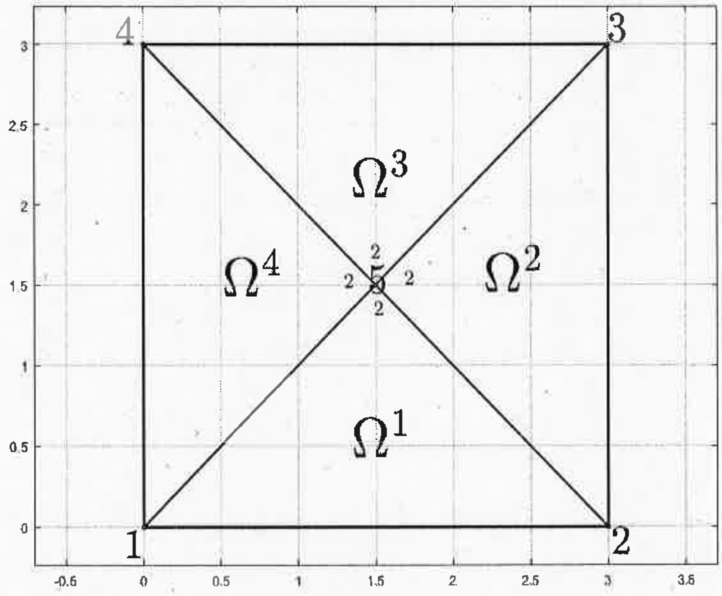
MÈTODES NUMÈRICS: **Ex-Final Q2-2018-19** (a)

Name and surnames:

(2) Consider the domain shown in the figure below, and with the elements, nodes and local and global numbering plotted there.



We want to solve the thermal problem defined by the equation

$$-4\frac{\partial^2 T}{\partial x^2} - 4\frac{\partial^2 T}{\partial y^2} = 0$$

with boundary conditions given by:

- At the lines joining the nodes 3,4 and 1, there is convection BC with $\beta = 1$ and $T_\infty = 30$

- $T = 100$ at the lines joining the nodes 1,2 and 3.

Then, compute:

(i) The stiffness matrix of the first element K^1.

(ii) The 5th row of the global stiff matrix before applying the BC.

(iii) The value of the coefficients for Q^4_{33} of the local system for Ω^4 as a function of T_1 and T_4.

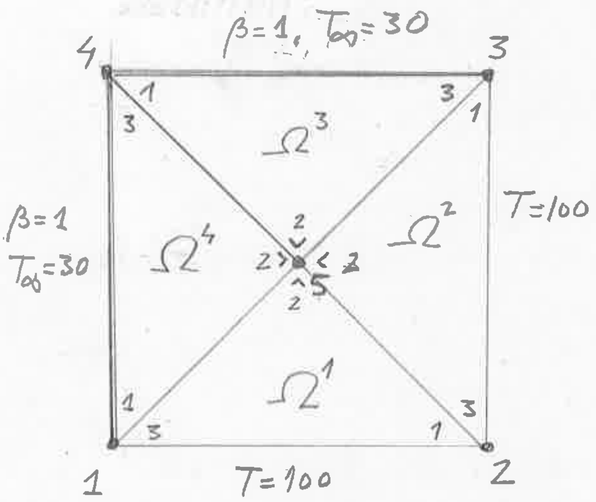
(iv) The value of the coefficients for Q^3_{13} of the local system for Ω^3 as a function of T_3 and T_4.

(v) The value of T_4.

Results:	
$[K^1] =$	$2\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$
5th row :	$2\begin{pmatrix} -2 & -2 & -2 & -2 & 8 \end{pmatrix}$
$Q^4_{33} = a_1T_1 + a_2T_4 + a_3$	coeff a_i: $a_1 = -\frac{1}{2}$, $a_2 = -1$, $a_3 = 45$
$Q^3_{13} = b_1T_4 + b_2T_3 + b_3$	coeff b_i: $b_1 = -1$, $b_2 = -\frac{1}{2}$, $b_3 = 45$
$T_4=$	58

(Hint: a_i and b_i are among the values: $2, -2, 1, -1, 1/2, -1/2, 1/6, -1/6, 35, 40, 45, 50, 85, 90, 95, 100, 105$.)

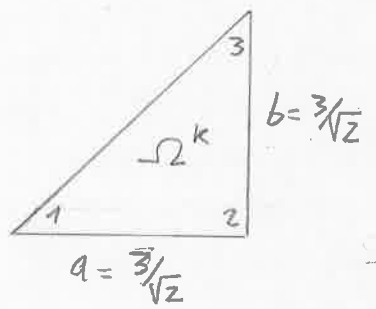
(2 points)



$$N = \begin{pmatrix} 0 & 0 \\ 3 & 0 \\ 3 & 3 \\ 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 5 & 2 \\ 4 & 5 & 3 \\ 1 & 5 & 4 \end{pmatrix}$$

$$k_c = 4 = c$$

(i)



$$K^k = \frac{c}{2ab} \begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}.$$

So: $$K^1 = \frac{4}{2 \frac{3}{\sqrt{2}} \cdot \frac{3}{\sqrt{2}}} \begin{pmatrix} 9/2 & -9/2 & 0 \\ -9/2 & 9 & -9/2 \\ 0 & -9/2 & 9/2 \end{pmatrix} = \frac{4}{9} \begin{pmatrix} 9/2 & -9/2 & 0 \\ -9/2 & 9 & -9/2 \\ 0 & -9/2 & 9/2 \end{pmatrix} = 2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\boxed{K^1 = 2 \cdot \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = K^2 = K^3 = K^4}$$

(ii) 5th Row of the global stiff matrix.

$$K_{5,\cdot} = \left(K^1_{23}+K^4_{21} \quad K^1_{21}+K^2_{23} \quad K^2_{21}+K^3_{23} \quad K^3_{21}+K^4_{23} \quad K^1_{22}+K^2_{22}+K^3_{22}+K^4_{22} \right)$$

$$= 2\left(-2 \quad -2 \quad -2 \quad -2 \quad 8\right) = \boxed{\left(-4 \quad -4 \quad -4 \quad -4 \quad 16\right)}$$

We shall need the 4th row as well...

$$K_{4,\cdot} = \left(K^4_{31} \quad 0 \quad K^3_{13} \quad K^3_{11}+K^4_{33} \quad K^3_{12}+K^4_{32} \right) = \boxed{\left(0 \quad 0 \quad 0 \quad 4 \quad -4\right)}$$

(iii) $$Q_{33}^{4} = \left(-\frac{\beta T_4}{3} - \frac{\beta T_1}{6} + \frac{\beta T_\infty}{2}\right) h_3^4 = \left\{\begin{matrix}\beta = 1 \\ T_\infty = 30 \\ h_3^4 = 3\end{matrix}\right\} = \boxed{-\frac{1}{2}T_2 - T_4 + 45}$$

So: $\boxed{a_1 = -\frac{1}{2},\ a_2 = -1,\ a_3 = 45}$

(iv) $$Q_{13}^{3} = \left(-\frac{\beta T_4}{3} - \frac{\beta T_3}{6} + \frac{\beta T_\infty}{2}\right) h_3^3 = \left\{\begin{matrix}\beta = 1 \\ T_\infty = 30 \\ h_3^3 = 3\end{matrix}\right\} = \boxed{-\frac{1}{2}T_3 - T_4 + 45}$$

So: $\boxed{b_1 = -1,\ b_2 = -\frac{1}{2},\ T_\infty = 45}$

(v) Boundary Conditions:

Essential: $T_1 = T_2 = T_3 = 100$

Natural: $Q_3 = Q_{33}^4 + Q_{13}^3 = -\frac{1}{2}T_2 - \frac{1}{2}T_3 - 2T_4 + 90 = -10 - 2T_4$

$Q_5 = 0$ (because of the flows balance on the internal edges)

Reduced system:

$$\begin{pmatrix} 4 & -4 \\ -4 & 16 \end{pmatrix}\begin{pmatrix} T_4 \\ T_5 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ -4 & -4 & -4 \end{pmatrix}\begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix} + \begin{pmatrix} -10-2T_4 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{matrix} 4T_4 - 4T_5 = -10 - 2T_4 \\ -4T_4 + 16T_5 = 1200 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} 6T_4 - 4T_5 = -10 \\ -4T_4 + 16T_5 = 1200 \end{matrix}$$

and solve by reduction:

$$\begin{array}{l} 24T_4 - 16T_5 = -40 \\ \underline{-4T_4 + 16T_5 = 1200} \\ 20T_4 \qquad\quad = 1160 \end{array} \Longrightarrow T_4 = \frac{116}{2} = \boxed{58}$$

$$T_5 = \frac{1}{4}(6T_4 + 10) = \frac{1}{4}(348 + 10) = \frac{358}{4} = \frac{179}{2} = \boxed{89.5}$$