Rational Approximations to $\sqrt{2}$

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In Chapter 1, Examples[*] III, no. 2, from G. H. Hardy's 'A Course of Pure Mathematics', Hardy has us examine a sequence of rational approximations to $\sqrt{2}$, which are:

$$\frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70}$$

These numbers squared are:

$$\frac{1}{1},\ \frac{9}{4},\ \frac{49}{25},\ \frac{289}{144},\ \frac{1681}{841},\ \frac{9801}{4900}$$

And we discover that their differences from 2 are:

$$-1,\ \frac{1}{4},\ -\frac{1}{25},\ \frac{1}{144},\ -\frac{1}{841},\ \frac{1}{4900}$$

In the next 'Example no. 3' we are asked to prove a theorem relating to the above sequence, noting it is only an *example* of an application of the theorem below.

Theorem

Show that if $\frac{m}{m}$ is a good approximation to $\sqrt{2}$, then $\frac{(m+2n)}{(m+n)}$ is a better one, and that the errors in the two cases are in opposite directions.

The proof follows.

Lemma

Let m, n and a be integers, where m and n are positive and $a \neq 0$, then

$$\frac{m^2}{n^2} + \frac{a}{n^2} = 2 \qquad \Leftrightarrow \qquad \frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} = 2$$

[*]Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, exercises for the student.

Proof of Lemma

$$\begin{array}{rrcl} & \dfrac{(m+2n)^2}{(m+n)^2} - \dfrac{a}{(m+n)^2} & = & 2 \\ \Leftrightarrow & (m+2n)^2 - a & = & 2(m+n)^2 \\ \Leftrightarrow & m^2 + 4mn + 4n^2 & = & 2m^2 + 4mn + 2n^2 + a \\ \Leftrightarrow & 4n^2 - 2n^2 & = & 2m^2 - m^2 + a \\ \Leftrightarrow & 2n^2 & = & m^2 + a \\ \Leftrightarrow & 2 & = & \dfrac{m^2}{n^2} + \dfrac{a}{n^2} \end{array}$$

QED

Proof of Theorem

Define the 'error' of a rational approximation to $\sqrt{2}$ as the difference between the approximation-squared and 2. So we learn from the Lemma that the error for the $\frac{m}{n}$ approximation of $\sqrt{2}$ is

$$\left|\tfrac{a}{n^2}\right| = \left|2 - \tfrac{m^2}{n^2}\right|,$$

and the error for the $\frac{(m+2n)}{(m+n)}$ approximation of $\sqrt{2}$ is

$$\left|\tfrac{-a}{(m+n)^2}\right| = \left|2 - \tfrac{(m+2n)^2}{(m+n)^2}\right|.$$

The denominator $(m+n)^2$ is always larger than n^2 therefore,

$$\left|\frac{-a}{(m+n)^2}\right| < \left|\frac{a}{n^2}\right|,$$

showing that the approximation $\frac{(m+2n)}{(m+n)}$ for $\sqrt{2}$ has a smaller error that that of $\frac{m}{n}$ and is therefore a better approximation to $\sqrt{2}$.

Furthermore, we also learn from the Lemma that the successive approximations flip to either side of 2 because:

If $\frac{m^2}{n^2} < 2$ then a is positive therefore $\frac{(m+2n)^2}{(m+n)^2} > 2$, similarly,

If $\frac{m^2}{n^2} > 2$ then a is negative therefore $\frac{(m+2n)^2}{(m+n)^2} < 2$.

QED