Theorem I.27

Every nonempty set S that is bounded below has a greatest lower bound; that is, there is a real number L such that $L = \inf S$.

Proof. Chapter_I_3.Real.exists_isGLB

□

Theorem I.29

For every real x there exists a positive integer n such that $n > x$.

Proof. Chapter_I_3.Real.exists_pnat_geq_self

□

Theorem I.30 (Archimedean Property of the Reals)

If $x > 0$ and if y is an arbitrary real number, there exists a positive integer n such that $nx > y$.

Proof. Chapter_I_3.Real.exists_pnat_mul_self_geq_of_pos

□

Theorem I.31

If three real numbers a, x, and y satisfy the inequalities

$$a \leq x \leq a + \frac{y}{n}$$

for every integer $n \geq 1$, then $x = a$.

Proof. Chapter_I_3.Real.forall_pnat_leq_self_leq_frac_imp_eq

□

Theorem I.32

Let h be a given positive number and let S be a set of real numbers.

(a) If S has a supremum, then for some x in S we have

$$x > \sup S - h.$$

(b) If S has an infimum, then for some x in S we have

$$x < \inf S + h.$$

Proof.

(a) Chapter_I_3.Real.sup_imp_exists_gt_sup_sub_delta

(b) Chapter_I_3.Real.inf_imp_exists_lt_inf_add_delta

□

Theorem I.33 (Additive Property)

Given nonempty subsets A and B of $\mathbb{R}$, let C denote the set

$$C = \{a + b : a \in A, b \in B\}.$$

(a) If each of A and B has a supremum, then C has a supremum, and

$$\sup C = \sup A + \sup B.$$

(b) If each of A and B has an infimum, then C has an infimum, and

$$\inf C = \inf A + \inf B.$$

Proof.

(a) Chapter_I_3.Real.sup_minkowski_sum_eq_sup_add_sup

(b) Chapter_I_3.Real.inf_minkowski_sum_eq_inf_add_inf

□

Theorem I.34

Given two nonempty subsets S and T of $\mathbb{R}$ such that

$$s \leq t$$

for every s in S and every t in T. Then S has a supremum, and T has an infimum, and they satisfy the inequality

$$\sup S \leq \inf T.$$

Proof. Chapter_I_3.Real.forall_mem_le_forall_mem_imp_sup_le_inf

□