

# Normal–Inverse–Wishart Distribution

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# Inverse Wishart and Matrix Gaussian Distributions

For  $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$ , positive definite matrices. Parameters: degrees of freedom  $\nu > n - 1$ , and scale matrix  $\mathbf{\Psi} > 0$ .

$$f_{IW}(\mathbf{\Sigma}; \nu, \mathbf{\Psi}) \propto |\mathbf{\Sigma}|^{-\frac{\nu+n+1}{2}} \exp\left(-\frac{1}{2}\text{tr}(\mathbf{\Psi} \mathbf{\Sigma}^{-1})\right),$$

For  $\mathbf{B} \in \mathbb{R}^{m \times n}$ , if  $\mathbf{\Sigma}$  and  $\mathbf{\Omega}$  are p.d. matrices of dimensions  $n$  and  $m$  respectively,

$$f_{MG}(\mathbf{B}; \mu, \mathbf{\Sigma} \otimes \mathbf{\Omega}) \propto |\mathbf{\Sigma}|^{-\frac{m}{2}} \exp\left[-\frac{1}{2}\text{tr}\left((\mathbf{B} - \mu)' \mathbf{\Omega}^{-1}(\mathbf{B} - \mu) \mathbf{\Sigma}^{-1}\right)\right].$$

# Vector Autoregression (VAR) Setup

Let with

$$\mathbf{x}'_t = [\mathbf{y}'_{t-1} \dots \mathbf{y}'_{t-p} \ 1] \text{ and } \mathbf{u}_t \sim \mathcal{N}(0, \mathbf{\Sigma}).$$

Reduced form:

$$\mathbf{y}'_t = \mathbf{x}'_t \mathbf{B} + \mathbf{u}'_t$$

## Reduced-Form Likelihood

$$p(\mathbf{Y} \mid \mathbf{B}, \mathbf{\Sigma}) \propto |\mathbf{\Sigma}|^{-\frac{T}{2}} \exp \left[ -\frac{1}{2} \sum_{t=1}^T (\mathbf{y}'_t - \mathbf{x}'_t \mathbf{B}) \mathbf{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{B}' \mathbf{x}_t) \right].$$

Using

$$\mathbf{Y}' = [\mathbf{y}_1 \dots \mathbf{y}_T], \quad \mathbf{X}' = [\mathbf{x}_1 \dots \mathbf{x}_T],$$

we get

$$p(\mathbf{Y} \mid \mathbf{B}, \mathbf{\Sigma}) \propto |\mathbf{\Sigma}|^{-\frac{T}{2}} \exp \left[ -\frac{1}{2} \text{tr}((\mathbf{Y} - \mathbf{X} \mathbf{B})' (\mathbf{Y} - \mathbf{X} \mathbf{B}) \mathbf{\Sigma}^{-1}) \right].$$

## Reduced-Form Likelihood

The likelihood of the data given parameters  $(\mathbf{B}, \mathbf{\Sigma})$  is:

$$p(\mathbf{Y} \mid \mathbf{B}, \mathbf{\Sigma}) = \prod_{t=1}^T p(\mathbf{y}_t \mid \mathbf{x}_t, \mathbf{B}, \mathbf{\Sigma}).$$

Since  $\mathbf{y}_t \mid \mathbf{x}_t, \mathbf{B}, \mathbf{\Sigma} \sim \mathcal{N}(\mathbf{B}' \mathbf{x}_t, \mathbf{\Sigma})$ ,

$$p(\mathbf{y}_t \mid \mathbf{x}_t, \mathbf{B}, \mathbf{\Sigma}) \propto |\mathbf{\Sigma}|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\mathbf{y}_t' - \mathbf{x}_t' \mathbf{B}) \mathbf{\Sigma}^{-1}(\mathbf{y}_t - \mathbf{B}' \mathbf{x}_t)\right].$$

Combining terms over  $t = 1, \dots, T$ :

$$p(\mathbf{Y} \mid \mathbf{B}, \mathbf{\Sigma}) \propto |\mathbf{\Sigma}|^{-\frac{T}{2}} \exp\left[-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t' - \mathbf{x}_t' \mathbf{B}) \mathbf{\Sigma}^{-1}(\mathbf{y}_t - \mathbf{B}' \mathbf{x}_t)\right].$$

## Algebra: Expanding the Likelihood Trace Term

We start from the log-likelihood quadratic form:

$$\sum_{t=1}^T (\mathbf{y}'_t - \mathbf{x}'_t \mathbf{B}) \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{B}' \mathbf{x}_t) = \text{tr} \left( \sum_{t=1}^T (\mathbf{y}_t - \mathbf{B}' \mathbf{x}_t) (\mathbf{y}'_t - \mathbf{x}'_t \mathbf{B}) \boldsymbol{\Sigma}^{-1} \right)$$

where we have used  $\text{tr}(\mathbf{X} \mathbf{Y}) = \text{tr}(\mathbf{Y} \mathbf{X})$ , then

$$\sum_{t=1}^T (\mathbf{y}'_t - \mathbf{x}'_t \mathbf{B}) \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{B}' \mathbf{x}_t) = \text{tr}((\mathbf{Y} - \mathbf{X} \mathbf{B})' (\mathbf{Y} - \mathbf{X} \mathbf{B}) \boldsymbol{\Sigma}^{-1})$$

where we have used  $\mathbf{X}' \mathbf{X} = \sum_{i=1}^n \mathbf{x}_i \mathbf{x}'_i$  and  $\mathbf{X}' = [\mathbf{x}_1, \dots, \mathbf{x}_n]$ .

## Likelihood Simplification

Rewrite in matrix form:

$$\begin{aligned}\text{tr}[(\mathbf{Y} - \mathbf{X}\mathbf{B})'(\mathbf{Y} - \mathbf{X}\mathbf{B})\mathbf{\Sigma}^{-1}] &= \text{tr}\left[\left(\hat{\mathbf{\Psi}} + (\mathbf{B} - \hat{\boldsymbol{\mu}})' \hat{\mathbf{\Omega}}^{-1}(\mathbf{B} - \hat{\boldsymbol{\mu}})\right) \mathbf{\Sigma}^{-1}\right] = \\ &\text{tr}\left[\hat{\mathbf{\Psi}} \mathbf{\Sigma}^{-1}\right] + \text{tr}\left[(\mathbf{B} - \hat{\boldsymbol{\mu}})' \hat{\mathbf{\Omega}}^{-1}(\mathbf{B} - \hat{\boldsymbol{\mu}}) \mathbf{\Sigma}^{-1}\right]\end{aligned}$$

Define:

$$\hat{\mathbf{\Omega}} = (\mathbf{X}'\mathbf{X})^{-1}, \quad \hat{\boldsymbol{\mu}} = \hat{\mathbf{\Omega}}\mathbf{X}'\mathbf{Y}, \quad \hat{\mathbf{\Psi}} = \mathbf{Y}'\mathbf{Y} - \hat{\boldsymbol{\mu}}'\hat{\mathbf{\Omega}}^{-1}\hat{\boldsymbol{\mu}}.$$

Then:

$$p(\mathbf{Y} | \mathbf{B}, \mathbf{\Sigma}) \propto |\mathbf{\Sigma}|^{-\frac{T}{2}} \exp\left[-\frac{1}{2}\text{tr}\left(\hat{\mathbf{\Psi}} \mathbf{\Sigma}^{-1}\right) - \frac{1}{2}\text{tr}\left((\mathbf{B} - \hat{\boldsymbol{\mu}})' \hat{\mathbf{\Omega}}^{-1}(\mathbf{B} - \hat{\boldsymbol{\mu}}) \mathbf{\Sigma}^{-1}\right)\right].$$

Degrees of freedom. We need  $|\mathbf{\Sigma}|^{-\frac{m}{2}}$  for the normal, hence we have  $|\mathbf{\Sigma}|^{-\frac{T-m}{2}} |\mathbf{\Sigma}|^{-\frac{m}{2}}$ .

This means  $\hat{v} + n + 1 = T - m$ , hence  $\hat{v} = T - m - n - 1$ .

## Gaussian Inverse Wishart Prior

A likelihood:

$$p(\mathbf{Y} \mid \mathbf{B}, \boldsymbol{\Sigma}) = f_{IW}(\boldsymbol{\Sigma}; \hat{\nu}, \hat{\boldsymbol{\Psi}}) f_{MG}(\mathbf{B}; \hat{\boldsymbol{\mu}}, \boldsymbol{\Sigma} \otimes \hat{\boldsymbol{\Omega}}).$$

A conjugate prior:

$$p(\mathbf{B}, \boldsymbol{\Sigma}) = f_{IW}(\boldsymbol{\Sigma}; \bar{\nu}, \bar{\boldsymbol{\Psi}}) f_{MG}(\mathbf{B}; \bar{\boldsymbol{\mu}}, \boldsymbol{\Sigma} \otimes \bar{\boldsymbol{\Omega}}).$$

Implied posterior:

$$p(\mathbf{B}, \boldsymbol{\Sigma} \mid \mathbf{Y}) = f_{IW}(\boldsymbol{\Sigma}; \tilde{\nu}, \tilde{\boldsymbol{\Psi}}) f_{MG}(\mathbf{B}; \tilde{\boldsymbol{\mu}}, \boldsymbol{\Sigma} \otimes \tilde{\boldsymbol{\Omega}}),$$

where

$$\begin{aligned} \tilde{\nu} &= T + \bar{\nu}, \quad \tilde{\boldsymbol{\Omega}} = (\hat{\boldsymbol{\Omega}}^{-1} + \bar{\boldsymbol{\Omega}}^{-1})^{-1}, \quad \tilde{\boldsymbol{\mu}} = \tilde{\boldsymbol{\Omega}}(\hat{\boldsymbol{\Omega}}^{-1}\hat{\boldsymbol{\mu}} + \bar{\boldsymbol{\Omega}}^{-1}\bar{\boldsymbol{\mu}}), \\ \tilde{\boldsymbol{\Psi}} &= \hat{\boldsymbol{\Psi}} + \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Omega}}^{-1}\hat{\boldsymbol{\mu}} + \bar{\boldsymbol{\Psi}} + \bar{\boldsymbol{\mu}}'\bar{\boldsymbol{\Omega}}^{-1}\bar{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}'\tilde{\boldsymbol{\Omega}}^{-1}\tilde{\boldsymbol{\mu}}. \end{aligned}$$

## Completing the Square in the Matrix Case

Let  $\mathbf{\Lambda}_i = \mathbf{\Omega}_i^{-1}$  (symmetric p.d.). Then

$$(\mathbf{B} - \boldsymbol{\mu}_1)' \mathbf{\Lambda}_1 (\mathbf{B} - \boldsymbol{\mu}_1) + (\mathbf{B} - \boldsymbol{\mu}_2)' \mathbf{\Lambda}_2 (\mathbf{B} - \boldsymbol{\mu}_2) =$$

$$\mathbf{B}' (\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \mathbf{B} - 2 \mathbf{B}' (\mathbf{\Lambda}_1 \boldsymbol{\mu}_1 + \mathbf{\Lambda}_2 \boldsymbol{\mu}_2) + \boldsymbol{\mu}_1' \mathbf{\Lambda}_1 \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2' \mathbf{\Lambda}_2 \boldsymbol{\mu}_2$$

Choose  $\mathbf{\Lambda}^* = \mathbf{\Lambda}_1 + \mathbf{\Lambda}_2$  and  $\boldsymbol{\mu}^* = (\mathbf{\Lambda}^*)^{-1} (\mathbf{\Lambda}_1 \boldsymbol{\mu}_1 + \mathbf{\Lambda}_2 \boldsymbol{\mu}_2)$ . Then

$$(\mathbf{B} - \boldsymbol{\mu}^*)' \mathbf{\Lambda}^* (\mathbf{B} - \boldsymbol{\mu}^*) = \mathbf{B}' \mathbf{\Lambda}^* \mathbf{B} - 2 \mathbf{B}' \mathbf{\Lambda}^* \boldsymbol{\mu}^* + (\boldsymbol{\mu}^*)' \mathbf{\Lambda}^* \boldsymbol{\mu}^*,$$

so  $(\mathbf{B} - \boldsymbol{\mu}_1)' \mathbf{\Lambda}_1 (\mathbf{B} - \boldsymbol{\mu}_1) + (\mathbf{B} - \boldsymbol{\mu}_2)' \mathbf{\Lambda}_2 (\mathbf{B} - \boldsymbol{\mu}_2) = (\mathbf{B} - \boldsymbol{\mu}^*)' \mathbf{\Lambda}^* (\mathbf{B} - \boldsymbol{\mu}^*) + \mathbf{C}$ , with  $\mathbf{C} = \boldsymbol{\mu}_1' \mathbf{\Lambda}_1 \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2' \mathbf{\Lambda}_2 \boldsymbol{\mu}_2 - (\boldsymbol{\mu}^*)' \mathbf{\Lambda}^* \boldsymbol{\mu}^*$ .

# The Degrees of Freedom

Finally, collect the powers of  $|\mathbf{\Sigma}|$ . The product has

$$-\frac{\nu_1 + n + 1}{2} - \frac{\nu_2 + n + 1}{2} - \frac{m}{2} - \frac{m}{2} = -\frac{\nu_1 + \nu_2 + (n + 1) + m + (n + 1) + m}{2}.$$

To express the result as one IW part and one matrix-normal part (i.e., one  $|\mathbf{\Sigma}|^{-m/2}$  kept with the matrix-normal), set

$$-\frac{\nu^* + n + 1}{2} - \frac{m}{2} = -\frac{\nu_1 + \nu_2 + n + 1 + m + n + 1}{2} - \frac{m}{2},$$

which gives the updated degrees of freedom  $\nu^* = \nu_1 + \nu_2 + n + 1 + m$ . In our case  $\nu_1 = \bar{\nu}$ ,  $\nu_2 = \hat{\nu} = T - m - n - 1$ , and  $\nu^* = \tilde{\nu}$ , hence  $\tilde{\nu} = T + \bar{\nu}$ .