Magic Numbers

James Rowell

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In the delightful film "School of Rock" Jack Black's character Dewey Finn, pretending to be substitute teacher Ned Schneebly, is put on the spot when Miss. Mullins, the school principle, comes into class and demands that he show her his teaching methods which apparently involve an electric guitar. So Mr. S sings "The Math Song" wherein the final line of his improvised tune he sings "...yes it's 9, and that's a magic-number..."

It seems that Jack Black was making a tip-of-the-hat to the "Schoolhouse Rock" song "3 Is A Magic Number" which most kids from the 70's/80's know. However referring to 9 as a magic-number isn't as well known*.

However it's not a total stretch, there is precedent for using the term "Magic Number" when referring to to the number 9. There are a host of mathematical-magic-tricks that rely on properties of the number 9. There is also a process called "casting out nines" and a related one called calculating the "digital root" for a number where you take its "digital sum". 9 plays a similar role in all these processes - so within these contexts sometimes 9 is referred to as a magic-number.

So Mr. S's musical-math-lesson is legit, if indeed he is referring to this fact about 9: If you add up the digits of any number then if that sum is divisible by 9, then the original number must be divisible by 9. It works the other way too, that is, if we know that a number is divisible by 9 then the sum of its digits must also be divisible by 9.

For example:

$$9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59049 \text{ and } 5+9+0+4+9 = 27 \text{ and 9 divides 27 because } 27 = 3 \times 9.$$

But 3 is also has this magic-number property, for example:

$$3 \times 67 = 201 \text{ and } 2+0+1 = 3$$

In this case, 9 does not divide 201† but 3 does (67 times). We can see that adding up the digits of 201 gives us 3, which of course is divisible by 3 (but not by 9).

So given that the term "magic number" hasn't got a formal definition in the math world, I think it's time to give it one. How about if we call ANY number that has qualities like 3 and 9 (as described above) a magic-number? After all both "Schoolhouse Rock" AND "The School of Rock" sanction using "magic number" for 3 and 9, so we may be on to something big here!

Too bad 3 and 9 are the only magic-numbers in base-10. Darn - perhaps our new term isn't going to be all that useful after all, ... or maybe it will! We happen to use base-10 because we

*According to my own *highly scientific* informal poll conducted Facebook, "9" is NOT the first thing that pops into people's heads when they hear the phrase "magic number" - in fact no one answered "9" in my poll.

†Why is it immediately apparent that 9 does not divide 201? Hint: both 3 and 67 are prime numbers. Check the "Fundamental Theorem of Arithmetic" wiki page for further insight.

have ten fingers* but what about other bases? The heptapods in "Arrival" have seven arms/legs with seven fingers each, so I'll be dollars to donuts they use base-49.

Let's examine some magic-numbers (according to our proposed usage of the term) in other bases. In the way that 9 is a magic-number in base-10, then 4 is a magic-number in base-5; Also 7 is a magic-number in base-8; 15 is a magic-number in base-16; ... and 30 is a magic-number in base-31, as you can see with these examples†:

- $4 \times 67 = 268$ written in base-5 is $(2033)_5$ and $2 + 0 + 3 + 3 = 8$ (4 divides 8).
- $7 \times 181 = 1267$ written in base-8 is $(2363)_8$ and $2 + 3 + 6 + 3 = 14$ (7 divides 14).
- $15 \times 23 = 345$ written in base-16 is $(159)_{16}$ and $1 + 5 + 9 = 15$.
- $30 \times 35951 = 1078530$ written in base-31 is $(15699)_{31}$ and $1 + 5 + 6 + 9 + 9 = 30$

Notice the pattern: the biggest single-digit in any base (which is always one less than the base) seems to qualify as a magic-number. For example, 4 is the biggest digit in base-5, 7 is the biggest digit in base-8, and 9 is the biggest digit in base-10; To state this in a general way we could say that if b is our base, then $(b - 1)$ is a magic-number in base-b.

So what about 3 also being a magic-number in base-10? Let's take a look at the interesting (but probably never used outside this paper) base-31. It appears that 2, 3, 5, 6, 10, 15 (along with 30) are *all* magic-numbers in base-31, as we we can see with the following examples:

- $2 \times 409 \times 1129 = 923522$ written in base-31 is $(10001)_{31}$ and $1 + 0 + 0 + 0 + 1 = 2$.
- $3 \times 641 = 1923$ written in base-31 is $(201)_{31}$ and $2 + 0 + 1 = 3$.
- $5 \times 139 \times 2659 = 1848005$ written in base-31 is $(20102)_{31}$ and $2 + 0 + 1 + 0 + 2 = 5$.
- $6 \times 10091 = 60546$ written in base-31 is $(2103)_{31}$ and $2 + 1 + 0 + 3 = 6$.
- $10 \times 197 \times 941 = 1853770$ written in base-31 is $(20701)_{31}$ and $2 + 0 + 7 + 0 + 1 = 10$.
- $15 \times 71503 = 1072545$ written in base-31 is $(15027)_{31}$ and $1 + 5 + 0 + 2 + 7 = 15$.

See the pattern? All the divisors of 30 are magic-numbers in base-31. It turns out that in base-b, then any divisor of $(b - 1)$ is a magic-number in base-b.

I guess that begs the question: What about 1? Isn't 1 a magic-number in all bases? Since we already know that all numbers are divisible by 1, then we don't need to go to any trouble of adding up digits to find this out. So let's exclude 1 from being considered as a magic-number - it's not helpful.

So let's come up with a nice formal definition for a "magic number". But first we need to remind ourselves exactly what it means to write a number in a given base - as described in much greater detail in the paper "SesameStreet++":

* To be honest, base-10 is not "universal" among humans, there are plenty of other bases in use, albeit not as widely used nor as mathematically useful as base-10. This doesn't count octal, hexadecimal and binary which are used in the computer world so by necessity are, mathematically speaking, useful.

† We are going to use the conventions for specifying numbers in alternate bases as outlined in the paper "Sesame Street++"

Basis Representation Theorem

Let b be a positive integer greater than 1.

For every positive integer n there is a unique sequence of integers $d_0, d_1, d_2, \ldots, d_k$ such that:

$$n = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b^1 + d_0 b^0,$$

where $0 \leq d_i < b$ for all i in $\{0, 1, 2, \ldots, k\}$ and $d_k \neq 0$.

Definition: n is represented in base-b by the string of base-b-digits $(d_k d_{k-1} \cdots d_2 d_1 d_0)_b$

Let's also add some useful notation to our discussion. In general, to say an integer a is divisible by another integer b (not zero) means there is a third integer c such that*:

$$a = b \cdot c$$

We express the fact that a is divisible by b, and say "b divides a" using this notation:

$$b \mid a$$

So the following is always true:

$$1 \mid a, \text{ and } a \mid a;$$

also

$$b \mid 0 \text{ for every } b \text{ except } 0$$

So given the Basis-Representation-Theorem and our new notation for divisability then here's our proposed theorem and definition for "magic number":

Magic Number Theorem

Let n be a positive integer represented in base-b.

If m is a positive integer such that $m \mid (b-1)$ then,

$$m \mid n \qquad \text{if and only if} \qquad m \mid (d_k + \cdots + d_2 + d_1 + d_0)$$

Definition: We call $m \neq 1$ a "magic-number" in base-b.

As an interesting side note - according to our definition, binary, or base-2, doesn't have a magic-number. No great loss, as we said above 1 isn't helpful to consider as a magic-number.

Let's prove the theorem. The proof is quite simple when using modular arithmetic. However if you aren't familiar with modular arithmetic then I recommend that you check out Khan Academy's "What is modular arithmetic?" to get a clear-cut-crash-course in the subject, it's excellent, as is the entire Khan Academy site in case you've never checked it out.

In summary, modular arithmetic is the kind of arithmetic you use when you tell time with a 12 hour clock.

*Paraphrased from "An Introduction to the Theory of Numbers" by G. H. Hardy and E. M. Wright. Pg 1.

For example, if it's 7 o'clock and I ask you to "meet me in 7 hours" then you know I mean "meet me at 2 o'clock". Suppose instead I said "meet me in 31 hours" then you would know that our meeting time is also going to be 2 o'clock (albeit 1 day hence) because:

$$7 \text{ o'clock} + 31 \text{ hours} = 7 \text{ o'clock} + (7 + 24) \text{ hours} = 2 \text{ o'clock}$$

When we do "clock arithmetic" we add some number of hours to the current time then divide that sum by 12; The remainder of the division then tells us the clock time.

Any time on the clock, say T o'clock, plus 12 hours is also T o'clock. Furthermore, adding ANY multiple of 12 hours (eg 24, 36, 48, ...) to T o'clock gets us back to T o'clock. We can say that $T, T+12, T+24, T+36, T+48, \ldots$ are all equivalent times on the clock, or that these numbers are all equivalent modulo 12. It just means the hour hand has spun around the clock exactly 1, 2, or 3, etc. more times.

So numbers being equivalent modulo 12, means that they have the same remainder when divided by 12. For example 1 is equivalent to 13 modulo 12. If your drill sargent barks an order at you to meet him at 13:00 hours, you'd better show up at 1 o'clock in the afternoon!

So to say that a is equivalent to b modulo c (where $c > 1$) mean

where a, b and c are integers such that $c > 1$.

(a+b) mod c=(a mod c+b mod c) mod c ab mod c=((a mod c)(b mod c)) mod c

As a generalization of 2), we get:

3) a mod c=b mod c akmod c=bk mod c

Importantly, a mod c=0 c divides a.

In other words, there exists a number n such that nc=a. Lemma: if d¿1 is a divisor of m then: (m+1)k mod d=1 Proof of Lemma: Let d be a divisor of m. Soc such that cd=m. Hence, (cd + 1) mod d = cd mod d + 1 mod d = (c mod d) (d mod d) + 1 mod d = (c mod d) 0 + 1 mod d = 1 mod d By property 3) above, since (cd+1) mod d=1 mod d then: (m + 1)k mod d = (cd + 1)k mod d = (cd+1) mod d = 1 QED

Proof of "Magic Number Theorem":

Let N=M+1.

That is; we are going to consider representations of numbers in base-M+1, so think of M as our magic-number.

Also let M=1, so our smallest possible base will be 2. Then every integer n is uniquely expressible in base-M+1 as follows: n=ak(M+1)k+ak-1(M+1)k-1+?+a2(M+1)2+a1(M+1)+a0 where 0=ai=M for all the "digits" ai where 0=i=k. n is divisible by M if and only if n mod M=0. Substitute the base-M+1 expression of n into the above relationship, therefor 0=(ak(M+1)k+ak-1(M+1)k-1+?+a2(M+1)2+a1(M+1)+a0)mod M