One-Variable Calculus, with an Introduction to Linear Algebra

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Contents

Chapter 1

A Set of Axioms for the Real-Number System

1.1 ✓ Lemma 1

Nonempty set S has supremum L if and only if set $-S$ has infimum $-L$.

Proof. ∃ – Apostol.Chapter_I_03.is_lub_neg_set_iff_is_glb_set_neg

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Suppose $L = \sup S$ and fix $x \in S$. By definition of the supremum, $x \leq L$ and L is the smallest value satisfying this inequality. Negating both sides of the inequality yields $-x \geq -L$. Furthermore, $-L$ must be the largest value satisfying this inequality. Therefore $-L = \inf -S$.

□

1.2 ✓ Theorem I.27

Every nonempty set S that is bounded below has a greatest lower bound; that is, there is a real number L such that $L = \inf S$.

Proof. ∃ – Apostol.Chapter_I_03.exists_isGLB

Let S be a nonempty set bounded below by x. Then $-S$ is nonempty and bounded above by x. By the completeness axiom, there exists a supremum L of $-S$. By ✓ Lemma 1, L is a supremum of $-S$ if and only if $-L$ is an infimum of S.

□

1.3 ✓ Theorem I.29

For every real x there exists a positive integer n such that $n > x$.

Proof. ∃ – Apostol.Chapter_I_03.exists_pnat_geq_self

Let $n = |\lceil x \rceil| + 1$. It is trivial to see n is a positive integer satisfying $n \geq 1$. Thus all that remains to be shown is that $n > x$. If x is nonpositive, $n > x$ immediately follows from $n \geq 1$. If x is positive,

$$x = |x| \leq |\lceil x \rceil| < |\lceil x \rceil| + 1 = n.$$

□

1.4 ✓ Theorem I.30

If $x > 0$ and if y is an arbitrary real number, there exists a positive integer n such that $nx > y$.

Note: This is known as the "Archimedean Property of the Reals."

Proof. ∃ – Apostol.Chapter_I_03.exists_pnat_mul_self_geq_of_pos

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Let $x > 0$ and y be an arbitrary real number. By ✓ Theorem I.29, there exists a positive integer n such that $n > y/x$. Multiplying both sides of the inequality yields $nx > y$ as expected.

□

1.5 ✓ Theorem I.31

If three real numbers a, x, and y satisfy the inequalities

$$a \leq x \leq a + \frac{y}{n}$$

for every integer $n \geq 1$, then $x = a$.

Proof. ∃ – Apostol.Chapter_I_03.forall_pnat_leq_self_leq_frac_imp_eq

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By the trichotomy of the reals, there are three cases to consider:

Case 1 Suppose $x = a$. Then we are immediately finished.

Case 2 Suppose $x < a$. But by hypothesis, $a \leq x$. Thus $a < a$, a contradiction.

Case 3 Suppose $x > a$. Then there exists some $c > 0$ such that $a + c = x$. By Theorem I.30, there exists an integer $n > 0$ such that $nc > y$. Rearranging terms, we see $y/n < c$. Therefore $a + y/n < a + c = x$. But by hypothesis, $x \leq a + y/n$. Thus $a + y/n < a + y/n$, a contradiction.

Conclusion Since these cases are exhaustive and both case 2 and 3 lead to contradictions, $x = a$ is the only possibility.

□

1.6 Lemma 2

If three real numbers a, x, and y satisfy the inequalities

$$a - y/n \leq x \leq a$$

for every integer $n \geq 1$, then $x = a$.

Proof. ∃ – Apostol.Chapter_I_03.forall_pnat_frac_leq_self_leq_imp_eq

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By the trichotomy of the reals, there are three cases to consider:

Case 1 Suppose $x = a$. Then we are immediately finished.

Case 2 Suppose $x < a$. Then there exists some $c > 0$ such that $x = a - c$. By Theorem I.30, there exists an integer $n > 0$ such that $nc > y$. Rearranging terms, we see that $y/n < c$. Therefore $a - y/n > a - c = x$. But by hypothesis, $x \geq a - y/n$. Thus $a - y/n < a - y/n$, a contradiction.

Case 3 Suppose $x > a$. But by hypothesis $x \leq a$. Thus $a < a$, a contradiction.

Conclusion Since these cases are exhaustive and both case 2 and 3 lead to contradictions, $x = a$ is the only possibility.

□

1.7 Theorem I.32

Let h be a given positive number and let S be a set of real numbers.

1.7.1 Theorem I.32a

If S has a supremum, then for some x in S we have $x > \sup S - h$.

Proof. ∃ – Apostol.Chapter_I_03.sup_imp_exists_gt_sup_sub_delta

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By definition of a supremum, $\sup S$ is the least upper bound of S. For the sake of contradiction, suppose for all $x \in S$, $x \leq \sup S - h$. This immediately implies $\sup S - h$ is an upper bound of S. But $\sup S - h < \sup S$, contradicting $\sup S$ being the *least* upper bound. Therefore our original hypothesis was wrong. That is, there exists some $x \in S$ such that $x > \sup S - h$.

□

1.7.2 Theorem I.32b

If S has an infimum, then for some x in S we have $x < \inf S + h$.

Proof. ∃ – Apostol.Chapter_I_03.inf_imp_exists_lt_inf_add_delta

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By definition of an infimum, $\inf S$ is the greatest lower bound of S. For the sake of contradiction, suppose for all $x \in S$, $x \geq \inf S + h$. This immediately implies $\inf S + h$ is a lower bound of S. But $\inf S + h > \inf S$, contradicting $\inf S$ being the *greatest* lower bound. Therefore our original hypothesis was wrong. That is, there exists some $x \in S$ such that $x < \inf S + h$.

□

1.8 Theorem I.33

Given nonempty subsets A and B of $\mathbb{R}$, let C denote the set

$$C = \{a + b : a \in A, b \in B\}.$$

Note: This is known as the "Additive Property."

1.8.1 Theorem I.33a

If each of A and B has a supremum, then C has a supremum, and

$$\sup C = \sup A + \sup B.$$

Proof. ∃ – Apostol.Chapter_I_03.sup_minkowski_sum_eq_sup_add_sup

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We prove (i) $\sup A + \sup B$ is an upper bound of C and (ii) $\sup A + \sup B$ is the *least* upper bound of C.

(i) Let $x \in C$. By definition of C, there exist elements $a' \in A$ and $b' \in B$ such that $x = a' + b'$. By definition of a supremum, $a' \leq \sup A$. Likewise, $b' \leq \sup B$. Therefore $a' + b' \leq \sup A + \sup B$. Since $x = a' + b'$ was arbitrarily chosen, it follows $\sup A + \sup B$ is an upper bound of C.

(ii) Since A and B have supremums, C is nonempty. By (i), C is bounded above. Therefore the completeness axiom tells us C has a supremum. Let $n > 0$ be an integer. We now prove that

$$\sup C \leq \sup A + \sup B \leq \sup C + 1/n. \tag{1.1}$$

Left-Hand Side First consider the left-hand side of (1.1). By (i), $\sup A + \sup B$ is an upper bound of C. Since $\sup C$ is the *least* upper bound of C, it follows $\sup C \leq \sup A + \sup B$.

Right-Hand Side Next consider the right-hand side of (1.1). By Theorem I.32a, there exists some $a' \in A$ such that $\sup A < a' + 1/(2n)$. Likewise, there exists some $b' \in B$ such that $\sup B < b' + 1/(2n)$. Adding these two inequalities together shows

$$\begin{aligned} \sup A + \sup B &< a' + b' + 1/n \\ &\leq \sup C + 1/n. \end{aligned}$$

Conclusion Applying Theorem I.31 to (1.1) proves $\sup C = \sup A + \sup B$ as expected.

□

1.8.2 Theorem I.33b

If each of A and B has an infimum, then C has an infimum, and

$$\inf C = \inf A + \inf B.$$

Proof. ∃ − Apostol.Chapter_I_03.inf_minkowski_sum_eq_inf_add_inf

---

We prove (i) $\inf A + \inf B$ is a lower bound of C and (ii) $\inf A + \inf B$ is the *greatest* lower bound of C.

(i) Let $x \in C$. By definition of C, there exist elements $a' \in A$ and $b' \in B$ such that $x = a' + b'$. By definition of an infimum, $a' \geq \inf A$. Likewise, $b' \geq \inf B$. Therefore $a' + b' \geq \inf A + \inf B$. Since $x = a' + b'$ was arbitrarily chosen, it follows $\inf A + \inf B$ is a lower bound of C.

(ii) Since A and B have infimums, C is nonempty. By (i), C is bounded below. Therefore Theorem I.27 tells us C has an infimum. Let $n > 0$ be an integer. We now prove that

$$\inf C - 1/n \leq \inf A + \inf B \leq \inf C. \tag{1.2}$$

Right-Hand Side First consider the right-hand side of (1.2). By (i), $\inf A + \inf B$ is a lower bound of C. Since $\inf C$ is the *greatest* upper bound of C, it follows $\inf C \geq \inf A + \inf B$.

Left-Hand Side Next consider the left-hand side of (1.2). By Theorem I.32b, there exists some $a' \in A$ such that $\inf A > a' - 1/(2n)$. Likewise, there exists some $b' \in B$ such that $\inf B > b' - 1/(2n)$. Adding these two inequalities together shows

$$\begin{aligned} \inf A + \inf B &> a' + b' - 1/n \\ &\geq \inf C - 1/n. \end{aligned}$$

Conclusion Applying Lemma 2 to (1.2) proves $\inf C = \inf A + \inf B$ as expected.

□

1.9 Theorem I.34

Given two nonempty subsets S and T of $\mathbb{R}$ such that

$$s \leq t$$

for every s in S and every t in T. Then S has a supremum, and T has an infimum, and they satisfy the inequality

$$\sup S \leq \inf T.$$

Proof. ∃ – Apostol.Chapter_I_03.forall_mem_le_forall_mem_imp_sup_le_inf

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By hypothesis, S and T are nonempty sets. Let $s \in S$ and $t \in T$. Then t is an upper bound of S and s is a lower bound of T. By the completeness axiom, S has a supremum. By Theorem I.27, T has an infimum. All that remains is showing $\sup S \leq \inf T$.

For the sake of contradiction, suppose $\sup S > \inf T$. Then there exists some $c > 0$ such that $\sup S = \inf T + c$. Therefore $\inf T < \sup S - c/2$. By Theorem I.32a, there exists some $x \in S$ such that $\sup S - c/2 < x$. Thus

$$\inf T < \sup S - c/2 < x.$$

But by hypothesis, $x \in S$ is a lower bound of T meaning $x \leq \inf T$. Therefore $x < x$, a contradiction. Out original assumption is incorrect; that is, $\sup S \leq \inf T$.

□

Chapter 2

The Concept of Area as a Set Function

We assume there exists a class $\mathscr{M}$ of measurable sets in the plane and a set function a, whose domain is $\mathscr{M}$, with the following properties:

2.1 ¶ Nonnegative Property

For each set S in $\mathscr{M}$, we have $a(S) \geq 0$.

Axiom. ∃ – Nonnegative Property

□

2.2 ¶ Additive Property

If S and T are in $\mathscr{M}$, then $S \cup T$ and $S \cap T$ are in $\mathscr{M}$, and we have $a(S \cup T) = a(S) + a(T) - a(S \cap T)$.

Axiom. ∃ – Additive Property

□

2.3 ¶ Difference Property

If S and T are in $\mathscr{M}$ with $S \subseteq T$, then $T - S$ is in $\mathscr{M}$, and we have $a(T - S) = a(T) - a(S)$.

Axiom. ∃ – Difference Property

□

2.4 ¶ Invariance Under Congruence

If a set S is in $\mathscr{M}$ and if T is congruent to S, then T is also in $\mathscr{M}$ and we have $a(S) = a(T)$.

Axiom. ∃ – Invariance Under Congruence

□

2.5 ¶ Choice of Scale

Every rectangle R is in $\mathscr{M}$. If the edges of R have lengths h and k, then $a(R) = hk$.

Axiom. ∃ – Choice of Scale

□

2.6 Exhaustion Property

Let Q be a set that can be enclosed between two step regions S and T, so that

$$S \subseteq Q \subseteq T. \tag{2.1}$$

If there is one and only one number c which satisfies the inequalities

$$a(S) \leq c \leq a(T)$$

for all step regions S and T satisfying (1.1), then Q is measurable and $a(Q) = c$.

Axiom. ∃ – Exhaustion Property

□

Chapter 3

Exercises 1.7

3.1 Exercise 1.7.1

Prove that each of the following sets is measurable and has zero area:

3.1.1 Exercise 1.7.1a

A set consisting of a single point.

Proof. Let S be a set consisting of a single point. By definition of a Point, S is a rectangle in which all vertices coincide. By ¶ Choice of Scale, S is measurable with area its width times its height. The width and height of S is trivially zero. Therefore $a(S) = (0)(0) = 0$.

□

3.1.2 Exercise 1.7.1b

A set consisting of a finite number of points in a plane.

Proof. Define predicate $P(n)$ as "A set consisting of n points in a plane is measurable with area 0". We use induction to prove $P(n)$ holds for all $n > 0$.

Base Case Consider a set S consisting of a single point in a plane. By Exercise 1.7.1a, S is measurable with area 0. Thus $P(1)$ holds.

Induction Step Assume induction hypothesis $P(k)$ holds for some $k > 0$. Let S_{k+1} be a set consisting of $k + 1$ points in a plane. Pick an arbitrary point of S_{k+1}. Denote the set containing just this point as T. Denote the remaining set of points as S_k. By construction, $S_{k+1} = S_k \cup T$. By the induction hypothesis, S_k is measurable with area 0. By Exercise 1.7.1a, T is measurable with area

0. By the ¶ Additive Property, $S_k \cup T$ is measurable, $S_k \cap T$ is measurable, and

$$\begin{aligned} a(S_{k+1}) &= a(S_k \cup T) \\ &= a(S_k) + a(T) - a(S_k \cap T) \\ &= 0 + 0 - a(S_k \cap T). \end{aligned} \tag{3.1}$$

There are two cases to consider:

Case 1 $S_k \cap T = \emptyset$. Then it trivially follows that $a(S_k \cap T) = 0$.

Case 2 $S_k \cap T \neq \emptyset$. Since T consists of a single point, $S_k \cap T = T$. By Exercise 1.7.1a, $a(S_k \cap T) = a(T) = 0$.

In both cases, (3.1) evaluates to 0, implying $P(k+1)$ as expected.

Conclusion By mathematical induction, it follows for all $n > 0$, $P(n)$ is true. □

3.1.3 Exercise 1.7.1c

The union of a finite collection of line segments in a plane.

Proof. Define predicate $P(n)$ as "A set consisting of n line segments in a plane is measurable with area 0". We use induction to prove $P(n)$ holds for all $n > 0$.

Base Case Consider a set S consisting of a single line segment in a plane. By definition of a Line Segment, S is a rectangle in which one side has dimension 0. By ¶ Choice of Scale, S is measurable with area its width w times its height h. Therefore $a(S) = wh = 0$. Thus $P(1)$ holds.

Induction Step Assume induction hypothesis $P(k)$ holds for some $k > 0$. Let S_{k+1} be a set consisting of $k+1$ line segments in a plane. Pick an arbitrary line segment of S_{k+1}. Denote the set containing just this line segment as T. Denote the remaining set of line segments as S_k. By construction, $S_{k+1} = S_k \cup T$. By the induction hypothesis, S_k is measurable with area 0. By the base case, T is measurable with area 0. By the ¶ Additive Property, $S_k \cup T$ is measurable, $S_k \cap T$ is measurable, and

$$\begin{aligned} a(S_{k+1}) &= a(S_k \cup T) \\ &= a(S_k) + a(T) - a(S_k \cap T) \\ &= 0 + 0 - a(S_k \cap T). \end{aligned} \tag{3.2}$$

There are two cases to consider:

Case 1 $S_k \cap T = \emptyset$. Then it trivially follows that $a(S_k \cap T) = 0$.

Case 2 $S_k \cap T \neq \emptyset$. Since T consists of a single point, $S_k \cap T = T$. By the base case, $a(S_k \cap T) = a(T) = 0$.

In both cases, (3.2) evaluates to 0, implying $P(k+1)$ as expected.

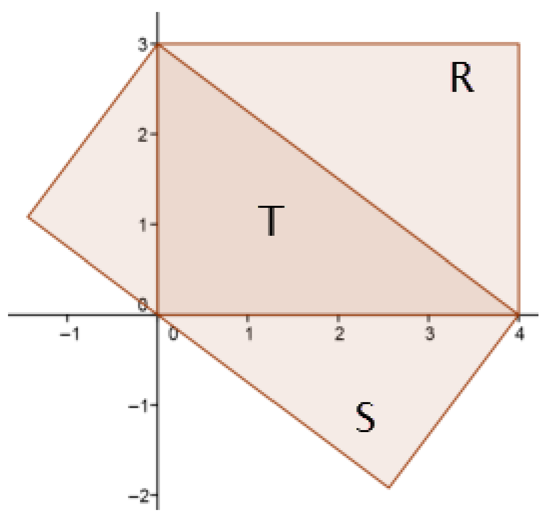
Conclusion By mathematical induction, it follows for all $n > 0$, $P(n)$ is true. □

3.2 ❗ Exercise 1.7.2

Every right triangular region is measurable because it can be obtained as the intersection of two rectangles. Prove that every triangular region is measurable and that its area is one half the product of its base and altitude.

Proof. Let T' be a triangular region with base of length a, height of length b, and hypotenuse of length c. Consider the translation and rotation of T', say T, such that its hypotenuse is entirely within quadrant I and the vertex opposite the hypotenuse is situated at point $(0, 0)$.

Let R be a rectangle of width a, height b, and bottom-left corner at $(0, 0)$. By construction, R covers all of T. Let S be a rectangle of width c and height $a \sin \theta$, where θ is the acute angle measured from the bottom-right corner of T relative to the x-axis. As an example, consider the image below of triangle T with width 4 and height 3:



By ¶ Choice of Scale, both R and S are measurable. By this same axiom, $a(R) = ab$ and $a(S) = ca \sin \theta$. By the ¶ Additive Property, $R \cup S$ and $R \cap S$ are both measurable. $a(R \cap S) = a(T)$ and $a(R \cup S)$ can be determined by noting that R's construction implies identity $a(R) = 2a(T)$. Therefore

$$\begin{aligned} a(T) &= a(R \cap S) \\ &= a(R) + a(S) - a(R \cup S) \\ &= ab + ca\sin\theta - a(R \cup S) \\ &= ab + ca\sin\theta - (ca\sin\theta + \frac{1}{2}a(R)) \\ &= ab + ca\sin\theta - ca\sin\theta - a(T). \end{aligned}$$

Solving for $a(T)$ gives the desired identity:

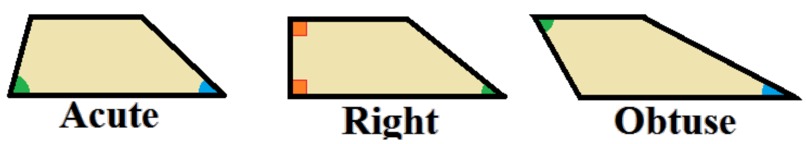
$$a(T) = \frac{1}{2}ab.$$

By ¶ Invariance Under Congruence, $a(T') = a(T)$, concluding our proof. □

3.3 ❗ Exercise 1.7.3

Prove that every trapezoid and every parallelogram is measurable and derive the usual formulas for their areas.

Proof. We begin by proving the formula for a trapezoid. Let S be a trapezoid with height h and bases b_1 and b_2, $b_1 < b_2$. There are three cases to consider:



Case 1 Suppose S is a right trapezoid. Then S is the union of non-overlapping rectangle R of width b_1 and height h with right triangle T of base $b_2 - b_1$ and height h. By ¶ Choice of Scale, R is measurable. By ❗ Exercise 1.7.2, T is measurable. By the ¶ Additive Property, $R \cup T$ and $R \cap T$ are both measurable

and

$$\begin{aligned}
a(S) &= a(R \cup T) \\
&= a(R) + a(T) - a(R \cap T) \\
&= a(R) + a(T) && \text{by construction} \\
&= b_1 h + a(T) && \text{Choice of Scale} \\
&= b_1 h + \frac{1}{2}(b_2 - b_1)h && \text{Exercise 1.7.2} \\
&= \frac{b_1 + b_2}{2} h.
\end{aligned}$$

Case 2 Suppose S is an acute trapezoid. Then S is the union of non-overlapping triangle T and right trapezoid R. Let c denote the length of base T. Then R has longer base edge of length $b_2 - c$. By Exercise 1.7.2, T is measurable. By Case 1, R is measurable. By the ¶ Additive Property, $R \cup T$ and $R \cap T$ are both measurable and

$$\begin{aligned}
a(S) &= a(T) + a(R) - a(R \cap T) \\
&= a(T) + a(R) && \text{by construction} \\
&= \frac{1}{2}ch + a(R) && \text{Exercise 1.7.2} \\
&= \frac{1}{2}ch + \frac{b_1 + b_2 - c}{2} h && \text{Case 1} \\
&= \frac{b_1 + b_2}{2} h.
\end{aligned}$$

Case 3 Suppose S is an obtuse trapezoid. Then S is the union of non-overlapping triangle T and right trapezoid R. Let c denote the length of base T. Reflect T vertically to form another right triangle, say T'. Then $T' \cup R$ is an acute trapezoid. By ¶ Invariance Under Congruence,

$$a(T' \cup R) = a(T \cup R). \tag{3.1}$$

By construction, $T' \cup R$ has height h and bases $b_1 - c$ and $b_2 + c$ meaning

$$\begin{aligned}
a(T \cup R) &= a(T' \cup R) && (3.1) \\
&= \frac{b_1 - c + b_2 + c}{2} h && \text{Case 2} \\
&= \frac{b_1 + b_2}{2} h.
\end{aligned}$$

Conclusion These cases are exhaustive and in agreement with one another. Thus S is measurable and

$$a(S) = \frac{b_1 + b_2}{2} h.$$

Let P be a parallelogram with base b and height h. Then P is the union of non-overlapping triangle T and right trapezoid R. Let c denote the length of base T. Reflect T vertically to form another right triangle, say T'. Then $T' \cup R$ is an acute trapezoid. By ¶ Invariance Under Congruence,

$$a(T' \cup R) = a(T \cup R). \tag{3.2}$$

By construction, $T' \cup R$ has height h and bases $b - c$ and $b + c$ meaning

$$\begin{aligned} a(T \cup R) &= a(T' \cup R) && (3.2) \\ &= \frac{b - c + b + c}{2} h && \text{Area of Trapezoid} \\ &= bh. \end{aligned}$$

□

3.4 Exercise 1.7.4

Let P be a polygon whose vertices are lattice points. The area of P is $I + \frac{1}{2}B - 1$, where I denotes the number of lattice points inside the polygon and B denotes the number on the boundary.

3.4.1 Exercise 1.7.4a

Prove that the formula is valid for rectangles with sides parallel to the coordinate axes.

Proof. Let P be a rectangle with sides parallel to the coordinate axes, with width w, height h, and lattice points for vertices. We assume P has three non-collinear points, ruling out any instances of points or line segments.

By ¶ Choice of Scale, P is measurable with area $a(P) = wh$. By construction, P has $I = (w-1)(h-1)$ interior lattice points and $B = 2(w+h)$ lattice points on its boundary. The following shows the lattice point area formula is in agreement with the expected result:

$$\begin{aligned} I + \frac{1}{2}B - 1 &= (w-1)(h-1) + \frac{1}{2}\left[2(w+h)\right] - 1 \\ &= (wh - w - h + 1) + \frac{1}{2}\left[2(w+h)\right] - 1 \\ &= (wh - w - h + 1) + (w + h) - 1 \\ &= wh. \end{aligned}$$

□

3.4.2 ❗ Exercise 1.7.4b

Prove that the formula is valid for right triangles and parallelograms.

Proof. Let P be a right triangle with width $w > 0$, height $h > 0$, and lattice points for vertices. Let T be the triangle P translated, rotated, and reflected such that the its vertices are $(0,0)$, $(0,w)$, and (w,h). Let I_T and B_T be the number of interior and boundary points of T respectively. Let H_L denote the number of lattice points on T's hypotenuse.

Let R be the overlapping rectangle of width w and height h, situated with bottom-left corner at $(0,0)$. Let I_R and B_R be the number of interior and boundary points of R respectively.

By construction, T shares two sides with R. Therefore

$$B_T = \frac{1}{2}B_R - 1 + H_L. \tag{3.3}$$

Likewise,

$$I_T = \frac{1}{2}(I_R - (H_L - 2)). \tag{3.4}$$

The following shows the lattice point area formula is in agreement with the expected result:

$$\begin{aligned}
I_T + \frac{1}{2}B_T - 1 &= \frac{1}{2}(I_R - (H_L - 2)) + \frac{1}{2}B_T - 1 && (3.4) \\
&= \frac{1}{2}\left[I_R - H_L + B_T\right] \\
&= \frac{1}{2}\left[I_R - H_L + \frac{1}{2}B_R - 1 + H_L\right] && (3.3) \\
&= \frac{1}{2}\left[I_R + \frac{1}{2}B_R - 1\right] \\
&= \frac{1}{2}\left[wh\right] && \text{❗ Exercise 1.7.4a.}
\end{aligned}$$

We do not prove this formula is valid for parallelograms here. Instead, refer to ❗ Exercise 1.7.4c below.

□

3.4.3 ❗ Exercise 1.7.4c

Use induction on the number of edges to construct a proof for general polygons.

Proof. Define predicate $P(n)$ as "An n-polygon with vertices on lattice points has area $I + \frac{1}{2}B - 1$." We use induction to prove $P(n)$ holds for all $n \geq 3$.

Base Case A 3-polygon is a triangle. By ❗ Exercise 1.7.4b, the lattice point area formula holds. Thus $P(3)$ holds.

Induction Step Assume induction hypothesis $P(k)$ holds for some $k \geq 3$. Let P be a $(k+1)$-polygon with vertices on lattice points. Such a polygon is equivalent to the union of a k-polygon S with a triangle T. That is, $P = S \cup T$.

Let I_P be the number of interior lattice points of P. Let B_P be the number of boundary lattice points of P. Similarly, let I_S, I_T, B_S, and B_T be the number of interior and boundary lattice points of S and T. Let c denote the number of boundary points shared between S and T.

By our induction hypothesis, $a(S) = I_S + \frac{1}{2}B_S - 1$. By our base case, $a(T) = I_T + \frac{1}{2}B_T - 1$. By construction, it follows:

$$\begin{aligned} I_P &= I_S + I_T + c - 2 \\ B_P &= B_S + B_T - (c-2) - c \\ &= B_S + B_T - 2c + 2. \end{aligned}$$

Applying the lattice point area formula to P yields the following:

$$\begin{aligned} &I_P + \frac{1}{2}B_P - 1 \\ &= (I_S + I_T + c - 2) + \frac{1}{2}(B_S + B_T - 2c + 2) - 1 \\ &= I_S + I_T + c - 2 + \frac{1}{2}B_S + \frac{1}{2}B_T - c + 1 - 1 \\ &= (I_S + \frac{1}{2}B_S - 1) + (I_T + \frac{1}{2}B_T - 1) \\ &= a(S) + (I_T + \frac{1}{2}B_T - 1) && \text{induction hypothesis} \\ &= a(S) + a(T). && \text{base case} \end{aligned}$$

By the ¶ Additive Property, $S \cup T$ is measurable, $S \cap T$ is measurable, and

$$\begin{aligned} a(P) &= a(S \cup T) \\ &= a(S) + a(T) - a(S \cap T) \\ &= a(S) + a(T). && \text{by construction} \end{aligned}$$

This shows the lattice point area formula is in agreement with our axiomatic definition of area. Thus $P(k+1)$ holds.

Conclusion By mathematical induction, it follows for all $n \geq 3$, $P(n)$ is true.

□

3.5 ❗ Exercise 1.7.5

Prove that a triangle whose vertices are lattice points cannot be equilateral.

[*Hint:* Assume there is such a triangle and compute its area in two ways, using Exercises 2 and 4.]

Proof. Proceed by contradiction. Let T be an equilateral triangle whose vertices are lattice points. Assume each side of T has length a. Then T has height $h = (a\sqrt{3})/2$. By ❗ Exercise 1.7.2,

$$a(T) = \frac{1}{2}ah = \frac{a^2\sqrt{3}}{4}. \tag{5.1}$$

Let I and B denote the number of interior and boundary lattice points of T respectively. By Exercise 1.7.4,

$$a(T) = I + \frac{1}{2}B - 1. \tag{5.2}$$

But (5.1) is irrational whereas (5.2) is not. This is a contradiction. Thus, there is *no* equilateral triangle whose vertices are lattice points.

□

3.6 ❗ Exercise 1.7.6

Let $A = \{1, 2, 3, 4, 5\}$, and let $\mathscr{M}$ denote the class of all subsets of A. (There are 32 altogether, counting A itself and the empty set $\emptyset$.) For each set S in $\mathscr{M}$, let $n(S)$ denote the number of distinct elements in S. If $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$, compute $n(S \cup T)$, $n(S \cap T)$, $n(S - T)$, and $n(T - S)$. Prove that the set function n satisfies the first three axioms for area.

Proof. Let $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$. Then

$$\begin{aligned}
n(S \cup T) &= n(\{1, 2, 3, 4\} \cup \{3, 4, 5\}) \\
&= n(\{1, 2, 3, 4, 5\}) \\
&= 5. \\
n(S \cap T) &= n(\{1, 2, 3, 4\} \cap \{3, 4, 5\}) \\
&= n(\{3, 4\}) \\
&= 2. \\
n(S - T) &= n(\{1, 2, 3, 4\} - \{3, 4, 5\}) \\
&= n(\{1, 2\}) \\
&= 2. \\
n(T - S) &= n(\{3, 4, 5\} - \{1, 2, 3, 4\}) \\
&= n(\{5\}) \\
&= 1.
\end{aligned}$$

We now prove n satisfies the first three axioms for area.

Nonnegative Property n returns the length of some member of $\mathscr{M}$. By hypothesis, the smallest possible input to n is $\emptyset$. Since $n(\emptyset) = 0$, it follows $n(S) \geq 0$ for all $S \subset A$.

Additive Property Let S and T be members of $\mathscr{M}$. It trivially follows that both $S \cup T$ and $S \cap T$ are in $\mathscr{M}$. Consider the value of $n(S \cup T)$. There are two cases to consider:

Case 1 Suppose $S \cap T = \emptyset$. That is, there is no common element shared between S and T. Thus

$$\begin{aligned} n(S \cup T) &= n(S) + n(T) \\ &= n(S) + n(T) - 0 \\ &= n(S) + n(T) - n(S \cap T). \end{aligned}$$

Case 2 Suppose $S \cap T \neq \emptyset$. Then $n(S) + n(T)$ counts each element of $S \cap T$ twice. Therefore $n(S \cup T) = n(S) + n(T) - n(S \cap T)$.

Conclusion These cases are exhaustive and in agreement with one another. Thus $n(S \cup T) = n(S) + n(T) - n(S \cap T)$.

Difference Property Suppose $S, T \in \mathscr{M}$ such that $S \subseteq T$. That is, every member of S is a member of T. By definition, $T - S$ consists of members in T but not in S. Thus $n(T - S) = n(T) - n(S)$.

$\square$

Chapter 4

Partitions and Step Functions

4.1 ¶ Partition

Let $[a, b]$ be a closed interval decomposed into n subintervals by inserting $n-1$ points of subdivision, say x_1, x_2, …, x_{n-1}, subject only to the restriction

$$a < x_1 < x_2 < \cdots < x_{n-1} < b. \tag{4.1}$$

It is convenient to denote the point a itself by x_0 and the point b by x_n. A collection of points satisfying (4.1) is called a **partition** P of $[a, b]$, and we use the symbol

$$P = \{x_0, x_1, \ldots, x_n\}$$

to designate this partition.

Definition. ∃ − Set.Intervals.Partition

□

4.2 ¶ Step Function

A function s, whose domain is a closed interval $[a, b]$, is called a step function if there is a ¶ Partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[ab]$ such that s is constant on each open subinterval of P. That is to say, for each $k = 1, 2, \ldots, n$, there is a real number s_k such that

$$s(x) = s_k \quad \text{if} \quad x_{k-1} < x < x_k.$$

Step functions are sometimes called piecewise constant functions.

Note: At each of the endpoints x_{k-1} and x_k the function must have some well-defined value, but this need not be the same as s_k.

Definition. $\exists$ − Set.Intervals.StepFunction

□

Chapter 5

Exercises 1.11

5.1 Exercise 1.11.4

Prove that the greatest-integer function has the properties indicated:

5.1.1 Exercise 1.11.4a

$\lfloor x+n \rfloor = \lfloor x \rfloor + n$ for every integer n.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4a

Let x be a real number and n an integer. Let $m = \lfloor x+n \rfloor$. By definition of the floor function, m is the unique integer such that $m \leq x+n < m+1$. Then $m-n \leq x < (m-n)+1$. That is, $m-n = \lfloor x \rfloor$. Rearranging terms we see that $m = \lfloor x \rfloor + n$ as expected.

□

5.1.2 Exercise 1.11.4b

$$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \text{ is an integer,} \\ -\lfloor x \rfloor - 1 & \text{otherwise.} \end{cases}$$

Proof.

∃ – Apostol.Chapter_1_11.exercise_4b_1
∃ – Apostol.Chapter_1_11.exercise_4b_2

There are two cases to consider:

Case 1 Suppose x is an integer. Then $x = \lfloor x \rfloor$ and $-x = \lfloor -x \rfloor$. It immediately follows that

$$\lfloor -x \rfloor = -x = -\lfloor x \rfloor.$$

Case 2 Suppose x is not an integer. Let $m = \lfloor -x \rfloor$. By definition of the floor function, m is the unique integer such that $m \leq -x < m + 1$. Equivalently, $-m - 1 < x \leq -m$. Since x is not an integer, it follows $-m - 1 \leq x < -m$. Then, by definition of the floor function, $\lfloor x \rfloor = -m - 1$. Solving for m yields

$$\lfloor -x \rfloor = m = -\lfloor x \rfloor - 1.$$

Conclusion The above two cases are exhaustive. Thus

$$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \text{ is an integer,} \\ -\lfloor x \rfloor - 1 & \text{otherwise.} \end{cases}$$

□

5.1.3 ✔ Exercise 1.11.4c

$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4c

Rewrite x and y as the sum of their floor and fractional components: $x = \lfloor x \rfloor + \{x\}$ and $y = \lfloor y \rfloor + \{y\}$. Now

$$\begin{aligned} \lfloor x + y \rfloor &= \lfloor \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} \rfloor \\ &= \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \rfloor \\ &= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor && \text{✔ Exercise 1.11.4a} \qquad (5.1) \end{aligned}$$

There are two cases to consider:

Case 1 Suppose $\{x\} + \{y\} < 1$. Then $\lfloor \{x\} + \{y\} \rfloor = 0$. Substituting this value into (5.1) yields

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor.$$

Case 2 Suppose $\{x\} + \{y\} \geq 1$. Because $\{x\}$ and $\{y\}$ are both less than 1, $\{x\} + \{y\} < 2$. Thus $\lfloor \{x\} + \{y\} \rfloor = 1$. Substituting this value into (5.1) yields

$$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1.$$

Conclusion Since the above two cases are exhaustive, it follows $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

□

5.1.4 Exercise 1.11.4d

$\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4d

This is immediately proven by applying Hermite's Identity as shown in Exercise 1.11.5.

□

5.1.5 Exercise 1.11.4e

$\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4e

This is immediately proven by applying Hermite's Identity as shown in Exercise 1.11.5.

□

5.2 Exercise 1.11.5

The formulas in Exercises 4(d) and 4(e) suggest a generalization for $\lfloor nx \rfloor$. State and prove such a generalization.

> Note: The stated generalization is known as "Hermite's Identity."

Proof. ∃ – Apostol.Chapter_1_11.exercise_5

We prove that for all natural numbers n and real numbers x, the following identity holds:

$$\lfloor nx \rfloor = \sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor \tag{5.2}$$

By definition of the floor function, $x = \lfloor x \rfloor + r$ for some $r \in [0, 1)$. Define S as the partition of non-overlapping subintervals

$$\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), \ldots, \left[\frac{n-1}{n}, 1\right).$$

By construction, $\cup S = [0, 1)$. Therefore there exists some $j \in \mathbb{N}$ such that

$$r \in \left[\frac{j}{n}, \frac{j+1}{n}\right). \tag{5.3}$$

With these definitions established, we now show the left- and right-hand sides of (5.2) evaluate to the same number.

Left-Hand Side Consider the left-hand side of identity (5.2). By (5.3), $nr \in [j, j+1)$. Therefore $\lfloor nr \rfloor = j$. Thus

$$\begin{aligned}
\lfloor nx \rfloor &= \lfloor n(\lfloor x \rfloor + r) \rfloor \\
&= \lfloor n \lfloor x \rfloor + nr \rfloor \\
&= \lfloor n \lfloor x \rfloor \rfloor + \lfloor nr \rfloor . && \text{Exercise 1.11.4a} \\
&= \lfloor n \lfloor x \rfloor \rfloor + j \\
&= n \lfloor x \rfloor + j. && (5.4)
\end{aligned}$$

Right-Hand Side Now consider the right-hand side of identity (5.2). We note each summand, by construction, is the floor of x added to a nonnegative number less than one. Therefore each summand contributes either $\lfloor x \rfloor$ or $\lfloor x \rfloor + 1$ to the total. Letting z denote the number of summands that contribute $\lfloor x \rfloor + 1$, we have

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = n \lfloor x \rfloor + z. \tag{5.5}$$

The value of z corresponds to the number of indices i that satisfy

$$\frac{i}{n} \geq 1 - r.$$

By (5.3), it follows

$$\begin{aligned}
1 - r &\in \left(1 - \frac{j+1}{n}, 1 - \frac{j}{n}\right] \\
&= \left(\frac{n-j-1}{n}, \frac{n-j}{n}\right].
\end{aligned}$$

Thus we can determine the value of z by instead counting the number of indices i that satisfy

$$\frac{i}{n} \geq \frac{n-j}{n}.$$

Rearranging terms, we see that $i \geq n - j$ holds for $z = (n-1) - (n-j) + 1 = j$ of the n summands. Substituting the value of z into (5.5) yields

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = n \lfloor x \rfloor + j. \tag{5.6}$$

Conclusion Since (5.4) and (5.6) agree with one another, it follows identity (5.2) holds.

□

5.3 ❗ Exercise 1.11.6

Recall that a lattice point (x, y) in the plane is one whose coordinates are integers. Let f be a nonnegative function whose domain is the interval $[a, b]$, where a and b are integers, $a < b$. Let S denote the set of points (x, y) satisfying $a \leq x \leq b$, $0 < y \leq f(x)$. Prove that the number of lattice points in S is equal to the sum

$$\sum_{n=a}^{b} \lfloor f(n) \rfloor .$$

Proof. Let $i = a, \ldots, b$ and define $S_i = \mathbb{N} \cap (0, f(i)]$. By construction, the number of lattice points in S is

$$\sum_{n=a}^{b} |S_n| . \tag{5.7}$$

All that remains is to show $|S_i| = \lfloor f(i) \rfloor$. There are two cases to consider:

Case 1 Suppose $f(i)$ is an integer. Then the number of integers in $(0, f(i)]$ is $f(i) = \lfloor f(i) \rfloor$.

Case 2 Suppose $f(i)$ is not an integer. Then the number of integers in $(0, f(i)]$ is the same as that of $(0, \lfloor f(i) \rfloor]$. Once again, that number is $\lfloor f(i) \rfloor$.

Conclusion By cases 1 and 2, $|S_i| = \lfloor f(i) \rfloor$. Substituting this identity into (5.7) finishes the proof.

□

5.4 Exercise 1.11.7

If a and b are positive integers with no common factor, we have the formula

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

When $b = 1$, the sum on the left is understood to be 0.

Note: When $b = 1$, the proofs of (a) and (b) are trivial. We continue under the assumption $b > 1$.

5.4.1 ❗ Exercise 1.11.7a

Derive this result by a geometric argument, counting lattice points in a right triangle.

Proof. Let $f\colon [1, b-1] \to \mathbb{R}$ be given by $f(x) = ax/b$. Let S denote the set of points (x, y) satisfying $1 \le x \le b-1$, $0 < y \le f(x)$. By ❗ Exercise 1.11.6, the number of lattice points of S is equal to the sum

$$\sum_{n=1}^{b-1} \lfloor f(n) \rfloor = \sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor . \tag{5.8}$$

Define T to be the triangle of width $w = b$ and height $h = f(b) = a$ as

$$T = \{(x, y) : 0 < x < b, 0 < y \le f(x)\}.$$

By construction, T does not introduce any additional lattice points. Thus S and T have the same number of lattice points. Let H_L denote the number of boundary points on T's hypotenuse. We prove that (i) $H_L = 2$ and (ii) that T has $\frac{(a-1)(b-1)}{2}$ lattice points.

(i) Consider the line L overlapping the hypotenuse of T. By construction, T's hypotenuse has endpoints $(0, 0)$ and (b, a). By hypothesis, a and b are positive, excluding the possibility of L being vertical. Define the slope of L as

$$m = \frac{a}{b}.$$

H_L coincides with the number of indices $i = 0, \ldots, b$ such that $(i, i * m)$ is a lattice point. But a and b are coprime by hypothesis and $i \le b$. Thus $i * m$ is an integer if and only if $i = 0$ or $i = b$. Thus $H_L = 2$.

(ii) Next we count the number of lattice points in T. Let R be the overlapping retangle of width w and height h, situated with bottom-left corner at $(0, 0)$. Let I_R denote the number of interior lattice points of R. Let I_T and B_T denote the interior and boundary lattice points of T respectively. By ❗ Exercise 1.7.4b,

$$\begin{aligned} I_T &= \frac{1}{2}(I_R - (H_L - 2)) \\ &= \frac{1}{2}(I_R - (2 - 2)) && \text{(i)} \\ &= \frac{1}{2} I_R. \end{aligned} \tag{5.9}$$

Furthermore, since both the adjacent and opposite side of T are not included in T and there exist no lattice points on T's hypotenuse besides the endpoints, it follows

$$B_T = 0. \tag{5.10}$$

Thus the number of lattice points of T equals

$$\begin{aligned}
I_T + B_T &= I_T && (5.10)\\
&= \frac{1}{2} I_R && (5.9)\\
&= \frac{(b-1)(a-1)}{2}. && \text{Exercise 1.7.4a} \qquad (5.11)
\end{aligned}$$

Conclusion By (5.8) the number of lattice points of S is equal to the sum

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor.$$

But the number of lattice points of S is the same as that of T. By (5.11), the number of lattice points in T is equal to

$$\frac{(b-1)(a-1)}{2}.$$

Thus

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

□

5.4.2 Exercise 1.11.7b

Derive the result analytically as follows: By changing the index of summation, note that $\sum_{n=1}^{b-1} \lfloor na/b \rfloor = \sum_{n=1}^{b-1} \lfloor a(b-n)/b \rfloor$. Now apply Exercises 4(a) and (b) to the bracket on the right.

Proof. ∃ – Apostol.Chapter_1_11.exercise_7b

Let $n = 1, \ldots, b-1$. By hypothesis, a and b are coprime. Furthermore, $n < b$ for all values of n. Thus an/b is not an integer. By Exercise 1.11.4b,

$$\left\lfloor -\frac{an}{b} \right\rfloor = -\left\lfloor \frac{an}{b} \right\rfloor - 1. \qquad (5.12)$$

Consider the following:

$$\begin{aligned}
\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor &= \sum_{n=1}^{b-1} \left\lfloor \frac{a(b-n)}{b} \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor \frac{ab-an}{b} \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor -\frac{an}{b} + a \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor -\frac{an}{b} \right\rfloor + a. && \text{Exercise 1.11.4a} \\
&= \sum_{n=1}^{b-1} - \left\lfloor \frac{an}{b} \right\rfloor - 1 + a && (5.12) \\
&= -\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor - \sum_{n=1}^{b-1} 1 + \sum_{n=1}^{b-1} a \\
&= -\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor - (b-1) + a(b-1).
\end{aligned}$$

Rearranging the above yields

$$2\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor = (a-1)(b-1).$$

Dividing both sides of the above identity concludes the proof.

□

5.5 Exercise 1.11.8

Let S be a set of points on the real line. The *characteristic function* of S is, by definition, the function $\mathcal{X}_S$ such that $\mathcal{X}_S(x) = 1$ for every x in S, and $\mathcal{X}_S(x) = 0$ for those x not in S. Let f be a step function which takes the constant value c_k on the kth open subinterval I_k of some partition of an interval $[a, b]$. Prove that for each x in the union $I_1 \cup I_2 \cup \cdots \cup I_n$ we have

$$f(x) = \sum_{k=1}^{n} c_k \mathcal{X}_{I_k}(x).$$

This property is described by saying that every step function is a linear combination of characteristic functions of intervals.

Proof. Let $x \in I_1 \cup I_2 \cup \cdots \cup I_n$ and $N = \{1, \ldots, n\}$. Let $k \in N$ such that $x \in I_k$. Consider an arbitrary $j \in N - \{k\}$. By definition of a partition, $I_j \cap I_k = \emptyset$.

That is, I_j and I_k are disjoint for all $j \in N - \{k\}$. Therefore, by definition of the characteristic function, $\mathcal{X}_{I_k}(x) = 1$ and $\mathcal{X}_{I_j}(x) = 0$ for all $j \in N - \{k\}$. Thus

$$\begin{aligned} f(x) &= c_k \\ &= (c_k)(1) + \sum\nolimits_{j \in N - \{k\}} (c_j)(0) \\ &= c_k \mathcal{X}_{I_k}(x) + \sum\nolimits_{j \in N - \{k\}} c_j \mathcal{X}_{I_j}(x) \\ &= \sum_{k=1}^{n} c_k \mathcal{X}_{I_k}(x). \end{aligned}$$

$\square$