Mètodes Numèrics 2$^{\text{on}}$ 20 de Gener de 2016

Nom i cognoms:

Problema 2. *(3.0 punts)*

Considerem un domini triangular de vèrtexs, $(0,1)$, $(0,0)$, $(1,0)$ i, en el seu interior, el problema donat per l'equació,

$$-\frac{\partial}{\partial x}\left((1+y)\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left((1+x)\frac{\partial u}{\partial y}\right)=0.$$

Suposem que les condicions de contorn d'aquest problema vénen donades per:

$\frac{\partial u}{\partial x}(x,y)=-y$, per $x=0$, $0\leq y\leq 1$. (costat que uneix els vèrtex $(0,1)$ i $(0,0)$).

$\frac{\partial u}{\partial y}(x,y)=-1$, per $y=0$, $0\leq x\leq 1$. (costat que uneix els vèrtex $(0,0)$ i $(1,0)$).

$u=0$, en el costat del triangle que uneix els vèrtex $(1,0)$ i $(0,1)$.

Si resolem el problema usant un únic element finit triangular lineal, es demana:

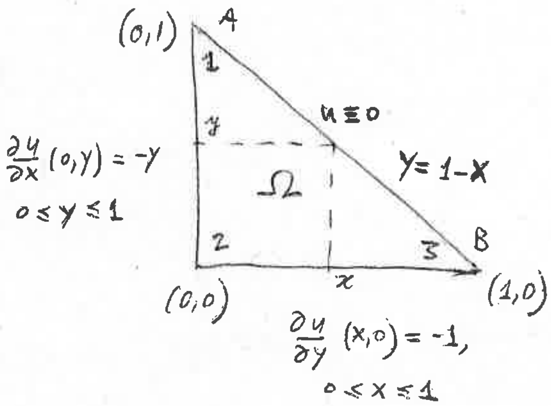
(a) Les funcions de forma d'aquest element. *(0.5 punts)*

(b) El sistema d'equacions elementals. *(1.0 punt)*

(c) Les condicions de contorn naturals i essencials que cal imposar. *(1.0 punt)*

(d) Els valors aproximats de la solució del problema en els punts $(0,0)$ i $(\frac{1}{4},\frac{1}{4})$. *(0.5 punts)*

Solució



$$-\frac{\partial}{\partial x}\left((1+y)\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left((1+x)\frac{\partial u}{\partial y}\right)=0$$

$$a_{11}(x,y)=1+y\,,\quad a_{22}(x,y)=1+x\,,$$

$$a_{12}=a_{21}=a_{00}=f\equiv 0$$

B.C.

$$\frac{\partial u}{\partial x}(0,y)=-y\,,\quad 0\leq y\leq 1$$

$$\frac{\partial u}{\partial y}(x,0)=-1\,,\quad 0\leq x\leq 1$$

$u(x,y)=0$ al llarg de $\overline{AB}$

(a) $\psi_1(x,y)=a_1+b_1x+c_1y$:
$$\left.\begin{array}{l}\psi_1(0,1)=a_1+c_1=1\\ \psi_1(0,0)=a_1=0\\ \psi_1(1,0)=a_1+b_1=0\end{array}\right\}:\quad \begin{array}{l}a_1=0=b_1,\ c_1=1,\\ \psi_1(x,y)=y\end{array}$$

$\psi_2(x,y)=a_2+b_2x+c_2y$:
$$\left.\begin{array}{l}\psi_2(0,1)=a_2+c_2=0\\ \psi_2(0,0)=a_2=1\\ \psi_2(1,0)=a_2+b_2=0\end{array}\right\}:\quad \begin{array}{l}a_2=1,\ b_2=c_2=-1\\ \psi_2(x,y)=1-x-y.\end{array}$$

$\psi_3(x,y)=a_3+b_3x+c_3y$:
$$\left.\begin{array}{l}\psi_3(0,1)=a_3+c_3=0\\ \psi_3(0,0)=a_3=0\\ \psi_3(1,0)=a_3+b_3=1\end{array}\right\}:\quad \begin{array}{l}a_3=0=c_3,\ b_3=1\\ \psi_3(x,y)=x.\end{array}$$

(b) T3-MN-FEM2D.pdf pàg. 45

$$K_{ij}^{11}=\iint_\Omega a_{11}(x,y)\,\frac{\partial\psi_i}{\partial x}(x,y)\,\frac{\partial\psi_j}{\partial x}(x,y)\,dx\,dy\,,\quad K_{ij}^{22}=\iint_\Omega a_{22}(x,y)\,\frac{\partial\psi_i}{\partial y}(x,y)\,\frac{\partial\psi_j}{\partial y}(x,y)\,dx\,dy$$

amb: $\frac{\partial\psi_1}{\partial x}(x,y)=0,\quad \frac{\partial\psi_2}{\partial x}(x,y)=-1,\quad \frac{\partial\psi_3}{\partial x}(x,y)=+1$

$\frac{\partial\psi_1}{\partial y}(x,y)=1,\quad \frac{\partial\psi_2}{\partial y}(x,y)=-1,\quad \frac{\partial\psi_3}{\partial y}(x,y)=0,$

$$K^{11}=\iint_\Omega(1+y)\,dx\,dy\cdot\begin{pmatrix}0&0&0\\0&1&-1\\0&-1&1\end{pmatrix}\overset{(*)}{=}\frac{2}{3}\begin{pmatrix}0&0&0\\0&1&-1\\0&-1&1\end{pmatrix},\quad K^{22}=\iint_\Omega(1+x)\,dx\,dy\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix}\overset{(**)}{=}\frac{2}{3}\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&0\end{pmatrix}$$

$$(*)\quad \iint_\Omega(1+y)\,dx\,dy=\int_0^1dy\int_0^{1-y}(1+y)\,dx=\int_0^1(1-y^2)\,dy=\left(y-\frac{y^3}{3}\right)\Big|_0^1=1-\frac{1}{3}=\frac{2}{3}$$

$$(**)\quad \iint_\Omega(1+x)\,dx\,dy=\int_0^1dx\int_0^{1-x}(1+x)\,dy=\frac{2}{3}$$

$$K = K^{11} + K^{22} = \frac{2}{3}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

— *Sistema d'equacions elemental:*

$$\frac{2}{3}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}$$

(c) Essential B.C. $U_1 = U_3 = 0$.

$$q_{n1} = \left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \nabla u, \vec{n} \right\rangle \Bigg|_{\Gamma_1} = \left\langle \begin{pmatrix} 1+y & 0 \\ 0 & 1+x \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\rangle = -(1+y)\frac{\partial u}{\partial x}\Bigg|_{\Gamma_1} = y(1+y), \quad 0 \le y \le 1$$

Parametrització de Γ_1: $\Gamma_1 = \{(x,y) = (0, 1-s),\ 0 \le s \le 1\}$. Llavors, $q_{n1}(s) = (1-s)(2-s)$, $0 \le s \le 1$

$$q_{n2} = \dots = \left\langle \begin{pmatrix} 1+y & 0 \\ 0 & 1+x \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\rangle \Bigg|_{\Gamma_2} = -(1+x)\frac{\partial u}{\partial y}\Bigg|_{\Gamma_2} = 1+x, \quad 0 \le x \le 1$$

flux lineal!

Parametrització de Γ_2: $\Gamma_2 = \{(x,y) = (s, 0),\ 0 \le s \le 1\}$. Llavors: $q_{n2}(s) = 1+s$, $0 \le s \le 1$.

$Q_2 = Q_{21} + Q_{22}$,

$(1-s)(1+1-s)(1-(1-s)) = (1-s)(1-(1-s)^2) = (1-s)-(1-s)^3$.

$$Q_{21} = \int_0^{h_1=1} q_{n1}(s)\, \underbrace{\Psi_{21}(s)}_{s/h_1}\, ds \underset{h_1=1}{=} \int_0^1 \underbrace{(1-s)(2-s)s}\, ds = \int_0^1 \left[(1-s)-(1-s)^3\right] ds = \left\{ \begin{matrix} \text{C.V.} \\ \xi = 1-s \end{matrix} \right\}$$

T3-MN-FEM2D.pdf pàg. 60

$$= \int_0^1 (\xi - \xi^3)\, d\xi = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

flux lineal! $Q_{22} = \left(\frac{1}{2} + \frac{1}{6}(2-1)\right) h_2^{1} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

Natural B.C.: $Q_2 = Q_{21} + Q_{22} = \frac{1}{4} + \frac{2}{3} = \frac{11}{12}$

(d) *Sistema reduït:* $\frac{4}{3} U_2 = \frac{11}{12} + \frac{2}{3} \overset{=0}{U_1} + \frac{2}{3} \overset{=0}{U_3} = \frac{11}{12} \Rightarrow U_2 = \frac{11}{16}$

Així:

$u(0,0) \simeq \boxed{U_2 = \frac{11}{16}}$

$$u\left(\tfrac{1}{4}, \tfrac{1}{4}\right) \simeq U_1 \Psi_1\left(\tfrac{1}{4}, \tfrac{1}{4}\right) + U_2 \underbrace{\Psi_2\left(\tfrac{1}{4}, \tfrac{1}{4}\right)}_{1-\frac{1}{4}-\frac{1}{4} = \frac{1}{2}} + \overset{=0}{U_3}\, \Psi_3\left(\tfrac{1}{4}, \tfrac{1}{4}\right) = \frac{11}{16} \times \frac{1}{2} = \boxed{\frac{11}{32}}$$