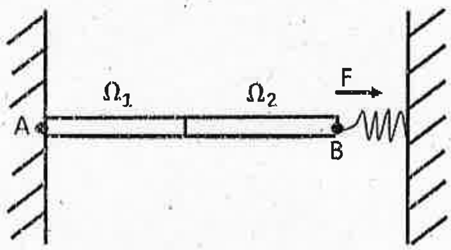
MÈTODES NUMÈRICS: Ex-Parcial (a) Q1-2018-19

Name and surnames:

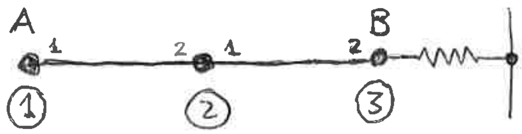
(1) We consider a piece of cross sectional area 3 cm^2 and length 20 cm clamped at x=0 (point A) and with a spring attached (initially at rest) at the other end, $x = 20$ cm (point B), as it is shown in the figure. The Young modulus of the material of the piece is given by $E(x) = 6x+10$ N/cm^2 and the constant of the spring is $k_s = 12$ N/cm.



Assuming that we apply a longitudinal force $F = 12$ N on point B, in the direction of increasing x, and using a mesh of two linear elements (each one of length 10 cm) to discretize the piece, compute:

$\psi_1^2(x) =$	$2 - \frac{x}{10}$
$[K^1]$ and $[K^2]$:	$[K^1] = \begin{pmatrix} 12 & -12 \\ -12 & 12 \end{pmatrix}$, $[K^2] = \begin{pmatrix} 30 & -30 \\ -30 & 30 \end{pmatrix}$
Assembled system:	$\begin{pmatrix} 12 & -12 & 0 \\ -12 & 42 & -30 \\ 0 & -30 & 30 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$
Boundary conditions:	$U_1 = 0, \quad Q_2 = 0, \quad Q_3 = F - k_s U_3 = 12 - 12U_3$
Displacement of nodes 2 and 3:	$U_2 = 5/12$ cm, $\quad U_3 = 7/12$ cm

◁ Solució.



$x_1^1 = x_1 = 0$ $\quad x_2^1 = x_1^2 = x_2 = 10\ (cm)$ $\quad x_2^2 = x_3 = 20\ (cm)$

$A = 3 cm^2$

$E(x) = 6x + 10\ (N/cm^2)$

$K_s = 12\ N/cm$

$F = 12$

$h_k := x_{k+1} - x_k\ ,\ k = 1,2$

Funcions de forma:

$$\psi_1^k(x) = \frac{x - x_2^k}{x_1^k - x_2^k} = \frac{x - x_{k+1}}{x_k - x_{k+1}} = -\frac{1}{h_k}(x - x_{k+1})$$

$$\psi_2^k(x) = \frac{x - x_1^k}{x_2^k - x_1^k} = \frac{x - x_k}{x_{k+1} - x_k} = \frac{1}{h_k}(x - x_k)$$

Aleshores: $\frac{d\psi_1^k}{dx}(x) = -1/h_k$, $\frac{d\psi_2^k(x)}{dx} = 1/h_k$, $\frac{d\psi_i^k}{dx} = (-1)^i \frac{1}{h_k},\ i = 1,2.$

(i) Llavors, per a $k=2$: $\psi_1^2(x) = \frac{x - x_3}{x_2 - x_3} = -\frac{1}{10}(x - 20) = 2 - \frac{x}{10}$

(ii) Per a calcular $[K^1]$ i $[K^2]$ podríem fer servir els resultats del problema 1 (com vam fer a la resolució del problema 10,(*) per exemple). Aquí però farem aquests càlculs explícitament:

$$K_{ij}^{k,1} = \int_{x_1^k}^{x_N^k} a_1(x) \frac{d\psi_i^k}{dx} \cdot \frac{d\psi_j^k}{dx}\, dx \underset{\substack{x_1^k = x_k \\ x_2^k = x_{k+1} \\ a_1(x) = E(x)\cdot A}}{\overset{N=2}{=}} \int_{x_k}^{x_{k+1}} (18x + 30) \underbrace{\frac{d\psi_i^k}{dx}}_{= \frac{(-1)^i}{h_k}} \cdot \underbrace{\frac{d\psi_j^k}{dx}}_{= \frac{(-1)^j}{h_k}}\, dx$$

$$= \frac{(-1)^{i+j}}{h_k^2} \cdot \int_{x_k}^{x_{k+1}} (18x + 30)\, dx = \frac{(-1)^{i+j}}{h_k^2} (9x^2 + 30x) \Big|_{x = x_k}^{x = x_{k+1}}$$

$$= \frac{(-1)^{i+j}}{h_k^2} \left[9(x_{k+1}^2 - x_k^2) + 30(x_{k+1} - x_k) \right] = \frac{(-1)^{i+j}}{h_k^2} \left[9(x_{k+1} - x_k)(x_{k+1} + x_k) + 30(x_{k+1} - x_k) \right]$$

$$= \frac{(-1)^{i+j}}{h_k} \left[9(x_{k+1} + x_k) + 30 \right].$$

(*) i la fórmula per a $K^{1,1}$ quan $a_1(x) \equiv ctnt$, que tenim als pdf's de teoria.

* per a $k=1$:

$$K^{1,1}_{ij} = \frac{(-1)^{i+j}}{10}\left[9(10+0)+30\right] = (-1)^{i+j}\,12,\ (i,j=1,2);\ \text{d'on:}\ K^{1,1} = \begin{pmatrix} 12 & -12 \\ -12 & 12 \end{pmatrix}.$$

$h_1 = x_2 - x_1 = 10$ $(x_2=10, x_1=0)$.

* per a $k=2$:

$$K^{2,1}_{ij} = \frac{(-1)^{i+j}}{10}\left[9(20+10)+30\right] = (-1)^{i+j}\,30,\ (i,j=1,2);\ \text{d'on:}\ K^{2,1} = \begin{pmatrix} 30 & -30 \\ -30 & 30 \end{pmatrix}.$$

$h_2 = x_3 - x_2 = 10$ $(x_3=20, x_1=10)$.

D'altra banda: $K^{1,0}=0$, $K^{2,0}=0$. Notem que l'equació de l'elasticitat és $-\frac{d}{dx}\left(E(x)A(x)\frac{du}{dx}\right) = f(x)$ i l'equació 'model' és $-\frac{d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)u = f(x)$. Comparant: $a_1(x) = E(x)A(x)$, $a_0(x) \equiv 0$. Per tant, $K^{k,0}_{ij} = \int_{x_1^k}^{x_N^k} a_0(x)\,\psi_i^k(x)\,\psi_j^k(x)\,dx = 0$ $\forall i,j, \forall k$. En aquest cas, a més, $f(x)\equiv 0$. Aleshores, també: $F_i^k = \int_{x_1^k}^{x_N^k} f(x)\,\psi_i^k(x)\,dx = 0$ $\forall i, \forall k$.

D'aquesta manera: $K^1 = K^{1,1} + K^{1,0} = \begin{pmatrix} 12 & -12 \\ -12 & 12 \end{pmatrix}$, $K^2 = K^{2,1} + K^{2,0} = \begin{pmatrix} 30 & -30 \\ -30 & 30 \end{pmatrix}$.

(iii) La matriu de connectivitat és, $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, amb la qual cosa, acoplant les matrius de rigidesa locals K^1 i K^2 s'obté, per a la matriu de rigidesa global, K:

$$K = \begin{pmatrix} 12 & -12 & \\ -12 & 42 & -30 \\ & -30 & 30 \end{pmatrix},\ \text{i per al sistema acoplat:}\ \begin{pmatrix} 12 & -12 & \\ -12 & 42 & -30 \\ & -30 & 30 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} Q_1 = Q_1^1 \\ Q_2 = Q_1^2 + Q_2^1 \\ Q_3 = Q_2^2 \end{pmatrix}$$

(iv) Boundary conditions (B.C.)

- Essential (sobre les variables primàries, les U's): $U_1 = 0$
- Natural (" " " secundàries, les Q's): $Q_2 = Q_1^2 + Q_2^1 = 0$, $Q_3 = Q_2^2 = F - k_s U_3$

Remarca. Notem que —a diferència del problema 10— aquí la molla està 'initially at rest' (inicialment relaxada). Per tant, la força de reacció de la molla estarà dirigida cap a l'esquerra si $U_3 > 0$, o cap a la dreta si $U_3 < 0$. Aleshores, la força de recuperació de la molla actuant sobre el node (global) 3 s'escriurà com: $F_R = -kU_3$. Així, la força puntual total que actua sobre el node global 3 serà $Q_3 = F - k_s U_3$.

(v) Sistema reduït: $\begin{pmatrix} 42 & -30 \\ -30 & 30 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 12 - 12U_3 \end{pmatrix} \Leftrightarrow \begin{matrix} 42U_2 - 30U_3 = 0 \\ -30U_2 + 30U_3 = 12 - 12U_3 \end{matrix} \Leftrightarrow \begin{matrix} 7U_2 - 5U_3 = 0 \\ -5U_2 + 7U_3 = 2 \end{matrix}$

solució: $U_3 = \frac{7}{5}U_2$, $-5U_2 + \frac{49}{5}U_2 = \frac{24}{5}U_2 = 2 \Leftrightarrow \boxed{U_2 = \frac{5}{12}}$, $\boxed{U_3 = \frac{7}{5}\cdot U_2 = \frac{7}{5}\cdot\frac{5}{12} = \frac{7}{12}}$. ▷