Factoradic Representation of Rational Numbers

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From 'A Course of Pure Mathematics' by G. H. Hardy. Chapter 1, Miscellaneous Examples.

Miscellaneous example[*] #2 at the end of chapter 1 in Hardy's 'Pure Mathematics' presents us with a fascinating result. The theorem feels like what the 'basis-representation-theorem' is for integers, but for rational numbers, ... beautiful!

Factorial Representation Theorem[†]

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \le a_1, \quad 0 \le a_2 < 2, \quad 0 \le a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Observations that lead to the proof.

We know that any rational number[‡], say $\frac{m}{q}$, can be written as an integer part, i, PLUS a fractional part, $\frac{p}{q}$, such that $\frac{m}{q} = i + \frac{p}{q}$, where $0 \le \frac{p}{q} < 1$ (note that i can be zero).

So if we're trying to represent any positive rational number $\frac{m}{q}$ in terms of the theorem then the a_1 term wants to play the role of the integer part, i, and the remainder of the expression $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ looks to be playing the roll of the rational part, $\frac{p}{q}$, where,

$$0 \le \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} < 1$$

It seemed to me a good idea to forget about the a_1 term and just focus on the $a_2, a_3, \ldots, a_k$ terms. In other words, prove the theorem for rational numbers $\frac{p}{q}$, where $0 \le \frac{p}{q} < 1$, then it should be trivial to extend it to ALL rational numbers by tacking the a_1 term back on at the end of the proof. Also, it started to become clear that including zero (that is, not JUST positive rational numbers) was going to simplify the task[§].

[*]Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, math exercises for the student.

[†]...it's not named in the text, so I named the theorem.

[‡]Every variable, coefficient or constant (eg. a_1, a_k, m, n, i, p, q) in this paper is going to represent a non-negative integer. We aren't dealing with 'real numbers' here, just rational numbers which we will always discuss in terms of one integer divided by another integer, like $\frac{p}{q}$.

[§]Did you notice how the theorem restricts the last term, a_k, to be strictly greater than zero, unlike all the other terms? We loosen up that restriction allow a_k to be equal to zero so all the terms are treated the same. At the very end of the proof it's trivial to reintroduce that restriction on the a_k term.

At first glance it wasn't remotely obvious to me how I'd go about finding such an assignment of coefficients $a_2, a_3, \ldots, a_k$ for a given rational number $\frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$, let alone that it would be unique.

After playing around for a while, and finally figuring out a way to calculate the a_i terms for a given rational number $\frac{p}{q}$, a few thing started to jump out at me. For example, look at these rational numbers,

$$\frac{1}{2} = \frac{1}{1 \cdot 2} = \frac{2! - 1}{2!}$$
$$\frac{5}{6} = \frac{1}{1 \cdot 2} + \frac{2}{1 \cdot 2 \cdot 3} = \frac{1 \cdot 3}{1 \cdot 2 \cdot 3} + \frac{2}{1 \cdot 2 \cdot 3} = \frac{3+2}{6} = \frac{3! - 1}{3!}$$
$$\frac{23}{24} = \frac{1}{1 \cdot 2} + \frac{2}{1 \cdot 2 \cdot 3} + \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{1 \cdot 3 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{2 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{12+8+3}{24} = \frac{4! - 1}{4!}$$

Good clue! It seemed to be the case that if we assign the largest possible values to the coefficients, from a_2 up to say a_k we get this rational number, $\frac{k!-1}{k!}$. This number is as close to 1 as you can get with a denominator of $k!$ without actually hitting 1. (What happens if you add $\frac{1}{k!}$ to $\frac{k!-1}{k!}$?) This turned out to be a pretty useful observation, and it became my 'Lemma-1' in the proof below.

Also, if we assign zeros to all the coefficients then naturally we get $\frac{0}{k!}$, plus it's pretty simple to figure out how to make the smallest non-zero such rational number $\frac{1}{k!}$. Then thinking about continually adding $\frac{1}{k!}$ to the result gives us an idea about how the a_i terms change as you keep incrementing by $\frac{1}{k!}$.

So we we restrict ourselves to using only $a_2, a_3, \ldots, a_k$, then can generate the smallest rational number ($\frac{0}{k!}$) and the largest ($\frac{k!-1}{k!}$) where $0 \leq \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} < 1$.

One further observation to help understand the motivation behind this proof is that by using combinatorics we can count how many possible combinations of a_i's there are. So, we have two choices for the a_2 term (0, 1), combined with three choices for the a_3 term (0, 1, 2), combined with four choices for the a_4 term (0, 1, 2, 3), ... combined with k choices for the a_k term (0, 1, 2, ..., $k-1$), that gives us $2 \cdot 3 \cdot 4 \cdot \cdot \cdot k = k!$ possibly different sums.

Hmmmmm, the following set has $k!$ members, $\{\frac{0}{k!}, \frac{1}{k!}, \frac{2}{k!}, \ldots, \frac{k!-1}{k!}\}$. So that, plus one or two other thoughts is what lead me to the proof below. I won't spoil the rest of it, to find out, go ahead and read the rest of the paper!

Lemma-1

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}, \quad \text{for integer } k \geq 2$$

Proof of Lemma-1

This equality is fairly trivial to demonstrate by induction, since $\frac{1}{2!} = \frac{2!-1}{2!}$ and,

$$\begin{aligned}
& \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\
&= \frac{(k-1)!-1}{(k-1)!} + \frac{k-1}{k!} \\
&= \frac{k((k-1)!-1)}{k(k-1)!} + \frac{k-1}{k!} \\
&= \frac{k!-k+k-1}{k!} \\
&= \frac{k!-1}{k!}
\end{aligned}$$

…thus establishing lemma-1 for all values of $k \geq 2$. QED

Lemma-2

For integers i, k where $2 \leq i < k$,

$$\frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!}$$

Proof of Lemma-2

$$\begin{aligned}
& \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \\
&= (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!}) - (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!}) \\
&= \frac{k!-1}{k!} - \frac{i!-1}{i!} && \text{(by lemma-1)} \\
&= \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\
&= \frac{1}{i!} - \frac{1}{k!} \\
&< \frac{1}{i!}
\end{aligned}$$

QED

Definitions

For integer $k \geq 2$, and integers $a_2, a_3, \ldots, a_k$,

$$\mathcal{S}_k = \{\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \quad | \quad 0 \leq a_2 < 2, \; 0 \leq a_3 < 3, \; \ldots, \; 0 \leq a_k < k\},$$
$$\mathcal{F}_k = \{\frac{0}{k!}, \; \frac{1}{k!}, \; \frac{2}{k!}, \; \ldots, \; \frac{k!-1}{k!}\}$$

Lemma-3

$$\mathcal{S}_k = \mathcal{F}_k$$

Corollary: For every rational number $\frac{p}{k!}$, where $k \geq 2$ and $0 \leq \frac{p}{k!} < 1$, then there is a unique sequence of integers $a_2, a_3, \ldots a_k$ such that,

$$\frac{p}{k!} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!},$$

where $0 \leq a_2 < 2, \; 0 \leq a_3 < 3, \; \ldots, \; 0 \leq a_k < k$.

Proof of Lemma-3

To show that the set $\mathcal{S}_k$ is the same as $\mathcal{F}_k$, it suffices to show that if $\frac{a}{b} \in \mathcal{S}_k$ then $0 \leq \frac{a}{b} < 1$ and $\frac{a}{b} = \frac{p}{k!}$ for some p, and that the size of $\mathcal{S}_k$ is the same as $\mathcal{F}_k$.

It's clear that the set $\mathcal{F}_k$ contains every rational number with denominator $k!$ where p is an integer and $0 \leq \frac{p}{k!} < 1$ and that the size of $\mathcal{F}_k$ is $k!$.

The smallest member of the set $\mathcal{S}_k$ is $\frac{0}{k!}$ and occurs when all the coefficients of the sum are zero. Furthermore, the largest member of the set occurs when all the coefficients of the sum are set to their maximum value, which gives us $\frac{k!-1}{k!}$ as shown in lemma-1.

We also note that every member of $\mathcal{S}_k$ can be written as a rational number with $k!$ as the denominator, like so,

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_{k-1}}{(k-1)!} + \frac{a_k}{k!} = \frac{k \cdot (k-1) \cdot \cdot \cdot 3 \cdot a_2}{k!} + \frac{k \cdot (k-1) \cdot \cdot \cdot 4 \cdot a_2}{k!} + \ldots + \frac{k \cdot a_{k-1}}{k!} + \frac{a_k}{k!}$$

Therefore any member of the set $\mathcal{S}_k$ can be written as $\frac{p}{k!}$ for some integer p, where

$$0 = \frac{0}{k!} \leq \frac{p}{k!} \leq \frac{k!-1}{k!} < \frac{k!}{k!} = 1$$

Furthermore, each possible assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ produce a unique member of the set $\mathcal{S}_k$.

For if this weren't true and both $\frac{p}{k!} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ and $\frac{p}{k!} = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$ for different coefficients $a_2, a_3, \ldots, a_k$ and $b_2, b_3, \ldots, b_k$, then we can arrive at a contradiction as follows.

First suppose that $a_i \neq b_i$ is the first such pair of coefficients that differ from each other. In other words, $a_2 = b_2, a_3 = b_3, \ldots, a_{i-1} = b_{i-1}$. Also, without loss of generality we can assume that $a_i > b_i$, and because of the equality of the two different representations for $\frac{p}{q}$ we can write,

$$\begin{aligned} \frac{a_i}{i!} + \frac{a_{i+1}}{(i+1)!} + \ldots + \frac{a_k}{k!} &= \frac{b_i}{i!} + \frac{b_{i+1}}{(i+1)!} + \ldots + \frac{b_k}{k!} \\ \Leftrightarrow \qquad \frac{a_i - b_i}{i!} &= \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!} \end{aligned}$$

Since $a_i - b_i \geq 1$, then $\frac{a_i - b_i}{i!} \geq \frac{1}{i!}$.

Also, $\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \geq \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!}$ regardless of the values of the coefficients on the right side of the inequality*.

Because lemma-2 tells us that $\frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!}$ then we can string all our inequalities together like so,

$$\frac{a_i - b_i}{i!} \geq \frac{1}{i!} > \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \geq \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!},$$

But we had also deduced that $\frac{a_i - b_i}{i!} = \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!}$ which contradicts the strict inequality above.

Therefore our assumption that there can be a second set of coefficients to produce the same rational number $\frac{p}{k!}$ must be false. Therefore any assignment of values to the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ must produce a unique member of the set $\mathcal{S}_k$.

Now we can count the number of members of $\mathcal{S}_k$, by looking at all the possible combinations of values for the coefficients $a_2, a_3, \ldots, a_k$. There are 2 choices for the coefficient a_2, combined with 3 choices for a_3, combined with 4 choices for a_4, ..., combined with k choices for a_k.

Therefore the total number of combinations of values that can be assigned to all the coefficients of $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is $2 \cdot 3 \cdot 4 \cdots k = k!$, and since each set of assignments creates a unique member of the set, then the size of $\mathcal{S}_k$ is $k!$, which is the same size as $\mathcal{F}_k$. Since any member of the set $\mathcal{S}_k$, say $\frac{a}{b}$, can be written as $\frac{a}{b} = \frac{p}{k!}$, for some p where $0 \leq \frac{p}{k!} < 1$ then $\mathcal{S}_k = \mathcal{F}_k$.

Furthermore the corollary is a direct result of the fact that there is a one-to-one mapping between the two sets. QED

Lemma-4

Any non-negative rational number that is less than one, can be expressed in one and only one way in the form

$$\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$$

where $a_2, \ldots, a_k$ are integers, and

$$0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 \leq a_k < k$$

*Letting all the b's be their maximum value, and all the a's be zero will produce the largest numerators in each term of the sum, any other possibility will result in a smaller term for the sum.

Proof Lemma-4

If $\frac{p}{q}$ is a rational number where $0 \leq \frac{p}{q} < 1$ and $2 \leq q \leq k$ *, then $\frac{p}{q} \in \mathcal{S}_k$ because,

$$\begin{aligned}
\frac{p}{q} &= \frac{2 \cdot 3 \cdot \cdot \cdot (q-1) \cdot (q+1) \cdot \cdot \cdot k}{2 \cdot 3 \cdot \cdot \cdot (q-1) \cdot (q+1) \cdot \cdot \cdot k} \cdot \frac{p}{q} \\
&= \frac{2 \cdot 3 \cdot \cdot \cdot (q-1) \cdot p \cdot (q+1) \cdot \cdot \cdot k}{2 \cdot 3 \cdot \cdot \cdot (q-1) \cdot q \cdot (q+1) \cdot \cdot \cdot k} \\
&= \frac{2 \cdot 3 \cdot \cdot \cdot (q-1) \cdot p \cdot (q+1) \cdot \cdot \cdot k}{k!}
\end{aligned}$$

Therefore $\frac{p}{q}$ is an element of $\mathcal{F}_k$ which is the same as the set $\mathcal{S}_k$. In other words, all non-negative rational numbers $\frac{p}{q}$ with denominators q from 2 up to k where $0 \leq \frac{p}{q} < 1$ are uniquely represented by the expression,

$$\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!},$$

where $a_2, a_3, \ldots, a_k$ are integers, and $0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 \leq a_k < k$

If we fix q and let k go to infinity then it's clear that we can make a set $\mathcal{S}_k$ that contains all rational numbers between zero and one. Since $\frac{p}{q}$ is uniquely represented by our expression within $\mathcal{S}_k$ then it is unique for all rational numbers between zero and one.

QED

Alternate observation: Another way to see this is to look at what happens to the expression for $\frac{p}{q} \in \mathcal{S}_k$ when we let $\mathcal{S}_k$ increase in size.

$$\begin{aligned}
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} \\
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} + \frac{0}{(k+1)!} + \ldots + \frac{0}{n!} \\
&\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} + \frac{a_{k+1}}{(k+1)!} + \ldots + \frac{a_n}{n!} \\
&0 \leq a_2 < 2,\ \ 0 \leq a_3 < 3, \ldots, 0 \leq a_k < k, a_{k+1} = 0, \ldots, a_n = 0.
\end{aligned}$$

Therefore, $\frac{p}{q} \in \mathcal{S}_n$ for all $n \geq k$.

*The list of rational numbers being $\frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{0}{5}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots, \frac{0}{k}, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-2}{k}, \frac{k-1}{k}$.

Factorial Representation Theorem (restated)

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Proof of Theorem

Thanks to Euclid we know that for all integers $j \geq 0$ and $q > 0$, there exist unique integers i and p such that,

$$\begin{aligned} & j = i \cdot q + p\ ; \quad 0 \leq p < q \\ \Leftrightarrow \quad & \frac{j}{q} = i + \frac{p}{q}\ ; \quad 0 \leq \frac{p}{q} < 1 \end{aligned}$$

Which tells us that all rational numbers $\frac{j}{q}$ can be written as an integer part, i, plus a fractional part $\frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$.

In our theorem, the a_1 coefficient plays the role of the integer part i, and the rest of the expression, $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ plays the role of the fractional part $\frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$.

Therefore to express any rational number in the form of the theorem, first apply the Euclidean Division Theorem to $\frac{j}{q}$ and let $a_1 = i$. If there is no fractional remainder, then the theorem is trivially true, however if there is a fractional remainder $\frac{p}{q}$, then by lemma-4 we know that the sum $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with $\frac{p}{q}$ so clearly $\frac{j}{q} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with all rational numbers $\frac{j}{q}$.

To meet the precise requirements of the theorem, we throw away that fact that we can represent zero, and suppose that if $\frac{p}{q} \in \mathcal{S}_n$ for some $n \geq q$ that we take for the coefficients $a_2, a_3, \ldots, a_k$ in the sum for $\frac{p}{q} \in \mathcal{S}_n$ all those for which $a_k \neq 0$ but $a_{k+1} = a_{k+2} = \ldots = a_n = 0$.

QED

Additional Observations

While it's true that $\frac{p}{q} \in \mathcal{S}_q$, $\mathcal{S}_q$ is not necessarily the smallest such set for which $\frac{p}{q}$ is a member.

For example, the smallest set containing $\frac{p}{5}$, where $0 \leq \frac{p}{5} < 1$, is $\mathcal{S}_5$ however the smallest set containing $\frac{p}{6}$, where $0 \leq \frac{p}{6} < 1$ is $\mathcal{S}_3$.

Which is easy to see when we list the contents of a couple of sets,

$$\begin{aligned}
\mathcal{S}_4 &= \{\frac{0}{24}, \frac{1}{24}, \frac{2}{24}, \frac{3}{24}, \frac{4}{24}, \frac{5}{24}, \frac{6}{24}, \frac{7}{24}, \frac{8}{24}, \frac{9}{24}, \frac{10}{24}, \frac{11}{24}, \frac{12}{24}, \frac{13}{24}, \frac{14}{24}, \frac{15}{24}, \frac{16}{24}, \frac{17}{24}, \frac{18}{24}, \frac{19}{24}, \frac{20}{24}, \frac{21}{24}, \frac{22}{24}, \frac{23}{24}\} \\
&= \{\frac{0}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4}, \frac{7}{24}, \frac{1}{3}, \frac{3}{8}, \frac{5}{12}, \frac{11}{24}, \frac{1}{2}, \frac{13}{24}, \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{17}{24}, \frac{3}{4}, \frac{19}{24}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{23}{24}\}
\end{aligned}$$

Which clearly doesn't contain $\frac{1}{5}$. We've established that $\frac{1}{5}$ is definitely in $\mathcal{S}_5$ but it's interesting to see what it looks like:

$$\frac{1}{5} = \frac{0}{2} + \frac{1}{2 \cdot 3} + \frac{0}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{6} + \frac{1}{30} = \frac{5+1}{30} = \frac{6}{30} = \frac{1}{5}$$

Also, $\mathcal{S}_3 = \{\frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\} = \{\frac{0}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$, which demonstrates the claim above that $\mathcal{S}_3$ contains $\frac{p}{6}$, where $0 \leq \frac{p}{6} < 1$.

I believe that for a given $q \geq 2$ then the smallest set for which the rational number $\frac{p}{q} \in \mathcal{S}_k$, is to pick k such that it is the smallest value for which q divides $k!$.

However, I'll leave that proof for another day.