MS&E349 Homework 1

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Theoretical Questions

Question 1

Given the ACF of the undifferenced data, the time series does not appear stationary.

When we difference the data, both the ACF and PACF indicate that the model depicts a moving average process. In particular, the fact that the ACF "dies off" past lag 1 indicates that the model is of the form
ARIMA(0, 1, 1).

As a rule of thumb, we know that the likely signs of an $MA(q)$ model consist in an ACF that "dies off" after lag q and a PACF that exhibits exponential decay. On the other hand, the likely signs of an $AR(p)$ model consist in an ACF that exponentially decays and a PACF that "dies off" after lag p.

Question 2

2.1

Note that

$$\begin{aligned}\mathbb{E}[h_1(X_t, \phi)] &= \mathbb{E}[Y_{t-1}Y_t - \phi Y_{t-1}^2] \\ &= \mathbb{E}[Y_{t-1}Y_t] - \phi\mathbb{E}[Y_{t-1}^2] \\ &= \gamma(0) - \phi\gamma(1)\end{aligned}$$

This is only satisfied when $\phi = \phi_0$

Now seeing that $Y_t = \epsilon_t + \phi_0 Y_{t-1}$

$$\begin{aligned}\mathbb{E}[h_2(X_t, \phi)] &= \mathbb{E}[Y_{t-2}(\epsilon_t + \phi_0 Y_{t-1} - \phi Y_{t-1})] \\ &= \mathbb{E}[Y_{t-2}(\phi_0 Y_{t-1} - \phi Y_{t-1})] \\ &= \phi_0\gamma(1) - \phi\gamma(1)\end{aligned}$$

This is only satisfied when $\phi = \phi_0$

We see that in this case $\Phi = \Phi_1 \cap \Phi_2 = \{\phi_0\}$

2.2

We see that

$$\hat{\phi}_1 = \arg\min \phi(\frac{1}{T}\sum_{t=1}^{T} Y_{t-1}Y_t - \phi Y_{t-1}^2))^2$$

First order conditions give us that

$$\hat{\phi}_1 = \frac{\sum_{t=1}^{T} Y_{t-1}Y_t}{\sum_{t=1}^{T} Y_{t-1}^2}$$

We now see that

$$\begin{aligned}\sqrt{T}(\hat{\phi}_1 - \phi_0) &= \sqrt{T}\frac{\sum_{t=1}^{T} Y_{t-1}Y_t - \phi_0\sum_{t=1}^{T} Y_{t-1}^2}{\sum_{t=1}^{T} Y_{t-1}^2} \\ &= \sqrt{T}\frac{\sum_{t=1}^{T} Y_{t-1}(Y_t - \phi_0 Y_{t-1})}{\sum_{t=1}^{T} Y_{t-1}^2} \\ &= \sqrt{T}\frac{\sum_{t=1}^{T} Y_{t-1}(\epsilon_t)}{\sum_{t=1}^{T}(Y_{t-1}^2} \\ &= \frac{\frac{1}{\sqrt{T}}\sum_{t=1}^{T} Y_{t-1}(\epsilon_t)}{\frac{1}{T}\sum_{t=1}^{T} Y_{t-1}^2}\end{aligned}$$

We note that the numerator converges in distribution to $\mathcal{N}(0, Var(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} Y_{t-1}\epsilon_t))$ and the denominator converges in probability to γ_0 so

$$\sqrt{T}(\hat{\phi}_1 - \phi_0) \xrightarrow{d} \mathcal{N}(0, \frac{1}{\gamma(0)^2} Var(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} Y_{t-1}\epsilon_t))$$

At this point, let's get a better handle on that variance term
We see that $\mathbb{E}[Y_{t-1}\epsilon_t] = 0$ so we have

$$\mathbb{E}[\sum_{t=1}^{T} Y_{t-1}\epsilon_t] = \mathbb{E}[\sum_{t=1}^{T} Y_{t-2}\epsilon_t] = 0$$

This gives us

$$Var(\frac{1}{\sqrt{T}}\sum_{t=1}^{T})\begin{pmatrix} Y_{t-1} \\ Y_{t-1} \end{pmatrix} = \begin{pmatrix} \mathbb{E}[(\sum_{t=1}^{T} Y_{t-1}\epsilon_t)^2] & \mathbb{E}[(\sum_{t=1}^{T} Y_{t-1}\epsilon_t)(\sum_{t=1}^{T} Y_{t-2}\epsilon_t)] \\ \mathbb{E}[(\sum_{t=1}^{T} Y_{t-1}\epsilon_t)(\sum_{t=1}^{T} Y_{t-2}\epsilon_t)] & \mathbb{E}[(\sum_{t=1}^{T} Y_{t-1}\epsilon_t)^2] \end{pmatrix}$$

Now we see

$$\begin{aligned}
\mathbb{E}[(\sum_{t=1}^{T} Y_{t-1}\epsilon_t)^2] &= \mathbb{E}[(\sum_{t=1}^{T} Y_{t-1}\epsilon_t)(\sum_{t=1}^{T} Y_{t-1}\epsilon_t)] \\
&= \mathbb{E}[(\sum_{t_1=1}^{T}\sum_{t_2=1}^{T} Y_{t_1-1}\epsilon_{t_1} Y_{t_2-1}\epsilon_{t_2})] \\
&= (\sum_{t_1=1}^{T}\sum_{t_2=1}^{T} \mathbb{E}[Y_{t_1-1}\epsilon_{t_1} Y_{t_2-1}\epsilon_{t_2})] \\
&= (\sum_{t_1=1}^{T} \mathbb{E}[Y_{t_1-1}\epsilon_{t_1} Y_{t_1-1}\epsilon_{t_1})] \\
&= \sum_{t_1=1}^{T} \gamma(0)
\end{aligned}$$

Similar steps give us

$$\mathbb{E}[(\sum_{t=1}^{T} Y_{t-2}\epsilon_t)^2] = \sum_{t=1}^{T} \gamma(0)$$

$$\mathbb{E}[(\sum_{t=1}^{T} Y_{t-1}\epsilon_t)(\sum_{t=1}^{T} Y_{t-2}\epsilon_t)] = \sum_{t_1=1}^{T} \gamma(1)$$

We can also see that

$$\gamma(0) = \sum_{i=0}^{\infty}(\phi_0^i)^2 = \frac{1}{1-\phi_0^2}$$

$$\gamma(1) = \sum_{i=0}^{\infty}(\phi_0^i)\phi_0^{i-1} = (1/\phi_0)\sum_{i=0}^{\infty}(\phi_0^i)^2 = \frac{1}{\phi_0(1-\phi_0^2)}$$

2.3

Similarly we note

$$\hat{\phi}_2 = \arg\min \phi(\frac{1}{T}\sum_{t=1}^{T}(Y_{t-2}Y_t - \phi Y_{t-1}Y_{t-2}))^2$$

First order conditions give us that

$$\hat{\phi}_2 = \frac{\sum_{t=1}^{T} Y_{t-2}Y_t}{\sum_{t=1}^{T} Y_{t-1}Y_{t-2}}$$

so we have

$$\begin{aligned}
\sqrt{T}(\hat{\phi}_2 - \phi_0) &= \frac{\sum_{t=1}^{T} Y_{t-2}Y_t - \phi_0 \sum_{t=1}^{T} Y_{t-1}Y_{t-2}}{\sum_{t=1}^{T} Y_{t-1}Y_{t-2}} \\
&= \frac{\sum_{t=1}^{T} Y_{t-2}(Y_t - \phi_0 Y_{t-1})}{\sum_{t=1}^{T} Y_{t-1}Y_{t-2}} \\
&= \frac{\sum_{t=1}^{T} Y_{t-2}\epsilon_t}{\sum_{t=1}^{T} Y_{t-1}Y_{t-2}}
\end{aligned}$$

2.4

A sufficient condition for $g = 0$ would be that the estimates for ϕ obtained in 2.2 and 2.3 are actually equal, that is, $\phi_1 = \phi_2$. When this is the case, the first order conditions give that the vector g is indeed zero, and while this imposes no structure on the matrix W, it imposes a pathological amount of structure on Y_t.