1 Exercise 4

Prove that the greatest-integer function has the properties indicated:

1.1 Exercise 4a

$\lfloor x+n \rfloor = \lfloor x \rfloor + n$ for every integer n.

Proof. Chapter_1_11.exercise_4a

∎

1.2 Exercise 4b

$$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \text{ is an integer,} \\ -\lfloor x \rfloor - 1 & \text{otherwise.} \end{cases}$$

Proof.

(a) Chapter_1_11.exercise_4b_1

(b) Chapter_1_11.exercise_4b_2

∎

1.3 Exercise 4c

$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proof. Chapter_1_11.exercise_4c

∎

1.4 Exercise 4d

$\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$.

Proof. Chapter_1_11.exercise_4d

∎

1.5 Exercise 4e

$\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor$.

Proof. Chapter_1_11.exercise_4e

∎

2 Exercise 5

The formulas in Exercises 4(d) and 4(e) suggest a generalization for $\lfloor nx \rfloor$. State and prove such a generalization.

Proof. Chapter_1_11.exercise_5

□

3 Exercise 6

Recall that a lattice point (x, y) in the plane is one whose coordinates are integers. Let f be a nonnegative function whose domain is the interval $[a, b]$, where a and b are integers, $a < b$. Let S denote the set of points (x, y) satisfying $a \leq x \leq b$, $0 < y \leq f(x)$. Prove that the number of lattice points in S is equal to the sum

$$\sum_{n=a}^{b} \lfloor f(n) \rfloor .$$

Proof. TODO

□

4 Exercise 7

If a and b are positive integers with no common factor, we have the formula

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

When $b = 1$, the sum on the left is understood to be 0.

4.1 Exercise 7a

Derive this result by a geometric argument, counting lattice points in a right triangle.

Proof. TODO

□

4.2 Exercise 7b

Derive the result analytically as follows: By changing the index of summation, note that $\sum_{n=1}^{b-1} \lfloor na/b \rfloor = \sum_{n=1}^{b-1} \lfloor a(b-n)/b \rfloor$. Now apply Exercises 4(a) and (b) to the bracket on the right.

Proof. Chapter_1_11.exercise_7b

□

5 Exercise 8

Let S be a set of points on the real line. The *characteristic function* of S is, by definition, the function χ_S such that $\chi_S(x) = 1$ for every x in S, and $\chi_S(x) = 0$ for those x not in S. Let f be a step function which takes the constant value c_k on the kth open subinterval I_k of some partition of an interval $[a, b]$. Prove that for each x in the union $I_1 \cup I_2 \cup \cdots \cup I_n$ we have

$$f(x) = \sum_{k=1}^{n} c_k \chi_{I_k}(x).$$

This property is described by saying that every step function is a linear combination of characteristic functions of intervals.

Proof. TODO

$\square$