Mètodes Numèrics 2^{on} 20 de Gener de 2016

Nom i cognoms:

Problema 1. *(4.0 punts)*

Considerem el problema $-\Delta u = f$ en $\Omega = (0,4) \times (0,2)$, amb les condicions de contorn:

$$u(0,y) = 0 \text{ , per a tot } y \in (0,2).$$
$$\frac{\partial u}{\partial y}(x,0) = 0 \text{ i } \frac{\partial u}{\partial y}(x,2) = 2 \text{ , per a tot } x \in (0,4).$$
$$\frac{\partial u}{\partial x}(4,y) \text{ lineal en } y \in (0,2) \text{ , } \frac{\partial u}{\partial x}(4,0) = 0 \text{ i } \frac{\partial u}{\partial x}(4,2) = 2.$$

El resolem usant tres elements finits lineals triangulars Ω^1, Ω^2 i Ω^3 que tenen els vèrtexs següents:

$$\Omega^1 = \{(0,0),(4,0),(4,1)\}, \quad \Omega^2 = \{(0,0),(4,1),(0,2)\}, \quad \Omega^3 = \{(4,1),(4,2),(0,2)\}.$$

Globalment, enumerem els nodes: $p_1 = (0,0)$, $p_2 = (0,2)$, $p_3 = (4,0)$, $p_4 = (4,1)$ i $p_5 = (4,2)$.

(a) Escriviu la matriu de connectivitat. *(0.5 punts)*

(b) Trobeu les matrius de rigidesa locals i, per a f constant, els vectors de càrregues locals. *(1.0 punt)*

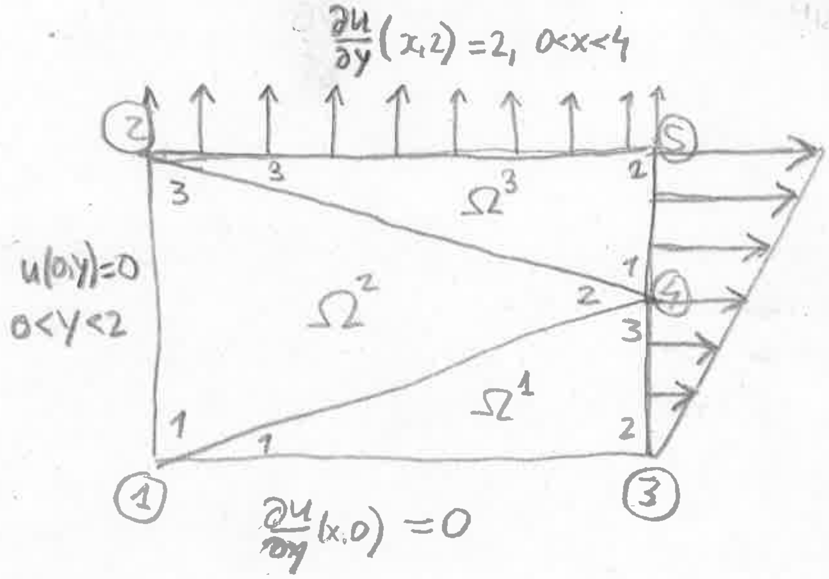
(c) Escriviu el sistema acoblat. *(1.0 punt)*

(d) Expliciteu les condicions de contorn. *(1.0 punt)*

(e) Si $f = 1$ i $U_3 = \frac{32084}{1683}$, trobeu U_4 i U_5. *(0.5 punts)*

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f, \quad \text{on } \Omega = (0,4)\times(0,2)$$

So: $a_{11} = a_{22} = 1$
$a_{12} = a_{21} = a_{00} \equiv 0$

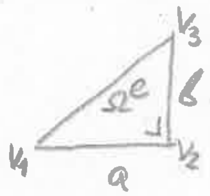


(a) Connectivity matrix $B = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 4 & 2 \\ 4 & 5 & 2 \end{pmatrix}$ nodes $= \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 4 & 0 \\ 4 & 1 \\ 4 & 2 \end{pmatrix}$

(b) T3-MN-FEM2D, page 48

Ω^1: Linear triangular rectangle element for Poisson equation

$a_{11} = a_{22} = c, \; a_{12} = a_{21} = a_{00} = 0$



$$K^1 = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix} = \left\{\begin{array}{l} a=4 \\ b=1 \\ c=1=a_{11}=a_{22} \end{array}\right\} = \frac{1}{16}\begin{pmatrix} 2 & -2 & 0 \\ -2 & 34 & -32 \\ 0 & -32 & 32 \end{pmatrix}$$

Ω^2: T3-MN-FEM2D page 47.

$$K^2_{ij} = \frac{a_{11}}{4A_2}\left(\beta_i^2\beta_j^2 + \gamma_i^2\gamma_j^2\right); \; i,j = 1,2,3.$$

$$\begin{pmatrix} 0 & 0 \\ 4 & 1 \\ 0 & 2 \end{pmatrix}: \quad \begin{array}{l|l} \beta_1^2 = y_2^2 - y_3^2 = 1-2 = -1 & \gamma_1^2 = -(x_2^2 - x_3^2) = -(4-0) = -4 \\ \beta_2^2 = y_3^2 - y_1^2 = 2-0 = 2 & \gamma_2^2 = -(x_3^2 - x_1^2) = -(0-0) = 0 \\ \beta_3^2 = y_1^2 - y_2^2 = 0-1 = -1 & \gamma_3^2 = -(x_1^2 - x_2^2) = -(0-4) = 4 \end{array}$$

$$\text{So: } K^2 = \frac{a_{11}}{4A_2}\left[\begin{pmatrix}\beta_1^2 \\ \beta_2^2 \\ \beta_3^2\end{pmatrix}\left(\beta_1^2\;\beta_2^2\;\beta_3^2\right) + \begin{pmatrix}\gamma_1^2 \\ \gamma_2^2 \\ \gamma_3^2\end{pmatrix}\left(\gamma_1^2\;\gamma_2^2\;\gamma_3^2\right)\right] = \left\{\begin{array}{l} A_2 = 4 \\ a_{11} = 1 \end{array}\right\}$$

$$= \frac{1}{16}\left[\begin{pmatrix}-1 \\ 2 \\ 1\end{pmatrix}(-1, 2, 1) + \begin{pmatrix}-4 \\ 0 \\ 4\end{pmatrix}(-4\;0\;4)\right]$$

$$= \frac{1}{16}\left[\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 16 & 0 & -16 \\ 0 & 0 & 0 \\ -16 & 0 & 16 \end{pmatrix}\right] = \frac{1}{16}\cdot\begin{pmatrix} 17 & -2 & -15 \\ -2 & 4 & -2 \\ -15 & -2 & 17 \end{pmatrix}$$

K^3: Same as K^1, but now with $a=1, b=4$. Therefore:

$$K^3 = \frac{1}{16}\begin{pmatrix} 32 & -32 & 0 \\ -32 & 34 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

- Load vectors: T3-MN-FEM2D.pdf page 47. We assume $f_e = f \equiv$ ctnt for all $e=1,2,3$. Hence:

$$F^e = \frac{f_e A_e}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}, \text{ and so } F^1 = \frac{2}{3}f\begin{pmatrix}1\\1\\1\end{pmatrix},\ F^2 = \frac{2}{3}f\begin{pmatrix}2\\2\\2\end{pmatrix},\ F^3 = \frac{4}{3}f\begin{pmatrix}1\\1\\1\end{pmatrix}$$

(c) Global stiffness matrix:
$$K = \begin{pmatrix} K^1_{11}+K^2_{11} & K^2_{13} & K^1_{12} & K^1_{13}+K^2_{12} & 0 \\ K_{21} & K^2_{33}+K^3_{33} & 0 & K^2_{32}+K^3_{31} & K^3_{32} \\ K_{31} & K_{32} & K^1_{22} & K^1_{23} & 0 \\ K_{41} & K_{42} & K_{43} & K^1_{33}+K^2_{22}+K^3_{11} & K^3_{12} \\ K_{51} & K_{52} & K_{53} & K_{54} & K^3_{22} \end{pmatrix}$$

symmetric!

with $K^T = K$. Moreover:

$$F = \begin{pmatrix}F_1\\F_2\\F_3\\F_4\\F_5\end{pmatrix} = \begin{pmatrix}F^1_1+F^2_1\\F^2_3+F^3_3\\F^1_2\\F^1_3+F^2_2+F^3_1\\F^3_2\end{pmatrix},\quad Q = \begin{pmatrix}Q^1_1+Q^2_1\\Q^2_3+Q^3_3\\Q^1_2\\Q^1_3+Q^2_2+Q^3_1\\Q^3_2\end{pmatrix}$$

So, the coupled system casts:

$$\frac{1}{16}\begin{pmatrix} 19 & -15 & -2 & -2 & 0 \\ -15 & 19 & 0 & -2 & -2 \\ -2 & 0 & 34 & -32 & 0 \\ -2 & -2 & -32 & 68 & -32 \\ 0 & -2 & 0 & -32 & 34 \end{pmatrix}\cdot\begin{pmatrix}U_1\\U_2\\U_3\\U_4\\U_5\end{pmatrix} = \frac{f}{3}\begin{pmatrix}6\\6\\2\\8\\2\end{pmatrix} + \begin{pmatrix}Q_1\\Q_2\\Q_3\\Q_4\\Q_5\end{pmatrix}$$

d) Essential B.C.: $U_1 = 0,\ U_2 = 0$

Natural B.C.: $Q_3 = Q^1_{22},\ Q_4 = Q^1_{32} + Q^3_{11},\ Q_5 = Q^3_{21} + Q^3_{22}$

$$q^1_{n,2}(s) = \left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\nabla u, \hat{n}\right\rangle\Big|_{\Gamma^1_2} \underset{\substack{a_{11}=a_{22}=1\\ a_{12}=a_{21}=0}}{=} \langle \nabla u, \hat{n}\rangle\Big|_{\Gamma^1_2} = \left(\frac{\partial u}{\partial x}(4,s), \frac{\partial u}{\partial y}(4,s)\right)\cdot\begin{pmatrix}1\\0\end{pmatrix} =$$

$$= \frac{\partial u}{\partial y}(4,s) = s,\quad 0 \le s \le 1.$$

$$q^3_{n,1}(s) = \dots = \langle \nabla u, \hat{n}\rangle\Big|_{\Gamma^3_1} = \left(\frac{\partial u}{\partial x}(4,1+s), \frac{\partial u}{\partial y}(4,1+s)\right)\begin{pmatrix}1\\0\end{pmatrix} = \frac{\partial u}{\partial x}(4,1+s) = 1+s,\ 0\le s\le 1$$

$$q^3_{n,2}(s) = \dots = \langle \nabla u, \hat{n}\rangle\Big|_{\Gamma^3_2} = \left(\frac{\partial u}{\partial x}(4-s,2)\ \frac{\partial u}{\partial y}(4-s,2)\right)\begin{pmatrix}0\\1\end{pmatrix} = \frac{\partial u}{\partial y}(4-s,2) = 2,\ 0\le s\le 4$$

$$Q^1_{22} = \int_0^{h^1_2} q^1_{n,2}(s)\,\psi^1_{22}(s)\,ds \underset{h^1_2=1}{=} \int_0^{h^1_2=1} s\left(1-\frac{s}{h^1_2}\right)ds = \boxed{\frac{1}{6}}$$: see T3-MN-FEM2D.pdf, page 60.

Alternatively (*) $Q^1_{22} = \frac{1}{6}\times 1\times 1\,(1-0) = \frac{1}{6}$.

$$Q^1_{32} = \int_0^{h^1_2} q^1_{n,2}(s)\,\psi^1_{32}(s)\,ds = \int_0^{h^1_2=1} s\cdot \frac{s}{h^1_2}\,ds = \boxed{\frac{1}{3}}$$. Alternatively: $Q^1_{32} = \frac{1}{3}\times 1\,(1-0) = \frac{1}{3}$.

$$Q^3_{11} = \int_0^{h^3_1} q^3_{n,1}(s)\cdot\psi^3_{11}(s)\,ds = \int_0^{h^3_1=1} (1+s)\left(1-\frac{s}{h^3_1}\right)ds = \int_0^1 (1-s^2)\,ds = \left(s-\frac{s^3}{3}\right)\Big|_0^1 = \boxed{\frac{2}{3}}$$

Alternatively: $Q^3_{11} = \frac{1}{2}\times 1\times 1 + \frac{1}{6}(2-1) = \frac{3}{6}+\frac{1}{6} = \frac{2}{3}$.

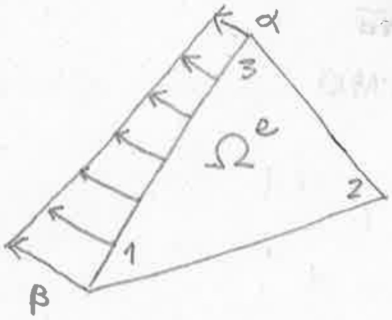
$$Q^3_{21} = \int_0^{h^3_1} q^3_{n,1}(s)\cdot\psi^3_{21}(s)\,ds = \int_0^{h^3_1=1} (1+s)\,\frac{s}{h^3_1}\,ds = \int_0^1 s(1+s)\,ds = \left(\frac{s^2}{2}+\frac{s^3}{3}\right)\Big|_0^1 = \boxed{\frac{5}{6}}$$

Alternatively: $Q^3_{21} = \frac{1}{2}\times 1\times 1 + \frac{1}{3}(2-1) = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}$.

$$Q^3_{12} = \int_0^{h^3_2} q^3_{n,2}(s)\,\psi^3_{12}(s)\,ds = \int_0^{h^3_2=4} 2\cdot\left(1-\frac{s}{h^3_2}\right)ds = 2\int_0^4\left(1-\frac{s}{4}\right)ds = 2\left(s-\frac{s^2}{8}\right)\Big|_0^4$$

$$= 2\left(4-\frac{16}{8}\right) = 2(4-2) = \boxed{4}$$. Alternatively: $Q^3_{12} = \frac{1}{2}\times 2\times 4 = 4$.

(*) Remmark. We apply the "rule for linear flows". For example



$$q^e_{n,3}(s) = \alpha\left(1-\frac{s}{h^e_3}\right) + \beta\,\frac{s}{h^e_3} = \alpha\,\psi^e_{33}(s) + \beta\,\psi^e_{13}(s)$$

$$Q^e_{33} = \int_0^{h^e_3} q^e_{n,3}(s)\,\psi^e_{33}(s) = \int_0^{h^e_3}\left(\alpha\psi^e_{33}(s)+\beta\psi^e_{13}(s)\right)\psi^e_{33}(s)\,ds$$

$$= \alpha\int_0^{h^e_3}\left(\psi^e_{33}(s)\right)^2 ds + \beta\int_0^{h^e_3}\psi^e_{13}(s)\,\psi^e_{33}(s)\,ds$$

$$\overset{(\&)}{=} \alpha h^e_3\,\frac{2!\,0!}{(2+0+1)!} + \beta h^e_3\,\frac{1!\,1!}{3!} = h^e_3\left(\frac{\alpha}{3}+\frac{\beta}{6}\right) = \underbrace{h^e_3\left(\frac{\alpha}{2}+\frac{1}{6}(\beta-\alpha)\right)}_{\text{"linear flows rule" for the "initial mode"}}$$

(&) Let $p, q = 0,1,2,3,\dots$. Therefore

$$\int_0^{h^e_\ell}\left(\psi^e_{n\ell}(s)\right)^p\left(\psi^e_{m\ell}(s)\right)^q ds = \begin{cases} 0, & \text{if either } \psi^e_{n\ell}(s)=0 \text{ or } \psi^e_{m\ell}(s)=0 \\ h^e_\ell\,\dfrac{p!\,q!}{(p+q+1)!} & \end{cases}$$

For, in the 2nd case: $\int_0^{h^e_\ell}\left(\psi^e_{n\ell}(s)\right)^p\cdot\left(\psi^e_{m\ell}(s)\right)^q ds = \left\{ \begin{matrix} x = s/h^e_\ell \\ \dots \end{matrix}\right\} = h^e_\ell\int_0^1 x^{p+1-1}(1-x)^{q+1-1}\,dx = h^e_\ell\,B(p+1,q+1)$

$$= h^e_\ell\,\frac{\Gamma(p+1)\,\Gamma(q+1)}{\Gamma(p+q+2)} = h^e_\ell\,\frac{p!\,q!}{(p+q+1)!}$$

$$Q_{13}^e = \int_0^{h_3^e} q_{n,13}^e(s)\,\psi_{13}^e(s)\,ds = \int_0^{h_3^e}\left(\alpha\psi_{33}^e(s)+\beta\psi_{13}^e(s)\right)\psi_{13}^e(s)\,ds$$

$$= \alpha\int_0^{h_3^e}\psi_{33}^e(s)\,\psi_{13}^e(s)\,ds + \beta\int_0^{h_3^e}\left(\psi_{13}^e(s)\right)^2 ds$$

$$= \alpha h_3^e\,\frac{1!\,1!}{3!} + \beta h_3^e\,\frac{2!\,0!}{(2+0+1)!} = h_3^e\left(\frac{\alpha}{6}+\frac{\beta}{3}\right) = \underbrace{h_3^e\left(\frac{\alpha}{2}+\frac{1}{3}(\beta-\alpha)\right)}$$

"Linear flow's rule" for the "final" node.

——— End of Remark ———

So, the essential B.C. turn out to be,

$$Q_3 = Q_{22}^1 = \frac{1}{6}$$

$$Q_4 = Q_{32}^1 + Q_{11}^3 = \frac{1}{3} + \frac{2}{3} = 1$$

$$Q_5 = Q_{21}^3 + Q_{22}^3 = \frac{5}{6} + 4 = \frac{29}{6}$$

(e) Take $f\equiv 1$. If $U_3 = \frac{32084}{1683}$, we must solve the corresponding reduced system, i.e.:

$$\frac{1}{16}\begin{pmatrix}68 & -32\\ -32 & 34\end{pmatrix}\begin{pmatrix}U_4\\ U_5\end{pmatrix} = \frac{1}{3}\begin{pmatrix}8\\ 2\end{pmatrix} + \begin{pmatrix}1\\ \frac{29}{6}\end{pmatrix} \underbrace{- \frac{1}{16}\begin{pmatrix}-2 & -2 & -32\\ 0 & -2 & 0\end{pmatrix}\cdot\begin{pmatrix}U_1 = 0\\ U_2 = 0\\ U_3 = \frac{32084}{1683}\end{pmatrix}}$$

$$\Leftrightarrow \begin{pmatrix}17/4 & -2\\ -2 & \frac{17}{8}\end{pmatrix}\begin{pmatrix}U_4\\ U_5\end{pmatrix} = \begin{pmatrix}\frac{70339}{1683}\\ 11/2\end{pmatrix}$$

MATLAB:
-K(freeNods, fixedNods)*U(fixedNods)

$$\Delta = \begin{vmatrix}17/4 & -2\\ -2 & \frac{17}{8}\end{vmatrix} = \frac{161}{32},\quad \Delta_{U_4} = \begin{vmatrix}\frac{70339}{1683} & -2\\ \frac{11}{2} & \frac{17}{8}\end{vmatrix} = \frac{79051}{792},\quad \Delta_{U_5} = \begin{vmatrix}\frac{17}{4} & \frac{70339}{1683}\\ -2 & \frac{11}{2}\end{vmatrix} = \frac{1440145}{13464}$$

$$\therefore\quad U_4 = \frac{\Delta_{U_4}}{\Delta} = \frac{1964}{99},\quad U_5 = \frac{\Delta_{U_5}}{\Delta} = \frac{35780}{1683}.$$

Finally, the complete solution is given by:

$$U_1 = 0,\quad U_2 = 0,\quad U_3 = \frac{32084}{1683},\quad U_4 = \frac{1964}{99},\quad U_5 = \frac{35780}{1683}. \qquad \square$$