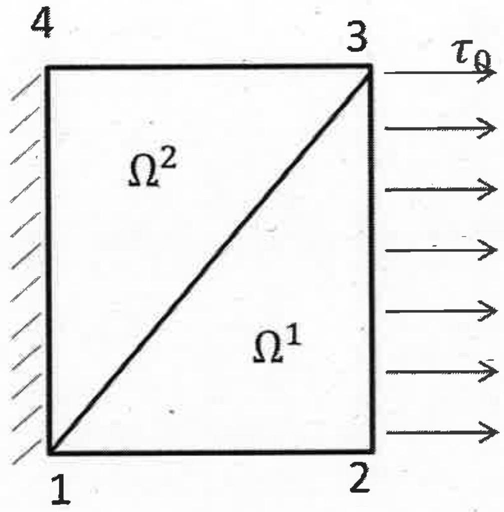
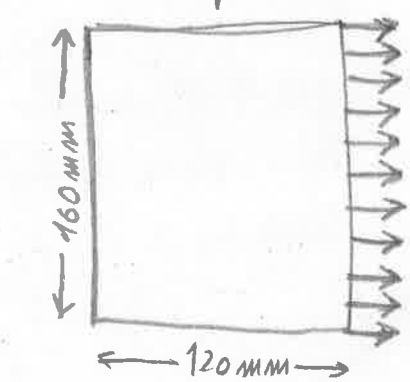
Example:

Consider a rectangular piece of 120x160 mm and thickness 0.036mm. It is fixed to the wall (left) and pulled by a constant traction τ_0 =1000N/mm. Compute the **displacements** if the material has $E = 30 \cdot 10^6 N/mm^2$ and $\nu = 0.25$

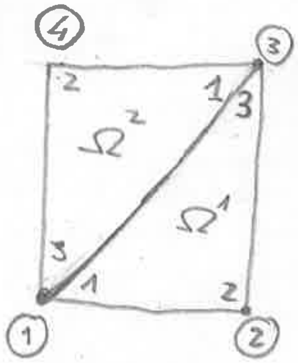


Stress Example



$$\tau_0 = 10^3 \frac{N}{mm}, \quad E = 3.0\times 10^7 \frac{N}{mm^2}, \quad \nu = 0.25, \quad t_h = 3.6\times 10^{-2}\, mm$$

Left edge fixed.



$$\text{nodes} = \begin{pmatrix} 0 & 0 \\ 120 & 0 \\ 120 & 160 \\ 0 & 160 \end{pmatrix}$$

$$\text{elem} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{pmatrix}$$

Stress problem:

$$c_{11} = c_{22} = \frac{E}{1-\nu^2} = \frac{3\times 10^7}{1-0.25^2} = \frac{16}{5}\times 10^7 = 3.2\times 10^7$$

$$c_{12} = c_{21} = \nu c_{11} = 0.8\times 10^7$$

$$c_{33} = \frac{E}{2(1+\nu)} = \frac{3\times 10^7}{2(1+0.25)} = \frac{3\times 10^7}{2.5} = \frac{6}{5}\times 10^7 = 1.2\times 10^7, \text{ hence: } C = \left(\begin{array}{cc|c} c_1 & c_{12} & \\ c_2 & c_{22} & \\ \hline & & c_{33} \end{array}\right) = 4\cdot 10^6 \left(\begin{array}{cc|c} 8 & 2 & \\ 2 & 8 & \\ \hline & & 3 \end{array}\right)$$

- Local stiffness matrices:

$$K^1 : \begin{pmatrix} 0 & 0 \\ 120 & 0 \\ 120 & 160 \end{pmatrix}$$

$$\beta_1^1 = y_2^1 - y_3^1 = 0 - 160 = -160, \qquad \gamma_1^1 = -(x_2^1 - x_3^1) = -(120-120) = 0,$$

$$\beta_2^1 = y_3^1 - y_1^1 = 160 - 0 = 160, \qquad \gamma_2^1 = -(x_3^1 - x_1^1) = -(120-0) = -120,$$

$$\beta_3^1 = y_1^1 - y_2^1 = 0 - 0 = 0. \qquad \gamma_3^1 = -(x_1^1 - x_2^1) = -(0-120) = 120.$$

$$\text{Area}_1 : A_1 = \tfrac{1}{2}\, 120\times 160 = 9600\ mm^2.$$

$$B_1 = \frac{1}{2A_1}\begin{pmatrix} \beta_1^1 & 0 & \beta_2^1 & 0 & \beta_3^1 & 0 \\ 0 & \gamma_1^1 & 0 & \gamma_2^1 & 0 & \gamma_3^1 \\ \gamma_1^1 & \beta_1^1 & \gamma_2^1 & \beta_2^1 & \gamma_3^1 & \beta_3^1 \end{pmatrix} = \frac{1}{480}\begin{pmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -4 & -3 & 4 & 3 & 0 \end{pmatrix},$$

$$K^1 = A_1 t_h B_1^T C B_1 = 6\cdot 10^3 \cdot \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & -4 \\ 4 & 0 & -3 \\ 0 & -3 & 4 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} \cdot \left(\begin{array}{cc|c} 8 & 2 & \\ 2 & 8 & \\ \hline & & 3 \end{array}\right) \cdot \begin{pmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -4 & -3 & 4 & 3 & 0 \end{pmatrix}$$

Therefore:

$$K^1 = 6\times 10^3 \cdot \left(\begin{array}{cc|cc|cc} 128 & 0 & -128 & 24 & 0 & -24 \\ 0 & 48 & 36 & -48 & -36 & 0 \\ \hline -128 & 36 & 155 & -60 & -27 & 24 \\ 24 & -48 & -60 & 120 & 36 & -72 \\ \hline 0 & -36 & -27 & 36 & 27 & 0 \\ -24 & 0 & 24 & -72 & 0 & 72 \end{array}\right) = K^2$$

since, for the given mesh $\beta_i^2 = -\beta_i^1$, and $\gamma_i^2 = -\gamma_i^1$, $i = 1,2,3$.

(*)

- Global Stiffness Matrix

$$\mathbb{K} = \begin{pmatrix} K_{11}^1 + K_{33}^2 & K_{12}^1 & K_{13}^1 + K_{31}^2 & K_{32}^2 \\ * & K_{22}^1 & K_{23}^1 & 0 \\ * & * & K_{33}^1 + K_{11}^2 & K_{12}^2 \\ * & * & * & * \end{pmatrix}$$

Remark 1. $\mathbb{K} = \mathbb{K}^T$ and now K_{ij}^e $(e=1,2)$ are the 2x2 blocs in the local stiffness matrices (*)

$$= 6\times 10^3 \cdot \left(\begin{array}{cc|cc|cc|cc} 155 & 0 & -128 & 24 & 0 & -60 & -27 & 36 \\ 0 & 120 & 36 & -48 & -60 & 0 & 24 & -72 \\ \hline -128 & 36 & 155 & -60 & -27 & 24 & 0 & 0 \\ 24 & -48 & -60 & 120 & 36 & -72 & 0 & 0 \\ \hline 0 & -60 & -27 & 36 & 155 & 0 & -128 & 24 \\ -60 & 0 & 24 & -72 & 0 & 120 & 36 & -48 \\ \hline -27 & 24 & 0 & 0 & -128 & 36 & 155 & -60 \\ 36 & -72 & 0 & 0 & 24 & -48 & -60 & 120 \end{array}\right)$$

- Boundary Conditions
  - Essential B.C. : $(U_1, V_1) = (0,0)$, $(U_4, V_4) = (0,0)$.
  - Natural B.C.

$$Q^1 = \begin{pmatrix} Q_1^1 \\ \hline Q_2^1 \\ \hline Q_3^1 \end{pmatrix} = \frac{h_2^1}{2} \tau_0 \begin{pmatrix} 0 \\ 0 \\ \hline 1 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix} = 8\times 10^4 \begin{pmatrix} 0 \\ 0 \\ \hline 1 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} Q_2^1 \\ \hline Q_2^2 \\ \hline Q_2^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \hline Q_2^2 \\ \hline Q_2^3 \end{pmatrix}$$

$$Q = \begin{pmatrix} Q_1 \\ Q_2 = Q_2^1 \\ Q_3 = Q_3^1 + Q_1^2 \\ Q_4 = Q_2^2 \end{pmatrix} \overset{(*)}{=} \begin{pmatrix} Q_1 \\ \hline 8\times 10^4 \\ 0 \\ \hline 8\times 10^4 \\ 0 \\ \hline Q_4 \end{pmatrix}$$

$$Q_3 = Q_3^1 + Q_1^2$$
$$= \underbrace{Q_{31}^1}_{=0} + Q_{32}^1 + \underbrace{Q_{33}^1}_{=0} + \underbrace{Q_{11}^2}_{=0} + \underbrace{Q_{12}^2}_{=0} + \underbrace{Q_{13}^2}_{=0} = Q_{32}^1 = \int_0^{h_2^1} \psi_{32}^{1(s)} \begin{pmatrix} \tau_0 \\ 0 \end{pmatrix} ds = \begin{pmatrix} \tau_0 \\ 0 \end{pmatrix} \int_0^{h_2^1} \frac{s}{h_2^1} ds = \frac{h_2^1}{2} \begin{pmatrix} \tau_0 \\ 0 \end{pmatrix}$$
$$= \frac{160}{2} \times 10^3 = 8\times 10^4 \quad (\text{Id. for } Q_2 = Q_2^1)$$

Reduced system

Remark 2. $F=0$: no internal forces (for ex. the weight) are taken into account.

$$\cancel{6\times10^{9}}\times\left(\begin{array}{cc|cc}155 & -60 & -27 & 24\\ -60 & 120 & 36 & -72\\ \hline -27 & 36 & 155 & 0\\ 24 & -72 & 0 & 120\end{array}\right)\cdot\begin{pmatrix}U_2\\ V_2\\ \hline U_3\\ V_3\end{pmatrix} = \cancel{8\times10^{4}}\ \frac{40}{3}\begin{pmatrix}1\\ 0\\ \hline 1\\ 0\end{pmatrix}$$

Solution:
$U_2 = 1.1291\times10^{-1}$ mm
$V_2 = 1.9637\times10^{-2}$ mm
$U_3 = 1.0113\times10^{-1}$ mm
$V_3 = -1.0800\times10^{-2}$ mm

So the displacements are (in mm),

$$\begin{array}{ll} U_1 = 0 & V_1 = 0\\ U_2 = 1.1291\times10^{-1} & V_2 = 1.9637\times10^{-2}\\ U_3 = 1.0113\times10^{-1} & V_3 = -1.0800\times10^{-2}\\ U_4 = 0 & V_4 = 0\end{array}$$

- Stress:

Remark 3. When using linear triangular elements, $\sigma=\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{pmatrix}$ (and strains $\gamma=\begin{pmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{pmatrix}$) are constants on each element.

$$\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{pmatrix} = \left(\begin{array}{cc|c}c_{11} & c_{12} & \\ c_{12} & c_{22} & \\ \hline & & c_{33}\end{array}\right)\cdot\begin{pmatrix}\partial_x u\\ \partial_y v\\ \partial_y u+\partial_x v\end{pmatrix}$$

$$\begin{pmatrix}\sigma_x^e\\ \sigma_y^e\\ \sigma_{xy}^e\end{pmatrix} = C\cdot B_e\cdot\begin{pmatrix}u_1^e\\ v_1^e\\ u_2^e\\ v_2^e\\ u_3^e\\ v_3^e\end{pmatrix}$$

$e=1$:
$$\begin{pmatrix}\sigma_x^1\\ \sigma_y^1\\ \sigma_{xy}^1\end{pmatrix} = \frac{4\times10^{6}}{480}\left(\begin{array}{cc|c}8 & 2 & \\ 2 & 8 & \\ \hline & & 3\end{array}\right)\cdot\begin{pmatrix}-4 & 0 & 4 & 0 & 0 & 0\\ 0 & 0 & 0 & -3 & 0 & 3\\ 0 & -4 & -3 & 4 & 3 & 0\end{pmatrix}\cdot\begin{pmatrix}u_1^1 = U_1 = 0\\ v_1^1 = V_1 = 0\\ u_2^1 = U_2 = 1.1291\times10^{-1}\\ v_2^1 = V_2 = 1.9637\times10^{-2}\\ u_3^1 = U_3 = 1.0113\times10^{-1}\\ v_3^1 = V_3 = -1.0800\times10^{-2}\end{pmatrix}$$

$$\frac{25\times10^{3}}{3}\cdot\begin{pmatrix}-32 & 0 & 32 & -6 & 0 & 6\\ -8 & 0 & 8 & -24 & 0 & 24\\ 0 & -12 & -9 & 12 & 9 & 0\end{pmatrix}\cdot\begin{pmatrix}0\\ 0\\ \hline 1.1291\times10^{-1}\\ 1.9637\times10^{-2}\\ \hline 1.0113\times10^{-1}\\ -1.0800\times10^{-2}\end{pmatrix} = \begin{pmatrix}2.8588\times10^{4}\\ 1.4400\times10^{3}\\ 1.0800\times10^{3}\end{pmatrix} \text{ (in N/mm}^2\text{)}$$

$e=2$
$$\begin{pmatrix}\sigma_x^2\\ \sigma_y^2\\ \sigma_{xy}^2\end{pmatrix} = -\frac{25\times10^{3}}{3}\begin{pmatrix}-32 & 0 & 32 & -6 & 0 & 6\\ -8 & 0 & 8 & -24 & 0 & 24\\ 0 & -12 & -9 & 12 & 9 & 0\end{pmatrix}\cdot\begin{pmatrix}u_1^2 = U_3 = 1.0113\times10^{-1}\\ v_1^2 = V_3 = -1.0800\times10^{-2}\\ u_2^2 = U_4 = 0\\ v_2^2 = V_4 = 0\\ u_3^2 = U_1 = 0,\ v_3^2 = V_1 = 0\end{pmatrix} = \begin{pmatrix}2.6968\times10^{4}\\ 6.7419\times10^{3}\\ -1.0800\times10^{3}\end{pmatrix}$$

(in N/mm²)

- Von Mises Stress

$e=1:\ \sigma_{VM}^{1} = \sqrt{(\sigma_x^1)^2+(\sigma_y^1)^2-\sigma_x^1\sigma_y^1+3(\sigma_{xy}^1)^2} = 2.7958\times10^5\ N/mm^2$

$e=2:\ \sigma_{VM}^{2} = \sqrt{(\sigma_x^2)^2+(\sigma_y^2)^2-\sigma_x^2\sigma_y^2+3(\sigma_{xy}^2)^2} = 2.4380\times10^4\ N/mm^2$

$\square$