Extracting Cost Recurrences from Sequential and Parallel Functional Programs

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April 28, 2016

Overview

- Complexity analysis aims to predict the resources, most often time and space, which a program requires
- Previous work by Danner et al. [2013] and Danner et al. [2015] formalizes the extraction of recurrences for the cost of higher-order functional programs
- Contribution:
  - We use the method of Danner et al. [2015] to analyze the complexity of higher-order functional programs
  - We extend the method to parallel cost semantics to help answer the question: how long will this program take to run on an arbitrary number of processors?
  - We prove an interesting fact about the recurrences for the cost of programs

Traditional Complexity Analysis

Source program:

```
fold f z xs =
  match xs with
    [] -> z
  | x::xs' -> f x (fold f z xs')
```

Write down recurrence for cost:

$$T(n) = \begin{cases} c_0 & \text{if } n = 0 \\ c_1 + T(n-1) & \text{otherwise} \end{cases}$$

Obtain a closed form solution:

$$T(n) = c_1 n + c_1 = \mathcal{O}(n)$$

Higher-Order Complexity Analysis

- Write programs in a "source language"
- Translate the programs to a "complexity language"
- The translated programs are recurrences for the complexity of the source language program
- **complexity** = cost $\times$ potential
- **cost**: steps to run a program
- **potential**: size of the result of evaluating program

Source Language

- Variant of System-T
  Types:

$$\tau ::= \mathtt{unit} \mid \tau \times \tau \mid \tau \to \tau \mid \mathtt{susp}\ \tau \mid \delta$$

  Expressions:

$$\begin{aligned} e ::= {} & x \mid \langle\rangle \mid \lambda x.e \mid e\ e \mid \langle e, e\rangle \mid \mathtt{split}(e, x.x.e) \\ & \mid \mathtt{delay}(e) \mid \mathtt{force}(e) \mid C^{\delta}\ e \mid \mathtt{rec}^{\delta}(e, \overline{C \mapsto x.e_C}) \\ & \mid \mathtt{map}^{\phi}(x.v, v) \mid \mathtt{let}(e, x.e) \end{aligned}$$

- Programmer defined datatypes:

```
datatype list = Nil | Cons int×list
```

Source Language - Cost Semantics

- Tuples:

$$\frac{e_0 \downarrow^{n_0} v_0 \qquad e_1 \downarrow^{n_1} v_1}{\langle e_0, e_1 \rangle \downarrow^{n_0+n_1} \langle v_0, v_1 \rangle}$$

- Application:

$$\frac{e_0 \downarrow^{n_0} \lambda x.e_0' \qquad e_1 \downarrow^{n_1} v_1 \qquad e_0'[v_1/x] \downarrow^{n} v}{e_0\ e_1 \downarrow^{1+n_0+n_1+n} v}$$

- Structural Recursion:

$$\frac{e \downarrow^{n_0} C v_0 \qquad \mathtt{map}^{\phi_C}(y.\langle y, \mathtt{delay}(rec(y, \overline{C \mapsto x.e_C}))\rangle, v_0) \downarrow^{n_1} v_1 \qquad e_C[v_1/x] \downarrow^{n_2} v}{rec(e, \overline{C \mapsto x.e_C}) \downarrow^{1+n_0+n_1+n_2} v}$$

Source Language

- OCaml:

```
length xs =
  match xs with
    [] -> 0
  | (x::xs) -> 1 + length xs
```

- Source language:

$$\lambda xs.\mathtt{rec}(xs, \mathtt{Nil} \mapsto 0, \mathtt{Cons} \mapsto \langle x, \langle xs, r\rangle\rangle.1 + \mathtt{force}(r))$$

Complexity Language

- Types

$$T ::= \mathbf{C} \mid \texttt{unit} \mid \Delta \mid T \times T \mid T \to T$$
$$\mathbf{C} ::= 0 \mid 1 \mid 2 \mid \ldots$$
$$\texttt{datatype}\Delta = C_0^\Delta \texttt{of}\Phi_{C_0}[\Delta] \mid \ldots \mid C_{n-1}^\Delta \texttt{of}\Phi_{C_{n-1}}[\Delta]$$

- Expressions

$$E ::= x \mid 0 \mid 1 \mid E + E \mid \langle\rangle \mid \langle E, E\rangle \mid$$
$$\pi_0 E \mid \pi_1 E \mid \lambda x.E \mid E\ E \mid C^\delta\ E \mid \mathrm{rec}^\Delta(E, \overline{C \mapsto x.E_C})$$

- No longer need mechanisms for controlling cost

Translation

- Translate source language programs of type τ to complexity language programs of type $\mathbf{C} \times \langle\langle \tau \rangle\rangle$
- $\mathbf{C}$ bound on the steps to evaluate the program
- $\langle\langle \tau \rangle\rangle$ expression for the size of the value
- Types of the translation function $\| \cdot \|$:

$$\begin{aligned}
\|\tau\| &= \mathbf{C} \times \langle\langle \tau \rangle\rangle \\
\langle\langle \mathtt{unit} \rangle\rangle &= \mathtt{unit} \\
\langle\langle \sigma \times \tau \rangle\rangle &= \langle\langle \sigma \rangle\rangle \times \langle\langle \tau \rangle\rangle \\
\langle\langle \sigma \to \tau \rangle\rangle &= \langle\langle \sigma \rangle\rangle \to \|\tau\| \\
\langle\langle \mathtt{susp}\ \tau \rangle\rangle &= \|\tau\| \\
\langle\langle \delta \rangle\rangle &= \delta
\end{aligned}$$

Translation

- Some cases of the translation function:

$$
\begin{aligned}
\|x\| &= \langle 0, x \rangle \\
\|\langle e_0, e_1 \rangle\| &= \langle \|e_0\|_c + \|e_1\|_c, \langle \|e_0\|_p, \|e_1\|_p \rangle \rangle \\
\|\lambda x.e\| &= \langle 0, \lambda x.\|e\| \rangle \\
\|e_0 \; e_1\| &= (1 + \|e_0\|_c + \|e_1\|_c) +_c \|e_0\|_p \|e_1\|_p
\end{aligned}
$$

Size-Based Denotational Semantics

- Complexity translation does not lose any information
- Abstract values to their sizes using a denotational semantics
- **denotational semantics** - assign meanings to programs by interpreting types as sets and terms as elements of sets
- We use a standard denotational semantics
- Must define the following for programmer defined datatypes:
  - $[\![\delta]\!]$ - the set in which to interpret the type
  - D^{δ} - sum type representing the unfolding of the datatype
  - $size : D^{\delta} \to [\![\delta]\!]$ - function from sum type to interpretation
- The interpretation of the recursor is non-standard

$$[\![rec(E, \overline{C \mapsto x.E_C})]\!]\xi = \bigvee_{size(z) \leq [\![E]\!]\xi} case(z, \overline{f_C})$$

Fast Reverse - Specification and Implementation

- datatype list $=$ Nil of unit $|$ Cons of int $\times$ list
- Specification: rev $[x_0, \ldots, x_{n-1}]$ = $[x_{n-1}, \ldots, x_0]$

  Implementation:

```
rev ys =
  let go xs =
    match xs with
      [] -> fun ys -> ys
    | (x::xs') -> fun a -> (go xs') (x::a)
  in go ys []
```

```
rev xs = foldl (fun xs x -> x::xs) [] xs
```

$$\texttt{rev} = \lambda \texttt{xs}.\texttt{rec}(\texttt{xs},$$
$$\texttt{Nil} \mapsto \lambda \texttt{a}.\texttt{a},$$
$$\texttt{Cons} \mapsto \langle x, \langle xs, r\rangle\rangle.\lambda \texttt{a}.\texttt{force}(\texttt{r})\ \texttt{Cons}\langle \texttt{x},\texttt{a}\rangle)))\ \texttt{Nil}$$

Fast Reverse - Implementation

- $\texttt{rev (Cons}\langle\texttt{0,Cons}\langle\texttt{1, Nil}\rangle\rangle\texttt{)}$
- $\rightarrow_\beta$ $\texttt{rec(Cons}\langle\texttt{0,Cons}\langle\texttt{1,Nil}\rangle\rangle\texttt{, Nil} \mapsto \lambda\texttt{a.a}$ $\texttt{Cons}\mapsto \langle x, \langle xs, r\rangle\rangle.\lambda\texttt{a.force(r) Cons}\langle\texttt{x,a}\rangle\texttt{) Nil}$
- $\rightarrow_\beta^*$ $(\lambda\texttt{a0.(}\lambda\texttt{a1.(}\lambda\texttt{a2.a2) Cons}\langle\texttt{1,a1}\rangle\texttt{) Cons}\langle\texttt{0,a0}\rangle\texttt{) Nil}$
- $\rightarrow_\beta$ $(\lambda \mathsf{a1}.(\lambda \mathsf{a2.a2})\ \mathsf{Cons}\langle 1,\mathsf{a1}\rangle)\ \mathsf{Cons}\langle 0,\mathsf{Nil}\rangle$
- $\rightarrow_\beta$ $(\lambda \mathsf{a2.a2})\ \mathsf{Cons}\langle 1,\mathsf{Cons}\langle 0,\mathsf{Nil}\rangle\rangle$
- $\rightarrow_\beta$ $\mathsf{Cons}\langle 1,\mathsf{Cons}\langle 0,\mathsf{Nil}\rangle\rangle$

Fast Reverse - Translation

- $\|\texttt{rev}\|$

$$\langle 0, \lambda xs.1 +_c \texttt{rec}(xs, \texttt{Nil} \mapsto \langle 1, \lambda a.\langle 0, a\rangle\rangle$$
$$\texttt{Cons} \mapsto \langle x, \langle xs', r\rangle\rangle.\langle 1, \lambda a.(1 + r_c) +_c r_p\ \texttt{Cons}\langle \pi_1 x, a\rangle\rangle)\ \texttt{Nil}\rangle$$

Fast Reverse - Interpretation

We need to provide an interpretation for programmer-defined datatypes

$$\begin{aligned} [\![\texttt{list}]\!] &= \mathbb{N} \\ D^{list} &= \{*\} + \{1\} \times \mathbb{N} \\ size_{list}(*) &= 1 \\ size_{list}(1, n) &= 1 + n \end{aligned}$$

Fast Reverse - Interpretation

Interpretation of the helper function

$$g(n) = \bigvee_{size\ \ z \leq n} case(z, f_C, f_N)$$

where

$$f_{Nil}(x) = (1, \lambda a.(0, a))$$

$$f_{Cons}(b) = (1, \lambda a.(1 + g_c(\pi_1 b)) +_c g_p(\pi_1 b)\ (a+1))$$

Fast Reverse - Interpretation

- Let $h(n, a) = g_p(n)\ a$.
- The recurrence for the cost:

$$h_c(n, a) = \begin{cases} 0 & n = 0 \\ 2 + h_c(n-1, a+1) & n > 0 \end{cases} \tag{1}$$

- $h_c(n, a) = 2n$
- The recurrence for the potential:

$$h_p(n, a) = \begin{cases} a & n = 0 \\ h_p(n-1, a+1) & n > 0 \end{cases} \tag{2}$$

- $h_p(n, a) = n + a$

Parametric Insertion Sort - Source Language Insert

- list datatype:
  data list $=$ Nil of unit | Cons of int $\times$ list
- Specification:
  insert f x $[x_0, \ldots, x_{n-1}] = [x_0, \ldots, x_i, x, x_{i+1}, \ldots., x_{n-1}]$ where f x x_{i+1} = True.
- OCaml:

```
insert f y xs =
  match xs with
    [] -> [y]
  | x::xs' | f y x -> y::xs
  | x::xs' -> x::insert f y xs'
```

- Source language:

$$
\begin{aligned}
\mathtt{insert} = \lambda f.\lambda x.\lambda xs.\mathtt{rec}(xs, \mathtt{Nil} \mapsto \mathtt{Cons}\langle x, \mathtt{Nil}\rangle,\\
\mathtt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.\mathtt{rec}(f\ x\ y, \mathtt{True} \mapsto \mathtt{Cons}\langle x, \mathtt{Cons}\langle y, ys\rangle\rangle,\\
\mathtt{False} \mapsto \mathtt{Cons}\langle y, \mathtt{force}(r)\rangle))
\end{aligned}
$$

Parametric Insertion Sort - Source Language Sort

- Specification:
  $\mathtt{sort\ f}\ [x_0, \ldots, x_{n-1}] = [y_0, \ldots, y_{n-1}]$ such that $\forall y_i, x_{i+1}.\mathtt{f}\ y_i\ y_{i+1} = \mathtt{True}$ and $[y_0, \ldots, y_{n-1}]$ is a sorted permutation of $[x_0, \ldots, x_{n-1}]$
- OCaml:

```
sort f xs =
  match xs with
    [] -> []
  | x::xs' -> insert f x (sort f xs')
```

- Source language:

$$\mathtt{sort} = \lambda f.\lambda xs.\mathtt{rec}(xs, \mathtt{Nil} \mapsto \mathtt{Nil}, \\ \mathtt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.\mathtt{insert}\ f\ y\ \mathtt{force}(r))$$

Parametric Insertion Sort - Complexity Language

$$
\begin{aligned}
\|\mathtt{insert}\| = \langle 0, \lambda f.\langle 0, \lambda x.\langle 0, \lambda xs.\mathtt{rec}(xs, \mathtt{Nil} \mapsto \langle 1, \mathtt{Cons}\langle x, \mathtt{Nil}\rangle\rangle,\\
\mathtt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.(3 + (f\ \ x)_c) +_c \mathtt{rec}(((f\ \ x)_p\ \ y)_p,\\
\mathtt{True} \mapsto \langle 1, \mathtt{Cons}\langle x, \mathtt{Cons}\langle y, ys\rangle\rangle\rangle,\\
\mathtt{False} \mapsto \langle 1 + r_c, \mathtt{Cons}\langle y, r_p\rangle\rangle))))
\end{aligned}
$$

$$
\begin{aligned}
\|\mathtt{sort}\| = \langle 0, \lambda f.\langle 0, \lambda xs.\mathtt{rec}(xs, \mathtt{Nil} \mapsto 1 +_c \langle 0, \mathtt{Nil}\rangle,\\
\mathtt{Cons} \mapsto \langle y, \langle ys, r\rangle\rangle.(4 + r_c) +_c ((\|\mathtt{insert}\|_p f)_p y)_p r_p)\rangle\rangle
\end{aligned}
$$

Parametric Insertion Sort - Denotational Semantics

We interpret lists as a pair of their largest element and their length.

$$\begin{aligned}
[\![\texttt{list}]\!] &= \mathbb{Z} \times \mathbb{N} \\
D^{list} &= \{*\} + \mathbb{Z} \times \mathbb{N} \\
size_{list}(*) &= (0, 0) \\
size_{list}((i, (j, n))) &= (max\{i, j\}, 1 + n)
\end{aligned}$$

Parametric Insertion Sort - Insert Interpretation

$$
\begin{aligned}
g(i,n) &= \bigvee_{size(z) \le (i,n)} case(z, f_{Nil}, f_{Cons}) \\
&\quad \text{where} \\
f_{Nil}(*) &= (1, (i, 1)) \\
f_{Cons}(j,m) &= (3 + ((f\ \ x)_p\ \ i)_c) +_c ((1, (max(x,j), 2+m)) \\
&\qquad \vee (1 + g_c(j,m), (max(j, \pi_0 r_p), 1 + \pi_1 g_p(j,m))))
\end{aligned}
$$

- Closed form solution for the cost:

$$g_c(i,n) \le (4 + ((f\ \ x)_p\ \ i)_c n + 1$$

- Closed form solution for the potential:

$$g_p(i,n) \le (max\{x, i\}, n+1)$$

Parametric Insertion Sort - Sort Interpretation

$$g(i, n) = \bigvee_{size(z) \leq n} case(z, f_{Nil}, f_{Cons})$$

where

$$f_{Nil} = (1, (0, 0))$$

$$f_{Cons} = (5 + g_c(j, m) + (f\ \ i)_p\ j)_c g_p(j, m), (max\{i, j\}, g_p(j, m) + 1))$$

- Closed form solution for potential:

$$g_p(i, n) \leq (i, n)$$

- Closed form solution for the cost:

$$g_c(i, n) \leq (3 + ((f\ \ i)_p\ \ i)_c n^2 + 5n + 1$$

Parallel Cost Semantics

- We can reduce the running time of a program by allowing independent computations to be executed simultaneously
- Extent to which parallelism can be exploited is limited by the dependencies between subcomputations
- We add *implicit binary fork-join* parallelism to the source language
- Analysis should not be tied to a number of processors
- Need a new measure of cost which reflects potential parallelism opportunities

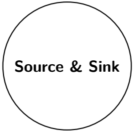
Parallel Cost Semantics

- **Cost graphs**

$$\mathcal{C} ::= 0 \mid 1 \mid \mathcal{C} \oplus \mathcal{C} \mid \mathcal{C} \otimes \mathcal{C}$$

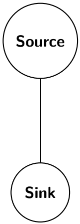
Parallel Cost Semantics

- **0** graph



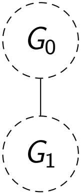
Parallel Cost Semantics

- $\mathbf{1}$ graph

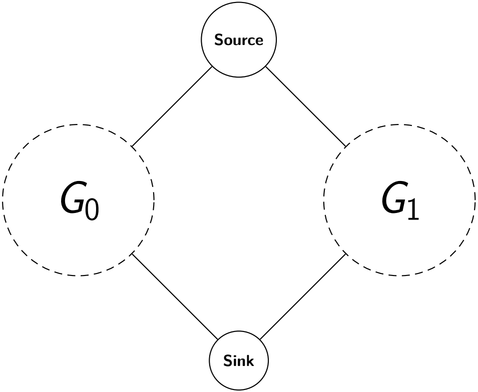


Parallel Cost Semantics

- $G_0 \oplus G_1$:



Parallel Cost Semantics



- $G_0 \otimes G_1$:

Parallel Cost Semantics

- Assign cost graphs to the evaluation of expressions

$$\frac{e_0 \downarrow^{n_0} v_0 \qquad e_1 \downarrow^{n_1} v_1}{\langle e_0, e_1 \rangle \downarrow^{n_0 \otimes n_1} \langle v_0, v_1 \rangle}$$

$$\frac{e_0 \downarrow^{n_0} \lambda x.e_0' \qquad e_1 \downarrow^{n_1} v_1 \qquad e_0'[v_1/x] \downarrow^{n} v}{e_0\ e_1 \downarrow^{(n_0 \otimes n_1) \oplus n \oplus 1} v}$$

Parallel Complexity Translation

$$
\begin{aligned}
\|\langle e_0, e_1\rangle\| &= \langle \|e_0\|_c \otimes \|e_1\|_c, \langle \|e_0\|_p, \|e_1\|_p\rangle\rangle \\
\|\lambda x.e\| &= \langle 0, \lambda x.\|e\|\rangle \\
\|e_0\ \ e_1\| &= 1 \oplus (\|e_0\|_c \otimes \|e_1\|_c) \oplus_c \|e_0\|_p\ \|e_1\|_p \\
\|delay(e)\| &= \langle 0, \|e\|\rangle \\
\|force(e)\| &= \|e\|_c \oplus_c \|e\|_p
\end{aligned}
$$

Bounding Theorem: If $\gamma \vdash e : \tau$, then $e \sqsubseteq_\tau \|e\|$
proof: by logical relations.

Work and Span

- **Work** total steps to run program

$$work(c) = \begin{cases} 0 & \text{if } c = 0 \\ 1 & \text{if } c = 1 \\ work(c_0) + work(c_1) & \text{if } c = c_0 \oplus c_1 \\ work(c_0) + work(c_1) & \text{if } c = c_0 \otimes c_1 \end{cases}$$

- **Span** critical path of program

$$span(c) = \begin{cases} 0 & \text{if } c = 0 \\ 1 & \text{if } c = 1 \\ span(c_0) + span(c_1) & \text{if } c = c_0 \oplus c_1 \\ max(span(c_0), span(c_1)) & \text{if } c = c_0 \otimes c_1 \end{cases}$$

Brent's Theorem

- A program with work w and span s may be evaluated on p processors in $O(max(w/p, s))$ steps.

Parallel Tree Map

- Tree definition:

$$\mathtt{datatype\ tree} = \mathtt{E\ of\ Unit} \mid \mathtt{N\ of\ int}\times\mathtt{tree}\times\mathtt{tree}$$

- OCaml:

```
map f t =
  match t with
    E -> E
  | N(x,l,r) -> N(f x, map f l, map f r)
```

- Source Language:

$$\begin{aligned}\mathtt{map} = \lambda f.\lambda t.\mathtt{rec}(t, &\mathrm{E} \mapsto \mathrm{E},\\ &\mathrm{N} \mapsto \langle x, \langle t_0, r_0\rangle, \langle t_1, r_1\rangle\rangle.\mathrm{N}\langle f\ x, \mathtt{force}(r_0), \mathtt{force}(r_1)\rangle)\end{aligned}$$

Parallel Tree Map

- Translation:

$$\begin{aligned}\|\mathtt{map}\| = \langle 0.\lambda f.\langle 0, \lambda t.\mathtt{rec}(t_p, \mathtt{E} \mapsto \langle 1, \mathtt{E}\rangle,\\ \mathtt{N} \mapsto \langle y, \langle t_0, r_0\rangle\langle t_1, r_1\rangle\rangle.\\ \langle 2 \oplus (f\ \ x)_c \otimes r_{0c} \otimes r_{1c}, \mathtt{N}\langle (f\ \ x)_p, r_{0p}, r_{1p}\rangle\rangle\end{aligned}$$

Parallel Tree Map

- Interpret trees as largest label and number of nodes:

$$
\begin{aligned}
[\![tree]\!] &= \mathbb{Z} \times \mathbb{Z} \\
D_{\texttt{tree}} &= \{*\} + \mathbb{Z} \times [\![\texttt{tree}]\!] \times [\![\texttt{tree}]\!] \\
size_{\texttt{tree}}(*) &= (0, 0) \\
size_{\texttt{tree}}(x, (m_0, n_0), (m_1, n_1)) &= (max(x, m_0, m_1), 1 + n_0 + n_1)
\end{aligned}
$$

- Interpret cost graphs as work and span:

$$
\begin{aligned}
[\![0]\!]\xi &= (0, 0) \\
[\![1]\!]\xi &= (1, 1) \\
[\![c_0 \oplus c_1]\!]\xi &= ([\![c_0]\!]\xi + [\![c_1]\!]\xi, [\![c_0]\!]\xi + [\![c_0]\!]\xi) \\
[\![c_0 \otimes c_1]\!]\xi &= ([\![c_0]\!]\xi + [\![c_1]\!]\xi, max([\![c_0]\!]\xi, [\![c_1]\!]\xi))
\end{aligned}
$$

Parallel Tree Map

- $$g(i,n) = \bigvee_{size(z) \leq (i,n)} case(z, f_E, f_N)$$

  where

$$\begin{aligned}
f_E(*) &= [\![\langle 1, \mathtt{E}\rangle]\!]\xi = ((1,1),(0,0)) \\
f_N(j,(j_0,n_0),(j_1,n_1)) &= [\![\langle 2 \oplus (f_p\ x)_c \otimes r_{0c} \otimes r_{1c}, \mathtt{N}\langle (f_p\ x)_p, r_{0p}, r_{1p}\rangle\rangle]\!] \\
&\quad \{t \mapsto (i,n), f \mapsto f, x \mapsto j, r_0 \mapsto g(j_0,n_0), r_1 \mapsto g(j_1,n_1)\} \\
&= ((2 + (f_p\ j)_c + \pi_0 g_c(j_0,n_0) + \pi_0 g_c(j_1,n_1), \\
&\qquad 2 + max((f_p\ j)_c, \pi_1 g_c(j_0,n_0), \pi_1 g_c(j_1,n_1))), \\
&\quad (max((f_p\ i)_p, \pi_0 g_p(j_0,n_0), \pi_1 g_p(j_1,n_1)), \\
&\qquad 1 + \pi_1 g_p(i_0,n_0) + \pi_1 g_p(i_1,n_1)))
\end{aligned}$$

Parallel Tree Map

- Work:

$$\pi_0 g_c(i, n) = (3 + (f\ \ i)_c)n + 1$$

- Span:

$$\pi_1 g_c(i, n) \leq 2 + (f\ \ i)_c + n$$

- Brent's Theorem:

$$\mathcal{O}(max\left(\frac{(f\ \ i)_c n}{p}, (f\ \ i)_c + n\right))$$

Mutual Recurrences

Pure Potential Translation

$$
\begin{aligned}
|\langle e_0, e_1\rangle| &= \langle |e_0|, |e_1|\rangle \\
|\lambda x.e| &= \lambda x.|e| \\
|e_0\ \ e_1| &= |e_0|\ \ |e_1| \\
|delay(e)| &= |e| \\
|force(e)| &= |e|
\end{aligned}
$$

Theorem: For all $\gamma \vdash e : \tau$, $|e| : \langle\!\langle \tau \rangle\!\rangle \sim_\tau \|e\| : \|\tau\|$
proof: by logical relations

Bibliography

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