


SesameStreet++






Most of us think about “whole numbers” not too differently from the way we learned to count by watching Sesame Street. The way we’ve trained ourselves, it’s automatic to think that how we write a number or say a number IS the number.

If I say “I owe you 13 cents” and I give you one dime and three pennies then we agree those coins are equal to 13 pennies. We identify the symbol “13” very strongly with this particular number - it would be tough to get through life in the modern world without such an automatic process running in our brains. This example highlights what this particular symbol “13” actually means - one dime (1×10) plus three pennies (3×1).


Let’s look at the number 13 in some alternative ways - it’s the number of months in a year *plus* one month; what I’m suggesting is that there is no need for the symbol “13” in order to think about this particular number of months. Better yet, 13 is this many apples

 ; or 13 is the sixth prime number (i.e., $\langle 1^{\text{st}}: 2 \rangle$, $\langle 2^{\text{nd}}: 3 \rangle$, $\langle 3^{\text{rd}}: 5 \rangle$, $\langle 4^{\text{th}}: 7 \rangle$, $\langle 5^{\text{th}}: 11 \rangle$, $\langle 6^{\text{th}}: 13 \rangle$); or perhaps 13 is “the number that follows the 2^{nd} prime-number multiplied by the 1^{st} prime-number squared”, also written as $(3 \times 2^2) + 1$. None of these ways of thinking about the number 13 require that we represent it using the digits 1 and 3 butted up next to each other.

Each number exists independently from any symbol or word that might represent it. Numbers are an idea - perhaps such a strong idea that the universe wouldn’t exist without it! Anyway, for our purposes whole numbers exist in some abstract realm - Each number is one whole unit more than the previous number, starting at nothing, that is “zero”, and jumping to something, that is “one”, then one more, which gets us to “two”, then “three”, etc. Continuing in this way... then we get them all.

To get the idea of what a whole number *really* is, try to forget about the symbols or words we use and picture a pile of apples. There’s zero apples (it’s hard to show no-apples), then we introduce an “” to get our very first, and smallest, non-empty pile of apples. Then add another apple to get a pile of “”, then “”, then “” then some big pile of “” after we’ve been adding apples for a while. Each successively bigger pile of apples corresponds with each successive whole number.

We also want the idea of a “negative number”, which can be thought as “missing something” or “owed apples” (in the sense of a debt). Each whole number (except zero) has a corresponding negative number, such that adding the positive number to its corresponding negative number results in a sum of zero. We expand the above set of whole numbers to include their negative-counterparts and call this larger set “integers”. We denote the set of integers with this symbol: \mathbb{Z} . If we only want to talk about positive integers *along with zero*, we use this symbol: \mathbb{Z}^+ . When we talk about numbers in mathematics we always treat them as belonging to some kind of set (like the integers) in this pure, abstract sort of way.

However... using a “1” followed by a “3” to represent “” is VERY handy. So we use Hindu-Arabic numbers and the positional notation of “base-ten” or “decimal” to represent each specific integer. We slap a “-” on the front if we need to talk about a negative integer.

Base-ten representation of an integer is far superior to ancient Roman numerals. Try adding two numbers together in ancient Rome, or worse, multiplying or dividing them. What’s XI times IX? Would you believe me if I told you it’s XCIX? Unless you convert those to Hindu-Arabic numerals, you’re just gonna have to trust me. Truth is - I don’t know how to multiply using Roman numerals - nor did most Romans! Not only that, but I’ll bet that any kid who graduates from Sesame Street can count higher than any Roman could - as Romans only really counted up to 4999.

As we discussed above, using base-ten for us is pretty automatic, we barely think about it when we’re adding numbers or multiplying them - but it’s worth looking *very carefully* at how base-ten works - so let’s examine it from the ground up.

The Hindus and Arabs created beautiful and simple symbols to represent each of the numbers from one to nine, namely our familiar numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9.

They also introduced a tenth relatively modern number “zero”, or “0”, standing for no-thing. Zero allows for the creation of our base-ten positional numbering system, without zero base-ten wouldn’t work

We all know what the word “ten” stands for, it means the next integer after nine, or your last finger when you’re using your fingers to count. Instead of making up a new symbol for this integer, the base-ten approach strings two digits together (a digit being one of 0, 1, 2, ..., 9), one after the other. The first digit tells us how many tens we have and the last digit says how many additional units are needed to get the desired integer.

So our very first two digit (digit means finger!) number “10” means “one set of ten - plus zero units” (that zero is pretty handy!) When we see “11” - we interpret it to mean “one set of ten - plus one unit”, and “12” is “one set of ten - plus two units”, etc.

Continuing on; “20” - we interpret it to mean “two sets of ten, plus zero units”, etc. up to “90” meaning “nine sets of ten, plus zero units”.

Following this line of reasoning then “100” must mean “ten sets of ten, plus zero units” - which is exactly what it means. We have a special word for this number we call it “one hundred” or “one set of a hundred, plus zero sets of tens, plus zero units”. Similarly “200” means “two sets of a hundred, plus zero sets of ten, plus zero units”, etc., etc.

We can keep going by one-hundred until we similarly get to “1000” or “ten hundred, plus zero sets of ten, plus zero units” otherwise known as “a thousand” or more specifically “one set of a thousand, plus zero sets of a hundred, plus zero sets of ten, plus zero units”.

It get's a little tedious to be so specific when reading out a number so our language has developed quite a few verbal shortcuts. Furthermore it doesn't take long before we run out of fancy names for these "powers of ten" like, million, billion, trillion, zillion etc.. So let's introduce some nice clean mathematical notation to describe these powers of ten and let's forget the fancy words.

$$\begin{aligned} 100 &= 10 \times 10 = 10^2, \\ 1000 &= 10 \times 10 \times 10 = 10^3, \\ 10000 &= 10 \times 10 \times 10 \times 10 = 10^4, \\ &\dots \\ 10\cdots000 &= 10 \times 10 \times 10 \times \cdots \times 10 = 10^k \end{aligned}$$

The last line, $10^k = 10 \times 10 \times \cdots \times 10$, means there are k 10's multiplied together - also written as a 1 followed by k zeros. The above list explicitly shows the cases for $k = 2, 3$ and 4. Using the k like that is just a way to show that we can pick ANY nonnegative number, i.e., there is no limit on how big k can be.

The notation of 10^k is very handy, in fact it extends to the case when $k = 0$ and $k = 1$. So 10^1 means that there is only one 10 multiplied together, in other words just the number 10. That $k = 1$ also tells us how many zero's follow the "1", so in this case there is one zero following the "1" - again just 10.

How about when $k = 0$. Examining the pattern of how the power k relates to how many zeros follow the "1" (eg, 10, 100, 1000, etc.) it makes sense that $10^0 = 1$, which is exactly correct. Actually any number raised to the 0^{th} power is 1, but we'll leave that discussion for another time.

Let's look at an example. Reading the number 46307 out according to our technique we can see that it's "four sets of ten-thousand, plus six sets of a thousand, plus three sets of a hundred, plus zero sets of ten, plus seven units":

$$\begin{array}{rcl} 4 \times 10000 & & 40000 \\ + 6 \times 1000 & & + 6000 \\ + 3 \times 100 & = & + 300 \\ + 0 \times 10 & & + 00 \\ + 7 \times 1 & & + 7 \\ \hline & & = 46307 \end{array}$$

Written in terms of powers of ten: $46307 = 4 \times 10^4 + 6 \times 10^3 + 3 \times 10^2 + 0 \times 10^1 + 7 \times 10^0$.

You can think of each digit as being a little dial that controls how much of a specific power of ten will contribute to the value of the integer. Given that we can use as high as power of ten as we like and we can string together as LONG A LIST of digits as pleases us, that means that we can create ANY INTEGER WE WANT no matter how big it is.

That's a pretty tall claim. How do we know that we can create ALL the integers with this scheme?

We're going to jump into the deep end and make our claim in a careful "mathy" kind of way. Such a careful statement is called a theorem - stay tuned - theorems require proof, which we're going to supply!

Theorem: Unique Representation of Nonnegative Integers in Base-Ten

Suppose $n, k \in \mathbb{Z}^+$ Then every n can be *uniquely* expressed as follows:

$$n = d_k \times 10^k + d_{k-1} \times 10^{k-1} + \dots + d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0$$

such that $0 \leq d_i \leq 9$ for each $d_i, i \in \mathbb{Z}^+, 0 \leq i \leq k$.

Furthermore $d_k \neq 0$ except when $n = 0$.

and we represent n thusly: $d_k d_{k-1} \dots d_2 d_1 d_0$ and call the d_i 's the "digits" of n .

A difficulty many folks have with math is the notation - it's kind of a language unto itself - like a computer program is a language. Let's take our theorem statement by statement and turn it into English.

i) "Suppose $n, k \in \mathbb{Z}^+$ "

This means we are going to talk about two distinct numbers that we are labelling n and k . That strange looking \in means "is an element of" (or "is a member of") and is always followed by something that is a "set". We talked above about the symbol \mathbb{Z}^+ which we defined as being the set of nonnegative integers. So, in other words, n can be one of 0 or 1 or 2 or 3 or ... any number - no matter how large - and the same goes for k .

ii) "Then every n can be uniquely expressed as follows"

We mean that what we are about to say applies to ALL nonnegative integers and furthermore the statement is going to be *unique* for each integer.

iii) " $n = d_k \times 10^k + d_{k-1} \times 10^{k-1} + \dots + d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0$ "

This is the expression in question. It equates n with a series of multiplications of some numbers times descending powers of 10, and adds them all together. Some important clarification of what those " d " symbols mean follows in the next statement and is critical to the theorem.

iv) "such that $0 \leq d_i \leq 9$ for each $d_i, i \in \mathbb{Z}^+, 0 \leq i \leq k$ "

This means that those d 's are nonnegative integers, and specifically can ONLY take on the values 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. It also is very fastidiously pointing out that the little ' i ' we just introduced in the subscript of the d 's is also an integer and can be as small as zero but only as large as our k - whatever it might be. This is very picky stuff -

like a computer program spelling things out very precisely so the computer knows exactly what you mean. Think of the subscript i 's as indices into our array called " d ".

v) "Furthermore $d_k \neq 0$..."

This is spelling out one more important (fastidious!) detail. We want to make sure that the "most significant d ", that is, our d_k that goes along with the highest power 10^k is **not** 0, in other words it must be one of 1, 2, 3, 4, 5, 6, 7, 8 or 9. This is necessary so that we can get our uniqueness property, otherwise we could say $13 = 013 = 0000013$ which are all the integer 13, so let's outlaw this uninteresting and annoying possibility.

vi) "...except when $n = 0$ "

completing that last statement which allows for *one exception* to the case where the "most significant digit" could be zero, and that's exactly when the integer n in question IS zero.

vii) "and we represent n thusly: $d_k d_{k-1} \dots d_2 d_1 d_0$ and call the d_i 's the "digits" of n ."

This is introducing what it means to write the number out in base-ten; that is, we toss out all the extraneous stuff from our expression in (iii) above, and string all the "digits" one after another, from most significant digit d_k on the left down to least significant digit d_0 on the right.

Before we prove our theorem, consider that base-ten is not the only base in use these days. Since the introduction of the EDVAC computer, around 1950, there have been many orders of magnitude more calculations done in base-two (otherwise known as binary) by computers than have EVER been done by people in base-ten for the entirety of human history.

These binary-computer logic gates (the building blocks of the modern computer) can only take one of two states, that is; "off" or "on". We interpret these two states to represent these two numerals: "0" and "1". By doing so, in the same way that base-ten uses ten numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9; we can represent integers in base-two with just "0" and "1". How is this possible? ... first let's take a trip into deep space.

Consider distant Planet-Nova on which the emergent intelligent species only have nine fingers. They have three hands with three fingers each - anyway, they use base-nine, so they only use the numerals 0, 1, 2, 3, 4, 5, 6, 7 and 8. So like we Earthlings did for the integer ten, instead of making up a new numeral for nine, they use "10" to represent the integer nine - which for them means "One set of nine, plus zero units".

Similarly on Planet-Ocho, since they only have eight fingers, then they use base-eight and only use numerals 0, 1, 2, 3, 4, 5, 6 and 7. For them "10" means "1 set of eight, plus zero units".

Finally we come upon Planet-Claire, where the poor bastards only have two fingers so they only use the numerals 0 and 1 and base-two, so for them "10" means "one set of two and zero units". So on Planet-Claire 10 means two. Recall above how we arrived at our 100 in base-ten, being "ten sets of ten, plus zero units" - similarly on Planet-Claire 100 in base-two for them means "Two sets of two

plus zero units” in other words four! What is 11 in base-two? Using our technique to describe the digits we see that it’s “One set of two, plus one unit”, in other words three.

Here’s how they count on Planet-Claire using base-two:

base-two	base-ten	base-two	base-ten
0	0	(...cont)	
1	1	1001	9
+ 1		1010	10
= 10	2	1011	11
+ 1		1100	12
= 11	3	1101	13
+ 1		1110	14
= 100	4	1111	15
+ 1		10000	16
= 101	5	10001	17
+ 1		...	
= 110	6	11111	31
+ 1		100000	32
= 111	7	...	
+ 1		1000000	64
= 1000	8	10000000	128
(cont...)		100000000	256 (...etc.)

Note something interesting in the list above - the powers of two, written in base-two, resemble our powers of 10 in base-ten! That is: $2^0(1) = 1$, $2^1(2) = 10$, $2^2(4) = 100$, $2^3(8) = 1000$, $2^4(16) = 10000$, $2^5(32) = 100000$, $2^6(64) = 1000000$, $2^7(128) = 10000000$, $2^8(256) = 100000000$, etc.

Let’s look at the binary number 11010 for example. Using our wordy technique to describe the number we can see that it’s “One set of sixteen, plus one set of eight, plus zero sets of four, plus one set of two, plus zero units”:

$$\begin{array}{rcl}
 1 \times 10000 & & 10000 \quad (16) \\
 + 1 \times 1000 & & + 1000 \quad (8) \\
 + 0 \times 100 & = & + 000 \\
 + 1 \times 10 & & + 10 \quad (2) \\
 + 0 \times 1 & & + 0 \\
 \hline
 & = & 11010 \quad (26)
 \end{array}$$

Written in terms of powers of two: $11010 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$,

Each digit in base-two can be thought of as a little switch that turns on or off the contribution of it’s corresponding power of two. Given that we can use as high a power of two as we like, and that we can string together as LONG A LIST of binary-digits as pleases us, that means that we can create ANY INTEGER WE WANT no matter how big it is.

Sound familiar? Let's restate our theorem for base-ten but rewritten for base-two.

Theorem: Unique Representation of Integers in Base-Two

Suppose, $n, k \in \mathbb{Z}^+$. Then every n can be uniquely expressed as follows:

$$n = d_k \times 2^k + d_{k-1} \times 2^{k-1} + \dots + d_2 \times 2^2 + d_1 \times 2^1 + d_0 \times 2^0$$

such that $0 \leq d_i \leq 1$ for each $d_i, i \in \mathbb{Z}^+, 0 \leq i \leq k$.

Furthermore $d_k \neq 0$ except when $n = 0$.

and we represent n thusly: $d_k d_{k-1} \dots d_2 d_1 d_0$ and call the d_i 's the "binary-digits" of n .

Before we go on, I want to introduce a little notation to help avoid confusion. How do you know what I'm talking about if I just write "1000"? Do I mean 10^3 or 2^3 ? If there is any possibility for confusion we write the number like this $(1000)_{10}$ for the base-ten version of 1000, and $(1000)_2$ to mean the binary version. That goes for any other base we might care to delve into like $(1000)_{16}$ which would mean 16^3 or $(4096)_{10}$.

We can use ANY positive integer greater than 1 for a base. In fact computer graphics artists are known to stumble upon numbers written in hexadecimal, which is base 16 (usually relating to specifying a color-channel). Base 16 introduces some new symbols to the usual numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9, to represent the base-ten integers 10, 11, 12, 13, 14 and 15, namely: A, B, C, D, E and F. So if you see this number $(80FB)_{16}$ then I bet you can figure out that it's $(33019)_{10}$. If we omit the parentheses and subscript from our number, it means we're talking about a number in base-ten - our "default" base. Case in point... the subscripts that we use to denote the base (like the "16" in $(80FB)_{16}$) ARE written in base-ten!

We still need to prove our two theorems above. What about proving the "base 9" version of the theorem for the aliens on Planet-Nova, or the "base 8" version for the inhabitants of Planet-Ocho? In fact, since we can use ANY integer greater than or equal to 2 as a base, then let's restate our theorem for the general case, call it "base b ", where b is any number, two or higher. If we can prove *that* theorem, then we'll automatically get *all* the individual cases of specific bases essentially for free.

Theorem: Unique Representation of Integers in an Arbitrary Base

Suppose, $n, k, b \in \mathbb{Z}^+$ s.t. $b \geq 2$. Then every integer n can be uniquely expressed as follows:

$$n = d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0$$

such that $0 \leq d_i \leq (b-1)$ for each $d_i, i \in \mathbb{Z}^+, 0 \leq i \leq k$.

Furthermore $d_k \neq 0$ except when $n = 0$.

and we represent n thusly: $(d_k d_{k-1} \dots d_2 d_1 d_0)_b$ and call the d_i 's the "digits" of n .

Definition:

d_k is called the "most significant digit of n " and

d_0 is called the "least significant digit of n ".

There are a some subtle changes in the way this theorem is stated, compared to the previous two theorems. The first and most important is that we substituted " b " for " 10 " and " 2 " above. Secondly, we did away with the " \times " symbol for multiplication. It's a notational convention that when two "terms" are written next to each other (a "term" being a number, constant, or a variable, e.g., 7 or 13 , or n, k or b) that multiplication is the *implied* operation (E.g., cy means $c \times y$ and $13b$ means $13 \times b$. Naturally, this notational convenience doesn't work for "terms" that are only numbers like 73 which of course *doesn't* mean 7×3). Also, we introduced another common abbreviation in math-speak - the "s.t." which is just a short form for "such that".

We are going to make use of a powerful 2-step technique to do our proof - namely "induction":

1. We begin by showing that the theorem is true for the "base case", when $n = 1$;
2. We then we *assume* that the proposed theorem is true for some arbitrary number n , and using *this* fact (as well as other established properties of numbers, axioms, etc.) we show that the proposed theorem is also true for the *next* larger number (i.e., $n + 1$).

If we can do those two things, then we've proven it FOR ALL numbers n .

Why would this approach actually prove the theorem for *all* numbers? Suppose $P(n)$ - means "some property of n ". If we establish the following statement as true: "If $P(n)$ is true, then $P(n+1)$ is true" and we also show that $P(1)$ is true - that implies that $P(2)$ is true, which implies that $P(3)$ is true, which in turn leads to the truth of $P(4)$, etc., over and over so that we eventually reach ANY number.

Some people picture an infinite row of dominoes. Step 1 is like being able to knock over the first domino. Step 2 is like the fact that any one domino has the ability to knock over the next. Once you've knocked over the first domino, they all fall!

Ok we're ready to finally jump in. In order to prove our theorem - first we will prove that there is such a representation for all integers n (existence). Meaning that every integer has a way of being written in the form described by the theorem - especially as relates to the restrictions on the values that the "digits" can take on. After that has been established we will use another technique to prove

that each such representation is unique - in other words there aren't two (or more) ways to represent the same integer in our base.

Existence Proof (by induction on n):

Base case:

For clarity, let's prove both $n = 0$ and $n = 1$. (This is like showing that we can knock over both the 1st and the 2nd dominoes.)

Suppose $n = 0$. We can choose $k = 0$ and $d_0 = 0$. (This is the one exception spelled out in the theorem in which the most significant digit of n is allowed to be zero.) Then,

$$n = d_0 b^0 = 0 \times b^0 = 0,$$

showing that we have a valid representation for 0 in base b since our only digit $d_0 = 0 \leq (b - 1)$, $\forall b \geq 2$. (The " \forall " symbol is shorthand for "for all".)

Now suppose $n = 1$. In this case, we can choose $k = 0$ and $d_0 = 1$. Then,

$$n = d_0 b^0 = 1 \times b^0 = 1 \times 1 = 1,$$

showing that we have a valid representation for 1 in base b since $d_0 = 1 \leq (b - 1)$, $\forall b \geq 2$.

$n+1$ st step:

Assume that n has a valid representation in base b . That is, we assume that n can be written thusly:

$$n = d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0$$

with all the appropriate conditions holding for the values of d_i and b and k ; and we will prove that $n + 1$ also has a valid representation in base b .

We're going to break this step into two cases which between the two covers all possibilities.

Case 1) $0 \leq d_0 \leq (b - 2)$

This case looks at the situation where the *least significant digit* of n is strictly-less-than the LARGEST value it can take in base b .

For example, in base-two, d_0 can only be 0. In base-five, d_0 can be at most 3. In base-ten, d_0 can be at most 8, etc.

So we can express $n + 1$ as follows:

$$\begin{aligned}
n + 1 &= (d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0) + 1 \\
&= d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0 + 1 \times 1 \\
&= d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0 + 1 \times b^0 \\
&= d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + (d_0 + 1) b^0
\end{aligned}$$

and since $d_0 \leq (b - 2)$ (our assumption for “case 1”), then

$$(d_0 + 1) \leq (b - 2) + 1 \leq (b - 1),$$

therefore the “least significant digit” (i.e. $(d_0 + 1)$ which corresponds the lowest power b^0) of $n + 1$ is less than or equal to $(b - 1)$, which means it is a valid digit in base b .

Since all other d_i terms (i.e. d_1, d_2, \dots, d_k) for $n + 1$ are unchanged from their representations for n and n is assumed to be a valid representation in base b then **all** the digits of $n + 1$ are valid in base b .

Therefore we’ve established the truth of “Case 1” for the integer $n + 1$.

Case 2) $d_0 = (b - 1)$

Now we’ll look at the case when the least significant digit of n is equal to the largest value it can take in base b , that is, $d_0 = (b - 1)$. (Note that between “Case 2” here and “Case 1” above, we’re covering *all* the possibilities for what d_0 can be.)

For example in base-two, $d_0 = 1$; in base 5, $d_0 = 4$; in base-ten $d_0 = 9$, etc.

Let’s temporarily introduce a new digit, $d_{k+1} = 0$ to our representation of n . The motivation being that in the situation where we’re adding 1 to 999, that we need another “digit” to “carry” the 1 over into so that we can get 1000.

$$\begin{aligned}
n &= d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0 \\
&= 0 + d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0 \\
&= 0 \cdot b^{k+1} + d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0 \\
&= d_{k+1} b^{k+1} + d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0
\end{aligned}$$

Also, let $j \in \mathbb{Z}^+$ be the lowest power of b such that $d_j < (b - 1)$, $(k + 1) \geq j \geq 1$, meaning we write n as follows for some j .

$$n = d_{k+1} b^{k+1} + \dots + d_j b^j + (b - 1) b^{j-1} + \dots + (b - 1) b^1 + (b - 1) b^0$$

For example, if $n = 69412999$, then $j = 3$, since 10^3 is the lowest power of 10 such that its digit $d_3 = 2$ is less than 9.

Let's get picky for a minute and think carefully about some boundary conditions, to make sure that our new expression for n is always possible to write this way.

What about when $k = 0$ meaning that n is only one digit long? Since we specified that $(k + 1) \geq j \geq 1$ or substituting zero for k then $(0 + 1) = 1 \geq j \geq 1$, which means that j must be 1. Note that we tacked on the extra digit d_{k+1} above, and by substituting $k = 0$ we see that $d_{k+1} = d_{0+1} = d_1 = 0$. Recall that we are in "Case 2" of our proof, so we know that $d_0 = (b - 1)$ so it's true that b^1 is the lowest power of b such that $d_1 = 0 < (b - 1)$, $\forall b \geq 2$. So we're OK here!

What about the case when ALL the digits of n , namely d_0, d_1, \dots, d_k are equal to $(b - 1)$? (For example $n = 999999$) Then $j = (k + 1)$, and $d_j = d_{k+1} = 0 < (b - 1)$.

Good! We're covered in terms of our new expression for n being valid for all possible values of j i.e.; $1 \leq j \leq (k + 1)$. So let's do some algebraic manipulations on n .

$$\begin{aligned}
 n &= d_{k+1}b^{k+1} + \dots + d_jb^j + (b - 1)b^{j-1} + \dots + (b - 1)b^1 + (b - 1)b^0 \\
 &= d_{k+1}b^{k+1} + \dots + d_jb^j + (b - 1)(b^{j-1} + \dots + b^1 + b^0) \\
 &= d_{k+1}b^{k+1} + \dots + d_jb^j + b(b^{j-1} + \dots + b^1 + b^0) - (b^{j-1} + \dots + b^1 + b^0) \\
 &= d_{k+1}b^{k+1} + \dots + d_jb^j + (bb^{j-1} + \dots + bb^1 + bb^0) - (b^{j-1} + \dots + b^1 + b^0) \\
 &= d_{k+1}b^{k+1} + \dots + d_jb^j + (b^j + \dots + b^2 + b^1) - b^{j-1} - \dots - b^1 - b^0 \\
 &= d_{k+1}b^{k+1} + \dots + d_jb^j + b^j + (b^{j-1} - b^{j-1}) + \dots + (b^2 - b^2) + (b^1 - b^1) - b^0 \\
 &= d_{k+1}b^{k+1} + \dots + (d_j + 1)b^j - b^0 \\
 &= d_{k+1}b^{k+1} + \dots + (d_j + 1)b^j - 1
 \end{aligned}$$

Therefore

$$\begin{aligned}
 n + 1 &= d_{k+1}b^{k+1} + \dots + (d_j + 1)b^j - 1 + 1 \\
 &= d_{k+1}b^{k+1} + \dots + (d_j + 1)b^j
 \end{aligned}$$

Since we picked j such that $d_j < (b - 1)$, less restate the inequality as $d_j \leq (b - 2)$ therefore,

$$(d_j + 1) \leq (b - 2) + 1 \leq (b - 1),$$

meaning the digit $(d_j + 1)$ for the base b representation of $n + 1$ is valid in base b .

All digits d_{j+1} and higher remain unchanged from what they were for the base b representation of n , and all digits d_{j-1} and lower are 0. Therefore all the digits of the base b representation of $n + 1$ are valid in base b .

To cleanly button this up we note that if we were dealing with the case that $j = k + 1$, (eg. n was something like 99999) then we can see that $n + 1 = b^{k+1}$ because:

$$\begin{aligned} n + 1 &= (d_j + 1)b^j \\ &= (d_{k+1} + 1)b^{k+1} \\ &= (0 + 1)b^{k+1} \\ &= b^{k+1} \end{aligned}$$

Meaning that $n + 1$ has a $(k+1)^{\text{st}}$ digit and it's equal to 1, with all the rest of the digits following being 0.

In all other cases (i.e.; $j < (k + 1)$) we discard the extra “temporary” digit $d_{k+1} = 0$ that we created from our expression, and are left with an expression for $n + 1$ that valid for base b .

QED - existence proof

Uniqueness Proof for n :

Let's assume that our base b representation for n is not unique, Showing that this assumption leads to a logical contradiction, then the only possibility remains is that our assumption was wrong, thereby proving uniqueness.

Suppose n is not unique and that,

$$n = d_k b^k + d_{k-1} b^{k-1} + \dots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

also suppose,

$$n = c_k b^k + c_{k-1} b^{k-1} + \dots + c_2 b^2 + c_1 b^1 + c_0 b^0$$

Let's further suppose that the index j is the first power such that the digits $d_j \neq c_j$ and without any loss of generality, let's assume that $d_j > c_j$.

Therefore