Magic Numbers

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In the delightful film "School of Rock" Jack Black's character Dewey Finn, pretending to be substitute teacher Ned Schneebly, is put on the spot when Miss. Mullins, the school principle, comes into class and demands that he show her his teaching methods which apparently involve an electric guitar. So "Mr. S" sings "The Math Song" - in the final line of his improvised tune he sings "...yes it's 9, and that's a magic number..."

He's right - he's referring to the fact that if you add up the digits of any number then if that sum is divisible by 9, then the original number must be divisible by 9. It works the other way too, that is, if we know that a number is divisible by 9 then the sum of its digits must also be divisible by 9.

For example:

$$9^5 = 59049 \text{ and } 5 + 9 + 0 + 4 + 9 = 27 \text{ and } 9 \text{ divides } 27 \text{ because } 27 = 3 \times 9.$$

9 is a magic number in base-ten but if we used another base, say 8, then 7 is a magic number. 4 is a magic number in base-5; 15 is a magic number in base-16; and 30 is a magic number in base-31. The pattern being that one-less-than the base, is a magic number.

Let's look at a couple of examples*.

- $7 \times 7 = 49$ written in base-8 is $(61)_8$ and $6 + 1 = 7$.
- $7^3 \times 5 = 1715$ written in base-8 is $(3263)_8$ and $3 + 2 + 6 + 3 = 14$ (7 divides 14).
- $4 \times 67 = 268$ written in base-5 is $(2033)_5$ and $2 + 0 + 3 + 3 = 8$ (4 divides 8).
- $15 \times 23 = 345$ written in base-16 is $(159)_{16}$ and $1 + 5 + 9 = 15$.
- $30 \times 29 \times 113 = 98310$ written in base-31 is $(3999)_{31}$ and $3 + 9 + 9 + 9 = 30$

But what about this example?

$$3 \times 67 = 201 \text{ and } 2 + 0 + 1 = 3$$

Clearly 9 does not divide 201† but 3 does divide 201. It's also the case that adding up the digits of 201 gives us 3, which is trivially divisible by three. So, we can also call 3 a magic number in base 10. That is; the sum of the digits of any number is divisible by 3 if and only if that original number is divisible by 3.

*We are going to use the conventions for specifying numbers in alternate bases as outlined on page 7 of the paper entitled "Sesame Street++" by James Rowell.

†Why is it immediately apparent that 9 does not divide 201? Hint: both 3 and 67 are prime numbers. Google the "Fundamental Theorem of Arithmetic".

Let's look at another few examples, in base-31.

- $2 \times 13 \times 37 = 962$ written in base-31 is $(101)_{31}$ and $1 + 0 + 1 = 2$.

2, 3, 5, 6, 10, 15 and 30 are all magic numbers in base-31.

See the pattern? It looks like any divisors of the biggest numeral in our base are magic numbers.

What about 1? Isn't 1 a magic number in all bases? Since we already know that all numbers are divisible by 1, then we don't need to go to any trouble of adding up digits to find this out. So let's exclude 1 from being considered as a magic number. (Akin to 1 not being a prime number).

Here's a precise statement defining a magic number m for any base-N and declaring what its properties are:

"Magic Number" Theorem:

Let m,N ? Z+and let m be a divisor of N-1.

Then any integer represented in base-N is divisible by m if and only if the sum of its digits (in base-N) is divisible by m. Let's define all such m?1 as a 'magic number'.

As an interesting side note - according to our definition, binary, or base-2, doesn't have a magic number. No great loss, as we said above 1 isn't helpful to consider as a magic number.

Let's look at the definition of a number represented in a given base.

...wip... break down what a number is (refer to baseTheorem paper) and explain motivation for mod

We assume the following properties of modular arithmetic where a,b,c ? Zsuch that c¿1.

(a+b) mod c=(a mod c+b mod c) mod c ab mod c=((a mod c)·(b mod c)) mod c

As a generalization of 2), we get:

3) a mod c=b mod c akmod c=bk mod c

Importantly, a mod c=0 c divides a.

In other words, there exists a number n such that nc=a. Lemma: if d¿1 is a divisor of m then: (m+1)k mod d=1 Proof of Lemma: Let d be a divisor of m. Soc such that cd=m. Hence, (cd + 1) mod d = cd mod d + 1 mod d = (c mod d) (d mod d) + 1 mod d = (c mod d) 0 + 1 mod d = 1 mod d By property 3) above, since (cd+1) mod d=1 mod d then: (m + 1)k mod d = (cd + 1)k mod d = (cd+1) mod d = 1 QED

Proof of "Magic Number" Theorem:

Let N=M+1.

That is; we are going to consider representations of numbers in base-M+1, so think of M as our magic number.

Also let M=1, so our smallest possible base will be 2. Then every integer n is uniquely expressible in base-M+1 as follows: n=ak(M+1)k+ak-1(M+1)k-1+?+a2(M+1)2+a1(M+1)+a0 where 0=ai=M for all the "digits" ai where 0=i=k. n is divisible by M if and only if n mod M=0. Substitute the base-M+1 expression of n into the above relationship, therefor 0=(ak(M+1)k+ak-1(M+1)k-1+?+a2(M+1)2+a1(M+1)+a0)mod M

TEST

Let $n, k \in \mathbb{Z}_{\geq 0}$. Then every n can be uniquely expressed as follows:

$$n = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_2 10^2 + d_1 10^1 + d_0 10^0$$

for some k such that $0 \leq d_i \leq 9$ where $d_i, i \in \mathbb{Z}$ and $0 \leq i \leq k$.

Furthermore $d_k \neq 0$ except when $n = 0$.

Definition: n is represented in base-ten as $d_k d_{k-1} \ldots d_2 d_1 d_0$