Framework

Consider the Normal-Inverse-Wishart (NIW) prior for the pair $(\boldsymbol{B}, \boldsymbol{\Sigma})$, where:

$$\boldsymbol{\Sigma} \sim \mathcal{IW}_n(\nu, \boldsymbol{\Phi})$$

and

$$\boldsymbol{B} \mid \boldsymbol{\Sigma} \sim \mathcal{MN}_{n\times m}(\boldsymbol{\Psi}, \boldsymbol{\Omega}, \boldsymbol{\Sigma})$$

where the density of the Inverse Wishart is:

$$\mathcal{IW}_n(\boldsymbol{\Sigma}; \nu, \boldsymbol{\Phi}) \propto |\boldsymbol{\Sigma}|^{-(\nu+n+1)/2} \exp\left(-\frac{1}{2}\operatorname{tr}(\boldsymbol{\Phi}\boldsymbol{\Sigma}^{-1})\right)$$

and the density of $\boldsymbol{B}$ conditional on $\boldsymbol{\Sigma}$ is:

$$\begin{aligned}
\mathcal{MN}_{n\times m}(\boldsymbol{B}; \boldsymbol{\Psi}, \boldsymbol{\Omega}, \boldsymbol{\Sigma}) &\propto |\boldsymbol{\Sigma}|^{-m/2} \exp\left(-\frac{1}{2}\operatorname{vec}(\boldsymbol{B}-\boldsymbol{\Psi})^\top (\boldsymbol{\Sigma}\otimes\boldsymbol{\Omega})^{-1}\operatorname{vec}(\boldsymbol{B}-\boldsymbol{\Psi})\right) \\
&= |\boldsymbol{\Sigma}|^{-m/2} \exp\left(-\frac{1}{2}\operatorname{tr}\left[(\boldsymbol{B}-\boldsymbol{\Psi})^\top \boldsymbol{\Omega}^{-1}(\boldsymbol{B}-\boldsymbol{\Psi})\boldsymbol{\Sigma}^{-1}\right]\right) \\
&= |\boldsymbol{\Sigma}|^{-m/2} \exp\left(-\frac{1}{2}\operatorname{tr}\left[\boldsymbol{\Omega}^{-1}(\boldsymbol{B}-\boldsymbol{\Psi})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{B}-\boldsymbol{\Psi})\right]\right)
\end{aligned}$$

"Weak" Normal-Inverse-Wishart Prior

Also known as Uligh or Jeffrey Prior

We specify the following parameters:

$$\begin{aligned}
\nu &= 0 \\
\boldsymbol{\Phi} &= \boldsymbol{0}_{n\times n} \\
\boldsymbol{\Psi} &= \boldsymbol{0}_{n\times m} \\
\boldsymbol{\Omega}^{-1} &= \boldsymbol{0}_{m\times m} \quad \Rightarrow \quad \boldsymbol{\Omega} = \operatorname{diag}(\infty, \dots, \infty)
\end{aligned}$$

The joint prior density reduces to:

$$\pi(\boldsymbol{B}, \boldsymbol{\Sigma}) \propto |\boldsymbol{\Sigma}|^{-(\nu+n+1)/2} = |\boldsymbol{\Sigma}|^{-(n+1)/2}$$

Uniform Prior on join Impulse Responses

Given the parameterization:

$$\begin{aligned}
\nu &= -(m-3) - (n+1) \\
\boldsymbol{\Phi} &= \boldsymbol{0}_{n\times n} \\
\boldsymbol{\Psi} &= \boldsymbol{0}_{n\times m} \\
\boldsymbol{\Omega}^{-1} &= \boldsymbol{0}_{m\times m} \quad \Rightarrow \quad \boldsymbol{\Omega} = \operatorname{diag}(\infty, \dots, \infty)
\end{aligned}$$

the implied prior over $(\boldsymbol{B}, \boldsymbol{\Sigma})$ is:

$$\pi(\boldsymbol{B}, \boldsymbol{\Sigma}) \propto |\boldsymbol{\Sigma}|^{-(\nu+n+1)/2} = |\boldsymbol{\Sigma}|^{(m-3)/2}$$

Minnesota Prior

$$
\begin{aligned}
\nu &= n + 2 \\
\mathbf{\Phi} &= \mathrm{diag}(\phi_1, \ldots, \phi_n), \text{ where } \phi_i = \widehat{\sigma}_i^2(\nu - n - 1) \\
\mathbf{\Psi} = (\mathbf{\Psi})_{ij} &= \begin{cases} 1 & \text{if } i = j \text{ and } i = \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases} \\
\mathbf{\Omega} &= \begin{bmatrix} \widetilde{\mathbf{\Omega}} & \mathbf{0} \\ \mathbf{0} & V_c \end{bmatrix}, \quad \widetilde{\mathbf{\Omega}} = \boldsymbol{K} \otimes \mathbf{\Phi}^{-1}
\end{aligned}
$$

with $\boldsymbol{K} = \mathrm{diag}\left(\frac{(\nu-n-1)\lambda_1^2}{1^{\lambda_3}}, \ldots, \frac{(\nu-n-1)\lambda_1^2}{p^{\lambda_3}}\right)$ and where $\widehat{\sigma}_i^2$ is the residual variance from fitting an AR(p) to variable i. This is related to the fact that

$$
\mathbb{E}(\mathbf{\Sigma}) = \mathbf{\Phi}/(\nu - \mathbf{n} - \mathbf{1}) \text{ hence } \mathbb{E}(\mathbf{\Sigma}) = \mathrm{diag}(\widehat{\sigma}_{\mathbf{1}}^{\mathbf{2}}, \ldots, \widehat{\sigma}_{\mathbf{n}}^{\mathbf{2}}).
$$

Minnesota Prior with Dummy Observations

The Minnesota prior with dummy observations combines $T^\star$ dummy observations, whose likelihood is

$$
p(\mathbf{Y}^\star | \mathbf{B}, \mathbf{\Sigma}),
$$

where $\mathbf{Y}^\star$ is a $T \times n$ matrix, with a prior with the following parameters

$$
\begin{aligned}
\nu &= 2 \\
\mathbf{\Phi} &= \mathbf{0}_{n \times n} \\
\mathbf{\Psi} &= \mathbf{0}_{n \times m} \\
\mathbf{\Omega}^{-1} &= \mathbf{0}_{m \times m} \quad \Rightarrow \quad \mathbf{\Omega} = \mathrm{diag}(\infty, \ldots, \infty)
\end{aligned}
$$

Del Negro and Schorfheide's (2010) Handbook Chapter use a prior over $(\mathbf{B}, \mathbf{\Sigma})$ that is proportional to the prior $|\mathbf{\Sigma}|^{-(n+1)/2}$. We use a different one to guarantee equality with the implementation of the Minnesota prior described above.

The dummy observations are as follows:

$$
\mathbf{Y}^* = \mathbf{X}^* \mathbf{B} + \mathbf{U}
$$

$$
\mathbf{Y}^* = \begin{bmatrix} \mathbf{y}_1^{*\prime} \\ \vdots \\ \mathbf{y}_{T^*}^{*\prime} \end{bmatrix}, \quad \mathbf{X}^* = \begin{bmatrix} \mathbf{x}_1^{*\prime} \\ \vdots \\ \mathbf{x}_{T^*}^{*\prime} \end{bmatrix}, \quad \mathbf{x}_t^{*\prime} = [\mathbf{y}_{t-1}^{*\prime}, \ldots, \mathbf{y}_{t-p}^{*\prime}, 1], \quad \mathbf{U} = \begin{bmatrix} \mathbf{u}_1^{*\prime} \\ \vdots \\ \mathbf{u}_{T^*}^{*\prime} \end{bmatrix}
$$

where $\mathbf{B} = \left[\mathbf{B}_1', \ldots, \mathbf{B}_p', \mathbf{c}'\right]'$.

Illustration with 2 variables and 2 lags Based on Del Negro and Schorfheide's (2010) handbook chapter. The prior for $\mathbf{B}_1$ is implemented via

$$\begin{bmatrix} \frac{\sqrt{\phi_1}}{\lambda_1} & 0 \\ 0 & \frac{\sqrt{\phi_2}}{\lambda_1} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{\phi_1}}{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{\phi_2}}{\lambda_1} & 0 & 0 & 0 \end{bmatrix} \mathbf{B}_1 + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

The prior for $\mathbf{B}_2$ is implemented via

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{\phi_1}}{\lambda_1} 2^{\frac{\lambda_3}{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{\phi_2}}{\lambda_1} 2^{\frac{\lambda_3}{2}} & 0 \end{bmatrix} \mathbf{B}_2 + \mathbf{U},$$

Then, the combination of the likelihood of the dummy observations and the weak prior yields:

$$\begin{aligned} \mathbf{\Sigma} &\sim \mathcal{IW}_n(\nu, \mathbf{\Phi}) \\ \mathbf{B} \mid \mathbf{\Sigma} &\sim \mathcal{MN}_{n\times m}(\boldsymbol{B}; \mathbf{\Psi}, \mathbf{\Omega}, \mathbf{\Sigma}) \end{aligned}$$

where:

$$\begin{aligned} \nu &= n+2 \\ \mathbf{\Omega} &= (\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1} \\ \mathbf{\Psi} &= \mathbf{\Omega}(\mathbf{X}^{*\prime}\mathbf{Y}^*) \\ \mathbf{\Phi} &= \mathbf{Y}^{*\prime}\mathbf{Y}^{*\prime} - \mathbf{\Psi}'\mathbf{\Omega}^{-1}\mathbf{\Psi} \end{aligned}$$

References

Del Negro, M. and F. Schorfheide (2010). Bayesian Macroeconometrics. In *The Oxford Handbook of Bayesian Econometrics.*