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CME 241: Assignment 8

8.1

State Space

Our state space will consist in the amount of cash on hand c_t and the amount invested x_t at the end of the day *prior to borrowing or investing any money for the day.*

$$\mathcal{S} = \{(s_t, x_t) | s_t \in \mathbb{R}, x_t \in \mathbb{R}_0^+\}$$

Action Space

The decision to borrow β_t or invest α_t will make up our action space. This decision is made at the end of the day, so it is therefore known how much cash we will have at the start of the next day, and therefore whether we will pay a regulator penalty.

In particular, we will start the next day with $\tilde{c}_{t+1} = c_t + \beta_t - \alpha_t$, and this is the amount subject to penalty. We must be able to pay the penalty at the beginning of the day. This constraints the state of available actions to pairs (α_t, β_t) that satisfy $c_t + \beta_t - \alpha_t \geq K \cot(\frac{\pi \tilde{c}_t}{2C})$. Our action space at time t is therefore

$$\mathcal{A}_t = \left\{(\alpha_t.\beta_t) \in \mathbb{R} \times \mathbb{R}_0^+ | \beta_t - \alpha_t \geq K \cot\left(\frac{\pi \tilde{c}_t}{2C}\right) - c_t\right\}$$

State Transition Function

We define the *door open* cash amount $c_{t+1}^{do} = \tilde{c}_{t+1} - K \cot(\frac{\pi \tilde{c}_{t+1}}{2C}) \cdot \mathbf{1}_{\tilde{c}_t < C}$ as the initial amount of cash made available to customers, that is, after the doors open.

Let ν_t denote the net customer demand for time period t. This is the net amount of cash requested from all withdrawals/deposits throughout the day. In addition, ϵ_t denotes the growth factor of our investment over time period t. The function $f_t(u, v)$ denotes the joint density function of the pair (ϵ_t, ν_t). It is possible that the net customer demand ν_t exceeds c_{t+1}^{do}, but we can simply turn customers away without penalty. In any event, the amount of cash at the end of time $t + 1$ (but before any decision is made) is given by

$$c_{t+1} = c_t^{do} - (1 + R)\beta_t + \tilde{\nu}_t$$

where $\tilde{\nu}_t = \max\{-c_t^{do}\}$. Not that c_{t+1} can be negative, in which case we constraint our action space $(\alpha_{t+1}, \beta_{t+1})$ to those decisions that give us sufficient cash to pay any penalty at the start of the next day. In addition, our invested amount at this time is given by

$$x_{t+1} = (x_t + \alpha_t)\epsilon_t$$

This lets us define our state-transition function as a conditional probability

$$\mathbb{P}[s_{t+1} \in (c_1, c_2) \times (x_1, x_2) | a_t = (\alpha_t, \beta_t), s_t = (c_t, x_t)]$$

$$= \int_{x_1/(x_t+\alpha_t)}^{x_2/(x_t+\alpha_t} \left(\int_{\max\{-c_{t+1}^{do}, c_1 - \Delta c_{t+1}\}}^{c_2 - \Delta c_{t+1}} f_t(u,v)dv + \mathbf{1}_{\{-c_{t+1}^{do} \geq c_1 - \Delta c_{t+1}\}} \int_{-\infty}^{-c_{t+1}^{do}} f_t(u,v)dv \right) du$$

where $\Delta c_{t+1} = c_{t+1}^{do} - (1+R)\beta_t$

Reward

We are interested in maximizing the expected utility of our terminal wealth $W_T = c_T + x_T$. From the state (c_T, x_T), there are no decisions to be made for the next day, so we have

$$\mathcal{R}(c_t, x_t) = \begin{cases} U(c_t + x_t) & t = T \\ 0 & \text{otherwise} \end{cases}$$

Implementation Ideas

If we have a closed form of our joint density function $f_t(u,v)$ for each timestep, we could proceed with backwards induction to find the optimal $(\alpha_{T-1}, \beta_{T-1})$, which is a function of the state (s_{T-1}, x_{T-1}). If the integral of our joint density function also has a closed form, backwards induction would be tractable.

Now suppose that $f_t(u,v)$ is very expensive to evaluate, and therefore to integrate. Our best bet would then be approximate dynamic programming, by which we sample a random number of states of the form (s_{T-1}, x_{T-1}) and create an approximate value function V_{T-1} for this timestep. Then we step backwards and work with a approximate form of our state-transition function obtained through discretizing over point masses of our random sample, and repeat.

8.2

Let F denote the cumulative distribution function of f. Starting from $g(S) = p \cdot g_1(S) + h \cdot g_2(S)$, we have that

$$\frac{\partial g_1}{\partial S} = -(1 - F(S)) = F(S) - 1$$

$$\frac{\partial g_2}{\partial S} = F(S)$$

whence

$$\frac{\partial g}{\partial S} = p \cdot F(S) - p + h \cdot F(S) = 0 \Longleftrightarrow F(S) = \frac{p}{p+h}$$

This is indeed a minimum, because

$$g''(S) = (p+h) \cdot f(s) \geq 0$$

by non-negativity of the density function. Therefore, the value of S that minimizes the expected cost is $F^{-1}(\frac{p}{p+h})$.

The Option Interpretation

$g_1(S)$ is equal to the expected payoff of a call option with strike price equal to S, while $g_2(S)$ equals that of a put option. This problem may therefore also be understood as finding the strike price that minimizes the expected payout of a portfolio consisting of p call options and h put options.