \geq 2$. QED.

The following Lemma 2 captures an idea that is best described by analogy to the basis representation theorem for integers. For base-ten numbers we can say,

$$1 \cdot 10^k > 9 \cdot 10^{k-1} + 9 \cdot 10^{k-2} + \ldots + 9 \cdot 10^2 + 9 \cdot 10^1 + 9 \cdot 10^0$$

The above inequality is merely stating that any single power of ten is bigger than the sum of every lower power of ten each times 9. For example, 1000 is bigger than 999. Read the statement of the inequality in Lemma 2 with this idea in mind.

Lemma 2

For integers i, k where $2 \leq i < k$,

$$\frac{1}{i!} > \frac{i}{(i+1)!} + \frac{i+1}{(i+2)!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!}$$

Proof

$$\begin{aligned}
&\frac{i}{(i+1)!} + \frac{i+1}{(i+2)!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\
=& (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!}) - (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!}) \\
=& \frac{k!-1}{k!} - \frac{i!-1}{i!} \qquad \text{(by Lemma 1)} \\
=& \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\
=& \frac{1}{i!} - \frac{1}{k!} \\
<& \frac{1}{i!}
\end{aligned}$$

QED.

Definitions

For integer $k \geq 2$, and integers $a_2, a_3, \ldots, a_k$, we define the following sets,

$$\mathcal{S}_k = \{\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \quad | \quad 0 \leq a_2 < 2,\ \ 0 \leq a_3 < 3,\ \ \ldots,\ \ 0 \leq a_k < k\},$$
$$\mathcal{F}_k = \{\frac{0}{k!},\ \ \frac{1}{k!},\ \ \frac{2}{k!},\ \ \ldots,\ \ \frac{k!-1}{k!}\}$$

Lemma 3

$$\mathcal{S}_k = \mathcal{F}_k$$

Proof

To show that the set $\mathcal{S}_k$ is the same as $\mathcal{F}_k$, it suffices to show that if $\frac{a}{b} \in \mathcal{S}_k$ then $0 \leq \frac{a}{b} < 1$ and $\frac{a}{b} = \frac{p}{k!}$ for some p, and that the size of $\mathcal{S}_k$ is the same as $\mathcal{F}_k$.

It's clear that the set $\mathcal{F}_k$ contains every rational number with denominator $k!$ where p is an integer and $0 \leq \frac{p}{k!} < 1$ and that the size of $\mathcal{F}_k$ is $k!$.

The smallest member of the set $\mathcal{S}_k$ is $\frac{0}{k!}$ and occurs when all the variables of the sum are set to zero. Furthermore, the largest member of the set occurs when all the variables of the sum are set to their maximum value, which gives us $\frac{k!-1}{k!}$ as shown in Lemma 1.

We also note that every member of $\mathcal{S}_k$ can be written as a rational number with $k!$ as the denominator, like so,

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_{k-1}}{(k-1)!} + \frac{a_k}{k!} = \frac{k \cdot (k-1) \cdot \cdot \cdot 3 \cdot a_2}{k!} + \frac{k \cdot (k-1) \cdot \cdot \cdot 4 \cdot a_2}{k!} + \ldots + \frac{k \cdot a_{k-1}}{k!} + \frac{a_k}{k!}$$

Therefore any member of the set $\mathcal{S}_k$ can be written as $\frac{p}{k!}$ for some integer p, where

$$0 = \frac{0}{k!} \leq \frac{p}{k!} \leq \frac{k!-1}{k!} < \frac{k!}{k!} = 1$$

Furthermore, each possible assignment of values to the variables of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produces a *unique* member of the set $\mathcal{S}_k$.

For if this weren't true and both $\frac{p}{k!} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ and $\frac{p}{k!} = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$ for different variables $a_2, a_3, \ldots, a_k$ and $b_2, b_3, \ldots, b_k$, then we can arrive at a contradiction as follows.

First suppose that $a_i \neq b_i$, where $i \leq k$, is the first such pair of variables that differ. In other words, $a_2 = b_2,\ a_3 = b_3, \ldots,\ a_{i-1} = b_{i-1},\ a_i \neq b_i$. Without loss of generality, further suppose that $a_i > b_i$. Because of the equality of the two different representations for $\frac{p}{q}$ we can now write,

$$\begin{aligned} \frac{a_i}{i!} + \frac{a_{i+1}}{(i+1)!} + \frac{a_{i+2}}{(i+2)!} + \ldots + \frac{a_k}{k!} &= \frac{b_i}{i!} + \frac{b_{i+1}}{(i+1)!} + \frac{b_{i+2}}{(i+2)!} + \ldots + \frac{b_k}{k!} \\ \Leftrightarrow \qquad \frac{a_i - b_i}{i!} &= \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \frac{b_{i+2} - a_{i+2}}{(i+2)!} + \ldots + \frac{b_k - a_k}{k!} \end{aligned} \tag{1}$$

But $a_i - b_i \geq 1$, so

$$\tfrac{a_i - b_i}{i!} \geq \tfrac{1}{i!}.$$

Furthermore, examining any single term on the right-side of (1), say the first one $\frac{b_{i+1}-a_{i+1}}{(i+1)!}$, we can see that since $0 \leq b_{i+1} \leq i$ and $0 \leq a_{i+1} \leq i$ that,

$$\tfrac{i-0}{(i+1)!} \geq \tfrac{b_{i+1}-a_{i+1}}{(i+1)!},$$

and hence,

$$\tfrac{i}{(i+1)!} + \tfrac{i+1}{(i+2)!} + \ldots + \tfrac{k-1}{k!} \geq \tfrac{b_{i+1}-a_{i+1}}{(i+1)!} + \tfrac{b_{i+2}-a_{i+2}}{(i+2)!} + \ldots + \tfrac{b_k - a_k}{k!}.$$

Furthermore, Lemma 2 tells us that $\frac{1}{i!} > \frac{i}{(i+1)!} + \frac{i+1}{(i+2)!} + \ldots + \frac{k-1}{k!}$, so we can string all our inequalities together as follows,

$$\tfrac{a_i-b_i}{i!} \geq \tfrac{1}{i!} > \tfrac{i}{(i+1)!} + \ldots + \tfrac{k-1}{k!} \geq \tfrac{b_{i+1}-a_{i+1}}{(i+1)!} + \ldots + \tfrac{b_k-a_k}{k!},$$

and hence,

$$\tfrac{a_i-b_i}{i!} > \tfrac{b_{i+1}-a_{i+1}}{(i+1)!} + \ldots + \tfrac{b_k-a_k}{k!},$$

But in equation (1) we had deduced that $\frac{a_i-b_i}{i!} = \frac{b_{i+1}-a_{i+1}}{(i+1)!} + \ldots + \frac{b_k-a_k}{k!}$ which contradicts the strict inequality above.

Therefore our assumption that there can be a second set of variables representing the same rational number $\frac{p}{k!}$ must be false. Therefore any assignment of values to the variables of the sum $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produces a *unique* number.

Now we can count the number of members of $\mathcal{S}_k$, by looking at all the possible combinations of values for the variables $a_2, a_3, \ldots, a_k$. There are 2 choices for the variable a_2, combined with 3 choices for a_3, combined with 4 choices for a_4, $\ldots$, combined with k choices for a_k.

Therefore the total number of combinations of values that can be assigned to all the variables of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is $2 \cdot 3 \cdot 4 \cdots k = k!$, and since each set of assignments creates a *unique* member of the set, then the size of $\mathcal{S}_k$ is $k!$, which is the same size as $\mathcal{F}_k$. Recalling from above that any member of the set $\mathcal{S}_k$, say $\frac{a}{b}$, can be written as $\frac{a}{b} = \frac{p}{k!}$, for some p where $0 \leq \frac{p}{k!} < 1$ then $\mathcal{S}_k = \mathcal{F}_k$.

QED.

Definitions

$$\mathcal{S}_\infty = \lim_{n\to\infty} \mathcal{S}_n \qquad \text{and} \qquad \mathcal{F}_\infty = \lim_{n\to\infty} \mathcal{F}_n$$

$a \in \mathcal{F}_\infty$ means that there exists an integer $k \geq 2$ such that $a \in \mathcal{F}_k$. Similarly for $a \in \mathcal{S}_\infty$.

Lemma 4

The set $\mathcal{F}_\infty$ contains all non-negative rational numbers less than one.

Proof

For all integers $k \geq 2, \mathcal{F}_k \subset \mathcal{F}_{k+1}$ because if $\frac{p}{k!} \in \mathcal{F}_k$ then $\frac{(k+1)\cdot p}{(k+1)\cdot k!} = \frac{(k+1)\cdot p}{(k+1)!} \in \mathcal{F}_{k+1}$. By induction it is simple to conclude that $\mathcal{F}_k \subset \mathcal{F}_{k+1} \subset \mathcal{F}_{k+2} \subset \ldots \subset \mathcal{F}_n$, for all $n > k$, and ultimately $\mathcal{F}_k \subset \mathcal{F}_\infty$ for all $k \geq 2$.

If $\frac{p}{k}$ is a rational number where $0 \leq \frac{p}{k} < 1$ then $\frac{p}{k} \in \mathcal{S}_k$ because,

$$\frac{p}{k} = \frac{2\cdot 3 \cdot \cdot \cdot (k-1)}{2\cdot 3 \cdot \cdot \cdot (k-1)} \cdot \frac{p}{k} = \frac{2\cdot 3 \cdot \cdot \cdot (k-1)\cdot p}{k!}$$

Therefore $\frac{p}{k}$ is an element of $\mathcal{F}_k$, hence an element of $\mathcal{F}_\infty$ for all integers $k \geq 2$.

To summarize; All rational numbers $\frac{p}{k}$ where $k \geq 2$ and $0 \leq \frac{p}{k} < 1$ are in the set $\mathcal{F}_\infty$

QED.

Lemma 5

Any positive rational number *less than one*, can be expressed in one and only one way in the form

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$$

where $a_2, \ldots, a_k$ are integers, and

$$0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Proof

Suppose $\frac{p}{q}$ is a positive rational number less than one. We note that its denominator is at least 2, so by Lemma 4 we can say that $\frac{p}{q} \in \mathcal{F}_\infty$, and Lemma 3 tells us that $\mathcal{S}_k = \mathcal{F}_k$ for all $k \geq 2$, then $\frac{p}{q} \in \mathcal{S}_\infty$.

From this we can conclude that there is a unique sequence of variables

QED.

Factorial Representation Theorem (restated)

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Proof

Thanks to Euclid we know that for all integers $j \geq 0$ and $q > 0$, there exist *unique* integers i and p such that,

$$\begin{aligned} & j = i \cdot q + p\ ; \quad 0 \leq p < q \\ \Leftrightarrow \quad & \frac{j}{q} = i + \frac{p}{q}\ ; \quad 0 \leq \frac{p}{q} < 1 \end{aligned}$$

Which tells us that all positive rational numbers $\frac{j}{q}$ can be *uniquely* written as an integer part, i, plus a fractional part $\frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$.

In our theorem, the a_1 variable plays the role of the integer part i, and the rest of the expression, $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ plays the role of the fractional part $\frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$.

Therefore to express any positive rational number in the form of the theorem, first apply the Euclidean Division Theorem to $\frac{j}{q}$ and let $a_1 = i$. If there is no fractional remainder, then the theorem is trivially true, however if there is a fractional remainder $\frac{p}{q}$, then by Lemma 5 we know that the sum $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with $\frac{p}{q}$ so clearly $\frac{j}{q} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with all positive rational numbers $\frac{j}{q}$.

To meet the precise requirements of the theorem, we suppose that if $\frac{p}{q} \in \mathcal{S}_n$ for some $n \geq q$ that we take for the variables $a_2, a_3, \ldots, a_k$ in the sum for $\frac{p}{q} \in \mathcal{S}_n$ all those for which $a_k \neq 0$ but $a_{k+1} = a_{k+2} = \ldots = a_n = 0$.

QED.

Additional Observations

While it's true that $\frac{p}{q} \in \mathcal{S}_q$, $\mathcal{S}_q$ is not necessarily the smallest such set for which $\frac{p}{q}$ is a member.

For example, the smallest set containing $\frac{p}{5}$, where $0 \leq \frac{p}{5} < 1$, is $\mathcal{S}_5$ however the smallest set containing $\frac{p}{6}$, where $0 \leq \frac{p}{6} < 1$ is $\mathcal{S}_3$.

Which is easy to see when we list the contents of a couple of sets,

$$\begin{aligned}\mathcal{S}_4 &= \{\frac{0}{24}, \frac{1}{24}, \frac{2}{24}, \frac{3}{24}, \frac{4}{24}, \frac{5}{24}, \frac{6}{24}, \frac{7}{24}, \frac{8}{24}, \frac{9}{24}, \frac{10}{24}, \frac{11}{24}, \frac{12}{24}, \frac{13}{24}, \frac{14}{24}, \frac{15}{24}, \frac{16}{24}, \frac{17}{24}, \frac{18}{24}, \frac{19}{24}, \frac{20}{24}, \frac{21}{24}, \frac{22}{24}, \frac{23}{24}\} \\ &= \{\frac{0}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4}, \frac{7}{24}, \frac{1}{3}, \frac{3}{8}, \frac{5}{12}, \frac{11}{24}, \frac{1}{2}, \frac{13}{24}, \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{17}{24}, \frac{3}{4}, \frac{19}{24}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{23}{24}\}\end{aligned}$$

Which clearly doesn't contain $\frac{1}{5}$. We've established that $\frac{1}{5}$ is definitely in $\mathcal{S}_5$ but it's interesting to see what it looks like:

$$\frac{1}{5} = \frac{0}{2} + \frac{1}{2 \cdot 3} + \frac{0}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{6} + \frac{1}{30} = \frac{5+1}{30} = \frac{6}{30} = \frac{1}{5}$$

Also, $\mathcal{S}_3 = \{\frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\} = \{\frac{0}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$, which demonstrates the claim above that $\mathcal{S}_3$ contains $\frac{p}{6}$, where $0 \leq \frac{p}{6} < 1$.

I believe that for a given $q \geq 2$ then the smallest set for which the rational number $\frac{p}{q} \in \mathcal{S}_k$, is to pick k such that it is the smallest value for which q divides $k!$.

However, I'll leave that proof for another day.