Examen_2a_Gener_2016_Problema1.pdf

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Mètodes Numèrics 2on 20 de Gener de 2016

Nom i cognoms:

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**Problema 1.** *(4.0 punts)*

Considerem el problema $-\Delta u = f$ en $\Omega = (0,4) \times (0,2)$, amb les condicions de contorn:

$$u(0,y) = 0 \text{ , per a tot } y \in (0,2),$$
$$\frac{\partial u}{\partial y}(x,0) = 0 \text{ i } \frac{\partial u}{\partial y}(x,2) = 2 \text{ , per a tot } x \in (0,4),$$
$$\frac{\partial u}{\partial x}(4,y) \text{ lineal en } y \in (0,2) \text{ , } \frac{\partial u}{\partial x}(4,0) = 0 \text{ i } \frac{\partial u}{\partial x}(4,2) = 2.$$

El resolem usant tres elements finits lineals triangulars $\Omega^1$, $\Omega^2$ i $\Omega^3$ que tenen els vèrtexs següents:

$$\Omega^1 = \{(0,0),(4,0),(4,1)\}, \quad \Omega^2 = \{(0,0),(4,1),(0,2)\}, \quad \Omega^3 = \{(4,1),(4,2),(0,2)\}.$$

Globalment, enumerem els nodes: $p_1 = (0,0)$, $p_2 = (0,2)$, $p_3 = (4,0)$, $p_4 = (4,1)$ i $p_5 = (4,2)$.

(a) Escriviu la matriu de connectivitat. *( 0.5 punts)*

(b) Trobeu les matrius de rigidesa locals i, per a $f$ constant, els vectors de càrregues locals. *( 1.0 punt)*

(c) Escriviu el sistema acoblat. *( 1.0 punt)*

(d) Expliciteu les condicions de contorn. *( 1.0 punt)*

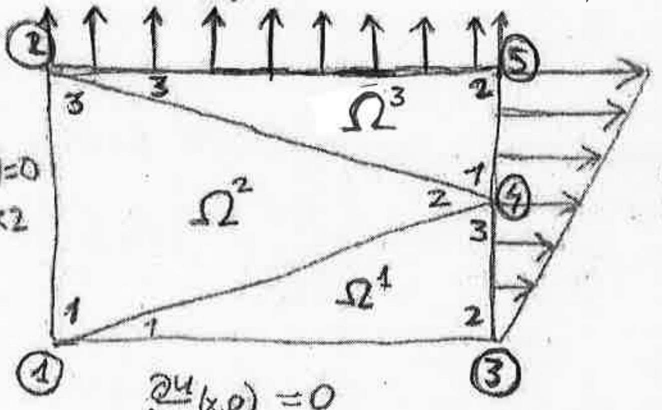
(e) Si $f = 1$ i $U_3 = \frac{32084}{1683}$, trobeu $U_4$ i $U_5$. *( 0.5 punts)*

(1)

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f, \quad \text{on } \Omega = (0,4)\times(0,2)$$

$\frac{\partial u}{\partial y}(x,2) = 2,\ 0<x<4$

So: $a_{11} = a_{22} = 1$
$a_{12} = a_{21} = a_{00} \equiv 0$




$u(0,y)=0$
$0<y<2$

$\frac{\partial u}{\partial x}(4,y) = y$

$\frac{\partial u}{\partial y}(x,0) = 0$

(a) Connectivity matrix $B = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 4 & 2 \\ 4 & 5 & 2 \end{pmatrix}$ nodes $= \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 4 & 0 \\ 4 & 1 \\ 4 & 2 \end{pmatrix}$

(b) TZ-MN-FEM2D, page 48

$\Omega^1$: Linear triangular rectangle element for Poisson equation

$a_{11} = a_{22} = c,\ a_{12} = a_{21} = a_{00} = 0$

$$K^1 = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix} = \left\{\begin{matrix} a=4 \\ b=1 \\ c=1=a_{11}=a_{22} \end{matrix}\right\} = \frac{1}{16}\begin{pmatrix} 2 & -2 & 0 \\ -2 & 34 & -32 \\ 0 & -32 & 32 \end{pmatrix}$$

$\Omega^2$: TZ-MN-FEM2D page 47. Note that, in <u>this</u> case $a_{11} = a_{22} = 1$

$$K^2_{ij} = \frac{1}{4A_2}\left(\overset{(a_{11})}{}\beta_i^2\beta_j^2 + \overset{(a_{22})}{}\gamma_i^2\gamma_j^2\right);\ i,j = 1,2,3.$$

$$\begin{pmatrix} 0 & 0 \\ 4 & 1 \\ 0 & 2 \end{pmatrix}: \quad \begin{aligned} \beta_1^2 &= y_2^2 - y_3^2 = 1-2 = -1 \\ \beta_2^2 &= y_3^2 - y_1^2 = 2-0 = 2 \\ \beta_3^2 &= y_1^2 - y_2^2 = 0-1 = -1 \end{aligned} \quad \Bigg| \quad \begin{aligned} \gamma_1^2 &= -(x_2^2 - x_3^2) = -(4-0) = -4 \\ \gamma_2^2 &= -(x_3^2 - x_1^2) = -(0-0) = 0 \\ \gamma_3^2 &= -(x_1^2 - x_2^2) = -(0-4) = 4 \end{aligned}$$

$$\text{So: } K^2 = \frac{c}{4A_2}\left[\begin{pmatrix}\beta_1^2\\ \beta_2^2\\ \beta_3^2\end{pmatrix}(\beta_1^2\ \beta_2^2\ \beta_3^2) + \begin{pmatrix}\gamma_1^2\\ \gamma_2^2\\ \gamma_3^2\end{pmatrix}(\gamma_1^2\ \gamma_2^2\ \gamma_3^2)\right] = \left\{\begin{matrix} A_2 = 4 \\ c = a_{11} = a_{22} = 1 \end{matrix}\right.$$

$(c=1)$

$$= \frac{1}{16}\left[\begin{pmatrix}-1\\2\\-1\end{pmatrix}(-1,2,-1) + \begin{pmatrix}-4\\0\\4\end{pmatrix}(-4\ 0\ 4)\right]$$

$$= \frac{1}{16}\left[\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 16 & 0 & -16 \\ 0 & 0 & 0 \\ -16 & 0 & 16 \end{pmatrix}\right] = \frac{1}{16}\begin{pmatrix} 17 & -2 & -15 \\ -2 & 4 & -2 \\ -15 & -2 & 17 \end{pmatrix}$$

$K^3$: Same as $K^1$, but now with $a=1, b=4$. Therefore:

$$K^3 = \frac{1}{16}\cdot\begin{pmatrix} 32 & -32 & 0 \\ -32 & 34 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

- Load vectors: T2-MN-FEM2D.pdf page 47. We assume $f_e = f \equiv$ ctnt for all $e=1,2,3$.

Hence: $F^e = \frac{f_e A_e}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$, and so $F^1 = \frac{2}{3}f\begin{pmatrix}1\\1\\1\end{pmatrix}$, $F^2 = \frac{2}{3}f\begin{pmatrix}2\\2\\2\end{pmatrix}$, $F^3 = \frac{2}{3}f\begin{pmatrix}1\\1\\1\end{pmatrix}$

(c) Global stiffness matrix:

$$K = \begin{pmatrix} K_{11}^1 + K_{11}^2 & K_{13}^2 & K_{12}^1 & K_{13}^1 + K_{12}^2 & 0 \\ K_{21} & K_{33}^2 + K_{33}^3 & 0 & K_{32}^2 + K_{31}^3 & K_{32}^3 \\ K_{31} & K_{32} & K_{22}^1 & K_{23}^1 & 0 \\ K_{41} & K_{42} & K_{43} & K_{33}^1 + K_{22}^2 + K_{11}^3 & K_{12}^3 \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{22}^3 \end{pmatrix}$$

← symmetric!

with $K^T = K$. Moreover:

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} = \begin{pmatrix} F_1^1 + F_1^2 \\ F_3^2 + F_3^3 \\ F_2^1 \\ F_3^1 + F_2^2 + F_1^3 \\ F_2^3 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1^1 + Q_1^2 \\ Q_3^2 + Q_3^3 \\ Q_2^1 \\ Q_3^1 + Q_2^2 + Q_1^3 \\ Q_2^3 \end{pmatrix}$$

So, the coupled system casts:

$$\frac{1}{16}\begin{pmatrix} 19 & -15 & -2 & -2 & 0 \\ -15 & 19 & 0 & -2 & -2 \\ -2 & 0 & 34 & -32 & 0 \\ -2 & -2 & -32 & 68 & -32 \\ 0 & -2 & 0 & -32 & 34 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \frac{f}{3}\begin{pmatrix} 6 \\ 6 \\ 2 \\ 8 \\ 2 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix}$$

d) Essential B.C.: $U_1 = 0$, $U_2 = 0$

Natural B.C: $Q_3 = Q_{22}^1$, $Q_4 = Q_{32}^1 + Q_{11}^3$, $Q_5 = Q_{21}^3 + Q_{22}^3$

$$q_{n,2}^1(s) = \left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \nabla u, \hat{n} \right\rangle \Big|_{\Gamma_2^1} \underset{\substack{a_{11}=a_{22}=1 \\ a_{12}=a_{21}=0}}{=} \langle \nabla u, \hat{n} \rangle \Big|_{\Gamma_2^1} = \left(\frac{\partial u}{\partial x}(4,s), \frac{\partial u}{\partial y}(4,s)\right)\cdot\begin{pmatrix}1\\0\end{pmatrix} =$$

$$= \frac{\partial u}{\partial y}(4,s) = s, \quad 0 \le s \le 1.$$

$$q_{n,1}^3(s) = \dots = \langle \nabla u, \hat{n} \rangle \Big|_{\Gamma_1^3} = \left(\frac{\partial u}{\partial x}(4,1+s), \frac{\partial u}{\partial y}(4,1+s)\right)\begin{pmatrix}1\\0\end{pmatrix} = \frac{\partial u}{\partial x}(4,1+s) = 1+s, \quad 0 \le s \le 1$$

$$q_{n,2}^3(s) = \dots = \langle \nabla u, \hat{n} \rangle \Big|_{\Gamma_2^3} = \left(\frac{\partial u}{\partial x}(4-s,2) \;\; \frac{\partial u}{\partial y}(4-s,2)\right)\begin{pmatrix}0\\1\end{pmatrix} = \frac{\partial u}{\partial y}(4-s,2) = 2, \quad 0 \le s \le 4$$

$$Q^1_{22} = \int_0^{h^1_2} q^1_{m,2}(s)\,\psi^1_{22}(s)\,ds \underset{h^1_2=1}{=} \int_0^{h^1_2=1} s\left(1-\frac{s}{h^1_2}\right)ds = \boxed{\frac{1}{6}}$$

: see T3-MN-FEM2D.pdf, page 60.

Alternatively (*) $Q^1_{22} = \frac{1}{3}\cdot 0 + \frac{1}{6}\times 1 = \frac{1}{6}$.

$$Q^1_{32} = \int_0^{h^1_2} q^1_{m,2}(s)\,\psi^1_{32}(s)\,ds = \int_0^{h^1_2=1} s\cdot\frac{s}{h^1_2}\,ds = \boxed{\frac{1}{3}}$$

Alternatively: $Q^1_{32} = \frac{1}{3}\times 1 + \frac{1}{6}\,0 = \frac{1}{3}$

$$Q^3_{11} = \int_0^{h^3_1} q^3_{m,1}(s)\cdot\psi^3_{11}(s)\,ds = \int_0^{h^3_1=1} (1+s)\left(1-\frac{s}{h^3_1}\right)ds = \int_0^1 (1-s^2)\,ds = \left(s-\frac{s^3}{3}\right)\Big|_0^1 = \boxed{\frac{2}{3}}$$

Alternatively: $Q^3_{11} = \frac{1}{3} + \frac{2}{6} = \frac{2}{3}$

$$Q^3_{21} = \int_0^{h^3_1} q^3_{m,1}(s)\cdot\psi^3_{21}(s)\,ds = \int_0^{h^3_1=1} (1+s)\,\frac{s}{h^3_1}\,ds = \int_0^1 s(1+s)\,ds = \left(\frac{s^2}{2}+\frac{s^3}{3}\right)\Big|_0^1 = \boxed{\frac{5}{6}}$$

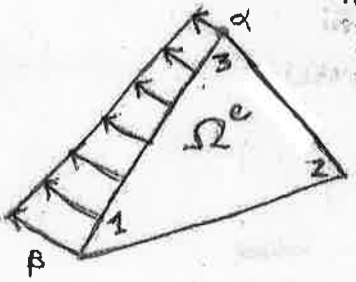
Alternatively: $Q^3_{21} = \frac{1}{6}\,1 + \frac{1}{3}\,2 = \frac{5}{6}$

$$Q^3_{12} = \int_0^{h^3_2} q^3_{m,2}(s)\,\psi^3_{12}(s)\,ds = \int_0^{h^3_2=4} 2\cdot\left(1-\frac{s}{h^3_2}\right)ds = 2\int_0^4\left(1-\frac{s}{4}\right)ds = 2\left(s-\frac{s^2}{8}\right)\Big|_0^4$$

$$= 2\left(4-\frac{16}{8}\right) = 2(4-2) = \boxed{4}$$

Alternatively: $Q^3_{12} = \frac{1}{2}\times 2\times 4 = 4.$

(*) Remark: We apply the "rule for linear flows". For example




$$q^e_{m,3}(s) = \alpha\left(1-\frac{s}{h^e_3}\right) + \beta\,\frac{s}{h^e_3} = \alpha\,\psi^e_{33}(s) + \beta\,\psi^e_{13}(s)$$

$$Q^e_{33} = \int_0^{h^e_3} q^e_{m,3}(s)\,\psi^e_{33}(s)\,ds = \int_0^{h^e_3}\left(\alpha\,\psi^e_{33}(s) + \beta\,\psi^e_{13}(s)\right)\psi^e_{33}(s)\,ds$$

$$= \alpha\int_0^{h^e_3}\left(\psi^e_{33}(s)\right)^2 ds + \beta\int_0^{h^e_3}\psi^e_{13}(s)\,\psi^e_{33}(s)\,ds$$

$$\overset{(\&)}{=} \alpha\, h^e_3\,\frac{2!\,0!}{(2+0+1)!} + \beta\, h^e_3\,\frac{1!\,1!}{3!} = h^e_3\left(\frac{\alpha}{3}+\frac{\beta}{6}\right).$$

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(&) Let $p, q = 0, 1, 2, 3, \dots$ Therefore

$$\int_0^{h^e_\ell}\left(\psi^e_{n\ell}(s)\right)^p\left(\psi^e_{m\ell}(s)\right)^q ds = \begin{cases} 0, \text{ if either } \psi^e_{n\ell}(s)=0 \text{ or } \psi^e_{m\ell}(s)=0 \\ h^e_\ell\,\dfrac{p!\,q!}{(p+q+1)!} \end{cases}$$

For, in the 2nd case: $\int_0^{h^e_\ell}(\psi^e_{n\ell}(s))^p\cdot(\psi^e_{m\ell}(s))^q\,ds = \left\{ x = s/h^e_\ell \; \dots \right\} = h^e_\ell\int_0^1 x^{p+1-1}(1-x)^{q+1-1}\,dx = h^e_\ell\, B(p+1, q+1)$

$$= h^e_\ell\,\frac{\Gamma(p+1)\,\Gamma(q+1)}{\Gamma(p+q+2)} = h^e_\ell\,\frac{p!\,q!}{(p+q+1)!}$$

$$Q_{13}^{e} = \int_0^{h_3^e} q_{n,13}^{e}(s)\,\psi_{13}^{e}(s)\,ds = \int_0^{h_3^e}\left(\alpha\,\psi_{33}^{e}(s) + \beta\,\psi_{13}^{e}(s)\right)\psi_{13}^{e}(s)\,ds$$

$$= \alpha\int_0^{h_3^e}\psi_{33}^{e}(s)\,\psi_{13}^{e}(s)\,ds + \beta\int_0^{h_3^e}\left(\psi_{13}^{e}(s)\right)^2 ds$$

$$= \alpha\, h_3^e\,\frac{1!\,1!}{3!} + \beta\, h_3^e\,\frac{2!\,0!}{(2+0+1)!} = h_3^e\left(\frac{\alpha}{6}+\frac{\beta}{3}\right)$$

———— End of Remark ————

So, the essential B.C. turn out to be,

$$Q_3 = Q_{22}^{1} = \tfrac{1}{6}$$

$$Q_4 = Q_{32}^{1} + Q_{11}^{3} = \tfrac{1}{3} + \tfrac{2}{3} = 1$$

$$Q_5 = Q_{21}^{3} + Q_{22}^{5} = \tfrac{5}{6} + 4 = \tfrac{29}{6}$$

(e) Take $f=1$. If $U_3 = \frac{32084}{1683}$, we must solve the corresponding reduced system, i.e.:

$$\frac{1}{16}\begin{pmatrix} 68 & -32 \\ -32 & 34 \end{pmatrix}\begin{pmatrix} U_4 \\ U_5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{29}{6} \end{pmatrix} \underbrace{- \frac{1}{16}\begin{pmatrix} -2 & -2 & -32 \\ 0 & -2 & 0 \end{pmatrix}\cdot\begin{pmatrix} U_1 = 0 \\ U_2 = 0 \\ U_3 = \frac{32084}{1683} \end{pmatrix}}$$

$$\Leftrightarrow \begin{pmatrix} 17/4 & -2 \\ -2 & \frac{17}{8} \end{pmatrix}\begin{pmatrix} U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} \frac{70339}{1683} \\ 11/2 \end{pmatrix}$$

in MATLAB:

-K( freeNods, fixedNods) * u(fixedNods)

$$\Delta = \begin{vmatrix} 17/4 & -2 \\ -2 & \frac{17}{8} \end{vmatrix} = \frac{161}{32}, \quad \Delta_{U_4} = \begin{vmatrix} \frac{70339}{1683} & -2 \\ \frac{11}{2} & \frac{17}{8} \end{vmatrix} = \frac{79051}{792}, \quad \Delta_{U_5} = \begin{vmatrix} \frac{17}{4} & \frac{70339}{1683} \\ -2 & \frac{11}{2} \end{vmatrix} = \frac{144024}{13464}$$

$$\therefore\quad U_4 = \frac{\Delta_{U_4}}{\Delta} = \frac{1964}{99}, \quad U_5 = \frac{\Delta_{U_5}}{\Delta} = \frac{35780}{1683}$$

Finally, the complete solution is given by:

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = \frac{32084}{1683}, \quad U_4 = \frac{1964}{99}, \quad U_5 = \frac{35780}{1683}. \qquad \square$$