Axiomatic Framework of Area

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We assume there exists a class $\mathscr{M}$ of measurable sets in the plane and a set function a, whose domain is $\mathscr{M}$, with the following properties:

Nonnegative Property

For each set S in $\mathscr{M}$, we have $a(S) \geq 0$.

Axiom. Nonnegative Property

$\square$

Additive Property

If S and T are in $\mathscr{M}$, then $S \cup T$ and $S \cap T$ are in $\mathscr{M}$, and we have $a(S \cup T) = a(S) + a(T) - a(S \cap T)$.

Axiom. Additive Property

$\square$

Difference Property

If S and T are in $\mathscr{M}$ with $S \subseteq T$, then $T - S$ is in $\mathscr{M}$, and we have $a(T - S) = a(T) - a(S)$.

Axiom. Difference Property

$\square$

Invariance Under Congruence

If a set S is in $\mathscr{M}$ and if T is congruent to S, then T is also in $\mathscr{M}$ and we have $a(S) = a(T)$.

Axiom. Invariance Under Congruence

$\square$

Choice of Scale

Every rectangle R is in $\mathscr{M}$. If the edges of R have lengths h and k, then $a(R) = hk$.

Axiom. Choice of Scale

□

Exhaustion Property

Let Q be a set that can be enclosed between two step regions S and T, so that

$$S \subseteq Q \subseteq T. \tag{1}$$

If there is one and only one number c which satisfies the inequalities

$$a(S) \leq c \leq a(T)$$

for all step regions S and T satisfying (1.1), then Q is measurable and $a(Q) = c$.

Axiom. Exhaustion Property

□