Ex_Final_Q1-2020-21_Problema 1.pdf

(1)

P1 - Ex-Final 1Q-2020-21

We want to solve $\cancel{\Delta u = 0}$ $-\Delta u = 0$ with boundary conditions given by

* At the lines that joints nodes 4 and 5, $\dfrac{\partial u}{\partial y} = u$

* At the lines that joints nodes 3 and 4, $\dfrac{\partial u}{\partial x} = y$

* $u = 0$ at the nodes 1,2,3,5.

Compute the following values with $c = 2$ and $L = 4$

(a, pts=2) The element K^1_{12} of the local stiff matrix K^1 of the element Ω^1 is $\boxed{-2.5000e-01 \pm 0.000050 \quad \checkmark}$.

(b, pts=2) The element K^2_{12} of the local stiff matrix K^2 of the element Ω^2 is

$\boxed{-2.5000e-01 \pm 0.000050 \quad \checkmark}$.

(c, pts=1) Is K^2 equal to K^3?

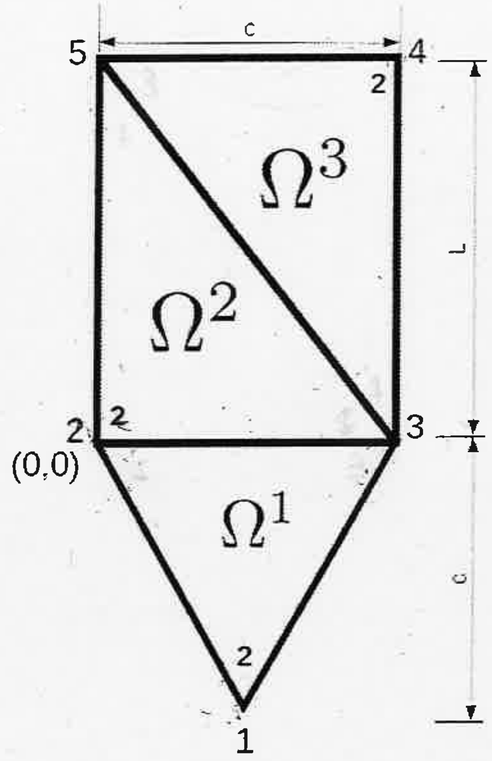
- Yes. ✓
- Equal but with opposite sign.
- No.
- K^2 and K^3 have different size.
- Leave it empty

(d, pts=2) The element K_{32} of the global stiff matrix K is (hint: $K_{13} =$ -2.5000e-01) $\boxed{-1.3750e+00 \pm 0.000500 \quad \checkmark}$.

(e, pts=2) When computing the value of Q_4 for the fourth equation of this problem, it can be mainly expressed as $a+bu_4$ (a constant term plus another one depending on u_4). The value of b is: $\boxed{6.6667e-01 \pm 0.000500 \quad \checkmark}$.

(f, pts=1) After solving the global system, the value of u_4 is

- 6.8098e+00
- 9.1429e+00 ✓
- 3.6227e+00
- 8.9543e+00
- Leave it empty



Solució.

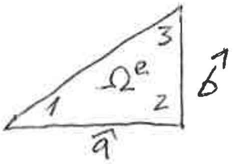
$A_1 = \frac{1}{2}c^2$

(a) $$K^1_{12} = \frac{a^1_{11}}{4A_1}\beta^1_1\beta^1_2 + \frac{a^1_{22}}{4A_1}\gamma^1_1\gamma^1_2 = \{a_{11}=a_{22}=K_c=1\} = \frac{1}{2c^2}\left(\beta^1_1\beta^1_2+\gamma^1_1\gamma^1_2\right)$$

$$= \frac{1}{2c^2}\left(-\frac{c^2}{2}\right) = \boxed{-\frac{1}{4}} = -0.25$$

$$\beta^1_1 = y^1_2 - y^1_3 = -c-0 = -c\,; \quad \gamma^1_1 = -(x^1_2 - x^1_3) = -(c/2 - c) = c/2$$
$$\beta^1_2 = y^1_3 - y^1_1 = 0-0 = 0\,; \quad \gamma^1_2 = -(x^1_3 - x^1_1) = -(c-0) = -c$$

(b) $e=2,3$

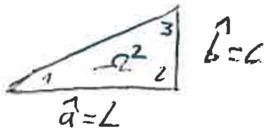


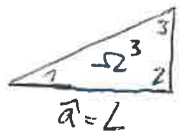
$$a_{11}=a_{22}=K_c, \quad K^e = \frac{K_c}{2\hat{a}\hat{b}}\begin{pmatrix} \hat{b}^2 & -\hat{b}^2 & 0 \\ -\hat{b}^2 & \hat{a}^2+\hat{b}^2 & -\hat{a}^2 \\ 0 & -\hat{a}^2 & \hat{a}^2 \end{pmatrix}$$

Here: $K_c = 1$, $\hat{a} = L$, $\hat{b} = c$

$$K^2_{12} = \frac{1}{2cL}(-c^2) = -\frac{c}{2L} = \boxed{-\frac{1}{4}} = -0.25 \qquad (L=4,\ c=2)$$

(c)





& Poisson equation, so $K^2 = K^3$

(d) $$K_{32} = K^1_{31} + K^2_{32} = -\frac{3}{8} - 1 = \boxed{-\frac{11}{8}} = -1.375$$

$$K^1_{31} = \frac{1}{2c^2}\left(\beta^1_3\beta^1_1 + \gamma^1_3\gamma^1_1\right) = \frac{1}{2c^2}\left[c\cdot(-c) + \frac{c}{2}\cdot\frac{c}{2}\right] = \frac{1}{2}\left(-1+\frac{1}{4}\right) = -\frac{3}{8}$$

$$\beta^1_3 = y^1_1 - y^1_2 = 0-(-c) = c, \quad \gamma^1_3 = -(x^1_1 - x^1_2) = -\left(-\frac{c}{2}\right) = \frac{c}{2}$$

$$K^2_{32} = \frac{1}{2Lc}(-L^2) = -\frac{L}{2c} = -\frac{4}{4} = -1$$

(e) $$Q_4 = Q^3_{21} + Q^3_{22} = a + b u_4$$

$$q^3_{n,2}(s) = \left\langle \begin{pmatrix} K_c & \\ & K_c \end{pmatrix}\begin{pmatrix} u_x(c-s,L) \\ u_y(c-s,L) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \overset{=1}{K_c}\frac{\partial u}{\partial y}(c-s,L) = u(c-s,L), \quad 0\le s\le c$$

$$Q^3_{22} = c\cdot\left(\frac{1}{3}q^3_{n,2}(0) + \frac{1}{6}q^3_{n,2}(s)\right) = c\left(\frac{1}{3}u(c,L) + \frac{1}{6}u(0,L)\right) = c\left(\frac{1}{3}u_4 + \frac{1}{6}\overset{=0}{u_5}\right) =$$
$$\underset{c=2}{=} \frac{2}{3}u_4$$ (recall that, acording with the essential BC $u_5=0$)

$$q^3_{n1}(s) = \left\langle \begin{pmatrix} K_c & \\ & K_c \end{pmatrix}\begin{pmatrix} u_x(c,s) \\ u_y(c,s) \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \overset{=1}{K_c}\frac{\partial u}{\partial x}(c,s) = s, \quad 0\le s\le L$$

$$Q^3_{21} = L\left(\frac{1}{6}q^3_{n1}(0) + \frac{1}{3}q^3_{n1}(L)\right) = L\left(\frac{1}{6}\cdot 0 + \frac{1}{3}L\right) = \frac{L^2}{3} \underset{L=4}{=} \frac{16}{3}.$$

Therefore:

$$Q_4 = Q_{21}^3 + Q_{22}^3 = a + b u_4 = \frac{16}{3} + \frac{2}{3} u_4 \text{ , and } B = \boxed{2/3} = 0.\hat{6}$$

(f) Natural B.C.: $Q_4 = 2/3\, u_4 + 16/3$

Essential B.C.: $u_1 = u_2 = u_3 = u_5 = 0.$

Reduced system: $K_{44} u_4 = 2/3\, u_4 + 16/3 \Leftrightarrow \left(K_{44} - 2/3\right) u_4 = \frac{16}{3}$

$$K_{44} = K_{22}^3 = \frac{1}{2cL}\left(c^2 + L^2\right) = \frac{1}{16}\left(4 + 16\right) = \frac{20}{16} = \frac{5}{4}$$

$$\left(5/4 - 2/3\right) u_4 = \frac{16}{3} \Leftrightarrow \frac{7}{12} u_4 = \frac{16}{3} \Leftrightarrow u_4 = \boxed{\frac{64}{7}} = 9.1429$$