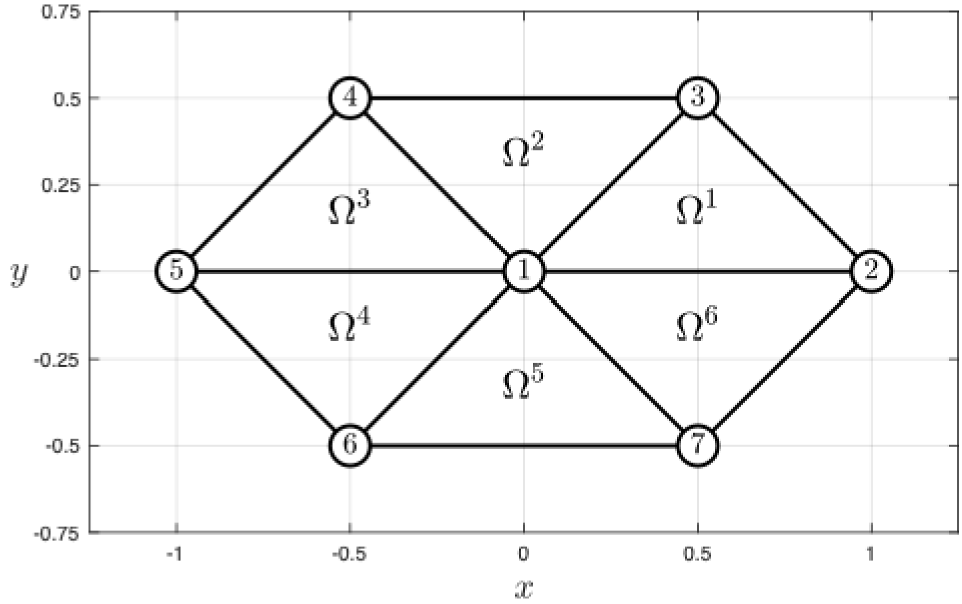
240032-ex-reava-2023-24

P1

Consider the hexagonal domain shown in the figure, with nodes $(0,0)$, $(1,0)$, $(1/2,1/2)$, $(-1/2,1/2)$, $(-1,0)$, $(-1/2,-1/2)$, $(1/2,-1/2)$. We want to solve the Poisson equation

$$-c\Delta u = 5$$

with $c = 11$ using 6 triangular elements by the right isosceles triangles of the figure.



Part (a)

(a) (2 points) Let us choose a local numbering in element Ω^1 given by the global nodes 2, 3 and 1. Let K^1 be the local stiffness matrix at this element. Which is the value of $K^1_{1,1}$?

- -1.10e+01
- 5.50e+00 ☐
- Empty answer (no penalty)
- 1.10e+01
- -5.50e+00

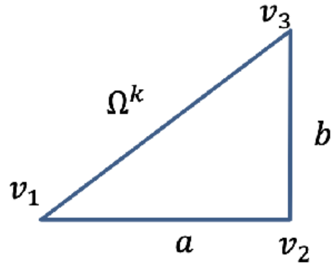
Solution

To compute the local stiffness matrix of Ω^1, we can use the explicit formula that apply when the element Ω^k is a right triangle, with local node 2 (global node 3) placed at the right angle's vertex (see the figure above) and the coefficients of the model equation are constants and equal to $a^k_{1,1} = a^k_{2,2} = c$, $a^k_{1,2} = a^k_{2,1} = a^k_{0,0} = 0$. This formula has been discussed in class and can be found in the notes of the FEM available at the *Numerical Factory*, page 28:

Computing the Integrals: Triangles

- In the case of a general **linear triangular rectangle element** for the **Poisson's Equation**

$$(a_{11} = a_{22} = c, \quad a_{12} = a_{21} = a_{00} = 0)$$



The formula:

$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

Simplifies to:

$$K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}$$

$$F^k = \frac{f_k A_k}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{f_k ab}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Here with $a = b = \sqrt{\frac{1}{2^2} + \frac{1}{2^2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, and $c = 11$. Thus, application of the above formula for K^k (here, for $k = 1$) yields,

$$K^1 = \frac{11}{2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}}\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix} = 11\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix} = \frac{11}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

and therefore $K^1_{1,1} = \frac{11}{2} = 5.50$.

```
clearvars
close all

a = sqrt(2.0)/2.0;
b = a;
c = 11;
f = 5;

nodes = [
    0,0;
    1,0;
    1/2, 1/2;
    -1/2, 1/2;
    -1,0;
    -1/2,-1/2;
```

```
    1/2,-1/2
    ];

elem = [
    2, 3, 1;
    4, 1, 3;
    1, 4, 5;
    5, 6, 1;
    7, 1, 6;
    1, 7, 2
    ];
numNodes = size(nodes,1); %Number of nodes
numElem = size(elem,1);   %Number of elements


%numbering = 1;
%plotElements(nodes,elem,numbering)
%axis equal
%hold off
Ke = c * [
    b^2, -b^2, 0;
    -b^2, a^2 + b^2, -a^2;
    0, -a^2, a^2
    ]/a/b/2;

Fe = f*a*b*[1;1;1]/6;

fprintf('%12.8f%12.8f%12.8f%12.8f\n',[Ke,Fe].')
```

```
  5.50000000 -5.50000000  0.00000000  0.41666667
 -5.50000000 11.00000000 -5.50000000  0.41666667
  0.00000000 -5.50000000  5.50000000  0.41666667
```

```
fprintf('(a) K1(1,1) = %.2f\n', Ke(1,1))
```

```
(a) K1(1,1) = 5.50
```

Part (b)

(b) (2 points) Give the value of $K_{1,1}$ of the assembled stiffness matrix K.

- Empty answer (no penalty)
- 1.10e+01
- -1.10e+01
- 1.65e+01
- 4.40e+01 ☐

Hint: The sum of any row or column is zero.

Solution

It is clear from the figure that the local stiffness matrices are all aqual to K^1, i.e., $K^e = K^1$ for $e = 2, 3, 4, 5, 6$. We can compute the whole stiffness matrix:

$$K = \begin{pmatrix} K^1_{3,3}+K^2_{2,2}+K^3_{1,1}+K^4_{3,3}+K^5_{2,2}+K^6_{1,1} & K^1_{3,1}+K^6_{1,3} & K^1_{3,2}+K^2_{2,3} & K^2_{2,1}+K^3_{1,2} & K^3_{1,3}+K^4_{3,1} & K^4_{3,2}+K^5_{2,3} & K^5_{2,1}+K^6_{1,2} \\ K^1_{1,3}+K^6_{3,1} & K^1_{1,1}+K^6_{3,3} & K^1_{1,2} & 0 & 0 & 0 & K^6_{3,2} \\ K^1_{2,3}+K^2_{3,2} & K^1_{2,1} & K^1_{2,2}+K^2_{3,3} & K^2_{3,1} & 0 & 0 & 0 \\ K^2_{1,2}+K^3_{2,1} & 0 & K^2_{1,3} & K^2_{1,1}+K^3_{2,2} & K^3_{2,3} & 0 & 0 \\ K^3_{3,1}+K^4_{1,3} & 0 & 0 & K^3_{3,2} & K^3_{3,3}+K^4_{1,1} & K^4_{1,2} & 0 \\ K^4_{2,3}+K^5_{3,2} & 0 & 0 & 0 & K^4_{2,1} & K^4_{2,2}+K^5_{3,3} & K^5_{3,1} \\ K^5_{1,2}+K^6_{2,1} & K^6_{2,3} & 0 & 0 & 0 & K^5_{1,3} & K^5_{1,1}+K^6_{2,2} \end{pmatrix}$$

$$= \begin{pmatrix} 44 & 0 & -11 & -11 & 0 & -11 & -11 \\ 0 & 11 & -11/2 & 0 & 0 & 0 & -11/2 \\ -11 & -11/2 & 33/2 & 0 & 0 & 0 & 0 \\ -11 & 0 & 0 & 33/2 & -11/2 & 0 & 0 \\ 0 & 0 & 0 & -11/2 & 11 & -11/2 & 0 \\ -11 & 0 & 0 & 0 & -11/2 & 33/2 & 0 \\ -11 & -11/2 & 0 & 0 & 0 & 0 & 33/2 \end{pmatrix}.$$

However, we recall that we're only asked for the $K_{1,1}$ component of the global (i.e., the assembled) stiffness matrix. Hence,

$$K_{1,1} = K^1_{3,3}+K^2_{2,2}+K^3_{1,1}+K^4_{3,3}+K^5_{2,2}+K^6_{1,1} = \frac{11}{2}+11+\frac{11}{2}+\frac{11}{2}+11+\frac{11}{2} = 22+22 = 44.$$

is the answer to the question.

```
K = zeros(numNodes);
F = zeros(numNodes,1);
Q = zeros(numNodes,1);

for e = 1:numElem
    rows = [elem(e,1); elem(e,2); elem(e,3)];
    cols = rows;
    K(rows,cols) = K(rows,cols) + Ke;
    F(rows) = F(rows) + Fe;
end

fprintf('%10.5f%10.5f%10.5f%10.5f%10.5f%10.5f%10.5f%10.5f\n',[K, F].')
```

```
  44.00000   0.00000 -11.00000 -11.00000   0.00000 -11.00000 -11.00000   2.50000
   0.00000  11.00000  -5.50000   0.00000   0.00000   0.00000  -5.50000   0.83333
 -11.00000  -5.50000  16.50000   0.00000   0.00000   0.00000   0.00000   0.83333
 -11.00000   0.00000   0.00000  16.50000  -5.50000   0.00000   0.00000   0.83333
   0.00000   0.00000   0.00000  -5.50000  11.00000  -5.50000   0.00000   0.83333
 -11.00000   0.00000   0.00000   0.00000  -5.50000  16.50000   0.00000   0.83333
 -11.00000  -5.50000   0.00000   0.00000   0.00000   0.00000  16.50000   0.83333
```

Part (c)

(c) (3 points) In this section we consider the essential boundary conditions givren by only $u = 7$ at the boundary of the hexagon, that is, at the edges that join the vertices 2, 3, 4, 5, 6, and 7. Which is the value of u at node 1?

- Empty answer (no penalty)
- 7.3432e+00
- 7.2789e+00
- 6.7936e+00
- 7.0568e+00 ☐

Solution

First we shall assemble the local force vectors. To compute the local force vectors F^e, $e = 1,2,3,4,5,6,7$, we shall use the formulas shown in Part (a), i.e.,

$$F^e = \frac{f^e ab}{6}\begin{pmatrix}1\\1\\1\end{pmatrix},$$

with $f^e = 5$ for $e = 1,2,3,4,5,6$; $a = b = \frac{\sqrt{2}}{2}$, so $F^e = \frac{5}{12}\begin{pmatrix}1\\1\\1\end{pmatrix}$, $e = 1,2,3,4,5,6$. Then,

$$F = \begin{pmatrix}F_1\\F_2\\F_3\\F_4\\F_5\\F_6\end{pmatrix} = \begin{pmatrix}F_3^1 + F_2^2 + F_1^3 + F_3^4 + F_2^5 + F_1^6\\ F_1^1 + F_3^6\\ F_2^1 + F_3^2\\ F_1^2 + F_2^3\\ F_3^3 + F_1^4\\ F_2^4 + F_3^5\\ F_1^5 + F_2^6\end{pmatrix} = \frac{5}{6}\begin{pmatrix}3\\1\\1\\1\\1\\1\\1\end{pmatrix}.$$

Note that, actually, we need only the first component of F, i.e.,

$$F_1 = F_3^1 + F_2^2 + F_1^3 + F_3^4 + F_2^5 + F_1^6 = \frac{5}{2}.$$

As node 1 is the only free node, the reduced system is derived straight from the first equation, i.e.,

$$K_{1,1}U_1 = F_1 + Q_1 - K_{1,2}U_2 - K_{1,3}U_3 - K_{1,4}U_4 - K_{1,5}U_5 - K_{16}U_6 - K_{1,7}U_7$$

and note that the natural B.C. reduces to $Q_1 = 0$, for this is an internal node, and flux cancellation applies. Substitution of $K_{1,1} = 44$, $K_{1,2} = 0$, $K_{1,3} = -11$, $K_{1,4} = -11$, $K_{1,5} = 0$, $K_{1,6} = -11$, $K_{1,7} = -11$; $U_2 = U_3 = U_4 = U_5 = U_6 = U_7 = 7$; $F_1 = 5/2$; $Q_1 = 0$, gives the equation

$$44U_1 = 5/2 + 0 - 0\times 7 - 11\times 7 - 11\times 7 - 0\times 7 - 11\times 7 - 11\times 7 = \frac{5}{2} + 44\times 7,$$

which yields

$$U_1 = \frac{5}{88} + 7 = \frac{621}{88} = 7.056\overline{81}.$$

```
fixedNods = [2,3,4,5,6,7];
freeNods = setdiff(1:numNodes,fixedNods);
```

```
%Boundary conditions

%Essential B.C.
Q(1) = 0;

%Natural B.C.
u = zeros(numNodes,1);
u(fixedNods) = 7;

%Reduced system
Fm = F(freeNods) + Q(freeNods) - K(freeNods,fixedNods)*u(fixedNods);
Km =K(freeNods,freeNods);

%Solve the reduced system
um = Km\Fm;

u(freeNods) = um;

fprintf('(c) U(1) = %.12f\n',u(1))
```

```
(c) U(1) = 7.056818181818
```

Part (d)

(d) (3 points) In this section, we replace the essential boundary conditions only on edges 2-3 and 3-4 by the following natural conditions. Fisrt assume that on the edge 2-3 we take an insulating condition $q_n = 0$.

Second, on the edge 3-4 we consider the following condition $\frac{\partial u}{\partial y} = 1 + x$. Which is the value of Q_3 in the assembled system?

- Empty answer (no penalty)
- 6.4167e+00 □
- 4.5833e+00
- 8.2500e+00
- 6.6819e+00

Solution

- $q^1_{n,1} \equiv 0$ on Γ^1_1 (edge 2-3), so $Q^1_{2,1} = 0$.
- $Q^2_{3,3} = h^2_3\left(\frac{1}{3}q^2_{n,3}(1/2, 1/2) + \frac{1}{6}q^2_{n,3}(-1/2, 1/2)\right) = 1 \cdot \left(\frac{1}{3} \times \frac{33}{2} + \frac{1}{6} \times \frac{11}{2}\right) = \frac{11}{2}\left(1 + \frac{1}{6}\right) = \frac{77}{12}$. Since $h^2_3 = 1$, $c = 11$, and:

$$q^2_{n,3}\left(\frac{1}{2}, \frac{1}{2}\right) = \left\langle c\begin{pmatrix} \frac{\partial u}{\partial x}\left(\frac{1}{2}, \frac{1}{2}\right) \\ \frac{\partial u}{\partial y}\left(\frac{1}{2}, \frac{1}{2}\right) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = c\frac{\partial u}{\partial y}\left(\frac{1}{2}, \frac{1}{2}\right) = c\left(1 + \frac{1}{2}\right) = 11 \times \frac{3}{2} = \frac{33}{2},$$

and

$$q_{n,3}^2\left(-\frac{1}{2},\frac{1}{2}\right) = \left\langle c\begin{pmatrix}\frac{\partial u}{\partial x}\left(-\frac{1}{2},\frac{1}{2}\right)\\ \frac{\partial u}{\partial y}\left(-\frac{1}{2},\frac{1}{2}\right)\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\rangle = c\frac{\partial u}{\partial y}\left(-\frac{1}{2},\frac{1}{2}\right) = c\left(1-\frac{1}{2}\right) = \frac{11}{2}.$$

Therefore,

$$Q_3 = Q_{2,1}^1 + Q_{3,3}^2 = 0 + \frac{77}{12} = 6.41\overline{6}.$$

```
qn23 = @(x,y) c*(1+x);
qn11 = 0;
h11 = a;
h23 = 1.0;
Q121 = h11*qn11/2;
Q233 = h23 * (qn23(1/2,1/2)/3 + qn23(-1/2,1/2)/6);
Q(3) = Q121 + Q233;

fprintf('(d) Q(3) = %.12f\n',Q(3))
```

```
(d) Q(3) = 6.416666666667
```