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CME 241: Assignment 9

9.2

The LPT model is defined by the dynamics

$$P_{t+1} = P_t e^Z$$
$$X_{t+1} = \rho X_t + \eta_t$$
$$Q_t = P_t \cdot (1 - \beta N_t - \theta X_t)$$

In general, the optimal value function is defined by

$$V_t^*((P_t, R_t)) = \max_{N_t}\{N_t \cdot Q_t + \mathbb{E}[V_{t+1}^*((P_{t+1}, R_{t+1}))]\}$$

but for the penultimate time $t = T-1$, we have that $R_{T-1} = N_{T-1}$ and $V_{T-1}^* = N_{T-1}Q_{T-1} = N_{T-1}P_{T-1} \cdot (1 - \beta N_{T-1} - \theta X_{T-1})$, whence

$$\begin{aligned} V_{T-2}^*((P_{T-2}, R_{T-2})) &= \max_{N_{T-2}}\{N_{T-2}Q_{T-2} + \mathbb{E}[N_{T-1}Q_{T-1}]\} \\ &= \max_{N_{T-2}}\{aN_{T-2}^2 + bN_{T-2} + c\} \end{aligned}$$

where

$$\begin{aligned} a &= -\beta P_{T-2}(1 + q_Z) \\ b &= P_{T-2}(1 - \theta X_{T-2} - q_Z(1 - 2\beta R_{T-2} - \rho\theta X_{T-2})) \\ c &= P_{T-2} \cdot q_Z(R_{T-2} - \beta R_{T-2} - \rho\theta X_{T-2}R_{T-2}) \\ q_Z &= e^{\mu_Z + \sigma_Z^2/2} \end{aligned}$$

The minimum of this quadratic function of N_{T-2} is attained at the vertex

$$-\frac{b}{2a} = \frac{1 - \theta X_{T-2} - q_Z(1 - 2\beta R_{T-2} - \rho\theta X_{T-2})}{2\beta(1 + q_Z)}$$

therefore, the optimal sale is a linear function of X_{T-2} and R_{T-2}:

$$N_{T-2}^* = \nu_{1,2} + \nu_{X,2}X_{T-2} + \nu_{R,2}R_{T-2}$$

where

$$\nu_{1,2} = \frac{1 - q_Z}{2\beta(1 + q_Z)}; \nu_{X,2} = \frac{\theta(q_Z\rho - 1)}{2\beta(1 + q_Z)}; \nu_{R,2} = \frac{q_Z}{1 + q_Z}$$

and the optimal value function is given by

$$V_{T-2}^*((P_{T-2}, R_{T-2}, X_{T-2}))$$

$$= q_Z P_{T-2}(c_2^{(0)} + c_2^{(1)} X_{T-2} + c_2^{(2)} X_{T-2}^2 + c_2^{(3)} X_{T-2} R_{T-2} + c_2^{(4)} R_{T-2} + c_2^{(5)} R_{T-2}^2)$$

where

$$c_2^{(0)} = \nu_{1,T-2}(1 - \theta\nu_{1,T-2}) - q_Z \nu_{1,T-2}(1 + \theta\nu_{1,T-2})$$
$$c_2^{(1)} = (1 - q_Z)\nu_{X,2}$$
$$c_2^{(2)} = \nu_{X,2}(\theta\nu_{X,2} - \beta) + q_Z \nu_{X,2}(\beta\rho - \theta\nu_{X,2})$$
$$c_2^{(3)} = -\beta(1+\rho)\nu_{R,2}$$
$$c_2^{(4)} = 2\nu_{R,2}$$
$$c_2^{(5)} = -\theta\nu_{R,2}$$

For the remaining time-steps, some painful algebra gives that

$$N_{T-k}^* = \nu_{1,k} + \nu_{X,k} X_{T-k} + \nu_{R,k} R_{T-k}$$

where

$$\nu_{X,k} = \frac{q_Z \rho c_{k-1}^{(2)} + \beta}{2(q_Z c_{k-1}^{(5)} - \theta)}$$
$$\nu_{R,k} = \frac{q_Z c_{k-1}^{(5)}}{q_Z c_{k-1}^{(5)} - \theta}$$
$$\nu_{1,k} = \frac{q_Z c_{k-1}^{(4)} - 1}{2(q_Z c_{k-1}^{(5)} - \theta)}$$

from which we obtain the value function

$$V_{T-k}^*((P_{T-k}, R_{T-k}, X_{T-k}))$$
$$= q_Z P_{T-k}(c_k^{(0)} + c_k^{(1)} X_{T-k} + c_k^{(2)} X_{T-k}^2 + c_k^{(3)} X_{T-k} R_{T-k}^2 + c_k^{(4)} R_{T-k} + c_k^{(5)} R_{T-k}^2)$$

with coefficients given by

$$c_k^{(0)} = \nu_{1,k}(1 - \theta\nu_{1,k}) + q_Z(c_{k-1}^{(0)} + \sigma_\eta^2 c_{k-1}^{(2)}) - q_Z \nu_{1,k}(c_{k-1}^{(4)} - \nu_{1,k} c_{k-1}^{(5)})$$
$$c_k^{(1)} = q_Z \rho c_{k-1}^{(1)} - \nu_{X,k}(q_Z c_{k-1}^{(4)} - 1)$$
$$c_k^{(2)} = -\nu_{X,k}(\theta\nu_{X,k} + \beta) + q_Z \rho^2 c_{k-1}^{(2)} - q_Z \nu_{X,k}(\rho c_{k-1}^{(3)} - \nu_{X,k} c_{k-1}^{(5)})$$
$$c_k^{(3)} = -\beta\nu_{R,k} + q_Z \rho c_{k-1}^{(3)}(1 - \nu_{R,k})$$
$$c_k^{(4)} = \nu_{R,k} + q_Z(1 - \nu_{R,k}) c_{k-1}^{(4)}$$
$$c_k^{(5)} = -\theta\nu_{R,k}$$

where σ_η^2 denotes the variance of the noise η.