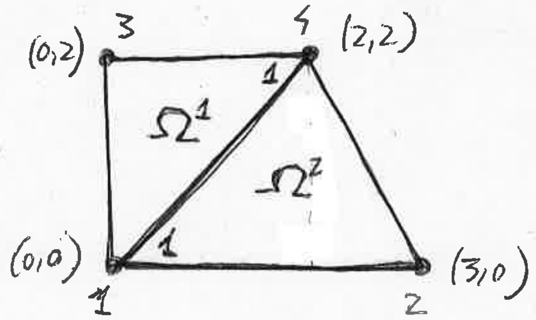
MÈTODES NUMÈRICS : Ex-Final —— (a) Q1-2017-18

(4) On the trapezoidal domain, D, shown in the figure below, that has been meshed using two triangles, we consider the problem



$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = 1+\frac{y}{2}, \text{ a } D$$

$u = 1+2y$, on the segment $\overline{3,1}$, $\frac{\partial u}{\partial \vec{n}} = 2$, on the segment $\overline{2,4}$

$\frac{\partial u}{\partial y} = 2+2x$, on the segment $\overline{1,2}$, $\frac{\partial u}{\partial \vec{n}} = 0$, on the segment $\overline{4,3}$

(i) For the elemental system, $[K^1]u^1 = F^1+Q^1$, associated to element Ω^1, compute

(a) $[K^1]$, (b) F_3^1

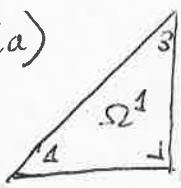
(ii) For the assembled system, $[K]U = F+Q$ write below the boundary conditions that must be applied

(c) Essential

(d) Natural

Solution:

(ia)



$b=2$ $\quad K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}$; $\quad k=1$, $c=1$, $a=b=2$, then $K^1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

(ib) $\psi_3^1(x,y) = 1-y/2$. Indeed: $\psi_3^1(2,2) = 0$ $\psi_3^1(0,2) = 0$, $\psi_3^1(0,0) = 1$

$$F_3^1 = \iint_{\Omega^1} f(x,y)\,\psi_3^1(x,y) = \int_0^2 dy \int_0^y \left(1+\frac{y}{2}\right)\left(1-\frac{y}{2}\right) dy = \int_0^2 dy \int_0^y \left(1-\frac{y^2}{4}\right) dx$$

$$= \int_0^2 y\left(1-\frac{y^2}{4}\right) dy = \left(\frac{y^2}{2} - \frac{y^4}{16}\right)\Bigg|_{y=0}^{y=2} = 2-1 = \boxed{1}$$

(iic) Essential BC: $U_1 = 1$, $U_3 = 5$.

Natural BC: $Q_2 = Q_{21}^2 + Q_{22}^2$, $Q_4 = Q_{32}^2 + Q_{11}^1$

$Q_{21}^2 = \left(-\frac{2}{6} - \frac{8}{3}\right)\cdot 3 = -1-8 = -9$ (linear flow on segment $\overline{1,2}$)

$Q_{22}^2 = \frac{2}{2}\sqrt{1^2+2^2} = \sqrt{5}$ (constant flow on segment $\overline{2,4}$: $\frac{\partial u}{\partial \vec{n}} = 2$)

$Q_{32}^2 = \frac{2}{2}\sqrt{1^2+2^2} = \sqrt{5}$ (" " " ").

$Q_{11}^1 = 0$: $\left(\frac{\partial u}{\partial \vec{n}} = 0 \text{ on the segment } \overline{4,3}\right)$

Hence: $Q_2 = Q_{21}^2 + Q_{22}^2 = \sqrt{5} - 9$, $Q_4 = Q_{32}^2 + Q_{11}^1 = \sqrt{5}$. □