Elements of Set Theory

Herbert B. Enderton

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Chapter R

Reference

R.1 ¶ Powerset

The **powerset** of some set A is the set of all subsets of A.

Definition. $\exists$ – Set.powerset

□

R.2 ¶ Principle of Extensionality

If A and B are sets such that for every object t,

$$t \in A \quad \text{iff} \quad t \in B,$$

then $A = B$.

Axiom. $\exists$ – Set.ext

□

Chapter 1

Introduction

1.1 Baby Set Theory

1.1.1 Exercise 1.1

Which of the following become true when "$\in$" is inserted in place of the blank? Which become true when "$\subseteq$" is inserted?

Exercise 1.1a

$\{\varnothing\} \_\_\_\_ \{\varnothing, \{\varnothing\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1a

Because the *object* $\{\varnothing\}$ is a member of the right-hand set, the statement is **true** in the case of "$\in$".

Because the *members* of $\{\varnothing\}$ are all members of the right-hand set, the statement is also **true** in the case of "$\subseteq$".

□

Exercise 1.1b

$\{\varnothing\} \_\_\_\_ \{\varnothing, \{\{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1b

Because the *object* $\{\varnothing\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\varnothing\}$ are all members of the right-hand set, the statement is **true** in the case of "$\subseteq$".

□

Exercise 1.1c

$\{\{\varnothing\}\}\_\_\_\_\{\varnothing, \{\varnothing\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1c

Because the *object* $\{\{\varnothing\}\}$ is not a member of the right-hand set, the statement is **false** in the case of ”$\in$”.

Because the *members* of $\{\{\varnothing\}\}$ are all members of the right-hand set, the statement is **true** in the case of ”$\subseteq$”.

□

Exercise 1.1d

$\{\{\varnothing\}\}\_\_\_\_\{\varnothing, \{\{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1d

Because the *object* $\{\{\varnothing\}\}$ is a member of the right-hand set, the statement is **true** in the case of ”$\in$”.

Because the *members* of $\{\{\varnothing\}\}$ are not all members of the right-hand set, the statement is **false** in the case of ”$\subseteq$”.

□

Exercise 1.1e

$\{\{\varnothing\}\}\_\_\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1e

Because the *object* $\{\{\varnothing\}\}$ is not a member of the right-hand set, the statement is **false** in the case of ”$\in$”.

Because the *members* of $\{\{\varnothing\}\}$ are not all members of the right-hand set, the statement is **false** in the case of ”$\subseteq$”.

□

1.1.2 Exercise 1.2

Show that no two of the three sets $\varnothing$, $\{\varnothing\}$, and $\{\{\varnothing\}\}$ are equal to each other.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_2

By the ¶ Principle of Extensionality, $\varnothing$ is only equal to $\varnothing$. This immediately shows it is not equal to the other two. Now consider object $\varnothing$. This object is a member of $\{\emptyset\}$ but is not a member of $\{\{\emptyset\}\}$. Again, by the ¶ Principle of Extensionality, these two sets must be different.

□

1.1.3 Exercise 1.3

Show that if $B \subseteq C$, then $\mathscr{P}\, B \subseteq \mathscr{P}\, C$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_3

Let $x \in \mathscr{P}\, B$. By definition of the ¶ Powerset, x is a subset of B. By hypothesis, $B \subseteq C$. Then $x \subseteq C$. Again by definition of the ¶ Powerset, it follows $x \in \mathscr{P}\, C$.

□

1.1.4 Exercise 1.4

Assume that x and y are members of a set B. Show that $\{\{x\}, \{x, y\}\} \in \mathscr{P}\, \mathscr{P}\, B$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_4

Let x and y be members of set B. Then $\{x\}$ and $\{x, y\}$ are subsets of B. By definition of the ¶ Powerset, $\{x\}$ and $\{x, y\}$ are members of $\mathscr{P}\, B$. Then $\{\{x\}, \{x, y\}\}$ is a subset of $\mathscr{P}\, B$. By definition of the ¶ Powerset, $\{\{x\}, \{x, y\}\}$ is a member of $\mathscr{P}\, \mathscr{P}\, B$.

□

1.2 Sets - An Informal View

1.2.1 Exercise 2.1

Define the rank of a set c to be the least α such that $c \subseteq V_\alpha$. Compute the rank of $\{\{\emptyset\}\}$. Compute the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

Proof. We first compute the values of V_n for $0 \leq n \leq 3$ under the assumption the set of atoms A at the bottom of the hierarchy is empty.

$$\begin{aligned}
V_0 &= \varnothing \\
V_1 &= V_0 \cup \mathscr{P}\, V_0 \\
&= \varnothing \cup \{\varnothing\} \\
&= \{\varnothing\} \\
V_2 &= V_1 \cup \mathscr{P}\, V_1 \\
&= \{\varnothing\} \cup \mathscr{P}\, \{\varnothing\} \\
&= \{\varnothing\} \cup \{\varnothing, \{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\} \\
V_3 &= V_2 \cup \mathscr{P}\, V_2 \\
&= \{\varnothing, \{\varnothing\}\} \cup \mathscr{P}\, \{\varnothing, \{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\} \cup \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
&= \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}
\end{aligned}$$

It then immediately follows $\{\{\varnothing\}\}$ has rank 2 and $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ has rank 3.

□

1.2.2 Exercise 2.2

We have stated that $V_{\alpha+1} = A \cup \mathscr{P}\, V_\alpha$. Prove this at least for $\alpha < 3$.

Proof. Let A be the set of atoms in our set hierarchy. Let $P(n)$ be the predicate, "$V_{n+1} = A \cup \mathscr{P}\, V_n$." We prove $P(n)$ holds true for all natural numbers $n \geq 1$ via induction.

Base Case Let $n = 1$. By definition, $V_1 = V_0 \cup \mathscr{P}\, V_0$. By definition, $V_0 = A$. Therefore $V_1 = A \cup \mathscr{P}\, V_0$. This proves $P(1)$ holds true.

Induction Step Suppose $P(n)$ holds true for some $n \geq 1$. Consider V_{n+1}. By definition, $V_{n+1} = V_n \cup \mathscr{P}\, V_n$. Therefore, by the induction hypothesis,

$$\begin{aligned}
V_{n+1} &= V_n \cup \mathscr{P}\, V_n \\
&= (A \cup \mathscr{P}\, V_{n-1}) \cup \mathscr{P}\, V_n \\
&= A \cup (\mathscr{P}\, V_{n-1} \cup \mathscr{P}\, V_n) \qquad (1.1)
\end{aligned}$$

But V_{n-1} is a subset of V_n. Exercise 1.3 then implies $\mathscr{P}\, V_{n-1} \subseteq \mathscr{P}\, V_n$. This means (1.1) can be simplified to

$$V_{n+1} = A \cup \mathscr{P}\, V_n,$$

proving $P(n+1)$ holds true.

Conclusion By mathematical induction, it follows for all $n \geq 1$, $P(n)$ is true. $\square$

1.2.3 ✎ Exercise 2.3

List all the members of V_3. List all the members of V_4. (It is to be assumed here that there are no atoms.)

Proof. As seen in the proof of ✎ Exercise 2.1,

$$V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}.$$

By ✎ Exercise 2.2, $V_4 = \mathscr{P}\, V_3$ (since it is assumed there are no atoms). Thus

$$\begin{aligned} V_4 = \{ & \\ & \varnothing, \\ & \{\varnothing\}, \\ & \{\{\varnothing\}\}, \\ & \{\{\{\varnothing\}\}\}, \\ & \{\{\varnothing, \{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing\}\}, \\ & \{\varnothing, \{\{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\{\varnothing\}, \{\{\varnothing\}\}\}, \\ & \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\ & \{\{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \\ & \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\ \}. & \end{aligned}$$

$\square$