problema_1.pdf

1. Donada l'equació $-\frac{d}{dx}\left(a_1(x)\frac{du}{dx}\right)+a_0(x)u=f(x)$, $0<x<1$, prenem una malla d'elements finits lineals $\Omega^1,\dots,\Omega^N$ amb $\Omega^k=[x_k, x_{k+1}]$, $h_k=x_{k+1}-x_k$, $a_1(x)=a_1^k x$, $a_0(x)=a_0^k$, i f^k són constants que depenen de cada element Ω^k. Vegeu que les matrius de rigidesa elementals i els vectors de càrrega elementals són:

$$[K^k]=\frac{a_1^k}{h_k}\left(\frac{x_k+x_{k+1}}{2}\right)\cdot\begin{pmatrix}1&-1\\-1&1\end{pmatrix}+\frac{a_0^k h_k}{6}\begin{pmatrix}2&1\\1&2\end{pmatrix},\quad f^k=\frac{1}{2}f_k h_k\begin{pmatrix}1\\1\end{pmatrix}.$$

◁ Solució. Suposem que $\Omega^k=[x_k, x_{k-1}]$, $k=0,1,2,\dots,N$. Les funcions de forma són $\psi_1^k(x)=\frac{x-x_{k+1}}{x_k-x_{k+1}}=-\frac{1}{h_k}(x-x_{k+1})$, $\psi_2^k(x)=\frac{x-x_k}{x_{k+1}-x_k}=\frac{1}{h_k}(x-x_k)$.

Llavors:

$$K_{1,1}^{k,1}=\int_{x_k}^{x_{k+1}}a_1(x)\overbrace{\frac{d\psi_1^k}{dx}}^{-1/h_k}(x)\cdot\frac{d\psi_1^k}{dx}(x)\,dx=\int_{x_k}^{x_{k+1}}a_1^k x\,\frac{dx}{h_k}=\frac{a_1^k}{2h_k^2}\left(x_{k+1}^2-x_k^2\right)$$

$$=a_1^k\,\frac{x_{k+1}^2-x_k^2}{2(x_{k+1}-x_k)^2}=\boxed{\frac{a_1^k}{2h_k}(x_{k+1}+x_k)=K_{22}^{k,1}}\quad(*)$$

$$K_{1,2}^{k,1}=\int_{x_k}^{x_{k+1}}a_1(x)\frac{d\psi_1^k}{dx}(x)\frac{d\psi_2^k}{dx}(x)\,dx=-\int_{x_k}^{x_{k+1}}a_1^k x\,\frac{dx}{h_k^2}=\boxed{-\frac{1}{2h_k}(x_{k+1}+x_k)=K_{21}^{k,1}}$$

(*) Notem que: $\frac{d\psi_2^k}{dx}\cdot\frac{d\psi_2^k}{dx}=\frac{1}{h_k^2}=\frac{d\psi_1^k}{dx}\cdot\frac{d\psi_1^k}{dx}$

Aleshores:

$$[K^{k,1}]=\frac{1}{2h_k}(x_{k+1}+x_k)\cdot\begin{pmatrix}1&-1\\-1&1\end{pmatrix},$$

D'altra banda:

$$K_{11}^{k,0}=\int_{x_k}^{x_{k+1}}a_0(x)\,\psi_1^k(x)\,\psi_1^k(x)\,dx=\frac{a_0^k}{h_k^2}\int_{x_k}^{x_{k+1}}(x-x_{k+1})^2\,dx=\frac{a_0^k}{3h_k^2}h_k^3=\boxed{\frac{a_0^k h_k}{3}}$$

$$K_{12}^{k,0}=\int_{x_k}^{x_{k+1}}a_0(x)\psi_1^k(x)\,\psi_2^k(x)\,dx=-\frac{a_0^k}{h_k^2}\int_{x_k}^{x_{k+1}}(x-x_{k+1})(x-x_k)\,dx$$

$$=\left\{\begin{array}{l}\text{c.v.}\\ s=x-x_k\\ \Leftrightarrow x=s+x_k\end{array}\right\}=-\frac{a_0^k}{h_k^2}\int_0^{h_k}(s-h_k)\,s=-\frac{a_0^k}{h_k^2}\left(\frac{s^3}{3}-h_k s\right)\Bigg|_{s=0}^{s=h_k}$$

$$=\boxed{\frac{2a_0^k h}{3}=K_{21}^k}$$

$$K_{22}^{k,0}=\int_{x_k}^{x_{k+1}}a_0(x)\,\psi_2^k(x)\,\psi_2(x)\,dx=\frac{a_0^k}{h_k^2}\int_{x_k}^{x_{k+1}}(x-x_k)^2\,dx=\frac{a_0^k}{3h_k^2}h_k^3=\boxed{\frac{a_0^k h_k}{3}}$$

Aleshores:

$$[K^{k,0}] = \frac{a_0^k h_k}{6}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Per últim:

$$F_1^K = \int_{x_k}^{x_{k+1}} f(x)\,\psi_1^k(x)\,dx = \int_{x_k}^{x_{k+1}} f^k \cdot \frac{x - x_{k+1}}{x_k - x_{k+1}}\,dx = -\frac{f^k}{2h_k}(x - x_{k+1})^2\Big|_{x_k}^{x_{k+1}} = \frac{f^k h_k}{2}$$

$$F_2^K = \int_{x_k}^{x_{k+1}} f(x)\,\psi_2^k(x)\,dx = \int_{x_k}^{x_{k+1}} f^k\,\frac{x - x_k}{x_{k+1} - x_k}\,dx = \frac{f^k}{2h_k}(x - x_k)^2\Big|_{x_k}^{x_{k+1}} = \frac{f^k h_k}{2}$$

Aleshores:

$$F^K = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

La matriu de rigidesa i el vector de càrrega elementals de l'element Ω^K són doncs:

$$[K^K] = [K^{k,1}] + [K^{k,0}] = \frac{a_1^k}{h_k}\left(\frac{x_k + x_{k+1}}{2}\right)\cdot\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{a_0^k h_k}{6}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$F^K = \frac{1}{2} f^k h_k \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \triangleright$$