

# Importance Sampling Homework

## Model Setup

We will study the performance of Importance Sampling (IS) when the proposal distribution is not well suited for the target distribution.

- Target density:  $\pi(\theta)$  follows a Student- $t_\nu$  distribution with degrees of freedom  $\nu \in \{3, 4, 10, 100\}$ .
- Proposal density:  $g(\theta)$  is a standard normal distribution  $N(0, 1)$ .
- Function of interest:  $h(\theta) = \theta$ , so that the goal is to estimate  $\mathbb{E}_\pi[\theta]$ .
- Importance Sampling weights:

$$\omega(\theta) = \frac{\pi(\theta)}{g(\theta)} = \frac{t_\nu(\theta)}{\phi(\theta)},$$

where  $t_\nu(\cdot)$  is the Student- $t$  pdf and  $\phi(\cdot)$  is the standard normal pdf.

## Questions

1. For each  $\nu = 3, 4, 10, 100$ , draw  $m = 10,000$  samples  $\theta_j \sim N(0, 1)$ .
2. Compute the IS estimate of the mean:

$$\widehat{\mathbb{E}}[\theta] = \frac{1}{m} \sum_{j=1}^m \theta_j \omega(\theta_j).$$

3. Compute the estimated variance of the IS mean estimator:

$$\widehat{\text{Var}}(\widehat{\mathbb{E}}[\theta]) = \frac{1}{m} \sum_{j=1}^m \left( \theta_j - \widehat{\mathbb{E}}[\theta] \right)^2 \omega(\theta_j)^2.$$

4. Report your results in the following table:

$\nu$	3	4	10	100
Estimated Mean				
Estimated Var(Mean)				

5. Discuss briefly why the estimated variance is extremely large for small  $\nu$ , and what this says about using a light-tailed proposal (Normal) for a heavy-tailed target (Student- $t$ ).