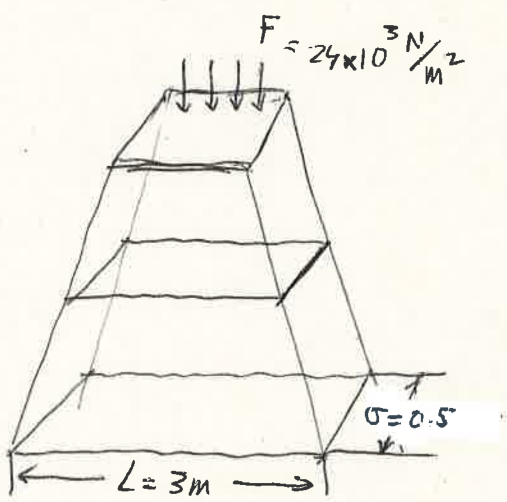
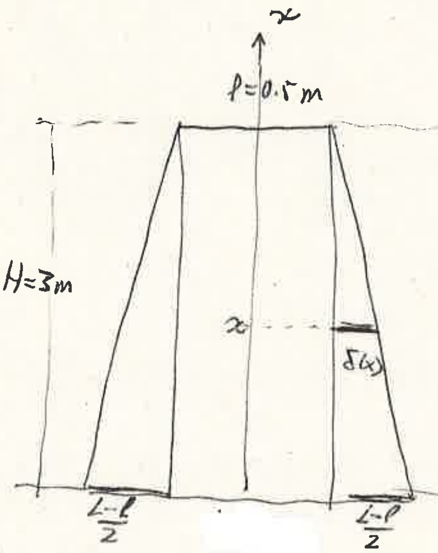
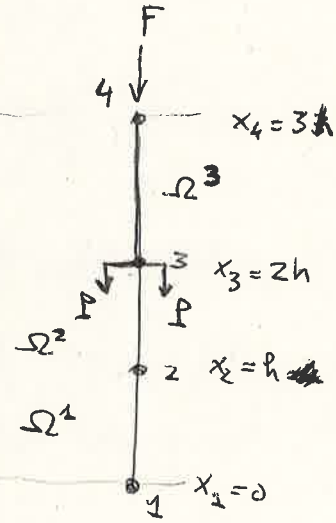
Example 2. Linear Elasticity 1D. Concrete pyramidal column with variable section area







$P = 2\times10^3\ N$, $h = 1\ m$, $E = 28\times10^9\ N/m^2$, $w = 25\times10^3\ N/m^3$

$$-\frac{d}{dx}\left(EA(x)\frac{du}{dx}\right) = f(x) \equiv \text{pes per unitat de longitud } f(x) = \frac{dW}{dx}$$

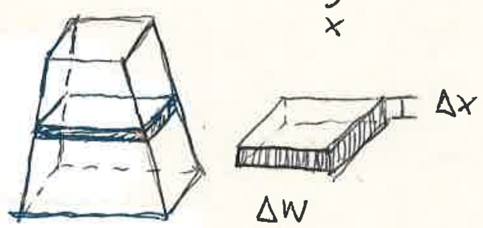
$a_0(x) = 0 \ \forall\ 0 \le x \le 3 \qquad a_1(x) = EA(x),\ f(x) = \frac{dW}{dx}$

$$\frac{\delta(x)}{H-x} = \frac{\frac{L-l}{2}}{H} \Leftrightarrow \delta(x) = \frac{L-l}{2H}(H-x) = \frac{\frac{3}{2}-\frac{1}{2}}{2\times3}(3-x) = \frac{1}{6}(3-x)$$

$$A(x) = \sigma\left(l + 2\delta(x)\right) = \frac{1}{2}\left(\frac{1}{2} + 1 - \frac{x}{3}\right) = \frac{3}{4} - \frac{x}{6}$$

$$a_1(x) = EA(x) = E\left(\frac{3}{4} - \frac{x}{3}\right)$$

Weight: $W(x) = \int_x^H w A(x)\, dx$, so: $f(x) = \frac{dW}{dx} = -w \cdot A(x) = -w\left(\frac{3}{4} - \frac{x}{3}\right)$



$\Delta W = w\, A(x) \cdot \Delta x$

Prob 1 (and also practice 2.2).

$$a_1(x) = \alpha x :\ K^{e,1} = \frac{\alpha}{h_e}\left(\frac{x_1^e + x_2^e}{2}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Theory notes:

$$a_1(x) = \beta \equiv ctnt :\ K^{e,1} = \frac{\beta}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Therefore; if $a_1(x) = \alpha x + \beta = \frac{\alpha}{h_e}\left(\frac{x_1^e + x_2^e}{2}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{\beta}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

In our problem: $\alpha = -E/6$, $\beta = \frac{3}{4}E$, $h_e = 1$. Hence:

$$K^1 = -\frac{E}{6}\left(\frac{0+1}{2}\right)\begin{pmatrix}1&-1\\-1&1\end{pmatrix} + \frac{3}{4}E\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \frac{2}{3}E\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \frac{E}{6}\begin{pmatrix}4&-4\\-4&4\end{pmatrix}$$

$$k^2 = -\frac{E}{6}\left(\frac{1+2}{2}\right)\begin{pmatrix}1&-1\\-1&1\end{pmatrix} + \frac{3}{4}E\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \frac{1}{2}E\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \frac{E}{6}\begin{pmatrix}3&-3\\-3&3\end{pmatrix}$$

$$k^3 = -\frac{E}{6}\left(\frac{2+3}{2}\right)\begin{pmatrix}1&-1\\-1&1\end{pmatrix} + \frac{3}{4}E\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \frac{1}{3}E\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \frac{E}{6}\begin{pmatrix}2&-2\\-2&2\end{pmatrix}$$

$$f(x) = \frac{3}{4}W - \frac{W}{6}x$$

Practice 2.2: If $f(x) = ax$, then $F^e = \frac{ah_e}{6}\begin{pmatrix}2x_1^e + x_2^e\\ x_1^e + 2x_2^e\end{pmatrix}$

Theory notes: If $f(x) = b \equiv ctnt$, then $F^e = \frac{bh_e}{2}\begin{pmatrix}1\\1\end{pmatrix}$

Hence, if $f(x) = ax+b$, then: $F^e = \frac{ah_e}{6}\begin{pmatrix}2x_1^e + x_2^e\\ x_1^e + 2x_2^e\end{pmatrix} + \frac{bh^e}{2}\begin{pmatrix}1\\1\end{pmatrix}$

In this case in point $a = W/6$, $b = -\frac{3}{4}W$, $h_e = 1$. Therefore:

$$F^1 = \frac{W}{36}\begin{pmatrix}1\\2\end{pmatrix} - \frac{3}{8}W\begin{pmatrix}1\\1\end{pmatrix} = W\begin{pmatrix}\frac{1}{36}-\frac{3}{8}\\ \frac{2}{36}-\frac{3}{8}\end{pmatrix} = -\frac{W}{72}\begin{pmatrix}25\\33\end{pmatrix}$$

$$F^2 = \frac{W}{36}\begin{pmatrix}4\\5\end{pmatrix} - \frac{3}{8}W\begin{pmatrix}1\\1\end{pmatrix} = W\begin{pmatrix}3/36-3/8\\ 5/36-3/8\end{pmatrix} = -\frac{W}{72}\begin{pmatrix}19\\17\end{pmatrix}$$

$$F^3 = \frac{W}{36}\begin{pmatrix}7\\8\end{pmatrix} - \frac{3}{8}W\begin{pmatrix}1\\1\end{pmatrix} = W\begin{pmatrix}7/36-3/8\\ 8/36-3/8\end{pmatrix} = -\frac{W}{72}\begin{pmatrix}13\\11\end{pmatrix}$$

Sistema acoplat.

$$\frac{E}{6}\begin{pmatrix}4&-4&&\\-4&7&-3&\\&-3&5&-2\\&&-2&2\end{pmatrix}\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix} = -\frac{W}{72}\begin{pmatrix}25\\42\\30\\11\end{pmatrix} + \begin{pmatrix}Q_1\\Q_2\\Q_3\\Q_4\end{pmatrix}$$

Natural B.C. $Q_2 = Q_1^2 + Q_2^1 = 0$,

$Q_3 = Q_2^2 + Q_1^3 = -2P = -4\times10^3\,N$

$Q_4 = Q_3^2 = -24\times10^3 \times \underbrace{A(3)}_{1/4} = -6\times10^3\,N$

Essential B.C. $U_1 = 0$.

Reduced system:

$$\frac{E}{6}\begin{pmatrix} 7 & -3 & \\ -3 & 5 & -2 \\ & -2 & 2 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \\ U_4 \end{pmatrix} = -\frac{w}{72}\begin{pmatrix} 42 \\ 30 \\ 11 \end{pmatrix} - 10^3\begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} - \frac{E}{6}\overset{\overset{0}{\shortparallel}}{\bar{U}_1}\begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 7 & -3 & \\ -3 & 5 & -2 \\ & -2 & 2 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \\ U_4 \end{pmatrix} = -\frac{W}{12E}\begin{pmatrix} 42 \\ 30 \\ 11 \end{pmatrix} - \frac{6\times 10^3}{E}\begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} \qquad (*)$$

with $w = 29\times 10^3 \, \frac{N}{m^3}$, $E = 28\times 10^9 \, \frac{N}{m^2}$

Solution of $(*)$

$$\boxed{\begin{aligned} U_2 &= -2.0796\times 10^{-6} \, m \\ U_3 &= -3.8108\times 10^{-6} \, m \\ U_4 &= -4.8628\times 10^{-6} \, m \end{aligned}}$$

Post-process: $Q_1 = \frac{E}{6}\left(4\bar{U}_1 - 4U_2\right) + \frac{25}{72}w = \boxed{4.75\times 10^4 \, N}$

Reaction force.