

Gibbs Sampling

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Gibbs Sampling — Systematic Scan

Let $x^{(1)} = (x_1^{(1)}, \dots, x_d^{(1)})$ be initial. For $t = 2, 3, \dots$

$$\text{Sample } X_1^{(t)} \sim \pi(x_1 \mid X_2^{(t-1)}, \dots, X_d^{(t-1)}),$$

$$\vdots$$

$$\text{Sample } X_j^{(t)} \sim \pi(x_j \mid X_{1:j-1}^{(t)}, X_{j+1:d}^{(t-1)}),$$

$$\vdots$$

$$\text{Sample } X_d^{(t)} \sim \pi(x_d \mid X_{1:d-1}^{(t)}).$$

Questions: Do conditionals define a unique joint π ? Is π invariant? Does the chain converge?

For $d = 2$, the Gibbs kernel is

$$K(x^{(t-1)}, x^{(t)}) = \pi(x_1^{(t)} \mid x_2^{(t-1)}) \pi(x_2^{(t)} \mid x_1^{(t)}).$$

Proposition. Systematic-scan Gibbs admits π as invariant distribution.

We have

$$\begin{aligned}\int K(x, y) \pi(x) dx &= \int \pi(y_2 | y_1) \pi(y_1 | x_2) \pi(x_1, x_2) dx_1 dx_2 \\ &= \pi(y_2 | y_1) \int \pi(y_1 | x_2) \pi(x_2) dx_2 \\ &= \pi(y_2 | y_1) \pi(y_1) = \pi(y_1, y_2) = \pi(y).\end{aligned}$$

Example: Bivariate Normal

Let $X = (X_1, X_2) \sim \mathcal{N}(\mu, \Sigma)$, $\mu = (\mu_1, \mu_2)$,

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{pmatrix}.$$

Gibbs updates:

$$X_1^{(t)} \sim \mathcal{N}\left(\mu_1 + \frac{\rho}{\sigma_2^2}(X_2^{(t-1)} - \mu_2), \sigma_1^2 - \frac{\rho^2}{\sigma_2^2}\right),$$
$$X_2^{(t)} \sim \mathcal{N}\left(\mu_2 + \frac{\rho}{\sigma_1^2}(X_1^{(t)} - \mu_1), \sigma_2^2 - \frac{\rho^2}{\sigma_1^2}\right).$$

Large $|\rho| \Rightarrow$ slow mixing.