Graph Decomposition

Definition: A **matching** M in a graph G is a subset of edges of G that share no vertices.

Definition: A **perfect matching** M in a graph G is a matching such that every vertex of G is incident with one of the edges of M. Another name for a perfect matching is a 1-**factor**.

Definition: A **decomposition** of a graph G is a set of subgraphs $H_1, \ldots, H_k$ that partition of the edges of G. That is, for all i and j,
$$\bigcup_{1 \le i \le k} H_i = G \text{ and } E(H_i) \cap E(H_j) = \emptyset.$$

Definition: An H-**decomposition** is a decomposition of G such that each subgraph H_i in the decomposition is isomorphic to H.

Perfect Matching Decomposition

Definition: A **perfect matching decomposition** is a decomposition such that each subgraph H_i in the decomposition is a perfect matching.

Theorem: For a k-regular graph G,
G has a perfect matching decomposition if and only if $\chi'(G) = k$.

Proof:

There exists a decomposition of G into a set of k perfect matchings.

$$\Updownarrow$$

There exists a coloring of the edges of G where each vertex is incident to edges of each of k different colors.

$$\Updownarrow$$

$$\chi'(G) = k$$

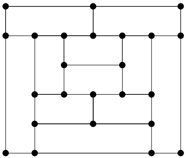
Corollary: K_{2n+1} has a perfect matching decomposition.
Corollary: A snark has no perfect matching decomposition.

Hamiltonian Cycles

Definition: A **Hamiltonian cycle** C in a graph G is a cycle containing every vertex of G.

Definition: A **Hamiltonian path** P in a graph G is a path containing every vertex of G.

⋆ Important: Paths and cycles do not use any vertex or edge twice. ⋆



An arbitrary graph may or may not contain a Hamiltonian cycle/path.

Theorem: If G has a Ham'n cycle, then G has a Ham'n path.

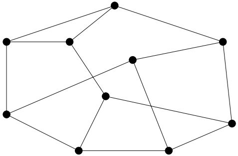
Proof:

Hamiltonian Cycles

Theorem 2.3.5: A snark has no Hamiltonian cycle.

Fact: A snark has an even number of vertices.

Proof: Suppose that a graph G is a snark and contains a Hamiltonian cycle. That is, G is:

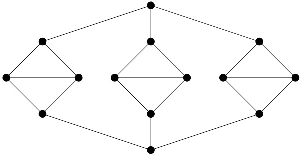


and G contains C, visiting each vertex once. Remove the edges of C; what remains?

Consider the coloring of G where the remaining edges are colored yellow and the edges in the cycle are colored alternating between blue and red. This is a proper 3-edge-coloring of G, a contradiction.

The converse is not true!

Example: Book Figure 2.3.4.



Cycle Decompositions

Definition: A **cycle decomposition** is a decomposition such that each subgraph H_i in the decomposition is a cycle.

Theorem: A graph that has a cycle decomposition is such that every vertex has even degree.

Proof: Each cycle of the cycle decomposition contributes two to the degree of each vertex in the cycle. The degree of each vertex v in G is the sum of the degrees of v over all subgraphs H_i, so it must be even.

Definition: A **Hamiltonian cycle decomposition** is a decomposition such that each subgraph H_i is a Hamiltonian cycle.

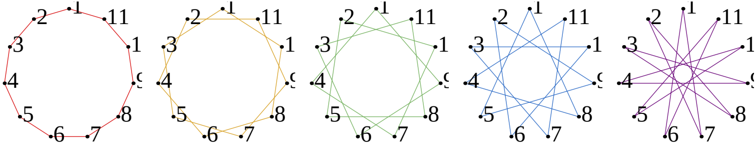
Question: Which graphs have a Hamiltonian cycle decomposition?

Which complete graphs?

Hamiltonian Cycle Decomposition

Example: K_7 has a Hamiltonian cycle decomposition.

Example: K_{11} has a Hamiltonian cycle decomposition.

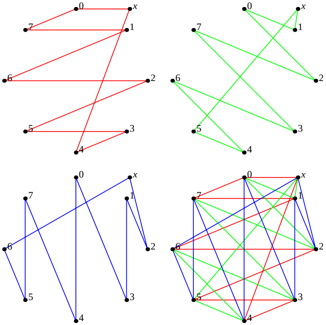


However: This construction does not work with K_9.

Hamiltonian Cycle Decomposition

Theorem 2.3.1: K_{2n+1} has a Hamiltonian cycle decomposition.

Proof: This proof uses another instance of a "turning trick". Place vertices 0 through $2n$ in a circle and draw a zigzag path visiting all the vertices in the circle. Connect the ends of the path to vertex x to form a Ham. cycle. As you rotate the zigzag path n times, you visit each edge of K_{2n+1} once to form a Ham'n cycle decomposition.

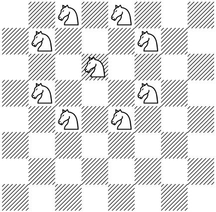


As a corollary:

Theorem 2.3.3: K_{2n} has a Hamiltonian path decomposition.

Knight's Tours

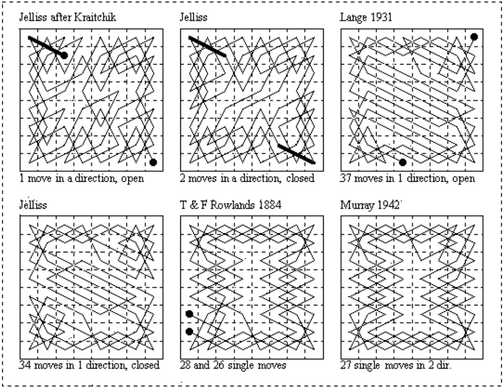
In chess, a knight (♘) is a piece that moves in an "L": two spaces over and one space to the side.



Question Is it possible for a knight to start on some square and, by a series of valid knight moves, visit each square on an 8×8 chessboard once? (How about return to where it started?)

Definition: A path of the first kind is called an **open knight's tour**. A cycle of the second kind is called a **closed knight's tour**.

8×8 Knight's Tour

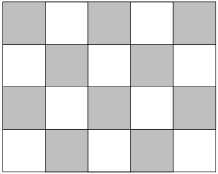


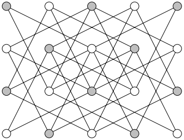
Source: `http://www.ktn.freeuk.com/ga.htm`

Question Are there any knight's tours on an $m \times n$ chessboard?

The Graph Theory of Knight's Tours

For any board we can draw a corresponding knight move graph: Create a vertex for every square on the board and create edges between vertices that are a knight's move away.





An open/closed knight's tour
on the board

A knight move always alternates between white and black squares. Therefore, a knight move graph is always ______________.

Question Are there any knight's tours on an $m \times n$ chessboard?

Knight's Tour Theorem

Theorem An $m \times n$ chessboard with $m \leq n$ has a *closed* knight's tour unless one or more of these conditions holds:

1. m and n are both odd.
2. $m = 1$, 2, or 4.
3. $m = 3$ and $n = 4$, 6, or 8.

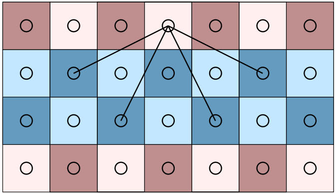
"Proof" We will only show that it is impossible in these cases.

Case 1. When m and n are both odd,

Case 2. When $m = 1$ or 2, the knight move graph is not connected.

Knight's Tour Theorem

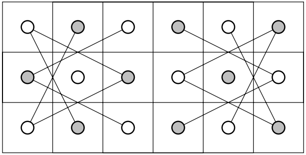
Case 2. When $m = 4$, consider:



Suppose we can find a Hamiltonian cycle C in the graph.
Then, C must alternate between white and black vertices.
Also, every red vertex in C is adjacent to only blue vertices.
There are the same number of red and blue vertices.
So, C must alternate between red and blue vertices. This means:
All vertices of C are "white and red" or "black and blue".

Knight's Tour Theorem

Case 3. 3×4 is covered by Case 2. Consider the 3×6 board:



Assume that there is a Hamiltonian cycle C in G.
Then, C visits each vertex v and uses two of v's incident edges.
If $\deg(v) = 2$, then both of v's incident edges are in C.
Draw in all these "forced edges" above. With just these forced edges, there is already a cycle of length four. This cycle cannot be a subgraph of any Hamiltonian cycle, contradicting its existence. □

The 3×8 case is similar, and part of your homework.

See also: "Knight's Tours on a Torus", by J. J. Watkins, R. L. Hoenigman