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# CME 241: Assignment 6

## 6.1

(a) When $x \sim \mathcal{N}(\mu, \sigma^2)$, $\mathbb{E}[x^2] = \mu^2 + \sigma^2$. Therefore,

$$\mathbb{E}[U(x)] = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)$$

(b) Our utility function is not monotonically increasing, so we will focus only on the increasing part of $U(x)$, in which case $U^{-1}(x) = \frac{1-\sqrt{1-2\alpha x}}{\alpha}$. Hence

$$x_{CE} = \frac{1 - \sqrt{1 - 2\alpha\mu + \alpha^2(\mu^2 + \sigma^2)}}{\alpha}$$

(c) $\pi_A = \mathbb{E}[x] - x_{CE} = \mu - \frac{1-\sqrt{1-2\alpha\mu+\alpha^2(\mu^2+\sigma^2)}}{\alpha}$

(d) We will denote the proportion of our wealth invested in the risky asset by $z$. After one year, our wealth as a function of $z$ is the random variable

$$W(z) = (1 + r)(1 - z) + z(1 + w)$$

where $x \sim \mathcal{N}(\mu, \sigma^2)$. Hence

$$\mathbb{E}[W(z)] = (1 + r)(1 - z) + z(1 + \mu)$$

$$\mathbb{E}[W(z)^2] = (1 + r)^2(1 - z)^2 + 2(1 + r)(1 - z)z(1 + \mu) + z^2(1 + 2\mu\sigma^2 + \mu^2)$$

Now

$$\frac{\partial\mathbb{E}[U(W(z)]}{\partial z} = 0 \iff z^*(\alpha) = \frac{(1 - (r + 1)\alpha)(\mu - r)}{\alpha((\mu - r)^2 + \sigma^2}$$

Under the constraint that $0 \leq z \leq 1$, we obtain the following plot for $r = 0.06$, $\mu = 0.10$, $\sigma = 0.25$ (next page):

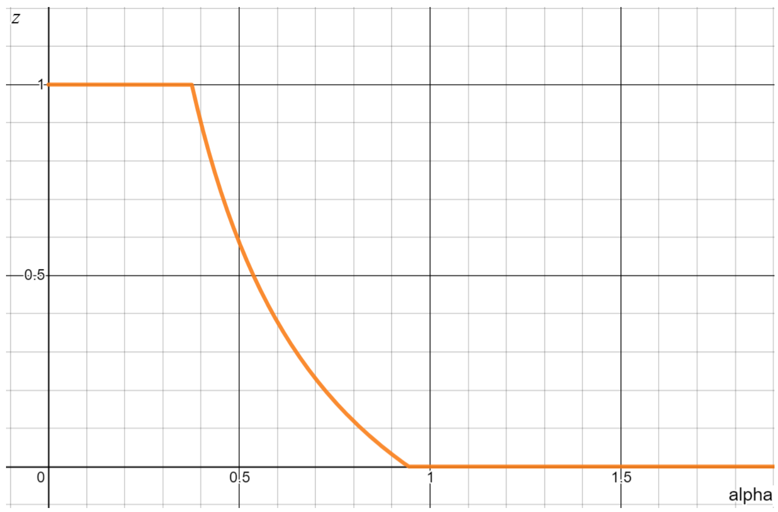


Figure 1: $z^*(\alpha)$

## 6.2

The two possible outcomes after wagering $fW_0$ of our wealth are $fW_0(1+\alpha)$ with probability $p$, and $fW_0(1-\beta)$ with probability $1-p$. We will denote our resulting wealth by the random variable $W(f)$, to call attention to its dependence on the wagered fraction $f$. The expected utility is given by

$$\mathbb{E}[\log(W(f))] = \log(W_0) + \log(1 + f\alpha)p + \log(1 - f\beta)q$$

whence

$$\frac{\partial}{\partial f}\mathbb{E}[\log(W(f))] = \frac{\alpha p}{1 + f\alpha} + \frac{q\beta}{1 - f\beta} = 0 \iff \alpha p(1 - f\beta) - q\beta(1 + f\alpha) = 0$$

$$\implies f^* = \frac{\alpha p - q\beta}{\alpha\beta}$$

to verify that this is the global maximum, taking the second derivative yields

$$\frac{\partial^2}{\partial f^2}\mathbb{E}[\log(W(f))] = -\frac{\alpha^2 p}{(1 + f\alpha)^2} - \frac{q\beta^2}{(1 - f\beta)^2} < 0$$

therefore, $f^*$ is indeed the global maximizer.

If $\alpha >> \beta$, then $f^* \approx \frac{p}{\beta}$, so that if $\beta < p$, the Kelly criterion would advise wagering close to all of our wealth. If $\beta >> \alpha$, then $f^*$ is negative, so that we should not wager any of our wealth. Interestingly, if $p = q = 0.5$ and $\alpha = \beta$, then we are playing what could be considered a “fair game”, but we should not wager any fraction of our wealth. This is due to the high risk-aversion of our utility function.