Exercises 1.11

Tom M. Apostol

Exercise 4

Prove that the greatest-integer function has the properties indicated:

Exercise 4a

$\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for every integer n.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4a

Let x be a real number and n an integer. Let $m = \lfloor x + n \rfloor$. By definition of the floor function, m is the unique integer such that $m \leq x + n < m + 1$. Then $m - n \leq x < (m - n) + 1$. That is, $m - n = \lfloor x \rfloor$. Rearranging terms we see that $m = \lfloor x \rfloor + n$ as expected.

□

Exercise 4b

$$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \text{ is an integer,} \\ -\lfloor x \rfloor - 1 & \text{otherwise.} \end{cases}$$

Proof.

∃ – Apostol.Chapter_1_11.exercise_4b_1
∃ – Apostol.Chapter_1_11.exercise_4b_2

There are two cases to consider:

Case 1 Suppose x is an integer. Then $x = \lfloor x \rfloor$ and $-x = \lfloor -x \rfloor$. It immediately follows that

$$\lfloor -x \rfloor = -x = -\lfloor x \rfloor .$$

Case 2 Suppose x is not an integer. Let $m = \lfloor -x \rfloor$. By definition of the floor function, m is the unique integer such that $m \leq -x < m + 1$. Equivalently, $-m - 1 < x \leq -m$. Since x is not an integer, it follows $-m - 1 \leq x < -m$. Then, by definition of the floor function, $\lfloor x \rfloor = -m - 1$. Solving for m yields

$$\lfloor -x \rfloor = m = -\lfloor x \rfloor - 1.$$

Conclusion The above two cases are exhaustive. Thus

$$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \text{ is an integer,} \\ -\lfloor x \rfloor - 1 & \text{otherwise.} \end{cases}$$

□

✓ Exercise 4c

$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4c

Rewrite x and y as the sum of their floor and fractional components: $x = \lfloor x \rfloor + \{x\}$ and $y = \lfloor y \rfloor + \{y\}$. Now

$$\begin{aligned} \lfloor x + y \rfloor &= \lfloor \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} \rfloor \\ &= \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \rfloor \\ &= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor && \text{✓ Exercise 4a} \quad (1) \end{aligned}$$

There are two cases to consider:

Case 1 Suppose $\{x\} + \{y\} < 1$. Then $\lfloor \{x\} + \{y\} \rfloor = 0$. Substituting this value into (1) yields

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor .$$

Case 2 Suppose $\{x\} + \{y\} \geq 1$. Because $\{x\}$ and $\{y\}$ are both less than 1, $\{x\} + \{y\} < 2$. Thus $\lfloor \{x\} + \{y\} \rfloor = 1$. Substituting this value into (1) yields

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1.$$

Conclusion Since the above two cases are exhaustive, it follows $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

□

⊙ Exercise 4d

$\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4d

This is immediately proven by applying Hermite's Identity as shown in ⊙ Exercise 5.

□

⊙ Exercise 4e

$\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4e

This is immediately proven by applying Hermite's Identity as shown in ⊙ Exercise 5.

□

⊙ Exercise 5

The formulas in Exercises 4(d) and 4(e) suggest a generalization for $\lfloor nx \rfloor$. State and prove such a generalization.

> Note: The stated generalization is known as "Hermite's Identity."

Proof. ∃ – Apostol.Chapter_1_11.exercise_5

We prove that for all natural numbers n and real numbers x, the following identity holds:

$$\lfloor nx \rfloor = \sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor \tag{2}$$

By definition of the floor function, $x = \lfloor x \rfloor + r$ for some $r \in [0, 1)$. Define S as the partition of non-overlapping subintervals

$$\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), \ldots, \left[\frac{n-1}{n}, 1\right).$$

By construction, $\cup S = [0, 1)$. Therefore there exists some $j \in \mathbb{N}$ such that

$$r \in \left[\frac{j}{n}, \frac{j+1}{n}\right). \tag{3}$$

With these definitions established, we now show the left- and right-hand sides of (2) evaluate to the same number.

Left-Hand Side Consider the left-hand side of identity (2). By (3), $nr \in [j, j+1)$. Therefore $\lfloor nr \rfloor = j$. Thus

$$\begin{aligned}
\lfloor nx \rfloor &= \lfloor n(\lfloor x \rfloor + r) \rfloor \\
&= \lfloor n \lfloor x \rfloor + nr \rfloor \\
&= \lfloor n \lfloor x \rfloor \rfloor + \lfloor nr \rfloor . && \text{Exercise 4a} \\
&= \lfloor n \lfloor x \rfloor \rfloor + j \\
&= n \lfloor x \rfloor + j. && (4)
\end{aligned}$$

Right-Hand Side Now consider the right-hand side of identity (2). We note each summand, by construction, is the floor of x added to a nonnegative number less than one. Therefore each summand contributes either $\lfloor x \rfloor$ or $\lfloor x \rfloor + 1$ to the total. Letting z denote the number of summands that contribute $\lfloor x \rfloor + 1$, we have

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = n \lfloor x \rfloor + z. \tag{5}$$

The value of z corresponds to the number of indices i that satisfy

$$\frac{i}{n} \geq 1 - r.$$

By (3), it follows

$$\begin{aligned}
1 - r &\in \left(1 - \frac{j+1}{n}, 1 - \frac{j}{n}\right] \\
&= \left(\frac{n-j-1}{n}, \frac{n-j}{n}\right].
\end{aligned}$$

Thus we can determine the value of z by instead counting the number of indices i that satisfy

$$\frac{i}{n} \geq \frac{n-j}{n}.$$

Rearranging terms, we see that $i \geq n - j$ holds for $z = (n-1) - (n-j) + 1 = j$ of the n summands. Substituting the value of z into (5) yields

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = n \lfloor x \rfloor + j. \tag{6}$$

Conclusion Since (4) and (6) agree with one another, it follows identity (2) holds.

□

Exercise 6

Recall that a lattice point (x, y) in the plane is one whose coordinates are integers. Let f be a nonnegative function whose domain is the interval $[a, b]$, where a and b are integers, $a < b$. Let S denote the set of points (x, y) satisfying $a \leq x \leq b$, $0 < y \leq f(x)$. Prove that the number of lattice points in S is equal to the sum

$$\sum_{n=a}^{b} \lfloor f(n) \rfloor .$$

Proof. Let $i = a, \ldots, b$ and define $S_i = \mathbb{N} \cap (0, f(i)]$. By construction, the number of lattice points in S is

$$\sum_{n=a}^{b} |S_n| . \tag{7}$$

All that remains is to show $|S_i| = \lfloor f(i) \rfloor$. There are two cases to consider:

Case 1 Suppose $f(i)$ is an integer. Then the number of integers in $(0, f(i)]$ is $f(i) = \lfloor f(i) \rfloor$.

Case 2 Suppose $f(i)$ is not an integer. Then the number of integers in $(0, f(i)]$ is the same as that of $(0, \lfloor f(i) \rfloor]$. Once again, that number is $\lfloor f(i) \rfloor$.

Conclusion By cases 1 and 2, $|S_i| = \lfloor f(i) \rfloor$. Substituting this identity into (7) finishes the proof.

□

Exercise 7

If a and b are positive integers with no common factor, we have the formula

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

When $b = 1$, the sum on the left is understood to be 0.

Note: When $b = 1$, the proofs of (a) and (b) are trivial. We continue under the assumption $b > 1$.

❗ Exercise 7a

Derive this result by a geometric argument, counting lattice points in a right triangle.

Proof. Let $f\colon [1, b-1] \to \mathbb{R}$ be given by $f(x) = ax/b$. Let S denote the set of points (x, y) satisfying $1 \leq x \leq b-1$, $0 < y \leq f(x)$. By ❗ Exercise 6, the number of lattice points of S is equal to the sum

$$\sum_{n=1}^{b-1} \lfloor f(n) \rfloor = \sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor. \tag{8}$$

Define T to be the triangle of width $w = b$ and height $h = f(b) = a$ as

$$T = \{(x, y) : 0 < x < b, 0 < y \leq f(x)\}.$$

By construction, T does not introduce any additional lattice points. Thus S and T have the same number of lattice points. Let H_L denote the number of boundary points on T's hypotenuse. We prove that (i) $H_L = 2$ and (ii) that T has $\frac{(a-1)(b-1)}{2}$ lattice points.

(i) Consider the line L overlapping the hypotenuse of T. By construction, T's hypotenuse has endpoints $(0, 0)$ and (b, a). By hypothesis, a and b are positive, excluding the possibility of L being vertical. Define the slope of L as

$$m = \frac{a}{b}.$$

H_L coincides with the number of indices $i = 0, \ldots, b$ such that $(i, i * m)$ is a lattice point. But a and b are coprime by hypothesis and $i \leq b$. Thus $i * m$ is an integer if and only if $i = 0$ or $i = b$. Thus $H_L = 2$.

(ii) Next we count the number of lattice points in T. Let R be the overlapping retangle of width w and height h, situated with bottom-left corner at $(0, 0)$. Let I_R denote the number of interior lattice points of R. Let I_T and B_T denote the interior and boundary lattice points of T respectively. By ❗ Exercise 4b,

$$\begin{aligned}
I_T &= \frac{1}{2}(I_R - (H_L - 2)) \\
&= \frac{1}{2}(I_R - (2 - 2)) && \text{(i)} \\
&= \frac{1}{2} I_R. && (9)
\end{aligned}$$

Furthermore, since both the adjacent and opposite side of T are not included in T and there exist no lattice points on T's hypotenuse besides the endpoints, it follows

$$B_T = 0. \tag{10}$$

Thus the number of lattice points of T equals

$$\begin{aligned}
I_T + B_T &= I_T && \text{(10)} \\
&= \frac{1}{2} I_R && \text{(9)} \\
&= \frac{(b-1)(a-1)}{2}. && \text{Exercise 4a} \qquad (11)
\end{aligned}$$

Conclusion By (8) the number of lattice points of S is equal to the sum

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor.$$

But the number of lattice points of S is the same as that of T. By (11), the number of lattice points in T is equal to

$$\frac{(b-1)(a-1)}{2}.$$

Thus

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

□

Exercise 7b

Derive the result analytically as follows: By changing the index of summation, note that $\sum_{n=1}^{b-1} \lfloor na/b \rfloor = \sum_{n=1}^{b-1} \lfloor a(b-n)/b \rfloor$. Now apply Exercises 4(a) and (b) to the bracket on the right.

Proof. ∃ – Apostol.Chapter_1_11.exercise_7b

Let $n = 1, \ldots, b-1$. By hypothesis, a and b are coprime. Furthermore, $n < b$ for all values of n. Thus an/b is not an integer. By Exercise 4b,

$$\left\lfloor -\frac{an}{b} \right\rfloor = -\left\lfloor \frac{an}{b} \right\rfloor - 1. \tag{12}$$

Consider the following:

$$
\begin{aligned}
\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor &= \sum_{n=1}^{b-1} \left\lfloor \frac{a(b-n)}{b} \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor \frac{ab-an}{b} \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor -\frac{an}{b} + a \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor -\frac{an}{b} \right\rfloor + a. && \text{Exercise 4a} \\
&= \sum_{n=1}^{b-1} - \left\lfloor \frac{an}{b} \right\rfloor - 1 + a && (12) \\
&= -\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor - \sum_{n=1}^{b-1} 1 + \sum_{n=1}^{b-1} a \\
&= -\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor - (b-1) + a(b-1).
\end{aligned}
$$

Rearranging the above yields

$$2\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor = (a-1)(b-1).$$

Dividing both sides of the above identity concludes the proof.

□

Exercise 8

Let S be a set of points on the real line. The *characteristic function* of S is, by definition, the function $\mathcal{X}_S$ such that $\mathcal{X}_S(x) = 1$ for every x in S, and $\mathcal{X}_S(x) = 0$ for those x not in S. Let f be a step function which takes the constant value c_k on the kth open subinterval I_k of some partition of an interval $[a, b]$. Prove that for each x in the union $I_1 \cup I_2 \cup \cdots \cup I_n$ we have

$$f(x) = \sum_{k=1}^{n} c_k \mathcal{X}_{I_k}(x).$$

This property is described by saying that every step function is a linear combination of characteristic functions of intervals.

Proof. Let $x \in I_1 \cup I_2 \cup \cdots \cup I_n$ and $N = \{1, \ldots, n\}$. Let $k \in N$ such that $x \in I_k$. Consider an arbitrary $j \in N - \{k\}$. By definition of a partition, $I_j \cap I_k = \emptyset$.

That is, I_j and I_k are disjoint for all $j \in N - \{k\}$. Therefore, by definition of the characteristic function, $\mathcal{X}_{I_k}(x) = 1$ and $\mathcal{X}_{I_j}(x) = 0$ for all $j \in N - \{k\}$. Thus

$$\begin{aligned} f(x) &= c_k \\ &= (c_k)(1) + \sum\nolimits_{j \in N - \{k\}} (c_j)(0) \\ &= c_k \mathcal{X}_{I_k}(x) + \sum\nolimits_{j \in N - \{k\}} c_j \mathcal{X}_{I_j}(x) \\ &= \sum_{k=1}^{n} c_k \mathcal{X}_{I_k}(x). \end{aligned}$$

$\square$