Magic Numbers

James Rowell

February 15, 2017

In the delightful film "School of Rock" Jack Black's character Dewey Finn, pretending to be substitute teacher Ned Schneebly, is put on the spot when Miss. Mullins, the school principle, comes into class and demands that he show her his teaching methods which apparently involve an electric guitar. So "Mr. S" sings "The Math Song" - in the final line of his improvised tune he sings "...yes it's 9, and that's a magic-number..."

I believe Jack Black was making a clever tip-of-the-hat to the "Schoolhouse Rock" song "3 Is A Magic Number" which most kids from the 70's/80's are familiar with, however referring to 9 as a magic number not common or well known*.

Never-the-less, there is precedent for using the term "Magic Number" when referring to to the number 9. There are a host of math-magic-tricks that you can play on folks which rely on properties of the number 9. There is also a process called "casting out nines" and a related one called calculating the "digital root" for a number where you take its "digital sum". 9 plays a special role in all these processes - so within these contexts sometimes 9 is referred to as a "magic number".

So let's say for the sake of argument that Mr. S. was referring to this fact about 9; which is that if you add up the digits of any number then if that sum is divisible by 9, then the original number must be divisible by 9. It works the other way too, that is, if we know that a number is divisible by 9 then the sum of its digits must also be divisible by 9.

For example:

$$9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59049 \text{ and } 5+9+0+4+9 = 27 \text{ and 9 divides 27 because } 27 = 3 \times 9.$$

But 3 is also has this magic-number property, for example:

$$3 \times 67 = 201 \text{ and } 2+0+1 = 3$$

In this case, 9 does not divide 201† but 3 does (67 times). We can see that adding up the digits of 201 gives us 3, which of course is divisible by three but not by 9.

Anytime I want to check to see if a number is divisible by 3 (but don't feel like busting out a calculator) I use this little trick of checking to see if the digits add up to some multiple of 3. Along with the tricks to see if the number is divisible by 2 (last digit is even) or is divisible by 5 (last digit is 0 or 5) - you can decide if you can factor out 2, 3 or 5 from any number pretty quickly. 7 takes real work ... but I digress.

*According to my own highly scientific informal poll conducted on my Facebook page 9 is NOT the first thing that pops into people's heads when they hear the phrase "Magic Number" - in fact no one answered 9.

†Why is it immediately apparent that 9 does not divide 201? Hint: both 3 and 67 are prime numbers. Google the "Fundamental Theorem of Arithmetic" for further insight.

Let's make the term "Magic Number" official! How about if we call ANY number that behaves like 3 and 9 (as described above) a "Magic Number". After all both "Schoolhouse Rock" AND "The School of Rock" sanction the usage - so we're on to something big here.

Too bad 3 and 9 are the only magic-numbers. Our nifty new official definition might not have any "legs". But wait, we're only considering base-ten (or decimal), what about other bases?

By our definition: 7 is a magic-number in base-8! 4 is a magic-number in base-5; 15 is a magic-number in base-16; ... and 30 is a magic-number in base-31.

Here are some examples*:

- $7 \times 7 = 49$ written in base-8 is $(61)_8$ and $6 + 1 = 7$.
- $7^3 \times 5 = 1715$ written in base-8 is $(3263)_8$ and $3 + 2 + 6 + 3 = 14$ (7 divides 14).
- $4 \times 67 = 268$ written in base-5 is $(2033)_5$ and $2 + 0 + 3 + 3 = 8$ (4 divides 8).
- $15 \times 23 = 345$ written in base-16 is $(159)_{16}$ and $1 + 5 + 9 = 15$.
- $30 \times 35951 = 1078530$ written in base-31 is $(15699)_{31}$ and $1 + 5 + 6 + 9 + 9 = 30$

You may notice a pattern: the biggest "digit" in any base (which is always one less than the base) seems to qualify as a magic-number. Using our examples, 9 is the biggest digit in base-10; 7 is the biggest digit in base-8; and 4 is the biggest digit in base-5. To write this out in a general way we would say that if b is our base, then $(b - 1)$ is a magic-number in base-b.

So what about 3 also being a magic-number in base-10? Let's take a look at the interesting (but probably never used outside this paper) base-31. It appears that 2, 3, 5, 6, 10, 15 (along with 30) are *all* magic-numbers in base-31, as we we can see with the following examples:

- $2 \times 13 \times 37 = 962$ written in base-31 is $(101)_{31}$ and $1 + 0 + 1 = 2$.
- $3 \times 641 = 1923$ written in base-31 is $(201)_{31}$ and $2 + 0 + 1 = 3$.
- $5 \times 139 \times 2659 = 1848005$ written in base-31 is $(20102)_{31}$ and $2 + 0 + 1 + 0 + 2 = 5$.
- $6 \times 10091 = 60546$ written in base-31 is $(2103)_{31}$ and $2 + 1 + 0 + 3 = 6$.
- $10 \times 197 \times 941 = 1853770$ written in base-31 is $(20701)_{31}$ and $2 + 0 + 7 + 0 + 1 = 10$.
- $15 \times 71503 = 1072545$ written in base-31 is $(15027)_{31}$ and $1 + 5 + 0 + 2 + 7 = 15$.

See the pattern? All the divisors of 30 are magic-numbers in base-31. It turns out that in base-b, then any divisor of $(b - 1)$ is a magic-number in base-b.

I guess that begs the question: What about 1? Isn't 1 a magic-number in all bases? Since we already know that all numbers are divisible by 1, then we don't need to go to any trouble of adding up digits to find this out. So let's exclude 1 from being considered as a magic-number - it's not helpful.

To make it official, we introduce a theorem defining the new term. First we need to remind ourselves what it means to write a number in a given base - we take the following Basis Representation Theorem as proven.

*We are going to use the conventions for specifying numbers in alternate bases as outlined in the paper "Sesame Street++".

Basis Representation Theorem

Let b be a positive integer greater than 1.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

where $0 \leq d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-b by the string of base-b-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$

Now our new theorem.

Magic Number Theorem

Let n be a positive integer represented in base-b.

If m is an integer such that $m \Big| (b-1)$ then,

$$m \Big| n \Leftrightarrow m \Big| (d_k + \cdots + d_2 + d_1 + d_0)$$

Definition: We call $m \neq 1$ a "magic-number" in base-b.

As an interesting side note - according to our definition, binary, or base-2, doesn't have a magic-number. No great loss, as we said above 1 isn't helpful to consider as a magic-number.

Let's look at the definition of a number represented in a given base.

...wip... break down what a number is (refer to baseTheorem paper) and explain motivation for mod

We assume the following properties of modular arithmetic where a,b,c ? Zsuch that c¿1.

(a+b) mod c=(a mod c+b mod c) mod c ab mod c=((a mod c)·(b mod c)) mod c

As a generalization of 2), we get:

3) a mod c=b mod c akmod c=bk mod c

Importantly, a mod c=0 c divides a.

In other words, there exists a number n such that nc=a. Lemma: if d¿1 is a divisor of m then: (m+1)k mod d=1 Proof of Lemma: Let d be a divisor of m. Soc such that cd=m. Hence, (cd + 1) mod d = cd mod d + 1 mod d = (c mod d) (d mod d) + 1 mod d = (c mod d) 0 + 1 mod d = 1 mod d By property 3) above, since (cd+1) mod d=1 mod d then: (m + 1)k mod d = (cd + 1)k mod d = (cd+1) mod d = 1 QED

Proof of "Magic Number" Theorem:

Let N=M+1.

That is; we are going to consider representations of numbers in base-M+1, so think of M as our magic-number.

Also let M=1, so our smallest possible base will be 2. Then every integer n is uniquely expressible in base-M+1 as follows: n=ak(M+1)k+ak-1(M+1)k-1+?+a2(M+1)2+a1(M+1)+a0 where 0=ai=M for all the "digits" ai where 0=i=k. n is divisible by M if and only if n mod M=0. Substitute the base-M+1 expression of n into the above relationship, therefor 0=(ak(M+1)k+ak-1(M+1)k-1+?+a2(M+1)2+a1(M+1)+a0)mod M

TEST

Let $n, k \in \mathbb{Z}_{\geq 0}$. Then every n can be uniquely expressed as follows:

$$n = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_2 10^2 + d_1 10^1 + d_0 10^0$$

for some k such that $0 \leq d_i \leq 9$ where $d_i, i \in \mathbb{Z}$ and $0 \leq i \leq k$.

Furthermore $d_k \neq 0$ except when $n = 0$.

Definition: n is represented in base-ten as $d_k d_{k-1} \ldots d_2 d_1 d_0$