Likelihood Inference

Juan Rubio-Ramírez

Emory University

Likelihood Inference

- We focus on likelihood-based inference.
- Why?
  1. Likelihood principle (Berger and Wolpert, 1988).
  2. Attractive asymptotic and good small-sample behavior.
  3. Parameters for policy/welfare analysis.
  4. Simple handling of misspecification (Monfort, 1996).
  5. Enables model comparison.

Likelihood-Based Inference

1. Maximum likelihood: $\hat{\theta} = \arg\max_{\theta} l_x(\theta)$.
2. Bayesian: $\pi(\theta \mid x) = \dfrac{l_x(\theta)\,\pi(\theta)}{\int l_x(u)\,\pi(u)\,\mathrm{d}u}$.

Bayesian Ingredients

- Data $y_T = \{y_t\}_{t=1}^{T}$.
- Model $i \in \mathcal{M}$; parameter space Θ_i.
- Likelihood $f(y_T \mid \theta, i)$, prior $\pi(\theta \mid i)$, posterior

$$\pi(\theta \mid y_T, i) = \frac{f(y_T \mid \theta, i)\, \pi(\theta \mid i)}{\int_{\Theta_i} f(y_T \mid u, i)\, \pi(u \mid i)\, \mathrm{d}u}.$$

The Likelihood

Let $y_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, 1)$, $t = 1, \ldots, n$. Then

$$f(y_T \mid \theta) \propto \exp\Big[-\tfrac{1}{2} \sum (y_i - \theta)^2 \Big] = \exp\Big[-\tfrac{1}{2}\big(ns^2 + n(\theta - \bar{y})^2\big)\Big].$$

where $\bar{y} = \frac{1}{n}\sum y_i$ and $s^2 = \frac{1}{n}\sum (y_i - \bar{y})^2$ are sample mean and sample variance.

The Prior

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right).$$

Combine Likelihood and Prior

$$\pi(\theta \mid y_T) \propto f(y_T \mid \theta)\, \pi(\theta) \propto \exp\left[-\tfrac{1}{2}\left(ns^2 + n(\theta - \bar{y})^2\right) - \tfrac{1}{2\sigma^2}(\theta - \mu)^2\right].$$

$$\pi(\theta \mid y_T) \propto \exp\left\{-\tfrac{1}{2}\left[n(\theta - \bar{y})^2 + \frac{(\theta - \mu)^2}{\sigma^2}\right]\right\}.$$

Expand and Collect Terms

Expand the exponent:

$$-\frac{1}{2}\left[n(\theta^2 - 2\bar{y}\theta + \bar{y}^2) + \frac{1}{\sigma^2}(\theta^2 - 2\mu\theta + \mu^2)\right].$$

Keep only terms involving θ:

$$-\frac{1}{2}\left[(n + \frac{1}{\sigma^2})\theta^2 - 2(n\bar{y} + \frac{\mu}{\sigma^2})\theta\right].$$

Let

$$a = \frac{n\bar{y} + \mu/\sigma^2}{n + 1/\sigma^2}.$$

Complete the Square

Rewrite inside the exponent:

$$(n + \tfrac{1}{\sigma^2})\theta^2 - 2(n\bar{y} + \tfrac{\mu}{\sigma^2})\theta = (n + \tfrac{1}{\sigma^2})\big[(\theta - a)^2 - a^2\big].$$

Constants that do not depend on θ can be dropped:

$$\pi(\theta \mid y_T) \propto \exp\big[-\tfrac{1}{2}(n + \tfrac{1}{\sigma^2})(\theta - a)^2\big]\,.$$

Express in Compact Form

Note that

$$n + \frac{1}{\sigma^2} = \frac{\sigma^2 n + 1}{\sigma^2} = \frac{\sigma^2 + 1/n}{\sigma^2/n}, \qquad a = \frac{n\bar{y} + \mu/\sigma^2}{n + 1/\sigma^2} = \frac{\bar{y}\,\sigma^2 + \mu/n}{\sigma^2 + 1/n}.$$

Substitute into the exponent:

$$\pi(\theta \mid y_T) \propto \exp\left[-\tfrac{1}{2}\,\frac{\sigma^2 + 1/n}{\sigma^2/n}\left(\theta - \frac{\bar{y}\,\sigma^2 + \mu/n}{\sigma^2 + 1/n}\right)^2\right].$$

Interpretation

- Posterior is normal:

$$\theta \mid y_T \sim \mathcal{N}\left(\frac{\bar{y}\,\sigma^2 + \mu/n}{\sigma^2 + 1/n}, \frac{\sigma^2/n}{\sigma^2 + 1/n}\right).$$

- As $n \to \infty$, posterior mean $\to \bar{y}$ and variance $\to 0$.
- As $\sigma^2 \to \infty$, prior becomes flat $\Rightarrow \bar{y}$ dominates.