Counting

James Rowell

July 24, 2017

There are 10 sorts of people in the world: those who understand binary and those who don't.

What does "10" mean?

We got it drilled into us watching Sesame Street that "10" is the symbol for the number "ten" which is this many apples "🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎" or the number of fingers on a typical person's two hands.

Once we are trained to automatically think of "10" as representing ten things, we quickly move past it to learn about 100 and 1000 and how to interpret a string of digits like 92507. Even at a young age we'd be able to accurately count out a pile of ninety-two-thousand-five-hundred-and-seven apples as time consuming and agonizing as it might be. Furthermore, learning how to add and multiply is easy once you can count in base-ten since the techniques are simple and straight-forward.

What about kids in ancient Rome, was it as easy for them? Try adding two numbers together in ancient Rome, or worse, multiplying or dividing them. What's XI times IX? Would you believe me if I told you it's XCIX?

Unless you convert those to Hindu-Arabic decimal or base-ten numerals to check, you're just gonna have to trust me. Truth is - I don't know how to multiply using Roman numerals - nor did most Romans. Not only that, but I'll bet that most modern eight-year-olds can count higher than any Roman could - as the Roman system only effectively allowed counting up to 4999.

Even though we use different symbols, the ancient Romans and us are talking about the same abstract set of numbers underneath, which we call integers[*]. Mathematics deals with numbers in this pure sort of way, divorced from the symbols used to represent each number. When we talk about positive integers in mathematics, it's best to remind yourself that we are really talking about the set of numbers that represent successively larger piles of apples, forgetting the symbols.

However we use numbers written out in base-ten all the time in mathematics, rarely thinking in terms of piles of apples. We take it for granted that we can use base-ten to represent the set of positive integers. *Caution*: the only thing modern mathematics takes for granted are axioms and the fact that we can use base-ten to represent the integers is NOT among the list of axioms.

Briefly; the axioms describe a few simple properites about addition and multiplication. These properites are *so simple* that they can't be expressed in yet other even-simpler ideas. The

[*]Integers are the set of all the postive whole numbers, as well as zero and all the negative counterparts to each positive number.

axioms are the minimal set of simple, obvious, irrefutable ideas from which everything else in mathematics is built*.

Since our ability to count in base-ten is not axiomatic, then to properly ground it in modern mathematics we should define what it means to write out a number in base-ten, state its properties in a theorem, then provide a proof of that theorem - The proof being a series of arguments that logically connects it directly† to the axioms. In doing so, the only way that the theorem could be false is if the axioms themselves are false.

Here's what that theorem looks like.

Basis Representation Theorem

> Let b be a positive integer greater than 1.
>
> For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:
>
> $$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$
>
> where $0 \le d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.
>
> Definition: n is represented in base-b by the string of base-b-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$

"That's nuts!" you might say, I don't even see a "ten" in there so how could that describe how we learned to count watching Sesame Street? If you let $b =$ ten in the above theorem, then we have the "Base-Ten Representation Theorem". We could let $b = 2$, then we'd have "Base-Two Representation Theorem" which states how we count in binary.

This theorem is a good example of what mathematicians like to do - they generalize ideas. Why restrict ourselves to ten when the idea applies equally well to two, three, four, five, ... etc.? The heptapods in "Arrival" have seven limbs with seven fingers each, perhaps they use base-fourty-nine. By generalizing the idea to a base b, where b is any number two or higher, we gain deeper understanding about the subject in question.

Even though doing arithmetic in base-ten has been going on for almost two-thousand years, formalizing it and generalizing it into a theorem is fairly recent. The earliest reference I've found to our theorem is in "Elementary Number Theory" by E. Landau in 1958. We probably don't need to look further back than Leibniz time when he introduced the idea of binary arithmetic in 1679. So our theorem is fairly recent on the world stage.

Anyway, if you're not a "math-person" don't fear, by the end of this paper you will know how to read the "Basis Representation Theorem" above so that it makes sense plus you'll know how to prove it.

Let's take a big leap back and work up to the statement of that theorem step by step, using our familiar base-ten for discussion.

---

*The axioms: For every integer a, b and c: Associativity: $(a+b)+c = a+(b+c)$ and $a(bc) = (ab)c$; Commutativity: $a+b = b+a$ and $ab = ba$; Distributive: $a(b+c) = (b+c)a = ab+ac$; Identities: There are integers 0 and 1 such that, $a+0 = 0+a = a$ and $a \cdot 1 = 1 \cdot a = a$ and Additive Inverse: $a + (-a) = 0$. Note: in general integers do NOT have multiplicative inverses that are also integers. (eg. $\frac{1}{2}$ is the multiplicative inverse of 2 because $\frac{1}{2} \cdot 2 = 1$ but $\frac{1}{2}$ is not an integer.)

†directly ... or indirectly via other previously proven theorems.

When using some arbitrary base-b to count with, it's useful to have simple symbols to represent each of the integers from zero to up to $(b-1)$. So in our base-ten system we use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Base-ten strings a series of these digits together one after the other to be able to represent each positive integer. Let's look at the first two-digit number, that is, ten, which as you well know looks like this: "10". This extra digit on the left tells us how many tens we have and the last, or rightmost digit says how many additional units to add to it.

So our very first two-digit number 10 means "one lot of ten - plus zero units". When we see "11" - we interpret it to mean "one lot of ten - plus one unit", and "12" is "one lot of ten - plus two units", etc.

Continuing on; "20" - we interpret to mean "two lots of ten, plus zero units", etc. up to "90" meaning "nine lots of ten, plus zero units".

Following this line of reasoning since "10" now means the integer ten, then "100" must mean "ten lots of ten, plus zero units"- which is exactly what it means. We have a special word for this number we call it "one hundred" or "one lot of a hundred, plus zero lots of tens, plus zero units". Similarly "200" means "two lots of a hundred, plus zero lots of ten, plus zero units", etc.

We can keep going by one-hundred until we similarly get to "1000" or "ten lots of a hundred, plus zero lots of ten, plus zero units" otherwise known as "a thousand" or more specifically "one lot of a thousand, plus zero lots of a hundred, plus zero lots of ten, plus zero units".

It gets a little tedious to be so specific when reading out a number so our language has developed quite a few verbal shortcuts. Furthermore it doesn't take long before we run out of fancy names for these "powers-of-ten" like, million, billion, trillion, zillion etc. So let's introduce some nice clean mathematical notation to describe these powers-of-ten and let's forget the fancy words.

$$\begin{aligned}
100 &= 10 \times 10 = 10^2, \\
1000 &= 10 \times 10 \times 10 = 10^3, \\
10000 &= 10 \times 10 \times 10 \times 10 = 10^4, \\
&\dots \\
\underbrace{10\dots000}_{\text{k zeros}} &= \underbrace{10 \times 10 \times 10 \times 10 \times \dots \times 10}_{\text{k 10s}} = 10^k
\end{aligned}$$

10^k means there are k tens multiplied together - also written as a 1 followed by k zeros*. The above list explicitly shows the cases for $k = 2, 3$ and 4. Using the k like that is just a way to show that we can pick ANY whole number, i.e., there is no limit on how big k can be.

The notation of 10^k is handy and extends to the case when $k = 0$ and $k = 1$.

So 10^1 means† that there is only one ten multiplied together, or one "0" following the "1", in other words just the number ten itself.

---

*k is called the "exponent" and you should read the symbol 10^k as "ten-raised-to-the-k^{th}-power" or "ten-to-the-k", so 10^2 is "ten raised to the second power" or 10^4 is "ten-to-the-fourth". You may also see 10^2 referred to as "ten squared", similarly 10^3 as "ten cubed" - but since we don't live in 4 dimensional hyperspace, we don't have a way of saying 10^4 that has geometric meaning.

†Don't forget to read 10^1 as "ten-to-the-one".

How about when $k = 0$? Examining the pattern of how the power k relates to how many zeros follow the "1" (eg, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc.) then it must be the case that $10^0 = 1$, i.e., no zeros follow the "1", which is exactly right. Furthermore every number raised to the 0^{th} power is 1.[*]

Let's look at an example. Reading the number 92507 out according to our technique we can see that it's "nine lots of ten-thousand, plus two lots of a thousand, plus five lots of a hundred, plus zero lots of ten, plus seven units":

$$
\begin{array}{rcrr} & 9 & \times & 10000 \\ + & 2 & \times & 1000 \\ + & 5 & \times & 100 \\ + & 0 & \times & 10 \\ + & 7 & \times & 1 \end{array}
\quad = \quad
\begin{array}{cr} & 90000 \\ + & 2000 \\ + & 500 \\ + & 00 \\ + & 7 \\ \hline = & 92507 \end{array}
$$

Written[†] in terms of powers-of-ten: $92507 = 9{\times}10^4 + 2{\times}10^3 + 5{\times}10^2 + 0{\times}10^1 + 7{\times}10^0$.

This way of breaking down the base-ten representation of a number into an algebraic expression can be done for EVERY string of decimal digits. It's the key to understanding what a string of decimal digits means.

Recalling that $10^0 = 1$ you might wonder why we bother to multiply $7{\times}10^0 = 7{\times}1 = 7$ since there is no actual effect when multiplying by one. Even though it's not necessary, including the 10^0 in the expression reveals a kind of mathematical symmetry. Each successive digit is multiplied by an ever decreasing power-of-ten, including the units digit, which is just some number from 0 to 9 times a power-of-ten like any of the other digits.

Our example number 92507 only has five digits and it's biggest power-of-ten is 10^4, but there's no limit on how big a power-of-ten could be involved in our expression. Look at "a trillion and one", i.e.; 1,000,000,000,001 which can be expressed as:

$$1{\times}10^{12} + 0{\times}10^{11} + 0{\times}10^{10} + \cdots + 0{\times}10^2 + 0{\times}10^1 + 1{\times}10^0$$

Or pushing that limit to silly heights we can also describe this next ludicrous number. It's twenty-thousand-and-one digits long, a "7" followed by 9999 zeros, then a "3" followed by 9999 more zeros, then a "5", which means this:

$$7{\times}10^{20000} + 0{\times}10^{19999} + \cdots + 0{\times}10^{10001} + 3{\times}10^{10000} + 0{\times}10^{9999} + \cdots + 0{\times}10^1 + 5{\times}10^0$$

Clearly we can keep going as high as we like.

[*] Proof: Since $a^{b+c} = a^b a^c$ consider when $c = 0$; that is, $a^b = a^{b+0} = a^b a^0$ so because of the uniqueness of the multiplicative identity "1", then a^0 *must* be 1 since it's behaving like a "1" in the expression $a^b = a^b a^0$.

[†] Recall the mnemonic "bedmas" for the "Order of Operations" in evaluating an expression, which is no different from what we did in our table above the expression.