Mètodes Numèrics 2$^{\text{on}}$ 20 de Gener de 2016

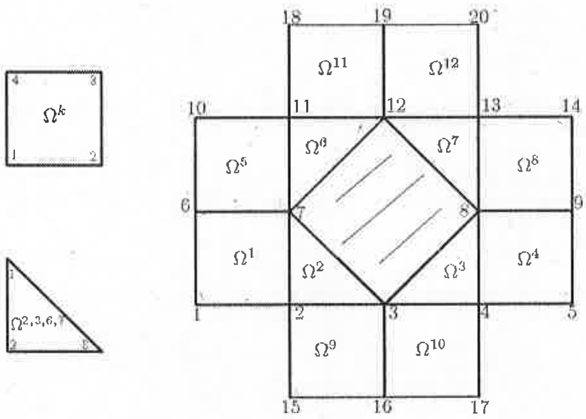
Nom i cognoms:

Problema 3. *(3.0 punts)*

Usant el mètode dels elements finits, volem aproximar la solució de l'equació,

$$-\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y^2} = f_0,$$

on f_0 és una constant, en la regió ja mallada i representada en la figura (un domini en forma de creu a la qual s'ha extret el quadrat interior de vèrtexs els nodes: 7, 3, 8, 12).



El problema està sotmès a la condició de contorn constant, $u = u_0$, a la part superior de la frontera, determinada unint els nodes: 10, 11, 18, 19, 20, 13, 14, i a la part inferior, determinada a l'unir, 1, 2, 15, 16, 17, 4, 5. La condició de contorn a les parts de frontera verticals, 1, 6, 10 i 5, 9, 14, així com a la frontera interior, 7, 3, 8, 12, és $q_n = q_0$, també constant.

Suposant que els costats dels elements quadrats tenen longitud de costat igual a 1, es demana:

(a) Calcular les matrius de rigidesa i vectors de càrrega elementals. (Indicació: els valors resultants per K^1_{32} i K^2_{21} són $-1/2$ i -1 respectivament). *(1.0 punt)*

(b) Calcular els elements diferents de zero de la fila 7 de la matriu de rigidesa global. *(1.0 punt)*

(c) Establir les condicions de contorn pel sistema global i calcular U_7. *(1.0 punt)*

Problema 3: $-\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial y^2} = f_0$,

$a_{11}=1,\ a_{12}=a_{21}=0,\ a_{22}=2,\ a_{00}=0,\ f=f_0$

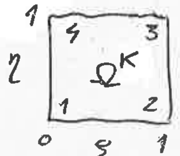
(a) T3-MN-FEM2D.pdf pàg. 49

quadrat: $K^{K,11} = \dfrac{1}{6}\begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix}$, $\boxed{K^{K,22} = \dfrac{1}{6}\begin{pmatrix} 4 & 2 & -2 & -4 \\ 2 & 4 & -4 & -2 \\ -2 & -4 & 4 & 2 \\ -4 & -2 & 2 & 4 \end{pmatrix}}$

$a=b=1$

$$\boxed{K^{1,5,11,12,8,4,10,9} = \frac{1}{6}\begin{pmatrix} 6 & 0 & -3 & -3 \\ 0 & 6 & -3 & -3 \\ -3 & -3 & 6 & 0 \\ -3 & -3 & 0 & 6 \end{pmatrix}}, \quad \boxed{F^{1,5,11,12,8,4,10,9\,(*)} = \frac{f_0}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

(*)



$\psi_1^K(\xi,\eta) = (1-\xi)(1-\eta)$

$\psi_2^K(\xi,\eta) = \xi(1-\eta)$

$\psi_3^K(\xi,\eta) = \xi\eta$

$\psi_4^K(\xi,\eta) = (1-\xi)\eta$

$F_i^K = \iint_{\Omega^K} f_i^K \psi_i^K \, d\xi d\eta$. (T3-MN-FEM2D.pdf pàg. 43)

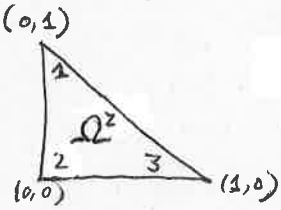
$$F_1^K = f_0 \int_0^1 (1-\xi)d\xi \int_0^1 (1-\eta)d\eta = f_0\left(\xi-\xi^2/2\right)\Big|_{\xi=0}^{\xi=1} \cdot \left(\eta-\eta^2/2\right)\Big|_{\eta=0}^{\eta=1}$$

$$= f_0/4 = F_2^K = F_3^K = F_4^K, \text{ per } K = 1,5,11,12,8,4,10,9.$$

es comprova

triangles:

1 2 3
2 3 1
3 1 2



$\beta_1^2 = 0 = \beta_1^7 \qquad \gamma_1^2 = 1 = -\gamma_1^7$

$\beta_2^2 = -1 = -\beta_2^7 \qquad \gamma_2^2 = -1 = -\gamma_2^7$

$\beta_3^2 = 1 = -\beta_3^7 \qquad \gamma_3^2 = 0 = \gamma_3^7$

Recordem (T3-MN-FEM2D.pdf, pàg. 47)

$K_{ij}^{e,11} = a_{11}^e \dfrac{1}{4A_K}\beta_i^e\beta_j^e$

$K_{ij}^{e,22} = a_{22}^e \dfrac{1}{4A_K}\gamma_i^e\gamma_j^e$

essent:

$\beta_i^e = y_j^e - y_k^e$

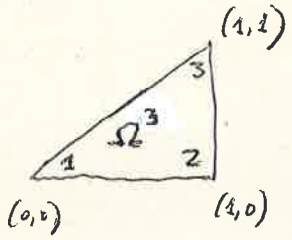
$\gamma_i^e = -(x_j^e - x_k^e)$,

amb $i,j,k = 1,2,3$ + permutacions cícliques.

$$K^{2,7} = \frac{1}{4\underbrace{A_{27}}_{1/2}}\left[\underbrace{a_{11}^{2,7}}_{1}\ \beta^{2,7}\left(\beta^{2,7}\right)^T + \overbrace{a_{22}^{2,7}}^{2}\ \gamma^{2,7}\left(\gamma^{2,7}\right)^T\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\right] =$$

$$= \frac{1}{2}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \boxed{\frac{1}{2}\begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}}$$



$$\beta^3_1 = -1 = -\beta^6_1 \qquad \gamma^3_1 = 0 = \gamma^6_1$$
$$\beta^3_2 = 1 = \beta^6_2 \qquad \gamma^3_2 = -1 = -\gamma^6_2$$
$$\beta^3_3 = 0 = \beta^6_3 \qquad \gamma^3_3 = 1 = -\gamma^6_3$$

$$K^{3,6} = \frac{1}{4\underbrace{A_{3,6}}_{\frac{1}{2}}}\left[\underbrace{a^{3,6}_{11}}_{1}\,\beta^{3,6}\cdot(\beta^{3,6})^T + \underbrace{a^{3,6}_{22}}_{2}\,\gamma^{3,6}\cdot(\gamma^{3,6})^T\right] = \frac{1}{2}\left[\begin{pmatrix}-1\\1\\0\end{pmatrix}\begin{pmatrix}-1 & 1 & 0\end{pmatrix} + 2\begin{pmatrix}0\\-1\\1\end{pmatrix}\begin{pmatrix}0 & -1 & 1\end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\0 & 2 & -2\\0 & -2 & 2\end{pmatrix}\right] = \boxed{\frac{1}{2}\begin{pmatrix}1 & -1 & 0\\-1 & 3 & -2\\0 & -2 & 2\end{pmatrix}}$$

$$F^{2,7,3,6} = \frac{f_0}{6}\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \text{(veure T3-MN-FEM2D.pdf pàg. 47)}$$

(b)
$$K_{7,1} = K^1_{3,1} = -\tfrac{1}{2}, \qquad K_{7,7} = K^5_{2,2} + K^1_{3,3} + K^2_{1,1} + K^6_{3,3} = 1+1+1+1 = 4$$
$$K_{7,2} = K^2_{1,2} + K^1_{3,2} = -1 - \tfrac{1}{2} = -\tfrac{3}{2} \qquad K_{7,10} = K^5_{2,4} = -\tfrac{1}{2}$$
$$K_{7,3} = K^2_{1,3} = 0 \qquad K_{7,11} = K^5_{2,3} + K^6_{3,2} = -\tfrac{1}{2} - 1 = -\tfrac{3}{2}$$
$$K_{7,6} = K^5_{2,1} + K^1_{3,4} = 0 + 0 = 0 \qquad K_{7,12} = K^6_{3,1} = 0$$

(c) Condicions de contorn <u>essencials</u>:

$$U_1 = U_2 = U_{15} = U_{16} = U_{17} = U_4 = U_5 = U_{10} = U_{11} = U_{18} = U_{19} = U_{20} = U_{13} = U_{14} = u_0$$

Condicions de contorn <u>naturals</u>:

$$Q_6 = Q^5_1 + Q^1_4 = \tfrac{q_0}{2} + \tfrac{q_0}{2} = q_0 = Q_9 \quad \left(\text{Notem que: } Q^5_{14} = \int_0^1 q_0\,\psi^5_{14}(s)\,ds = q_0\int_0^1 s\,ds = \tfrac{q_0}{2}\right)$$

$$Q_7 = \underbrace{\boxed{Q^5_2}}_{0} + Q^6_3 + \underbrace{\boxed{Q^1_3}}_{0} + Q^2_1 = \tfrac{q_0}{2}\sqrt{2} + \tfrac{q_0}{2}\sqrt{2} = q_0\sqrt{2} = Q_8 = Q_3 = Q_{12}$$
$$\left(\text{Notem que: } Q^6_{33} = \tfrac{q_0}{2}\sqrt{2} = Q^2_{11}, \text{ veure: T3-MN-FEM2D.pdf pàg. 61}\right)$$

Càlcul de U_7:

$$-\tfrac{1}{2}u_0 - \tfrac{3}{2}u_0 + 4\cdot U_7 - \tfrac{1}{2}u_0 - \tfrac{3}{2}u_0 = \left(-\tfrac{1}{2} - \tfrac{1}{2} - \tfrac{3}{2} - \tfrac{3}{2}\right)u_0 + 4U_7 = Q_7 + F_7$$

Amb $Q_7 = q_0\sqrt{2}$, $F_7 = F^5_2 + F^6_3 + F^1_3 + F^2_1 = \left(\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{6}\right)f_0 = \left(\tfrac{1}{2} + \tfrac{1}{3}\right) = \tfrac{5}{6}f_0$;

és a dir:

$$-4u_0 + 4U_7 = q_0\sqrt{2} + \tfrac{5}{6}f_0 \iff \boxed{U_7 = u_0 + q_0\tfrac{\sqrt{2}}{4} + \tfrac{5}{24}f_0} \quad \square$$