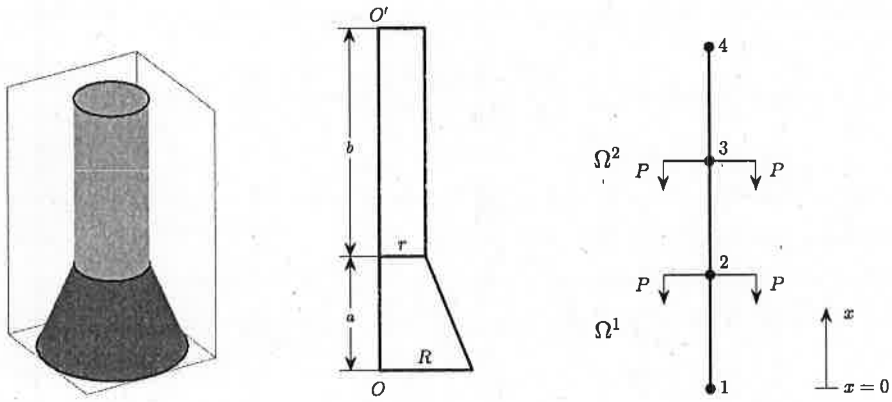
MÈTODES NUMÈRICS: **Ex-Final Q1-2018-19 (a)** (I)

1/3

Name and surnames:

1. We consider a pillar made of two pieces (see Fig. (a)). In some normalized units, the upper part (shaded in light grey) is a cylinder of length $b = 4$ and radius $r = 1$, and the lower part (in dark grey) is a truncated cone of height $a = 1$ whose area is given as a function of the height x by $A(x) = \pi\,(x-2)^2$, with $0 \leq x \leq 1$. The pillar is generated by revolving the half-section shown on Fig. (b) about the edge $\overline{OO'}$.



(a) (b) (c)

At the junction of the two pieces, and at the middle point of the cylinder (nodes 2 and 3 on Fig. (c)), it is applied a force downwards of $2P$, with $P = 2$. Furthermore, the pillar is fixed at its top and at its basis (nodes 1 and 4 on Fig. (c)).

We study this problem as a FEM1D problem using a mesh of two elements (see Fig.(c)). A **linear** element, Ω^1, for the truncated cone, and a **quadratic** element, Ω^2, for the cylinder. The model for 1D elasticity problems is

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(EA(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0,$$

where the Young modulus is taken $E = \dfrac{12}{\pi}$, in order to simplify the calculations.

(i) Give the two shape functions $\psi_1^1(x)$, $\psi_2^1(x)$ of element Ω^1.

(ii) Write the local stiffness matrices for both Ω^1 and Ω^2.

(iii) Write the assembled system.

(iv) Set up boundary conditions and compute the displacement of nodes 2 and 3.

(v) Finally, compute the reaction forces (Q_1 and Q_4).

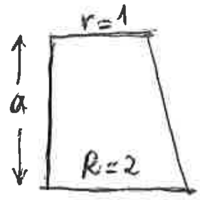
Results:	
$\psi_1^1(x)$ and $\psi_2^1(x)$:	$\psi_1^1(x) = 1 - x$, $\psi_2^1(x) = x$.
$[K^1]$ and $[K^2]$:	$[K^1] = 28\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $[K^2] = \begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}$.
Assembled system:	$\begin{pmatrix} 28 & -28 & 0 & 0 \\ -28 & 35 & -8 & 1 \\ 0 & -8 & 16 & -8 \\ 0 & 1 & -8 & 7 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}$
Displacement of nodes 2 and 3:	$\begin{pmatrix} 35 & -8 \\ -8 & 16 \end{pmatrix} \begin{pmatrix} U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$. Sol.: $\left\{ \begin{array}{l} U_2 = -\dfrac{6}{31} \\ U_3 = -\dfrac{43}{124} \end{array} \right\}$. $U_2 = -0'19354$ $U_3 = -0'346774$
Reaction forces (Q's):	$Q_1 = \dfrac{168}{31}$, $Q_4 = \dfrac{80}{31}$. $Q_1 = 5'419\,355$, $Q_4 = 2'580\,645$

(Hint. U_2 is one of these numbers: $-\frac{6}{11}, -\frac{6}{13}, -\frac{6}{17}, -\frac{6}{19}, -\frac{6}{23}, -\frac{6}{29}, -\frac{6}{31}, -\frac{6}{37}, -\frac{6}{42}, -\frac{6}{43}$) (2 points)

Problema 1

$b=4,\ r=1,\ R=2,\ E=\frac{\pi}{12}$. (I) $a=1$ (II) $a=2$

S.



$$r(x) = R - \frac{R-r}{a}x \longrightarrow A(x) = \pi\left(R-\frac{R-r}{a}x\right)^2,\quad 0\le x\le a$$

(i) $\psi_1^1(x) = 1-\frac{x}{a},\ \psi_2^1(x) = \frac{x}{a}$

(I): $\psi_1^1(x) = 1-x,\ \psi_2^1(x) = x$.

(II): $\psi_1^1(x) = 1-\frac{x}{2},\ \psi_2^1(x) = \frac{x}{2}$.

(ii)

$$K_{ij}^k = \int_{x_A^k}^{x_B^k} EA(x)\,\frac{d\psi_i^k}{dx}(x)\cdot\frac{d\psi_j^k}{dx}(x)\,dx:$$

$$K_{11}^1 = \frac{E\pi}{a^2}\int_0^a\left(R-\frac{R-r}{a}x\right)^2dx = \frac{E\pi}{a^2}\,\frac{-a}{R-r}\cdot\frac{1}{3}\left[\left(R-\frac{R-r}{a}x\right)^3\right]_{x=0}^{x=a} = \frac{E\pi}{3a}\,\frac{R^3-r^3}{R-r} = \frac{E\pi}{3a}\left(r^2+rR+R^2\right)$$

$$\underset{E=\frac{12}{\pi},\,r=1,\,R=2}{=\ldots=} \frac{28}{a}.$$ Hence: $K^1 = \frac{28}{a}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$. On the other hand: $K^2 = \frac{E\pi r^2}{3b}\begin{pmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{pmatrix} \underset{\substack{b=4\\r=1\\E=\frac{12}{\pi}}}{=} \begin{pmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{pmatrix}$

(I): $K^1 = 28\begin{pmatrix}1&-1\\-1&1\end{pmatrix},\ K^2 = \begin{pmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{pmatrix}$ (II): $K^1 = 14\begin{pmatrix}1&-1\\-1&1\end{pmatrix},\ K^2 = \begin{pmatrix}7&-8&1\\-8&16&-8\\1&-8&7\end{pmatrix}$

(iii)

(I): $\begin{pmatrix}28&-28&0&0\\-28&35&-8&1\\0&-8&16&-8\\0&1&-8&7\end{pmatrix}\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix} = \begin{pmatrix}Q_1\\Q_2\\Q_3\\Q_4\end{pmatrix}$, (II): $\begin{pmatrix}14&-14&0&0\\-14&14&-8&1\\0&-8&16&-8\\0&1&-8&7\end{pmatrix}\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix} = \begin{pmatrix}Q_1\\Q_2\\Q_3\\Q_4\end{pmatrix}$

(iv)

(I): $\begin{pmatrix}35&-8\\-8&16\end{pmatrix}\begin{pmatrix}U_2\\U_3\end{pmatrix} = \begin{pmatrix}-2P=-4\\-2P=-4\end{pmatrix}$ $\Delta = 35\cdot16-64 = 560-64 = 496,$

$\Delta_{U_2} = \begin{vmatrix}-4&-8\\-4&16\end{vmatrix} = -64-32 = -96,\ \Delta_{U_3} = \begin{vmatrix}35&-4\\-8&-4\end{vmatrix} = -140-32 = -172$

$U_2 = -\frac{96}{496} = \boxed{-\frac{6}{31}},\ U_3 = -\frac{172}{496} = \boxed{-\frac{43}{124}}$

(II): $\begin{pmatrix}21&-8\\-8&16\end{pmatrix}\begin{pmatrix}U_2\\U_3\end{pmatrix} = \begin{pmatrix}-4\\-4\end{pmatrix}$, $\Delta = 21\cdot16-64 = 336-64 = 272$.

$\Delta_{U_2} = \begin{vmatrix}-4&-8\\-4&16\end{vmatrix} = -64-32 = -96,\ \Delta_{U_3} = \begin{vmatrix}21&-4\\-8&-4\end{vmatrix} = -84-32 = -116$

$U_2 = -\frac{96}{272} = -\frac{6}{17},\ U_3 = -\frac{116}{272} = -\frac{29}{68}$.

(v) (I): $Q_1 = -28\,U_2 = -28\cdot\left(-\frac{6}{31}\right) = \boxed{\frac{168}{31}},\ Q_4 = U_2-8U_3 = -\frac{6}{31}+\frac{86}{31} = \boxed{\frac{80}{31}}$

(II): $Q_1 = -14\cdot U_2 = -14\cdot\left(-\frac{6}{17}\right) = -\frac{84}{17},\ Q_4 = U_2-8U_3 = -\frac{6}{17}+\frac{58}{17} = \frac{52}{17}$.