# Factorial Basis Representation of Rational Numbers

James Philip Rowell

December 3, 2018 (v60)

Miscellaneous example-2* at the end of chapter 1 in G. H. Hardy's 'A Course of Pure Mathematics' presents us with, and asks us to prove, a fascinating result. The theorem feels like what the 'basis-representation-theorem' is for integers, but for rational numbers ... beautiful!

## Factorial Representation Theorem†

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \leq a_1, \quad 0 \leq a_2 < 2, \quad 0 \leq a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

## Observations that led me to the proof.

Any positive rational number‡, say $\frac{m}{q}$, can be written as an integer part, $i$, plus a fractional part, $\frac{p}{q}$, such that $\frac{m}{q} = i + \frac{p}{q}$, where $0 \leq \frac{p}{q} < 1$.

So trying to represent any positive rational number $\frac{m}{q}$ in the form of the theorem, the integer $a_1$ wants to play the role of the integer part, $i$, and the remainder of the expression $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ looks to be playing the role of the rational part, $\frac{p}{q}$, where,

$$0 \leq \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!} < 1$$

It seemed a good idea to forget about the integer $a_1$ and just focus on the integers $a_2, a_3, \ldots, a_k$. In other words, first prove a restricted form of the theorem for rational numbers between zero and one, then later prove the full theorem by re-introducing the $a_1$ to get all positive rational numbers. Furthermore, including zero (that is, not just *positive* rational numbers) was also going to make things easier with this approach.

At first, it wasn't remotely obvious how I'd go about calculating the values of the integers $a_2, a_3, \ldots, a_k$ for a given number $\frac{p}{q}$. However, after playing around for a while I figured it out; it's kinda like doing long-division. At this point a few patterns started to jump out at me. For example, look at these sums using the maximum-values for each $a_i$,

---

*'Examples' is the term Hardy uses for 'Exercises' or 'Questions' in his textbook.

†The theorem is not named in the textbook, so I named it.

‡Almost every variable, or constant (eg. $a_1, a_k, m, n, i, p, q$) in this paper is going to represent a non-negative integer, or *very* occasionally a rational-number. However, we will almost always describe a rational-number in terms of one integer divided by another integer, like $\frac{p}{q}$.

$$\frac{1}{1\cdot 2} = \frac{1}{2} = \frac{2!-1}{2!}$$

$$\frac{1}{1\cdot 2} + \frac{2}{1\cdot 2\cdot 3} = \frac{1\cdot 3}{1\cdot 2\cdot 3} + \frac{2}{1\cdot 2\cdot 3} = \frac{3+2}{6} = \frac{5}{6} = \frac{3!-1}{3!}$$

$$\frac{1}{1\cdot 2} + \frac{2}{1\cdot 2\cdot 3} + \frac{3}{1\cdot 2\cdot 3\cdot 4} = \frac{1\cdot 3\cdot 4}{1\cdot 2\cdot 3\cdot 4} + \frac{2\cdot 4}{1\cdot 2\cdot 3\cdot 4} + \frac{3}{1\cdot 2\cdot 3\cdot 4} = \frac{12+8+3}{24} = \frac{23}{24} = \frac{4!-1}{4!}$$

An obvious pattern emerged! By assigning the largest possible values to the variables, the sum is $\frac{k!-1}{k!}$, which is as close to 1 as possible without actually hitting 1. (What is $\frac{1}{k!} + \frac{k!-1}{k!}$?) This turned out to be a pretty useful observation, and it became 'Lemma 1' in what follows.

On the other end of the scale, assigning zeros to all the variables gives us a sum of zero. Furthermore I could make the smallest *positive* number, $\frac{1}{k!}$, by letting all the variables be zero *except* for $a_k = 1$. Then I considered what happens to the variables by adding $\frac{1}{k!}$ to it, repeatedly, over and over.

So by making various assignments of values to the variables $a_2, a_3, \ldots, a_k$ I could generate the smallest numbers, $\frac{0}{k!}, \frac{1}{k!}, \frac{2}{k!}$, etc. as well as the largest, $\frac{k!-1}{k!}$. Then I realized that I could count the number of possible sums that might be formed with the expression.

There are two choices for the $a_2$ variable (0 and 1), combined with three choices for the $a_3$ variable (0, 1 and 2), combined with four choices for the $a_4$ variable (0, 1, 2, 3), ... combined with $k$ choices for the $a_k$ variable (0, 1, 2, ..., $k-1$), which gives us $2 \cdot 3 \cdot 4 \cdots k = k!$ possibly different sums.

Hmmm, the following set has $k!$ members, $\{\frac{0}{k!}, \ \frac{1}{k!}, \ \frac{2}{k!}, \ \ldots, \ \frac{k!-1}{k!}\}$. By forming another set out of all possible sums from the expression it seemed like it would be fairly easy to show that the two sets are identical.

Another insight was that in order to be able to represent a fraction with a prime-number denominator (say $p$) then the sum in our expression would *have to contain* this term, $\frac{a_p}{p!}$, with a non-zero value of $a_p$.

This led me to the realization that any fraction $\frac{m}{k}$ should be able to be represented by the expression if the terms went so far as to include $\frac{a_k}{k!}$. For that matter by using terms up to $\frac{a_k}{k!}$, then I'd also be able to represent all fractions with $(k-1)$ as the denominator or $(k-2)$ as the denominator, or ... 4, 3, or 2 as the denominator. In other words, $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots, \frac{1}{k-1}, \frac{2}{k-1}, \frac{3}{k-1}, \frac{4}{k-1}, \ldots, \frac{k-3}{k-1}, \frac{k-2}{k-1}, \frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \frac{4}{k}, \ldots, \frac{k-2}{k}, \frac{k-1}{k}$ should be representable with the expression $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$.

So by using a set to collect all the numbers generated by the expression $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$, then letting $k$ grow without bound I should get a set that contains *all* the rational numbers between zero and one!

Those ideas are behind the proof that follows.

# Lemma 1

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!} = \frac{k!-1}{k!}, \quad \text{for all integers } k \geq 2$$

### Proof

By induction: When $k = 2$ it's clear that $\frac{1}{2!} = \frac{2!-1}{2!}$.

So assuming the lemma is true for $k = n$, we show that it must be true for $k = n + 1$:

$$\begin{aligned}
\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n-1}{n!} + \frac{n}{(n+1)!} &= \frac{n!-1}{n!} + \frac{n}{(n+1)!} \\
&= \frac{(n+1)(n!-1)}{(n+1) \cdot n!} + \frac{n}{(n+1)!} \\
&= \frac{(n+1)! - (n+1) + n}{(n+1)!} \\
&= \frac{(n+1)! - 1}{(n+1)!}
\end{aligned}$$

...thus establishing Lemma 1 for all values of $k \geq 2$. QED.

The following lemma captures an idea that is perhaps most easily grasped by analogy to the basis-representation-theorem for integers. For base-ten numbers we can say,

$$1 \cdot 10^k > 9 \cdot 10^{k-1} + 9 \cdot 10^{k-2} + \ldots + 9 \cdot 10^2 + 9 \cdot 10^1 + 9 \cdot 10^0$$

The above inequality is merely stating that any single power of ten is bigger than nine (the largest 'digit') times the sum of every smaller power of ten. For example, 1000 is bigger than 999. Read the statement of the inequality in Lemma 2 with this idea in mind.

# Lemma 2

For integers $i, k$ where $2 \leq i < k$,

$$\frac{1}{i!} > \frac{i}{(i+1)!} + \frac{i+1}{(i+2)!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!}$$

### Proof

$$\begin{aligned}
\frac{i}{(i+1)!} + \frac{i+1}{(i+2)!} + \ldots + \frac{k-2}{(k-1)!} + \frac{k-1}{k!} &= (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{k-1}{k!}) - (\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{i-1}{i!}) \\
&= \frac{k!-1}{k!} - \frac{i!-1}{i!} \qquad \text{(by Lemma 1)} \\
&= \frac{k!}{k!} - \frac{1}{k!} - \frac{i!}{i!} + \frac{1}{i!} \\
&= \frac{1}{i!} - \frac{1}{k!} \\
&< \frac{1}{i!} \qquad \text{QED.}
\end{aligned}$$

## Definitions

For integer $k \geq 2$, and integers $a_2, a_3, \ldots, a_k$, we define the following sets,

$$\mathcal{S}_k = \{\ s\ \mid\ s = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!},\ \text{for all } 0 \leq a_2 < 2,\ \ 0 \leq a_3 < 3,\ \ \ldots,\ \ 0 \leq a_k < k\},$$
$$\mathcal{F}_k = \{\frac{0}{k!},\ \ \frac{1}{k!},\ \ \frac{2}{k!},\ \ \ldots,\ \ \frac{k!-1}{k!}\}$$

Naming $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$:

> Let $s \in \mathcal{S}_k$ such that $s = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$, for some $a_2, a_3, \ldots, a_k$, where $k \geq 2$, and $0 \leq a_2 < 2,\ 0 \leq a_3 < 3,\ \ldots,\ 0 \leq a_k < k$. The expression $\frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is called "The Factorial-Representation of $s$".

## Lemma 3

$$\mathcal{S}_k = \mathcal{F}_k$$

### Proof

Two sets are equal if and only if they have the same elements*.

We're going to establish the equality of $\mathcal{S}_k$ and $\mathcal{F}_k$ by showing that some key properties of the two sets are the same; then by the pigeon-hole principle each element from one set must be equal to exactly one element from the other set Note that we will not describe how to equate one specific element to the other, just that it must be true that there is a one-to-one equality between such pairs.

To begin with, it's clear that the set $\mathcal{F}_k$ contains *every* rational number of the form $\frac{p}{k!}$ where $p$ is an integer and $0 \leq \frac{p}{k!} < 1$ and that the size of $\mathcal{F}_k$ is $k!$ (i.e.; $|\mathcal{F}_k| = k!$).

That takes care of everything we need to know about $\mathcal{F}_k$; now on to show that $\mathcal{S}_k$ has these same qualities.

The smallest member of the set $\mathcal{S}_k$ is $\frac{0}{k!}$ and occurs when all the variables are set to zero. Furthermore, the largest member of the set occurs when all the variables are set to their maximum value, which sums to $\frac{k!-1}{k!}$ as shown in Lemma 1.

We also note that every member $s \in \mathcal{S}_k$ can be written as a rational number with $k!$ as the denominator, like so,

$$s = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_{k-1}}{(k-1)!} + \frac{a_k}{k!} = \frac{k \cdot (k-1) \cdot \cdot \cdot 3 \cdot a_2}{k!} + \frac{k \cdot (k-1) \cdot \cdot \cdot 4 \cdot a_2}{k!} + \ldots + \frac{k \cdot a_{k-1}}{k!} + \frac{a_k}{k!}$$

*If that isn't clear: $(\mathcal{X} = \mathcal{Y}) \Leftrightarrow (\forall a; a \in \mathcal{X} \Leftrightarrow a \in \mathcal{Y})$.

Therefore any member $s \in \mathcal{S}_k$ can be written as $s = \frac{p}{k!}$ for some integer $p$, where

$$0 = \frac{0}{k!} \leq \frac{p}{k!} \leq \frac{k!-1}{k!} < \frac{k!}{k!} = 1, \quad \text{hence,} \quad 0 \leq \frac{p}{k!} < 1 \quad \text{i.e.;} \quad 0 \leq s < 1.$$

We now show that each possible assignment of values to the variables of the factorial-representation of $s \in \mathcal{S}_k$ produces a *unique* member of the set $\mathcal{S}_k$.

For if this weren't true and both $s = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ and $s = \frac{b_2}{2!} + \frac{b_3}{3!} + \ldots + \frac{b_k}{k!}$ for different variables $a_2, a_3, \ldots, a_k$ and $b_2, b_3, \ldots, b_k$, then we can arrive at a contradiction as follows.

Suppose $a_i \neq b_i$, where $i \leq k$, is the first such pair of variables that differ. In other words, $a_2 = b_2,\ a_3 = b_3, \ldots,\ a_{i-1} = b_{i-1},\ a_i \neq b_i$. Without loss of generality, further suppose that $a_i > b_i$. It follows that,

$$\begin{aligned} & \frac{a_i}{i!} + \frac{a_{i+1}}{(i+1)!} + \frac{a_{i+2}}{(i+2)!} + \ldots + \frac{a_k}{k!} = \frac{b_i}{i!} + \frac{b_{i+1}}{(i+1)!} + \frac{b_{i+2}}{(i+2)!} + \ldots + \frac{b_k}{k!} \\ \Leftrightarrow \quad & \frac{a_i - b_i}{i!} = \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \frac{b_{i+2} - a_{i+2}}{(i+2)!} + \ldots + \frac{b_k - a_k}{k!} \end{aligned} \tag{1}$$

But $a_i - b_i \geq 1$, so

$$\tfrac{a_i - b_i}{i!} \geq \tfrac{1}{i!}.$$

Furthermore, let's examine one of the terms on the right-side of (1), say the first one $\frac{b_{i+1} - a_{i+1}}{(i+1)!}$. We can see that since $0 \leq b_{i+1} \leq i$ and $0 \leq a_{i+1} \leq i$, so $(i - 0) \geq (b_{i+1} - a_{i+1})$, therefore,

$$\tfrac{i-0}{(i+1)!} \geq \tfrac{b_{i+1} - a_{i+1}}{(i+1)!}.$$

Similarly by extending this idea to the other terms,

$$\tfrac{i}{(i+1)!} + \tfrac{i+1}{(i+2)!} + \ldots + \tfrac{k-1}{k!} \geq \tfrac{b_{i+1} - a_{i+1}}{(i+1)!} + \tfrac{b_{i+2} - a_{i+2}}{(i+2)!} + \ldots + \tfrac{b_k - a_k}{k!}.$$

Lastly, Lemma 2 tells us that $\frac{1}{i!} > \frac{i}{(i+1)!} + \frac{i+1}{(i+2)!} + \ldots + \frac{k-1}{k!}$, so we can string all our inequalities together as follows,

$$\tfrac{a_i - b_i}{i!} \geq \tfrac{1}{i!} > \tfrac{i}{(i+1)!} + \ldots + \tfrac{k-1}{k!} \geq \tfrac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \tfrac{b_k - a_k}{k!},$$

and hence,

$$\tfrac{a_i - b_i}{i!} > \tfrac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \tfrac{b_k - a_k}{k!},$$

But in equation (1), $\frac{a_i - b_i}{i!} = \frac{b_{i+1} - a_{i+1}}{(i+1)!} + \ldots + \frac{b_k - a_k}{k!}$, which provides the desired contradiction.

Therefore our assumption that there can be a different list of variables representing the same rational number $s \in \mathcal{S}_k$ must be false. Therefore any assignment of values to the variables of the factorial-representation of $s$ produces a *unique* member of the set $\mathcal{S}_k$.

Now we can count the number of members of $\mathcal{S}_k$, by looking at all the possible combinations of values for the variables $a_2, a_3, \ldots, a_k$. There are 2 choices for the variable $a_2$, combined with 3 choices for $a_3$, combined with 4 choices for $a_4$, $\ldots$, combined with $k$ choices for $a_k$.

Therefore the total number of combinations of values that can be assigned to the variables of the factorial-representation of $s$ is $2 \cdot 3 \cdot 4 \cdots k = k!$. Since each possible assignment of values to the variables creates a *unique* member of $\mathcal{S}_k$, then $|\mathcal{S}_k| = k! = |\mathcal{F}_k|$.

Recalling from above that any member $s \in \mathcal{S}_k$, can be written as $s = \frac{p}{k!}$, for some $p$ where $0 \leq \frac{p}{k!} < 1$, and that $\mathcal{F}_k$ contains *every* such number, then by the pigeon-hole principle it follows that $\mathcal{S}_k = \mathcal{F}_k$.

QED.

## Corollary to Lemma 3

The factorial-representation for every non-negative rational number $\frac{p}{k!} \in \mathcal{F}_k$ is *unique*.

The corollary is a direct result of the fact that the sets $\mathcal{F}_k$ and $\mathcal{S}_k$ are equal.

## Definitions

$$\mathcal{F} = \bigcup_{k=2}^{\infty} \mathcal{F}_k$$

$$\mathcal{S} = \bigcup_{k=2}^{\infty} \mathcal{S}_k$$

$$\mathbb{Q}_{01} = \{0\} \cup \{\ \frac{p}{q}\ \mid p, q \in \mathbb{Z},\ \text{where } q \geq 2,\ 1 \leq p < q,\ \text{and } \gcd(p, q) = 1\}$$

$\mathcal{F}$ is the set of ALL non-negative rational numbers, less than one, that could be formed with any factorial as the denominator.

$\mathcal{S}$ is the set of ALL non-negative rational numbers, less than one, formed by every factorial-representation possible.

Finally, $\mathbb{Q}_{01}$ is the set of ALL non-negative rational numbers $0 \leq \frac{p}{q} < 1$, where $p$ and $q$ are co-prime. It is convenient to specify the co-prime condition so that our *description* of the contents uniquely identifies each member of the set. Also, we want zero to be included so we toss it back into the set with the union operator. We are not going to prove it here, but it is straightforward to show that $\mathbb{Q}_{01}$ contains ALL the rational numbers between zero and one, and we will take this as established.

Recall that two sets, $\mathcal{X}$ and $\mathcal{Y}$ are equal if and only if, for all $a$,

$$a \in \mathcal{X} \Leftrightarrow a \in \mathcal{Y}.$$

# Lemma 4

$$\mathbb{Q}_{01} = \mathcal{S}$$

**Proof**

Lemma 3 tells us that the sets $\mathcal{F}_k$ and $\mathcal{S}_k$ are equal, so clearly,

$$\mathcal{F} = \bigcup_{k=2}^{\infty} \mathcal{F}_k = \bigcup_{k=2}^{\infty} \mathcal{S}_k = \mathcal{S},$$

hence,

$$\mathcal{F} = \mathcal{S}. \tag{2}$$

For all $\frac{p}{k} \in \mathbb{Q}_{01}$ we have $\frac{p}{k} \in \mathcal{F}_k$ because,

$$\frac{p}{k} = \frac{2 \cdot 3 \cdot \cdot \cdot (k-1)}{2 \cdot 3 \cdot \cdot \cdot (k-1)} \cdot \frac{p}{k} = \frac{2 \cdot 3 \cdot \cdot \cdot (k-1) \cdot p}{k!},$$

and because $\mathcal{F}_k \subset \mathcal{F}$, then $\frac{p}{k} \in \mathcal{F}$.

Conversely for all $\frac{p}{k!} \in \mathcal{F}$, then $\frac{p}{k!} \in \mathbb{Q}_{01}$ as follows,

Let $m = \frac{p}{\gcd(p,k!)}$ and $n = \frac{k!}{\gcd(p,k!)}$.

$$\frac{p}{k!} = \frac{p * \frac{1}{\gcd(p,k!)}}{k! * \frac{1}{\gcd(p,k!)}} = \frac{m}{n},$$

and since $\gcd(m,n) = 1$ then $\frac{m}{n} \in \mathbb{Q}_{01}$, that is; $\frac{p}{k!} \in \mathbb{Q}_{01}$.

Therefore, for all $a$; $a \in \mathcal{F} \Leftrightarrow a \in \mathbb{Q}_{01}$, which means, $\mathcal{F} = \mathbb{Q}_{01}$, hence by the transitivity of equality, and equation (2) above,

$$\mathcal{S} = \mathbb{Q}_{01}.$$

QED.

# Corollary to Lemma 4

The factorial-representation for every rational number in $\mathbb{Q}_{01}$ is *unique*.

The corollary is a direct result of the fact that the sets $\mathcal{S}$ and $\mathbb{Q}_{01}$ are equal.

That statement is sufficient to prove the corollary, so you can skip to the actual proof of the "Factorial Representation Theorem" if you like, however to shed a little more light on why this corollary is true, it's worth looking at a few details.

We first note that $\mathcal{F}_2 \subset \mathcal{F}_3 \subset \mathcal{F}_4 \subset \ldots$ because, for all $\frac{p}{k!} \in \mathcal{F}_k$, since $\frac{p}{k!} = \frac{(k+1) \cdot p}{(k+1) \cdot k!} = \frac{(k+1) \cdot p}{(k+1)!}$, and $\frac{(k+1) \cdot p}{(k+1)!} \in \mathcal{F}_{k+1}$, therefore $\frac{p}{k!} \in \mathcal{F}_{k+1}$, hence $\mathcal{F}_k \subset \mathcal{F}_{k+1}$.

Similarly $\mathcal{S}_2 \subset \mathcal{S}_3 \subset \mathcal{S}_4 \subset \ldots$, because for all $\frac{a_2}{2!}+\frac{a_3}{3!}+\ldots+\frac{a_k}{k!} \in \mathcal{S}_k$, since $\frac{a_2}{2!}+\frac{a_3}{3!}+\ldots+\frac{a_k}{k!} = \frac{a_2}{2!}+\frac{a_3}{3!}+\ldots+\frac{a_k}{k!}+\frac{0}{(k+1)!}$ and $\frac{a_2}{2!}+\frac{a_3}{3!}+\ldots+\frac{a_k}{k!}+\frac{0}{(k+1)!} \in \mathcal{S}_{k+1}$ then, $\frac{a_2}{2!}+\frac{a_3}{3!}+\ldots+\frac{a_k}{k!} \in \mathcal{S}_{k+1}$. Hence $\mathcal{S}_k \subset \mathcal{S}_{k+1}$.

These infinite chains of super-sets helps us to think about the meaning of our new sets $\mathcal{F}$ and $\mathcal{S}$, each of which are defined as a union of an infinite sequence of $\mathcal{F}_i$'s and $\mathcal{S}_i$'s.

We can imagine that as we build up either master-set by including each successive $\mathcal{F}_i$ or $\mathcal{S}_i$, that either we're only adding new elements to the ones that we've already gathered up, or we're simply tossing the whole previous collection out, then using the entire contents of the new largest set in its place. Either way of thinking about building up our infinite union of sets is valid.

So once a particular rational number $\frac{p}{q} \in \mathbb{Q}_{01}$ finds it's way into one of the sets $\mathcal{S}_i$ for some $i$, then it will forever be in all successive sets $\mathcal{S}_n$ for $n > i$, and it's factorial-representation is unique within each of those successively larger sets.

## Factorial Representation Theorem

Any positive rational number can be expressed in one and only one way in the form

$$a_1 + \frac{a_2}{1 \cdot 2} + \frac{a_3}{1 \cdot 2 \cdot 3} + \ldots + \frac{a_k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k},$$

where $a_1, a_2, \ldots, a_k$ are integers, and

$$0 \le a_1, \quad 0 \le a_2 < 2, \quad 0 \le a_3 < 3, \quad \ldots, \quad 0 < a_k < k$$

### Proof

Thanks to Euclid we know that for all integers $j \ge 0$ and $q > 0$, there exist *unique* integers $i$ and $p$ such that,

$$\begin{aligned} & j = i \cdot q + p\ ; \quad 0 \le p < q \\ \Leftrightarrow \quad & \frac{j}{q} = i + \frac{p}{q}\ ; \quad 0 \le \frac{p}{q} < 1 \end{aligned}$$

Which tells us that *all* positive rational numbers $\frac{j}{q}$ can be *uniquely* written as an integer part, $i$, plus a fractional part $\frac{p}{q}$, where $0 \le \frac{p}{q} < 1$.

Apply the Euclidean Division Theorem to $\frac{j}{q}$ and let $a_1 = i$. If there is no fractional remainder, then the theorem has been proven.

When there is a non-zero fractional remainder $\frac{p}{q}$, then by the Corollary to Lemma 4 we know that there is a unique factorial-representation for $\frac{p}{q}$.

So $\frac{p}{q} = \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_n}{n!} \in \mathcal{S}_n$, for some $n \ge 2$. If we choose $k$ such that $a_k \ne 0$ but $a_{k+1} = a_{k+2} = a_{k+3} = \ldots = a_{n-1} = a_n = 0$ then we can satisfy the condition that the last term in the sum is non-zero, and hence:

$a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \ldots + \frac{a_k}{k!}$ is uniquely associated with the rational number $\frac{j}{q}$.

QED.

## Additional Observations

We're guaranteed that $\frac{p}{q} \in \mathcal{S}_q$, but $\mathcal{S}_q$ is not necessarily the smallest such set for which $\frac{p}{q}$ is a member.

For example, the smallest set containing $\frac{p}{5}$, where $0 \leq \frac{p}{5} < 1$, is $\mathcal{S}_5$ however the smallest set containing $\frac{p}{6}$, where $0 \leq \frac{p}{6} < 1$ is $\mathcal{S}_3$, which is easy to see when we list the contents of a couple of sets,

$$\begin{aligned}\mathcal{S}_4 &= \{\frac{0}{24}, \frac{1}{24}, \frac{2}{24}, \frac{3}{24}, \frac{4}{24}, \frac{5}{24}, \frac{6}{24}, \frac{7}{24}, \frac{8}{24}, \frac{9}{24}, \frac{10}{24}, \frac{11}{24}, \frac{12}{24}, \frac{13}{24}, \frac{14}{24}, \frac{15}{24}, \frac{16}{24}, \frac{17}{24}, \frac{18}{24}, \frac{19}{24}, \frac{20}{24}, \frac{21}{24}, \frac{22}{24}, \frac{23}{24}\} \\ &= \{\frac{0}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}, \frac{1}{4}, \frac{7}{24}, \frac{1}{3}, \frac{3}{8}, \frac{5}{12}, \frac{11}{24}, \frac{1}{2}, \frac{13}{24}, \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{17}{24}, \frac{3}{4}, \frac{19}{24}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{23}{24}\}\end{aligned}$$

By examination $\mathcal{S}_4$ doesn't contain $\frac{1}{5}$, but it's definitely in $\mathcal{S}_5$ because,

$$\frac{0}{2} + \frac{1}{2 \cdot 3} + \frac{0}{2 \cdot 3 \cdot 4} + \frac{4}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{6} + \frac{1}{30} = \frac{5+1}{30} = \frac{6}{30} = \frac{1}{5}$$

Also, $\mathcal{S}_3 = \{\frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\}$, which demonstrates the claim that $\mathcal{S}_3$ contains $\frac{p}{6}$, where $0 \leq \frac{p}{6} < 1$.

I believe that for a given $q \geq 2$ the smallest set $\mathcal{S}_k$ for which $\frac{p}{q}$ is a member will be found by choosing the smallest $k$ such that $q$ divides $k!$.

However, I'll leave that examination for another day.

The number $2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$ can't be a rational number, as its fractional-part is not in $\mathcal{S}$ because we can't point to any particular $\mathcal{S}_k$ that the fractional-part would be a member of. Not surprising really, since this number is the well known constant $e$. Is this a proof that $e$ is irrational?