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CME 241: Assignment 7

7.1

We will start from the general form of the HJB equation:

$$\max_{\pi_t, c_t} \left(V_t + V_w((\pi_t(\mu - r) + r)W_t - c_t) + V_{ww}\frac{\pi_t^2\sigma^2 W_t^2}{2} + U(c_t) \right)$$

with a $U(\cdot) = \log(\cdot)$, we take the partial derivatives with respect to π_t and c_t, as we did previously. The same result emerges for the optimal portfolio allocation π_t^*:

$$\pi_t^* = -\frac{V_w(\mu - r)}{V_{ww}\sigma^2 W_t}$$

Meanwhile, the optimal level of consumption satisfies

$$c_t^* = V_w^{-1}$$

Plugging these values into the HJB equation gives

$$V_t + rW_tV_w - \frac{(\mu - r)^2}{2\sigma^2}\frac{V_w^2}{V_{ww}} - 1 + \log(V_w) = \rho V$$

After some thought, a promising form of the value function to surmise is $V(t, W_t) = f(t)(\log(W_t) + h(t))$, from which we obtain

$$V_t = f'(t)\log(W_t) + f(t)h'(t)$$

$$V_w = \frac{f(t)}{W_t}$$

$$V_{ww} = -\frac{f(t)}{W_t^2}$$

which yields

$$f'(t)\log(W_t) + f'(t)h(t) + f(t)h'(t) + rf(t) - \frac{(\mu - r)^2}{2\sigma^2} - 1 + \log(W_t) - \log(f(t)) = \rho f(t)\log(W_t) + \rho f(t)h(t)$$

Since W_t is a random variable, we obtain two separate differential equations

$$f'(t) + 1 = \rho f(t)$$

$$f'(t)h(t) + f(t)h'(t) + rf(t) - \frac{(\mu - r)^2}{2\sigma^2} - 1 + \log(f(t)) = \rho f(t)h(t)$$

Now we impose the terminal conditions $f(T) = \epsilon << 1$, $h(T) = 0$. This fully characterizes $f(t)$ from the first equation:

$$f(t) = \frac{\epsilon\rho - 1}{\rho} e^{-\rho(T-t)} + \frac{1}{\rho}$$

At this point, we have an explicit form of c_t^* assuming that there exists an h satisfying both equations above and the zero terminal condition. Adapting a result from Kogan and Uppal [1], we may express $h(t)$ to be of the form

$$h(t) = -\frac{1}{f(t)} \int_t^T e^{-\rho(s-t)} \ln(f(s)) ds + \frac{1}{f(t)} \int_t^T \left(f(t) - \frac{1}{\rho}(1 - e^{-\rho(s-t)}) \right) \left(-\frac{1}{f(s)} + r + \frac{(\mu - r)^2}{2\sigma^2} \right) ds$$

which we can verify satisfies the zero terminal condition, and after taking a few painful derivatives, we see that h also satisfies this pair of differential equations.

We now have that

$$c_t^* = \frac{W_t}{f(t)} = \frac{\rho W_t}{(\epsilon\rho - 1)e^{-\rho(T-t)} + 1}$$

$$\pi_t^* = \frac{(\mu - r)^2}{\sigma^2}$$

and the optimal value function $V^*(t, W_t)$ is given by

$$V^*(t, W_t) = f(t)(\log(W_t) + h(t))$$

with $f(t)$ and $h(t)$ defined as above.

7.3

Simple Case: Single Job and Skill Level

Our state space is defined by

$$\mathcal{S} = \{(s_t, J_t) | s_t \in \mathbb{R}^+, J_t \in \{E, U\}\}$$

that is, our state records the current skill level s_t, and whether we are currently employed (E) or unemployed (U).

Our space of possible actions consists in the allocation of our day to working versus learning, hence $\mathcal{A} = [0, 1] \subseteq \mathbb{R}$

Meanwhile, our immediate reward is simply the utility of the wealth we made in a day, so $\mathcal{R}((s, J), \alpha) = U(\alpha f(s)) \cdot \mathbf{1}_{J=E}$

Finally, our state transition function $\mathcal{P}$ is defined by

$$\mathcal{P}((s_t, U), \alpha, (s_{t+1} = 2^{-1/\lambda} s_t, E)) = h(s_t)$$
$$\mathcal{P}((s_t, U), \alpha, (s_{t+1} = 2^{-1/\lambda} s_t, U)) = 1 - h(s_t)$$
$$\mathcal{P}\Bigg((s_t, E), \alpha, \Bigg(s_t(1 - \alpha_t) \int_t^{t+1} g(s_u) du, U\Bigg)\Bigg) = p$$
$$\mathcal{P}\Bigg((s_t, E), \alpha, \Bigg(s_t(1 - \alpha_t) \int_t^{t+1} g(s_u) du, E\Bigg)\Bigg) = 1 - p$$

If we are to maximize the discounted expected utility of our lifetime earnings over a finite time horizon, one would expect that the optimal policy would invest initial time in developing one's skill level, and as time nears its terminal point, less priority is given to getting rehired so less time is spent increasing skill.

Over an infinite horizon, we would expect a consistent amount of time invested in maintaining or increasing one's skill level at all times, because we would expect that the optimal policy would seek keeping the probability of rehiring high at all times to avoid long expanses of unemployment.

Introducing Multiple Jobs, Daily Expenses, and Leisure Time

Let's consider a more interesting setup: Suppose there are n jobs, each with their own rate of pay, skill levels, and probabilities of loss. Each skill has its own half-life when atrophying under unemployment, and we assume for convenience that events of loss / re-offer for each job are independent.

Our daily schedule can incorporate time spent on these jobs and skills, in addition to leisure time, whose utility we define to be our reward. During our leisure time we consume a given amount of wealth W_t.

We therefore add our current wealth W_t to our state space, and introduce a fixed daily expense d. If for whatever reason we don't have enough money to fulfill this expense, we will just hit rock-bottom and pay what we have so that we have, say, \$1.00 left to avoid infinite utility.

$$\mathcal{S} = \{(\vec{s}_t, \vec{J}_t, W_t) | s_t^i \in \mathbb{R}_0^+, J_t^i \in \{E, U\}, W_t \in \mathbb{R}_0^+\}$$

We let $\vec{\alpha}_t$ denote the allocation of time to each job, while $\vec{\beta}_t$ denotes the allocation of time to each skill. This leaves $\ell_t = \mathbf{1}^T(\vec{\alpha}_t + \vec{\beta}_t)$ time for leisure, and the chosen amount of money to expend during this time is c_t. Our action space is therefore a simplex with $2n+1$ vertices along with a positive real number c_t:

$$\mathcal{A} = \{\alpha_t^1, \ldots \alpha_t^n, \beta_t^1, \ldots \beta_t^n, \ell_t, c_t | \ell_t + \mathbf{1}^T(\vec{\alpha} + \vec{\beta}) = 1, c_t \in (0, W_t - d]\}$$

where d denotes our fixed daily expenses. Our reward is defined in terms of leisure time multiplied by our consumption, that is,

$$\mathcal{R}((\vec{s}_t, \vec{J}_t, W_t), (\vec{\alpha}_t, \vec{\beta}_t, \ell_t, c_t)) = U(\ell_t \cdot c_t)$$

and finally, our state transitions are defined by

$$\mathcal{P}\Bigg((\vec{s}_t, \vec{J}_t, W_t), (\vec{\alpha}_t, \vec{\beta}_t, \ell_t, c_t), \Bigg(\vec{s}_{t+1}, \vec{J}_{t+1}, W_t + \sum_{i=1}^{n} \alpha_i f(s_t^i) - d - c_t\Bigg)\Bigg) = \prod_{i=1}^{n} \rho(J_t^i, J_{t+1}^i)$$

with

$$\begin{aligned} s_{t+1}^i = \beta_t^i s_t^i \int_t^{t+1} g(s_u^i) du & \qquad J_t^i = E \\ s_{t+1}^i = s_t^i 2^{-1/\lambda_i} & \qquad J_t^i = U \end{aligned}$$

with the function ρ defined by

$$\begin{aligned} \rho(E_t^i, U_t^i) &= p_i \\ \rho(E_t^i, U_t^i) &= 1 - p_i \\ \rho(U_t^i, E_t^i) &= h(s_t^i) \\ \rho(U_t^i, U_t^i) &= 1 - h(s_t^i) \end{aligned}$$

References

[1] Leonid Kogan and Raman Uppal. Risk aversion and optimal portfolio policies in partial and general equilibrium economies. Working Paper 8609, National Bureau of Economic Research, November 2001.