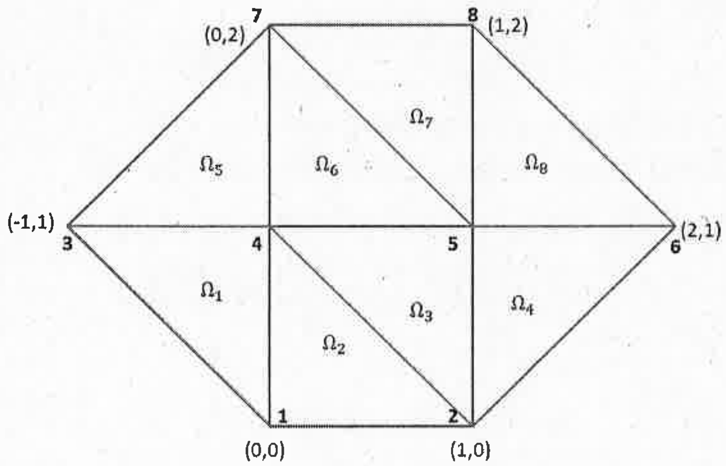
MÈTODES NUMÈRICS: Ex-Final

(a) Q2-2016-17

T(p)=	6.3462e+01
lowest T=	3.0177e+01

4. Consider the hexagonal domain shown in the figure below, and with the elements and nodes plotted there.



We want to solve the problem defined by the equation

$$-\Delta u = 2$$

with boundary conditions given by:

- $u = 3$ at the edges that joint the vertices $(0,2)$, $(-1,1)$, $(0,0)$, $(1,0)$ and $(2,1)$.
- The natural boundary condition $q_n = 0$ at the edge that joints the vertices $(2,1)$ and $(1,2)$
- The natural boundary condition $\frac{\partial u}{\partial y}(x,2) = 1 - x$ at the edge that joints the vertices $(0,2)$ and $(1,2)$

Meshing this domain with eight triangular elements as it is shown in the figure and using the proposing numbering for this mesh, compute:

(a) The coefficient K_{36}, K_{45} of the global assembled system.

K_{36}	K_{45}
0	-1

(b) The essential and natural boundary conditions needed to solve the global assembled system (write them just below)

Ex_Final_Q2-2016-17_Problema2.pdf

$$u_1 = u_2 = u_3 = u_6 = u_7 = 3,\ Q_8 = 1/6,\ Q_4 = Q_5 = 0.$$

(c) The value of $u(1,2)$

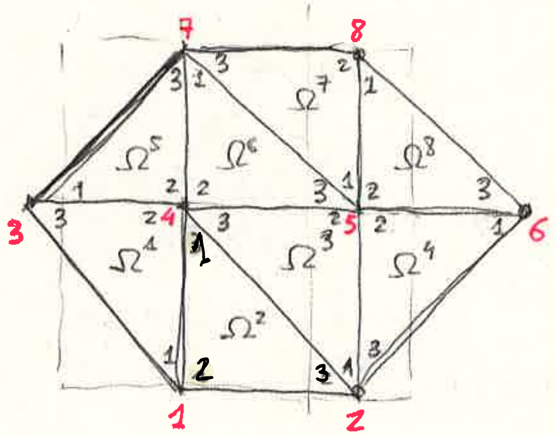
$\boxed{u(1,2) = \quad 4.1261}$

(*Hint:* It must be a value among the following ones:
10.297, 2.0180, 3.0045, 4.7568, 4.1261, 6.1532, 2.1631, 3.1511)

(d) The value of $u(0.5, 0.8)$

$\boxed{u(0.5, 0.8) = \quad 3.5721}$

(4 points)



(a) T2-MN-FEM2D, page 48 : $K^k = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, $F^k = \frac{2\cdot\frac{1}{2}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$, $k=1,\dots,8$

$$K_{36} = 0, \quad K_{45} = K^3_{32} + K^6_{23} = \tfrac{1}{2}(-1-1) = -1$$

(b) $U_1 = U_2 = U_3 = U_6 = U_7 = 3$

$$Q_4 = Q_5 = 0, \quad Q_8 = \underbrace{Q^8_{13}}_{=0} + Q^7_{22} = \frac{1}{3}\,0\cdot 1 + \frac{1}{6}\,1\cdot 1 = \frac{1}{6}$$

	①	②	③	④	⑤	⑥	⑦	⑧
④	$K^2_{12}+K^1_{21}$	$K^3_{31}+K^2_{13}$	$K^5_{21}+K^1_{23}$	$K^1_{22}+K^2_{11}+K^3_{33}+K^6_{22}+K^5_{22}$	$K^3_{32}+K^6_{23}$	0	$K^6_{21}+K^5_{23}$	0
⑤	0	$K^3_{21}+K^4_{23}$	0	$K^6_{32}+K^3_{23}$	$K^3_{22}+K^6_{33}+K^7_{11}+K^8_{22}+K^4_{22}$	$K^4_{21}+K^8_{23}$	$K^6_{31}+K^7_{13}$	$K^7_{12}+K^8_{21}$
⑧	0	0	0	0	$K^7_{21}+K^8_{12}$	K^8_{13}	K^7_{23}	$K^7_{22}+K^8_{11}$

$$\begin{array}{c} \begin{array}{cccccccc} ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \end{array} \\ \begin{pmatrix} -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \end{array} \cdot \begin{pmatrix} U_1 \\ \vdots \\ U_8 \end{pmatrix} \overset{(*)}{=} \begin{pmatrix} F_4 = \frac{5}{3} \\ F_5 = \frac{5}{3} \\ F_8 = \frac{2}{3} \end{pmatrix} + \begin{pmatrix} Q_4 = 0 \\ Q_5 = 0 \\ Q_8 = \frac{1}{6} \end{pmatrix}$$

(*)

$$\boxed{\begin{aligned} F_4 &= F^1_2 + F^2_1 + F^3_3 + F^6_2 + F^5_2 = \tfrac{5}{3} \\ F_5 &= F^3_2 + F^4_2 + F^8_2 + F^7_1 + F^6_3 = \tfrac{5}{3} \\ F_8 &= F^7_2 + F^8_1 = \tfrac{2}{3} \end{aligned}}$$

So the reduced system casts:

$$\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} U_4 \\ U_5 \\ U_8 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ \frac{3}{2} \end{pmatrix} + \begin{pmatrix} \frac{5}{3} \\ \frac{5}{3} \\ \frac{2}{3} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{32}{3} \\ \frac{23}{3} \\ \frac{7}{3} \end{pmatrix}$$

or, equivalently:

$$\begin{pmatrix} 12 & -3 & 0 \\ -3 & 12 & -3 \\ 0 & -6 & 9 \end{pmatrix} \begin{pmatrix} U_4 \\ U_5 \\ U_8 \end{pmatrix} = \begin{pmatrix} 32 \\ 23 \\ 14 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 12 & -3 & 0 & 32 \\ -3 & 12 & -3 & 23 \\ 0 & -6 & 9 & 14 \end{array}\right) \underset{4\times 2^a f + 1^a f}{\sim} \left(\begin{array}{ccc|c} 12 & -3 & 0 & 32 \\ 0 & 45 & -12 & 124 \\ 0 & -6 & 9 & 14 \end{array}\right) \underset{\frac{45}{6} 3^a f + 2^a f}{\sim} \left(\begin{array}{ccc|c} 12 & -3 & 0 & 32 \\ 0 & 45 & -12 & 124 \\ 0 & 0 & \frac{111}{2} & 229 \end{array}\right)$$

So:

$$U_8 = \frac{2\times 229}{111} = \frac{458}{111}$$

$$U_5 = \frac{1}{45}\left(124 + 12\times \frac{458}{111}\right) = \frac{19260}{45\times 111} = \frac{428}{111}$$

$$U_4 = \frac{1}{12}\left(32 + 3\times \frac{428}{111}\right) = \frac{4836}{12\times 111} = \frac{403}{111}$$

FEM Solution:

$U_1 = 3$

$U_2 = 3$

$U_3 = 3$

$U_4 = \frac{403}{111}$

$U_5 = \frac{428}{111}$

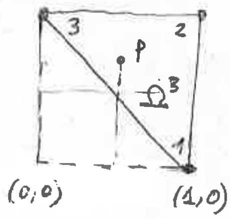
$U_6 = 3$

$U_7 = 3$

$U_8 = \frac{458}{111}$

(c) $u(1,2) \simeq U_8 = \frac{458}{111} = \boxed{4.\overline{1261}}$

(d) Clearly $P = (0.5, 0.8) \in \Omega^3$



By inspection (or we can use formulas in T1-MN-Interpolation-Shape-Functions.pdf, page 30):

$$\psi_1^3(x,y) = 1-y, \quad \psi_2^3(x,y) = x+y-1, \quad \psi_3^3(x,y) = 1-x$$

Therefore:

$$u(0.5, 0.8) \simeq U_2\, \psi_1^3(0.5,0.8) + U_5\, \psi_2^3(0.5,0.8) + U_4\, \psi_3^3(0.5,0.8)$$

$$= 3\,(1-0.8) + \frac{428}{111}\times 0.3 + \frac{403}{111}\times 0.5 = \frac{793}{222}$$

$$= \boxed{3.57\overline{207}}$$