problema_1.pdf

① Donada l'equació $-\frac{d}{dx}\left(a_1(x)\frac{du}{dx}\right)+a_0(x)u=f(x)$, $0<x<1$, prenem una malla d'elements finits lineals $\Omega^1,\dots,\Omega^N$ amb $\Omega^K=[x_K,x_{K+1}]$, $h_K=x_{K+1}-x_K$, $a_1(x)=a_1^K x$, $a_0(x)=a_0^K$, i $f^K$ són constants que depenen de cada element $\Omega^K$. Vegeu que les matrius de rigidesa elementals i els vectors de càrrega elementals són:

$$[K^K]=\frac{a_1^K}{h_K}\left(\frac{x_K+x_{K+1}}{2}\right)\cdot\begin{pmatrix}1&-1\\-1&1\end{pmatrix}+\frac{a_0^K h_K}{6}\begin{pmatrix}2&1\\1&2\end{pmatrix},\quad F^K=\frac{1}{2}f^K h_K\begin{pmatrix}1\\1\end{pmatrix}.$$

◁ Solució. Suposem que $\Omega^K=[x_K,x_{K-1}]$, $K=0,1,2,\dots,N$. Les funcions de forma són $\psi_1^K(x)=\frac{x-x_{K+1}}{x_K-x_{K+1}}=-\frac{1}{h_K}(x-x_{K+1})$, $\psi_2^K(x)=\frac{x-x_K}{x_{K+1}-x_K}=\frac{1}{h_K}(x-x_K)$.

Llavors:

$$K_{1,1}^{K,1}=\int_{x_K}^{x_{K+1}}a_1(x)\overbrace{\frac{d\psi_1^K}{dx}}^{-1/h_K}(x)\cdot\frac{d\psi_1^K}{dx}(x)\,dx=\int_{x_K}^{x_{K+1}}a_1^K x\frac{dx}{h_K}=\frac{a_1^K}{2h_K^2}\left(x_{K+1}^2-x_K^2\right)$$

$$=a_1^K\frac{x_{K+1}^2-x_K^2}{2(x_{K+1}-x_K)^2}=\boxed{\frac{a_1^K}{2h_K}\left(x_{K+1}+x_K\right)=K_{22}^{K,1}}\quad(*)$$

$$K_{1,2}^{K,1}=\int_{x_K}^{x_{K+1}}a_1(x)\frac{d\psi_1^K}{dx}(x)\frac{d\psi_2^K}{dx}(x)\,dx=-\int_{x_K}^{x_{K+1}}a_1^K x\frac{dx}{h_K^2}=\boxed{-\frac{1}{2h_K}\left(x_{K+1}+x_K\right)=K_{21}^{K,1}}$$

(*) Notem que: $\frac{d\psi_2^K}{dx}\cdot\frac{d\psi_2^K}{dx}=\frac{1}{h_K^2}=\frac{d\psi_1^K}{dx}\cdot\frac{d\psi_1^K}{dx}$

Aleshores:

$$[K^{K,1}]=\frac{1}{2h_K}(x_{K+1}+x_K)\cdot\begin{pmatrix}1&-1\\-1&1\end{pmatrix},$$

D'altra banda:

$$K_{11}^{K,0}=\int_{x_K}^{x_{K+1}}a_0(x)\psi_1^K(x)\psi_1^K(x)\,dx=\frac{a_0^K}{h_K^2}\int_{x_K}^{x_{K+1}}\left(x-x_{K+1}\right)^2dx=\frac{a_0^K}{3h_K^2}h_K^3=\boxed{\frac{a_0^K h_K}{3}}$$

$$K_{12}^{K,0}=\int_{x_K}^{x_{K+1}}a_0(x)\psi_1^K(x)\psi_2^K(x)\,dx=-\frac{a_0^K}{h_K^2}\int_{x_K}^{x_{K+1}}(x-x_{K+1})(x-x_K)\,dx$$

$$=\left\{\begin{array}{c}\text{C.V.}\\ s=x-x_K\\ \Leftrightarrow x=s+x_K\end{array}\right\}=-\frac{a_0^K}{h_K^2}\int_0^{h_K}(s-h_K)s=-\frac{a_0^K}{h_K^2}\left(\frac{s^3}{3}-\frac{h_K s^2}{2}\right)\Bigg|_{s=0}^{s=h_K}$$

$$=\boxed{\frac{a_0^K h}{6}=K_{21}^K}$$

$$K_{22}^{K,0}=\int_{x_K}^{x_{K+1}}a_0(x)\psi_2^K(x)\psi_2^K(x)\,dx=\frac{a_0^K}{h_K^2}\int_{x_K}^{x_{K+1}}(x-x_K)^2dx=\frac{a_0^K}{3h_K^2}h_K^3=\boxed{\frac{a_0^K h_K}{3}}$$

Aleshores:

$$[K^{k,0}] = \frac{a_0^k h_k}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Per últim:

$$F_1^k = \int_{x_k}^{x_{k+1}} f(x)\,\psi_1^k(x)\,dx = \int_{x_k}^{x_{k+1}} f^k \cdot \frac{x - x_{k+1}}{x_k - x_{k+1}}\,dx = -\frac{f^k}{2h_k}(x - x_{k+1})^2\Big|_{x_k}^{x_{k+1}} = \frac{f^k h_k}{2}$$

$$F_2^k = \int_{x_k}^{x_{k+1}} f(x)\,\psi_2^k(x)\,dx = \int_{x_k}^{x_{k+1}} f^k\,\frac{x - x_k}{x_{k+1} - x_k}\,dx = \frac{f^k}{2h_k}(x - x_k)^2\Big|_{x_k}^{x_{k+1}} = \frac{f^k h_k}{2}$$

Aleshores:

$$F^k = \frac{f^k h_k}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

La matriu de rigidesa i el vector de càrrega elementals de l'element $\Omega^k$ són doncs:

$$[K^k] = [K^{k,1}] + [K^{k,0}] = \frac{a_1^k}{h_k}\left(\frac{x_k + x_{k+1}}{2}\right)\cdot\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{a_0^k h_k}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$F^k = \frac{1}{2} f^k h_k \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \triangleright$$