Ex_Final_Q1-2020-21_Problema 1.pdf

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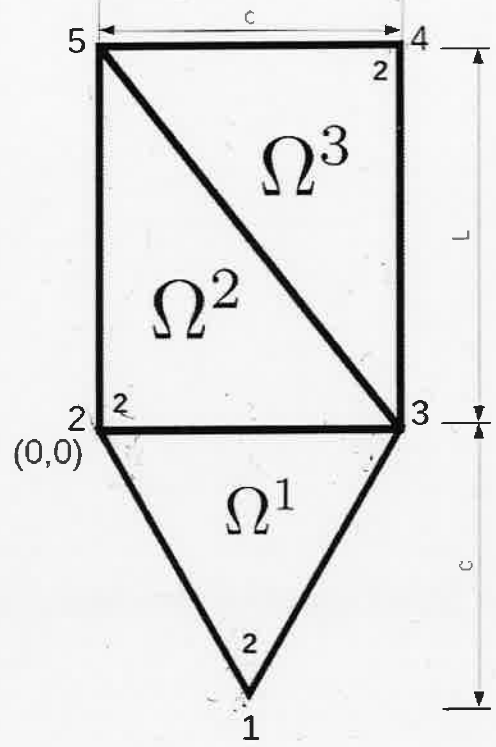
P1 - Ex-Final 1Q-2020-21

We want to solve $\Delta u = 0$ with boundary conditions given by

* At the lines that joints nodes 4 and 5, $\dfrac{\partial u}{\partial y} = u$

* At the lines that joints nodes 3 and 4, $\dfrac{\partial u}{\partial x} = y$

* $u = 0$ at the nodes 1,2,3,5.



Compute the following values with $c = 2$ and $L = 4$

(a, pts=2) The element K^1_{12} of the local stiff matrix K^1 of the element Ω^1 is $\boxed{-2.5000e-01 \pm 0.000050 \quad \checkmark}$.

(b, pts=2) The element K^2_{12} of the local stiff matrix K^2 of the element Ω^2 is

$\boxed{-2.5000e-01 \pm 0.000050 \quad \checkmark}$.

(c, pts=1) Is K^2 equal to K^3?

- Yes. ✓
- Equal but with opposite sign.
- No.
- K^2 and K^3 have different size.
- Leave it empty

(d, pts=2) The element K_{32} of the global stiff matrix K is (hint: K_{13} = -2.5000e-01) $\boxed{-1.3750e+00 \pm 0.000500 \quad \checkmark}$.

(e, pts=2) When computing the value of Q_4 for the fourth equation of this problem, it can be mainly expressed as $a+bu_4$ (a constant term plus another one depending on u_4). The value of b is: $\boxed{6.6667e-01 \pm 0.000500 \quad \checkmark}$.

(f, pts=1) After solving the global system, the value of u_4 is

- 6.8098e+00
- 9.1429e+00 ✓
- 3.6227e+00
- 8.9543e+00
- Leave it empty

Solució :

(a) $K^1_{12} = \frac{a^1_{11}}{4A_1}\beta^1_1\beta^1_2 + \frac{a^1_{22}}{4A_1}\gamma^1_1\gamma^1_2 \overset{(*)}{=} \{a_{11}=a_{22}=1\} = \frac{1}{2c^2}\left(\beta^1_1\beta^1_2+\gamma^1_1\gamma^1_2\right) = \frac{1}{2c^2}\left(-\frac{c^2}{2}\right) = \boxed{-\frac{1}{4}}$

$= -0.25$

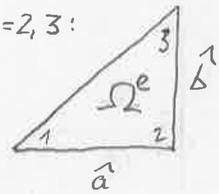
(*) $\Delta u = 0 \Leftrightarrow -\Delta u = -\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = 0$

(**) $A_1 = \frac{1}{2}c^2$

$\beta^1_1 = y^1_2 - y^1_3 = -c-0 = -c \quad \Big| \quad \gamma^1_1 = -(x^1_2 - x^1_3) = -(c/2 - c) = c/2$

$\beta^1_2 = y^1_3 - y^1_1 = 0-0 = 0 \quad \Big| \quad \gamma^1_2 = -(x^1_3 - x^1_1) = -(c-0) = -c$

(b) e=2,3:



$a_{11}=a_{22}=K_c$:

$$K^e = \frac{K_c}{2\hat{a}\hat{b}}\begin{pmatrix} \hat{b}^2 & -\hat{b}^2 & 0 \\ -\hat{b}^2 & \hat{a}^2+\hat{b}^2 & -\hat{a}^2 \\ 0 & -\hat{a}^2 & \hat{a}^2 \end{pmatrix} :$$

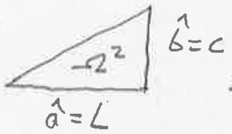
$K_c = a_{11} = a_{22} = 1,$

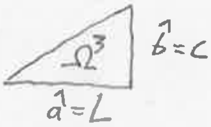
$\hat{a} = L = 4$

$\hat{b} = c = 2$

$K^2_{12} = \frac{1}{2cL}(-c^2) = -\frac{c}{2L} = -\frac{2}{8} = \boxed{-\frac{1}{4}} = -0.25$

(c)





: $K^2 = K^3$.

(d) $K_{32} = K^2_{32} + K^1_{31} =$

$K^2_{32} = \frac{1}{2cL}(-L^2) = -\frac{L}{2c} = -\frac{4}{4} = -1-\frac{3}{8} = \boxed{-\frac{11}{8}} = -1.375$

$K^1_{31} = \frac{1}{2c^2}\left(\beta^1_3\beta^1_1 + \gamma^1_3\gamma^1_1\right) = \frac{1}{2c^2}\left[c\cdot(-c) + \frac{c}{2}\cdot\frac{c}{2}\right] = \frac{1}{2c^2}\left(-c^2+\frac{c^2}{4}\right) = -\frac{3}{8}$

$\beta^1_3 = y^1_1 - y^1_2 = 0-(-c) = c, \quad \gamma^1_3 = -(x^1_1 - x^1_2) = -(0-c/2) = c/2.$

Convection with $\beta=1$, $u_0=0$

(e) $Q_4 = Q^3_{21} + Q^3_{22} = \frac{2}{3}u_4 + \frac{16}{3}$. So if $Q_4 = \check{a} + \check{b}u_4$, then $\check{b} = \boxed{2/3} = 0.\hat{6}$

$q^3_1(s) = \left\langle \begin{pmatrix} K_c & \\ & K_c \end{pmatrix}\begin{pmatrix} u_x(s,L) \\ u_y(s,L) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = K_c\, u_y(s,L) = K_c\, u(s,L) = u(s,L), \quad 0 \le s \le c$

$q^3_2(s) = \left\langle \begin{pmatrix} K_c & \\ & K_c \end{pmatrix}\cdot\begin{pmatrix} u_x(c,L-s) \\ u_y(c,L-s) \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = K_c\, u_x(c,L-s) = K_c(L-s) = L-s, \quad 0 \le s \le L$

$Q^3_{21} = c\left(\frac{1}{6}q^3_1(0) + \frac{1}{3}q^3_1(c)\right) = c\left(\frac{1}{6}u(0,L) + \frac{1}{3}u(c,L)\right) = c\left(\frac{1}{6}u_5 + \frac{1}{3}u_4\right) = \frac{2}{3}u_4$

($u_5 = 0$, according to the essential B.C.)

$Q^3_{22} = L\left(\frac{1}{3}q^3_2(0) + \frac{1}{6}q^3_2(L)\right) = L\left(\frac{1}{3}L + \frac{1}{6}0\right) = \frac{L^2}{3} = \frac{16}{3}$

(f) Natural B.C.: $Q_4 = \frac{2}{3} u_4 + \frac{16}{3}$

Essential B.C.: $u_1 = u_2 = u_3 = u_5 = 0$

Reduced system: $K_{44} u_4 = \frac{2}{3} u_4 + \frac{16}{3} \Leftrightarrow \left(K_{44} - \frac{2}{3}\right) u_4 = \frac{16}{3}$

$$K_{44} = K_{22}^{3} = \frac{1}{2cL}\left(c^2 + L^2\right) = \frac{1}{16}\left(4 + 16\right) = \frac{20}{16} = \frac{5}{4}$$

$$\left(\frac{5}{4} - \frac{2}{3}\right) u_4 = \frac{16}{3} \Leftrightarrow \frac{7}{12} u_4 = \frac{16}{3} \Leftrightarrow u_4 = \boxed{\frac{64}{7}} = 9.1429$$