Mètodes Numèrics 2^{on} 20 de Gener de 2016

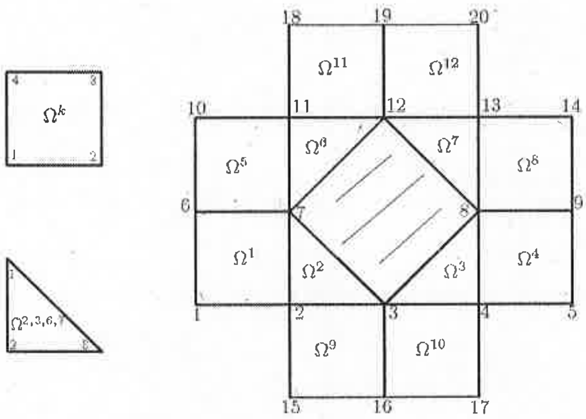
Nom i cognoms:

Problema 3. *(3.0 punts)*

Usant el mètode dels elements finits, volem aproximar la solució de l'equació,

$$-\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y^2} = f_0,$$

on f_0 és una constant, en la regió ja mallada i representada en la figura (un domini en forma de creu a la qual s'ha extret el quadrat interior de vèrtexs els nodes: 7, 3, 8, 12).



El problema està sotmès a la condició de contorn constant, $u = u_0$, a la part superior de la frontera, determinada unint els nodes: 10, 11, 18, 19, 20, 13, 14, i a la part inferior, determinada a l'unir, 1, 2, 15, 16, 17, 4, 5. La condició de contorn a les parts de frontera verticals, 1, 6, 10 i 5, 9, 14, així com a la frontera interior, 7, 3, 8, 12, és $q_n = q_0$, també constant.

Suposant que els costats dels elements quadrats tenen longitud de costat igual a 1, es demana:

(a) Calcular les matrius de rigidesa i vectors de càrrega elementals. (Indicació: els valors resultants per K^1_{32} i K^2_{21} són $-1/2$ i -1 respectivament). *(1.0 punt)*

(b) Calcular els elements diferents de zero de la fila 7 de la matriu de rigidesa global. *(1.0 punt)*

(c) Establir les condicions de contorn pel sistema global i calcular U_7. *(1.0 punt)*

Problema 3: $\quad -\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial y^2} = f_0 \,,$

$$a_{11} = 1,\ a_{12} = a_{21} = 0,\ a_{22} = 2,\ a_{00} = 0,\ f = f_0$$

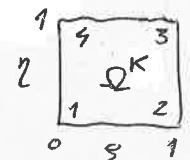
(a) T2-MN-FEM2D.pdf pàg. 49

quadrats: $a = b = 1$

$$K^{K,11} = \frac{1}{6}\begin{pmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{pmatrix}, \quad \boxed{K^{K,22} = \frac{1}{6}\begin{pmatrix} 4 & 2 & -2 & -4 \\ 2 & 4 & -4 & -2 \\ -2 & -4 & 4 & 2 \\ -4 & -2 & 2 & 4 \end{pmatrix}}$$

$$\boxed{K^{1,5,11,12,8,4,10,9} = \frac{1}{6}\begin{pmatrix} 6 & 0 & -3 & -3 \\ 0 & 6 & -3 & -3 \\ -3 & -3 & 6 & 0 \\ -3 & -3 & 0 & 6 \end{pmatrix}}, \quad \boxed{F^{1,5,11,12,8,4,10,9} \overset{(*)}{=} \frac{f_0}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

(*)



$$\psi_1^K(\xi,\eta) = (1-\xi)(1-\eta)$$
$$\psi_2^K(\xi,\eta) = \xi(1-\eta)$$
$$\psi_3^K(\xi,\eta) = \xi\eta$$
$$\psi_4^K(\xi,\eta) = (1-\xi)\eta$$

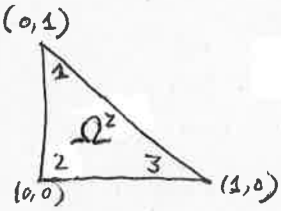
$$F_i^K = \iint_{\Omega^K} f_i^K \psi_i^K \, d\xi \, d\eta. \quad \text{(T2-MN-FEM2D.pdf pàg. 43)}$$

$$F_1^K = f_0\int_0^1 (1-\xi)\,d\xi \int_0^1 (1-\eta)\,d\eta = f_0\left(\xi - \xi^2/2\right)\Big|_{\xi=0}^{\xi=1} \cdot \left(\eta - \eta^2/2\right)\Big|_{\eta=0}^{\eta=1}$$

$$= f_0/4 \underset{\text{es comprova}}{=} F_2^K = F_3^K = F_4^K \text{, per } K = 1,5,11,12,8,4,10,9.$$

triangles:

1 2 3
2 3 1
3 1 2



Recordem (T2-MN-FEM2D.pdf, pàg. 47)

$$K_{ij}^{e,11} = a_{11}^e \frac{1}{4A_K} \beta_i^e \beta_j^e$$

$$K_{ij}^{e,22} = a_{22}^e \frac{1}{4A_K} \gamma_i^e \gamma_j^e$$

essent:

$$\beta_i^e = y_j^e - y_k^e$$
$$\gamma_i^e = -(x_j^e - x_k^e),$$

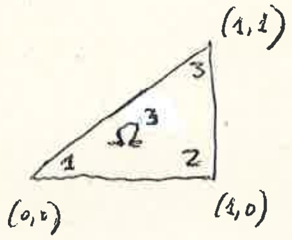
amb $i,j,k = 1,2,3$ + permutacions cícliques.

$$\beta_1^2 = 0 = \beta_1^7 \qquad \gamma_1^2 = 1 = -\gamma_1^7$$
$$\beta_2^2 = -1 = -\beta_2^7 \qquad \gamma_2^2 = -1 = -\gamma_2^7$$
$$\beta_3^2 = 1 = -\beta_3^7 \qquad \gamma_3^2 = 0 = \gamma_3^7$$

$$K^{2,7} = \frac{1}{4\underbrace{A_{27}}_{1/2}}\Big[\underbrace{a_{11}^{2,7}}_{1} \beta^{2,7}\left(\beta^{2,7}\right)^T + \overbrace{a_{22}^{2,7}}^{=2} \gamma^{2,7}\left(\gamma^{2,7}\right)^T\Big]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\right] =$$

$$= \frac{1}{2}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \boxed{\frac{1}{2}\begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}}$$



$$\beta_1^3 = -1 = -\beta_1^6 \qquad \gamma_1^3 = 0 = \gamma_1^6$$
$$\beta_2^3 = 1 = \beta_2^6 \qquad \gamma_2^3 = -1 = -\gamma_2^6$$
$$\beta_3^3 = 0 = \beta_3^6 \qquad \gamma_3^3 = 1 = -\gamma_3^6$$

$$K^{3,6} = \frac{1}{4\underbrace{A_{3,6}}_{\frac{1}{2}}}\left[\underbrace{a_{11}^{3,6}}_{1}\,\beta^{3,6}\cdot\left(\beta^{3,6}\right)^T + \underbrace{a_{22}^{3,6}}_{2}\,\gamma^{3,6}\cdot\left(\gamma^{3,6}\right)^T\right] = \frac{1}{2}\left[\begin{pmatrix}-1\\1\\0\end{pmatrix}\begin{pmatrix}-1 & 1 & 0\end{pmatrix} + 2\begin{pmatrix}0\\-1\\1\end{pmatrix}\begin{pmatrix}0 & -1 & 1\end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 2 & -2\\ 0 & -2 & 2\end{pmatrix}\right] = \boxed{\frac{1}{2}\begin{pmatrix}1 & -1 & 0\\ -1 & 3 & -2\\ 0 & -2 & 2\end{pmatrix}}$$

$$F^{2,7,3,6} = \frac{f_0}{6}\begin{pmatrix}1\\1\\1\end{pmatrix}$$ (veure T2-MN-FEM2D.pdf pàg. 47)

(b) $K_{7,1} = K_{3,1}^1 = -\frac{1}{2}$, $\qquad K_{7,7} = K_{2,2}^5 + K_{3,3}^1 + K_{1,1}^2 + K_{3,3}^6 = 1+1+1+1 = 4$

$K_{7,2} = K_{1,2}^2 + K_{3,2}^1 = -1 - \frac{1}{2} = -3/2 \qquad K_{7,10} = K_{2,4}^5 = -\frac{1}{2}$

$K_{7,3} = K_{1,3}^2 = 0 \qquad K_{7,11} = K_{2,3}^5 + K_{3,2}^6 = -\frac{1}{2} - 1 = -3/2$

$K_{7,6} = K_{2,1}^5 + K_{3,4}^1 = 0 + 0 = 0 \qquad K_{7,12} = K_{3,1}^6 = 0$

(c) Condicions de contorn essencials:

$$U_1 = U_2 = U_{15} = U_{16} = U_{17} = U_4 = U_5 = U_{10} = U_{11} = U_{18} = U_{19} = U_{20} = U_{13} = U_{14} = u_0$$

Condicions de contorn naturals:

$$Q_6 = Q_1^5 + Q_4^1 = q_0/2 + q_0/2 = q_0 = Q_9$$ (Notem que: $Q_{14}^5 = \int_0^1 q_0 \psi_{14}^5(s)\,ds = q_0 \int_0^1 s\,ds = q_0/2$)

$$Q_7 = \boxed{\underbrace{Q_2^5}_{0}} + Q_3^6 + \boxed{\underbrace{Q_3^1}_{0}} + Q_1^2 = \frac{q_0}{2}\sqrt{2} + \frac{q_0}{2}\sqrt{2} = q_0\sqrt{2} = Q_8 = Q_3 = Q_{12}$$ (Notem que: $Q_{33}^6 = \frac{q_0}{2}\sqrt{2} = Q_{11}^2$, veure: T2-MN-FEM2D.pdf pàg. 61)

Càlcul de U_7:

$$-\frac{1}{2}u_0 - \frac{3}{2}u_0 + 4\cdot U_7 - \frac{1}{2}u_0 - \frac{3}{2}u_0 = \left(-\frac{1}{2}-\frac{1}{2}-\frac{3}{2}-\frac{3}{2}\right)u_0 + 4U_7 = Q_7 + F_7$$

Amb $Q_7 = q_0\sqrt{2}$, $F_7 = F_2^5 + F_3^6 + F_3^1 + F_1^2 = \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\frac{1}{6}\right)f_0 = \left(\frac{1}{2}+\frac{1}{3}\right) = \frac{5}{6}f_0$;

és a dir:

$$-4u_0 + 4U_7 = q_0\sqrt{2} + \frac{5}{6}f_0 \iff \boxed{U_7 = u_0 + q_0\frac{\sqrt{2}}{4} + \frac{5}{24}f_0}$$ □