

Likelihood Inference

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Likelihood Inference

- We focus on likelihood-based inference.
- Why?
 1. Likelihood principle (Berger and Wolpert, 1988).
 2. Attractive asymptotic and good small-sample behavior.
 3. Parameters for policy/welfare analysis.
 4. Simple handling of misspecification (Monfort, 1996).
 5. Enables model comparison.

Likelihood-Based Inference

1. Maximum likelihood: $\hat{\theta} = \arg \max_{\theta} l_x(\theta)$.
2. Bayesian: $\pi(\theta | x) = \frac{l_x(\theta) \pi(\theta)}{\int l_x(u) \pi(u) du}$.

Bayesian Ingredients

- Data $y_T = \{y_t\}_{t=1}^T$.
- Model $i \in \mathcal{M}$; parameter space Θ_i .
- Likelihood $f(y_T | \theta, i)$, prior $\pi(\theta | i)$, posterior

$$\pi(\theta | y_T, i) = \frac{f(y_T | \theta, i) \pi(\theta | i)}{\int_{\Theta_i} f(y_T | u, i) \pi(u | i) du}.$$

The Likelihood

Let $y_t \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, 1)$, $t = 1, \dots, n$. Then

$$f(y_T | \theta) \propto \exp\left[-\frac{1}{2} \sum (y_i - \theta)^2\right] = \exp\left[-\frac{1}{2}(ns^2 + n(\theta - \bar{y})^2)\right].$$

where $\bar{y} = \frac{1}{n} \sum y_i$ and $s^2 = \frac{1}{n} \sum (y_i - \bar{y})^2$ are sample mean and sample variance.

The Prior

$$\pi(\theta) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(\theta - \mu)^2}{2\sigma^2}\right).$$

Combine Likelihood and Prior

$$\pi(\theta \mid y_T) \propto f(y_T \mid \theta) \pi(\theta) \propto \exp\left[-\frac{1}{2}(ns^2 + n(\theta - \bar{y})^2) - \frac{1}{2\sigma^2}(\theta - \mu)^2\right].$$

$$\pi(\theta \mid y_T) \propto \exp\left\{-\frac{1}{2}\left[n(\theta - \bar{y})^2 + \frac{(\theta - \mu)^2}{\sigma^2}\right]\right\}.$$

Expand and Collect Terms

Expand the exponent:

$$-\frac{1}{2} \left[n(\theta^2 - 2\bar{y}\theta + \bar{y}^2) + \frac{1}{\sigma^2}(\theta^2 - 2\mu\theta + \mu^2) \right].$$

Keep only terms involving θ :

$$-\frac{1}{2} \left[(n + \frac{1}{\sigma^2})\theta^2 - 2(n\bar{y} + \frac{\mu}{\sigma^2})\theta \right].$$

Let

$$a = \frac{n\bar{y} + \mu/\sigma^2}{n + 1/\sigma^2}.$$

Complete the Square

Rewrite inside the exponent:

$$(n + \frac{1}{\sigma^2})\theta^2 - 2(n\bar{y} + \frac{\mu}{\sigma^2})\theta = (n + \frac{1}{\sigma^2})[(\theta - a)^2 - a^2].$$

Constants that do not depend on θ can be dropped:

$$\pi(\theta | y_T) \propto \exp\left[-\frac{1}{2}(n + \frac{1}{\sigma^2})(\theta - a)^2\right].$$

Express in Compact Form

Note that

$$n + \frac{1}{\sigma^2} = \frac{\sigma^2 n + 1}{\sigma^2} = \frac{\sigma^2 + 1/n}{\sigma^2/n}, \quad a = \frac{n\bar{y} + \mu/\sigma^2}{n + 1/\sigma^2} = \frac{\bar{y}\sigma^2 + \mu/n}{\sigma^2 + 1/n}.$$

Substitute into the exponent:

$$\pi(\theta | y_T) \propto \exp \left[-\frac{1}{2} \frac{\sigma^2 + 1/n}{\sigma^2/n} \left(\theta - \frac{\bar{y}\sigma^2 + \mu/n}{\sigma^2 + 1/n} \right)^2 \right].$$

Interpretation

- Posterior is normal:

$$\theta \mid y_T \sim \mathcal{N}\left(\frac{\bar{y} \sigma^2 + \mu/n}{\sigma^2 + 1/n}, \frac{\sigma^2/n}{\sigma^2 + 1/n}\right).$$

- As $n \rightarrow \infty$, posterior mean $\rightarrow \bar{y}$ and variance $\rightarrow 0$.
- As $\sigma^2 \rightarrow \infty$, prior becomes flat $\Rightarrow \bar{y}$ dominates.