UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria Industrial de Barcelona
ETSEIB

Mètodes Numèrics:

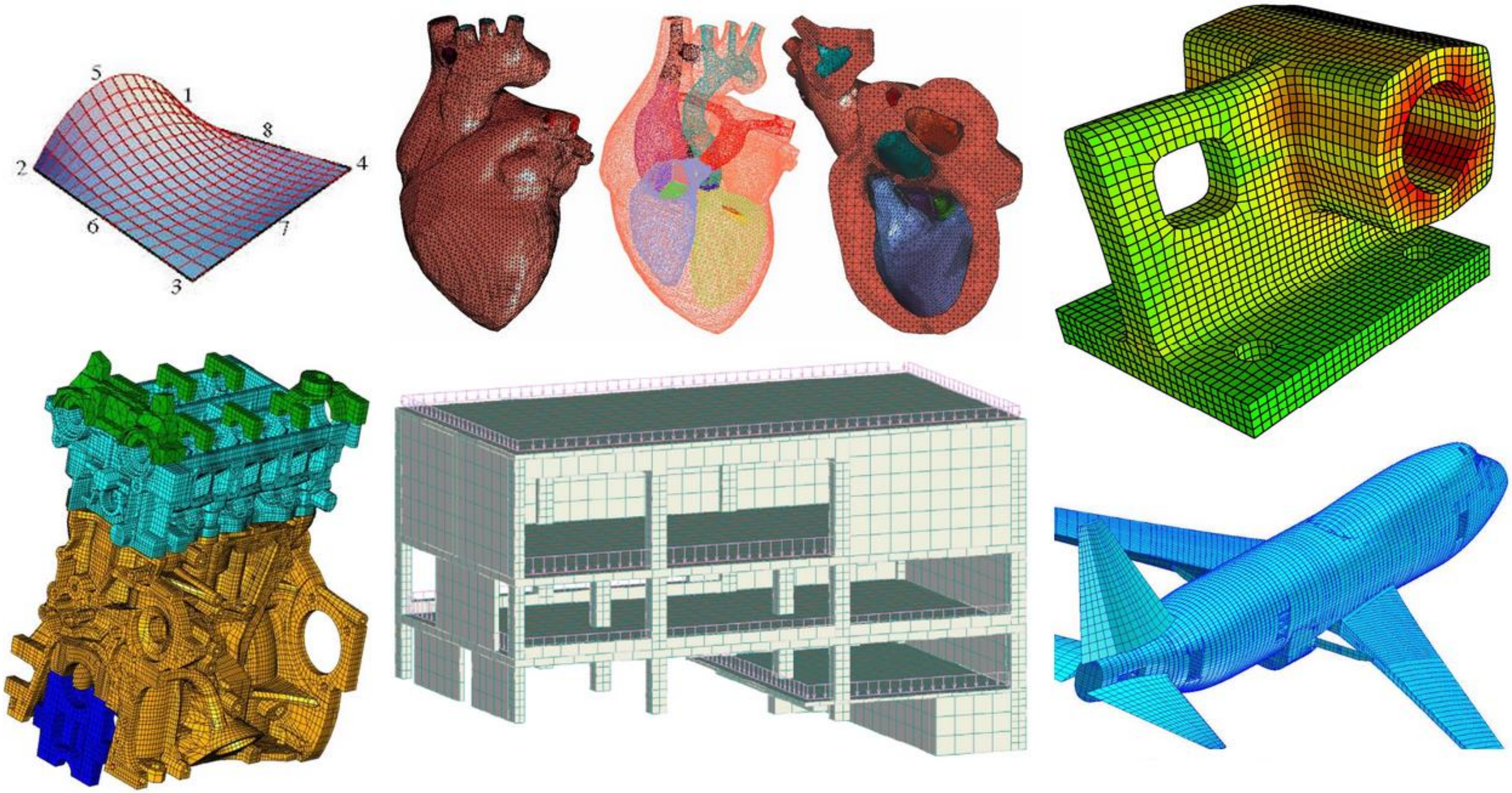
A First Course on Finite Elements

Finite Elements (I)

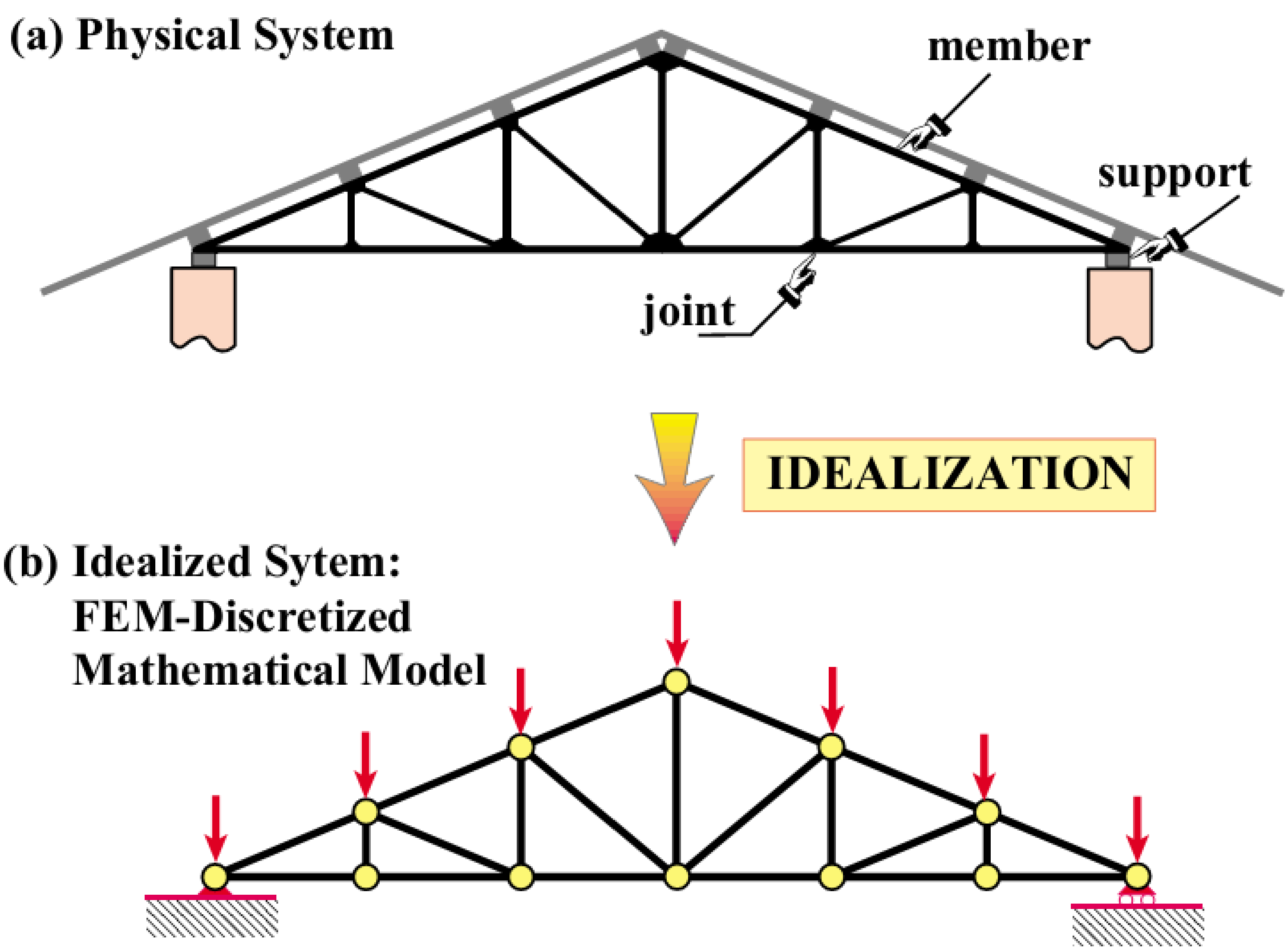
Following: *Curs d'Elements Finits amb Aplicacions* (J. Masdemont)
http://hdl.handle.net/2099.3/36166

Dept. Matemàtiques ETSEIB - UPC BarcelonaTech

Only nice Images?



NO!! …Nice computations



NO!! …Nice computations

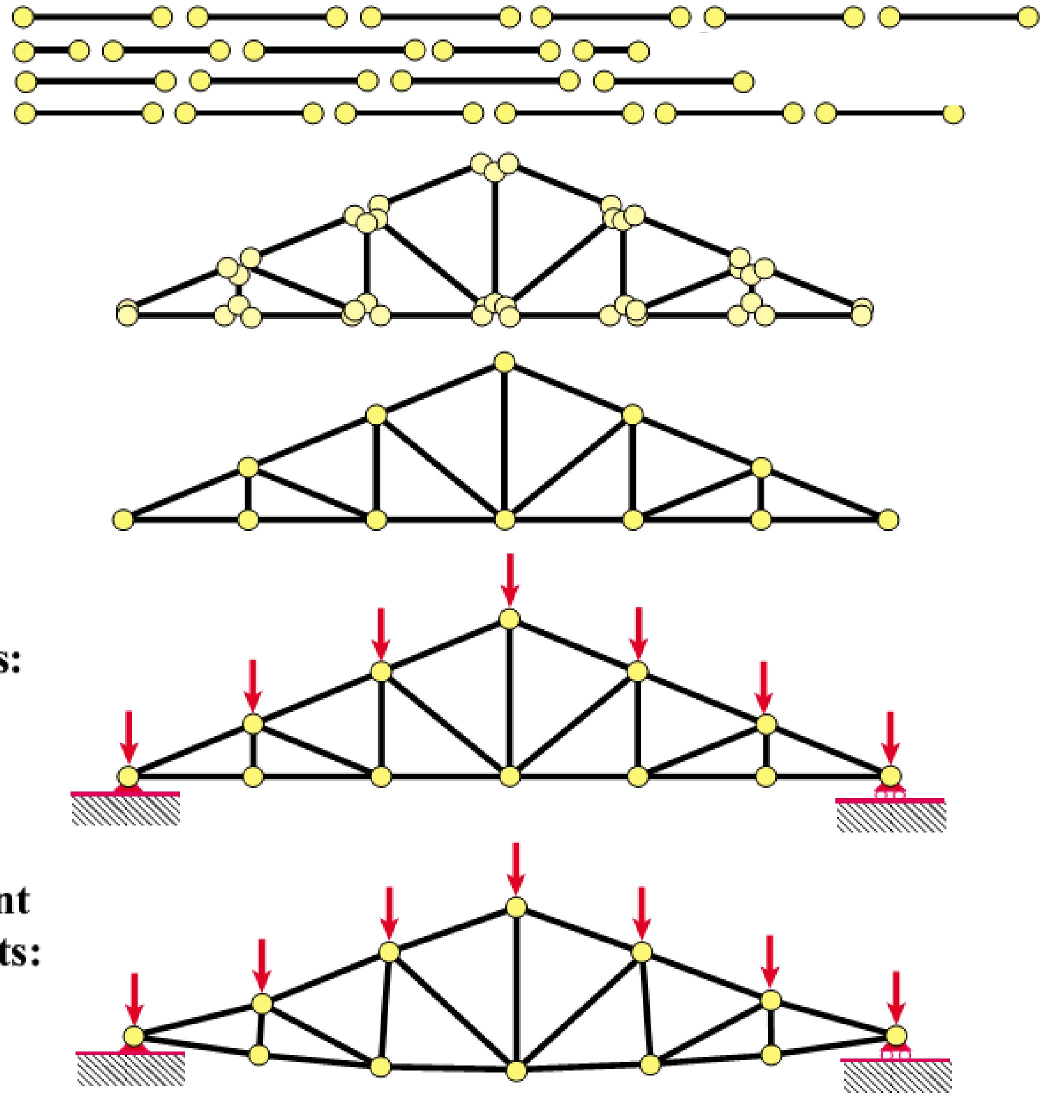
Form elements:

Globalize:

Merge:

Apply loads and supports:

Solve for joint displacements:



Generalized Approach

Physical Problem
(Modeling Equations)

←

Mathematical tools

Partial Differential Equations

Weak Problem Formulation

Shape functions (interpolation)

Numerical Integration

$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

1D Model Equation

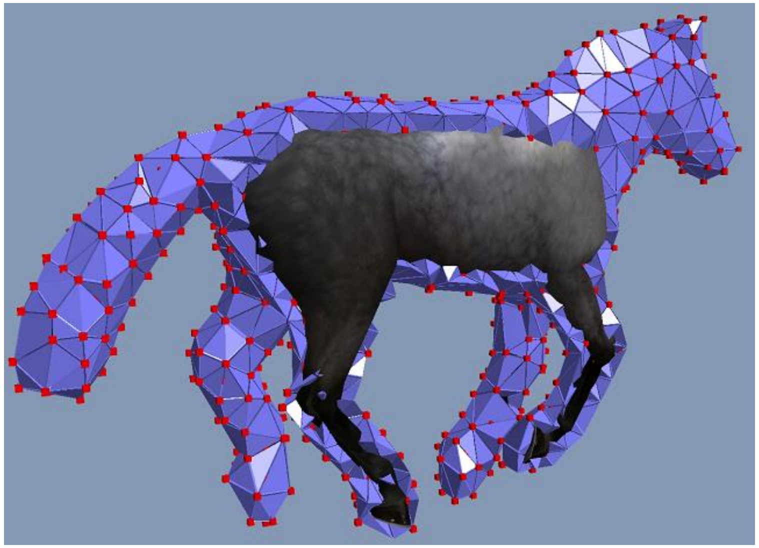
2D Model Equation

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}u = f,$$

Generalized Approach

Physical Problem
(Modeling Equations)

Domain discretization
(Elements, Nodes)



Mathematical steps

Element dimension and shape

Meshing general domains

Impose **Boundary Conditions**

Obtain element matrices

Enumerate Element and Nodes

Global Assembly

Generalized Approach

Physical Problem
(Modeling Equations)

↓

Domain discretization
(Elements, Nodes)

↓

Problem Solution
(Linear Systems)

←

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ y_n \end{pmatrix}$$

$$A \cdot x = y$$

<u>Mathematical steps</u>

Matrix shape and reordering

Vector and Matrix Manipulation

Numerical methods for Linear Systems

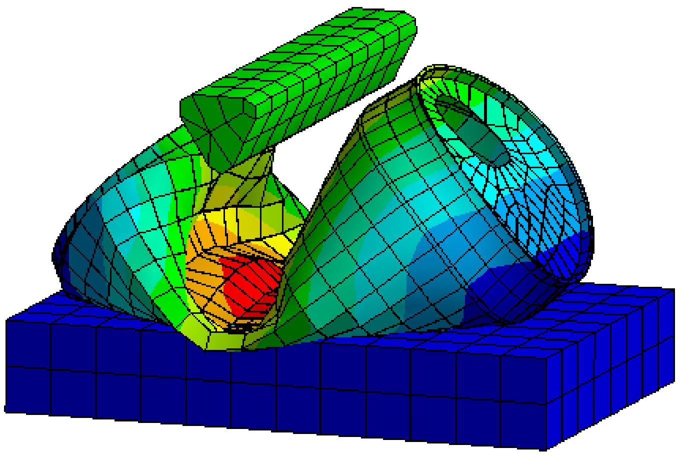
Generalized Approach

Physical Problem
(Modeling Equations)

Domain discretization
(Elements, Nodes)

Problem Solution
(Linear Systems)

Solution Postprocess
(Show new results)



Mathematical steps

Compute secondary variables

Plot results

Evaluate solution: critical points, remeshing, estimate errors, etc.

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Mètodes Numèrics:

A First Course on Finite Elements

Finite Elements (II)

1D Finite Elements

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Weak Formulation: intuition

Idea of the Weak Formulation

- We need the solution of the **model equation**:

(1) $$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right)+a_0(x)u=f(x),$$

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That's hard!!!

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- We need the solution of the **model equation**:

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$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

- Instead, we use an **integral equation**:

(2)
$$\int_{\Omega^k} \omega\left[-\frac{d}{dx}\left(a_1\frac{du}{dx}\right) + a_0 u - f\right] dx = 0$$

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Question:

If we have $\int_{\Omega^k} F(u,x)dx = 0$ $\Longrightarrow$ $F(u,x) = 0$

Idea of the Weak Formulation

- We need the solution of the **model equation**:

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$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

(2)
$$\int_{\Omega^k} \omega \left[-\frac{d}{dx}\left(a_1 \frac{du}{dx}\right) + a_0 u - f\right] dx = 0$$

Answer: NO!!!

If we have $\int_{\Omega^k} F(u,x)dx = 0$ $\quad \cancel{\xrightarrow{?}} \quad$ $F(u,x) = 0$

Idea of the Weak Formulation

- We need the solution of the **model equation**:

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$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

(2)
$$\int_{\Omega^k} \omega \left[-\frac{d}{dx}\left(a_1\frac{du}{dx}\right) + a_0 u - f\right] dx = 0$$

What about if we have?

$$\left.\begin{array}{l} \displaystyle\int_{\Omega^k} \omega_1(x)\cdot F(u,x)dx = 0 \\ \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ \displaystyle\int_{\Omega^k} \omega_n(x)\cdot F(u,x)dx = 0 \end{array}\right\} \xrightarrow{\boxed{?}} F(u,x) = 0$$

Idea of the Weak Formulation

- We need the solution of the **model equation**:

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$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

(2)
$$\int_{\Omega^k} \omega \left[-\frac{d}{dx}\left(a_1\frac{du}{dx}\right) + a_0 u - f\right] dx = 0$$

Answer: ….almost!!!

$$\left.\begin{array}{l} \displaystyle\int_{\Omega^k} \omega_1(x)\cdot F(u,x)dx = 0 \\ \qquad\qquad\vdots \qquad\qquad\qquad \vdots \\ \displaystyle\int_{\Omega^k} \omega_n(x)\cdot F(u,x)dx = 0 \end{array}\right\} \xrightarrow{\boxed{?}} F(u,x) = 0$$

Idea of the Weak Formulation

- We need the solution of the **model equation**:

(1) $$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

(2) $$\int_{\Omega^k} \omega \left[-\frac{d}{dx}\left(a_1 \frac{du}{dx}\right) + a_0 u - f\right] dx = 0$$

If the functions $\omega_i(x)$ are the **shape functions** (the base of the polynomials) the solution of (2) is a good approximation of the solution of (1).
This solution is known as the **weak solution** of equation (1)

Weak Formulation: Computation

The 1D case: Weak Formulation

The **integral equation** for the 1D case is:

$$\int_{\Omega^k} \omega \left[-\frac{d}{dx}\left(a_1 \frac{du}{dx} \right) + a_0 u - f \right] dx = 0$$

The 1D case: Weak Formulation

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Considering only the first term, we can use the Integration by parts formula:

$$-\int \omega \left[\frac{d}{dx}\left(a_1 \frac{du}{dx} \right) \right] dx = -\omega a_1 \frac{du}{dx} + \int a_1 \frac{d\omega}{dx}\frac{du}{dx} dx$$

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That, for a 1D domain $\Omega^{\mathrm{k}} = [x_A, x_B]$, give us the expression:

$$\int_{x_A}^{x_B} \left(a_1 \frac{d\omega}{dx}\frac{du}{dx} + a_0 \omega u - \omega f \right) dx - \left[\omega a_1 \frac{du}{dx} \right]_{x_A}^{x_B} = 0$$

The 1D case: Weak Formulation

Some **notation**:

Variables:

The unknown function $u = u(x)$ is known as the **primary variable**

and it's **derivative** $a_1 \dfrac{du}{dx}$ is known as the **secondary variable**

The 1D case: Weak Formulation

Some **notation**:

Variables:

The unknown function $u = u(x)$ is known as the **primary variable**

and it's **derivative** $a_1 \dfrac{du}{dx}$ is known as the **secondary variable**

Boundary Conditions:

To fix the value at the primary variable $u(x_A) = u_A$ or $u(x_B) = u_B$

is known as to give **an essential BC**

To fix the value at the secondary variable is known as to give **a natural BC**

$$Q_A = -\left(a_1 \frac{du}{dx}\right)\Big|_{x=x_A} \qquad Q_B = \left(a_1 \frac{du}{dx}\right)\Big|_{x=x_B}$$

The 1D case: Weak Formulation

Some **notation**:

Variables:

The unknown function $u = u(x)$ is known as the **primary variable**

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$$Q_A = -\left(a_1 \frac{du}{dx}\right)_{|x=x_A} \quad \text{i,} \quad Q_B = \left(a_1 \frac{du}{dx}\right)_{|x=x_B}$$

Notice the negative sign on the left boundary point

The 1D case: Weak Formulation

Now for a 1D element $\Omega^{\mathrm{k}} = [x_A, x_B]$, we can choose a certain degree n for interpolating the values of a function $u = u(x)$

$$u(x) = \sum_{j=1}^{n} u_j^k \psi_j^k(x)$$ and the derivative $$\frac{du}{dx} = \sum_{j=1}^{n} u_j^k \frac{d\psi_j^k}{dx}$$

where the funtions $\psi_j^k(x)$ are the corresponding **shape functions** (Lagrange's Polynomials) and u_j^k are the unknowns values at the **nodes** (interpolation points)

The 1D case: Weak Formulation

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where the funtions $\psi_j^k(x)$ are the corresponding **shape functions** (Lagrange's Polynomials) and u_j^k are the unknowns values at the **nodes** (interpolation points)

$$\int_{x_A}^{x_B} \left(a_1 \frac{d\omega}{dx}\frac{du}{dx} + a_0 \omega u - \omega f \right) dx - \left[\omega a_1 \frac{du}{dx} \right]_{x_A}^{x_B} = 0$$

As we mention, the **weight functions** on the Weak integral equation, can be substituted by $\omega = \psi_j^k(x),\ j = 1 \ldots n$

The 1D case: Weak Formulation

Substituing now, both the interpolated function, and writing the integral equation for each $\omega = \psi_j^k(x),\ j = 1 \dots n$ we get a system of equations where the unknowns are u_j^k:

$$
\left.
\begin{array}{l}
\displaystyle\int_{x_A}^{x_B} \left[a_1 \frac{d\psi_1^k}{dx} \left(\sum_{j=1}^{n} u_j^k \frac{d\psi_j^k}{dx} \right) + \psi_1^k a_0 \left(\sum_{j=1}^{n} u_j^k \psi_j^k(x) \right) - \psi_1^k f \right] dx - \sum_{j=1}^{n} \psi_1^k(x_j^k) Q_j^k = 0 \\
\displaystyle\int_{x_A}^{x_B} \left[a_1 \frac{d\psi_2^k}{dx} \left(\sum_{j=1}^{n} u_j^k \frac{d\psi_j^k}{dx} \right) + \psi_2^k a_0 \left(\sum_{j=1}^{n} u_j^k \psi_j^k(x) \right) - \psi_2^k f \right] dx - \sum_{j=1}^{n} \psi_2^k(x_j^k) Q_j^k = 0 \\
\hfill \cdots\cdots\cdots\cdots \\
\displaystyle\int_{x_A}^{x_B} \left[a_1 \frac{d\psi_i^k}{dx} \left(\sum_{j=1}^{n} u_j^k \frac{d\psi_j^k}{dx} \right) + \psi_i^k a_0 \left(\sum_{j=1}^{n} u_j^k \psi_j^k(x) \right) - \psi_i^k f \right] dx - \sum_{j=1}^{n} \psi_i^k(x_j^k) Q_j^k = 0 \\
\hfill \cdots\cdots\cdots\cdots \\
\displaystyle\int_{x_A}^{x_B} \left[a_1 \frac{d\psi_n^k}{dx} \left(\sum_{j=1}^{n} u_j^k \frac{d\psi_j^k}{dx} \right) + \psi_n^k a_0 \left(\sum_{j=1}^{n} u_j^k \psi_j^k(x) \right) - \psi_n^k f \right] dx - \sum_{j=1}^{n} \psi_n^k(x_j^k) Q_j^k = 0
\end{array}
\right\}
$$

The 1D case: Weak Formulation

Passing the integral inside the summation term, each equation can be written as:

$$\sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx\right]u_j^k + \sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_0\psi_i^k\psi_j^k\,dx\right]u_j^k - \int_{x_A}^{x_B} f\psi_i^k\,dx - Q_i^k = 0\,,\;\; i = 1,\dots n$$

Therefore we can write the system with the matrix notation

$$[K^k]u^k - F^k - Q^k = 0$$

$[K^k]$ is an $n \times n$ matrix named **Stiffness Matrix**
u^k is the unknown vector
F^k is related to the internal forces of the problem
Q^k is related to the natural Boundary Conditions

The 1D case: Weak Formulation

- Where the terms come from?

$$\sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx\right]u_j^k + \sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_0\psi_i^k\psi_j^k\,dx\right]u_j^k - \int_{x_A}^{x_B} f\psi_i^k\,dx - Q_i^k = 0,$$

$$\sum_{j=1}^{n} K_{ij}^k u_j^k - F_i^k - Q_i^k = 0\,, \quad i = 1\ldots n$$

The 1D case: Weak Formulation

- Where the terms come from?

$$\sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx\right]u_j^k + \sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_0 \psi_i^k \psi_j^k\,dx\right]u_j^k - \int_{x_A}^{x_B} f\psi_i^k\,dx - Q_i^k = 0,$$

$$\sum_{j=1}^{n} K_{ij}^k u_j^k - F_i^k - Q_i^k = 0\,, \quad i = 1\ldots n$$

$$\boxed{K_{ij}^k = K_{ij}^{k,1} + K_{ij}^{k,0}}$$

$$K_{ij}^{k,1} = \int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx, \qquad K_{ij}^{k,0} = \int_{x_A}^{x_B} a_0 \psi_i^k \psi_j^k\,dx, \qquad F_i^k = \int_{x_A}^{x_B} f\psi_i^k\,dx.$$

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$$\sum_{j=1}^{n} K_{ij}^k u_j^k - F_i^k - Q_i^k = 0\,, \quad i = 1\ldots n$$

$$\boxed{K_{ij}^k = K_{ij}^{k,1} + K_{ij}^{k,0}}$$

Notice that only polynomials are involved in the integrals

$$K_{ij}^{k,1} = \int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx, \qquad K_{ij}^{k,0} = \int_{x_A}^{x_B} a_0\psi_i^k\psi_j^k\,dx, \qquad F_i^k = \int_{x_A}^{x_B} f\psi_i^k\,dx.$$

The 1D case: Weak Formulation

- As a linear equations system it is written:

$$\sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_1 \frac{d\psi_i^k}{dx}\frac{d\psi_j^k}{dx}\,dx\right]u_j^k + \sum_{j=1}^{n}\left[\int_{x_A}^{x_B} a_0\psi_i^k\psi_j^k\,dx\right]u_j^k - \int_{x_A}^{x_B} f\psi_i^k\,dx - Q_i^k = 0,$$

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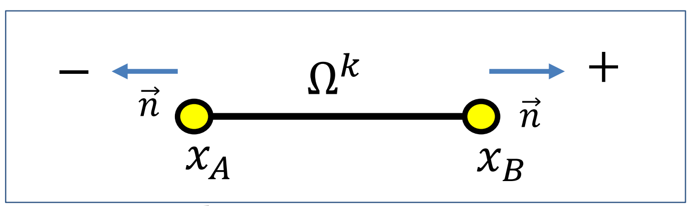
$$\sum_{j=1}^{n} K_{ij}^k u_j^k - F_i^k - Q_i^k = 0\,, \quad i = 1\ldots n$$



Like in 2D, the term $Q \equiv a_1 \frac{du}{dx}\cdot\vec{n}$, and because the outer normal orientation,

we have $Q_i^k = -a_1 \frac{du}{dx}|_{x=x_A}$ and $Q_i^k = a_1 \frac{du}{dx}|_{x=x_B}$.

Notice the minus sign on the left node.

Computing the integrals

(Linear and Quadratic Elements)

The 1D Linear element

Consider now the **linear reference element** $\Omega^R = [-1,1]$

The shape functions can be written for $\xi \in [-1,1]$

$$\psi_1^R(\xi) = \frac{1}{2}(1-\xi) \qquad \psi_2^R(\xi) = \frac{1}{2}(1+\xi)$$

The 1D Linear element

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The idea is to use **integral properties** to pass every other linear element to the reference one.

The 1D Linear element

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The shape functions can be written for $\xi \in [-1,1]$

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The idea is to use **integral properties** to pass every other linear element to the reference one.

For a **general element** $\Omega^k = [x_A, x_B]$ with $x \in [x_A, x_B]$ we have the change of variables with the interval $[-1,1]$ is:

$$x = \frac{h_k}{2}(\xi + 1) + X_A \qquad \text{for } h_k = x_B - x_A$$

Or the inverse:

$$\xi = \frac{2}{h_k}(x - x_A) - 1$$

The 1D Linear element

- Therefore, the change of variables in the integrals gives:

$$K_{ij}^{k,0} = \int_{x_A}^{x_B} a_0(x)\, \psi_i^k(x) \psi_j^k(x)\, dx$$

The 1D Linear element

- Therefore, the change of variables in the integrals gives:

$$K_{ij}^{k,0} = \int_{x_A}^{x_B} a_0(x)\,\psi_i^k(x)\psi_j^k(x)\,dx = \int_{-1}^{1} a_0(\phi_k(\xi))\,\psi_i^R(\xi)\psi_j^R(\xi)\frac{h_k}{2}d\xi.$$

The 1D Linear element

- For the second integral $K_{ij}^{k,1}$

$$K_{ij}^{k,1} = \int_{x_A}^{x_B} a_1(x) \frac{d\psi_i^k(x)}{dx} \frac{d\psi_j^k(x)}{dx} \, dx$$

The 1D Linear element

- For the second integral $K_{ij}^{k,1}$

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with

$$\psi_1^R(\xi) = \frac{1}{2}(1-\xi), \qquad \psi_2^R(\xi) = \frac{1}{2}(1+\xi).$$

The 1D Linear element

- Therefore, the change of variables in the integrals gives:

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with

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The 1D Linear element

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with

$$\psi_1^R(\xi) = \frac{1}{2}(1-\xi), \qquad \psi_2^R(\xi) = \frac{1}{2}(1+\xi).$$

$$= \int_{-1}^{1} a_1(\phi(\xi)) \underbrace{\frac{d\psi_i^R(\xi)}{d\xi}}_{= \pm\frac{1}{2}} \frac{2}{h_k} \underbrace{\frac{d\psi_j^R(\xi)}{d\xi}}_{= \pm\frac{1}{2}} \frac{2}{h_k} \frac{h_k}{2} d\xi$$

The 1D Linear element

- Therefore, the change of variables in the integrals gives:

$$K_{ij}^{k,1} = \int_{x_A}^{x_B} a_1(x)\frac{d\psi_i^k(x)}{dx}\frac{d\psi_j^k(x)}{dx}dx = \int_{-1}^{1} a_1(\phi_k(\xi))\,\frac{d\psi_i^R(\xi)}{d\xi}\frac{2}{h_k}\frac{d\psi_j^R(\xi)}{d\xi}\frac{2}{h_k}\frac{h_k}{2}\,d\xi,$$

with

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The 1D Linear element

- The **constant case**:

Consider now the case where $a_1(x) = a_1^k, a_o(x) = a_0^k,\ \ f(x) = f^k$

$$K_{11}^{k,1} = \int_{-1}^{1} \frac{a_1^k}{h_k^2} \frac{h_k}{2} \, d\xi = \frac{a_1^k}{h_k},$$

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$$\Longrightarrow \quad [K^{k,1}] = \frac{a_1^k}{h_k}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The 1D Linear element

- The **constant case**:

Consider now the case where $a_1(x) = a_1^k, a_o(x) = a_0^k,\ \ f(x) = f^k$

$$K_{11}^{k,0} = \int_{-1}^{1} a_0^k \frac{(1-\xi)^2}{4} \frac{h_k}{2} \, d\xi = \frac{a_0^k h_k}{3},$$

$$K_{12}^{k,0} = K_{21}^{k,0} = \int_{-1}^{1} a_0^k \frac{(1-\xi)(1+\xi)}{4} \frac{h_k}{2} \, d\xi = \frac{a_0^k h_k}{6},$$

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The 1D Linear element

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$$K_{22}^{k,0} = \int_{-1}^{1} a_0^k \frac{(1+\xi)^2}{4} \frac{h_k}{2}\, d\xi = \frac{a_0^k h_k}{3}.$$

The 1D Linear element

- The **constant case**:

Consider now the case where $a_1(x) = a_1^k, a_o(x) = a_0^k,\ f(x) = f^k$

$$F_1^k = \int_{-1}^{1} \left(f^k \frac{1-\xi}{2} \right) \frac{h_k}{2} d\xi = \frac{1}{2} f^k h_k,$$

$$F_2^k = \int_{-1}^{1} \left(f^k \frac{1+\xi}{2} \right) \frac{h_k}{2} d\xi = \frac{1}{2} f^k h_k.$$

$$\Longrightarrow \quad F^k = \frac{f^k h_k}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The 1D Linear element

- The **constant case**:

collecting all the terms we have

$$\sum_{j=1}^{n} K_{ij}^{k} u_j^k = F_i^k + Q_i^k \qquad i = 1 \ldots n$$

$$[K^{k,1}] = \frac{a_1^k}{h_k} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad [K^{k,0}] = \frac{a_0^k h_k}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$F^k = \frac{f^k h_k}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The 1D **Quadratic element**

- When we consider quadratic elements, the shape functions are

$$\psi_1^R(\xi) = \frac{1}{2}\xi(\xi - 1), \qquad \psi_2^R(\xi) = (1+\xi)(1-\xi), \qquad \psi_3^R(\xi) = \frac{1}{2}\xi(1+\xi)$$

- In the **constant coefficient** case we obtain:

$$[K^{k,1}] = \frac{a_1^k}{3h_k}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}$$

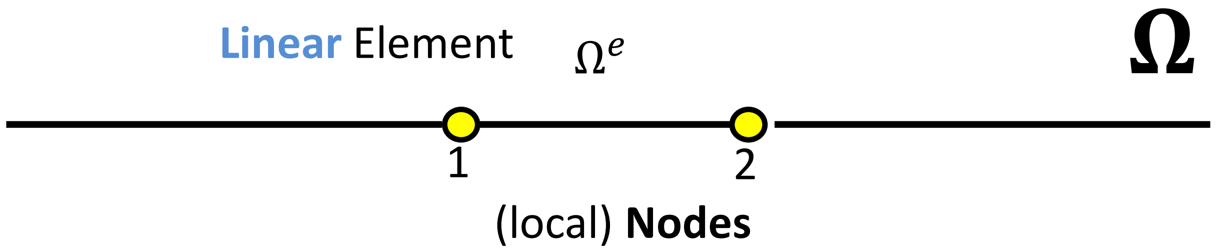
$$F^k = \frac{f^k h_k}{6}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

$$[K^{k,0}] = \frac{a_0^k h_k}{30}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}$$

1D Finite Elements
Meshing and Assembly

Finite Elements 1D

- For 1D domains, generically, the **elements** are defined as segments $\Omega^e = [x_i, x_{i+1}]$ that covers de complete domain $\boldsymbol{\Omega}$.



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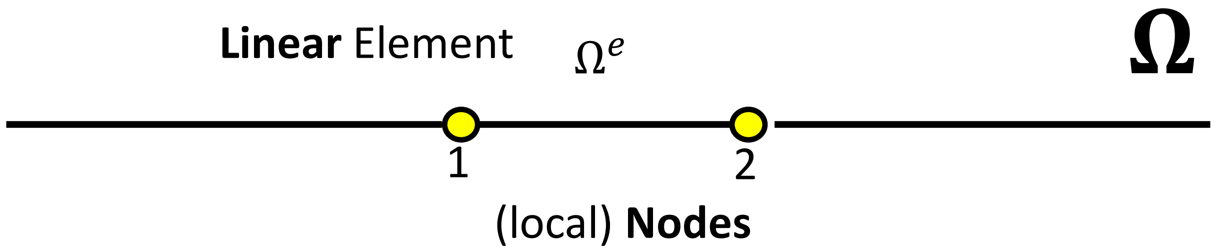
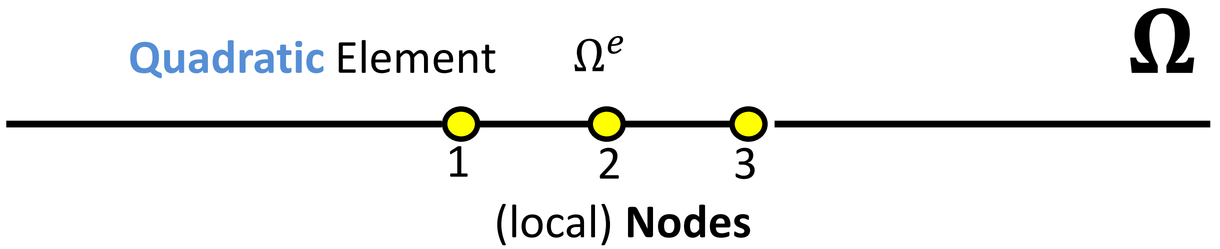


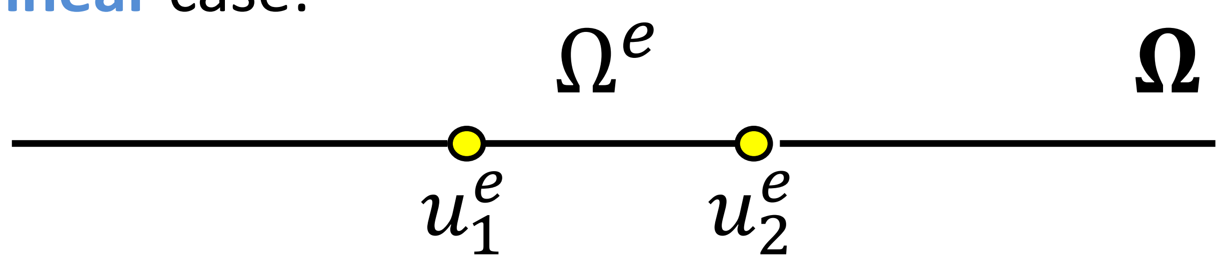


Finite Elements 1D

Let's assume that $u(x)$ is a **magnitude** (temperature, displacement, etc.) that we want to compute in the nodes n_i of one element Ω^e the usual FEM **notation** is:

$$\boxed{u(n_i) = u_i^e}$$

For the **linear** case:



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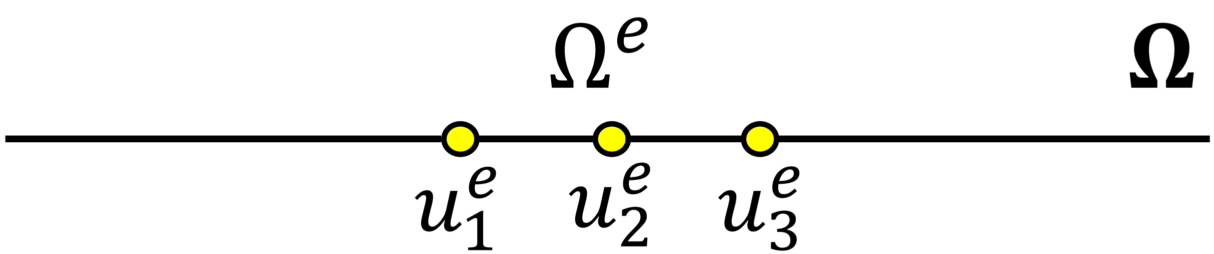

Nodes enumeration

When we consider the total domain, nodes of **consecutive elements** must be identify in order to obtain *continuous solutions*: (last node equals the next first one)

$$u_N^{e-1} = u_1^e, \qquad u_N^e = u_1^{e+1}$$

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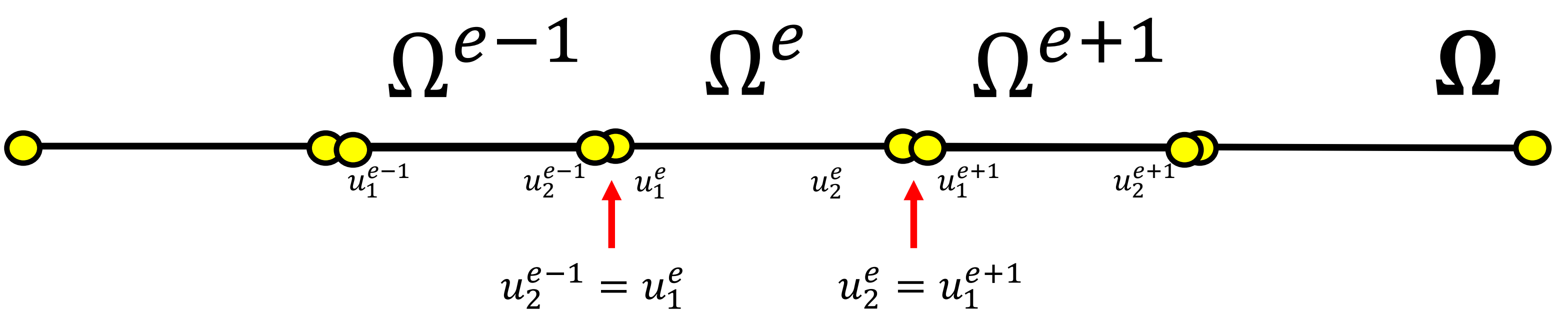
For the **linear** case (N =2):



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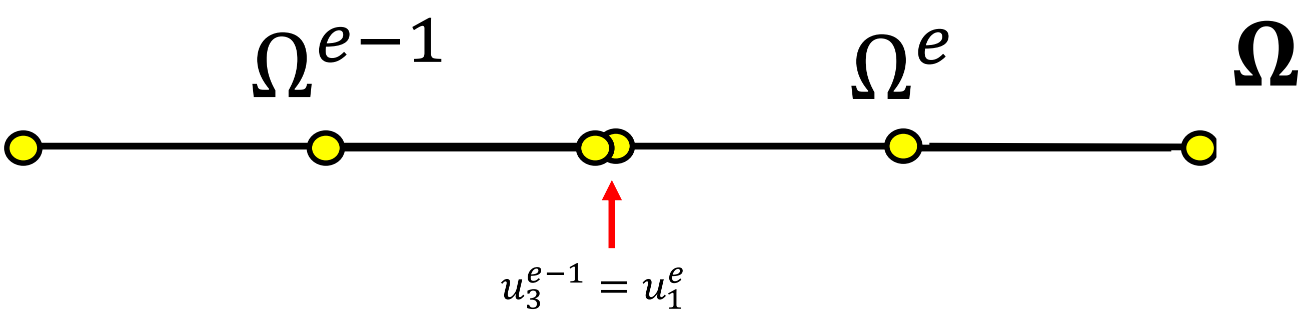
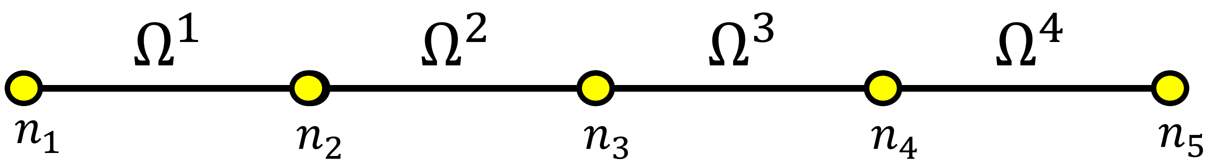


Nodes enumeration

- **Global enumeration**: Once we have identify the connected nodes, we rename them using a global enumeration.



Global Nodes

Connectivity Matrix

- **Connectivity Matrix**: Says the global nodes attached to each element.

Connectivity Matrix

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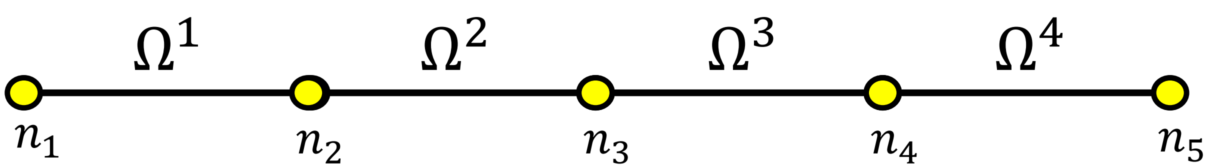
- Example:

Node numbers

$$elem = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{pmatrix}$$

One row for each element



Global Nodes

Stiff Matrix

Element Stiff Matrix ($\boldsymbol{K^e}$): Is the one related to the physical problem stated for each element (this is the *thought* part of the method).

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Because it is associated to each element, its size agrees with the number of nodes in each element.

1-dim linear element (two nodes) $\longrightarrow$ $\boldsymbol{K^e}$ is a 2x2 matrix

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1-dim linear element (two nodes) $\Longrightarrow$ $\boldsymbol{K^e}$ is a 2x2 matrix

1-dim quadratic element (three nodes) $\Longrightarrow$ $\boldsymbol{K^e}$ is a 3x3 matrix

$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{pmatrix}$$

Global Stiff Matrix

- **Global Stiff Matrix** $(\boldsymbol{K})$: Is the assembled matrix of all $\boldsymbol{K}^{\boldsymbol{e}}$ element stiff matrices.

In a generic way, for a 1D problem, $\boldsymbol{K}$ is square matrix of dimension the number of global nodes.

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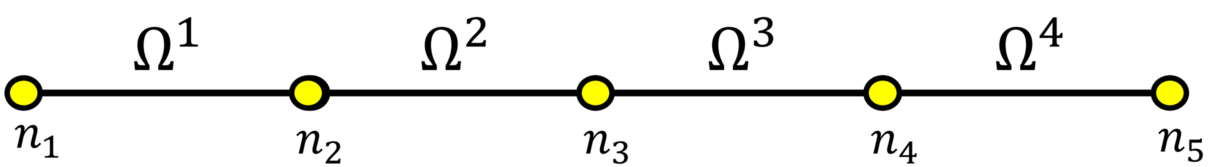
Example:



$$\boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$

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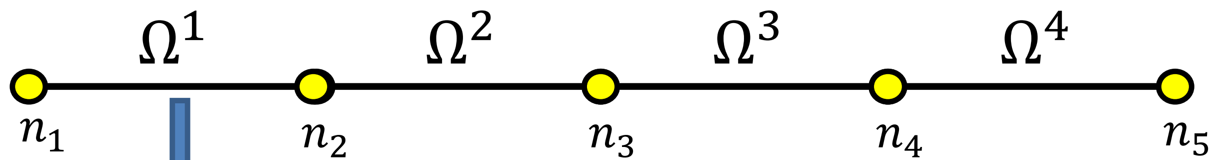
Element Assembly

1D Elements Assembly

- **Assembly**



$$elem = \begin{pmatrix} 1 & 2 \\ \vdots & \vdots \\ 4 & 5 \end{pmatrix}$$

$$K^1 = \begin{pmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{pmatrix} \Rightarrow$$

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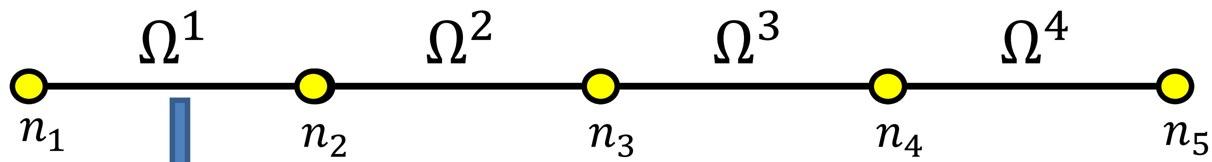

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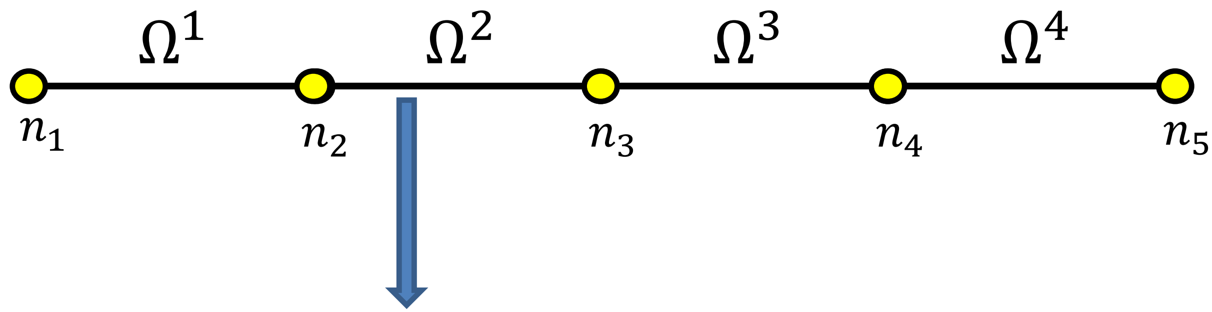

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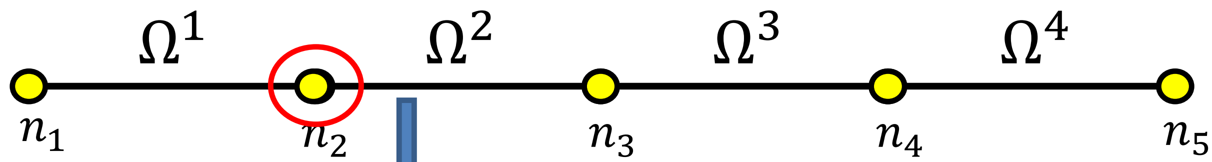

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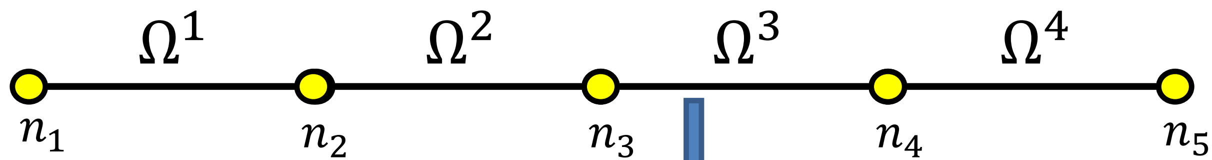
Contribution of **two** elements

$$k_{22} = k_{22}^1 + k_{11}^2$$

1D Elements Assembly

- **Assembly**



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1D Elements Assembly

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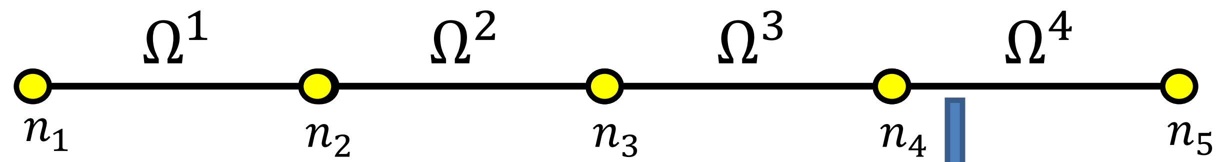

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The rest of the elements in $\boldsymbol{K}$ are zero

1D Elements Assembly

- Example: Suppose that for each element its local Stiff matrix is constant (*we'll see later how to compute it*)

$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{pmatrix} = C \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

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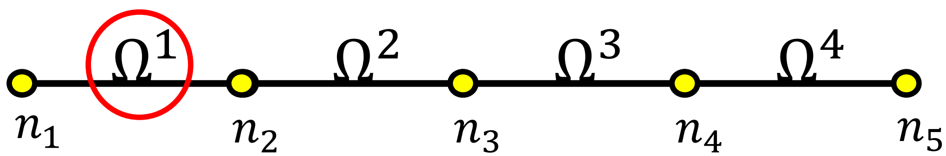
For the bar:



$$\boldsymbol{K} = C \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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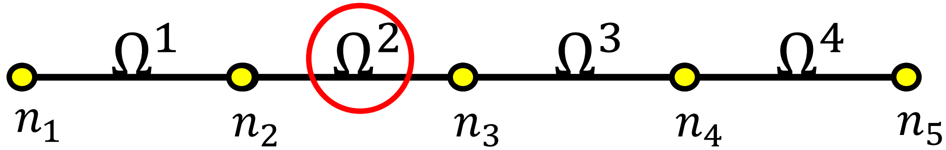

$$\boldsymbol{K} = C \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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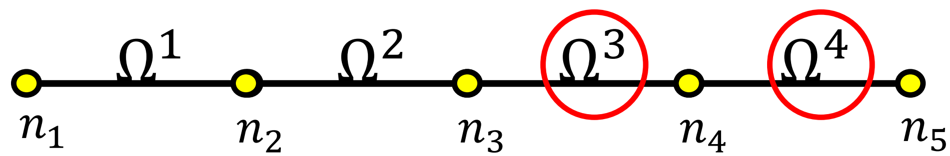
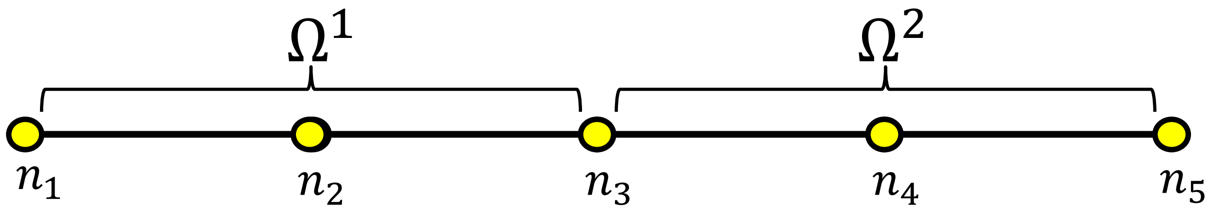


$$\boldsymbol{K} = C \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

1D Elements Assembly

- **Exercise**: Consider the problem of the previous bar, but using now **quadratic elements.**



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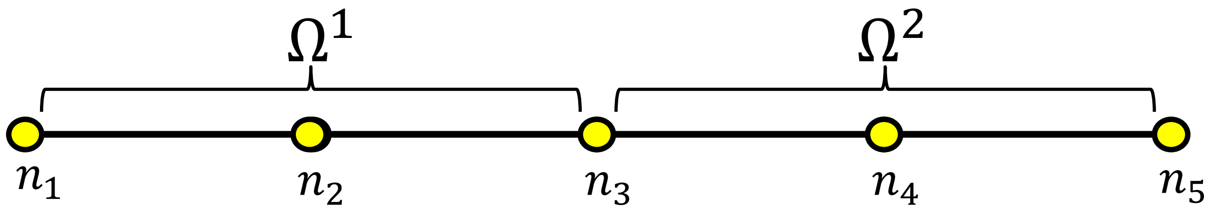

Modify the previous steps in order to obtain the assembly matrix when only two elements are taken

$$K^e = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix} \Rightarrow \boldsymbol{K} = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13} & k_{14} & k_{15} \\ k_{21}^1 & k_{22}^1 & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{pmatrix}$$

1D Finite Elements Examples

Linear Elasticity 1D

Linear Elasticity 1D equation:

$$-\frac{d}{dx}\left(E(x)A(x)\frac{du(x)}{dx}\right) = 0,$$

with $E(x)$ the material elasticity function (**Young modulus**), $A(x)$ the **section area** and $u(x)$ the **displacement**.

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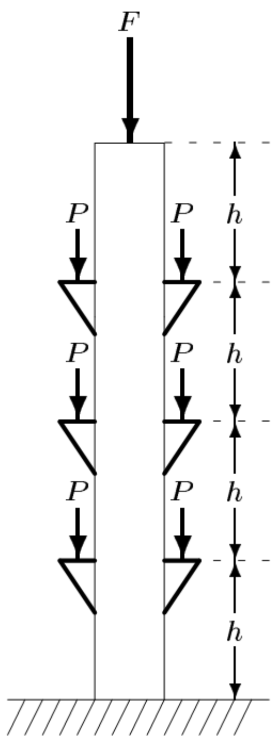
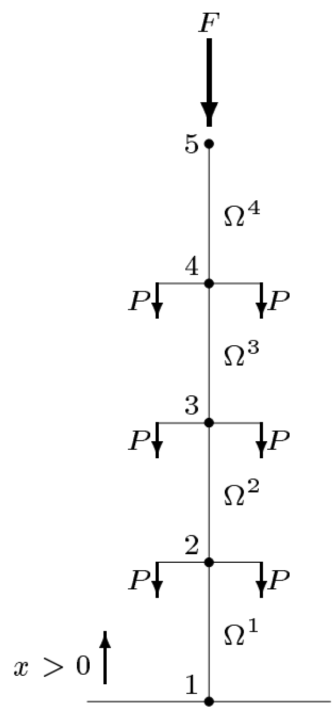
Constant Pillar

Linear Elasticity 1D

- **Example 1:** Constant loaded column

Let's assume *E·A* constant (homogeneous column).





$h = 4.5\, m$

$P = 11 \times 10^4 N$

$F = 3 \times 10^5 N$

$E = 2.0 \times 10^{11} N/m^2$

$A = 250\, cm^2$

Linear Elasticity 1D

- This is a particular case of the model equation

$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right)+a_0(x)u=f(x),$$

with $a_1 = EA$ constant, $a_0 \equiv 0$ i $f(x) \equiv 0$ (if the column weight is not consider).

Linear Elasticity 1D

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$$\frac{-d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x),$$

with $a_1 = EA$ constant, $a_0 \equiv 0$ i $f(x) \equiv 0$ (if the column weight is not consider).

as we learned before, the problem can be stated as

$$[K^k]u^k = F^k + Q^k$$

$$[K^k] = [K^{k,1}] = \frac{EA}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad F^k = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{per} \quad k = 1, 2, 3, 4.$$

Linear Elasticity 1D

After assembly the system we obtain

$$\frac{EA}{h}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{pmatrix}$$

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Linear Elasticity 1D

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Linear Elasticity 1D

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Linear Elasticity 1D

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$$\frac{EA}{h}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{pmatrix}$$

Finally, we impose

$$U_1 = 0\,m, \quad Q_2^1 + Q_1^2 = Q_2^2 + Q_1^3 = Q_2^3 + Q_1^4 = -2.2\times 10^5\,N, \quad Q_2^4 = -3\times 10^5\,N.$$

Unknowns primary: U_2, U_3 , U_4 , U_5 and also secondary Q_1^1

Linear Elasticity 1D

After assembly the system we obtain

$$\frac{EA}{h}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{pmatrix}.$$

Finally, we impose

$$U_1 = 0\,m, \quad Q_2^1 + Q_1^2 = Q_2^2 + Q_1^3 = Q_2^3 + Q_1^4 = -2.2 \times 10^5\,N, \quad Q_2^4 = -3 \times 10^5\,N.$$

$$1.11 \times 10^9 \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = -\begin{pmatrix} 2.2 \\ 2.2 \\ 2.2 \\ 3.0 \end{pmatrix} \times 10^5$$

Linear Elasticity 1D

After assembly the system we obtain

$$\frac{EA}{h}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{pmatrix}$$

Finally, we impose

$$U_1 = 0\,m, \quad Q_2^1 + Q_1^2 = Q_2^2 + Q_1^3 = Q_2^3 + Q_1^4 = -2.2 \times 10^5\,N, \quad Q_2^4 = -3 \times 10^5\,N.$$

$$1.11 \times 10^9 \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = -\begin{pmatrix} 2.2 \\ 2.2 \\ 2.2 \\ 3.0 \end{pmatrix} \times 10^5$$

Solution:

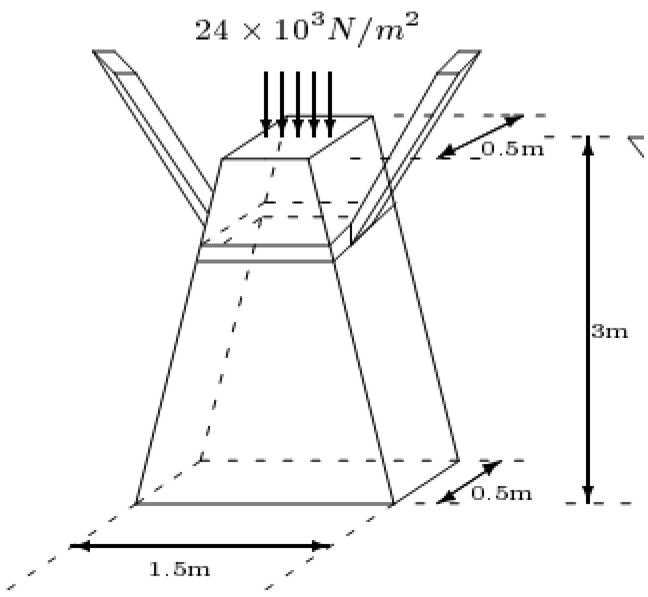
$$U_1 = 0\,m, \quad U_2 = -0.86\,mm, \quad U_3 = -1.53\,mm, \quad U_4 = -2.00\,mm, \quad U_5 = -2.27\,mm.$$

$$Q_1^1 = \frac{EA}{h}(U_1 - U_2) = 9.6 \times 10^5\,N$$ (Reaction force on the ground)

Variable Pillar

Linear Elasticity 1D

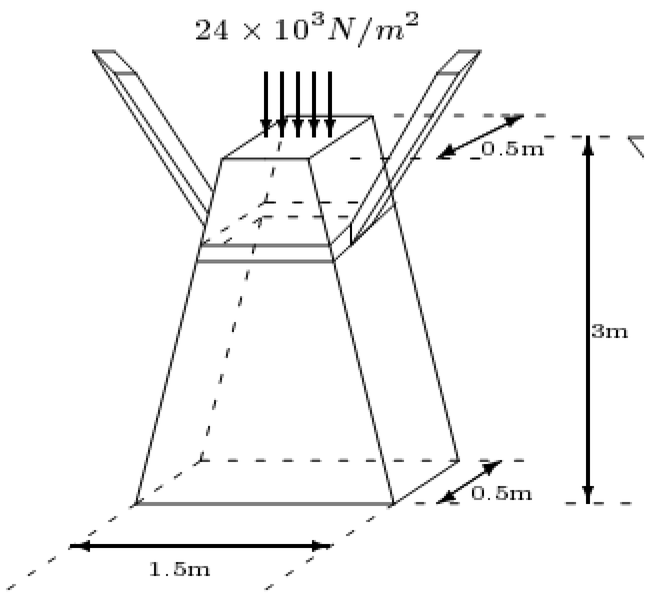
- **Example 2:** Concrete Pyramidal Column with variable section area



$$-\frac{d}{dx}\left(E(x)A(x)\frac{du(x)}{dx}\right) = f(x)$$

Linear Elasticity 1D

- **Example 2:** Concrete Pyramidal Column with variable section area



$$-\frac{d}{dx}\left(E(x)A(x)\frac{du(x)}{dx}\right) = f(x)$$

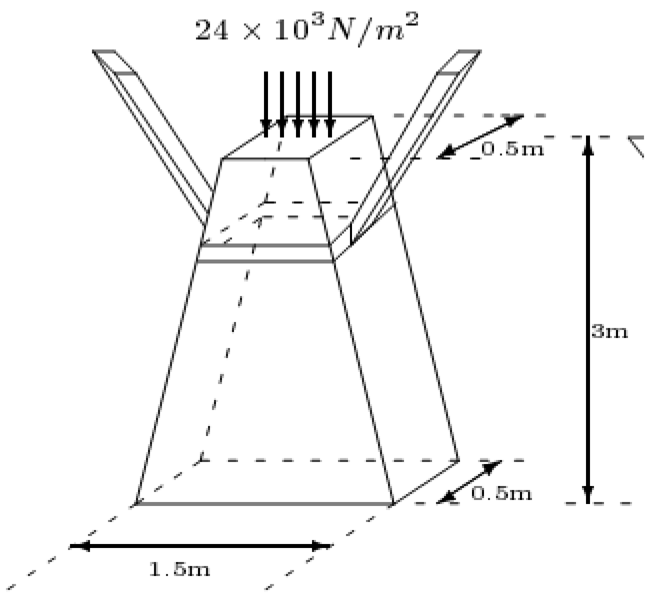
Now the inner weight of the column is also taken into account

$$f(x) = -\omega \frac{dV}{dx}$$

ω been the concrete specific weight

Linear Elasticity 1D

- **Example 2:** Concrete Pyramidal Column with variable section area



$$-\frac{d}{dx}\left(E(x)A(x)\frac{du(x)}{dx}\right) = f(x)$$

Model equation:

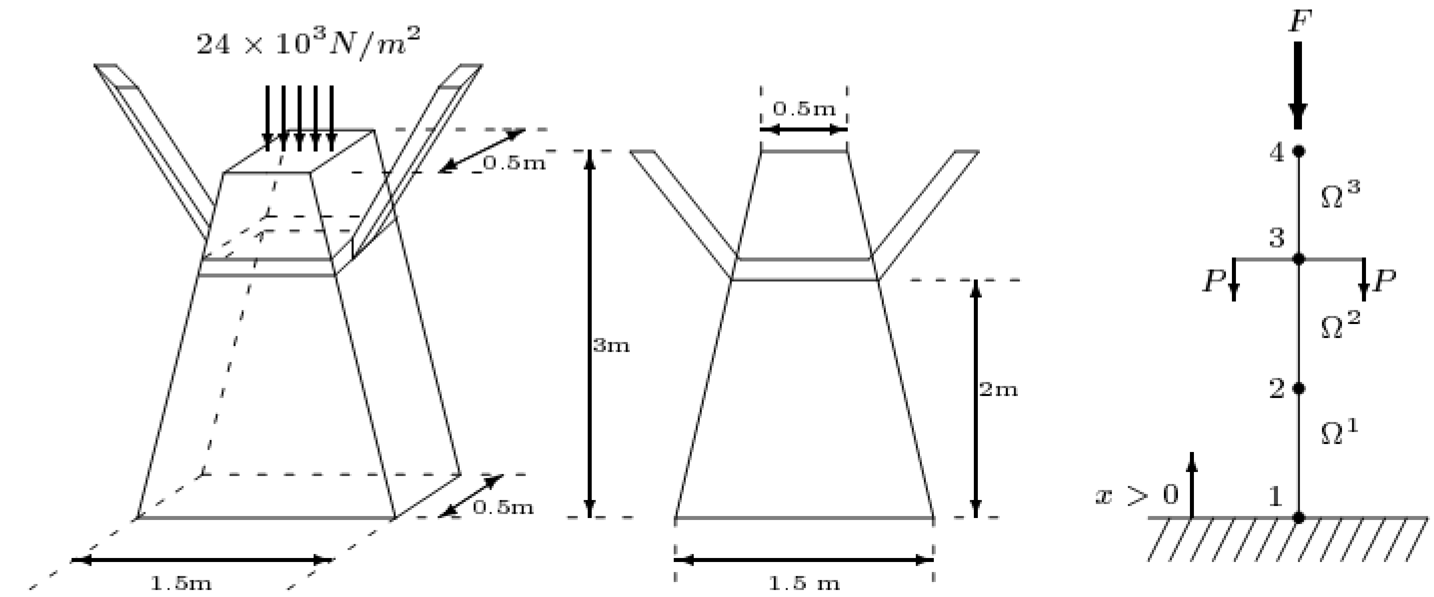
$a_1(x) = E(x) \cdot A(x)$ **non constant**

$a_0(x) = 0$

$f(x) = -\omega \frac{dV}{dx}$ internal weight force

Linear Elasticity 1D

Example 2: Concrete Pyramidal Column

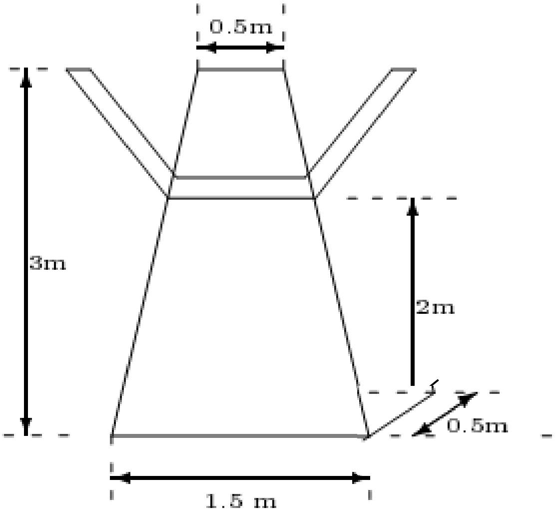


Consider $P = 2 \times 10^3\, N$ $\quad h = 1\, m$

$E = 28 \times 10^9\, N/m^2$ $\quad w = 25 \times 10^3\, N/m^3$ (concrete specific weight)

Linear Elasticity 1D

Now, the section area is not constant

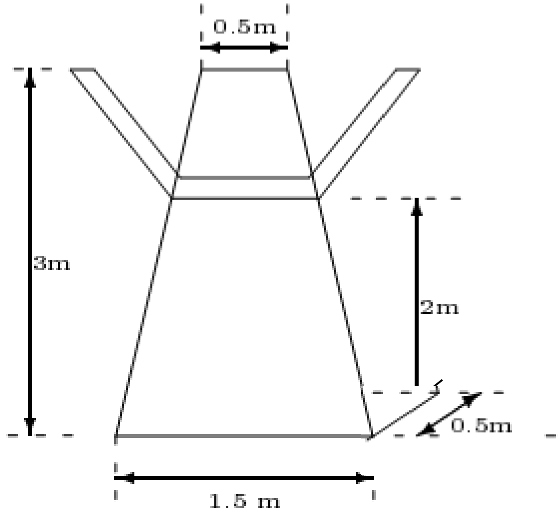


Wide value as a function $b(x)$

x	0	3
b	1.5	0.5

Linear Elasticity 1D

Now, the section area is not constant



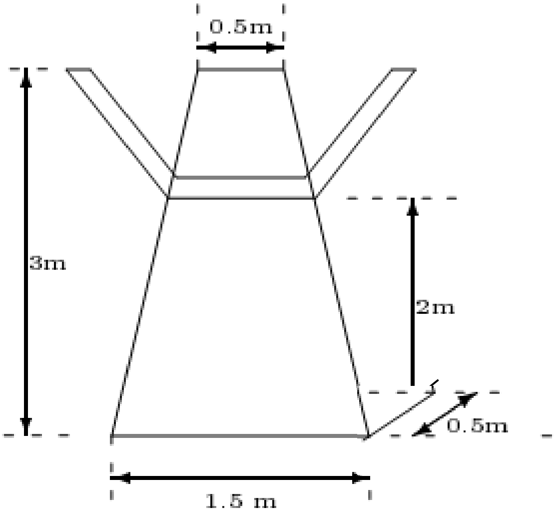
Wide value as a function $b(x)$

x	0	3
b	1.5	0.5

$$b(x) = \left(1.5\,\frac{3-x}{3} + 0.5\,\frac{x}{3}\right)$$

Linear Elasticity 1D

Now, the section area is not constant



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Each section is a rectangle:

$$A(x) = 0.5 \cdot b(x)$$

Linear Elasticity 1D

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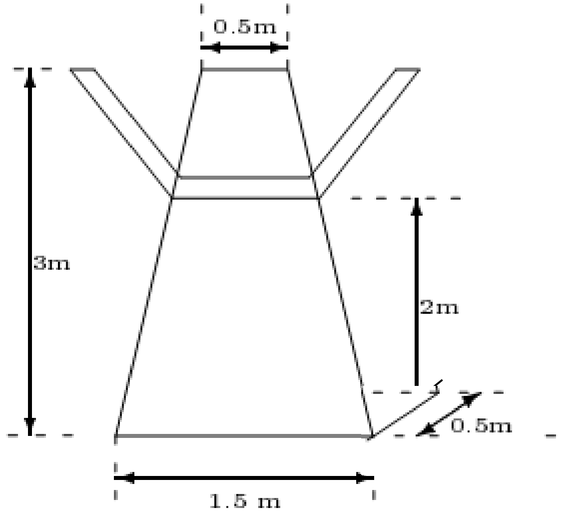

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$$b(x) = \left(1.5\,\frac{3-x}{3} + 0.5\,\frac{x}{3}\right)$$

Each section is a rectangle:

$$A(x) = 0.5 \cdot b(x)$$

$$A(x) = 0.5\left(1.5\,\frac{3-x}{3} + 0.5\,\frac{x}{3}\right) = \frac{3}{4} - \frac{x}{6}\,m^2.$$

Linear Elasticity 1D

The **internal forces** corresponding to the pillar weight can be computed from the specific weight

$$f(x) = -\omega \frac{dV}{dx}$$

for unit length can be expressed by the product

$$f(x) = -w\,A(x) = \left(\frac{25x}{6} - \frac{75}{4}\right) \times 10^3\,N/m,$$

(*Hint*: This can be obtained by the derivative of the formula for the volume of column above a level x.

$$V(x) = \frac{1}{2}\big(A(3) + A(x)\big) \cdot h$$

)

Linear Elasticity 1D

- Here, to obtain the linear system $[K^k]u^k = F^k + Q^k$

we need to compute

$$K_{ij}^k = K_{ij}^{k,1} = E\int_{x_A}^{x_B} A(x)\,\frac{d\psi_i^k}{dx}(x)\,\frac{d\psi_j^k}{dx}(x)\,dx, \qquad F_i^k = \int_{x_A}^{x_B} f(x)\,\psi_i^k(x)\,dx.$$

$$\psi_1^k(x) = \frac{x - x_B}{x_A - x_B} = \frac{x - x_B}{-h_k}$$

$$\psi_2^k(x) = \frac{x - x_A}{x_B - x_A} = \frac{x - x_A}{h_k}$$

Linear Elasticity 1D

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$$\psi_1^k(x) = \frac{x - x_B}{x_A - x_B} = \frac{x - x_B}{-h_k} \Longrightarrow \frac{d\psi_1^k}{dx}(x) = \frac{-1}{h_k}$$

$$\psi_2^k(x) = \frac{x - x_A}{x_B - x_A} = \frac{x - x_A}{h_k} \Longrightarrow \frac{d\psi_2^k}{dx}(x) = \frac{1}{h_k}$$

Linear Elasticity 1D

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That gives **different values** for each element.
Considering the first one, $\Omega^1 = [0,1]$, here $h_k = 1$, we obtain:

$$K_{11}^1 = K_{22}^1 = E\int_0^1 A(x)\,dx = \frac{2E}{3},$$

$$A(x) = \frac{3}{4} - \frac{x}{6}m^2.$$

Linear Elasticity 1D

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That gives **different values** for each element.
Considering the first one, $\Omega^1 = [0,1]$, we obtain:

$$K_{11}^1 = K_{22}^1 = E\int_0^1 A(x)\,dx = \frac{2E}{3}, \qquad K_{12}^1 = K_{21}^1 = -E\int_0^1 A(x)\,dx = -\frac{2E}{3},$$

Linear Elasticity 1D

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we need to compute

$$K_{ij}^k = K_{ij}^{k,1} = E\int_{x_A}^{x_B} A(x)\,\frac{d\psi_i^k}{dx}(x)\frac{d\psi_j^k}{dx}(x)\,dx, \qquad F_i^k = \int_{x_A}^{x_B} f(x)\,\psi_i^k(x)\,dx.$$

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$$[K^1] = \frac{2E}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Linear Elasticity 1D

- Here, to obtain the linear system $[K^k]u^k = F^k + Q^k$

we need to compute

$$K_{ij}^k = K_{ij}^{k,1} = E \int_{x_A}^{x_B} A(x) \frac{d\psi_i^k}{dx}(x) \frac{d\psi_j^k}{dx}(x)\, dx, \qquad F_i^k = \int_{x_A}^{x_B} f(x)\, \psi_i^k(x)\, dx.$$

That gives **different values** for each element.
Considering the first one, $\Omega^1 = [0,1]$, we obtain:

$$F_1^1 = -w \int_0^1 A(x)(1-x)\, dx = -\frac{25\, w}{72}, \qquad F_2^1 = -w \int_0^1 xA(x)\, dx = -\frac{23\, w}{72}.$$

$$A(x) = \frac{3}{4} - \frac{x}{6}\, m^2. \qquad F^1 = -\frac{w}{72} \begin{pmatrix} 25 \\ 23 \end{pmatrix}$$

Linear Elasticity 1D

- For the three elements we have:

$$[K^1] = \frac{2E}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad [K^2] = \frac{E}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad [K^3] = \frac{E}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$F^1 = -\frac{w}{72}\begin{pmatrix} 25 \\ 23 \end{pmatrix}, \qquad F^2 = -\frac{w}{72}\begin{pmatrix} 19 \\ 17 \end{pmatrix}, \qquad F^3 = -\frac{w}{72}\begin{pmatrix} 13 \\ 11 \end{pmatrix}.$$

Linear Elasticity 1D

- For the three elements we have:

$$[K^1] = \frac{2E}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad [K^2] = \frac{E}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad [K^3] = \frac{E}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

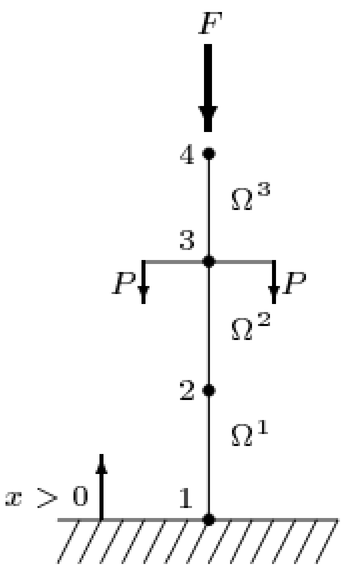
$$F^1 = -\frac{w}{72}\begin{pmatrix} 25 \\ 23 \end{pmatrix}, \qquad F^2 = -\frac{w}{72}\begin{pmatrix} 19 \\ 17 \end{pmatrix}, \qquad F^3 = -\frac{w}{72}\begin{pmatrix} 13 \\ 11 \end{pmatrix}.$$

and next, assemble the system

$$\frac{E}{6}\begin{pmatrix} 4 & -4 & 0 & 0 \\ -4 & 7 & -3 & 0 \\ 0 & -3 & 5 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = -\frac{w}{72}\begin{pmatrix} 25 \\ 42 \\ 30 \\ 11 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}$$

Linear Elasticity 1D

$$\frac{E}{6}\begin{pmatrix} 4 & -4 & 0 & 0 \\ -4 & 7 & -3 & 0 \\ 0 & -3 & 5 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = -\frac{w}{72}\begin{pmatrix} 25 \\ 42 \\ 30 \\ 11 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}$$

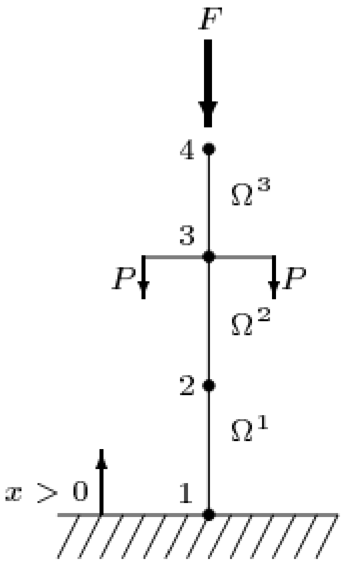


With BC

$U_1 = 0,$ **essential condition**

Linear Elasticity 1D

$$\frac{E}{6}\begin{pmatrix} 4 & -4 & 0 & 0 \\ -4 & 7 & -3 & 0 \\ 0 & -3 & 5 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = -\frac{w}{72}\begin{pmatrix} 25 \\ 42 \\ 30 \\ 11 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}$$



With BC

essential BC: $U_1 = 0,$

natural BC:

$$Q_2 = Q_2^1 + Q_1^2 = 0, \qquad Q_3 = Q_2^2 + Q_1^3 = -2P, \qquad Q_4 = Q_2^3 = -F,$$

$$F = (0.5)^2 \times 24 \times 10^3 = 6000\,N.$$

Solution:

$$U_1 = 0\,m, \qquad U_2 = -2.08 \times 10^{-6}\,m, \qquad U_3 = -3.81 \times 10^{-6}\,m, \qquad U_4 = -4.86 \times 10^{-6}\,m.$$

$$Q_1 = \frac{E}{6}(4U_1 - 4U_2) + \frac{25\omega}{72} = -4.3586e + 04$$

1D Finite Elements Transient solution

Thermal Problems

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Consider the following **thermal problem** defined on a 1D bar of section area *A* and material *conductivity coefficient* k_c.

$$-\frac{d}{dx}\left(k_c A \frac{du}{dx}\right) = 0, \quad u(0) = 10, u(1) = 60$$

It corresponds to the *model equation* for thermal problems with

$a_1 = k_c A, a_0 = 0, f = 0$

If we compute the final temperature distribution on the bar, in fact, we are computing the **stationary solution** of the problem.

Transient Solution

If we now want to include the variation along the time (**transient problem**), then we have to include an extra term and the dependency of $u = u(t, x)$.
Plus a time dependent derivative term, with coefficient $a_2 = k_c A.$

$$a_2 \frac{du}{dt} - \frac{d}{dx}\left(k_c A \frac{du}{dx}\right) = 0, \quad u(t,0) = 10, \;\; u(t,1) = 60$$

For each element $\Omega^k = [x_i \;\; , \;\; x_{i+1}]$, we get a linear system of equations, similar to the one for the **Steady problem**, that we can express in general as

$$[M^k]\frac{du}{dt} + [K^k]u = \widehat{F}^k \quad \text{where} \quad \widehat{F}^k = F^k + Q^k \quad \textbf{(eq.1)}$$

Obs. We assume that none of the matrices depen on the time variable.

Transient Solution

Linear Elements

In our case, because $a_1 = k_c A$ (constant) , $a_0 = 0$ and $f = 0$, for **linear elements** we have:

$$K^k = \frac{a_1}{h_k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F^k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$M^k = \frac{a_2 h_k}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ because is defined by $M_{ij}{}^k = \int_{x_i}^{x_{i+1}} a_2 \varphi_i \varphi_j$ (notice that is similar to the matrix for the usual constant a_0 term)

Obs. Usually, for easy inversion, we consider an approxiamtion known as the **row-sum lumping-mass matrix**:

$$M^k = \frac{a_2 h_k}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Transient Solution

Solution of the transient problem:

To solve the transient problem, we consider the time derivative $\frac{du}{dt} \equiv \dot{u}_t$ not as the usual numerical approximation:

$$\dot{u} \approx \frac{u_{t+dt} - u_t}{dt}$$

being u_t the present time solution and u_{t+dt} the solution for the next step (after an amount of time dt), but as a linear combination of the present and next derivatives:

$$\alpha \cdot \dot{u}_{t+dt} + (1-\alpha) \cdot \dot{u}_t = \frac{u_{t+dt} - u_t}{dt} \quad \text{where } \alpha \in [0,1] \quad \textbf{(eq. 2)}$$

For different value of α, we get different numerical methods:

$$\alpha = 0 \;\; Forward \;\; Differences \;\; (conditionally \;\; stable). \;\; Precission \;\; O(dt)$$
$$\alpha = 1/2 \;\; Crank-Nicolson \;\; (stable). \;\; Precission \;\; O(dt^2)$$
$$\alpha = 2/3 \;\; Galerkin \;\; method \;\; (stable). \;\; Precission \;\; O(dt^2)$$
$$\alpha = 1 \;\; Backward \;\; Differences \;\; (stable). \;\; Precission \;\; O(dt)$$

Transient Solution

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The values for approximating the derivatives comes again from **(eq. 1)**

$$[M]\dot{u}_t + [K]u_t = \widehat{F}$$ **(eq.3a)**

and

$$[M]\dot{u}_{t+dt} + [K]u_{t+dt} = \widehat{F}$$ **(eq.3b)**

Multipliying **(eq.2)** by $[M]$ and substituing the derivatives using **(eq.3a)** and **(eq.3b)** we get

$$\alpha \cdot [M]\ \dot{u}_{t+dt} + (1-\alpha)\cdot[M]\ \dot{u}_t = [M]\ \frac{u_{t+dt} - u_t}{dt}$$

$$\alpha \cdot (\widehat{F} - [K]u_{t+dt}) + (1-\alpha)\cdot(\widehat{F} - [K]u_t) = [M]\ \frac{u_{t+dt} - u_t}{dt}$$

Rearranging the terms we get the final equation

$$\Big([M] + \alpha \cdot dt \cdot [K]\Big)u_{t+dt} = \Big([M] - (1-\alpha)\cdot dt \cdot [K]\Big)u_t + \widehat{F}$$ **(eq. 4)**

Therefore, the problem is to compute the value u_{t+dt} , which means **to solve a linear system of equations for each time step**.

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Mètodes Numèrics:

A First Course on Finite Elements

Finite Elements (III)

FEM 2D

Following: *Curs d'Elements Finits amb Aplicacions* (J. Masdemont)
http://hdl.handle.net/2099.3/36166

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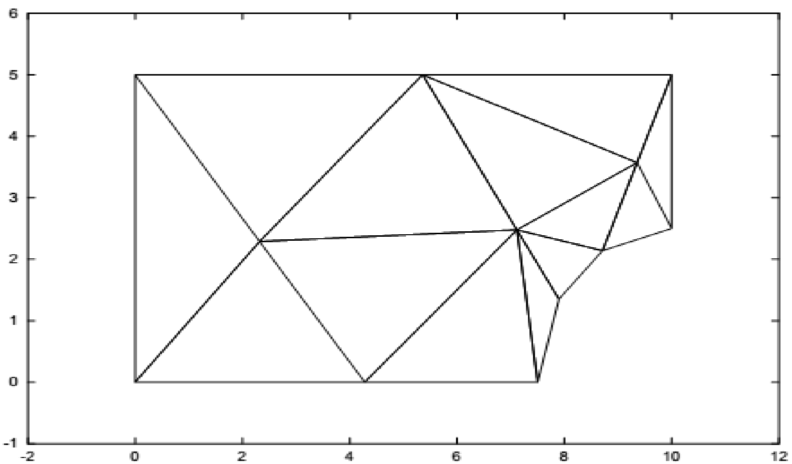
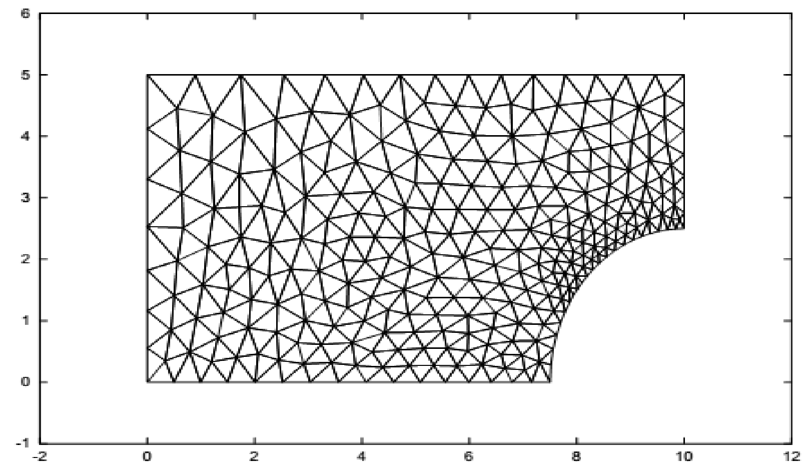
Meshing

Meshing 2D domains

- Meshing a general domain is a **difficult problem**. We'll not study it in depth in our course.
- Two main concerns when meshing a domain are:
  - Good fitting of the domain
  - Good Numerical properties (stability)

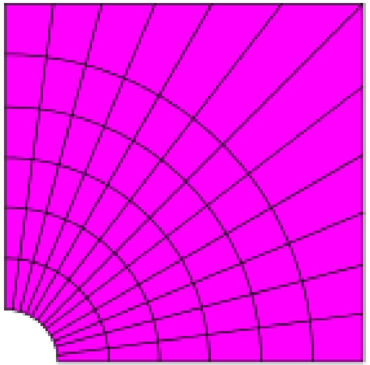




Meshing 2D domains

- Classification:
  - Structured Mesh:

  are identified by regular connectivity



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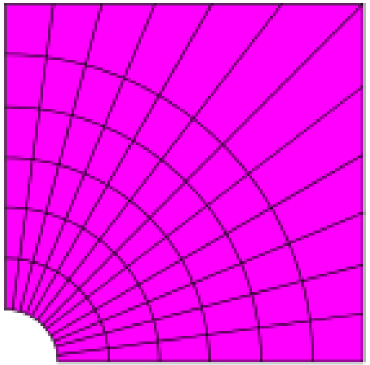
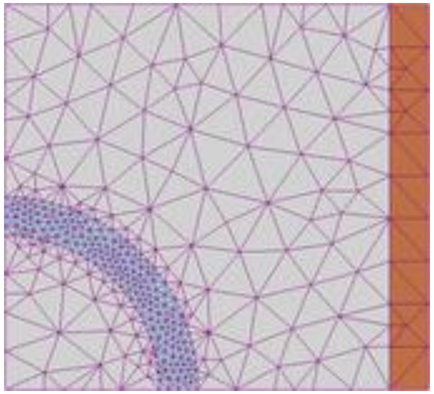
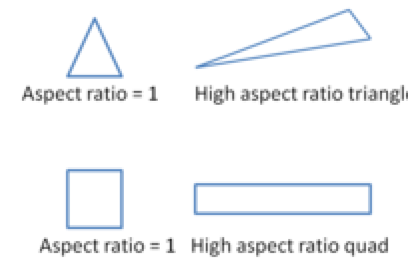


  - Unstructured Mesh



Meshing 2D domains

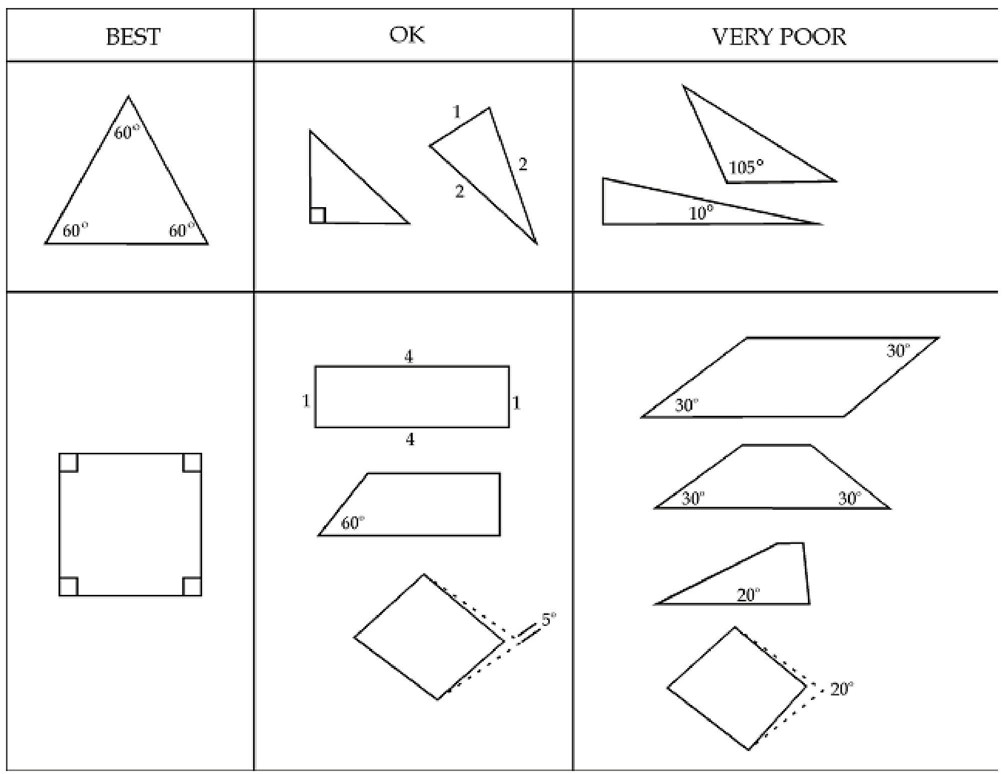


Mesh quality:

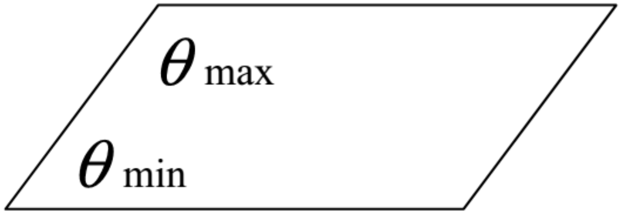
– **Aspect Ratio**: It is the ratio of *longest* to the *shortest* side in an element.

Best = 1

Acceptable < 5



Meshing 2D domains



Mesh quality:

– Skewness:

Another common measure of quality is based on equiangular skew.

$$\text{Equiangle Skew} = \max\left[\frac{\theta_{max}-\theta_e}{180-\theta_e}, \frac{\theta_e-\theta_{min}}{\theta_e}\right]$$

where:

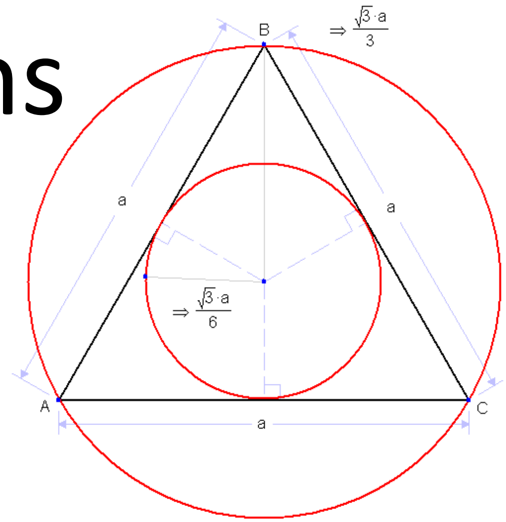
θ_{max} is the largest angle in a face or cell,

θ_{min} is the smallest angle in a face or cell,

θ_e is the angle for equi-angular face or cell i.e. 60 for a triangle and 90 for a square.

Value of Skewness	0-0.25	0.25-0.50	0.50-0.80	0.80-0.95	0.95-0.99	0.99-1.00
Cell Quality	excellent	good	acceptable	poor	sliver	degenerate

Meshing 2D domains



Mesh quality:

- **Inscribed-Circumscribed ratio**:

$$q = 2\frac{r_{\text{in}}}{r_{\text{out}}} = \frac{(b+c-a)(c+a-b)(a+b-c)}{abc}$$

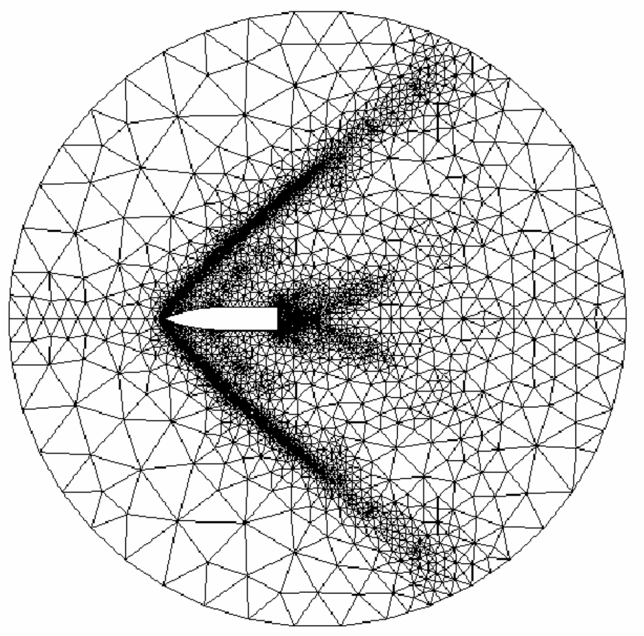
where a, b, c are the side lengths.

An equilateral triangle has $q = 1$

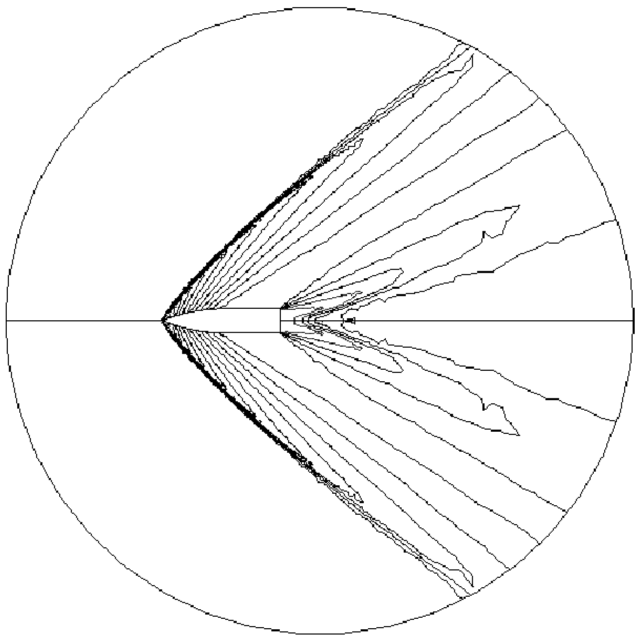
As a rule of thumb, if all triangles have $q > 0.5$ the results are good.

Meshing 2D domains

- **Mesh refinement**: More elements where physical features are changing



2D planar shell - final grid



2D planar shell - contours of pressure final grid

Model Equation

2D-Model Equation

For 2D problems we will use the **model equation**. A 2on order PDE for $\boxed{u = u(x,y)}$ *(primary variable)*

$$\boxed{-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}u = f,}$$

Defined on a 2-dim domain Ω, with $a_{ij}(x,y)$ and $f(x,y)$ known functions.

2D-Model Equation

- **Notation**: In many books you can find the expressions

$$\nabla \cdot u \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\ , \qquad \text{if} \ \ u = u(x, y)$$

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$$\nabla \cdot (u_1, u_2) \equiv div(u) \equiv \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\,, \qquad \text{if} \quad u = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix}$$

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$$\nabla u \equiv grad(u) \equiv \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \qquad \text{if } \; u = u(x,y)$$

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$$\nabla u \equiv grad(u) \equiv \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \qquad \text{if} \;\; u = u(x,y)$$

- **Example:** Poisson equation

$$-\nabla \cdot (a \nabla u) = f$$

If a =const, $\quad -a\nabla \cdot (\nabla u) \equiv -a\, \nabla^2 u \equiv -a\Delta u = f$

Laplacian Operator

2D-Model Equation

- **Poisson equation**:

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a\frac{\partial u}{\partial y}\right)=f.$$

It corresponds to the 2D ***model equation***

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+a_{00}u=f,$$

with

$$a_{11}=a_{22}=a, \quad a_{12}=a_{21}=a_{00}=0$$

Weak Formulation

Weak Formulation

Then, for each element you need to compute two terms to build the element linear system:

- The stiff matrix

$$\begin{pmatrix} k_{11}^k & \cdots & k_{1n}^k \\ \vdots & \ddots & \vdots \\ k_{n1}^k & \cdots & k_{nn}^k \end{pmatrix} \begin{pmatrix} u_1^k \\ \vdots \\ u_n^k \end{pmatrix} = \begin{pmatrix} F_1^k \\ \vdots \\ F_n^k \end{pmatrix} + \begin{pmatrix} Q_1^k \\ \vdots \\ Q_n^k \end{pmatrix}$$

Weak Formulation

Then, for each element you need to compute two terms to build the element linear system:

- The stiff matrix
- The F's vector

$$\begin{pmatrix} k_{11}^k & \cdots & k_{1n}^k \\ \vdots & \ddots & \vdots \\ k_{n1}^k & \cdots & k_{nn}^k \end{pmatrix} \begin{pmatrix} u_1^k \\ \vdots \\ u_n^k \end{pmatrix} = \begin{pmatrix} F_1^k \\ \vdots \\ F_n^k \end{pmatrix} + \begin{pmatrix} Q_1^k \\ \vdots \\ Q_n^k \end{pmatrix}$$

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The Q's vector depends on the Boundary Conditions and is computed once you assemble the global system

Weak Formulation

Remember $$q_n \equiv n_x\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+n_y\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)$$

Notation:

$u = u(x, y)$ is named **primary variable**

q_n is named **secondary variable**

Weak Formulation

Remember $q_n \equiv n_x\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + n_y\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)$

Notation:

$u = u(x, y)$ is named **primary variable**

q_n is named **secondary variable**

Boundary Conditions (BC):

$u_A = u(x_A)$ is an **essential BC** (fix the primary variable)

$q_n = Q_0$ is a **natural BC** (fix the secondary variable)

Weak Formulation

Notation from the global system of equations: $[K^k]u^k = F^k + Q^k.$

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$$K_{ij}^k = \int_{\Omega^k}\left[\frac{\partial\psi_i^k}{\partial x}\left(a_{11}\frac{\partial\psi_j^k}{\partial x} + a_{12}\frac{\partial\psi_j^k}{\partial y}\right) + \frac{\partial\psi_i^k}{\partial y}\left(a_{21}\frac{\partial\psi_j^k}{\partial x} + a_{22}\frac{\partial\psi_j^k}{\partial y}\right) + a_{00}\psi_i^k\psi_j^k\right] dx\, dy,$$

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We will denote the different terms by:

$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

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Computing the integrals

Computing the Integrals

To build the linear system you need to compute terms like these ones:

$$K_{ij}^{k,00} = \int_{\Omega^k} a_{00}\, \psi_i^k \psi_j^k dx\, dy\,, \qquad K_{ij}^{k,11} = \int_{\Omega^k} a_{11}\, \frac{\partial \psi_i^k}{\partial x} \frac{\partial \psi_j^k}{\partial x} dx\, dy\,,$$

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In general you need to compute **numerically** these 2D integrals. For that we will use **Gauss integration methods** that will be introduced later.

For some easy cases there are some **explicit formulas** that we present next.

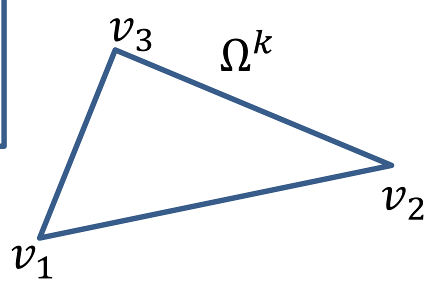
Triangles

Computing the Integrals: Triangles

- If we consider **constant coefficients** for the *model equation*

We have to compute

$$\boxed{\boldsymbol{K}_{ij}^{k,11} = a_{11} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial x}(x,y)\,dxdy}$$

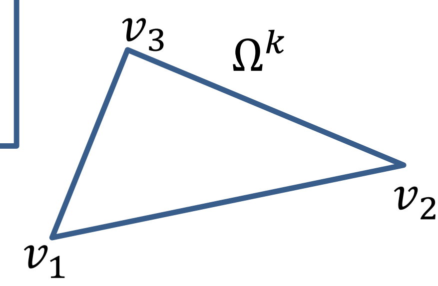


Computing the Integrals: Triangles

- If we consider **constant coefficients** for the *model equation*

We have to compute

$$\boxed{\boldsymbol{K}_{ij}^{k,11} = a_{11} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y)\frac{\partial \psi_j^k}{\partial x}(x,y)dxdy}$$



In the case of a general **linear triangular element**

$$\psi_i^k(x,y) = \frac{a_i + \beta_i x + \gamma_i y}{2A_k}, \; i = 1,2,3$$

$$a_i = x_j y_k - x_k y_j$$

$$\beta_i = y_j - y_k$$

$$\gamma_i = x_k - x_j$$

(i,j,k) cyclic permutations

Computing the Integrals: Triangles

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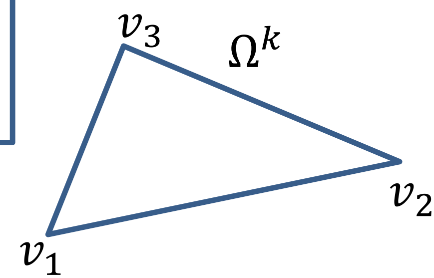


In the case of a general **linear triangular element**

$$\psi_i^k(x,y) = \frac{a_i + \beta_i x + \gamma_i y}{2A_k}, \quad i = 1,2,3$$

$$a_i = x_j y_k - x_k y_j$$

$$\beta_i = y_j - y_k$$

$$\gamma_i = x_k - x_j$$

(i,j,k) cyclic permutations

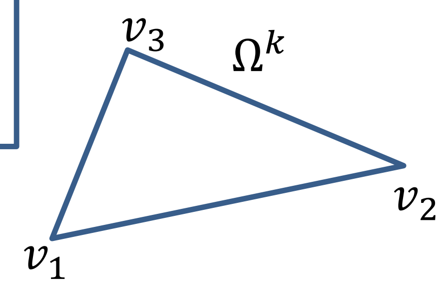
$$\Rightarrow \quad \boxed{\frac{\partial \psi_i^k}{\partial x}(x,y) = \frac{\beta_i}{2A_k}} \qquad \boxed{\frac{\partial \psi_i^k}{\partial y}(x,y) = \frac{\gamma_i}{2A_k}}$$

Computing the Integrals: Triangles

- If we consider **constant coefficients** for the *model equation*

We have to compute

$$\boxed{\boldsymbol{K}_{ij}^{k,11} = a_{11} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial x}(x,y)\,dxdy}$$



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$$\psi_i^k(x,y) = \frac{a_i + \beta_i x + \gamma_i y}{2A_k}, \quad i = 1,2,3$$

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$$\gamma_i = x_k - x_j$$

(i,j,k) cyclic permutations

$$\boxed{\frac{\partial \psi_i^k}{\partial x}(x,y) = \frac{\beta_i}{2A_k}}$$

$$\boxed{\frac{\partial \psi_i^k}{\partial y}(x,y) = \frac{\gamma_i}{2A_k}}$$

$$\boxed{\boldsymbol{K}_{ij}^{k,11} = a_{11} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial x}(x,y)\,dxdy = a_{11} \frac{1}{4A_k} \beta_i \beta_j}$$

Computing the Integrals: Triangles

All together:

$$K_{ij}^{k,11} = a_{11} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial x}(x,y) dxdy = \frac{a_{11}}{4A_k} \beta_i \beta_j$$

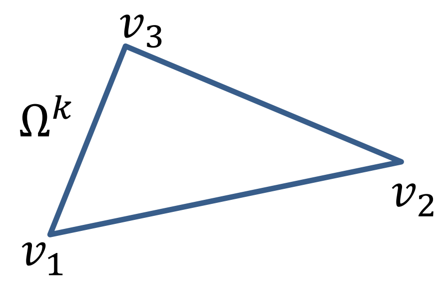
$$K_{ij}^{k,12} = a_{12} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial x}(x,y) \frac{\partial \psi_j^k}{\partial y}(x,y) dxdy = \frac{a_{12}}{4A_k} \beta_i \gamma_j$$

$$K_{ij}^{k,21} = \frac{a_{21}}{4A_k} \gamma_i \beta_j$$

$$K_{ij}^{k,22} = a_{22} \iint_{\Omega_k} \frac{\partial \psi_i^k}{\partial y}(x,y) \frac{\partial \psi_j^k}{\partial y}(x,y) dxdy = \frac{a_{22}}{4A_k} \gamma_i \gamma_j$$

$$K_{ij}^{k,00} = a_{00} \iint_{\Omega_k} \psi_i^k(x,y) \psi_j^k(x,y) dxdy = \frac{a_{00} A_k}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$F^k = \frac{f_k A_k}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$



If the vertices of the triangle are $v_i = (x_i, y_i)$ we define:

$$\beta_i = y_j - y_k$$
$$\gamma_i = -(x_j - x_k)$$

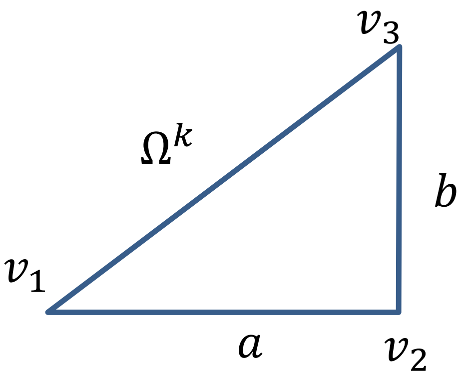
(i,j,k) cyclic permutations

A_k is triangle area

Computing the Integrals: Triangles

- In the case of a general **linear** **triangular rectangle** **element** for the **Poisson's Equation**

$$(\mathrm{a}_{11} = \mathrm{a}_{22} = c, \quad \mathrm{a}_{12} = \mathrm{a}_{21} = \mathrm{a}_{00} = 0)$$



Computing the Integrals: Triangles

- In the case of a general **linear triangular rectangle element** for the **Poisson's Equation**

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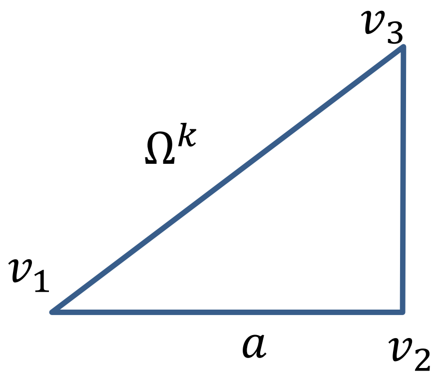


The formula:

$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

Simplifies to:

$$K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2 + b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}$$

$$F^k = \frac{f_k A_k}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{f_k ab}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

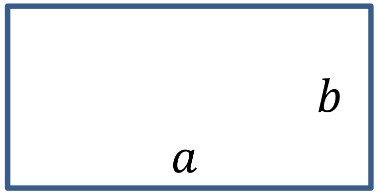
Rectangles

Computing the Integrals: Rectangles

- If we consider **constant coefficients** for the *model equation*

In the case of a **rectangular quadrilateral**

Ω^k

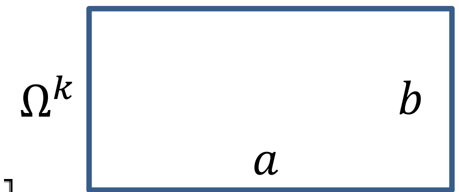


$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

Computing the Integrals: Rectangles

- If we consider **constant coefficients** for the *model equation*

In the case of a **rectangular quadrilateral**



$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

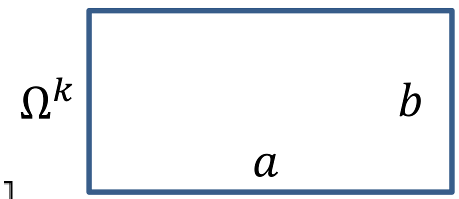
$$[K^{k,11}] = \frac{b\,a_{11}^k}{6a}\begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix}, \quad [K^{k,12}] = \frac{a_{12}^k}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$[K^{k,22}] = \frac{a\,a_{22}^k}{6b}\begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix}, \quad [K^{k,00}] = \frac{ab\,a_{00}^k}{36}\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}.$$

Computing the Integrals: Rectangles

- If we consider **constant coefficients** for the *model equation*

In the case of a **rectangular quadrilateral**



$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

$$[K^{k,11}] = \frac{b\,a_{11}^k}{6a}\begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix}, \quad [K^{k,12}] = \frac{a_{12}^k}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

It does NOT apply to general quadrilaterals!!

$$[K^{k,22}] = \frac{a\,a_{22}^k}{6b}\begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix}, \quad [K^{k,00}] = \frac{ab\,a_{00}^k}{36}\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}.$$

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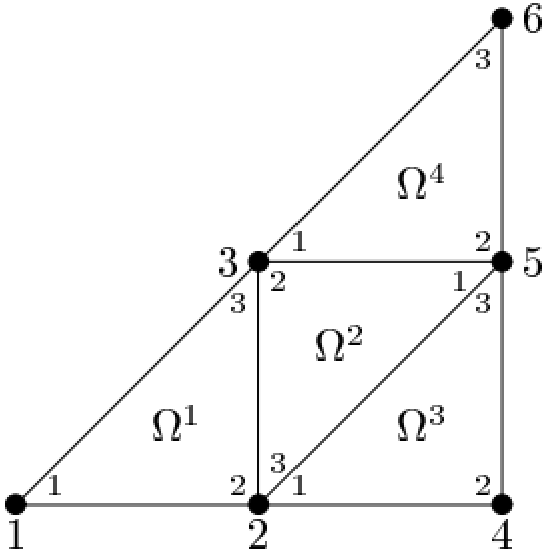
2D Element Assembly

2D Elements Assembly

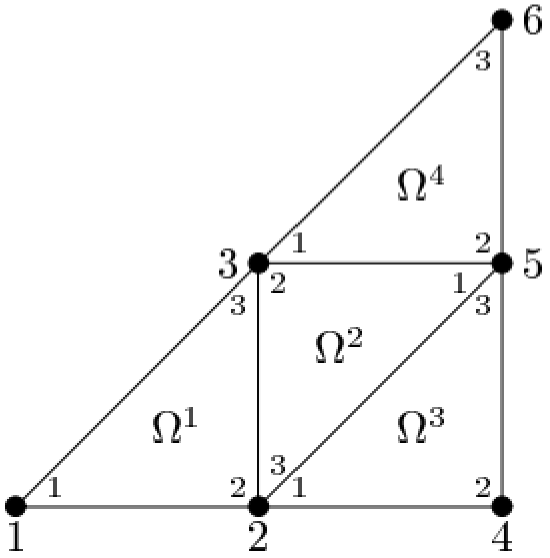
- Example of **local** and **global** 2D nodes enumeration for linear triangular elements



Hint: Notice that the local enumeration must be counter-clockwise in order to preserve orientation

2D Elements Assembly

- Example of **local** and **global** 2D nodes enumeration for linear triangular elements

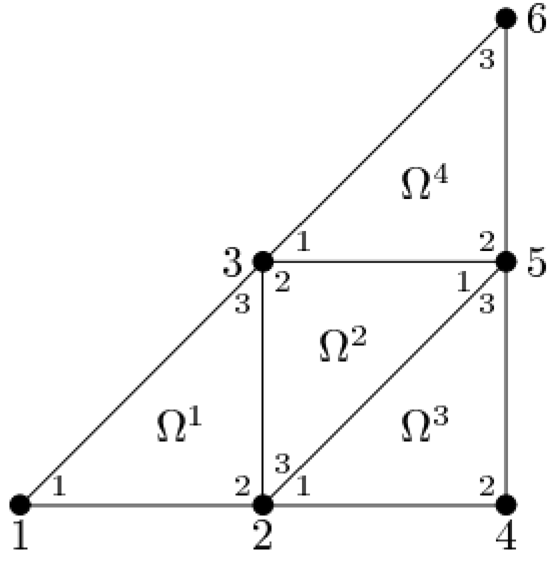


Connectivity matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

2D Elements Assembly

- Example of **local** and **global** 2D nodes enumeration for linear triangular elements

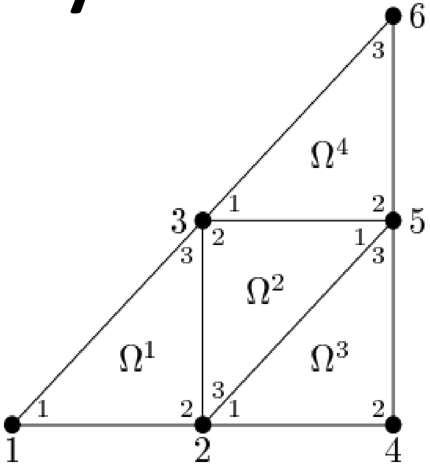


Connectivity matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

Compute the global stiff matrix for this example: $K^e = \begin{pmatrix} k_{11} & \cdots & k_{16} \\ \vdots & \ddots & \vdots \\ k_{61} & \cdots & k_{66} \end{pmatrix}$?

2D Elements Assembly



For linear triangular elements:

If $u(x)$ is a **1D magnitude** (temperature)
For each element the **Stiffness Matrix** is a 3x3 matrix

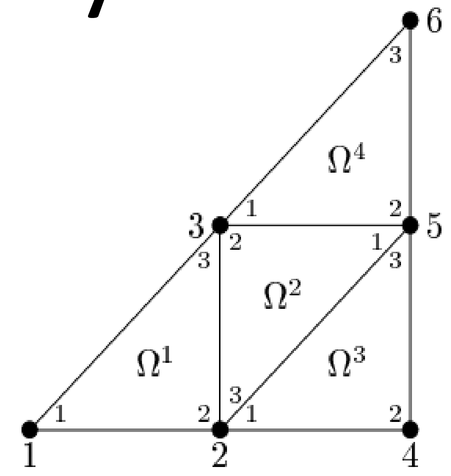
$$K^e = \begin{pmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{pmatrix}, \qquad u = \begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \end{pmatrix}$$

Let's do the assembly process for this example.

2D Elements Assembly

The stiff matrix is a 6x6 matrix

$$K = \begin{pmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{pmatrix}$$



Each row is associated to the corresponding node: row 1 with node 1
We will fill the matrix row by row

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2D Elements Assembly

<u>First row:</u>

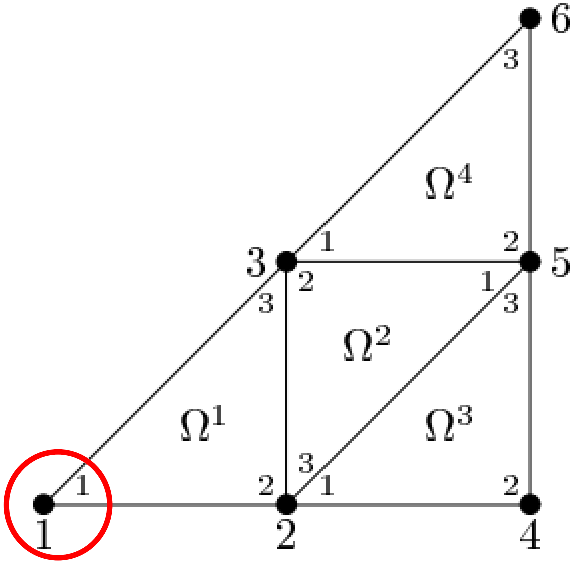
The k_{11} element is the relation of global node 1 with itself

this means the relationship between:

local node 1 of the element Ω^1 and itself: $\boxed{k_{11}^1}$

Row 1 :

$$K = \begin{pmatrix} k_{11}^1 & & \\ & & \\ & & \end{pmatrix}$$



2D Elements Assembly

First row:

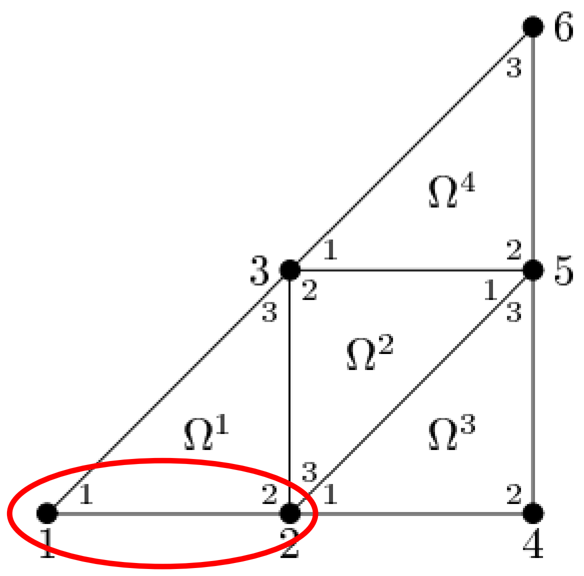
The k_{12} element is the relation of global node 1 with node 2

this means the relationship between:

local node 1 of Ω^1 and local node 2 of Ω^1: $\boxed{k^1_{12}}$

Row 1 :

$$K = \begin{pmatrix} k^1_{11} & k^1_{12} & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}$$



2D Elements Assembly

First row:

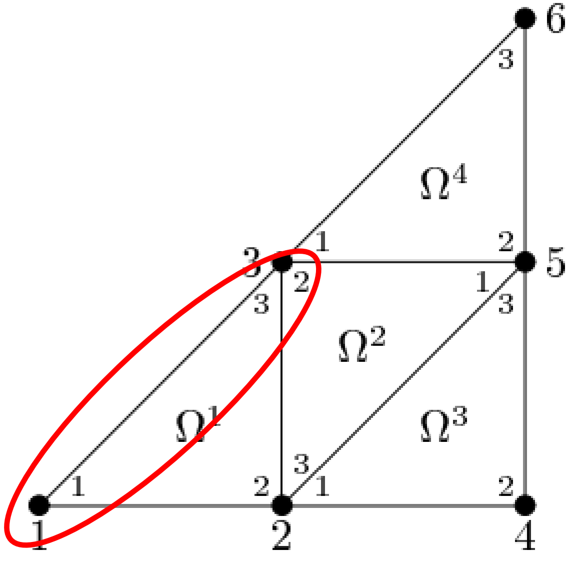
The k_{13} element is the relation of global node 1 with node 3

this means the relationship between:

local node 1 of Ω^1 and local node 3 of Ω^1: $\boxed{k_{13}^1}$

Row 1 :

$$K = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}$$



2D Elements Assembly

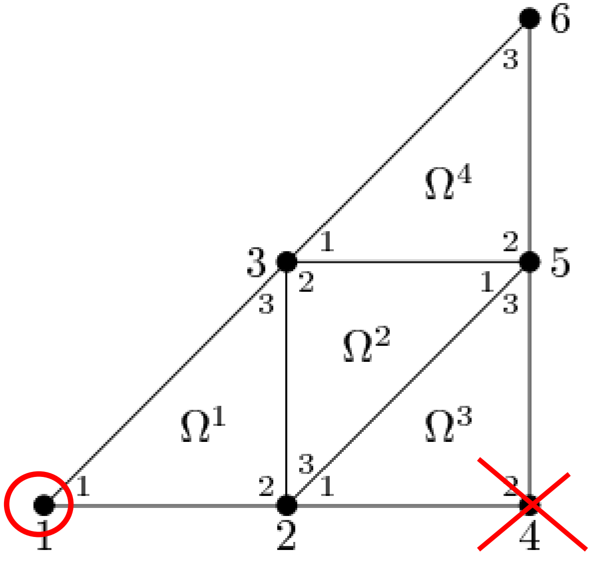
First row:

The k_{14} element is the relation of global node 1 with node 4

No connection through an element exist !!

Row 1 :

$$K = \begin{pmatrix} k_{11}^{1} & k_{12}^{1} & k_{13}^{1} & 0 & & \\ & & & & & \\ & & & & & \end{pmatrix}$$



2D Elements Assembly

First row:

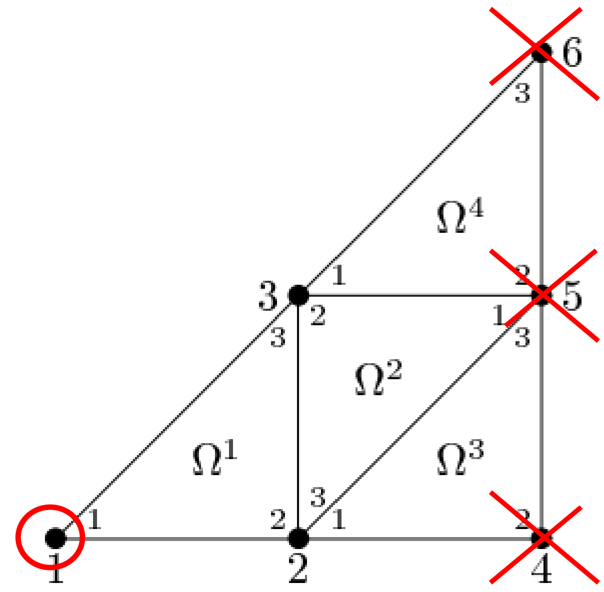
The k_{14} element is the relation of global node 1 with node 4

No connection through an element exist !!

The same happens with nodes 5 and 6

Row 1 :

$$K = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}$$



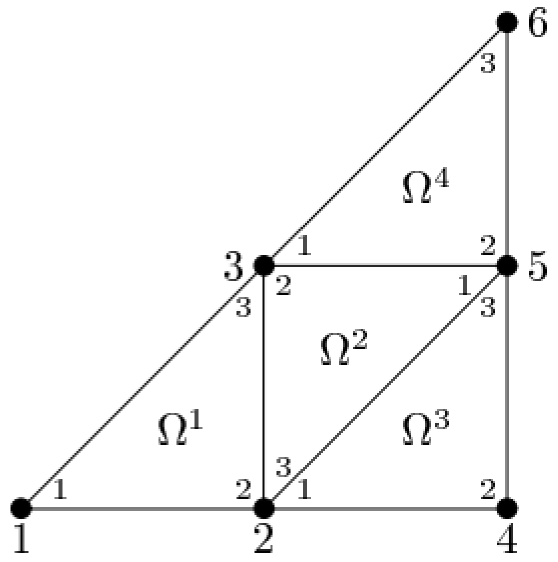
2D Elements Assembly

<u>Symmetries:</u>

Because of stiff matrix symmetries, we can fill the first column.

col 1 :

$$K = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 \\ k_{21}^1 & & & & & \\ k_{31}^1 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}$$



2D Elements Assembly

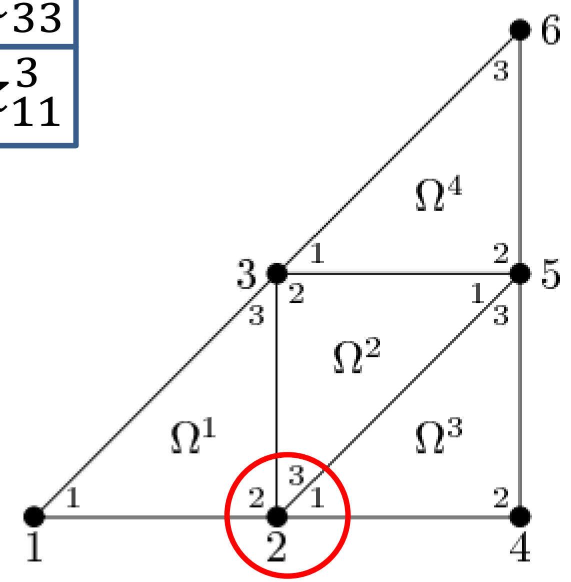
Second row:

The k_{22} element is the relation of global node 2 with itself:

local node 2 of the element Ω^1 and itself: $\boxed{k^1_{22}}$

local node 3 of the element Ω^2 and itself: $\boxed{k^2_{33}}$

local node 1 of the element Ω^3 and itself: $\boxed{k^3_{11}}$



Row 2 :

$$k_{22} = k^1_{22} + k^2_{33} + k^3_{11}$$

$$K = \begin{pmatrix} k^1_{11} & k^1_{12} & k^1_{13} & 0 & 0 & 0 \\ k^1_{21} & k_{22} & & & & \\ k^1_{31} & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}$$

2D Elements Assembly

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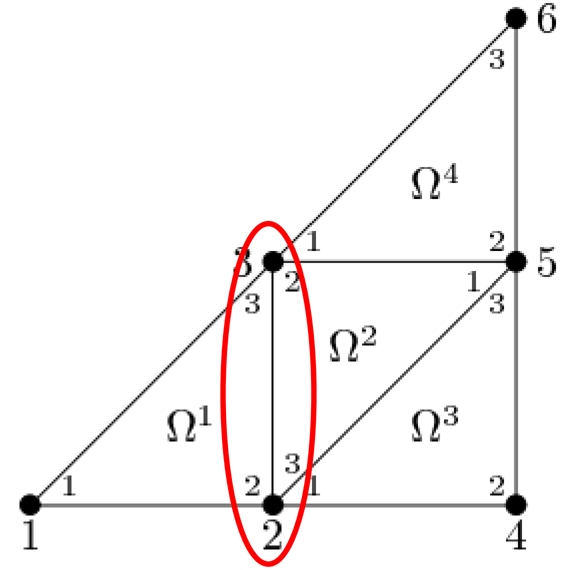
<u>Second row:</u>

The k_{23} element is the relation of global node 2 with node 2:

local node 2 of Ω^1 and node 3 of Ω^1: $\boxed{k_{23}^1}$

local node 3 of Ω^2 and node 2 of Ω^2: $\boxed{k_{32}^2}$



Row 2 :

$$k_{23} = k_{23}^1 + k_{32}^2$$

$$K = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 \\ k_{21}^1 & k_{22} & k_{23} & & & \\ k_{31}^1 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}$$

2D Elements Assembly

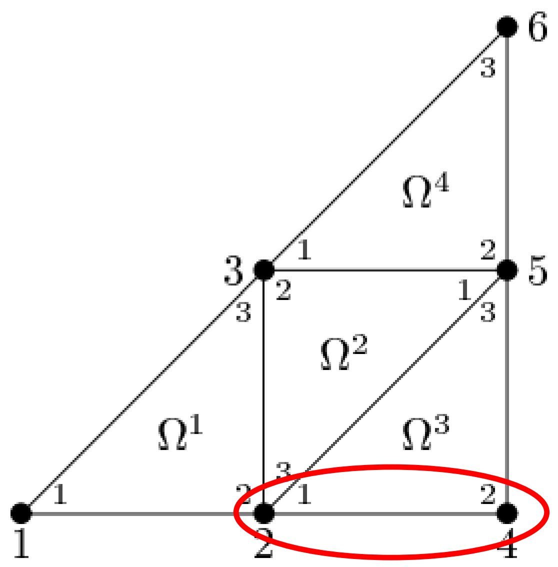
<u>Second row:</u>

The k_{24} element is the relation of global node 2 with node 4: local node 1 of Ω^3 and node 2 of Ω^3: $\boxed{k^3_{12}}$

Row 2 :

$$k_{24} = k^3_{12}$$

$$K = \begin{pmatrix} k^1_{11} & k^1_{12} & k^1_{13} & 0 & 0 & 0 \\ k^1_{21} & k_{22} & k_{23} & k_{24} & & \\ k^1_{31} & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}$$



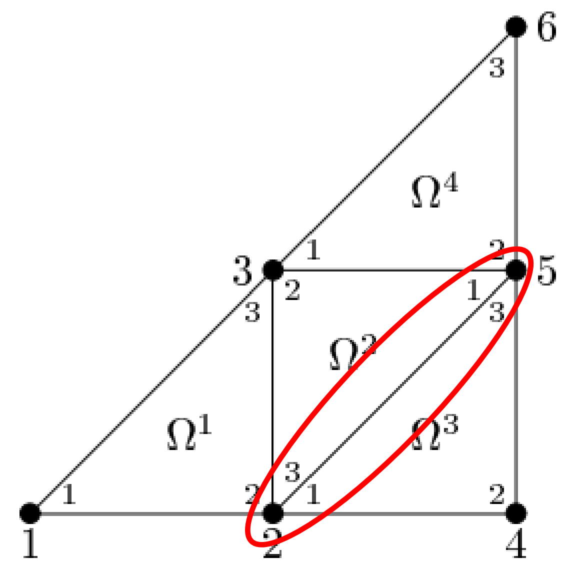
2D Elements Assembly

Second row:

The k_{25} element is the relation of global node 2 with node 5:

local node 3 of Ω^2 and node 1 of Ω^2: $\boxed{k_{31}^2}$

local node 1 of Ω^3 and node 3 of Ω^3: $\boxed{k_{13}^3}$



Row 2 :

$$k_{25} = k_{31}^2 + k_{13}^3$$

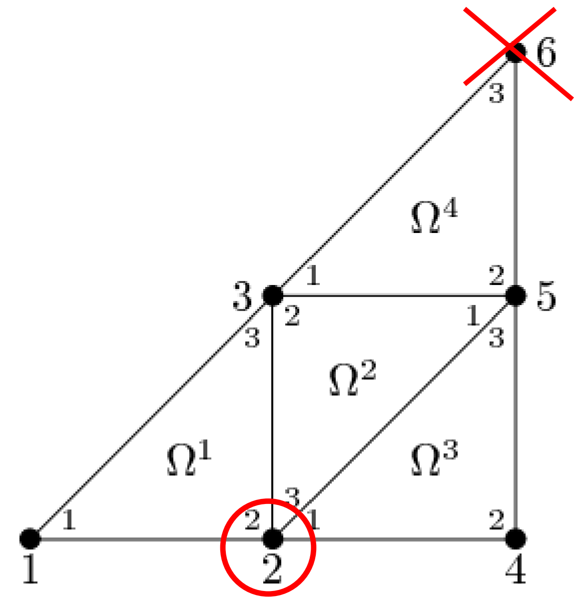
$$K = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 \\ k_{21}^1 & k_{22} & k_{23} & k_{24} & k_{25} & \\ k_{31}^1 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}$$

2D Elements Assembly

<u>Second row:</u>

The k_{26} element is the relation of global node 2 with node 6:

No connection exist



Row 2 :

$$k_{26} = 0$$

$$K = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 \\ k_{21}^1 & k_{22} & k_{23} & k_{24} & k_{25} & 0 \\ k_{31}^1 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix}$$

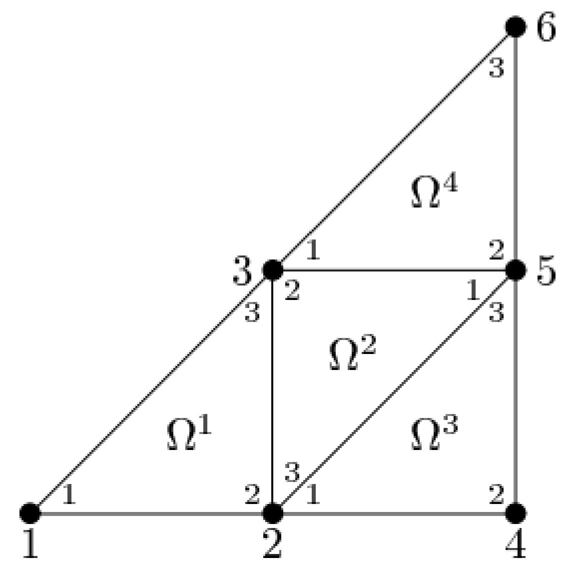
2D Elements Assembly

Symmetries:

Because of stiff matrix symmetries, we can fill the first column.

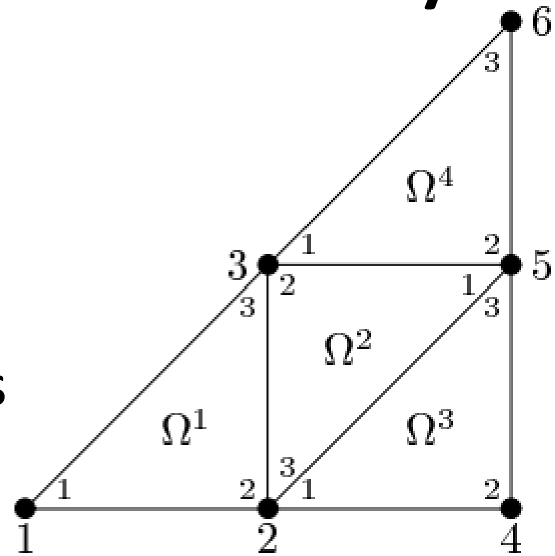
col 2 :

$$K = \begin{pmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 \\ k_{21}^1 & k_{22} & k_{23} & k_{24} & k_{25} & 0 \\ k_{31}^1 & k_{23} & \square & \square & \square & \square \\ 0 & k_{24} & \square & \square & \square & \square \\ 0 & k_{25} & \square & \square & \square & \square \\ 0 & 0 & \square & \square & \square & \square \end{pmatrix}$$



We left as an exercise to fill all the matrix!!

Matlab: 2D Elements Assembly

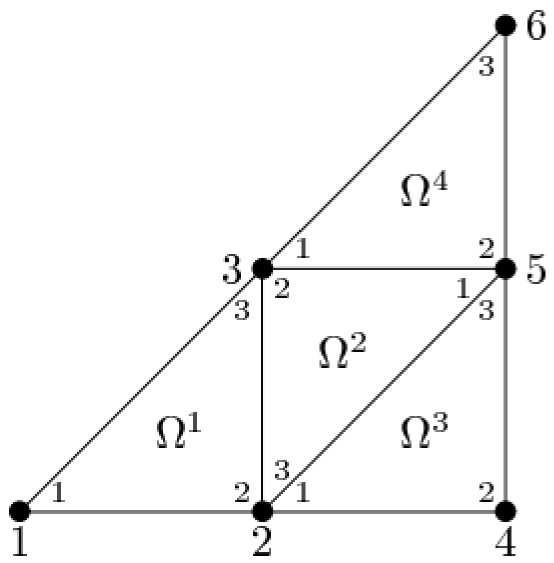


- For linear triangular elements

If $\boldsymbol{u}(\boldsymbol{x})$ is a **1D magnitude** (temperature)
For each element the **Stiffness Matrix** is a 3x3 matrix

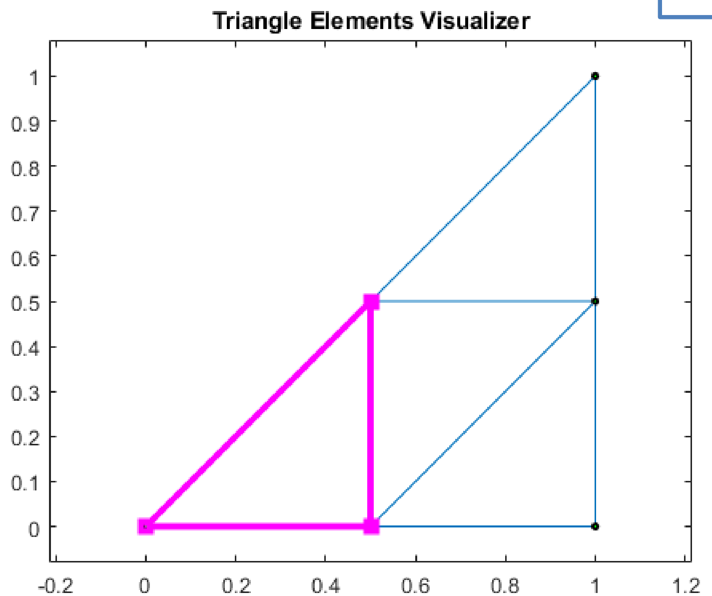
$$K^e = \begin{pmatrix} k^e_{11} & k^e_{12} & k^e_{13} \\ k^e_{21} & k^e_{22} & k^e_{23} \\ k^e_{31} & k^e_{32} & k^e_{33} \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

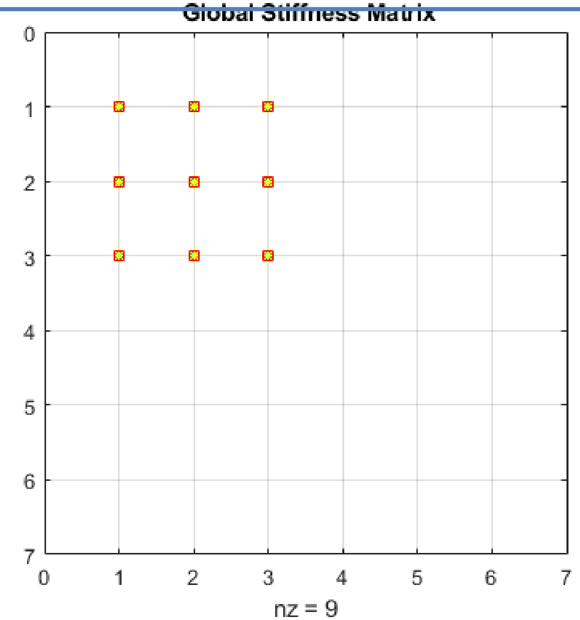
```
rows  = [elem(e,1);  elem(e,2);  elem(e,3)];
colums =  rows;
K(rows,colums) = K(rows,colums) + Ke; %assembly
```

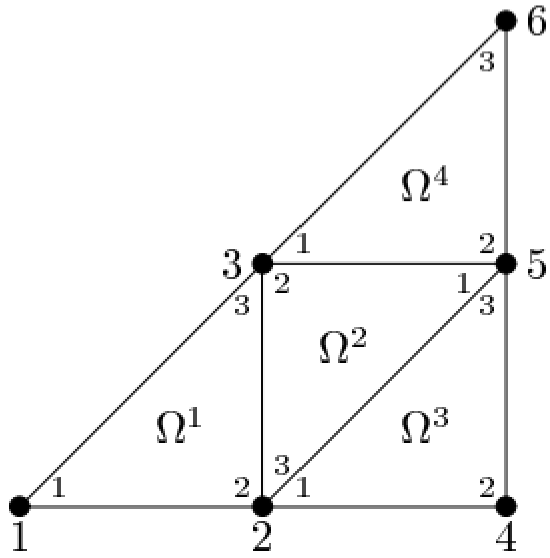


$$B = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

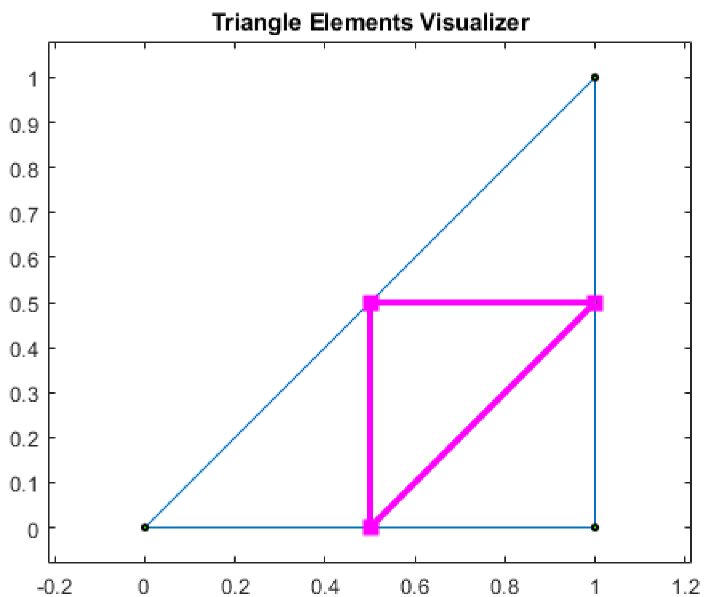
```
rows  = [elem(e,1);  elem(e,2);  elem(e,3)];
colums =  rows;
K(rows,colums) = K(rows,colums) + Ke; %assembly
```

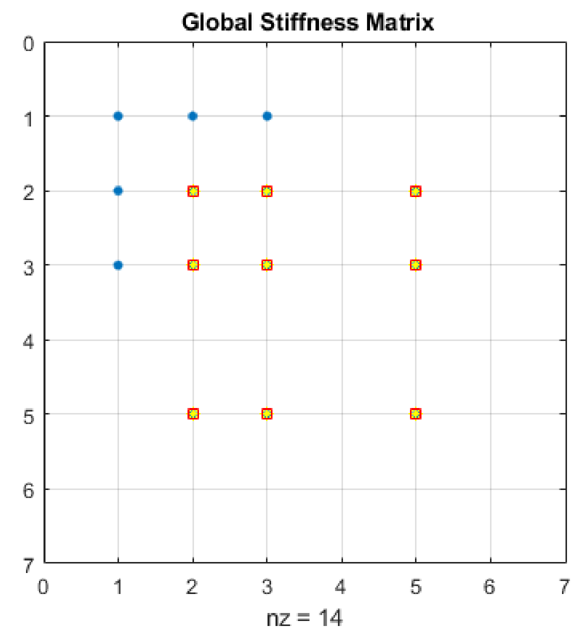






$$B = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$





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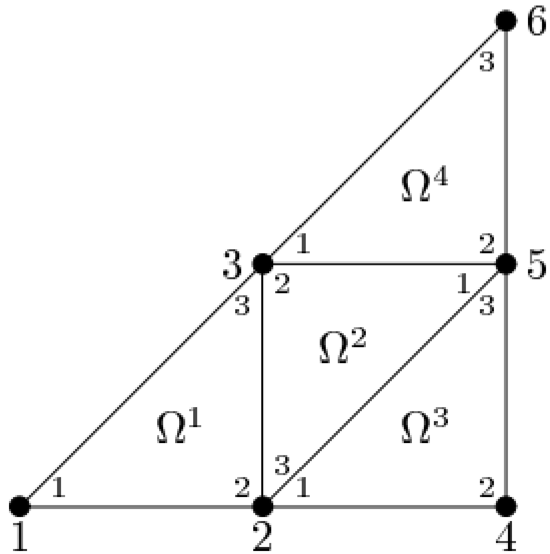
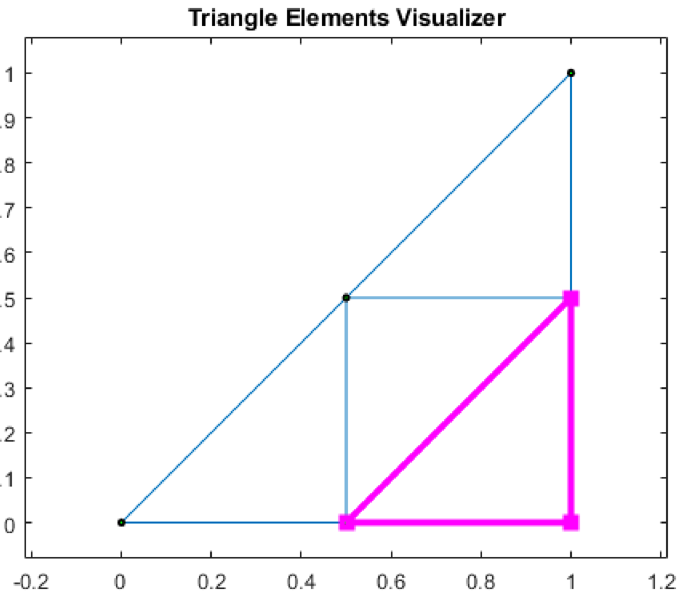
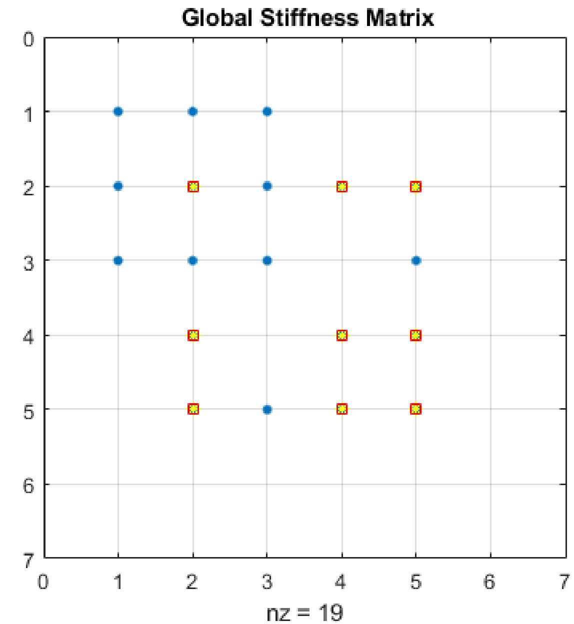
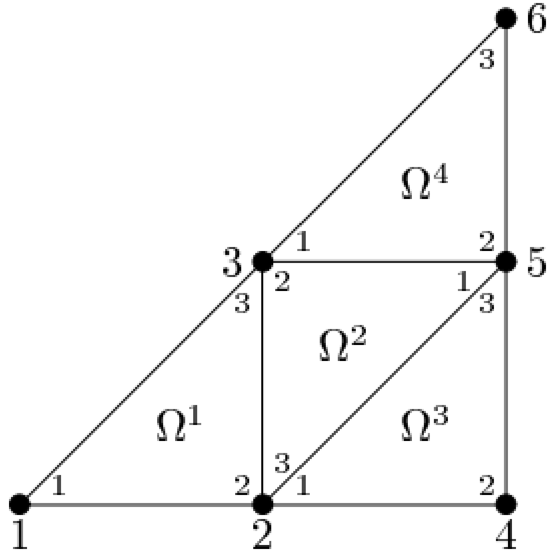


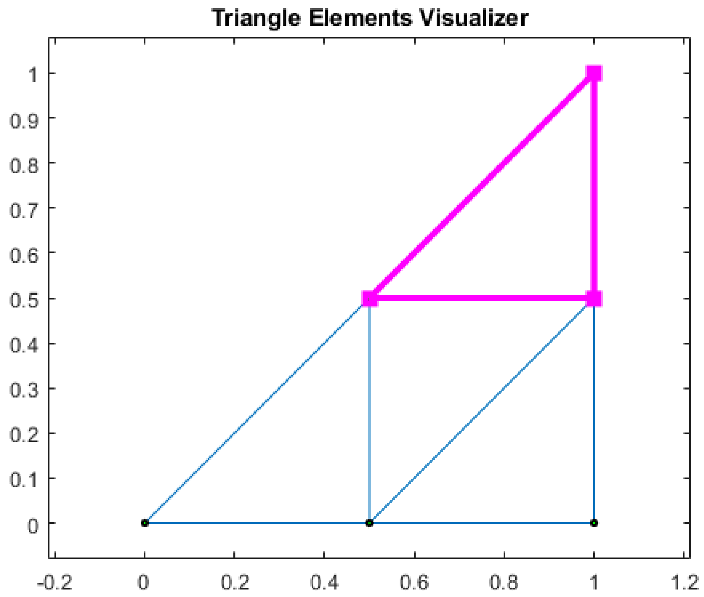
$$B = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

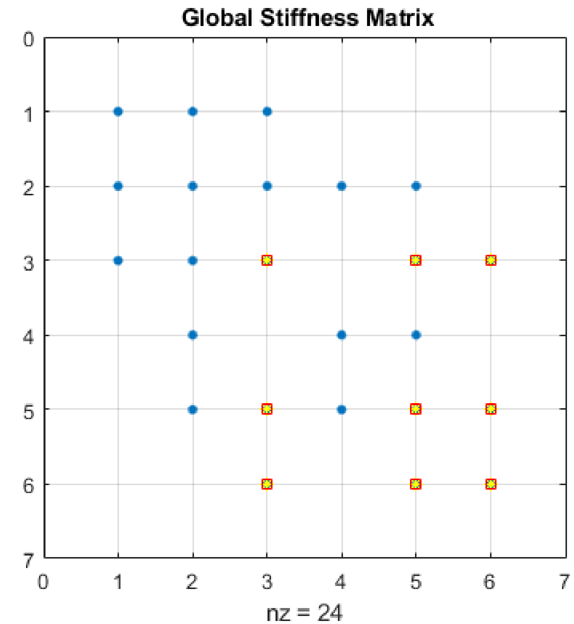






$$B = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

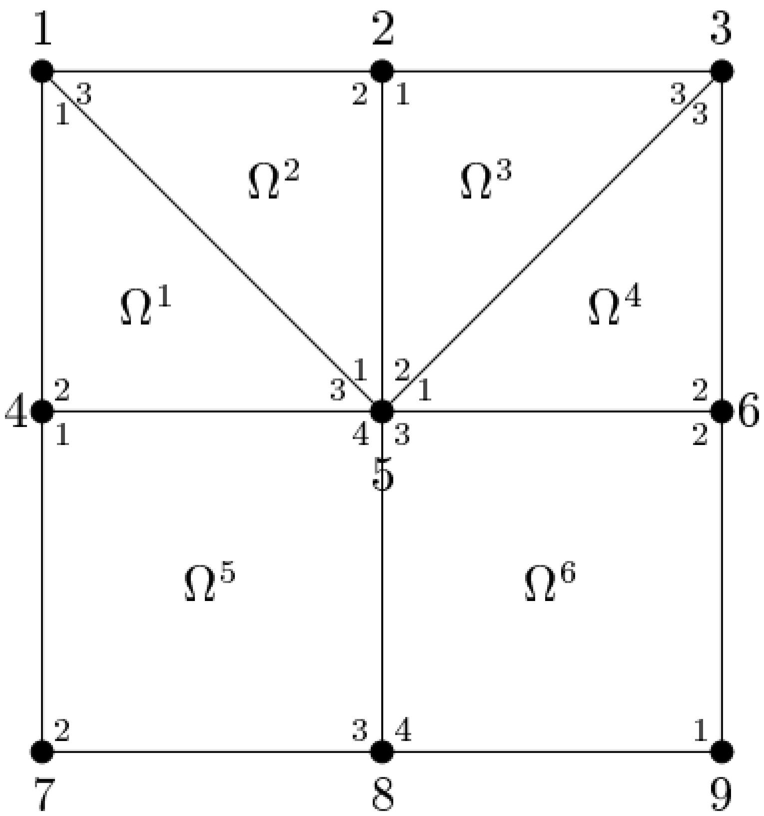




2D Assembly of mixed elements

- Let's consider the example:

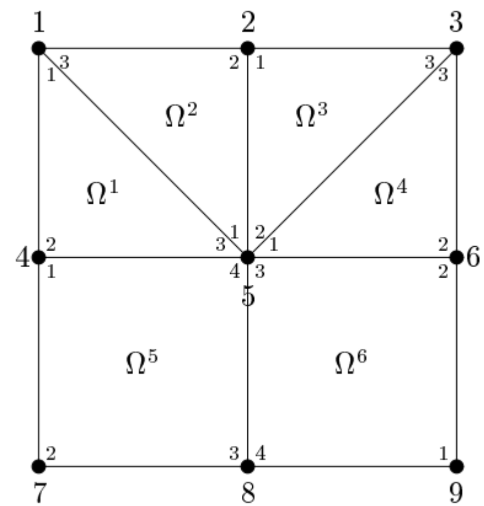
Although it is **not usual**,
we can mix different type of elements:
Triangular + Rectangular



2D Assembly of element equations

- The connectivity matrix in this case is not uniform

$$C = \begin{pmatrix} 1 & 4 & 5 & * \\ 5 & 2 & 1 & * \\ 2 & 5 & 3 & * \\ 5 & 6 & 3 & * \\ 4 & 7 & 8 & 5 \\ 9 & 6 & 5 & 8 \end{pmatrix}$$



Triangular

$$K^e = \begin{pmatrix} k^e_{11} & k^e_{12} & k^e_{13} \\ k^e_{21} & k^e_{22} & k^e_{23} \\ k^e_{31} & k^e_{32} & k^e_{33} \end{pmatrix}, \quad e=1,2,3,4$$

Rectangular

$$K^e = \begin{pmatrix} k^e_{11} & k^e_{12} & k^e_{13} & k^e_{14} \\ k^e_{21} & k^e_{22} & k^e_{23} & k^e_{24} \\ k^e_{31} & k^e_{32} & k^e_{33} & k^e_{34} \\ k^e_{41} & k^e_{42} & k^e_{43} & k^e_{44} \end{pmatrix}, \quad e=5,6$$

2D Assembly of element equations

- The global **stiffness matrix** is

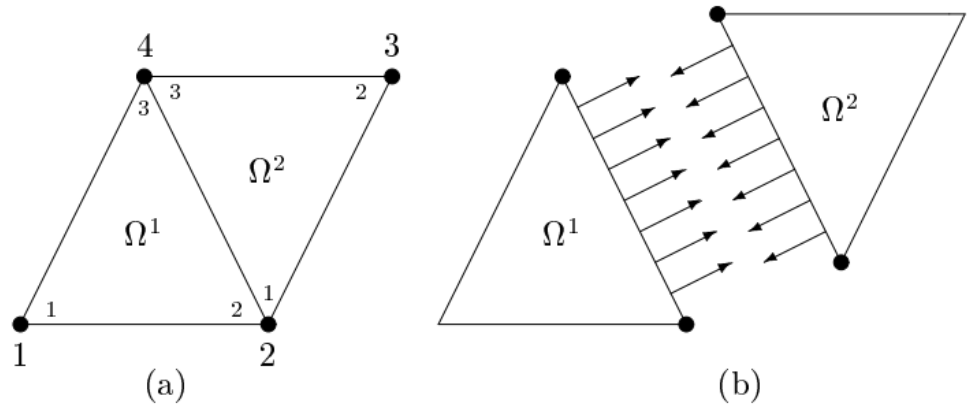
$$
\begin{pmatrix}
K^1_{11}+K^2_{33} & K^2_{11} & 0 & K^1_{12} & K^1_{13}+K^2_{31} & 0 & 0 & 0 & 0 \\
K^2_{13} & K^2_{22}+K^3_{11} & K^3_{13} & 0 & K^2_{21}+K^3_{12} & 0 & 0 & 0 & 0 \\
0 & K^3_{31} & K^3_{33}+K^4_{33} & 0 & K^3_{32}+K^4_{31} & K^4_{32} & 0 & 0 & 0 \\
K^1_{21} & 0 & 0 & K^1_{22}+K^5_{11} & K^1_{23}+K^5_{14} & 0 & K^5_{12} & K^5_{13} & 0 \\
K^1_{31}+K^2_{13} & K^2_{12}+K^3_{21} & K^3_{23}+K^4_{13} & K^1_{32}+K^5_{41} & K_{55} & K^4_{12}+K^6_{32} & K^5_{42} & K^5_{43}+K^6_{34} & K^6_{31} \\
0 & 0 & K^4_{23} & 0 & K^4_{21}+K^6_{23} & K^4_{22}+K^6_{22} & 0 & K^6_{24} & K^6_{21} \\
0 & 0 & 0 & K^5_{21} & K^5_{24} & 0 & K^5_{22} & K^5_{23} & 0 \\
0 & 0 & 0 & K^5_{31} & K^5_{34}+K^6_{43} & K^6_{42} & K^5_{32} & K^5_{33}+K^6_{44} & K^6_{41} \\
0 & 0 & 0 & 0 & K^6_{13} & K^6_{12} & 0 & K^6_{14} & K^6_{11}
\end{pmatrix}
$$

with

$$K_{55} = K^1_{33} + K^2_{11} + K^3_{22} + K^4_{11} + K^5_{44} + K^6_{33}.$$

2D Assembly of element equations

- Let's consider a simple example to explain flux balance and BC for the assembled system $[K]\,U = F + Q$,



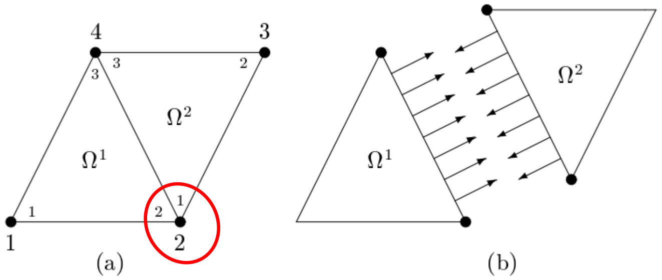
$$\begin{pmatrix} K^1_{11} & K^1_{12} & 0 & K^1_{13} \\ K^1_{21} & K^1_{22}+K^2_{11} & K^2_{12} & K^1_{23}+K^2_{13} \\ 0 & K^2_{21} & K^2_{22} & K^2_{32} \\ K^1_{31} & K^1_{32}+K^2_{31} & K^2_{32} & K^1_{33}+K^2_{33} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} F^1_1 \\ F^1_2+F^2_1 \\ F^2_2 \\ F^1_3+F^2_3 \end{pmatrix} + \begin{pmatrix} Q^1_1 \\ Q^1_2+Q^2_1 \\ Q^2_2 \\ Q^1_3+Q^2_3 \end{pmatrix}$$

2D Assembly of element equations

- Here the balance must be imposed on nodes 2 and 4 remember that Q^k_{ij} means

the flux on node i corresponding to

the contribution of edge j

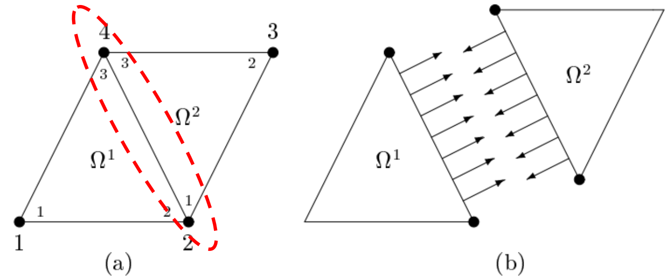


Consider node 2:

$$Q_2 = Q^1_2 + Q^2_1 = (Q^1_{21} + Q^1_{22} + Q^1_{23}) + (Q^2_{11} + Q^2_{12} + Q^2_{13})$$

2D Assembly of element equations

- Here the balance must be imposed on nodes 2 and 4 remember that Q_{ij}^{k} means



the flux on node i corresponding to

the contribution of edge j

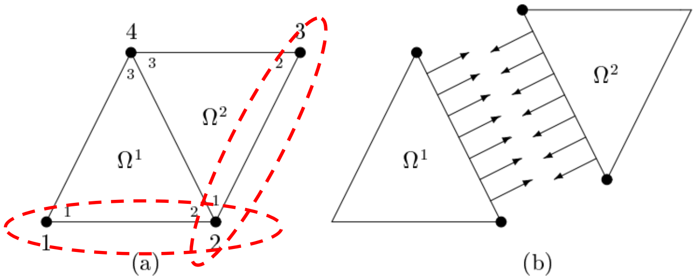
Consider node 2:

$$Q_2 = Q_2^1 + Q_1^2 = (Q_{21}^1 + Q_{22}^1 + Q_{23}^1) + (Q_{11}^2 + Q_{12}^2 + Q_{13}^2) = Q_{21}^1 + Q_{23}^1 + \underbrace{(Q_{22}^1 + Q_{13}^2)}_{=0} + Q_{11}^2 + Q_{12}^2.$$

2D Assembly of element equations

- Here the balance must be imposed on nodes 2 and 4 remember that Q_{ij}^{k} means

the flux on node i corresponding to the contribution of edge j



Consider node 2:

$$Q_2 = Q_2^1 + Q_1^2 = (Q_{21}^1 + Q_{22}^1 + Q_{23}^1) + (Q_{11}^2 + Q_{12}^2 + Q_{13}^2) = Q_{21}^1 + Q_{23}^1 + \underbrace{(Q_{22}^1 + Q_{13}^2)}_{=0} + Q_{11}^2 + Q_{12}^2.$$

by construction we also have $Q_{23}^1 = Q_{12}^2 = 0,$ therefore

$Q_2 = Q_{21}^1 + Q_{11}^2.$, that have to be **defined on the BC** of the problem.

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Mètodes Numèrics:

A First Course on Finite Elements

Finite Elements (V)

Boundary Conditions

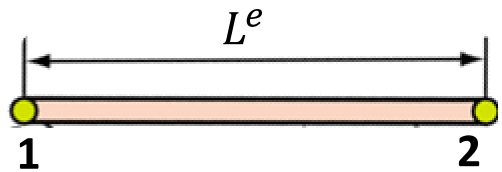
Following: *Curs d'Elements Finits amb Aplicacions* (J. Masdemont)
http://hdl.handle.net/2099.3/36166

Dept. Matemàtiques ETSEIB - UPC BarcelonaTech

FEM 2D
Natural Boundary Conditions

Boundary Conditions

1D – Boundary Conditions (BC)



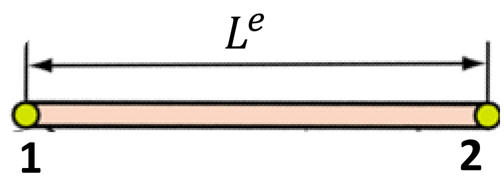
$$Q_i^k = \left(a_1 \frac{du}{dx}\right) \cdot \vec{n}$$

$$Q_1^k = -a_1 \frac{du}{dx}$$

$$Q_2^k = +a_1 \frac{du}{dx}$$

Boundary Conditions

1D – Boundary Conditions (BC)



$$Q_i^k = \left(a_1 \frac{du}{dx} \right) \cdot \vec{n}$$

$$Q_1^k = -a_1 \frac{du}{dx} \qquad Q_2^k = +a_1 \frac{du}{dx}$$

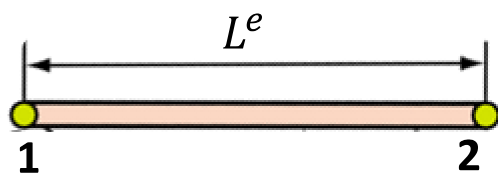
2D – Boundary Conditions (BC)

$$Q_i^k = \int_{\Gamma^k} q_n \psi_i(x,y)\, ds$$ where $$q_n \equiv n_x \left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + n_y \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right)$$

and $\Gamma^k \equiv \partial\Omega^{\mathrm{k}}$ is the **boundary** of the element Ω^k .

Boundary Conditions

1D – Boundary Conditions (BC)

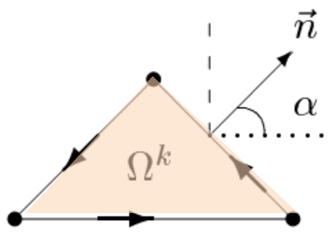


$$Q_i^k = \left(a_1 \frac{du}{dx}\right) \cdot \vec{n} \qquad Q_1^k = -a_1 \frac{du}{dx} \qquad Q_2^k = +a_1 \frac{du}{dx}$$

2D – Boundary Conditions (BC)

$$Q_i^k = \int_{\partial\Omega} q_n \psi_i(x,y)\, ds \quad \text{where} \quad q_n \equiv n_x \left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + n_y \left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)$$

and $\Gamma^k \equiv \partial\Omega^k$ is the **boundary** of the element Ω^k .

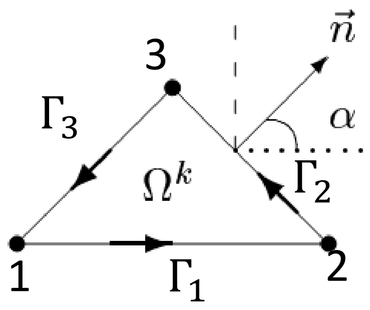


Triangular element

$$\vec{n} = (n_x, n_y) = (\cos\alpha, \sin\alpha)$$

Boundary Conditions

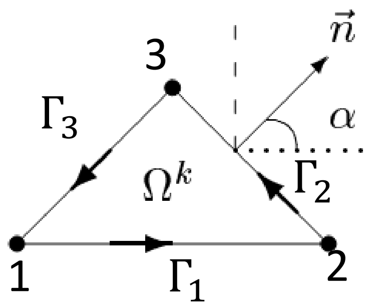
If we consider a **triangular element**



$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell$$

Boundary Conditions

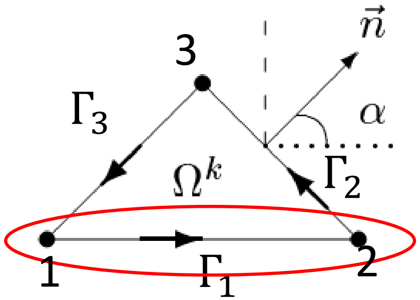
If we consider a **triangular element**



$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = \int_{\Gamma_1^k} q_{n1}^k(s)\psi_{i1}^k(s)\, ds + \int_{\Gamma_2^k} q_{n2}^k(s)\psi_{i2}^k(s)\, ds + \int_{\Gamma_3^k} q_{n3}^k(s)\psi_{i3}^k(s)\, ds,$$

Boundary Conditions

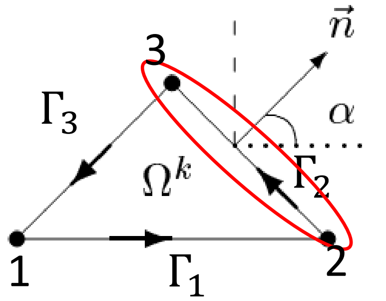
If we consider a **triangular element**



$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = \int_{\Gamma_1^k} q_{n1}^k(s)\psi_{i1}^k(s)\, ds + \int_{\Gamma_2^k} q_{n2}^k(s)\psi_{i2}^k(s)\, ds + \int_{\Gamma_3^k} q_{n3}^k(s)\psi_{i3}^k(s)\, ds,$$

Boundary Conditions

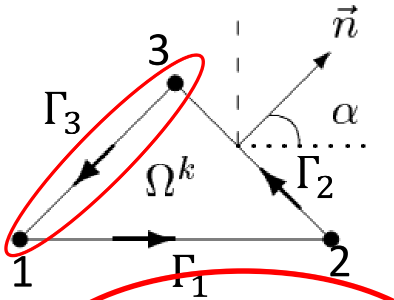
If we consider a **triangular element**



$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = \int_{\Gamma_1^k} q_{n1}^k(s)\psi_{i1}^k(s)\, ds + \int_{\Gamma_2^k} q_{n2}^k(s)\psi_{i2}^k(s)\, ds + \int_{\Gamma_3^k} q_{n3}^k(s)\psi_{i3}^k(s)\, ds,$$

Boundary Conditions

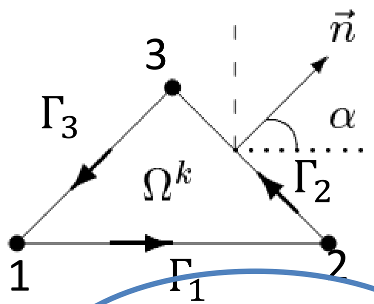
If we consider a **triangular element**



$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = \int_{\Gamma_1^k} q_{n1}^k(s)\psi_{i1}^k(s)\, ds + \int_{\Gamma_2^k} q_{n2}^k(s)\psi_{i2}^k(s)\, ds + \int_{\Gamma_3^k} q_{n3}^k(s)\psi_{i3}^k(s)\, ds,$$

Boundary Conditions

If we consider a **triangular element**



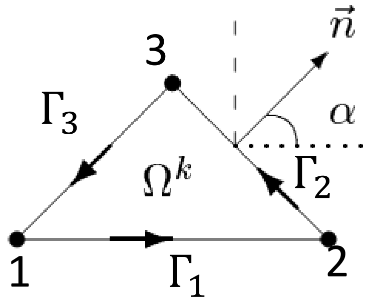
$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = \int_{\Gamma_1^k} q_{n1}^k(s)\psi_{i1}^k(s)\, ds + \int_{\Gamma_2^k} q_{n2}^k(s)\psi_{i2}^k(s)\, ds + \int_{\Gamma_3^k} q_{n3}^k(s)\psi_{i3}^k(s)\, ds,$$

$$Q_i^k \equiv Q_{i1}^k + Q_{i2}^k + Q_{i3}^k, \quad \text{with} \quad Q_{ij}^k = \int_{\Gamma_j^k} q_{nj}^k(s)\psi_{ij}^k(s)\, ds,$$

(Q_{ij}^k means the flux on node i corresponding to the contribution of edge j)

Boundary Conditions

If we consider a **triangular element**



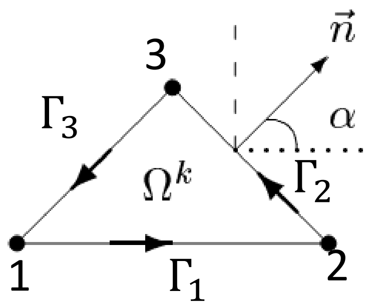
$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = \int_{\Gamma_1^k} q_{n1}^k(s)\psi_{i1}^k(s)\, ds + \int_{\Gamma_2^k} q_{n2}^k(s)\psi_{i2}^k(s)\, ds + \int_{\Gamma_3^k} q_{n3}^k(s)\psi_{i3}^k(s)\, ds,$$

$$Q_i^k \equiv Q_{i1}^k + Q_{i2}^k + Q_{i3}^k, \quad \text{with} \quad Q_{ij}^k = \int_{\Gamma_j^k} q_{nj}^k(s)\psi_{ij}^k(s)\, ds,$$

(Q_{ij}^k means the flux on node i corresponding to the contribution of edge j)

Boundary Conditions

If we consider a **triangular element**



$$Q_i^k = \int_{\Gamma^k} q_n^k \psi_i^k \, d\ell = \int_{\Gamma_1^k} q_{n1}^k(s)\psi_{i1}^k(s)\, ds + \int_{\Gamma_2^k} q_{n2}^k(s)\psi_{i2}^k(s)\, ds + \int_{\Gamma_3^k} q_{n3}^k(s)\psi_{i3}^k(s)\, ds,$$

$$Q_i^k \equiv Q_{i1}^k + Q_{i2}^k + Q_{i3}^k, \quad \text{with} \quad Q_{ij}^k = \int_{\Gamma_j^k} q_{nj}^k(s)\psi_{ij}^k(s)\, ds,$$

(Q_{ij}^k means the flux on node i corresponding to the contribution of edge j)

BC on Triangles

Boundary Conditions

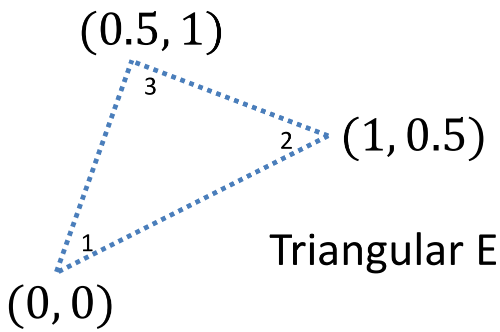
Triangular Shape Functions (a plane)

$$\psi_i^k(x, y) = \alpha + \beta x + \gamma y$$

Boundary Conditions

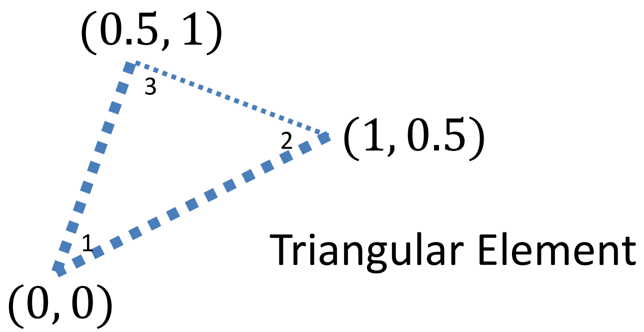
Triangular Shape Functions (a plane)

$$\psi_i^k(x,y) = \alpha + \beta x + \gamma y$$



Triangular Element

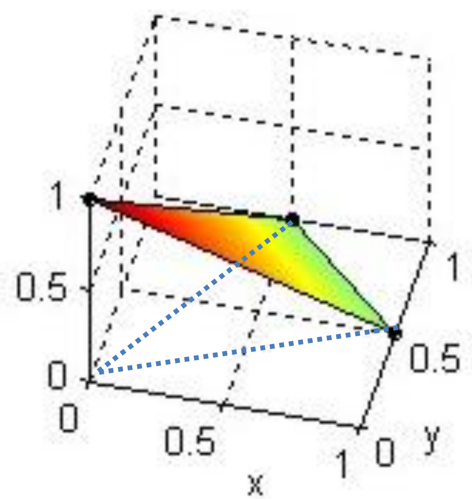
Boundary Conditions



Triangular Shape Functions

$$\psi_i^k(x, y) = \alpha + \beta x + \gamma y$$

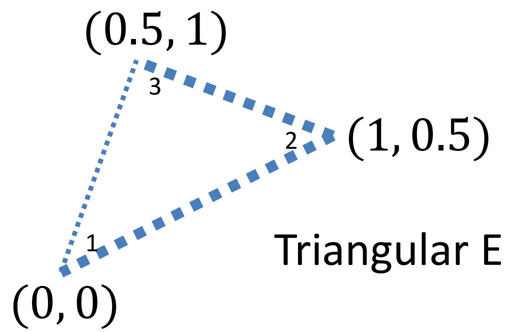
$$\psi_1^k(x, y)$$



Boundary Conditions

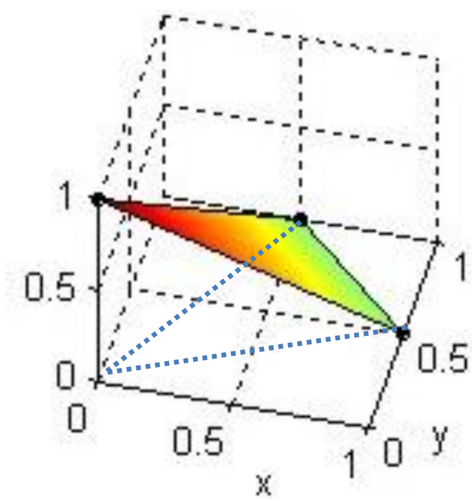
Triangular Shape Functions

$$\psi_i^k(x,y) = \alpha + \beta x + \gamma y$$

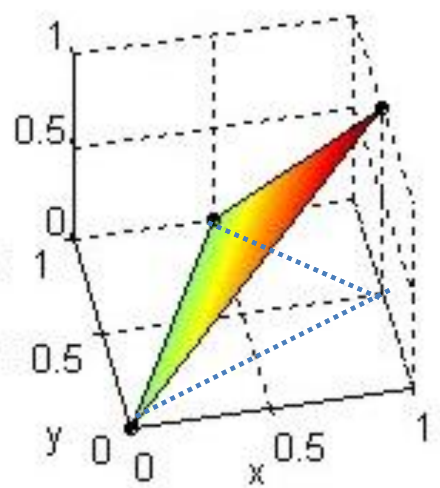


Triangular Element

$\psi_1^k(x,y)$



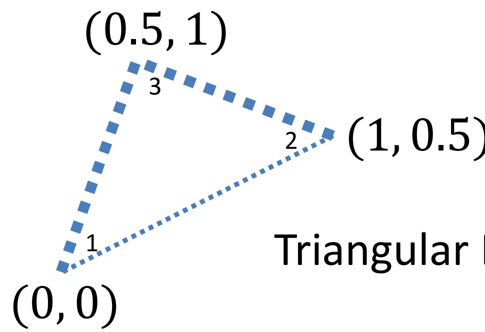
$\psi_2^k(x,y)$



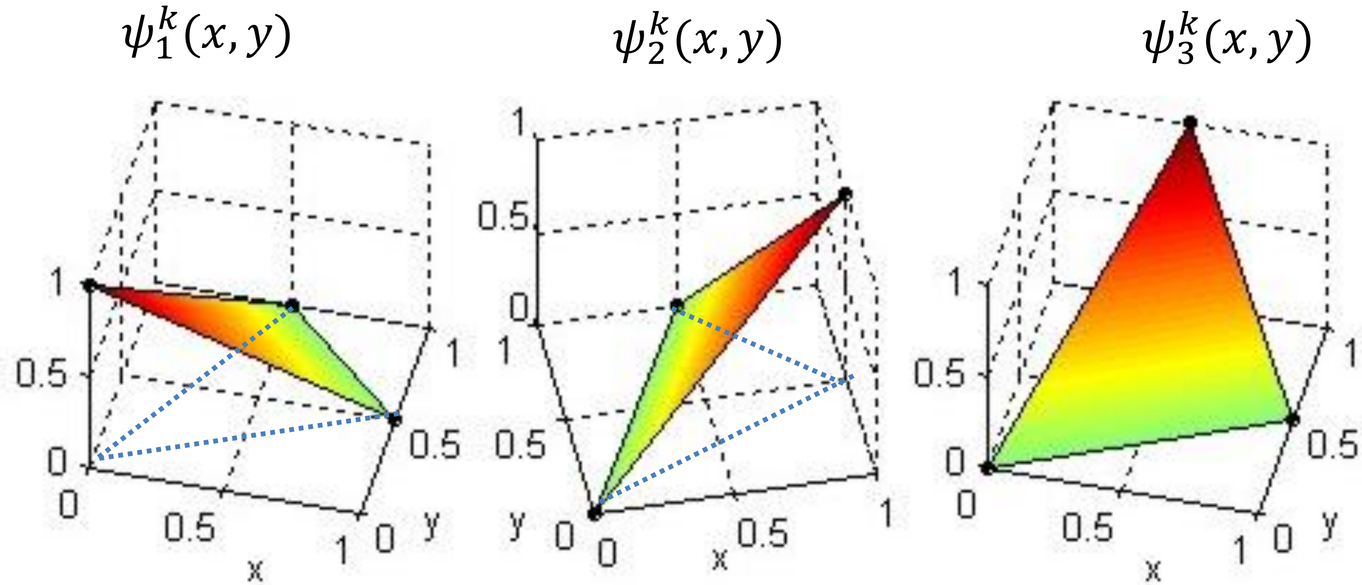
Boundary Conditions

Triangular Shape Functions

$$\psi_i^k(x,y) = \alpha + \beta x + \gamma y$$



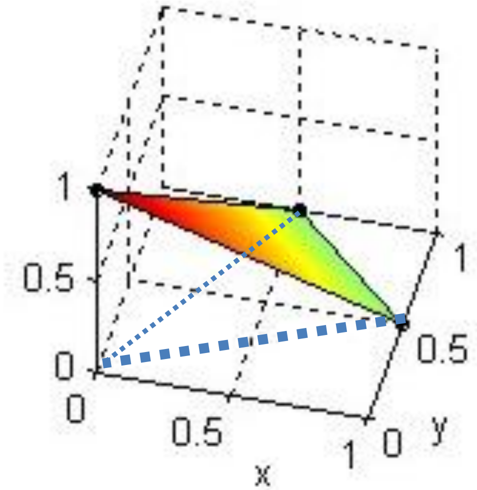
Triangular Element



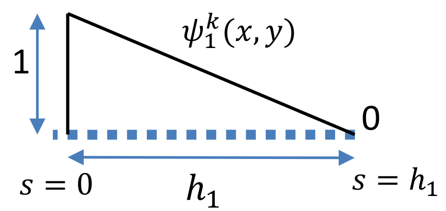
Boundary Conditions

- Restriction of $\psi_i^k(x,y)$ to the boundaries

<u>First function:</u> $\psi_1^k(x,y)$



Restriction to Γ_1

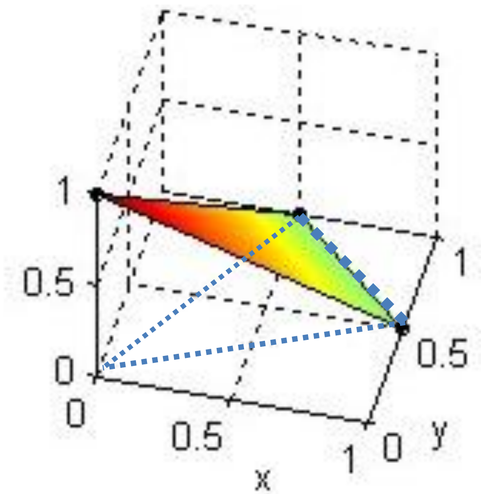


$$\psi_{11}^k(x,y)\Big|_{\Gamma_1} = 1 - \frac{s}{h_1}\,, \qquad s \in [0,h_1]$$

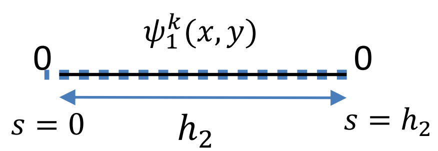
Boundary Conditions

- Restriction of $\psi_i^k(x,y)$ to the boundaries

First function: $\psi_1^k(x,y)$



Restriction to Γ_2

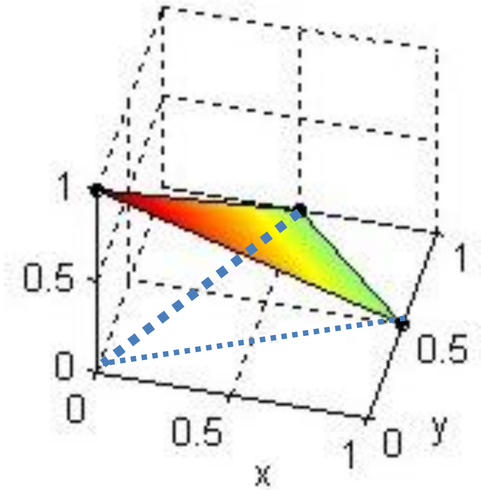


$$\psi_{12}^k(x,y)\Big|_{\Gamma_2} = 0$$

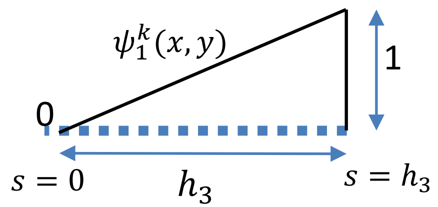
Boundary Conditions

- Restriction of $\psi_i^k(x,y)$ to the boundaries

<u>First function:</u> $\psi_1^k(x,y)$



Restriction to Γ_3

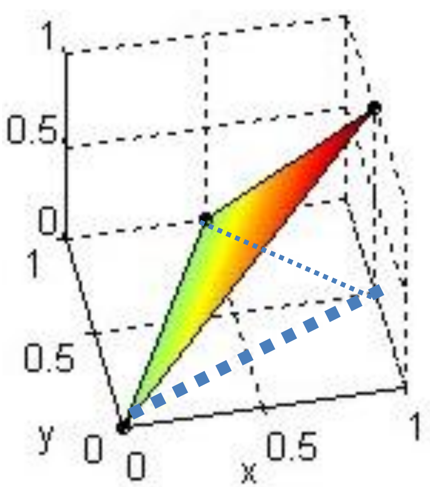


$$\psi_{13}^k(x,y)\Big|_{\Gamma_3} = \frac{s}{h_3}, \qquad s \in [0,h_3]$$

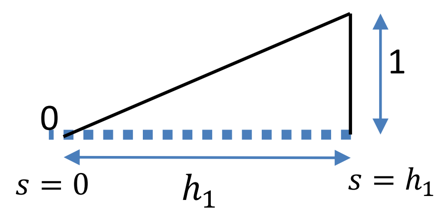
Boundary Conditions

- Restriction of $\psi_i^k(x,y)$ to the boundaries

<u>Second function:</u> $\psi_2^k(x,y)$



Restriction to Γ_1

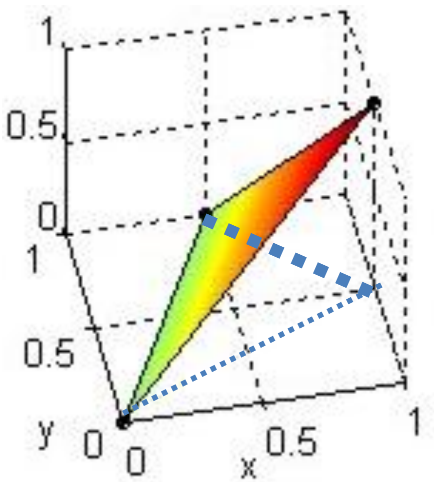


$$\psi_{21}^k(x,y)\Big|_{\Gamma_1} = \frac{s}{h_1}, \qquad s \in [0,h_1]$$

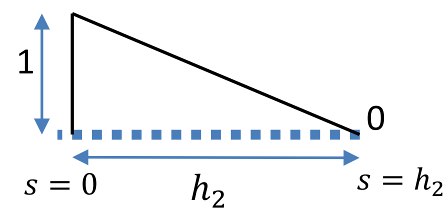
Boundary Conditions

- Restriction of $\psi_i^k(x, y)$ to the boundaries

<u>Second function:</u> $\psi_2^k(x, y)$



Restriction to Γ_2



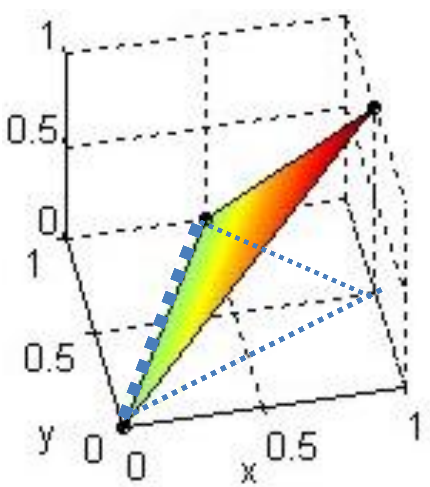
$$\psi_{22}^k(x, y)\Big|_{\Gamma_2} = 1 - \frac{s}{h_2}, \qquad s \in [0, h_2]$$

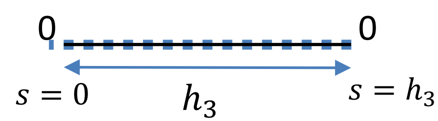
Boundary Conditions

- Restriction of $\psi_i^k(x,y)$ to the boundaries

<u>Second function:</u> $\psi_2^k(x,y)$

Restriction to Γ_3



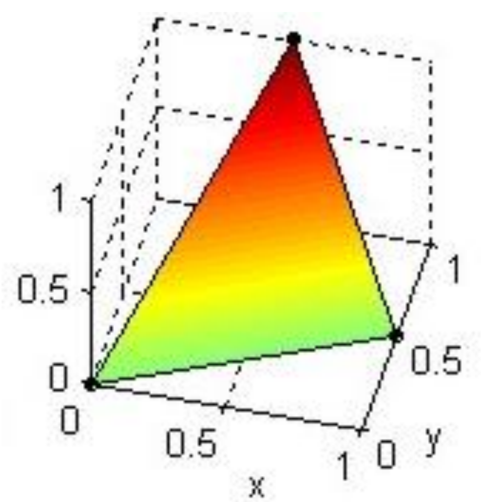


$$\psi_{23}^k(x,y)\Big|_{\Gamma_3} = 0$$

Boundary Conditions

- Restriction of $\psi_i^k(x,y)$ to the boundaries

Third function: $\psi_3^k(x,y)$



Restriction to Γ_1

$$\psi_{31}^k(x,y)\Big|_{\Gamma_1} = 0$$

Restriction to Γ_2

$$\psi_{32}^k(x,y)\Big|_{\Gamma_2} = \frac{s}{h_2}\,, \qquad s \in [0,h_2]$$

Restriction to Γ_3

$$\psi_{33}^k(x,y)\Big|_{\Gamma_3} = 1 - \frac{s}{h_3}\,, \qquad s \in [0,h_3]$$

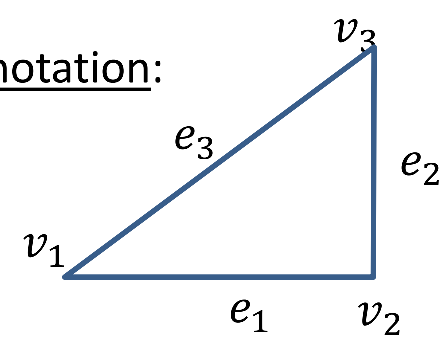
Boundary Conditions

- For the shape functions, they can be seen as the 1D **Lagrange's polynomial associated to the edge**

$$
\begin{array}{lll}
\psi^k_{11}(s) = 1 - \frac{s}{h^k_1}, & \psi^k_{12}(s) = 0, & \psi^k_{13}(s) = \frac{s}{h^k_3}, \\
\psi^k_{21}(s) = \frac{s}{h^k_1}, & \psi^k_{22}(s) = 1 - \frac{s}{h^k_2}, & \psi^k_{23}(s) = 0, \\
\psi^k_{31}(s) = 0, & \psi^k_{32}(s) = \frac{s}{h^k_2}, & \psi^k_{33}(s) = 1 - \frac{s}{h^k_3}.
\end{array}
$$

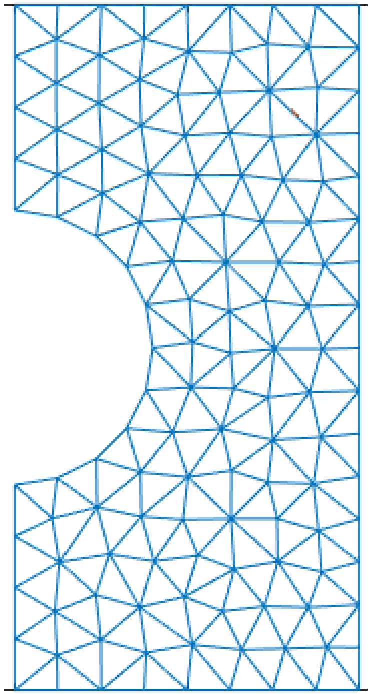
h^k_j is the length of the j-th edge of the triangle

Edge notation:



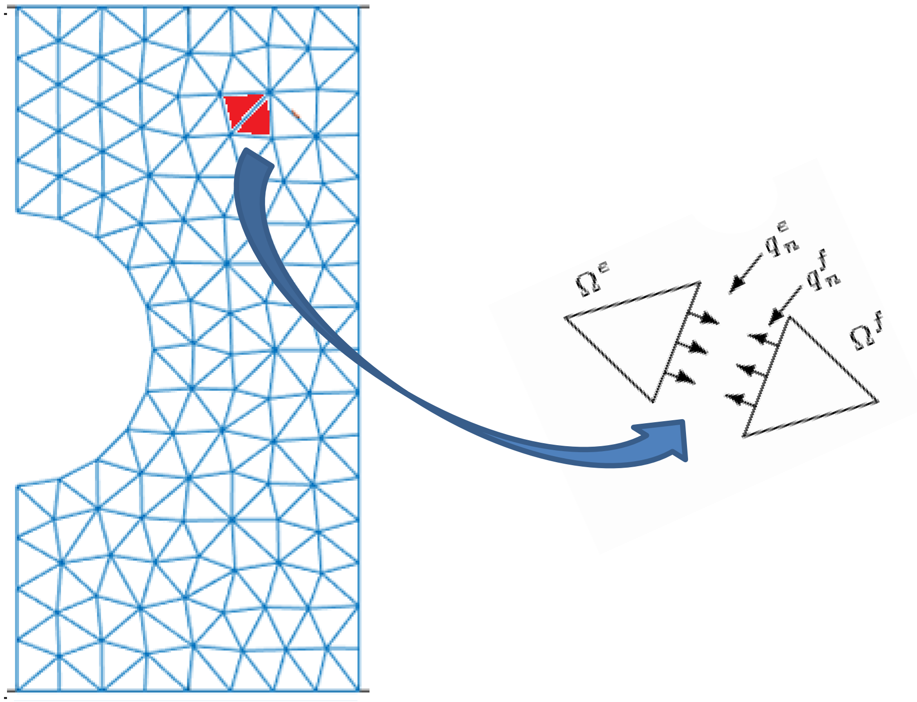
Boundary Conditions

- Balance, of course, applies to **interior faces**



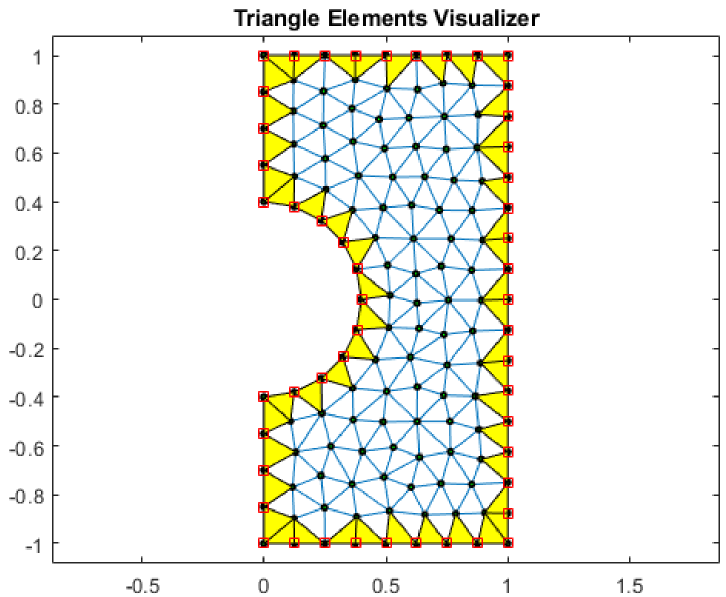
Boundary Conditions

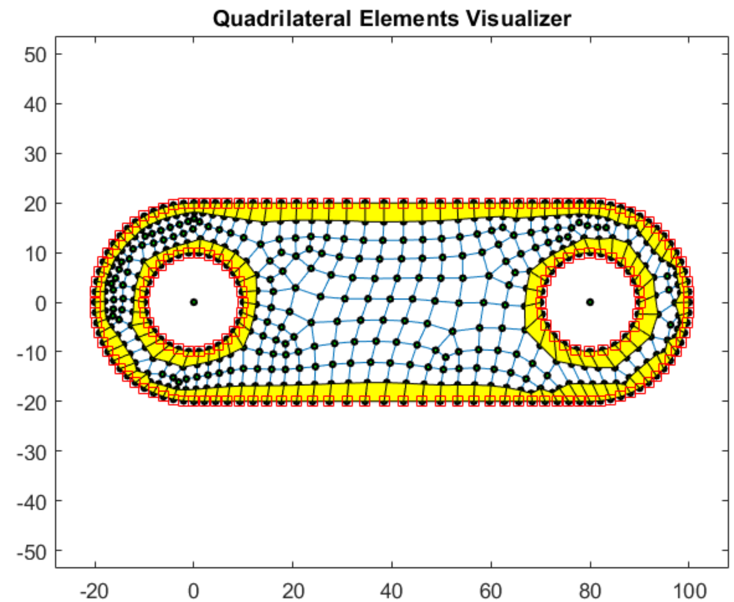
- **Balance**, of course, applies to **interior faces**



Boundary Conditions

- Balance, of course, applies to **interior faces** and only the ones on the **boundary** have to be consider





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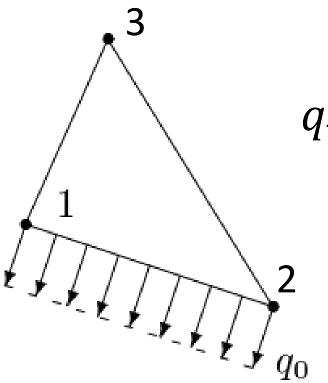
Constant BC on Triangles

Constant Boundary Conditions

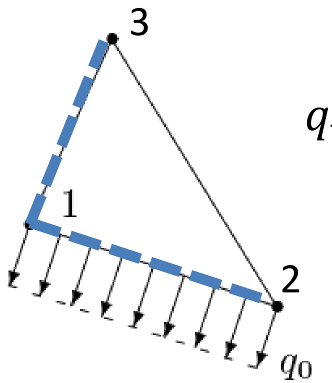
- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



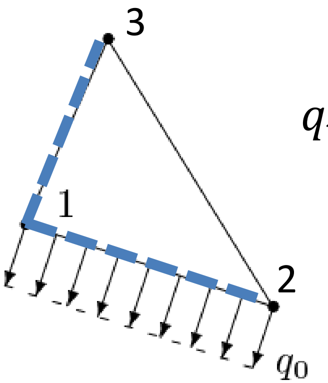
$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

<u>Node 1</u>

$$\boxed{Q_1^k = Q_{11}^k + Q_{13}^k}$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

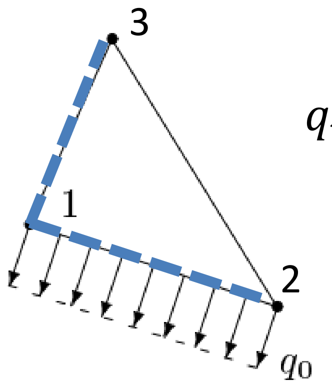
<u>Node 1</u>

$$Q_1^k = Q_{11}^k + Q_{13}^k$$

$$Q_{11}^k = \int_{\Gamma_1} q_n \psi_{11}^k(s)\, ds =$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

<u>Node 1</u>

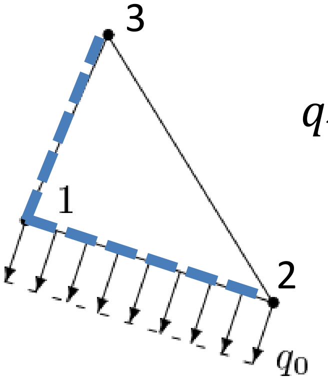
$$Q_1^k = Q_{11}^k + Q_{13}^k$$

$$Q_{11}^k = \int_{\Gamma_1} q_n \psi_{11}^k(s)\, ds =$$

$$= q_0 \int_0^{h_1} \left(1 - \frac{s}{h_1}\right) ds = \frac{1}{2}\, q_0 h_1$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

<u>Node 1</u>

$$Q_1^k = Q_{11}^k + Q_{13}^k$$

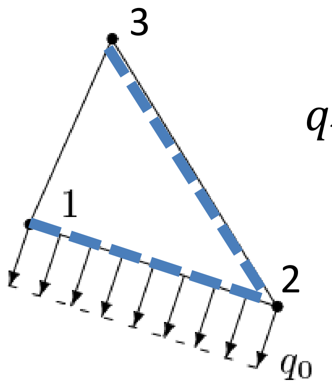
$$Q_{11}^k = \int_{\Gamma_1} q_n \psi_{11}^k(s)\, ds =$$

$$= q_0 \int_0^{h_1} \left(1 - \frac{s}{h_1}\right) ds = \frac{1}{2}\, q_0 h_1$$

$$Q_{13}^k = \int_{\Gamma_3} q_n \psi_{13}^k(s)\, ds = 0$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



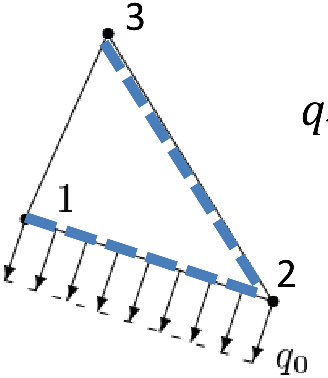
$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

<u>Node 2</u>

$$\boxed{Q_2^k = Q_{21}^k + Q_{22}^k}$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

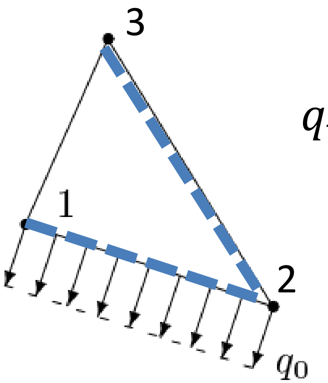
<u>Node 2</u>

$$Q_2^k = Q_{21}^k + Q_{22}^k$$

$$Q_{21}^k = \int_{\Gamma_1} q_n \psi_{21}^k(s)\, ds =$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

<u>Node 2</u>

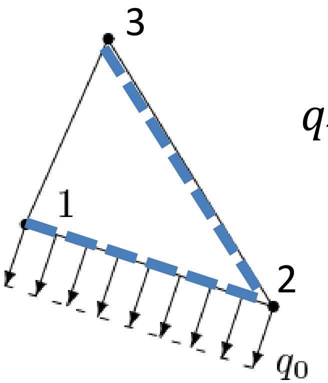
$$Q_2^k = Q_{21}^k + Q_{22}^k$$

$$Q_{21}^k = \int_{\Gamma_1} q_n \psi_{21}^k(s)\, ds =$$

$$= q_0 \int_0^{h_1} \left(\frac{s}{h_1}\right) ds = \frac{1}{2}\, q_0 h_1$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

Node 2

$$Q_2^k = Q_{21}^k + Q_{22}^k$$

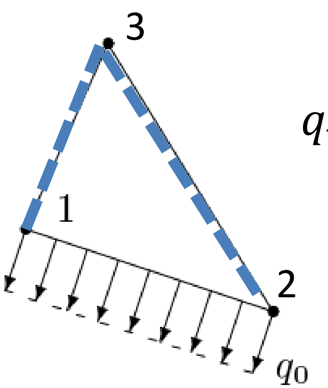
$$Q_{21}^k = \int_{\Gamma_1} q_n \psi_{21}^k(s)\, ds =$$

$$= q_0 \int_0^{h_1} \left(\frac{s}{h_1}\right) ds = \frac{1}{2}\, q_0 h_1$$

$$Q_{22}^k = \int_{\Gamma_2} q_n \psi_{22}^k(s)\, ds = 0$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



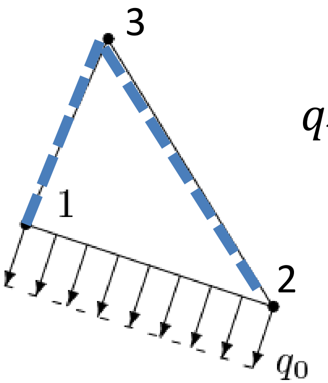
$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

Node 3

$$\boxed{Q_3^k = Q_{32}^k + Q_{33}^k}$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

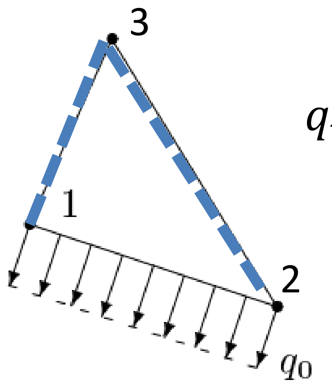
<u>Node 3</u>

$$Q_3^k = Q_{32}^k + Q_{33}^k$$

$$Q_{32}^k = \int_{\Gamma_2} q_n \psi_{32}^k(s)\, ds = 0$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

Node 3

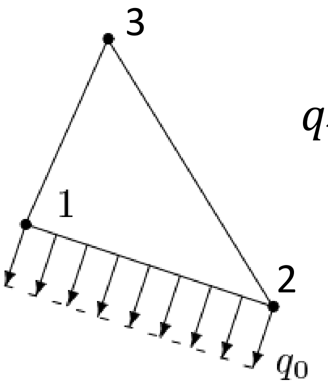
$$Q_3^k = Q_{32}^k + Q_{33}^k$$

$$Q_{32}^k = \int_{\Gamma_2} q_n \psi_{32}^k(s)\, ds = 0$$

$$Q_{33}^k = \int_{\Gamma_3} q_n \psi_{33}^k(s)\, ds = 0$$

Constant Boundary Conditions

- Consider the case where a **constant BC** is applied to one edge of the triangle



$$q_n^k = \begin{cases} q_0 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases} \quad \Rightarrow \quad \boxed{\begin{pmatrix} Q_1^k \\ Q_2^k \\ Q_3^k \end{pmatrix} = \frac{q_0 h_1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

The **constant value is distributed** between nodes 1 and 2

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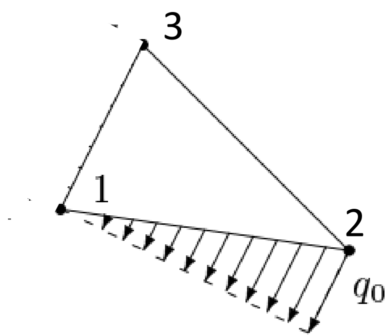
Linear BC on Triangles

Linear Boundary Conditions

- Consider now a **linear function** applied at edge 1

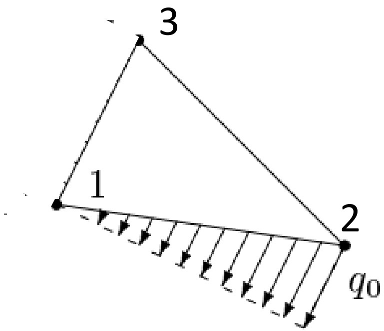


$$q_n^k = \begin{cases} q_0 \dfrac{s}{h_1} & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

$$\boxed{Q_1^k = Q_{11}^k + Q_{13}^k}$$

Linear Boundary Conditions

- Consider now a **linear function** applied at edge 1



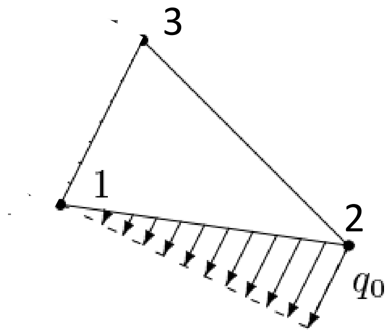
$$q_n^k = \begin{cases} q_0 \dfrac{s}{h_1} & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

$$Q_1^k = Q_{11}^k + Q_{13}^k$$

$$Q_{11}^k = \int_0^{h_1^k} q_0 \frac{s}{h_1^k}\left(1 - \frac{s}{h_1^k}\right) ds = \frac{1}{6} h_1^k q_0$$

Linear Boundary Conditions

- Consider now a **linear function** applied at edge 1



$$q_n^k = \begin{cases} q_0 \dfrac{s}{h_1} & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

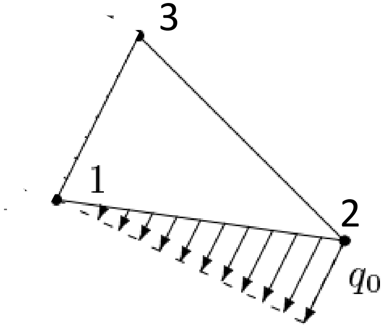
$$Q_1^k = Q_{11}^k + Q_{13}^k$$

$$Q_{11}^k = \int_0^{h_1^k} q_0 \frac{s}{h_1^k} \left(1 - \frac{s}{h_1^k}\right) ds = \frac{1}{6} h_1^k q_0$$

$$Q_{13}^k = 0$$

Linear Boundary Conditions

- Consider now a **linear function** applied at edge 1



$$q_n^k = \begin{cases} q_0 \dfrac{s}{h_1} & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

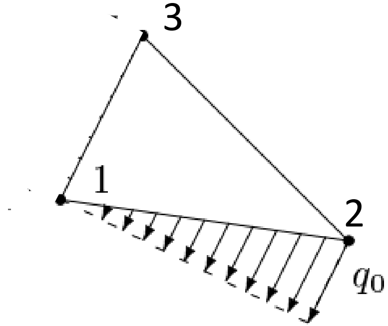
$$Q_1^k = Q_{11}^k + Q_{13}^k$$

$$Q_{11}^k = \int_0^{h_1^k} q_0 \frac{s}{h_1^k}\left(1 - \frac{s}{h_1^k}\right) ds = \frac{1}{6} h_1^k q_0$$

$$\boxed{Q_2^k = Q_{21}^k + Q_{22}^k}$$

Linear Boundary Conditions

- Consider now a **linear function** applied at edge 1



$$q_n^k = \begin{cases} q_0 \dfrac{s}{h_1} & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

$$Q_1^k = Q_{11}^k + Q_{13}^k$$

$$Q_{11}^k = \int_0^{h_1^k} q_0 \frac{s}{h_1^k} \left(1 - \frac{s}{h_1^k}\right) ds = \frac{1}{6} h_1^k q_0;$$

$$Q_2^k = Q_{21}^k + Q_{22}^k$$

$$Q_{21}^k = \int_0^{h_1^k} q_0 \frac{s}{h_1^k} \left(\frac{s}{h_1^k}\right) ds = \frac{1}{3} h_1^k q_0.$$

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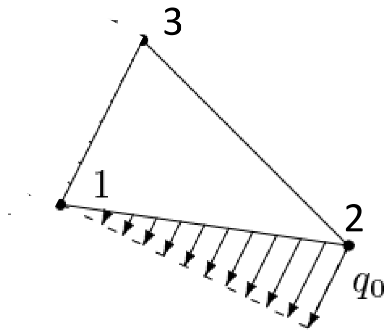
Linear Boundary Conditions

- Consider now a **linear function** applied at edge 1



$$q_n^k = \begin{cases} q_0 \dfrac{s}{h_1} & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \end{cases}$$

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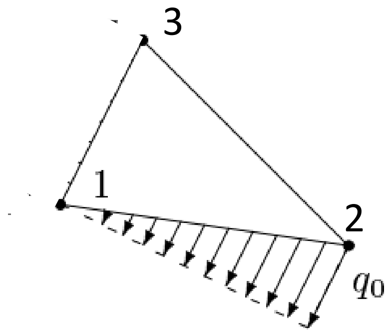

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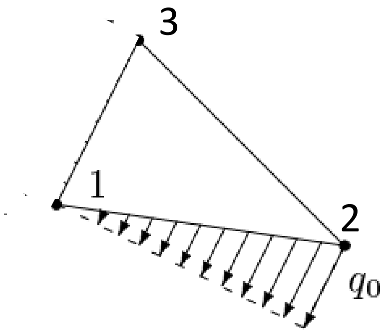

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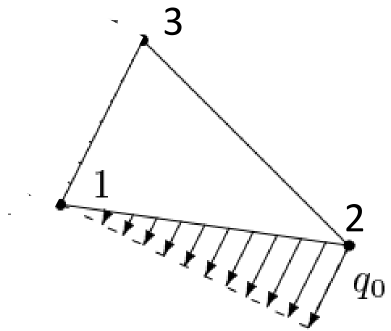

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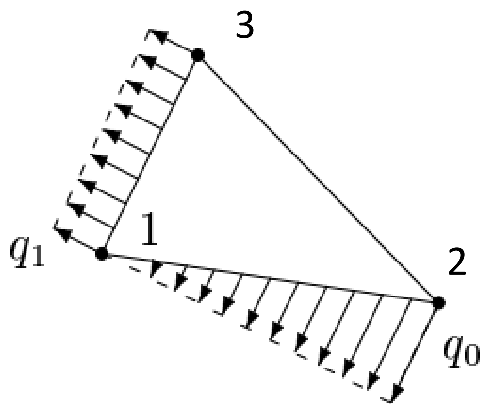
Different Boundary Conditions

- As a final case now we have de contribution of the two previous cases. In principle,



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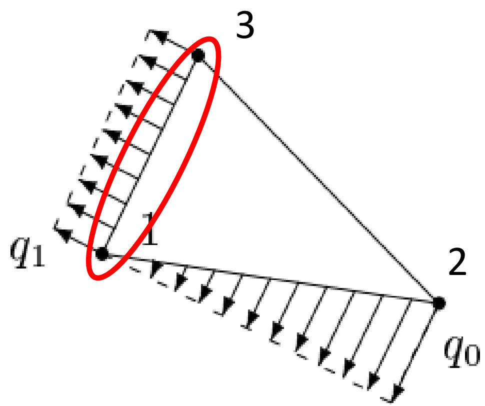

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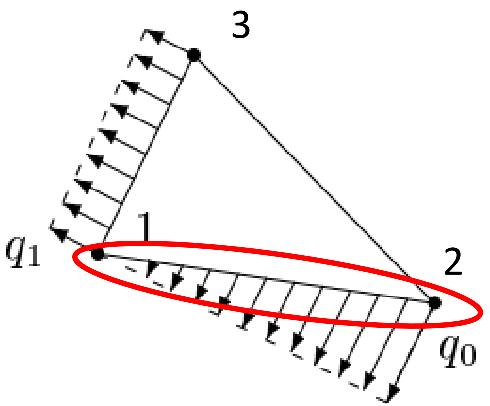

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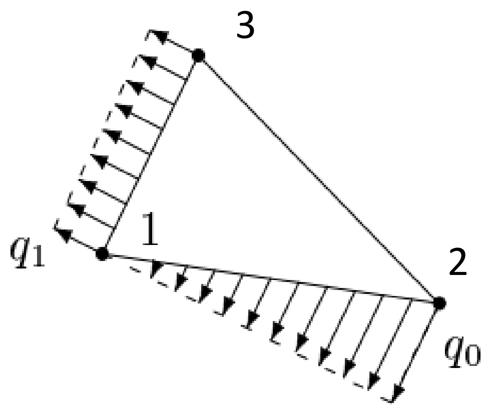

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