

Kalman Filter

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State Space Form

- Transition equation:

$$x_{t+1} = Fx_t + G\omega_{t+1}, \quad \omega_{t+1} \sim N(0, Q)$$

- Measurement equation

$$z_t = H'x_t + v_t, \quad v_t \sim N(0, R)$$

- Where x_t are the states and z_t are the observables
- Assume we want to write the likelihood function of $z^T = \{z_t\}_{t=1}^T$.
- A necessary and sufficient condition for the system to be stable is that:

$$|\lambda_i(F)| < 1$$

for all i , where $\lambda_i(F)$ stands for eigenvalue of F .

The State Space Representation is Not Unique

- Let the following system
 - Transition equation

$$x_{t+1} = Fx_t + G\omega_{t+1}, \quad \omega_{t+1} \sim N(0, Q)$$

- Measurement equation

$$z_t = H'x_t + v_t, \quad v_t \sim N(0, R)$$

- Let B be a non-singular squared matrix conforming with F . Then, if $x_t^* = Bx_t$, $F^* = BFB^{-1}$, $G^* = BG$, and $H^* = (H'B)'$, we can write:
 - Transition equation

$$x_{t+1}^* = F^*x_t^* + G^*\omega_{t+1}, \quad \omega_{t+1} \sim N(0, Q)$$

- Measurement equation

$$z_t = H^{*'}x_t^* + v_t, \quad v_t \sim N(0, R)$$

Example

- Assume the following AR(2) process $z_t = \rho_1 z_{t-1} + \rho_2 z_{t-2} + v_t$, $v_t \sim N(0, \sigma_v^2)$.
- State Space I

- Transition equation

$$x_t = \begin{bmatrix} \rho_1 & 1 \\ \rho_2 & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_t$$

- Where $x_t = [y_t \quad \rho_2 y_{t-1}]'$ and measurement equation $z_t = [1 \quad 0] x_t$

- State Space II

- Transition equation

$$x_t = \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_t$$

- Where $x_t = [y_t \quad y_{t-1}]'$ and measurement equation $z_t = [1 \quad 0] x_t$
- Try $B = \text{diag}(\ 1 \quad \rho_2 \)$ on the second system to get the first system.

What is the Kalman Filter trying to do?

- Let $x_{t|t-1} = E(x_t | z^{t-1})$.
- Let $z_{t|t-1} = E(z_t | z^{t-1}) = H'x_{t|t-1}$.
- Let $x_{t|t} = E(x_t | z^t)$.
- Let assume we have $x_{t|t-1}$ and $z_{t|t-1}$.
- We observe a new z_t .
- We need to obtain $x_{t|t}$.
- Note that $x_{t+1|t} = Fx_{t|t}$ and $z_{t+1|t} = H'x_{t+1|t}$, so we can wait for z_{t+1} .
- Therefore, the key question is how to obtain from $x_{t|t-1}$ and z_t .

Kalman Filter Gain to find $x_{t|t}$

- Assume we use the following equation to get $x_{t|t}$ from z_t and $x_{t|t-1}$:

$$x_{t|t} = x_{t|t-1} + K_t (z_t - z_{t|t-1}) = x_{t|t-1} + K_t (z_t - H' x_{t|t-1})$$

- This formula will have some probabilistic justification (to follow).
- K_t is called the Kalman filter gain and it measures how much we update $x_{t|t-1}$ as a function in our error in predicting z_t .
- Let $\Sigma_{t|t-1} = E \left((x_t - x_{t|t-1}) (x_t - x_{t|t-1})' | z^{t-1} \right)$.
- Let $\Omega_{t|t-1} = E \left((z_t - z_{t|t-1}) (z_t - z_{t|t-1})' | z^{t-1} \right)$.
- Let $\Sigma_{t|t} = E \left((x_t - x_{t|t}) (x_t - x_{t|t})' | z^t \right)$.
- Let $\Phi_{t|t-1} = E \left((z_t - z_{t|t}) (x_t - x_{t|t})' | z^{t-1} \right)$.
- It can be shown that:

$$K_t = \Sigma_{t|t-1} H (H' \Sigma_{t|t-1} H + R)^{-1}$$

- The gain:

$$K_t = \Sigma_{t|t-1} H \left(H' \Sigma_{t|t-1} H + R \right)^{-1}$$

- Update of $x_{t|t}$ given z_t and $x_{t|t-1}$ being:

$$x_{t|t} = x_{t|t-1} + K_t (z_t - H' x_{t|t-1})$$

- If we did a big mistake forecasting $x_{t|t-1}$ using past information ($\Sigma_{t|t-1}$ large) we give a lot of weight to the new information (K_t large).
- If the new information is noise (R large) we give a lot of weight to the old prediction (K_t small)

Example

- Assume the following model in State Space form:

- Transition equation

$$x_t = \mu + v_t, v_t \sim N(0, \sigma_v^2)$$

- Measurement equation

$$z_t = x_t + \xi_t, \xi_t \sim N(0, \sigma_\xi^2)$$

- Let $\sigma_\xi^2 = q\sigma_v^2$.
- Then, if $\Sigma_{t|t-1} = \sigma_v^2$, we have

$$K_t = \sigma_v^2 \frac{1}{1+q} \propto \frac{1}{1+q}.$$

- The bigger σ_ξ^2 relative to σ_v^2 (the bigger q) the lower K_t and the less we trust z_t .

The Algorithm

- Given $\Sigma_{t|t-1}$, z_t , and $x_{t|t-1}$, we can now set the Kalman filter algorithm:
- Let $\Sigma_{t|t-1}$, then we compute: $\Omega_{t|t-1} = E \left((z_t - z_{t|t-1}) (z_t - z_{t|t-1})' | z^{t-1} \right)$
$$= E \begin{pmatrix} H' (x_t - x_{t|t-1}) (x_t - x_{t|t-1})' H + \\ v_t (x_t - x_{t|t-1})' H + H' (x_t - x_{t|t-1}) v_t' + \\ v_t v_t' | z^{t-1} \end{pmatrix} = H' \Sigma_{t|t-1} H + R$$
- Let $\Sigma_{t|t-1}$, then we compute: $\Phi_{t|t-1} = E \left((z_t - z_{t|t-1}) (x_t - x_{t|t-1})' | z^{t-1} \right)$
$$= E \left(H' (x_t - x_{t|t-1}) (x_t - x_{t|t-1})' + v_t (x_t - x_{t|t-1})' | z^{t-1} \right) = H' \Sigma_{t|t-1}$$
- Let $\Sigma_{t|t-1}$, then we compute $K_t = \Sigma_{t|t-1} H (H' \Sigma_{t|t-1} H + R)^{-1}$.
- Let $\Sigma_{t|t-1}$, $x_{t|t-1}$, K_t , and z_t then we compute $x_{t|t} = x_{t|t-1} + K_t (z_t - H' x_{t|t-1})$

The Algorithm

- Let $\Sigma_{t|t-1}$ and K_t , then we compute: $\Sigma_{t|t} = E \left((x_t - x_{t|t}) (x_t - x_{t|t})' | z^t \right)$
$$= E \begin{pmatrix} (x_t - x_{t|t-1}) (x_t - x_{t|t-1})' - \\ (x_t - x_{t|t-1}) (z_t - H' x_{t|t-1})' K_t' - \\ K_t (z_t - H' x_{t|t-1}) (x_t - x_{t|t-1})' + \\ K_t (z_t - H' x_{t|t-1}) (z_t - H' x_{t|t-1})' K_t' | z^t \end{pmatrix} = \Sigma_{t|t-1} - K_t H' \Sigma_{t|t-1}$$
- Let $\Sigma_{t|t}$, then we compute $\Sigma_{t+1|t} = F \Sigma_{t|t} F' + G Q G'$
- Let $x_{t|t}$, then we compute:
 - $x_{t+1|t} = F x_{t|t}$
 - $z_{t+1|t} = H' x_{t+1|t}$
- Therefore, from $x_{t|t-1}$, $\Sigma_{t|t-1}$, and z_t we compute $x_{t|t}$ and $\Sigma_{t|t}$.
- We also compute $z_{t|t-1}$, $\Omega_{t|t-1}$, and $\Phi_{t|t-1}$. Why for? As you will see later, they are necessary to calculate the likelihood function of $z^T = \{z_t\}_{t=1}^T$ (to follow).

Summary

We start with $x_{t|t-1}$ and $\Sigma_{t|t-1}$, the we observe z_t and....

- $\Omega_{t|t-1} = H'\Sigma_{t|t-1}H + R$
- $z_{t|t-1} = H'x_{t|t-1}$
- $K_t = \Sigma_{t|t-1}H(H'\Sigma_{t|t-1}H + R)^{-1}$
- $\Sigma_{t|t} = \Sigma_{t|t-1} - K_tH'\Sigma_{t|t-1}$
- $x_{t|t} = x_{t|t-1} + K_t(z_t - H'x_{t|t-1})$
- $\Sigma_{t+1|t} = F\Sigma_{t|t}F' + GQG'$
- $x_{t+1|t} = Fx_{t|t}$

We finish with $x_{t+1|t}$ and $\Sigma_{t+1|t}$.

A Probabilistic Approach to K_t

- Assume

$$Z|w = [X'|_w \ Y'|_w]' \sim N \left(\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \right)$$

- then

$$X|y, w \sim N \left(\bar{x} + \Sigma_{xy} \Sigma_{yy}^{-1} (y - \bar{y}), \Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx} \right)$$

- But $x_{t|t-1} = E(x_t|z^{t-1})$ and $\Sigma_{t|t-1} = E((x_t - x_{t|t-1})(x_t - x_{t|t-1})' | z^{t-1})$
- If $z_t|z^{t-1}$ is the random variable z_t (observable) conditional on z^{t-1} , then:

- Let $z_{t|t-1} = H' x_{t|t-1}$
- Let $\Omega_{t|t-1} = H' \Sigma_{t|t-1} H + R$
- Let $\Phi_{t|t-1} = H' \Sigma_{t|t-1}$

A Probabilistic Approach to K_t

- Assume we know $x_{t|t-1}$ and $\Sigma_{t|t-1}$, then

$$\begin{pmatrix} x_t \\ z_t \end{pmatrix} | z^{t-1} \sim N \left(\begin{bmatrix} x_{t|t-1} \\ H'x_{t|t-1} \end{bmatrix}, \begin{bmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1}H \\ H'\Sigma_{t|t-1} & H'\Sigma_{t|t-1}H + R \end{bmatrix} \right)$$

- Remember that:

$$X|y, w \sim N(\bar{x} + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \bar{y}), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})$$

- Then, we can write $x_t|z_t, z^{t-1} = x_t|z^t \sim N(x_{t|t}, \Sigma_{t|t})$, where

$$x_{t|t} = x_{t|t-1} + \underbrace{\Sigma_{t|t-1}H(H'\Sigma_{t|t-1}H + R)^{-1}}_{K_t}(z_t - H'x_{t|t-1})$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \underbrace{\Sigma_{t|t-1}H(H'\Sigma_{t|t-1}H + R)^{-1}H'\Sigma_{t|t-1}}_{K_t}$$