# Factoradic Representation of Rational Numbers

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*From 'A Course in Pure Mathematics' by G. H. Hardy. Chapter 1, Miscellaneous Examples.*

Miscellaneous example* #2 at the end of chapter 1 in Hardy's 'Pure Mathematics' presents us with a fascinating result (which was new to me). It feels like a kind of basis-representation-theorem, but for rational numbers, ... beautiful!

Here it is, followed by my proof which starts out with some lemmas to get us rolling.

## Theorem

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

## Lemma-1

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}$$

## Proof of Lemma-1

This equality is fairly trivial to demonstrate by induction, since $\frac{1}{2!} = \frac{2!-1}{2!}$ and,

$$\begin{aligned} &\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\ &= \frac{(k-1)!-1}{(k-1)!} + \frac{k-1}{k!} \\ &= \frac{k((k-1)!-1)}{k(k-1)!} + \frac{k-1}{k!} \\ &= \frac{k! - k + k - 1}{k!} \\ &= \frac{k!-1}{k!} \end{aligned}$$

...thus establishing lemma-1 for all values of k. QED

*Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, math exercises like calculations to perform, theorems to prove etc.

## Lemma-2

For integers $i, k$ where $2 \leq i < k$ such that,

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!} + \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!},$$

then

$$\frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!}$$

## Proof of Lemma-2

$$\begin{aligned}
&\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \\
&= (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!}) - (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!}) \\
&= \frac{k!-1}{k!} - \frac{i!-1}{i!} \\
&= \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\
&= \frac{1}{i!} - \frac{1}{k!} \\
&< \frac{1}{i!}
\end{aligned}$$

QED

## Lemma-3

The set of rational numbers,

$$\mathcal{S}(k) = \{\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \quad | \quad 0 \leq a_2 < 2, \;\; 0 \leq a_3 < 3, \;\; \ldots, \;\; 0 \leq a_k < k\}$$

is identical to the set of rational numbers,

$$\mathcal{F}(k) = \{\frac{0}{k!}, \;\; \frac{1}{k!}, \;\; \frac{2}{k!}, \;\; \ldots, \;\; \frac{k!-1}{k!}\}$$

## Proof of Lemma-3

It's clear that the set $\mathcal{F}(k)$ contains every rational number with denominator $k!$ where $0 \leq \frac{p}{k!} < 1$ and also clear that the size of $\mathcal{F}(k)$ is $k!$. To show that the set $\mathcal{S}(k)$ is the same as $\mathcal{F}(k)$, it suffices to show that every member of $\mathcal{S}(k)$ is also of the form $0 \leq \frac{p}{k!} < 1$, and that the size of $\mathcal{S}(k)$ is also $k!$.

The smallest member of the set $\mathcal{S}(k)$ is $\frac{0}{k!}$ and occurs when all the coefficients of the sum are zero. Furthermore, the largest member of the set occurs when all the coefficients of the sum are set to their maximum value, which gives us $\frac{k!-1}{k!}$ as shown in lemma-1.

We also note that every member of $\mathcal{S}(k)$ can be written as a rational number with $k!$ as the denominator, like so,

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_{k-1}}{(k-1)!} + \frac{a_k}{k!} = \frac{k \cdot (k-1) \cdot \cdot \cdot 3 \cdot a_2}{k!} + \frac{k \cdot (k-1) \cdot \cdot \cdot 4 \cdot a_2}{k!} + \ldots + \frac{k \cdot a_{k-1}}{k!} + \frac{a_k}{k!}$$

Therefor any member of the set $\mathcal{S}(k)$ is of the form $0 \le \frac{p}{k!} < 1$, where $p$ in an integer in the range $0 \le p \le k! - 1$.

Furthermore, each possible assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produce a unique member of the set $\mathcal{S}(k)$.

For this weren't true and both $\frac{p}{k!} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ and $\frac{p}{k!} = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$ for different coefficients $a_2, a_3, \ldots, a_k$ and $b_2, b_3, \ldots, b_k$, then we can arrive at a contradiction as follows.

First suppose that $a_i \ne b_i$ is the first such pair of coefficients that differ from each other. In other words, $a_2 = b_2, a_3 = b_3, \ldots, a_{i-1} = b_{i-1}$. Also, without loss of generality we can assume that $a_i > b_i$ and state the following equality:

$$\frac{a_i}{i!} + \frac{a_{i+1}}{(i+1)!} + \ldots + \frac{a_k}{k!} = \frac{b_i}{i!} + \frac{b_{i+1}}{(i+1)!} + \ldots + \frac{b_k}{k!}$$

$$\Leftrightarrow \qquad \frac{a_i - b_i}{i!} = \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!}$$

Since $a_i - b_i \ge 1$, then $\frac{a_i - b_i}{i!} \ge \frac{1}{i!}$.

Also, $\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \ge \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!}$ regardless of the values of the coefficients on the right side of the inequality*.

However, lemma-2 tells us,

$$\frac{a_i - b_i}{i!} \ge \frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \ge \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!},$$

demonstrating that equality two expressions at either end of the inequality is impossible, so our assumption that there can be a second set of coefficients to produce the same rational number $\frac{p}{k!}$ is false. Therefor any assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produces a unique member of the set $\mathcal{S}(k)$.

Now we can count the number of members of the set $\mathcal{S}(k)$, by looking at all the possible combinations of values for the coefficients $a_2, a_3, \ldots, a_k$. There are 2 choices for the coefficient $a_2$, multiplied by the 3 choices for $a_3$, multiplied by the 4 choices for $a_4$, $\ldots$, up to multiplying by $k$ values that $a_k$ can assume.

Therefore the total number of combinations of values that can be assigned to all the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is $2 \cdot 3 \cdot 4 \cdots k = k!$, which means the size of the set $\mathcal{S}(k)$ is $k!$. This plus our conclusion above that all members of the set $\mathcal{S}(k)$ are of the form $0 \le \frac{p}{k!} < 1$ means that $\mathcal{S}(k) = \mathcal{F}(k)$. QED

*Letting all the $b$'s be their maximum value, and all the $a$'s be zero will produce the largest numerators in each term of the sum, any other possibility will result in a smaller term for the sum.

## Lemma-4

All rational numbers $\frac{p}{q}$ such that $0 \leq p < q$ and $2 \leq q \leq k^{*}$ are members of the set $\mathcal{S}(k)$.

## Proof of Lemma-4

Since $0 \leq p < q$ then $0 \leq \frac{p}{q} < 1$ and

$$\begin{aligned} \frac{p}{q} &= \frac{2 \cdot 3 \cdots (q-1) \cdot (q+1) \cdots k}{2 \cdot 3 \cdots (q-1) \cdot (q+1) \cdots k} \cdot \frac{p}{q} \\ &= \frac{2 \cdot 3 \cdots (q-1) \cdot p \cdot (q+1) \cdots k}{2 \cdot 3 \cdots (q-1) \cdot q \cdot (q+1) \cdots k} \\ &= \frac{2 \cdot 3 \cdots (q-1) \cdot p \cdot (q+1) \cdots k}{k!} \end{aligned}$$

Therefor $\frac{p}{q}$ is an element of $\mathcal{F}(k)$ which is the same as the set $\mathcal{S}(k)$.

QED

*That list of rational numbers would look like this, $\frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{0}{5}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots, \frac{0}{k}, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-2}{k}, \frac{k-1}{k}$.