Final 26-01-2022 Qüestió 1-BVP

Consider the 1D Boundary Value Problem (BVP)

$$\left.\begin{aligned} -\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) &= f(x), \quad 0 < x < \pi. \\ \frac{du}{dx}(0) &= u_0' \\ u(\pi) &= 0 \end{aligned}\right\} \qquad (1)$$

With $a(x) = \begin{cases} 1, & 0 \le x \le \pi/2 \\ \sin x, & \frac{\pi}{2} \le x \le \pi \end{cases}$

In (1) set $f(x) = 1.0 \;\forall x \in [0,\pi]$ and let $\{u_n\}_{n=1\div 4}$ be the nodal solution of the BVP computed by the FEM using two elements: a quadratic one, $\Omega^1 = [0, \pi/2]$, and a linear one, $\Omega^2 = [\pi/2, \pi]$, where global nodes are numbered in ascending order from left to right (so node 1 is the leftmost and node 4 is the rightmost). Therefore:

(a) The component F_3 of the global load vector is.

(b) The value of u_0' such that the value u_2 is twice the value u_3 is

(c) For the value of u_0' found in (b), the corresponding value of Q_4 is

Solution:

$$F^1 = \frac{\pi}{12}\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$: from the formulas for quadratic elements with constant coefficients, here with $f^1 = 1.0$, $h_1 = \pi/2$

$$\psi_1^2(x) = \frac{x-\pi}{\pi/2-\pi} = \frac{2}{\pi}(\pi - x); \quad \psi_2^2(x) = \frac{x-\pi/2}{\pi-\pi/2} = \frac{2}{\pi}\left(x - \frac{\pi}{2}\right)$$

$$F_1^2 = \int_{\pi/2}^{\pi} f(x)\,\psi_1^2(x)\,dx \underset{f(x)\equiv 1}{=} \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\,dx = -\frac{1}{\pi}(\pi - x)^2\Big|_{\pi/2}^{\pi} = \frac{\pi}{4},$$

$$F_2^2 = \int_{\pi/2}^{\pi} f(x)\,\psi_2^2(x)\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi}\left(x - \frac{\pi}{2}\right)dx = \frac{1}{\pi}\left(x - \frac{\pi}{2}\right)^2\Big|_{\pi/2}^{\pi} = \frac{\pi}{4}.$$

So: $F^2 = \frac{\pi}{4}\begin{pmatrix}1\\1\end{pmatrix} = \frac{\pi}{12}\begin{pmatrix}3\\3\end{pmatrix}$

And hence: $F = \frac{\pi}{12}\begin{pmatrix}1\\4\\4\\3\end{pmatrix}$, and the solution to part (a) is $\boxed{F_3 = \frac{\pi}{3} = 1.0472\ldots}$

$$K^1 = \frac{2}{3\pi}\begin{pmatrix}7 & -8 & 1\\ -8 & 16 & -8\\ 1 & -8 & 7\end{pmatrix}$$: from the formulas for quadratic elements with constant coefficients, being $a_1^1 = 1$ and $h_1 = \pi/2$.

$$K_{11}^2 = K_{11}^{2,1} = \int_{\pi/2}^{\pi} a(x)\frac{d\psi_1^2}{dx}\frac{d\psi_1^2}{dx}dx = \frac{4}{\pi^2}\int_{\pi/2}^{\pi}\sin x\,dx = \frac{4}{\pi^2}$$

$$K_{12}^2 = K_{12}^{2,1} = \int_{\pi/2}^{\pi} a(x)\frac{d\psi_1^2}{dx}\cdot\frac{d\psi_2^2}{dx}dx = -\frac{4}{\pi^2}\int_{\pi/2}^{\pi}\sin x\,dx = -\frac{4}{\pi^2} = K_{21}$$

$$K_{22}^2 = K_{22}^{2,1} = \int_{\pi/2}^{\pi} a(x)\frac{d\psi_2^2}{dx}\frac{d\psi_2^2}{dx}dx = \frac{4}{\pi^2}\int_{\pi/2}^{\pi}\sin x\,dx = \frac{4}{\pi^2}$$

Therefore:

$$K^2 = \frac{4}{\pi^2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

So the coupled system is:

$$\frac{2}{3\pi^2}\begin{pmatrix}7\pi & -8\pi & \pi & \\ -8\pi & 16\pi & -8\pi & \\ \pi & -8\pi & 7\pi+6 & -6\\ & & -6 & 6\end{pmatrix}\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix} = \frac{\pi}{12}\begin{pmatrix}1\\4\\4\\3\end{pmatrix} + \begin{pmatrix}Q_1\\Q_2\\Q_3\\Q_4\end{pmatrix} \qquad (2)$$

Boundary conditions:

- Natural: $Q_1 = -a(0)\frac{du}{dx}(0) = -u_0'$, $Q_2 = 0$, $Q_3 = 0$
- Essential: $U_4 = u(\pi) = 0$

Reduced system:

$$\frac{2}{3\pi^2}\begin{pmatrix}7\pi & -8\pi & \pi\\ -8\pi & 16\pi & -8\pi\\ \pi & -8\pi & 7\pi+6\end{pmatrix}\cdot\begin{pmatrix}U_1\\U_2\\U_3\end{pmatrix} = \frac{\pi}{12}\begin{pmatrix}1\\4\\4\end{pmatrix} + \begin{pmatrix}-u_0'\\0\\0\end{pmatrix} - \frac{2}{3\pi^2}\cdot\begin{pmatrix}0\\0\\-6\end{pmatrix}\overset{=0}{U_4}$$

We want u_0' s.t. $U_2 = 2U_3$, so the system that we have to solve is:

$$\frac{2}{3\pi^2}\begin{pmatrix} 7\pi & -8\pi & \pi & +\frac{3\pi^2}{2} \\ -8\pi & 16\pi & -8\pi & 0 \\ \pi & -8\pi & 7\pi+6 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ u_0' \end{pmatrix} = \frac{\pi}{12}\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

Solution (Matlab):

$U_1 = -4.6888$, $U_2 = -2.7147$, $\boxed{U_3 = -1.3573\ldots}$, $U_4 = 0$ (from the essential B.C)

and $\boxed{u_0' = 2.9063\ldots}$, which is the solution of part (b).

Finally, from the coupled system $^{(2)}$ we can find Q_4, and thus the solution of part (c) is

$$Q_4 = \frac{2}{3\pi^2}(-6)U_3 + \frac{2}{3\pi^2}\cdot 6\cdot \overset{=0}{U_4} - \frac{\pi}{12}\cdot 3 = -\frac{4}{\pi^2}U_3 - \frac{\pi}{4} = \boxed{-0.23526\ldots}$$