# Factoradic Representation of Rational Numbers

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*From 'A Course of Pure Mathematics' by G. H. Hardy. Chapter 1, Miscellaneous Examples.*

Miscellaneous example[*] #2 at the end of chapter 1 in Hardy's 'Pure Mathematics' presents us with a fascinating result (which was new to me). The theorem feels like what the basis-representation-theorem is for integers, but this one is for rational numbers, ... beautiful!

Here it is, followed by my proof which starts out with some lemmas to get us rolling.

## Theorem

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

## Lemma-1

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}$$

## Proof of Lemma-1

This equality is fairly trivial to demonstrate by induction, since $\frac{1}{2!} = \frac{2!-1}{2!}$ and,

$$\begin{aligned}
&\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\
&= \frac{(k-1)!-1}{(k-1)!} + \frac{k-1}{k!} \\
&= \frac{k((k-1)!-1)}{k(k-1)!} + \frac{k-1}{k!} \\
&= \frac{k! - k + k - 1}{k!} \\
&= \frac{k!-1}{k!}
\end{aligned}$$

...thus establishing lemma-1 for all values of k. QED

[*] Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, math exercises like calculations to perform, theorems to prove etc.

## Lemma-2

For integers $i, k$ where $2 \leq i < k$ such that,

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!} + \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!},$$

then

$$\frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!}$$

## Proof of Lemma-2

$$\begin{aligned}
&\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \\
&= \left(\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!}\right) - \left(\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!}\right) \\
&= \frac{k!-1}{k!} - \frac{i!-1}{i!} \\
&= \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\
&= \frac{1}{i!} - \frac{1}{k!} \\
&< \frac{1}{i!}
\end{aligned}$$

QED

## Lemma-3

The set of rational numbers,

$$\mathcal{S}_k = \{\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \quad | \quad 0 \leq a_2 < 2, \;\; 0 \leq a_3 < 3, \;\; \ldots, \;\; 0 \leq a_k < k\}$$

is identical to the set of rational numbers,

$$\mathcal{F}_k = \{\frac{0}{k!}, \;\; \frac{1}{k!}, \;\; \frac{2}{k!}, \;\; \ldots, \;\; \frac{k!-1}{k!}\}$$

## Proof of Lemma-3

It's clear that the set $\mathcal{F}_k$ contains every rational number with denominator $k!$ where $0 \leq \frac{p}{k!} < 1$ and also clear that the size of $\mathcal{F}_k$ is $k!$. To show that the set $\mathcal{S}_k$ is the same as $\mathcal{F}_k$, it suffices to show that every member of $\mathcal{S}_k$ is also of the form $0 \leq \frac{p}{k!} < 1$, and that the size of $\mathcal{S}_k$ is also $k!$.

The smallest member of the set $\mathcal{S}_k$ is $\frac{0}{k!}$ and occurs when all the coefficients of the sum are zero. Furthermore, the largest member of the set occurs when all the coefficients of the sum are set to their maximum value, which gives us $\frac{k!-1}{k!}$ as shown in lemma-1.

We also note that every member of $\mathcal{S}_k$ can be written as a rational number with $k!$ as the denominator, like so,

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_{k-1}}{(k-1)!} + \frac{a_k}{k!} = \frac{k \cdot (k-1) \cdots 3 \cdot a_2}{k!} + \frac{k \cdot (k-1) \cdots 4 \cdot a_2}{k!} + \ldots + \frac{k \cdot a_{k-1}}{k!} + \frac{a_k}{k!}$$

Therefore any member of the set $\mathcal{S}_k$ is of the form $0 \leq \frac{p}{k!} < 1$, where $p$ in some integer in the range $0 \leq p \leq k! - 1$.

Furthermore, each possible assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produce a unique member of the set $\mathcal{S}_k$.

For this weren't true and both $\frac{p}{k!} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ and $\frac{p}{k!} = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$ for different coefficients $a_2, a_3, \ldots, a_k$ and $b_2, b_3, \ldots, b_k$, then we can arrive at a contradiction as follows.

First suppose that $a_i \neq b_i$ is the first such pair of coefficients that differ from each other. In other words, $a_2 = b_2, a_3 = b_3, \ldots, a_{i-1} = b_{i-1}$. Also, without loss of generality we can assume that $a_i > b_i$ and state the following equality:

$$\begin{aligned} & \frac{a_i}{i!} + \frac{a_{i+1}}{(i+1)!} + \ldots + \frac{a_k}{k!} = \frac{b_i}{i!} + \frac{b_{i+1}}{(i+1)!} + \ldots + \frac{b_k}{k!} \\ \Leftrightarrow \qquad & \frac{a_i - b_i}{i!} = \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!} \end{aligned}$$

Since $a_i - b_i \geq 1$, then $\frac{a_i - b_i}{i!} \geq \frac{1}{i!}$.

Also, $\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \geq \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!}$ regardless of the values of the coefficients on the right side of the inequality*.

However, lemma-2 tells us,

$$\frac{a_i - b_i}{i!} \geq \frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \geq \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!},$$

demonstrating that equality between the two expressions at either end of the inequality is impossible, so our assumption that there can be a second set of coefficients to produce the same rational number $\frac{p}{k!}$ is false. Therefore any assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produces a unique member of the set $\mathcal{S}_k$.

Now we can count the number of members of the set $\mathcal{S}_k$, by looking at all the possible combinations of values for the coefficients $a_2, a_3, \ldots, a_k$. There are 2 choices for the coefficient $a_2$, multiplied by the 3 choices for $a_3$, multiplied by the 4 choices for $a_4$, ..., up to multiplying by $k$ values that $a_k$ can assume.

Therefore the total number of combinations of values that can be assigned to all the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is $2 \cdot 3 \cdot 4 \cdots k = k!$, which means the size of the set $\mathcal{S}_k$ is $k!$. Recalling our previous conclusion that all members of the set $\mathcal{S}_k$ are of the form $0 \leq \frac{p}{k!} < 1$ we can assert that $\mathcal{S}_k = \mathcal{F}_k$. QED

*Letting all the $b$'s be their maximum value, and all the $a$'s be zero will produce the largest numerators in each term of the sum, any other possibility will result in a smaller term for the sum.

## Corollary to Lemma-3

The coefficients $a_2, a_3, \ldots, a_k$ uniquely determine the value of $\frac{p}{q} \in \mathcal{S}_k$.

## Proof of Corollary to Lemma-3

This fact was used in the proof of lemma-3 to show the one-to-one correspondence between $\mathcal{S}_k$ and $\mathcal{F}_k$, but more simply, every element of $\mathcal{F}_k$ is clearly unique and each set of coefficients is associated with one element in $\mathcal{F}_k$, which proves the Corollary.

## Lemma-4

If $\frac{p}{q} \in \mathcal{S}_k$ then $\frac{p}{q} \in \mathcal{S}_n$ for all $n \geq k$. Furthermore, the sum associated with $\frac{p}{q}$ is unchanged for all $\mathcal{S}_n$, which implies that the sum is uniquely associated with $\frac{p}{q}$.

## Proof Lemma-4

if $\frac{p}{q} \in \mathcal{S}_k$ then,

$$\begin{aligned}
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \\
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} + \frac{0}{(k+1)!} + \ldots + \frac{0}{n!} \\
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} + \frac{a_{k+1}}{(k+1)!} + \ldots + \frac{a_n}{n!} \\
&0 \leq a_2 < 2,\ \ 0 \leq a_3 < 3, \ldots, 0 \leq a_k < k, a_{k+1} = 0, \ldots, a_n = 0.
\end{aligned}$$

Therefore, $\frac{p}{q} \in \mathcal{S}_n$ for all $n \geq k$, also demonstrating that the sum for $\frac{p}{q}$ is the same for all $n \geq k$, establishing it's unique association with $\frac{p}{q}$.

## Theorem (restated)

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

## Proof of Theorem

Thanks to Euclid we know that for all integers $j \geq 0$ and $q > 0$, there exist unique integers $i$ and $p$ such that,

$$\begin{aligned} & j = i \cdot q + p\ ; \quad 0 \leq p < q \\ \Leftrightarrow \quad & \frac{j}{q} = i + \frac{p}{q}\ ; \quad 0 \leq \frac{p}{q} < 1 \end{aligned}$$

Which tells us that all rational numbers $\frac{j}{q}$ can be written as an integer part, $i$, plus a fractional part $0 \leq \frac{p}{q} < 1$.

In our theorem, the $a_1$ coefficient plays the role of the integer part $i$, and the rest of the expression, $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ plays the role of the fractional part $0 \leq \frac{p}{q} < 1$.

Therefore to express any rational number in the form of the theorem, first apply the Euclidean Division Theorem to $\frac{j}{q}$ and let $a_1 = i$. If there is no fractional remainder, then the theorem is trivially true, however if there is a fractional remainder $\frac{p}{q}$, then it is a member of all sets $\mathcal{S}_n$ such that $n \geq q$.

We take for the coefficients $a_2, a_3, \ldots, a_k$ in the sum for $\frac{p}{q} \in \mathcal{S}_n$ all those for which $a_k \neq 0$ but $a_{k+1} = a_{k+2} = \ldots = a_n = 0$.

By lemma-5 we know that the sum $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with $\frac{p}{q}$ then clearly $\frac{j}{q} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with all rational numbers $\frac{j}{q}$.

QED

## Additional Observations

While it's true that $\frac{p}{q} \in \mathcal{S}_q$, $\mathcal{S}_q$ is not necessarily the smallest such set for which $\frac{p}{q}$ is a member.

For example, the smallest set containing $0 \leq \frac{p}{5} < 1$ is $\mathcal{S}_5$ however the smallest set containing $0 \leq \frac{p}{6} < 1$ is $\mathcal{S}_3$.

Which is easy to see when we list the contents of a couple of sets,

$$\begin{aligned} \mathcal{S}_4 &= \{\frac{0}{24}, \frac{1}{24}, \frac{2}{24}, \frac{3}{24}, \frac{4}{24}, \frac{5}{24}, \frac{6}{24}, \frac{7}{24}, \frac{8}{24}, \frac{9}{24}, \frac{10}{24}, \frac{11}{24}, \frac{12}{24}, \frac{13}{24}, \frac{14}{24}, \frac{15}{24}, \frac{16}{24}, \frac{17}{24}, \frac{18}{24}, \frac{19}{24}, \frac{20}{24}, \frac{21}{24}, \frac{22}{24}, \frac{23}{24}\} \\ &= \{\frac{0}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4}, \frac{7}{24}, \frac{1}{3}, \frac{3}{8}, \frac{5}{12}, \frac{11}{24}, \frac{1}{2}, \frac{13}{24}, \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{17}{24}, \frac{3}{4}, \frac{19}{24}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{23}{24}\} \end{aligned}$$

Which clearly doesn't contain $\frac{1}{5}$. We've established that $\frac{1}{5}$ is definitely in $\mathcal{S}_5$ but it's interesting to see what it looks like:

$$\frac{1}{5} = \frac{0}{2} + \frac{1}{2 \cdot 3} + \frac{0}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{6} + \frac{1}{30} = \frac{5+1}{30} = \frac{6}{30} = \frac{1}{5}$$

Also, $\mathcal{S}_3 = \{\frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\} = \{\frac{0}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$, which demonstrates the claim above that $\mathcal{S}_3$ contains $0 \le \frac{p}{6} < 1$.

I believe that for a given $q \ge 2$ then the smallest set for which $0 \le \frac{p}{q} < 1$ are members is the set $\mathcal{S}_k$ such that $k$ is the smallest value for which $q$ divides $k!$.

However, I'll leave that proof for another day.