

1 Continuation method

Our goal is to continue *numerically* a curve $\mathcal{C} \subset \mathbb{R}^{n+1}$, defined implicitly by the equation $F(z) = 0$, being $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ a smooth function. Let us assume that $z^j \in \mathbb{R}^{n+1}$, is a *regular* point of \mathcal{C} , so $F(z^j) = 0$, and $\text{rank } DF(z^j) = n$. Moreover, let $v^j \in \mathbb{R}^{n+1}$ be an unitary vector tangent to the curve \mathcal{C} at the point z^j , $v^j \in T_{z^j}\mathcal{C}$, so $\|v^j\| = 1$, and $DF(z^j)v^j = 0$.

Then, it is possible to find a new point on the curve, $z^{j+1} \in \mathcal{C}$, and a new unitary tangent vector, to \mathcal{C} at z^{j+1} , $v^{j+1} \in T_{z^{j+1}}\mathcal{C}$, $\|v^{j+1}\| = 1$. If, on its turn, z^{j+1} is a regular point of \mathcal{C} , then one can look for yet another point on \mathcal{C} , $z^{j+2} \in \mathcal{C}$, and a new unitary tangent vector to \mathcal{C} at z^{j+2} , $v^{j+2} \in T_{z^{j+2}}\mathcal{C}$, $\|v^{j+2}\| = 1$, and so on.

Of course, there are several numerical methods to do this *step-by-step* continuation of \mathcal{C} from an initial point on the curve, $z^j \in \mathcal{C}$, and a (normalised) tangent direction at that point, $v^j \in T_{z^j}\mathcal{C}$. The one we outline here is the so called *pseudo-arc continuation method* (see [1], Chap. 10, Sect. 2, for a complete description). In a nutshell, it consists in the three stages discussed below.

Now, this process can be iterated until we reach a point $z^\ell \in \mathcal{C}$ such that eventually $\text{Rank } DF(z^\ell) < n$

As described in [1] (see chap. 10, sect. 2),

points on the curve can be approximated by means of the *pseudo arc* method, following these three stages:

1. *Stage 1: Prediction.* Take $\hat{z}^{j+1} = z^j + h_j v^j \in z^j + \langle v^j \rangle$ as an approximation for another new point $z^{j+1} \in \mathcal{C}$. Here $h_j > 0$ is the pseudo-arc length, and can be conveniently adapted at each step.
2. *Stage 2: Correction.* Refine the approximation \hat{z}^{j+1} to find $z^{j+1} \in \mathbb{R}^{n+1}$ such that $F(z^{j+1}) = 0$. However, as the system $F(z) = 0$ has n equations and $n + 1$ unknowns $z_1, z_2, \dots, z_n, z_{n+1}$, we need to ask for an additional condition: in particular, we shall require that $z^{j+1} \in \hat{z}^{j+1} + \langle v^j \rangle^\perp$, i.e., that z^{j+1} belongs to the hyperplane orthogonal to the vector v^j that holds \hat{z}^{j+1} (see Figure 1). The corresponding equation can be formulated as

$$\begin{aligned} \langle v^j, z^{j+1} - \hat{z}^{j+1} \rangle &= \langle v^j, z^{j+1} - z^j - h_j v^j \rangle \\ &= \langle v^j, z^{j+1} \rangle - \langle v^j, z^j \rangle - h_j \langle v^j, v^j \rangle \\ &= \langle v^j, z^{j+1} \rangle - \langle v^j, z^j \rangle - h_j = 0, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the *inner* (or dot) product $\langle \xi, \eta \rangle := \xi_1 \eta_1 + \dots + \xi_m \eta_m$, for $\xi, \eta \in \mathbb{R}^m$. Hence z^{j+1} will be given by the solution of the nonlinear system,

$$\begin{aligned} F(z) &= 0, \\ \langle v^j, z \rangle &= \langle v^j, z^j \rangle - h_j \end{aligned}$$

that can be solved by some iterative method (for example, Newton method) taking from some the initial approximation. $z = \hat{z}^{j+1}$ is usually a good choice.

3. *Step 2. Compute a new approximation.* For this we need a tangent vector, v^{j+1} , to the curve \mathcal{C} at the point z^{j+1} found at step one. To find v^{j+1} , first we solve the *appended* linear system

$$\begin{aligned} DF(z^{j+1})v &= 0, \\ \langle v^j, v \rangle &= 1. \end{aligned} \tag{1}$$

We note two points remarked in [1]. On the one hand, if \mathcal{C} is a regular curve and z^j, z^{j+1} are close enough, the system (1) is nonsingular. On the other hand, the last equation assures that the direction along the curve is preserved.

v^{j+1} must be a solution of the equation $DF(z^{j+1})v = 0$. Clearly, it is not unique, since $\text{rank } DF(z^{j+1}) \leq n$ equations defines. The second condition, in terms of the inner product, can be written as

4. *Predicition:* we shall take $\hat{z}^{j+1} = z^j + h_j v^j$ as an approximation of the new point of z^{j+1} , where $h_j \in \mathbb{R}$ is the *pseudo-arc* step (the *step* in what follows), and $v^j \in \mathbb{R}^{n+1}$, $\|v^j\| = 1$, is the tangent vector to the

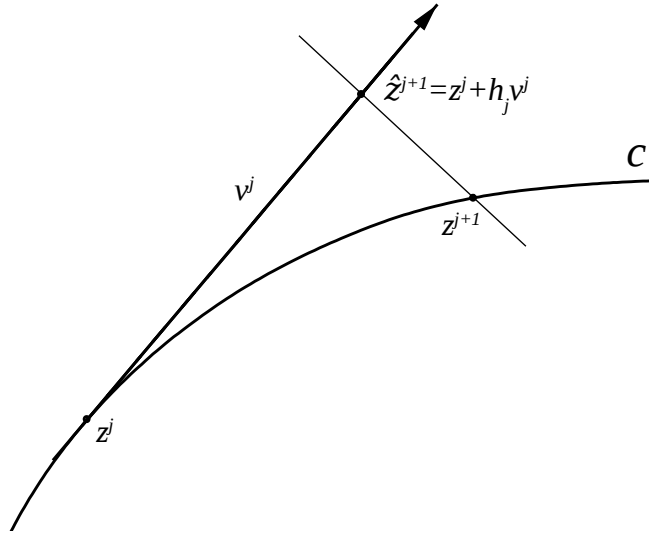


Figure 1

\mathcal{C} at point z^j , that will be find as the solution of és el vector tangent a la corba \mathcal{C} al punt z^j , el qual determinarem resolent el *sistema ampliat*,

$$\begin{aligned} DF(z^j)v &= 0, \\ \langle v^{j-1}, v \rangle &= 1, \end{aligned} \quad (2)$$

on $v^{j-1} \in \mathbb{R}^{n+1}$, $\|v^{j-1}\| = 1$, és el vector tangent a la corba \mathcal{C} al punt z^{j-1} , tots dos (v^{j-1} i z^{j-1}) prèviament calculats. Com s'observa a [1]:

- (i) El sistema lineal (2) és no singular si \mathcal{C} és una corba regular (i.e., si $\text{rang } DF(z) = n$, $z \in \mathcal{C}$) i els punts z^{j-1} i z^j estan suficientment a prop.
- (ii) La solució $v^* \in \mathbb{R}^{n+1}$ satisfà la condició $\langle v^{j-1}, v^* \rangle = 1$, per tant es preserva la direcció al llarg de la corba.

Per últim, normalitzem per tenir $v^j = v^* / \|v^*\|$. *Nota:* a l'inici, quan $j = 0$, no podem escriure el sistema (2), sinó que resoldrem el sistema $n \times n$ que s'obté de seleccionar n columnes linealment independents (siguin les columnes $1, 2, \dots, i-1, i+1, \dots, n, n+1$) de $DF(z^j)$ a la primera equació de (2) i fixar $v_i = 1$. D'aquesta manera trobarem un vector $v^* \in \mathbb{R}^n$, $v_i^* = 1$, t.q. $DF(z^0)v^* = 0$. Llavors $v^0 = \pm v^* / \|v^*\|$, on la tria del signe determinarà la direcció en què es continua la corba.

5. *Correcció.* Per a “refinar” el valor aproximat $\hat{z}^{j+1} = z^j + h_j v^j$ del pas predictiu pel mètode de Newton i determinar el nou punt sobre la corba, $z^{j+1} \in \mathcal{C}$, s'ha d'afegir alguna equació addicional al sistema $F(z) = 0$. Al mètode del pseudo-arc, s'imposa que $z^{j+1} \in \hat{z}^{j+1} + \langle v^j \rangle^\perp$; això és, que el punt z^{j+1} pertanyi també al hiperplà perpendicular al vector v^j que conté \hat{z}^{j+1} . Usant el producte escalar aquesta condició geomètrica s'escriu com,

$$\langle z^{j+1} - \hat{z}^{j+1}, v^j \rangle = \langle z^{j+1} - z^j - h_j v^j, v^j \rangle = \langle z^{j+1} - z^j, v^j \rangle - h_j = 0$$

(vegeu la figura 1). Aleshores aplicarem el mètode de Newton al sistema no lineal

$$\begin{aligned} F(z) &= 0, \\ \langle z - z^j, v^j \rangle &= h_j, \end{aligned}$$

prenent $z^{(0)} = \hat{z}^{j+1}$ com a aproximació inicial.

2 References

- [1] Yuri A. Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathematical Sciences*. Springer-Verlag, New York, third edition, 2004. 1, 2