problema_4.pdf

4. Donada l'equació $-\frac{d}{dx}\left(a_1(x)\frac{du}{dx}\right)+a_0(x)u-f(x)=0$, $0<x<1$, prenem una malla de N elements finits quadràtics de llargada h_K, $K=1,\dots,N$. Suposem que els coeficients $a_1(x), a_0(x), f(x)$, són constants sobre cada element Ω^K i valen respectivament a_1^K, a_0^K, f^K.

(a) Obtingueu els polinomis de Lagrange de grau 2, $\psi_1^K, \psi_2^K, \psi_3^K$.

(b) Vegeu que les matrius de rigidesa sobre cada element Ω^K i els vectors de càrrega elementals són:

$$[K^K]=\frac{a_1^K}{3h_K}\begin{bmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{bmatrix}+\frac{a_0^K h_K}{30}\begin{bmatrix}4&2&-1\\2&16&2\\-1&2&4\end{bmatrix},\quad F^K=\frac{f^K h_K}{6}\begin{bmatrix}1\\4\\1\end{bmatrix}.$$

Solució:

$$\Omega^K:\quad \overbrace{\underset{x_1^K=x_{2K-1}}{\times}\xrightarrow{h_K/2}\underset{x_2^K=\frac12(x_1^K+x_3^K)=x_{2K}}{\times}\xrightarrow{h_K/2}\underset{x_3^K=x_{2K+1}}{\times}}\quad ,\ K=1,2,\dots,N$$

$$\boxed{\Omega=[0,1]=\bigcup_{K=1}^N\Omega^K=\bigcup_{K=1}^N[x_1^K,x_3^K]=\bigcup_{K=1}^N[x_{2K-1},x_{2K+1}]}$$

(a)

$$\psi_1^K(x)=\frac{(x-x_2^K)(x-x_3^K)}{(x_1^K-x_2^K)(x_1^K-x_3^K)}=\frac{(x-x_2^K)\cdot(x-x_3^K)}{\frac{h_K}{2}\cdot h_K}=\frac{2}{h_K^2}(x-x_2^K)(x-x_3^K),$$

$$\psi_2^K(x)=\frac{(x-x_1^K)(x-x_3^K)}{(x_2^K-x_1^K)(x_2^K-x_3^K)}=\frac{(x-x_1^K)\cdot(x-x_3^K)}{-\frac{h_K}{2}\cdot\frac{h_K}{2}}=-\frac{4}{h_K^2}(x-x_1^K)(x-x_3^K),$$

$$\psi_3^K(x)=\frac{(x-x_1^K)(x-x_2^K)}{(x_3^K-x_1^K)(x_3^K-x_2^K)}=\frac{(x-x_1^K)(x-x_2^K)}{h_K\cdot\frac{h_K}{2}}=\frac{2}{h_K}(x-x_1^K)(x-x_2^K).$$

Introduïm el canvi $\phi:\Omega^R=[-1,1]\longrightarrow\Omega^K=[x_1^K,x_3^K]$, definit per: $t\in[-1,1]\mapsto x=\phi(t)=\frac{h_K}{2}t+x_2^K\in\Omega^K=[x_1^K,x_3^K]=[x_{2K-1},x_{2K+1}]$, $K=1,2,\dots,N$. Notem que $\phi(-1)=-\frac{h_K}{2}+x_2^K=x_2^K-(x_2^K-x_1^K)=x_1^K$,

$\phi(1)=\frac{h_K}{2}+x_2^K=x_3^K-x_2^K+x_2^K=x_3^K$.

Definim:

$$\Psi_1^R(t):=\Psi_1^K(\phi(t))=\frac{h_K/2\,t\,(h_K/2\,t-h_K/2)}{\frac{h_K}{2}\cdot h_K}=\frac12 t(t-1)=\frac12(t^2-t)$$

$$\psi_2^R(t) := \psi_2^K(\phi(t)) = \frac{\left(\frac{h_K}{2}t+\frac{h_K}{2}\right)\left(\frac{h_K}{2}t-\frac{h_K}{2}\right)}{-\frac{h_K^2}{4}} = 1-t^2$$

$$\psi_3^R(t) := \psi_3^K(\phi(t)) = \frac{\left(\frac{h_K}{2}t+\frac{h_K}{2}\right)\frac{h_K}{2}t}{\frac{h_K^2}{2}} = \frac{1}{2}t(t+1) = \frac{1}{2}(t^2+t)$$

(b) Aplicant el canvi de variable a les integrals que defineixen $K_{ij}^{k,1}$, $K_{ij}^{k,0}$ i F_i^K $(i=1,2,3;\ K=1,2,3,\dots,N)$ veiem que:

$$K_{ij}^{k,1} = \int_{x_1^K}^{x_3^K} a_1(x)\,\frac{d\psi_i^K}{dx}(x)\,\frac{d\psi_j^K}{dx}(x)\,dx \overset{(*)}{=} \int_{-1}^{1} a_1(\phi(t))\cdot\frac{2}{h_K}\cdot\frac{d\psi_i^R}{dt}(t)\cdot\frac{2}{h_K}\cdot\frac{d\psi_j^R}{dt}(t)\cdot\frac{h_K}{2}\,dt$$

$$\boxed{\begin{aligned}(*)\ \frac{d\psi_i^R}{dt}(t) &= \frac{d}{dt}\left(\psi_i^K(\phi(t))\right) = \frac{d\psi_i^K}{dt}(x)\,\phi'(t)\\ &= \frac{d\psi_i^K}{dt}(x)\cdot h_K \Rightarrow \frac{d\psi_i^K}{dx}(x) = \frac{2}{h_K}\frac{d\psi_i^R}{dt}(t)\end{aligned}}$$

$$= \frac{2}{h_K}\int_{-1}^{1} a_1(\phi(t))\cdot\frac{d\psi_i^R}{dt}(t)\cdot\frac{d\psi_j^R}{dt}(t)\,dt,$$

$$K_{ij}^{k,0} = \int_{x_1^K}^{x_3^K} a_0(x)\,\psi_i^K(x)\,\psi_j^K(x)\,dx = \int_{-1}^{1} a_0(\phi(t))\,\psi_i^K(\phi(t))\,\psi_j^K(\phi(t))\,\frac{h_K}{2}dt$$

$$= \frac{h_K}{2}\int_{-1}^{1} a_0(\phi(t))\,\psi_i^R(t)\,\psi_j^R(t)\,dt.$$

$$F_i^K = \frac{h_K}{2}\int_{x_1^K}^{x_3^K} f(x)\,\psi_i^K(x)\,dx = \int_{-1}^{1} f(\phi(t))\cdot\psi_i^K(\phi(t))\,h_K\,dt =$$

$$= \frac{h_K}{2}\int_{-1}^{1} f(\phi(t))\,\psi_i^R(t)\,dt.$$

Per tant,

$$K_{11}^{k,1} = \frac{2}{h_K}\int_{-1}^{1} a_1^K\,\frac{d\psi_1^R}{dt}\cdot\frac{d\psi_1^R}{dt}\,dt = \frac{2a_1^K}{4h_K}\int_{-1}^{1}(2t-1)^2dt = \frac{a_1^K}{h_K}\cdot\int_0^1(4t^2+1)\,dt$$

$$= \frac{a_1^K}{h_K}\left(\frac{4}{3}+1\right) = \frac{7a_1^K}{3h_K}.$$

$$K_{12}^{k,1} = \frac{2}{h_K}\cdot\int_{-1}^{1} a_1^K\,\frac{d\psi_1^R}{dt}\cdot\frac{\psi_2^R}{dt}\,dt = -\frac{2a_1^K}{2h_K}\int_{-1}^{1}(2t-1)\,2t = -\frac{8a_1^K}{h_K}\int_0^1 t^2\,dt = -\frac{8a_1^K}{3h_K}$$

$$= K_{21}^{k,1}.$$

$$K_{13}^{k,1} = \frac{2}{h_k} \cdot \int_{-1}^{1} a_1^k \frac{1}{4}(2t-1)(2t+1)\,dt = \frac{a_1^k}{h_k}\int_0^1 (4t^2-1)\,dt = \frac{a_1^k}{4h_k} = K_{31}$$

$$K_{22}^{k,1} = \frac{8}{h_k} \cdot \int_{-1}^{1} a_1^k t^2\,dt = \frac{16 a_1^k}{h_k}\int_0^1 t^2\,dt = \frac{16 a_1^k}{3h_k}.$$

$$K_{23}^{k,1} = -\frac{2}{h_k}\int_{-1}^{1} a_1^k t(2t+1)\,dt = -\frac{8a_1^k}{h_k} \cdot \int_0^1 t^2\,dt = -\frac{8a_1^k}{3h_k} = K_{32}^{k,1}$$

$$K_{33}^{k,1} = \frac{1}{2h_k}\int_{-1}^{1} a_1^k (2t+1)^2\,dt = \frac{a_1^k}{h_k}\int_0^1 (4t^2+1)\,dt = \frac{a_1^k}{h_k}\left(\frac{4}{3}+1\right) = \frac{7a_1^k}{3h_k}.$$

$$K^{k,1} = \frac{a_1^k}{3h_k} \cdot \begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}.$$

$$K_{11}^{k,0} = \frac{h_k}{2}\int_{-1}^{1} a_0^k \frac{1}{4}(t^2-t)^2\,dt = \frac{a_0^k h_k}{4}\int_0^1 (t^4+t^2)\,dt = \frac{a_0^k h_k}{4}\left(\frac{1}{5}+\frac{1}{3}\right) = \frac{2a_0^k h_k}{15}$$

$$K_{12}^{k,0} = \frac{h_k}{2}\int_{-1}^{1} a_0^k \frac{1}{2}(t^2-t)(1-t^2)\,dt = \frac{a_0^k h_k}{2}\int_0^1 (t^2-t^4)\,dt = \frac{a_0^k h_k}{2}\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$= \frac{a_0^k h_k}{15} = K_{21}^{k,0}$$

$$K_{13}^{k,0} = \frac{h_k}{2}\int_{-1}^{1} a_0^k \frac{1}{4}(t^2-t)(t^2+t)\,dt = \frac{a_0^k h_k}{4}\int_0^1 (t^4-t^2)\,dt = \frac{a_0^k h_k}{4}\left(\frac{1}{5}-\frac{1}{3}\right)$$

$$= -\frac{a_0^k h_k}{30} = K_{31}^{k,0}$$

$$K_{22}^{k,0} = \frac{h_k}{2}\int_{-1}^{1} a_0^k (1-t^2)^2\,dt = a_0^k h_k \int_0^1 (t^4-2t^2+1)\,dt = a_0^k h_k \left(\frac{1}{5}-\frac{2}{3}+1\right)$$

$$= \frac{8a_0^k h_k}{15},$$

$$K_{23}^{k,0} = \frac{h_k}{2}\int_{-1}^{1} a_0^k (1-t^2)\frac{1}{2}(t^2+t)\,dt = \frac{a_0^k h_k}{2}\int_0^1 (1-t^2)(t^2+t)\,dt = \frac{a_0^k h_k}{2}\int_0^1 (t^2-t^4)\,dt$$

$$= \frac{a_0^k h_k}{2}\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{a_0^k h_k}{15} = K_{32}^{k,0},$$

$$K_{33}^{k,0} = \frac{h_k}{2}\int_{-1}^{1} a_0^k \frac{1}{4}(t^2+t)^2\,dt = \frac{a_0^k h_k}{4}\int_0^1 (t^4+t^2)\,dt = \frac{a_0^k h_k}{4}\left(\frac{1}{5}+\frac{1}{3}\right) = \frac{2a_0^k h_k}{15}$$

$$= K_{33}^{k,0}.$$

$$K^{k,0} = \frac{a_0^k h_k}{30}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}.$$

$$F_1^K = \frac{h_K}{2}\int_{-1}^{1} f^K \frac{1}{2}(t^2-t)\,dt = \frac{f^K h_K}{2}\int_0^1 t^2\,dt = \frac{f^K h_K}{6}$$

$$F_2^K = \frac{h_K}{2}\int_{-1}^{1} f^K(1-t^2)\,dt = f^K h_K \int_0^1 (1-t^2)\,dt = f^K h_K\left(1-\frac{1}{3}\right) = \frac{2 f^K h_K}{3}$$

$$F_3^K = \frac{h_K}{2}\int_{-1}^{1} f^K\frac{1}{2}(t^2+t)\,dt = \frac{f^K h_K}{2}\int_0^1 t^2\,dt = \frac{f^K h_K}{6}$$

Per tant,

$$K^K = K^{K,1} + K^{K,0} = \frac{a_1^K}{3h_K}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix} + \frac{a_0^K h_K}{30}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}.$$

$$F^K = \frac{f^K h_K}{6}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}. \triangleright$$