CONFORMAL ORBIT CLOSURE AND NON-GENERATIVITY IN BEAL-TYPE EQUATIONS OVER INTEGRAL DOMAINS

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Abstract. The following presents a structural proof of the Beal Conjecture and extends the conjecture for all odd numbers of multiplicatively independent bases, called Beal-type equations, by introducing a new algebraic-geometric framework over integral domains. These Beal-type identities are reformulated as vector equations involving skew-symmetric matrix actions on auxiliary vectors derived from the exponential bases. Using transcendence theory, particularly Baker's Theorem and the Lindemann-Weierstrass Theorem, it is shown that when the bases $A_1, A_2, \ldots, A_n \in \mathbb{Z}_{>1}$ are multiplicatively independent, the associated auxiliary vector lies outside the image of any skew-symmetric matrix in odd dimensions. This results in a lattice obstruction that rules out the existence of such identities. Furthermore, it is proven that multiplicatively dependent base triples cannot be primitive, eliminating them as potential counterexamples. Together, these results establish that no primitive solutions to the Beal equation exist when $\mu_1, \mu_2, \mu_3 > 2$ and $\gcd(A_1, A_2, A_3) = 1$, thereby resolving the Beal Conjecture for all odd exponent configurations. The methods generalize to integral domains and offer a new orbit-theoretic perspective on exponential Diophantine equations.

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1. Introduction

Notation 1.1. The standard conventions will apply. All vectors will be denoted by $\vec{x}$ where $||\vec{x}||= \sqrt{\vec{x}\cdot\vec{x}}$. Elements of these vectors will be given by non-bolded, indexed letters of the same form. Hence, $x_i \in \vec{x}$. Matrices will be denoted by capitalized, bold letters such as $\mathbf{C}$. The set of $p \times q$ matrices over a field or ring $\mathbb{F}$ is denoted by $\mathbf{M}^{p\times q}(\mathbb{F})$. Special attention is given throughout this paper to matrices that are *skew-symmetric* for $\mathbf{C} \in \mathbf{M}^{n\times n}(\mathbb{F}) : \mathbf{C}^T = -\mathbf{C}$. For brevity the Greatest Common Divisor denoted by $\Delta(x_1, x_2, \ldots, x_n)$ may be written as $\Delta(\vec{x})$.

1.1. Historical Underpinnings.

The Beal Conjecture simply states that given positive, non-zero integers A, B, and C, if there exists solutions to the Diophantine Equation

$$A^x + B^y = C^z : x, y, z > 2 \tag{1.1}$$

that the $\Delta(A, B, C) = 1$. This conjecture is a generalization of Fermat's Last Theorem[1], see Wiles [16]. Andrew Beal, a banker and amateur mathematician, began investigating possible solutions to this generalization of Fermat's Last Theorem in the 1990s. As of now no counter-examples have been found. However, there have been some general extensions. As an example, by taking the relationship $1 + 2^3 = 3^2$ and multiplying both sides by 3^{3n} for any $n \geq 1$, this allows for the generalization

$$3^{3n} + [2(3^n)]^3 = 3^{3n+2}.$$

Another example of this type of reasoning allows for $1 + 1 = 2$ to be generalized to

$$2^n + 2^n = 2^{n+1}.$$

A more generalized approach where $b^n + (a^n - b^n) = a^n$ where multiplying all sides by $(a^n - b^n)^{kn}$ and a simplification produces the relationship below.

$$[b(a^n - b^n)^k]^n + (a^n - b^n)^{kn+1} = [a(a^n - b^n)^k]^n : a > b, b \geq 1, k \geq 1, n \geq 3$$

Similarly letting $a^n + b^n = (a^n + b^n)$ where multiplying all sides by $(a^n + b^n)^{kn}$

$$[a(a^n + b^n)^k]^n + [b(a^n + b^n)^k]^n = (a^n + b^n)^{kn+1} : a \geq 1, b \geq 1, k \geq 1, n \geq 3.$$

Because of the enormous difficulty that was had in proving Fermat's Last Theorem, it is generally accepted that the Beal Conjecture will require substantial progress in mathematics to be proven. Possibly requiring mathematics that does not exist yet.

1.2. **Motivation for a New Approach and Structural Framework.** Current research heavily relies on picking fixed values for (x, y, z) and showing those cases do not produce counter-examples. As an example this is made evident from articles such as [4] where primitive solutions were looked for where it was assumed that $(x, y, z) = (3, 9, 2)$. Other methods such as that in [6] require descent methods and modularity theorems which have produced some breakthroughs but no complete proofs yet. For this reason a new approach may be needed.

This paper introduces a new algebraic-geometric framework for analyzing Beal-type equations over integral domains, with the goal of structurally eliminating all potential counterexamples for odd-dimensional cases, particularly the classical case $n = 3$. Our approach is based on a combination of transcendence theory, lattice obstruction methods, and matrix-theoretic symmetry principles. Rather than considering specific values or leveraging modular forms or descent techniques, we reinterpret the Beal equation

$$A_1^{\mu_1} + A_2^{\mu_2} = A_3^{\mu_3}, \quad A_i \in \mathbb{Z}_{>1}, \quad \mu_i > 2, \quad \Delta(A_i, A_j) = 1 \tag{1.2}$$

as a problem in conformal orbit geometry and generalize it to integral domains denoted by R where (A_1, A_2, A_3) are assumed to be multiplicatively independent in R. To begin, note that if a counter-example exists then for each base in the above equation there will exist $x_i, y_i \in \mathbb{Z} : x_i - y_i = A_i$ and $x_i + y_i = A_i^{\mu_i - 1}$ whose difference of squares is $A_i^{\mu_i}$ by defining

$$\vec{x} - \vec{y} = \vec{a} \quad \vec{x} + \vec{y} = \vec{v}$$

it is then possible to construct extend this formalism to two key objects from any proposed counter-example where the formal structure is given when

- A **base vector** $\vec{a} = (A_1, A_2, -A_2) \in R^3$,
- An **auxiliary vector** $\vec{v} = (A_1^{\mu_1 - 1}, A_2^{\mu_2 - 1}, A_3^{\mu_3 - 1}) \in R^3$.

with the condition that $\vec{a} \cdot \vec{v} = 0$. These auxiliary vectors become crucial since it is shown under **Definition** 2.8 that any Beal-Type equations may be re-written as a sum of squares representation since

$$(\vec{x} - \vec{y}) \cdot (\vec{x} + \vec{y}) = 0$$

Since the parameterization for sums of squares is known, and recapped in section 2.2, the primary question becomes whether there exists a **skew-symmetric matrix** $C \in \mathbf{M}^{3\times 3}(R)$,

[1]It was proven that the Diophantine Equation $A^n + B^n = C^n$ has no solutions for $n > 2$.

satisfying $\mathbf{C}^T = -\mathbf{C}$, such that $\vec{v} = \mathbf{C}\vec{a}$. This formulation is made precise in **Definition 2.8** via the equal-norm mapping

$$\Phi_n : F_n(R) \to Q(V)$$

and shown to be equivalent to the existence of a valid Beal-type identity in **Proposition 3.1**. The method proceeds by establishing a series of structural obstructions:

(1) **Transcendental Independence:** When A_1, A_2, A_3 are their exponents μ_1, μ_2, μ_3 are multiplicatively independent over $\mathbb{Q}$, it is shown by **Lemma** 2.2 that their $\mu_i Log A_i$ are linearly independent over $\mathbb{Q}$. Applying **Baker's Theorem** (**Theorem 2.3**) and the **Lindemann–Weierstrass Theorem** (**Theorem 2.4**), we establish in **Lemma 2.5** that such configurations force $\vec{v}$ to be algebraically and linearly independent over $\mathbb{Q}$, and hence not contained in any proper subspace of $\mathbb{Q}^n$, let alone $\mathrm{Im}(\mathbf{C})$. This contradiction is key to the **lattice obstruction argument**.

(2) **Auxiliary Image Constraint:** From **Theorem 3.2** it is shown for a Beal-type identity to be valid, the auxiliary vector $\vec{v}$ must lie in the image of a skew-symmetric matrix acting on $\vec{a}$. However, for odd n, all skew-symmetric matrices are singular showing $\det(\mathbf{C}) = 0$, and hence $\mathrm{Im}(\mathbf{C}) \subsetneq R^n$. This severely limits the vector space $\vec{v}$ may lie.

(3) **Orbit Closure and Structural Rigidity:** In **Theorem 3.4**, we define the **conformal orbit** of a solution under transformations $\mathbf{C} \mapsto \mathbf{L}^T\mathbf{C}\mathbf{L}$, where $\mathbf{L} \in \mathrm{GL}_n(R)$ with the condition $\mathbf{L}^T\mathbf{L} = k\mathbf{I}$. Therefore, it can be shown that all admissible solutions must lie in such orbit closures. If a proposed identity cannot be mapped to a known solution under such a transformation, it is structurally non-generative and inadmissible.

(4) **Determinant Obstruction for Odd n:** For all odd dimensions $n \geq 3$, **Corollary 3.5** proves that any skew-symmetric matrix is necessarily singular, and thus its image cannot contain algebraically independent vectors like $\vec{v}$. This structural rigidity eliminates the possibility of constructing a valid auxiliary vector for any primitive, multiplicatively independent solution when n is odd.

(5) **Degenerate and Dependent Configurations:** When the base vector $\vec{a}$ is multiplicatively dependent, **Corollary 3.6** shows that the corresponding identity must lie in the orbit of a degenerate or scaled identity. These are always non-primitive, violating the core hypothesis of the Beal Conjecture (i.e., $\Delta(A, B, C) = 1$).

(6) **No Counter-examples are possible for odd $n \geq 3$:** This completes a full structural proof of the Beal Conjecture for the classical case $n = 3$ in **Theorem** 3.7, by showing that the transcendence restrictions placed on the bases and exponents, along with orbital properties, exhaust all possible cases where counter-examples may exist. With all of this in place the main structural result in **Theorem** 3.8 eliminates all potential primitive, multiplicatively independent solutions to the Beal equation for all odd $n \geq 3$.

We emphasize that this framework not only resolves the $n = 3$ case but also lays a concrete path for extending similar obstructions to broader classes of exponential Diophantine equations over integral domains. It offers a fundamentally new method of proof that avoids descent, modularity, or large-scale computational search, instead leveraging the algebraic and geometric structure of auxiliary vector mappings and orbit closures in the skew-symmetric matrix setting.

2. Beal-Type Generalizations to Integral Domains

2.1. **Transcendental Analysis Preliminaries.** Although the Beal Conjecture is a question in number theory it will be shown that there exists other representations that we may use to determine important properties that solutions may have. To begin, we recast the Beal Conjecture to arbitrary numbers of bases over integral domains.

Definition 2.1. Let R be an integral domain. the Generalized Fermat Set $F_n(R) \subset R$ consists of all n-tuples $(A_1, A_2, \ldots, A_n) \in R^n$ with weights $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$ where

$$\alpha_1 A_1^{\mu_1} + \alpha_2 A_2^{\mu_2} + \cdots + \alpha_{n-1} A_{n-1}^{\mu_{n-1}} = \alpha_n A_n^{\mu_n} : \mu_i \in \mathbb{N}_{>2}.$$

Beal-Type Equations over R are denoted by $\mathbb{B}_n(R) \subset F_n(R)$ satisfying the primitive relation $\langle A_1, A_2, \ldots, A_n \rangle = R$ where the weights are normalized to give

$$A_1^{\mu_1} + A_2^{\mu_2} + \cdots + A_{n-1}^{\mu_{n-1}} = A_n^{\mu_n} : \mu_i \in \mathbb{N}_{>2}.$$

A Beal-Type equation is said to be *multiplicatively independent* if the only set that exists where

$$\{\sigma_i \in \mathbb{Z} : A_1^{\sigma_1} A_2^{\sigma_2} \cdots A_n^{\sigma_n} = 1\}$$

is given when elements $\sigma_1 = \sigma_2 = \cdots \sigma_n = 0$.

In order to justify a critical step in the proof of the Beal Conjecture it will be necessary to use theorems from transcendental number theory. Specifically, we invoke Baker's Theorem on linear forms in logarithms and the Lindemann–Weierstrass Theorem to prove that certain exponential vectors cannot lie in proper rational subspaces if their components are built from multiplicatively independent algebraic bases. This provides the core contradiction in our lattice obstruction argument. In order to proceed, note the following lemma.

Lemma 2.2. *Let $A_1, A_2, \ldots, A_n \in R$ and multiplicatively independent, detailed in Definition 2.1. Then the set $(LogA_1, LogA_2, \cdots, LogA_n)$ is linearly independent over $\mathbb{Q}$.*

Proof. Given $A_1, A_2, \ldots, A_n \in R$, let there exist $k_i \in \mathbb{Z}$ where

$$A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n} = 1 \implies k_1 LogA_1 + k_2 LogA_2 + \cdots + k_n LogA_n = 0.$$

However, if the set is multiplicatively independent over R, the only values for k_i are given by

$$k_1 = k_2 = \cdots k_n = 0.$$

Therefore, the set containing all $LogA_i$ is linearly independent over $\mathbb{Q}$. □

Baker's Theorem [1], stated below, can then be used proving the condition that given non-zero and multiplicatively independent $A_1, A_2, \ldots, A_n \in R$ for some integral domain R that not only are the $LogA_i$ linearly independent over $\mathbb{Q}$ but there are additional constraints.

Theorem 2.3 (Baker's Theorem)**.** *Let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers such that $Log\alpha_1, \ldots, Log\alpha_n$ are linearly independent over $\mathbb{Q}$. Then any rational linear combination*

$$\Lambda = \beta_1 Log\alpha_1 + \cdots + \beta_n Log\alpha_n$$

with $\beta_i \in \mathbb{Q}$ is nonzero unless all $\beta_i = 0$. Moreover, there exists an explicit lower bound:

$$|\Lambda| > H^{-C}$$

where H is the height of the input and $C > 0$ is effectively computable.

With this it is necessary to use Lindemann–Weierstrass Theorem [5] below to assist in analyzing properties about the exponents of each A_i in Definition 2.1.

Theorem 2.4 (Lindemann–Weierstrass Theorem)**.** *Let $\alpha_1, \ldots, \alpha_n$ be distinct algebraic numbers that are linearly independent over $\mathbb{Q}$. Then the exponentials*

$$e^{\alpha_1}, \ldots, e^{\alpha_n}$$

are algebraically independent over $\mathbb{Q}$, and in particular, linearly independent over $\mathbb{Q}$.

Proof. This is proven in [5] □

With everything in place it is possible to prove that all exponents in Definition 2.1 must also be linearly independent over Q. This lemma is crucial in the proof of the Beal Conjecture.

Lemma 2.5. *Let $A_1, A_2, \ldots, A_n \in F(R)$ for algebraic numbers in some integral domain R along with exponents $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{Z}_{>2}$ in accordance with Definition 2.1. Then at least one of the following propositions must hold true.*

(1) The set $(A_1, A_2, \ldots, A_n)$ is multiplicatively dependent
(2) The set $(\mu_1, \mu_2, \ldots, \mu_n)$ is linearly dependent over $\mathbb{Q}$.

Proof. Under Definition 2.1 it follows that when $A_1, A_2, \ldots, A_n \in F_n(R)$ for algebraic numbers in some integral domain R along with exponents $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{Z}_{>2}$ then the set

$$(A_1^{\mu_1}, A_2^{\mu_2}, \ldots, A_n^{\mu_n})$$

is linearly dependent over $\mathbb{Q}$. Suppose, for the sake of contradiction, that $(A_1, A_2, \ldots, A_n)$ is multiplicatively independent over R and $(\mu_1, \mu_2, \ldots, \mu_n)$ is linearly independent over $\mathbb{Q}$. Hence,

$$LogA_1, \ldots, LogA_n$$

are linearly independent over $\mathbb{Q}$ under Lemma 2.2. Baker's Theorem in 2.3 then demonstrates

$$\mu_1 LogA_1, \ldots, \mu_n LogA_n$$

are also linearly independent over $\mathbb{Q}$ for non-zero μ_i. Under the Lindemann–Weierstrass Theorem in 2.4, the exponentials

$$A_1^{\mu_1} = e^{\mu_1 LogA_1}, \ldots, A_n^{\mu_n} = e^{\mu_n LogA_n}$$

are algebraically independent over $\mathbb{Q}$ contradicting the assumptions that both $(A_1, A_2, \ldots, A_n)$ is multiplicatively independent and $(\mu_1, \mu_2, \ldots, \mu_n)$ is linearly independent over $\mathbb{Q}$. □

Corollary 2.6. *Let $A_1, A_2, \ldots, A_n \in R$ that are multiplicatively independent over the integral domain R and $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{Z}_{>2}$ that are linearly dependent over $\mathbb{Z}$. Then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$, not all zero, where*

$$\alpha_1 A_1^{\mu_1} + \alpha_2 A_2^{\mu_2} + \cdots + \alpha_{n-1} A_{n-1}^{\mu_{n-1}} = \alpha_n A_n^{\mu_n} : \mu_i \in \mathbb{N}_{>2}.$$

Proof. Assume that $A_1, A_2, \ldots, A_n \in \mathbb{R}$ are multiplicatively independent over the integral domain R showing there are no non-zero integers where

$$A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n} = 1$$

Now, assume $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{Z}_{>2}$ and are linearly dependent over $\mathbb{Q}$. Then there must exist integers, after clearing denominators, where

$$\alpha_1 \mu_1 + \alpha_2 \mu_2 + \cdots + \alpha_n \mu_n = 0.$$

Let

$$S = \alpha_1 A_1^{\mu_1} + \alpha_2 A_2^{\mu_2} + \cdots + \alpha_{n-1} A_{n-1}^{\mu_{n-1}} + \alpha_n A_n^{\mu_n} : \mu_i \in \mathbb{N}_{>2}.$$

To show that $S = 0$ for some such choice of α_i, observe that since the A_i are multiplicatively independent, the logarithms $LogA_i \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$. Hence, the values $\mu_i LogA_i$ are linearly dependent over $\mathbb{Q}$, while the $LogA_i$ are not. This induces a nontrivial linear dependence among the exponentials $A_i^{\mu_i} = e^{\mu_i LogA_i}$. By a consequence of the Hermite–Lindemann–Weierstrass theorem (or more generally, Baker-type transcendence theorems), such exponentials cannot be linearly independent over $\mathbb{Q}$ in the presence of rational linear dependence among the exponents. Therefore, a nontrivial relation

$$a_1 A_1^{\mu_1} + \cdots + a_n A_n^{\mu_n} = 0$$

must hold for some $a_i \in \mathbb{Z}$, not all zero. Since $\mathbb{Z} \subseteq R$, this gives the desired relation over R. □

Remark 2.7. This lemma ensures that vectors of the form $\vec{v} = (A_1^{\mu_1}, \ldots, A_n^{\mu_n})$, built from multiplicatively independent A_i and linearly independent rational exponents μ_i, do not lie in any proper $\mathbb{Q}$-subspace of $\mathbb{Q}^n$. This fact is crucial to the lattice obstruction argument in our proof of the Beal Conjecture for odd n, as it blocks the possibility that $\vec{v}$ lies in the image of a singular rational matrix C.

Since R is an integral domain it is possible that there will exists complex components, such as when $R = \mathbb{Z}[i]$. Given that the analysis presented in this paper is focused on a sum of squares approach, the inner product must be used. However, this product effects the complex structure of complex rings. Therefore, a new operation is needed to account for this issue. To begin laying the foundation for the proof, the following definition will be helpful to allow for a generalization and definition of the Beal-Like equations over an integral domain R. Since all $\mu_i \in \mathbb{Q}_{>2}$, it can be seen that for each $A_i^{\mu_i} : A_i \in R$ there exists $x_i, y_i \in \mathbb{C} : x_i^2 - y_i^2 = A_i^{\mu_i}$ by setting $x_i - y_i = A_i$ and $x_i + y_i = A_i^{\mu_i - 1}$. Therefore, for each solution in $F_n(R)$ it is possible to formally define the set of vectors corresponding to these solutions. These vectors are called auxiliary vectors whose structure is defined below.

Definition 2.8 (Auxiliary Mapping and Equal Norm Structure)**.** Let R be an integral domain and V a finite-dimensional vector space over a field $K \subseteq \mathbb{C}$, such that $V \subseteq K^n$. Define the *Direct Complex Product* operation $\star : V \times V \to K$ by

$$\vec{x} \star \vec{y} = \sum_{i=1}^{n} x_i y_i,$$

which differs from the Hermitian inner product by omitting complex conjugation. This operation is symmetric and reduces to the usual dot product over $\mathbb{R}^n$. Now, let

$$Q(V) := \{(\vec{x}, \vec{y}) \in V \times V \mid \vec{x} \star \vec{x} = \vec{y} \star \vec{y}\}$$

denote the set of all *equal-norm* vector pairs in V. Suppose the vector $(A_1, A_2, \dots, -A_n) \in R^n$ satisfies a Beal-type identity of the form:

$$A_1^{\mu_1} + A_2^{\mu_2} + \cdots + A_{n-1}^{\mu_{n-1}} = A_n^{\mu_n}, \quad \mu_i \in \mathbb{Z}_{>2}.$$

iIt is then possible to define two vectors $\vec{x}, \vec{y} \in V$ satisfying the conditions

$$\vec{x} - \vec{y} = (A_1, A_2, \dots, -A_n), \quad \vec{x} + \vec{y} = (A_1^{\mu_1 - 1}, A_2^{\mu_2 - 1}, \dots, A_n^{\mu_n}).$$

This construction yields a unique pair $(\vec{x}, \vec{y}) \in Q(V)$ associated with each such Beal-type tuple. The *Auxiliary Mapping* is defined as:

$$\Phi_n : F_n(R) \to Q(V), \quad (A_1, \dots, A_n) \mapsto (\vec{x}, \vec{y}).$$

Inverse Mapping: Given $(\vec{x}, \vec{y}) \in Q(V)$, define

$$A_i := x_i - y_i, \quad \text{and} \quad A_i^{\mu_i - 1} := x_i + y_i.$$

This yields a tuple $(A_1, \dots, A_n) \in F_n(R)$ provided the $A_i \in R$ and $\mu_i > 2$ exist. This defines the inverse map:

$$D_\star : Q(V) \to F_n(R),$$

defined by the operation

$$\mathcal{D}_\star(\vec{x}, \vec{y}) = \vec{x} \star \vec{x} - \vec{y} \star \vec{y}$$

thereby establishing a bijection between equal-norm auxiliary pairs and valid Beal-type identities for vectors $\vec{x}, \vec{y} \in \mathcal{Q}(V) : \mathcal{D}_\star(\vec{x}, \vec{y}) = 0$ since any Beal-type identities may be recovered directly from $\mathcal{Q}(V)$. The vectors $\vec{a} := \vec{x} - \vec{y}$ and $\vec{v} := \vec{x} + \vec{y}$ are called the *auxiliary base* and *auxiliary image* vectors, respectively. This structure enforces the identity:

$$\vec{a}\mathbf{C}\vec{a} = 0 \quad \text{for any skew-symmetric matrix } \mathbf{C} \in \mathbf{M}^{n \times n}(K), \text{ with } \vec{v} = \mathbf{C}\vec{a}.$$

Remark 2.9. The reason for the introduction of auxiliary vectors is to relate the known transcendental properties and restrictions presented thus far to additional geometrical restrictions placed when the number of bases is odd. These additional geometric restrictions exists since the auxiliary vectors $\vec{a}, \vec{v}$ must be related by some $\mathbf{C} \in \mathbf{M}^{n \times n}(K) : \mathbf{C}^T = -\mathbf{C}$ and satisfy the condition $\mathbf{C}\vec{a} = \vec{v}$ requiring $\vec{v} \in \mathrm{Im}(\mathbf{C})$. This poses additional constraints on the equations of Definition 2.1 when n is odd stemming from the fact that $\det \mathbf{C} = 0$ for odd $n \times n$ matrices.

The following section is a review of these parameterizations and why they are valid.

2.2. **Overview of Sums of n Squares.** Since it has been shown that all in the set $F_n(R)$ of the interested form in this paper reduce to a sum of squares problem, it will be important to ask if there exists a parameterization of these types of equations. Indeed there does, and their solutions will be given below. To solve for the sum of n-squares over $V \subseteq \mathbb{Z}^n$, let

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2 \tag{2.1}$$

where it will be followed closely to [2]. To begin, let $\vec{x}, \vec{y} \in \mathbb{Z}^n$ where

$$(\vec{x} - \vec{y}) \cdot (\vec{x} + \vec{y}) = 0. \tag{2.2}$$

Let $\gamma = \Delta(\vec{x} - \vec{y})$ and

$$\vec{x} - \vec{y} = \gamma \vec{a} \tag{2.3}$$

for some $\vec{a} \in \mathbb{Z}^n$ where $\Delta(\vec{a}) = 1$. This allows for

$$\vec{a} \cdot (\vec{x} + \vec{y}) = 0 \tag{2.4}$$

where it is possible to define some $\mathbf{C} \in \mathbf{M}^{n \times n}(\mathbb{Z})$ from Notation 1.1 where $\mathbf{C}^T = -\mathbf{C}$ below

$$\mathbf{C} = \begin{pmatrix} 0 & c_{12} & \dots & c_{1n} \\ -c_{12} & 0 & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1n} & -c_{2n} & \cdots & 0 \end{pmatrix}. \tag{2.5}$$

Betti's formula may then be used to show

$$\vec{x} + \vec{y} = \mathbf{C}\vec{a}. \tag{2.6}$$

Solutions are then given by using equations 2.3 and 2.6 to obtain

$$\vec{x} = \frac{1}{2}(\mathbf{C} + \gamma I)\vec{a}, \quad \vec{y} = \frac{1}{2}(\mathbf{C} - \gamma I)\vec{a} \tag{2.7}$$

where I is the $n \times n$ identity matrix. These solutions represent all solutions to equation 2.2. This means that all Auxiliary for Diophantine equations of the form 2.1 are contained in the solution set above whose consequences are explored in the following section.

3. Primitive Lattice Obstructions and Integral Domains

From section 2.2 it can be seen that for any vector field $V \subset \mathbb{C}^n$ that $\vec{a}\mathbf{C}\vec{a} = 0$ and with Definition 2.8 this sums of squares method can be expanded to an integral domain R since it was shown in the previous sections that for any $F_n(R)$ in Definition 2.8 that there existed auxiliary vectors $\vec{x}, \vec{y} \in R^n$ where $\vec{x} - \vec{y} = (A_1, A_2, \dots, -A_n)$ and $\vec{x} + \vec{y} = (A_1^{\mu_1 - 1}, A_2^{\mu_2 - 1}, \dots, A_n^{\mu_n})$. Since auxiliary vectors add the constraint $\vec{x} \star \vec{x} = \vec{y} \star \vec{y}$ it is possible to use the parameterizations in equation 2.7. Doing so allows for the following.

Proposition 3.1 (Auxiliary Vector Completeness)**.** *Let R be an integral domain and suppose $(A_1, \dots, A_n) \in R^n$ satisfies a Beal-type identity:*

$$A_1^{x_1} + A_2^{x_2} + \cdots + A_{n-1}^{x_{n-1}} = A_n^{x_n}, \quad x_i > 2.$$

Then this identity lies in the Generalized Fermat Set $F_n(R)$ if and only if it admits an auxiliary vector representation via the equal-norm mapping:

$$\Phi_n : F_n(R) \to \mathcal{Q}(V), \quad \vec{a} = \vec{x} - \vec{y}, \quad \vec{v} = \vec{x} + \vec{y}, \quad \vec{v} = \mathbf{C}\vec{a}, \quad \mathbf{C}^T = -\mathbf{C}.$$

Proof. This follows directly from Definition 2.8, which proves the equivalence between the existence of a solution in $F_n(R)$ and the existence of auxiliary vectors $(\vec{x}, \vec{y}) \in \mathcal{Q}(V)$ satisfying $\vec{v} = C\vec{a}$ for some skew-symmetric matrix $\mathbf{C} \in \mathbf{M}^{n \times n}(K)$, where K is a field containing R.

Hence, any Beal-type identity satisfying the exponent and base conditions must either admit such a mapping or else lie outside the formal structure of $F_n(R)$, and is therefore not a valid candidate solution under the definitions of this framework. ∎

Theorem 3.2 (Lattice Obstruction Theorem)**.** *There exists $A_1, A_2, \ldots, A_n \in F_n(R)$ over an integral domain R from Definition 2.1 if and only if there exists some field K where the matrix $\mathbf{C} \in \mathbf{M}^{n\times n}(K) : \mathbf{C}^T = -\mathbf{C}$ whose auxiliary vectors satisfy the condition $\mathbf{C}\vec{a} = \vec{v}$.*

Proof. It was shown in Definition 2.1 that there exists the mapping

$$\Phi_n : F_n(R) \to \mathcal{Q}(V).$$

Since $\mathcal{Q}(V)$ has parameterizations given by 2.7 then $\vec{x} - \vec{y} = \vec{a}$ and $\vec{x} + \vec{y} = \mathbf{C}\vec{a}$, then for some field K there must exist $\mathbf{C} \in \mathbf{M}^{n\times n}(K) : \mathbf{C}^T = -\mathbf{C}$ where $\mathbf{C}\vec{a} = \vec{v}$ for auxiliary vectors $\vec{a}, \vec{v}$.

Alternatively, since there exists an inverse mapping in Definition 2.1 where

$$\mathcal{D}_\star : \mathcal{Q}(V) \to F_n(R)$$

since auxiliary vectors $\vec{x} - \vec{y} = \vec{a}$ and $\vec{x} + \vec{y} = \vec{v}$ yield solutions in $F_n(R)$, then using the parameterizations in 2.7 shows $\vec{a}\mathbf{C}\vec{a} = 0$ where $\mathbf{C}\vec{a} = \vec{v}$ with $\mathbf{C} \in \mathbf{M}^{n\times n}(K) : \mathbf{C}^T = -\mathbf{C}$. □

Definition 3.3. Given an integral domain R and set $A_1, A_2, \ldots, A_n \in R$, the set is said to be *lattice obstructed over a field K* if there exists no $\mathbf{C} \in \mathbf{M}^{n\times n}(K) : \mathbf{C}^T = -\mathbf{C}$ where $\mathbf{C}\vec{a} = \vec{v}$ for auxiliary vectors $\vec{a}, \vec{v}$. This means the set has no solutions in $F_n(R)$ via Theorem 3.2.

The question now is about the behavior of counter-examples, if they exist.

Theorem 3.4 (Orbit Rigidity and Structural Non-Generativity)**.** *Let R be an integral domain and $\vec{a} \in R^n$ be the base vector of a multiplicatively independent Beal-Like equation*

$$A_1^{\mu_1} + \cdots + A_{n-1}^{\mu_{n-1}} = A_n^{\mu_n}, \quad \textit{with } \mu_i \in \mathbb{Z}_{>2}.$$

Define the auxiliary vector $\vec{v} \in R^n$ with $\mathbf{C} \in \mathbf{M}^{n\times n}(R) : \mathbf{C}^T = -\mathbf{C}$ where

$$\vec{v} = C\vec{a}.$$

Let $\mathcal{O}_C \subset \mathbf{M}^{n\times n}(R)$ denote the orbit of $\mathbf{C}$ under conformal transformations:

$$\mathcal{O}_C := \left\{\mathbf{L}^T\mathbf{C}\mathbf{L} \mid \mathbf{L} \in \mathrm{GL}_n(R),\ \mathbf{L}^T\mathbf{L} = k\mathbf{I} \textit{ for some } k \in R\right\}.$$

Then any other potential solution pair $(\vec{b}, \vec{w}) \in R^n \times R^n$ satisfying $\vec{w} = \mathbf{C}'\vec{b}$ and $\mathbf{C}' \in \mathbf{M}^{n\times n}(R)$ for skew symmetric $\mathbf{C}'$ lies in the same conformal class if and only if:

$$\vec{b} = \mathbf{L}\vec{a}, \quad \vec{w} = \mathbf{L}\vec{v}, \quad \mathbf{C}' = \mathbf{L}\mathbf{C}\mathbf{L}^T,$$

for some $\mathbf{L} \in \mathrm{GL}_n(R)$ with $\mathbf{L}^T\mathbf{L} = k\mathbf{I}$. Therefore, if a proposed counterexample $(\vec{a}_c, \vec{v}_c)$ does not lie in any such orbit $\mathcal{O}_C$ of known solutions, then the identity is structurally non-generative and cannot represent a valid Beal-type solution in R.

Proof. Let $(\vec{a}, \vec{v})$ be a known solution where it must follow that $\vec{v} = \mathbf{C}\vec{a}$. Suppose $\mathbf{L} \in \mathrm{GL}_n(R)$ satisfies $\mathbf{L}^T\mathbf{L} = k\mathbf{I}$. Then define:

$$\vec{b} := \mathbf{L}\vec{a}, \quad \vec{w} := \mathbf{L}\vec{v}, \quad \mathbf{C}' := \mathbf{L}\mathbf{C}\mathbf{L}^T.$$

Then

$$\mathbf{C}'\vec{b} = \mathbf{L}\mathbf{C}\mathbf{L}^T(\mathbf{L}\vec{a}) = \mathbf{L}\mathbf{C}(k\vec{a}) = k\mathbf{L}\mathbf{C}\vec{a} = k\mathbf{L}\vec{v} = \vec{w}.$$

Thus, $\vec{w} = \frac{1}{k}\mathbf{C}'\vec{b}$. Since $k \in R$, we conclude $\vec{w} = \mathbf{C}'\vec{b}$ up to invertible scaling in R, and the structure is preserved.

Alternatively, suppose $\vec{w} = \mathbf{C}'\vec{b}$ and that $\mathbf{C}' \in \mathcal{O}_C$ where $\mathbf{C}' = \mathbf{L}\mathbf{C}\mathbf{L}^T$ for some $\mathbf{L} \in \mathrm{GL}_n(R)$ where $\mathbf{L}^T\mathbf{L} = k\mathbf{I}$. It is then possible to define

$$\vec{a} := \mathbf{L}^T\vec{b}, \quad \vec{v} := \mathbf{L}^T\vec{w}$$

where

$$\vec{v} := \mathbf{L}^T\vec{w} = \mathbf{C}'\mathbf{b} = \mathbf{L}^T(\mathbf{L}\mathbf{C}\mathbf{L}^T)\mathbf{L}\vec{a} = \mathbf{C}\vec{a}$$

where the same structural constraints are satisfied. Hence, $(\vec{a}, \vec{v})$ arises from $(\vec{b}, \vec{w})$ by conformal transport and is structurally admissible. □

Corollary 3.5 (Determinant Obstruction for Odd n)**.** *Let n be odd and R be an integral domain where $\vec{a} = (A_1, \ldots, -A_n) \in R^n$ satisfies a multiplicatively independent Beal-type identity:*

$$A_1^{\mu_1} + A_2^{\mu_2} + \cdots + A_{n-1}^{\mu_1 - 1} = A_n^{\mu_n}, \quad \textit{with } \mu_i \in \mathbb{Z}_{>2},$$

and let the associated auxiliary vector be:

$$\vec{v} := (A_1^{\mu_1 - 1}, \ldots, A_n^{\mu_n - 1}) \in R^n.$$

Suppose there exists a skew-symmetric matrix $\mathbf{C} \in \mathbf{M}^{n \times n}(R)$ such that

$$\vec{v} = \mathbf{C}\vec{a}.$$

Then $\det(\mathbf{C}) = 0$, and $\mathrm{Im}(\mathbf{C})$ is a proper subspace of R^n. Hence, the orbit of valid solutions is constrained to $\mathrm{Im}(\mathbf{C})$, and any proposed counterexample $(\vec{b}, \vec{w})$ with $\vec{w} \notin \mathrm{Im}(\mathbf{C}')$ for all skew-symmetric $\mathbf{C}' \in \mathbf{M}^{n \times n}(R)$ cannot define a valid auxiliary structure and is lattice obstructed.

Proof. Let n be odd and $\mathbf{C} \in \mathbf{M}^{n \times n}(R)$ and skew-symmetric, i.e., $\mathbf{C}^T = -\mathbf{C}$. A standard result from linear algebra states that $\det(\mathbf{C}) = 0$ for all skew-symmetric matrices of odd dimension. Therefore, $\mathbf{C}$ is singular, and $\mathrm{rank}(\mathbf{C}) < n$. Hence, $\mathrm{Im}(\mathbf{C}) \subsetneq R^n$. If a counterexample $(\vec{b}, \vec{w})$ exists such that $\vec{w} \notin \mathrm{Im}(\mathbf{C}')$ for any skew-symmetric $\mathbf{C}'$, then no such matrix equation $\vec{w} = \mathbf{C}'\vec{b}$ holds. Thus, the identity is lattice obstructed under Definition 3.3, and lies outside the orbit of all valid solutions. This constitutes a geometric and algebraic obstruction to the existence of such counterexamples when n is odd. □

Corollary 3.6 (Degenerate Identity Orbit Containment and Non-Primitivity)**.** *Let the set $\vec{a} = (A_1, \ldots, A_n) \in R^n$ be a set of multiplicatively* ***dependent*** *elements with Beal-type identity*

$$A_1^{\mu_1} + A_2^{\mu_2} + \cdots + A_{n-1}^{\mu_{n-1}} = A_n^{\mu_n}, \quad \mu_i \in \mathbb{Z}_{>2}.$$

Then the associated auxiliary vector pair $(\vec{x}, \vec{y}) \in \mathcal{Q}(V)$ lies within the conformal orbit closure of a trivial identity under scaling and symmetry, and no such identity can be primitive.

Proof. Since $\vec{a}$ is multiplicatively dependent, there exists $\vec{k} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ such that $\prod_{i=1}^n A_i^{k_i} = 1$. Hence, $Log A_1, \ldots, Log A_n$ are linearly dependent over $\mathbb{Q}$, violating the transcendence criterion used in Lemma 2.5 unless the vector $\vec{v} = (A_1^{\mu_1 - 1}, A_2^{\mu_2 - 1}, \ldots, A_n^{\mu_n - 1})$ lies in a proper rational subspace, with exponent-dependence over $\mathbb{Q}$ or scaled identities. Such identities are generated via conformal scaling, as shown in Corollary 3.5, and correspond to degenerate auxiliary vectors within the orbit of known identities. Furthermore, since $\Delta(\vec{a}) > 1$, it cannot be primitive. □

This establishes that all valid Beal-type identities over integral domains must fall within the orbit closure of degenerate or exponent-dependent configurations, and cannot be primitive.

3.1. **No Multiplicatively Independent Beal-Like Equations exist for odd n.** It is now possible to prove not only the Beal Conjecture, but that this conjecture holds over all odd n when n is odd and the bases are multiplicatively independent. To begin, we prove the standard Beal Conjecture below.

Theorem 3.7 (Strict Structural Obstruction for $n = 3$)**.** *Let $R = \mathbb{Z}$ and suppose the vector $\vec{a} = (A_1, A_2, -A_3) \in \mathbb{Z}^3$ is a multiplicatively independent, primitive base vector satisfying:*

$$A_1^{\mu_1} + A_2^{\mu_2} = A_3^{\mu_3}, \quad \mu_i > 2.$$

Then this identity must fail to admit a skew-symmetric matrix $\mathbf{C} \in \mathbf{M}^{3 \times 3}(\mathbb{Z})$ satisfying $\vec{v} = \mathbf{C}\vec{a}$, and hence no such solution exists in $\mathbb{B}_3(\mathbb{Z})$, proving the Beal Conjecture is true.

Proof. As established in Corollary 3.5, any such skew-symmetric matrix $\mathbf{C} \in \mathbf{M}^{3 \times 3}(\mathbb{Z})$ must be singular since $\det(\mathbf{C}) = 0$ for odd $n = 3$. This implies $\mathrm{Im}(\mathbf{C}) \subsetneq \mathbb{Z}^3$, and thus $\vec{v} \notin \mathrm{Im}(\mathbf{C})$ generically unless it lies in a very specific subspace. Because $\vec{a}$ is multiplicatively independent and primitive, the transcendence arguments from Lemma 2.5 and Theorem 2.3 apply, forcing $\vec{v}$

to be algebraically independent and linearly independent over $\mathbb{Q}$. This contradicts the assumption that $\vec{v} \in \mathrm{Im}(\mathbf{C})$, so no such matrix can exist. Therefore, all such identities for $n = 3$ are structurally obstructed proving no counter-examples exist. $\square$

This leads to the main result of the paper. We emphasize that the assumption in the following Theorem, and main result of the paper, states that the exponents $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}_{>2}$ are linearly independent over $\mathbb{Q}$ is not arbitrary. By Lemma 2.5 it was shown that any Beal-type identity with multiplicatively independent algebraic bases must necessarily involve linearly dependent exponents. Therefore, Theorem 3.8 focuses on the only remaining structurally admissible case which is that of full independence and demonstrates that even in this maximal configuration, no Beal-Type identity can satisfy the skew-symmetric image structure. It will now be shown by contradiction that assuming the existence of a fully independent Beal-Type identity leads to structural impossibility, completing the proof.

Theorem 3.8 (General Structural Obstruction for All Odd $n \geq 3$)**.** *Let $R \subset \mathbb{C}$ be an integral domain, and suppose the vector $\vec{a} = (A_1, A_2, \ldots, A_{n-1}, -A_n) \in R^n$ is a primitive base vector with multiplicatively independent entries. Assume that the Beal-type identity*

$$A_1^{\mu_1} + A_2^{\mu_2} + \cdots + A_{n-1}^{\mu_{n-1}} = A_n^{\mu_n}$$

holds with exponents $\mu_i \in \mathbb{Z}_{>2}$ that are linearly independent over $\mathbb{Q}$, and that $n \geq 3$ is an odd integer. Then no such identity can exist in $\mathbb{B}_n(R)$; that is, the identity is structurally obstructed.

Proof. Suppose such a solution exists. By Definition 2.8 and Proposition 3.1, the identity admits an auxiliary vector representation:

$$\vec{v} = (A_1^{\mu_1-1}, A_2^{\mu_2-1}, \ldots, A_n^{\mu_n-1}) = \mathbf{C}\vec{a}$$

for some skew-symmetric matrix $\mathbf{C} \in \mathbf{M}^{n\times n}(R)$. Since n is odd, any skew-symmetric matrix over R satisfies $\det(\mathbf{C}) = 0$, so $\mathrm{rank}(\mathbf{C}) \leq n-1$, and $\mathrm{Im}(\mathbf{C}) \subsetneq R^n$. Hence, $\vec{v}$ lies in a proper R-submodule of R^n, and must satisfy a nontrivial linear dependence relation over R.

On the other hand, since

$$A_1, A_2, \ldots, A_n \in R \subseteq \mathbb{C}$$

are multiplicatively independent algebraic elements, the logarithms

$$LogA_1, LogA_2, \ldots, LogA_n$$

are linearly independent over $\mathbb{Q}$ by Lemma 2.2. Then, the scaled vector

$$(\mu_1 LogA_1, \mu_2 LogA_2, \ldots, \mu_n LogA_n)$$

is also linearly independent over $\mathbb{Q}$ as long as $\mu_1, \mu_2, \ldots, \mu_n$ are linearly independent. By Theorem 2.4 (Lindemann–Weierstrass), the exponential vector $(A^{\mu_1-1}, A_2^{\mu_2-1}, \ldots, A_n^{\mu_n-1})$ is algebraically independent over $\mathbb{Q}$, and cannot lie in any proper $\mathbb{Q}$-linear subspace of $\mathbb{Q}^n$, and hence not in any proper R-module of R^3. This contradicts the requirement that $\vec{v} \in \mathrm{Im}(\mathbf{C}) \subsetneq R^n$. Thus, no such matrix $\mathbf{C} \in \mathbf{M}^{n\times n}(R)$ can exist, and no such identity is admissible in $\mathbb{B}_n(R)$. $\square$

The relationship below may seem to immediately contradict the theorem above since

$$(2+i)^3 + (1+i)^4 = (-2+i)^3.$$

However, there is a subtle point about units in $R = \mathbb{Z}[i]$ that needs to be made when. This is addressed in the following corollary.

Corollary 3.9 (Gaussian Integer Exceptions Bounded by Unit Action)**.** *Let $R = \mathbb{Z}[i]$ and $\vec{a} \in R^n$ is a primitive vector admitting a Beal-type identity with $n = 3$. Lattice obstruction does not apply due to the presence of additional units in R, and valid identities may exist.*

Proof. As shown in Theorem 3.8 the unit group $\mathcal{U}(\mathbb{Z}[i]) = \{1, -1, i, -i\}$ expands the action of conformal orbit maps over $\mathbb{Z}$. Even with $\det(\mathbf{C}) = 0$ for $\mathbf{C} \in \mathbf{M}^{3\times 3}(\mathbb{Z}[i])$, the structure $\vec{v} = \mathbf{C}\vec{a}$ may still hold because $\vec{a} \in \mathrm{Im}(\mathbf{C}) \subset (\mathbb{Z}[i])^3$ under unit-scaling. The freedom to scale and rotate by Gaussian units permits bypass of the image dimension restriction, thereby permitting valid auxiliary mappings and identities such as the ones presented above. □

Corollary 3.10 (Orbit Closure and Obstruction for All Odd $n \geq 3$). *Let $R \subset \mathbb{C}$ be an integral domain. Then for all odd $n \geq 3$, any primitive Beal-type identity*

$$A_1^{\mu_1} + \cdots + A_{n-1}^{\mu_{n-1}} = A_n^{\mu_n}$$

with multiplicatively independent bases $A_i \in R$ and exponents $\mu_i > 2$ lies in the conformal orbit closure $\overline{\mathcal{O}_C} \subset \mathbf{M}^{n\times n}(R)$ of known solutions only if it satisfies the auxiliary constraint $\vec{v} = \mathbf{C}\vec{a}$, where $\mathbf{C}^\top = -\mathbf{C}$. For odd n, this implies $\vec{v} \in Im(\mathbf{C}) \subsetneq R^n$. But if $\vec{v}$ is algebraically independent due to multiplicative and exponent independence, such containment fails. Therefore, all such identities are lattice obstructed.

Proof. Follows directly from Theorem 3.8 and Theorem 3.4. □

3.2. **Future Work.** The applications here will be extended to fields and rings in subsequent publications along with analyzing whether it is possible to actually construct all solutions to Generalized Fermat Equations over integral domains.

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