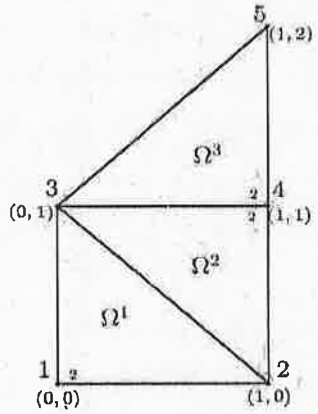
MÈTODES NUMÈRICS: **Ex-Final Q1-2018-19 (a)**

Name and surnames:

- **2.** Consider the domain shown in the figure below, and with the elements, nodes and local and global numbering plotted there.



We want to solve the thermal problem defined by the equation

$$-\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = 0$$

with boundary conditions given by:

- At the line that joints nodes 2 and 5, there is convection BC with $\beta = 3$ and $T_\infty = 30$
- $T = 100$ at the nodes 5, 3, 1 and 2.

Then, compute:

(i) The stiffness matrix of the first element K^1.

(ii) The 4th row of the global stiff matrix before applying the BC.

(iii) The value of the coefficients for Q^2_{21} of the local system for Ω^2 as a function of T_2 and T_4.

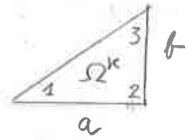
(iv) The value of the coefficients for Q^3_{22} of the local system for Ω^3 as a function of T_4 and T_5.

(v) The value of T_4.

Results:	
$[K^1] =$	$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$
4th row :	$\frac{1}{2}\begin{pmatrix} 0 & -1 & -2 & 4 & -1 \end{pmatrix}$
$Q^2_{21} = a_1T_2 + a_2T_4 + a_3$	coeff a_i: $a_1 = -\frac{1}{2}$, $a_2 = -1$, $a_3 = 45$
$Q^3_{22} = b_1T_4 + b_2T_5 + b_3$	coeff b_i: $b_1 = -1$, $b_2 = -\frac{1}{2}$, $b_3 = 45$
a_i and b_i are among the values: $-2, 2, 1, -1, 1/2, -1/2, 1/6, -1/6, 35, 40, 45, 50, 85, 90, 95, 100, 105$	
$T_4=$	47.5

(2 points)

(i) T2-MN-FEM2D.pdf pàg. 48



For Poisson equation ($a_{11}=a_{22}=c=\text{ctnt.}$, $a_{12}=a_{21}=a_{00}=0$)

$$K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix} \; ; \; \underset{\substack{a=b=1\\ c=1}}{K^1 =} \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = K^2 = K^3$$

(ii) $$K(4,:) = \begin{pmatrix} 0 & K^2_{21} & K^2_{23}+K^3_{21} & K^2_{22}+K^3_{22} & K^3_{23} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0, & -1, & -2, & 4, & -1 \end{pmatrix}$$

(iii) $$Q^2_{21} = \frac{\beta T_\infty}{2}h^2_1 - \frac{\beta h^2_1}{6}T_2 - \frac{\beta h^2_1}{3}T_4 \overset{(*)}{\underset{\substack{\beta=3\\ T_\infty=30\\ h^2_1=1}}{=}} 45 - \frac{1}{2}T_2 - T_4 \qquad \begin{cases} q^k_a = -\beta T_2 \\ q^k_b = -\beta T_4 \\ h = h^2_1 \end{cases}$$

(iv) $$Q^3_{22} = \frac{\beta T_\infty}{2}h^2_2 - \frac{\beta h^2_2}{3}T_4 - \frac{\beta h^2_2}{6}T_5 \underset{h^2_2=1}{=} 45 - T_4 - \frac{1}{2}T_5 \qquad \begin{cases} q^k_a = -\beta T_4 \\ q^k_b = -\beta T_5 \\ h = h^3_2 \end{cases}$$

(v) B.C.

- Natural: $Q_4 = Q^2_{21} + Q^3_{22} = 90 - \frac{1}{2}T_2 - 2T_4 - \frac{1}{2}T_5$.
- Essential: $T_1 = T_2 = T_3 = T_5 = 100$

4th equation of the global system: $\cancel{-\frac{1}{2}T_2} - T_3 + 2T_4 \cancel{-\frac{1}{2}T_5} = 90 \cancel{-\frac{1}{2}T_2} - 2T_4 \cancel{-\frac{1}{2}T_5}$

Reduced system: $4T_4 = 90 + \underset{\uparrow}{T_3} = 90 + 100 = 190 \Rightarrow T_4 = \frac{190}{4} = \frac{95}{2} = \boxed{47'5^\circ C}$

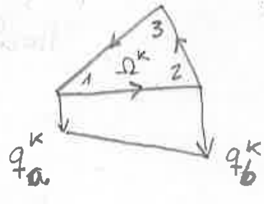
$\leftarrow$ Essential B.C. $T_3=100$

Alternatively, we could have used matrices on T2-MN-FEM2D-applications.pdf, page 33. Therefore:

$$K^{2,c} = \frac{\beta h^2_1}{6}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K^{3,c} = \frac{\beta h^3_2}{6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

So: $$\bar{K}^1 = K^1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \bar{K}^2 = K^2 + K^{2,c} = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \bar{K}^3 = K^3 + K^{3,c} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(*) Recall. For linear flows on linear triangles:



$$q^k_{n,1}(s) = \frac{q^k_b - q^k_a}{h^k_1}s + q^k_a \qquad q^k_{n,1}(0) = q^k_a, \quad q^k_{n,1}(h^k_1) = q^k_b$$

$$Q^k_{11} = \int_0^{h^k_1} q^k_{n,1}(s)\,\psi^k_{11}(s)\,ds = \frac{q^k_a h^k_1}{3} + \frac{q^k_b h^k_1}{6}$$

$$Q^k_{21} = \int_0^{h^k_1} q^k_{n,1}(s)\,\psi^k_{21}(s)\,ds = \frac{q^k_a h^k_1}{6} + \frac{q^k_b h^k_1}{3}.$$

$$\bar{K}(:,4) = \left(0,\ \bar{K}^2_{21},\ \bar{K}^2_{23} + \bar{K}^3_{21},\ \bar{K}^2_{22} + \bar{K}^3_{22},\ \bar{K}^3_{23}\right)$$

$$= \frac{1}{2}\left(0,\ 0,\ -2,\ 8,\ 0\right)$$

$$Q^{2,c} = \frac{\beta T_\infty h_1^2}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = 45\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad Q^{3,c} = \frac{\beta T_\infty h_2^3}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} = 45\begin{pmatrix}0\\1\\1\end{pmatrix}$$

So: $Q_4 = Q_2^{2,c} + Q_2^{3,c} = 90.$

Then the 4^{th} equation of the global system casts:

$$-T_3 + 4T_2 = 90$$

And the reduced system: $4T_2 = 90 + T_3 = 90 + 100 = 190 \Rightarrow \boxed{T_2 = 47'5\,°C}$

Essential B.C. $T_3 = 100$