Elements of Set Theory

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Contents

Chapter R

Reference

R.1 Empty Set Axiom

There is a set having no members:

$$\exists B, \forall x, x \notin B.$$

R.2 ¶ Extensionality Axiom

If two sets have exactly the same members, then they are equal:

$$\forall A, \forall B, [\forall x, (x \in A \iff x \in B) \Rightarrow A = B].$$

Axiom. ∃ – Set.ext

□

R.3 Pair Set

For any sets u and v, the **pair set** $\{u, v\}$ is the set whose only members are u and v.

R.4 Pairing Axiom

For any sets u and v, there is a set having as members just u and v:

$$\forall u, \forall v, \exists B, \forall x, (x \in B \iff x = u \text{ or } x = v).$$

R.5 ¶ Power Set

For any set a, the **power set** $\mathscr{P}a$ is the set whose members are exactly the subsets of a.

Definition. ∃ – Set.powerset

□

R.6 Power Set Axiom

For any set a, there is a set whose members are exactly the subsets of a:

$$\forall a, \exists B, \forall x, (x \in B \iff x \subseteq a).$$

R.7 Subset Axioms

For each formula ϕ not containing B, the following is an axiom:

$$\forall t_1, \cdots \forall t_k, \forall c, \exists B, \forall x, (x \in B \iff x \in c \wedge \phi).$$

R.8 Union Axiom

For any set A, there exists a set B whose elements are exactly the members of the members of A:

$$\forall A, \exists B, \forall x\, [x \in B \iff (\exists b \in A) x \in b]$$

R.9 Union Axiom, Preliminary Form

For any sets a and b, there is a set whose members are those sets belonging either to a or to b (or both):

$$\forall a, \forall b, \exists B, \forall x, (x \in B \iff x \in a \text{ or } x \in b).$$

Chapter 1

Introduction

1.1 Baby Set Theory

1.1.1 Exercise 1.1

Which of the following become true when "$\in$" is inserted in place of the blank? Which become true when "$\subseteq$" is inserted?

Exercise 1.1a

$\{\varnothing\}$____$\{\varnothing, \{\varnothing\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1a

Because the *object* $\{\varnothing\}$ is a member of the right-hand set, the statement is **true** in the case of "$\in$".

Because the *members* of $\{\varnothing\}$ are all members of the right-hand set, the statement is also **true** in the case of "$\subseteq$".

□

Exercise 1.1b

$\{\varnothing\}$____$\{\varnothing, \{\{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1b

Because the *object* $\{\varnothing\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\varnothing\}$ are all members of the right-hand set, the statement is **true** in the case of "$\subseteq$".

□

Exercise 1.1c

$\{\{\varnothing\}\}$____$\{\varnothing, \{\varnothing\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1c

Because the *object* $\{\{\varnothing\}\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are all members of the right-hand set, the statement is **true** in the case of "$\subseteq$".

□

Exercise 1.1d

$\{\{\varnothing\}\}$____$\{\varnothing, \{\{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1d

Because the *object* $\{\{\varnothing\}\}$ is a member of the right-hand set, the statement is **true** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are not all members of the right-hand set, the statement is **false** in the case of "$\subseteq$".

□

Exercise 1.1e

$\{\{\varnothing\}\}$__$\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_1e

Because the *object* $\{\{\varnothing\}\}$ is not a member of the right-hand set, the statement is **false** in the case of "$\in$".

Because the *members* of $\{\{\varnothing\}\}$ are not all members of the right-hand set, the statement is **false** in the case of "$\subseteq$".

□

1.1.2 Exercise 1.2

Show that no two of the three sets $\varnothing$, $\{\varnothing\}$, and $\{\{\varnothing\}\}$ are equal to each other.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_2

By the ¶ Extensionality Axiom, $\varnothing$ is only equal to $\varnothing$. This immediately shows it is not equal to the other two. Now consider object $\varnothing$. This object is a member of $\{\varnothing\}$ but is not a member of $\{\{\varnothing\}\}$. Again, by the ¶ Extensionality Axiom, these two sets must be different.

$\square$

1.1.3 Exercise 1.3

Show that if $B \subseteq C$, then $\mathscr{P}B \subseteq \mathscr{P}C$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_3

Let $x \in \mathscr{P}B$. By definition of the ¶ Power Set, x is a subset of B. By hypothesis, $B \subseteq C$. Then $x \subseteq C$. Again by definition of the ¶ Power Set, it follows $x \in \mathscr{P}C$.

$\square$

1.1.4 Exercise 1.4

Assume that x and y are members of a set B. Show that $\{\{x\}, \{x, y\}\} \in \mathscr{P}\mathscr{P}B$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_1_4

Let x and y be members of set B. Then $\{x\}$ and $\{x, y\}$ are subsets of B. By definition of the ¶ Power Set, $\{x\}$ and $\{x, y\}$ are members of $\mathscr{P}B$. Then $\{\{x\}, \{x, y\}\}$ is a subset of $\mathscr{P}B$. By definition of the ¶ Power Set, $\{\{x\}, \{x, y\}\}$ is a member of $\mathscr{P}\mathscr{P}B$.

$\square$

1.2 Sets - An Informal View

1.2.1 Exercise 2.1

Define the rank of a set c to be the least α such that $c \subseteq V_\alpha$. Compute the rank of $\{\{\varnothing\}\}$. Compute the rank of $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.

Proof. We first compute the values of V_n for $0 \leq n \leq 3$ under the assumption the set of atoms A at the bottom of the hierarchy is empty.

$$\begin{aligned}
V_0 &= \varnothing \\
V_1 &= V_0 \cup \mathscr{P}V_0 \\
&= \varnothing \cup \{\varnothing\} \\
&= \{\varnothing\} \\
V_2 &= V_1 \cup \mathscr{P}V_1 \\
&= \{\varnothing\} \cup \mathscr{P}\{\varnothing\} \\
&= \{\varnothing\} \cup \{\varnothing, \{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\} \\
V_3 &= V_2 \cup \mathscr{P}V_2 \\
&= \{\varnothing, \{\varnothing\}\} \cup \mathscr{P}\{\varnothing, \{\varnothing\}\} \\
&= \{\varnothing, \{\varnothing\}\} \cup \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
&= \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}
\end{aligned}$$

It then immediately follows $\{\{\varnothing\}\}$ has rank 2 and $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ has rank 3.

□

1.2.2 Exercise 2.2

We have stated that $V_{\alpha+1} = A \cup \mathscr{P}V_\alpha$. Prove this at least for $\alpha < 3$.

Proof. Let A be the set of atoms in our set hierarchy. Let $P(n)$ be the predicate, "$V_{n+1} = A \cup \mathscr{P}V_n$." We prove $P(n)$ holds true for all natural numbers $n \geq 1$ via induction.

Base Case Let $n = 1$. By definition, $V_1 = V_0 \cup \mathscr{P}V_0$. By definition, $V_0 = A$. Therefore $V_1 = A \cup \mathscr{P}V_0$. This proves $P(1)$ holds true.

Induction Step Suppose $P(n)$ holds true for some $n \geq 1$. Consider V_{n+1}. By definition, $V_{n+1} = V_n \cup \mathscr{P}V_n$. Therefore, by the induction hypothesis,

$$\begin{aligned}
V_{n+1} &= V_n \cup \mathscr{P}V_n \\
&= (A \cup \mathscr{P}V_{n-1}) \cup \mathscr{P}V_n \\
&= A \cup (\mathscr{P}V_{n-1} \cup \mathscr{P}V_n) \qquad (1.1)
\end{aligned}$$

But V_{n-1} is a subset of V_n. Exercise 1.3 then implies $\mathscr{P}V_{n-1} \subseteq \mathscr{P}V_n$. This means (1.1) can be simplified to

$$V_{n+1} = A \cup \mathscr{P}V_n,$$

proving $P(n+1)$ holds true.

Conclusion By mathematical induction, it follows for all $n \geq 1$, $P(n)$ is true. □

1.2.3 Exercise 2.3

List all the members of V_3. List all the members of V_4. (It is to be assumed here that there are no atoms.)

Proof. As seen in the proof of Exercise 2.1,

$$V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}.$$

By Exercise 2.2, $V_4 = \mathscr{P}V_3$ (since it is assumed there are no atoms). Thus

$$\begin{array}{l}
V_4 = \{ \\
\qquad \varnothing, \\
\qquad \{\varnothing\}, \\
\qquad \{\{\varnothing\}\}, \\
\qquad \{\{\{\varnothing\}\}\}, \\
\qquad \{\{\varnothing, \{\varnothing\}\}\}, \\
\qquad \{\varnothing, \{\varnothing\}\}, \\
\qquad \{\varnothing, \{\{\varnothing\}\}\}, \\
\qquad \{\varnothing, \{\varnothing, \{\varnothing\}\}\}, \\
\qquad \{\{\varnothing\}, \{\{\varnothing\}\}\}, \\
\qquad \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\
\qquad \{\{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \\
\qquad \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}, \\
\qquad \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \\
\qquad \{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
\qquad \{\{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}, \\
\qquad \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\} \\
\}.
\end{array}$$

□

Chapter 2

Axioms and Operations

2.1 Axioms

2.1.1 Theorem 2A

Theorem 2A. *There is no set to which every set belongs.*

Proof. TODO ☐

2.1.2 Theorem 2B

Theorem 2B. *For any nonempty set A, there exists a unique set B such that for any x,*

$$x \in B \iff x \text{ belongs to every member of } A.$$

Proof. TODO ☐

2.2 Exercises 3

2.2.1 Exercise 3.1

Assume that A is the set of integers divisible by 4. Similarly assume that B and C are the sets of integers divisible by 9 and 10, respectively. What is in $A \cap B \cap C$?

Answer. ∃ – Enderton.Set.Chapter_1.exercise_3_1

The set of integers divisible by 4, 9, and 10.

□

2.2.2 Exercise 3.2

Give an example of sets A and B for which $\bigcup A = \bigcup B$ but $A \neq B$.

Answer. ∃ – Enderton.Set.Chapter_1.exercise_3_2

Let $A = \{\{1\}, \{2\}\}$ and $B = \{\{1, 2\}\}$.

□

2.2.3 Exercise 3.3

Show that every member of a set A is a subset of $\bigcup A$. (This was stated as an example in this section.)

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_3

Let $x \in A$. By definition,

$$\bigcup A = \{y \mid (\exists b \in A) y \in b\}.$$

Then $\{y \mid y \in x\} \subseteq \bigcup A$. But $\{y \mid y \in x\} = x$. Thus $x \subseteq \bigcup A$.

□

2.2.4 Exercise 3.4

Show that if $A \subseteq B$, then $\bigcup A \subseteq \bigcup B$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_4

Let A and B be sets such that $A \subseteq B$. Let $x \in \bigcup A$. By definition of the union, there exists some $b \in A$ such that $x \in b$. By definition of the subset, $b \in B$. This immediatley implies $x \in \bigcup B$. Since this holds for all $x \in \bigcup A$, it follows $\bigcup A \subseteq \bigcup B$.

□

2.2.5 Exercise 3.5

Assume that every member of $\mathscr{A}$ is a subset of B. Show that $\bigcup \mathscr{A} \subseteq B$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_5

Let $x \in \bigcup \mathscr{A}$. By definition,

$$\bigcup \mathscr{A} = \{y \mid (\exists b \in A) y \in b\}.$$

Then there exists some $b \in A$ such that $x \in b$. By hypothesis, $b \subseteq B$. Thus x must also be a member of B. Since this holds for all $x \in \bigcup \mathscr{A}$, it follows $\bigcup \mathscr{A} \subseteq B$.

□

2.2.6 Exercise 3.6a

Show that for any set A, $\bigcup \mathscr{P} A = A$.

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_6a

We prove that (i) $\bigcup \mathscr{P} A \subseteq A$ and (ii) $A \subseteq \bigcup \mathscr{P} A$.

(i) By definition, the ¶ Power Set of A is the set of all subsets of A. In other words, every member of $\mathscr{P} A$ is a subset of A. By Exercise 3.5, $\bigcup \mathscr{P} A \subseteq A$.

(ii) Let $x \in A$. By definition of the power set of A, $\{x\} \in \mathscr{P} A$. By definition of the union,

$$\bigcup \mathscr{P} A = \{y \mid (\exists b \in \mathscr{P} A), y \in b).$$

Since $x \in \{x\}$ and $\{x\} \in \mathscr{P} A$, it follows $x \in \bigcup \mathscr{P} A$. Thus $A \subseteq \bigcup \mathscr{P} A$.

Conclusion By (i) and (ii), $\bigcup \mathscr{P} A = A$.

□

2.2.7 Exercise 3.6b

Show that $A \subseteq \mathscr{P} \bigcup A$. Under what conditions does equality hold?

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_6b

Let $x \in A$. By Exercise 3.3, x is a subset of $\bigcup A$. By the definition of the ¶ Power Set,

$$\mathscr{P} \bigcup A = \{y \mid y \subseteq \bigcup A\}.$$

Therefore $x \in \mathscr{P} \bigcup A$. Since this holds for all $x \in A$, $A \subseteq \mathscr{P} \bigcup A$.

♠ ♠ ♠

We show equality holds if and only if there exists some set B such that $A = \mathscr{P} B$.

(⇒) Suppose $A = \mathscr{P} \bigcup A$. Then our statement immediately follows by settings $B = \bigcup A$.

(⇐) Suppose there exists some set B such that $A = \mathscr{P} B$. Therefore

$$\begin{aligned}
\mathscr{P} \bigcup A &= \mathscr{P} \left(\bigcup \mathscr{P} B \right) \\
&= \mathscr{P} B && \text{Exercise 3.6a} \\
&= A.
\end{aligned}$$

Conclusion By (⇒) and (⇐), $A = \mathscr{P} \bigcup A$ if and only if there exists some set B such that $A = \mathscr{P} B$.

□

2.2.8 Exercise 3.7a

Show that for any sets A and B,

$$\mathscr{P} A \cap \mathscr{P} B = \mathscr{P}(A \cap B).$$

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_7a

Let A and B be arbitrary sets. We show that $\mathscr{P} A \cap \mathscr{P} B \subseteq \mathscr{P}(A \cap B)$ and then show that $\mathscr{P} A \cap \mathscr{P} B \supseteq \mathscr{P}(A \cap B)$.

(⊆) Let $x \in \mathscr{P} A \cap \mathscr{P} B$. That is, $x \in \mathscr{P} A$ and $x \in \mathscr{P} B$. By the definition of the ¶ Power Set,

$$\begin{aligned}
\mathscr{P} A &= \{y \mid y \subseteq A\} \\
\mathscr{P} B &= \{y \mid y \subseteq B\}
\end{aligned}$$

Thus $x \subseteq A$ and $x \subseteq B$, meaning $x \subseteq A \cap B$. But then $x \in \mathscr{P}(A \cap B)$, the set of all subsets of $A \cap B$. Since this holds for all $x \in \mathscr{P} A \cap \mathscr{P} B$, it follows

$$\mathscr{P} A \cap \mathscr{P} B \subseteq \mathscr{P}(A \cap B).$$

(⊇) Let $x \in \mathscr{P}(A \cap B)$. By the definition of the ¶ Power Set,

$$\mathscr{P}(A \cap B) = \{y \mid y \subseteq A \cap B\}.$$

Thus $x \subseteq A \cap B$, meaning $x \subseteq A$ and $x \subseteq B$. But this implies $x \in \mathscr{P} A$, the set of all subsets of A. Likewise $x \in \mathscr{P} B$, the set of all subsets of B. Thus $x \in \mathscr{P} A \cap \mathscr{P} B$. Since this holds for all $x \in \mathscr{P}(A \cap B)$, it follows

$$\mathscr{P}(A \cap B) \subseteq \mathscr{P} A \cap \mathscr{P} B.$$

Conclusion Since each side of our identity is a subset of the other,

$$\mathscr{P}(A \cap B) = \mathscr{P}A \cap \mathscr{P}B.$$

□

2.2.9 Exercise 3.7b

Show that $\mathscr{P}A \cup \mathscr{P}B \subseteq \mathscr{P}(A \cup B)$. Under what conditions does equality hold?

Proof.
∃ – Enderton.Set.Chapter_1.exercise_3_7b_i
∃ – Enderton.Set.Chapter_1.exercise_3_7b_ii

Let $x \in \mathscr{P}A \cup \mathscr{P}B$. By definition, $x \in \mathscr{P}A$ or $x \in \mathscr{P}B$ (or both). By the definition of the ¶ Power Set,

$$\mathscr{P}A = \{y \mid y \subseteq A\}$$
$$\mathscr{P}B = \{y \mid y \subseteq B\}.$$

Thus $x \subseteq A$ or $x \subseteq B$. Therefore $x \subseteq A \cup B$. But then $x \in \mathscr{P}(A \cup B)$, the set of all subsets of $A \cup B$.

♠♠♠

We show equality holds if and only if one of A or B is a subset of the other.

(⇒) Suppose

$$\mathscr{P}A \cup \mathscr{P}B = \mathscr{P}(A \cup B). \tag{2.1}$$

By the definition of the ¶ Power Set, $A \cup B \in \mathscr{P}(A \cup B)$. Then (2.1) implies $A \cup B \in \mathscr{P}A \cup \mathscr{P}B$. That is, $A \cup B \in \mathscr{P}A$ or $A \cup B \in \mathscr{P}B$ (or both).

For the sake of contradiction, suppose $A \not\subseteq B$ and $B \not\subseteq A$. Then there exists an element $x \in A$ such that $x \notin B$ and there exists an element $y \in B$ such that $y \notin A$. But then $A \cup B \notin \mathscr{P}A$ since y cannot be a member of a member of $\mathscr{P}A$. Likewise, $A \cup B \notin \mathscr{P}B$ since x cannot be a member of a member of $\mathscr{P}B$. Therefore our assumption is incorrect. In other words, $A \subseteq B$ or $B \subseteq A$.

(⇐) WLOG, suppose $A \subseteq B$. Then, by Exercise 1.3, $\mathscr{P}A \subseteq \mathscr{P}B$. Thus

$$\begin{aligned}\mathscr{P}A \cup \mathscr{P}B &= \mathscr{P}B \\ &= \mathscr{P}A \cup B.\end{aligned}$$

Conclusion By (⇒) and (⇐), it follows $\mathscr{P}A \cup \mathscr{P}B \subseteq \mathscr{P}(A \cup B)$ if and only if $A \subseteq B$ or $B \subseteq A$.

□

2.2.10 Exercise 3.8

Show that there is no set to which every singleton (that is, every set of the form $\{x\}$) belongs. [*Suggestion*: Show that from such a set, we could construct a set to which every set belonged.]

Proof. We proceed by contradiction. Suppose there existed a set A consisting of every singleton. Then the Union Axiom suggests $\bigcup A$ is a set. But this set is precisely the class of all sets, which is *not* a set. Thus our original assumption was incorrect. That is, there is no set to which every singleton belongs.

□

2.2.11 Exercise 3.9

Give an example of sets a and B for which $a \in B$ but $\mathscr{P}a \notin \mathscr{P}B$.

Answer. ∃ – Enderton.Set.Chapter_1.exercise_3_9

Let $a = \{1\}$ and $B = \{\{1\}\}$. Then

$$\begin{aligned}\mathscr{P}a &= \{\varnothing, \{1\}\} \\ \mathscr{P}B &= \{\varnothing, \{\{1\}\}\}.\end{aligned}$$

It immediately follows that $\mathscr{P}a \notin \mathscr{P}B$.

□

2.2.12 Exercise 3.10

Show that if $a \in B$, then $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$. [*Suggestion*: If you need help, look in the Appendix.]

Proof. ∃ – Enderton.Set.Chapter_1.exercise_3_10

Suppose $a \in B$. By Exercise 3.3, $a \subseteq \bigcup B$. By Exercise 1.3, $\mathscr{P}a \subseteq \mathscr{P}\bigcup B$. By the definition of the ¶ Power Set,

$$\mathscr{P}\mathscr{P}\bigcup B = \{y \mid y \subseteq \mathscr{P}\bigcup B\}.$$

Therefore $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$.

□