PHY607 - Computational Physics

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0.1 Describing an RC circuit with an ODE

For my ODE problem, I chose a basic RC circuit that can be described by exponential growth and decay. The circuit is a very simple loop with current flowing through a resistor and capacitor in series. Using the following ODE, we can describe the discharging of the capacitor with some initial charge.

$$R\frac{dQ}{dT} + \frac{Q(t)}{RC} = Q_0$$

We can use simple calculus to solve this ODE:

$$\int \frac{d}{dt}(e^{\frac{t}{RC}}Q(t)) = \int \frac{Q_0}{R}e^{\frac{t}{RC}}$$

it follows that

$$Q(t) = Q_0C - Q_0Ce^{\frac{-t}{RC}}$$

However, we can find approximate solutions with **Euler's** method, with the necessary equations

$$\frac{dQ}{dt} = f(Q,t)$$

$$\Delta Q = f(Q,t)\Delta t$$

$$Q(t+\Delta t) = Q(t) + f(Q,t)\Delta t$$

Applying this to our RC circuit, we have

$$\frac{dQ}{dt} = -\frac{Q}{RC}$$

$$Q(t+\Delta t) = Q(t) - \frac{Q}{RC}\Delta t$$

Given initial conditions and a value for our time increments Δt, we can find how much charge is on our capacitor while it is discharging over a fixed period. I designed an algorithm that yields a result for this problem which is shown in figure 1.

We may also approximate the ODE with **4th Order Runge-Kutta**, with necessary equations

$$k_1 = -\frac{Q}{RC}\Delta t$$

$$k_1 = -\frac{Q}{RC} + k1\frac{\Delta t}{2}$$

$$k_1 = -\frac{Q}{RC} + k2\frac{\Delta t}{2}$$

$$k_1 = -\frac{Q}{RC} + k3\Delta t$$

$$Q(t+\Delta t) = Q(t) + (\frac{k1}{6} + \frac{k2}{3} + \frac{k3}{3} + \frac{k1}{6})$$

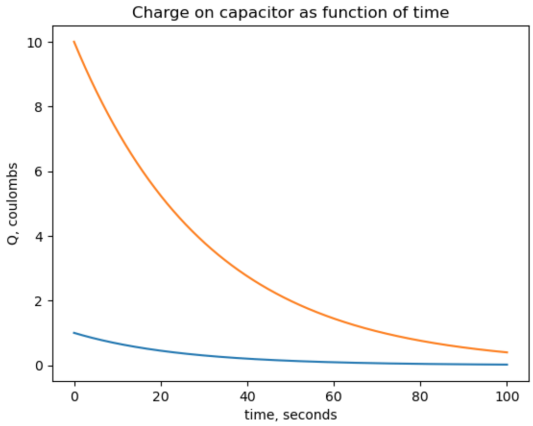

Figure 1: My Code, Blue: $R = 10\Omega, C = 15F, Q_0 = 1C$, Orange: $R = 30, C = 1, Q_0 = 10$

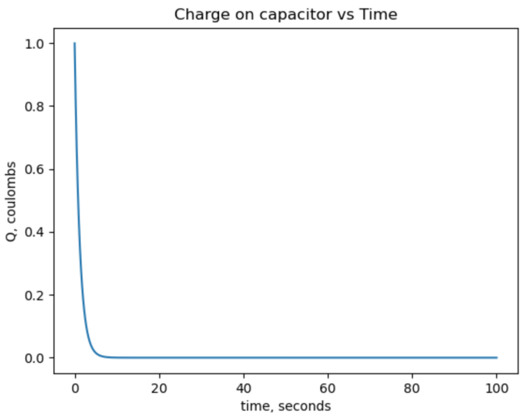

Figure 2: My RK Code $R = 10\Omega, C = 15F, Q_0 = 1C$

I wrote an algorithm that implemented this method. My results with the same corresponding initial conditions above are in figures 2 and 3.

Lastly, the analytical solutions are shown in figure 4, which when compared to my methods, show the error in using these approximations.

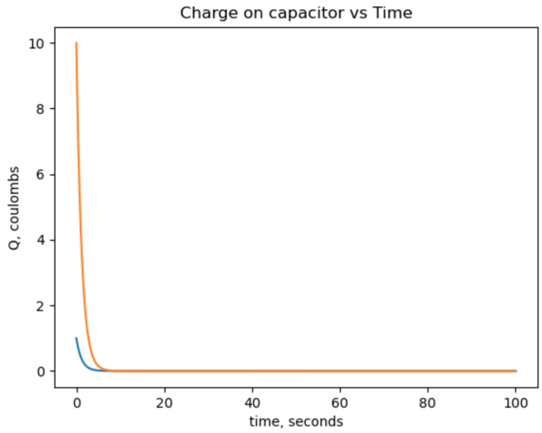

Figure 3: My RK Code: $R = 30, C = 1, Q_0 = 10$

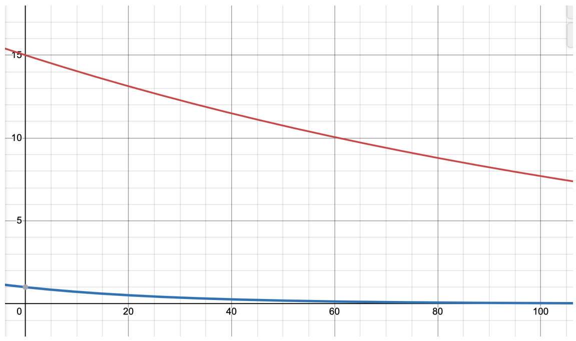

Figure 4: Analytic, Blue: $R = 10\Omega, C = 15F, Q_0 = 1C$, Orange: $R = 30, C = 1, Q_0 = 10$

0.2 Integrating a 3D charge distribution

As a continuation of our study of approximation methods for integral, I picked to calculate the total charge in a given volume with the following volume charge density:

$$\rho(x, y, z) = \frac{4C}{a^4} \frac{xz}{(y+a)^2}$$

I used three separate methods of integrations, including the following: We can write a 3D **Riemann Sum** as

$$\int\int\int f(x, y, z)dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f(x_{ijk}, y_{ijk}, z_{ijk}) \Delta x \Delta y \Delta z$$

The trapezoidal method uses the following statement in 3D:

$$\int\int\int f(x, y, z)dV = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \frac{f(x_{ijk-1}, y_{ijk-1}, z_{ijk-1}) + f(x_{ijk}, y_{ijk}, z_{ijk})}{2} \Delta x \Delta y \Delta z$$

And Simpson's rule utilizes this statement:

$$\int\int\int f(x, y, z)dV = \frac{lmn}{3}(f(x_{ijk-1}, y_{ijk-1}, z_{ijk-1}) + 4f(x_{ijk}, y_{ijk}, z_{ijk}) + f(x_{ijk+1}, y_{ijk+1}, z_{ijk+1})$$

The results of each method is given in Table 1, using the boundary conditions $0 \leq x \leq 20, 0 \leq y \leq 30, 0 \leq z \leq 50$.

Method	Result (Coulombs)
Riemann Sum	7.4341 C
Simpson's Rule	7.4331
Trapezoidal Sum	7.3740
Analytical Soln	7.5

Table 1: Results, $x \leq 20cm, y \leq 30, z \leq 50$

I then used another set of conditions with $0 \leq x \leq 64, 0 \leq y \leq 72, 0 \leq z \leq 30$, results given in table 2.

Method	Result (Coulombs)
Riemann Sum	32.5866 C
Simpson's Rule	32.6178
Trapezoidal Sum	32.9093
Analytical Soln	33

Table 2: Results, $x \leq 64cm, y \leq 72, z \leq 30$