Perturbation Methods Extension

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A Baby Example: A Basic RBC

$$\max_{\{c_t, k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha},$$

$$z_{t+1} = \rho z_t + \sigma \varepsilon_{t+1}, \ \varepsilon_t \sim \mathcal{N}(0,1).$$

FOC of this problem

$$\frac{1}{c_t} = \beta \, \mathbb{E}_t \left[\frac{1}{c_{t+1}} \left(\alpha e^{\rho z_t + \chi \sigma \varepsilon_{t+1}} k_{t+1}^{\alpha-1} \right) \right]$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha},$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t, \; \varepsilon_t \sim \mathcal{N}(0,1).$$

Let $H(c_t, c_{t+1}, k_t, k_{t+1}, z_{t-1}, z_t, \chi)$, where

$$H_1 = \begin{bmatrix} \frac{-1}{c_{ss}^2} \\ 1 \\ 0 \end{bmatrix}, H_2 = \begin{bmatrix} \frac{1}{c_{ss}^2} \\ 0 \\ 0 \end{bmatrix}, H_3 = \begin{bmatrix} 0 \\ -\alpha k_{ss}^{\alpha-1} \\ 0 \end{bmatrix}, H_4 = \begin{bmatrix} \frac{\alpha-1}{k_{ss} c_{ss}} \\ 1 \\ 0 \end{bmatrix}, H_5 = \begin{bmatrix} 0 \\ 0 \\ -\rho \end{bmatrix}, H_6 = \begin{bmatrix} \frac{\rho}{c_{ss}} \\ -k_{ss}^{\alpha} \\ 1 \end{bmatrix}$$

H_7 is a column vector of zeros as $\mathbb{E}_t \varepsilon_{t+1} = 0$

A Re-parametrized Policy Function

We search for policy functions

$$c_t = c(k_t, z_{t-1}, \varepsilon_t; \chi).$$

$$c_{t+1} = c(k_{t+1}, z_t, \chi\varepsilon_{t+1}; \chi).$$

$$k_{t+1} = k(k_t, z_{t-1}, \varepsilon_t; \chi).$$

Taylor's Theorem (assume $\delta = 1$)

Equilibrium conditions:

$$
\begin{aligned}
0 &= \mathbb{E}_t \left[\frac{1}{c(k_t, z_{t-1}, \varepsilon_t; \chi)} - \beta \frac{\alpha e^{\rho z_t + \chi \sigma \varepsilon_{t+1}}, k(k_t, z_{t-1}, \varepsilon_t; \chi)^{\alpha-1}}{c(k(k_t, z_{t-1}, \varepsilon_t; \chi), z_t, \chi \varepsilon_{t+1}; \chi)} \right], \\
0 &= c(k_t, z_{t-1}, \varepsilon_t; \chi) + k(k_t, z_{t-1}, \varepsilon_t; \chi) - e^{z_t} k_t^{\alpha} \\
0 &= z_t - \rho z_{t-1} - \sigma \varepsilon_t.
\end{aligned}
$$

This defines $F(k_t, z_{t-1}, \varepsilon_t; \chi)$. We will take derivatives w.r.t. k_t, z_{t-1}, ε_t and χ and apply Taylor's theorem around the deterministic steady state.

Zero-Order Approximation

Evaluate at $(k_t, z_{t-1}, \varepsilon_t; \chi) = (k, 0, 0, 0)$:

$$F(k, 0, 0; 0) = 0.$$

Steady state:

$$\frac{1}{c} = \beta\,\alpha(k)^{\alpha-1} \quad \Leftrightarrow \quad 1 = \alpha\beta(k)^{\alpha-1}.$$

Thus

$$k = (\alpha\beta)^{\frac{1}{1-\alpha}}, \qquad c = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}. \quad \text{(or using the resource constraint)}$$

How good is this approximation?

First-Order Approximation

Take derivatives of $F(k_t, z_{t-1}, \varepsilon_t; \chi)$ and evaluate at $(k, 0, 0; 0)$. First these two.

$$F_k(k, 0, 0; 0) = 0 \text{ and } F_z(k, 0, 0; 0) = 0$$

Solving the System I

Recall

$$F(k_t, z_{t-1}, \varepsilon_t; \chi) = \mathbb{E}_t$$

$$H\big(c(k_t, z_{t-1}, \varepsilon_t; \chi), c(k(k_t, z_{t-1}, \varepsilon_t; \chi), \rho z_{t-1} + \sigma\varepsilon_t, \chi\varepsilon_{t+1}), k_t, k(k_t, z_{t-1}, \varepsilon_t; \chi), z_{t-1}, \rho z_{t-1} + \sigma\varepsilon_t, \chi\big)$$

$$F_k = H_1 c_k + H_2 c_k k_k + H_3 + H_4 k_k = 0,$$

$$F_z = H_1 c_z + H_2 (c_k k_z + c_z \rho) + H_4 k_z + H_5 + H_6 \rho = 0$$

We can solve these as they do not depend on derivatives with respect to ε_t or χ.

Solving the System II

Unknowns at first order: c_k, c_z, k_k, k_z. We have a system to determine them. How to solve it efficiently?

How to do it?

Rewrite as

$$\begin{bmatrix} H_4 & H_6 & H_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{bmatrix} \begin{bmatrix} k_k \\ 0 \end{bmatrix} = -\begin{bmatrix} H_3 & H_5 & H_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ c_k \end{bmatrix},$$

$$\begin{bmatrix} H_4 & H_6 & H_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{bmatrix} \begin{bmatrix} k_z \\ \rho \end{bmatrix} = -\begin{bmatrix} H_3 & H_5 & H_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ c_z \end{bmatrix}.$$

Solving the System III

Now we take derivative with respect to ε_t

$$F_\varepsilon = H_1 c_\varepsilon + H_2(c_k k_\varepsilon + c_z \sigma) + H_4 k_\varepsilon + H_6 \sigma = \begin{bmatrix} 0 \\ 0 \\ \sigma \end{bmatrix}$$

We can solve these given c_k and c_z. Let

$$C = \begin{bmatrix} H1 & H_2 c_k + H_4 & H_2 c_z + H_6 \end{bmatrix}$$

then

$$C \begin{bmatrix} c_\varepsilon \\ k_\varepsilon \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma \end{bmatrix} \text{ or } \begin{bmatrix} c_\varepsilon \\ k_\varepsilon \\ \sigma \end{bmatrix} = C^{-1} \begin{bmatrix} 0 \\ 0 \\ \sigma \end{bmatrix}$$

Solving the System IV

Now we take derivative with respect to χ

$$F_{\varepsilon} = H_1 c_{\chi} + H_2(c_k k_{\chi} + c_z 0 + c_{\chi}) + H_4 k_{\chi} + H_6 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can solve these given c_k and c_z. Let

$$D = \begin{bmatrix} H1 + H2 & H_2 c_k + H_4 & H_2 c_z + H_6 \end{bmatrix}$$

then

$$D \begin{bmatrix} c_{\chi} \\ k_{\chi} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} c_{\chi} \\ k_{\chi} \\ 0 \end{bmatrix} = D^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$