Ex-Reavaluacio-2017-18_Problema 2.pdf

MÈTODES NUMÈRICS

Ex Reava 2018 (a)

$\psi_1^1(x) =$	$\dfrac{3-x}{3}$
$[K^1] =$	$\frac{E}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
$[F^2] =$	$-\frac{\omega 2}{4}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
Assembled system:	$\frac{E}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} - \frac{w2}{4}\begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}$
Boundary conditions:	$U_1 = 0$, Q_2=0, $Q_3 = -P$
Displacement of the central point:	$-\frac{2}{E}(P + 2\omega 2)$

(*Hint2:* The element K_{11} of the global matrix is one the values: E, E/2, E/3, E/4, E/5, E/6, E/7, E/8, E/9, E/10).

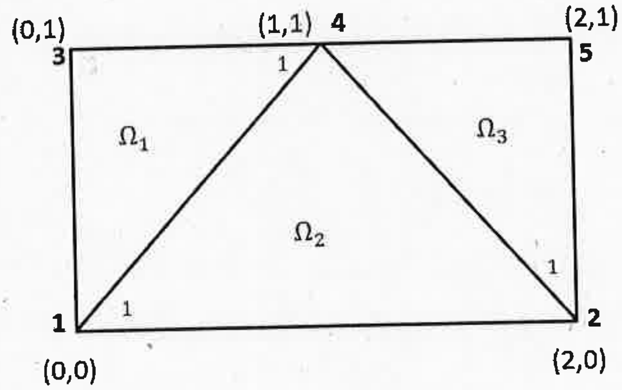
(*Hint3:* The component F_3 of the load vector is one the values: -ω/2, -ω, -3ω/2, -2ω, -5ω/2, -3ω, -7ω/2, -4ω, -9ω/2, -5ω).

(2 points)

(2) Let's consider the following problem in the domain $(0,2)\times(0,1)$:

$$\begin{cases} -\Delta u = f, & \text{in } (0,2)\times(0,1) \\ u(x,0) = 2, & \text{for } 0 \le x \le 2 \\ u(2,y) = 2, & \text{for } 0 \le y \le 1 \\ \dfrac{\partial u}{\partial x}(0,y) = 4, & \text{for } 0 \le y \le 1 \\ \dfrac{\partial u}{\partial y}(x,1) = 6x, & \text{for } 0 \le x \le 2 \end{cases}$$

where $f(x,y) = 6$.



(2 points)

(i) Considering the mesh of the figure, fill the following table:

$[K^1],\quad F^1 =$	$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},\quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$[K^2],\quad F^2 =$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$
$[K^3],\quad F^3 =$	$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},\quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
Assembled system:	$\frac{1}{2}\begin{pmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix}$

(*Hint1:*The element K_{22} of the global matrix is one the values: 0, 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 9/2).
(*Hint2:*The component F_2 of the load vector is one the values: 0,1,2,3,4,5,6,7,8,9).

(ii) Compute the values for Q_3 and Q_4 and the solution at the points U_3, U_4.

$Q_3 =$	-1
$Q_4 =$	6
$U_3 =$	34/7
$U_4 =$	54/7

(3) Consider the mesh `meshPlaca4foratsQuad.m`.

(i) The `quality` of a quadrilateral element from a numerical point of view is defined as the ratio

$$R = \frac{longuestedge}{shortestedge},$$

it is considered good if $1 \leq R \leq 5$. Compute the maximum and minimum value of R for all the elements in this mesh.

idElemMax=	22	MaxRatio=	4.5572e+00
idElemMin=	483	MinRatio=	1.0159e+00

(*Hint:* For element 27 we obtain $R = 1.5344e + 00$).

(ii) Compute the area of this object as the sum of the area of each element. Give the relative error obtained comparing this value with the true area.

computedArea=	2.8875e+01	relError=	5.59311e-04

(*Hint:* For element 45 we obtain area = 1.9725e-02).

(2 points)

(4) Consider again `meshPlaca4foratsQuad.m` as a four-holes aluminium plate, with thermal conductivity $k_c = 230.5$. The temperature at the boundaries of the holes is fixed at $T = 37.78$°C, whereas through the straight lines that form the upper and lower boundaries there is convection with the outer environment, at a bulk temperature $T_\infty = -17.78$°C and convection coefficient $\beta = 205.4$ (all units in the IS).

(i) Compute the maximum value on the convection boundaries.

idMaxTconv=	63	MaxTempConv=	1.7268e+01

(*Hint:* For node 159 we obtain Temp= 3.7780e+01).

(ii) Now we assign to each element a constant temperature T_e obtained from interpolation of the computed values in the center of the element (defined as the point with equal barycentric coordinates).

(a) Compute for the element $e = 17$ the coordinates of its center, (x_c, y_c) and the interpolated temperature value $T_e = T_{\text{interp}}(x_c, y_c)$ at that point.

x_c	y_c	T_e
5.4840e+00	2.0719e+00	3.4144e+01

(*Hint:* For the element 22 the center is at $(x_c, y_c) = (6.9147\text{e+}00, 2.9503\text{e+}00)$).

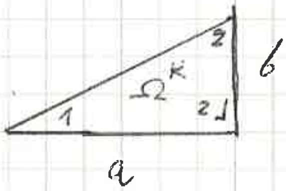
(b) Compute the element that have maximum temperature value:

$idMaxT_e$=	653	MaxTempElem=	3.7413e+01

(*Hint:* For element 67 we obtain Temp= 2.4298e+01).

Solution Exercise 2:

(i) $$\boxed{K^1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} ; \quad F^1 = \frac{f_1 A_1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \left\{ \begin{matrix} 6 \\ A_1 = \frac{1}{2} \end{matrix} \right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$



$$K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix} : \text{ here } a = b = 1, \quad c = k_c = 1$$

$$F^k = \frac{f_k A_k}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

k	x_k^2	y_k^2
1	0	0
2	2	0
3	1	1

$$\beta_1^2 = y_2^2 - y_3^2 = 0 - 1 = -1 \qquad \gamma_1^2 = -(x_2^2 - x_3^2) = -(2-1) = -1$$

$$\beta_2^2 = y_3^2 - y_1^2 = 1 - 0 = 1 \qquad \gamma_2^2 = -(x_3^2 - x_1^2) = -(1-0) = -1$$

$$\beta_3^2 = y_1^2 - y_2^2 = 0 - 0 = 0 \qquad \gamma_3^2 = -(x_1^2 - x_2^2) = -(0-2) = 2$$

$$K^{2,11} = \frac{a_{11}}{4A_2}\begin{pmatrix} \beta_2^1 \\ \beta_2^2 \\ \beta_3^2 \end{pmatrix}\begin{pmatrix} \beta_1^2 & \beta_2^2 & \beta_3^2 \end{pmatrix} = \{a_{11} = 1, A_2 = 1\} = \frac{1}{4}\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} -1 & -1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$K^{2,22} = \frac{a_{22}}{4A_2}\begin{pmatrix} \gamma_1^2 \\ \gamma_2^2 \\ \gamma_3^2 \end{pmatrix} \cdot \begin{pmatrix} \gamma_1^2 & \gamma_2^2 & \gamma_3^2 \end{pmatrix} = \{a_{22} = a_{11} = 1\} = \frac{1}{4}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & -1 & 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}$$

$K^{2,12} = K^{2,21} = K^{2,00} = 0$, since $a_{12} = a_{21} = a_{00} = 0$.

Therefore $$K^2 = K^{2,11} + K^{2,12} + K^{2,21} + K^{2,22} + K^{2,00} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -1 & -1 & 2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix};$$

$$\boxed{K^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad F^2 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}$$

and: $$F^2 = \frac{f_2 A_2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{6 \cdot 1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

(iii) $$\boxed{K^3 = K^1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} ; \quad F^3 = F^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.}$$

$$K = \begin{pmatrix} K^1_{33}+K^2_{11} & K^2_{12} & K^1_{32} & K^1_{31}+K^2_{13} & 0 \\ * & K^2_{22}+K^3_{11} & 0 & K^2_{23}+K^3_{13} & K^3_{12} \\ * & * & K^1_{22} & K^1_{21} & 0 \\ * & * & * & K^1_{11}+K^2_{33}+K^3_{33} & K^3_{32} \\ * & * & * & * & K^3_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

symmetric.

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} = \begin{pmatrix} F^1_3 + F^2_1 \\ F^2_2 + F^3_1 \\ F^1_2 \\ F^1_1 + F^2_3 + F^3_3 \\ F^3_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 4 \\ 1 \end{pmatrix}$$

Assembled system:

$$\boxed{\frac{1}{2}\begin{pmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix}}$$

(ii) $$q^1_{n1}(s) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x}(s,1) \\ \frac{\partial u}{\partial y}(s,1) \end{pmatrix} \cdot \vec{e}_2 = \left(\frac{\partial u}{\partial x}(s,1), \frac{\partial u}{\partial y}(s,1)\right)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{\partial u}{\partial y}(s,1) = 6s, \quad 0 \le s \le 1.$$

$$q^3_{n2}(s) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x}(2-s,1) \\ \frac{\partial u}{\partial y}(2-s,1) \end{pmatrix} \cdot \vec{e}_2 = \left(\frac{\partial u}{\partial x}(2-s,1), \frac{\partial u}{\partial y}(2-s,1)\right)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{\partial u}{\partial y}(2-s,1) = 6(2-s) = 12-6s, \quad 0 \le s \le 1$$

$$q_{n2}^{1}(s) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x}(0,s) \\ \frac{\partial u}{\partial y}(0,s) \end{pmatrix} \cdot (-\vec{e}_1) = \left(\frac{\partial u}{\partial x}(0,s), \frac{\partial u}{\partial y}(0,s) \right) \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$= -\frac{\partial u}{\partial x}(0,s) = -4, \quad 0 \leq s \leq 1. \quad (*)$$

$$Q_3 = Q_{21}^{1} + Q_{22}^{1}, \quad Q_4 = Q_{11}^{1} + Q_{32}^{3}.$$

$$Q_{21}^{1} = \left(\frac{q_{n1}^{1}(0)}{3} + \frac{q_{n1}^{1}(1)}{6} \right) \underbrace{h_1^1}_{=1} = \left(\frac{0}{3} + \frac{6}{6} \right) 1 = 1.$$

$$Q_{22}^{1} = \frac{q_{n2}^{1}(0)}{2} \underbrace{h_2^1}_{=1} = -\frac{4}{2} = -2$$

$$Q_{11}^{1} = \left(\frac{q_{n1}^{1}(1)}{3} + \frac{q_{n1}^{1}(0)}{6} \right) \cdot \underbrace{h_1^1}_{=1} = \left(\frac{6}{3} \cdot 1 + \frac{0}{6} \right) \cdot 1 = 2$$

$$Q_{32}^{3} = \left(\frac{q_{n2}^{3}(1)}{3} + \frac{q_{n2}^{3}(0)}{6} \right) \cdot \underbrace{h_2^2}_{=1} = \left(\frac{6}{3} + \frac{12}{6} \right) \cdot 1 = 4.$$

Boundary conditions:

- Natural: $\boxed{Q_3 = Q_{21}^{1} + Q_{22}^{1} = 1-2 = -1, \ Q_4 = Q_{11}^{1} + Q_{32}^{3} = 2+2 = 6}$
- Essential: $U_1 = U_2 = U_5 = 2$ (since $u(x,0) = 2$, for $0 \leq x \leq 2$, and $u(2,y) = 2$, for $0 \leq y \leq 1$, $U_1 = u(0,0) = 2$, $U_2 = u(2,0)$, $U_5 = u(2,1) = 2$).

Reduced system:

$$\frac{1}{2} \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 6 \end{pmatrix} - \underbrace{\frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} U_1 = 2 \\ U_2 = 2 \\ U_5 = 2 \end{pmatrix}}_{= \begin{pmatrix} -1 \\ -3 \end{pmatrix}}$$

(*) These parameterisations are not really necessary. Since the flows are either linear or constant, it's enough to know the values of q_n at the points coming from the corresponding edges.

$$\Longleftrightarrow \quad \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 26 \end{pmatrix}$$

Solució:

$$\Delta := \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} = 7, \quad \Delta_{U_1} = \begin{vmatrix} 2 & -1 \\ 26 & 4 \end{vmatrix} = 8+26 = 34,$$

$$\Delta_{U_2} = \begin{vmatrix} 2 & 2 \\ -1 & 26 \end{vmatrix} = 52+2 = 54.$$

$$\boxed{U_1 = \frac{\Delta_{U_1}}{\Delta} = \frac{34}{7}, \quad U_2 = \frac{\Delta_{U_2}}{\Delta} = \frac{54}{7}} \quad \square$$