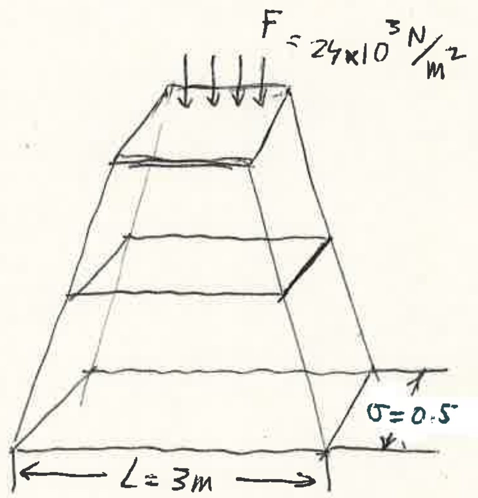
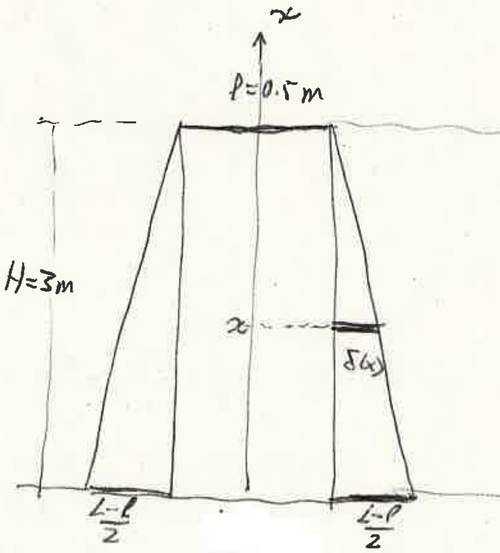
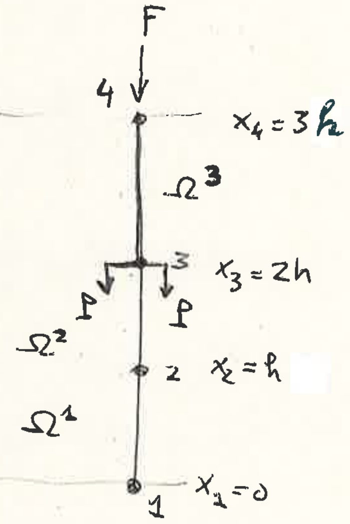
Example 2. Linear Elasticity 1D. Concrete pyramidal column with variable section area







$P = 2\times10^3\ N$, $h = 1\ m$, $E = 28\times10^9\ N/m^2$, $w = 25\times10^3\ N/m^3$

$$-\frac{d}{dx}\left(EA(x)\frac{du}{dx}\right) = f(x) \equiv \text{pes per unitat de longitud} \quad f(x) = \frac{dW}{dx}$$

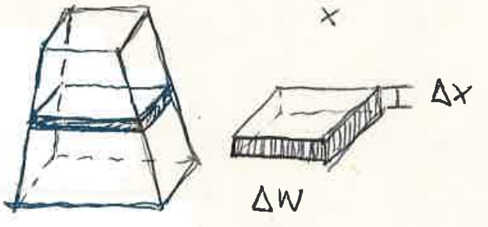
$a_0(x) = 0 \ \forall \ 0 \le x \le 3 \qquad a_1(x) = EA(x), \ f(x) = \frac{dW}{dx}$

$$\frac{\delta(x)}{H-x} = \frac{\frac{L-\ell}{2}}{H} \Leftrightarrow \delta(x) = \frac{L-\ell}{2H}(H-x) = \frac{\frac{3}{2}-\frac{1}{2}}{2\times3}(3-x) = \frac{1}{6}(3-x)$$

$$A(x) = \sigma\left(\ell + 2\delta(x)\right) = \frac{1}{2}\left(\frac{1}{2} + 1 - \frac{x}{3}\right) = \frac{3}{4} - \frac{x}{6}$$

$$a_1(x) = EA(x) = E\left(\frac{3}{4} - \frac{x}{3}\right)$$

Weight: $W(x) = \int_x^H w A(x)\, dx$, so: $f(x) = \frac{dW}{dx} = -w \cdot A(x) = -w\left(\frac{3}{4} - \frac{x}{3}\right)$



$\Delta W = w\, A(x) \cdot \Delta x$

Prob 1 (and also practice 2.2)

$a_1(x) = \alpha x$: $K^{e,1} = \frac{\alpha}{h_e}\left(\frac{x_1^e + x_2^e}{2}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Theory notes:

$a_1(x) = \beta \equiv \text{ctnt}$: $K^{e,1} = \frac{\beta}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Therefore; if $a_1(x) = \alpha x + \beta$, then $K^{e,1} = \frac{\alpha}{h_e}\left(\frac{x_1^e + x_2^e}{2}\right)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{\beta}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (exercise)

In our problem: $\alpha = -\frac{E}{6}$, $\beta = \frac{3}{4}E$, $h_e = 1$. Hence:

$$K^1 = -\frac{E}{6}\left(\frac{0+1}{2}\right)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \frac{3}{4}E\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{2}{3}E\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{E}{6}\begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}$$

$$K^2 = -\frac{E}{6}\left(\frac{1+2}{2}\right)\cdot\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \frac{3}{4}E\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{2}E\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{E}{6}\begin{pmatrix}3 & -3\\ -3 & 3\end{pmatrix}$$

$$K^3 = -\frac{E}{6}\left(\frac{2+3}{2}\right)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + \frac{3}{4}E\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{3}E\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{E}{6}\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix}$$

$$f(x) = -\frac{3}{4}W + \frac{W}{3}x$$

Practice 2.2: If $f(x) = ax$, then $F^e = \frac{ah_e}{6}\begin{pmatrix}2x_1^e + x_2^e\\ x_1^e + 2x_2^e\end{pmatrix}$

Theory notes: If $f(x) = b \equiv ct$, then $F^e = \frac{bh_e}{2}\begin{pmatrix}1\\ 1\end{pmatrix}$

Hence, if $f(x) = ax+b$, then: $F^e = \frac{ah_e}{6}\begin{pmatrix}2x_1^e + x_2^e\\ x_1^e + 2x_2^e\end{pmatrix} + \frac{bh_e}{2}\begin{pmatrix}1\\ 1\end{pmatrix}$ (exercise)

In this case in point $a = W/3$, $b = -\frac{3}{4}W$, $h_e = 1$. Therefore:

$$F^1 = \frac{W}{36}\begin{pmatrix}1\\ 2\end{pmatrix} - \frac{3}{8}W\begin{pmatrix}1\\ 1\end{pmatrix} = W\begin{pmatrix}\frac{1}{36} - \frac{3}{8}\\ \frac{2}{36} - \frac{3}{8}\end{pmatrix} = -\frac{W}{72}\begin{pmatrix}25\\ 33\end{pmatrix}$$

$$F^2 = \frac{W}{36}\begin{pmatrix}4\\ 5\end{pmatrix} - \frac{3}{8}W\begin{pmatrix}1\\ 1\end{pmatrix} = W\begin{pmatrix}3/36 - 3/8\\ 5/36 - 3/8\end{pmatrix} = -\frac{W}{72}\begin{pmatrix}19\\ 17\end{pmatrix}$$

$$F^3 = \frac{W}{36}\begin{pmatrix}7\\ 8\end{pmatrix} - \frac{3}{8}W\begin{pmatrix}1\\ 1\end{pmatrix} = W\begin{pmatrix}7/36 - 3/8\\ 8/36 - 3/8\end{pmatrix} = -\frac{W}{72}\begin{pmatrix}13\\ 11\end{pmatrix}$$

Sistema acoplat.

$$\frac{E}{6}\begin{pmatrix}4 & -4 & & \\ -4 & 7 & -3 & \\ & -3 & 5 & -2\\ & & -2 & 2\end{pmatrix}\begin{pmatrix}U_1\\ U_2\\ U_3\\ U_4\end{pmatrix} = -\frac{W}{72}\begin{pmatrix}25\\ 42\\ 30\\ 11\end{pmatrix} + \begin{pmatrix}Q_1\\ Q_2\\ Q_3\\ Q_4\end{pmatrix}$$

Natural B.C. $Q_2 = Q_1^2 + Q_2^1 = 0$,

$$Q_3 = Q_2^2 + Q_1^3 = -2P = -4\times10^3\,N$$

$$Q_4 = Q_3^2 = -24\times10^3 \times \underbrace{A(3)}_{1/4} = -6\times10^3\,N$$

Essential B.C. $U_1 = 0$.

Reduced system:

$$\frac{E}{6}\begin{pmatrix} 7 & -3 & \\ -3 & 5 & -2 \\ & -2 & 2 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \\ U_4 \end{pmatrix} = -\frac{W}{72}\begin{pmatrix} 42 \\ 30 \\ 11 \end{pmatrix} - 10^3\begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} - \frac{E}{6}\overset{0}{U_1}\begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 7 & -3 & \\ -3 & 5 & -2 \\ & -2 & 2 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \\ U_4 \end{pmatrix} = -\frac{W}{12E}\begin{pmatrix} 42 \\ 30 \\ 11 \end{pmatrix} - \frac{6\times10^3}{E}\begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} \qquad (*)$$

with $W = 29\times10^3 \, \frac{N}{m^3}$, $E = 28\times10^7 \, \frac{N}{m^2}$

Solution of $(*)$

$$\boxed{\begin{aligned} U_2 &= -2.0796\times10^{-6} \text{ m} \\ U_3 &= -3.8108\times10^{-6} \text{ m} \\ U_4 &= -4.8628\times10^{-6} \text{ m} \end{aligned}}$$

Post-process: $Q_1 = \frac{E}{6}\left(4U_1 - 4U_2\right) + \frac{25}{72}W = \boxed{4.75\times10^4 \text{ N}}$

Reaction force.