EX_PARCIAL_Q1_2016_17_Problema_3.pdf

Parcial Q1 - 2016-17. Problema 3

We want to study the problem

$$\left.\begin{aligned} -\frac{d}{dx}\left((x^2+1)\frac{du}{dx}\right) &= 1 \\ u(0) = 0, \quad \left((x^2+1)\frac{du}{dx}\right)(2) &= 1 \end{aligned}\right\}$$

using FEM with 2 linear elements: $\Omega^1 = [0,1]$ and $\Omega^2 = [1,2]$.

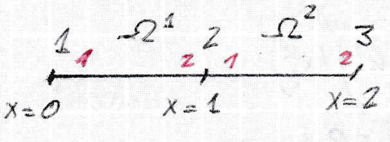
(a) Give the explicit expressions for the shape functions for each element $\psi_j^k(x)$, $j = 1,2$

(b) Write the system of equations associated to each element.

(c) Using the usual 1D lineal and global enumeration of nodes, write the connectivity matrix.

(d) Assemble the global system and compute the results for the primary variable in all nodes and compute also the secondary variable en $x=0$.

◁ Solució.




(a) $\psi_1^1(x) = \dfrac{x-1}{-1} = 1-x$, $\psi_2^1(x) = \dfrac{x}{1} = x$,

$\psi_1^2(x) = \dfrac{x-2}{1-2} = 2-x$, $\psi_2^2(x) = \dfrac{x-1}{2-1} = x-1$.

(b) $K_{1,1}^{1,1a} = \displaystyle\int_0^1 x^2 \underbrace{\frac{d\psi_1^1}{dx}\cdot\frac{d\psi_1^1}{dx}}_{=1} = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} = K_{22}^{1,1a}$.

$\Omega^1$:

Aquesta ve de la part ctnt, 1 en $x^2+1$

$K_{1,2}^{1,1a} = -\dfrac{1}{3} = K_{2,1}^{1,1a}$, d'on: $K^1 = \dfrac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \underbrace{\dfrac{1}{1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} = \dfrac{4}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$F^1 = \dfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ [Vegeu T2-MN-FEM2D.pdf pàg. 56, amb $f^k = 1$, $h_k = 1$].

$\Omega^2$:

$K_{11}^{2,1a} = \displaystyle\int_1^2 x^2 \underbrace{\frac{d\psi_1^2}{dx}\frac{d\psi_1^2}{dx}}_{=1} dx = \int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = K_{22}^{2,1a}$

$$K^{2,1a}_{12} = K^{2,1a}_{21} = -7/3 \; ; \; \text{d'on } K^2 = 7/3 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 10 & -10 \\ -10 & 10 \end{pmatrix}$$

$$F^2 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- Equations associated to element $\Omega^1$: $\frac{1}{3} \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} \cdot \begin{pmatrix} U_1^1 \\ U_2^1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} Q_1^1 \\ Q_2^1 \end{pmatrix}$

- Equations associated to element $\Omega^2$: $\frac{1}{3} \begin{pmatrix} 10 & -10 \\ -10 & 10 \end{pmatrix} \cdot \begin{pmatrix} U_1^2 \\ U_2^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} Q_1^2 \\ Q_2^2 \end{pmatrix}$

on $U_1^1 = U_1$, $U_2^1 = U_1^2 = U_2$, $U_2^2 = U_2$

(c) Connectivity matrix $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

(d) Assembled system:

$$\frac{1}{3} \begin{pmatrix} 4 & -4 & \\ -4 & 14 & -10 \\ & -10 & 10 \end{pmatrix} \cdot \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} Q_1 = Q_1^1 \\ Q_2 = Q_2^1 + Q_1^2 \\ Q_3 = Q_2^2 \end{pmatrix}.$$

Boundary conditions:

- Essential: $U_1 = \underline{0}$;
- Natural: $Q_2 = Q_2^1 + Q_1^2 = +\left((x^2+1)u'\right)(x_2^1) - \left((x^2+1)u'\right)(x_1^2)$

$$= \left((x^2+1)u'\right)(1) - \left((x^2+1)u'\right)(1) = \underline{0}.$$

Remark: we could have set $Q_2 = 0$ outright, for there're not point forces applied at node 2.

$Q_3 = Q_2^2 = a_1(x_2^2)\, u'(x_2^2) = \left((x^2+1)u'\right)(2) = \underline{1}$ [Vegeu T2-MN-FEM2D-pdf pàg. 50].

Reduced system:

$$\left.\begin{aligned} 14\,U_2 - 10\,U_3 &= 3 \\ -10\,U_2 + 10\,U_3 &= 9/2 \end{aligned}\right\}$$

Solution:

$$4\,U_2 = 15/2 \Rightarrow U_2 = \boxed{\frac{15}{8}}$$

$$U_3 = \frac{1}{10}\left(14 U_2 - 3\right) = \frac{1}{10}\left(14 \frac{15}{8} - 3\right) = \boxed{\frac{93}{40}}$$

$$Q_1 = \frac{4}{3} U_1 - \frac{4}{3} U_2 + 0 \cdot U_3 - \frac{1}{2} = -\frac{4}{3} \cdot \frac{15}{8} - \frac{1}{2} = -\frac{5}{2} - \frac{1}{2} = \boxed{-3}. \triangleright$$