

# Sequences

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## 1 Summations

### 1.1 ✔ Arithmetic Series

Let  $(a_i)_{i \geq 0}$  be an arithmetic sequence with common difference  $d$ . Then for some  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^n a_i = \frac{(n+1)(a_0 + a_n)}{2}. \quad (1)$$

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*Proof.* [✚ – Real.Arithmetic.sum-recursive-closed](#)

Let  $(a_i)_{i \geq 0}$  be an arithmetic sequence with common difference  $d$ . By definition, for all  $k \in \mathbb{N}$ ,

$$a_k = (a_0 + kd). \quad (2)$$

Define predicate  $P(n)$  as "identity (1) holds for value  $n$ ." We use induction to prove  $P(n)$  holds for all  $n \geq 0$ .

**Base Case** Let  $k = 0$ . Then

$$\sum_{i=0}^k a_i = a_0 = \frac{2a_0}{2} = \frac{(k+1)(a_0 + a_k)}{2}.$$

Therefore  $P(0)$  holds.

**Induction Step** Assume induction hypothesis  $P(k)$  holds for some  $k \geq 0$ . Then

$$\begin{aligned}
\sum_{i=0}^{k+1} a_i &= \sum_{i=0}^k a_i + a_{k+1} \\
&= \frac{(k+1)(a_0 + a_k)}{2} + a_{k+1} && \text{induction hypothesis} \\
&= \frac{(k+1)(a_0 + (a_0 + kd))}{2} + (a_0 + (k+1)d) && (2) \\
&= \frac{(k+1)(2a_0 + kd)}{2} + (a_0 + (k+1)d) \\
&= \frac{(k+1)(2a_0 + kd) + 2a_0 + 2(k+1)d}{2} \\
&= \frac{2ka_0 + k^2d + 4a_0 + kd + 2kd + 2d}{2} \\
&= \frac{(k+2)(2a_0 + kd + d)}{2} \\
&= \frac{(k+2)(a_0 + a_0 + (k+1)d)}{2} \\
&= \frac{(k+2)(a_0 + a_{k+1})}{2} && (2) \\
&= \frac{((k+1)+1)(a_0 + a_{k+1})}{2}.
\end{aligned}$$

Thus  $P(k)$  implies  $P(k+1)$  holds true.

**Conclusion** By mathematical induction, it follows for all  $n \geq 0$ ,  $P(n)$  is true.  $\square$

## 1.2 Geometric Series

Let  $(a_i)_{i \geq 0}$  be a geometric sequence with common ratio  $r \neq 1$ . Then for some  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^n a_i = \frac{a_0(1 - r^{n+1})}{1 - r}. \quad (3)$$

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*Proof.* [↗ – Real.Geometric.sum\\_recursive\\_closed](#)

Let  $(a_i)_{i \geq 0}$  be a geometric sequence with common ratio  $r \neq 1$ . By definition, for all  $k \in \mathbb{N}$ ,

$$a_k = a_0 r^k. \quad (4)$$

Define predicate  $P(n)$  as "identity (3) holds for value  $n$ ." We use induction to prove  $P(n)$  holds for all  $n \geq 0$ .

**Base Case** Let  $k = 0$ . Then

$$\sum_{i=0}^k a_i = a_0 = \frac{a_0(1-r)}{1-r} = \frac{a_0(1-r^{k+1})}{1-r}$$

Therefore  $P(0)$  holds.

**Induction Step** Assume induction hypothesis  $P(k)$  holds for some  $k \geq 0$ . Then

$$\begin{aligned} \sum_{i=0}^{k+1} a_i &= \sum_{i=0}^k a_i + a_{k+1} \\ &= \frac{a_0(1-r^{k+1})}{1-r} + a_{k+1} && \text{induction hypothesis} \\ &= \frac{a_0(1-r^{k+1})}{1-r} + a_0 r^{k+1} && (4) \\ &= \frac{a_0(1-r^{k+1}) + a_0 r^{k+1}(1-r)}{1-r} \\ &= \frac{a_0(1-r^{k+1} + r^{k+1}(1-r))}{1-r} \\ &= \frac{a_0(1-r^{k+1} + r^{k+1} - r^{k+2})}{1-r} \\ &= \frac{a_0(1-r^{k+2})}{1-r} \\ &= \frac{a_0(1-r^{(k+1)+1})}{1-r}. \end{aligned}$$

Thus  $P(k)$  implies  $P(k+1)$  holds true.

**Conclusion** By mathematical induction, it follows for all  $n \geq 0$ ,  $P(n)$  is true.  $\square$