Factoradic Representation of Rational Numbers

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From 'A Course in Pure Mathematics' by G. H. Hardy. Chapter 1, Miscellaneous Examples.

Miscellaneous example* #2 at the end of chapter 1 in Hardy's 'Pure Mathematics' presents us with a fascinating result. I had never seen it before, but upon seeing it felt like I was looking at a kind of basis-representation-theorem but for rational numbers, ... beautiful!

Here it is, followed by my proof starting with the lemma.

Theorem

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \le a_1, \quad 0 \le a_2 < 2, \quad 0 \le a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

Lemma

The set of rational numbers,

$$\mathcal{S} = \{ \tfrac{a_2}{2!} + \tfrac{a_3}{3!} + \ldots + \tfrac{a_k}{k!} \;\; | \;\; 0 \le a_2 < 2, \; 0 \le a_3 < 3, \; \ldots, \; 0 \le a_k < k \},$$

is identical to the set of rational numbers,

$$\mathcal{F} = \{ \tfrac{0}{k!}, \; \tfrac{1}{k!}, \; \tfrac{2}{k!}, \; \ldots, \; \tfrac{k!-1}{k!} \}$$

*Hardy doesn't call them 'Exercises' or 'Questions', but that's what they are, math exercises like calculations to perform, theorems to prove etc.

Proof of Lemma

It's useful to establish that,

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}$$

Which is fairly trivial to demonstrate by induction, since $\frac{1}{2!} = \frac{2!-1}{2!}$ and,

$$\begin{aligned}
&\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} \\
&= \frac{(k-1)!-1}{(k-1)!} + \frac{k-1}{k!} \\
&= \frac{k((k-1)!-1)}{k(k-1)!} + \frac{k-1}{k!} \\
&= \frac{k!-k+k-1}{k!} \\
&= \frac{k!-1}{k!}
\end{aligned}$$

Now we make note of the fact that the smallest member of the set $\mathcal{S}$ occurs when all the coefficients of the sum are zero, i.e.; $\frac{0}{k!}$. Furthermore, the largest member of the set occurs when all the coefficients are set to their maximum value, which we have just seen gives us $\frac{k!-1}{k!}$.

We also note that all the members of $\mathcal{S}$ can be written as a rational number with $k!$ as the denominator, like so:

$$\tfrac{a_2}{2!} + \tfrac{a_3}{3!} + \ldots + \tfrac{a_k}{k!} = \tfrac{k \cdot (k-1) \cdots 3 \cdot a_2}{k!} + \tfrac{k \cdot (k-1) \cdots 4 \cdot a_2}{k!} + \ldots + \tfrac{a_k}{k!}$$

Also when $i < k$ such that,

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!} + \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!}$$

Then we can conclude that,

$$\frac{1}{i!} - \frac{1}{k!} = \frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!}$$

Because,

$$\begin{aligned}
&\frac{i}{(i+1)!} + \ldots + \frac{k-1}{k!} \\
&= (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!}) - (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!}) \\
&= \frac{k!-1}{k!} - \frac{i!-1}{i!} \\
&= \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\
&= \frac{1}{i!} - \frac{1}{k!}
\end{aligned}$$

From here we can deduce that any assignment of values to the coefficients of a member of $\mathcal{S}$ produces a unique member of the set, for if it didn't and $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is not uniquely determined by the coefficients $a_2, a_3, \ldots, a_k$. That is, suppose there is a second DIFFERENT sequence of coefficients $b_2, b_3, \ldots, b_k$ such that $\frac{p}{q} = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$.

The number of values that the coefficient a_2 can assume is 2, a_3 can take on 3 values, $\ldots$, up to a_k which can take on k values. So the total number of combinations of values that can be assigned to all the coefficients is $2 \cdot 3 \cdot 4 \cdot \ldots \cdot k = k!$.

The size of $\mathcal{F}$ is clearly $k!$