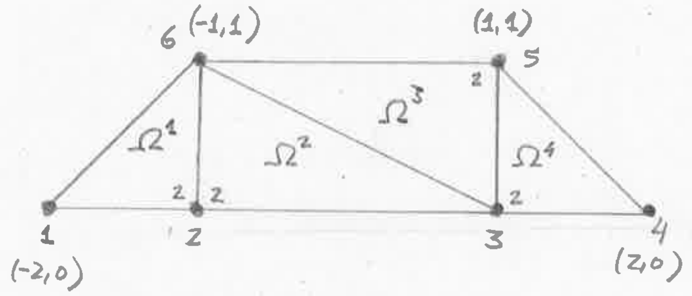
Ex_Final_Q1_2016_17_Problema 5.pdf

MÈTODES NUMÈRICS: Ex. Final ______________________ Q1-2016-17

5. Consider the trapezoidal domain shown in the figure below, with vertices $(-2,0)$, $(2,0)$, $(1,1)$, $(-1,1)$. We want to solve the problem defined by the equation $-\Delta u = 1$ with BC defined as:

* $u = 0$ on the polygonal made by the edges that joins vertices $(-2,0)$, $(-1,1)$, $(1,1)$ and $(2,0)$.
* $\frac{\partial u}{\partial y}(x,y) = x+2$ for $-2 \leq x \leq 2$.



Meshing this domain with four triangular elements as it is shown in the figure and using the proposed numbering for this mesh, compute:

(i) The stiffness and load vectors corresponding to elements $\Omega^1, \Omega^2, \Omega^3$ and Ω^4.

(ii) The connectivity matrix of the system.

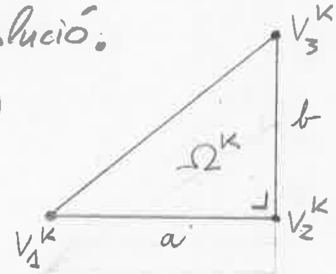
(iii) The global assembled system $KU = F + Q$.

(iv) The essential and natural boundary conditions needed to solve the system.

(v) The reduced system to compute the solution on the nodes 2, 3 and its solution (you can use Matlab to solve the system).

Solució.

(i)



$a_{11} = a_{22} \equiv c$

$a_{12} = a_{21} = a_{00} \equiv 0$

$$K^K = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & b^2 \end{pmatrix}, \quad F^K = \frac{f_K A_K}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

A_K: area of the triangle.

$$K^1 = K^4 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad F^1 = F^4 = \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad K^2 = K^3 = \frac{1}{4}\begin{pmatrix} 4 & -4 & 0 \\ -4 & 5 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad F^2 = F^3 = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(ii) Connectivity matrix : $elem = \begin{pmatrix} 1 & 2 & 6 \\ 6 & 2 & 3 \\ 3 & 5 & 6 \\ 5 & 3 & 4 \end{pmatrix}$

(iii)

$$K = \begin{pmatrix} K^1_{11} & K^1_{12} & 0 & 0 & 0 & K^1_{13} \\ * & K^1_{22}+K^2_{22} & K^2_{23} & 0 & 0 & K^1_{23}+K^2_{21} \\ * & * & K^2_{33}+K^3_{11}+K^4_{22} & K^4_{23} & K^3_{12}+K^4_{21} & K^2_{31}+K^3_{13} \\ * & * & * & K^4_{33} & K^4_{31} & 0 \\ * & * & * & * & K^4_{11}+K^3_{22} & K^3_{23} \\ * & * & * & * & * & K^1_{33}+K^2_{11}+K^3_{33} \end{pmatrix}$$

symmetric

$$= \frac{1}{4}\begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -2 & 9 & -1 & 0 & 0 & -6 \\ 0 & -1 & 9 & -2 & -6 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & -6 & 0 & 7 & -1 \\ 0 & -6 & 0 & 0 & -1 & 7 \end{pmatrix};$$

and: $F_1 = F^1_1 = 1/6$

$F_2 = F^1_2 + F^2_2 = 3/6$

$F_3 = F^2_3 + F^3_1 + F^4_2 = 5/6$

$F_4 = F^4_3 = 1/6$

$F_5 = F^3_2 + F^4_1 = 3/6$

$F_6 = F^4_3 + F^2_1 + F^3_3 = 5/6$

$$\bar{F} \equiv \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{pmatrix} \equiv \begin{pmatrix} F^1_1 \\ F^1_2 + F^2_2 \\ F^2_3 + F^3_1 + F^4_2 \\ F^4_3 \\ F^3_2 + F^4_1 \\ F^1_3 + F^2_1 + F^3_3 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 \\ 3 \\ 5 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

So the global assembled system can be written as:

$$\frac{1}{4}\begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -2 & 9 & -1 & 0 & 0 & -6 \\ 0 & -1 & 9 & -2 & -6 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & -6 & 0 & 7 & -1 \\ 0 & -6 & 0 & 0 & -1 & 7 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 \\ 3 \\ 5 \\ 1 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{pmatrix}$$

(iv) 1^{st}: Let us parameterise the q_n's wrt the edges' length

* For $\Gamma_1^1 = \{(x,y)\in\mathbb{R}^2 : y=0,\ -2\le x\le -1\}$: $(x_1^1(s), y_1^1(s)) = (-2+s, 0)$, $0\le s\le h_1^1 = 1$.

$$q_{n1}^1(s) = A(-2+s,0)\nabla u(-2+s,0)\cdot(-\vec{e}_2) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x}(-2+s,0)\\ \frac{\partial u}{\partial y}(-2+s,0)\end{pmatrix}\cdot(-\vec{e}_2)$$

$$= \left(\frac{\partial u}{\partial x}(-2+s,0), \frac{\partial u}{\partial y}(-2+s,0)\right)\cdot\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{\partial u}{\partial y}(-2+s,0) = -(\cancel{-2}+s+\cancel{2}) = -s,\ 0\le s\le h_1^1=1$$

* For $\Gamma_2^2 = \{(x,y)\in\mathbb{R}^2 : y=0,\ -1\le x\le 1\}$: $(x_2^2(s), y_2^2(s)) = (-1+s, 0)$, $0\le s\le h_2^2 = 2$

$$q_{n2}^2(s) = A(-1+s,0)\nabla u(-1+s,0)\cdot(-\vec{e}_2) = \left(*, \frac{\partial u}{\partial y}(-1+s,0)\right)\cdot\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{\partial u}{\partial y}(-1+s,0)$$

$$= -(-1+s+2) = -1-s,\quad 0\le s\le 2.$$

* For $\Gamma_2^4 = \{(x,y)\in\mathbb{R}^2 : y=0,\ 1\le x\le 2\}$: $(x_2^4(s), y_2^4(s)) = (1+s, 0)$, $0\le s\le h_2^4 = 1$

$$q_{n2}^4(s) = A(1+s,0)\nabla u(1+s,0)\cdot(-\vec{e}_2) = \left(*, \frac{\partial u}{\partial y}(1+s,0)\right)\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{\partial u}{\partial y}(1+s,0)$$

$$= -(1+s+2) = -s-3,\quad 0\le s\le 1$$

We want to compute: $Q_2 = Q_{21}^1 + Q_{22}^2$, and $Q_3 = Q_{32}^2 + Q_{22}^4$. So:

$$Q_{21}^1 = \int_0^{h_1^1} q_{n1}^1(s)\,\psi_{21}^1(s)\,ds = \int_0^{h_1^1}(-s)\cdot\frac{s}{h_1^1}\,ds \overset{(h_1^1=1)}{=} -\int_0^1 s^2\,ds = \boxed{-\tfrac{1}{3}}$$

$$Q_{22}^2 = \int_0^{h_2^2} q_{n2}^2(s)\,\psi_{22}^2(s)\,ds = \int_0^{h_2^2}(-1-s)\left(1-\frac{s}{h_2^2}\right)ds \overset{(h_2^2=2)}{=} -\int_0^2(1+s)(1-s/2)\,ds = \left\{\begin{matrix}C.V\\ \tau = 1-s/2\end{matrix}\right\}$$

$$= -2\int_0^1 \tau(3-2\tau)\,d\tau = -3+\tfrac{4}{3} = \boxed{-\tfrac{5}{3}}$$

$$Q_{32}^2 = \int_0^{h_2^2} q_{n2}^2(s)\,\psi_{32}^2(s)\,ds = \int_0^{h_2^2}(-1-s)\cdot\frac{s}{h_2^2}\,ds \overset{(h_2^2=2)}{=} -\frac{1}{2}\int_0^2(1+s)\,s\,ds = -\frac{1}{2}\left(\frac{s^2}{2}+\frac{s^3}{3}\right)\Big|_0^2$$

$$= -\frac{1}{2}\left(2+\frac{8}{3}\right) = \boxed{-\tfrac{7}{3}}$$

$$Q_{22}^4 = \int_0^{h_2^4} q_{n2}^4(s)\,\psi_{22}^4(s)\,ds = \int_0^{h_2^4}(-s-3)\left(1-\frac{s}{h_2^4}\right)ds \overset{(h_2^4=1)}{=} -\int_0^1(s+3)(1-s)\,ds = \left\{\begin{matrix}C.V.\\ \tau = 1-s\end{matrix}\right\}$$

$$= -\int_0^1(4-\tau)\tau\,d\tau = -2+\tfrac{1}{3} = \boxed{-5/3}$$

Remark. We note that $q_n(x,y) = -\frac{\partial u}{\partial y}(x,y) = -x-2$, at $(x,y)\in\Gamma = \{(x,y)\in\mathbb{R}^2;\ y=0,\ -2\le x\le 2\}$ is an affine function, so we've "linear flow" and then:

$$Q_{21}^1 = \Big(\frac{1}{6}\underbrace{q_n(-2,0)}_{=0} + \frac{1}{3}\underbrace{q_n(-1,0)}_{=-1}\Big)\cdot \underbrace{h_1^1}_{=1} = \Big(\frac{0}{6}+\frac{1}{3}(-1)\Big)1 = \boxed{-\frac{1}{3}}$$

$$Q_{22}^2 = \Big(\frac{1}{3}\underbrace{q_n(-1,0)}_{=-1} + \frac{1}{6}\underbrace{q_n(1,0)}_{=-3}\Big)\cdot \underbrace{h_2^2}_{=2} = \Big(\frac{1}{3}(-1)+\frac{1}{6}(-3)\Big)2 = \boxed{-5/3}$$

$$Q_{32}^2 = \Big(\frac{1}{6}\underbrace{q_n(-1,0)}_{=-1} + \frac{1}{3}\underbrace{q_n(1,0)}_{=-3}\Big)\underbrace{h_2^2}_{=2} = \Big(\frac{1}{6}(-1)+\frac{1}{3}(-3)\Big)2 = \boxed{-\frac{7}{3}}$$

$$Q_{22}^4 = \Big(\frac{1}{3}\underbrace{q_n(1,0)}_{=-3} + \frac{1}{6}\underbrace{q_n(2,0)}_{=-4}\Big)\underbrace{h_2^4}_{=1} = \Big(\frac{1}{3}(-3)+\frac{1}{6}(-4)\Big)1 = \boxed{-\frac{5}{3}}$$

... this save us from computing a lot of integrals.

Thus, the natural BC are:

$$Q_2 = Q_{21}^1 + Q_{22}^2 = -\frac{1}{3}-\frac{5}{3} = -2,\quad Q_3 = Q_{32}^2 + Q_{22}^4 = -7/3 - 5/3 = -4,$$

Whereas the essential BC are:

$$U_1 = U_4 = U_5 = U_6 = 0$$

(V) Reduced system: $\frac{1}{4}\begin{pmatrix}9 & -1\\ -1 & 9\end{pmatrix}\begin{pmatrix}U_2\\ U_3\end{pmatrix} = \frac{1}{6}\begin{pmatrix}3\\ 5\end{pmatrix} + \begin{pmatrix}-2\\ -4\end{pmatrix} \Leftrightarrow \begin{pmatrix}9 & -1\\ -1 & 9\end{pmatrix}\begin{pmatrix}U_2\\ U_3\end{pmatrix} = \begin{pmatrix}-6\\ -38/3\end{pmatrix}$

Solution of the reduced system

$$\left(\begin{array}{cc|c}9 & -1 & -6\\ -1 & 9 & -\frac{38}{3}\end{array}\right) \sim \left(\begin{array}{cc|c}9 & -1 & -6\\ 0 & 80 & -120\end{array}\right) \sim \left(\begin{array}{cc|c}9 & -1 & -6\\ 0 & 2 & -3\end{array}\right) \sim \left(\begin{array}{cc|c}18 & 0 & -15\\ 0 & 2 & -3\end{array}\right) \sim \left(\begin{array}{cc|c}6 & 0 & -5\\ 0 & 2 & -3\end{array}\right).$$

$\therefore$ $\boxed{U_2 = -5/6 = -0.8\overline{3}}$, $\boxed{U_3 = -3/2 = -1.5}$ $\square$