Rational Approximations to $\sqrt{2}$

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In Chapter 1, Examples* III, no. 2, from G. H. Hardy's 'A Course of Pure Mathematics', Hardy has us examine a sequence of rational approximations to $\sqrt{2}$, which are:

$$\frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70}$$

These numbers squared are:

$$\frac{1}{1},\ \frac{9}{4},\ \frac{49}{25},\ \frac{289}{144},\ \frac{1681}{841},\ \frac{9801}{4900}$$

And we discover that their differences from 2 are:

$$-1,\ \frac{1}{4},\ -\frac{1}{25},\ \frac{1}{144},\ -\frac{1}{841},\ \frac{1}{4900}$$

In the next 'Example no. 3' we are asked to prove a theorem relating to the above sequence, noting it is only an *example* of an application of the theorem below.

Theorem

Show that if $\frac{m}{n}$ is a good approximation to $\sqrt{2}$, then $\frac{(m+2n)}{(m+n)}$ is a better one, and that the errors in the two cases are in opposite directions.

The proof follows.

Lemma

Let m, n and a be integers, where m and n are positive and $a \neq 0$, then

$$\frac{m^2}{n^2} + \frac{a}{n^2} = 2 \qquad \Leftrightarrow \qquad \frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} = 2$$

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*Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, exercises for the student.

Proof of Lemma

$$
\begin{aligned}
\frac{(m+2n)^2}{(m+n)^2} - \frac{a}{(m+n)^2} &= 2 \\
\Leftrightarrow \quad (m+2n)^2 - a &= 2(m+n)^2 \\
\Leftrightarrow \quad m^2 + 4mn + 4n^2 &= 2m^2 + 4mn + 2n^2 + a \\
\Leftrightarrow \quad 4n^2 - 2n^2 &= 2m^2 - m^2 + a \\
\Leftrightarrow \quad 2n^2 &= m^2 + a \\
\Leftrightarrow \quad 2 &= \frac{m^2}{n^2} + \frac{a}{n^2}
\end{aligned}
$$

QED

Proof of Theorem

Define the 'error' of a rational approximation to $\sqrt{2}$ as the difference between the approximation-squared and 2. So we learn from the Lemma that the error for the $\frac{m}{n}$ approximation of $\sqrt{2}$ is

$$
\left| \tfrac{a}{n^2} \right| = \left| 2 - \tfrac{m^2}{n^2} \right|,
$$

and the error for the $\frac{(m+2n)}{(m+n)}$ approximation of $\sqrt{2}$ is

$$
\left| \tfrac{-a}{(m+n)^2} \right| = \left| 2 - \tfrac{(m+2n)^2}{(m+n)^2} \right|.
$$

The denominator $(m+n)^2$ is always larger than n^2 therefore,

$$
\left| \frac{-a}{(m+n)^2} \right| < \left| \frac{a}{n^2} \right|,
$$

showing that the approximation $\frac{(m+2n)}{(m+n)}$ for $\sqrt{2}$ has a smaller error than that of $\frac{m}{n}$ and is therefore a better approximation to $\sqrt{2}$.

Furthermore, we also learn from the Lemma that the successive approximations flip to either side of 2 because:

If $\frac{m^2}{n^2} < 2$ then a is positive therefore $\frac{(m+2n)^2}{(m+n)^2} > 2$, similarly,

If $\frac{m^2}{n^2} > 2$ then a is negative therefore $\frac{(m+2n)^2}{(m+n)^2} < 2$.

QED