Rational Approximations to $\sqrt{2}$

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Examples from 'A Course in Pure Mathematics' by G. H. Hardy. Chapter 1, Section 5.

"Example 2" in Hardy's "Pure Mathematics" (Ch. 1, Section 5, Examples III) looks at a sequence of rational approximations to $\sqrt{2}$, that is:

$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \tfrac{99}{70}$$

So these numbers squared are:

$$\tfrac{1}{1},\ \tfrac{9}{4},\ \tfrac{49}{25},\ \tfrac{289}{144},\ \tfrac{1681}{841},\ \tfrac{9801}{4900}$$

And their differences from 2 are:

$$-1,\ \tfrac{1}{4},\ -\tfrac{1}{25},\ \tfrac{1}{144},\ -\tfrac{1}{841},\ \tfrac{1}{4900}$$

Theorem: Examples III, Number 3.

Show that if $\frac{m}{m}$ is a good approximation to $\sqrt{2}$, then $\frac{(m+2n)}{(m+n)}$ is a better one, and that the errors in the two cases are in opposite directions.

Lemma

Let m, n and a be positive integers

$$\frac{m^2}{n^2} + \frac{a}{n^2} = 2 \qquad \Leftrightarrow \qquad \frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} = 2$$

Proof of Lemma

$$\begin{array}{rrcl} & \dfrac{(m+2n)^2}{(m+n)^2} - \dfrac{a}{(m+n)^2} & = & 2 \\ \Leftrightarrow & (m+2n)^2 - a & = & 2(m+n)^2 \\ \Leftrightarrow & m^2 + 4mn + 4n^2 & = & 2m^2 + 4mn + 2n^2 + a \\ \Leftrightarrow & 4n^2 - 2n^2 & = & 2m^2 - m^2 + a \\ \Leftrightarrow & 2n^2 & = & m^2 + a \\ \Leftrightarrow & 2 & = & \dfrac{m^2}{n^2} + \dfrac{a}{n^2} \end{array}$$

QED

Proof of Theorem

Define the "error" of a rational approximation to $\sqrt{2}$ as the difference between the approximation-squared and 2. So we learn from the Lemma that the error for the $\frac{m}{n}$ approximation of $\sqrt{2}$ is

$$\left|\tfrac{a}{n^2}\right| = \left|2 - \tfrac{m^2}{n^2}\right|,$$

and the error for the $\frac{(m+2n)}{(m+n)}$ approximation of $\sqrt{2}$ is

$$\left|\tfrac{-a}{(m+n)^2}\right| = \left|2 - \tfrac{(m+2n)^2}{(m+n)^2}\right|.$$

The denominator $(m+n)^2$ is always larger than n^2 therefore,

$$\left|\frac{-a}{(m+n)^2}\right| < \left|\frac{a}{n^2}\right|,$$

showing that the approximation $\frac{(m+2n)}{(m+n)}$ for $\sqrt{2}$ has a smaller error that that of $\frac{m}{n}$ and is therefore a better approximation to $\sqrt{2}$.

Furthermore, we also learn from the Lemma that the successive approximations flip to either side of 2 because:

If $\frac{m^2}{n^2} < 2$ then a is positive therefore $\frac{(m+2n)}{(m+n)} > 2$, similarly,

If $\frac{m^2}{n^2} > 2$ then a is negative therefore $\frac{(m+2n)}{(m+n)} < 2$.

QED