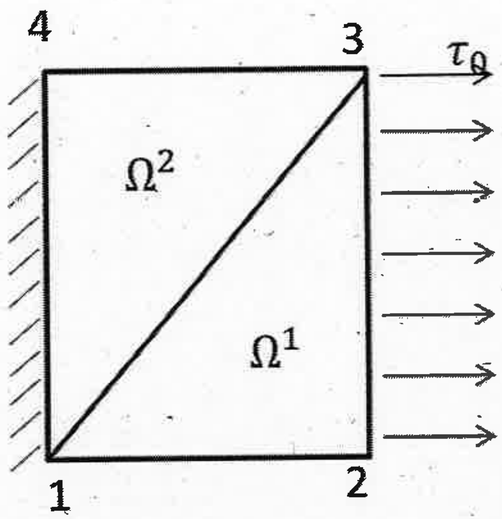
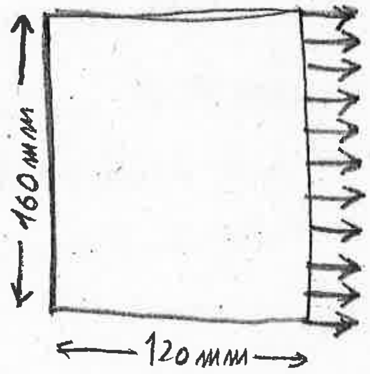
Example:

Consider a rectangular piece of 120x160 mm and thickness 0.036mm. It is fixed to the wall (left) and pulled by a constant traction τ_0 =1000N/mm. Compute the **displacements** if the material has $E = 30 \cdot 10^6 N/mm^2$ and $\nu = 0.25$



planarElasticity2Triangles.pdf

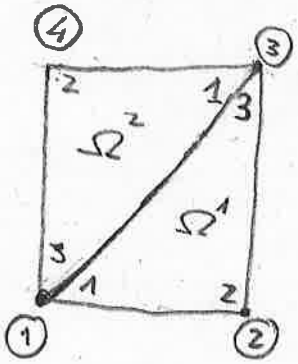
Stress example



dimensions

$\tau_0 = 10^3 \frac{N}{mm^2}$, $E = 3.0 \times 10^7 \frac{N}{mm^2}$, $\nu = 0.25$, $\boxed{t_h = 3.6 \times 10^{-2}\ mm}$

Left edge fixed.



$$nodes = \begin{pmatrix} 0 & 0 \\ 120 & 0 \\ 120 & 160 \\ 0 & 160 \end{pmatrix}$$

$$elem = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{pmatrix}$$

Stress problem:

$$c_{11} = c_{22} = \frac{E}{1-\nu^2} = \frac{3 \times 10^7}{1-0.25^2} = \frac{16}{5} \times 10^7 = 3.2 \times 10^7$$

$$c_{12} = c_{21} = \nu c_{11} = 0.8 \times 10^7$$

$$c_{33} = \frac{E}{2(1+\nu)} = \frac{3 \times 10^7}{2(1+0.25)} = \frac{3 \times 10^7}{2.5} = \frac{6}{5} \times 10^7 = 1.2 \times 10^7, \text{ hence: } C = \begin{pmatrix} c_{11} & c_{12} & \\ c_{12} & c_{22} & \\ & & c_{33} \end{pmatrix} = 4 \cdot 10^6 \begin{pmatrix} 8 & 2 & \\ 2 & 8 & \\ & & 3 \end{pmatrix}$$

- Local stiffness matrices:

K^1: $\begin{pmatrix} 0 & 0 \\ 120 & 0 \\ 120 & 160 \end{pmatrix}$

$$\beta_1^1 = y_2^1 - y_3^1 = 0 - 160 = -160, \qquad \gamma_1^1 = -(x_2^1 - x_3^1) = -(120-120) = 0,$$

$$\beta_2^1 = y_3^1 - y_1^1 = 160 - 0 = 160, \qquad \gamma_2^1 = -(x_3^1 - x_1^1) = -(120-0) = -120,$$

$$\beta_3^1 = y_1^1 - y_2^1 = 0 - 0 = 0. \qquad \gamma_3^1 = -(x_1^1 - x_2^1) = -(0-120) = 120.$$

Area$_1$: $A_1 = \frac{1}{2}\, 120 \times 160 = 9600\ mm^2$.

$$B_1 = \frac{1}{2A_1} \begin{pmatrix} \beta_1^1 & 0 & \beta_2^1 & 0 & \beta_3^1 & 0 \\ 0 & \gamma_1^1 & 0 & \gamma_2^1 & 0 & \gamma_3^1 \\ \gamma_1^1 & \beta_1^1 & \gamma_2^1 & \beta_2^1 & \gamma_3^1 & \beta_3^1 \end{pmatrix} = \frac{1}{480} \begin{pmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -4 & -3 & 4 & 3 & 0 \end{pmatrix},$$

$$K^1 = A_1 t_h B_1^T C B_1 = 6 \cdot 10^3 \cdot \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & -4 \\ 4 & 0 & -3 \\ 0 & -3 & 4 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 8 & 2 & \\ 2 & 8 & \\ & & 3 \end{pmatrix} \cdot \begin{pmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -4 & -3 & 4 & 3 & 0 \end{pmatrix}$$

Therefore:

$$K^1 = 6 \times 10^3 \cdot \left(\begin{array}{cc|cc|cc} 128 & 0 & -128 & 24 & 0 & -24 \\ 0 & 48 & 36 & -48 & -36 & 0 \\ \hline -128 & 36 & 155 & -60 & -27 & 24 \\ 24 & -48 & -60 & 120 & 36 & -72 \\ \hline 0 & -36 & -27 & 36 & 27 & 0 \\ -24 & 0 & 24 & -72 & 0 & 72 \end{array}\right) = K^2$$

since, for the given mesh $\beta_i^2 = -\beta_i^1$, and $\gamma_i^2 = -\gamma_i^1$, $i = 1,2,3$. (*)

- Global Stiffness Matrix

$$\mathbb{K} = \begin{pmatrix} K_{11}^1 + K_{33}^2 & K_{12}^1 & K_{13}^1 + K_{31}^2 & K_{32}^2 \\ * & K_{22}^1 & K_{23}^1 & 0 \\ * & * & K_{33}^1 + K_{11}^2 & K_{12}^2 \\ * & * & * & * \end{pmatrix}$$

Remark 1. $\mathbb{K} = \mathbb{K}^T$ and now K_{ij}^e $(e=1,2)$ are the 2x2 blocs in the local stiffness matrices (*)

$$= 6\times 10^3 \cdot \left(\begin{array}{cc|cc|cc|cc} 155 & 0 & -128 & 24 & 0 & -60 & -27 & 36 \\ 0 & 120 & 36 & -48 & -60 & 0 & 24 & -72 \\ \hline -128 & 36 & 155 & -60 & -27 & 24 & 0 & 0 \\ 24 & -48 & -60 & 120 & 36 & -72 & 0 & 0 \\ \hline 0 & -60 & -27 & 36 & 155 & 0 & -128 & 24 \\ -60 & 0 & 24 & -72 & 0 & 120 & 36 & -48 \\ \hline -27 & 24 & 0 & 0 & -128 & 36 & 155 & -60 \\ 36 & -72 & 0 & 0 & 24 & -48 & -60 & 120 \end{array}\right)$$

- Boundary Conditions
  - Essential B.C. : $(U_1, V_1) = (0,0)$, $(U_4, V_4) = (0,0)$.
  - Natural B.C.

$$Q^1 = \begin{pmatrix} Q_1^1 \\ \hline Q_2^1 \\ \hline Q_3^1 \end{pmatrix} = t_h \frac{h_2^1}{2} \tau_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = t_h 8\times 10^4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad Q^2 = \begin{pmatrix} Q_1^2 \\ \hline Q_2^2 \\ \hline Q_3^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \hline Q_2^2 \\ \hline Q_3^2 \end{pmatrix}$$

$$Q = \begin{pmatrix} Q_1 \\ Q_2 = Q_2^1 \\ Q_3 = Q_3^1 + Q_1^2 \\ Q_4 = Q_2^2 \end{pmatrix} \overset{(*)}{=} t_h \begin{pmatrix} Q_1 \\ \hline 8\times 10^4 \\ 0 \\ \hline 8\times 10^4 \\ 0 \\ \hline Q_4 \end{pmatrix}$$

$$\begin{aligned} Q_3 &= Q_3^1 + Q_1^2 \\ &= \underbrace{Q_{31}^1}_{=0} + Q_{32}^1 + \underbrace{Q_{33}^1}_{=0} \\ &+ \underbrace{Q_{11}^2}_{=0} + \underbrace{Q_{12}^2}_{=0} + Q_{13}^2 = Q_{32}^1 = t_h \int_0^{h_2^1} \psi_{32}^1(s) \begin{pmatrix} \tau_0 \\ 0 \end{pmatrix} ds = t_h \begin{pmatrix} \tau_0 \\ 0 \end{pmatrix} \int_0^{h_2^1} \frac{s}{h_2^1} ds = \frac{h_2^1}{2} \begin{pmatrix} \tau_0 \\ 0 \end{pmatrix} \\ &= t_h \frac{160}{2} \times 10^3 = 8\times 10^4\, t_h \quad (\text{Id. for } Q_2 = Q_2^1) \end{aligned}$$

Reduced system

Remark. $F=0$: no internal forces (for ex. weight, inertial forces, ...) are taken into account

$$\cancel{6\times10^3}\times\left(\begin{array}{cc|cc} 155 & -60 & -27 & 24 \\ -60 & 120 & 36 & -72 \\ \hline -27 & 36 & 155 & 0 \\ 24 & 72 & 0 & 120 \end{array}\right)\left(\begin{array}{c} U_2 \\ V_2 \\ \hline U_3 \\ V_3 \end{array}\right) = t_h \times \cancel{8\times10^4}\ \frac{40}{3}\left(\begin{array}{c} 1 \\ 0 \\ \hline 1 \\ 0 \end{array}\right)$$

$$t_h = 0.036\,mm$$
$$= 0.48\left(\begin{array}{c} 1 \\ 0 \\ \hline 1 \\ 0 \end{array}\right)$$

Solution: $U_2 = 4.0648\times10^{-3}\,mm$

$V_2 = 7.0692\times10^{-4}\,mm$

$U_3 = 3.6406\times10^{-3}\,mm$

$V_3 = -3.8881\times10^{-4}\,mm$

So, the displacements (in mm) are:

$U_1 = 0$	$V_1 = 0$
$U_2 = 4.0648\times10^{-3}$	$V_2 = 7.0692\times10^{-4}$
$U_3 = 3.6406\times10^{-3}$	$V_3 = -3.8881\times10^{-4}$
$U_4 = 0$	$V_4 = 0$

Stress:

Remark 3. When using linear triangular elements, $\sigma = \begin{pmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{pmatrix}$ $\gamma = \begin{pmatrix}\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \partial u/\partial y + \frac{\partial v}{\partial x}\end{pmatrix}$ (i.e. the stress and the strains respectively) are constant on each element.

$$\begin{pmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{pmatrix} = \left(\begin{array}{cc|c} c_{11} & c_{12} & \\ c_{21} & c_{22} & \\ \hline & & c_{33}\end{array}\right)\begin{pmatrix}\partial_x u \\ \partial_y v \\ \partial_y u + \partial_x v\end{pmatrix} ; \quad \begin{pmatrix}\sigma_x^e \\ \sigma_y^e \\ \tau_{xy}^e\end{pmatrix} = C\cdot B_e\begin{pmatrix}u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \\ u_3^e \\ v_3^e\end{pmatrix}, \; e=1,2.$$

for $e=1$

$$\begin{pmatrix} \sigma_x^1 \\ \sigma_y^1 \\ \tau_{xy}^1 \end{pmatrix} = \frac{4\times 10^6}{480} \left(\begin{array}{cc|c} 8 & 2 & \\ 2 & 8 & \\ \hline & & 3 \end{array}\right) \overbrace{\begin{pmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -4 & -3 & 4 & 3 & 0 \end{pmatrix}}^{B_1} \begin{pmatrix} u_1^1 = U^1 = 0 \\ V_1^1 = V^1 = 0 \\ u_2^1 = U_2 = 4.0648\times 10^{-3} \\ V_2^1 = V_2 = 7.0692\times 10^{-4} \\ u_3^1 = U_3 = 3.6406\times 10^{-3} \\ V_3^1 = V_3 = -3.8881\times 10^{-4} \end{pmatrix}$$

$$= \begin{pmatrix} 1.0292\times 10^{3} \\ 5.1844\times 10^{1} \\ 3.8881\times 10^{1} \end{pmatrix}, \text{ in } N/mm^2$$

for $e=2$

$$\begin{pmatrix} \sigma_x^2 \\ \sigma_y^2 \\ \tau_{xy}^2 \end{pmatrix} = \frac{4\times 10^6}{480} \left(\begin{array}{cc|c} 8 & 2 & \\ 2 & 8 & \\ \hline & & 3 \end{array}\right) \overbrace{\begin{pmatrix} 4 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -3 \\ 0 & 4 & 3 & -4 & -3 & 0 \end{pmatrix}}^{B_2} \begin{pmatrix} u_1^2 = U_3 = 3.6406\times 10^{-3} \\ V_1^2 = V_3 = -3.8881\times 10^{-4} \\ u_2^2 = U_4 = 0 \\ V_2^2 = V_4 = 0 \\ u_3^2 = U_1 = 0 \\ V_3^2 = V_1 = 0 \end{pmatrix}$$

$$= \begin{pmatrix} 9.7084\times 10^{-2} \\ 2.4271\times 10^{-2} \\ -3.8881\times 10^{-2} \end{pmatrix}, \text{ in } N/mm^2 \quad \square$$