MÈTODES NUMÈRICS: **Ex-Parcial Q1-2019-20 (c)**

Name and surnames:

1. Let us consider the following problem for $x \in (0, \pi)$:

$$\begin{cases} -\dfrac{d}{dx}\left(a_1(x)\dfrac{du}{dx}\right) = \dfrac{12}{\pi}, \\ u(0) = 0, \quad u(\pi) = 8, \end{cases} \quad \text{with} \quad a_1(x) = \begin{cases} \dfrac{3\pi}{2}, & x \in (0, \frac{\pi}{2}). \\ \dfrac{\pi}{2}\sin x + \dfrac{\pi - 2}{2}, & x \in (\frac{\pi}{2}, \pi), \end{cases}$$

We approximate the solution of this problem using the FEM in a mesh of two elements: $\Omega^1 = [0, \frac{\pi}{2}]$ being a **quadratic** element and $\Omega^2 = [\frac{\pi}{2}, \pi]$ being a **linear** element.

Assuming that all node numberings go from left to right, compute and fill the following table:

$\psi_1^1(x) =$	$\frac{8}{\pi^2}(x-\frac{\pi}{4})(x-\frac{\pi}{2})$
$[K^1]$ and $[K^2]$:	$K^1 = \begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}, K^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
Assembled system:	$\begin{pmatrix} 7 & -8 & 1 & \\ -8 & 16 & -8 & \\ 1 & -8 & 8 & -1 \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}$
Nodal Values U_2 and U_3:	$U_2 = 2, \; U_3 = 7/2$
Approximation for $u(3\pi/4)$:	$23/4$

(**Hint:** For the global system, the value of K_{33} =8)

(4 points)

$$\psi_1^1(x) = \frac{(x-\frac{\pi}{4})(x-\frac{\pi}{2})}{\frac{\pi}{4}\cdot\frac{\pi}{2}} = \frac{8}{\pi^2}(x-\frac{\pi}{4})(x-\frac{\pi}{2})$$

$$K^1 = \frac{3\pi/2}{3\pi/2}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}, \quad f^1 = \frac{\frac{12}{\pi}\cdot\frac{\pi}{2}}{6}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

$$\psi_1^{(2)}(x) = \frac{x-\pi}{\frac{\pi}{2}-\pi} = \frac{2}{\pi}(\pi - x); \quad \frac{d\psi_1^{(2)}}{dx} = -2/\pi$$

$$\psi_2^{(2)}(x) = \frac{x-\frac{\pi}{2}}{\pi-\frac{\pi}{2}} = \frac{2}{\pi}(x-\frac{\pi}{2}); \quad \frac{d\psi_2^{(2)}}{dx} = \frac{2}{\pi}$$

$$K_{11}^2 = \int_{\pi/2}^{\pi} a_1(x)\frac{d\psi_1^2}{dx}(x)\frac{d\psi_1^2}{dx}(x)\,dx = \frac{4}{\pi^2}\int_{\pi/2}^{\pi}\left(\frac{\pi}{2}\sin x + \frac{\pi-2}{2}\right)dx$$

$$= \frac{4}{\pi^2}\left[-\frac{\pi}{2}\cos x\Big|_{\pi/2}^{\pi} + \frac{\pi-2}{2}x\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{4}{\pi^2}\left(\frac{\pi}{2} + \frac{\pi-2}{2}\cdot\frac{\pi}{2}\right) = \frac{4}{\pi^2}\cdot\frac{\pi^2}{4} = 1$$

Clarament $K^2_{22} = K^2_{11} = 1$

$K^2_{12} = K^2_{21} = -1$

Llavors: $K^2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$$f^2_1 = \int_{\pi/2}^{\pi} \frac{12}{\pi} \cdot \frac{2}{\pi}(\pi - x)\,dx = \left\{\begin{array}{l} t = \pi - x : dx = -dt \\ x = \pi : t = 0 \\ x = \pi/2 : t = \pi/2 \end{array}\right\} = \frac{24}{\pi^2}\int_0^{\pi/2} t\,dt = \frac{24}{\pi^2}\cdot\frac{\pi^2}{8} = \boxed{3}$$

$$f^2_2 = \int_{\pi/2}^{\pi} \frac{12}{\pi} \cdot \frac{2}{\pi}\left(x - \frac{\pi}{2}\right)dx = \left\{\begin{array}{l} t = x - \frac{\pi}{2} : dx = dt \\ x = \pi,\ t = \pi/2 \\ x = \pi/2,\ t = 0 \end{array}\right\} = \frac{24}{\pi^2}\int_0 t\,dt = \frac{24}{\pi^2}\cdot\frac{\pi^2}{8} = \boxed{3}$$

$f^2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

Assembled system:

$$\begin{pmatrix} 7 & -8 & 1 & \\ -8 & 16 & -8 & \\ 1 & -8 & 8 & -1 \\ & & -1 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 = 0 \\ Q_3 = 0 \\ Q_4 \end{pmatrix}$$

B.C. (Essential)

$u(0) = U_1 = 0$

$u(\pi) = U_8 = 8$

Reduced system: $-8\cdot \overset{=0}{U_1} + 16U_2 - 8U_3 + 0\cdot \overset{=8}{U_4} = 4$

$1\cdot \overset{=0}{U_1} - 8U_2 + 8U_3 - 1\cdot \overset{=8}{U_4} = 4$

$$\left.\begin{array}{l} 16U_2 - 8U_3 = 4 \\ -8U_2 + 8U_3 = 12 \end{array}\right\} ; \quad \begin{array}{l} 8U_2 = 16 \Rightarrow \boxed{U_2 = 2} \\ 8U_3 = 12 + 8U_2 = 12 + 8\cdot 2 = 28 \\ \text{d'on } \boxed{U_3 = \frac{28}{8} = \frac{7\cdot\cancel{4}}{2\cdot\cancel{4}} = \frac{7}{2}} \end{array}$$

Interpolació:

$$u\left(\frac{3\pi}{4}\right) = U_3\,\psi^2_1\left(\frac{3\pi}{4}\right) + U_4\,\psi^2_2\left(\frac{3\pi}{4}\right)$$

$$= \frac{7}{2}\left(-\frac{2}{\pi}\left(\frac{3\pi}{4} - \pi\right)\right) + 8\,\frac{2}{\pi}\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) = \frac{7}{4} + \frac{16}{4} = \boxed{\frac{23}{4}}$$