A Set of Axioms for the Real-Number System

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Lemma 1

Nonempty set S has supremum L if and only if set $-S$ has infimum $-L$.

Proof. Apostol.Chapter_I_03.is_lub_neg_set_iff_is_glb_set_neg

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Suppose $L = \sup S$ and fix $x \in S$. By definition of the supremum, $x \leq L$ and L is the smallest value satisfying this inequality. Negating both sides of the inequality yields $-x \geq -L$. Furthermore, $-L$ must be the largest value satisfying this inequality. Therefore $-L = \inf -S$.

□

Theorem I.27

Every nonempty set S that is bounded below has a greatest lower bound; that is, there is a real number L such that $L = \inf S$.

Proof. Apostol.Chapter_I_03.exists_isGLB

Let S be a nonempty set bounded below by x. Then $-S$ is nonempty and bounded above by x. By the completeness axiom, there exists a supremum L of $-S$. By Lemma 1, L is a supremum of $-S$ if and only if $-L$ is an infimum of S.

□

Theorem I.29

For every real x there exists a positive integer n such that $n > x$.

Proof. Apostol.Chapter_I_03.exists_pnat_geq_self

Let $n = |\lceil x \rceil| + 1$. It is trivial to see n is a positive integer satisfying $n \geq 1$. Thus all that remains to be shown is that $n > x$. If x is nonpositive, $n > x$ immediately follows from $n \geq 1$. If x is positive,

$$x = |x| \leq |\lceil x \rceil| < |\lceil x \rceil| + 1 = n.$$

□

Theorem I.30

If $x > 0$ and if y is an arbitrary real number, there exists a positive integer n such that $nx > y$.

Note: This is known as the "Archimedean Property of the Reals."

Proof. Apostol.Chapter_I_03.exists_pnat_mul_self_geq_of_pos

Let $x > 0$ and y be an arbitrary real number. By Theorem I.29, there exists a positive integer n such that $n > y/x$. Multiplying both sides of the inequality yields $nx > y$ as expected.

□

Theorem I.31

If three real numbers a, x, and y satisfy the inequalities

$$a \leq x \leq a + \frac{y}{n}$$

for every integer $n \geq 1$, then $x = a$.

Proof. Apostol.Chapter_I_03.forall_pnat_leq_self_leq_frac_imp_eq

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By the trichotomy of the reals, there are three cases to consider:

Case 1 Suppose $x = a$. Then we are immediately finished.

Case 2 Suppose $x < a$. But by hypothesis, $a \leq x$. Thus $a < a$, a contradiction.

Case 3 Suppose $x > a$. Then there exists some $c > 0$ such that $a + c = x$. By Theorem I.30, there exists an integer $n > 0$ such that $nc > y$. Rearranging terms, we see $y/n < c$. Therefore $a + y/n < a + c = x$. But by hypothesis, $x \leq a + y/n$. Thus $a + y/n < a + y/n$, a contradiction.

Conclusion Since these cases are exhaustive and both case 2 and 3 lead to contradictions, $x = a$ is the only possibility.

□

Lemma 2

If three real numbers a, x, and y satisfy the inequalities

$$a - y/n \leq x \leq a$$

for every integer $n \geq 1$, then $x = a$.

Proof. Apostol.Chapter_I_03.forall_pnat_frac_leq_self_leq_imp_eq

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By the trichotomy of the reals, there are three cases to consider:

Case 1 Suppose $x = a$. Then we are immediately finished.

Case 2 Suppose $x < a$. Then there exists some $c > 0$ such that $x = a - c$. By Theorem I.30, there exists an integer $n > 0$ such that $nc > y$. Rearranging terms, we see that $y/n < c$. Therefore $a - y/n > a - c = x$. But by hypothesis, $x \geq a - y/n$. Thus $a - y/n < a - y/n$, a contradiction.

Case 3 Suppose $x > a$. But by hypothesis $x \leq a$. Thus $a < a$, a contradiction.

Conclusion Since these cases are exhaustive and both case 2 and 3 lead to contradictions, $x = a$ is the only possibility.

□

Theorem I.32

Let h be a given positive number and let S be a set of real numbers.

Theorem I.32a

If S has a supremum, then for some x in S we have $x > \sup S - h$.

Proof. Apostol.Chapter_I_03.sup_imp_exists_gt_sup_sub_delta

By definition of a supremum, $\sup S$ is the least upper bound of S. For the sake of contradiction, suppose for all $x \in S$, $x \leq \sup S - h$. This immediately implies $\sup S - h$ is an upper bound of S. But $\sup S - h < \sup S$, contradicting $\sup S$ being the *least* upper bound. Therefore our original hypothesis was wrong. That is, there exists some $x \in S$ such that $x > \sup S - h$.

□

Theorem I.32b

If S has an infimum, then for some x in S we have $x < \inf S + h$.

Proof. Apostol.Chapter_I_03.inf_imp_exists_lt_inf_add_delta

By definition of an infimum, $\inf S$ is the greatest lower bound of S. For the sake of contradiction, suppose for all $x \in S$, $x \geq \inf S + h$. This immediately implies $\inf S + h$ is a lower bound of S. But $\inf S + h > \inf S$, contradicting $\inf S$ being the *greatest* lower bound. Therefore our original hypothesis was wrong. That is, there exists some $x \in S$ such that $x < \inf S + h$.

□

Theorem I.33

Given nonempty subsets A and B of $\mathbb{R}$, let C denote the set

$$C = \{a + b : a \in A, b \in B\}.$$

Note: This is known as the "Additive Property."

Theorem I.33a

If each of A and B has a supremum, then C has a supremum, and

$$\sup C = \sup A + \sup B.$$

Proof. Apostol.Chapter_I_03.sup_minkowski_sum_eq_sup_add_sup

We prove (i) $\sup A + \sup B$ is an upper bound of C and (ii) $\sup A + \sup B$ is the *least* upper bound of C.

(i) Let $x \in C$. By definition of C, there exist elements $a' \in A$ and $b' \in B$ such that $x = a' + b'$. By definition of a supremum, $a' \leq \sup A$. Likewise, $b' \leq \sup B$. Therefore $a' + b' \leq \sup A + \sup B$. Since $x = a' + b'$ was arbitrarily chosen, it follows $\sup A + \sup B$ is an upper bound of C.

(ii) Since A and B have supremums, C is nonempty. By (i), C is bounded above. Therefore the completeness axiom tells us C has a supremum. Let $n > 0$ be an integer. We now prove that

$$\sup C \leq \sup A + \sup B \leq \sup C + 1/n. \tag{1}$$

Left-Hand Side First consider the left-hand side of (1). By (i), $\sup A + \sup B$ is an upper bound of C. Since $\sup C$ is the *least* upper bound of C, it follows $\sup C \leq \sup A + \sup B$.

Right-Hand Side Next consider the right-hand side of (1). By Theorem I.32a, there exists some $a' \in A$ such that $\sup A < a' + 1/(2n)$. Likewise, there exists some $b' \in B$ such that $\sup B < b' + 1/(2n)$. Adding these two inequalities together shows

$$\begin{aligned}\sup A + \sup B &< a' + b' + 1/n \\ &\leq \sup C + 1/n.\end{aligned}$$

Conclusion Applying Theorem I.31 to (1) proves $\sup C = \sup A + \sup B$ as expected.

□

Theorem I.33b

If each of A and B has an infimum, then C has an infimum, and

$$\inf C = \inf A + \inf B.$$

Proof. Apostol.Chapter_I_03.inf_minkowski_sum_eq_inf_add_inf

We prove (i) $\inf A + \inf B$ is a lower bound of C and (ii) $\inf A + \inf B$ is the *greatest* lower bound of C.

(i) Let $x \in C$. By definition of C, there exist elements $a' \in A$ and $b' \in B$ such that $x = a' + b'$. By definition of an infimum, $a' \geq \inf A$. Likewise, $b' \geq \inf B$. Therefore $a' + b' \geq \inf A + \inf B$. Since $x = a' + b'$ was arbitrarily chosen, it follows $\inf A + \inf B$ is a lower bound of C.

(ii) Since A and B have infimums, C is nonempty. By (i), C is bounded below. Therefore Theorem I.27 tells us C has an infimum. Let $n > 0$ be an integer. We now prove that

$$\inf C - 1/n \leq \inf A + \inf B \leq \inf C. \tag{2}$$

Right-Hand Side First consider the right-hand side of (2). By (i), $\inf A + \inf B$ is a lower bound of C. Since $\inf C$ is the *greatest* upper bound of C, it follows $\inf C \geq \inf A + \inf B$.

Left-Hand Side Next consider the left-hand side of (2). By Theorem I.32b, there exists some $a' \in A$ such that $\inf A > a' - 1/(2n)$. Likewise, there exists some $b' \in B$ such that $\inf B > b' - 1/(2n)$. Adding these two inequalities together shows

$$\begin{aligned}\inf A + \inf B &> a' + b' - 1/n \\ &\geq \inf C - 1/n.\end{aligned}$$

Conclusion Applying Lemma 2 to (2) proves $\inf C = \inf A + \inf B$ as expected.

$\square$

Theorem I.34

Given two nonempty subsets S and T of $\mathbb{R}$ such that

$$s \leq t$$

for every s in S and every t in T. Then S has a supremum, and T has an infimum, and they satisfy the inequality

$$\sup S \leq \inf T.$$

Proof. Apostol.Chapter_I_03.forall_mem_le_forall_mem_imp_sup_le_inf

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By hypothesis, S and T are nonempty sets. Let $s \in S$ and $t \in T$. Then t is an upper bound of S and s is a lower bound of T. By the completeness axiom, S has a supremum. By Theorem I.27, T has an infimum. All that remains is showing $\sup S \leq \inf T$.

For the sake of contradiction, suppose $\sup S > \inf T$. Then there exists some $c > 0$ such that $\sup S = \inf T + c$. Therefore $\inf T < \sup S - c/2$. By Theorem I.32a, there exists some $x \in S$ such that $\sup S - c/2 < x$. Thus

$$\inf T < \sup S - c/2 < x.$$

But by hypothesis, $x \in S$ is a lower bound of T meaning $x \leq \inf T$. Therefore $x < x$, a contradiction. Out original assumption is incorrect; that is, $\sup S \leq \inf T$.

$\square$