Rational Approximation to $\sqrt{2}$

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From 'A Course in Pure Mathematics' by G. H. Hardy. Chapter 1, Section 5.

Examples III, Number 3.

Show that if $\frac{m}{m}$ is a good approximation to $\sqrt{2}$, then $\frac{(m+2n)}{(m+n)}$ is a better one, and that the errors in the two cases are in opposite directions.

Lemma

Let m, n and a be positive integers

$$\frac{m^2}{n^2} + \frac{a}{n^2} = 2 \qquad \Leftrightarrow \qquad \frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} = 2$$

Proof of Lemma

$$\begin{array}{rrcl}
 & \frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} & = & 2 \\
\Leftrightarrow & (m+2n)^2 - a & = & 2(m+n)^2 \\
\Leftrightarrow & m^2 + 4mn + 4n^2 & = & 2m^2 + 4mn + 2n^2 + a \\
\Leftrightarrow & 4n^2 - 2n^2 & = & 2m^2 - m^2 + a \\
\Leftrightarrow & 2n^2 & = & m^2 + a \\
\Leftrightarrow & 2 & = & \frac{m^2}{n^2} + \frac{a}{n^2} \qquad \text{QED}
\end{array}$$

Proof of Examples III, Number 3.

The error in any rational approximation to $\sqrt{2}$ is the difference between the approximation-squared and 2. Let e_1 represent the error for the $\frac{m}{n}$ approximation of $\sqrt{2}$, and let e_2 represent the error for the $\frac{(m+2n)}{(m+n)}$ approximation of $\sqrt{2}$, in other words:

$$e_1 = \left|\frac{a}{n^2}\right| = \left|2 - \frac{m^2}{n^2}\right|, \quad \text{and} \quad e_2 = \left|\frac{-a}{(m+n)^2}\right| = \left|2 - \frac{(m+2n)^2}{(m+n)^2}\right|$$

Since the denominator $(m+n)^2$ is always larger than n^2 therefore,

$$e_1 = \left|\frac{a}{n^2}\right| > \left|\frac{-a}{(m+n)^2}\right| = e_2,$$

showing that $\frac{(m+2n)}{(m+n)}$ is a better approximation to $\sqrt{2}$ than $\frac{m}{n}$.