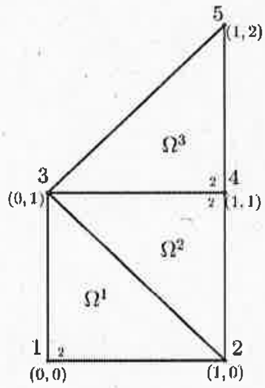
Examen_ReAvaluacio-2019_Problema2.pdf

Name and surnames:

(2) Consider the domain shown in the figure below, and with the elements, nodes and local and global numbering plotted there.



We want to solve the problem defined by the equation

$$-\frac{\partial}{\partial x}(a_{11}(x,y)\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(a_{22}(x,y)\frac{\partial u}{\partial y}) \;=\; 0$$

where the coefficients a_{11} and a_{22} are different depending on the element and are given by:

$$\begin{aligned} a_{11}(x,y) &= a_{22}(x,y) = 2, \text{ on } \Omega^1, \Omega^2 \\ a_{11}(x,y) &= 21y, \text{ and } a_{22}(x,y) = \frac{6}{5}(1+x) \text{ on } \Omega^3 \end{aligned}$$

with **boundary conditions** given by:

- At the line that joints nodes 2 and 5, we have: $\dfrac{\partial u}{\partial x}(1,y) = 1$
- $u = 3$ at the nodes 1,2,3,5.

Hint. You can use that $\iint_{\Omega^3} y\,dx\,dy = \frac{2}{3}$ and $\iint_{\Omega^3} (1+x)\,dx\,dy = \frac{5}{6}$

Then, compute:

(i) The stiffness matrix of the elements Ω^1, Ω^2.

(ii) The shape functions of element Ω^3.

(iii) The stiffness matrix of the element Ω^3

(iv) The 4th row of the global stiff matrix.

(v) The value of the coefficient Q^2_{21}.

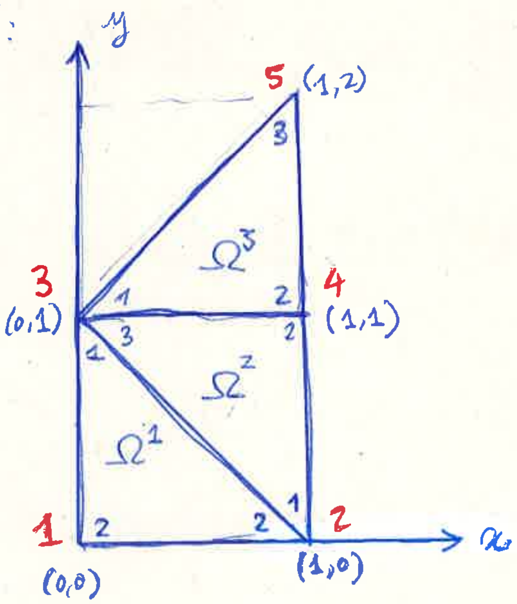
(vi) The value of the coefficients for Q^3_{22}.

(vii) The value of U_4.

Results:	
$[K^1] =$	$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$
$[K^2] =$	$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$
Shape functions =	$\psi_1^3(x,y) = 1 - x, \quad \psi_2^3(x,y) = 1 + x - y, \quad \psi_3^3(x,y) = y - 1$
$[K^3] =$	$\begin{pmatrix} 14 & -14 & 0 \\ -14 & 15 & -1 \\ 0 & -1 & 1 \end{pmatrix}$
4th row :	$0,\ -1,\ -15,\ 17,\ -1$
$Q_{21}^2 =$	1
$Q_{22}^3 =$	14
$U_4 =$	198/51=3.884

Hint. For the 4th row $K_{43} = -15$ (2 points)

Solució:



(i) Stiffness matrix of the elements Ω^1, Ω^2:

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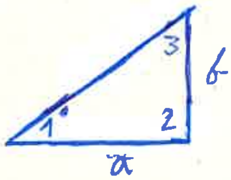
$$K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}$$



$$K^1 = \frac{2}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = K^2$$

ii) The shape functions of the elements Ω^3,

$$\psi_i^3(x,y) = a_i + \frac{\beta_i}{2A_3}x + \frac{\gamma_i}{2A_3}y, \quad i=1,2,3.$$

* A_3 = area of triangle Ω^3 : $A_3 = m(\Omega^3) = \frac{1\cdot 1}{2} = \frac{1}{2}$

* $\beta_i = y_j - y_k$, $\gamma_i = -(x_j - x_k)$, with $i,j,k = 1,2,3$ and cyclic permutations

x	y
0	1
1	1
1	2

$\beta_1 = y_2 - y_3 = 1-2 = -1,$ | $\gamma_1 = -(x_2 - x_3) = -(1-1) = 0,$

$\beta_2 = y_3 - y_1 = 2-1 = 1,$ | $\gamma_2 = -(x_3 - x_1) = -(1-0) = -1,$

$\beta_3 = y_1 - y_2 = 1-1 = 0,$ | $\gamma_3 = -(x_1 - x_2) = -(0-1) = 1.$

$\psi_1^3(x,y) = a_1 - x$: $\psi_1^3(0,1) = a_1 - 0 = 1 \Rightarrow \psi_1^3(x,y) = 1-x$

$\psi_2^3(x,y) = a_2 + x - y$: $\psi_2^3(1,1) = a_2 + 0 = 1 \Rightarrow \psi_2^3(x,y) = 1+x-y$

$\psi_3^3(x,y) = a_3 + y$: $\psi_3^3(1,2) = a_3 + 2 = 1 \Rightarrow \psi_3^3(x,y) = y-1$

(iii) T2-MN-FEM2D.pdf page 45.

$$K_{ij}^{3,11} = \iint_{\Omega^3} a_{11}(x,y)\frac{\partial \psi_i^k}{\partial x}(x,y)\frac{\partial \psi_j^k}{\partial y}(x,y)\,dx\,dy = \frac{21}{4A_3^2}\beta_i\beta_j \iint_{\Omega^3} y\,dx\,dy$$

$$\underset{A_3=\frac{1}{2}}{\overset{(*)}{=}} \frac{21}{1}\cdot\frac{2}{3}\cdot\beta_i\beta_j = 14\,\beta_i\beta_j$$

$$(*)\ \iint_{\Omega^3} y\,dx\,dy = \int_0^1 dx\int_1^{1+x} y\,dy = \frac{1}{2}\int_0^1 [(1+x)^2 - 1]\,dx = \frac{1}{2}\int_0^1 (x^2+2x)\,dx = \frac{1}{2}\left(\frac{x^3}{3}+x^2\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3}+1\right) = \frac{2}{3}$$

$$K^{3,11} = 14\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}(\beta_1 \cdot \beta_2\ \beta_3) = 14\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}(-1\ 1\ 0) = \begin{pmatrix}14 & -14 & 0\\ -14 & 14 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$K^{3,22}_{ij} = \iint_{\Omega^3} a_{22}(x,y)\,\frac{\partial \psi_i^3}{\partial y}(x,y)\cdot\frac{\partial \psi_j^3}{\partial y}(x,y)\,dx\,dy = \frac{6}{5}\,\frac{\gamma_i\gamma_j}{4A_3^2}\iint_{\Omega^3}(1+x)\,dx\,dy$$

$$\underset{A_3=1/2}{\overset{(*)}{=}} \frac{6}{5}\cdot\frac{5}{6}\,\gamma_i\gamma_j = \gamma_i\gamma_j.$$

$$(*)\quad \iint_{\Omega^3}(1+x)\,dx\,dy = \int_0^1(1+x)\,dx\int_1^{1+x}dy = \int_0^1(1+x)\cdot[(1+x)-1]\,dx = \left[\frac{(1+x)^3}{3}-\frac{(1+x)^2}{2}\right]_0^1$$

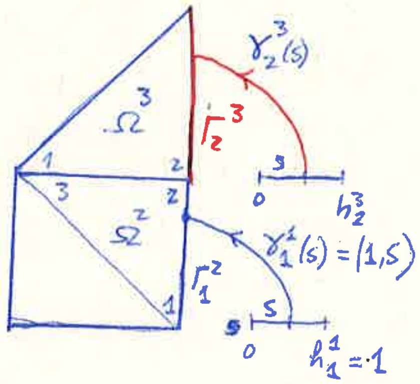
$$= \frac{1}{6}\left(2\cdot 2^3 - 3\cdot 2^2 - 2 + 3\right) = \frac{5}{6}$$

$$K^{3,22} = \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}(\gamma_1\ \gamma_2\ \gamma_3) = \begin{pmatrix}0\\ -1\\ 1\end{pmatrix}(0\ -1\ 1) = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}$$

$$K^3 = K^{3,11} + K^{3,22} = \begin{pmatrix}14 & -14 & 0\\ -14 & 15 & -1\\ 0 & -1 & 1\end{pmatrix}$$

(iv) $$K(4,:) = \left(0\quad K^2_{21}\quad K^2_{23}+K^3_{21}\quad K^2_{22}+K^3_{22}\quad K^3_{23}\right)$$

$$= \left(0\quad -1\quad -15\quad 17\quad -1\right)$$



(v) $$s\in[0,h_1^1]=[0,1] \longmapsto \gamma_1^1(s)=(1,s)\in\Gamma_1^2$$

$$q^2_{n,1}(s) = \begin{pmatrix}a_{11}(1,s) & 0\\ 0 & a_{22}(1,s)\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x}(1,s)\\ \frac{\partial u}{\partial y}(1,s)\end{pmatrix}\cdot\vec{e}_1$$

$$\left\{\begin{matrix}a_{11}(1,s)=a_{22}(1,s)\equiv 2\\ \frac{\partial u}{\partial x}(1,s)=1\end{matrix}\right\} = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1\\ *\end{pmatrix}\cdot\vec{e}_1 = (2,*)\cdot\begin{pmatrix}1\\ 0\end{pmatrix} = 2$$

$$Q^1_{21} = \int_{\Gamma_1^2} q^2_{n,1}(s)\,\psi^2_{21}(s)\,ds = \int_0^{h_1^2=1} 2\cdot s\,ds = 1$$

Remark: it's a linear flow, so $Q^2_{21} = \frac{1}{2}q^2_{n,1}h^2_1 = \frac{1}{2}\cdot 2\cdot 1 = 1$

(vi) $s \in [0, h_2^3] = [0,1] \longrightarrow \gamma_2^3(s) = (1, 1+s)$

$$q_{n,2}^3(s) = \begin{pmatrix} a_{11}(1,1+s) & 0 \\ 0 & a_{22}(1,1+s) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x}(1,1+s) \\ \frac{\partial u}{\partial y}(1,1+s) \end{pmatrix} \cdot \vec{e}_1$$

$$= \Big(a_{11}(1,1+s) \cdot \underbrace{\frac{\partial u}{\partial x}(1,1+s)}_{=1}, \ * \Big) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 21(1+s), \quad 0 \leq s \leq h_2^3 = 1$$

$$Q_{22}^3 = \int_{\Gamma_2^3} q_{n,2}^3(s) \Psi_{2,2}^3(s)\, ds = 21 \int_0^{h_2^3=1} (1+s)(1-s)\, ds = 21 \int_0^1 (1-s^2)\, ds$$

$$= 21\left(1 - \frac{1}{3}\right) = 14$$

Remark: actually it's a linear flow, so $Q_{22}^3 = \frac{1}{3} q_{n,2}^3(0) \cdot h_2^3 + \frac{1}{6} q_{n,2}^3(h_2^3)\, h_2^3$

$$= \frac{21}{3}(1+0) \cdot 1 + \frac{21}{6}(1+1) \cdot 1 = 7 + 7 = 14.$$

(vii)

$$\begin{pmatrix} 0 & -1 & -15 & 17 & -1 \end{pmatrix} \cdot \begin{pmatrix} U_1 = 3 \\ U_2 = 3 \\ U_3 = 3 \\ U_4 \\ U_5 = 3 \end{pmatrix} = Q_4 = Q_{21}^2 + Q_{22}^3 = 1 + 14 = 15$$

$$\Leftrightarrow \quad -3 - 45 + 17U_3 - 3 = -51 + 17U_4 = 15 \Leftrightarrow 17U_4 = 66$$

Therefore: $U_4 = \frac{66}{17} = 3.882$ $\square$