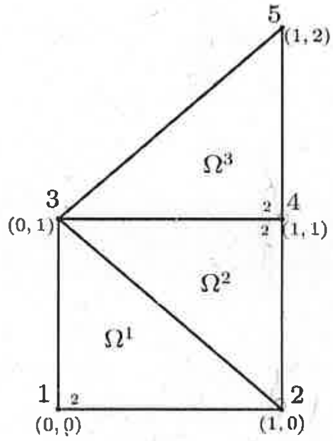
**Name and surnames:**

**2.** Consider the domain shown in the figure below, and with the elements, nodes and local and global numbering plotted there.




We want to solve the thermal problem defined by the equation

$$-\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = 0$$

with boundary conditions given by:

- At the line that joints nodes 2 and 5, there is convection BC with $\beta = 3$ and $T_\infty = 30$
- $T = 100$ at the nodes 5, 3, 1 and 2.

Then, compute:

(i) The stiffness matrix of the first element $K^1$.

(ii) The 4th row of the global stiff matrix before applying the BC.

(iii) The value of the coefficients for $Q^2_{21}$ of the local system for $\Omega^2$ as a function of $T_2$ and $T_4$.

(iv) The value of the coefficients for $Q^3_{22}$ of the local system for $\Omega^3$ as a function of $T_4$ and $T_5$.

(v) The value of $T_4$.

| **Results:** | |
|---|---|
| $[K^1] =$ | $\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ |
| 4th row : | $\frac{1}{2}\begin{pmatrix} 0 & -1 & -2 & 4 & -1 \end{pmatrix}$ |
| $Q_{21}^2 = a_1 T_2 + a_2 T_4 + a_3$ | coeff $a_i$: $a_1 = -\frac{1}{2}$, $a_2 = -1$, $a_3 = 45$ |
| $Q_{22}^3 = b_1 T_4 + b_2 T_5 + b_3$ | coeff $b_i$: $b_1 = -1$, $b_2 = -\frac{1}{2}$, $b_3 = 45$ |
| $a_i$ and $b_i$ are among the values: $-2, 2, 1, -1, 1/2, -1/2, 1/6, -1/6, 35, 40, 45, 50, 85, 90, 95, 100, 105$ | |
| $T_4 =$ | 47.5 |

(2 points)

Data: $T_\infty = 30$, $\beta = 3$, $T = T_{1,3,5,2} = 100°C$

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i $K^1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = K^2 = K^3$

ii $K(4,:) = \left(0, K^2_{21}, K^2_{23} + K^3_{21}, K^2_{22} + K^3_{22}, K^3_{23}\right)$

$= \frac{1}{2}\left(0, -1, -2, 4, -1\right)$

iii $Q^2_{21} = \int_0^{h^2_1} -\beta\left(T_2 \psi^2_{11}(s) + T_4 \psi^2_{21}(s) - T_\infty\right)\psi^2_{21}(s)\, ds$

$= -\beta T_2 \int_0^{h^2_1} \psi^2_{11}(s)\psi^2_{21}(s)\, ds - \beta T_4 \int_0^{h^2_1}\left(\psi^2_{21}(s)\right)^2 ds + \beta T_\infty \int_0^{h^2_1} \psi^2_{21}(s)\, ds$

$= -\beta T_2 h^2_1 \frac{1!1!}{3!} - \beta T_4 h^2_1 \frac{2!0!}{3!} + \beta T_\infty h^2_1 \cdot \frac{1!0!}{2!}$

$= -\beta h^2_1\left(\frac{T_2}{6} + \frac{T_4}{3} - \frac{T_\infty}{2}\right) \underset{h^2_1=1}{=} -3\left(\frac{T_2}{6} + \frac{T_4}{3} - 15\right) = \boxed{-\frac{1}{2}T_2 - T_4 + 45}$

*Alternatively, using the "linear flow's rule":*

$Q^2_{21} = -\beta h^2_1\left(\frac{T_2}{2} + \frac{1}{3}(T_4 - T_2)\right) + \beta h^2_1 \frac{T_\infty}{2} = -\beta h^2_1\left(\frac{T_2}{6} + \frac{T_4}{3} - \frac{T_\infty}{2}\right) = -3\left(\frac{T_2}{6} + \frac{T_4}{3} - 15\right)$

iv $Q^3_{22} = \int_0^{h^3_2} -\beta\left(T_4 \psi^3_{22}(s) + T_5 \psi^3_{32}(s) - T_\infty\right)\psi^3_{22}(s)\, ds$

$= -\beta T_4 \int_0^{h^3_2}\left(\psi^3_{22}(s)\right)^2 ds - \beta T_5 \int_0^{h^3_2} \psi^3_{32}(s)\psi^3_{22}(s)\, ds + \beta T_\infty \int_0^{h^3_2} \psi^3_{22}(s)\, ds$

$= -\beta T_4 h^3_2 \frac{2!0!}{(2+0+1)!} - \beta T_5 h^3_2 \frac{1!1!}{(1+1+1)!} + \beta T_\infty h^3_2 \frac{1!0!}{(1+0+1)!}$

$= -\beta h^3_2\left(\frac{T_4}{3} + \frac{T_5}{6} - \frac{T_\infty}{2}\right) \underset{h^3_2=1}{=} -3\left(\frac{T_4}{3} + \frac{T_5}{6} - 15\right) = \boxed{-T_4 - \frac{1}{2}T_5 + 45}$

*Alternatively:*

$-\beta h^3_2\left(\frac{T_4}{2} + \frac{1}{6}(T_5 - T_4)\right) + \beta h^3_2 \frac{T_\infty}{2} = -\beta h^3_2\left(\frac{T_4}{3} + \frac{T_5}{6}\right) + \beta h^3_2 \frac{T_\infty}{2}$

$= -\beta h^3_2\left(\frac{T_4}{3} + \frac{T_5}{6} - \frac{T_\infty}{2}\right) = -3 \cdot \left(\frac{T_4}{3} + \frac{T_5}{6} - 15\right).$

V B.C.

Essentials: $T_2 = T_1 = T_3 = T_5 = T = 100$

Natural : $Q_4 = Q^2_{21} + Q^3_{22} = -\beta\left(\frac{T_2}{6} + \frac{2T_4}{3} + \frac{T_5}{6} - T_\infty\right)$

$$\underset{T_2=T_5=T}{=} -\frac{\beta}{3}\left(T + 2T_4 - 3T_\infty\right)$$

Reduced system :

$$K_{44}T_4 = -\beta/3\left(T + 2T_4 - 3T_\infty\right) - K_{41}\overset{=T}{T_1} - K_{42}\overset{=T}{T_2} - K_{43}\overset{=T}{T_3} - K_{45}\overset{=T}{T_5}$$

$$= -\left(\beta/3 + K_{41} + K_{42} + K_{43} + K_{45}\right)T + \beta T_\infty - 2\beta/3\, T_4$$

$$\Leftrightarrow \quad 2\,T_4 = -\left(\beta/3 - 0 - 1/2 - 1 - 1/2\right)\cdot T + \beta T_\infty - 2\beta/3\, T_4$$

$$T_4 = \frac{(2 - \beta/3)T + \beta T_\infty}{2(1 + \beta/3)} \underset{\substack{\beta=3 \\ T=100 \\ T_\infty=30}}{=} \frac{(2-1)\times 100 + 3\times 30}{2(1+1)} = \frac{190}{4} = \frac{95}{2} = \boxed{47'5}$$