# Extracting Cost Recurrences from Sequential and Parallel Functional Programs

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# Overview

- Previous work by Danner et al. [2013] and Danner et al. [2015] formalizes the extraction of recurrences for the cost of higher-order functional programs
- Contribution:
  - We use the method of Danner et al. [2015] to analyze the complexity of higher-order functional programs
  - We extend the method to parallel cost semantics to help answer the question: how long will this program take to run on an arbitrary number of processors?
  - We prove an interesting fact about the recurrences for the cost of programs

# Traditional Complexity Analysis

Source program:

```
fold f z xs =
  match xs with
    [] -> z
  | x::xs' -> f x (fold f z xs')
```

Write down recurrence for cost:

$$T(n) = \begin{cases} c_0 & \text{if } n = 0 \\ c_1 + T(n-1) & \text{otherwise} \end{cases}$$

Obtain a closed form solution:

$$T(n) = c_1 n + c_0 = \mathcal{O}(n)$$

# Method of Danner et al. [2015]

- Write programs in a "source language"
- Translate the programs to a "complexity language"
- The translated programs are recurrences for the complexity of the source language program
- **complexity** $=$ cost $\times$ potential
- **cost**: steps to run a program
- **potential**: size of the result of evaluating program

# Source Language

- Variant of System-T
  Types:

$$\tau ::= \texttt{unit} \mid \tau \times \tau \mid \tau \to \tau \mid \texttt{susp}\ \tau \mid \delta$$

  Expressions:

$$\begin{aligned} e ::= {} & x \mid \langle\rangle \mid \lambda x.e \mid e\ e \mid \langle e, e\rangle \mid \texttt{split}(e, x.x.e) \\ & \mid \texttt{delay}(e) \mid \texttt{force}(e) \mid C^{\delta}\ e \mid \texttt{rec}^{\delta}(e, \overline{C \mapsto x.e_C}) \\ & \mid \texttt{map}^{\phi}(x.v, v) \mid \texttt{let}(e, x.e) \end{aligned}$$

- Programmer defined datatypes:

```
datatype list = Nil | Cons int×list
```

# Source Language - Cost Semantics

- Tuples:

$$\frac{e_0 \downarrow^{n_0} v_0 \qquad e_1 \downarrow^{n_1} v_1}{\langle e_0, e_1 \rangle \downarrow^{n_0+n_1} \langle v_0, v_1 \rangle}$$

- Application:

$$\frac{e_0 \downarrow^{n_0} \lambda x.e_0' \qquad e_1 \downarrow^{n_1} v_1 \qquad e_0'[v_1/x] \downarrow^{n} v}{e_0\ e_1 \downarrow^{1+n_0+n_1+n} v}$$

- Structural Recursion:

$$\frac{e \downarrow^{n_0} Cv_0 \qquad \mathtt{map}^{\phi C}(y.\langle y, \mathtt{delay}(rec(y, \overline{C \mapsto x.e_C}))\rangle, v_0) \downarrow^{n_1} v_1 \qquad e_C[v_1/x] \downarrow^{n_2} v}{rec(e, \overline{C \mapsto x.e_C}) \downarrow^{1+n_0+n_1+n_2} v}$$

# Source Language

- OCaml:

```
length xs =
  match xs with
    [] -> 0
  | (x::xs) -> 1 + length xs
```

- Source language:

$$\lambda xs.\texttt{rec}(xs, \texttt{Nil} \mapsto 0, \texttt{Cons} \mapsto \langle x, \langle xs, r\rangle\rangle.1 + \texttt{force}(r))$$

# Complexity Language

- Types

$$T ::= \mathbf{C} \mid \mathtt{unit} \mid \Delta \mid T \times T \mid T \to T$$
$$\mathbf{C} ::= 0 \mid 1 \mid 2 \mid \ldots$$
$$\mathtt{datatype}\Delta = C_0^{\Delta}\mathtt{of}\Phi_{C_0}[\Delta] \mid \ldots \mid C_{n-1}^{\Delta}\mathtt{of}\Phi_{C_{n-1}}[\Delta]$$

- Expressions

$$E ::= x \mid 0 \mid 1 \mid E + E \mid \langle\rangle \mid \langle E, E\rangle \mid$$
$$\pi_0 E \mid \pi_1 E \mid \lambda x.E \mid E\ E \mid C^{\delta}\ E \mid \mathrm{rec}^{\Delta}(E, \overline{C \mapsto x.E_C})$$

- No longer need mechanisms for controlling cost

# Translation

- Translate source language programs of type $\tau$ to complexity language programs of type $\mathbf{C} \times \langle\!\langle \tau \rangle\!\rangle$
- $\mathbf{C}$ bound on the steps to evaluate the program
- $\langle\!\langle \tau \rangle\!\rangle$ expression for the size of the value
- Types of the translation function $\| \cdot \|$:

$$
\begin{aligned}
\|\tau\| &= \mathbf{C} \times \langle\!\langle \tau \rangle\!\rangle \\
\langle\!\langle \mathtt{unit} \rangle\!\rangle &= \mathtt{unit} \\
\langle\!\langle \sigma \times \tau \rangle\!\rangle &= \langle\!\langle \sigma \rangle\!\rangle \times \langle\!\langle \tau \rangle\!\rangle \\
\langle\!\langle \sigma \to \tau \rangle\!\rangle &= \langle\!\langle \sigma \rangle\!\rangle \to \|\tau\| \\
\langle\!\langle \mathtt{susp}\ \tau \rangle\!\rangle &= \|\tau\| \\
\langle\!\langle \delta \rangle\!\rangle &= \delta
\end{aligned}
$$

# Translation

- Some cases of the translation function:

$$
\begin{aligned}
\|x\| &= \langle 0, x \rangle \\
\|\langle e_0, e_1 \rangle\| &= \langle \|e_0\|_c + \|e_1\|_c, \langle \|e_0\|_p, \|e_1\|_p \rangle \rangle \\
\|\lambda x.e\| &= \langle 0, \lambda x.\|e\| \rangle \\
\|e_0\ e_1\| &= (1 + \|e_0\|_c + \|e_1\|_c) +_c \|e_0\|_p \|e_1\|_p
\end{aligned}
$$

# Size-Based Denotational Semantics

- Complexity translation does not lose any information
- Abstract values to their sizes using a denotational semantics
- **denotational semantics** - assign meanings to programs by interpreting types as sets and terms as elements of sets
- We use a standard denotational semantics
- Must define the following for programmer defined datatypes:
  - $[\![\delta]\!]$ - the set in which to interpret the type
  - $D^\delta$ - sum type representing the unfolding of the datatype
  - $size : D^\delta \to [\![\delta]\!]$ - function from sum type to interpretation
- The interpretation of the recursor is non-standard

$$[\![rec(E, \overline{C \mapsto x.E_C})]\!]\xi = \bigvee_{size(z) \leq [\![E]\!]\xi} case(z, \overline{f_C})$$

# Fast Reverse - Specification and Implementation

- datatype list = Nil of unit | Cons of int × list
- Specification: rev $[x_0, \ldots, x_{n-1}]$ = $[x_{n-1}, \ldots, x_0]$

  Implementation:

```
rev ys =
  let go xs =
    match xs with
      [] -> fun ys -> ys
    | (x::xs') -> fun a -> (go xs') (x::a)
  in go ys []

rev xs = foldl (fun xs x -> x::xs) [] xs
```

```
rev = λxs.rec(xs,
  Nil ↦ λa.a,
  Cons↦ ⟨x, ⟨xs, r⟩⟩.λa.force(r) Cons⟨x,a⟩))) Nil
```

# Fast Reverse - Implementation

- $\texttt{rev (Cons}\langle\texttt{0,Cons}\langle\texttt{1, Nil}\rangle\rangle\texttt{)}$
- $\rightarrow_\beta \texttt{rec(Cons}\langle\texttt{0,Cons}\langle\texttt{1,Nil}\rangle\rangle\texttt{, Nil} \mapsto \lambda\texttt{a.a}$
  $\texttt{Cons} \mapsto \langle x, \langle xs, r\rangle\rangle.\lambda\texttt{a.force(r) Cons}\langle\texttt{x,a}\rangle\texttt{) Nil}$
- $\rightarrow_\beta^* (\lambda\texttt{a0.(}\lambda\texttt{a1.(}\lambda\texttt{a2.a2) Cons}\langle\texttt{1,a1}\rangle\texttt{) Cons}\langle\texttt{0,a0}\rangle\texttt{) Nil}$
- $\rightarrow_\beta (\lambda\text{a1.}(\lambda\text{a2.a2}) \text{ Cons}\langle\text{1,a1}\rangle) \text{ Cons}\langle\text{0,Nil}\rangle$
- $\rightarrow_\beta (\lambda\text{a2.a2}) \text{ Cons}\langle\text{1,Cons}\langle\text{0,Nil}\rangle\rangle$
- $\rightarrow_\beta \text{Cons}\langle\text{1,Cons}\langle\text{0,Nil}\rangle\rangle$

# Fast Reverse - Translation

- $\|\texttt{rev}\|$

$$\langle 0, \lambda xs.1 +_c \texttt{rec}(xs, \texttt{Nil} \mapsto \langle 1, \lambda a.\langle 0, a\rangle\rangle$$
$$\texttt{Cons} \mapsto \langle x, \langle xs', r\rangle\rangle.\langle 1, \lambda a.(1 + r_c) +_c r_p\ \texttt{Cons}\langle x, a\rangle\rangle)\ \texttt{Nil}\rangle$$

# Fast Reverse - Interpretation

We need to provide an interpretation for programmer-defined datatypes

$$\begin{aligned} [\![\texttt{list}]\!] &= \mathbb{N} \\ D^{list} &= \{*\} + \{1\} \times \mathbb{N} \\ size_{list}(*) &= 1 \\ size_{list}(1, n) &= 1 + n \end{aligned}$$

# Fast Reverse - Interpretation

Interpretation of the helper function

$$g(n) = \bigvee_{size\ z \leq n} case(z, f_C, f_N)$$

where

$$f_{Nil}(x) = (1, \lambda a.(0, a))$$

$$f_{Cons}(b) = (1, \lambda a.(1 + g_c(\pi_1 b)) +_c g_p(\pi_1 b)\ (a + 1))$$

# Fast Reverse - Interpretation

- Let $h(n, a) = g_p(n)\ a$.
- The recurrence for the cost:

$$h_c(n, a) = \begin{cases} 0 & n = 0 \\ 2 + h_c(n-1, a+1) & n > 0 \end{cases} \tag{1}$$

- $h_c(n, a) = 2n$
- The recurrence for the potential:

$$h_p(n, a) = \begin{cases} a & n = 0 \\ h_p(n-1, a+1) & n > 0 \end{cases} \tag{2}$$

- $h_p(n, a) = n + a$

# Parametric Insertion Sort - Source Language Insert

- `list` datatype:
  `data list` $=$ `Nil of unit | Cons of int` $\times$ `list`
- Specification:
  `insert f x` $[x_0, ..., x_{n-1}] = [x_0, ..., x_i, x, x_{i+1}, ...., x_{n-1}]$ where `f x` $x_{i+1}$ `= True`.
- OCaml:

```
insert f y xs =
  match xs with
    [] -> [y]
  | x::xs' | f y x -> y::xs
  | x::xs' -> x::insert f y xs'
```

- Source language:

$$\texttt{insert} = \lambda f.\lambda x.\lambda xs.\texttt{rec}(xs, \texttt{Nil} \mapsto \texttt{Cons}\langle x, \texttt{Nil}\rangle,$$
$$\texttt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.\texttt{rec}(f\ x\ y, \texttt{True} \mapsto \texttt{Cons}\langle x, \texttt{Cons}\langle y, ys\rangle\rangle,$$
$$\texttt{False} \mapsto \texttt{Cons}\langle y, \texttt{force}(r)\rangle))$$

# Parametric Insertion Sort - Source Language Sort

- Specification:
  $\texttt{sort f } [x_0, ..., x_{n-1}] = [y_0, ..., y_{n-1}]$ such that
  $\forall y_i, x_{i+1}.\texttt{f } y_i \ y_{i+1} = \texttt{True}$ and $[y_0, ..., y_{n-1}]$ is a sorted permutation of $[x_0, ..., x_{n-1}]$

- OCaml:

```
sort f xs =
  match xs with
    [] -> []
  | x::xs' -> insert f x (sort f xs')
```

- Source language:

$$\texttt{sort} = \lambda f.\lambda xs.\texttt{rec}(xs, \texttt{Nil} \mapsto \texttt{Nil},$$
$$\texttt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.\texttt{insert } f \ y \ \texttt{force}(r))$$

# Parametric Insertion Sort - Complexity Language

$$\begin{aligned}\|\texttt{insert}\| = \langle 0, \lambda f.\langle 0, \lambda x.\langle 0, \lambda xs.\texttt{rec}(xs, \texttt{Nil} \mapsto \langle 1, \texttt{Cons}\langle x, \texttt{Nil}\rangle\rangle,\\ \texttt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.(3 + (f\ x)_c) +_c \texttt{rec}(((f\ x)_p\ y)_p,\\ \texttt{True} \mapsto \langle 1, \texttt{Cons}\langle x, \texttt{Cons}\langle y, ys\rangle\rangle\rangle,\\ \texttt{False} \mapsto \langle 1 + r_c, \texttt{Cons}\langle y, r_p\rangle\rangle))))\end{aligned}$$

$$\begin{aligned}\|\texttt{sort}\| = \langle 0, \lambda f.\langle 0, \lambda xs.\texttt{rec}(xs, \texttt{Nil} \mapsto 1 +_c \langle 0, \texttt{Nil}\rangle,\\ \texttt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.(4 + r_c) +_c (((\|\texttt{insert}\|_p f)_p y)_p r_p)\rangle\rangle\end{aligned}$$

# Parametric Insertion Sort - Denotational Semantics

We interpret lists as a pair of their largest element and their length.

$$\begin{aligned} [\![\texttt{list}]\!] &= \mathbb{Z} \times \mathbb{N} \\ D^{list} &= \{*\} + \mathbb{Z} \times \mathbb{N} \\ size_{list}(*) &= (0, 0) \\ size_{list}((i, (j, n))) &= (max\{i, j\}, 1 + n) \end{aligned}$$

# Parametric Insertion Sort - Insert Interpretation

$$g(i,n) = \bigvee_{size(z) \leq (i,n)} case(z, f_{Nil}, f_{Cons})$$

where

$$f_{Nil}(*) = (1,(i,1))$$

$$f_{Cons}(j,m) = (3 + ((f\ x)_p\ i)_c) +_c ((1,(max(x,j),2+m)) \vee (1 + g_c(j,m),(max(j,\pi_0 r_p),1+\pi_1 g_p(j,m))))$$

- Closed form solution for the cost:

$$g_c(i,n) \leq (4 + ((f\ x)_p\ i)_c n + 1$$

- Closed form solution for the potential:

$$g_p(i,n) \leq (max\{x,i\}, n+1)$$

# Parametric Insertion Sort - Sort Interpretation

$$g(i,n) = \bigvee_{size(z) \leq n} case(z, f_{Nil}, f_{Cons})$$

where

$$f_{Nil} = (1,(0,0))$$

$$f_{Cons} = (5 + g_c(j,m) + (f\ \ i)_p\ j)_c g_p(j,m), (max\{i,j\}, g_p(j,m)+1))$$

- Closed form solution for potential:

$$g_p(i,n) \leq (i,n)$$

- Closed form solution for the cost:

$$g_c(i,n) \leq (3 + ((f\ \ i)_p\ i)_c n^2 + 5n + 1$$

# Parallel Cost Semantics

- We can reduce the running time of a program by allowing independent computations to be executed simultaneously
- Extent to which parallelism can be exploited is limited by the dependencies between subcomputations
- We add *implicit binary fork-join* parallelism to the source language
- Analysis should not be tied to a number of processors
- Need a new measure of cost which reflects potential parallelism opportunities

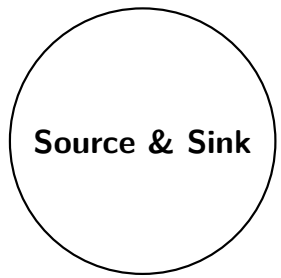
# Parallel Cost Semantics

- **Cost graphs**

$$\mathcal{C} ::= 0 \mid 1 \mid \mathcal{C} \oplus \mathcal{C} \mid \mathcal{C} \otimes \mathcal{C}$$

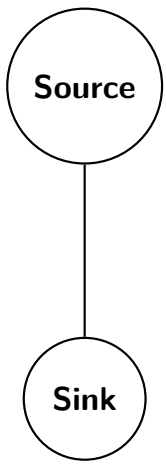
# Parallel Cost Semantics

- **0** graph



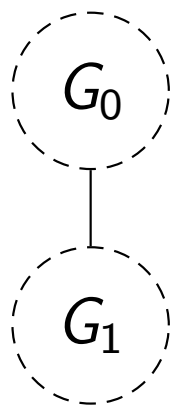
# Parallel Cost Semantics

- **1** graph

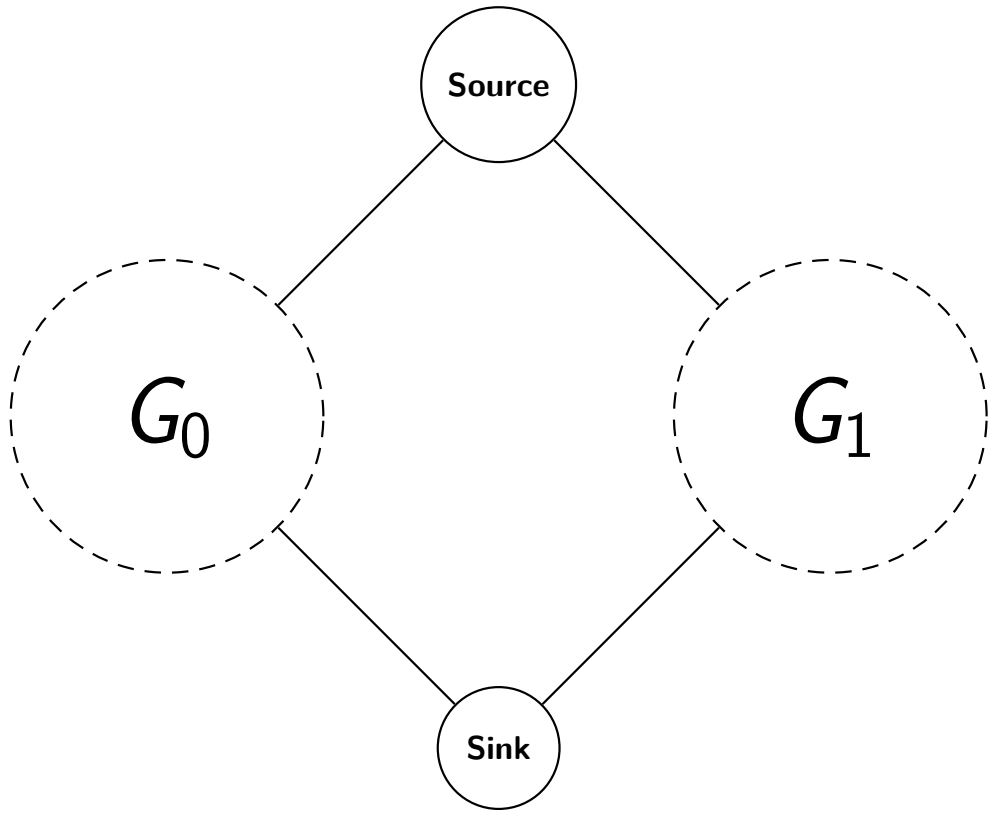


# Parallel Cost Semantics

- $G_0 \oplus G_1$:



# Parallel Cost Semantics




- $G_0 \otimes G_1$:

# Parallel Cost Semantics

- Assign cost graphs to the evaluation of expressions
- Tuples:

$$\frac{e_0 \downarrow^{n_0} v_0 \qquad e_1 \downarrow^{n_1} v_1}{\langle e_0, e_1 \rangle \downarrow^{n_0 \otimes n_1} \langle v_0, v_1 \rangle}$$

- Application:

$$\frac{e_0 \downarrow^{n_0} \lambda x.e_0' \qquad e_1 \downarrow^{n_1} v_1 \qquad e_0'[v_1/x] \downarrow^{n} v}{e_0 \ e_1 \downarrow^{(n_0 \otimes n_1) \oplus n \oplus 1} v}$$

# Parallel Complexity Translation

$$
\begin{aligned}
\|\langle e_0, e_1\rangle\| &= \langle \|e_0\|_c \otimes \|e_1\|_c, \langle \|e_0\|_p, \|e_1\|_p\rangle\rangle \\
\|\lambda x.e\| &= \langle 0, \lambda x.\|e\|\rangle \\
\|e_0\ e_1\| &= 1 \oplus (\|e_0\|_c \otimes \|e_1\|_c) \oplus_c \|e_0\|_p\ \|e_1\|_p \\
\|\mathit{delay}(e)\| &= \langle 0, \|e\|\rangle \\
\|\mathit{force}(e)\| &= \|e\|_c \oplus_c \|e\|_p
\end{aligned}
$$

**Bounding Theorem:** If $\gamma \vdash e : \tau$, then $e \sqsubseteq_\tau \|e\|$
*proof:* by logical relations.

# Work and Span

- **Work** total steps to run program

$$work(c) = \begin{cases} 0 & \text{if } c = 0 \\ 1 & \text{if } c = 1 \\ work(c_0) + work(c_1) & \text{if } c = c_0 \oplus c_1 \\ work(c_0) + work(c_1) & \text{if } c = c_0 \otimes c_1 \end{cases}$$

- **Span** critical path of program

$$span(c) = \begin{cases} 0 & \text{if } c = 0 \\ 1 & \text{if } c = 1 \\ span(c_0) + span(c_1) & \text{if } c = c_0 \oplus c_1 \\ max(span(c_0), span(c_1)) & \text{if } c = c_0 \otimes c_1 \end{cases}$$

# Brent's Theorem

- A program with work $w$ and span $s$ may be evaluated on $p$ processors in $O(max(w/p, s))$ steps.

# Parallel Tree Map

- Tree definition:

$$\texttt{datatype tree} = \texttt{E of Unit} \mid \texttt{N of int} \times \texttt{tree} \times \texttt{tree}$$

- OCaml:

```
map f t =
  match t with
    E -> E
  | N(x,l,r) -> N(f x, map f l, map f r)
```

- Source Language:

$$\texttt{map} = \lambda f.\lambda t.\texttt{rec}(t, \texttt{E} \mapsto \texttt{E},$$
$$\texttt{N} \mapsto \langle x, \langle t_0, r_0 \rangle, \langle t_1, r_1 \rangle \rangle.\texttt{N}\langle f\ x, \texttt{force}(r_0), \texttt{force}(r_1) \rangle)$$

# Parallel Tree Map

- Translation:

$$\begin{aligned}\|\mathtt{map}\| = \langle 0.\lambda f.\langle 0, \lambda t.\mathtt{rec}(t_p, \mathtt{E} \mapsto \langle 1, \mathtt{E}\rangle,\\ \mathtt{N} \mapsto \langle y, \langle t_0, r_0\rangle\langle t_1, r_1\rangle\rangle.\\ \langle 2 \oplus (f\ x)_c \otimes r_{0c} \otimes r_{1c}, \mathtt{N}\langle (f\ x)_p, r_{0p}, r_{1p}\rangle\rangle\end{aligned}$$

# Parallel Tree Map

- Interpret trees as largest label and number of nodes:

$$\begin{aligned}
[\![tree]\!] &= \mathbb{Z} \times \mathbb{Z} \\
D_{\texttt{tree}} &= \{*\} + \mathbb{Z} \times [\![\texttt{tree}]\!] \times [\![\texttt{tree}]\!] \\
size_{\texttt{tree}}(*) &= (0, 0) \\
size_{\texttt{tree}}(x, (m_0, n_0), (m_1, n_1)) &= (max(x, m_0, m_1), 1 + n_0 + n_1)
\end{aligned}$$

- Interpret cost graphs as work and span:

$$\begin{aligned}
[\![0]\!]\xi &= (0, 0) \\
[\![1]\!]\xi &= (1, 1) \\
[\![c_0 \oplus c_1]\!]\xi &= ([\![c_0]\!]\xi + [\![c_1]\!]\xi, [\![c_0]\!]\xi + [\![c_0]\!]\xi) \\
[\![c_0 \otimes c_1]\!]\xi &= ([\![c_0]\!]\xi + [\![c_1]\!]\xi, max([\![c_0]\!]\xi, [\![c_1]\!]\xi))
\end{aligned}$$

# Parallel Tree Map

- $$g(i,n) = \bigvee_{size(z) \leq (i,n)} case(z, f_E, f_N)$$

where

$$\begin{aligned}
f_E(*) &= [\![\langle 1, \mathbb{E} \rangle]\!]\xi = ((1,1),(0,0)) \\
f_N(j,(j_0,n_0),(j_1,n_1)) &= [\![\langle 2 \oplus (f_p\ x)_c \otimes r_{0c} \otimes r_{1c}, \mathbb{N}\langle (f_p\ x)_p, r_{0p}, r_{1p} \rangle \rangle]\!] \\
&\quad \{t \mapsto (i,n), f \mapsto f, x \mapsto j, r_0 \mapsto g(j_0,n_0), r_1 \mapsto g(j_1,n_1)\} \\
&= ((2 + (f_p\ j)_c + \pi_0 g_c(j_0,n_0) + \pi_0 g_c(j_1,n_1), \\
&\qquad 2 + max((f_p\ j)_c, \pi_1 g_c(j_0,n_0), \pi_1 g_c(j_1,n_1))), \\
&\qquad (max((f_p\ i)_p, \pi_0 g_p(j_0,n_0), \pi_1 g_p(j_1,n_1)), \\
&\qquad 1 + \pi_1 g_p(i_0,n_0) + \pi_1 g_p(i_1,n_1)))
\end{aligned}$$

# Parallel Tree Map

- Work:

$$\pi_0 g_c(i, n) = (3 + (f\ \ i)_c)n + 1$$

- Span:

$$\pi_1 g_c(i, n) \leq 2 + (f\ \ i)_c + n$$

- Brent's Theorem:

$$\mathcal{O}(max\left(\frac{(f\ \ i)_c n}{p}, (f\ \ i)_c + n\right))$$

# Mutual Recurrences

Pure Potential Translation

$$
\begin{aligned}
|\langle e_0, e_1 \rangle| &= \langle |e_0|, |e_1| \rangle \\
|\lambda x.e| &= \lambda x.|e| \\
|e_0\ e_1| &= |e_0|\ |e_1| \\
|\mathit{delay}(e)| &= |e| \\
|\mathit{force}(e)| &= |e|
\end{aligned}
$$

**Theorem:** For all $\gamma \vdash e : \tau$, $|e| : \langle\!\langle \tau \rangle\!\rangle \sim_\tau \|e\| : \|\tau\|$
*proof*: by logical relations

# Bibliography


Norman Danner, Jennifer Paykin, and James S. Royer. A static cost analysis for a higher-order language. In *Proceedings of the 7th workshop on Programming languages meets program verification*, pages 25–34. ACM Press, 2013. doi: 10.1145/2428116.2428123. URL `http://arxiv.org/abs/1206.3523`.

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