Extracting Cost Recurrences from Sequential and Parallel Functional Programs

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April 26, 2016

Abstract

- Complexity analysis aims to predict the resources, most often time and space, which a program requires
- Previous work by Danner et al. [2013] and Danner et al. [2015] formalizes the analysis of higher-order function programs
- We use the method of Danner et al. [2015] to analyze higher order functional programs
- We extend the method to parallel cost semantics
- We prove an interesting fact about the recurrences for the cost of programs

Introduction

- Write programs in a "source language"
- Translate the programs to a "complexity language"
- The translated programs are recurrences for the complexity of the source language program
- **complexity** $=$ cost $\times$ potential
- **cost**: steps to run a program
- **potential**: size of the result of evaluating program

Source Language

- Variant of System-T

$$
\begin{aligned}
e ::= x \mid \langle\rangle \mid \lambda x.e \mid e\ e \mid \langle e, e\rangle \mid \mathtt{split}(e, x.x.e) \\
\mid \mathtt{delay}(e) \mid \mathtt{force}(e) \mid C^{\delta}\ e \mid \mathtt{rec}^{\delta}(e, \overline{C \mapsto x.e_C}) \\
\mid \mathtt{map}^{\phi}(x.v, v) \mid \mathtt{let}(e, x.e)
\end{aligned}
$$

- Programmer defined datatypes
  - `datatype list = Nil | Cons int×list`
- Structural Recursion
  - OCaml:

```
length xs =
  match xs with
   [] -> 0
   (x::xs) -> 1 + length xs
```

  - $\lambda xs.\mathtt{rec}(xs, \mathtt{Nil} \mapsto 0, \mathtt{Cons} \mapsto \langle x, \langle xs, r\rangle\rangle.1 + \mathtt{force}(r))$

Complexity Language

- Language for recurrences
- Source language without syntactic constructs for controlling costs
- No `let`, `delay`, `split`

Translation

- Translate source language programs of type τ to complexity language programs of type $\mathbf{C} \times \langle\langle\tau\rangle\rangle$
- $\mathbf{C}$ bound on the steps to evaluate the program
- $\langle\langle\tau\rangle\rangle$ expression for the size of the value
- Some cases of the translation function $\| \cdot \|$:
  - $\|x\| = \langle 0, x\rangle$
  - $\|\langle e_0, e_1\rangle\| = \langle \|e_0\|_c + \|e_1\|_c, \langle \|e_0\|_p, \|e_1\|_p\rangle\rangle$
  - $\|\lambda x.e\| = \langle 0, \lambda x.\|e\|\rangle$
  - $\|e_0\ e_1\| = (1 + \|e_0\|_c + \|e_1\|_c) +_c \|e_0\|_p \|e_1\|_p$

Fast Reverse - Specification and Implementation

- $\texttt{datatype list} = \texttt{Nil of unit} \mid \texttt{Cons of int} \times \texttt{list}$
- Specification: `rev` $[x_0, \ldots, x_{n-1}]$ `=` $[x_{n-1}, \ldots, x_0]$
- Implementation:

  $\texttt{rev xs = }\lambda\texttt{xs.rec(xs,}$
  $\quad\texttt{Nil} \mapsto \lambda\texttt{a.a,}$
  $\quad\texttt{Cons} \mapsto \langle x, \langle xs, r\rangle\rangle.\lambda\texttt{a.force(r) Cons}\langle\texttt{x,a}\rangle\texttt{))) Nil}$
- Specification of auxilary function:

  $\texttt{rec(}[x_0, \ldots, x_{n-1}], \ldots\texttt{)}\ [y_0, \ldots, y_{m-1}]\ \texttt{=}$
  $[x_{n-1}, \ldots, x_0, y_0, \ldots, y_{m-1}]$

Fast Reverse - Specification and Implementation

- $\texttt{rev (Cons}\langle\texttt{0,Cons}\langle\texttt{1, Nil}\rangle\rangle\texttt{)}$
- $\rightarrow_\beta$ $\texttt{rec(Cons}\langle\texttt{0,Cons}\langle\texttt{1,Nil}\rangle\rangle\texttt{, Nil} \mapsto \lambda\texttt{a.a}$ $\texttt{Cons}\mapsto \langle x, \langle xs, r\rangle\rangle.\lambda\texttt{a.force(r) Cons}\langle\texttt{x,a}\rangle\texttt{) Nil}$
- $\rightarrow_\beta^*$ $(\lambda\texttt{a0.(}\lambda\texttt{a1.(}\lambda\texttt{a2.a2) Cons}\langle\texttt{1,a1}\rangle\texttt{) Cons}\langle\texttt{0,a0}\rangle\texttt{) Nil}$
- $\rightarrow_\beta$ $(\lambda \mathsf{a1}.(\lambda \mathsf{a2}.\mathsf{a2})\ \mathsf{Cons}\langle 1,\mathsf{a1}\rangle)\ \mathsf{Cons}\langle 0,\mathsf{Nil}\rangle$
- $\rightarrow_\beta$ $(\lambda \mathsf{a2}.\mathsf{a2})\ \mathsf{Cons}\langle 1,\mathsf{Cons}\langle 0,\mathsf{Nil}\rangle\rangle$
- $\rightarrow_\beta$ $\mathsf{Cons}\langle 1,\mathsf{Cons}\langle 0,\mathsf{Nil}\rangle\rangle$

Fast Reverse - Translation

- $\|\texttt{rev}\|$

$$\langle 0, \lambda xs.1 +_c \texttt{rec}(xs, \texttt{Nil} \mapsto \langle 1, \lambda a.\langle 0, a\rangle\rangle$$
$$\texttt{Cons} \mapsto \langle x, \langle xs', r\rangle\rangle.\langle 1, \lambda a.(1 + r_c) +_c r_p\ \texttt{Cons}\langle \pi_1 x, a\rangle\rangle)\ \texttt{Nil}\rangle$$

- $\|\texttt{rev xs}\|$

$$1 +_c (\lambda xs.\texttt{rec}(xs, \texttt{Nil} \mapsto \langle 1, \lambda a.\langle 0, a\rangle\rangle$$
$$\texttt{Cons} \mapsto \langle x, \langle xs', r\rangle\rangle.\langle 1, \lambda a.(1 + r_c) +_c r_p\ \texttt{Cons}\langle x, a\rangle\rangle)\ \texttt{Nil})\ xs$$

Fast Reverse - Interpretation

We need to provide an interpretation for programmer-defined datatypes

$$
\begin{aligned}
[\![\texttt{list}]\!] &= \mathbb{N}^{\infty} \\
D^{list} &= \{*\} + \{1\} \times \mathbb{N}^{\infty} \\
size_{list}(\texttt{Nil}) &= 1 \\
size_{list}(\texttt{Cons(1,n)}) &= 1 + n
\end{aligned}
$$

The interpretation of the recursor is

$$
g(n) = \bigvee_{size\ \ z \leq n} case(z, f_C, f_N)
$$

where

$$
\begin{aligned}
f_{Nil}(x) &= (1, \lambda a.(0, a)) \\
f_{Cons}(b) &= (1, \lambda a.(1 + g_c(\pi_1 b)) +_c g_p(\pi_1 b)\ (a + 1))
\end{aligned}
$$

Fast Reverse - Interpretation

- Let $h(n, a) = g_p(n)\ a$.
- The recurrence for the cost:

$$h_c(n, a) = \begin{cases} 0 & n = 0 \\ 2 + h_c(n-1, a+1) & n > 0 \end{cases} \tag{1}$$

- $h_c(n, a) = 2n$
- The recurrence for the potential:

$$h_p(n, a) = \begin{cases} a & n = 0 \\ h_p(n-1, a+1) & n > 0 \end{cases} \tag{2}$$

- $h_p(n, a) = n + a$

Parametric Insertion Sort - Source Language

$$\mathtt{data\ list} = \mathtt{Nil\ of\ unit\ \mid\ Cons\ of\ int\ \times\ list}$$

$$\begin{aligned}\mathtt{insert} = \lambda f.\lambda x.\lambda xs.\mathtt{rec}(xs, \mathtt{Nil} \mapsto \mathtt{Cons}\langle x, \mathtt{Nil}\rangle,\\ \mathtt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.\mathtt{rec}(f\ x\ y, \mathtt{True} \mapsto \mathtt{Cons}\langle x, \mathtt{Cons}\langle y, ys\rangle\rangle,\\ \mathtt{False} \mapsto \mathtt{Cons}\langle y, \mathtt{force}(r)\rangle))\end{aligned}$$

$$\begin{aligned}\mathtt{sort} = \lambda f.\lambda xs.\mathtt{rec}(xs, \mathtt{Nil} \mapsto \mathtt{Nil},\\ \mathtt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.\mathtt{insert}\ f\ y\ \mathtt{force}(r))\end{aligned}$$

Parametric Insertion Sort - Complexity Language

$$
\begin{aligned}
\|\texttt{insert}\| = \langle 0, \lambda f.\langle 0, \lambda x.\langle 0, \lambda xs.\texttt{rec}(xs, \texttt{Nil} &\mapsto \langle 1, \texttt{Cons}\langle x, \texttt{Nil}\rangle\rangle, \\
\texttt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.(3 + (fx)_c) +_c \texttt{rec}(((f\ \ x)_p\ \ y)_p&, \\
\texttt{True} &\mapsto \langle 1, \texttt{Cons}\langle x, \texttt{Cons}\langle y, ys\rangle\rangle\rangle, \\
\texttt{False} &\mapsto \langle 1 + r_c, \texttt{Cons}\langle y, r_p\rangle\rangle)))
\end{aligned}
$$

$$
\begin{aligned}
\|\texttt{sort}\| = \langle 0, \lambda f.\langle 0, \lambda xs.\texttt{rec}(xs, \texttt{Nil} &\mapsto 1 +_c \langle 0, \texttt{Nil}\rangle, \\
\texttt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.(4 + r_c) +_c ((\|\texttt{insert}\|_p f)_p y)_p r_p)\rangle\rangle&
\end{aligned}
$$

Parametric Insertion Sort - Interpretation

$$
\begin{aligned}
\|\texttt{insert}\| &= (0, \lambda f.(0, \lambda x.(0, \lambda(i,n).g(i,n)))) \\
&\quad \text{where} \\
g(i,n) &= \bigvee_{size(z) \leq (i,n)} case(z, f_{Nil}, f_{Cons}) \\
&\qquad \text{where} \\
f_{Nil}(*) &= (1, (x, 1)) \\
f_{Cons}(j,m) &= (4 + (f\ x)_c) +_c ((1, (max(x,j), 2+m)) \\
&\qquad \vee (1 + g_c(j,m), (max(j, \pi_0 r_p), 1 + \pi_1 g_p(j,m))))
\end{aligned}
$$

Sequential Cost Semantics

$$\frac{e_0 \downarrow^{n_0} v_0 \qquad e_1 \downarrow^{n_1} v_1}{\langle e_0, e_1 \rangle \downarrow^{n_0+n_1} \langle v_0, v_1 \rangle}$$

$$\frac{e_0 \downarrow^{n_0} \lambda x.e_0' \qquad e_1 \downarrow^{n_1} v_1 \qquad e_0'[v_1/x] \downarrow^{n} v}{e_0\ e_1 \downarrow^{1+n_0+n_1+n} v}$$

$$\frac{}{\texttt{delay}(e) \downarrow^{0} \texttt{delay}(e)} \qquad \frac{e \downarrow^{n_0} \texttt{delay}(e_0) \qquad e_0 \downarrow^{n_1} v}{\texttt{force}(e) \downarrow^{n_0+n_1} v}$$

Parallel Cost Semantics

- Cost graphs

$$\mathcal{C} ::= 0 \mid 1 \mid \mathcal{C} \oplus \mathcal{C} \mid \mathcal{C} \otimes \mathcal{C}$$

Evaluation Semantics

$$\frac{e_0 \downarrow^{n_0} v_0 \qquad e_1 \downarrow^{n_1} v_1}{\langle e_0, e_1 \rangle \downarrow^{n_0 \otimes n_1} \langle v_0, v_1 \rangle}$$

$$\frac{e_0 \downarrow^{n_0} \lambda x.e_0' \qquad e_1 \downarrow^{n_1} v_1 \qquad e_0'[v_1/x] \downarrow^{n} v}{e_0 \; e_1 \downarrow^{(n_0 \otimes n_1) \oplus n \oplus 1} v}$$

Work and Span

- **Work** total steps to run program

$$
work(c) = \begin{cases} 0 & \text{if } c = 0 \\ 1 & \text{if } c = 1 \\ work(c_0) + work(c_1) & \text{if } c = c_0 \otimes c_1 \\ work(c_0) + work(c_1) & \text{if } c = c_0 \oplus c_1 \end{cases}
$$

- **Span** critical path of program

$$
span(c) = \begin{cases} 0 & \text{if } c = 0 \\ 1 & \text{if } c = 1 \\ max(span(c_0), span(c_1)) & \text{if } c = c_0 \otimes c_1 \\ span(c_0) + span(c_1) & \text{if } c = c_0 \oplus c_1 \end{cases}
$$

Parallel Complexity Translation

$$
\begin{aligned}
\|\langle e_0, e_1 \rangle\| &= \langle \|e_0\|_c \otimes \|e_1\|_c, \langle \|e_0\|_p, \|e_1\|_p \rangle \rangle \\
\|\lambda x.e\| &= \langle 0, \lambda x.\|e\| \rangle \\
\|e_0 \ e_1\| &= 1 \oplus (\|e_0\|_c \otimes \|e_1\|_c) \oplus_c \|e_0\|_p \ \|e_1\|_p \\
\|\mathit{delay}(e)\| &= \langle 0, \|e\| \rangle \\
\|\mathit{force}(e)\| &= \|e\|_c \oplus_c \|e\|_p
\end{aligned}
$$

Bounding relation: if $\gamma \vdash e : \tau$, then $e \sqsubseteq_\tau \|e\|$. Proof by logical relations.

Mutual Recurrences

$$\begin{aligned}
|\langle e_0, e_1\rangle| &= \langle |e_0|, |e_1|\rangle \\
|\lambda x.e| &= \lambda x.|e| \\
|e_0 \; e_1| &= |e_0| \; |e_1| \\
|\mathit{delay}(e)| &= |e| \\
|\mathit{force}(e)| &= |e|
\end{aligned}$$

For all $\gamma \vdash e : \tau$, $|e| : \langle\!\langle \tau \rangle\!\rangle \sim_\tau \|e\| : \|\tau\|$ Proof by logical relations

Bibliography

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