

Magic Numbers

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In the delightful film “[School of Rock](#)” Jack Black’s character Dewey Finn, pretending to be substitute teacher Ned Schneebly, is put on the spot when Miss. Mullins, the school principle, comes into class and demands that he show her his teaching methods which apparently involve an electric guitar. So Mr. S sings “[The Math Song](#)” wherein the final line of his improvised tune he sings “...yes it’s 9, and that’s a magic-number...”

It seems that Jack Black was making a tip-of-the-hat to the “Schoolhouse Rock” song “[3 Is A Magic Number](#)” which most kids from the 70’s/80’s know. However referring to 9 as a magic-number isn’t as well known*.

However it’s not a total stretch, there is precedent for using the term “[Magic Number](#)” when referring to the number 9. There are a host of [mathematical-magic-tricks](#) that rely on properties of the number 9. There is also a process called “[casting out nines](#)” and a related one called calculating the “[digital root](#)” for a number where you take its “digital sum”. 9 plays a similar role in all these processes - so within these contexts sometimes 9 is referred to as a magic-number.

So Mr. S’s musical-math-lesson is legit, if indeed he is referring to this fact about 9: If you add up the digits of any number then if that sum is divisible by 9, then the original number must be divisible by 9. It works the other way too, that is, if we know that a number is divisible by 9 then the sum of its digits must also be divisible by 9.

For example:

$$9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59049 \text{ and } 5 + 9 + 0 + 4 + 9 = 27 \text{ and } 9 \text{ divides } 27 \text{ because } 27 = 3 \times 9.$$

But 3 is also has this magic-number property, for example:

$$3 \times 67 = 201 \text{ and } 2 + 0 + 1 = 3$$

In this case, 9 does not divide 201^\dagger but 3 does (67 times). We can see that adding up the digits of 201 gives us 3, which of course is divisible by 3 (but not by 9).

So given that the term “magic number” hasn’t got a formal definition in the math world, I think it’s time to give it one. How about if we call ANY number that has qualities like 3 and 9 (as described above) a magic-number? After all both “Schoolhouse Rock” AND “The School of Rock” sanction using the term “magic number” for 3 and 9, so we may be on to something big here!

*According to my own *highly scientific* informal poll conducted Facebook, “9” is NOT the first thing that pops into people’s heads when they hear the phrase “magic number” - in fact no one answered “9” in my poll.

[†]Why is it immediately apparent that 9 does not divide 201? Hint: both 3 and 67 are prime numbers. Check the “[Fundamental Theorem of Arithmetic](#)” wiki page for further insight.

Too bad 3 and 9 are the only magic-numbers in base-10. Darn - perhaps our new term isn't going to be all that useful after all, ... or maybe it will! We happen to use base-10 because we have ten fingers* but what about other bases? The heptapods in the film "Arrival" have 7 limbs, 3 arms and 4 legs, with 7 fingers/toes each. I'm going to guess that at least at some point in their history they used positional notation like base-10, but instead of 10 they probably base-7 or base-21, maybe even base-49. So perhaps an understanding of magic-numbers in alternate bases could come in handy.

Let's examine some magic-numbers in other bases. In the way that 9 is a magic-number in base-10, then 4 is a magic-number in base-5; Also 7 is a magic-number in base-8; 15 is a magic-number in base-16; ... and 30 is a magic-number in base-31, as you can see with these examples[†]:

- $4 \times 67 = 268$ written in base-5 is $(2033)_5$ and $2 + 0 + 3 + 3 = 8$ (4 divides 8).
- $7 \times 181 = 1267$ written in base-8 is $(2363)_8$ and $2 + 3 + 6 + 3 = 14$ (7 divides 14).
- $15 \times 23 = 345$ written in base-16 is $(159)_{16}$ and $1 + 5 + 9 = 15$.
- $30 \times 35951 = 1078530$ written in base-31 is $(15699)_{31}$ and $1 + 5 + 6 + 9 + 9 = 30$

Notice the pattern: the biggest single-digit in any base (which is always one less than the base) seems to qualify as a magic-number. For example, 4 is the biggest digit in base-5, 7 is the biggest digit in base-8, and 9 is the biggest digit in base-10; To state this in a general way we could say that if b is our base, then $(b - 1)$ is a magic-number in base- b .

So what about 3 also being a magic-number in base-10? Let's take a look at the interesting (but probably never used outside this paper) base-31. It appears that 2, 3, 5, 6, 10, 15 (along with 30) are *all* magic-numbers in base-31, as we can see with the following examples:

- $2 \times 409 \times 1129 = 923522$ written in base-31 is $(10001)_{31}$ and $1 + 0 + 0 + 0 + 1 = 2$.
- $3 \times 641 = 1923$ written in base-31 is $(201)_{31}$ and $2 + 0 + 1 = 3$.
- $5 \times 139 \times 2659 = 1848005$ written in base-31 is $(20102)_{31}$ and $2 + 0 + 1 + 0 + 2 = 5$.
- $6 \times 10091 = 60546$ written in base-31 is $(2103)_{31}$ and $2 + 1 + 0 + 3 = 6$.
- $10 \times 197 \times 941 = 1853770$ written in base-31 is $(20701)_{31}$ and $2 + 0 + 7 + 0 + 1 = 10$.
- $15 \times 71503 = 1072545$ written in base-31 is $(15027)_{31}$ and $1 + 5 + 0 + 2 + 7 = 15$.

See the pattern? All the divisors of 30 are magic-numbers in base-31. It turns out that in base- b , then any divisor of $(b - 1)$ is a magic-number in base- b .

I guess that begs the question: What about 1? Isn't 1 a magic-number in all bases? Since we already know that all numbers are divisible by 1, then we don't need to go to any trouble of adding up digits to find this out. So let's exclude 1 from being considered as a magic-number - it's not helpful.

*To be honest, base-10 is not "universal" among humans, there are plenty of [other bases](#) in use, albeit not as widely used nor as mathematically useful as base-10. This doesn't count octal, hexadecimal and binary which are used in the computer world so by necessity are, mathematically speaking, useful.

[†]We are going to use the conventions for specifying numbers in alternate bases as outlined in the paper "[SesameStreet++](#)"

Let's quickly review what it means for a number to be divisible by another number. Recall from the paper "[SesameStreet++](#)":

Euclidean Division Theorem

For all integers a and b such that $b \neq 0$, there exist *unique* integers q and r such that:

$$a = qb + r \text{ such that } 0 \leq r < |b|$$

Definition: In the above equation*:

a is the *dividend* ("the number being divided")
 b is the *divisor* ("the number doing the dividing")
 q is the *quotient* ("the result of the division")
 r is the *remainder* ("the leftover")

So to say that a number a is divisible by b simply means that r is zero.

In other words, to say an integer a is divisible by another integer b (not zero) means there is a third integer q such that[†]:

$$a = q \cdot b \quad \text{or} \quad \frac{a}{b} = q$$

We express the fact that a is divisible by b , and say " b divides a " using this notation:

$$b \mid a$$

So the following is always true:

$$1 \mid a, \text{ and } a \mid a;$$

also

$$b \mid 0 \text{ for every } b \text{ except } 0$$

In order to create our theorem with a nice definition for "magic number" we need to remind ourselves what it means to write a number in a given base[‡]:

Basis Representation Theorem

Let b be a positive integer greater than 1.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \dots, d_k$ such that:

$$n = d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

where $0 \leq d_i < b$ for all i in $\{0, 1, 2, \dots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base- b by the string of base- b -digits $(d_k d_{k-1} \dots d_2 d_1 d_0)_b$

*Recall that $|b|$ means the "absolute value" of b and is always positive. For example $|-13| = 13$.

[†]Paraphrased from "An Introduction to the Theory of Numbers" by G. H. Hardy and E. M. Wright. Pg 1.

[‡]Also from "[SesameStreet++](#)"

Magic Number Theorem

Let n be a positive integer represented in base- b .

If m is a positive integer such that $m \mid (b-1)$ then,

$$m \mid n \quad \text{if and only if} \quad m \mid (d_k + \cdots + d_2 + d_1 + d_0)$$

Definition: We call $m \neq 1$ a “magic-number” in base- b .

As an interesting side note - according to our definition, binary, or base-2, doesn't have a magic-number. No great loss, as we said above 1 isn't helpful to consider as a magic-number.

Let's prove the theorem. All we need to do is show that if you divide n by m that the remainder of the division is always zero.

The proof is quite simple when using modular arithmetic. However if you aren't familiar with modular arithmetic then I recommend that you check out Khan Academy's "[What is modular arithmetic?](#)" to get a clear-cut-crash-course in the subject, it's excellent, as is the entire Khan Academy site in case you've never checked it out.

Assuming you're now familiar with modular-arithmetic you understand that it gives us very powerful tools to deal with divisibility and remainders of integers. To say that the remainder of dividing n by m is zero is the same as saying:

$$n \equiv 0 \pmod{m}$$

where a, b and c are integers such that $c > 1$.

$$(a+b) \bmod c = (a \bmod c + b \bmod c) \bmod c \quad ab \bmod c = ((a \bmod c)(b \bmod c)) \bmod c$$

As a generalization of 2), we get:

$$3) \quad a \bmod c = b \bmod c \implies a^k \bmod c = b^k \bmod c$$

Importantly, $a \bmod c = 0 \implies c$ divides a .

In other words, there exists a number n such that $nc=a$. Lemma: if $d \nmid 1$ is a divisor of m then: $(m+1)^k \bmod d = 1$ Proof of Lemma: Let d be a divisor of m . So c such that $cd=m$. Hence, $(cd+1) \bmod d = cd \bmod d + 1 \bmod d = (c \bmod d)(d \bmod d) + 1 \bmod d = (c \bmod d)0 + 1 \bmod d = 1 \bmod d$ By property 3) above, since $(cd+1) \bmod d = 1 \bmod d$ then: $(m+1)^k \bmod d = (cd+1)^k \bmod d = 1 \bmod d$ QED

Proof of “Magic Number Theorem”:

Let $N=M+1$.

That is; we are going to consider representations of numbers in base- $M+1$, so think of M as our magic-number.

Also let $M=1$, so our smallest possible base will be 2. Then every integer n is uniquely expressible in base- $M+1$ as follows: $n=a_k(M+1)^k+a_{k-1}(M+1)^{k-1}+\cdots+a_2(M+1)^2+a_1(M+1)+a_0$ where $0=a_i=M$ for all the “digits” a_i where $0=i=k$. n is divisible by M if and only if $n \bmod M=0$. Substitute the base- $M+1$ expression of n into the above relationship, therefore $0=(a_k(M+1)^k+a_{k-1}(M+1)^{k-1}+\cdots+a_2(M+1)^2+a_1(M+1)+a_0) \bmod M$