Elements of Set Theory

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Contents

Chapter R

Reference

R.1 ¶ Empty Set Axiom

There is a set having no members:

$$\exists B, \forall x, x \notin B.$$

Axiom. ∃ – Set.emptyCollection

□

R.2 ¶ Extensionality Axiom

If two sets have exactly the same members, then they are equal:

$$\forall A, \forall B, [\forall x, (x \in A \iff x \in B) \Rightarrow A = B].$$

Axiom. ∃ – Set.ext

□

R.3 ¶ Ordered Pair

For any sets u and v, the **ordered pair** $\langle u, v \rangle$ is the set $\{\{u\}, \{u, v\}\}$.

Definition. ∃ – OrderedPair

□

R.4 ¶ Pair Set

For any sets u and v, the **pair set** $\{u, v\}$ is the set whose only members are u and v.

Definition.

∃ – Set.insert
∃ – Set.singleton

□

R.5 ¶ Pairing Axiom

For any sets u and v, there is a set having as members just u and v:

$$\forall u, \forall v, \exists B, \forall x, (x \in B \iff x = u \text{ or } x = v).$$

Axiom.

∃ – Set.insert
∃ – Set.singleton

□

R.6 ¶ Power Set

For any set a, the **power set** $\mathscr{P}a$ is the set whose members are exactly the subsets of a.

Definition. ∃ – Set.powerset

□

R.7 ¶ Power Set Axiom

For any set a, there is a set whose members are exactly the subsets of a:

$$\forall a, \exists B, \forall x, (x \in B \iff x \subseteq a).$$

Axiom. ∃ – Set.powerset

□

R.8 ¶ Subset Axioms

For each formula ϕ not containing B, the following is an axiom:

$$\forall t_1, \cdots \forall t_k, \forall c, \exists B, \forall x, (x \in B \iff x \in c \wedge \phi).$$

Axiom. ∃ − Set.Subset

□

R.9 ¶ Symmetric Difference

The **symmetric difference** $A+B$ of sets A and B is the set $(A-B)\cup(B-A)$.

Definition. ∃ − symmDiff_def

□

R.10 ¶ Union Axiom

For any set A, there exists a set B whose elements are exactly the members of the members of A:

$$\forall A, \exists B, \forall x\, [x \in B \iff (\exists b \in A) x \in b]$$

Axiom. ∃ − Set.sUnion

□

R.11 ¶ Union Axiom, Preliminary Form

For any sets a and b, there is a set whose members are those sets belonging either to a or to b (or both):

$$\forall a, \forall b, \exists B, \forall x, (x \in B \iff x \in a \text{ or } x \in b).$$

Axiom. ∃ − Set.union

□

Chapter 1

Introduction

1.1 Baby Set Theory

1.1.1 Exercise 1.1

Which of the following become true when "$\in$" is inserted in place of the blank? Which become true when "$\subseteq$" is inserted?

Exercise 1.1a

$\{\emptyset\}$ ____ $\{\emptyset, \{\emptyset\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1a

Because the *object* $\{\emptyset\}$ is a member of the right-hand set, the statement is **true** in the case of "$\in$".

Because the *members* of $\{\emptyset\}$ are all members of the right-hand set, the statement is also **true** in the case of "$\subseteq$".

□

Exercise 1.1b

$\{\emptyset\}$ ____ $\{\emptyset, \{\{\emptyset\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1b

Because the *object* $\{\emptyset\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\emptyset\}$ are all members of the right-hand set, the statement is **true** in the case of "$\subseteq$".

□

Exercise 1.1c

$\{\{\varnothing\}\}$____$\{\varnothing, \{\varnothing\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1c

Because the *object* $\{\{\varnothing\}\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are all members of the right-hand set, the statement is **true** in the case of "$\subseteq$".

□

Exercise 1.1d

$\{\{\varnothing\}\}$____$\{\varnothing, \{\{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1d

Because the *object* $\{\{\varnothing\}\}$ is a member of the right-hand set, the statement is **true** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are not all members of the right-hand set, the statement is **false** in the case of "$\subseteq$".

□

Exercise 1.1e

$\{\{\varnothing\}\}$__$\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1e

Because the *object* $\{\{\varnothing\}\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are not all members of the right-hand set, the statement is **false** in the case of "$\subseteq$".

□

1.1.2 Exercise 1.2

Show that no two of the three sets $\varnothing$, $\{\varnothing\}$, and $\{\{\varnothing\}\}$ are equal to each other.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_2

By the ¶ Extensionality Axiom, $\varnothing$ is only equal to $\varnothing$. This immediately shows it is not equal to the other two. Now consider object $\varnothing$. This object is a member of $\{\varnothing\}$ but is not a member of $\{\{\varnothing\}\}$. Again, by the ¶ Extensionality Axiom, these two sets must be different.

□

1.1.3 Exercise 1.3

Show that if $B \subseteq C$, then $\mathscr{P}B \subseteq \mathscr{P}C$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_3

Let $x \in \mathscr{P}B$. By definition of the ¶ Power Set, x is a subset of B. By hypothesis, $B \subseteq C$. Then $x \subseteq C$. Again by definition of the ¶ Power Set, it follows $x \in \mathscr{P}C$.

□

1.1.4 Exercise 1.4

Assume that x and y are members of a set B. Show that $\{\{x\}, \{x, y\}\} \in \mathscr{P}\mathscr{P}B$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_4

Let x and y be members of set B. Then $\{x\}$ and $\{x, y\}$ are subsets of B. By definition of the ¶ Power Set, $\{x\}$ and $\{x, y\}$ are members of $\mathscr{P}B$. Then $\{\{x\}, \{x, y\}\}$ is a subset of $\mathscr{P}B$. By definition of the ¶ Power Set, $\{\{x\}, \{x, y\}\}$ is a member of $\mathscr{P}\mathscr{P}B$.

□

1.2 Sets - An Informal View

1.2.1 Exercise 2.1

Define the rank of a set c to be the least α such that $c \subseteq V_\alpha$. Compute the rank of $\{\{\varnothing\}\}$. Compute the rank of $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.

Proof. We first compute the values of V_n for $0 \leq n \leq 3$ under the assumption the set of atoms A at the bottom of the hierarchy is empty.

$$\begin{aligned}
V_0 &= \varnothing \\
V_1 &= V_0 \cup \mathscr{P} V_0 \\
&= \varnothing \cup \{\varnothing\} \\
&= \{\varnothing\} \\
V_2 &= V_1 \cup \mathscr{P} V_1 \\
&= \{\varnothing\} \cup \mathscr{P}\{\varnothing\} \\
&= \{\varnothing\} \cup \{\varnothing, \{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\} \\
V_3 &= V_2 \cup \mathscr{P} V_2 \\
&= \{\varnothing, \{\varnothing\}\} \cup \mathscr{P}\{\varnothing, \{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\} \cup \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
&= \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}
\end{aligned}$$

It then immediately follows $\{\{\varnothing\}\}$ has rank 2 and $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ has rank 3.

□

1.2.2 Exercise 2.2

We have stated that $V_{\alpha+1} = A \cup \mathscr{P} V_\alpha$. Prove this at least for $\alpha < 3$.

Proof. Let A be the set of atoms in our set hierarchy. Let $P(n)$ be the predicate, "$V_{n+1} = A \cup \mathscr{P} V_n$." We prove $P(n)$ holds true for all natural numbers $n \geq 1$ via induction.

Base Case Let $n = 1$. By definition, $V_1 = V_0 \cup \mathscr{P} V_0$. By definition, $V_0 = A$. Therefore $V_1 = A \cup \mathscr{P} V_0$. This proves $P(1)$ holds true.

Induction Step Suppose $P(n)$ holds true for some $n \geq 1$. Consider V_{n+1}. By definition, $V_{n+1} = V_n \cup \mathscr{P} V_n$. Therefore, by the induction hypothesis,

$$\begin{aligned}
V_{n+1} &= V_n \cup \mathscr{P} V_n \\
&= (A \cup \mathscr{P} V_{n-1}) \cup \mathscr{P} V_n \\
&= A \cup (\mathscr{P} V_{n-1} \cup \mathscr{P} V_n) \qquad (1.1)
\end{aligned}$$

But V_{n-1} is a subset of V_n. Exercise 1.3 then implies $\mathscr{P} V_{n-1} \subseteq \mathscr{P} V_n$. This means (1.1) can be simplified to

$$V_{n+1} = A \cup \mathscr{P} V_n,$$

proving $P(n+1)$ holds true.

Conclusion By mathematical induction, it follows for all $n \geq 1$, $P(n)$ is true. $\square$

1.2.3 Exercise 2.3

List all the members of V_3. List all the members of V_4. (It is to be assumed here that there are no atoms.)

Proof. As seen in the proof of Exercise 2.1,

$$V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}.$$

By Exercise 2.2, $V_4 = \mathscr{P}V_3$ (since it is assumed there are no atoms). Thus

$$\begin{aligned}
V_4 = \{ & \\
& \varnothing, \\
& \{\varnothing\}, \\
& \{\{\varnothing\}\}, \\
& \{\{\{\varnothing\}\}\}, \\
& \{\{\varnothing, \{\varnothing\}\}\}, \\
& \{\varnothing, \{\varnothing\}\}, \\
& \{\varnothing, \{\{\varnothing\}\}\}, \\
& \{\varnothing, \{\varnothing, \{\varnothing\}\}\}, \\
& \{\{\varnothing\}, \{\{\varnothing\}\}\}, \\
& \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\
& \{\{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \\
& \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}, \\
& \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\
& \{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
& \{\{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \\
& \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
\}. &
\end{aligned}$$

$\square$

Chapter 2

Axioms and Operations

2.1 Axioms

2.1.1 ✎ Theorem 2A

Theorem 2A. *There is no set to which every set belongs.*

> *Note: This was revisited after reading Enderton's proof prior.*

Proof. Let A be an arbitrary set. Define $B = \{x \in A \mid x \notin x\}$. By the ¶ Subset Axioms, B is a set. Then

$$B \in B \iff B \in A \wedge B \notin B.$$

If $B \in A$, then $B \in B \iff B \notin B$, a contradiction. Thus $B \notin A$. Since this process holds for any set A, there must exist no set to which every set belongs. □

2.1.2 ✎ Theorem 2B

Theorem 2B. *For any nonempty set A, there exists a unique set B such that for any x,*

$$x \in B \iff x \text{ belongs to every member of } A.$$

Proof. Suppose A is a nonempty set. This ensures the statement we are trying to prove does not vacuously hold for all sets x (which would yield a contradiction due to ✎ Theorem 2B). By the ¶ Union Axiom, $\bigcup A$ is a set. Define

$$B = \{x \in \bigcup A \mid (\forall b \in A), x \in b\}.$$

By the ¶ Subset Axioms, B is indeed a set. By construction,

$$\forall x, x \in B \iff x \text{ belongs to every member of } A.$$

By the ¶ Extensionality Axiom, B is unique.

□

2.2 Exercises 3

2.2.1 Exercise 3.1

Assume that A is the set of integers divisible by 4. Similarly assume that B and C are the sets of integers divisible by 9 and 10, respectively. What is in $A \cap B \cap C$?

Answer. ∃ – Enderton.Set.Chapter_2.exercise_3_1

The set of integers divisible by 4, 9, and 10.

□

2.2.2 Exercise 3.2

Give an example of sets A and B for which $\bigcup A = \bigcup B$ but $A \neq B$.

Answer. ∃ – Enderton.Set.Chapter_2.exercise_3_2

Let $A = \{\{1\}, \{2\}\}$ and $B = \{\{1, 2\}\}$.

□

2.2.3 Exercise 3.3

Show that every member of a set A is a subset of $\bigcup A$. (This was stated as an example in this section.)

Proof. ∃ – Enderton.Set.Chapter_2.exercise_3_3

Let $x \in A$. By definition,

$$\bigcup A = \{y \mid (\exists b \in A) y \in b\}.$$

Then $\{y \mid y \in x\} \subseteq \bigcup A$. But $\{y \mid y \in x\} = x$. Thus $x \subseteq \bigcup A$.

□

2.2.4 Exercise 3.4

Show that if $A \subseteq B$, then $\bigcup A \subseteq \bigcup B$.

Proof. ∃ – Enderton.Set.Chapter_2.exercise_3_4

Let A and B be sets such that $A \subseteq B$. Let $x \in \bigcup A$. By definition of the union, there exists some $b \in A$ such that $x \in b$. By definition of the subset, $b \in B$. This immediatley implies $x \in \bigcup B$. Since this holds for all $x \in \bigcup A$, it follows $\bigcup A \subseteq \bigcup B$.

□

2.2.5 Exercise 3.5

Assume that every member of $\mathscr{A}$ is a subset of B. Show that $\bigcup \mathscr{A} \subseteq B$.

Proof. ∃ – Enderton.Set.Chapter_2.exercise_3_5

Let $x \in \bigcup \mathscr{A}$. By definition,

$$\bigcup \mathscr{A} = \{y \mid (\exists b \in A) y \in b\}.$$

Then there exists some $b \in A$ such that $x \in b$. By hypothesis, $b \subseteq B$. Thus x must also be a member of B. Since this holds for all $x \in \bigcup \mathscr{A}$, it follows $\bigcup \mathscr{A} \subseteq B$.

□

2.2.6 Exercise 3.6a

Show that for any set A, $\bigcup \mathscr{P} A = A$.

Proof. ∃ – Enderton.Set.Chapter_2.exercise_3_6a

We prove that (i) $\bigcup \mathscr{P} A \subseteq A$ and (ii) $A \subseteq \bigcup \mathscr{P} A$.

(i) By definition, the ¶ Power Set of A is the set of all subsets of A. In other words, every member of $\mathscr{P} A$ is a subset of A. By Exercise 3.5, $\bigcup \mathscr{P} A \subseteq A$.

(ii) Let $x \in A$. By definition of the power set of A, $\{x\} \in \mathscr{P} A$. By definition of the union,

$$\bigcup \mathscr{P} A = \{y \mid (\exists b \in \mathscr{P} A), y \in b).$$

Since $x \in \{x\}$ and $\{x\} \in \mathscr{P} A$, it follows $x \in \bigcup \mathscr{P} A$. Thus $A \subseteq \bigcup \mathscr{P} A$.

Conclusion By (i) and (ii), $\bigcup \mathscr{P} A = A$.

$\square$

2.2.7 Exercise 3.6b

Show that $A \subseteq \mathscr{P} \bigcup A$. Under what conditions does equality hold?

Proof. ∃ – Enderton.Set.Chapter_2.exercise_3_6b

Let $x \in A$. By Exercise 3.3, x is a subset of $\bigcup A$. By the definition of the ¶ Power Set,

$$\mathscr{P} \bigcup A = \{y \mid y \subseteq \bigcup A\}.$$

Therefore $x \in \mathscr{P} \bigcup A$. Since this holds for all $x \in A$, $A \subseteq \mathscr{P} \bigcup A$.

♠♠♠

We show equality holds if and only if there exists some set B such that $A = \mathscr{P} B$.

(⇒) Suppose $A = \mathscr{P} \bigcup A$. Then our statement immediately follows by settings $B = \bigcup A$.

(⇐) Suppose there exists some set B such that $A = \mathscr{P} B$. Therefore

$$\begin{aligned} \mathscr{P} \bigcup A &= \mathscr{P} \left(\bigcup \mathscr{P} B \right) \\ &= \mathscr{P} B && \text{Exercise 3.6a} \\ &= A. \end{aligned}$$

Conclusion By (⇒) and (⇐), $A = \mathscr{P} \bigcup A$ if and only if there exists some set B such that $A = \mathscr{P} B$.

$\square$

2.2.8 Exercise 3.7a

Show that for any sets A and B,

$$\mathscr{P} A \cap \mathscr{P} B = \mathscr{P}(A \cap B).$$

Proof. ∃ – Enderton.Set.Chapter_2.exercise_3_7a

Let A and B be arbitrary sets. We show that $\mathscr{P} A \cap \mathscr{P} B \subseteq \mathscr{P}(A \cap B)$ and then show that $\mathscr{P} A \cap \mathscr{P} B \supseteq \mathscr{P}(A \cap B)$.

(⊆) Let $x \in \mathscr{P}A \cap \mathscr{P}B$. That is, $x \in \mathscr{P}A$ and $x \in \mathscr{P}B$. By the definition of the ¶ Power Set,

$$\mathscr{P}A = \{y \mid y \subseteq A\}$$
$$\mathscr{P}B = \{y \mid y \subseteq B\}$$

Thus $x \subseteq A$ and $x \subseteq B$, meaning $x \subseteq A \cap B$. But then $x \in \mathscr{P}(A \cap B)$, the set of all subsets of $A \cap B$. Since this holds for all $x \in \mathscr{P}A \cap \mathscr{P}B$, it follows

$$\mathscr{P}A \cap \mathscr{P}B \subseteq \mathscr{P}(A \cap B).$$

(⊇) Let $x \in \mathscr{P}(A \cap B)$. By the definition of the ¶ Power Set,

$$\mathscr{P}(A \cap B) = \{y \mid y \subseteq A \cap B\}.$$

Thus $x \subseteq A \cap B$, meaning $x \subseteq A$ and $x \subseteq B$. But this implies $x \in \mathscr{P}A$, the set of all subsets of A. Likewise $x \in \mathscr{P}B$, the set of all subsets of B. Thus $x \in \mathscr{P}A \cap \mathscr{P}B$. Since this holds for all $x \in \mathscr{P}(A \cap B)$, it follows

$$\mathscr{P}(A \cap B) \subseteq \mathscr{P}A \cap \mathscr{P}B.$$

Conclusion Since each side of our identity is a subset of the other,

$$\mathscr{P}(A \cap B) = \mathscr{P}A \cap \mathscr{P}B.$$

□

2.2.9 Exercise 3.7b

Show that $\mathscr{P}A \cup \mathscr{P}B \subseteq \mathscr{P}(A \cup B)$. Under what conditions does equality hold?

Proof.

∃ – Enderton.Set.Chapter_2.exercise_3_7b_i
∃ – Enderton.Set.Chapter_2.exercise_3_7b_ii

Let $x \in \mathscr{P}A \cup \mathscr{P}B$. By definition, $x \in \mathscr{P}A$ or $x \in \mathscr{P}B$ (or both). By the definition of the ¶ Power Set,

$$\mathscr{P}A = \{y \mid y \subseteq A\}$$
$$\mathscr{P}B = \{y \mid y \subseteq B\}.$$

Thus $x \subseteq A$ or $x \subseteq B$. Therefore $x \subseteq A \cup B$. But then $x \in \mathscr{P}(A \cup B)$, the set of all subsets of $A \cup B$.

♠♠♠

We show equality holds if and only if one of A or B is a subset of the other.

(⇒) Suppose

$$\mathscr{P}A \cup \mathscr{P}B = \mathscr{P}(A \cup B). \tag{2.1}$$

By the definition of the ¶ Power Set, $A \cup B \in \mathscr{P}(A \cup B)$. Then (2.1) implies $A \cup B \in \mathscr{P}A \cup \mathscr{P}B$. That is, $A \cup B \in \mathscr{P}A$ or $A \cup B \in \mathscr{P}B$ (or both).

For the sake of contradiction, suppose $A \not\subseteq B$ and $B \not\subseteq A$. Then there exists an element $x \in A$ such that $x \notin B$ and there exists an element $y \in B$ such that $y \notin A$. But then $A \cup B \notin \mathscr{P}A$ since y cannot be a member of a member of $\mathscr{P}A$. Likewise, $A \cup B \notin \mathscr{P}B$ since x cannot be a member of a member of $\mathscr{P}B$. Therefore our assumption is incorrect. In other words, $A \subseteq B$ or $B \subseteq A$.

(⇐) WLOG, suppose $A \subseteq B$. Then, by Exercise 1.3, $\mathscr{P}A \subseteq \mathscr{P}B$. Thus

$$\begin{aligned} \mathscr{P}A \cup \mathscr{P}B &= \mathscr{P}B \\ &= \mathscr{P}A \cup B. \end{aligned}$$

Conclusion By (⇒) and (⇐), it follows $\mathscr{P}A \cup \mathscr{P}B \subseteq \mathscr{P}(A \cup B)$ if and only if $A \subseteq B$ or $B \subseteq A$.

□

2.2.10 Exercise 3.8

Show that there is no set to which every singleton (that is, every set of the form $\{x\}$) belongs. [*Suggestion*: Show that from such a set, we could construct a set to which every set belonged.]

Proof. We proceed by contradiction. Suppose there existed a set A consisting of every singleton. Then the ¶ Union Axiom suggests $\bigcup A$ is a set. But this set is precisely the class of all sets, which is *not* a set. Thus our original assumption was incorrect. That is, there is no set to which every singleton belongs.

□

2.2.11 Exercise 3.9

Give an example of sets a and B for which $a \in B$ but $\mathscr{P}a \notin \mathscr{P}B$.

Answer. ∃ – Enderton.Set.Chapter_2.exercise_3_9

Let $a = \{1\}$ and $B = \{\{1\}\}$. Then

$$\begin{aligned} \mathscr{P}a &= \{\varnothing, \{1\}\} \\ \mathscr{P}B &= \{\varnothing, \{\{1\}\}\}. \end{aligned}$$

It immediately follows that $\mathscr{P}a \notin \mathscr{P}B$.

□

2.2.12 Exercise 3.10

Show that if $a \in B$, then $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$. [*Suggestion*: If you need help, look in the Appendix.]

Proof. ∃ – Enderton.Set.Chapter_2.exercise_3_10

Suppose $a \in B$. By Exercise 3.3, $a \subseteq \bigcup B$. By Exercise 1.3, $\mathscr{P}a \subseteq \mathscr{P}\bigcup B$. By the definition of the ¶ Power Set,

$$\mathscr{P}\mathscr{P}\bigcup B = \{y \mid y \subseteq \mathscr{P}\bigcup B\}.$$

Therefore $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$.

□

2.3 Algebra of Sets

2.3.1 Commutative Laws

For any sets A and B,

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{aligned}$$

Proof.

∃ – Set.union_comm
∃ – Set.inter_comm

Let A and B be sets. We prove that

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$.

(i) By the definition of the union of sets,

$$\begin{aligned} A \cup B &= \{x \mid x \in A \lor x \in B\} \\ &= \{x \mid x \in B \lor x \in A\} \\ &= B \cup A. \end{aligned}$$

(ii) By the definition of the intersection of sets,

$$\begin{aligned}
A \cap B &= \{x \mid x \in A \wedge x \in B\} \\
&= \{x \mid x \in B \wedge x \in A\} \\
&= B \wedge A.
\end{aligned}$$

□

2.3.2 Associative Laws

For any sets A, B and C,

$$\begin{aligned}
A \cup (B \cup C) = (A \cup B) \cup C \\
A \cap (B \cap C) = (A \cap B) \cap C
\end{aligned}$$

Proof.

∃ – Set.union_assoc
∃ – Set.inter_assoc

Let A, B, and C be sets. We prove that

(i) $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

(i) By the definition of the union of sets,

$$\begin{aligned}
A \cup (B \cup C) &= \{x \mid x \in A \vee x \in (B \cup C)\} \\
&= \{x \mid x \in A \vee x \in \{y \mid y \in B \vee C\}\} \\
&= \{x \mid x \in A \vee (x \in B \vee x \in C)\} \\
&= \{x \mid (x \in A \vee x \in B) \vee x \in C\} \\
&= \{x \mid x \in \{y \mid y \in A \vee y \in B\} \vee x \in C\} \\
&= \{x \mid x \in (A \cup B) \vee x \in C\} \\
&= (A \cup B) \cup C.
\end{aligned}$$

(ii) By the definition of the intersection of sets,

$$\begin{aligned}
A \cap (B \cap C) &= \{x \mid x \in A \wedge x \in (B \cap C)\} \\
&= \{x \mid x \in A \wedge x \in \{y \mid y \in B \wedge y \in C\}\} \\
&= \{x \mid x \in A \wedge (x \in B \wedge x \in C)\} \\
&= \{x \mid (x \in A \wedge x \in B) \wedge x \in C\} \\
&= \{x \mid x \in \{y \mid y \in A \wedge y \in B\} \wedge x \in C\} \\
&= \{x \mid x \in (A \cap B) \wedge x \in C\} \\
&= (A \cap B) \cap C.
\end{aligned}$$

□

2.3.3 Distributive Laws

For any sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof.

∃ – Set.inter_distrib_left
∃ – Set.union_distrib_left

Let A, B, and C be sets. We prove that

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(i) By the definition of the union and intersection of sets,

$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A \wedge x \in B \cup C\} \\ &= \{x \mid x \in A \wedge x \in \{y \mid y \in B \vee y \in C\}\} \\ &= \{x \mid x \in A \wedge (x \in B \vee x \in C)\} \\ &= \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\ &= \{x \mid x \in A \cap B \vee x \in A \cap C\} \\ &= (A \cap B) \cup (A \cap C). \end{aligned}$$

(ii) By the definition of the union and intersection of sets,

$$\begin{aligned} A \cup (B \cap C) &= \{x \mid x \in A \vee x \in B \cap C\} \\ &= \{x \mid x \in A \vee x \in \{y \mid y \in B \wedge y \in C\}\} \\ &= \{x \mid x \in A \vee (x \in B \wedge x \in C)\} \\ &= \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} \\ &= \{x \mid x \in A \cup B \wedge x \in A \cup C\} \\ &= (A \cup B) \cap (A \cup C). \end{aligned}$$

□

2.3.4 De Morgan's Laws

For any sets A, B, and C,

$$C - (A \cup B) = (C - A) \cap (C - B)$$
$$C - (A \cap B) = (C - A) \cup (C - B)$$

Proof.

∃ – Set.diff_inter_diff
∃ – Set.diff_inter

Let A, B, and C be sets. We prove that

(i) $C - (A \cup B) = (C - A) \cap (C - B)$

(ii) $C - (A \cap B) = (C - A) \cup (C - B)$

(i) By definition of the union, intersection, and relative complements of sets,

$$\begin{aligned} C - (A \cup B) &= \{x \mid x \in C \land x \notin A \cup B\} \\ &= \{x \mid x \in C \land x \notin \{y \mid y \in A \lor y \in B\}\} \\ &= \{x \mid x \in C \land \neg(x \in A \lor x \in B)\} \\ &= \{x \mid x \in C \land (x \notin A \land x \notin B)\} \\ &= \{x \mid (x \in C \land x \notin A) \land (x \in C \land x \notin B)\} \\ &= \{x \mid x \in (C - A) \land x \in (C - B)\} \\ &= (C - A) \cap (C - B). \end{aligned}$$

(ii) By definition of the union, intersection, and relative complements of sets,

$$\begin{aligned} C - (A \cap B) &= \{x \mid x \in C \land x \notin A \cap B\} \\ &= \{x \mid x \in C \land x \notin \{y \mid y \in A \land y \in B\}\} \\ &= \{x \mid x \in C \land \neg(x \in A \land x \in B)\} \\ &= \{x \mid x \in C \land (x \notin A \lor x \notin B)\} \\ &= \{x \mid (x \in C \land x \notin A) \lor (x \in C \land x \notin B)\} \\ &= \{x \mid x \in (C - A) \lor x \in (C - B)\} \\ &= (C - A) \cup (C - B). \end{aligned}$$

□

2.3.5 Identities Involving $\varnothing$

For any set A,

$$\begin{aligned} A \cup \varnothing &= A \\ A \cap \varnothing &= \varnothing \\ A \cap (C - A) &= \varnothing \end{aligned}$$

Proof.

$\exists$ – Set.union_empty
$\exists$ – Set.inter_empty
$\exists$ – Set.inter_diff_self

Let A be an arbitrary set. We prove that

(i) $A \cup \varnothing = A$

(ii) $A \cap \varnothing = \varnothing$

(iii) $A \cap (C - A) = \varnothing$

(i) By definition of the emptyset and union of sets,

$$\begin{aligned} A \cup \varnothing &= \{x \mid x \in A \lor x \in \varnothing\} \\ &= \{x \mid x \in A \lor F\} \\ &= \{x \mid x \in A\} \\ &= A. \end{aligned}$$

(ii) By definition of the emptyset and intersection of sets,

$$\begin{aligned} A \cap \varnothing &= \{x \mid x \in A \land x \in \varnothing\} \\ &= \{x \mid x \in A \land F\} \\ &= \{x \mid F\} \\ &= \{x \mid x \neq x\} \\ &= \varnothing. \end{aligned}$$

(iii) By definition of the emptyset, and the intersection and relative complement of sets,

$$\begin{aligned}
A \cap (C - A) &= \{x \mid x \in A \land x \in C - A\} \\
&= \{x \mid x \in A \land x \in \{y \mid y \in C \land y \notin A\}\} \\
&= \{x \mid x \in A \land (x \in C \land x \notin A)\} \\
&= \{x \mid x \in C \land F\} \\
&= \{x \mid F\} \\
&= \{x \mid x \neq x\} \\
&= \varnothing.
\end{aligned}$$

□

2.3.6 Monotonicity

For any sets A, B, and C,

$$\begin{aligned}
&A \subseteq B \Rightarrow A \cup C \subseteq B \cup C \\
&A \subseteq B \Rightarrow A \cap C \subseteq B \cap C \\
&A \subseteq B \Rightarrow \bigcup A \subseteq \bigcup B
\end{aligned}$$

Proof.

∃ – Set.union_subset_union_left
∃ – Set.inter_subset_inter_left
∃ – Set.sUnion_mono

Let A, B, and C be arbitrary sets. We prove that

(i) $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$

(ii) $A \subseteq B \Rightarrow A \cap C \subseteq B \cap C$

(iii) $A \subseteq B \Rightarrow \bigcup A \subseteq \bigcup B$

(i) Suppose $A \subseteq B$. Let $x \in A \cup C$. There are two cases to consider.

Case 1 Suppose $x \in A$. Then, by definition of the subset, $x \in B$. Therefore $x \in B \cup C$.

Case 2 Suppose $x \in C$. Then x is trivially a member of $B \cup C$.

Conclusion Since these cases are exhaustive and both imply $x \in B \cup C$, it follows $A \cup C \subseteq B \cup C$.

(ii) Suppose $A \subseteq B$. Let $x \in A \cap C$. Then, by definition of the intersection of sets, $x \in A$ and $x \in C$. By definition of the subset, $x \in A$ implies $x \in B$. Therefore $x \in B$ and $x \in C$. That is, $x \in B \cap C$. Since this holds for arbitrary $x \in A \cap C$, it follows $A \cap C \subseteq B \cap C$.

(iii) Suppose $A \subseteq B$. Let $x \in \bigcup A$. Then, by definition of the union of sets, there exists some $b \in A$ such that $x \in b$. By definition of the subset, $b \in B$ as well. Another application of the definition of the union of sets immediately implies that x is a member of $\bigcup B$.

$\square$

2.3.7 Anti-monotonicity

For any sets A, B, and C,

$$A \subseteq B \Rightarrow C - B \subseteq C - A$$
$$\varnothing \neq A \subseteq B \Rightarrow \bigcap B \subseteq \bigcap A.$$

Proof.

∃ – Set.diff_subset_diff_right
∃ – Set.sInter_subset_sInter

Let A, B, and C be arbitrary sets. We prove that

(i) $A \subseteq B \Rightarrow C - B \subseteq C - A$

(ii) $\varnothing \neq A \subseteq B \Rightarrow \bigcap B \subseteq \bigcap A$

(i) Suppose $A \subseteq B$. Let $x \in C - B$. By definition of the relative complement, $x \in C$ and $x \notin B$. Then x cannot be a member of A, since otherwise this would contradict our subset hypothesis. That is, $x \in C$ and $x \notin A$. Therefore $x \in C - A$. Since this holds for arbitrary $x \in C - B$, it follows that $C - B \subseteq C - A$.

(ii) Suppose $A \neq \varnothing$ and $A \subseteq B$. Then $B \neq \varnothing$. Let $x \in \bigcap B$. By definition of the intersection of sets, for all $b \in B$, $x \in b$. But then, by definition of the subset, for all $a \in A$, $x \in a$. Therefore $x \in \bigcap A$. Since this holds for arbitrary $x \in \bigcap B$, it follows that $\bigcap B \subseteq \bigcap A$.

$\square$

2.3.8 General Distributive Laws

For any sets A and $\mathscr{B}$,

$$A \cup \bigcap \mathscr{B} = \bigcap \{A \cup X \mid X \in \mathscr{B}\} \quad \text{for} \quad \mathscr{B} \neq \varnothing$$
$$A \cap \bigcup \mathscr{B} = \bigcup \{A \cap X \mid X \in \mathscr{B}\}$$

Proof. Let A and $\mathscr{B}$ be sets. We prove that

(i) For $\mathscr{B} \neq \varnothing$, $A \cup \bigcap \mathscr{B} = \bigcap \{A \cup X \mid X \in \mathscr{B}\}$.

(ii) $A \cap \bigcup \mathscr{B} = \bigcup \{A \cap X \mid X \in \mathscr{B}\}$

(i) Suppose $\mathscr{B}$ is nonempty. Then $\bigcap \mathscr{B}$ is defined. By definition of the union and intersection of sets,

$$\begin{aligned}
A \cup \bigcap \mathscr{B} &= \{x \mid x \in A \vee x \in \bigcap \mathscr{B}\} \\
&= \{x \mid x \in A \vee x \in \{y \mid (\forall b \in \mathscr{B}), y \in b\}\} \\
&= \{x \mid x \in A \vee (\forall b \in \mathscr{B}), x \in b\} \\
&= \{x \mid \forall b \in \mathscr{B}, x \in A \vee x \in b\} \\
&= \{x \mid \forall b \in \mathscr{B}, x \in A \cup b\} \\
&= \{x \mid x \in \bigcap \{A \cup X \mid X \in \mathscr{B}\}\} \\
&= \bigcap \{A \cup X \mid X \in \mathscr{B}\}.
\end{aligned}$$

(ii) By definition of the intersection and union of sets,

$$\begin{aligned}
A \cap \bigcup \mathscr{B} &= \{x \mid x \in A \wedge x \in \bigcup \mathscr{B}\} \\
&= \{x \mid x \in A \wedge x \in \{y \mid (\exists b \in \mathscr{B}), y \in b\}\} \\
&= \{x \mid x \in A \wedge (\exists b \in \mathscr{B}), x \in b\} \\
&= \{x \mid \exists b \in \mathscr{B}, x \in A \wedge x \in b\} \\
&= \{x \mid \exists b \in \mathscr{B} x \in A \cap b\} \\
&= \{x \mid x \in \bigcup \{A \cap X \mid X \in \mathscr{B}\}\} \\
&= \bigcup \{A \cap X \mid X \in \mathscr{B}\}.
\end{aligned}$$

$\square$

2.3.9 General De Morgan's Laws

For any set C and $\mathscr{A} \neq \varnothing$,

$$C - \bigcup \mathscr{A} = \bigcap \{C - X \mid X \in \mathscr{A}\}$$
$$C - \bigcap \mathscr{A} = \bigcup \{C - X \mid X \in \mathscr{A}\}$$

Proof. Let C and $\mathscr{A}$ be sets such that $\mathscr{A} \neq \varnothing$. We prove that

(i) $C - \bigcup \mathscr{A} = \bigcap \{C - X \mid X \in \mathscr{A}\}$

(ii) $C - \bigcap \mathscr{A} = \bigcup \{C - X \mid X \in \mathscr{A}\}$

(i) By definition of the relative complement, union, and intersection of sets,

$$\begin{aligned}
C - \bigcup \mathscr{A} &= \{x \mid x \in C \wedge x \notin \bigcup \mathscr{A}\} \\
&= \{x \mid x \in C \wedge x \notin \{y \mid (\exists b \in \mathscr{A}) y \in b\}\} \\
&= \{x \mid x \in C \wedge \neg(\exists b \in \mathscr{A}, x \in b)\} \\
&= \{x \mid x \in C \wedge (\forall b \in \mathscr{A}, x \notin b)\} \\
&= \{x \mid \forall b \in \mathscr{A}, x \in C \wedge x \notin b\} \\
&= \{x \mid \forall b \in \mathscr{A}, x \in C - b\} \\
&= \{x \mid x \in \bigcap \{C - X \mid X \in \mathscr{A}\} \\
&= \bigcap \{C - X \mid X \in \mathscr{A}\}.
\end{aligned}$$

(ii) By definition of the relative complement, union, and intersection of sets,

$$\begin{aligned}
C - \bigcap \mathscr{A} &= \{x \mid x \in C \wedge x \notin \bigcap \mathscr{A}\} \\
&= \{x \mid x \in C \wedge x \notin \{y \mid (\forall b \in \mathscr{A}) y \in b\}\} \\
&= \{x \mid x \in C \wedge \neg(\forall b \in \mathscr{A}, x \in b)\} \\
&= \{x \mid x \in C \wedge \exists b \in \mathscr{A}, x \notin b\} \\
&= \{x \mid \exists b \in \mathscr{A}, x \in C \wedge x \notin b\} \\
&= \{x \mid \exists b \in \mathscr{A}, x \in C - b\} \\
&= \{x \mid x \in \bigcup \{C - X \mid X \in \mathscr{A}\}\} \\
&= \bigcup \{C - X \mid X \in \mathscr{A}\}.
\end{aligned}$$

□

2.3.10 $\cap/-$ Associativity

Let A, B, and C be sets. Then $A \cap (B - C) = (A \cap B) - C$.

Proof. ∃ – Set.inter_diff_assoc

Let A, B, and C be sets. By definition of the intersection and relative complement of sets,

$$\begin{aligned}
A \cap (B - C) &= \{x \mid x \in A \wedge x \in B - C\} \\
&= \{x \mid x \in A \wedge (x \in B \wedge x \notin C)\} \\
&= \{x \mid (x \in A \wedge x \in B) \wedge x \notin C\} \\
&= \{x \mid x \in A \cap B \wedge x \notin C\} \\
&= (A \cap B) - C.
\end{aligned}$$

□

2.3.11 Nonmembership of Symmetric Difference

Let A and B be sets. $x \notin A + B$ if and only if either $x \in A \cap B$ or $x \notin A \cup B$.

Proof. ∃ – Set.not_mem_symm_diff_inter_or_not_union

By definition of the ¶ Symmetric Difference,

$$\begin{aligned}
x \notin A + B &= \neg(x \in A + B) \\
&= \neg[x \in (A - B) \cup (B - A)] \\
&= \neg[x \in (A - B) \vee x \in (B - A)] \\
&= \neg[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)] \\
&= \neg(x \in A \wedge x \notin B) \wedge \neg(x \in B \wedge x \notin A) \\
&= (x \notin A \vee x \in B) \wedge (x \notin B \vee x \in A) \\
&= ((x \notin A \vee x \in B) \wedge x \notin B) \vee ((x \notin A \vee x \in B) \wedge x \in A) \\
&= (x \notin A \wedge x \notin B) \vee (x \in B \wedge x \in A) \\
&= \neg(x \in A \vee x \in B) \vee (x \in B \wedge x \in A) \\
&= x \notin A \cup B \text{ or } x \in A \cap B.
\end{aligned}$$

□

2.4 Exercises 4

2.4.1 Exercise 4.11

Show that for any sets A and B,

$$A = (A \cap B) \cup (A - B) \quad \text{and} \quad A \cup (B - A) = A \cup B.$$

Proof.

∃ – Enderton.Set.Chapter_2.exercise_4_11_i
∃ – Enderton.Set.Chapter_2.exercise_4_11_ii

Let A and B be sets. We prove that

(i) $A = (A \cap B) \cup (A - B)$

(ii) $A \cup (B - A) = A \cup B$

(i) By definition of the intersection, union, and relative complements of sets,

$$\begin{aligned}
(A \cap B) \cup (A - B) &= \{x \mid x \in A \cap B \lor x \in A - B\} \\
&= \{x \mid x \in \{y \mid y \in A \land y \in B\} \lor x \in A - B\} \\
&= \{x \mid (x \in A \land x \in B) \lor x \in A - B\} \\
&= \{x \mid (x \in A \land x \in B) \lor x \in \{y \mid y \in A \land y \notin B\}\} \\
&= \{x \mid (x \in A \land x \in B) \lor (x \in A \land x \notin B)\} \\
&= \{x \mid x \in A \lor (x \in B \land x \notin B)\} \\
&= \{x \mid x \in A \lor F\} \\
&= \{x \mid x \in A\} \\
&= A.
\end{aligned}$$

(ii) By definition of the union and relative complements of sets,

$$\begin{aligned}
A \cup (B - A) &= \{x \mid x \in A \lor x \in B - A\} \\
&= \{x \mid x \in A \lor x \in \{y \mid y \in B \land y \notin A\}\} \\
&= \{x \mid x \in A \lor (x \in B \land x \notin A)\} \\
&= \{x \mid (x \in A \lor x \in B) \land (x \in A \lor x \notin A)\} \\
&= \{x \mid (x \in A \lor x \in B) \land T\} \\
&= \{x \mid x \in A \lor x \in B\} \\
&= \{x \mid x \in A \cup B\} \\
&= A \cup B.
\end{aligned}$$

□

2.4.2 ✓ Exercise 4.12

Verify the following identity (one of De Morgan's laws):

$$C - (A \cap B) = (C - A) \cup (C - B).$$

Proof. Refer to De Morgan's Laws.

□

2.4.3 Exercise 4.13

Show that if $A \subseteq B$, then $C - B \subseteq C - A$.

Proof. Refer to Anti-monotonicity.

□

2.4.4 Exercise 4.14

Show by example that for some sets A, B, and C, the set $A - (B - C)$ is different from $(A - B) - C$.

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_14

Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, and $C = \{3, 4, 5\}$. Then

$$\begin{aligned}
A - (B - C) &= \{1, 2, 3\} - (\{2, 3, 4\} - \{3, 4, 5\}) \\
&= \{1, 2, 3\} - \{2\} \\
&= \{1, 3\}
\end{aligned}$$

but

$$\begin{aligned}
(A - B) - C &= (\{1, 2, 3\} - \{2, 3, 4\}) - \{3, 4, 5\} \\
&= \{1\} - \{3, 4, 5\} \\
&= \{1\}.
\end{aligned}$$

□

2.4.5 Exercise 4.15a

Show that $A \cap (B + C) = (A \cap B) + (A \cap C)$.

Proof. ∃ – Set.inter_symmDiff_distrib_left

By definition of the intersection, ¶ Symmetric Difference, and relative complement of sets,

$$
\begin{aligned}
&(A \cap B) + (A \cap C) \\
&\quad = [(A \cap B) - (A \cap C)] \cup [(A \cap C) - (A \cap B)] \\
&\quad = [(A \cap B) - A] \\
&\qquad \cup [(A \cap B) - C] \\
&\qquad \cup [(A \cap C) - A] \\
&\qquad \cup [(A \cap C) - B] && \text{De Morgan's Laws} \\
&\quad = [A \cap (B - A)] \\
&\qquad \cup [A \cap (B - C)] \\
&\qquad \cup [A \cap (C - A)] \\
&\qquad \cup [A \cap (C - B)] && \cap/- \text{ Associativity} \\
&\quad = \varnothing \\
&\qquad \cup [A \cap (B - C)] \\
&\qquad \cup \varnothing \\
&\qquad \cup [A \cap (C - B)] && \text{Identities Involving } \varnothing \\
&\quad = [A \cap (B - C)] \cup [A \cap (C - B)] \\
&\quad = A \cap [(B - C) \cup (C - B)] && \text{Distributive Laws} \\
&\quad = A \cap (B + C).
\end{aligned}
$$

□

2.4.6 Exercise 4.15b

Show that $A + (B + C) = (A + B) + C$.

Proof. ∃ – Set.symmDiff_assoc

Let A, B, and C be sets. We prove that

(i) $A + (B + C) \subseteq (A + B) + C$

(ii) $(A + B) + C \subseteq A + (B + C)$

(i) Let $x \in A + (B + C)$. Then x is in A or in $B + C$, but not both. There are two cases to consider:

Case 1 Suppose $x \in A$ and $x \notin B + C$. Then, by Nonmembership of Symmetric Difference, (a) $x \in B \cap C$ or (b) $x \notin B \cup C$. Suppose (a) was true. That is, $x \in B$ and $x \in C$. Since x is a member of A and B, $x \notin (A + B)$. Since x is not a member of $A + B$ but is a member of C, $x \in (A + B) + C$. Now suppose (b) was true. That is, $x \notin B$ and $x \notin C$. Since x is a member of A but not B, $x \in (A + B)$. Since x is a member of $A + B$ but not C, $x \in (A + B) + C$.

Case 2 Suppose $x \in B + C$ and $x \notin A$. Then (a) $x \in B$ or (b) $x \in C$ but not both. Suppose (a) was true. That is, $x \in B$ and $x \notin C$. Since x is not a member of A and is a member of B, $x \in A + B$. Since x is a member of $A + B$ but not C, $x \in (A + B) + C$. Now suppose (b) was true. That is, $x \notin B$ and $x \in C$. Since x is not a member of A nor a member of B, $x \notin A + B$. Since x is not a member of $A + B$ but is a member of C, $x \in (A + B) + C$.

(ii) Let $x \in (A + B) + C$. Then x is in $A + B$ or in C, but not both. There are two cases to consider:

Case 1 Suppose $x \in A + B$ and $x \notin C$. Then (a) $x \in A$ or (b) $x \in B$ but not both. Suppose (a) was true. That is, $x \in A$ and $x \notin B$. Since x is not a member of B nor C, $x \notin B + C$. Since x is not a member of $B + C$ but is a member of A, $x \in A + (B + C)$. Now Suppose (b) was true. That is, $x \notin A$ and $x \in B$. Since x is a member of B and not of C, then $x \in B + C$. Since x is a member of $B + C$ and not of A, $x \in A + (B + C)$.

Case 2 Suppose $x \notin A + B$ and $x \in C$. Then, by ✓ Nonmembership of Symmetric Difference, (a) $x \in A \cap B$ or (b) $x \notin A \cup B$. Suppose (a) was true. That is, $x \in A \wedge x \in B$. Since x is a member of B and C, $x \notin B + C$. Since x is not a member of $B + C$ but is a member of A, $x \in A + (B + C)$. Now suppose (b) was true. That is, $x \notin A$ and $x \notin B$. Since x is not a member of B but is a member of C, $x \in B + C$. Since x is a member of $B + C$ but not of A, $x \in A + (B + C)$.

Conclusion In both (i) and (ii), the subcases are exhaustive and prove the desired subset relation. Therefore $A + (B + C) = (A + B) + C$.

□

2.4.7 ✓ Exercise 4.16

Simplify:

$$[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A].$$

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_16

Let A, B, and C be arbitrary sets. Then

$$\begin{aligned}
&[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A] \\
&\quad = [A \cup B] - [A] \\
&\quad = \{x \mid x \in (A \cup B) \land x \notin A\} \\
&\quad = \{x \mid x \in \{y \mid y \in A \lor y \in B\} \land x \notin A\} \\
&\quad = \{x \mid (x \in A \lor x \in B) \land x \notin A\} \\
&\quad = \{x \mid (x \in A \land x \notin A) \lor (x \in B \land x \notin A)\} \\
&\quad = \{x \mid F \lor (X \in B \land x \notin A)\} \\
&\quad = \{x \mid x \in B \land x \notin A\} \\
&\quad = B - A.
\end{aligned}$$

□

2.4.8 Exercise 4.17

Show that the following four conditions are equivalent.

(a) $A \subseteq B$,

(b) $A - B = \varnothing$,

(c) $A \cup B = B$,

(d) $A \cap B = A$.

Proof.

∃ – Enderton.Set.Chapter_2.exercise_4_17_i
∃ – Enderton.Set.Chapter_2.exercise_4_17_ii
∃ – Enderton.Set.Chapter_2.exercise_4_17_iii
∃ – Enderton.Set.Chapter_2.exercise_4_17_iv

Let A and B be arbitrary sets. We show that (i) $(a) \Rightarrow (b)$, (ii) $(b) \Rightarrow (c)$, (iii) $(c) \Rightarrow (d)$, and (iv) $(d) \Rightarrow (a)$.

(i) Suppose $A \subseteq B$. That is, $\forall t, t \in A \Rightarrow t \in B$. Then there is no element such that $t \in A$ and $t \notin B$. By definition of the relative complement, this immediately implies $A - B = \varnothing$.

(ii) Suppose $A - B = \varnothing$. By definition of the relative complement,

$$A - B = \varnothing = \{x \mid x \in A \land x \notin B\}.$$

Then, for all t, $\neg(t \in A \land t \notin B) = t \notin A \lor t \in B$. This implies, by definition of the subset, that $A \subseteq B$. It then immediately follows that $A \cup B = B$.

(iii) Suppose $A \cup B = B$. Then there is no member of A that is not a member of B. In other words, $A \subseteq B$. This immediately implies $A \cap B = A$.

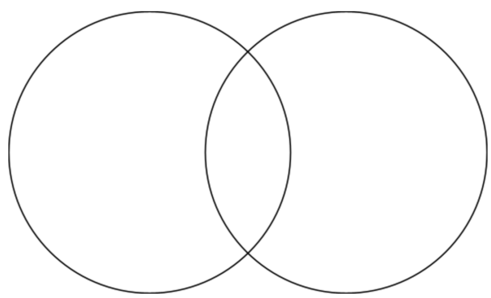
(iv) Suppose $A \cap B = A$. Then every member of A is a member of B. This immediately implies $A \subseteq B$.

□

2.4.9 Exercise 4.18

Assume that A and B are subsets of S. List all of the different sets that can be made from these three by use of the binary operations $\cup$, $\cap$, and $-$.

Proof. We can reason about this diagrammatically:



In the above diagram, we assume the left circle corresponds to set A and the right circle corresponds to B. The the possible sets we can make via the specified operators are:

- $A - B$, the left circle excluding the overlapping region.
- $A \cap B$, the overlapping region.
- $B - A$, the right circle excluding the overlapping region.
- $(A \cup B) \cap A$, the left circle.
- $(A \cup B) \cap B$, the right circle.
- $(A - B) \cup (B - A)$, the symmetric difference.
- $A \cup B$, the entire diagram.

□

2.4.10 ✓ Exercise 4.19

Is $\mathscr{P}(A - B)$ always equal to $\mathscr{P}A - \mathscr{P}B$? Is it ever equal to $\mathscr{P}A - \mathscr{P}B$?

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_19

Let A and B be arbitrary sets. We show (i) that $\varnothing \in \mathscr{P}(A - B)$ and (ii) $\varnothing \notin \mathscr{P}A - \mathscr{P}B$.

(i) By definition of the ¶ Power Set,

$$\mathscr{P}(A - B) = \{x \mid x \subseteq A - B\}.$$

But $\varnothing$ is a subset of *every* set. Thus $\varnothing \in \mathscr{P}(A - B)$.

(ii) By the same reasoning found in (i), $\varnothing \in \mathscr{P}A$ and $\varnothing \in \mathscr{P}B$. But then, by definition of the relative complement, $\varnothing \notin \mathscr{P}A - \mathscr{P}B$.

Conclusion By the ¶ Extensionality Axiom, the two sets are never equal.

□

2.4.11 ✓ Exercise 4.20

Let A, B, and C be sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_20

Let A, B, and C be arbitrary sets. By the ¶ Extensionality Axiom, $B = C$ if and only if for all sets x, $x \in B \iff x \in C$. We prove both directions of this biconditional.

($\Rightarrow$) Suppose $x \in B$. Then there are two cases to consider:

Case 1 Assume $x \in A$. Then $x \in A \cap B$. By hypothesis, $A \cap B = A \cap C$. Thus $x \in A \cap C$ immediately implying $x \in C$.

Case 2 Assume $x \notin A$. Then $x \in A \cup B$. By hypothesis, $A \cup B = A \cup C$. Thus $x \in A \cup C$. Since $x \notin A$, it follows $x \in C$.

($\Leftarrow$) Suppose $x \in C$. Then there are two cases to consider:

Case 1 Assume $x \in A$. Then $x \in A \cap C$. By hypothesis, $A \cap B = A \cap C$. Thus $x \in A \cap B$, immediately implying $x \in B$.

Case 2 Assume $x \notin A$. Then $x \in A \cup C$. By hypothesis, $A \cup B = A \cup C$. Thus $x \in A \cup B$. Since $x \notin A$, it follows $x \in B$.

□

2.4.12 Exercise 4.21

Show that $\bigcup(A \cup B) = \bigcup A \cup \bigcup B$.

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_21

Let A and B be arbitrary sets. By the ¶ Extensionality Axiom, the specified equality holds if and only if for all sets x,

$$x \in \bigcup(A \cup B) \iff x \in \bigcup A \cup \bigcup B.$$

We prove both directions of this biconditional.

($\Rightarrow$) Suppose $x \in \bigcup(A \cup B)$. By definition of the union of sets, there exists some $b \in A \cup B$ such that $x \in b$. If $b \in A$, then $x \in \bigcup A$ and $x \in \bigcup A \cup \bigcup B$. Alternatively, if $b \in B$, then $x \in \bigcup B$ and $x \in \bigcup A \cup \bigcup B$. Regardless, x is in the target set.

($\Leftarrow$) Suppose $x \in \bigcup A \cup \bigcup B$. Then $x \in \bigcup A$ or $x \in \bigcup B$. WLOG, suppose $x \in \bigcup A$. By definition of the union of sets, there exists some $b \in A$ such that $x \in b$. But then $b \in A \cup B$ meaning x is also a member of $\bigcup(A \cup B)$.

□

2.4.13 Exercise 4.22

Show that if A and B are nonempty sets, then $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$.

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_22

Let A and B be arbitrary, nonempty sets. By the ¶ Extensionality Axiom, the specified equality holds if and only if for all sets x,

$$x \in \bigcap(A \cup B) \iff x \in \bigcap A \cap \bigcap B. \tag{2.2}$$

We prove both directions of this biconditional.

($\Rightarrow$) Suppose $x \in \bigcap(A \cup B)$. Then for all $b \in A \cup B$, $x \in B$. In other words, for every member b_1 of A and every member b_2 of B, x is a member of both b_1 and b_2. But that implies $x \in \bigcap A$ and $x \in \bigcap B$.

(⇐) Suppose $x \in \bigcap A \cap \bigcap B$. That is, $x \in \bigcap A$ and $x \in \bigcap B$. By definition of the intersection of sets, forall sets b, if $b \in A$, then $x \in b$. Likewise, if $b \in B$, then $x \in b$. In other words, if b is a member of either A or B, $x \in b$. That immediately implies $x \in \bigcap(A \cup B$.

□

2.4.14 ✎ Exercise 4.23

Show that if $\mathscr{B}$ is nonempty, then $A \cup \bigcap \mathscr{B} = \bigcap \{A \cup X \mid X \in \mathscr{B}\}$.

Proof. Refer to ✎ General Distributive Laws.

□

2.4.15 ✔ Exercise 4.24a

Show that if $\mathscr{A}$ is nonempty, then $\mathscr{P} \bigcap \mathscr{A} = \bigcap \{\mathscr{P} X \mid X \in \mathscr{A}\}$.

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_24a

Suppose $\mathscr{A}$ is a nonempty set. Then $\bigcap \mathscr{A}$ is well-defined. Therefore

$$\begin{aligned}
\mathscr{P} \bigcap \mathscr{A} &= \{x \mid x \subseteq \bigcap \mathscr{A}\} && \text{¶ Power Set} \\
&= \{x \mid x \subseteq \{y \mid \forall X \in \mathscr{A}, y \in X\}\} && \text{def'n intersection} \\
&= \{x \mid \forall t \in x, t \in \{y \mid \forall X \in \mathscr{A}, y \in X\}\} && \text{def'n subset} \\
&= \{x \mid \forall t \in x, (\forall X \in \mathscr{A}, t \in X)\} \\
&= \{x \mid \forall X \in \mathscr{A}, (\forall t \in x, t \in X)\} \\
&= \{x \mid \forall X \in \mathscr{A}, x \subseteq X\} \\
&= \{x \mid \forall X \in \mathscr{A}, x \in \mathscr{P} X\} && \text{¶ Power Set Axiom} \\
&= \{x \mid \forall t \in \{\mathscr{P} X \mid X \in \mathscr{A}\}, x \in t\} \\
&= \bigcap \{\mathscr{P} X \mid X \in \mathscr{A}\}.
\end{aligned}$$

□

2.4.16 ✔ Exercise 4.24b

Show that

$$\bigcup \{\mathscr{P} X \mid X \in \mathscr{A}\} \subseteq \mathscr{P} \bigcup \mathscr{A}. \tag{2.3}$$

Under what conditions does equality hold?

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_24b

We first prove (2.3). Let $x \in \bigcup \{\mathscr{P}X \mid X \in \mathscr{A}\}$. By definition of the union of sets, $(\exists X \in \mathscr{A}), x \in \mathscr{P}X$. By definition of the ¶ Power Set, $x \subseteq X$. By Exercise 3.3, $X \subseteq \bigcup \mathscr{A}$. Therefore $x \subseteq \bigcup \mathscr{A}$, proving $x \in \mathscr{P}\mathscr{A}$ as expected.

♠ ♠ ♠

We show $\mathscr{P} \bigcup A \subseteq \bigcup \{\mathscr{P}X \mid X \in \mathscr{A}\}$ if and only if $\bigcup \mathscr{A} \in \mathscr{A}$.

($\Rightarrow$) Suppose $\mathscr{P} \bigcup \mathscr{A} \subseteq \bigcup \{\mathscr{P}X \mid X \in \mathscr{A}\}$. By definition of the ¶ Power Set, $\bigcup \mathscr{A} \in \mathscr{P} \bigcup \mathscr{A}$. By hypothesis, $\bigcup \mathscr{A} \in \bigcup \{\mathscr{P}X \mid X \in \mathscr{A}\}$. By definition of the union of sets, there exists some $X \in \mathscr{A}$ such that $\bigcup \mathscr{A} \in \mathscr{P}X$. That is, $\bigcup \mathscr{A} \subseteq X$. But $\bigcup \mathscr{A}$ cannot be a proper subset of X since $X \in \mathscr{A}$. Thus $\bigcup \mathscr{A} = X$. This proves $\bigcup \mathscr{A} \in \bigcup \{\mathscr{P}X \mid X \in \mathscr{A}\}$.

($\Leftarrow$) Suppose $\bigcup \mathscr{A} \in A$. Let $x \in \mathscr{P} \bigcup \mathscr{A}$. Since $\bigcup \mathscr{A} \in \mathscr{A}$, it immediately follows that $x \in \{\mathscr{P}X \mid X \in \mathscr{A}\}$.

Conclusion Equality follows immediately from this fact in conjunction with the proof of (2.3).

□

2.4.17 Exercise 4.25

Is $A \cup \bigcup \mathscr{B}$ always the same as $\bigcup \{A \cup X \mid X \in \mathscr{B}\}$? If not, then under what conditions does equality hold?

Proof. ∃ – Enderton.Set.Chapter_2.exercise_4_25

We prove that

$$A \cup \bigcup \mathscr{B} = \bigcup \{A \cup X \mid X \in \mathscr{B}\} \tag{2.4}$$

if and only if $A = \varnothing$ or $\mathscr{B} \neq \varnothing$. We prove both directions of this biconditional.

($\Rightarrow$) Suppose (2.4) holds true. There are two cases to consider:

Case 1 Suppose $B \neq \varnothing$. Then $A = \varnothing \vee \mathscr{B} \neq \varnothing$ holds trivially.

Case 2 Suppose $B = \varnothing$. Then

$$A \cup \bigcup \mathscr{B} = A \cup \bigcup \varnothing = A$$

and

$$\bigcup \{A \cup X \mid X \in \mathscr{B}\} = \bigcup \varnothing = \varnothing.$$

Then by hypothesis (2.4), $A = \varnothing$. Then $A = \varnothing \vee \mathscr{B} \neq \varnothing$ holds trivially.

($\Leftarrow$) Suppose $A = \varnothing$ or $\mathscr{B} \neq \varnothing$. There are two cases to consider:

Case 1 Suppose $A = \varnothing$. Then $A \cup \bigcup \mathscr{B} = \bigcup \mathscr{B}$. Likewise,

$$\bigcup\{A \cup X \mid X \in \mathscr{B}\} = \bigcup\{X \mid X \in \mathscr{B}\} = \bigcup \mathscr{B}.$$

Therefore (2.4) holds.

Case 2 Suppose $B \neq \varnothing$. Then

$$\begin{aligned}
A \cup \bigcup \mathscr{B} &= \{x \mid x \in A \vee x \in \bigcup \mathscr{B}\} \\
&= \{x \mid x \in A \vee (\exists b \in \mathscr{B}) x \in b\} \\
&= \{x \mid (\exists b \in \mathscr{B}) x \in A \vee x \in b\} \\
&= \{x \mid (\exists b \in \mathscr{B}) x \in A \cup b\} \\
&= \{x \mid x \in \bigcup\{A \cup X \mid X \in \mathscr{B}\}\} \\
&= \bigcup\{A \cup X \mid X \in \mathscr{B}\}.
\end{aligned}$$

Therefore (2.4) holds.

$\square$

Chapter 3

Relations and Functions

3.1 Ordered Pairs

3.1.1 Theorem 3A

Theorem 3. *For any sets x, y, u, and v,*

$$\langle u, v\rangle = \langle x, y\rangle \iff u = x \land v = y. \tag{3.1}$$

Proof. ∃ – Set.OrderedPair.ext_iff

Let x, y, u, and v be arbitrary sets.

($\Leftarrow$) This follows trivially.

($\Rightarrow$) Suppose $\langle u, v\rangle = \langle x, y\rangle$. Then, by definition of an ¶ Ordered Pair,

$$\{\{u\}, \{u, v\}\} = \{\{x\}, \{x, y\}\}. \tag{3.2}$$

By the ¶ Extensionality Axiom, it follows $\{u\} \in \{\{x\}, \{x, y\}\}$ and $\{u, v\} \in \{\{x\}, \{x, y\}\}$. That is,

$$\{u\} = \{x\} \quad \text{or} \quad \{u\} = \{x, y\}$$

and

$$\{u, v\} = \{x\} \quad \text{or} \quad \{u, v\} = \{x, y\}.$$

There are 4 cases to consider:

Case 1 Suppose $\{u\} = \{x\}$ and $\{u, v\} = \{x\}$. The former identity implies $u = x$. The latter identity implies $u = v = x$. Then (3.2) simplifies to

$$\{\{u\}\} = \{\{x\}, \{x, y\}\},$$

meaning $x = y$. Thus $v = y$ as well.

Case 2 Suppose $\{u\} = \{x\}$ and $\{u, v\} = \{x, y\}$. The former identity implies $u = x$. Substituting into the latter identity yields $\{u, v\} = \{u, y\}$. This holds if and only if $v = y$.

Case 3 Suppose $\{u\} = \{x, y\}$ and $\{u, v\} = \{x\}$. The former identity implies $x = y = u$. Substituting into the latter yields $\{u, v\} = \{u\}$. Thus $u = v$ which in turn implies $v = y$.

Case 4 Suppose $\{u\} = \{x, y\}$ and $\{u, v\} = \{x, y\}$. The former identity implies $x = y = u$. Substituting into the latter yields $\{u, v\} = \{u\}$. This implies $v = u$ which in turn implies $v = y$.

Conclusion These cases are exhaustive and each implies that $u = x$ and $v = y$.

□