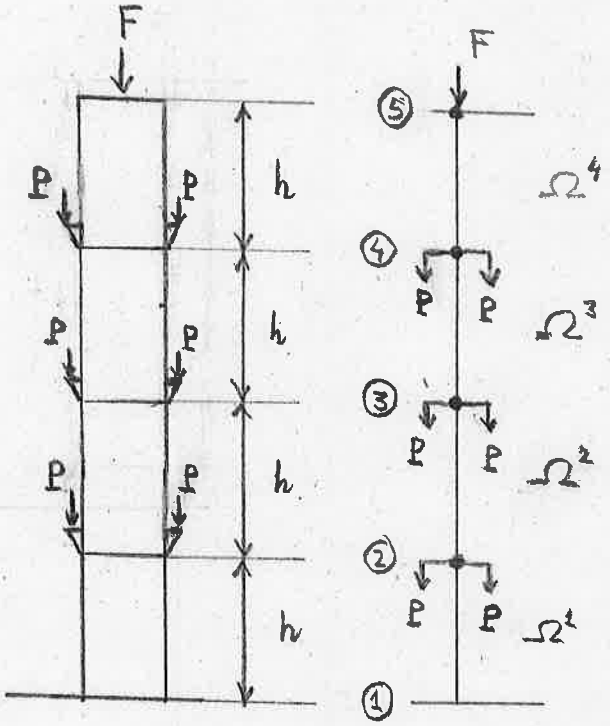
Linear Elasticity 1D

Example 1. Loaded Column



Data:

$h = 4.5\,m$

$P = 11 \times 10^4\,N$

$E = 2.0 \times 10^{11}\,\frac{N}{m^2}$

$A = 250\,cm^2 = 0.025\,m^2$

$F = 3 \times 10^5\,N$

Elasticity model equation: $-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x)$, $f \equiv$ force by unit of length

General model equation: $-\frac{d}{dx}\left(a_1(x)\frac{du}{dx}\right) + a_0(x)\,u = f(x)$

Identifying coeficients: $a_1(x) = EA$, $a_0(x) = 0$, $f(x) = 0$ (the weight of the column is not consider); $\forall x$. So we have constant coefficients and constant r.h.s.

Linear 1D elements (see T2-MN-FEM2D.pdf) p7:

$$K^{e,1} = \frac{a_1^e}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\quad K^{e,0} = (0)\ (a_0 \equiv 0),\quad F^e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ (f \equiv 0).$$

$$K^e = K^{e,1} + K^{e,0} \underset{\substack{a_1^e = AE \\ h_e = h}}{=} \frac{AE}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\quad F^e = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\quad e = 1,2,3,4$$

In <u>this</u> case the local stiffness matrices (K^e, $e=1,2,3,4$) and the local load vector (F_e, $e=1,2,3,4$) are the same for all the elements (this will not be true for all the problems).

Assembly: in the comon nodes, we add the "forces" (Q's and F's) and impose the continuity on the displacements (the U's):

with $a_{11}^e = \frac{EA}{h} = a_{22}^e$, $a_{21}^e = a_{12}^e = -\frac{EA}{h}$ (&)

$$\text{①}: \begin{cases} a_{11}^1 U_1^1 + a_{12}^1 U_2^1 = Q_1^1 + F_1^1 \\ a_{21}^1 U_1^1 + a_{22}^1 U_2^1 = Q_2^1 + F_2^1 \end{cases} \qquad Q_1^1 = Q_1,\ F_1^1 = F_1,\ U_1^1 = U_1$$

$\oplus$ $\qquad Q_2^1 + Q_1^2 = Q_2,\ F_2^1 + F_1^2 = F_2,\ U_2^1 = U_1^2 = U_2$

$$\text{②}: \begin{cases} a_{11}^2 U_1^2 + a_{12}^2 U_2^2 = Q_1^2 + F_1^2 \\ a_{21}^2 U_1^2 + a_{22}^2 U_2^2 = Q_2^2 + F_2^2 \end{cases}$$

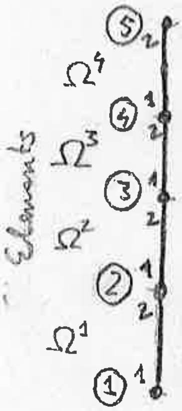
$\oplus$ $\qquad Q_2^2 + Q_1^3 = Q_3,\ F_2^2 + F_1^3 = F_3,\ U_2^2 = U_1^3 = U_3$

$$\text{③}: \begin{cases} a_{11}^3 U_1^3 + a_{12}^3 U_2^3 = Q_1^3 + F_1^3 \\ a_{21}^3 U_1^3 + a_{22}^3 U_2^3 = Q_2^3 + F_2^3 \end{cases}$$

$\oplus$ $\qquad Q_2^3 + Q_1^4 = Q_4,\ F_2^3 + F_1^4 = F_4,\ U_2^3 = U_1^4 = U_4$

$$\text{④}: \begin{cases} a_{11}^4 U_1^4 + a_{12}^4 U_2^4 = Q_1^4 + F_1^4 \\ a_{21}^4 U_1^4 + a_{22}^4 U_2^4 = Q_2^4 + F_2^4 \end{cases} \qquad Q_2^4 = Q_5,\ F_2^4 = F_5,\ U_2^4 = U_5$$

Global Nodes



⑤ $U_4^2 = U_5 \qquad Q_2^4 = Q_5$

④ $U_2^3 = U_1^4 = U_4 \qquad Q_2^3 + Q_1^4 = Q_4,\ F_2^3 + F_1^4 = F_4$

③ $U_2^2 = U_1^3 = U_3 \qquad Q_2^2 + Q_1^3 = Q_3,\ F_2^2 + F_1^3 = F_3$

② $U_2^1 = U_1^2 = U_2 \qquad Q_2^1 + Q_1^2 = Q_2,\ F_2^1 + F_1^2 = F_2$

① $U_1^1 = U_1 \qquad Q_1^1 = Q_1,\ F_1^1 = F_1$

Thence, the coupled system takes the form:

$$\left.\begin{aligned} a_{11}^1 U_1 + a_{12}^1 U_2 &= Q_1 + F_1 \\ a_{21}^1 U_1 + (a_{22}^1 + a_{11}^2) U_2 + a_{12}^2 U_3 &= Q_2 + F_2 \\ a_{21}^2 U_2 + (a_{22}^2 + a_{11}^3) U_3 + a_{12}^3 U_4 &= Q_3 + F_3 \\ a_{21}^3 U_3 + (a_{22}^3 + a_{11}^4) U_4 + a_{12}^4 U_5 &= Q_4 + F_4 \\ a_{21}^4 U_4 + a_{22}^4 U_5 &= Q_5 + F_5 \end{aligned}\right\} (*)$$

with $G_1 = G_1^1$, $G_2 = G_2^1 + G_1^2$, $G_3 = G_2^2 + G_1^3$, $G_4 = G_2^3 + G_1^4$, $G_5 = G_2^4$, $\{G = Q, F\}$.

Now, using (&) and writing the coupled sytem (*) in the matrix notation:

$$\frac{EA}{h}\begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} Q_1 = Q_1^1 \\ Q_2 = Q_2^1 + Q_1^2 \\ Q_3 = Q_2^2 + Q_1^3 \\ Q_4 = Q_2^3 + Q_1^4 \\ Q_5 = Q_2^4 \end{pmatrix}$$

Recall that $F_i^e = 0 \ \forall i = 1,2, \forall e = 1,2,3,4$ so $F_1 = F_1^1 = 0$, $F_2 = F_2^1 + F_1^2 = 0$, $F_3 = F_2^2 + F_1^3 = 0$, $F_4 = F_2^3 + F_1^4 = 0$ and $F_5 = F_2^4 = 0$

B.C.

Natural: Set at the "Q's": Q_2, Q_3, Q_4 and Q_5 the total point forces, applied at nodes 2, 3, 4 and 5. Hence $Q_2 = Q_3 = Q_4 = 2P$ and $Q_5 = -F$.

Essential: $U_1 = 0$

Reduced System:

$$\frac{EA}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} -2P \\ -2P \\ -2P \\ -F \end{pmatrix} - \frac{EA}{h}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \overset{=0}{U_1} \qquad (**)$$

$$\frac{EA}{h} = \frac{250 \times 2.0 \times 10^{11}}{10^4 \times 4.5} = \frac{500}{4.5} \times 10^7 = \frac{1}{9} \times 10^{10} = 1.\hat{1} \times 10^{10}$$

$2P = 22 \times 10^4$

$F = 3 \times 10^5$

Solution of the reduced system:

$U_2 = -0.86$ mm,

$U_3 = -1.53$ mm

$U_4 = -1.998$ mm

$U_5 = -2.268$ mm

FEM solution:

$$U_1 = 0.0, \ U_2 = -0.86, \ U_3 = -1.53, \ U_4 = -1.998, \ U_5 = -2.268 \text{ mm}$$

Post-Process:

$$Q = Ku - \underset{0}{F} = (9.60, -2.2, -2.2, -2.2, -3)^T \times 10^5$$

Therefore: $Q_1 = 9.60 \times 10^5$ N which corresponds to the reaction force. $\square$