Mètodes Numèrics (240032)
Plane elasticity. Weak Formulation

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Model equation for plane elasticity (I)

(In Reddy, 2006, chap. 11) for the plane elasticity problems, the the normal stress in the x direction, $\sigma_{xx} = \sigma_{xx}(x, y)$, the normal stress in the y direction, $\sigma_{yy} = \sigma_{yy}(x, y)$, and the shear stress, $\sigma_{xy} = \sigma_{xy}(x, y)$, satisfy the BVP given by the two coupled system of PDE,

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x(x, y) = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y(x, y) = 0 \end{aligned}\right\} \quad \text{on } \Omega \subset \mathbb{R}^2, \tag{1}$$

the *natural* B.C.

$$\left.\begin{aligned} t_x \equiv \sigma_{xx} n_x + \sigma_{xy} n_y = \hat{t}_x \\ t_y \equiv \sigma_{xy} n_x + \sigma_{yy} n_y = \hat{t}_y \end{aligned}\right\} \quad \text{on } \Gamma_\sigma, \tag{2}$$

and the *essential* B.C.

$$u = \hat{u}, \quad v = \hat{v} \quad \text{on } \Gamma_u \tag{3}$$

where

Model equation for plane elasticity (II)

f_x, f_y are the components of the body force vector (per unit volume) along the x and the y direction respectively.

n_x, n_y denote the components (or the direction cosines) of the unit normal vector on the boundary of Γ.

Γ_σ, Γ_u are two disjoint pieces of Γ.

$\hat{t}_x$, $\hat{t}_y$ denote the components of the specified traction vector.

$\hat{u}$, $\hat{v}$ are the components of the specified displacement vector.

Only one element of each pair, (u, t_x) and (v, t_y), may be specified at a boundary point.

Relations Strain-displacement

On the one hand the components of the strain tensor

$$\varepsilon_{xx} = \varepsilon_{xx}(x, y), \quad \varepsilon_{xy} = \varepsilon_{xy}(x, y), \quad \varepsilon_{yy} = \varepsilon_{yy}(x, y),$$

are given by the components of the displacement field $u = u(x, y)$, $v = v(x, y)$ through the relations

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y},$$

or, in vector form,

$$\varepsilon \equiv \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \tag{4}$$

Stress-Strain equation

On the other hand, the stress tensor is related by the strain tensor through the Stress-Strain equation

Example: an algorithm with caption

Data: $n \geq 0$
Result: $y = x^n$

$y \leftarrow 1$
$X \leftarrow x$
$N \leftarrow n$
while $N \neq 0$ **do**
    if N *is even* **then**
        $X \leftarrow X \times X$
        $N \leftarrow \frac{N}{2}$ `# This is a comment`
    else
        if N *is odd* **then**
            $y \leftarrow y \times X$
            $N \leftarrow N - 1$
        end
    end
end

Algorithm 1: An algorithm with caption

References (I)

J. N. Reddy. *An Introduction to the Finite Element Method*. McGraw-Hill series in mechanical engineering. McGraw-Hill Higher Education, New York, NY, third edition. edition, 2006. ISBN 1-76042-211-8.