Factoradic Representation of Rational Numbers

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From 'A Course of Pure Mathematics' by G. H. Hardy. Chapter 1, Miscellaneous Examples.

Miscellaneous example* #2 at the end of chapter 1 in Hardy's 'Pure Mathematics' presents us with a fascinating result (which was new to me). The theorem feels like what the basis-representation-theorem is for integers, but this one is for rational numbers, ... beautiful!

Factorial Representation Theorem†

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Observations to get us started.

The first thing to note are the ranges of the coefficients $a_1, a_2, a_3, \ldots, a_k$.

The first one, a_1, is unrestricted in terms of how big it can be. It's just a plain-old non-negative integer, which is pretty obvious because it has no other number dividing it.

We know that any rational number‡, say $\frac{m}{q}$, can be written as an integer i, part PLUS a fractional part $\frac{p}{q}$ such that $\frac{m}{q} = i + \frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$ (note that i can be zero).

So if we're trying to represent any positive rational number $\frac{m}{q}$ in terms of the theorem then the integer part, i, will be the a_1 term, and the remainder of the expression should be the rational part, $\frac{p}{q}$, such that,

$$0 \leq \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot k} < 1$$

So it seemed to me a good idea to forget about the a_1 term and just focus on the $a_1, a_2, a_3, \ldots, a_k$ terms. So I began to only focus on proving the theorem for rational numbers $\frac{p}{q}$, such that $0 \leq \frac{p}{q} < 1$, as it would be trivial to extend it to all rational numbers later. Also, it started

*Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, math exercises for the student, like calculations to perform, theorems to prove etc.

†...anyway, that's what I'm calling the theorem.

‡Every variable, coefficient or constant (eg. a_1, a_k, m, n, i, p, q) in this paper is going to represent a non-negative integer. We aren't dealing with 'real numbers' here, just rational numbers which we will always discuss in terms of one integer divided by another integer, like $\frac{p}{q}$.

to become clear that including zero (that is, not JUST positive rational numbers) was going to simplifiy the task too.

At first glance it not even remotely obvious how you'd go about finding such an assignment of coefficients $a_2, a_3, \ldots, a_k$ for a given rational number let alone that it would be unique.

After playing around for a good chunk of time, and finally figuring out a way to calculate the a_i terms for a given rational number $\frac{p}{q}$, a few thing started to jump out at me. For starters look at these particular rational numbers,

$$\frac{1}{2} = \frac{1}{1\cdot 2} = \frac{2!-1}{2!}$$
$$\frac{5}{6} = \frac{1}{1\cdot 2} + \frac{2}{1\cdot 2\cdot 3} = \frac{1\cdot 3}{1\cdot 2\cdot 3} + \frac{2}{1\cdot 2\cdot 3} = \frac{3+2}{6} = \frac{3!-1}{3!}$$
$$\frac{23}{24} = \frac{1}{1\cdot 2} + \frac{2}{1\cdot 2\cdot 3} + \frac{3}{1\cdot 2\cdot 3\cdot 4} = \frac{1\cdot 3\cdot 4}{1\cdot 2\cdot 3\cdot 4} + \frac{2\cdot 4}{1\cdot 2\cdot 3\cdot 4} + \frac{3}{1\cdot 2\cdot 3\cdot 4} = \frac{12+8+3}{24} = \frac{4!-1}{4!}$$

This was a pretty strong hint! It seemed to be the case that if we assign the biggest possible values to the coefficients, from a_2 up to say a_k we get this rational number, $\frac{k!-1}{k!}$. This turned out to be a pretty useful observation, and it became my 'Lemma 1' in the proof below.

Also, if we assign zeros to all the coefficients then naturally we get $\frac{0}{k!}$. So for a given sum This seems pretty useful because for a given to cover the range for our $\frac{p}{q}$ part of our specific rational number where $0 \leq \frac{p}{q} < 1$.

Lemma-1

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}$$

Proof of Lemma-1

This equality is fairly trivial to demonstrate by induction, since $\frac{1}{2!} = \frac{2!-1}{2!}$ and,

$$\begin{aligned}
&\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\
&= \frac{(k-1)!-1}{(k-1)!} + \frac{k-1}{k!} \\
&= \frac{k((k-1)!-1)}{k(k-1)!} + \frac{k-1}{k!} \\
&= \frac{k!-k+k-1}{k!} \\
&= \frac{k!-1}{k!}
\end{aligned}$$

...thus establishing lemma-1 for all values of k. QED

Lemma-2

For integers i, k where $2 \leq i < k$ such that,

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!} + \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!},$$

then

$$\frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!}$$

Proof of Lemma-2

$$\begin{aligned}
&\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \\
&= (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!}) - (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!}) \\
&= \frac{k!-1}{k!} - \frac{i!-1}{i!} \\
&= \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\
&= \frac{1}{i!} - \frac{1}{k!} \\
&< \frac{1}{i!}
\end{aligned}$$

QED

Lemma-3

For any integer $k \geq 2$, and integers $a_2, a_3, \ldots a_k$, the set of rational numbers,

$$\mathcal{S}_k = \{\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \quad | \quad 0 \leq a_2 < 2, \ \ 0 \leq a_3 < 3, \ \ \ldots, \ \ 0 \leq a_k < k\},$$

is identical to the set of rational numbers,

$$\mathcal{F}_k = \{\frac{0}{k!}, \ \ \frac{1}{k!}, \ \ \frac{2}{k!}, \ \ \ldots, \ \ \frac{k!-1}{k!}\}$$

Which implies that for every rational number $0 \leq \frac{p}{k!} < 1$ there is a unique sequence of integers $a_2, a_3, \ldots a_k$ such that,

$$\frac{p}{k!} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!},$$

where $0 \leq a_2 < 2, \ \ 0 \leq a_3 < 3, \ \ \ldots, \ \ 0 \leq a_k < k$.

Proof of Lemma-3

It's clear that the set $\mathcal{F}_k$ contains every rational number with denominator $k!$ where p is an integer and $0 \leq \frac{p}{k!} < 1$ and that the size of $\mathcal{F}_k$ is $k!$. To show that the set $\mathcal{S}_k$ is the same as $\mathcal{F}_k$, it suffices to show that every member of $\mathcal{S}_k$ is also of the form $0 \leq \frac{p}{k!} < 1$, and that the size of $\mathcal{S}_k$ is also $k!$.

The smallest member of the set $\mathcal{S}_k$ is $\frac{0}{k!}$ and occurs when all the coefficients of the sum are zero. Furthermore, the largest member of the set occurs when all the coefficients of the sum are set to their maximum value, which gives us $\frac{k!-1}{k!}$ as shown in lemma-1.

We also note that every member of $\mathcal{S}_k$ can be written as a rational number with $k!$ as the denominator, like so,

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_{k-1}}{(k-1)!} + \frac{a_k}{k!} = \frac{k \cdot (k-1) \cdot \cdot \cdot 3 \cdot a_2}{k!} + \frac{k \cdot (k-1) \cdot \cdot \cdot 4 \cdot a_2}{k!} + \ldots + \frac{k \cdot a_{k-1}}{k!} + \frac{a_k}{k!}$$

Therefore any member of the set $\mathcal{S}_k$ is of the form $0 \leq \frac{p}{k!} < 1$, where p in some integer in the range $0 \leq p \leq k! - 1$.

Furthermore, each possible assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produce a unique member of the set $\mathcal{S}_k$.

For if this weren't true and both $\frac{p}{k!} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ and $\frac{p}{k!} = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$ for different coefficients $a_2, a_3, \ldots, a_k$ and $b_2, b_3, \ldots, b_k$, then we can arrive at a contradiction as follows.

First suppose that $a_i \neq b_i$ is the first such pair of coefficients that differ from each other. In other words, $a_2 = b_2, a_3 = b_3, \ldots, a_{i-1} = b_{i-1}$. Also, without loss of generality we can assume that $a_i > b_i$ and state the following equality:

$$\begin{aligned} & \frac{a_i}{i!} + \frac{a_{i+1}}{(i+1)!} + \ldots + \frac{a_k}{k!} = \frac{b_i}{i!} + \frac{b_{i+1}}{(i+1)!} + \ldots + \frac{b_k}{k!} \\ \Leftrightarrow \quad & \frac{a_i - b_i}{i!} = \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!} \end{aligned}$$

Since $a_i - b_i \geq 1$, then $\frac{a_i - b_i}{i!} \geq \frac{1}{i!}$.

Also, $\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \geq \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!}$ regardless of the values of the coefficients on the right side of the inequality*.

However, lemma-2 tells us,

$$\frac{a_i - b_i}{i!} \geq \frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \geq \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!},$$

demonstrating that equality between the two expressions at either end of the inequality is impossible, so our assumption that there can be a second set of coefficients to produce the same rational number $\frac{p}{k!}$ is false. Therefore any assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produces a unique member of the set $\mathcal{S}_k$.

Now we can count the number of members of the set $\mathcal{S}_k$, by looking at all the possible combinations of values for the coefficients $a_2, a_3, \ldots, a_k$. There are 2 choices for the coefficient a_2,

*Letting all the b's be their maximum value, and all the a's be zero will produce the largest numerators in each term of the sum, any other possibility will result in a smaller term for the sum.

multiplied by the 3 choices for a_3, multiplied by the 4 choices for a_4, ..., up to multiplying by k values that a_k can assume.

Therefore the total number of combinations of values that can be assigned to all the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is $2 \cdot 3 \cdot 4 \cdots k = k!$, which means the size of the set $\mathcal{S}_k$ is $k!$. Recalling our previous conclusion that all members of the set $\mathcal{S}_k$ are of the form $0 \leq \frac{p}{k!} < 1$ we can assert that $\mathcal{S}_k = \mathcal{F}_k$.

Futhermore we've shown above that there is a unique sequence of integers $a_2, a_3, \ldots a_k$, (with the appropriate ranges of values for each integer and $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$) for each value of $\frac{p}{k!} \in \mathcal{F}_k$. QED

Corollary to Lemma-3

If $\frac{p}{q} \in \mathcal{S}_k$ then there is a unique sequence of integers $0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$ such that $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$.

Proof of Corollary to Lemma-3

Apart from the fact that this was part of the proof of lemma-3 The immediately follows from lemma-3 because the sets $\mathcal{S}_k$ and $\mathcal{S}_k$ are identicaal and there are k! possible combinations of coefficients, one for each of the k! unique members of the set $\mathcal{F}_k$.

Lemma-4

If $\frac{p}{q} \in \mathcal{S}_k$ then $\frac{p}{q} \in \mathcal{S}_n$ for all $n \geq k$. Furthermore, the sum associated with $\frac{p}{q}$ is unchanged for all $\mathcal{S}_n$, which implies that the sum is uniquely associated with $\frac{p}{q}$.

Proof Lemma-4

if $\frac{p}{q} \in \mathcal{S}_k$ then by the corollary to lemma-3 there is a unique sequence of integers,
$0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$ such that,

$$
\begin{aligned}
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \\
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} + \frac{0}{(k+1)!} + \ldots + \frac{0}{n!} \\
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} + \frac{a_{k+1}}{(k+1)!} + \ldots + \frac{a_n}{n!} \\
&0 \leq a_2 < 2,\ \ 0 \leq a_3 < 3, \ldots, 0 \leq a_k < k, a_{k+1} = 0, \ldots, a_n = 0.
\end{aligned}
$$

Therefore, $\frac{p}{q} \in \mathcal{S}_n$ for all $n \geq k$, also demonstrating that the sum for $\frac{p}{q}$ is the same for all $n \geq k$, establishing it's unique association with $\frac{p}{q}$.

Theorem (restated)

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \le a_1, \quad 0 \le a_2 < 2, \quad 0 \le a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Proof of Theorem

Thanks to Euclid we know that for all integers $j \ge 0$ and $q > 0$, there exist unique integers i and p such that,

$$\begin{aligned} & j = i \cdot q + p \; ; \quad 0 \le p < q \\ \Leftrightarrow \quad & \frac{j}{q} = i + \frac{p}{q} \; ; \quad 0 \le \frac{p}{q} < 1 \end{aligned}$$

Which tells us that all rational numbers $\frac{j}{q}$ can be written as an integer part, i, plus a fractional part $0 \le \frac{p}{q} < 1$.

In our theorem, the a_1 coefficient plays the role of the integer part i, and the rest of the expression, $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ plays the role of the fractional part $0 \le \frac{p}{q} < 1$.

Therefore to express any rational number in the form of the theorem, first apply the Euclidean Division Theorem to $\frac{j}{q}$ and let $a_1 = i$. If there is no fractional remainder, then the theorem is trivially true, however if there is a fractional remainder $\frac{p}{q}$, then it is a member of all sets $\mathcal{S}_n$ such that $n \ge q$.

We take for the coefficients $a_2, a_3, \ldots, a_k$ in the sum for $\frac{p}{q} \in \mathcal{S}_n$ all those for which $a_k \ne 0$ but $a_{k+1} = a_{k+2} = \ldots = a_n = 0$.

By lemma-5 we know that the sum $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with $\frac{p}{q}$ then clearly $\frac{j}{q} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with all rational numbers $\frac{j}{q}$.

QED

Additional Observations

While it's true that $\frac{p}{q} \in \mathcal{S}_q$, $\mathcal{S}_q$ is not necessarily the smallest such set for which $\frac{p}{q}$ is a member.

For example, the smallest set containing $0 \le \frac{p}{5} < 1$ is $\mathcal{S}_5$ however the smallest set containing $0 \le \frac{p}{6} < 1$ is $\mathcal{S}_3$.

Which is easy to see when we list the contents of a couple of sets,

$$\begin{aligned}\mathcal{S}_4 &= \{\frac{0}{24}, \frac{1}{24}, \frac{2}{24}, \frac{3}{24}, \frac{4}{24}, \frac{5}{24}, \frac{6}{24}, \frac{7}{24}, \frac{8}{24}, \frac{9}{24}, \frac{10}{24}, \frac{11}{24}, \frac{12}{24}, \frac{13}{24}, \frac{14}{24}, \frac{15}{24}, \frac{16}{24}, \frac{17}{24}, \frac{18}{24}, \frac{19}{24}, \frac{20}{24}, \frac{21}{24}, \frac{22}{24}, \frac{23}{24}\} \\ &= \{\frac{0}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4}, \frac{7}{24}, \frac{1}{3}, \frac{3}{8}, \frac{5}{12}, \frac{11}{24}, \frac{1}{2}, \frac{13}{24}, \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{17}{24}, \frac{3}{4}, \frac{19}{24}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{23}{24}\}\end{aligned}$$

Which clearly doesn't contain $\frac{1}{5}$. We've established that $\frac{1}{5}$ is definitely in $\mathcal{S}_5$ but it's interesting to see what it looks like:

$$\frac{1}{5} = \frac{0}{2} + \frac{1}{2 \cdot 3} + \frac{0}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{6} + \frac{1}{30} = \frac{5+1}{30} = \frac{6}{30} = \frac{1}{5}$$

Also, $\mathcal{S}_3 = \{\frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\} = \{\frac{0}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$, which demonstrates the claim above that $\mathcal{S}_3$ contains $0 \leq \frac{p}{6} < 1$.

I believe that for a given $q \geq 2$ then the smallest set for which $0 \leq \frac{p}{q} < 1$ are members is the set $\mathcal{S}_k$ such that k is the smallest value for which q divides $k!$.

However, I'll leave that proof for another day.