Importance Sampling Homework

Model Setup

We will study the performance of Importance Sampling (IS) when the proposal distribution is not well suited for the target distribution.

- Target density: $\pi(\theta)$ follows a Student-t_ν distribution with degrees of freedom $\nu \in \{3, 4, 10, 100\}$.
- Proposal density: $g(\theta)$ is a standard normal distribution $N(0, 1)$.
- Function of interest: $h(\theta) = \theta$, so that the goal is to estimate $\mathbb{E}_\pi[\theta]$.
- Importance Sampling weights:
$$\omega(\theta) = \frac{\pi(\theta)}{g(\theta)} = \frac{t_\nu(\theta)}{\phi(\theta)},$$
where $t_\nu(\cdot)$ is the Student-t pdf and $\phi(\cdot)$ is the standard normal pdf.

Questions

1. For each $\nu = 3, 4, 10, 100$, draw $m = 10{,}000$ samples $\theta_j \sim N(0, 1)$.
2. Compute the IS estimate of the mean:
$$\widehat{\mathbb{E}}[\theta] = \frac{1}{m} \sum_{j=1}^{m} \theta_j \, \omega(\theta_j).$$
3. Compute the estimated variance of the IS mean estimator:
$$\widehat{\mathrm{Var}}(\widehat{\mathbb{E}}[\theta]) = \frac{1}{m} \sum_{j=1}^{m} \left(\theta_j - \widehat{\mathbb{E}}[\theta]\right)^2 \omega(\theta_j)^2.$$
4. Report your results in the following table:

ν	3	4	10	100
Estimated Mean				
Estimated Var(Mean)				

5. Discuss briefly why the estimated variance is extremely large for small ν, and what this says about using a light-tailed proposal (Normal) for a heavy-tailed target (Student-t).