Perturbation Methods

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- How?

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- Very successful in physics.

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- In particular, second order approximations are easy to compute and improve accuracy notably.
- Perturbation theory is the generalization of the well-known linearization strategy.
- Sometimes perturbation is known as asymptotic methods.

Regular versus Singular Perturbations

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- Regular perturbation: a *small* change in the problem induces a *small* change in the solution.
- Singular perturbation: a *small* change in the problem induces a *large* change in the solution.
- Most problems in economics involve regular perturbations. Sometimes, however, we can have singularities (for example by introducing a new asset in an incomplete market model).

A Baby Example: A Basic RBC

$$\max_{c_t, k_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} + (1 - \delta)\, k_t$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t, \;\; \varepsilon_t \sim \mathcal{N}(0, 1)$$

FOC of this problem

$$\frac{1}{c_t} = \beta \mathbb{E}_t \frac{1}{c_{t+1}} \left(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} - \delta\right)$$

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- Solve the new problem for a particular choice of the perturbation parameter.
- This step is usually ambiguous since there are different ways to do so.
- Use the previous solution to approximate the solution of original the problem.

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- Hence, we want to transform the problem.
- Which perturbation parameter? Standard deviation σ.
- Set $\sigma = 0 \Rightarrow$ deterministic model, $z_t = 0$ and $e^{z_t} = 1$.
- We know how to solve the deterministic steady state.

A Re-parametrized Policy Function

- We search for policy function

$$c_t = c\left(k_t, z_t; \sigma\right)$$

and

$$k_{t+1} = k\left(k_t, z_t; \sigma\right)$$

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- We are building a local approximation around $\sigma = 0$.

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$$\mathbb{E}_t \left(\frac{1}{c\left(k_t, z_t; \sigma\right)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k\left(k_t, z_t; \sigma\right)^{\alpha-1}}{c\left(k\left(k_t, z_t; \sigma\right), \rho z_t + \sigma \varepsilon_{t+1}; \sigma\right)} \right) = 0$$

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- How do you do it in logs?

Important Question

- Is $c(k_t, z_t; \sigma)|_{k,0,0}$ different from $c(k_t, z_t; \sigma)|_{k,0,1}$?

Asymptotic Expansion

$c_t = c\left(k_t, z_t; \sigma\right)\big|_{k,0,0}$

$$
\begin{aligned}
c_t \;=\; & c\left(k,0;0\right) \\
& + c_k\left(k,0;0\right)\left(k_t - k\right) + c_z\left(k,0;0\right) z_t + c_\sigma\left(k,0;0\right)\sigma \\
& + \frac{1}{2} c_{kk}\left(k,0;0\right)\left(k_t - k\right)^2 + \frac{1}{2} c_{kz}\left(k,0;0\right)\left(k_t - k\right) z_t \\
& + \frac{1}{2} c_{k\sigma}\left(k,0;0\right)\left(k_t - k\right)\sigma + \frac{1}{2} c_{zk}\left(k,0;0\right) z_t\left(k_t - k\right) \\
& + \frac{1}{2} c_{zz}\left(k,0;0\right) z_t^2 + \frac{1}{2} c_{z\sigma}\left(k,0;0\right) z_t\sigma \\
& + \frac{1}{2} c_{\sigma k}\left(k,0;0\right)\sigma\left(k_t - k\right) + \frac{1}{2} c_{\sigma z}\left(k,0;0\right)\sigma z_t \\
& + \frac{1}{2} c_{\sigma\sigma}\left(k,0;0\right)\sigma^2 + \ldots
\end{aligned}
$$

Asymptotic Expansion

$$k_{t+1} = k\left(k_t, z_t; \sigma\right)\big|_{k,0,0}$$

$$\begin{aligned}
k_{t+1} = \; & k\left(k,0;0\right) \\
& + k_k\left(k,0;0\right)\left(k_t - k\right) + k_z\left(k,0;0\right)z_t + k_\sigma\left(k,0;0\right)\sigma \\
& + \frac{1}{2}k_{kk}\left(k,0;0\right)\left(k_t - k\right)^2 + \frac{1}{2}k_{kz}\left(k,0;0\right)\left(k_t - k\right)z_t \\
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\end{aligned}$$

Notation

- From now on, to save on notation, I will write

$$F\left(k_t, z_t; \sigma\right) = \mathbb{E}_t \left[\begin{array}{c} \frac{1}{c(k_t,z_t;\sigma)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t,z_t;\sigma)^{\alpha-1}}{c(k(k_t,z_t;\sigma),\rho z_t + \sigma \varepsilon_{t+1};\sigma)} \\ c\left(k_t, z_t; \sigma\right) + k\left(k_t, z_t; \sigma\right) - e^{z_t} k_t^{\alpha} \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

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- Note that:

$$\begin{gathered} F(k_t, z_t; \sigma) \\ = \mathbb{E}_t \mathcal{H}(c_t, c_{t+1}, k_t, k_{t+1}, z_t, z_{t+1}; \sigma) \\ = \mathbb{E}_t \mathcal{H}\left(\begin{array}{c} c(k_t, z_t; \sigma), c(k(k_t, z_t; \sigma), \rho z_t + \sigma \varepsilon_{t+1}; \sigma), \\ k_t, k(k_t, z_t; \sigma), z_t, \rho z_t + \sigma \varepsilon_{t+1} \end{array} \right) \end{gathered}$$

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- $\mathcal{H}_i$ represents the derivative of $\mathcal{H}$ with respect to the *ith* component

Zero-Order Approximation

- First, we evaluate $k_t = k$, $z_t = 0$, and $\sigma = 0$:

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- How good is this approximation?

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- With respect to σ:

$$F_\sigma(k, 0; 0) = 0$$

Solving the System I

- Remember that:

$$F\left(k_t, z_t; \sigma\right) =$$

$$\mathbb{E}_t \mathcal{H}\left(\begin{array}{c} c\left(k_t, z_t; \sigma\right), c\left(k\left(k_t, z_t; \sigma\right), \rho z_t + \sigma \varepsilon_{t+1}; \sigma\right), \\ k_t, k\left(k_t, z_t; \sigma\right), z_t, \rho z_t + \sigma \varepsilon_{t+1} \end{array}\right) = 0$$

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- Then:

$$F_k\left(k, 0; 0\right) = \mathcal{H}_1 c_k + \mathcal{H}_2 c_k k_k + \mathcal{H}_3 + \mathcal{H}_4 k_k = 0$$

$$F_z\left(k, 0; 0\right) = \mathcal{H}_1 c_z + \mathcal{H}_2\left(c_k k_z + c_z \rho\right) + \mathcal{H}_4 k_z + \mathcal{H}_5 + \mathcal{H}_6 \rho = 0$$

$$F_\sigma\left(k, 0; 0\right) = \mathcal{H}_1 c_\sigma + \mathcal{H}_2\left(c_k k_\sigma + c_z \mathbb{E}_t \varepsilon_{t+1} + c_\sigma\right) + \mathcal{H}_4 k_\sigma + \mathcal{H}_6 \mathbb{E}_t \varepsilon_{t+1}$$
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  is a quadratic system of four equations on four unknowns: c_k, c_z, k_k, and k_z.

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- Does not depend on c_σ or k_σ!!!!!
- Procedures to solve quadratic systems: Blanchard and Kahn (1980), Uhlig (1999), Sims (2000), and Klein (2000). All of them equivalent.
- Why quadratic? Stable and unstable manifold.

How to do it?

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$$\begin{pmatrix} H_4 & H_6 & H_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{pmatrix} \begin{pmatrix} k_k \\ 0 \end{pmatrix} = - \begin{pmatrix} H_3 & H_5 & H_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ c_k \end{pmatrix}$$

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How to do it?

- or

$$\begin{pmatrix} H_4 & H_6 & H_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{pmatrix} \begin{pmatrix} k_k & k_z \\ 0 & \rho \end{pmatrix} =$$

$$-\begin{pmatrix} H_3 & H_5 & H_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{pmatrix}$$

- Now let is try to find

$$h_x = \begin{pmatrix} k_k & k_z \\ 0 & \rho \end{pmatrix} \text{ and } \begin{pmatrix} c_k & c_z \end{pmatrix}$$

How to do it?

- Let Λ and P be the matrix of eigenvalues and eigenvectors of h_x respectively.

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- Hence

$$\begin{pmatrix} H_4 & H_6 & H_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{pmatrix} P\Lambda =$$

$$-\begin{pmatrix} H_3 & H_5 & H_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{pmatrix} P$$

How to do it?

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$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{pmatrix} P$$

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How to do it?

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- and

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- We have

$$AZ\Lambda = BZ$$

Generalized Eigenvalue Problem

- We can then map the above problem into a generalized eigenvalue problem.
- For given matrices A and B, there exists a matrix V and a diagonal matrix D such that

$$A \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} = B \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

- Where D_{11} has all the roots with absolute value less than one.

Generalized Eigenvalue Problem

- Assume now that the number of eigenvalues with absolute value less than one is equal to the number of states.
- Then we can say $\Lambda = D_{11}$ and $Z = V_1$.
- Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{pmatrix} P = V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}$$

- or

$$P = V_{11}$$

$$\begin{pmatrix} c_k & c_z \end{pmatrix} = V_{21} P^{-1}$$

and

$$h_x = P D_{11} P^{-1}$$

Solving the System III

- Now that we have c_k, c_z, k_k, and k_z.
- We need c_σ and k_σ.
- Note that, given, c_k, c_z, k_k, and k_z:

$$F_\sigma (k, 0; 0) = \mathcal{H}_1 c_\sigma + \mathcal{H}_2 (c_k k_\sigma + c_\sigma) + \mathcal{H}_4 k_\sigma = 0$$

  is a linear, and homogeneous system in c_σ and k_σ.
- This means

$$\left(\begin{array}{cc} \mathcal{H}_1 + \mathcal{H}_2 & \mathcal{H}_2 c_k + \mathcal{H}_4 \end{array} \right) \left(\begin{array}{c} c_\sigma \\ k_\sigma \end{array} \right) = 0$$

- Hence:

$$c_\sigma = k_\sigma = 0$$

Interpretation

- Since $c_\sigma = k_\sigma = 0$, the system is certainty equivalent.
- Interpretation$\Rightarrow$no precautionary behavior.
- Difference between risk-aversion and precautionary behavior. Leland (1968), Kimball (1990).
- Risk-aversion depends on the second derivative (concave utility).
- Precautionary behavior depends on the third derivative (convex marginal utility).
- Also, if we forget about numerical errors, there is not approximation errors in this solution.
- Value function has approximation errors.

Comparison with Linearization

- After Kydland and Prescott (1982) a popular method to solve economic models has been the use of a LQ approximation.
- Close relative: linearization of equilibrium conditions.
- When properly implemented linearization, LQ, and first-order perturbation are equivalent.
- Advantages of linearization:
  1. Theorems.
  2. Higher order terms.

Second-Order Approximation

- We take second-order derivatives of $F(k_t, z_t; \sigma)$ evaluated at k, 0, and 0:

$$
\begin{aligned}
F_{kk}(k, 0; 0) &= 0 \\
F_{kz}(k, 0; 0) &= 0 \\
F_{k\sigma}(k, 0; 0) &= 0 \\
F_{zz}(k, 0; 0) &= 0 \\
F_{z\sigma}(k, 0; 0) &= 0 \\
F_{\sigma\sigma}(k, 0; 0) &= 0
\end{aligned}
$$

- Remember Young's theorem!

Solving the System

- We substitute the coefficients that we already know.
- A linear system of 12 equations on 12 unknowns. Why linear?
- Cross-terms $k\sigma$ and $z\sigma$ are zero.
- Conjecture on all the terms with odd powers of σ.

Correction for Risk

- We have a term in σ^2.
- Captures precautionary behavior.
- We do not have certainty equivalence any more!
- Important advantage of second order approximation.

Higher Order Terms

- We can continue the iteration for as long as we want.
- Great advantage of procedure: it is recursive!
- Often, a few iterations will be enough.
- The level of accuracy depends on the goal of the exercise: Fernández-Villaverde, Rubio-Ramírez, and Santos (2006).