Problem

For example, for a parameter θ , you already have got the unbiased estimation $\hat{\theta}$ which means $E(\hat{\theta}) = \theta$. But what if you want to get a point estimation of θ^k ? In general, $\hat{\theta}^k$ is no an unbiased estimation of θ^k . i.e., $E(\hat{\theta}^k) \neq \theta^k$ in general. How to do this?

Method

A natural way to deal with this is to first see the difference between $E(\hat{\theta}^k)$ and θ^k . A (almost only) way is to use taylor expansion.

$$E(\hat{\theta}^k) = E[\ (\hat{\theta} - \theta + \theta)^k\ ] \approx E[\ \theta^k + k\ \theta^{k-1}(\hat{\theta} - \theta) + \tfrac{1}{2}k(k-1)\ \theta^{k-2}\ (\hat{\theta} - \theta)^2\ ]$$

$$= E\ [\theta^k] + k\ \theta^{k-1}E(\hat{\theta} - \theta) + \tfrac{1}{2}k(k-1)\ \theta^{k-2}\ E[(\hat{\theta} - \theta)^2\ ]$$

$$= \theta^k + 0 + \tfrac{1}{2}k(k-1)\ \theta^{k-2}\ E[(\hat{\theta} - \theta)^2\ ]$$

$$= \theta^k + \tfrac{1}{2}k(k-1)\ \theta^{k-2}\ E[\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2\ ] = \theta^k + \tfrac{1}{2}k(k-1)\ \theta^{k-2}\ var(\hat{\theta})$$

This means that, (approximately) , the point estimator $\hat{\theta}^k - \tfrac{1}{2}k(k-1)\ \theta^{k-2}\ var(\hat{\theta})$ can approximately provide a unbiased estimation of θ^k , that is (just rearrange the above results) $\theta^k = E[\hat{\theta}^k] - \tfrac{1}{2}k(k-1)\ \theta^{k-2}\ var(\hat{\theta})$. But, what is θ^{k-2} ? naturally, we need to do this recursively.

Procedure

We already know that

$$\theta = E(\hat{\theta}) \text{ (1)}$$

$$\theta^2 = E(\hat{\theta})^2 = E(\hat{\theta}^2) - var(\hat{\theta}) \text{ (2)}$$

$$\theta^3 = E(\hat{\theta}^3) - \tfrac{1}{2}3(3-1)\ \theta var(\hat{\theta}) \text{ (3)}$$

plug in the expression of θ in (1)

$$\theta^4 = E(\hat{\theta}^4) - \tfrac{1}{2}4(4-1)\ \theta^2 var(\hat{\theta}) \text{ (4)}$$

Plut in the expression of θ^2 in (2)....

Doing this recursively, we can finally get a corrected estimation of θ^k