Elements of Set Theory

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Chapter R

Reference

R.1 ¶ Empty Set Axiom

There is a set having no members:

$$\exists B, \forall x, x \notin B.$$

Axiom. ∃ – Set.emptyCollection

□

R.2 ¶ Extensionality Axiom

If two sets have exactly the same members, then they are equal:

$$\forall A, \forall B, [\forall x, (x \in A \iff x \in B) \Rightarrow A = B].$$

Axiom. ∃ – Set.ext

□

R.3 ¶ Pair Set

For any sets u and v, the **pair set** $\{u, v\}$ is the set whose only members are u and v.

Definition.

∃ – Set.insert
∃ – Set.singleton

□

R.4 ¶ Pairing Axiom

For any sets u and v, there is a set having as members just u and v:

$$\forall u, \forall v, \exists B, \forall x, (x \in B \iff x = u \text{ or } x = v).$$

Axiom.

∃ – Set.insert
∃ – Set.singleton

□

R.5 ¶ Power Set

For any set a, the **power set** $\mathscr{P}a$ is the set whose members are exactly the subsets of a.

Definition. ∃ – Set.powerset

□

R.6 ¶ Power Set Axiom

For any set a, there is a set whose members are exactly the subsets of a:

$$\forall a, \exists B, \forall x, (x \in B \iff x \subseteq a).$$

Axiom. ∃ – Set.powerset

□

R.7 ¶ Subset Axioms

For each formula ϕ not containing B, the following is an axiom:

$$\forall t_1, \cdots \forall t_k, \forall c, \exists B, \forall x, (x \in B \iff x \in c \land \phi).$$

Axiom. ∃ – Set.Subset

□

R.8 ¶ Union Axiom

For any set A, there exists a set B whose elements are exactly the members of the members of A:

$$\forall A, \exists B, \forall x \, [x \in B \iff (\exists b \in A) x \in b]$$

Axiom. $\exists$ − Set.sUnion

□

R.9 ¶ Union Axiom, Preliminary Form

For any sets a and b, there is a set whose members are those sets belonging either to a or to b (or both):

$$\forall a, \forall b, \exists B, \forall x, (x \in B \iff x \in a \text{ or } x \in b).$$

Axiom. $\exists$ − Set.union

□

Chapter 1

Introduction

1.1 Baby Set Theory

1.1.1 Exercise 1.1

Which of the following become true when "$\in$" is inserted in place of the blank? Which become true when "$\subseteq$" is inserted?

Exercise 1.1a

$\{\varnothing\}$ ____ $\{\varnothing, \{\varnothing\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1a

Because the *object* $\{\varnothing\}$ is a member of the right-hand set, the statement is **true** in the case of "$\in$".

Because the *members* of $\{\varnothing\}$ are all members of the right-hand set, the statement is also **true** in the case of "$\subseteq$".

□

Exercise 1.1b

$\{\varnothing\}$ ____ $\{\varnothing, \{\{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1b

Because the *object* $\{\varnothing\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\varnothing\}$ are all members of the right-hand set, the statement is **true** in the case of "$\subseteq$".

□

Exercise 1.1c

$\{\{\varnothing\}\}\_\_\_\_\{\varnothing, \{\varnothing\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1c

Because the *object* $\{\{\varnothing\}\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are all members of the right-hand set, the statement is **true** in the case of "$\subseteq$".

□

Exercise 1.1d

$\{\{\varnothing\}\}\_\_\_\_\{\varnothing, \{\{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1d

Because the *object* $\{\{\varnothing\}\}$ is a member of the right-hand set, the statement is **true** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are not all members of the right-hand set, the statement is **false** in the case of "$\subseteq$".

□

Exercise 1.1e

$\{\{\varnothing\}\}\_\_\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1e

Because the *object* $\{\{\varnothing\}\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are not all members of the right-hand set, the statement is **false** in the case of "$\subseteq$".

□

1.1.2 Exercise 1.2

Show that no two of the three sets $\varnothing$, $\{\varnothing\}$, and $\{\{\varnothing\}\}$ are equal to each other.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_2

By the ¶ Extensionality Axiom, $\varnothing$ is only equal to $\varnothing$. This immediately shows it is not equal to the other two. Now consider object $\varnothing$. This object is a member of $\{\varnothing\}$ but is not a member of $\{\{\varnothing\}\}$. Again, by the ¶ Extensionality Axiom, these two sets must be different.

□

1.1.3 Exercise 1.3

Show that if $B \subseteq C$, then $\mathscr{P}B \subseteq \mathscr{P}C$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_3

Let $x \in \mathscr{P}B$. By definition of the ¶ Power Set, x is a subset of B. By hypothesis, $B \subseteq C$. Then $x \subseteq C$. Again by definition of the ¶ Power Set, it follows $x \in \mathscr{P}C$.

□

1.1.4 Exercise 1.4

Assume that x and y are members of a set B. Show that $\{\{x\}, \{x, y\}\} \in \mathscr{P}\mathscr{P}B$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_4

Let x and y be members of set B. Then $\{x\}$ and $\{x, y\}$ are subsets of B. By definition of the ¶ Power Set, $\{x\}$ and $\{x, y\}$ are members of $\mathscr{P}B$. Then $\{\{x\}, \{x, y\}\}$ is a subset of $\mathscr{P}B$. By definition of the ¶ Power Set, $\{\{x\}, \{x, y\}\}$ is a member of $\mathscr{P}\mathscr{P}B$.

□

1.2 Sets - An Informal View

1.2.1 Exercise 2.1

Define the rank of a set c to be the least α such that $c \subseteq V_\alpha$. Compute the rank of $\{\{\varnothing\}\}$. Compute the rank of $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.

Proof. We first compute the values of V_n for $0 \leq n \leq 3$ under the assumption the set of atoms A at the bottom of the hierarchy is empty.

$$\begin{aligned}
V_0 &= \varnothing \\
V_1 &= V_0 \cup \mathscr{P} V_0 \\
&= \varnothing \cup \{\varnothing\} \\
&= \{\varnothing\} \\
V_2 &= V_1 \cup \mathscr{P} V_1 \\
&= \{\varnothing\} \cup \mathscr{P}\{\varnothing\} \\
&= \{\varnothing\} \cup \{\varnothing, \{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\} \\
V_3 &= V_2 \cup \mathscr{P} V_2 \\
&= \{\varnothing, \{\varnothing\}\} \cup \mathscr{P}\{\varnothing, \{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\} \cup \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
&= \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}
\end{aligned}$$

It then immediately follows $\{\{\varnothing\}\}$ has rank 2 and $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ has rank 3.

□

1.2.2 Exercise 2.2

We have stated that $V_{\alpha+1} = A \cup \mathscr{P} V_\alpha$. Prove this at least for $\alpha < 3$.

Proof. Let A be the set of atoms in our set hierarchy. Let $P(n)$ be the predicate, "$V_{n+1} = A \cup \mathscr{P} V_n$." We prove $P(n)$ holds true for all natural numbers $n \geq 1$ via induction.

Base Case Let $n = 1$. By definition, $V_1 = V_0 \cup \mathscr{P} V_0$. By definition, $V_0 = A$. Therefore $V_1 = A \cup \mathscr{P} V_0$. This proves $P(1)$ holds true.

Induction Step Suppose $P(n)$ holds true for some $n \geq 1$. Consider V_{n+1}. By definition, $V_{n+1} = V_n \cup \mathscr{P} V_n$. Therefore, by the induction hypothesis,

$$\begin{aligned}
V_{n+1} &= V_n \cup \mathscr{P} V_n \\
&= (A \cup \mathscr{P} V_{n-1}) \cup \mathscr{P} V_n \\
&= A \cup (\mathscr{P} V_{n-1} \cup \mathscr{P} V_n) \qquad (1.1)
\end{aligned}$$

But V_{n-1} is a subset of V_n. Exercise 1.3 then implies $\mathscr{P} V_{n-1} \subseteq \mathscr{P} V_n$. This means (1.1) can be simplified to

$$V_{n+1} = A \cup \mathscr{P} V_n,$$

proving $P(n+1)$ holds true.

Conclusion By mathematical induction, it follows for all $n \geq 1$, $P(n)$ is true. $\square$

1.2.3 Exercise 2.3

List all the members of V_3. List all the members of V_4. (It is to be assumed here that there are no atoms.)

Proof. As seen in the proof of Exercise 2.1,

$$V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}.$$

By Exercise 2.2, $V_4 = \mathscr{P}V_3$ (since it is assumed there are no atoms). Thus

$$\begin{aligned} V_4 = \{ & \\ & \varnothing, \\ & \{\varnothing\}, \\ & \{\{\varnothing\}\}, \\ & \{\{\{\varnothing\}\}\}, \\ & \{\{\varnothing, \{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing\}\}, \\ & \{\varnothing, \{\{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\{\varnothing\}, \{\{\varnothing\}\}\}, \\ & \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\ & \{\{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\ \}. & \end{aligned}$$

$\square$

Chapter 2

Axioms and Operations

2.1 Axioms

2.1.1 ✎ Theorem 2A

Theorem 2A. *There is no set to which every set belongs.*

> *Note: This was revisited after reading Enderton's proof prior.*

Proof. Let A be an arbitrary set. Define $B = \{x \in A \mid x \notin x\}$. By the ¶ Subset Axioms, B is a set. Then

$$B \in B \iff B \in A \wedge B \notin B.$$

If $B \in A$, then $B \in B \iff B \notin B$, a contradiction. Thus $B \notin A$. Since this process holds for any set A, there must exist no set to which every set belongs. □

2.1.2 ✎ Theorem 2B

Theorem 2B. *For any nonempty set A, there exists a unique set B such that for any x,*

$$x \in B \iff x \text{ belongs to every member of } A.$$

Proof. Suppose A is a nonempty set. This ensures the statement we are trying to prove does not vacuously hold for all sets x (which would yield a contradiction due to ✎ Theorem 2B). By the ¶ Union Axiom, $\bigcup A$ is a set. Define

$$B = \{x \in \bigcup A \mid (\forall b \in A), x \in b\}.$$

By the ¶ Subset Axioms, B is indeed a set. By construction,

$$\forall x, x \in B \iff x \text{ belongs to every member of } A.$$

By the ¶ Extensionality Axiom, B is unique.

$\square$

2.2 Exercises 3

2.2.1 Exercise 3.1

Assume that A is the set of integers divisible by 4. Similarly assume that B and C are the sets of integers divisible by 9 and 10, respectively. What is in $A \cap B \cap C$?

Answer. ∃ – Enderton.Set.Chapter_1.exercise_3_1

The set of integers divisible by 4, 9, and 10.

$\square$

2.2.2 Exercise 3.2

Give an example of sets A and B for which $\bigcup A = \bigcup B$ but $A \neq B$.

Answer. ∃ – Enderton.Set.Chapter_1.exercise_3_2

Let $A = \{\{1\}, \{2\}\}$ and $B = \{\{1, 2\}\}$.

$\square$

2.2.3 Exercise 3.3

Show that every member of a set A is a subset of $\bigcup A$. (This was stated as an example in this section.)

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_3

Let $x \in A$. By definition,

$$\bigcup A = \{y \mid (\exists b \in A) y \in b\}.$$

Then $\{y \mid y \in x\} \subseteq \bigcup A$. But $\{y \mid y \in x\} = x$. Thus $x \subseteq \bigcup A$.

$\square$

2.2.4 ✓ Exercise 3.4

Show that if $A \subseteq B$, then $\bigcup A \subseteq \bigcup B$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_4

Let A and B be sets such that $A \subseteq B$. Let $x \in \bigcup A$. By definition of the union, there exists some $b \in A$ such that $x \in b$. By definition of the subset, $b \in B$. This immediatley implies $x \in \bigcup B$. Since this holds for all $x \in \bigcup A$, it follows $\bigcup A \subseteq \bigcup B$.

□

2.2.5 ✓ Exercise 3.5

Assume that every member of $\mathscr{A}$ is a subset of B. Show that $\bigcup \mathscr{A} \subseteq B$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_5

Let $x \in \bigcup \mathscr{A}$. By definition,

$$\bigcup \mathscr{A} = \{y \mid (\exists b \in A) y \in b\}.$$

Then there exists some $b \in A$ such that $x \in b$. By hypothesis, $b \subseteq B$. Thus x must also be a member of B. Since this holds for all $x \in \bigcup \mathscr{A}$, it follows $\bigcup \mathscr{A} \subseteq B$.

□

2.2.6 ✓ Exercise 3.6a

Show that for any set A, $\bigcup \mathscr{P} A = A$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_6a

We prove that (i) $\bigcup \mathscr{P} A \subseteq A$ and (ii) $A \subseteq \bigcup \mathscr{P} A$.

(i) By definition, the ¶ Power Set of A is the set of all subsets of A. In other words, every member of $\mathscr{P} A$ is a subset of A. By ✓ Exercise 3.5, $\bigcup \mathscr{P} A \subseteq A$.

(ii) Let $x \in A$. By definition of the power set of A, $\{x\} \in \mathscr{P} A$. By definition of the union,

$$\bigcup \mathscr{P} A = \{y \mid (\exists b \in \mathscr{P} A), y \in b).$$

Since $x \in \{x\}$ and $\{x\} \in \mathscr{P} A$, it follows $x \in \bigcup \mathscr{P} A$. Thus $A \subseteq \bigcup \mathscr{P} A$.

Conclusion By (i) and (ii), $\bigcup \mathscr{P} A = A$.

$\square$

2.2.7 Exercise 3.6b

Show that $A \subseteq \mathscr{P} \bigcup A$. Under what conditions does equality hold?

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_6b

Let $x \in A$. By Exercise 3.3, x is a subset of $\bigcup A$. By the definition of the ¶ Power Set,

$$\mathscr{P} \bigcup A = \{y \mid y \subseteq \bigcup A\}.$$

Therefore $x \in \mathscr{P} \bigcup A$. Since this holds for all $x \in A$, $A \subseteq \mathscr{P} \bigcup A$.

♠ ♠ ♠

We show equality holds if and only if there exists some set B such that $A = \mathscr{P} B$.

(⇒) Suppose $A = \mathscr{P} \bigcup A$. Then our statement immediately follows by settings $B = \bigcup A$.

(⇐) Suppose there exists some set B such that $A = \mathscr{P} B$. Therefore

$$\begin{aligned} \mathscr{P} \bigcup A &= \mathscr{P} \left(\bigcup \mathscr{P} B \right) \\ &= \mathscr{P} B && \text{Exercise 3.6a} \\ &= A. \end{aligned}$$

Conclusion By (⇒) and (⇐), $A = \mathscr{P} \bigcup A$ if and only if there exists some set B such that $A = \mathscr{P} B$.

$\square$

2.2.8 Exercise 3.7a

Show that for any sets A and B,

$$\mathscr{P} A \cap \mathscr{P} B = \mathscr{P}(A \cap B).$$

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_7a

Let A and B be arbitrary sets. We show that $\mathscr{P} A \cap \mathscr{P} B \subseteq \mathscr{P}(A \cap B)$ and then show that $\mathscr{P} A \cap \mathscr{P} B \supseteq \mathscr{P}(A \cap B)$.

(⊆) Let $x \in \mathscr{P}A \cap \mathscr{P}B$. That is, $x \in \mathscr{P}A$ and $x \in \mathscr{P}B$. By the definition of the ¶ Power Set,

$$\mathscr{P}A = \{y \mid y \subseteq A\}$$
$$\mathscr{P}B = \{y \mid y \subseteq B\}$$

Thus $x \subseteq A$ and $x \subseteq B$, meaning $x \subseteq A \cap B$. But then $x \in \mathscr{P}(A \cap B)$, the set of all subsets of $A \cap B$. Since this holds for all $x \in \mathscr{P}A \cap \mathscr{P}B$, it follows

$$\mathscr{P}A \cap \mathscr{P}B \subseteq \mathscr{P}(A \cap B).$$

(⊇) Let $x \in \mathscr{P}(A \cap B)$. By the definition of the ¶ Power Set,

$$\mathscr{P}(A \cap B) = \{y \mid y \subseteq A \cap B\}.$$

Thus $x \subseteq A \cap B$, meaning $x \subseteq A$ and $x \subseteq B$. But this implies $x \in \mathscr{P}A$, the set of all subsets of A. Likewise $x \in \mathscr{P}B$, the set of all subsets of B. Thus $x \in \mathscr{P}A \cap \mathscr{P}B$. Since this holds for all $x \in \mathscr{P}(A \cap B)$, it follows

$$\mathscr{P}(A \cap B) \subseteq \mathscr{P}A \cap \mathscr{P}B.$$

Conclusion Since each side of our identity is a subset of the other,

$$\mathscr{P}(A \cap B) = \mathscr{P}A \cap \mathscr{P}B.$$

□

2.2.9 Exercise 3.7b

Show that $\mathscr{P}A \cup \mathscr{P}B \subseteq \mathscr{P}(A \cup B)$. Under what conditions does equality hold?

Proof.

∃ – Enderton.Set.Chapter_1.exercise_3_7b_i
∃ – Enderton.Set.Chapter_1.exercise_3_7b_ii

Let $x \in \mathscr{P}A \cup \mathscr{P}B$. By definition, $x \in \mathscr{P}A$ or $x \in \mathscr{P}B$ (or both). By the definition of the ¶ Power Set,

$$\mathscr{P}A = \{y \mid y \subseteq A\}$$
$$\mathscr{P}B = \{y \mid y \subseteq B\}.$$

Thus $x \subseteq A$ or $x \subseteq B$. Therefore $x \subseteq A \cup B$. But then $x \in \mathscr{P}(A \cup B)$, the set of all subsets of $A \cup B$.

♠♠♠

We show equality holds if and only if one of A or B is a subset of the other.

(⇒) Suppose
$$\mathscr{P}A \cup \mathscr{P}B = \mathscr{P}(A \cup B). \tag{2.1}$$
By the definition of the ¶ Power Set, $A \cup B \in \mathscr{P}(A \cup B)$. Then (2.1) implies $A \cup B \in \mathscr{P}A \cup \mathscr{P}B$. That is, $A \cup B \in \mathscr{P}A$ or $A \cup B \in \mathscr{P}B$ (or both).

For the sake of contradiction, suppose $A \not\subseteq B$ and $B \not\subseteq A$. Then there exists an element $x \in A$ such that $x \notin B$ and there exists an element $y \in B$ such that $y \notin A$. But then $A \cup B \notin \mathscr{P}A$ since y cannot be a member of a member of $\mathscr{P}A$. Likewise, $A \cup B \notin \mathscr{P}B$ since x cannot be a member of a member of $\mathscr{P}B$. Therefore our assumption is incorrect. In other words, $A \subseteq B$ or $B \subseteq A$.

(⇐) WLOG, suppose $A \subseteq B$. Then, by Exercise 1.3, $\mathscr{P}A \subseteq \mathscr{P}B$. Thus
$$\begin{aligned}\mathscr{P}A \cup \mathscr{P}B &= \mathscr{P}B \\ &= \mathscr{P}A \cup B.\end{aligned}$$

Conclusion By (⇒) and (⇐), it follows $\mathscr{P}A \cup \mathscr{P}B \subseteq \mathscr{P}(A \cup B)$ if and only if $A \subseteq B$ or $B \subseteq A$.

□

2.2.10 Exercise 3.8

Show that there is no set to which every singleton (that is, every set of the form $\{x\}$) belongs. [*Suggestion*: Show that from such a set, we could construct a set to which every set belonged.]

Proof. We proceed by contradiction. Suppose there existed a set A consisting of every singleton. Then the ¶ Union Axiom suggests $\bigcup A$ is a set. But this set is precisely the class of all sets, which is *not* a set. Thus our original assumption was incorrect. That is, there is no set to which every singleton belongs.

□

2.2.11 Exercise 3.9

Give an example of sets a and B for which $a \in B$ but $\mathscr{P}a \notin \mathscr{P}B$.

Answer. ∃ – Enderton.Set.Chapter_1.exercise_3_9

Let $a = \{1\}$ and $B = \{\{1\}\}$. Then
$$\begin{aligned}\mathscr{P}a &= \{\varnothing, \{1\}\} \\ \mathscr{P}B &= \{\varnothing, \{\{1\}\}\}.\end{aligned}$$
It immediately follows that $\mathscr{P}a \notin \mathscr{P}B$.

□

2.2.12 ✓ Exercise 3.10

Show that if $a \in B$, then $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$. [*Suggestion*: If you need help, look in the Appendix.]

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_10

Suppose $a \in B$. By ✓ Exercise 3.3, $a \subseteq \bigcup B$. By ✓ Exercise 1.3, $\mathscr{P}a \subseteq \mathscr{P}\bigcup B$. By the definition of the ¶ Power Set,

$$\mathscr{P}\mathscr{P}\bigcup B = \{y \mid y \subseteq \mathscr{P}\bigcup B\}.$$

Therefore $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$.

□

2.3 Algebra of Sets

2.3.1 ✓ Commutative Laws

For any sets A and B,

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}$$

Proof.

∃ – Set.union_comm
∃ – Set.inter_comm

Let A and B be sets. We show (i) $A \cup B = B \cup A$ and then (ii) $A \cap B = B \cap A$.

(i) By the definition of the union of sets,

$$\begin{aligned} A \cup B &= \{x \mid x \in A \lor x \in B\} \\ &= \{x \mid x \in B \lor x \in A\} \\ &= B \cup A. \end{aligned}$$

(ii) By the definition of the intersection of sets,

$$\begin{aligned} A \cap B &= \{x \mid x \in A \land x \in B\} \\ &= \{x \mid x \in B \land x \in A\} \\ &= B \land A. \end{aligned}$$

□

2.3.2 Associative Laws

For any sets A, B and C,

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Proof.

∃ – Set.union_assoc
∃ – Set.inter_assoc

Let A, B, and C be sets. We show (i) $A \cup (B \cup C) = (A \cup B) \cup C$ and then (ii) $A \cap (B \cap C) = (A \cap B) \cap C$.

(i) By the definition of the union of sets,

$$\begin{aligned} A \cup (B \cup C) &= \{x \mid x \in A \lor x \in (B \cup C)\} \\ &= \{x \mid x \in A \lor x \in \{y \mid y \in B \lor C\}\} \\ &= \{x \mid x \in A \lor (x \in B \lor x \in C)\} \\ &= \{x \mid (x \in A \lor x \in B) \lor x \in C\} \\ &= \{x \mid x \in \{y \mid y \in A \lor y \in B\} \lor x \in C\} \\ &= \{x \mid x \in (A \cup B) \lor x \in C\} \\ &= (A \cup B) \cup C. \end{aligned}$$

(ii) By the definition of the intersection of sets,

$$\begin{aligned} A \cap (B \cap C) &= \{x \mid x \in A \land x \in (B \cap C)\} \\ &= \{x \mid x \in A \land x \in \{y \mid y \in B \land y \in C\}\} \\ &= \{x \mid x \in A \land (x \in B \land x \in C)\} \\ &= \{x \mid (x \in A \land x \in B) \land x \in C\} \\ &= \{x \mid x \in \{y \mid y \in A \land y \in B\} \land x \in C\} \\ &= \{x \mid x \in (A \cap B) \land x \in C\} \\ &= (A \cap B) \cap C. \end{aligned}$$

□

2.3.3 Distributive Laws

For any sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof.

∃ – Set.inter_distrib_left
∃ – Set.union_distrib_left

Let A, B, and C be sets. We show (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and then (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(i) By the definition of the union and intersection of sets,

$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A \wedge x \in B \cup C\} \\ &= \{x \mid x \in A \wedge x \in \{y \mid y \in B \vee y \in C\}\} \\ &= \{x \mid x \in A \wedge (x \in B \vee x \in C)\} \\ &= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\ &= \{x \mid x \in A \cap B \vee x \in A \cap C\} \\ &= (A \cap B) \cup (A \cap C). \end{aligned}$$

(ii) By the definition of the union and intersection of sets,

$$\begin{aligned} A \cup (B \cap C) &= \{x \mid x \in A \vee x \in B \cap C\} \\ &= \{x \mid x \in A \vee x \in \{y \mid y \in B \wedge y \in C\}\} \\ &= \{x \mid x \in A \vee (x \in B \wedge x \in C)\} \\ &= \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} \\ &= \{x \mid x \in A \cup B \wedge x \in A \cup C\} \\ &= (A \cup B) \cap (A \cup C). \end{aligned}$$

□

2.3.4 De Morgan's Laws

For any sets A, B, and C,

$$\begin{aligned} C - (A \cup B) &= (C - A) \cap (C - B) \\ C - (A \cap B) &= (C - A) \cup (C - B) \end{aligned}$$

Proof.

∃ – Set.diff_inter_diff
∃ – Set.diff_inter

Let A, B, and C be sets. We show (i) $C - (A \cup B) = (C - A) \cap (C - B)$ and then (ii) $C - (A \cap B) = (C - A) \cup (C - B)$.

(i) By definition of the union, intersection, and relative complements of sets,

$$\begin{aligned}
C - (A \cup B) &= \{x \mid x \in C \land x \notin A \cup B\} \\
&= \{x \mid x \in C \land x \notin \{y \mid y \in A \lor y \in B\}\} \\
&= \{x \mid x \in C \land \neg(x \in A \lor x \in B)\} \\
&= \{x \mid x \in C \land (x \notin A \land x \notin B)\} \\
&= \{x \mid (x \in C \land x \notin A) \land (x \in C \land x \notin B)\} \\
&= \{x \mid x \in (C - A) \land x \in (C - B)\} \\
&= (C - A) \cap (C - B).
\end{aligned}$$

(ii) By definition of the union, intersection, and relative complements of sets,

$$\begin{aligned}
C - (A \cap B) &= \{x \mid x \in C \land x \notin A \cap B\} \\
&= \{x \mid x \in C \land x \notin \{y \mid y \in A \land y \in B\}\} \\
&= \{x \mid x \in C \land \neg(x \in A \land x \in B)\} \\
&= \{x \mid x \in C \land (x \notin A \lor x \notin B)\} \\
&= \{x \mid (x \in C \land x \notin A) \lor (x \in C \land x \notin B)\} \\
&= \{x \mid x \in (C - A) \lor x \in (C - B)\} \\
&= (C - A) \cup (C - B).
\end{aligned}$$

□

2.3.5 Identities Involving $\varnothing$

For any set A,

$$\begin{aligned}
A \cup \varnothing &= A \\
A \cap \varnothing &= \varnothing \\
A \cap (C - A) &= \varnothing
\end{aligned}$$

Proof.

∃ – Set.union_empty
∃ – Set.inter_empty
∃ – Set.inter_diff_self

Let A be an arbitrary set. We prove (i) that $A \cup \varnothing = A$, (ii) $A \cap \varnothing = \varnothing$, and (iii) $A \cap (C - A) = \varnothing$.

(i) By definition of the emptyset and union of sets,

$$\begin{aligned} A \cup \varnothing &= \{x \mid x \in A \vee x \in \varnothing\} \\ &= \{x \mid x \in A \vee F\} \\ &= \{x \mid x \in A\} \\ &= A. \end{aligned}$$

(ii) By definition of the emptyset and intersection of sets,

$$\begin{aligned} A \cap \varnothing &= \{x \mid x \in A \wedge x \in \varnothing\} \\ &= \{x \mid x \in A \wedge F\} \\ &= \{x \mid F\} \\ &= \{x \mid x \neq x\} \\ &= \varnothing. \end{aligned}$$

(iii) By definition of the emptyset, and the intersection and relative complement of sets,

$$\begin{aligned} A \cap (C - A) &= \{x \mid x \in A \wedge x \in C - A\} \\ &= \{x \mid x \in A \wedge x \in \{y \mid y \in C \wedge y \notin A\}\} \\ &= \{x \mid x \in A \wedge (x \in C \wedge x \notin A)\} \\ &= \{x \mid x \in C \wedge F\} \\ &= \{x \mid F\} \\ &= \{x \mid x \neq x\} \\ &= \varnothing. \end{aligned}$$

□

2.3.6 Monotonicity

For any sets A, B, and C,

$$\begin{aligned} A \subseteq B &\Rightarrow A \cup C \subseteq B \cup C \\ A \subseteq B &\Rightarrow A \cap C \subseteq B \cap C \\ A \subseteq B &\Rightarrow \bigcup A \subseteq \bigcup B \end{aligned}$$

Proof. TODO

□

2.3.7 ❗ Anti-monotonicity

For any sets A, B, and C,

$$A \subseteq B \Rightarrow C - B \subseteq C - A$$
$$\varnothing \neq A \subseteq B \Rightarrow \bigcap B \subseteq \bigcap A.$$

Proof. TODO ☐

2.3.8 ❗ General Distributive Laws

For any sets A and $\mathscr{B}$,

$$A \cup \bigcap \mathscr{B} = \bigcap \{A \cup X \mid X \in \mathscr{B}\} \quad \text{for} \quad \mathscr{B} \neq \varnothing$$
$$A \cap \bigcup \mathscr{B} = \bigcup \{A \cap X \mid X \in \mathscr{B}\}$$

Proof. TODO ☐

2.3.9 ❗ General De Morgan's Laws

For any set C and $\mathscr{A} \neq \varnothing$,

$$C - \bigcup \mathscr{A} = \bigcap \{C - X \mid X \in \mathscr{A}\}$$
$$C - \bigcap \mathscr{A} = \bigcup \{C - X \mid X \in \mathscr{A}\}$$

Proof. TODO ☐

2.4 Exercises 4

2.4.1 ❗ Exercise 4.11

Show that for any sets A and B,

$$A = (A \cap B) \cup (A - B) \quad \text{and} \quad A \cup (B - A) = A \cup B.$$

Proof. TODO ☐

2.4.2 Exercise 4.12

Verify the following identity (one of De Morgan's laws):

$$C - (A \cap B) = (C - A) \cup (C - B).$$

Proof. TODO ☐

2.4.3 Exercise 4.13

Show that if $A \subseteq B$, then $C - B \subseteq C - A$.

Proof. TODO ☐

2.4.4 Exercise 4.14

Show by example that for some sets A, B, and C, the set $A-(B-C)$ is different from $(A - B) - C$.

Proof. TODO ☐

2.4.5 Exercise 4.15

Define the symmetric difference $A + B$ of sets A and B to be the set $(A - B) \cup (B - A)$.

Exercise 4.15a

Show that $A \cap (B + C) = (A \cap B) + (A \cap C)$.

Proof. TODO ☐

Exercise 4.15b

Show that $A + (B + C) = (A + B) + C$.

Proof. TODO

$\square$

2.4.6 Exercise 4.16

Simplify:

$$[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A].$$

Proof. TODO

$\square$

2.4.7 Exercise 4.17

Show that the following four conditions are equivalent.

(a) $A \subseteq B$,

(b) $A - B = \varnothing$,

(c) $A \cup B = B$,

(d) $A \cap B = A$.

Proof. TODO

$\square$

2.4.8 Exercise 4.18

Assume that A and B are subsets of S. List all of the different sets that can be made from these three by use of the binary operations $\cup$, $\cap$, and $-$.

Proof. TODO

$\square$

2.4.9 Exercise 4.19

Is $\mathscr{P}(A - B)$ always equal to $\mathscr{P}A - \mathscr{P}B$? Is it ever equal to $\mathscr{P}A - \mathscr{P}B$?

Proof. TODO

$\square$

2.4.10 Exercise 4.20

Let A, B, and C be sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.

Proof. TODO $\square$

2.4.11 Exercise 4.21

Show that $\bigcup(A \cup B) = \bigcup A \cup \bigcup B$.

Proof. TODO $\square$

2.4.12 Exercise 4.22

Show that if A and B are nonempty sets, then $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$.

Proof. TODO $\square$

2.4.13 Exercise 4.23

Show that if $\mathscr{B}$ is nonempty, then $A \cup \bigcap \mathscr{B} = \bigcap \{A \cup X \mid X \in \mathscr{B}\}$.

Proof. TODO $\square$

2.4.14 Exercise 4.24a

Show that if $\mathscr{A}$ is nonempty, then $\mathscr{P} \bigcap A = \bigcap \{\mathscr{P}X \mid X \in \mathscr{A}\}$.

Proof. TODO $\square$

2.4.15 Exercise 4.24b

Show that

$$\bigcup \{\mathscr{P}X \mid X \in \mathscr{A}\} \subseteq \mathscr{P} \bigcup A.$$

Under what conditions does equality hold?

Proof. TODO

□

2.4.16 Exercise 4.25

Is $A \cup \bigcup \mathscr{B}$ always the same as $\bigcup \{A \cup X \mid X \in \mathscr{B}\}$? If not, then under what conditions does equality hold?

Proof. TODO

□