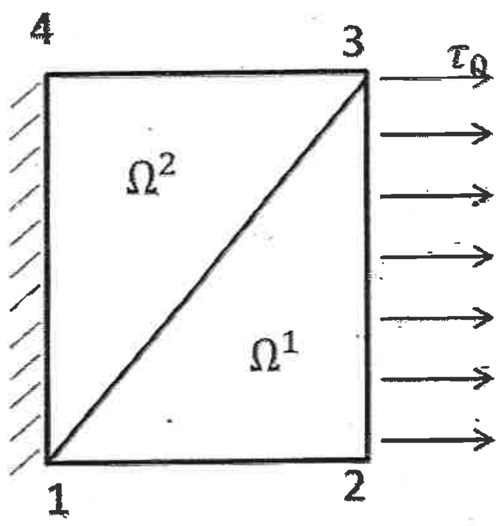
Example:

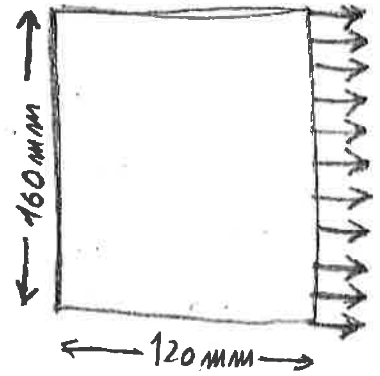
(AMENDED VERSION)

Consider a rectangular piece of 120x160 mm and thickness 0.036mm. It is fixed to the wall (left) and pulled by a constant traction τ_0 =1000N/mm. Compute the **displacements** if the material has $E = 30 \cdot 10^6 N/mm^2$ and $\nu = 0.25$



The traction is given in units of strength/length, so the Ω's must be computed without multiplying by thickness.

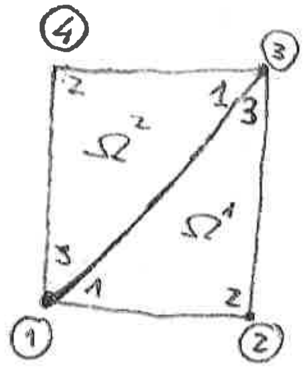
PlaneStress2Triang.pdf



EXAMPLE.

$\tau_0 = 10^3 \frac{N}{mm}$ $\quad E = 3.0 \times 10^7 \frac{N}{mm^2}$, $\nu = 0.25$, $(t_h = 3.6 \times 10^{-2}\, mm$

Left edge fixed.



$$\text{nodes} = \begin{pmatrix} 0 & 0 \\ 120 & 0 \\ 120 & 160 \\ 0 & 160 \end{pmatrix}$$

$$\text{elem} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{pmatrix}$$

Stress problem:

$$c_{11} = c_{22} = \frac{E}{1-\nu^2} = \frac{3\times10^7}{1-0.25^2} = \frac{16}{5}\times10^7 = 3.2\times10^7$$

$$c_{12} = c_{21} = \nu c_{11} = 0.8\times10^7$$

$$c_{33} = \frac{E}{2(1+\nu)} = \frac{3\times10^7}{2(1+0.25)} = \frac{3\times10^7}{2.5} = \frac{6}{5}\times10^7 = 1.2\times10^7, \text{ hence: } C = \begin{pmatrix} c_{11} & c_{12} & \\ c_{12} & c_{22} & \\ & & c_{33} \end{pmatrix} = 4\cdot10^6 \begin{pmatrix} 8 & 2 & \\ 2 & 8 & \\ & & 3 \end{pmatrix}$$

- Local stiffness matrices:

$$K^1: \begin{pmatrix} 0 & 0 \\ 120 & 0 \\ 120 & 160 \end{pmatrix}$$

$$\beta_1^1 = y_2^1 - y_3^1 = 0-160 = -160, \quad \gamma_1^1 = -(x_2^1 - x_3^1) = -(120-120) = 0,$$
$$\beta_2^1 = y_3^1 - y_1^1 = 160-0 = 160, \quad \gamma_2^1 = -(x_3^1 - x_1^1) = -(120-0) = -120,$$
$$\beta_3^1 = y_1^1 - y_2^1 = 0-0 = 0. \quad \gamma_3^1 = -(x_1^1 - x_2^1) = -(0-120) = 120.$$

Area$_1$: $A_1 = \frac{1}{2}\, 120 \times 160 = 9600\ mm^2$.

$$B_1 = \frac{1}{2A_1}\begin{pmatrix} \beta_1^1 & 0 & \beta_2^1 & 0 & \beta_3^1 & 0 \\ 0 & \gamma_1^1 & 0 & \gamma_2^1 & 0 & \gamma_3^1 \\ \gamma_1^1 & \beta_1^1 & \gamma_2^1 & \beta_2^1 & \gamma_3^1 & \beta_3^1 \end{pmatrix} = \frac{1}{480}\begin{pmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -4 & -3 & 4 & 3 & 0 \end{pmatrix}$$

$$K^1 = A_1 t_h B_1^T C B_1 = 6\cdot10^3 \cdot \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & -4 \\ 4 & 0 & -3 \\ 0 & -3 & 4 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 8 & 2 & \\ 2 & 8 & \\ & & 3 \end{pmatrix} \cdot \begin{pmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -4 & -3 & 4 & 3 & 0 \end{pmatrix}$$

Therefore:

$$K^1 = 6\times10^3 \cdot \left(\begin{array}{cc|cc|cc} 128 & 0 & -128 & 24 & 0 & -24 \\ 0 & 48 & 36 & -48 & -36 & 0 \\ \hline -128 & 36 & 155 & -60 & -27 & 24 \\ 24 & -48 & -60 & 120 & 36 & -72 \\ \hline 0 & -36 & -27 & 36 & 27 & 0 \\ -24 & 0 & 24 & -72 & 0 & 72 \end{array}\right) = K_1^2$$

since, for the given mesh $\beta_i^2 = -\beta_i^1$, and $\gamma_i^2 = -\gamma_i^1$, $i = 1,2,3$.

(*

- **Global Stiffness Matrix**

$$\mathbb{K} = \begin{pmatrix} K^1_{11}+K^2_{33} & K^1_{12} & K^1_{13}+K^2_{31} & K^2_{32} \\ * & K^1_{22} & K^1_{23} & 0 \\ * & * & K^1_{33}+K^2_{11} & K^2_{12} \\ * & * & * & * \end{pmatrix}$$

Remark1. $\mathbb{K}=\mathbb{K}^T$ and now K^e_{ij} $(e=1,2)$ are the 2x2 blocs in the local stiffness matrices (*)

$$= 6\times 10^3 \cdot \left(\begin{array}{cc|cc|cc|cc} 155 & 0 & -128 & 24 & 0 & -60 & -27 & 36 \\ 0 & 120 & 36 & -48 & -60 & 0 & 24 & -72 \\ \hline -128 & 36 & 155 & -60 & -27 & 24 & 0 & 0 \\ 24 & -48 & -60 & 120 & 36 & -72 & 0 & 0 \\ \hline 0 & -60 & -27 & 36 & 155 & 0 & -128 & 24 \\ -60 & 0 & 24 & -72 & 0 & 120 & 36 & -48 \\ \hline -27 & 24 & 0 & 0 & -128 & 36 & 155 & -60 \\ 36 & -72 & 0 & 0 & 24 & -48 & -60 & 120 \end{array}\right)$$

(*) Remark: the traction τ_0 is given in N/mm (units: Force/length), so we compute the Q's without multiplying by the thickness

- Boundary Conditions
  - Essential B.C.: $(U_1,V_1)=(0,0)$, $(U_4,V_4)=(0,0)$.
  - Natural B.C.

$$Q^1 = \begin{pmatrix} Q^1_1 \\ \hline Q^1_2 \\ \hline Q^1_3 \end{pmatrix} \overset{(*)}{=} \cancel{t}\,\frac{h^1_2}{2}\,\tau_0 \begin{pmatrix} 0\\0\\1\\0\\1\\0 \end{pmatrix} = \cancel{t}\,8\times10^4 \begin{pmatrix} 0\\0\\1\\0\\1\\0 \end{pmatrix} \qquad Q^2 = \begin{pmatrix} Q^2_1 \\ \hline Q^2_2 \\ \hline Q^2_3 \end{pmatrix} = \begin{pmatrix} 0\\0\\ \hline Q^2_2 \\ \hline Q^2_3 \end{pmatrix}$$

$$Q = \begin{pmatrix} Q_1 \\ Q_2 = Q^1_2 \\ Q_3 = Q^1_3 + Q^2_1 \\ Q_4 = Q^2_2 \end{pmatrix} \overset{(*)}{=} \cancel{t} \begin{pmatrix} Q_1 \\ \hline 8\times10^4 \\ 0 \\ \hline 8\times10^4 \\ 0 \\ \hline Q_4 \end{pmatrix}$$

$$Q_3 = Q^1_3 + Q^2_1$$

$$= \underbrace{Q^1_{31}}_{=0} + Q^1_{32} + Q^1_{33} + \underbrace{Q^2_{11}}_{0} + \underbrace{Q^2_{12}}_{0} + Q^2_{13} = Q^1_{32} = \cancel{t}\int_0^{h^1_2} \psi^1_{32}(s) \begin{pmatrix}\tau_0\\0\end{pmatrix} ds = \cancel{t}\begin{pmatrix}\tau_0\\0\end{pmatrix}\int_0^{h^1_2} \frac{s}{h^1_2}\, ds = \frac{h^1_2}{2}\,\cancel{t}\begin{pmatrix}\tau_0\\0\end{pmatrix}$$

$$= \cancel{t}\,\frac{160}{2}\times10^3 = 8\times10^4\,\cancel{t} \quad (\text{Id. for } Q_2 = Q^1_2)$$

Reduced system

Remark. $F=0$: no internal forces (for ex. weight, inertial forces,...) are taken into account

$$\cancel{6\times10^{3}}\times\left(\begin{array}{cc|cc} 155 & -60 & -27 & 24 \\ -60 & 120 & 36 & -72 \\ \hline -27 & 36 & 155 & 0 \\ 24 & -72 & 0 & 120 \end{array}\right)\begin{pmatrix} U_2 \\ V_2 \\ \hline U_3 \\ V_3 \end{pmatrix} = \cancel{t_h}\times\cancel{8\times10^{4}}\ \tfrac{40}{3}\begin{pmatrix} 1 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix}$$

$\cancel{t_h = 0.036\,mm}$

$$= \cancel{0.48}\ \tfrac{4}{3}\begin{pmatrix} 10 \\ 0 \\ \hline 10 \\ 0 \end{pmatrix}$$

Solution:
$U_2 = \cancel{4.0648\times10^{-3}}$ mm 1.1291×10^{-1} mm
$V_2 = \cancel{7.0692\times10^{-4}}$ mm 1.9637×10^{-2} mm
$U_3 = \cancel{3.6406\times10^{-3}}$ mm 1.0113×10^{-1} mm
$V_3 = \cancel{-3.8881\times10^{-4}}$ mm -1.0800×10^{-2} mm

So, the displacements (in mm) are:

~~$U_1=0$ $V_1=0$~~
~~$U_2=4.0648\times10^{-3}$ $V_2=7.0692\times10^{-4}$~~
~~$U_3=3.6406\times10^{-3}$ $V_3=-3.8881\times10^{-4}$~~
~~$U_4=0$ $V_4=0$~~

(Global)

Nod.	U (mm)	V (mm)
1	0	0
2	1.1291×10^{-1}	1.9637×10^{-2}
3	1.0113×10^{-1}	-1.0800×10^{-2}
4	0	0

Stress:

Remark 3. When using linear triangular elements, $\sigma=\begin{pmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{pmatrix}$ $\gamma=\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \partial u/\partial y+\frac{\partial v}{\partial x}\end{pmatrix}$ (i.e. the stress and the strains respectively) are constant on each element.

$$\begin{pmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{pmatrix}=\left(\begin{array}{cc|c} c_{11} & c_{12} & \\ c_{21} & c_{22} & \\ \hline & & c_{33}\end{array}\right)\begin{pmatrix}\partial_x u\\ \partial_y v\\ \partial_y u+\partial_x v\end{pmatrix};\quad \begin{pmatrix}\sigma_x^e\\ \sigma_y^e\\ \tau_{xy}^e\end{pmatrix}=C\cdot B_e\begin{pmatrix}u_1^e\\ v_1^e\\ u_2^e\\ v_2^e\\ u_3^e\\ v_3^e\end{pmatrix},\ e=1,2.$$

for $e=1$,

$$\begin{pmatrix}\sigma_x^1\\ \sigma_y^1\\ \sigma_{xy}^1\end{pmatrix} = \frac{4\times10^6}{480}\begin{pmatrix}8 & 2 & \\ 2 & 8 & \\ & & 3\end{pmatrix}\overbrace{\begin{pmatrix}-4 & 0 & 4 & 0 & 0 & 0\\ 0 & 0 & 0 & -3 & 0 & 3\\ 0 & -4 & -3 & 4 & 3 & 0\end{pmatrix}}^{\text{"}B_1}\begin{pmatrix}u_1^1 = U^1 = 0\\ V_1^1 = V^1 = 0\\ u_2^1 = U_2 = 4.0648\times10^{-3}\\ V_2^1 = V_2 = 7.0692\times10^{-4}\\ u_3^1 = U_3 = 3.6406\times10^{-3}\\ V_3^1 = V_3 = -3.8881\times10^{-4}\end{pmatrix}$$

$$= \begin{pmatrix}1.0292\times10^3\\ 5.1841\times10^1\\ 3.8881\times10^1\end{pmatrix}, \text{ in } N/mm^2$$

for $e=2$,

$$\begin{pmatrix}\sigma_x^2\\ \sigma_y^2\\ \sigma_{xy}^2\end{pmatrix} = \frac{4\times10^6}{380}\begin{pmatrix}8 & 2 & \\ 2 & 8 & \\ & & 3\end{pmatrix}\overbrace{\begin{pmatrix}4 & 0 & -4 & 0 & 0 & 0\\ 0 & 0 & 0 & 3 & 0 & -3\\ 0 & 4 & 3 & -4 & -3 & 0\end{pmatrix}}^{\text{"}B_2}\begin{pmatrix}u_1^2 = U_3 = 3.6406\times10^{-3}\\ V_1^2 = V_3 = -3.8881\times10^{-4}\\ u_2^2 = U_4 = 0\\ V_2^2 = V_4 = 0\\ u_3^2 = U_1 = 0\\ V_3^2 = V_1 = 0\end{pmatrix}$$

$$= \begin{pmatrix}9.7084\times10^2\\ 2.4271\times10^2\\ -3.8881\times10^1\end{pmatrix}, \text{ in } N/mm^2 \quad \square$$

for $e=1$

$$\begin{pmatrix}\sigma_x^1\\ \sigma_y^1\\ \sigma_{xy}^1\end{pmatrix} = \frac{10^5}{12}\cdot\begin{pmatrix}8 & 2 & \\ 2 & 8 & \\ & & 3\end{pmatrix}\cdot\begin{pmatrix}-4 & 0 & 4 & 0 & 0 & 0\\ 0 & 0 & 0 & -3 & 0 & 3\\ 0 & -4 & -3 & 4 & 3 & 0\end{pmatrix}\cdot\begin{pmatrix}u_1^1 = U_1 = 0\\ V_1^1 = V_1 = 0\\ u_2^1 = U_2 = 1.1291\times10^{-1}\\ V_2^1 = V_2 = 1.9637\times10^{-2}\\ u_3^1 = U_3 = 1.0113\times10^{-1}\\ V_3^1 = V_3 = -1.0800\times10^{-2}\end{pmatrix}$$

$$= \begin{pmatrix}2.8588\times10^4\\ 1.4400\times10^3\\ 1.0800\times10^3\end{pmatrix}, \text{ in } N/mm^2.$$

for $e=2$

$$\begin{pmatrix}\sigma_x^2\\ \sigma_y^2\\ \sigma_{xy}^2\end{pmatrix} = \frac{10^5}{12}\cdot\begin{pmatrix}8 & 2 & \\ 2 & 8 & \\ & & 3\end{pmatrix}\cdot\begin{pmatrix}4 & 0 & -4 & 0 & 0 & 0\\ 0 & 0 & 0 & 3 & 0 & -3\\ 0 & 4 & 3 & -4 & -3 & 0\end{pmatrix}\cdot\begin{pmatrix}u_1^2 = U_3 = 1.0113\times10^{-1}\\ V_1^2 = V_3 = -1.0800\times10^{-2}\\ u_2^2 = U_4 = 0\\ V_2^2 = V_4 = 0\\ u_3^2 = U_1 = 0\\ V_3^2 = V_1 = 0\end{pmatrix}$$

$$= \begin{pmatrix}2.6968\times10^4\\ 6.7419\times10^3\\ -1.0800\times10^3\end{pmatrix}, \text{ in } N/mm^2 \quad \square$$