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# CME 241: Assignment 7

## 7.1

We will start from the general form of the HJB equation:

$$\max_{\pi_t, c_t} \left( V_t + V_w((\pi_t(\mu - r) + r)W_t - c_t) + V_{ww}\frac{\pi_t^2\sigma^2 W_t^2}{2} + U(c_t) \right)$$

with a $U(\cdot) = \log(\cdot)$, we take the partial derivatives with respect to $\pi_t$ and $c_t$, as we did previously. The same result emerges for the optimal portfolio allocation $\pi_t^*$:

$$\pi_t^* = -\frac{V_w(\mu - r)}{V_{ww}\sigma^2 W_t}$$

Meanwhile, the optimal level of consumption satisfies

$$c_t^* = V_w^{-1}$$

Plugging these values into the HJB equation gives

$$V_t + rW_tV_w - \frac{(\mu - r)^2}{2\sigma^2}\frac{V_w^2}{V_{ww}} - 1 + \log(V_w) = \rho V$$

After some thought, a promising form of the value function to surmise is $V(t, W_t) = f(t)(\log(W_t) + h(t))$, from which we obtain

$$V_t = f'(t)\log(W_t) + f(t)h'(t)$$

$$V_w = \frac{f(t)}{W_t}$$

$$V_{ww} = -\frac{f(t)}{W_t^2}$$

which yields

$$f'(t)\log(W_t) + f'(t)h(t) + f(t)h'(t) + rf(t) - \frac{(\mu - r)^2}{2\sigma^2} - 1 + \log(W_t) - \log(f(t)) = \rho f(t)\log(W_t) + \rho f(t)h(t)$$

Since $W_t$ is a random variable, we obtain two separate differential equations

$$f'(t) + 1 = \rho f(t)$$

$$f'(t)h(t) + f(t)h'(t) + rf(t) - \frac{(\mu - r)^2}{2\sigma^2} - 1 + \log(f(t)) = \rho f(t)h(t)$$

Now we impose the terminal conditions $f(T) = \epsilon << 1$, $h(T) = 0$. This fully characterizes $f(t)$ from the first equation:

$$f(t) = \frac{\epsilon\rho - 1}{\rho} e^{-\rho(T-t)} + \frac{1}{\rho}$$

At this point, we have an explicit form of $c_t^*$ assuming that there exists an $h$ satisfying both equations above and the zero terminal condition. Adapting a result from Kogan and Uppal [1], we may express $h(t)$ to be of the form

$$h(t) = -\frac{1}{f(t)} \int_t^T e^{-\rho(s-t)} \ln(f(s)) ds + \frac{1}{f(t)} \int_t^T \left( f(t) - \frac{1}{\rho}(1 - e^{-\rho(s-t)}) \right) \left( -\frac{1}{f(s)} + r + \frac{(\mu - r)^2}{2\sigma^2} \right) ds$$

which we can verify satisfies the zero terminal condition, and after taking a few painful derivatives, we see that $h$ also satisfies this pair of differential equations.

We now have that

$$c_t^* = \frac{W_t}{f(t)} = \frac{\rho W_t}{(\epsilon\rho - 1)e^{-\rho(T-t)} + 1}$$

$$\pi_t^* = \frac{(\mu - r)^2}{\sigma^2}$$

and the optimal value function $V^*(t, W_t)$ is given by

$$V^*(t, W_t) = f(t)(\log(W_t) + h(t))$$

with $f(t)$ and $h(t)$ defined as above.

### 7.3

(WIP)

## References

[1] Leonid Kogan and Raman Uppal. Risk aversion and optimal portfolio policies in partial and general equilibrium economies. Working Paper 8609, National Bureau of Economic Research, November 2001.