Ex_Final_Q1-2019-20_Problema2.pdf

MÈTODES NUMÈRICS: **Ex-Final Q1-2019-20** (a)

Name and surnames:

(2) Let D be the domain meshed according to the following data (**nodes** and **connectivity** matrix):

1	(0,−1)	3	(2,0)
2	(0,0)	4	(1,$\sqrt{3}$)

$$C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

We consider the following problem,

$$\begin{cases} -k_c\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right) = f \quad \text{on D, with,} \quad \mathbf{k_c} = 2, \ (x,y)\in\Omega^1; \quad \mathbf{k_c} = \sqrt{3}, \ (x,y)\in\Omega^2. \\ u = 0, \ \text{on the line } \overline{3,4}. \qquad u(0,-1) = -2, \\ \dfrac{\partial u}{\partial x} = 9\,y, \ \text{on the line } \overline{2,1}. \quad \dfrac{\partial u}{\partial \vec{n}} = -\dfrac{\sqrt{3}\,u}{2}, \ \text{on the line } \overline{4,2}. \end{cases}$$

(*Hint:* The BC on the line $\overline{4,2}$ is equivalent to a convection condition with $T_\infty = 0$)

Fill the boxes and answer the questions:

$$[K^1] = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 4 \end{pmatrix}$$

$$[K^2] = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Assume that $\mathbf{f(x,y)} \equiv 30, \ (x,y)\in\Omega^1$, and $\mathbf{f(x,y)} \equiv 0, \ (x,y)\in\Omega^2$.

Write the assembled system:

$$\frac{1}{2}\begin{pmatrix} 4 & -4 & 0 & 0 \\ -4 & 7 & -2 & -1 \\ 0 & -2 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 10 \\ 0 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}$$

(*Hint:* $K_{32} = -1$)

$Q^1_{22} =$ 3 $\quad Q^2_{13} =$ $-U_2$

(*Hint:* Q^2_{13} depends on the U's.)

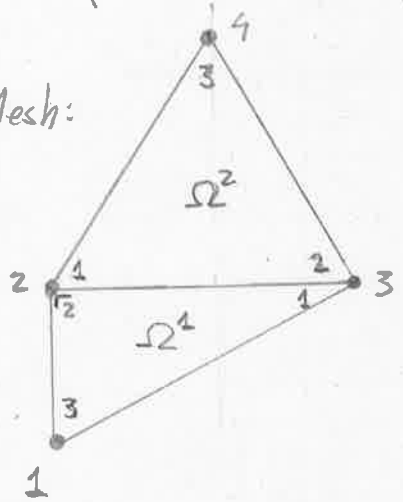
$u(0,0) \simeq$ 2

Solution:

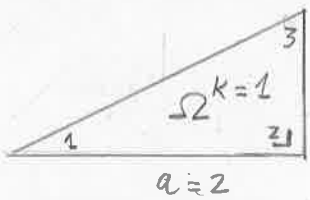
$$C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 4 \end{pmatrix} \quad N = \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 2 & 0 \\ 1 & \sqrt{3} \end{pmatrix}$$

Local Nod	x^1_k	y^1_k	x^2_k	y^2_k
1	2	0	0	0
2	0	0	2	0
3	0	-1	1	$\sqrt{3}$

Mesh:



(a)



$$b=1, \quad K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix}$$

$k=1$, $c = K^1_c = 2$, $a=2$, $b=1$. Hence:

$$\boxed{K^1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 4 \end{pmatrix}}$$

$$\beta^2_1 = y^2_2 - y^2_3 = 0 - \sqrt{3} = -\sqrt{3} \qquad \gamma^2_1 = -(x^2_2 - x^2_3) = -(2-1) = -1$$

$$\beta^2_2 = y^2_3 - y^2_1 = \sqrt{3} - 0 = \sqrt{3} \qquad \gamma^2_2 = -(x^2_3 - x^2_1) = -(1-0) = -1$$

$$\beta^2_3 = y^2_1 - y^2_2 = 0 - 0 = 0 \qquad \gamma^2_3 = -(x^2_1 - x^2_2) = -(0-2) = 2.$$

$$A_2 = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \sqrt{3} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{vmatrix} = \sqrt{3},$$

Therefore:
$$K^{2,11} = \frac{a^2_{11}}{4A_2}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \cdot (\beta_1 \; \beta_2 \; \beta_3) = \left\{ \begin{matrix} a^2_{11} = K^2_c = \sqrt{3} \\ A_2 = \sqrt{3} \end{matrix} \right\} = \frac{1}{4}\begin{pmatrix} -\sqrt{3} \\ \sqrt{3} \\ 0 \end{pmatrix} \cdot (-\sqrt{3} \; \sqrt{3} \; 0)$$

$$= \frac{1}{4}\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K^{2} = \frac{a_{22}^{2}}{4A_2}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}\begin{pmatrix}\gamma_1 & \gamma_2 & \gamma_3\end{pmatrix} = \left\{\begin{matrix} a_{22}^2 = a_{11}^2 = k_2^2 = \sqrt{3}\\ A_2 = \sqrt{3}\end{matrix}\right\} = \frac{1}{4}\begin{pmatrix}-1\\ -1\\ 2\end{pmatrix}\begin{pmatrix}-1 & -1 & 2\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1 & -2\\ 1 & 1 & -2\\ -2 & -2 & 4\end{pmatrix} \; ; \quad \text{and } K^{2,12} = K^{2,21} = K^{2,00} = 0 \; \left(\text{since } a_{12} = a_{21} = a_{00} \equiv 0\right)$$

$$K^{2} = K^{2,11} + K^{2,22} = \frac{1}{4}\begin{pmatrix}3 & -3 & 0\\ -3 & 3 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1 & 1 & -2\\ 1 & 1 & -2\\ -2 & -2 & 4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4 & 2 & -2\\ -2 & 4 & -2\\ -2 & -2 & 4\end{pmatrix}$$

$$\boxed{K^{2} = \frac{1}{2}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}}$$

$$f(x,y) \equiv 30, \; (x,y) \in \Omega^{1} \Rightarrow f_1 = 30, \text{ and } F^{1} = \frac{f_1 A_1}{3}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} = \left\{\begin{matrix} f_1 = 30\\ A_1 = 1\end{matrix}\right\} = \begin{pmatrix}10\\ 10\\ 10\end{pmatrix}$$

$$f(x,y) \equiv 0, \; (x,y) \in \Omega^{2} \Rightarrow f_2 = 0, \text{ and } F^{2} = \frac{f_2 A_2}{3}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix} = \left\{\begin{matrix} f_2 \equiv 0\\ \cdots\end{matrix}\right\} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$K = \begin{pmatrix} K_{33}^{1} & K_{32}^{1} & K_{31}^{1} & 0\\ * & K_{22}^{1} + K_{11}^{2} & K_{21}^{1} + K_{12}^{2} & K_{13}^{2}\\ * & * & K_{11}^{1} + K_{22}^{2} & K_{23}^{2}\\ * & * & * & K_{33}^{2}\end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix}4 & -4 & 0 & 0\\ -4 & 7 & -2 & -1\\ 0 & -2 & 3 & -1\\ 0 & -1 & -1 & 2\end{pmatrix}$$

Symmetric

$$F = \begin{pmatrix}F_1\\ F_2\\ F_3\\ F_4\end{pmatrix} = \begin{pmatrix}F_3^{1}\\ F_2^{1} + F_1^{2}\\ F_1^{1} + F_2^{2}\\ F_3^{2}\end{pmatrix} = \begin{pmatrix}10\\ 10\\ 10\\ 0\end{pmatrix}$$

Assembled system:

$$\boxed{\frac{1}{2}\begin{pmatrix}4 & -4 & 0 & 0\\ -4 & 7 & -2 & -1\\ 0 & -2 & 3 & -1\\ 0 & -1 & -1 & 2\end{pmatrix}\cdot\begin{pmatrix}U_1\\ U_2\\ U_3\\ U_4\end{pmatrix} = \begin{pmatrix}10\\ 10\\ 10\\ 0\end{pmatrix} + \begin{pmatrix}Q_1\\ Q_2\\ Q_3\\ Q_4\end{pmatrix}}$$

$$q_n^1(0,y) = \begin{pmatrix} k_c^1 & 0 \\ 0 & k_c^1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x}(0,y) \\ \frac{\partial u}{\partial y}(0,y) \end{pmatrix} \cdot (-\vec{e}_1) \underset{\substack{k_c^1 = 2 \\ -\vec{e}_1 = (-1,0)^T}}{=} 2 \left(\frac{\partial u}{\partial x}(0,y) \;\; \frac{\partial u}{\partial y}(0,y) \right) \begin{pmatrix} -1 \\ 0 \end{pmatrix} =$$

$$= -2 \frac{\partial u}{\partial y}(0,y) = -18y, \; -1 \leq y \leq 0;$$

So the flow on the edge Γ_2^1 is linear (q_n on this edge is an affine function).

Thus:

$$Q_{22}^1 = \left(\frac{q_n(0,0)}{3} + \frac{q_n(0,-1)}{6} \right) h_2^1 \underset{h_2^1 = 1}{=} \left(0 + \frac{18}{6} \right) \cdot 1 = \boxed{3}$$

On edge Γ_3^2 is as if there were convection with $\beta = k_c^2 \frac{\sqrt{3}}{2} = \frac{3}{2}^{(*)}$ and $T_\infty = 0$. Hence:

$$Q_{13}^2 = \left(-\frac{\beta U_2}{3} - \frac{\beta U_4}{6} \right) h_3^2 = \left\{ \begin{array}{l} \beta = \sqrt{3}/2, \\ \text{from the BC: } U_4 = 0 \end{array} \right\}$$

$$= \left(-\frac{3/2}{3} U_2 - 0 \right) \underbrace{\sqrt{1^2 + \sqrt{3}^2}}_{=2} = -\frac{1}{2} \cdot 2 \, U_2 = \boxed{-U_2}$$

Boundary Conditions:

- Natural: $Q_2 = Q_{22}^1 + Q_{13}^2 = 3 - U_2$
- Essential: $U_1 = -2, \; U_3 = U_4 = 0$

Then:

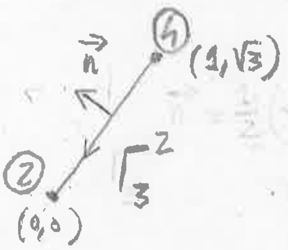
$$\frac{1}{2} \left(-4 \;\; 7 \;\; 2 \;\; -1 \right) \cdot \begin{pmatrix} -2 \\ U_2 \\ 0 \\ 0 \end{pmatrix} = 10 + 3 - U_2 \iff 4 + \frac{7}{2} U_2 = 13 - U_2$$

$\iff$ Reduced system: $\frac{9}{2} U_2 = 9$. Solution: $\boxed{U_2 = 2}$

(*) Indeed: $q_n(x,y) = \begin{pmatrix} k_c^2 & 0 \\ 0 & k_c^2 \end{pmatrix} \nabla u(x,y) \cdot \vec{n} = k_c^2 \nabla u(x,y) \cdot \vec{n} = k_c^2 \frac{\partial u}{\partial \vec{n}}(x,y), \; (x,y) \in \Gamma_3^2$

and,

$$q_n(x,y) = k_c \frac{\partial u}{\partial \vec{n}}(x,y) = -\underbrace{k_c \frac{\sqrt{3}}{2}}_{\beta} (u - \underbrace{0}_{T_\infty}), \; (x,y) \in \Gamma_3^2$$



$\vec{n} = \frac{1}{2}(-\sqrt{3}, 1)$