

Final

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Question 1: Use the MH Truncated Multivariate Normal in a Half-Space

Let $X = (X_1, X_2, X_3)^\top \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ with given $\boldsymbol{\mu} \in \mathbb{R}^3$ and positive definite $\Sigma \in \mathbb{R}^{3 \times 3}$. Consider the half-space truncation

$$\mathcal{A} = \{x \in \mathbb{R}^3 : \mathbf{1}^\top x = x_1 + x_2 + x_3 > 0\}.$$

Your task is to generate draws from the *truncated* distribution $X | \{\mathbf{1}^\top X > 0\}$ and to compare two approaches.

Target density (up to a constant):

$$f(x) \propto \phi_3(x; \boldsymbol{\mu}, \Sigma) \mathbf{1}\{\mathbf{1}^\top x > 0\},$$

where $\phi_3(\cdot; \boldsymbol{\mu}, \Sigma)$ is the $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ density on \mathbb{R}^3 .

What to do

(A) Direct MH for the truncated normal.

Implement a Metropolis–Hastings (MH) sampler that targets $\pi_{\text{trunc}}(x)$ directly using a *symmetric random-walk Gaussian* proposal

$$x' = x + s \varepsilon, \quad \varepsilon \sim \mathcal{N}(\mathbf{0}, I_3),$$

where $s > 0$ is a step-size tuning parameter.

- If $x' \notin \mathcal{A}$, reject automatically.
- Otherwise, accept with probability $\min\{1, \phi_3(x'; \boldsymbol{\mu}, \Sigma)/\phi_3(x; \boldsymbol{\mu}, \Sigma)\}$ (you should implement this on the log scale).

Report: acceptance rate, trace plots of $\mathbf{1}^\top X^{(t)}$, and an estimate of $\mathbb{E}[X | \mathbf{1}^\top X > 0]$ with Monte Carlo standard errors.

(B) Untruncated MH + Accept/Reject (A/R).

First, run an MH sampler that targets the *untruncated* $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ (same random-walk proposal). Then apply a simple A/R filter that *keeps* only the draws with $\mathbf{1}^\top X > 0$.

Report: the empirical keep-rate (i.e., $\Pr(\mathbf{1}^\top X > 0)$ estimated from the chain), how many total MH iterations you needed to collect the same number of truncated draws as in (A), and the resulting estimate of $\mathbb{E}[X | \mathbf{1}^\top X > 0]$ with Monte Carlo standard errors.

Deliverables

- Your MATLAB code (clearly organized and commented).
- A short writeup with: (i) a description of your proposal and tuning choice s , (ii) acceptance rates, (iii) effective sample sizes (ESS) for each component, (iv) computational cost comparison between (A) and (B), and (v) a brief discussion of which approach you would prefer here and why.

Default values for testing

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0.6 & 0.2 \\ 0.6 & 1 & 0.4 \\ 0.2 & 0.4 & 1 \end{bmatrix}, \quad \text{iterations} = 50,000.$$

Question 2: Solve RBC model with demand shock

We consider a standard infinite-horizon RBC model with the following features, except that the stochastic shock now affects the **utility function** rather than the production function.

- Infinite horizon representative agent.
- Preferences are affected by a stochastic demand (preference) shock:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{d_t} \frac{C_t^{1-\gamma}}{1-\gamma}.$$

- Budget Constraint:

$$C_t + K_{t+1} = K_t^\alpha.$$

- Stochastic *demand (preference)* shock:

$$d_{t+1} = \sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, 1).$$

Use $\alpha = 0.4$, $\gamma = 2$, $\sigma = 1$, and $\beta = 0.9$.

Things to do

1. Write the first-order conditions of the model. Remember there must be three equations since we are looking for policy functions for consumption, capital, and the demand shock.
 - (i) Euler equation,
 - (ii) Resource constraint,
 - (iii) Law of motion for d_t .
2. Find the steady-state analytically (set $d_t = 0$ and all variables constant over time).
3. Write the system of equilibrium conditions in the form

$$H(c_t, c_{t+1}, k_t, k_{t+1}, d_t, d_{t+1}) = 0.$$

4. Write the policy functions we are looking for as functions of (k_t, d_t, σ) :

$$c_t = g_c(k_t, d_t, \sigma), \quad k_{t+1} = g_k(k_t, d_t, \sigma).$$

5. Write H_i for $i = 1, \dots, 6$, that is, each of the six scalar equilibrium conditions that compose H .
6. Write H_i for $i = 1, \dots, 6$ evaluated at the steady-state.
7. Compute F_k , F_d , and F_σ .
8. Construct the matrices \mathbf{A} and \mathbf{B} from the linearized system.
9. Solve for the policy derivatives \mathbf{h}_x and \mathbf{c}_x both using dynare and the method we developed in class.

Question 3: A Gibbs Sampler for bayesian regression

Model and likelihood. Let $x_1, \dots, x_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with unknown (μ, σ^2) . Write $\tau \equiv \sigma^{-2}$, $\bar{x} = \frac{1}{n} \sum_i x_i$, and $S = \sum_{i=1}^n (x_i - \bar{x})^2$. The (kernel of the) likelihood is

$$L(\mathbf{x}; \mu, \tau) \propto \tau^{n/2} \exp\left(-\frac{\tau}{2}[n(\mu - \bar{x})^2 + S]\right) = \underbrace{\tau^{1/2} \exp\left(-\frac{n\tau}{2}(\mu - \bar{x})^2\right)}_{\mathcal{N}(\theta = \bar{x}, h = \frac{1}{\tau n})} \underbrace{\tau^{(n-1)/2} \exp\left(-\frac{S}{2}\tau\right)}_{\text{Ga}(a = \frac{n+1}{2}, b = \frac{S}{2})}.$$

where $\mathbf{x} = x_1, \dots, x_n$, remember that density of normal with mean θ and precision matrix h is $h^{1/2} e^{(-\frac{h}{2}(x-\theta)^2)}$ and the density of gamma is proportional to $\tau^{a-1} e^{-b\tau}$.

Conditionals

$$\mu \mid \tau, \mathbf{x} \sim \mathcal{N}\left(\theta = \bar{x}, h = \frac{1}{\tau n}\right)$$

$$\tau \mid \mathbf{x} \sim \text{Ga}\left(a = \frac{n+1}{2}, b = \frac{S}{2}\right)$$

so, the normal draw depends on τ but the draw of τ does not.

Gibbs (independent τ draw):

$$\tau^{(m+1)} \sim \text{Ga}\left(\frac{n+1}{2}, \frac{S}{2}\right), \quad \mu^{(m+1)} \sim \mathcal{N}\left(\bar{x}, 1/(\tau^{(m+1)} n)\right).$$

Tasks.

1. Simulate data with $\mu^* = 1.0$, $\sigma^* = 2.0$, $n = 200$.
2. Implement Gibbs and compare histograms of draws for μ and σ with the true values.