

Mètodes Numèrics (240032)

Plane elasticity. Weak Formulation

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Model equation for plane elasticity (I)

(In Reddy, 2006, chap. 11) for the plane elasticity problems, the the normal stress in the x direction, $\sigma_{xx} = \sigma_{xx}(x, y)$, the normal stress in the y direction, $\sigma_{yy} = \sigma_{yy}(x, y)$, and the shear stress, $\sigma_{xy} = \sigma_{xy}(x, y)$, satisfy the BVP given by the two coupled system of PDE,

$$\left. \begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x(x, y) &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y(x, y) &= 0 \end{aligned} \right\} \text{on } \Omega \subset \mathbb{R}^2, \quad (1)$$

the *natural* B.C.

$$\left. \begin{aligned} t_x &\equiv \sigma_{xx} n_x + \sigma_{xy} n_y = \hat{t}_x \\ t_y &\equiv \sigma_{xy} n_x + \sigma_{yy} n_y = \hat{t}_y \end{aligned} \right\} \text{on } \Gamma_\sigma, \quad (2)$$

and the *essential* B.C.

$$u = \hat{u}, \quad v = \hat{v} \quad \text{on } \Gamma_u \quad (3)$$

where

Model equation for plane elasticity (II)

f_x, f_y are the components of the body force vector (per unit volume) along the x and the y direction respectively.

n_x, n_y denote the components (or the direction cosines) of the unit normal vector on the boundary of Γ .

Γ_σ, Γ_u are two disjoint pieces of Γ .

\hat{t}_x, \hat{t}_y denote the components of the specified traction vector.

\hat{u}, \hat{v} are the components of the specified displacement vector.

Only one element of each pair, (u, t_x) and (v, t_y) , may be specified at a boundary point.

Relations Strain-displacement

On the one hand the components of the strain tensor

$$\varepsilon_{xx} = \varepsilon_{xx}(x, y), \quad \varepsilon_{xy} = \varepsilon_{xy}(x, y), \quad \varepsilon_{yy} = \varepsilon_{yy}(x, y),$$

are given by the components of the displacement field $u = u(x, y)$, $v = v(x, y)$ through the relations

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y},$$

or, in vector form,

$$\boldsymbol{\varepsilon} \equiv \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \quad (4)$$

Stress-Strain equation

On the other hand, the stress tensor, σ , is related by the strain tensor, ε , through the Stress-Strain equation

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{pmatrix} \quad (5)$$

or, in matrix form

$$\sigma = C\varepsilon, \quad (6)$$

with

$$\sigma = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{pmatrix}. \quad (7)$$

For stress problems

Formulas for c_{ij} for stress problems

For strain problems

Formulas for c_{ij} for strain problems

Example: an algorithm with caption

Data: $n \geq 0$
Result: $y = x^n$

```
1  $y \leftarrow 1$ 
2  $X \leftarrow x$ 
3  $N \leftarrow n$ 
4 while  $N \neq 0$  do
5   if  $N$  is even then
6      $X \leftarrow X \times X$ 
7      $N \leftarrow \frac{N}{2}$                                 # This is a comment
8   else
9     if  $N$  is odd then
10     $y \leftarrow y \times X$ 
11     $N \leftarrow N - 1$ 
12  end
13 end
14 end
```

Algorithm 1: An algorithm with caption

References (I)

J. N. Reddy. *An Introduction to the Finite Element Method*. McGraw-Hill series in mechanical engineering. McGraw-Hill Higher Education, New York, NY, third edition. edition, 2006. ISBN 1-76042-211-8.