Perturbation Methods Extension

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A Baby Example

A Baby Example: A Basic RBC

$$\max_{\{c_t, k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha},$$

$$z_{t+1} = \rho z_t + \sigma \varepsilon_{t+1}, \qquad \varepsilon_t \sim \mathcal{N}(0,1).$$

FOC of this problem

$$
\frac{1}{c_t} = \beta\,\mathbb{E}_t\left[\frac{1}{c_{t+1}}\left(\alpha e^{\rho z_t+\chi\sigma\varepsilon_{t+1}}k_{t+1}^{\alpha-1}\right)\right]
$$
$$
c_t + k_{t+1} = e^{z_t}k_t^{\alpha},
$$
$$
z_t = \rho z_{t-1} + \sigma\varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,1).
$$

Let $H(c_t, c_{t+1}, k_t, k_{t+1}, z_{t-1}, z_t)$, where

$$
H_1 = \begin{bmatrix} \frac{-1}{c_{ss}^2} \\ 1 \\ 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} \frac{1}{c_{ss}^2} \\ 0 \\ 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 \\ -\alpha k_{ss}^{\alpha-1} \\ 0 \end{bmatrix},
$$

$$
H_4 = \begin{bmatrix} \frac{1-\alpha}{k_{ss}c_{ss}} \\ 1 \\ 0 \end{bmatrix}, \quad H_5 = \begin{bmatrix} 0 \\ 0 \\ -\rho \end{bmatrix}, \quad H_6 = \begin{bmatrix} -\frac{\rho}{c_{ss}} \\ -k_{ss}^{\alpha} \\ 1 \end{bmatrix}.
$$

Re-parametrization and Taylor

A Re-parametrized Policy Function

We search for policy functions

$$c_t = c(k_t, z_{t-1}, \varepsilon_t; \chi).$$

$$c_{t+1} = c(k_{t+1}, z_t, \chi\varepsilon_{t+1}; \chi).$$

$$k_{t+1} = k(k_t, z_{t-1}, \varepsilon_t; \chi).$$

Taylor's Theorem (assume $\delta = 1$)

Equilibrium conditions:

$$
\begin{aligned}
0 &= \mathbb{E}_t \left[\frac{1}{c(k_t, z_{t-1}, \varepsilon_t; \chi)} - \beta \frac{\alpha\, e^{\rho z_t + \chi \sigma \varepsilon_{t+1}}\, k(k_t, z_{t-1}, \varepsilon_t; \chi)^{\alpha-1}}{c(k(k_t, z_{t-1}, \varepsilon_t; \chi), z_t, \chi\varepsilon_{t+1}; \chi)} \right], \\
0 &= c(k_t, z_{t-1}, \varepsilon_t; \chi) + k(k_t, z_{t-1}, \varepsilon_t; \chi) - e^{z_t} k_t^{\alpha}, \\
0 &= z_t - \rho z_{t-1} - \sigma \varepsilon_t.
\end{aligned}
$$

This defines $F(k_t, z_{t-1}, \varepsilon_t; \chi)$. We will take derivatives w.r.t. k_t, z_{t-1}, ε_t and χ and apply Taylor's theorem around the deterministic steady state.

Second-Order Approximation

Second-Order Approximation: What changes?

At first order you found (certainty equivalence in this log-utility RBC):

$$c_\chi = 0, \qquad k_\chi = 0.$$

At second order, two things happen:

- The policy rules become *curved*: terms like c_{kk}, c_{kz}, c_{zz} (and analogously for k).
- Risk matters through *even* powers of χ (notably χ^2): you get *precautionary* corrections.

Technically: second-order derivatives of the equilibrium map satisfy a *linear* system once first-order objects are known.

Second-Order Policy Expansions (your parametrization)

Let deviations be

$$\tilde{k}_t := k_t - k_{ss}, \qquad \tilde{z}_{t-1} := z_{t-1}, \qquad \varepsilon_t \text{ as before.}$$

Second-order Taylor expansions around $(k_{ss}, 0, 0; \chi = 0)$:

$$\begin{aligned} c_t = c_{ss} &+ c_k\,\tilde{k}_t + c_z\,\tilde{z}_{t-1} + c_\varepsilon\,\varepsilon_t + c_\chi\,\chi \\ &+ \frac{1}{2}\Big(c_{kk}\tilde{k}_t^2 + 2c_{kz}\tilde{k}_t\tilde{z}_{t-1} + 2c_{k\varepsilon}\tilde{k}_t\varepsilon_t + 2c_{k\chi}\tilde{k}_t\chi \\ &\qquad + c_{zz}\tilde{z}_{t-1}^2 + 2c_{z\varepsilon}\tilde{z}_{t-1}\varepsilon_t + 2c_{z\chi}\tilde{z}_{t-1}\chi \\ &\qquad + c_{\varepsilon\varepsilon}\varepsilon_t^2 + 2c_{\varepsilon\chi}\varepsilon_t\chi + c_{\chi\chi}\chi^2 \Big), \end{aligned}$$

Second-Order Conditions (what we set to zero)

Your equilibrium conditions define $F(k_t, z_{t-1}, \varepsilon_t; \chi) = 0$ for all admissible inputs. Hence, at $(k_{ss}, 0, 0; 0)$:

$$F_{xx} = 0 \quad \text{for } x \in \{k, z, \varepsilon, \chi\}.$$

Distinct second-order blocks (Young symmetry applies):

$$F_{kk},\ F_{kz},\ F_{k\varepsilon},\ F_{k\chi},\ F_{zz},\ F_{z\varepsilon},\ F_{z\chi},\ F_{\varepsilon\varepsilon},\ F_{\varepsilon\chi},\ F_{\chi\chi}.$$

Given first-order objects, these equations become *linear* in the second-order coefficients of (c, k).

Unknown second-order coefficients and linear system

Collect second-order coefficients (illustrative ordering):

$$x_2 := \left(c_{kk}, c_{kz}, c_{zz}, c_{k\varepsilon}, c_{z\varepsilon}, c_{\varepsilon\varepsilon}, c_{\chi\chi}, k_{kk}, k_{kz}, k_{zz}, k_{k\varepsilon}, k_{z\varepsilon}, k_{\varepsilon\varepsilon}, k_{\chi\chi}\right)^\top.$$

Given first-order objects, the second-order conditions can be written as

$$\mathcal{A}_2\, x_2 = b_2,$$

where:

- $\mathcal{A}_2$ depends only on steady-state objects and first-order derivatives (and on H_u, H_{uu} at the steady state).
- b_2 collects known curvature terms from composing policy rules and the expectations using $\mathbb{E}[\varepsilon_{t+1}^2] = 1$.

This is the second-order analogue of your first-order system for (c_k, c_z, k_k, k_z).

Implementation template (chain rule at second order)

Write the equilibrium mapping schematically as

$$F(\cdot) = \mathbb{E}_t\, H\big(c_t, c_{t+1}, k_t, k_{t+1}, z_{t-1}, z_t; \chi\big) + \begin{bmatrix} 0 \\ 0 \\ -\sigma\varepsilon_t \end{bmatrix}.$$

At the steady state, define

$$\Gamma_x := \frac{\partial(c_t, c_{t+1}, k_t, k_{t+1}, z_{t-1}, z_t, \chi)}{\partial x}, \quad x \in \{k, z, \varepsilon, \chi\},$$

and

$$\Gamma_{xy} := \frac{\partial^2(c_t, c_{t+1}, k_t, k_{t+1}, z_{t-1}, z_t, \chi)}{\partial x\, \partial y}.$$

Then a generic second derivative of F has the structure

$$F_{xy} = \mathbb{E}_t\Big[\, H_u\, \Gamma_{xy} + \Gamma_x^\top\, H_{uu}\, \Gamma_y \Big],$$

where H_u is the gradient and H_{uu} the Hessian of H w.r.t. its arguments.

Where does $F_{ab} = \mathbb{E}_t\left[H_y\, y_{ab} + y_a^\top H_{yy} y_b\right]$ come from? (1/2)

Define the equilibrium residual as a composition:

$$F(x) \;=\; \mathbb{E}_t\left[\, H(y(x)) \,\right], \qquad x = (k, z, \varepsilon, \chi), \quad y = y(x).$$

Notation:

$$y_a := \frac{\partial y}{\partial a}, \qquad y_{ab} := \frac{\partial^2 y}{\partial a\, \partial b},$$
$$H_y := \frac{\partial H}{\partial y}, \qquad H_{yy} := \frac{\partial^2 H}{\partial y\, \partial y^\top}.$$

Step 1 (first derivative). Chain rule:

$$F_a = \frac{\partial}{\partial a}\mathbb{E}_t[H(y(x))] = \mathbb{E}_t\left[\frac{\partial}{\partial a} H(y(x))\right] = \mathbb{E}_t[H_y\, y_a]\,.$$

Step 2 (start second derivative). Differentiate again and use product rule:

$$F_{ab} = \mathbb{E}_t\left[\frac{\partial}{\partial b}(H_y\, y_a)\right] = \mathbb{E}_t[(H_y)_b\, y_a + H_y\, y_{ab}]\,.$$

Where does $F_{ab} = \mathbb{E}_t\left[H_y\, y_{ab} + y_a^\top H_{yy} y_b\right]$ come from? (2/2)

Step 3 (differentiate H_y). Since H_y depends on x only through $y(x)$,

$$(H_y)_b = H_{yy}\, y_b.$$

Substitute into Step 2:

$$F_{ab} = \mathbb{E}_t[(H_{yy} y_b)\, y_a + H_y\, y_{ab}]\,.$$

Write the first term as a quadratic form (up to row/column conventions):

$$(H_{yy} y_b)\, y_a \;=\; y_a^\top H_{yy}\, y_b.$$

Therefore,

$$F_{ab} = \mathbb{E}_t\left[H_y\, y_{ab} + y_a^\top H_{yy}\, y_b\right].$$

Interpretation: $H_y\, y_{ab}$ captures curvature from the policy mapping $y(x)$; $y_a^\top H_{yy} y_b$ captures curvature of H interacting with first-order movements in y.

First-order derivatives: compact notation

Define the equilibrium residual after substitution as

$$F(x) = \mathbb{E}_t H(y(x)) + \begin{bmatrix} 0 \\ 0 \\ -\sigma\varepsilon_t \end{bmatrix} \in \mathbb{R}^3, \qquad x = (k,\ z_{t-1},\ \varepsilon,\ \chi).$$

Collect the arguments of H in

$$y = \left(c_t,\ c_{t+1},\ k_t,\ k_{t+1},\ z_{t-1},\ z_t\right) \in \mathbb{R}^6.$$

Jacobian (dimensions):

$$H_y := \frac{\partial H}{\partial y} \in \mathbb{R}^{3\times 6}.$$

At the steady state, write H_y by columns:

$$\boxed{H_y = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \end{bmatrix}, \qquad H_i \in \mathbb{R}^{3\times 1}}$$

First-order derivatives: the k_t direction (2/3)

Policy rules:

$$c_t = c(k_t, z_{t-1}, \varepsilon_t; \chi), \qquad k_{t+1} = k(k_t, z_{t-1}, \varepsilon_t; \chi), \qquad z_t = \rho z_{t-1} + \sigma \varepsilon_t .$$

Derivative of the H-argument vector w.r.t. k_t:

$$y_k = \frac{\partial y}{\partial k_t} = \begin{bmatrix} (c_t)_k \\ (c_{t+1})_k \\ (k_t)_k \\ (k_{t+1})_k \\ (z_{t-1})_k \\ (z_t)_k \end{bmatrix} = \begin{bmatrix} c_k \\ c_k\, k_k \\ 1 \\ k_k \\ 0 \\ 0 \end{bmatrix}, \qquad (c_{t+1})_k = c_k\,(k_{t+1})_k = c_k k_k .$$

Therefore,

$$F_k = H_y\, y_k = H_1 c_k + H_2 (c_k k_k) + H_3 + H_4 k_k .$$

First-order derivatives: the z_{t-1} direction (3/3)

Derivative of the H-argument vector w.r.t. z_{t-1}:

$$y_z = \frac{\partial y}{\partial z_{t-1}} = \begin{bmatrix} (c_t)_z \\ (c_{t+1})_z \\ (k_t)_z \\ (k_{t+1})_z \\ (z_{t-1})_z \\ (z_t)_z \end{bmatrix} = \begin{bmatrix} c_z \\ c_k k_z + c_z \rho \\ 0 \\ k_z \\ 1 \\ \rho \end{bmatrix}, \qquad (c_{t+1})_z = c_k k_z + c_z (z_t)_z = c_k k_z + c_z \rho.$$

Therefore,

$$F_z = H_y\, y_z = H_1 c_z + H_2 (c_k k_z + c_z \rho) + H_4 k_z + H_5 + H_6 \rho.$$

First-order derivatives: the ε_t direction

Derivative of the H-argument vector w.r.t. ε_t:

$$y_\varepsilon = \frac{\partial y}{\partial \varepsilon_t} = \begin{bmatrix} (c_t)_\varepsilon \\ (c_{t+1})_\varepsilon \\ (k_t)_\varepsilon \\ (k_{t+1})_\varepsilon \\ (z_{t-1})_\varepsilon \\ (z_t)_\varepsilon \end{bmatrix} = \begin{bmatrix} c_\varepsilon \\ c_k k_\varepsilon + c_z \sigma \\ 0 \\ k_\varepsilon \\ 0 \\ \sigma \end{bmatrix}, \qquad (c_{t+1})_\varepsilon = c_k k_\varepsilon + c_z (z_t)_\varepsilon.$$

Equivalently,

$$F_\varepsilon = H_1 c_\varepsilon + H_2(c_k k_\varepsilon + c_z \sigma) + H_4 k_\varepsilon + H_6 \sigma + \begin{bmatrix} 0 \\ 0 \\ -\sigma \end{bmatrix}.$$

First-order derivatives: the σ direction

Derivative of the H-argument vector w.r.t. σ:

$$y_\sigma = \frac{\partial y}{\partial \sigma} = \begin{bmatrix} (c_t)_\sigma \\ (c_{t+1})_\sigma \\ (k_t)_\sigma \\ (k_{t+1})_\sigma \\ (z_{t-1})_\sigma \\ (z_t)_\sigma \end{bmatrix} = \begin{bmatrix} c_\sigma \\ c_k k_\sigma + c_z \varepsilon_t \\ 0 \\ k_\sigma \\ 0 \\ \varepsilon_t \end{bmatrix}, \qquad (c_{t+1})_\sigma = c_k k_\sigma + c_z (z_t)_\sigma.$$

Equivalently,

$$F_\sigma = H_1 c_\sigma + H_2(c_k k_\sigma + c_z \mathbb{E}_t \varepsilon_t) + H_4 k_\sigma + H_6 \mathbb{E}_t \varepsilon_t = H_1 c_\sigma + H_2 c_k k_\sigma + H_4 k_\sigma$$

The χ^2 block: precautionary terms

Because $c_\chi = k_\chi = 0$ at first order, mixed terms with a single χ typically drop out. The risk correction is pinned down by

$$F_{\chi\chi}(k_{ss}, 0, 0; 0) = 0.$$

This equation is *linear* in $(c_{\chi\chi}, k_{\chi\chi})$ once the rest of x_2 is solved, and it shifts policies due to uncertainty:

$$c_t \approx c_{ss} + \cdots + \frac{1}{2}\, c_{\chi\chi}\, \chi^2, \qquad k_{t+1} \approx k_{ss} + \cdots + \frac{1}{2}\, k_{\chi\chi}\, \chi^2.$$

Setting $\chi = 1$ yields the second-order "risk-adjusted" constants used in simulations.

Practical recipe to compute the second-order solution

1. Compute steady state (k_{ss}, c_{ss}).
2. Solve first-order system for (c_k, c_z, k_k, k_z).
3. Solve the linear system for $(c_\varepsilon, k_\varepsilon)$.
4. Assemble $\mathcal{A}_2$ and b_2 from the second-order derivative conditions and solve $\mathcal{A}_2 x_2 = b_2$.
5. Set $\chi = 1$ to obtain the second-order approximation for simulations and moments.