UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria Industrial de Barcelona
ETSEIB

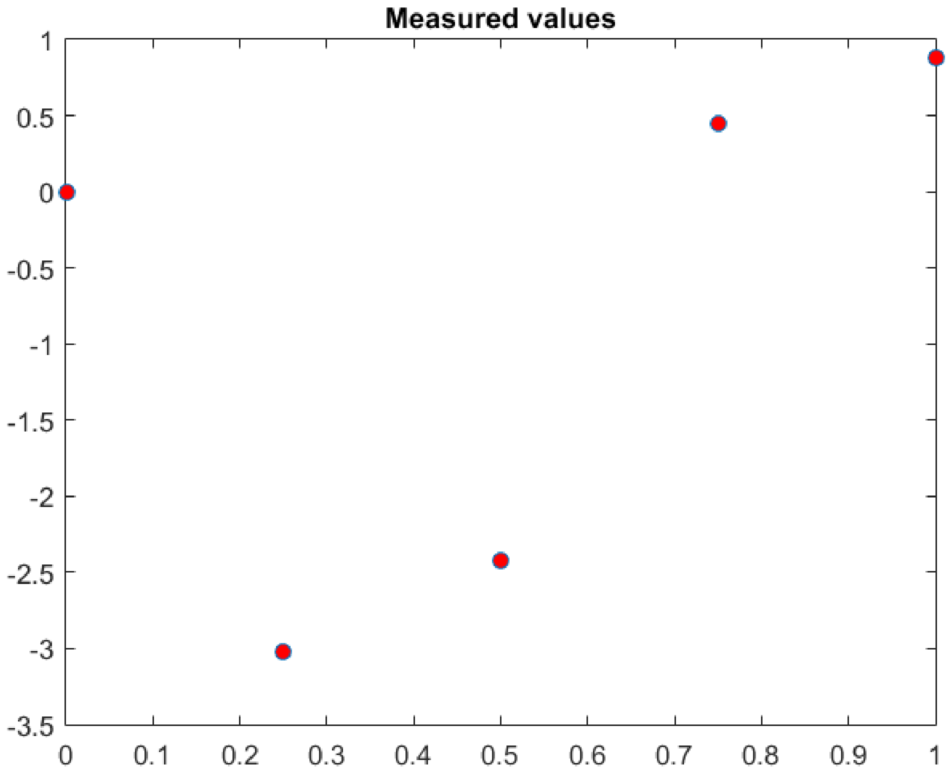
Numerical Methods in Engineering

Interpolation (Shape Functions)

Dept. Matemàtiques ETSEIB - UPC BarcelonaTech

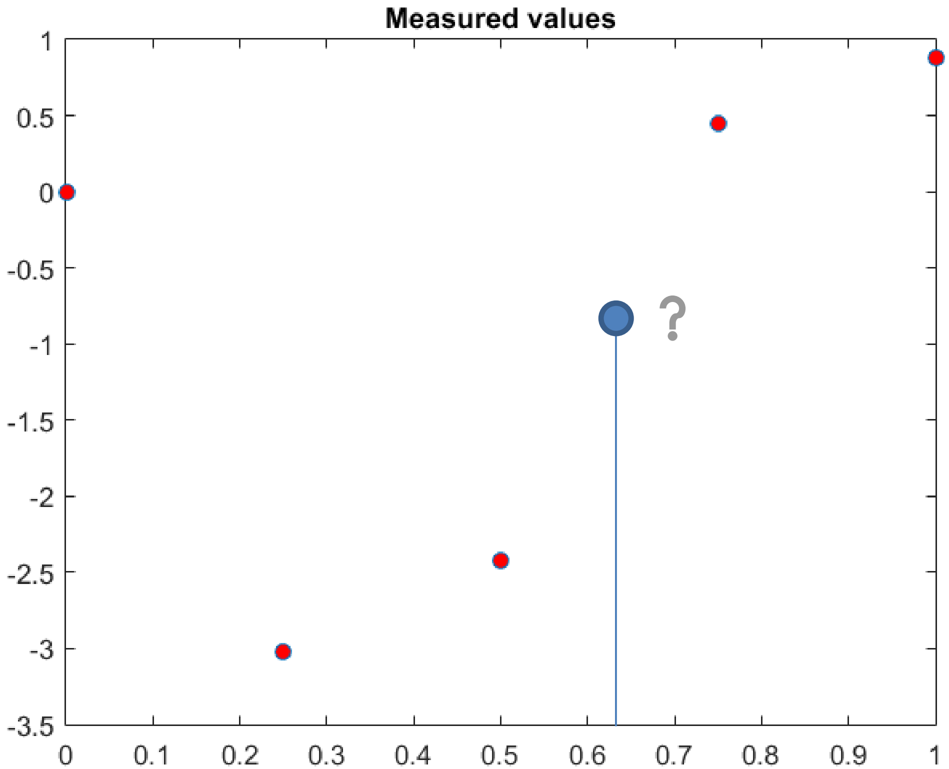
Interpolation Problem 1D

Let's assume that we have **measured** an unknown magnitude $u = u(x)$ on a set of data points $x = [x_1, x_2, \dots., x_N]$ obtaining a set of values $u = [u_1, u_2, \dots., u_N]$



Interpolation Problem 1D

Let's assume that we have **measured** an unknown magnitude $u = u(x)$ on a set of data points $x = [x_1, x_2, \dots, x_N]$ obtaining a set of values $u = [u_1, u_2, \dots, u_N]$

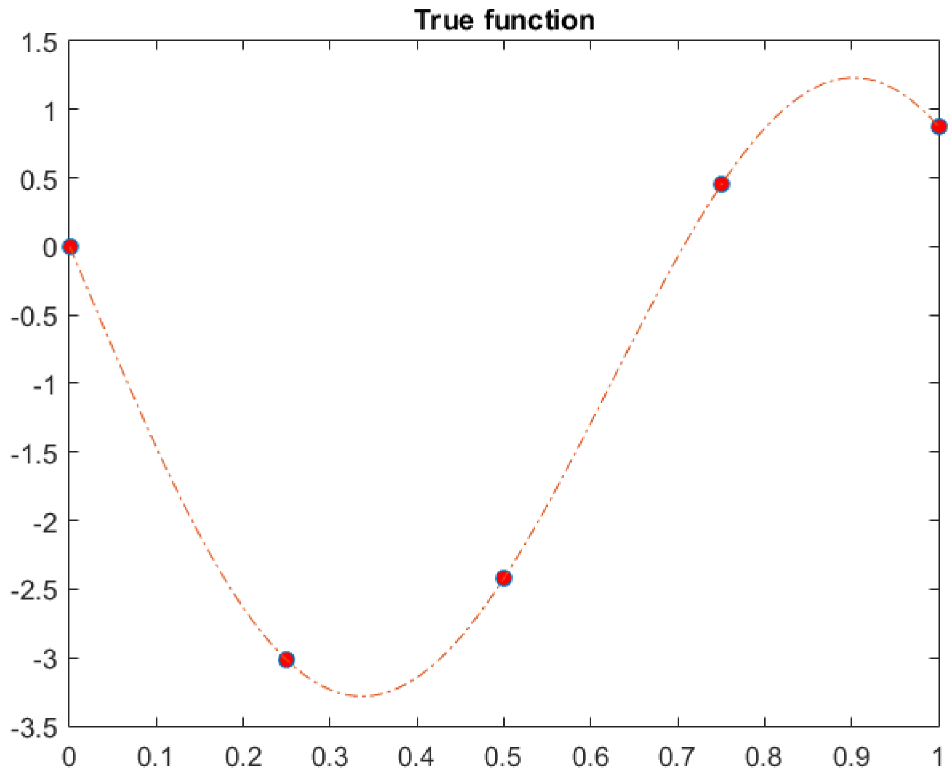


How can we guess the value of a new point?

Interpolation Problem 1D

Let's assume that we have **measured** an unknown magnitude $u = u(x)$ on a set of data points $x = [x_1, x_2, \dots, x_N]$ obtaining a set of values $u = [u_1, u_2, \dots, u_N]$

Actual function
$f(x) = x^3 - 3x^2 - 3\sin(5x)$



Interpolation Problem 1D

To **Interpolate** a set of data measured points (x_i, u_i) means to build a function $P_n(x)$ (usually a **polynomial**) passing through these points. That is

$$\boxed{u_i = P_n(x_i)\,,\ \ i = 1 \dots N}$$

Theorem:

The interpolation problem **has a unique** solution when $n = N - 1$.

Idea of the proof:

$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \longrightarrow n + 1 \equiv N$ unknowns

$u_i = P_n(x_i)\,,\ \ i = 1 \dots N \longrightarrow N$ linear equations

Unique solution if the system is compatible determined. That means the interpolation polynomial has **degree one less** than the number of points $\square$

Shape Functions 1D

Shape Functions 1D

Lagrange Polynomials (Shape functions)

One (N-1) degree Polynomial for each measure point $x_i, \;\; i = 1, \dots, N$

$$\psi_i(x) = \frac{(x - x_1)(x - x_2) \dots\dots \widehat{(x - x_i)} \dots\dots (x - x_{N-1})(x - x_N)}{(x_i - x_1)(x_i - x_2) \dots\dots \widehat{(x_i - x_i)} \dots\dots (x_i - x_{N-1})(x_i - x_N)}$$

Properties:

1.- Lagrange Polynomials are **a base** of the set $\mathbb{P}_n[x]$

2.- $\psi_i(x_j) = 0 \quad if \quad i \neq j$

3.- $\psi_i(x_i) = 1$

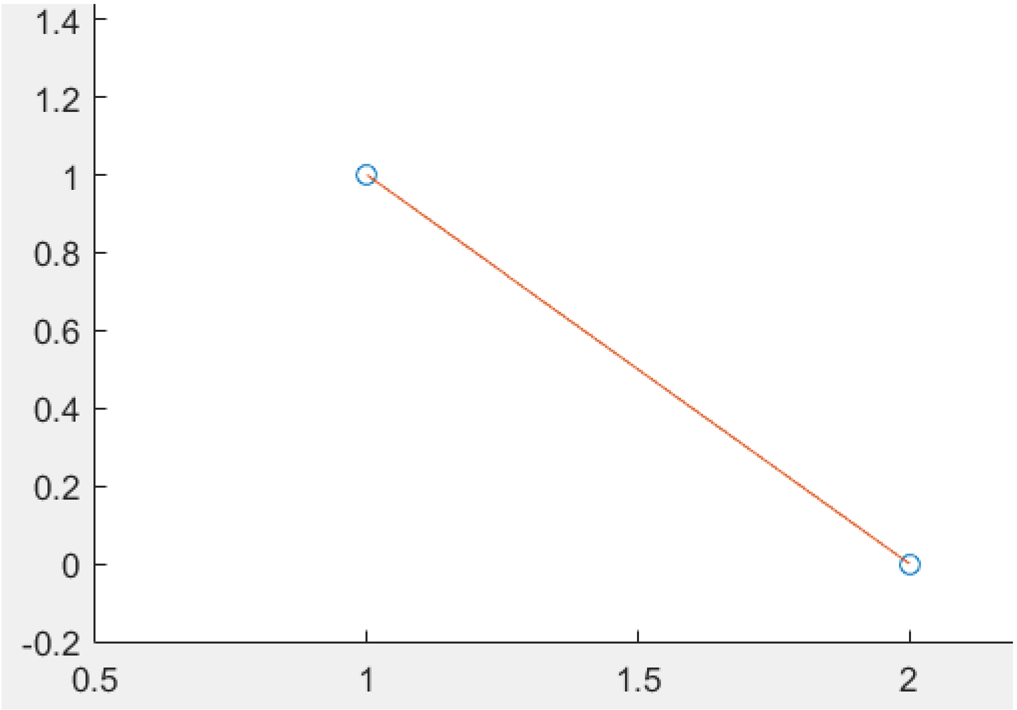
Interpolation Polynomial (a linear combination of the shape functions)

$$\boxed{P_n(x) = u_1\psi_1(x) + u_2\psi_2(x) + \;\; \dots \;\; + u_N\psi_N(x)}$$

Shape Functions 1D

N = 2 only two points: $x = [x_1, x_2]$ and their measures $u = [u_1, u_2]$

$$\psi_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}$$



$x = [1, 2]$

It is a **straight line**

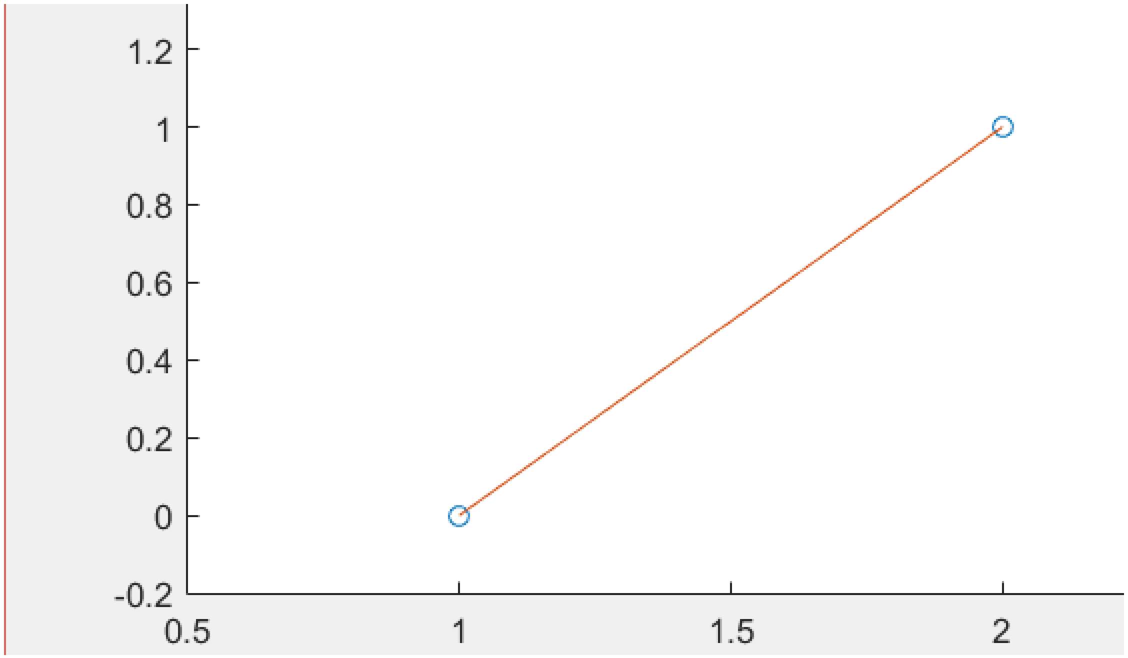
Shape Functions 1D

N = 2 only two points: $x = [x_1, x_2]$ and their measures $u = [u_1, u_2]$

$$\psi_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}$$

$$\psi_2(x) = \frac{(x - x_1)}{(x_2 - x_1)}$$

$x = [1, 2]$

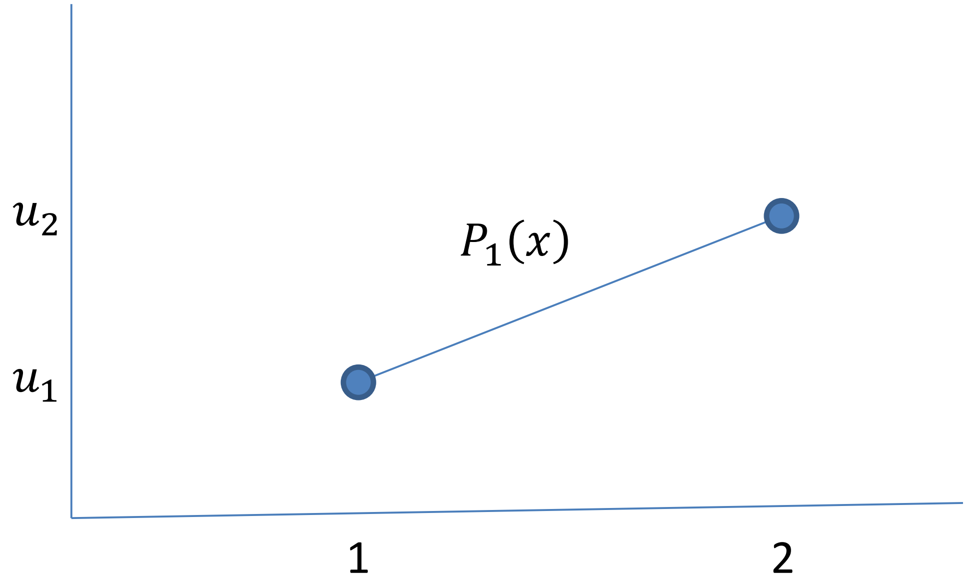


Shape Functions 1D

N = 2 only two points: $x = [x_1, x_2]$ and their measures $u = [u_1, u_2]$

$$\psi_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}$$

$$\psi_2(x) = \frac{(x - x_1)}{(x_2 - x_1)}$$



Interpolation Polynomial

$$P_1(x) = u_1\psi_1(x) + u_2\psi_2(x)$$

It is **also a straight line** **(linear interpolation)**

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria Industrial de Barcelona
ETSEIB

Shape Functions 1D

N = 2 only two points: $x = [x_1, x_2]$ and their measures $u = [u_1, u_2]$

$$\psi_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}$$

$$\psi_2(x) = \frac{(x - x_1)}{(x_2 - x_1)}$$

Interpolation Polynomial

$$P_1(x) = u_1\psi_1(x) + u_2\psi_2(x)$$

Example:

In the case $x = [1, 2]$ and $u = [0.7, \quad 1.8]$

$$\psi_1(x) = \frac{(x - 2)}{(1 - 2)} = -(x - 2)$$

$$\psi_2(x) = \frac{(x - 1)}{(2 - 1)} = (x - 1)$$

How to approximate the value for $x = 1.5$?

$$P_1(1.5) = 0.7\psi_1(1.5) + 1.8\psi_2(1.5)$$
$$= 0.7 * 0.5 + 1.8 * 0.5 = \mathbf{1.25}$$

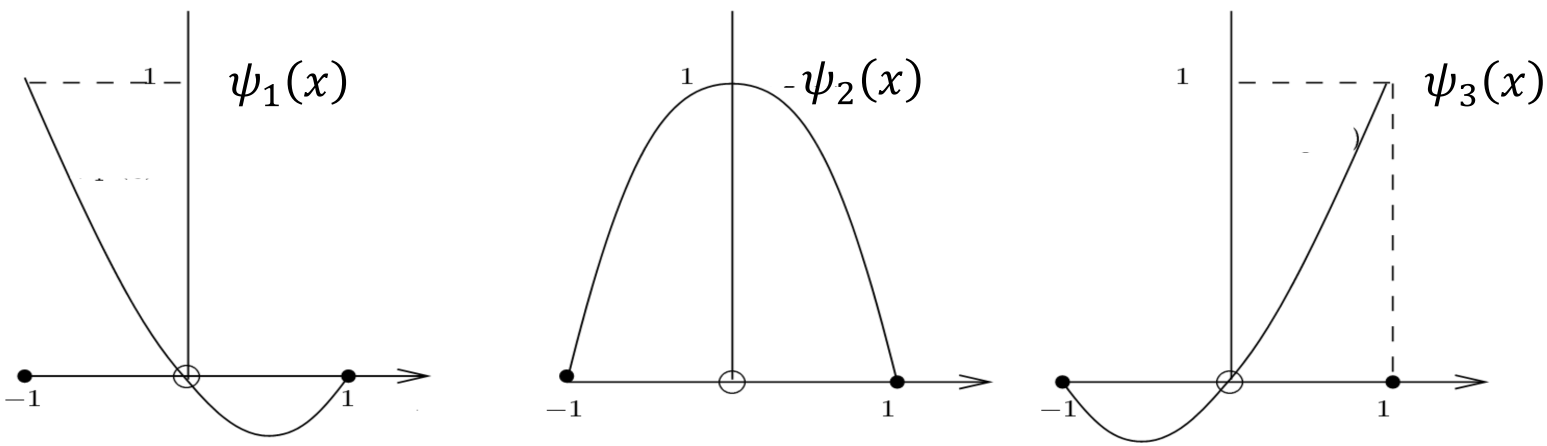
Shape Functions 1D

Three points (N=3): $x = [x_1, x_2, x_3\,]$ and measures $u = [u_1, u_2, u_3]$

$$\psi_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \qquad \psi_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$\psi_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

For $x = [-1, 0, 1]$ the quadratic shape functions are



Shape Functions 1D

N = 3 three points: $x = [x_1, x_2, x_3\,]$ and measures $u = [u_1, u_2, u_3]$

$$\psi_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \qquad \psi_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$\psi_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Interpolation Polynomial

$$P_2(x) = u_1\psi_1(x) + u_2\psi_2(x) + u_3\psi_3(x)$$

Now it is **a parabola (quadratic interpolation)**

Shape Functions 1D

Example:

Consider $x = [-1,0,1]$ and $u = [1.2, 0.7,\ 1.8]$. Compute $u(0.2)$?

$$\psi_1(x) = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}x(x-1)$$

$$\psi_2(x) = \frac{(x+1)(x-1)}{(0+1)(0-1)} = -(x+1)\,(x-1)$$

$$\psi_3(x) = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{1}{2}x(x+1)$$

$$P_2(0.2) = 1.2\psi_1(0.2) + 0.7\psi_2(0.2) + 1.8\psi_3(0.2) = 0.7920$$

Shape Functions 1D

- Matlab code:

```
x = 0:0.25:1; % measure points
f = @(x) x.^3-3*x.^2-3*sin(5*x); %inline exemple function
y = f(x);       %measured values
figure()
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
title('Measured values'); % plot only measured values
figure()
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
hold on;
xx = 0:0.01:1; % take many points for a better plot
yy = f(xx);
plot(xx, yy,'-.')
title('True function'); % plot the function
hold off
```

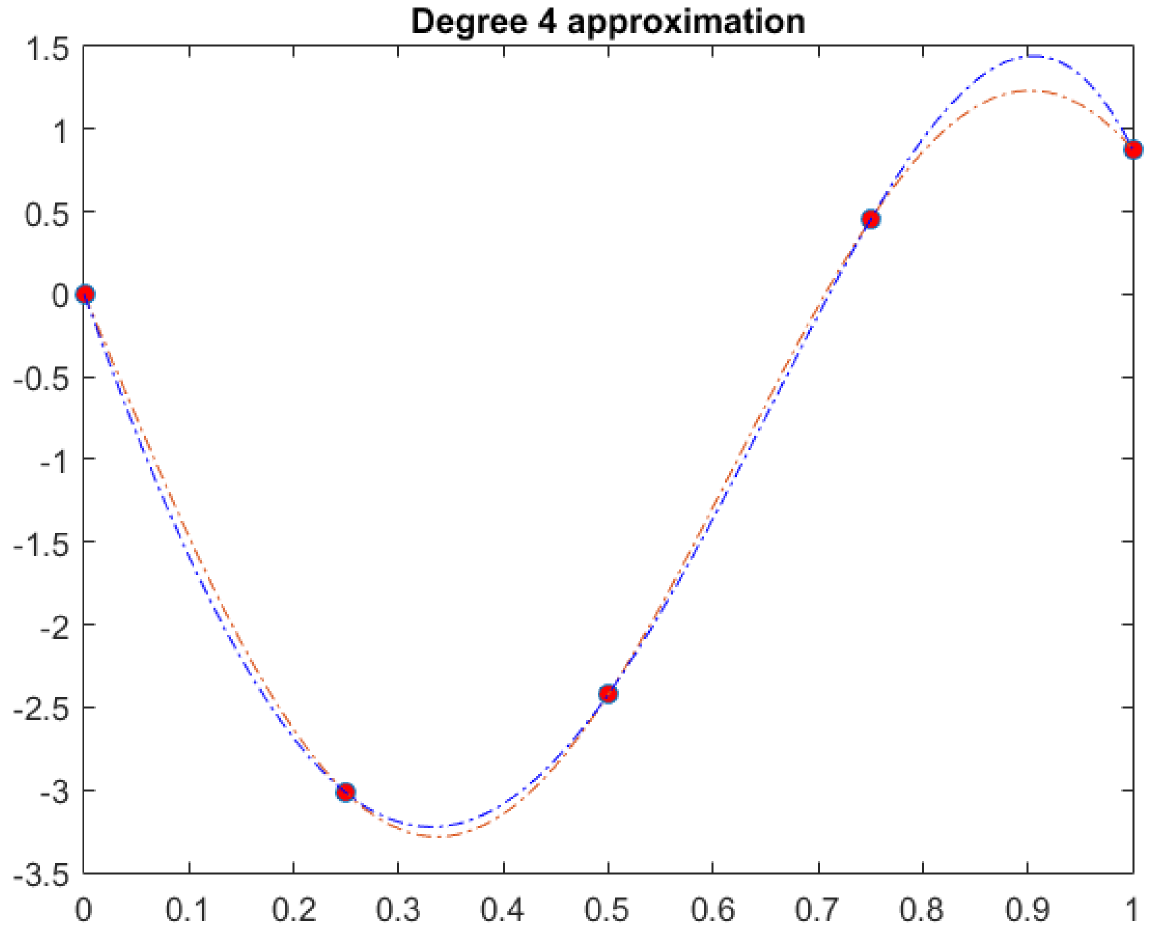
Shape Functions 1D

- Matlab code (continuation):

```
figure() % Polyfit degree 4 function (interpolator)
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
hold on;
plot(xx, yy, '-.')
p = polyfit(x,y,4);
yyy = polyval(p, xx);
plot(xx, yyy, '-.b')
title('Degree 4  approximation');
hold off
```

Shape Functions 1D

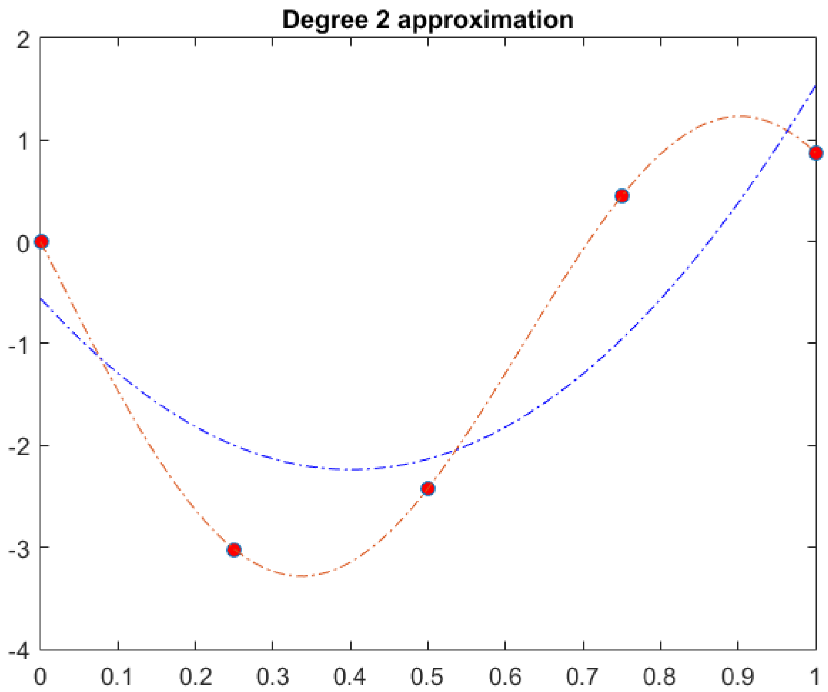
- **Interpolation Results:**

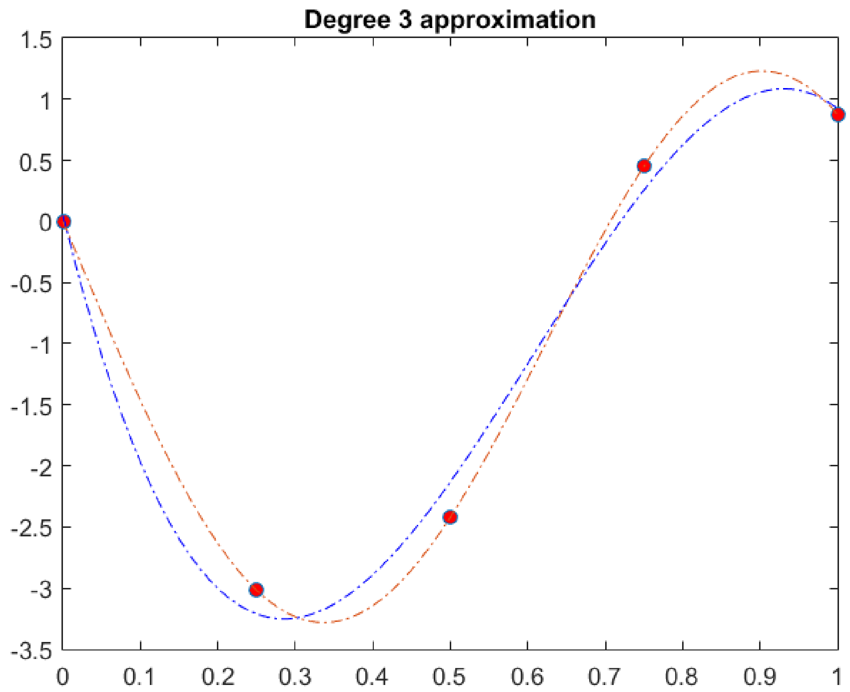


Shape Functions 1D

- **Approximation:** In case $n < (N-1)$ there is NO solution (incompatible system) because the polynomial can not pass through all the points. In that case: **mean square approx**

$$P_n(x) = \min(\| \tilde{P}_n(x) - f(x) \|)$$



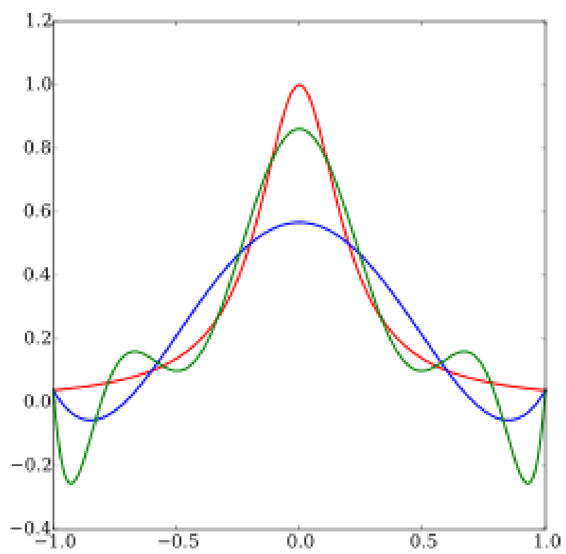


Shape Functions 1D

- **Higher interpolation:** When N is big (>8) there is a better solution than using a polynomial of high degree.
- **Runge's Phenomenon:**

$$f(x) = \frac{1}{1+25x^2}$$

  - The red curve is the Runge function $f(x)$.
  - The blue curve is a 5th-order interpolating polynomial (using six equally spaced interpolating points).
  - The green curve is a 9th-order interpolating polynomial (using ten equally spaced interpolating points).

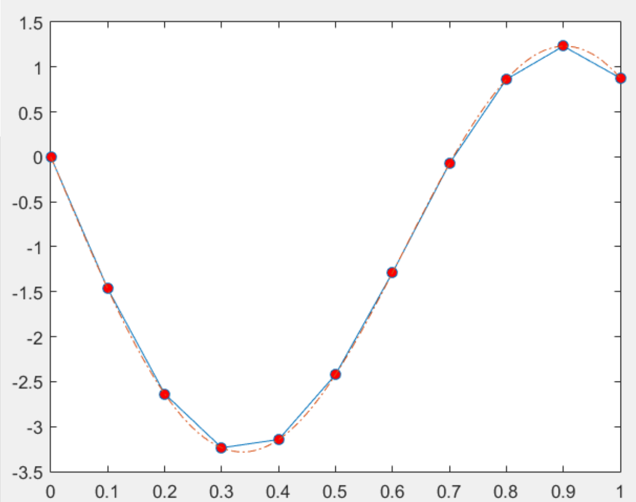


Shape Functions 1D

Two possible approaches:

<u>Polygonal curve</u>: Take the points in pairs and build a line segment (1^{st} order interp) in each case. **Continuous but not derivable.**

Matlab code:

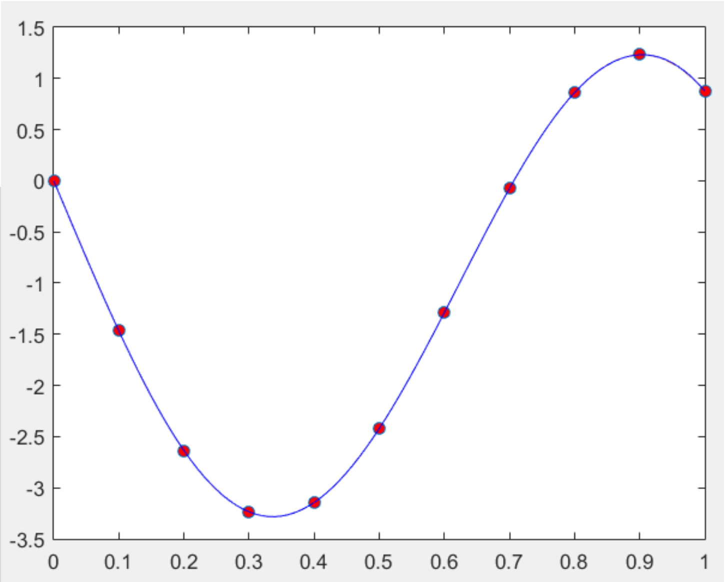



```
%% Polygonal
figure()
x=0:0.1:1;
y=f(x);
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
hold on;
plot(xx,yy,'-.')
xxx=0:0.01:1;
yyy=interp1(x,y,xx);  %only substitution is possible
plot(xxx,yyy,'b')
hold off
```

Shape Functions 1D

Two possible approaches:

Spline curve (cubic): Take the points in groups of 2 and build a third order Polynomial (cubic interp) in each case imposing $\mathbb{C}^2$ **continuity.**



Matlab code (continuation):

```
%% Spline
figure()
x=0:0.1:1;
y=f(x);
plot(x,y,'o','Marker','o','MarkerFaceColor','red');
hold on;
plot(xx,yy,'-.')
xxx = 0:0.01:1;
yyy = spline(x,y,xxx);  %only substitution is possible
plot(xxx,yyy,'b')
hold off
```

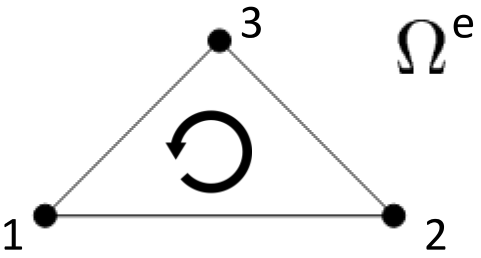
Shape Functions 2D

2D interpolation

How can we extend the interpolation problem to 2D?

That is, how can we approximate the value of $u(x, y)$ using a 2D polynomial $P_n(x, y)$?

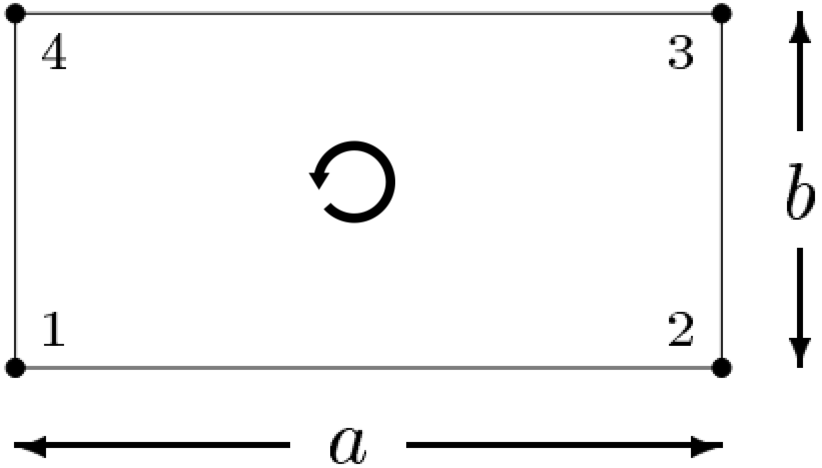
We restrict to fixed shapes:



Triangular element

Numbered

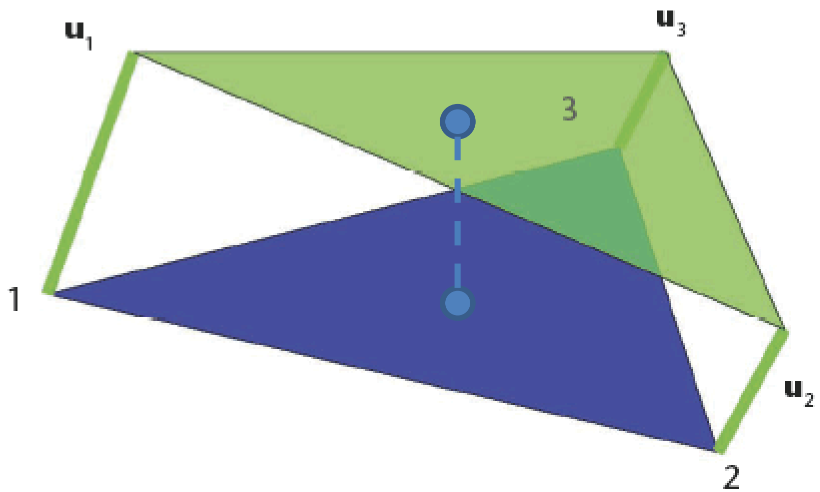
counterclockwise



Quadrilateral element

2D interpolation

From the measured values u_i at the vertices we can interpolate the value of $u(x,y)$ at any other point in the triangle.



measured values

$$u(x,y) = \sum_{i=1}^{3} u_i \cdot \psi_i(x,y)$$

interpolated value

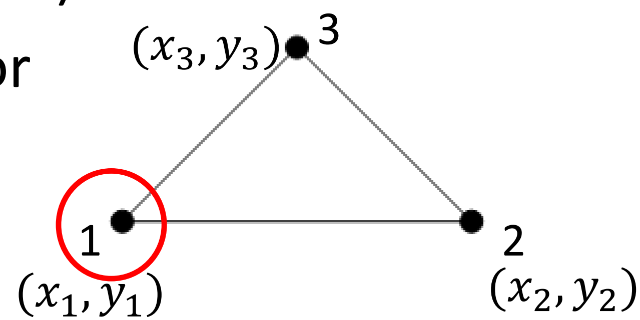
shape functions

Triangular Shape Functions

We need the equivalent to the Lagrange polynomials (2-dim)

$\psi_i^e(x,y) = a_i + \beta_i\, x + \gamma_i\, y$ (three unknowns)

such that, it's value is 1 for vertex i, and 0 for the other two.



For the first vertex:

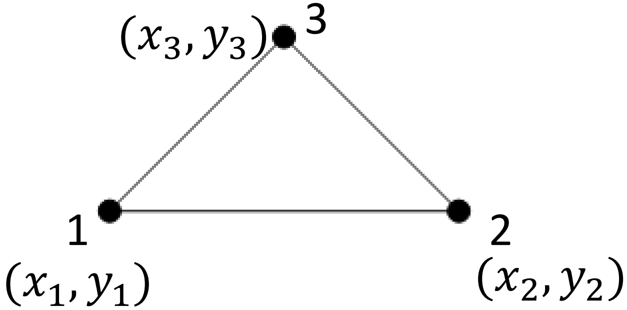
$$1 = \psi_1^e(x_1,y_1) = a_1 + \beta_1\, x_1 + \gamma_1\, y_1$$
$$0 = \psi_1^e(x_2,y_2) = a_1 + \beta_1\, x_2 + \gamma_1\, y_2$$
$$0 = \psi_1^e(x_3,y_3) = a_1 + \beta_1\, x_3 + \gamma_1\, y_3$$

$$\begin{array}{cccc} \mathbf{A} & \mathbf{c} & & \mathbf{b} \\ \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} & \begin{pmatrix} a_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} & = & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{array} \Longrightarrow \boldsymbol{c} = \boldsymbol{A} \backslash \boldsymbol{b} \text{ (matlab solution)}$$

Triangular Shape Functions

Explicitly

$$\psi_i^e(x,y) = a_i + \beta_i\, x + \gamma_i\, y\,, \qquad i = 1,2,3$$



$$a_i = \frac{x_j y_k - x_k y_j}{2A_e}$$

$$\beta_i = \frac{y_j - y_k}{2A_e}$$

$$\gamma_i = \frac{x_k - x_j}{2A_e}$$

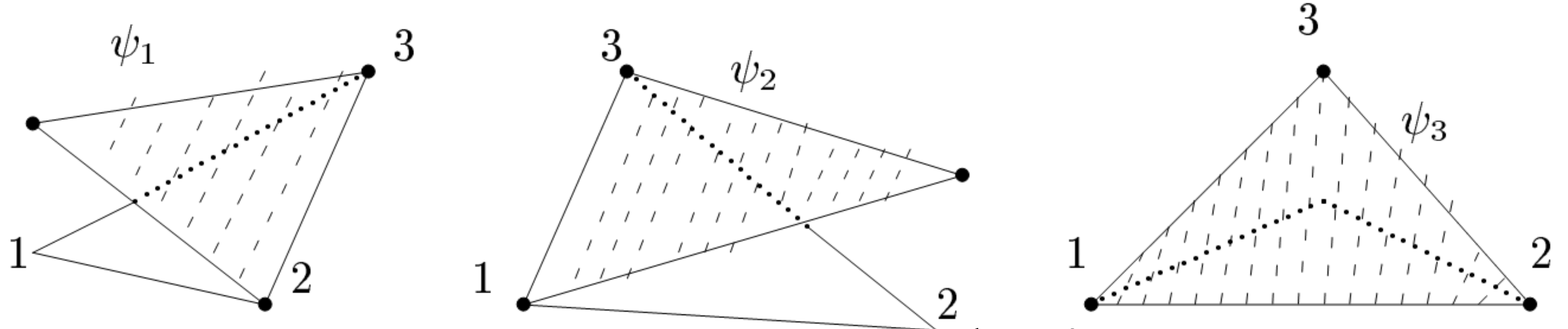
$where\ (i,j,k) = (1,2,3)\ cyclic$ and A_e is the **Area** of Ω_e

Remember:

$$\text{Area}=\frac{1}{2}\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

Triangular Shape Functions

Once we have the shape functions $\psi_i^e(x,y), \quad i = 1,2,3$
(geometrically they are planes passing through one of the edges)



finally, the interpolation value of a point $(\tilde{x},\tilde{y})$ is

$$u(\tilde{x},\tilde{y}) = u(x_1,y_1)\psi_1^e(\tilde{x},\tilde{y}) + u(x_2,y_2)\psi_2^e(\tilde{x},\tilde{y}) + u(x_3,y_3)\psi_3^e(\tilde{x},\tilde{y})$$

The terms

$\alpha_i^e = \psi_i^e(\tilde{x},\tilde{y})$ are known as **Barycentric Coordinates** of $(\tilde{x},\tilde{y})$

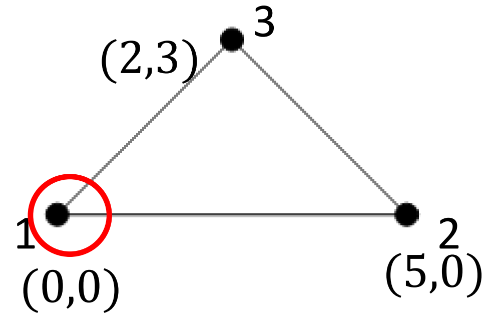
(We'll see more in the practical sessions)

Triangular Shape Functions

Example: The temperature of a thin triangular plate has been measured at the vertices $T(v_1) = 40^o, \;\; T(v_2) = 90^o, \;\; T(v_3) = 10^o$.

If the coordinates of the vertices are: $v_1 = (0{,}0), v_2 = (5{,}0), v_3 = (2{,}3)$, estimate the temperature at the point $p = (3{,}1)$?

Solution: using the **explicit** formulas



$$\psi_i^e(x, y) = a_i + \beta_i\, x + \gamma_i\, y\,, \qquad i = 1{,}2{,}3$$

$$a_i = \frac{x_j y_k - x_k y_j}{2A_e}$$

$$\beta_i = \frac{y_j - y_k}{2A_e}$$

$$\gamma_i = \frac{x_k - x_j}{2A_e}$$

$$\boxed{i = 1, j = 2, k = 3}$$

$$A_e = \frac{15}{2}$$

$$a_1 = \frac{x_2 y_3 - x_3 y_2}{2A_e} = 1$$

$$\beta_1 = \frac{y_2 - y_3}{2A_e} = \frac{-1}{5}$$

$$\gamma_1 = \frac{x_3 - x_2}{2A_e} = \frac{-1}{5}$$

$$\psi_1^e(x, y) = 1 - \frac{x}{5} - \frac{y}{5}$$

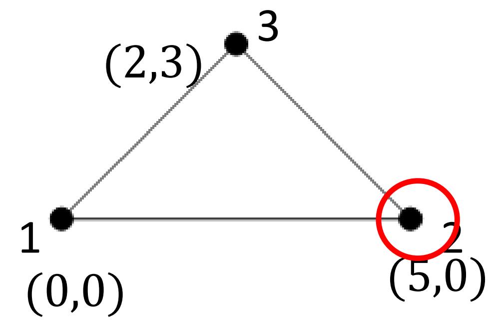
First Shape Function

Triangular Shape Functions

Example: The temperature of a thin triangular plate has been measured at the vertices $T(v_1) = 40^o, \;\; T(v_2) = 90^o, \;\; T(v_3) = 10^o$.

If the coordinates of the vertices are: $v_1 = (0,0), v_2 = (5,0), v_3 = (2,3)$, estimate the temperature at the point $p = (3,1)$?

Solution: using the **explicit** formulas



$$\psi_i^e(x,y) = a_i + \beta_i\, x + \gamma_i\, y\,, \qquad i = 1,2,3$$

$$a_i = \frac{x_j y_k - x_k y_j}{2A_e}$$

$$\beta_i = \frac{y_j - y_k}{2A_e}$$

$$\gamma_i = \frac{x_k - x_j}{2A_e}$$

$$\boxed{i = 2, j = 3, k = 1}$$

$$A_e = \frac{15}{2}$$

$$a_2 = \frac{x_3 y_1 - x_1 y_3}{2A_e} = 0$$

$$\beta_2 = \frac{y_3 - y_1}{2A_e} = \frac{1}{5}$$

$$\gamma_2 = \frac{x_1 - x_3}{2A_e} = \frac{-2}{15}$$

$$\psi_2^e(x,y) = 0 + \frac{x}{5} - \frac{2y}{15}$$

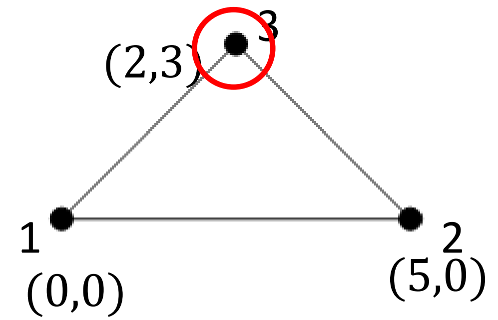
Second Shape Function

Triangular Shape Functions

Example: The temperature of a thin triangular plate has been measured at the vertices $T(v_1) = 40^o$, $T(v_2) = 90^o$, $T(v_3) = 10^o$.

If the coordinates of the vertices are: $v_1 = (0,0), v_2 = (5,0), v_3 = (2,3)$, estimate the temperature at the point $p = (3,1)$?

Solution: using the **explicit** formulas



$$\psi_i^e(x,y) = a_i + \beta_i\, x + \gamma_i\, y\,, \qquad i = 1,2,3$$

$$a_i = \frac{x_j y_k - x_k y_j}{2A_e}$$

$$\beta_i = \frac{y_j - y_k}{2A_e}$$

$$\gamma_i = \frac{x_k - x_j}{2A_e}$$

$$\boxed{i = 3, j = 1, k = 2}$$

$$A_e = \frac{15}{2}$$

$$a_3 = \frac{x_1 y_2 - x_2 y_1}{2A_e} = 0$$

$$\beta_3 = \frac{y_1 - y_2}{2A_e} = 0$$

$$\gamma_3 = \frac{x_2 - x_1}{2A_e} = \frac{1}{3}$$

$$\psi_3^e(x,y) = 0 + 0 \cdot x + \frac{y}{3}$$

Third Shape Function

Triangular Shape Functions

Example: The temperature of a thin triangular plate has been measured at the vertices $T(v_1) = 40^o$, $T(v_2) = 90^o$, $T(v_3) = 10^o$.

If the coordinates of the vertices are: $v_1 = (0,0), v_2 = (5,0), v_3 = (2,3)$, estimate the temperature at the point $p = (3,1)$?

Solution:

$$\psi_1^e(x,y) = 1 - \frac{x}{5} - \frac{y}{5} \longrightarrow \alpha_1 = \psi_1^e(3,1)$$

$$\psi_2^e(x,y) = 0 + \frac{x}{5} - \frac{2y}{15} \longrightarrow \alpha_2 = \psi_2^e(3,1)$$

$$\psi_3^e(x,y) = 0 - 0x + \frac{y}{3} \longrightarrow \alpha_3 = \psi_3^e(3,1)$$

$$T(p) = 40\alpha_1 + 90\alpha_2 + 10\alpha_3 = \boxed{53.3333^o}$$

Exercise: Compute the shape functions associated to the triangle $v_1 = (1,1)$, $v_2 = (3,2), v_3 = (2,4)$.

Solution:

$$\psi_1^e(x,y) = \frac{1}{5}(8 - 2x - y)$$

$$\psi_2^e(x,y) = \frac{1}{5}(-2 + 3x - y)$$

$$\psi_3^e(x,y) = \frac{1}{5}(-1 - x + 2y)$$

Quadrilateral Shape Functions

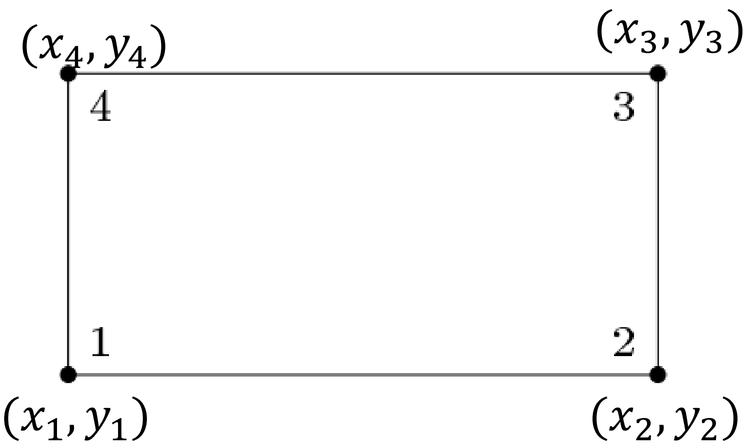
A similar situation is found when a **rectangular** element is used.

$\psi_i^e(x,y) = c_1 + c_2 x + c_3 y + c_4 xy$ (four unknowns)

such that, it's value is 1 for vertex i, and 0 for the other three:

For the first vertex:

$$1 = \psi_1^e(x_1,y_1) = c_1 + c_2 x_1 + c_3 y_1 + c_4 x_1 y_1$$
$$0 = \psi_1^e(x_2,y_2) = c_1 + c_2 x_2 + c_3 y_2 + c_4 x_2 y_2$$
$$0 = \psi_1^e(x_3,y_3) = c_1 + c_2 x_3 + c_3 y_3 + c_4 x_3 y_3$$
$$0 = \psi_1^e(x_4,y_4) = c_1 + c_2 x_4 + c_3 y_4 + c_4 x_4 y_4$$



Analogously for the other vertices

(This is not correct for general quadrilateral elements)

Quadrilateral Shape Functions

General expressions for **Rectangles:**

If we denote by

$a = x_2 - x_1$

$b = y_2 - y_1$

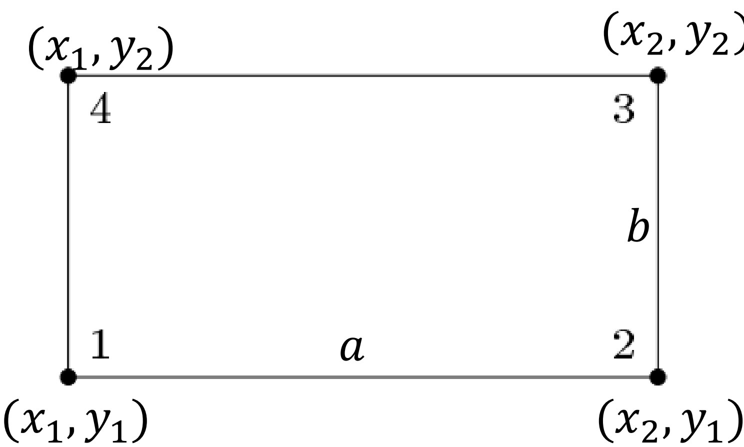
Then

$$\psi_1^e(x,y) = \frac{1}{ab}(x - x_2)(y - y_2)$$

$$\psi_2^e(x,y) = \frac{-1}{ab}(x - x_1)(y - y_2)$$

$$\psi_3^e(x,y) = \frac{1}{ab}(x - x_1)(y - y_1)$$

$$\psi_4^e(x,y) = \frac{-1}{ab}(x - x_2)(y - y_1)$$

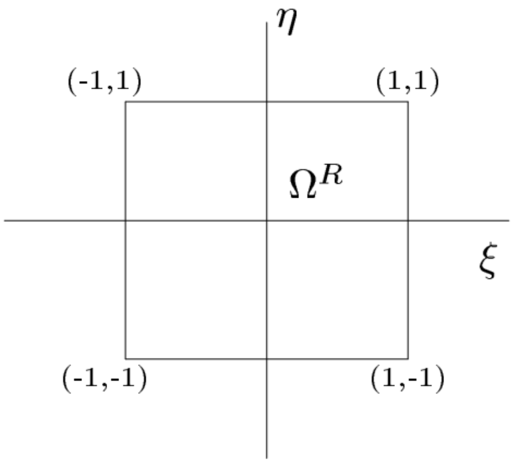


Quadrilateral Shape Functions

The **reference** bilinear quadrilateral element $\boldsymbol{\Omega^R}$: [-1,1]x[-1,1]

We can use matlab to compute the coeff.

$$\psi_j^e(x_i, y_i) = c_{1j} + c_{2j}x_i + c_{3j}y_i + c_{4j}x_i y_i = \delta_{ij}$$




```
punts=[
   -1,-1;
    1,-1;
    1,1;
    -1,1];
A=[ones(4,1), punts(:,1), punts(:,2), punts(:,1).*punts(:,2)]
b=[1;0;0;0];
C1= A\b  %coeff of the fisrt shape function
% exercise: do it for the other shape functions
```

Quadrilateral Shape Functions

The shape functions for the reference element $\mathbf{\Omega}^{\boldsymbol{R}}$, are:

$$\psi_1^R(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1-\eta)}{2}, \qquad \psi_2^R(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1-\eta)}{2},$$

$$\psi_3^R(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1+\eta)}{2}, \qquad \psi_4^R(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1+\eta)}{2}.$$

Or equivalently:

$$\psi_1^R(\xi,\eta) = \frac{1-\xi-\eta+\xi\eta}{4} \qquad \psi_2^R(\xi,\eta) = \frac{1+\xi-\eta-\xi\eta}{4}$$

$$\psi_3^R(\xi,\eta) = \frac{1+\xi+\eta+\xi\eta}{4} \qquad \psi_4^R(\xi,\eta) = \frac{1-\xi+\eta-\xi\eta}{4}$$

Quadrilateral Shape Functions

Example: The temperature of at the vertices of the **reference** quadrilateral has been measured as $T(v_1) = 10^o$, $T(v_2) = 20^o$, $T(v_3) = 30^o, T(v_4) = 40^o$.

Estimate the temperature at the point $p = (-0.3{,}0.5)$?

Solution:

$\psi_1^R(p) =$0.1625; $\psi_2^R(p) =$0.0875;

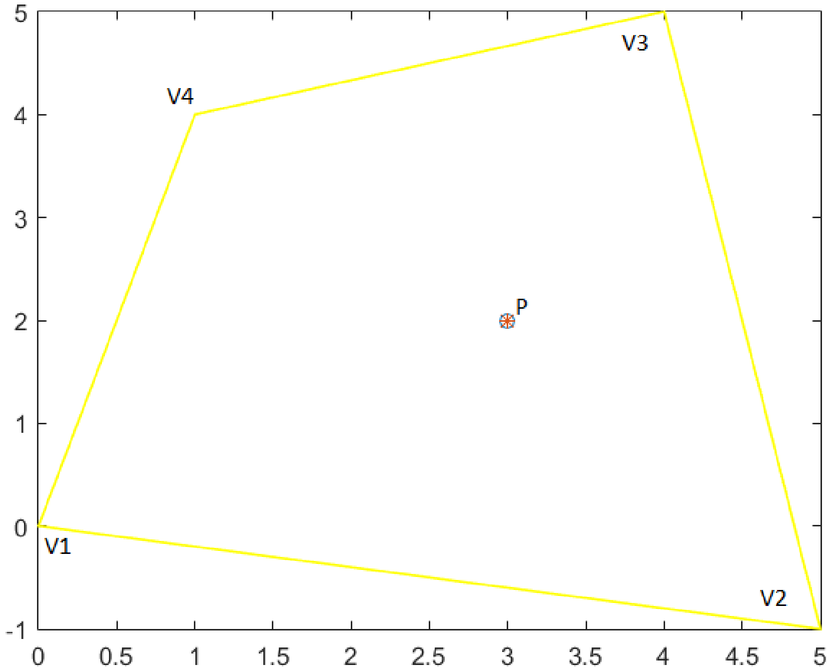
$\psi_3^R(p) =$0.2625; $\psi_4^R(p) = 0.4875$;

$$T(p) = T_1\psi_1^R(p) + T_2\psi_2^R(p) + T_3\psi_3^R(p) + T_4\psi_4^R(p) = 30.75^o$$

Quadrilateral Shape Functions

- For a General Quadrilateral element usually the shape function is NOT a 2 degree polynomial. Because of that it is not easy to compute these functions, but we still can compute the **Barycentric Coordinates.**

How to compute $\alpha_i^k = \psi_i^k(P)$?

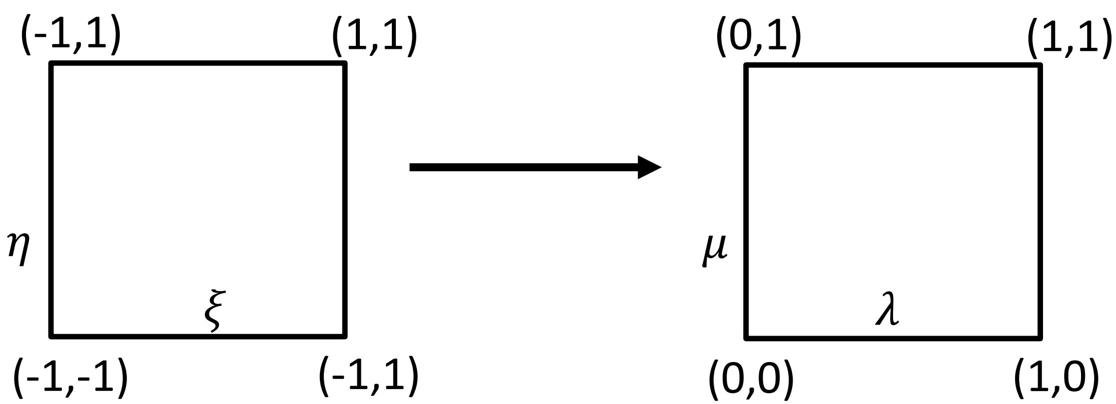


UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Tècnica Superior d'Enginyeria Industrial de Barcelona
ETSEIB

Quadrilateral Shape Functions

- Change from the Reference quadrilateral to [0,1]x[0,1]

$$\boxed{\lambda = \frac{\xi + 1}{2}, \qquad \mu = \frac{\eta + 1}{2}}$$



Shape functions on [0,1]x[0,1]:

$$\psi_1(\lambda, \mu) = (1 - \lambda)(1 - \mu) \qquad \psi_3(\lambda, \mu) = \lambda\mu$$

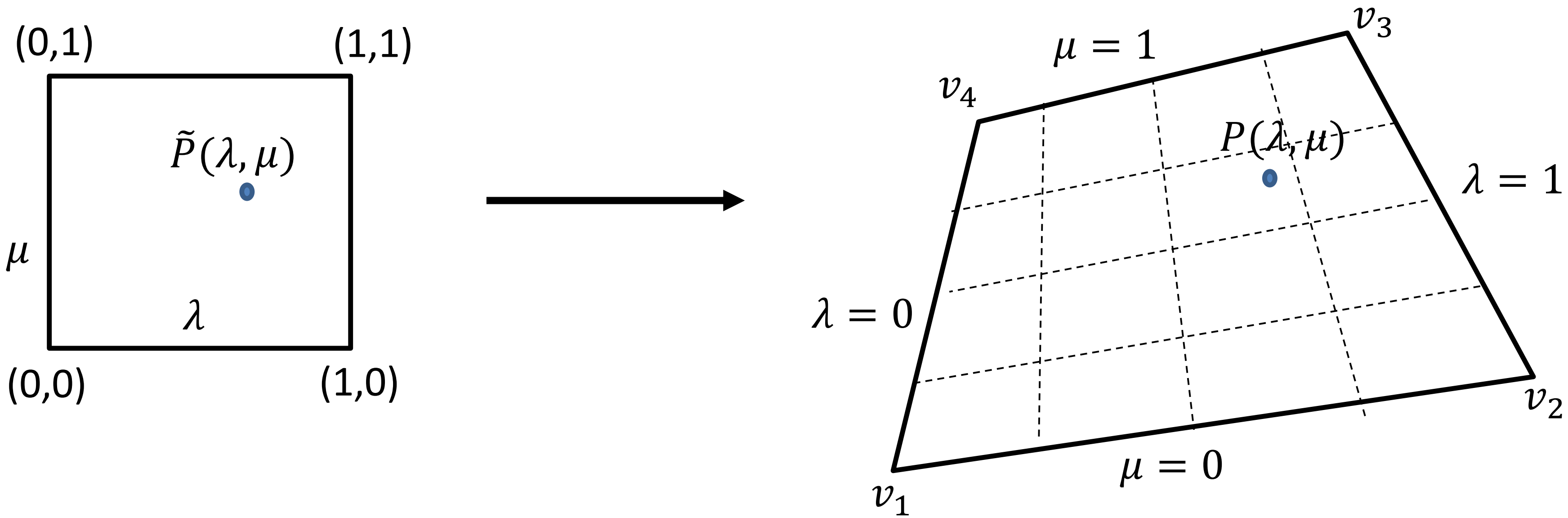
$$\psi_2(\lambda, \mu) = \lambda(1 - \mu) \qquad \psi_4(\lambda, \mu) = (1 - \lambda)\mu$$

Quadrilateral Shape Functions

- General Quadrilateral: **Isoparametric** transformation

$$(x,y) = \psi_1(\lambda,\mu)v_1 + \psi_2(\lambda,\mu)v_2 + \psi_3(\lambda,\mu)v_3 + \psi_4(\lambda,\mu)v_4$$

Change to another quadrilateral using the shape functions: for simplicity we will use here the rectangle [0,1]x[0,1]:



$$P(\lambda,\mu) = \underbrace{(1-\lambda)(1-\mu)}_{\alpha_1}\, v_1 + \underbrace{\lambda(1-\mu)}_{\alpha_2}\, v_2 + \underbrace{\lambda\mu}_{\alpha_3}\, v_3 + \underbrace{(1-\lambda)\mu}_{\alpha_4}\, v_4$$

Quadrilateral Shape Functions

We can compute them using $\lambda, \mu \in [0,1]$, as a parametrization of the quadrilateral edges.

$$P = (1-\lambda)(1-\mu)\, v_1 + \lambda(1-\mu)\, v_2 + \lambda\mu\, v_3 + (1-\lambda)\mu\, v_4$$

Rearranging the terms, the previous equation can be written as:

$$\boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c} + \lambda\mu\boldsymbol{d} = 0$$

where

$$\boldsymbol{a} = v_1 - P, \qquad \boldsymbol{b} = v_2 - v_1, \qquad \boldsymbol{c} = v_4 - v_1,$$
$$\boldsymbol{d} = v_1 - v_2 + v_3 - v_4$$

Quadrilateral Shape Functions

In fact, we have to solve a system of two non-linear equations. We will use **Newton's iterative method**:

Our system is:

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} + \lambda \begin{pmatrix} b_x \\ b_y \end{pmatrix} + \mu \begin{pmatrix} c_x \\ c_y \end{pmatrix} + \lambda\mu \begin{pmatrix} d_x \\ d_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} f(\lambda, \mu) = 0 \\ g(\lambda, \mu) = 0 \end{cases}$$

Newton's method

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix}_{n+1} = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}_{n} - \begin{pmatrix} \dfrac{\partial f}{\partial \lambda}(\lambda_n, \mu_n) & \dfrac{\partial f}{\partial \mu}(\lambda_n, \mu_n) \\ \dfrac{\partial g}{\partial \lambda}(\lambda_n, \mu_n) & \dfrac{\partial g}{\partial \mu}(\lambda_n, \mu_n) \end{pmatrix}^{-1} \begin{pmatrix} f(\lambda_n, \mu_n) \\ g(\lambda_n, \mu_n) \end{pmatrix}$$

Quadrilateral Shape Functions

In our case:

$$\frac{\partial f}{\partial \lambda} = b_x + \mu d_x \qquad \frac{\partial f}{\partial \mu} = c_x + \lambda d_x$$

$$\frac{\partial g}{\partial \lambda} = b_y + \mu d_y \qquad \frac{\partial g}{\partial \mu} = c_y + \lambda d_y$$

then

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix}_{n+1} = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}_{n} - \begin{pmatrix} b_x + \mu_n d_x & c_x + \lambda_n d_x \\ b_y + \mu_n d_y & c_y + \lambda_n d_y \end{pmatrix}^{-1} \begin{pmatrix} f(\lambda_n, \mu_n) \\ g(\lambda_n, \mu_n) \end{pmatrix}$$

Or

$$\begin{pmatrix} b_x + \mu_n d_x & c_x + \lambda_n d_x \\ b_y + \mu_n d_y & c_y + \lambda_n d_y \end{pmatrix} \begin{pmatrix} \Delta\lambda \\ \Delta\mu \end{pmatrix} = - \begin{pmatrix} f(\lambda_n, \mu_n) \\ g(\lambda_n, \mu_n) \end{pmatrix}$$

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix}_{n+1} = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}_{n} + \begin{pmatrix} \Delta\lambda \\ \Delta\mu \end{pmatrix}$$

Quadrilateral Shape Functions

- Example: Given a quadrilateral defined by vertices

$v_1 = (0{,}0), v_2 = (5,-1), v_3 = (4{,}5), v_4 = (1{,}4)$ compute the barycentric coordinates of point *P*=(3,2).

Compute: $$\boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c} + \lambda\mu \boldsymbol{d} = 0$$

where $\boldsymbol{a} = v_1 - P,\ \boldsymbol{b} = v_2 - v_1,\ \boldsymbol{c} = v_4 - v_1,\ \boldsymbol{d} = v_1 - v_2 + v_3 - v_4$

$$\begin{pmatrix} 5+\mu_n(-1) & 1+\lambda_n(-1) \\ -1+\mu_n 2 & 4+\lambda_n 2 \end{pmatrix} \begin{pmatrix} \Delta\lambda \\ \Delta\mu \end{pmatrix} = -\begin{pmatrix} f(\lambda_n, \mu_n) \\ g(\lambda_n, \mu_n) \end{pmatrix}$$

Initialization:

$\begin{pmatrix} \lambda_0 \\ \mu_0 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$ can be any value between 0 and 1.

Sol. $\lambda = 0.6250, \mu = 0.5$

$\alpha = [0.1875 \quad 0.3125 \quad 0.3125 \quad 0.1875]$

(see the implementation on the practices)