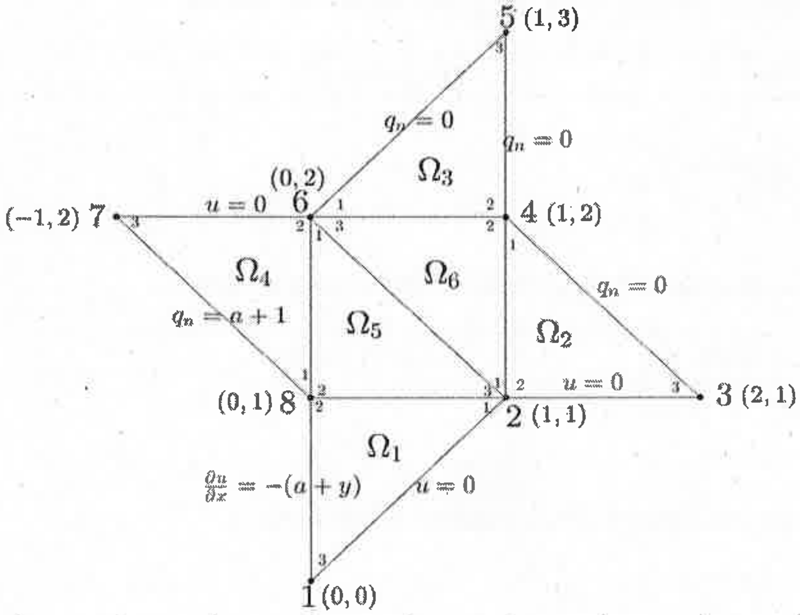
MÈTODES NUMÈRICS : Ex-Re Avaluació (a) 2016-17

Examen_ReAvaluacio-2016-17_Problema4.pdf



4. Consider the domain shown in the figure below, and with the elements,nodes and boundary conditions plotted there. **(4 points)**

We want to solve the problem defined by the equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 2$$

with boundary conditions given by:

- $u = 0$ at the edges joining the nodes 1,2 and 3.
- $u = 0$ at the edge that joint the nodes 6 and 7.
- $q_n = 0$ at the edges joining the nodes 3,4,5 and 6
- $q_n = a + 1$ at the edge that joint nodes 7 and 8.
- $\frac{\partial u}{\partial x}(0, y) = -(a + y), 0 \leq y \leq 1$ at the edge that joint nodes 8 and 1

Let $\mathbf{a} = \mathbf{1}$, and mesh this domain with six triangular elements as it is shown in the figure and using the proposing numbering for this mesh, compute:

(a) The coefficients $K(2,3), K(3,3)$ of the global assembled system.

$K(2,3)=$ -0.5	$K(3,3)=$ 0.5

(b) The values of coefficients Q^4_{13}, Q^1_{22} of the global assembled system.

$Q^4_{13}=$ 1.4142e+00	$Q^1_{22}=$ 8.3333e-01

(Hint: Be careful with the length of the edges of the elements)

(c) The value of $u(0,1)$

$u(0,1) =$ 1.2990e+00

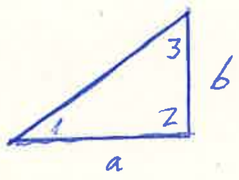
(Hint: It must be a value among the following ones:
0.9931e+00; 1.2990e+00; 1.4233e+00; 1.7819e+00; 1.9233e+00; 2.2647e+00; 2.4233e+00

(d) The value of $u(0.1,0.8)$

$u(0.1,0.8) =$ 9.0931e-01

(a) Coefficients K_{23}, K_{33} of the global assembled system

TZ-MN-FEM2D.pdf page 48



$$K^k = \frac{c}{2ab}\begin{pmatrix} b^2 & -b^2 & 0 \\ -b^2 & a^2+b^2 & -a^2 \\ 0 & -a^2 & a^2 \end{pmatrix} \underset{\substack{a=b=1\\ c=1}}{=} \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$F^k = \frac{f_k A_k}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \underset{\substack{f_k=2\\ A_k=\frac{1}{2}}}{=} \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad k = 1, 2, \ldots, 8.$$

$$K_{23} = K^2_{23} = -\tfrac{1}{2} = -0.5$$

$$K_{33} = K^2_{33} = \tfrac{1}{2} = 0.5$$

(b) $$Q^4_{13} = \frac{1}{2}(a+1)\cdot h^4_3 = \frac{a+1}{2}\sqrt{2} \underset{a=1}{=} \sqrt{2} = 1.4142$$

$$q_{n,2}(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x}(0, 1-s) \\ \frac{\partial u}{\partial y}(0, 1-s) \end{pmatrix} \cdot (-\vec{e}_1) = -\left(-(a+1-s), *\right)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= a+1-s \underset{a=1}{=} 2-s$$

$$Q^1_{22} = \frac{1}{3}(2-0)\cdot h^1_2 + \frac{1}{6}(2-1)\cdot h^1_2 = \frac{2}{3} + \frac{1}{6} = \frac{5}{6} = 0.8\overline{3}$$

(c) $$K(8,:) = \left(K^1_{23}, K^1_{21} + K^5_{23}, 0, 0, 0, K^4_{12} + K^5_{21}, K^4_{13}, K^1_{22} + K^5_{22} + K^4_{11}\right)$$

$$= \frac{1}{2}\left(-1, -2, 0, 0, 0, -2, 0, 5\right)$$

$$\frac{1}{2}\begin{pmatrix} -1 & -2 & 0 & 0 & 0 & -2 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} U_1 = 0 \\ U_2 = 0 \\ U_3 \\ U_4 \\ U_5 \\ U_6 = 0 \\ U_7 = 0 \\ U_8 \end{pmatrix} = Q_8 + F_8 = Q^4_{13} + Q^1_{22}$$

$$+ F^1_2 + F^5_2 + F^4_1$$

$$= \sqrt{2} + \frac{5}{6} + 1 = \frac{6\sqrt{2} + 11}{6}$$

$$= 1.2990$$

$$\tfrac{5}{2} U_8 = (6\sqrt{2}+11)/6 \,!$$

$$u(0,1) \simeq U_8 = \frac{6\sqrt{2}+11}{15} = 1.2990$$

(d) Value of $u(0.1, 0.8)$

$p = (0.1, 0.8) \in \Omega_1$

$$\psi_1^1(x,y) = x\,, \quad \psi_2^1(x,y) = y-x\,, \quad \psi_3^1(x,y) = 1-y$$

for instance, for $\psi_2^1(x,y) = a + bx + cy$

$$\left.\begin{array}{l} \psi_2^1(1,1) = a+b+c = 0 \Rightarrow b = -1 \\ \psi_2^1(0,1) = a+c = 1 \Rightarrow c = 1 \\ \psi_2^1(0,0) = a = 0 \end{array}\right\} \Rightarrow \psi_2^1(x,y) = y-x$$

And hence:

$$u(0.1,0.8) \simeq \underset{u_1^1}{\overset{=0}{U_2}}\, \psi_1^1(0.1,0.8) + \underset{u_2^1}{U_8}\, \psi_2^1(0.1,0.8) + \underset{u_3^1}{\overset{=0}{U_1}}\, \psi_3^1(0.1,0.8)$$

$$= 0 + \frac{6\sqrt{2}+11}{15}(0.8-0.1) + 0 = 0.7\,\frac{6\sqrt{2}+11}{15} = 0.90931. \;\square$$