Abstract. In traditional multigrid, the number of unknowns on refined domains can create a memory bottleneck. The memory requirements can also reduce the efficiency of the algorithm due to the communication required between cores. I analyze Segmental Refinement Multigrid (SRMG) a modification to multigrid methods that reduces communication and has logarithmic scaling memory complexity. While previous work has been done on this method, and there have been earlier implementations, they all failed to converge on an arbitrary number of levels. My work provides a theoretical analysis and practical implementation demonstrating the root cause of SRMG's previous failures was due to the underlying low-communication mechanisms of the algorithm. My concluding analysis demonstrates future potential for SRMG, although it imposes requirements that could not be met by the implementation accompanying this work.

Key words. Multigrid, Segmental Refinement

AMS subject classifications. Lookup comp math

1. Introduction. Here we study a low-memory modification of multigrid methods, which are a class of methods for solving partial differential equations. This modified multigrid method, Segmental Refinement (SRMG) [**?**], reduces concurrent memory requirements by creating an overlapping domain decomposition and independently solving the partial differential equation on each subset of the domain. Error imposed by this method exists only in the overlapping region, or buffer region, and requires only modest additional computation in the redundant overlap between domains. This overlapping decomposition is the reason SRMG has logarithmic scaling memory requirements rather than the linear scaling memory requirements of traditional multigrid.

Previous analysis and implementation demonstrated that SRMG could work for some fixed number of levels [**?**], but my work attempts to demonstrate that convergence holds for any number of levels of SRMG, so long as each level eliminates low-frequency error and further levels maintain this high-frequency accuracy.

There are two unanswered questions regarding SRMG. One is whether we have convergence guarantees in spite of communication no longer occuring between patches. I show error induced by no longer communicating with the entire domain is localized and does not extend past the buffer regions of the domain decomposition.

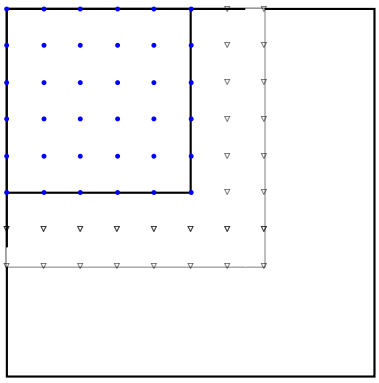


FIG. 1. *
Get better picture Significant error is constrainted to triangular black points

The second question is whether SRMG converges on an arbitrary number of refinement levels. I demonstrate a set of requirements which, when satisfied, guarantee convergence on an arbitrary number of levels. I accompanied my analysis by implementing the method with a spectral solver, to investigate the feasibility of these requirements.

2. Background. By 1982, Multigrid was an accepted numerical algorithm for solving elliptic equations [**?**]. The speed at which multigrid solves large problems is the primary reason for its use. One of the problems with multigrid, as well as many other numerical algorithms, is that memory requirements scales linearly.

For traditional multigrid methods, memory requirements scale linearly with the number of grid points on the highest-resolution domain. With the advent of multi-

core machines, large problems can be solved by distributing memory loads across several cores. Parallel multigrid is one implementation of traditional multigrid that can distribute massive problems across several cores by partitioning the full domain into smaller, interdependent subdomains. However, parallel multigrid requires communication [**?**] because the entire domain must be solved and the solution on each subdomain is dependent on the solution on neighboring subdomains.

Even in some communication-efficient implementations of parallel multigrid, the high-resolution domain is partitioned into non-overlapping subdomains that must communicate. The communication complexity of these algorithms scales linearly with the number of subdomains [**?**]. For especially difficult problems, the number of subdomains must be very large in order to fit each subdomain onto one core. Thus, the slowdown due to communication needs of parallel multigrid is especially noticeable.

An improvement upon parallel multigrid would be a method of solving partial differential equations that does not require communication between cores and requires less memory, all without sacrificing too much of the speed offered by traditional multigrid.

Segmental Refinement is an algorithm that finds a solution in only $O(\log N)$ memory, where N is the number of grid points on the full, highest resolution domain. Furthermore, the overlapping or buffer region of the decomposition into subdomains allows each subdomain to be solved indepedently, without communication with its neighbors. Thus, there is no communication required between the subdomains.

One may notice that since there are redundant grid points in the overlap of the SRMG subdomains, these redundant grid points increase the number of computations needed. However, my analysis shows that the size of this overlap is small relative to size of the problem, so the additional computations needed are a modest trade-off compared to the low-memory advantages. These additional computations are especially modest when one considers that they entirely eliminate the previously-discussed communication requirements of parallel multigrid, which scale with the number of subdomains distributed across the separate cores of a machine. In some instances, especially on machines with cores that are not efficient at horizontal communication, the overall computational time may be reduced [**?**].

Models of this SRMG algorithm were presented as early as 1994 [**?**]. However, even now there is no software package written to perform SRMG for a user-provided differential equation, and there is still debate over what conditions must be met for the algorithm to have convergence. One question that had been left unanswered by previous research is whether Segmental Refinement can be performed an arbitrary number of times with arbitrarily small error. [**?**].

There has also been, without a software implementation, proof of convergence via segmental refinement with a fixed number of levels. However this proof is unaccompanied by numerical implementation on any specific differential equations and lacks strong evidence that unlimited levels of SRMG are possible [**?**]. The theory presented showed that error induced by one level of SRMG decays within the buffer region of each subdomain, but did not demonstrate that a buffer region of fixed size can still allow further levels of SRMG without eventually inducing errors on the highest-resolution domains. However, this was an early work and still served to demonstrate that this algorithm had a potential use in reducing communication complexity during the last few stages of refinement when performing multigrid. In and of itself, this provided a significant, powerful use. Yet, it left unanswered the full potential of SRMG.

Marcus Mohr provided further analysis in 2000, particularly focusing on how it can be used to reduce communication complexity when used on a multi-core machine and

in some cases even reduce overall computation time. This continued to demonstrate the considerable advantages of such an algorithm. However, an implementation was still not given, and the model still did not account for arbitrary refinements. [**?**].

Finally in 2015, an actual implementation was investigated [**?**]. Analysis was done once again in depth, and with an accompanying numerical implementation, provided very strong support that SRMG had potential as a low-memory but reliable partial differential equation solver. Furthermore, code performing the algorithm was made publicly available, including all the data on which the researches experimented. While the code was experimental and not designed for use on new, user-defined problems, the research done by these authors demonstrated even further the potential of SRMG.

In their work, the researchers discovered that convergence depended on an additional parameter not previously considered [**?**]. In their implementation of SRMG, the authors found that increasing the resolution of the coarsest level on which SRMG is used proportionally decreased the error induced by SRMG and also the buffer region required to achieve convergence on the interior of each subdomain. Previous analysis made no connection between the resolution on the coarsest grid and the error induced by subsequent levels of SRMG.

While convergence was shown in the 2015 implementation for SRMG up to a few levels, there was no evidence suggesting segmental refinement could or could not be performed indefinitely. Errors grew beyond acceptable levels unless the buffer size grew as compensation. Increasing the buffer size also increases the memory requirements and computational complexity of SRMG. Therefore, a requirement that buffer size grow with the number of SRMG levels would undermine the advantages offered by using this algorithm in place of parallel multigrid. Even worse, it was unclear whether a large number of SRMG levels would necessitate a buffer region extending beyond the boundaries of the original problem. It was also not clear what was causing convergence failures under different circumstances, and whether they could be ameliorated while preserving the low-memory, high-speed properties of SRMG.

My work builds upon this body of previous work by approaching the analysis of the algorithm from a new perspective, showing that convergence depends on the number of unknowns solved for on the coarsest level as well as the on solver used. In order for SRMG to be effective, one must eliminate all low-frequency errors on coarser patches. That means, at a minimum, one must use a solver that does not induce interpolation error. Additionally, one must solve on a sufficiently high-resolution grid before commencing SRMG levels. Failure to meet both of these conditions can cause errors to propagate into the decomposed domain.

Our analysis shows that error induced by SRMG decays exponentially with respect to the frequency of error on the boundary. Thus, a sufficiently high-resolution domain must be solved before commencing SRMG so that low-frequency errors are eliminated. Furthermore, the solver used must not reintroduce low frequency errors when moving from low to high resolution grids. When combined, these two properties force errors to decay to the level of second order discretization error in the buffer region, so that our solution converges within the area of interest of each subdomain.

The results from my analysis and implementation support the work done in 2015 by demonstrating why the buffer needed to grow with SRMG in order to guarantee convergence. Because the coarsest grid does not solve for a sufficient number of unknowns when levels begin using SRMG, low-frequency errors are induced on each buffer. Furthermore, the solver they use induces interpolation error, thus low frequency errors are introduced at each new level of SRMG.

Since low-frequency errors are coming from the coarsest SRMG level and each

interpolation from high to low resolution domains, the method induces low frequency errors at the boundary of each buffer region. These low-frequency errors propagate further into each subdomain than high-frequency errors. Thus, the buffer size has to grow much larger to compensate for the long-distance propagation of these errors. My analysis shows that these low-frequency errors are a probable cause for convergence failures in the aforementioned implementation [**?**]. Furthermore, my analysis improves the robustness of the algorithm by deriving explicit error bounds.

With some information about the magnitude of frequencies of the true solution to a problem (more precisely, bounds on the high-frequency Fourier coefficients), one can find exactly how many unknowns must be solved for on the entire domain before one can switch from solving coarse grids via traditional multigrid to taking advantage of the low-memory SRMG algorithm on all subsequent high resolution domains. Through numerical experiments, I find that the number of unknowns which must be solved is tractable and will still require much less memory than traditional multigrid.

Note that my analysis demonstrates that it is critical for the method used not to induce any interpolation error. A method such as finite differences would induce interpolation error, which would allow low-frequency errors to enter highly refined grids.

Finally, I demonstrate that SRMG scales logarithmically with the number of unknowns. I show that my aforementioned proposals for adjustments to prior implementations do not increase the scaling of SRMG's memory requirements, even with an arbitrary number of levels.

To demonstrate the true value of SRMG, I implement an algorithm with a spectral solver in order to observe whether convergence can hold for a high number of levels. Note that this solver is not implemented in the true spirit of multigrid. Each solve is performed directly, rather than through an iterative method. However, the nature of this implementation was exploratory, intended to isolate possible sources of previous multi-level convergence failures. Furthermore, I take advantage of my preliminary analysis to determine requisite buffer sizes and coarse resolutions required to achieve multi-level convergence.

Upon review of the numerical results, it was clear that multi-level convergence was not occuring consistently. However, this eliminated the theory that interpolation error was the reason for SRMG requiring a buffer that grows with the number of levels desired.

There were many possible reasons for this multi-level convergence failure, so I performed a second, more abstract analysis in light of these results. Error induced by the first artificial boundary condition imposed on the first level of SRMG cannot not be eliminated by refinement, only by increasing coarsest resolution or buffer size. However, this is not the death of multi-level SRMG. My analysis also demonstrates how to set up SRMG to achieve a pre-determined level of accuracy on the highest-resolution domains. In fact, it is possible that guaranteeing error decays to machine-epsilon yields convergence on an arbitrary number of levels. While this is a stronger requirement than originally considered, it is not wholly infeasible. And it can be shown that, so long as subsequent levels continue to generate high-frequency accuracy on their interior, a fixed buffer size can allow convergence on an arbitrary number of levels of SRMG. Unfortunately, the implementation provided in, and computational resources available for, this work were not suitable for this newfound set of requirements.

The value of the analysis done in this thesis is two-fold: finding the cause of pre-

vious limitations on implementations of SRMG and also investigating the necessary modifications needed to remove these limitations. A next step would be to implement a more sophisticated method for Segmental Refinement, one capable of removing all sources of low-frequency error, and one that genereates overlapping domain decompositions with a buffer size large enough to accomodate the added requirements for Segmental Refinement to achieve multi-level convergence.

It remains to be shown that an implementation eliminating all sources of low-frequency error would still yield a reasonably fast application for solving Poisson's equation. However, software designed to meet these convergence requirements, if made publicly available, will allow one to solve Poisson's equation in $O(\log N)$ memory. This allows once-unsolvable problems to be solved, both for individual researchers with a dearth of resources and organizations with a plethora of computational power. Memory will no longer be a bottleneck for computations done via multigrid, and there will only be a small increase in the number of computations that must be done, but with significantly decreased communication complexity.

3. Methodology.

3.1. Theoretical Justification. Consider Poisson's equation on the domain $[0,1]$ x $[0,1]$

$$\begin{aligned} \triangle u(x,y) &= f \\ u(0,y) = u_{west}(y) \quad u(1,y) &= u_{east}(y) \\ u(x,0) = u_{south}(x) \quad u(x,1) &= u_{north}(x) \end{aligned} \tag{1}$$

We can attempt to solve this problem numerically using multigrid.

Discretizing (1) by a spectral method and solving using a multigrid method yields an approximate solution with a Fourier representation that is accurate up to the first k modes. To increase the value of this k, one can solve on a more refined grid using the approximation generated on a coarse grid as an initial guess for the solution on the refined discretization.

We consider a modification of multigrid: segmental refinement, which solves for the refined domain by locally solving on patches of an overlapping decomposition of the domain.

A brief outline of the SRMG method is:

1. Solve coarse grid
2. Quarter grid into four patches
3. Extend boundary of patch to include buffer region.
4. Refine each patch
5. Solve on patch, then repeat steps 2-4 on this new patch.

3.2. Justification of Convergence. The SRMG algorithm crucialy relies on the accuracy of the solution far from the boundary. If error decays sufficiently rapidly away from a boundary, then an inexact boundary condition combined with a buffer region can be used to obtain an accurate solution inside the domain.

To this end, we will look at the decay of the solution to the Laplace equation on a square domain with boundary values driven by high-frequency errors. Assume that step 1 of SRMG is completed, meaning a coarse solution $u^{(0)}(x,y)$ is generated as an approximation to the true solution. This coarse solution $u^{(0)}$ is used to impose boundary conditions on the patches generated by steps 2 and 3.

Consider in specific the patch solution $u^{(1)}$ generated on the lower left patch $[0, L]$ x $[0, L]$, which interfaces with the true boundary on the West and South but is given by artificially boundary conditions on the other two sides.

$$\begin{aligned} &\triangle u^{(1)}(x,y) = f \\ u^{(1)}(0,y) = u_{west}(y) \quad & u^{(1)}(L,y) = u^{(0)}(L,y) \\ u^{(1)}(x,0) = u_{south}(x) \quad & u^{(1)}(x,L) = u^{(0)}(x,L) \end{aligned} \tag{2}$$

Let the error between the true solution and the coarse solution be given by $E^{(0)}(x,y) = u^{(0)}(x,y) - u(x,y)$

We can therefore analyze the error on this lower left patch, $E^{(1)}(x,y) = u^{(1)}(x,y) - u(x,y)$, resulting from this first SRMG decomposition by expressing it as a solution to the following PDE:

$$\begin{aligned} &\triangle E^{(1)}(x,y) = \triangle u^{(1)} - \triangle u = f - f = 0 \\ E^{(1)}(0,y) = u_{west}(y) - u_{west}(y) = 0 \quad & E^{(1)}(L,y) = u^{(0)}(L,y) - u(L,y) = E^{(0)}(L,y) \\ E^{(1)}(x,0) = u_{south}(x) - u_{south}(x) = 0 \quad & E^{(1)}(x,L) = u^{(0)}(x,L) - u(x,L) = E^{(0)}(x,L) \end{aligned} \tag{3}$$

On $[0, L] \times [0, L]$. We can obtain a generic boundary condition by superposition of solutions satisfying one boundary conditions and three zero boundary value conditions.

Therefore, we further simplify our analysis by individually evaluating the error imposed by one boundary.

$$\begin{aligned} &\triangle E(x,y) = 0 \\ E(0,y) = 0 \quad & u(L,y) = 0 \\ u(x,0) = 0 \quad & u(x,L) = E^{(0)}(x,L) \end{aligned} \tag{4}$$

This partial differential equation has series solution [**?**]

$$\begin{aligned} E(x,y) &= \sum_{n=1}^{\infty} E_n(x,y) \\ &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right), \end{aligned} \tag{5}$$

where the coefficients B_n can be determined in terms of the inhomogeneous boundary condition, using the orthogonality of trigonometric functions.

$$B_n = \frac{2}{L \sinh(n\pi)} \int_0^L E^{(0)}(x,L) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

As one moves from the top boundary, the solution to this problem decays exponentially, which will motivate the size we require for our buffer used in SRMG.

Thus we have the bound

$$\begin{aligned}|E_n(x,y)| &\leq \frac{2B_n}{L\sinh(n\pi)}\sinh\left(\frac{n\pi y}{L}\right)\\ &\leq \frac{4B_n}{L\cdot exp(n\pi)}exp(\frac{n\pi y}{L})\\ &\leq \frac{4B_n}{L}exp(\frac{n\pi(y-L)}{L})\end{aligned}$$

Letting $d = L - y$ be the distance from the boundary, we have

$$|E_n(x,y)| \leq \frac{4C}{L}exp(\frac{-n\pi d}{L}), \tag{6}$$

and we see that the exponential decay away from the boundary is also exponential in the wavenumber of the perturbation.

For our solution procedure, we assume that we obtain the solution on an initial coarse grid to remove any low-frequency errors. Thus along any artificial boundary of some form analogous to $y \in [0, L]$ and $x = L$ we have accuracy in the first N_0 Fourier components. That is the error corresponding to each wavefrequency, (5) is 0 for $n \leq N_0$.

Further assume that each quarter of the domain contains $N_0 \times N_0$ unknowns.

We will number each grid starting with $l = 0$ for this coarsest grid.

If we use a second order method for solving each finite-resolution patch, then the discretization error [1] at each level is $C_D\left(\frac{1}{2^l n_0}\right)^2$ for some constant C_D. If we double the resolution of our initial grid in each direction, but divide it into four patches, then at level $l = 1$ we have $L = \frac{1}{2}$ [2].

Let $C = \max\left(|B_{n\leq 2N_0-1}|, \sum_{n=2N_0}^{\infty}|B_n|\right)$

$$\begin{aligned}|E(x,y)| &\leq \sum_{n=N_0+1}^{\infty}|\frac{4B_n}{L}exp(\frac{-n\pi d}{L})|\\ &= \sum_{n=N_0+1}^{2N_0-1}|\frac{4B_n}{L}exp(\frac{-n\pi d}{L})| + \sum_{n=2N_0}^{\infty}|\frac{4B_n}{L}exp(\frac{-n\pi d}{L})|\\ &\leq \frac{4C(N_0-1)}{L}exp(\frac{-N_0\pi d}{L}) + \frac{4C}{L}exp(\frac{-N_0\pi d}{L})\\ &= \frac{4CN_0}{L}exp(\frac{-N_0\pi d}{L})\end{aligned} \tag{7}$$

Since later levels of SRMG may generate subpatches with up to four artificial boundaries, we require that the total error imposed by up to four erroneous boundaries be less than the discretization error at this level,

[1] To clarify any ambiguity, here our second order discretization error is defined as second order with respect to the distance between points or number of unknowns per unit area on a boundary, not cell size, which would actually be$(\frac{1}{k})^4$. This is a trivial difference because the additional needed buffer size would at most double, or we could achieve this by doubling the coarse-solve accuracy n_0

[2] Technically, SRMG must create a buffer zone, so $L = \frac{1}{2} + \epsilon$. We disregard this detail in the analysis provided here.

$$
\begin{aligned}
4\frac{4CN_0}{L}exp(\frac{-N_0\pi d_1}{L}) &< C_D\left(\frac{L}{N_0}\right)^2 \\
\implies exp(\frac{-N_0\pi d_1}{L}) &< \frac{C_D}{C}\frac{L^3}{16N_0^3} \\
\implies d_1 &> -\frac{L}{N_0\pi}\ln\left(\frac{C_D}{C}\frac{L^3}{16N_0^3}\right) \\
\implies d_1 &> \frac{\ln(C)-\ln(C_D)+(4+3)\ln(2)+3\ln(N_0)}{2^1N_0\pi}. \qquad (8)
\end{aligned}
$$

This way, we can guarantee second-order discretization error on each patch. This is the same guarantee given by a second-order finite difference method, and multigrid with a finite difference discretization still eliminates low-frequency errors using the coarser grids.

We continue to assume that refining each patch before quartering assures that each subpatch is generated with higher-frequency accuracy. We interpret this as an initial approximation on on all four subpatches that is exact up to the first N_0 components of the true Fourier representation of the given subpatch's boundary. That is (5) is still 0 for $n \leq N_0$, but L has shrunken, meaning the expression for the boundary-imposed error decays faster.

On level l, let d_l refer to the distance requirement at l levels from the coarsest grid in order to have the error imposed by artificial boundaries decay to discretization error with respect to the given refinement. We assume that each level of SRMG is performed by first doubling the resolution of each patch, then quartering the domain. Note that this means discretization error on a patch at level l is given by $\frac{C_D}{2^lN_0}$

We can derive a requirement for d_l of the following form

$$
d_l > \frac{\ln(C)-\ln(C_D)+(4+3l)\ln(2)+3\ln(N_0)}{2^lN_0\pi} \qquad (9)
$$

The sum $d = \sum_{l=1}^{\infty} d_l$ converges. This means that a finite but suitably large buffer zone around the initial coarse patches will guarantee convergence of the method on all subsequent levels. Of course, we need that $d < 1/2$ and ideally much less, but we can achieve this by raising N_0, the initial grid refinement. For implementation, we are concerned with $\overline{d}_l$, the total buffer needed at level l which would allow convergence on all subsequent patches.

$$
\begin{aligned}
\overline{d}_l &= \sum_{m=l}^{\infty} d_m \\
&> \sum_{m=l}^{\infty} \frac{\ln(C)-\ln(C_D)+4\ln(2)+3\ln(N_0)}{2^mN_0\pi} + \frac{3m\ln(2)}{2^mN_0\pi}, \\
&> \frac{\ln(C)-\ln(C_D)+4\ln(2)+3\ln(N_0)}{2^{l-1}N_0\pi} + \frac{3(l+1)\ln(2)}{2^{l-1}N_0\pi} \qquad (10)
\end{aligned}
$$

so the number of cells $c_l = \overline{d}_l 2^l$ required in the buffer in each direction is

$$
c_l = 2\frac{\ln(C)-\ln(C_D)+(7+3l)\ln(2)+3\ln(N_0)}{N_0\pi},
$$

which means at each level we must increase the number of cells by a constant, but the overall distance from the boundary remains constant.

In summary, we require that the solution perturbation, driven by high-frequency errors in the artificially imposed boundary conditions, decays to discretization error within some distance d_l away from the patch boundary, where l is the level index for the patch. Continuing this process recursively by allowing an additional buffer of size d_l for each level, we can show that segmental refinement at each level finds solutions to a PDE up to discretization error. This is the same guarantee offered by traditional multigrid with a finite difference method.

Since the infinite sequence $\{d_l\}$is summable, a finite buffer $\sum_{i=1}^{\infty} d_i = \overline{d_1}$ on the first level $l = 1$ of SRMG is sufficient to guarantee an accurate solution on all successive patches without communication with the whole domain.

3.3. Implementation. Aspects of the implementation of this method was described in detail by John Boyd [**?**]

In order to solve the partial differential equation $\triangle u(x) = f(x)$ with boundary boundary conditions u(x) = g(x) along the boundary $\partial\Omega$

$$\begin{aligned} \triangle u(x) &= f(x) \\ u(x) &= g(x)\ \forall x \in \partial\Omega \end{aligned} \tag{11}$$

we can derive an equivalent formulation:

$$\begin{aligned} u(x) &= \phi(x) + B(x) \\ \triangle\phi(x) &= f - \triangle B(x) \\ B(x) &= g(x)\ \forall x \in \partial\Omega \\ \phi(x) &= 0\ \forall x \in \partial\Omega \end{aligned} \tag{12}$$

Therefore, we solve the boundary conditions with B(x) and add it to a function $\phi(x)$ satisfying homogenous boundary conditions such that $\triangle(\phi(x)+B(x)) = f = \triangle u$.

If ϕ is constructed from a set of basis elements that are equal to 0 on the boundary of Ω, then there are less computations required to solve for $\phi(x)$ such that $\triangle\phi(x) = f - \triangle B(x)$ on a discrete domain. In this case, we must only satisfy conditions for the interior points $\Omega \cap \partial\Omega^c$, since homogenous boundary conditions are satisfied by choice of basis.

Given this suitable choice of basis functions, and if the domain is two dimensional with N_0 points in each dimension of our discretization, this implementation requires inverting a system $(N_0 - 2) * (N_0 - 2)$ unknowns, rather than $N_0 \times N_0$ unknowns. For large N_0, the computational difference is negligible .

The primary benefit of this implementation is that it improves readability of the code and makes it simpler to enforce the proper boundary conditions, g(x), for a subdomain that shares a boundary with the full domain.

3.4. Construction from Chebyshev Basis. We construct a solution to (11) in the specific case where $\Omega = [-1, 1]x[-1, 1]$.

$\partial\Omega = \{(x, y) : x = 1\} \cup \{(x, y) : y = 1\} \cup \{(x, y) : x = -1\} \cup \{(x, y) : y = -1\}$

$$u(-1, y) = u_{west}(y),\ u(1, y) = u_{east}(y),\ u(x, -1) = u_{south}(y),\ u(x, 1) = u_{north}(y)$$

We can construct the function B(x, y) as follows:

$$\begin{aligned} B(x,y) = &\frac{1-x}{2}u_{west}(y) + \frac{1+x}{2}u_{east}(y) + \frac{1-y}{2}u_{south}(x) + \frac{1+y}{2}u_{north}(x) \\ &- \frac{(1-x)(1-y)}{4}u(-1,-1) - \frac{(1-x)(1+y)}{4}u(-1,1) \\ &- \frac{(1+x)(1-y)}{4}u(1,-1) - \frac{(1+x)(1+y)}{4}u(1,1) \end{aligned}$$

$$\begin{aligned} \triangle B(x,y) = &\frac{1-x}{2}\,\frac{\partial^2 u_{west}(y)}{\partial y^2} + \frac{1+x}{2}\,\frac{\partial^2 u_{east}(y)}{\partial y^2} \\ &+ \frac{1-y}{2}\,\frac{\partial^2 u_{south}(x)}{\partial x^2} + \frac{1+y}{2}\,\frac{\partial^2 u_{north}(x)}{\partial x^2} \end{aligned}$$

Our choice of ϕ is motivated by the Chebyshev polynomials, T_n. Note that Chebyshev polynomials satisfy the following property

$$\begin{aligned} T_{2n}(\pm 1) &= 1 \\ T_{2n+1}(\pm 1) &= \pm 1 \end{aligned}$$

It is therefore convenient to consider the recombination of Chebyshev polynomials

$$\begin{aligned} P_{2n}(x) &= T_{2n} - 1 \\ P_{2n+1}(x) &= T_{2n+1} - x \end{aligned}$$

Therefore $P_n(\pm 1) = 0$ and $P_i(x)P_j(y) = 0\ \forall i, j \in N_0 \times N_0\ \forall x, y \in \partial\Omega$

Also note that $P_0(x) = 0 = P_1(x)$

The construction of ϕ is generated as an element from the space of a tensor product of recombined Chebyshev polynomials.

So $\phi(x,y) = \sum_{i,j} c_{i,j}\ P_i(x)P_j(y)$ is a suitable choice to construct our solution via the implementation described in (12).

Assume that we discretize our domain with N_0 points in each dimension and $N_0 - 2$ points on the interior of the domain. Then ϕ is calculated by solving the linear system

$\sum_{i=2}^{N_0}\sum_{j=2}^{N_0} c_{i,j}\ (P_i''(x)P_j(y) + P_j''(y)P_i(x)) = f(x,y) - \triangle B(x,y)$

3.5. Segmental Refinement: Solving on Subdomains. The method for constructing the solution to (11) as described above is only valid for $\Omega = [-1,1]x[-1,1]$, but we are interested in the process of solving the PDE on subdomains with exact forcing function f given by the problem, and artificial boundary conditions imposed by previous levels of segmental refinement.

Therefore, the domain of each of the patches of Segmental Refinement are not $[-1,1] \times [-1,1]$ but are instead given by some $\tilde{\Omega} = [a,b] \times [c,d]$.

A simple modification allows us to use the same approach as above by considering two linear mappings M_x, M_y such that:

$$M_x([a,b]) = [-1,1]$$
$$M_x([c,d]) = [-1,1]$$

Therefore the solution to a PDE on a subdomain with boundary conditions imposed by the previous levels of SRMG is given by

$$\triangle u(x,y) = f(x,y) \tag{13}$$
$$u(x,y) = g(x,y)\ \forall (x,y) \in \partial(\tilde{\Omega} = \partial\{[a,b]\times[c,d]\})$$

with equivalent expression:

$$\triangle u(M_x(x), M_y(y)) = f(x,y) \tag{14}$$
$$u(M_x(x), M_y(y)) = g(x,y)\ \forall (x,y) \in \partial\tilde{\Omega}$$

Note that via the chain rule, $\triangle u(M_x(x), M_y(y)) = (M_y')^2 \frac{\partial^2 u(M_x(x),M_y(y))}{\partial M_y^2} + (M_x')^2 \frac{\partial^2 u_(M_x(x),M_y(y))}{\partial M_x^2}$

Therefore, with a very simple modification, we can use the same method to solve subdomains or patches generated by Segmental Refinement as we do to solve the coarse domain. The only difference is that boundary conditions will not be imposed by the problem, but by an approximate solution from a previous level of SRMG.

4. Results.

4.1. Accuracy of Method. The coarse resolution was set at $N_0 = 10$

L_1 error was computed with a ten-point Gaussian Quadrature on each patch. This error was summed up over all of the patches corresponding to a given level to achieve a total metric for L_1 error.

Here the L_1 error was calculated against the true solution $\sin 2\pi x \sin 2\pi y$

FIG. 2. *Error via Boyd box function plateus or increases after level 1*

FIG. 3. *Error via sinhsin box function plateus or increases after level 1*

Here the L_1 error was calculated against the true solution $u = \sin 2\pi y \frac{\sinh 2\pi(x-1)}{\sinh 4\pi}$

FIG. 4. *Error via Boyd box function plateus or increases after level 1*

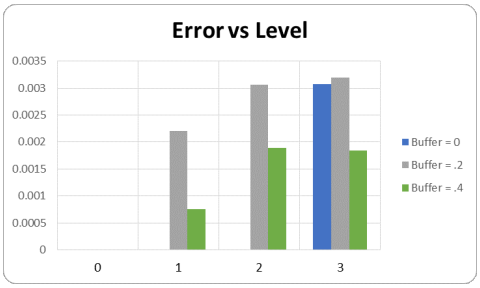


FIG. 5. *Error via sinhsin box function plateus or increases after level 1*

This implementation of SRMG could not consistently achieve error reduction on levels beyond the first level of SRMG, regardless of buffer size, coarse resolution, or number of levels.

5. Concluding Analysis. The underlying assumptions of SRMG require that subsequent levels generate higher-frequency approximations of the true solution. However, low-frequency errors coming from the first level of SRMG seemed to propagate into the interior of all subpatches generated at later levels.

A new, more careful analysis was done with this problem in mind, leading to a new condition for iterative SRMG.

5.1. Poisson's Equation on Subdomains. Consider the uniqueness theorem of Poisson's equation [**?**].

THEOREM 5.1. *Let $u(\Omega)$ be a solution to Poisson's equation*

$$\nabla^2 u = f$$

$$u = g \text{ on } \partial\Omega$$

$u(\Omega)$ is unique.

We can also conclude the following corrolary.

COROLLARY 5.2. *On any closed subset of the domain $\tilde{\Omega}_i \subset \Omega$, there is a unique solution v to the following equation.*

$$\nabla^2 v = f$$

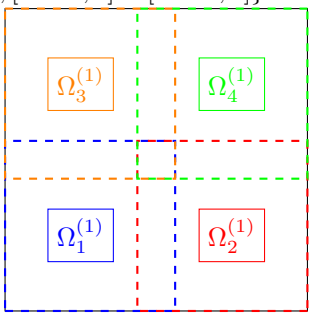
$$v = u \text{ on } \partial\tilde{\Omega}_i$$

$v(\tilde{\Omega}_i)$ is unique. Furthermore, u, the solution to Poisson's equation on the full domain, restricted to $\tilde{\Omega}_i$ satisfies this equation $\implies v = u$.

5.2. Domain Decomposition. The mechanics of segmental refinement assume that a coarse solve $u^{(0)}(x, y)$ is first done on the full domain. That is, we generate $u^{(0)}$ satisfying, for some discretization or approximation of the true forcing given by $f^{(0)}$:

$$\begin{aligned} \triangle u_i^{(0)} &= f^{(0)} \\ u_i^{(0)} &= g \in \partial\Omega \end{aligned} \tag{15}$$

Then an overlapping domain decomposition is formed as follows
$\{\Omega_1^1, \Omega_2^1, \Omega_3^1, \Omega_4^1\} = \{[0, .5+\epsilon] \times [0, .5+\epsilon]; [.5-\epsilon, 1] \times [0, .5+\epsilon]; [0, .5+\epsilon] \times [.5-\epsilon, 1]; [.5-\epsilon, 1] \times [.5-\epsilon, 1]\}$



A set of 4 solutions $u_i^{(1)}$ is constructed on each patch satisfying the following system of equations on a corresponding subdomain $\Omega_i^{(1)}$

$$\begin{aligned} \triangle u_i^{(1)} &= f \\ u_i^{(1)} &= u^{(0)} \in \partial\Omega_i^1 \end{aligned} \tag{16}$$

In other words, we impose an artificial boundary condition on each subdomain, extracted from the coarse approximation $u^{(0)}$. Our previous error analysis on SRMG holds for this first level.

5.3. Recursive SRMG. Assume the following about the mechanics of SRMG: each subdomain generated by the l^{th} level of SRMG, Ω_i^l, is a proper subset of some subdomain Ω_j^{l-1}.

Furthermore assume that on each level of SRMG for $l \geq 1$, a patch solution $u_i^{(l)}$ is approximated to the following Poisson equation:

$$\begin{aligned} \triangle u_i^{(l)} &= f \\ u_i^{(l)} &= u_j^{(l-1)} \in \partial\Omega_i^l \end{aligned} \tag{17}$$

Here $u_j^{(l-1)}$ is a solution to a corresponding Poisson equation from a subdomain generated by the previous level of SRMG. That is, each patch is a solution to Poisson's equation with artificial boundary conditions imposed by the solution generated via the previous level of SRMG.

THEOREM 5.3. *For $l \geq 2$, the analytical solution $u_i^{(l)} \in \Omega_i^l \subset \Omega_j^{l-1}$ is unique and equals: $u_j^{(l-1)} \in \Omega_j^{l-1}$.*

The above theorem follows directly from 5.2

COROLLARY 5.4. *The analytical solution to the equations governing any $u_i^{(l)}$ is given by the solution $u_j^{(1)}$ where $\Omega_i^l \subset \Omega_j^1$*

The direct result of this is that $e_i^{(l)} = u_i^{(l)} - u = u_j^{(1)} - u = e_j^{(1)}$. That is, the pointwise error induced by imposing artificial boundary conditions on level 1 has an effect on all subsequent levels of Segmental Refinement.

This contradicts one of our fundamental requirements of SRMG, that high-frequency errors on the interior of a patch could be eliminated by refinement. Unfortunately, error imposed by the first artificial boundary cannot be eliminated by anything other than a more refined coarse solve, or a suitably large buffer. The required size of this buffer grows to accommodate the stronger convergence requirements on higher-resolution grids.

5.4. Multi-level Error Analysis. In the previous section, we studied the analytical solution to patches governed by a PDE with artificially imposed boundary conditions coming from a previous level of Segmental Refinement.

Let us be more precise and consider that each patch also employs approximation error.

For simplicity, consider Poisson's equation $\triangle u = f$ with zero boundary value conditions on the full domain, so that any approximation or patch encodes the true boundary values where its domain intersects with the boundary of the full domain.

Let $u_j^{(l)}$ be the solution to the imposed patch PDE on the subdomain Ω_j^l (17).

The generated approximation of $u_j^{(1)}$ depends on the method used to solve the PDE, which, in general, will not be an analytic or exact solution to the boundary conditions or the forcing condition.

We deem the generated approximation to a patch PDE $\tilde{u}_j^{(l-1)}$ and note that the ensuing subpatches approximate the true solution to a modified PDE 17 as follows:

$$\begin{aligned} \triangle u_i^{(l)} &= f \\ u_i^{(l)} &= \tilde{u}_j^{(l-1)} \in \partial(\Omega_i^l) \end{aligned}$$

Let $\tilde{e}^{(0)} = \tilde{u}^{(0)} - u$ be the error of the coarse approximation, which is used to impose a new PDE on artificially imposed boundary.

Let $\tilde{e}_j^{(1)} = \tilde{u}_j^{(1)} - u_j^{(1)}$ be the error between the true solution to the artificially imposed PDE and the generated approximation to this artificially imposed PDE.

Recall the error between the true solution to the imposed patch PDE and the true solution on the full domain is given by: $e_j^{(1)} = u_j^{(1)} - u$.

The 1st level of SRMG generates errors of the following form:

$\tilde{u}_j^{(1)} - u = (\tilde{u}_j^{(1)} - u_j^{(1)}) + (u_j^{(1)} - u) = e_j^{(1)} + \tilde{e}_j^{(1)}$

To study the error induced by SRMG on later levels, note that the mechanics of SRMG and uniqueness of Poisson's equation guarantee that the true solution to the artificial PDE imposed on each patch satisfies boundary conditions corresponding to the parent patch's approximation. That is: $u_i^{(l)} = \tilde{u}_j^{(l-1)}$ on the boundaries.

So on a patch's boundary Ω_i^l, we have:

$$\begin{aligned}\tilde{u}_i^{(l)} - u &= (\tilde{u}_i^{(l)} - u_i^{(l)}) + (u_i^{(l)} - u) = \tilde{e}_i^{(l)} + e_i^{(l)} \\ &= \tilde{e}_i^{(l)} + (u_i^{(l)} - u) = \tilde{e}_i^{(l)} + (\tilde{u}_j^{(l-1)} - u) \\ &= \tilde{e}_i^{(l)} + (\tilde{u}_j^{(l-1)} - u_j^{(l-1)}) + (u_j^{(l-1)} - u) = \tilde{e}^{(l)} + \tilde{e}^{(l-1)} + e^{(l-1)} \\ &\ldots \\ &= \sum_{k=0}^{l} \tilde{e}^{(k)} + e^{(0)}\end{aligned}$$

On any patch and any level of SRMG, we can therefore express the error $\tilde{u}_i^{(l)} - u$, as a solution to a PDE where $\tilde{f}_i^{(l)} = \triangle\tilde{u}^{(l)}$, is a discretization or approximation of the forcing condition, which is not induced by parent patches but extracted from the true PDE:

$$\triangle(\tilde{u}_i^{(l)} - u) = \tilde{f}_i^{(l)} - f$$
$$\tilde{u}_i^{(l)} - u = \sum_{k=0}^{l} \tilde{e}^{(k)} + e^{(0)} \in \partial\Omega_{i_l}^{(l)}$$

It is clear from this analysis that the error induced by boundaries in the first level of SRMG does indeed have an effect on all subsequent patches. Refinement does nothing to eliminate this error or restrict it to higher frequencies.

5.5. Error Analysis Revisited. Note that our original analysis holds the for the first level of Segmental Refinement. If $u^{(0)} - u$ contains primarily high frequncies, then the first artificial boundaries induced by SRMG impose second-order discretization error with a suitably large buffer. The problem is that further levels of SRMG still encode this error, and this error decays no faster on later levels, regardless of refinement.

Therefore, SRMG requires not just that $e^{(0)} \leq \frac{C_D}{N_0^2}$, but that $e^{(0)} \leq \frac{C_D}{(2^S N_0)^2}$, where S is the deepest level of SRMG we desire. Reviewing our original analysis, it may seem this would impose a requirement on the buffer size that grows nearly-proportionally with respect to the levels desired.

To see this we revisit just the first-level buffer required, given by (10), but note that instead of our original requirement that SRMG induce some error bounded by $\frac{C_D}{N_0^2}$, we say that SRMG must induce some error bounded by $\frac{C_D}{S(2^S N_0)^2}$. The updated requirement on d_1 is that

$$\begin{aligned}d_1 &> \frac{\ln(C) - \ln\left(4^{-S} C_D \cdot S\right) + (4+3)\ln 2 + 3\ln N_0}{2N_0\pi} \\ &= \frac{\ln(C) - \ln C_D + 3\ln N_0 + (7+2S)\ln 2 + \ln S}{2N_0\pi}\end{aligned}$$

d_1 grows nearly proportionally as S increases.

However, taking into account the practical limitations of a computer, we can instead impose the more robust requirement $e^{(0)} < \epsilon_{mach}$, that the error imposed by an artificial boundary decay to machine error.

Now we revisit the buffer size requirements derived from our first analysis (9) (10) and note that this updated analysis yields the following on d_1 in order to achieve multi-level convergence.

$$d_1 > \frac{5\ln 2 + \ln C + \ln N_0 - \ln \epsilon_{mach}}{2N_0\pi}$$

And a general buffer requirement on each level.

$$d_l > \frac{\ln 2(4+l) + \ln C + \ln N_0 - \ln \epsilon_{mach}}{N_0\pi 2^l}$$

So that each artificially imposed boundary, at this distance, imposes practically 0 error due to the limitations of machine precision. SRMG can thus be performed for an arbitrary number of levels without revisiting the coarsest level and increasing its resolution or retroactively increasing the size of the first set of buffer regions.

We again study $\overline{d_l} = \sum_{i=l}^{\infty} d_i$, the total buffer required at each level to perform further refinements.

$$\begin{aligned}
\overline{d_l} &= \sum_{i=l}^{\infty} d_i \\
&= \frac{ln2}{2N_0\pi} 2^{1-l}(l+5) + 2^{1-l}\frac{\ln C + \ln N_0 - \ln \epsilon_{mach}}{N_0\pi} \\
&= \frac{2^{1-l}}{N_0\pi}\left(\frac{(l+5)\ln 2}{2} + \ln C + \ln N_0 - \ln \epsilon_{mach}\right)
\end{aligned}$$

Note that this is a nontrivial requirement. In the case of double-precision accuracy, $\epsilon_{mach} = 2^{-53}$

With this in mind, we again generate a table of required buffers regions in order to achieve the new convergence requirements. We take C to be 1.

N_0	$\overline{d}_1$	$\overline{d}_2$	$\overline{d}_3$
10	1.309	.660	.333
100	.138	.070	.035
1000	.015	.007	.004

An extremely important thing to note is that this analysis crucially relies on no low-frequency error being introduced by any subsequent level of SRMG. This may cause failure to converge not only on this given patch, but on any subsequent subdomain, since my analsis has shown that no method of refinement will eliminate this error.

Therefore, we still must consider an implementation that remedies all sources of low-frequency error including aliasing error, interpolation error, or limitations of the basis used to represent each patch solution.

6. Conclusion. Previous analysis and work relied on the assumption that if each level of SRMG could guarantee error on the same order of magnitude as traditional multigrid, then it would also inherit the multigrid property that this error would be restricted to higher frequencies. This property on the frequency of error does not hold in SRMG. Each artificial boundary condition imposes an error in a certain frequency

range, and the process of refinement does nothing to restrict this error to higher frequencies.

Computational resources and/or a suitably optimized spectral solver were not available to test the newfound convergence requirements. However, we have shown here several numerical experiments and a rigorous analysis of Segmental Refinement. All are consistent with previous work, where the buffer size had to grow to accommodate further levels of SRMG. I have also shown potential solutions for this issue, although the feasibility of these solutions is yet to be determined.

Iterative Segmental Refinement only guarantees convergence if there is no reintroduction of low-frequency error via interpolation, no low-frequency error due to aliasing, and a buffer size large enough to eliminate error induced by the first (and each subsequent) artificial boundary.

These are requirements that are much more difficult to satisfy than those posed by our original justification. However, in the case that these requirements are not feasible, Segmental Refinement is still suitable for a few of the finest levels assuming a large buffer is chosen. My analysis here demonstrates possible points of failure and methods to ameloriate them.