Ex_Final_Q2_2019_20. ProblemaT4.pdf

MÈTODES NUMÈRICS: Ex-Final Q2-2019-20

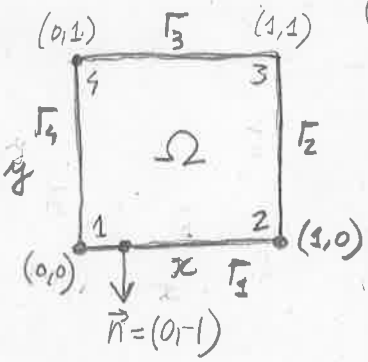
(T4) We consider the equation $-k_c \frac{\partial^2 T}{\partial x^2} - k_c \frac{\partial^2 T}{\partial y^2} = 1$. We want to use the Finite Element method with just one rectangular element, Ω defined by the nodes $(0,0), (1,0), (1,1), (0,1)$ locally numbered by 1, 2, 3, 4 respectively, and with boundary conditions given by:

* $T = 100$ at the edges 3 and 4.
* $\frac{\partial u}{\partial y}(x,0) = x^2$, at the edge 1.
* At the edge 2 there is convection BC with $\beta = n$ and $T_\infty = 30$

(a) Then the value of Q_{21} is: Solution: $Q_{21} = -k_c/4$

(b) Following the previous exercise but now fixing $k_c = \beta = 2$, when we express the term $Q_{22} = aT_2 + b$ being a, b constant values, the value of a is equal to: Solution: $a = -\frac{2}{3}$

Solution.



(a) $q^1_{n1}(s) \doteq -k_c \frac{\partial u}{\partial y}(s,0) = -k_c s^2,\ 0 \le s \le 1$

$\psi_2(x,y) = x(1-y)$, so $\psi_{21}(s) = \psi_2(s,0) = s$, and hence:

$$Q_{21} = \int_0^{h_1} q_{n1}(s)\,\psi_{21}(s)\,ds \overset{(h_1=1)}{=} -k_c \int_0^1 s^3\,ds = \boxed{-k_c/4}$$

(b) $q_{n2}(s) = -\beta(\hat{T}_2(s) - T_\infty) \overset{(*)}{=} -\beta\left(T_1\psi_{12}(s) + T_2\psi_{22}(s) + T_3\psi_{32}(s) + T_4\psi_{42}(s) - T_\infty\right)$

Let $(x_2(s), y_2(s)) \doteq (1, s)$, $0 \le s \le 1$ be the parameterisation of the edge 2 by the length s. Therefore:

- Since $\psi_1(x,y) = (1-x)(1-y)$, then: $\psi_{12}(s) = \psi_1(1,s) = (1-1)(1-s) = 0$ for all $0 \le s \le 1$.

(*) Here $\hat{T}_2(s)$ is the Temperature at the edge 2 parameterised by the length s. We approximate it by interpolation, i.e. $\hat{T}_2(s) = T_1\psi_{12}(s) + T_2\psi_{22}(s) + T_3\psi_{32}(s) + T_4\psi_{42}(s)$, being T_1, T_2, T_3, T_4 (the temperatures at nodes 1,2,3,4 respectively and $\psi_{i2}(s)$ the restriction of the shape function ψ_i $(i = 1,2,3,4)$ on the edge 2.

- Since: $\psi_2(x,y) = x(1-y)$, then $\psi_{22}(s) = \psi_2(1,s) = 1\cdot(1-s) = 1-s$, $0 \leq s \leq 1$.
- Since: $\psi_3(x,y) = xy$, then $\psi_{32}(s) = \psi_3(1,s) = s$, $0 \leq s \leq 1$
- Since: $\psi_4(x,y) = (1-x)y$, then $\psi_{42}(s) = \psi_4(1,s) = (1-1)s = 0$ for all $0 \leq s \leq 1$.

Hence:

$$q_{n2}(s) = -\beta\left(\hat{T}_2(s) - T_\infty\right) = -\beta\left(T_2(1-s) + T_3 s - T_\infty\right), \quad 0 \leq s \leq 1.$$

and then,

$$Q_{22} = \int_0^{h_2} q_{n2}(s)\,\psi_{22}(s)\,ds \overset{(h_2=1)}{=} \left(-\beta\int_0^1 (1-s)^2 ds\right) T_2 + (-\beta)\cdot T_3 \int_0^1 s(1-s)\,ds + \beta T_\infty \int_0^1 (1-s)\,ds$$

$$= \left\{ \begin{array}{l} T_3 = 100 \text{ (from the BC)} \\ \beta = 2,\ T_\infty = 30 \end{array} \right\} = \underbrace{\left(-2\int_0^1 (1-s)^2 ds\right)}_{= a} T_2 + \underbrace{(-200)\int_0^1 s(1-s)\,ds + 60\int_0^1 (1-s)\,ds}_{= b}$$

$$a = -2\int_0^1 (1-s)^2 ds = -2\int_0^1 \sigma^2 d\sigma = \boxed{-2/3}$$

$$b = -200\int_0^1 s(1-s)\,ds + 60\int_0^1 (1-s)\,ds = -\frac{200}{6} + 30 = \frac{-100}{3} + 30 = \frac{-10}{3} \text{ (not required)} \quad \square$$

Remark. If we do:

$$Q_{22} = \left(-\frac{\beta T_2}{3} - \frac{\beta T_3}{6}\right)\overset{=1}{h_2} + \frac{\beta T_\infty h_2}{2} = \left\{ \begin{array}{l} T_3 = 100 \\ \beta = 2 \\ T_\infty = 30 \end{array} \right\} = -\tfrac{2}{3}T_2 - \frac{200}{6} + 30 = -\tfrac{2}{3}T_2 - \frac{10}{3}$$

So: $a = -2/3$ and $b = -10/3$

question: do the rules for linear and constant flows, we got for linear triangles still work for bilinear rectangles? and for general bilinear quadrilaterals?
