1 Continuation method

Our goal is to continue *numerically* a curve $\mathcal{C} \subset \mathbb{R}^{n+1}$, defined implicitely by the equation $F(z) = 0$, being $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ a smooth function. Let us assume that $z^j \in \mathbb{R}^{n+1}$, is a *regular* point of $\mathcal{C}$, so $F\left(z^j\right) = 0$, and $rank\, DF\left(z^j\right) = n$. Moreover, let $v^j \in \mathbb{R}^{n+1}$ be an unitary vector tangent to the curve $\mathcal{C}$ at the point z^j, $v^j \in T_{z^j}\mathcal{C}$, so $\|v^j\| = 1$, and $DF\left(z^j\right) v^j = 0$.

Then, it is possible to find a new point on the curve, $z^{j+1} \in \mathcal{C}$, and a new unitary tangent vector, to $\mathcal{C}$ at z^{j+1}, $v^{j+1} \in T_{z^{j+1}}\mathcal{C}$, $\|v^{j+1}\| = 1$. If, on its turn, z^{j+1} is a regular point of $\mathcal{C}$, then one can look for yet another point on $\mathcal{C}$, $z^{j+2} \in \mathcal{C}$, and a new unitary tangent vector to $\mathcal{C}$ at z^{j+2}, $v^{j+2} \in T_{z^{j+2}}\mathcal{C}$, $\|v^{j+2}\| = 1$, and so on.

Of course, there are several numerical methods to do this *step-by-step* continuation of $\mathcal{C}$ from an inital point on the curve, $z^j \in \mathcal{C}$, and a (normalizsed) tangent direction at that point, $v^j \in T_{z^j}\mathcal{C}$. The one we outline here is the so called *pseudo-arc continuation method* (see [1], Chap. 10, Sect. 2, for a complete description). In a nutshell, it consists in the three stages discussed below.

Now, this process can be iterated until we reach a a point $z^\ell \in \mathcal{C}$ such that eventually $Rank\, DF(z^\ell) < n$
As described in [1] (see chap. 10, sect. 2),
points on the curve can be approximated by means of the *pseudo arc* method, following these three stages:

1. *Stage 1: Prediction.* Take $\hat{z}^{j+1} = z^j + h_j v^j \in z^j + \langle v^j \rangle$ as an approximation for another new point $z^{j+1} \in \mathcal{C}$. Here $h_j > 0$ is the pseudo-arc length, and can be conveniently adapted at each step.

2. *Stage 2: Correction.* Refine the approximation $\hat{z}^{j+1}$ to find $z^{j+1} \in \mathbb{R}^{n+1}$ such that $F\left(z^{j+1}\right) = 0$. However, as the system F(z) = 0 has n equations and $n+1$ unknowns $z_1, z_2, \ldots, z_n, z_{n+1}$, we need to ask for an additional condition: in particular, we shall require that $z^{j+1} \in \hat{z}^{j+1} + \langle v^j \rangle^\perp$, i.e., that z^{j+1} belongs to the hyperplane orthogonal to the vector v^j that holds $\hat{z}^{j+1}$ (see Figure 1). The corresponding equation con be formulated as
$$\begin{aligned}\left\langle v^j, z^{j+1} - \hat{z}^{j+1} \right\rangle &= \left\langle v^j, z^{j+1} - z^j - h_j v^j \right\rangle \\ &= \left\langle v^j, z^{j+1} \right\rangle - \left\langle v^j, z^j \right\rangle - h_j \left\langle v^j, v^j \right\rangle \\ &= \left\langle v^j, z^{j+1} \right\rangle - \left\langle v^j, z^j \right\rangle - h_j = 0,\end{aligned}$$
where $\langle \cdot, \cdot \rangle$ stands for the *inner* (or dot) product $\langle \xi, \eta \rangle := \xi_1 \eta 1 + \cdots + \xi_m \eta_r$, for $\xi, \eta \in \mathbb{R}^m$. Hence z^{j+1} will be the solution of the nonlinear system,
$$\begin{aligned}F(z) &= 0, \\ \left\langle v^j, z \right\rangle &= \left\langle v^j, z^j \right\rangle - h_j\end{aligned}$$
that can be solved by some iterative method (for example, Newton method) taking form some the initial approximation. $z = \hat{z}^{j+1}$ is usually a good choice.

3. *Step 2. Compute a new approximation.* For this we need a tangent vector, v^{j+1}, to the curve $\mathcal{C}$ at the point z^{j+1} found at step one. To find v^{j+1}, first we solve the *enlarged* linear system
$$\begin{aligned}DF\left(z^{j+1}\right) v &= 0, \\ \left\langle v^j, v \right\rangle &= 1.\end{aligned} \tag{1}$$
We quote here to remarks pointed in the reference above: *(i)* $\mathcal{C}$ is a regular curve, the system (1) is nonsingular, provided the points z^j and z^{j+1} are close enough, and *(ii)* if we denote v^* the solution of (1), then $\langle v^j, v^* \rangle = 1$, so the direction along the curve is preserved.

   v^{j+1} must be a solution of the equation $DF\left(z^{j+1}\right) v = 0$. Clearly, it is not unique, since rank$DF\left(z^{j+1}\right) \leq n$ equatons defines The second condition, in terms of the inner product, can be written as

4. *Prediciton:* we shall take $\hat{z}^{j+1} = z^j + h_j v^j$ as an approximation of the new point of z^{j+1}, where $h_j \in \mathbb{R}$ is the *pseudo-arc* step (the *step* in what follows), and $v^j \in \mathbb{R}^{n+1}$, $\|v^j\| = 1$, is the tangent vector to the $\mathcal{C}$ at point z^j, that will be find as the solution of és el vector tangent a la corba $\mathcal{C}$ al punt z^j, el qual determinarem resolent el *sistema ampliat*,
$$\begin{aligned}DF\left(z^j\right) v &= 0, \\ \left\langle v^{j-1}, v \right\rangle &= 1,\end{aligned} \tag{2}$$

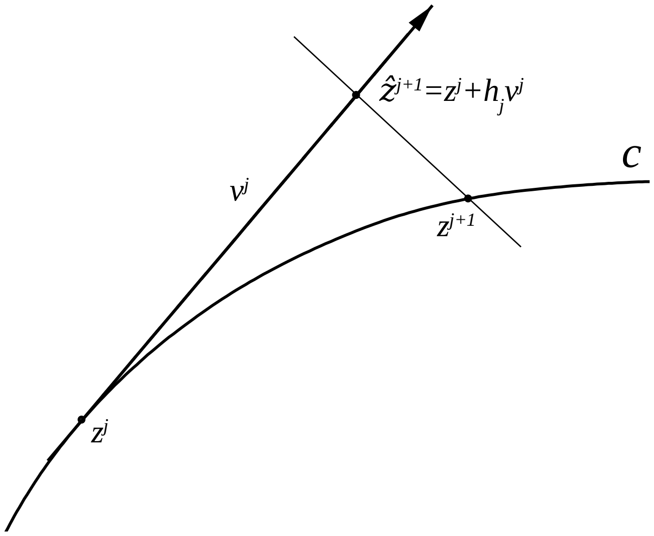

Figure 1

on $v^{j-1} \in \mathbb{R}^{n+1}$, $\|v^{j-1}\| = 1$, és el vector tangent a la corba $\mathcal{C}$ al punt z^{j-1}, tots dos (v^{j-1} i z^{j-1}) prèviament calculats. Com s'observa a [1]:

(i) El sistema lineal (2) és no singular si $\mathcal{C}$ és una corba regular (i.e., si $\operatorname{rang} DF(z) = n$, $z \in \mathcal{C}$) i els punts z^{j-1} i z^j estan suficientment a prop.

(ii) La solució $v^* \in \mathbb{R}^{n+1}$ satisfà la condició $\langle v^{j-1}, v^* \rangle = 1$, per tant es preserva la direcció al llarg de la corba.

Per últim, normalitzem per tenir $v^j = v^* / \|v^*\|$. *Nota:* a l'inici, quan $j = 0$, no podrem escriure el sistema (2), sinó que resoldrem el sistema $n \times n$ que s'obté de seleccionar n columnes linealment independents (siguin les columnes $1, 2, \ldots, i-1, i+1, \ldots, n, n+1$) de $DF(z^j)$ a la primera equació de (2) i fixar $v_i = 1$. D'aquesta manera trobarem un vector $v^* \in \mathbb{R}^n$, $v_i^* = 1$, t.q. $DF(z^0)v^* = 0$. Llavors $v^0 = \pm v^* / \|v^*\|$, on la tria del signe determinarà la direcció en què es continua la corba.

5. *Correcció.* Per a "refinar" el valor aproximat $\hat{z}^{j+1} = z^j + h_j v^j$ del pas predictiu pel mètode de Newton i determinar el nou punt sobre la corba, $z^{j+1} \in \mathcal{C}$, s'ha d'afegir alguna equació addicional al sistema $F(z) = 0$. Al mètode del pesudo-arc, s'imposa que $z^{j+1} \in \hat{z}^{j+1} + \langle v^j \rangle^{\perp}$; això és, que el punt z^{j+1} pertanyi també al hiperplà perpendicular al vector v^j que conté $\hat{z}^{j+1}$. Usant el producte escalar aquesta condició geomètrica s'escriu com,

$$\left\langle z^{j+1} - \hat{z}^{j+1}, v^j \right\rangle = \left\langle z^{j+1} - z^j - h_j v^j, v^j \right\rangle = \left\langle z^{j+1} - z^j, v^j \right\rangle - h_j = 0$$

(vegeu la figura 1). Aleshores aplicarem el mètode de Newton al sistema no lineal

$$\begin{aligned} F(z) &= 0, \\ \left\langle z - z^j, v^j \right\rangle &= h_j, \end{aligned}$$

prenent $z^{(0)} = \hat{z}^{j+1}$ com a aproximació inicial.

2 References

[1] Yuri A. Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathematical Sciences*. Springer-Verlag, New York, third edition, 2004. 1, 2