Magic-Numbers

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In the film "School of Rock", Jack Black's character is put on the spot when the school principle comes into class and demands that he show her his "teaching methods" involving an electric guitar. So he sings "The Math Song" wherein the final line of his improvised tune is ". . . yes it's nine, and that's a magic-number. . . "

It seems that Jack was tipping his hat to the "Schoolhouse Rock" song "3 Is A Magic Number" which most kids from the 70's/80's know - However referring to 9 as a magic-number isn't as well known*.

Mind you, there is precedent for using the term "magic-number" when referring to to the number 9. There are a host of mathematical-magic-tricks that rely on properties of the number 9. There is also a process called "casting out nines" and a related one called calculating the "digital root" for a number where you take its "digital sum". 9 plays a similar role in all these processes - so within these contexts sometimes 9 is referred to as a magic-number.

So the musical-math-lesson is legit, assuming that Jack is referring to this fact about 9:

The sum of the digits of a number divisible by 9,
if and only if,
the number is divisible by 9.

For example:

$$9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59049 \text{ and } 5+9+0+4+9 = 27 \text{ and 9 divides 27 because } 27 = 3 \times 9.$$

But 3 is also has this magic-number property, for example:

$$3 \times 67 = 201 \text{ and } 2 + 0 + 1 = 3$$

In this case, 9 does not divide 201† but 3 does (67 times). We can see that adding up the digits of 201 gives us 3, which of course is divisible by 3, but not by 9.

So given that the term "magic-number" hasn't got a formal definition in the math world, I think it's time to give it one. How about if we call ANY number that has this divisibility property (with respect to adding up the digits of some other number) a "magic-number"?

Too bad 3 and 9 are the only magic-numbers in base-10, but what about other bases? The Heptapods in the film "Arrival" have 7 limbs, 3 arms and 4 legs, with 7 fingers/toes each. I'm

*According to my own *highly scientific (cough cough)* informal poll conducted Facebook, "9" is NOT the first thing that pops into people's heads when they hear the phrase "magic-number" - in fact no one answered "9" in my poll.

†Why is it immediately apparent that 9 does not divide 201? Hint: both 3 and 67 are prime numbers. Check the "Fundamental Theorem of Arithmetic" wiki page for further insight.

guessing that they probably use some form of positional-notation, but instead of base-ten they probably use base-7, base-21, maybe even base-49*. What might the Heptapod's magic-numbers be?

Let's examine some numbers in other bases.

- $4 \times 67 = 268$ written in base-5 is $(2033)_5$ and $2 + 0 + 3 + 3 = 8$ (4 divides 8).
- $7 \times 181 = 1267$ written in base-8 is $(2363)_8$ and $2 + 3 + 6 + 3 = 14$ (7 divides 14).
- $15 \times 23 = 345$ written in base-16 is $(159)_{16}$ and $1 + 5 + 9 = 15$.
- $20 \times 908249 = 18164980$ written in base-21 is $(498991)_{21}$ and $4 + 9 + 8 + 9 + 9 + 1 = 40$
- $30 \times 35951 = 1078530$ written in base-31 is $(15699)_{31}$ and $1 + 5 + 6 + 9 + 9 = 30$

Notice the pattern in our examples: $(b - 1)$ acts like a magic-number in base-b.

Let's take a look at some other examples in base-31. It appears that 2, 3, 5, 6, 10, 15 (along with 30) might *all* be magic-numbers in base-31, as we we can see with the following examples:

- $2 \times 409 \times 1129 = 923522$ written in base-31 is $(10001)_{31}$ and $1 + 0 + 0 + 0 + 1 = 2$.
- $3 \times 641 = 1923$ written in base-31 is $(201)_{31}$ and $2 + 0 + 1 = 3$.
- $5 \times 139 \times 2659 = 1848005$ written in base-31 is $(20102)_{31}$ and $2 + 0 + 1 + 0 + 2 = 5$.
- $6 \times 10091 = 60546$ written in base-31 is $(2103)_{31}$ and $2 + 1 + 0 + 3 = 6$.
- $10 \times 197 \times 941 = 1853770$ written in base-31 is $(20701)_{31}$ and $2 + 0 + 7 + 0 + 1 = 10$.
- $15 \times 71503 = 1072545$ written in base-31 is $(15027)_{31}$ and $1 + 5 + 0 + 2 + 7 = 15$.

See the pattern? All the divisors of 30 seem to be magic-numbers in base-31. It turns out that in base-b, then any divisor of $(b - 1)$ is a magic-number in base-b; however, since we've only looked at some examples, we can't make that claim unless we prove it for ALL such situations.

What about 1? Wouldn't 1 a magic-number in all bases? Of course it would because we know that all numbers are divisible by 1, and adding up any set of digits is also divisible by 1, but this isn't very interesting so let's exclude 1 from being considered as a magic-number - it's not helpful information.

Let's review what it means for a number to be divisible by another number†.

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*For an understanding of alternate bases please see the paper "Counting" by James Philip Rowell

† "Counting" p. 21

Euclidean Division Theorem

For all integers a and b such that $b \neq 0$, there exist *unique* integers q and r such that:

$$a = qb + r \text{ such that } 0 \leq r < |b|$$

Definition: In the above equation*:

a is the *dividend*	("the number being divided")
b is the *divisor*	("the number doing the dividing")
q is the *quotient*	("the result of the division")
r is the *remainder*	("the leftover")

Which is how you learned to divide in primary school with "long-division", that is;

$$\frac{a}{b} = q, \text{ with remainder } r$$

So for example.

$$\frac{19}{6} = 3, \text{ with remainder } 1$$

So to say that a is divisible by b simply means† that r is zero so we can write:

$$a = q \cdot b \qquad \text{or} \qquad \frac{a}{b} = q$$

We express the fact that a is divisible by b, and say "b divides a" using this notation:

$$b \mid a$$

So the following is always true:

$$1 \mid a, \text{ and } a \mid a;$$

also

$$b \mid 0 \text{ for every } b \text{ except } 0$$

In order to create our theorem with a nice definition for "magic-number" we need to remind ourselves what it means to write a number in a given base‡:

Basis Representation Theorem

Let b be a positive integer greater than 1.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

where $0 \leq d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-b by the string of base-b-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$

Armed with our divisibility notation above plus what it means to express an arbitrary number in a given base we can state our new theorem as follows.

*Recall that $|b|$ means the "absolute value" of b and is always positive. For example $|-13| = 13$.

†Paraphrased from "An Introduction to the Theory of Numbers" by G. H. Hardy and E. M. Wright. Pg 1.

‡ "Counting p. 2"

Magic-Number Theorem

Let $n = d_k b^k + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0$ be the base-b-representation of n.

If m is a positive integer such that $m \mid (b-1)$ then*,

$$m \mid n \qquad \Leftrightarrow \qquad m \mid (d_k + \cdots + d_2 + d_1 + d_0)$$

Definition: We call $m \neq 1$ a "magic-number" in base-b.

As an interesting side note - according to our definition, binary, or base-2, doesn't have a magic-number. No great loss, as we said above 1 isn't helpful to consider as a magic-number.

Let's prove the magic-number-theorem using modular-arithmetic. If you aren't familiar with modular-arithmetic then check out Khan Academy's "What is modular-arithmetic?". Khan's introductory explanation is excellent, simple and clear, as is the entire Khan Academy site in case you've never checked it out.

If we take the a, b, q and r from the Euclidean Division Theorem above where:

$$a = qb + r \text{ such that } 0 \leq r < |b|$$

Then the "mod" operator is defined like this:

$$a \bmod b = r$$

To prove our theorem we need to prove that the remainder of n divided by m is zero, if and only if, the remainder of $d_k + \cdots + d_1 + d_0$ divided by m is also zero. Restated in terms of modular-arithmetic, we need to prove:

$$n \bmod m = 0 \qquad \Leftrightarrow \qquad (d_k + \cdots + d_1 + d_0) \bmod m = 0$$

I'm going to assume that you are familiar with how to use modular-arithmetic from this point forward; if not, make a quick detour to Khan Academy.

Lemma

Let $b > 1$ and m be positive integers such that $m \mid (b-1)$ then,

$$b^k \equiv 1 \pmod{m}, \text{ for all integers } k \geq 0$$

*Recall that the bidirectional arrow symbol $\Leftrightarrow$ means "if and only if" - it's like a logical "equals" sign

Proof of Lemma:

Let k, b and m be integers as described in the lemma, then there exists an integer q such that,

$$\begin{array}{lll} & m \cdot q = b - 1 & \text{(Definition of } m \mid (b-1)\text{)} \\ \Leftrightarrow & m \cdot q + 1 = b & \text{(Add 1 to both sides)} \end{array}$$

Given this equality we can state the following modular-congruence:

$$\begin{array}{lrll} & m \cdot q + 1 \equiv b & \pmod{m} & \\ \Leftrightarrow & 0 \cdot q + 1 \equiv b & \pmod{m} & \text{(Since } m \bmod m = 0\text{)} \\ \Leftrightarrow & 1 \equiv b & \pmod{m} & \end{array}$$

Furthermore,

$$\begin{array}{lrll} & 1 \equiv 1^k & \pmod{m} & \text{(For all } k >= 0\text{)} \\ \Leftrightarrow & 1 \equiv b^k & \pmod{m} & \text{(Substitute } b \text{ for 1 since they are congruent)} \end{array}$$

QED - Lemma.

Proof of Magic-Number Theorem

Let b be a positive integer greater than 1.

Let $n = d_k b^k + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0$ be the base-b-representation of n. Also let m be a positive integer such that $m \mid (b-1)$. Therefore,

$$\begin{array}{llll} & n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0 & & \\ \Rightarrow & n \equiv d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0 & \pmod{m} & \\ \Leftrightarrow & n \equiv d_k \cdot 1 + d_{k-1} \cdot 1 + \cdots + d_2 \cdot 1 + d_1 \cdot 1 + d_0 \cdot 1 & \pmod{m} & \text{(By lemma)} \\ \Leftrightarrow & n \equiv d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0 & \pmod{m} & \end{array}$$

Therefore n is congruent to the sum of its digits mod m, furthermore $m \mid n$ if and only if:

$$\begin{array}{llll} & 0 \equiv n & \pmod{m} & \text{(Definition of } m \mid n\text{)} \\ \Leftrightarrow & 0 \equiv d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0 & \pmod{m} & \text{(Substitute sum of digits for } n\text{)} \end{array}$$

QED