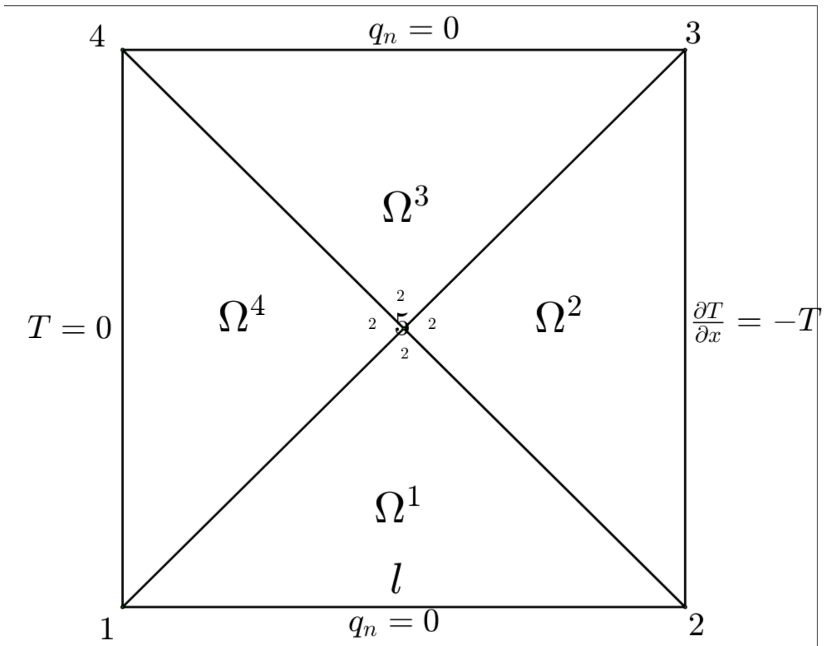
Final exam 21-06-2023

Problem 2

Consider the Poisson heat difussion on the square of side length $\ell = 3$ shown below, meshed by four rectangular triangular finite elements



with the local and global numbering plotted there. We consider that the thermal conductivity is $k_c = 1$ and the internal heating is $f = 2$. Also suppose that the temperature is $T = 0$ on the side determined by the nodes 1 and 4; $q_n = 0$, on the sides between nodes 1 and 2, and 3 and 4; and on the edge between nodes 2 and 3, the BC is $\dfrac{\partial T}{\partial x} = -T$. (you can also formulate this BC as a convection problem for suitables β and T_∞). Answer the following questions :

(a) (3 points) The value of F_5 of the global vector forces is:

Hint 1. The value of the entry K_{54} of the global stiffness matrix K is -1.0000e+00.

Solution.

First we compute the local stiffness matrices and the local force vectors for each element. On the one hand, as we consider the Poisson equation equation ($a_{11} = a_{22} = c$, $a_{12} = a_{21} = a_{00} = 0$), and all the 4 elements are (linear) *right triangles,* to find the stiffness matrices we can use the specific formula:

$$K^e = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}.$$

Recall that, for the above formula to hold, local node 2 must be placed at the right angle of the triangle Ω^e; then, a is the length of the triangle's edge joining local nodes1 and 2, and b is the lenght of the triangle's edges joining local nodes 2 and 3. In our problem at hand $c = k_c = 1/2$, and $a = b$ for all the triangles. Hence,

$$K^e = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \text{ for } e = 1,2,3,4,$$

and the local stiffness matrices are the same for all the four triangles. On the other hand, as the r.h.s. of the equation, the internal heating, is also constant for each element, we can use the formula,

$$F^e = \frac{f^e A^e}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Here $f^e = f = 2$, and the area $A^e = \ell^2/4 = 9/4$ are the same for all the triangles, $e = 1,2,3,4$. Thus, in this case in point,

$$F^e = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ for } e = 1,2,3,4,$$

and the local force vectors are the same for all the four triangles. Therefore,

$$F_3 = F_2^1 + F_2^2 + F_2^3 + F_2^4 = 4 \times \frac{3}{2} = 6, \text{ and } K_{54} = K_{23}^4 + K_{21}^3 = \frac{1}{2}(-1-1) = -1.$$

(b) (3 points) One knows that $Q_2 = c_2 T_2 + c_3 T_3$ for some c_2 and c_3. Then, the value of c_2 is

Hint 2. this coefficient for $Q_2 + Q_3$ is -1.5000e+00.

Solution.

First, we note that the BC $\dfrac{\partial T}{\partial x} = -T$ on the vertical right edge can be thought of as a convenction BC with $\beta = k_c$ and $T_\infty = 0$, so $q_n = -k_c T$ along the vertical right edge. Then, using the formulas for the linear flows:

$$\begin{aligned} Q_2 &= Q_{13}^1 + Q_{33}^2 = Q_{33}^2 = -\beta\ell\left(\frac{T_2}{3} + \frac{T_3}{6}\right) = -k_c\ell\left(\frac{T_2}{3} + \frac{T_3}{6}\right) = -3\left(\frac{T_2}{3} + \frac{T_3}{6}\right) \\ &= -T_2 - \frac{T_3}{2}, \\ Q_3 &= Q_{33}^3 + Q_{13}^2 = Q_{13}^2 = -\beta\ell\left(\frac{T_2}{6} + \frac{T_3}{3}\right) = -k_c\ell\left(\frac{T_2}{6} + \frac{T_3}{3}\right) = -3\left(\frac{T_2}{6} + \frac{T_3}{3}\right) \\ &= -\frac{T_2}{2} - T_3 \end{aligned}$$

(we stress that $Q_{13}^1 = 0$ and $Q_{33}^3 = 0$, since $q_n \equiv 0$ in the horizontal lower and upper edges respectively). Now,

- If we write $Q_2 = c_2T_2 + c_3T_3$, comparison of coeffcients leads to $c_2 = -1$.
- $Q_2 + Q_3 = -\frac{3}{2}T_2 - \frac{3}{2}T_3$ so, if we write $Q_2 + Q_3 = c_2T_2 + c_3T_3$, comparison of coefficients leads to $c_2 = -\frac{3}{2} = -1.5$.

(c) (4 points) The temperature of node 5, T_5, is :

Hint 3. $T_2 - T_3$ is 0.

Solution.

Since, from the (essential) BC we have, $T_1 = T_4 = 0$, the equations for the *free nodes*, nodes 2, 3, and 5,are:

$$
\begin{aligned}
K_{22}T_2 + K_{23}T_3 + K_{25}T_5 &= \left(K^1_{11} + K^2_{33}\right)T_2 + K^2_{31}T_3 + \left(K^1_{12} + K^2_{32}\right)T_5 \\
&= \frac{1}{2}(1+1)T_2 + 0 \cdot T_3 + \frac{1}{2}(-1-1)T_5 \\
&= T_2 - T_5 = Q_2 + F_2, \\
K_{32}T_2 + K_{33}T_3 + K_{35}T_5 &= K_{23}T_2 + \left(K^2_{11} + K^3_{33}\right)T_3 + \left(K^3_{32} + K^2_{12}\right)T_5 \\
&= 0 \cdot T_2 + \frac{1}{2}(1+1)T_3 + \frac{1}{2}(-1-1)T_5 \\
&= T_3 - T_5 = F_3 + Q_3, \\
K_{52}T_2 + K_{53}T_3 + K_{55}T_5 &= K_{25}T_2 + K_{35}T_3 + \left(K^1_{22} + K^2_{22} + K^3_{22} + K^4_{22}\right)T_5 \\
&= \frac{1}{2}(-1-1)T_2 + \frac{1}{2}(-1-1|)T_3 + \frac{1}{2}(2+2+2+2)T_5 \\
&= -T_2 - T_3 + 4T_5 = Q_5 + F_5.
\end{aligned}
$$

Remark: we note that, as T_1 and T_4 are both 0, then $K_{i1}T_1 + K_{i4}T_4 = 0$, for $i = 2, 3, 5$, and no term has to be moved to the rhs in the above equations.

With,

$$
\begin{aligned}
F_2 &= F^1_1 + F^2_3 = \frac{3}{2} + \frac{3}{2} = 3, \\
F_3 &= F^2_1 + F^3_3 = \frac{3}{2} + \frac{3}{2} = 3, \\
F_5 &= F^1_2 + F^2_2 + F^3_2 + F^4_2 = \frac{3}{2} + \frac{3}{2} + \frac{3}{2} + \frac{3}{2} = 6.
\end{aligned}
$$

and Q_2, Q_3, Q_5 given by the (natural) BC, i.e.,

$$
Q_2 = -T_2 - \frac{T_3}{2}, \qquad Q_3 = -\frac{T_2}{2} - T_3, \qquad Q_5 = 0,
$$

Hence, after substitution of these quantities in the above equations, we get,

$$T_2 - T_5 = -T_2 - \frac{T_3}{2} + 3,$$

$$T_3 - T_5 = -\frac{T_2}{2} - T_3 + 3,$$

$$-T_2 - T_3 + 4T_5 = 6,$$

Now moving to the lhs the terms with T_2 and T_3 that appear at the rhs of the first two equations, we see that the reduced system writes as

$$2T_2 + \frac{1}{2}T_3 - T_5 = 3,$$

$$\frac{1}{2}T_2 + 2T_3 - T_5 = 3,$$

$$-T_2 - T_3 + 4T_5 = 6.$$

Substracting the second equations to the first in this last system, it is readily seen that $T_2 = T_3$. Then we can reduce it to the following two linear equation system in T_2 and T_5,

$$5T_2 - 2T_5 = 6,$$

$$-2T_2 + 4T_5 = 6,$$

that can be solved straightforward, for example by Cramer's rule:

$$T_2 = \frac{\begin{vmatrix} 6 & -2 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ -2 & 4 \end{vmatrix}} = \frac{9}{4}, \qquad T_5 = \frac{\begin{vmatrix} 5 & 6 \\ -2 & 6 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ -2 & 4 \end{vmatrix}} = \frac{21}{8}.$$

So finally, the temperatures at the nodes are:

$$T_1 = 0, \qquad T_2 = T_3 = \frac{9}{4}, \qquad T_4 = 0, \qquad T_5 = \frac{21}{8}.$$

Let us check this result with Matlab...

```
T = zeros(5,1);
freeNodes = [2,3,5];
format rat; % We know that the solutions of the system is
            % rational,so show the numbers as rationals

% Reduced system
Km = [2, 1/2,-1; 1/2, 2, -1; -1, -1, 4]; Fm = [3; 3; 6];

% Solve the reduced system
Tm = Km\Fm;
T(freeNodes) = Tm;

% Set up a table showing temperature at each node
sz = [5, 2];
varTypes = ["int64", "double"];
```

```matlab
varNames = ["node", "T(ºC)"];
sols = table('Size', sz,...
    'VariableTypes', varTypes,...
    'VariableNames', varNames);
sols(:,1) = table((1:5)'); sols(:,2) = table(T);
sols
```

sols = 5×2 table

	node	T(ºC)
1	1	0
2	2	9/4
3	3	9/4
4	4	0
5	5	21/8

```matlab
format short e; format compact % back to floating point
```