1

MÈTODES NUMÈRICS: **Ex-Parcial Q2-2018-19** (a)

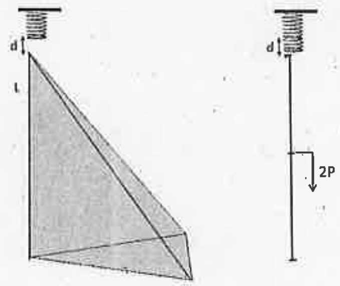
Name and surnames:

(1) Consider the piece of piramidal shape with triangular section as in the figure (on the left). Its hight is $L = 2mm$ and the triangle in its base has area $A_0\, mm^2$. It is made of a material whose Young modulus is $E\,\frac{N}{mm^2}$ and we consider it fixed on the bottom.

Over the piece, at distance d in mm, there is a spring of constant $K\,\frac{N}{mm}$ fixed at the upper end.

The piece is deformed as a consequence of the forces on it caused because we attach the piece to the spring and we put a load $2\,P$ in its midpoint (figure on the right).

We study this problem discretizing the piece in **two linear** 1-dimensional finite elements of equal length: Ω^1 at the lower part of the piece and Ω^2 at the upper part. We number the nodes from the bottom (where we take the origin $x = 0$).



Compute:

(a) The shape functions $\psi_1^k(x)$, $\psi_2^k(x)$, of each element Ω^k. $k = 1, 2$.

(b) Write the local stiffness matrices for both Ω^1 and Ω^2.

(c) Write the assembled system.

(d) Set up boundary conditions

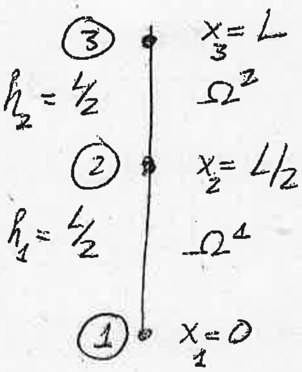
(e) The displacement of the nodes 2 and 3 of the piece when $A_0=12$, $E=10$, $K=10$, $d=0.5$, and $P=30$

Note: You can use that:

- The stregth produced by a string is $-k\,e$, where e is the extension of the spring.

Ex-Parcial Q2-2018-19. Exercise 1.

(a) Shape functions:



$L=2$

$$\psi_1^1(x) = \frac{x-x_2}{x_1-x_2} = \frac{2}{L}\left(\frac{L}{2}-x\right) = 1-x \Rightarrow \frac{d\psi_1^1}{dx} = -1$$

$$\psi_2^1(x) = \frac{x-x_1}{x_2-x_1} = \frac{2}{L}(x-0) = x \Rightarrow \frac{d\psi_2^1}{dx} = 1$$

$$\psi_1^2(x) = \frac{x-x_3}{x_2-x_3} = \frac{2}{L}(L-x) = 2-x \Rightarrow \frac{d\psi_1^2}{dx} = -1$$

$$\psi_2^2(x) = \frac{x-x_2}{x_3-x_2} = \frac{2}{L}\left(x-\frac{L}{2}\right) = x-1 \Rightarrow \frac{d\psi_2^2}{dx} = 1$$

(b) Equation of the elasticity: $-\frac{d}{dx}\left(A(x)E(x)\frac{du}{dx}\right) = f(x)$

Here: $A(x) = \alpha(x-L)^2$, $A(0) = \alpha(0-L)^2 = \alpha L^2 = A_0 \Rightarrow \alpha = \frac{A_0}{L^2} \overset{L=2}{=} \frac{A_0}{4}$. Hence $A(x) = \frac{A_0}{4}(x-2)^2$

$E(x) \equiv E$ constant $\forall\, 0 \le x \le L = 2$

$f(x) \equiv 0$ constant $\forall\, 0 \le x \le L = 2$ (neither weight nor other global forces are considered).

So, the corresponding coefficients on the model equation should be $a_1(x) = \frac{A_0 E}{4}(x-2)^2$

$a_0(x) = 0 \;\forall\, 0 \le x \le L = 2$, $f(x) = 0 \;\forall\, 0 \le x \le L = 2$.

$$K_{11}^{1,1} = \int_{x_1^1}^{x_2^1} a_1(x)\frac{d\psi_1^1}{dx}\frac{d\psi_1^1}{dx}dx = \int_0^{L/2=1} \frac{A_0E}{4}(x-2)^2(-1)\cdot(-1)\,dx = \frac{A_0E}{4}\underbrace{\int_0^1 (x-2)^2 dx}_{=7/3}$$

$$= \frac{A_0E}{4}\left[\frac{(x-2)^3}{3}\right]_{x=0}^{x=1} = \frac{A_0E}{4}\left(-\frac{1}{3}+\frac{8}{3}\right) = \frac{7A_0E}{12}$$

Notació: $x_i^e \equiv$ posició del node $i=1,2$ de l'element $e=1,2$.

$$K_{12}^{1,1} = \int_{x_1^1}^{x_2^1} a_1(x)\frac{d\psi_1^1}{dx}\cdot\frac{d\psi_2^1}{dx}dx = \int_0^{L/2=1} \frac{A_0E}{4}(x-2)^2(-1)\cdot(1)\,dx = -\frac{A_0E}{4}\int_0^1 (x-2)^2 dx = -\frac{7A_0E}{12}$$

$$K_{21}^{1,1} = \int_{x_1^1}^{x_2^1} a_1(x)\frac{d\psi_2^1}{dx}\cdot\frac{d\psi_1^1}{dx}dx = \int_0^{L/2=1} \frac{A_0E}{4}(x-2)^2(1)\cdot(-1)\,dx = -\frac{A_0E}{4}\int_0^1 (x-2)^2 dx = -\frac{7A_0E}{12}$$

$$K_{22}^{1,1} = \int_{x_1^1}^{x_2^1} a_1(x)\frac{d\psi_2^1}{dx}\cdot\frac{d\psi_2^1}{dx}dx = \int_0^{L/2=1} \frac{A_0E}{4}(x-2)^2(1)\cdot(1)\,dx = \frac{A_0E}{4}\int_0^1 (x-2)^2 dx = \frac{7A_0E}{12}$$

On the other hand: $K_{ij}^{1,0} = 0 \;\forall\, i,j = 1,2$, since $a_0(x) = 0 \;\forall\, 0 \le x \le L/2 = 1$. Therefore.

$$\boxed{K^1 = K^{1,1} = \frac{7A_0E}{12}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}.$$

Let us compute now the components of $K^{2,1}$:

$$K^{2,1}_{11} = \int_{x_1^2}^{x_2^2} a_1(x)\,\frac{d\psi_1^2}{dx}\cdot\frac{d\psi_1^2}{dx}\,dx = \int_{L/2=1}^{L=2} \frac{A_0E}{4}(x-2)^2(-1)(-1)\,dx = \frac{A_0E}{4}\int_1^2 (x-2)^2 dx \overset{(*)}{=} \frac{A_0E}{12}$$

$$(*)\quad \int_1^2 (x-2)^2 dx = \left[\frac{(x-2)^3}{3}\right]_1^2 = \frac{1}{3}$$

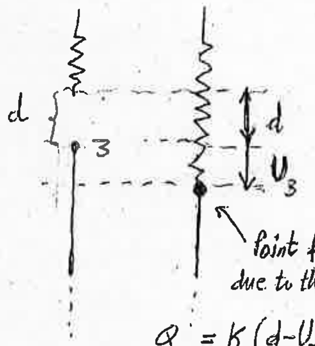
$$K^{2,1}_{12} = \int_{x_1^2}^{x_2^2} a_1(x)\,\frac{d\psi_1^2}{dx}\cdot\frac{d\psi_2^2}{dx}\,dx = \int_{L/2=1}^{L=2} \frac{A_0E}{4}(x-2)^2(-1)(1)\,dx = -\frac{A_0E}{4}\int_1^2 (x-2)^2 dx = -\frac{A_0E}{12}$$

$$K^{2,1}_{21} = \int_{x_1^2}^{x_2^2} a_1(x)\,\frac{d\psi_2^2}{dx}\cdot\frac{d\psi_1^2}{dx}\,dx = \int_{L/2=1}^{L=2} \frac{A_0E}{4}(x-2)^2(1)(-1)\,dx = -\frac{A_0E}{4}\int_1^2 (x-2)^2 dx = -\frac{A_0E}{12}$$

$$K^{2,1}_{22} = \int_{x_1^2}^{x_2^2} a_1(x)\,\frac{d\psi_2^2}{dx}\cdot\frac{d\psi_2^2}{dx}\,dx = \int_{L/2=1}^{L=2} \frac{A_0E}{4}(x-2)^2(1)\cdot(1)\,dx = \frac{A_0E}{4}\int_1^2 (x-2)^2 dx = \frac{A_0E}{12}$$

As for element 1, $K^{2,0} = 0$ since $a_0(x) = 0$ also for all $L/2 = 1 \le x \le L = 2$. Therefore

$$\boxed{K^2 = K^{2,1} = \frac{A_0E}{12}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}$$



Point force due to the spring:

$Q_3 = k(d - U_3)$

(c) Assembled system:

$$\frac{A_0E}{12}\begin{pmatrix} 7 & -7 & \\ -7 & 8 & -1 \\ & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

(d) Boundary Conditions:

Natural B.C.: $Q_2 = -2P$, $Q_3 = k(d - U_3)$ (see problem 10)

Essential B.C.: $U_1 = 0$

(e) $A_0 = 12$, $E = 10$, $k = 10$, $d = 1/2$ and $P = 30$:

Reduced system: $\begin{pmatrix} 80 & -10 \\ -10 & 10 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} -60 \\ 5 - 10U_3 \end{pmatrix} + \begin{pmatrix} 70 \\ 0 \end{pmatrix}\overset{0}{\overset{\shortparallel}{U_1}} \Leftrightarrow \begin{pmatrix} 80 & -10 \\ -10 & 20 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} -60 \\ 5 \end{pmatrix}$

$$\Delta = \begin{vmatrix} 80 & -10 \\ -10 & 20 \end{vmatrix} = 1600 - 100 = 1500,\quad \Delta_2 = \begin{vmatrix} -60 & -10 \\ 5 & 20 \end{vmatrix} = -1200 + 50 = -1150,\quad \Delta_3 = \begin{vmatrix} 80 & -60 \\ -10 & 5 \end{vmatrix} = 400 - 600 = -200.$$

Hence, the displacements at nodes 2 and 3 are:

$$\boxed{U_2 = \frac{\Delta_2}{\Delta} = -\frac{115\cancel{0}}{150\cancel{0}} = -\frac{23}{30},\quad U_3 = -\frac{200}{1500} = -\frac{2}{15}}$$