240032-24-ex-final-q1-2023-24

P2

Consider the equation $-c\Delta u = 0$ on the domain $\mathscr{D} = \Omega^1 \cup \Omega^2$ meshed with two elements and connectivity matrix $C = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 5 & * \end{pmatrix}$. Ω^1 is a rectangle with node 1 in $(0,0)$, whose edge 1-2 lies in the OX axis. Edge 1-2 of length 5 and edge 2-3 of length 1. Ω^2 is a right triangle with edge 3-4 of length 3 and edge 4-5 of length 4. The value of c is 30 in Ω^1 and 48 in Ω^2.

Hint. Due to the shape of the elements, there is no need to compute the nodes' coordinates.

Part (a)

(a) (2 points) The values of $K^1_{2,3}$ and $K^2_{1,2}$ are,

- -34 and -9
- -46 and -16
- Leave it empty (no penalty)
- -41 and -18
- -49 and -32

Solution

Taking into account the edges' lengths told in the the text and the connectivity matrix, $C = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 5 & * \end{pmatrix}$, we can sketch the tilling of the domain $\mathscr{D}$ into the elements Ω^1 (rectangle) and Ω^2 (right triangle), as shown in the figure below.

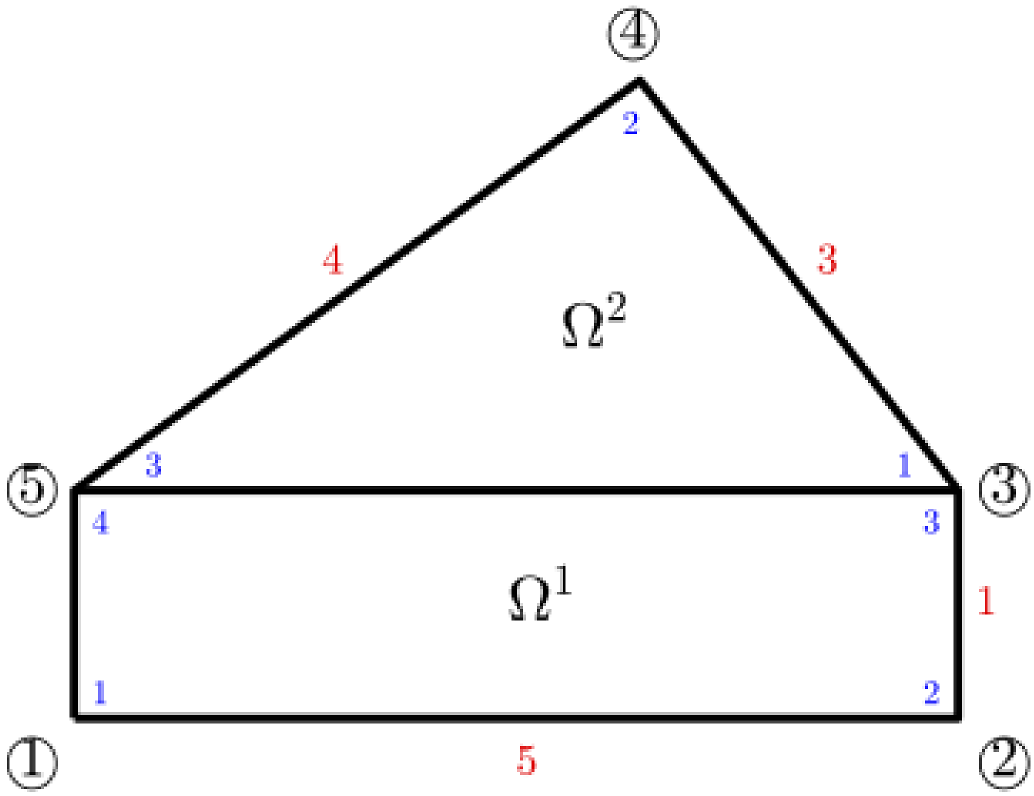


Figure 1

Local stiffness matrices. To compute the local stiffness matrix of Ω^1, we can use the explicit formulas for rectangular quadrilateral elements, when the coefficients of the model equation are constant. These formulas have been discussed in class, and can be found in the notes on the FEM available at the *Numerical Factory*, page 28:

Computing the Integrals: Rectangles

- If we consider **constant coefficients** for the *model equation*

In the case of a **rectangular quadrilateral**

$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

$$[K^{k,11}] = \frac{b\,a^k_{11}}{6a}\begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix}, \quad [K^{k,12}] = \frac{a^k_{12}}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$[K^{k,22}] = \frac{a\,a^k_{22}}{6b}\begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix}, \quad [K^{k,00}] = \frac{ab\,a^k_{00}}{36}\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}.$$

Here $k=1$ is the number of the element, the length of the edge joining the local nodes 1 and 2 is $a=5$, the length of the edge joining the local nodes 2 and 3 is $b=1$, $a^1_{1,1}=a^1_{2,2}=c=30$, $a^1_{1,2}=a^1_{2,1}=a^1_{0,0}=0$. Hence $K^{1,12}=K^{1,21}=K^{1,00}=0$ and then

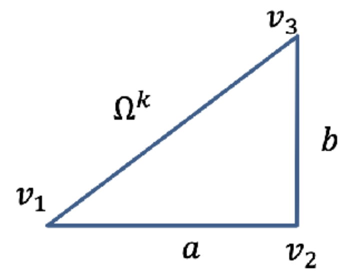
$$K^1 = K^{1,11}+K^{1,22} = \begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix} + 25\begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 52 & 23 & -26 & -49 \\ 23 & 52 & -49 & -26 \\ -26 & -49 & 52 & 23 \\ -49 & -26 & 23 & 52 \end{pmatrix}$$

To compute the local stiffness matrix of Ω^2, we can use the explicit formula that apply when the element Ω^k is a right triangle, with local node 2 placed at the right angle's vertex (see figure below) and the coefficients of the model equation are constants and equal to $a^k_{1,1}=a^k_{2,2}=c$, $a^k_{1,2}=a^k_{2,1}=a^k_{0,0}=0$. This formula has been discussed in class and can be found in the notes of the FEM available at the *Numerical Factory*, page 28:

Computing the Integrals: Triangles

- In the case of a general **linear triangular rectangle element** for the **Poisson's Equation**

$(a_{11} = a_{22} = c, \quad a_{12} = a_{21} = a_{00} = 0)$



The formula:

$$[K^k] = [K^{k,00}] + [K^{k,11}] + [K^{k,12}] + [K^{k,21}] + [K^{k,22}],$$

Simplifies to:

$$K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}$$

$$F^k = \frac{f_k A_k}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{f_k ab}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Here $k = 2$ is the number of the element, the length of the edge joining the local nodes 1 and 2 is $a = 3$, the length of the edge joining the nodes 2 and 3 is $b = 1$ (see the sketch of the meshed domain above, and note also that the local node 2 is at the right angle's vertex), and $c = 48$

$$K^2 = \begin{pmatrix} 32 & -32 & 0 \\ -32 & 50 & -18 \\ 0 & -18 & 18 \end{pmatrix}$$

Therefore, the solution of part (a) is: $K^1_{2,3} = -49$, $k^2_{1,2} = -32$.

```
clearvars
close all

nodes = [0,0;
    5,0;
    5,1;
    16/5, 14/5;
    0,1];

elem = [1,2,3,5;
    3,4,5,5];

%Element 1
```

```
a = 5; b = 1; c = 30;
K1 = b*c*[2, -2, -1, 1;
    -2, 2, 1, -1;
    -1, 1, 2, -2;
    1, -1, -2, 2]/6/a + a*c*[2,1,-1,-2;
      1,2,-2,-1;
      -1,-2,2,1;
      -2,-1,1,2]/6/b
```

```
K1 = 4×4
    52    23   -26   -49
    23    52   -49   -26
   -26   -49    52    23
   -49   -26    23    52
```

```
%Element 2
a = 3; b = 4; c = 48;
K2 =c*[b^2, -b^2, 0;
    -b^2, a^2 + b^2, -a^2;
    0, -a^2, a^2]/2/a/b
```

```
K2 = 3×3
    32   -32     0
   -32    50   -18
     0   -18    18
```

```
fprintf('(a) K1(2,3) = %f, K2(1,2) = %f\n',K1(2,3),K2(1,2))
```

```
(a) K1(2,3) = -49.000000, K2(1,2) = -32.000000
```

Part (b)

(b) (3 points) The value of $K_{3,5}$ is,

- -7
- Leave it empty (no penalty)
- 7
- 17
- 23

Solution

The global assembled matrix is:

$$K = \begin{pmatrix} K^1_{1,1} & K^1_{1,2} & K^1_{1,3} & 0 & K^1_{1,4} \\ K^1_{2,1} & K^1_{2,2} & K^1_{2,3} & 0 & K^1_{2,4} \\ K^1_{3,1} & K^1_{3,2} & K^1_{3,3}+K^2_{1,1} & K^2_{1,2} & K^1_{3,4}+K^2_{1,3} \\ 0 & 0 & K^2_{2,1} & K^2_{2,2} & K^2_{2,3} \\ K^1_{4,1} & K^1_{4,2} & K^1_{4,3}+K^2_{3,1} & K^2_{3,2} & K^1_{4,4}+K^2_{3,3} \end{pmatrix} = \begin{pmatrix} 52 & 23 & -26 & 0 & -49 \\ 23 & 52 & -49 & 0 & -26 \\ -26 & -49 & 84 & -32 & 23 \\ 0 & 0 & -32 & 50 & -18 \\ -49 & -26 & 23 & -18 & 70 \end{pmatrix}.$$

Per tant $K_{3,5} = 23$.

```
numNodes = size(nodes,1);
numElem = size(elem,1);
K = zeros(numNodes);

%Assemble matrices K1 & K2
rows = elem(1,:); cols = rows;
K(rows,cols) = K(rows,cols) + K1;

rows = elem(2,1:3); cols = rows;
K(rows,cols) = K(rows,cols) + K2;

fprintf('%6.1f %6.1f %6.1f %6.1f %6.1f\n',K.')
```

```
  52.0   23.0  -26.0    0.0  -49.0
  23.0   52.0  -49.0    0.0  -26.0
 -26.0  -49.0   84.0  -32.0   23.0
   0.0    0.0  -32.0   50.0  -18.0
 -49.0  -26.0   23.0  -18.0   70.0
```

```
fprintf('(b) K(3,5) = %f',K(3,5))
```

```
(b) K(3,5) = 23.000000
```

Part (c)

(c) (3 points) Assume that we have a boundary conditions $u(x,y) \equiv 1$ on the boundaries 4-5, 5-1, $q_n^1(x,y) \equiv 0$ on 2-3, and $q_n^2(x,y) \equiv 2$ on 3-4. Then, the value we obtain for the approximate solution at the node 3 (i.e. U_3) is,

- Leave it empty (no penalty)
- 1.043956
- 1.035714
- 1.046512
- 1.040541

Solution

- Natural boundary conditions: $Q_3 = Q_{3,2}^1 + Q_{1,1}^2 = 0 + \frac{q_1^2 h_1^2}{2} = \frac{2 \times 3}{2} = 3$ (note that $Q_{3,2}^1 = 0$, since $q_n^1(x,y) = 0$ for $(x,y) \in \Gamma_2^1$.
- Essential boundary conditions $U_1 = U_2 = U_4 = U_5 = 1$.

Therefore, the reduced system is,

$$84U_3 = F_3 + Q_3 + 26U_1 + 49U_2 + 32U_4 - 23U_5 = 0 + 3 + 26 + 49 + 32 - 23 = 87$$

We stress that $F_3 = F_3^1 + F_1^2 = 0$, since the r.h.s. of the equation is $f(x, y) = 0$ for all $(x, y) \in \mathscr{D}$, so $F^1 = F^2 = 0$. Finally, from the last equation we, see that the approximate soution at the node 3 is,

$$u(5,1) \approx U_3 = \frac{87}{84} = 1.03\overline{571428}.$$

```
Q = zeros(numNodes,1);
F = zeros(numNodes,1);
u = zeros(numNodes,1);

%Boundary conditions
fixedNods = [1,2,4,5];
freeNods = setdiff(1:numNodes,fixedNods);

%Natural B.C.
Q(3) = 3;

%Essential B.C.
u(fixedNods) = 1;

%Reduced system
Fm = F(freeNods) + Q(freeNods) - K(freeNods,fixedNods)*u(fixedNods);
Km = K(freeNods, freeNods);

um = Km\Fm;
u(freeNods) = um;
fprintf('(c) The approximate solution at the node 3 is u(%d,%d) %s U(3) =
%.9f\n',...
    nodes(3,:),char(8776),u(3))
```

```
(c) The approximate solution at the node 3 is u(5,1) ≈ U(3) = 1.035714286
```

Part (d)

(d) (2 points) Same as (c), but now, $\frac{\partial u}{\partial x}(x, y) = 2u(x, y)$ on 2-3. Then, the values of U_3 is,

- Leave it empty (no penalty)
- -1.113537
- 1.515625
- -1.914894
- -19.5000

Hint. You can formulate this BC as a convection one for the suitable values of β and T_∞.

Solution

The boundary conditon on the edge Γ_2^1can be formulated as,

$$q^1_{n,2}(x,y) = c\frac{\partial u}{\partial x}(x,y) = -\beta(u(x,y) - u_\infty), \qquad (x,y) \in \Gamma^1_2,$$

with $\beta = -2c = -60$ and $u_\infty = 0$. Hence,

$$Q^1_{3,2} = -\beta h^1_2\left(\frac{U_3}{3} + \frac{U_2}{6}\right) + h^1_2\frac{\beta u_\infty}{2} = 2ch^1_2\left(\frac{U_3}{3} + \frac{U_2}{6}\right) - cu_\infty h^1_2 = 20U_3 + 10U_2 = 20U_3 + 10,$$

where substitution $U_2 = 1$ has alredy been made. So, now,

$$Q_3 = Q^1_{3,2} + Q^2_{1,1} = 20U_3 + 10 + 3 = 20U_3 + 23$$

and the new reduced system writes as,

$$84U_3 = 26 + 49 + 32 - 23 + 20U_3 + 13 = 97 + 20U_3.$$

Moving the term $20U_3$ at the r.h.s back to the l.h.s. of the equation, we can solve for U_3. Indeed,

$$64U_3 = 97 \Longrightarrow U_3 = \frac{97}{64} = 1.515625.$$

So, with the new boundary condition on the edge Γ^1_2, the approximate solution at the node 3 is given by, $u(5,1) \approx U_3 = 1.515625$.

```
%The same computations, using Matlab
% Km = Km - 20
% Qm = Q(freeNods) + 10

Km = Km - 20;
Fm = F(freeNods) + Q(freeNods) + 10 - K(freeNods,fixedNods)*u(fixedNods);

um = Km\Fm;
u(freeNods) = um;
fprintf('(d) u(%d,%d) %s U(3) = %.6f\n',nodes(3,:),char(8776),u(3))
```

```
(d) u(5,1) ≈ U(3) = 1.515625
```