1 Continuation method

Our goal is to continue *numerically* a curve $\mathcal{C} \subset \mathbb{R}^{n+1}$, defined implicitely by the equation $F(z) = 0$, being $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ a smooth function. Let us assume that $z^j \in \mathbb{R}^{n+1}$, is a *regular* point of $\mathcal{C}$, so $F\left(z^j\right) = 0$, and $rank\, DF\left(z^j\right) = n$. Moreover, let $v^j \in \mathbb{R}^{n+1}$ be an unitary vector tangent to the curve $\mathcal{C}$ at the point z^j, $v^j \in T_{z^j}\mathcal{C}$, so $\|v^j\| = 1$, and $DF\left(z^j\right) v^j = 0$.

Then, it is possible to find a new point on the curve, $z^{j+1} \in \mathcal{C}$, and a new unitary tangent vector, to $\mathcal{C}$ at z^{j+1}, $v^{j+1} \in T_{z^{j+1}}\mathcal{C}$, $\|v^{j+1}\| = 1$. If, on its turn, z^{j+1} is a regular point of $\mathcal{C}$, then one can look for yet another point on $\mathcal{C}$, $z^{j+2} \in \mathcal{C}$, and a new unitary tangent vector to $\mathcal{C}$ at z^{j+2}, $v^{j+2} \in T_{z^{j+2}}\mathcal{C}$, $\|v^{j+2}\| = 1$, and so on.

Of course, there are several numerical methods to do this *step-by-step* continuation of $\mathcal{C}$ from an inital point on the curve, $z^j \in \mathcal{C}$, and a (normalizsed) tangent direction at that point, $v^j \in T_{z^j}\mathcal{C}$. The one we outline here is the so called *pseudo-arc continuation method* (see [1], Chap. 10, Sect. 2, for a complete description). In a nutshell, it consists in the three stages discussed below.

Stage 1: Prediction. Take $\hat{z}^{j+1} = z^j + h_j v^j \in z^j + \langle v^j \rangle$ as an approximation for another new point $z^{j+1} \in \mathcal{C}$. Here $h_j > 0$ is the pseudo-arc length, and can be conveniently adapted at each step.

Stage 2: Correcton. Refine the approximation $\hat{z}^{j+1}$ to find $z^{j+1} \in \mathbb{R}^{n+1}$ such that $F\left(z^{j+1}\right) = 0$. However, as the system $F(z) = 0$ has n equations and $n+1$ unknowns $z_1, z_2, \ldots, z_n, z_{n+1}$, we need to ask for an additional condition: in particular, we shall require that $z^{j+1} \in \hat{z}^{j+1} + \langle v^j \rangle^{\perp}$, i.e., that z^{j+1} belongs to the hyperplane orthogonal to the vector v^j that holds $\hat{z}^{j+1}$ (see Figure 1). The corresponding equation con be formulated as

$$\begin{aligned}
\left\langle v^j, z^{j+1} - \hat{z}^{j+1} \right\rangle &= \left\langle v^j, z^{j+1} - z^j - h_j v^j \right\rangle \\
&= \left\langle v^j, z^{j+1} \right\rangle - \left\langle v^j, z^j \right\rangle - h_j \left\langle v^j, v^j \right\rangle \\
&= \left\langle v^j, z^{j+1} \right\rangle - \left\langle v^j, z^j \right\rangle - h_j = 0,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the *inner* (or dot) product $\langle \xi, \eta \rangle := \xi_1 \eta_1 + \cdots + \xi_m \eta_m$, $\xi, \eta \in \mathbb{R}^m$. Hence z^{j+1} will be given by the solution of the nonlinear system,

$$\begin{aligned}
F(z) &= 0, \\
\left\langle v^j, z \right\rangle &= \left\langle v^j, z^j \right\rangle - h_j
\end{aligned}$$

that can be solved by some iterative method (for example, Newton method) taking as initial approximation. $z^{(0)} = \hat{z}^{j+1}$.

Stage 3: Tangent vector. To find the tangent vector to the curve at the new point $z^{j+1} \in \mathcal{C}$ found at Stage 2, $v^{j+1} \in T_{z^{j+1}}\mathcal{C}$, first we solve the $(n+1)$-dimensional linear system

$$\begin{aligned}
DF\left(z^{j+1}\right) v &= 0, \\
\left\langle v^j, v \right\rangle &= 1.
\end{aligned} \tag{1}$$

As it is pointed out in [1]:

(i) If $\mathcal{C}$ is a regular curve and z^j, z^{j+1} are close enough, the system (1) is nonsingular.

(ii) The solution, $v^* \in \mathbb{R}^{n+1}$, of (1) satisfies $\langle v^j, v^* \rangle = 1$, so the direction along the curve is preserved.

Next, we normalize. If $v^* \in \mathbb{R}^{n+1}$ denotes the solution of (1), we divide by its norm, so $v^{j+1} = v^* / \|v^*\|$. This is the tangent vector we look for. ♢

Now, the process can be iterated using the output of Stage 3, the tangent vector v^{j+1} to feed Stage 1 and find a another point close to the curve $\mathcal{C}$, and so on. If for the new computed point curve C, and so on. If at Stage 2, $Rank\, DF\left(z^k\right) < n$, for the new computed point, $z^k \in \mathcal{C}$, then one has to stop the process and analyse for the possible appearing of branches (bifurcations).

Neverteless, to start the process we need an initial point on the curve and its corresponding tangent vector, $z^0 \in \mathcal{C}$ and $v^0 \in T_{z^0}$. To find z^0 we can proceed as follows: let us assume that we have an approximation for

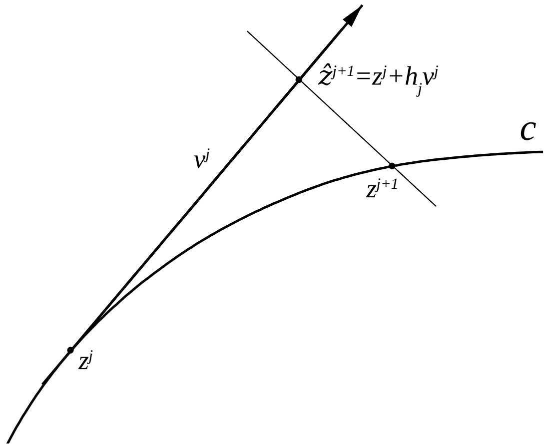

Figure 1: We add an extra condition: $z^{j+1} \in \hat{z}^{ji+1} + \langle v^j \rangle^{\perp}$. See [1], Figure 10.6(b).

the initial solution, $\hat{z}^0 = (\hat{z}^0_1, \ldots, \hat{z}^0_n, \hat{z}^0_{n+1})$, then we fix one of its components, suppose that we take the last one, $\hat{z}^0_{n+1}$ and solve the n-dimensional system,

$$\begin{array}{rl} F_1\left(z_1, \ldots, z_{n+1}, \hat{z}^0_{n+1}\right) & = 0, \\ F_2\left(z_1, \ldots, z_{n+1}, \hat{z}^0_{n+1}\right) & = 0, \\ \multicolumn{2}{c}{\dotfill} \\ F_n\left(z_1, \ldots, z_{n+1}, \hat{z}^0_{n+1}\right) & = 0, \end{array}$$

by the Newton method, starting with $z_1 = \hat{z}^0_1, \ldots, z_n = \hat{z}^0_n$. If the method converges, we have an initial point on the curve, $z^0 = (z^0_1, \ldots, z^0_n, \hat{z}^0_{n+1})$. Besides, to

2 References

[1] Yuri A. Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathematical Sciences*. Springer-Verlag, New York, third edition, 2004. 1, 2