One-Variable Calculus, with an Introduction to Linear Algebra

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Contents

Chapter R

Reference

R.1 ¶ Characteristic Function

Let S be a set of points on the real line. The **characteristic function** of S is the function $\mathcal{X}_S$ such that $\mathcal{X}_S(x) = 1$ for every x in S, and $\mathcal{X}_S(x) = 0$ for those x not in S.

Definition. ∃ – Set.characteristic

□

R.2 ¶ Infimum

A number B is called an **infimum** of a nonempty set S if B has the following two properties:

(a) B is a lower bound for S.

(b) No number greater than B is a lower bound for S.

Such a number B is also known as the **greatest lower bound**.

Definition. ∃ – IsGLB

□

R.3 ✎ Integrable

Let f be a function defined and bounded on $[a, b]$. f is said to be **integrable** if there exists one and only one number I such that (2) holds. If f is integrable on $[a, b]$, we say that the integral $\int_a^b f(x)\,dx$ **exists**.

R.4 Integral of a Bounded Function

Let f be a function defined and bounded on $[a, b]$. Let s and t denote arbitrary step functions defined on $[a, b]$ such that

$$s(x) \leq f(x) \leq t(x) \tag{1}$$

for every x in $[a, b]$. If there is one and only one number I such that

$$\int_a^b s(x)\,dx \leq I \leq \int_a^b t(x)\,dx \tag{2}$$

for every pair of step functions s and t satisfying (1), then this number I is called the **integral of f from a to b**, and is denoted by the symbol $\int_a^b f(x)\,dx$ or by $\int_a^b f$.

If $a < b$, we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, provided f is Integrable on $[a, b]$. We also define $\int_a^a f(x)\,dx = 0$.

The function f is called the **integrand**, the numbers a and b are called the **limits of integration**, and the interval $[a, b]$ the **interval of integration**.

R.5 Integral of a Step Function

Let s be a ¶ Step Function defined on $[a, b]$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a ¶ Partition of $[a, b]$ such that s is constant on the open subintervals of P. Denote by s_k the constant value that s takes in the kth open subinterval of P, so that

$$s(x) = s_k \quad \text{if} \quad x_{k-1} < x < x_k, \quad k = 1, 2, \ldots, n.$$

The **integral of s from a to b**, denoted by the symbol $\int_a^b s(x)\,dx$, is defined by the following formula:

$$\int_a^b s(x)\,dx = \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}).$$

If $a < b$, we define $\int_b^a s(x)\,dx = -\int_a^b s(x)\,dx$. We also define $\int_a^a s(x)\,dx = 0$.

R.6 Lower Integral

Let f be a function bounded on $[a, b]$ and S denote the set of numbers $\int_a^b s(x)\,dx$ obtained as s runs through all ¶ Step Functions below f. That is, let

$$S = \left\{ \int_a^b s(x)\,dx : s \leq f \right\}.$$

The number $\sup S$ is called the **lower integral of f**. It is denoted as $\underline{I}(f)$.

R.7 Monotonic

A function f is called **monotonic** on set S if it is increasing on S or if it is decreasing on S. f is said to be **strictly monotonic** if it is strictly increasing on S or strictly decreasing on S.

A function f is said to be **piecewise monotonic** on an interval if its graph consists of a finite number of monotonic pieces. In other words, f is piecewise monotonic on $[a, b]$ if there is a ¶ Partition of $[a, b]$ such that f is monotonic on each of the open subintervals of P.

R.8 ¶ Partition

Let $[a, b]$ be a closed interval decomposed into n subintervals by inserting $n-1$ points of subdivision, say x_1, x_2, …, x_{n-1}, subject only to the restriction

$$a < x_1 < x_2 < \cdots < x_{n-1} < b. \tag{3}$$

It is convenient to denote the point a itself by x_0 and the point b by x_n. A collection of points satisfying (3) is called a **partition** P of $[a, b]$, and we use the symbol

$$P = \{x_0, x_1, \ldots, x_n\}$$

to designate this partition.

Definition. ∃ – Set.Partition

□

R.9 Refinement

Let P be a ¶ Partition of closed interval $[a, b]$. A **refinement** P' of P is a partition formed by adjoining more subdivision points to those already in P.

P' is said to be **finer than** P. The union of two partitions P_1 and P_2 is called the **common refinement** of P_1 and P_2.

R.10 ¶ Step Function

A function s, whose domain is a closed interval $[a, b]$, is called a **step function** if there is a ¶ Partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that s is constant on each open subinterval of P. That is to say, for each $k = 1, 2, \ldots, n$, there is a real number s_k such that

$$s(x) = s_k \quad \text{if} \quad x_{k-1} < x < x_k.$$

Step functions are sometimes called **piecewise constant functions**.

Note: At each of the endpoints x_{k-1} and x_k the function must have some well-defined value, but this need not be the same as s_k.

Definition. ∃ – Geometry.StepFunction

□

R.11 ¶ Supremum

A number B is called a **supremum** of a nonempty set S if B has the following two properties:

(a) B is an upper bound for S.

(b) No number less than B is an upper bound for S.

Such a number B is also known as the **least upper bound**.

Definition. ∃ – IsLUB

□

R.12 Upper Integral

Let f be a function bounded on $[a, b]$ and T denote the set of numbers $\int_a^b t(x)\,dx$ obtained as t runs through all ¶ Step Functions above f. That is, let

$$T = \left\{ \int_a^b t(x)\,dx : f \leq t \right\}.$$

The number $\inf T$ is called the **upper integral of** f. It is denoted as $\bar{I}(f)$.

Chapter 1

A Set of Axioms for the Real-Number System

1.1 ¶ Completeness Axiom

Every nonempty set S of real numbers which is bounded above has a supremum; that is, there is a real number B such that $B = \sup S$.

Axiom. ∃ − Real.exists_isLUB

□

1.2 Lemma 1

Lemma 1. *Nonempty set S has supremum L if and only if set $-S$ has infimum $-L$.*

Proof. ∃ − Apostol.Chapter_I_03.is_lub_neg_set_iff_is_glb_set_neg

Suppose $L = \sup S$ and fix $x \in S$. By definition of the ¶ Supremum, $x \leq L$ and L is the smallest value satisfying this inequality. Negating both sides of the inequality yields $-x \geq -L$. Furthermore, $-L$ must be the largest value satisfying this inequality. Therefore $-L = \inf -S$.

□

1.3 Existence of a Greatest Lower Bound

Theorem I.27. *Every nonempty set S that is bounded below has a greatest lower bound; that is, there is a real number L such that $L = \inf S$.*

Proof. ∃ – Apostol.Chapter_I_03.exists_isGLB

Let S be a nonempty set bounded below by x. Then $-S$ is nonempty and bounded above by x. By the ¶ Completeness Axiom, there exists a ¶ Supremum L of $-S$. By ✓ Lemma 1, L is a supremum of $-S$ if and only if $-L$ is an infimum of S.

□

1.4 ✓ Positive Integers Unbounded Above

Theorem I.29. *For every real x there exists a positive integer n such that $n > x$.*

Proof. ∃ – Apostol.Chapter_I_03.exists_pnat_geq_self

Let $n = |\lceil x \rceil| + 1$. It is trivial to see n is a positive integer satisfying $n \geq 1$. Thus all that remains to be shown is that $n > x$. If x is nonpositive, $n > x$ immediately follows from $n \geq 1$. If x is positive,

$$x = |x| \leq |\lceil x \rceil| < |\lceil x \rceil| + 1 = n.$$

□

1.5 ✓ Archimedean Property of the Reals

Theorem I.30. *If $x > 0$ and if y is an arbitrary real number, there exists a positive integer n such that $nx > y$.*

Proof. ∃ – Apostol.Chapter_I_03.exists_pnat_mul_self_geq_of_pos

Let $x > 0$ and y be an arbitrary real number. By ✓ Positive Integers Unbounded Above, there exists a positive integer n such that $n > y/x$. Multiplying both sides of the inequality yields $nx > y$ as expected.

□

1.6 ✓ Theorem I.31

Theorem I.31. *If three real numbers a, x, and y satisfy the inequalities*

$$a \leq x \leq a + \frac{y}{n}$$

for every integer $n \geq 1$, then $x = a$.

Proof. ∃ – Apostol.Chapter_I_03.forall_pnat_leq_self_leq_frac_imp_eq

By the trichotomy of the reals, there are three cases to consider:

Case 1 Suppose $x = a$. Then we are immediately finished.

Case 2 Suppose $x < a$. But by hypothesis, $a \leq x$. Thus $a < a$, a contradiction.

Case 3 Suppose $x > a$. Then there exists some $c > 0$ such that $a + c = x$. By Archimedean Property of the Reals, there exists an integer $n > 0$ such that $nc > y$. Rearranging terms, we see $y/n < c$. Therefore $a + y/n < a + c = x$. But by hypothesis, $x \leq a + y/n$. Thus $a + y/n < a + y/n$, a contradiction.

Conclusion Since these cases are exhaustive and both case 2 and 3 lead to contradictions, $x = a$ is the only possibility.

□

1.7 Lemma 2

Lemma 2. *If three real numbers a, x, and y satisfy the inequalities*

$$a - y/n \leq x \leq a$$

for every integer $n \geq 1$, then $x = a$.

Proof. ∃ – Apostol.Chapter_I_03.forall_pnat_frac_leq_self_leq_imp_eq

By the trichotomy of the reals, there are three cases to consider:

Case 1 Suppose $x = a$. Then we are immediately finished.

Case 2 Suppose $x < a$. Then there exists some $c > 0$ such that $x = a - c$. By Archimedean Property of the Reals, there exists an integer $n > 0$ such that $nc > y$. Rearranging terms, we see that $y/n < c$. Therefore $a - y/n > a - c = x$. But by hypothesis, $x \geq a - y/n$. Thus $a - y/n < a - y/n$, a contradiction.

Case 3 Suppose $x > a$. But by hypothesis $x \leq a$. Thus $a < a$, a contradiction.

Conclusion Since these cases are exhaustive and both case 2 and 3 lead to contradictions, $x = a$ is the only possibility.

□

1.8 Theorem I.32

Let h be a given positive number and let S be a set of real numbers.

1.8.1 Theorem I.32a

Theorem I.32a. *If S has a supremum, then for some x in S we have $x > \sup S - h$.*

Proof. ∃ – Apostol.Chapter_I_03.sup_imp_exists_gt_sup_sub_delta

By definition of a ¶ Supremum, $\sup S$ is the least upper bound of S. For the sake of contradiction, suppose for all $x \in S$, $x \leq \sup S - h$. This immediately implies $\sup S - h$ is an upper bound of S. But $\sup S - h < \sup S$, contradicting $\sup S$ being the *least* upper bound. Therefore our original hypothesis was wrong. That is, there exists some $x \in S$ such that $x > \sup S - h$.

□

1.8.2 Theorem I.32b

Theorem I.32b. *If S has an infimum, then for some x in S we have $x < \inf S + h$.*

Proof. ∃ – Apostol.Chapter_I_03.inf_imp_exists_lt_inf_add_delta

By definition of an ¶ Infimum, $\inf S$ is the greatest lower bound of S. For the sake of contradiction, suppose for all $x \in S$, $x \geq \inf S + h$. This immediately implies $\inf S + h$ is a lower bound of S. But $\inf S + h > \inf S$, contradicting $\inf S$ being the *greatest* lower bound. Therefore our original hypothesis was wrong. That is, there exists some $x \in S$ such that $x < \inf S + h$.

□

1.9 Additive Property of Supremums and Infimums

Given nonempty subsets A and B of $\mathbb{R}$, let C denote the set

$$C = \{a + b : a \in A, b \in B\}.$$

Note: This is known as the "Additive Property."

1.9.1 Theorem I.33a

Theorem I.33a. *If each of A and B has a supremum, then C has a supremum, and*

$$\sup C = \sup A + \sup B.$$

Proof. ∃ − Apostol.Chapter_I_03.sup_minkowski_sum_eq_sup_add_sup

We prove (i) $\sup A + \sup B$ is an upper bound of C and (ii) $\sup A + \sup B$ is the *least* upper bound of C.

(i) Let $x \in C$. By definition of C, there exist elements $a' \in A$ and $b' \in B$ such that $x = a' + b'$. By definition of a ¶ Supremum, $a' \leq \sup A$. Likewise, $b' \leq \sup B$. Therefore $a' + b' \leq \sup A + \sup B$. Since $x = a' + b'$ was arbitrarily chosen, it follows $\sup A + \sup B$ is an upper bound of C.

(ii) Since A and B have supremums, C is nonempty. By (i), C is bounded above. Therefore the completeness axiom tells us C has a supremum. Let $n > 0$ be an integer. We now prove that

$$\sup C \leq \sup A + \sup B \leq \sup C + 1/n. \tag{1.1}$$

Left-Hand Side First consider the left-hand side of (1.1). By (i), $\sup A + \sup B$ is an upper bound of C. Since $\sup C$ is the *least* upper bound of C, it follows $\sup C \leq \sup A + \sup B$.

Right-Hand Side Next consider the right-hand side of (1.1). By Theorem I.32a, there exists some $a' \in A$ such that $\sup A < a' + 1/(2n)$. Likewise, there exists some $b' \in B$ such that $\sup B < b' + 1/(2n)$. Adding these two inequalities together shows

$$\begin{aligned} \sup A + \sup B &< a' + b' + 1/n \\ &\leq \sup C + 1/n. \end{aligned}$$

Conclusion Applying Theorem I.31 to (1.1) proves $\sup C = \sup A + \sup B$ as expected.

□

1.9.2 Theorem I.33b

Theorem I.33b. *If each of A and B has an infimum, then C has an infimum, and*

$$\inf C = \inf A + \inf B.$$

Proof. ∃ − Apostol.Chapter_I_03.inf_minkowski_sum_eq_inf_add_inf

We prove (i) $\inf A + \inf B$ is a lower bound of C and (ii) $\inf A + \inf B$ is the *greatest* lower bound of C.

(i) Let $x \in C$. By definition of C, there exist elements $a' \in A$ and $b' \in B$ such that $x = a' + b'$. By definition of an ¶ Infimum, $a' \geq \inf A$. Likewise, $b' \geq \inf B$. Therefore $a' + b' \geq \inf A + \inf B$. Since $x = a' + b'$ was arbitrarily chosen, it follows $\inf A + \inf B$ is a lower bound of C.

(ii) Since A and B have infimums, C is nonempty. By (i), C is bounded below. Therefore Existence of a Greatest Lower Bound tells us C has an infimum. Let $n > 0$ be an integer. We now prove that

$$\inf C - 1/n \leq \inf A + \inf B \leq \inf C. \tag{1.2}$$

Right-Hand Side First consider the right-hand side of (1.2). By (i), $\inf A + \inf B$ is a lower bound of C. Since $\inf C$ is the *greatest* upper bound of C, it follows $\inf C \geq \inf A + \inf B$.

Left-Hand Side Next consider the left-hand side of (1.2). By Theorem I.32b, there exists some $a' \in A$ such that $\inf A > a' - 1/(2n)$. Likewise, there exists some $b' \in B$ such that $\inf B > b' - 1/(2n)$. Adding these two inequalities together shows

$$\begin{aligned}\inf A + \inf B &> a' + b' - 1/n \\ &\geq \inf C - 1/n.\end{aligned}$$

Conclusion Applying Lemma 2 to (1.2) proves $\inf C = \inf A + \inf B$ as expected.

□

1.10 Theorem I.34

Theorem I.34. *Given two nonempty subsets S and T of $\mathbb{R}$ such that*

$$s \leq t$$

for every s in S and every t in T. Then S has a supremum, and T has an infimum, and they satisfy the inequality

$$\sup S \leq \inf T.$$

Proof. ∃ – Apostol.Chapter_I_03.forall_mem_le_forall_mem_imp_sup_le_inf

By hypothesis, S and T are nonempty sets. Let $s \in S$ and $t \in T$. Then t is an upper bound of S and s is a lower bound of T. By the completeness axiom, S has a supremum. By Existence of a Greatest Lower Bound, T has an infimum. All that remains is showing $\sup S \leq \inf T$.

For the sake of contradiction, suppose $\sup S > \inf T$. Then there exists some $c > 0$ such that $\sup S = \inf T + c$. Therefore $\inf T < \sup S - c/2$. By Theorem I.32a, there exists some $x \in S$ such that $\sup S - c/2 < x$. Thus

$$\inf T < \sup S - c/2 < x.$$

But by hypothesis, $x \in S$ is a lower bound of T meaning $x \leq \inf T$. Therefore $x < x$, a contradiction. Out original assumption is incorrect; that is, $\sup S \leq \inf T$.

□

Chapter 2

The Concepts of Integral Calculus

2.1 The Concept of Area as a Set Function

We assume there exists a class $\mathscr{M}$ of measurable sets in the plane and a set function a, whose domain is $\mathscr{M}$, with the following properties:

2.1.1 ¶ Nonnegative Property

For each set S in $\mathscr{M}$, we have $a(S) \geq 0$.

Axiom. ∃ – Nonnegative Property

□

2.1.2 ¶ Additive Property

If S and T are in $\mathscr{M}$, then $S \cup T$ and $S \cap T$ are in $\mathscr{M}$, and we have $a(S \cup T) = a(S) + a(T) - a(S \cap T)$.

Axiom. ∃ – Additive Property

□

2.1.3 ¶ Difference Property

If S and T are in $\mathscr{M}$ with $S \subseteq T$, then $T - S$ is in $\mathscr{M}$, and we have $a(T - S) = a(T) - a(S)$.

Axiom. ∃ – Difference Property

□

2.1.4 ¶ Invariance Under Congruence

If a set S is in $\mathscr{M}$ and if T is congruent to S, then T is also in $\mathscr{M}$ and we have $a(S) = a(T)$.

Axiom. ∃ – Invariance Under Congruence

□

2.1.5 ¶ Choice of Scale

Every rectangle R is in $\mathscr{M}$. If the edges of R have lengths h and k, then $a(R) = hk$.

Axiom. ∃ – Choice of Scale

□

2.1.6 Exhaustion Property

Let Q be a set that can be enclosed between two step regions S and T, so that

$$S \subseteq Q \subseteq T. \tag{2.1}$$

If there is one and only one number c which satisfies the inequalities

$$a(S) \leq c \leq a(T)$$

for all step regions S and T satisfying (2.1), then Q is measurable and $a(Q) = c$.

Axiom. ∃ – Exhaustion Property

□

2.2 Exercises 1.7

2.2.1 Exercise 1.7.1

Prove that each of the following sets is measurable and has zero area:

Exercise 1.7.1a

A set consisting of a single point.

Proof. Let S be a set consisting of a single point. By definition of a point, S is a rectangle in which all vertices coincide. By ¶ Choice of Scale, S is measurable with area its width times its height. The width and height of S is trivially zero. Therefore $a(S) = (0)(0) = 0$.

□

Exercise 1.7.1b

A set consisting of a finite number of points in a plane.

Proof. Define predicate $P(n)$ as "A set consisting of n points in a plane is measurable with area 0". We use induction to prove $P(n)$ holds for all $n > 0$.

Base Case Consider a set S consisting of a single point in a plane. By Exercise 1.7.1a, S is measurable with area 0. Thus $P(1)$ holds.

Induction Step Assume induction hypothesis $P(k)$ holds for some $k > 0$. Let S_{k+1} be a set consisting of $k+1$ points in a plane. Pick an arbitrary point of S_{k+1}. Denote the set containing just this point as T. Denote the remaining set of points as S_k. By construction, $S_{k+1} = S_k \cup T$. By the induction hypothesis, S_k is measurable with area 0. By Exercise 1.7.1a, T is measurable with area 0. By the ¶ Additive Property, $S_k \cup T$ is measurable, $S_k \cap T$ is measurable, and

$$\begin{aligned} a(S_{k+1}) &= a(S_k \cup T) \\ &= a(S_k) + a(T) - a(S_k \cap T) \\ &= 0 + 0 - a(S_k \cap T). \end{aligned} \tag{2.2}$$

There are two cases to consider:

Case 1 $S_k \cap T = \varnothing$. Then it trivially follows that $a(S_k \cap T) = 0$.

Case 2 $S_k \cap T \neq \varnothing$. Since T consists of a single point, $S_k \cap T = T$. By Exercise 1.7.1a, $a(S_k \cap T) = a(T) = 0$.

In both cases, (2.2) evaluates to 0, implying $P(k+1)$ as expected.

Conclusion By mathematical induction, it follows for all $n > 0$, $P(n)$ is true.

□

Exercise 1.7.1c

The union of a finite collection of line segments in a plane.

Proof. Define predicate $P(n)$ as "A set consisting of n line segments in a plane is measurable with area 0". We use induction to prove $P(n)$ holds for all $n > 0$.

Base Case Consider a set S consisting of a single line segment in a plane. By definition of a line segment, S is a rectangle in which one side has dimension 0. By ¶ Choice of Scale, S is measurable with area its width w times its height h. Therefore $a(S) = wh = 0$. Thus $P(1)$ holds.

Induction Step Assume induction hypothesis $P(k)$ holds for some $k > 0$. Let S_{k+1} be a set consisting of $k+1$ line segments in a plane. Pick an arbitrary line segment of S_{k+1}. Denote the set containing just this line segment as T. Denote the remaining set of line segments as S_k. By construction, $S_{k+1} = S_k \cup T$. By the induction hypothesis, S_k is measurable with area 0. By the base case, T is measurable with area 0. By the ¶ Additive Property, $S_k \cup T$ is measurable, $S_k \cap T$ is measurable, and

$$\begin{aligned} a(S_{k+1}) &= a(S_k \cup T) \\ &= a(S_k) + a(T) - a(S_k \cap T) \\ &= 0 + 0 - a(S_k \cap T). \end{aligned} \tag{2.3}$$

There are two cases to consider:

Case 1 $S_k \cap T = \varnothing$. Then it trivially follows that $a(S_k \cap T) = 0$.

Case 2 $S_k \cap T \neq \varnothing$. Since T consists of a single point, $S_k \cap T = T$. By the base case, $a(S_k \cap T) = a(T) = 0$.

In both cases, (2.3) evaluates to 0, implying $P(k+1)$ as expected.

Conclusion By mathematical induction, it follows for all $n > 0$, $P(n)$ is true. □

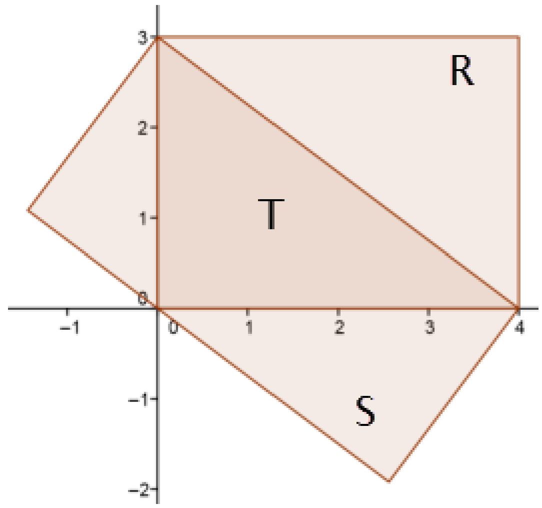
2.2.2 Exercise 1.7.2

Every right triangular region is measurable because it can be obtained as the intersection of two rectangles. Prove that every triangular region is measurable and that its area is one half the product of its base and altitude.

Proof. Let T' be a triangular region with base of length a, height of length b, and hypotenuse of length c. Consider the translation and rotation of T', say T, such that its hypotenuse is entirely within quadrant I and the vertex opposite the hypotenuse is situated at point $(0, 0)$.

Let R be a rectangle of width a, height b, and bottom-left corner at $(0, 0)$. By construction, R covers all of T. Let S be a rectangle of width c and height

$a \sin\theta$, where θ is the acute angle measured from the bottom-right corner of T relative to the x-axis. As an example, consider the image below of triangle T with width 4 and height 3:



By ¶ Choice of Scale, both R and S are measurable. By this same axiom, $a(R) = ab$ and $a(S) = ca \sin\theta$. By the ¶ Additive Property, $R \cup S$ and $R \cap S$ are both measurable. $a(R \cap S) = a(T)$ and $a(R \cup S)$ can be determined by noting that R's construction implies identity $a(R) = 2a(T)$. Therefore

$$\begin{aligned}
a(T) &= a(R \cap S) \\
&= a(R) + a(S) - a(R \cup S) \\
&= ab + ca \sin\theta - a(R \cup S) \\
&= ab + ca \sin\theta - (ca \sin\theta + \frac{1}{2}a(R)) \\
&= ab + ca \sin\theta - ca \sin\theta - a(T).
\end{aligned}$$

Solving for $a(T)$ gives the desired identity:

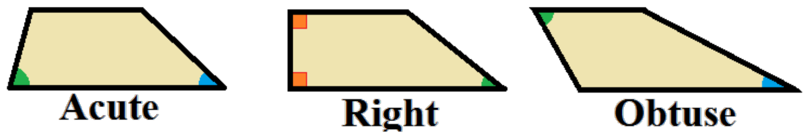
$$a(T) = \frac{1}{2}ab.$$

By ¶ Invariance Under Congruence, $a(T') = a(T)$, concluding our proof.

□

2.2.3 Exercise 1.7.3

Prove that every trapezoid and every parallelogram is measurable and derive the usual formulas for their areas.



Proof. We begin by proving the formula for a trapezoid. Let S be a trapezoid with height h and bases b_1 and b_2, $b_1 < b_2$. There are three cases to consider:

Case 1 Suppose S is a right trapezoid. Then S is the union of non-overlapping rectangle R of width b_1 and height h with right triangle T of base $b_2 - b_1$ and height h. By ¶ Choice of Scale, R is measurable. By Exercise 1.7.2, T is measurable. By the ¶ Additive Property, $R \cup T$ and $R \cap T$ are both measurable and

$$\begin{aligned} a(S) &= a(R \cup T) \\ &= a(R) + a(T) - a(R \cap T) \\ &= a(R) + a(T) && \text{by construction} \\ &= b_1 h + a(T) && \text{Choice of Scale} \\ &= b_1 h + \frac{1}{2}(b_2 - b_1)h && \text{Exercise 1.7.2} \\ &= \frac{b_1 + b_2}{2} h. \end{aligned}$$

Case 2 Suppose S is an acute trapezoid. Then S is the union of non-overlapping triangle T and right trapezoid R. Let c denote the length of base T. Then R has longer base edge of length $b_2 - c$. By Exercise 1.7.2, T is measurable. By Case 1, R is measurable. By the ¶ Additive Property, $R \cup T$ and $R \cap T$ are both measurable and

$$\begin{aligned} a(S) &= a(T) + a(R) - a(R \cap T) \\ &= a(T) + a(R) && \text{by construction} \\ &= \frac{1}{2} ch + a(R) && \text{Exercise 1.7.2} \\ &= \frac{1}{2} ch + \frac{b_1 + b_2 - c}{2} h && \text{Case 1} \\ &= \frac{b_1 + b_2}{2} h. \end{aligned}$$

Case 3 Suppose S is an obtuse trapezoid. Then S is the union of non-overlapping triangle T and right trapezoid R. Let c denote the length of base T. Reflect T vertically to form another right triangle, say T'. Then $T' \cup R$ is an acute trapezoid. By ¶ Invariance Under Congruence,

$$a(T' \cup R) = a(T \cup R). \tag{3.1}$$

By construction, $T' \cup R$ has height h and bases $b_1 - c$ and $b_2 + c$ meaning

$$\begin{aligned} a(T \cup R) &= a(T' \cup R) && (3.1) \\ &= \frac{b_1 - c + b_2 + c}{2} h && \text{Case 2} \\ &= \frac{b_1 + b_2}{2} h. \end{aligned}$$

Conclusion These cases are exhaustive and in agreement with one another. Thus S is measurable and

$$a(S) = \frac{b_1 + b_2}{2} h.$$

♠♠♠

Let P be a parallelogram with base b and height h. Then P is the union of non-overlapping triangle T and right trapezoid R. Let c denote the length of base T. Reflect T vertically to form another right triangle, say T'. Then $T' \cup R$ is an acute trapezoid. By ¶ Invariance Under Congruence,

$$a(T' \cup R) = a(T \cup R). \tag{2.4}$$

By construction, $T' \cup R$ has height h and bases $b - c$ and $b + c$ meaning

$$\begin{aligned} a(T \cup R) &= a(T' \cup R) && (2.4) \\ &= \frac{b - c + b + c}{2} h && \text{Area of Trapezoid} \\ &= bh. \end{aligned}$$

□

2.2.4 Exercise 1.7.4

Let P be a polygon whose vertices are lattice points. The area of P is $I + \frac{1}{2}B - 1$, where I denotes the number of lattice points inside the polygon and B denotes the number on the boundary.

Exercise 1.7.4a

Prove that the formula is valid for rectangles with sides parallel to the coordinate axes.

Proof. Let P be a rectangle with sides parallel to the coordinate axes, with width w, height h, and lattice points for vertices. We assume P has three non-collinear points, ruling out any instances of points or line segments.

By ¶ Choice of Scale, P is measurable with area $a(P) = wh$. By construction, P has $I = (w-1)(h-1)$ interior lattice points and $B = 2(w+h)$ lattice points on its boundary. The following shows the lattice point area formula is in agreement with the expected result:

$$\begin{aligned} I + \frac{1}{2}B - 1 &= (w-1)(h-1) + \frac{1}{2}\left[2(w+h)\right] - 1 \\ &= (wh - w - h + 1) + \frac{1}{2}\left[2(w+h)\right] - 1 \\ &= (wh - w - h + 1) + (w+h) - 1 \\ &= wh. \end{aligned}$$

□

✎ Exercise 1.7.4b

Prove that the formula is valid for right triangles and parallelograms.

Proof. Let P be a right triangle with width $w > 0$, height $h > 0$, and lattice points for vertices. Let T be the triangle P translated, rotated, and reflected such that the its vertices are $(0,0)$, $(0,w)$, and (w,h). Let I_T and B_T be the number of interior and boundary points of T respectively. Let H_L denote the number of lattice points on T's hypotenuse.

Let R be the overlapping rectangle of width w and height h, situated with bottom-left corner at $(0,0)$. Let I_R and B_R be the number of interior and boundary points of R respectively.

By construction, T shares two sides with R. Therefore

$$B_T = \frac{1}{2}B_R - 1 + H_L. \tag{2.5}$$

Likewise,

$$I_T = \frac{1}{2}(I_R - (H_L - 2)). \tag{2.6}$$

The following shows the lattice point area formula is in agreement with the expected result:

$$\begin{aligned}
I_T + \frac{1}{2}B_T - 1 &= \frac{1}{2}(I_R - (H_L - 2)) + \frac{1}{2}B_T - 1 && (2.6)\\
&= \frac{1}{2}\left[I_R - H_L + B_T\right]\\
&= \frac{1}{2}\left[I_R - H_L + \frac{1}{2}B_R - 1 + H_L\right] && (2.5)\\
&= \frac{1}{2}\left[I_R + \frac{1}{2}B_R - 1\right]\\
&= \frac{1}{2}\left[wh\right] && \text{Exercise 1.7.4a.}
\end{aligned}$$

We do not prove this formula is valid for parallelograms here. Instead, refer to Exercise 1.7.4c below.

□

Exercise 1.7.4c

Use induction on the number of edges to construct a proof for general polygons.

Proof. Define predicate $P(n)$ as "An n-polygon with vertices on lattice points has area $I + \frac{1}{2}B - 1$." We use induction to prove $P(n)$ holds for all $n \geq 3$.

Base Case A 3-polygon is a triangle. By Exercise 1.7.4b, the lattice point area formula holds. Thus $P(3)$ holds.

Induction Step Assume induction hypothesis $P(k)$ holds for some $k \geq 3$. Let P be a $(k+1)$-polygon with vertices on lattice points. Such a polygon is equivalent to the union of a k-polygon S with a triangle T. That is, $P = S \cup T$.

Let I_P be the number of interior lattice points of P. Let B_P be the number of boundary lattice points of P. Similarly, let I_S, I_T, B_S, and B_T be the number of interior and boundary lattice points of S and T. Let c denote the number of boundary points shared between S and T.

By our induction hypothesis, $a(S) = I_S + \frac{1}{2}B_S - 1$. By our base case, $a(T) = I_T + \frac{1}{2}B_T - 1$. By construction, it follows:

$$\begin{aligned}
I_P &= I_S + I_T + c - 2\\
B_P &= B_S + B_T - (c - 2) - c\\
&= B_S + B_T - 2c + 2.
\end{aligned}$$

Applying the lattice point area formula to P yields the following:

$$
\begin{aligned}
& I_P + \frac{1}{2}B_P - 1 \\
& = (I_S + I_T + c - 2) + \frac{1}{2}(B_S + B_T - 2c + 2) - 1 \\
& = I_S + I_T + c - 2 + \frac{1}{2}B_S + \frac{1}{2}B_T - c + 1 - 1 \\
& = (I_S + \frac{1}{2}B_S - 1) + (I_T + \frac{1}{2}B_T - 1) \\
& = a(S) + (I_T + \frac{1}{2}B_T - 1) && \text{induction hypothesis} \\
& = a(S) + a(T). && \text{base case}
\end{aligned}
$$

By the ¶ Additive Property, $S \cup T$ is measurable, $S \cap T$ is measurable, and

$$
\begin{aligned}
a(P) &= a(S \cup T) \\
&= a(S) + a(T) - a(S \cap T) \\
&= a(S) + a(T). && \text{by construction}
\end{aligned}
$$

This shows the lattice point area formula is in agreement with our axiomatic definition of area. Thus $P(k+1)$ holds.

Conclusion By mathematical induction, it follows for all $n \geq 3$, $P(n)$ is true. □

2.2.5 Exercise 1.7.5

Prove that a triangle whose vertices are lattice points cannot be equilateral.

[*Hint:* Assume there is such a triangle and compute its area in two ways, using Exercises 2 and 4.]

Proof. Proceed by contradiction. Let T be an equilateral triangle whose vertices are lattice points. Assume each side of T has length a. Then T has height $h = (a\sqrt{3})/2$. By Exercise 1.7.2,

$$a(T) = \frac{1}{2}ah = \frac{a^2\sqrt{3}}{4}. \tag{5.1}$$

Let I and B denote the number of interior and boundary lattice points of T respectively. By Exercise 1.7.4,

$$a(T) = I + \frac{1}{2}B - 1. \tag{5.2}$$

But (5.1) is irrational whereas (5.2) is not. This is a contradiction. Thus, there is *no* equilateral triangle whose vertices are lattice points. □

2.2.6 Exercise 1.7.6

Let $A = \{1, 2, 3, 4, 5\}$, and let $\mathscr{M}$ denote the class of all subsets of A. (There are 32 altogether, counting A itself and the empty set $\varnothing$.) For each set S in $\mathscr{M}$, let $n(S)$ denote the number of distinct elements in S. If $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$, compute $n(S \cup T)$, $n(S \cap T)$, $n(S - T)$, and $n(T - S)$. Prove that the set function n satisfies the first three axioms for area.

Proof. Let $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$. Then

$$\begin{aligned}
n(S \cup T) &= n(\{1, 2, 3, 4\} \cup \{3, 4, 5\}) \\
&= n(\{1, 2, 3, 4, 5\}) \\
&= 5. \\
n(S \cap T) &= n(\{1, 2, 3, 4\} \cap \{3, 4, 5\}) \\
&= n(\{3, 4\}) \\
&= 2. \\
n(S - T) &= n(\{1, 2, 3, 4\} - \{3, 4, 5\}) \\
&= n(\{1, 2\}) \\
&= 2. \\
n(T - S) &= n(\{3, 4, 5\} - \{1, 2, 3, 4\}) \\
&= n(\{5\}) \\
&= 1.
\end{aligned}$$

We now prove n satisfies the first three axioms for area.

Nonnegative Property n returns the length of some member of $\mathscr{M}$. By hypothesis, the smallest possible input to n is $\varnothing$. Since $n(\varnothing) = 0$, it follows $n(S) \geq 0$ for all $S \subset A$.

Additive Property Let S and T be members of $\mathscr{M}$. It trivially follows that both $S \cup T$ and $S \cap T$ are in $\mathscr{M}$. Consider the value of $n(S \cup T)$. There are two cases to consider:

Case 1 Suppose $S \cap T = \varnothing$. That is, there is no common element shared between S and T. Thus

$$\begin{aligned}
n(S \cup T) &= n(S) + n(T) \\
&= n(S) + n(T) - 0 \\
&= n(S) + n(T) - n(S \cap T).
\end{aligned}$$

Case 2 Suppose $S \cap T \neq \varnothing$. Then $n(S) + n(T)$ counts each element of $S \cap T$ twice. Therefore $n(S \cup T) = n(S) + n(T) - n(S \cap T)$.

Conclusion These cases are exhaustive and in agreement with one another. Thus $n(S \cup T) = n(S) + n(T) - n(S \cap T)$.

Difference Property Suppose $S, T \in \mathscr{M}$ such that $S \subseteq T$. That is, every member of S is a member of T. By definition, $T - S$ consists of members in T but not in S. Thus $n(T - S) = n(T) - n(S)$.

□

2.3 Exercises 1.11

2.3.1 Exercise 1.11.4

Prove that the greatest-integer function has the properties indicated:

Exercise 1.11.4a

$\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for every integer n.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4a

Let x be a real number and n an integer. Let $m = \lfloor x + n \rfloor$. By definition of the floor function, m is the unique integer such that $m \leq x + n < m + 1$. Then $m - n \leq x < (m - n) + 1$. That is, $m - n = \lfloor x \rfloor$. Rearranging terms we see that $m = \lfloor x \rfloor + n$ as expected.

□

Exercise 1.11.4b

$$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \text{ is an integer,} \\ -\lfloor x \rfloor - 1 & \text{otherwise.} \end{cases}$$

Proof.

∃ – Apostol.Chapter_1_11.exercise_4b_1
∃ – Apostol.Chapter_1_11.exercise_4b_2

There are two cases to consider:

Case 1 Suppose x is an integer. Then $x = \lfloor x \rfloor$ and $-x = \lfloor -x \rfloor$. It immediately follows that

$$\lfloor -x \rfloor = -x = -\lfloor x \rfloor.$$

Case 2 Suppose x is not an integer. Let $m = \lfloor -x \rfloor$. By definition of the floor function, m is the unique integer such that $m \leq -x < m + 1$. Equivalently,

$-m-1 < x \leq -m$. Since x is not an integer, it follows $-m-1 \leq x < -m$. Then, by definition of the floor function, $\lfloor x \rfloor = -m-1$. Solving for m yields

$$\lfloor -x \rfloor = m = -\lfloor x \rfloor - 1.$$

Conclusion The above two cases are exhaustive. Thus

$$\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \text{ is an integer,} \\ -\lfloor x \rfloor - 1 & \text{otherwise.} \end{cases}$$

□

Exercise 1.11.4c

$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4c

Rewrite x and y as the sum of their floor and fractional components: $x = \lfloor x \rfloor + \{x\}$ and $y = \lfloor y \rfloor + \{y\}$. Now

$$\begin{aligned} \lfloor x+y \rfloor &= \lfloor \lfloor x \rfloor + \{x\} + \lfloor y \rfloor + \{y\} \rfloor \\ &= \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} \rfloor \\ &= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \{x\} + \{y\} \rfloor \qquad \text{Exercise 1.11.4a} \end{aligned} \tag{2.7}$$

There are two cases to consider:

Case 1 Suppose $\{x\} + \{y\} < 1$. Then $\lfloor \{x\} + \{y\} \rfloor = 0$. Substituting this value into (2.7) yields

$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor.$$

Case 2 Suppose $\{x\} + \{y\} \geq 1$. Because $\{x\}$ and $\{y\}$ are both less than 1, $\{x\} + \{y\} < 2$. Thus $\lfloor \{x\} + \{y\} \rfloor = 1$. Substituting this value into (2.7) yields

$$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1.$$

Conclusion Since the above two cases are exhaustive, it follows $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$.

□

Exercise 1.11.4d

$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4d

This is immediately proven by applying Hermite's Identity.

□

Exercise 1.11.4e

$\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Proof. ∃ – Apostol.Chapter_1_11.exercise_4e

This is immediately proven by applying Hermite's Identity.

□

2.3.2 Hermite's Identity

The formulas in Exercises 4(d) and 4(e) suggest a generalization for $\lfloor nx \rfloor$. State and prove such a generalization.

Proof. ∃ – Apostol.Chapter_1_11.exercise_5

We prove that for all natural numbers n and real numbers x, the following identity holds:

$$\lfloor nx \rfloor = \sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor \tag{2.8}$$

By definition of the floor function, $x = \lfloor x \rfloor + r$ for some $r \in [0, 1)$. Define S as the partition of non-overlapping subintervals

$$\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), \ldots, \left[\frac{n-1}{n}, 1\right).$$

By construction, $\cup S = [0, 1)$. Therefore there exists some $j \in \mathbb{N}$ such that

$$r \in \left[\frac{j}{n}, \frac{j+1}{n}\right). \tag{2.9}$$

With these definitions established, we now show the left- and right-hand sides of (2.8) evaluate to the same number.

Left-Hand Side Consider the left-hand side of identity (2.8). By (2.9), $nr \in [j, j+1)$. Therefore $\lfloor nr \rfloor = j$. Thus

$$\begin{aligned}
\lfloor nx \rfloor &= \lfloor n(\lfloor x \rfloor + r) \rfloor \\
&= \lfloor n \lfloor x \rfloor + nr \rfloor \\
&= \lfloor n \lfloor x \rfloor \rfloor + \lfloor nr \rfloor . && \text{Exercise 1.11.4a} \\
&= \lfloor n \lfloor x \rfloor \rfloor + j \\
&= n \lfloor x \rfloor + j. && (2.10)
\end{aligned}$$

Right-Hand Side Now consider the right-hand side of identity (2.8). We note each summand, by construction, is the floor of x added to a nonnegative number less than one. Therefore each summand contributes either $\lfloor x \rfloor$ or $\lfloor x \rfloor + 1$ to the total. Letting z denote the number of summands that contribute $\lfloor x \rfloor + 1$, we have

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = n \lfloor x \rfloor + z. \tag{2.11}$$

The value of z corresponds to the number of indices i that satisfy

$$\frac{i}{n} \geq 1 - r.$$

By (2.9), it follows

$$\begin{aligned}
1 - r &\in \left(1 - \frac{j+1}{n}, 1 - \frac{j}{n}\right] \\
&= \left(\frac{n-j-1}{n}, \frac{n-j}{n}\right].
\end{aligned}$$

Thus we can determine the value of z by instead counting the number of indices i that satisfy

$$\frac{i}{n} \geq \frac{n-j}{n}.$$

Rearranging terms, we see that $i \geq n - j$ holds for $z = (n-1) - (n-j) + 1 = j$ of the n summands. Substituting the value of z into (2.11) yields

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = n \lfloor x \rfloor + j. \tag{2.12}$$

Conclusion Since (2.10) and (2.12) agree with one another, it follows identity (2.8) holds.

□

2.3.3 Exercise 1.11.6

Recall that a lattice point (x, y) in the plane is one whose coordinates are integers. Let f be a nonnegative function whose domain is the interval $[a, b]$, where a and b are integers, $a < b$. Let S denote the set of points (x, y) satisfying $a \leq x \leq b$, $0 < y \leq f(x)$. Prove that the number of lattice points in S is equal to the sum

$$\sum_{n=a}^{b} \lfloor f(n) \rfloor .$$

Proof. Let $i = a, \ldots, b$ and define $S_i = \mathbb{N} \cap (0, f(i)]$. By construction, the number of lattice points in S is

$$\sum_{n=a}^{b} |S_n| . \tag{2.13}$$

All that remains is to show $|S_i| = \lfloor f(i) \rfloor$. There are two cases to consider:

Case 1 Suppose $f(i)$ is an integer. Then the number of integers in $(0, f(i)]$ is $f(i) = \lfloor f(i) \rfloor$.

Case 2 Suppose $f(i)$ is not an integer. Then the number of integers in $(0, f(i)]$ is the same as that of $(0, \lfloor f(i) \rfloor]$. Once again, that number is $\lfloor f(i) \rfloor$.

Conclusion By cases 1 and 2, $|S_i| = \lfloor f(i) \rfloor$. Substituting this identity into (2.13) finishes the proof.

□

2.3.4 Exercise 1.11.7

If a and b are positive integers with no common factor, we have the formula

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

When $b = 1$, the sum on the left is understood to be 0.

> Note: When $b = 1$, the proofs of (a) and (b) are trivial. We continue under the assumption $b > 1$.

Exercise 1.11.7a

Derive this result by a geometric argument, counting lattice points in a right triangle.

Proof. Let $f\colon [1, b-1] \to \mathbb{R}$ be given by $f(x) = ax/b$. Let S denote the set of points (x, y) satisfying $1 \le x \le b-1$, $0 < y \le f(x)$. By Exercise 1.11.6, the number of lattice points of S is equal to the sum

$$\sum_{n=1}^{b-1} \lfloor f(n) \rfloor = \sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor. \tag{2.14}$$

Define T to be the triangle of width $w = b$ and height $h = f(b) = a$ as

$$T = \{(x, y) : 0 < x < b, 0 < y \le f(x)\}.$$

By construction, T does not introduce any additional lattice points. Thus S and T have the same number of lattice points. Let H_L denote the number of boundary points on T's hypotenuse. We prove that (i) $H_L = 2$ and (ii) that T has $\frac{(a-1)(b-1)}{2}$ lattice points.

(i) Consider the line L overlapping the hypotenuse of T. By construction, T's hypotenuse has endpoints $(0, 0)$ and (b, a). By hypothesis, a and b are positive, excluding the possibility of L being vertical. Define the slope of L as

$$m = \frac{a}{b}.$$

H_L coincides with the number of indices $i = 0, \ldots, b$ such that $(i, i * m)$ is a lattice point. But a and b are coprime by hypothesis and $i \le b$. Thus $i * m$ is an integer if and only if $i = 0$ or $i = b$. Thus $H_L = 2$.

(ii) Next we count the number of lattice points in T. Let R be the overlapping retangle of width w and height h, situated with bottom-left corner at $(0, 0)$. Let I_R denote the number of interior lattice points of R. Let I_T and B_T denote the interior and boundary lattice points of T respectively. By *Proof.*,

$$\begin{aligned}
I_T &= \frac{1}{2}(I_R - (H_L - 2)) \\
&= \frac{1}{2}(I_R - (2 - 2)) && \text{(i)} \\
&= \frac{1}{2} I_R. && (2.15)
\end{aligned}$$

Furthermore, since both the adjacent and opposite side of T are not included in T and there exist no lattice points on T's hypotenuse besides the endpoints, it follows

$$B_T = 0. \tag{2.16}$$

Thus the number of lattice points of T equals

$$\begin{aligned}
I_T + B_T &= I_T && (2.16) \\
&= \frac{1}{2} I_R && (2.15) \\
&= \frac{(b-1)(a-1)}{2}. && \text{Exercise 1.7.4a} \qquad (2.17)
\end{aligned}$$

Conclusion By (2.14) the number of lattice points of S is equal to the sum

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor.$$

But the number of lattice points of S is the same as that of T. By (2.17), the number of lattice points in T is equal to

$$\frac{(b-1)(a-1)}{2}.$$

Thus

$$\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

□

Exercise 1.11.7b

Derive the result analytically as follows: By changing the index of summation, note that $\sum_{n=1}^{b-1} \lfloor na/b \rfloor = \sum_{n=1}^{b-1} \lfloor a(b-n)/b \rfloor$. Now apply Exercises 4(a) and (b) to the bracket on the right.

Proof. ∃ – Apostol.Chapter_1_11.exercise_7b

Let $n = 1, \dots, b-1$. By hypothesis, a and b are coprime. Furthermore, $n < b$ for all values of n. Thus an/b is not an integer. By Exercise 1.11.4b,

$$\left\lfloor -\frac{an}{b} \right\rfloor = -\left\lfloor \frac{an}{b} \right\rfloor - 1. \tag{2.18}$$

Consider the following:

$$
\begin{aligned}
\sum_{n=1}^{b-1} \left\lfloor \frac{na}{b} \right\rfloor &= \sum_{n=1}^{b-1} \left\lfloor \frac{a(b-n)}{b} \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor \frac{ab-an}{b} \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor -\frac{an}{b} + a \right\rfloor \\
&= \sum_{n=1}^{b-1} \left\lfloor -\frac{an}{b} \right\rfloor + a. && \text{Exercise 1.11.4a} \\
&= \sum_{n=1}^{b-1} - \left\lfloor \frac{an}{b} \right\rfloor - 1 + a && (2.18) \\
&= -\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor - \sum_{n=1}^{b-1} 1 + \sum_{n=1}^{b-1} a \\
&= -\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor - (b-1) + a(b-1).
\end{aligned}
$$

Rearranging the above yields

$$2\sum_{n=1}^{b-1} \left\lfloor \frac{an}{b} \right\rfloor = (a-1)(b-1).$$

Dividing both sides of the above identity concludes the proof.

□

2.3.5 Exercise 1.11.8

Let S be a set of points on the real line. Let $\mathcal{X}_S$ denote the ¶ Characteristic Function of S. Let f be a ¶ Step Function which takes the constant value c_k on the kth open subinterval I_k of some partition of an interval $[a, b]$. Prove that for each x in the union $I_1 \cup I_2 \cup \cdots \cup I_n$ we have

$$f(x) = \sum_{k=1}^{n} c_k \mathcal{X}_{I_k}(x).$$

This property is described by saying that every step function is a linear combination of characteristic functions of intervals.

Proof. Let $x \in I_1 \cup I_2 \cup \cdots \cup I_n$ and $N = \{1, \ldots, n\}$. Let $k \in N$ such that $x \in I_k$. Consider an arbitrary $j \in N - \{k\}$. By definition of a namerefref:partition, $I_j \cap I_k = \varnothing$. That is, I_j and I_k are disjoint for all $j \in N - \{k\}$. Therefore, by definition of the characteristic function, $\mathcal{X}_{I_k}(x) = 1$ and $\mathcal{X}_{I_j}(x) = 0$ for all $j \in N - \{k\}$. Thus

$$\begin{aligned} f(x) &= c_k \\ &= (c_k)(1) + \sum\nolimits_{j \in N - \{k\}} (c_j)(0) \\ &= c_k \mathcal{X}_{I_k}(x) + \sum\nolimits_{j \in N - \{k\}} c_j \mathcal{X}_{I_j}(x) \\ &= \sum_{k=1}^{n} c_k \mathcal{X}_{I_k}(x). \end{aligned}$$

□

2.4 Properties of the Integral of a Step Function

2.4.1 ✎ Additive Property

Theorem 1.2. *Let s and t be ¶ Step Functions on closed interval $[a, b]$. Then*

$$\int_a^b [s(x) + t(x)]\, dx = \int_a^b s(x)\, dx + \int_a^b t(x)\, dx\,.$$

Proof. Let s and t be step functions on closed interval $[a, b]$. By definition of a step function, there exists a ¶ Partition P_s such that s is constant on each open subinterval of P_s. Likewise, there exists a partition P_t such that t is constant on each open subinterval of P_t. Therefore $s + t$ is a step function with step partition

$$P = P_s \cup P_t = \{x_0, x_1, \ldots, x_n\},$$

the common refinement of P_s and P_t with subdivision points x_0, x_1, …, x_n.

s and t remain constant on every open subinterval of P. Let s_k denote the constant value of s on the kth open subinterval of P. Let t_k denote the constant value of t on the kth open subinterval of P. By definition of the ✎ Integral of a Step Function,

$$\begin{aligned}
\int_a^b [s(x) + t(x)]\, dx &= \sum_{k=1}^{n} (s_k + t_k) \cdot (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} [s_k \cdot (x_k - x_{k-1}) + t_k \cdot (x_k - x_{k-1})] \\
&= \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}) + \sum_{k=1}^{n} t_k \cdot (x_k - x_{k-1}) \\
&= \int_a^b s(x)\, dx + \int_a^b t(x)\, dx\,.
\end{aligned}$$

□

2.4.2 Homogeneous Property

Theorem 1.3. *Let s be a ¶ Step Function on closed interval $[a, b]$. For every real number c, we have*

$$\int_a^b c \cdot s(x)\, dx = c \int_a^b s(x)\, dx\,.$$

Proof. Let s be a step function on closed interval $[a, b]$. By definition of a step function, there exists a ¶ Partition $P = \{x_0, x_1, \ldots, x_n\}$ such that s is constant on each open subinterval of P. Let s_k denote the constant value of s on the kth open subinterval of P. Then $c \cdot s$ is a step function with step partition P. By definition of the Integral of a Step Function,

$$\begin{aligned}
\int_a^b c \cdot s(x)\, dx &= \sum_{k=1}^{n} c \cdot s_k \cdot (x_k - x_{k-1}) \\
&= c \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}) \\
&= c \int_a^b s(x)\, dx\,.
\end{aligned}$$

□

2.4.3 Linearity Property

Theorem 1.4. *Let s and t be ¶ Step Functions on closed interval $[a, b]$. For every real c_1 and c_2, we have*

$$\int_a^b [c_1 s(x) + c_2 t(x)]\, dx = c_1 \int_a^b s(x)\, dx + c_2 \int_a^b t(x)\, dx\,.$$

Proof. Let s and t be step functions on closed interval $[a, b]$. Let c_1 and c_2 be real numbers. Then $c_1 \cdot s$ and $c_2 \cdot t$ are step functions. Then

$$\begin{aligned}
&\int_a^b [c_1 s(x) + c_2 t(x)]\, dx \\
&= \int_a^b c_1 s(x)\, dx + \int_a^b c_2 t(x)\, dx && \text{✎ Additive Property} \\
&= c_1 \int_a^b s(x)\, dx + c_2 \int_a^b t(x).\, dx && \text{✎ Homogeneous Property}
\end{aligned}$$

□

2.4.4 ✎ Comparison Theorem

Theorem 1.5. *Let s and t be ¶ Step Functions on closed interval $[a, b]$. If $s(x) < t(x)$ for every x in $[a, b]$, then*

$$\int_a^b s(x)\, dx < \int_a^b t(x)\, dx\,.$$

Proof. Let s and t be step functions on closed interval $[a, b]$. By definition of a step function, there exists a ¶ Partition P_s such that s is constant on each open subinterval of P_s. Likewise, there exists a partition P_t such that t is constant on each open subinterval of P_t. Let

$$P = P_s \cup P_t = \{x_0, x_1, \ldots, x_n\}$$

be the common refinement of P_s and P_t with subdivision points x_0, x_1, …, x_n.

By construction, P is a step partition for both s and t. Thus s and t remain constant on every open subinterval of P. Let s_k denote the constant value of s on the kth open subinterval of P. Let t_k denote the constant value of t on the kth open subinterval of P. By definition of the ✎ Integral of a Step Function,

$$\begin{aligned}
\int_a^b s(x)\, dx &= \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}) \\
&< \sum_{k=1}^{n} t_k \cdot (x_k - x_{k-1}) && \text{by hypothesis} \\
&= \int_a^b t(x)\, dx\,.
\end{aligned}$$

□

2.4.5 Additivity With Respect to the Interval of Integration

Theorem 1.6. *Let $a, b, c \in \mathbb{R}$ and s a ¶ Step Function on the smallest closed interval containing them. Then*

$$\int_a^c s(x)\,dx + \int_c^b s(x)\,dx + \int_b^a s(x)\,dx = 0.$$

Proof. WLOG, suppose $a < c < b$ and s be a step function on closed interval $[a, b]$. By definition of a step function, there exists a ¶ Partition P such that s is constant on each open subinterval of P.

Let $Q = \{x_0, x_1, \ldots, x_n\}$ be a refinement of P that includes c as a subdivision point. Then Q is a step partition of s and there exists some $0 < i < n$ such that $x_i = c$. Let s_k denote the constant value of s on the kth open subinterval of Q. By definition of the Integral of a Step Function,

$$\begin{aligned}
\int_a^b s(x)\,dx &= \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}) \\
&= \sum_{k=1}^{i} s_k \cdot (x_k - x_{k-1}) + \sum_{k=i+1}^{n} s_k \cdot (x_k - x_{k-1}) \\
&= \int_a^c s(x)\,dx + \int_c^b s(x)\,dx\,.
\end{aligned}$$

Rearranging terms shows

$$\begin{aligned}
0 &= \int_a^c s(x)\,dx + \int_c^b s(x)\,dx - \int_a^b s(x)\,dx \\
&= \int_a^c s(x)\,dx + \int_c^b s(x)\,dx + \int_b^a s(x)\,dx\,.
\end{aligned}$$

□

2.4.6 Invariance Under Translation

Theorem 1.7. *Let s be a step function on closed interval $[a, b]$. Then*

$$\int_a^b s(x)\,dx = \int_{a+c}^{b+x} s(x-c)\,dx \quad \textit{for every real } c.$$

Proof. Let s be a step function on closed interval $[a, b]$. By definition of a step function, there exists a ¶ Partition $P = \{x_0, x_1, \ldots, x_n\}$ such that s is constant on each open subinterval of P. Let s_k denote the constant value of s on the kth open subinterval of P.

Let c be a real number. Then $t(x) = s(x - c)$ is a step function on closed interval $[a+c, b+c]$ with partition $Q = \{x_0+c, x_1+c, \ldots, x_n+c\}$. Furthermore, t is constant on each open subinterval of Q. Let t_k denote the value of t on the kth open subinterval of Q. By construction, $t_k = s_k$.

By definition of the ✎ Integral of a Step Function,

$$\begin{aligned}
\int_{a+c}^{b+c} s(x-c)\,dx &= \int_{a+c}^{b+c} t(x)\,dx \\
&= \sum_{k=1}^{n} t_k \cdot ((x_k + c) - (x_{k-1} + c)) \\
&= \sum_{k=1}^{n} t_k \cdot (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}) \\
&= \int_{a}^{b} s(x)\,dx\,.
\end{aligned}$$

□

2.4.7 ✎ Expansion or Contraction of the Interval of Integration

Theorem 1.8. *Let s be a step function on closed interval $[a, b]$. Then*

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right) dx = k \int_{a}^{b} s(x)\,dx \quad \textit{for every } k \neq 0.$$

Proof. Let s be a step function on closed interval $[a, b]$. By definition of a step function, there exists a ¶ Partition $P = \{x_0, x_1, \ldots, x_n\}$ such that s is constant on each open subinterval of P. Let s_i denote the value of s on the ith open subinterval of P.

Let $k \neq 0$ be a real number. There are two cases to consider:

Case 1 Suppose $k > 0$. Then $t(x) = s(x/k)$ is a step function on closed interval $[ka, kb]$ with partition $Q = \{kx_0, kx_1, \ldots, kx_n\}$. Furthermore $t_i = s_i$.

By definition of the Integral of a Step Function,

$$\begin{aligned}
\int_{ka}^{kb} s(x/k)\,dx &= \int_{ka}^{kb} t(x)\,dx \\
&= \sum_{i=1}^{n} t_i \cdot (kx_i - kx_{i-1}) \\
&= k\sum_{i=1}^{n} t_i \cdot (x_i - x_{i-1}) \\
&= k\sum_{i=1}^{n} s_i \cdot (x_i - x_{i-1}) \\
&= k\int_{a}^{b} s(x)\,dx\,.
\end{aligned}$$

Case 2 Let $k < 0$ be a real number. Then $t(x) = s(x/k)$ is a step function on closed interval $[kb, ka]$ with partition $Q = \{kx_n, kx_{n-1}, \ldots, kx_0\}$. Furthermore $t_i = s_i$. By definition of the Integral of a Step Function,

$$\begin{aligned}
\int_{ka}^{kb} s(x/k)\,dx &= -\int_{kb}^{ka} s(x/k)\,dx \\
&= -\int_{kb}^{ka} t(x)\,dx \\
&= -\sum_{i=1}^{n} t_i \cdot (kx_{i-1} - kx_i) \\
&= -\sum_{i=1}^{n} s_i \cdot (kx_{i-1} - kx_i) \\
&= -\sum_{i=1}^{n} s_i \cdot (-k) \cdot (x_i - x_{i-1}) \\
&= k\sum_{i=1}^{n} s_i \cdot (x_i - x_{i-1}) \\
&= k\int_{a}^{b} s(x)\,dx\,.
\end{aligned}$$

□

2.4.8 Reflection Property

Let s be a step function on closed interval $[a, b]$. Then

$$\int_{a}^{b} s(x)\,dx = -\int_{-b}^{-a} s(-x)\,dx\,.$$

Proof. Let $k = -1$. By Expansion or Contraction of the Interval of Integration,

$$\int_{-a}^{-b} s\left(\frac{x}{-1}\right) = -\int_{a}^{b} s(x)\,dx\,.$$

Simplifying the left-hand side of the above identity, and multiplying both sides by -1 yields the desired result.

□

2.5 Exercises 1.15

2.5.1 Exercise 1.15.1

Compute the value of each of the following integrals.

Exercise 1.15.1a

$\int_{-1}^{3} \lfloor x \rfloor \, dx$.

Proof. Let $s(x) = \lfloor x \rfloor$ with domain $[-1, 3]$. By construction, s is a step function with partition $P = \{-1, 0, 1, 2, 3\} = \{x_0, x_1, x_2, x_3, x_4\}$. Let s_k denote the constant value s takes on the kth open subinterval of P. By definition of the Integral of a Step Function,

$$\begin{aligned}\int_{-1}^{3} \lfloor x \rfloor \, dx &= \sum_{k=1}^{4} s_k \cdot (x_k - x_{k-1}) \\ &= -1 + 0 + 1 + 2 \\ &= 2.\end{aligned}$$

□

Exercise 1.15.1c

$\int_{-1}^{3} \left(\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor\right) dx$.

Proof. Let $s(x) = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$ with domain $[-1, 3]$. By construction, s is a step function with partition

$$\begin{aligned}P &= \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\} \\ &= \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}.\end{aligned}$$

Let s_k denote the constant value s takes on the kth open subinterval of P. By definition of the Integral of a Step Function,

$$\begin{aligned}
\int_{-1}^{3} \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor &= \sum_{k=1}^{8} s_k \cdot (x_k - x_{k-1}) \\
&= \frac{1}{2} \sum_{k=1}^{8} s_k \\
&= \frac{1}{2} (-2 - 1 + 0 + 1 + 2 + 3 + 4 + 5) \\
&= 6.
\end{aligned}$$

□

Exericse 1.15.1e

$\int_{-1}^{3} \lfloor 2x \rfloor \, dx$.

Proof. Let $s(x) = \lfloor 2x \rfloor$. By Hermite's Identity, $s(x) = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$. Thus, by Exercise 1.15.1c,

$$\int_{-1}^{3} \lfloor 2x \rfloor \, dx = 6.$$

□

2.5.2 Exercise 1.15.3

Show that $\int_a^b \lfloor x \rfloor \, dx + \int_a^b \lfloor -x \rfloor \, dx = a - b$.

Proof. Let $s(x) = \lfloor x \rfloor$ and $t(x) = \lfloor -x \rfloor$, both with domain $[a, b]$. Let $x_1, \ldots, x_{n-1}$ denote the integers found in interval (a, b). Then $P = \{x_0, x_1, \ldots, x_n\}$, $x_0 = a$ and $x_n = b$, is a step ¶ Partition of both s and t. Let s_k and t_k denote the constant values s and t take on the kth open subinterval of P respectively. By Exercise 1.11.4b, $\lfloor -x \rfloor = - \lfloor x \rfloor - 1$ for all x in every open subinterval of

P. That is, $s_k = -t_k - 1$. By definition of the ✎ Integral of a Step Function,

$$\begin{aligned}
\int_a^b \lfloor x \rfloor\, dx + \int_a^b \lfloor -x \rfloor\, dx &= \sum_{k=1}^n s_k(x_k - x_{k-1}) + \sum_{k=1}^n t_k(x_k - x_{k-1}) \\
&= \sum_{k=1}^n (x_k - x_{k-1}) \cdot (s_k + t_k) \\
&= \sum_{k=1}^n (x_k - x_{k-1}) \cdot (-t_k - 1 + t_k) \\
&= \sum_{k=1}^n (x_{k-1} - x_k) \\
&= x_0 - x_n \\
&= a - b.
\end{aligned}$$

□

2.5.3 ✎ Exercise 1.15.5

✎ Exercise 1.15.5a

Prove that $\int_0^2 \lfloor t^2 \rfloor\, dt = 5 - \sqrt{2} - \sqrt{3}$.

Proof. Let $s(t) = \lfloor t^2 \rfloor$ with domain $[0, 2]$. Then s is a ¶ Step Function with partition $P = \{0, 1, \sqrt{2}, \sqrt{3}, 2\} = \{x_0, x_1, \dots, x_4\}$. Let s_k denote the constant value that s takes in the kth open subinterval of P. By the ✎ Integral of a Step Function,

$$\begin{aligned}
\int_0^2 \lfloor t^2 \rfloor\, dt &= \sum_{k=1}^4 s_k \cdot (x_k - x_{k-1}) \\
&= 0 \cdot (1 - 0) + 1 \cdot (\sqrt{2} - 1) + 2 \cdot (\sqrt{3} - \sqrt{2}) + 3 \cdot (2 - \sqrt{3}) \\
&= 5 - \sqrt{2} - \sqrt{3}.
\end{aligned}$$

□

✎ Exercise 1.15.5b

Compute $\int_{-3}^3 \lfloor t^2 \rfloor\, dt$.

Proof. Let $s(t) = \lfloor t^2 \rfloor$ with domain $[0, 3]$. Then s is a ¶ Step Function with ¶ Partition

$$\begin{aligned}
P &= \{\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{9}\} \\
&= \{x_0, x_1, \dots, x_9\}.
\end{aligned}$$

Let s_k denote the constant value that s takes in the kth open subinterval of P. By the ✎ Integral of a Step Function,

$$\int_0^3 \lfloor t^2 \rfloor \, dt = \sum_{k=1}^{9} s_k \cdot (x_k - x_{k-1})$$
$$= \sum_{k=0}^{8} k \cdot (\sqrt{k+1} - \sqrt{k}). \tag{2.19}$$

We notice $\lfloor t^2 \rfloor$ is symmetric about the y-axis. Thus

$$\int_{-3}^{0} \lfloor t^2 \rfloor \, dt = \int_0^3 \lfloor t^2 \rfloor. \tag{2.20}$$

By ✎ Additivity With Respect to the Interval of Integration,

$$\begin{aligned}
\int_{-3}^{3} \lfloor t^2 \rfloor \, dt &= \int_{-3}^{0} \lfloor t^2 \rfloor \, dt + \int_0^3 \lfloor t^2 \rfloor \, dt \\
&= 2 \int_0^3 \lfloor t^2 \rfloor \, dt && (2.20) \\
&= 2 \left[\sum_{k=0}^{8} k \cdot (\sqrt{k+1} - \sqrt{k}) \right]. && (2.19)
\end{aligned}$$

□

2.5.4 ✎ Exercise 1.15.7

✎ Exercise 1.15.7a

Compute $\int_0^9 \lfloor \sqrt{t} \rfloor \, dt$.

Proof. Let $s(t) = \lfloor \sqrt{t} \rfloor$ with domain $[0, 9]$. Then s is a ¶ Step Function with ¶ Partition $P = \{0, 1, 4, 9\} = \{x_0, x_1, x_2, x_3\}$. Let s_k denote the constant value that s takes in the kth open subinterval of P. By the ✎ Integral of a Step Function,

$$\begin{aligned}
\int_0^9 \lfloor \sqrt{t} \rfloor \, dt &= \sum_{k=1}^{3} s_k \cdot (x_k - x_{k-1}) \\
&= 0 \cdot (1 - 0) + 1 \cdot (4 - 1) + 2 \cdot (9 - 4) \\
&= 13.
\end{aligned}$$

□

✎ Exercise 1.15.7b

If n is a positive integer, prove that

$$\int_0^{n^2} \left\lfloor \sqrt{t} \right\rfloor dt = n(n-1)(4n+1)/6.$$

Proof. Define predicate $P(n)$ as

$$\int_0^{n^2} \left\lfloor \sqrt{t} \right\rfloor dt = \frac{n(n-1)(4n+1)}{6}. \tag{2.21}$$

We use induction to prove $P(n)$ holds for all integers satisfying $n > 0$.

Base Case Let $n = 1$. Define $s(t) = \left\lfloor \sqrt{t} \right\rfloor$ with domain $[0, 1]$. Then s is a ¶ Step Function with ¶ Partition $P = \{0, 1\} = \{x_0, x_1\}$. Let s_k denote the constant value of s on the kth open subinterval of P. By definition of the ✎ Integral of a Step Function, the left-hand side of (2.21) evaluates to

$$\begin{aligned}
\int_0^{n^2} \left\lfloor \sqrt{t} \right\rfloor dt &= \int_0^{1} \left\lfloor \sqrt{t} \right\rfloor dt \\
&= \sum_{k=1}^{1} s_k \cdot (x_k - x_{k-1}) \\
&= 0.
\end{aligned}$$

The right-hand side of (2.21) likewise evaluates to 0. Thus $P(1)$ holds.

Induction Step Let $n > 0$ be a positive integer and suppose $P(n)$ is true. Define $s(t) = \left\lfloor \sqrt{t} \right\rfloor$ with domain $[0, (n+1)^2]$. Then s is a ¶ Step Function with ¶ Partition

$$\begin{aligned}
P &= \{0, 1, 4, \dots, n^2, (n+1)^2\} \\
&= \{x_0, x_1, \dots, x_n, x_{n+1}\}.
\end{aligned}$$

Let s_k denote the constant value of s on the kth open subinterval of P. By definition of the Integral of a Step Function, it follows that

$$\begin{aligned}
&\int_0^{(n+1)^2} s(x)\,dx \\
&= \sum_{k=1}^{n+1} s_k \cdot (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}) + [s_{n+1} \cdot (x_{n+1} - x_n)] \\
&= \int_0^{n^2} s(x)\,dx + [s_{n+1} \cdot (x_{n+1} - x_n)] \\
&= \int_0^{n^2} s(x)\,dx + \left[n \cdot ((n+1)^2 - n^2)\right] \\
&= \int_0^{n^2} s(x)\,dx + \left[2n^2 + n\right] \\
&= \frac{n(n-1)(4n+1)}{6} + 2n^2 + n && \text{induction hypothesis} \\
&= \frac{n(n-1)(4n+1) + 12n^2 + 6n}{6} \\
&= \frac{4n^3 + 9n^2 + 5n}{6} \\
&= \frac{(n^2+n)(4n+5)}{6} \\
&= \frac{(n+1)((n+1)-1)(4(n+1)+1)}{6}.
\end{aligned}$$

Thus $P(n+1)$ holds.

Conclusion By mathematical induction, it follows for all positive integers n, $P(n)$ is true.

□

2.5.5 Exercise 1.15.9

Show that the following property is equivalent to Expansion or Contraction of the Interval of Integration:

$$\int_{ka}^{kb} f(x)\,dx = k \int_a^b f(kx)\,dx\,. \tag{2.22}$$

Proof. Let f be a step function on closed interval $[a, b]$ and $k \neq 0$. Applying Expansion or Contraction of the Interval of Integration to the right-hand side of (2.22) yields

$$k \int_{ka}^{kb} f(kx/k)\,dx = k\left[k \int_{a}^{b} f(kx)\,dx\right].$$

Simplifying the left-hand side and dividing both sides by k immediately yields the desired result.

□

2.5.6 Exercise 1.15.11

If we instead defined the integral of step functions as

$$\int_a^b s(x)\,dx = \sum_{k=1}^{n} s_k^3 \cdot (x_k - x_{k-1}),$$

a new and different theory of integration would result. Which of the following properties would remain valid in this new theory?

Exercise 1.15.11a

$\int_a^b s + \int_b^c s = \int_a^c s.$

> Note: This property mirrors Additivity With Respect to the Interval of Integration.

Proof. The above property is **valid**.

WLOG, suppose $a < b < c$. Let s be a step function defined on closed interval $[a, c]$. By definition of a ¶ Step Function, there exists a ¶ Partition such that s is constant on each open subinterval of P. Let $Q = \{x_0, x_1, \ldots, x_n\}$ be a refinement of P that includes b as a subdivision point. Then Q is a step partition of s and there exists some $0 < i < n$ such that $x_i = c$. Let s_k denote the constant value of s on the kth open subinterval of Q. By (2.5.6),

$$\begin{aligned}
\int_a^c s &= \sum_{k=1}^{n} s_k^3 \cdot (x_k - x_{k-1}) \\
&= \sum_{k=1}^{i} s_k^3 \cdot (x_k - x_{k-1}) + \sum_{k=i+1}^{n} s_k^3 \cdot (x_k - x_{k-1}) \\
&= \int_a^b s + \int_b^c s.
\end{aligned}$$

□

✎ Exercise 1.15.11b

$\int_a^b (s+t) = \int_a^b s + \int_a^b t$.

> Note: This property mirrors the ✎ Additive Property.

Proof. The above property is **invalid**.

Let s and t be step functions on closed interval $[a, b]$. By definition of a step function, there exists a ¶ Partition P_s such that s is constant on each open subinterval of P_s. Likewise, there exists a partition P_t such that t is constant on each open subinterval of P_t. Therefore $s + t$ is a step function with step partition

$$P = P_s \cup P_t = \{x_0, x_1, \ldots, x_n\},$$

the common refinement of P_s and P_t with subdivision points x_0, x_1, …, x_n.

s and t remain constant on every open subinterval of P. Let s_k denote the constant value of s on the kth open subinterval of P_s. Let t_k denote the constant value of t on the kth open subinterval of P_t. By (2.5.6),

$$\begin{aligned}
\int_a^b s + t &= \sum_{k=1}^{n} (s_k + t_k)^3 \cdot (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} \left[s_k^3 + 3s_k^2 t_k + 3s_k t_k^2 + t_k^3\right] \\
&= \sum_{k=1}^{n} s_k^3 \cdot (x_k - x_{k-1}) + \\
&\qquad \sum_{k=1}^{n} t_k^3 \cdot (x_k - x_{k-1}) + \\
&\qquad \sum_{k=1}^{n} (3s_k^2 t_k + 3s_k t_k^2) \cdot (x_k - x_{k-1}) \\
&= \int_a^b s + \int_a^b t + \sum_{k=1}^{n} (3s_k^2 t_k + 3s_k t_k^2) \cdot (x_k - x_{k-1}).
\end{aligned}$$

Since this last addend does not necessarily equal 0, the desired property is invalid.

□

✎ Exercise 1.15.11c

$\int_a^b c \cdot s = c \int_a^b s$.

> Note: This property mirrors the ✎ Homogeneous Property.

Proof. The above property is **invalid**.

Let s be a step function on closed interval $[a, b]$. By definition of a step function, there exists a ¶ Partition $P = \{x_0, x_1, \ldots, x_n\}$ such that s is constant on each open subinterval of P. Let s_k denote the constant value of s on the kth open subinterval of P. Then $c \cdot s$ is a step function with step partition P. By (2.5.6),

$$\begin{aligned}
\int_a^b c \cdot s &= \sum_{k=1}^{n} (c \cdot s_k)^3 \cdot (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} c^3 \cdot s_k^3 \cdot (x_k - x_{k-1}) \\
&= c^3 \sum_{k=1}^{n} s_k^3 \cdot (x_k - x_{k-1}) \\
&= c^3 \int_a^b s.
\end{aligned}$$

Since c^3 does not necessarily equal c, the desired property is invalid.

□

✎ Exercise 1.15.11d

$\int_{a+c}^{b+c} s(x)\,dx = \int_a^b s(x+c)\,dx$.

> Note: This property mirrors ✎ Invariance Under Translation.

Proof. The above property is **valid**.

Let s be a step function on closed interval $[a + c, b + c]$. By definition of a ¶ Step Function, there exists a ¶ Partition $P = \{x_0, x_1, \ldots, x_n\}$ such that s is constant on each open subinterval of P. Let s_k denote the constant value of s on the kth open subinterval of P.

Let c be a real number. Then $t(x) = s(x + c)$ is a step function on closed interval $[a, b]$ with partition $Q = \{x_0 - c, x_1 - c, \ldots, x_n - c\}$. Furthermore, t is constant on each open subinterval of Q. Let t_k denote the value of t on the kth open subinterval of Q. By construction, $t_k = s_k$. By (2.5.6),

$$\begin{aligned}
\int_{a+c}^{b+c} s(x)\,dx &= \sum_{k=1}^{n} s_k^3 \cdot (x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} s_k^3 \cdot ((x_k - c) - (x_{k-1} - c)) \\
&= \sum_{k=1}^{n} t_k^3 \cdot ((x_k - c) - (x_{k-1} - c)) \\
&= \int_a^b t(x)\,dx \\
&= \int_a^b s(x+c)\,dx\,.
\end{aligned}$$

□

Exercise 1.15.11e

If $s(x) < t(x)$ for each x in $[a, b]$, then $\int_a^b s < \int_a^b t$.

Note: This property mirrors the Comparison Theorem.

Proof. The above property is **valid**.

Let s and t be step functions on closed interval $[a, b]$. By definition of a ¶ Step Function, there exists a ¶ Partition P_s such that s is constant on each open subinterval of P_s. Likewise, there exists a partition P_t such that t is constant on each open subinterval of P_t. Let

$$P = P_s \cup P_t = \{x_0, x_1, \ldots, x_n\}$$

be the common refinement of P_s and P_t with subdivision points x_0, x_1, …, x_n.

By construction, P is a step partition for both s and t. Thus s and t remain constant on every open subinterval of P. Let s_k denote the constant value of s on the kth open subinterval of P. Let t_k denote the constant value of t on the kth open subinterval of P. By (2.5.6),

$$\begin{aligned}
\int_a^b s &= \sum_{k=1}^{n} s_k^3 \cdot (x_k - x_{k-1}) \\
&< \sum_{k=1}^{n} t_k^3 \cdot (x_k - x_{k-1}) \\
&= \int_a^b t.
\end{aligned}$$

$\square$

2.6 Upper and Lower Integrals

2.6.1 ✎ Theorem 1.9

Theorem 1.9. *Every function f which is bounded on $[a, b]$ has a lower integral $\underline{I}(f)$ and an upper integral $\bar{I}(f)$ satisfying the inequalities*

$$\int_a^b s(x)\,dx \leq \underline{I}(f) \leq \bar{I}(f) \leq \int_a^b t(x)\,dx \tag{2.23}$$

for all ¶ Step Functions s and t with $s \leq f \leq t$. The function f is ✎ Integrable on $[a, b]$ if and only if its upper and lower integrals are equal, in which case we have

$$\int_a^b f(x)\,dx = \underline{I}(f) = \bar{I}(f).$$

Proof. Let f be a function bounded on $[a, b]$. We prove that (i) f has a lower and upper integral satisfying (2.23) and (ii) that f is integrable on $[a, b]$ if and only if its lower and upper integrals are equal.

(i) Because f is bounded, there exists some $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

Let S denote the set of numbers $\int_a^b s(x)\,dx$ obtained as s runs through all step functions below f. That is, let

$$S = \left\{ \int_a^b s(x)\,dx : s \leq f \right\}.$$

Note S is nonempty since, e.g. constant function $c(x) = -M$ is a member.

Likewise, let T denote the set of numbers $\int_a^b t(x)\,dx$ obtained as t runs through all step functions above f. That is, let

$$T = \left\{ \int_a^b t(x)\,dx : f \leq t \right\}.$$

Note T is nonempty since e.g. constant function $c(x) = M$ is a member.

By construction, $s \leq t$ for every s in S and t in T. Therefore ✔ Theorem I.34 tells us S has a ¶ Supremum, T has an ¶ Infimum, and $\sup S \leq \inf T$. By definition of the ✎ Lower Integral, $\underline{I}(f) = \sup S$. By definition of the ✎ Upper Integral, $\bar{I}(f) = \inf S$. Thus (2.23) holds.

(ii) By definition of integrability, f is integrable on $[a, b]$ if and only if there exists one and only one number I such that

$$\int_a^b s(x)\,dx \leq I \leq \int_a^b t(x)\,dx$$

for every pair of step functions s and t satisfying (1). By (2.23) and the definition of the supremum/infimum, this holds if and only if $\underline{I}(f) = \bar{I}(f)$, concluding the proof.

□

2.7 The Area of an Ordinate Set Expressed as an Integral

2.7.1 Theorem 1.10

Theorem 1.10. *Let f be a nonnegative function, Integrable on an interval $[a, b]$, and let Q denote the ordinate set of f over $[a, b]$. Then Q is measurable and its area is equal to the integral $\int_a^b f(x)\,dx$.*

Proof. Let f be a nonnegative function, Integrable on $[a, b]$. By definition of integrability, there exists one and only one number I such that

$$\int_a^b s(x)\,dx \leq I \leq \int_a^b t(x)\,dx$$

for every pair of step functions s and t satisfying (1). In other words, I is the one and only number that satisfies

$$a(S) \leq I \leq a(T)$$

for every pair of step regions $S \subseteq Q \subseteq T$. By the Exhaustion Property, Q is measurable and its area is equal to $I = \int_a^b f(x)\,dx$.

□

2.7.2 Theorem 1.11

Theorem 1.11. *Let f be a nonnegative function, integrable on an interval $[a, b]$. Then the graph of f, that is, the set*

$$\{(x, y) \mid a \leq x \leq b, y = f(x)\}, \tag{2.24}$$

is measurable and has area equal to 0.

Proof. Let f be a nonnegative function, integrable on an interval $[a,b]$. Let

$$Q' = \{(x,y) \mid a \leq x \leq b, 0 \leq y < f(x)\}.$$

We show that (i) Q' is measurable with area equal to $\int_a^b f(x)\,dx$ and (ii) the graph of f is meaurable with area equal to 0.

(i) By definition of integrability, there exists one and only one number I such that

$$\int_a^b s(x)\,dx \leq I \leq \int_a^b t(x)\,dx$$

for every pair of step functions s and t satisfying (1). In other words, I is the one and only number that satisfies

$$a(S) \leq I \leq a(T)$$

for every pair of step regions $S \subseteq Q' \subseteq T$. By the ✎ Exhaustion Property, Q' is measurable and its area is equal to $I = \int_a^b f(x)\,dx$.

(ii) Let Q denote the ordinate set of f. By ✎ Theorem 1.10, Q is measurable with area equal to the integral $I = \int_a^b f(x)\,dx$. By (i), Q' is measurable with area also equal to I. We note the graph of f, (2.24), is equal to set $Q - Q'$. By the ¶ Difference Property, $Q - Q'$ is measurable and

$$a(Q - Q') = a(Q) - a(Q') = I - I = 0.$$

Thus the graph of f is measurable and has area equal to 0.

□

2.8 Integrability of Bounded Monotonic Functions

2.8.1 ✎ Theorem 1.12

Theorem 1.12. *If f is ✎ Monotonic on a closed interval $[a,b]$, then f is ✎ Integrable on $[a,b]$.*

Proof. Let f be a monotonic function on closed interval $[a,b]$. That is to say, either f is increasing on $[a,b]$ or f is decreasing on $[a,b]$. Because f is on a closed interval, it is bounded. By ✎ Theorem 1.9, f has a ✎ Lower Integral $\underline{I}(f)$, f has an ✎ Upper Integral $\bar{I}(f)$, and f is integrable if and only if $\underline{I}(f) = \bar{I}(f)$.

Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ in which $x_k - x_{k-1} = (b-a)/n$ for each $k = 1, \ldots, n$. There are two cases to consider:

Case 1 Suppose f is increasing. Let s be the step function below f with constant value $f(x_{k-1})$ on every kth open subinterval of P. Let t be the step function above f with constant value $f(x_k)$ on every kth open subinterval of P. Then, by (2.23), it follows

$$\int_a^b s(x)\,dx \leq \underline{I}(f) \leq \bar{I}(f) \leq \int_a^b t(x)\,dx\,. \tag{2.25}$$

By definition of the Integral of a Step Function,

$$\begin{aligned}\int_a^b s(x)\,dx &= \sum_{k=1}^n f(x_{k-1})\left[\frac{b-a}{n}\right]\\ \int_a^b t(x)\,dx &= \sum_{k=1}^n f(x_k)\left[\frac{b-a}{n}\right].\end{aligned}$$

Thus

$$\begin{aligned}\int_a^b t(x)\,dx - \int_a^b s(x)\,dx &= \sum_{k=1}^n f(x_k)\left[\frac{b-a}{n}\right] - \sum_{k=1}^n f(x_{k-1})\left[\frac{b-a}{n}\right]\\ &= \left[\frac{b-a}{n}\right]\sum_{k=1}^n f(x_k) - f(x_{k-1})\\ &= \frac{(b-a)(f(b)-f(a))}{n}.\end{aligned}$$

By (2.25),

$$\begin{aligned}\underline{I}(f) &\leq \bar{I}(f)\\ &\leq \int_a^b t(x)\,dx\\ &= \int_a^b s(x)\,dx + \frac{(b-a)(f(b)-f(a))}{n}\\ &\leq \underline{I}(f) + \frac{(b-a)(f(b)-f(a))}{n}.\end{aligned}$$

Since the above holds for all positive integers n, Theorem I.31 indicates $\underline{I}(f) = \bar{I}(f)$.

Case 2 Suppose f is decreasing. Let s be the step function below f with constant value $f(x_k)$ on every kth open subinterval of P. Let t be the step function above f with constant value $f(x_{k-1})$ on every kth open subinterval of P. Then, by (2.23), it follows

$$\int_a^b s(x)\,dx \leq \underline{I}(f) \leq \bar{I}(f) \leq \int_a^b t(x)\,dx\,. \tag{2.26}$$

By definition of the Integral of a Step Function,

$$\begin{aligned}\int_a^b s(x)\,dx &= \sum_{k=1}^{n} f(x_k)\left[\frac{b-a}{n}\right] \\ \int_a^b t(x)\,dx &= \sum_{k=1}^{n} f(x_{k-1})\left[\frac{b-a}{n}\right].\end{aligned}$$

Thus

$$\begin{aligned}\int_a^b t(x)\,dx - \int_a^b s(x)\,dx &= \sum_{k=1}^{n} f(x_{k-1})\left[\frac{b-a}{n}\right] - \sum_{k=1}^{n} f(x_k)\left[\frac{b-a}{n}\right] \\ &= \left[\frac{b-a}{n}\right]\sum_{k=1}^{n} f(x_{k-1}) - f(x_k) \\ &= \frac{(b-a)(f(a)-f(b))}{n}.\end{aligned}$$

By (2.26),

$$\begin{aligned}\underline{I}(f) &\leq \bar{I}(f) \\ &\leq \int_a^b t(x)\,dx \\ &= \int_a^b s(x)\,dx + \frac{(b-a)(f(a)-f(b))}{n} \\ &\leq \underline{I}(f) + \frac{(b-a)(f(a)-f(b))}{n}.\end{aligned}$$

Since the above holds for all positive integers n, Theorem I.31 indicates $\underline{I}(f) = \bar{I}(f)$.

□

2.8.2 Theorem 1.13

Theorem 1.13. *Assume f is increasing on a closed interval $[a, b]$. Let $x_k = a + k(b-a)/n$ for $k = 0, 1, \ldots, n$. If I is any number which satisfies the inequalities*

$$\frac{b-a}{n}\sum_{k=0}^{n-1} f(x_k) \leq I \leq \frac{b-a}{n}\sum_{k=1}^{n} f(x_k) \tag{2.27}$$

for every integer $n \geq 1$, then $I = \int_a^b f(x)\,dx$.

Proof. Let f be increasing on a closed interval $[a,b]$ and I be a number satisfying (2.27). Let s be the step function below f with constant value $f(x_{k-1})$ on every kth open subinterval of P. Let t be the step function above f with constant value $f(x_k)$ on every kth open subinterval of P. By definition of the Integral of a Step Function,

$$\begin{aligned}\int_a^b s(x)\,dx &= \sum_{k=1}^{n} f(x_{k-1})\left[\frac{b-a}{n}\right]\\ &= \sum_{k=0}^{n-1} f(x_k)\left[\frac{b-a}{n}\right]\\ \int_a^b t(x)\,dx &= \sum_{k=1}^{n} f(x_k)\left[\frac{b-a}{n}\right].\end{aligned}$$

Therefore (2.27) can alternatively be written as

$$\int_a^b s(x)\,dx \le I \le \int_a^b t(x)\,dx\,. \tag{2.28}$$

By Theorem 1.12, f is integrable. Therefore Theorem 1.9 indicates f satisfies

$$\int_a^b s(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b t(x)\,dx\,. \tag{2.29}$$

Manipulating (2.28) and (2.29) together yields

$$\begin{aligned}I - \int_a^b f(x)\,dx &\le \int_a^b t(x)\,dx - \int_a^b s(x)\,dx,\\ \int_a^b f(x)\,dx - I &\le \int_a^b t(x)\,dx - \int_a^b s(x)\,dx\,.\end{aligned}$$

Combining the above inequalities in turn yields

$$\begin{aligned}0 &\le \left|\int_a^b f(x)\,dx - I\right|\\ &\le \int_a^b t(x)\,dx - \int_a^b s(x)\,dx\\ &= \sum_{k=1}^{n} f(x_k)\left[\frac{b-a}{n}\right] - \sum_{k=1}^{n} f(x_{k-1})\left[\frac{b-a}{n}\right]\\ &= \frac{b-a}{n}\sum_{k=1}^{n} f(x_k) - f(x_{k-1})\\ &= \frac{(b-a)(f(b)-f(a))}{n}.\end{aligned}$$

The above chain of inequalities holds for all positive integers $n \geq 1$, meaning Theorem I.31 applies. Thus

$$\left| \int_a^b f(x)\,dx - I \right| = 0,$$

which immediately implies the desired result.

$\square$

2.8.3 Theorem 1.14

Theorem 1.14. *Assume f is descreasing on $[a,b]$. Let $x_k = a + k(b-a)/n$ for $k = 0, 1, \ldots, n$. If I is any number which satisfies the inequalities*

$$\frac{b-a}{n} \sum_{k=1}^{n} f(x_k) \leq I \leq \frac{b-a}{n} \sum_{k=0}^{n-1} f(x_k) \tag{2.30}$$

for every integer $n \geq 1$, then $I = \int_a^b f(x)\,dx$.

Proof. Let f be decreasing on a closed interval $[a,b]$ and I be a number satisfying (2.30). Let s be the step function below f with constant value $f(x_k)$ on every kth open subinterval of P. Let t be the step function above f with constant value $f(x_{k-1})$ on every kth open subinterval of P. By definition of the Integral of a Step Function,

$$\begin{aligned}
\int_a^b s(x)\,dx &= \sum_{k=1}^{n} f(x_k) \left[\frac{b-a}{n}\right] \\
\int_a^b t(x)\,dx &= \sum_{k=1}^{n} f(x_{k-1}) \left[\frac{b-a}{n}\right] \\
&= \sum_{k=0}^{n-1} f(x) \left[\frac{b-a}{n}\right].
\end{aligned}$$

Therefore (2.30) can alternatively be written as

$$\int_a^b s(x)\,dx \leq I \leq \int_a^b t(x)\,dx\,. \tag{2.31}$$

By Theorem 1.12, f is integrable. Therefore Theorem 1.9 indicates f satisfies

$$\int_a^b s(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b t(x)\,dx\,. \tag{2.32}$$

Manipulating (2.31) and (2.32) together yields

$$I - \int_a^b f(x)\,dx \le \int_a^b t(x)\,dx - \int_a^b s(x)\,dx,$$
$$\int_a^b f(x)\,dx - I \le \int_a^b t(x)\,dx - \int_a^b s(x)\,dx\,.$$

Combining the above inequalities in turn yields

$$\begin{aligned}
0 &\le \left|\int_a^b f(x)\,dx - I\right| \\
&\le \int_a^b t(x)\,dx - \int_a^b s(x)\,dx \\
&= \sum_{k=1}^{n} f(x_k)\left[\frac{b-a}{n}\right] - \sum_{k=1}^{n} f(x_{k-1})\left[\frac{b-a}{n}\right] \\
&= \frac{b-a}{n}\sum_{k=1}^{n} f(x_k) - f(x_{k-1}) \\
&= \frac{(b-a)(f(b)-f(a))}{n}.
\end{aligned}$$

The above chain of inequalities holds for all positive integers $n \ge 1$, meaning Theorem I.31 applies. Thus

$$\left|\int_a^b f(x)\,dx - I\right| = 0,$$

which immediately implies the desired result.

□

2.8.4 Integral of $\int_0^b x^p\,dx$ when p is a Positive Integer

Theorem 1.15. *If p is a positive integer and $b > 0$, we have*

$$\int_0^b x^p\,dx = \frac{b^{p+1}}{p+1}.$$

Proof. TODO

□