Problem-1

(a) $\psi_2^3(x,y) = a + bx + cy$ s.t.:

$$\psi_2^3(3,0) = a + 3b = 0$$
$$\psi_2^3(3,1) = a + 3b + c = 1$$
$$\psi_2^3(1,1) = a + b + c = 0$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 0 & 0 & -3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Solution:
$a = -3/2,\ b = 1/2,\ c = 1$

So the form function is,

$$\boxed{\psi_2^3(x,y) = x/2 + y - 3/2}$$

(b) Find the value of K_{52} (the (5,2) entry of the global stiff matrix K)

$$K^1 = \frac{k_c}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad K^2 = K^3 = \frac{k_c}{4}\begin{pmatrix} 4 & -4 & 0 \\ -4 & 5 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$K_{52} = K^1_{32} + K^2_{12} = -\frac{k_c}{2} - \frac{4k_c}{4}$$

$$= \left(-\tfrac{1}{2} - 1\right)k_c = -\tfrac{3}{2}k_c \underset{k_c=4}{=} \boxed{-6}$$

Hint. $K_{34} = K^3_{12} = \frac{k_c}{4}(-4) = \boxed{-4}$

(c) The value of F_5 (the global internal heating of global node 5),

$$F^1 = \frac{f_1 A_1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{f}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$F^2 = F^3 = \frac{f_{2,3} A_{2,3}}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{f}{6}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix},$$

So,

$$F_5 = F_3^1 + F_1^2 + F_3^3 = \frac{f}{6}(1+2+2)$$

$$= \frac{5}{6} f \underset{f=3}{=} 5/2 = \boxed{2.5}$$

Hint. $F_1 = F_1^1 = \frac{f}{6} = 3/6 = 1/2 = \boxed{0.5}$

(d) Now, suppose that a linear flow q_n on the edge between nodes 1 and 5 of values $q_n = 30$ and 0 respectively is applied. Then the value of Q_{33}^1 is,

$$Q_{33}^1 = \frac{0}{3}\sqrt{2} + \frac{30}{6}\sqrt{2} = 5\sqrt{2} = \boxed{7.0711e+00}$$

(e) When computing Q_4 as part of a convective edge, the expression relating T_3, T_4 and T_∞ is

$$Q_4 = -\frac{\beta T_3}{6} h_1^3 - \frac{\beta T_4}{3} h_1^3 + \frac{\beta T_\infty}{2} h_1^3$$

$$= \boxed{-T_3 - 2T_4 + 3T_\infty}$$

$h_1^3 = 1$

$\beta = 6.0$

(f) Suppose that $T_0 = 0$ on all the bottom boundary and also $q_n = 0$ on the top edge joining the nodes 4 and 5, besides the boundary conditions already done. The value of T_4 when $T_\infty = 0$ and T_5 is 1.4166e+00 is,

$$K_{41}\overset{=0}{T_1} + K_{42}\overset{=0}{T_2} + K_{43}\overset{=0}{T_3} + K_{44}T_4 + K_{45}T_5$$

$$\overset{(*)}{=} \frac{5}{4} K_c T_4 - \frac{K_c}{4} T_5 = F_4 + Q_4 \quad (\&)$$

$$(*)\ K_{44} = K_{22}^3 = \frac{5}{4} K_c \qquad F_4 = F_2^3 = \frac{f}{3} = 1$$

$$K_{45} = K_{23}^3 = -\frac{K_c}{4} \qquad Q_4 = -\overset{=0}{T_3} - 2T_4 + 3\overset{=0}{T_\infty}$$

$k_c = 4$, so the system (&) writes,

$$5T_4 - T_5 = 1 - 2T_4$$

$$\Leftrightarrow \quad 7T_4 = 1 + T_5$$

$$\Leftrightarrow \quad T_4 = \frac{1+T_5}{7} = \frac{2.4166}{7}$$

$$= \boxed{3.4523e\text{-}01} \qquad \square$$

$$\begin{array}{l|l} 2.4166 & 7 \\ \hline 31 & 0.345228 \\ \ \ 36 & \\ \ \ \ 16 & \\ \ \ \ \ 20 & \\ \ \ \ \ \ 60 & \\ \ \ \ \ \ \ 40 & \end{array}$$

$$\simeq 0.34523$$

$$= 3.4523e\text{-}01$$