Exercises 1.11

Tom M. Apostol

Exercise 4

Prove that the greatest-integer function has the properties indicated:

⦿ Exercise 4a

$\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for every integer n.

Proof. Apostol.Chapter_1_11.exercise_4a

□

⦿ Exercise 4b

$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \text{ is an integer,} \\ -\lfloor x \rfloor - 1 & \text{otherwise.} \end{cases}$

Proof.

(a) Apostol.Chapter_1_11.exercise_4b_1

(b) Apostol.Chapter_1_11.exercise_4b_2

□

⦿ Exercise 4c

$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proof. Apostol.Chapter_1_11.exercise_4c

□

⦿ Exercise 4d

$\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$.

Proof. Apostol.Chapter_1_11.exercise_4d

□

⊙ Exercise 4e

$\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Proof. Apostol.Chapter_1_11.exercise_4e

□

⊙ Exercise 5

The formulas in Exercises 4(d) and 4(e) suggest a generalization for $\lfloor nx \rfloor$. State and prove such a generalization.

> Note: The stated generalization is known as "Hermite's Identity."

Proof. Apostol.Chapter_1_11.exercise_5

We prove that for all natural numbers n and real numbers x, the following identity holds:

$$\lfloor nx \rfloor = \sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor \tag{1}$$

By definition of the floor function, $x = \lfloor x \rfloor + r$ for some $r \in [0, 1)$. Define S as the partition of non-overlapping subintervals

$$\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), \ldots, \left[\frac{n-1}{n}, 1\right).$$

By construction, $\cup S = [0, 1)$. Therefore there exists some $j \in \mathbb{N}$ such that

$$r \in \left[\frac{j}{n}, \frac{j+1}{n}\right). \tag{2}$$

With these definitions established, we now show the left- and right-hand sides of (1) evaluate to the same number.

Left-Hand Side Consider the left-hande side of identity (1) By (2), $nr \in [j, j+1)$. Therefore $\lfloor nr \rfloor = j$. Thus

$$\begin{aligned}
\lfloor nx \rfloor &= \lfloor n(\lfloor x \rfloor + r) \rfloor \\
&= \lfloor n \lfloor x \rfloor + nr \rfloor \\
&= \lfloor n \lfloor x \rfloor \rfloor + \lfloor nr \rfloor . && \text{⊙ Exercise 4a} \\
&= \lfloor n \lfloor x \rfloor \rfloor + j \\
&= n \lfloor x \rfloor + j. && (3)
\end{aligned}$$

Right-Hand Side Now consider the right-hand side of identity (1). We note each summand, by construction, is the floor of x added to a nonnegative number less than one. Therefore each summand contributes either $\lfloor x \rfloor$ or $\lfloor x \rfloor + 1$ to the total. Letting z denote the number of summands that contribute $\lfloor x \rfloor + 1$, we have

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = n \lfloor x \rfloor + z. \tag{4}$$

The value of z corresponds to the number of indices i that satisfy

$$\frac{i}{n} \geq 1 - r.$$

By (2), it follows

$$\begin{aligned} 1 - r &\in \left(1 - \frac{j+1}{n}, 1 - \frac{j}{n}\right] \\ &= \left(\frac{n-j-1}{n}, \frac{n-j}{n}\right]. \end{aligned}$$

Thus we can determine the value of z by instead counting the number of indices i that satisfy

$$\frac{i}{n} \geq \frac{n-j}{n}.$$

Rearranging terms, we see that $i \geq n - j$ holds for $z = (n-1) - (n-j) + 1 = j$ of the n summands. Substituting the value of z into (4) yields

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = n \lfloor x \rfloor + j. \tag{5}$$

Conclusion Since (3) and (5) agree with one another, it follows identity (1) holds.

□

Exercise 6

Recall that a lattice point (x, y) in the plane is one whose coordinates are integers. Let f be a nonnegative function whose domain is the interval $[a, b]$, where a and b are integers, $a < b$. Let S denote the set of points (x, y) satisfying $a \leq x \leq b$, $0 < y \leq f(x)$. Prove that the number of lattice points in S is equal to the sum

$$\sum_{n=a}^{b} \lfloor f(n) \rfloor.$$

Proof. Define $S_i = \mathbb{Z} \cap (0, f(i)]$ for all $i \in \mathbb{Z}$. By definition, the set of lattice points of S is given by

$$L = \{(i, j) : i = a, \dots, b \wedge j \in S_i\}.$$

By construction, it follows

$$\sum_{j \in S_i} 1 = \lfloor f(i) \rfloor .$$

Therefore

$$|L| = \sum_{i=a}^{b} \sum_{j \in S_i} 1 = \sum_{i=1}^{b} \lfloor f(i) \rfloor .$$

□

Exercise 7

If a and b are positive integers with no common factor, we have the formula

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

When $b = 1$, the sum on the left is understood to be 0.

Exercise 7a

Derive this result by a geometric argument, counting lattice points in a right triangle.

Proof. TODO

□

Exercise 7b

Derive the result analytically as follows: By changing the index of summation, note that $\sum_{n=1}^{b-1} \lfloor na/b \rfloor = \sum_{n=1}^{b-1} \lfloor a(b-n)/b \rfloor$. Now apply Exercises 4(a) and (b) to the bracket on the right.

Proof. Apostol.Chapter_1_11.exercise_7b

□

❗ Exercise 8

Let S be a set of points on the real line. The *characteristic function* of S is, by definition, the function χ_S such that $\chi_S(x) = 1$ for every x in S, and $\chi_S(x) = 0$ for those x not in S. Let f be a step function which takes the constant value c_k on the kth open subinterval I_k of some partition of an interval $[a, b]$. Prove that for each x in the union $I_1 \cup I_2 \cup \cdots \cup I_n$ we have

$$f(x) = \sum_{k=1}^{n} c_k \chi_{I_k}(x).$$

This property is described by saying that every step function is a linear combination of characteristic functions of intervals.

Proof. TODO

□