Magic Numbers

James Rowell

January 22, 2018

In the delightful film “School of Rock” Jack Black’s character Dewey Finn, pretending to be substitute teacher Ned Schneebly, is put on the spot when Miss. Mullins, the school principle, comes into class and demands that he show her his teaching methods which apparently involve an electric guitar. So Mr. S sings “The Math Song” wherein the final line of his improvised tune he sings “. . . yes it’s 9, and that’s a magic-number. . . ”

It seems that Jack Black was tipping his hat to the “Schoolhouse Rock” song “3 Is A Magic Number” which most kids from the 70’s/80’s know - However referring to 9 as a magic-number isn’t as well known*.

Mind you, it’s not a total stretch, there is precedent for using the term “Magic Number” when referring to to the number 9. There are a host of mathematical-magic-tricks that rely on properties of the number 9. There is also a process called “casting out nines” and a related one called calculating the “digital root” for a number where you take its “digital sum”. 9 plays a similar role in all these processes - so within these contexts sometimes 9 is referred to as a magic-number.

So Mr. S’s musical-math-lesson is legit, if he is referring to this fact about 9:

The sum of the digits of a number divisible by 9,
if and only if,
the original number is divisible by 9.

For example:

$$9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59049 \text{ and } 5+9+0+4+9 = 27 \text{ and 9 divides 27 because } 27 = 3 \times 9.$$

But 3 is also has this magic-number property, for example:

$$3 \times 67 = 201 \text{ and } 2 + 0 + 1 = 3$$

In this case, 9 does not divide 201† but 3 does (67 times). We can see that adding up the digits of 201 gives us 3, which of course is divisible by 3 (but not by 9).

So given that the term “magic number” hasn’t got a formal definition in the math world, I think it’s time to give it one. How about if we call ANY number that has this divisibility property with respect to adding up the digits etc. (like 3 and 9) a magic-number?

*According to my own *highly scientific* informal poll conducted Facebook, “9” is NOT the first thing that pops into people’s heads when they hear the phrase “magic number” - in fact no one answered “9” in my poll.

†Why is it immediately apparent that 9 does not divide 201? Hint: both 3 and 67 are prime numbers. Check the “Fundamental Theorem of Arithmetic” wiki page for further insight.

Too bad 3 and 9 are the only magic-numbers in base-10, but what about other bases? The heptapods in the film "Arrival" have 7 limbs, 3 arms and 4 legs, with 7 fingers/toes each. I'm going to guess that at least at some point in their history they used positional notation, like our base-10, but instead of 10 they probably base-7 or base-21, maybe even base-49? So an understanding of magic-numbers in alternate bases might come in handy when we're called upon to do alien arithmetic.

Let's examine some magic-numbers in other bases.

- $4 \times 67 = 268$ written in base-5 is $(2033)_5$ and $2 + 0 + 3 + 3 = 8$ (4 divides 8).
- $7 \times 181 = 1267$ written in base-8 is $(2363)_8$ and $2 + 3 + 6 + 3 = 14$ (7 divides 14).
- $15 \times 23 = 345$ written in base-16 is $(159)_{16}$ and $1 + 5 + 9 = 15$.
- $30 \times 35951 = 1078530$ written in base-31 is $(15699)_{31}$ and $1 + 5 + 6 + 9 + 9 = 30$

Notice the pattern: If b is our base, then $(b - 1)$ is a magic-number in base-b.

Let's take a look at some other examples in base-31. It appears that 2, 3, 5, 6, 10, 15 (along with 30) are *all* magic-numbers in base-31, as we we can see with the following examples:

- $2 \times 409 \times 1129 = 923522$ written in base-31 is $(10001)_{31}$ and $1 + 0 + 0 + 0 + 1 = 2$.
- $3 \times 641 = 1923$ written in base-31 is $(201)_{31}$ and $2 + 0 + 1 = 3$.
- $5 \times 139 \times 2659 = 1848005$ written in base-31 is $(20102)_{31}$ and $2 + 0 + 1 + 0 + 2 = 5$.
- $6 \times 10091 = 60546$ written in base-31 is $(2103)_{31}$ and $2 + 1 + 0 + 3 = 6$.
- $10 \times 197 \times 941 = 1853770$ written in base-31 is $(20701)_{31}$ and $2 + 0 + 7 + 0 + 1 = 10$.
- $15 \times 71503 = 1072545$ written in base-31 is $(15027)_{31}$ and $1 + 5 + 0 + 2 + 7 = 15$.

See the pattern? All the divisors of 30 are magic-numbers in base-31. It turns out that in base-b, then any divisor of $(b - 1)$ is a magic-number in base-b.

I guess that begs the question: What about 1? Isn't 1 a magic-number in all bases? Since we already know that all numbers are divisible by 1, then we don't need to go to any trouble of adding up digits to find this out. So let's exclude 1 from being considered as a magic-number - it's not helpful.

Let's quickly review what it means for a number to be divisible by another number. Recall from the paper "Counting"...

Euclidean Division Theorem

For all integers a and b such that $b \neq 0$, there exist *unique* integers q and r such that:

$$a = qb + r \text{ such that } 0 \leq r < |b|$$

Definition: In the above equation*:

a is the *dividend*	("the number being divided")
b is the *divisor*	("the number doing the dividing")
q is the *quotient*	("the result of the division")
r is the *remainder*	("the leftover")

So to say that a number a is divisible by b simply means that r is zero.

In other words, to say an integer a is divisible by another integer b (not zero) means there is a third integer q such that†:

$$a = q \cdot b \qquad \text{or} \qquad \frac{a}{b} = q$$

We express the fact that a is divisible by b, and say "b divides a" using this notation:

$$b \mid a$$

So the following is always true:

$$1 \mid a, \text{ and } a \mid a;$$

also

$$b \mid 0 \text{ for every } b \text{ except } 0$$

In order to create our theorem with a nice definition for "magic number" we need to remind ourselves what it means to write a number in a given base‡:

Basis Representation Theorem

Let b be a positive integer greater than 1.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

where $0 \leq d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-b by the string of base-b-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$

*Recall that $|b|$ means the "absolute value" of b and is always positive. For example $|-13| = 13$.

†Paraphrased from "An Introduction to the Theory of Numbers" by G. H. Hardy and E. M. Wright. Pg 1.

‡Also from "Counting"

Magic Number Theorem

Let n be a positive integer represented in base-b.

If m is a positive integer such that $m \mid (b-1)$ then,

$$m \mid n \qquad \text{if and only if} \qquad m \mid (d_k + \cdots + d_2 + d_1 + d_0)$$

Definition: We call $m \neq 1$ a "magic-number" in base-b.

As an interesting side note - according to our definition, binary, or base-2, doesn't have a magic-number. No great loss, as we said above 1 isn't helpful to consider as a magic-number.

Let's prove the magic-number-theorem using modular arithmetic. If you aren't familiar with modular arithmetic then I recommend that you check out Khan Academy's "What is modular arithmetic?". Khan's introductory explanation is excellent, simple and clear, as is the entire Khan Academy site in case you've never checked it out.

If we take the a, b, q and r from the Euclidean Division Theorem above where:

$$a = qb + r \text{ such that } 0 \leq r < |b|$$

Then the "mod" operator is defined like this:

$$a \bmod b = r$$

To prove our theorem we need to prove that the remainder of n divided by m is zero if and only if the remainder of $d_k + \cdots + d_1 + d_0$ divided by m is also zero. Those two desired results can be can be restated with modular-arithmetic by showing:

$$n \bmod m = 0 \qquad \text{if and only if} \qquad (d_k + \cdots + d_1 + d_0) \bmod m = 0$$

SKIP FURTHER EXPLANATION OF MOD UNTIL PROOF COMPLETED.

Proof of Magic Number Theorem

Let b be a positive integer greater than 1.

Let n be represented in base-b with the sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

where $0 \leq d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Also let m be a positive integer such that $m \mid (b-1)$, in other words there exists some integer q such that:

$$m \cdot q = b - 1$$

Lemma:

$$b^k \equiv 1 \pmod{m}, \text{ for all integers } k \geq 0$$

Proof of Lemma:

Let k be an integer such that $k \geq 0$.

$$\begin{aligned} & m \cdot q = b - 1 && \text{(Definition of } m\text{)} \\ \Leftrightarrow\quad & m \cdot q + 1 = b && \text{(Add 1 to both sides)} \end{aligned}$$

Given this expression for b we can state the following modular congruence:

$$\begin{aligned} & m \cdot q + 1 \equiv b \pmod{m} & \\ \Leftrightarrow\quad & 0 \cdot q + 1 \equiv b \pmod{m} & \text{(Since } m \bmod m = 0\text{)} \\ \Leftrightarrow\quad & 1 \equiv b \pmod{m} & \end{aligned}$$

Furthermore,

$$\begin{aligned} & 1 \equiv 1^k \pmod{m} & \\ \Leftrightarrow\quad & 1 \equiv b^k \pmod{m} & \text{(Substitute } b \text{ for 1)} \end{aligned}$$

QED - Lemma.

Again here's the base-b expression for n:

$$\begin{aligned} & n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0 \\ \Rightarrow\quad & n \equiv d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0 \pmod{m} \\ \Leftrightarrow\quad & n \equiv d_k \cdot 1 + d_{k-1} \cdot 1 + \cdots + d_2 \cdot 1 + d_1 \cdot 1 + d_0 \cdot 1 \pmod{m} \\ \Leftrightarrow\quad & n \equiv d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0 \pmod{m} \quad \text{(Note equivalence of sum of digits and } n\text{)} \end{aligned}$$

m divides n if and only if:

$$\begin{aligned} & 0 \equiv n \pmod{m} \\ \Leftrightarrow\quad & 0 \equiv d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0 \pmod{m} \quad \text{(Sum of digits and } n \text{ are equivalent)} \end{aligned}$$

QED