

# Metropolis–Hastings Homework: Bivariate Normal Sampling

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## Objective

In this homework, you will implement a Metropolis–Hastings (MH) algorithm to draw from a **bivariate normal** distribution

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \mathbf{\Sigma}\right),$$

where  $\mu_1, \mu_2$  and  $\mathbf{\Sigma}$  are known parameters. You will use a **flat prior** on  $\mathbf{\Sigma}$  restricted to the set of **symmetric positive definite** matrices.

## Model Setup

- Let the true parameters be:

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}.$$

- The target density is:

$$f(\mathbf{x}) \propto \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right) 1_{\mathbb{S}(2)}(\mathbf{\Sigma}).$$

where  $\mathbb{S}(2)$  is the set of definite positive matrices of dimension two.

- You will construct a Metropolis–Hastings chain  $\{\mathbf{x}_t\}_{t=1}^N$  whose stationary distribution is this  $f(\mathbf{x})$ .

## Flat Prior on Covariance Matrix

Although the target  $\mathbf{\Sigma}$  is known for this exercise, assume in your setup that the prior on  $\mathbf{\Sigma}$  is **flat** over the set:

$$\mathcal{S} = \{\mathbf{\Sigma} \in \mathbb{R}^{2 \times 2} : \mathbf{\Sigma} = \mathbf{\Sigma}', \mathbf{\Sigma} \succ 0\}.$$

This means you only accept covariance matrices that are symmetric positive definite (SPD). You may check SPD by verifying that  $\Sigma$  has positive eigenvalues.

## Algorithm

1. Initialize the chain at an arbitrary point  $\mathbf{x}_0$ .

2. At each iteration  $t$ :

(a) Propose a new draw:

$$\mathbf{x}^* = \mathbf{x}_{t-1} + \boldsymbol{\eta}, \quad \boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, c^2 I_2),$$

where  $c > 0$  controls the proposal variance.

(b) Compute the acceptance probability:

$$\alpha(\mathbf{x}_{t-1}, \mathbf{x}^*) = \min \left\{ 1, \frac{f(\mathbf{x}^*)}{f(\mathbf{x}_{t-1})} \right\}.$$

(c) Draw  $u \sim \text{Uniform}[0, 1]$ .

(d) If  $u \leq \alpha(\mathbf{x}_{t-1}, \mathbf{x}^*)$ , accept the candidate and set  $\mathbf{x}_t = \mathbf{x}^*$ ; otherwise set  $\mathbf{x}_t = \mathbf{x}_{t-1}$ .

3. Repeat steps 2(a)–2(d) for  $N$  iterations.

## Questions

1. Write the mathematical expression for the log target density  $\log f(\mathbf{x})$  (up to a constant).
2. Write the Metropolis–Hastings acceptance probability explicitly in terms of  $\boldsymbol{\mu}$  and  $\Sigma$ .
3. Implement the MH algorithm in **MATLAB** to draw  $N = 10,000$  samples from  $f(\mathbf{x})$ .
4. Plot the trajectory of the chain and the histogram of each marginal distribution.
5. Compute the empirical mean and covariance of your draws, and compare them with the true  $\boldsymbol{\mu}$  and  $\Sigma$ .
6. Experiment with different proposal variances  $c^2$  (e.g.,  $c = 0.1, 0.5, 1, 2$ ). How does it affect the acceptance rate and mixing?