

Mètodes Numèrics 2^{on}

20 de Gener de 2016

Nom i cognoms:

Problema 1.

(4.0 punts)

Considerem el problema $-\Delta u = f$ en $\Omega = (0, 4) \times (0, 2)$, amb les condicions de contorn:

$$u(0, y) = 0, \text{ per a tot } y \in (0, 2).$$

$$\frac{\partial u}{\partial y}(x, 0) = 0 \text{ i } \frac{\partial u}{\partial y}(x, 2) = 2, \text{ per a tot } x \in (0, 4).$$

$$\frac{\partial u}{\partial x}(4, y) \text{ lineal en } y \in (0, 2), \frac{\partial u}{\partial x}(4, 0) = 0 \text{ i } \frac{\partial u}{\partial x}(4, 2) = 2.$$

El resollem usant tres elements finits lineals triangulars Ω^1 , Ω^2 i Ω^3 que tenen els vèrtexs següents:

$$\Omega^1 = \{(0, 0), (4, 0), (4, 1)\}, \quad \Omega^2 = \{(0, 0), (4, 1), (0, 2)\}, \quad \Omega^3 = \{(4, 1), (4, 2), (0, 2)\}.$$

Globalment, enumerem els nodes: $p_1 = (0, 0)$, $p_2 = (0, 2)$, $p_3 = (4, 0)$, $p_4 = (4, 1)$ i $p_5 = (4, 2)$.

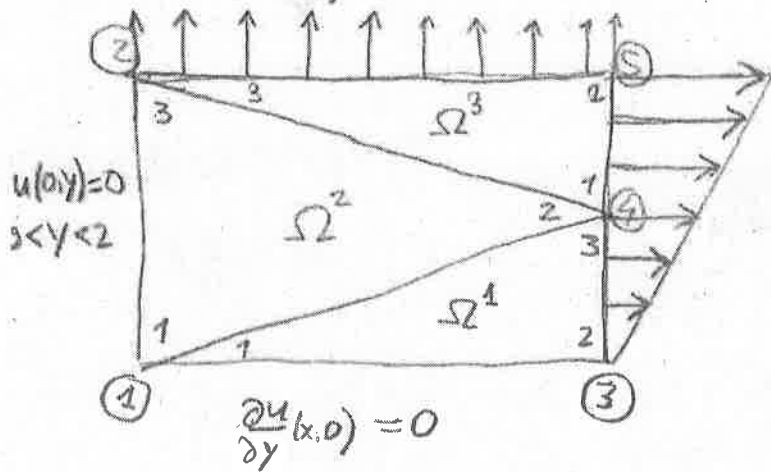
- (a) Escriviu la matriu de connectivitat. (0.5 punts)
- (b) Trobeu les matrius de rigidesa locals i, per a f constant, els vectors de càrregues locals. (1.0 punt)
- (c) Escriviu el sistema acoblat. (1.0 punt)
- (d) Expliciteu les condicions de contorn. (1.0 punt)
- (e) Si $f = 1$ i $U_3 = \frac{32084}{1683}$, trobeu U_4 i U_5 . (0.5 punts)

①

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f, \text{ on } \Omega = (0,4) \times (0,2)$$

$$\frac{\partial u}{\partial y}(x,2) = 2, 0 < x < 4$$

$$\text{so: } q_{11} = q_{22} = 1 \\ q_{12} = q_{21} = q_{00} = 0$$

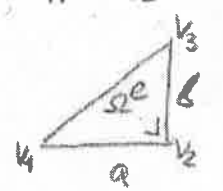


(a) Connectivity matrix $B = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 4 & 2 \\ 4 & 5 & 2 \end{pmatrix}$ modes $= \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$

(b) T2-MN-FEM2D, page 48

Ω^1 : Linear triangular rectangle element for Poisson equation

$$q_{11} = q_{22} = c, q_{12} = q_{21} = q_{00} = 0$$



$$K^1 = \frac{c}{2ab} \begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix} = \begin{cases} a=4 \\ b=1 \\ c=1=q_{11}=q_{22} \end{cases} = \frac{1}{16} \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix}$$

Ω^2 : T2-MN-FEM2D page 47.

$$K_{ij}^2 = \frac{1}{4A_2} \left(\overset{(a_{11})}{\beta_i^2 \beta_j^2} + \overset{(a_{22})}{\gamma_i^2 \gamma_j^2} \right); i,j = 1,2,3.$$

$$\begin{pmatrix} 0 & 0 \\ 4 & 1 \\ 0 & 2 \end{pmatrix} : \begin{array}{l} \beta_1^2 = y_2^2 - y_3^2 = 1 - 2 = -1 \\ \beta_2^2 = y_3^2 - y_1^2 = 2 - 0 = 2 \\ \beta_3^2 = y_1^2 - y_2^2 = 0 - 1 = -1 \end{array} \quad \begin{array}{l} \gamma_1^2 = -(x_2^2 - x_3^2) = -(4 - 0) = -4 \\ \gamma_2^2 = -(x_3^2 - x_1^2) = -(0 - 0) = 0 \\ \gamma_3^2 = -(x_1^2 - x_2^2) = -(0 - 4) = 4 \end{array}$$

$$\text{So: } K^2 = \frac{c}{4A_2} \left[\begin{pmatrix} \beta_1^2 \\ \beta_2^2 \\ \beta_3^2 \end{pmatrix} (\beta_1^2 \beta_2^2 \beta_3^2) + \begin{pmatrix} \gamma_1^2 \\ \gamma_2^2 \\ \gamma_3^2 \end{pmatrix} (\gamma_1^2 \gamma_2^2 \gamma_3^2) \right] = \left\{ \begin{array}{l} A_2 = 4 \\ c = q_{11} = q_{22} = 1 \end{array} \right\}$$

$$= \frac{1}{16} \left[\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} (-1, 2, 1) + \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} (-4, 0, 4) \right]$$

$$= \frac{1}{16} \left[\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 16 & 0 & -16 \\ 0 & 0 & 0 \\ -16 & 0 & 16 \end{pmatrix} \right] = \frac{1}{16} \begin{pmatrix} 17 & -2 & -15 \\ -2 & 4 & -2 \\ -15 & -2 & 17 \end{pmatrix}$$

K^3 : same as K^1 , but now with $a=1, b=4$. Therefore:

$$K^3 = \frac{1}{16} \begin{pmatrix} 32 & -32 & 0 \\ -32 & 32 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

• Load vectors: T2-MN-FEM2D.pdf page 47, We assume $f_e = f \equiv \text{const}$ for all $e=1,2,3$.

Hence:

$$F^e = \frac{f_e A_e}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ and so } F^1 = \frac{2}{3} f \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, F^2 = \frac{2}{3} f \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, F^3 = \frac{2}{3} f \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) Global stiffness matrix: $K =$

$$\begin{pmatrix} K_{11}^1 + K_{11}^2 & K_{13}^2 & K_{12}^1 & K_{13}^1 + K_{12}^2 & 0 \\ K_{21} & K_{33}^2 + K_{33}^3 & 0 & K_{32}^2 + K_{31}^3 & K_{32}^3 \\ K_{31} & K_{32} & K_{22}^1 & K_{23}^1 & 0 \\ K_{41} & K_{42} & K_{43} & K_{33}^1 + K_{22}^2 + K_{11}^3 & K_{12}^3 \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{22}^3 \end{pmatrix}$$

with $K^T = K$. Moreover:

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} = \begin{pmatrix} F_1^1 + F_1^2 \\ F_3^2 + F_3^3 \\ F_2^1 \\ F_3^1 + F_2^2 + F_1^3 \\ F_2^3 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1^1 + Q_1^2 \\ Q_3^2 + Q_3^3 \\ Q_2^1 \\ Q_3^1 + Q_2^2 + Q_1^3 \\ Q_2^3 \end{pmatrix}$$

← symmetric!

So, the coupled system casts:

$$\frac{1}{16} \begin{pmatrix} 19 & -15 & -2 & -2 & 0 \\ -15 & 19 & 0 & -2 & -2 \\ -2 & 0 & 34 & -32 & 0 \\ -2 & -2 & -32 & 68 & -32 \\ 0 & -2 & 0 & -32 & 34 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 6 \\ 6 \\ 2 \\ 8 \\ 2 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix}$$

d) Essential B.C.: $U_1 = 0, U_2 = 0$

Natural B.C.: $Q_3 = Q_{22}^1, Q_4 = Q_{32}^1 + Q_{11}^3, Q_5 = Q_{21}^3 + Q_{22}^3$

$$q_{n,2}^1(s) = \left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \nabla u, \hat{n} \right\rangle \Big|_{\Gamma_2^1}^{a_{11}=a_{22}=1, a_{12}=a_{21}=0} = \left\langle \nabla u, \hat{n} \right\rangle \Big|_{\Gamma_2^1} = \left(\frac{\partial u}{\partial x}(4, s), \frac{\partial u}{\partial y}(4, s) \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial u}{\partial x}(4, s) = s, \quad 0 \leq s \leq 1.$$

$$q_{n,1}^3(s) = \dots = \left\langle \nabla u, \hat{n} \right\rangle \Big|_{\Gamma_1^3} = \left(\frac{\partial u}{\partial x}(4, 1+s), \frac{\partial u}{\partial y}(4, 1+s) \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\partial u}{\partial x}(4, 1+s) = 1+s, \quad 0 \leq s \leq 1$$

$$q_{n,2}^3(s) = \dots = \left\langle \nabla u, \hat{n} \right\rangle \Big|_{\Gamma_2^3} = \left(\frac{\partial u}{\partial x}(4-s, 2), \frac{\partial u}{\partial y}(4-s, 2) \right) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial u}{\partial y}(4-s, 2) = 2, \quad 0 \leq s \leq 4$$

$$Q_{12}^1 = \int_0^{h_1} q_{m,2}^1(s) \psi_{22}^1(s) ds = \int_0^{h_2^1=1} s(1-\frac{s}{h_2^1}) ds = \boxed{\frac{1}{6}} : \text{see T3-MN-FEM2D.pdf, page 60.}$$

Alternatively (*) $Q_{22}^1 = \frac{1}{3} \cdot 0 + \frac{1}{6} \times 1 = \frac{1}{6}$

$$Q_{32}^1 = \int_0^{h_2^1} q_{m,2}^1(s) \psi_{32}^1(s) ds = \int_0^{h_2^1=1} s \cdot \frac{s}{h_2^1} ds = \boxed{\frac{1}{3}} \quad \text{Alternatively: } Q_{32}^1 = \frac{1}{3} \times 1 + \frac{1}{6} \cdot 0 = \frac{1}{3}$$

$$Q_{11}^3 = \int_0^{h_1^3} q_{m,1}^3(s) \cdot \psi_{11}^3(s) ds = \int_0^{h_1^3=1} (1+s)(1-\frac{s}{h_1^3}) ds = \int_0^1 (1-s^2) ds = \left(s - \frac{s^3}{3} \right) \Big|_0^1 = \boxed{\frac{2}{3}}$$

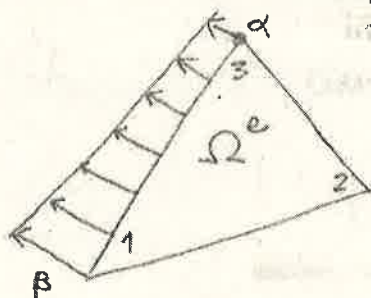
Alternatively: $Q_{11}^3 = \frac{1}{3} + \frac{2}{6} = \frac{2}{3}$

$$Q_{21}^3 = \int_0^{h_1^3} q_{m,1}^3(s) \cdot \psi_{21}^3(s) ds = \int_0^{h_1^3=1} (1+s) \cdot \frac{s}{h_1^3} ds = \int_0^1 s(1+s) ds = \left(\frac{s^2}{2} + \frac{s^3}{3} \right) \Big|_0^1 = \boxed{\frac{5}{6}}$$

Alternatively: $Q_{21}^3 = \frac{1}{6} \cdot 1 + \frac{1}{3} \cdot 2 = \frac{5}{6}$

$$Q_{12}^3 = \int_0^{h_2^3} q_{m,2}^3(s) \psi_{12}^3(s) ds = \int_0^{h_2^3=4} 2 \cdot \left(1 - \frac{s}{h_2^3}\right) ds = 2 \int_0^4 \left(1 - \frac{s}{4}\right) ds = 2 \left(s - \frac{s^2}{8} \right) \Big|_0^4 = 2(4 - \frac{16}{8}) = 2(4-2) = \boxed{4} \quad \text{Alternatively: } Q_{12}^3 = \frac{1}{2} \times 2 \times 4 = 4.$$

(*) Remark. We apply the "rule for linear flows". For example



$$q_{m,3}^e(s) = \alpha \left(1 - \frac{s}{h_3^e}\right) + \beta \frac{s}{h_3^e} = \alpha \psi_{33}^e(s) + \beta \psi_{13}^e(s)$$

$$Q_{33}^e = \int_0^{h_3^e} q_{m,3}^e(s) \psi_{33}^e(s) ds = \int_0^{h_3^e} (\alpha \psi_{33}^e(s) + \beta \psi_{13}^e(s)) \psi_{33}^e(s) ds$$

$$= \alpha \int_0^{h_3^e} (\psi_{33}^e(s))^2 ds + \beta \int_0^{h_3^e} \psi_{13}^e(s) \psi_{33}^e(s) ds$$

$$\stackrel{(2*)}{=} \alpha h_3^e \frac{2! \cdot 0!}{(2+0+1)!} + \beta h_3^e \frac{1! \cdot 1!}{3!} = h_3^e \left(\frac{\alpha}{3} + \frac{\beta}{6} \right).$$

(2*) Let $p, q = 0, 1, 2, 3, \dots$. Therefore

$$\int_0^{h_\ell^e} (\psi_{m\ell}^e(s))^p (\psi_{m\ell}^e(s))^q ds = \begin{cases} \alpha, & \text{if either } \psi_{m\ell}^e(s) = 0 \text{ or } \psi_{m\ell}^e(s) = 0 \\ h_\ell^e \frac{p! q!}{(p+q+1)!} \end{cases}$$

For, in the 2nd case: $\int_0^{h_\ell^e} (\psi_{m\ell}^e(s))^p (\psi_{m\ell}^e(s))^q ds = \int_0^1 x^p (1-x)^{q+1} dx = \frac{1}{(p+1)(q+2)} = \frac{p! q!}{(p+q+2)!} = h_\ell^e \frac{p! q!}{(p+q+1)!}$

$$\begin{aligned}
 Q_{13}^e &= \int_0^{h_3^e} q_{1,13}^e(s) \psi_{13}^e(s) ds = \int_0^{h_3^e} (\alpha \psi_{33}^e(s) + \beta \psi_{13}^e(s)) \psi_{13}^e(s) ds \\
 &= \alpha \int_0^{h_3^e} \psi_{33}^e(s) \psi_{13}^e(s) ds + \beta \int_0^{h_3^e} (\psi_{13}^e(s))^2 ds \\
 &= \alpha h_3^e \frac{1!1!}{3!} + \beta h_3^e \frac{2!0!}{(2+1)!} = h_3^e \left(\frac{\alpha}{6} + \frac{\beta}{3} \right)
 \end{aligned}$$

End of Remark

So, the essential B.C. turn out to be,

$$Q_3 = Q_{22}^1 = \frac{1}{6}$$

$$Q_4 = Q_{32}^1 + Q_{11}^3 = \frac{1}{3} + \frac{2}{3} = 1$$

$$Q_5 = Q_{21}^3 + Q_{22}^5 = \frac{5}{6} + 4 = \frac{29}{6}$$

(e) Take $f=1$. If $U_3 = \frac{32084}{1683}$, we must solve the corresponding reduced system, i.e.:

$$\frac{1}{16} \begin{pmatrix} 68 & -32 \\ -32 & 34 \end{pmatrix} \begin{pmatrix} U_4 \\ U_5 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 29 \\ 6 \end{pmatrix} - \frac{1}{16} \begin{pmatrix} -2 & -2 & -32 \\ 0 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} U_1=0 \\ U_2=0 \\ U_3 = \frac{32084}{1683} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \frac{17}{4} & -2 \\ -2 & \frac{17}{8} \end{pmatrix} \begin{pmatrix} U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} \frac{70339}{1683} \\ \frac{11}{2} \end{pmatrix}$$

in MATLAB:

$$-K(\text{free Nods}, \text{fixed Nods}) \times U(\text{fixed Nods})$$

$$\Delta = \begin{vmatrix} \frac{17}{4} & -2 \\ -2 & \frac{17}{8} \end{vmatrix} = \frac{161}{32}, \quad \Delta_{U_4} = \begin{vmatrix} \frac{70339}{1683} & -2 \\ \frac{11}{2} & \frac{17}{8} \end{vmatrix} = \frac{79051}{792}, \quad \Delta_{U_5} = \begin{vmatrix} \frac{17}{4} & \frac{70339}{1683} \\ -2 & \frac{11}{2} \end{vmatrix} = \frac{144024}{13464}$$

$$\therefore U_4 = \frac{\Delta_{U_4}}{\Delta} = \frac{1964}{99}, \quad U_5 = \frac{\Delta_{U_5}}{\Delta} = \frac{35780}{1683}$$

Finally, the complete solution is given by:

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = \frac{32084}{1683}, \quad U_4 = \frac{1964}{99}, \quad U_5 = \frac{35780}{1683}.$$

□