The Basic Idea of Hypothesis Testing

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Basic idea

Basically, for a parameter θ, you want to know whether it belongs Θ. To realize this, we often create a 'judge criterion':if a statistic $T(x_1, , , x_n) \in W$, then we reject the statement that $\theta \in \Theta$.

So, 'given our statistic' T, we want to find out the W, which we call 'reject area'. But how to find it? To illustrate this, we give a concrete example.

Suppose the sample comes from a normal distribution $N(\mu, \sigma^2)$. Suppose that we already know σ^2 but μ is unknown. suppose we want to test the hypothesis that $\mu \leq \mu_0$

Since $\overline{x}$ is an unbiased estimation of the mean, A natural and straight forward way to set the judge criterion is that: if $\overline{x} > c$ (i.e., the average of the sample is too large, larger than some threshold), then we reject $\mu \leq \mu_0$

but notice! since sample is random, then$\overline{x}$ is random, and it is totally possible that$\overline{x} > c$ EVEN when $\mu \leq \mu_0$ does hold. When such a thing happens, we call that we make a mistake, since we reject the hypothesis when hypothesis actually holds. To be specific, this is called 'Type 1 Error'. In general, we want the probability of such an error to be as small as possible. Therefore, we often want to control such error to be under a level, α, that is:

$$S(\mu, c) \equiv pr(\overline{x} > c|\mu) \leq \alpha, for \quad \forall \mu \leq \mu_0$$

the expression of $pr(\overline{x} > c|\mu)$ is reasonable, since the $\overline{x}$ is a random variable whose distribution is tied to μ. So here, we want to find out how these probability looks like. Again, by creating a (increasing)function $g(\overline{x}; \theta)$ such that $g(\overline{x}; \theta)$'s distribution function does not rely on θ, we can rewrite this expression.

under the context of the example, we can easily create $g(\overline{x}; \mu) = \frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$, and g here follows $N(0, 1)$

therefore we have the following:

$$S(\mu, c) \equiv pr(\overline{x} > c|\mu) = pr(g(\overline{x}, \mu) > g(c, \mu)|\mu) = \Phi(g(c, \mu)) = \Phi(\frac{c-\mu}{\frac{\sigma}{\sqrt{n}}})$$

In which Φ is the accumulative distribution function of standard normal distribution.We want $\Phi(\frac{c-\mu}{\frac{\sigma}{\sqrt{n}}}) < \alpha$ for any $\mu \leq \mu_0$. Since this function decreases with μ, to make this condition holds ,we can make $\Phi(\frac{c-\mu_0}{\frac{\sigma}{\sqrt{n}}}) = \alpha$. Through this expression, we get the value of c.

In summary, the basic procedure of doing the the hypothesis test is:

1. Specify the null hypothesis $\theta \in \Theta$ and alternative hypothesis.$\theta \notin \Theta$
2. Find (according to your intuition) a proper statistics $T(x_1, , , , x_n)$ and specify the reject area $T \in W$
3. Now you want $S(\theta, W) \equiv pr(T(x_1, , , , x_n) \in W|\theta) \leq \alpha$ holds for all $\theta \in \Theta$.(i.e., keep the type 1 error small.) the purpose to find the W.

4. To do this, create $g(T, \theta)$ such that the distribution of g does not rely on θ. we then have

$$pr(T(x_1, , , , x_n) \in W|\theta) = pr(g(T(x_1, , , , x_n), \theta) \in g(W, \theta)|\theta)$$

   in which $g(W, \theta)$ denotes the domain of $g(w, \theta)$ for any $w \in W$.

5.Find out the W, which is a range of T, s.t.$pr(g(T(x_1, , , , x_n), \theta) \in g(W, \theta)|\theta) \leq \alpha$ holds for any $\theta \in \Theta$.