The Metropolis–Hastings Algorithm

Juan F. Rubio-Ramírez

Emory University

Metropolis–Hastings Algorithm Definition

Let

$$P_{MH}(x, dy) = p_{MH}(x, y)dy + r_{MH}(x)\Upsilon_{\{x\}}(dy)$$

where

$$p_{MH}(x, y) = p(x, y)\alpha(x, y),$$

and

$$\alpha(x, y) = \min\left\{\frac{f(y)p(y, x)}{f(x)p(x, y)}, 1\right\},$$

and $p(x, y)$ is a density.

Corollary

$$\int_A f(y)\,dy = \int_{\mathbb{R}^d} P_{MH}(x, A) f(x)\,dx.$$

Proof.
We only need to show that

$$f(x) p_{MH}(x, y) = f(y) p_{MH}(y, x).$$

Assume without loss of generality that $\alpha(x, y) < 1 \Rightarrow \alpha(y, x) = 1$. Then

$$f(x) p_{MH}(x, y) = f(x) p(x, y) \alpha(x, y) = f(y) p(y, x) = f(y) p_{MH}(y, x).$$

$\square$

How Does it Work?

1. Initialize the algorithm with an arbitrary value x_0 and N.
2. Set $j = 1$.
3. Generate x_j^* from $p(x_{j-1}, x_j^*)$ and $u \sim \mathrm{Uniform}[0, 1]$.
4. If $u \leq \alpha(x_{j-1}, x_j^*)$, set $x_j = x_j^*$; otherwise $x_j = x_{j-1}$.
5. If $j \leq N$, set $j \leftarrow j + 1$ and go to step 3.

Remarks on Metropolis–Hastings

- We need to evaluate a function $g(x) \propto f(x)$.
- The algorithm is defined by $p(x, y)$.
- If the candidate is rejected, the current value becomes the next one.

Uphill and Downhill Moves

Assume $p(y, x) = p(x, y)$:

- If the jump is "uphill" $\left(\frac{f(y)}{f(x)} > 1\right)$, it is always accepted.
- If it goes "downhill" $\left(\frac{f(y)}{f(x)} < 1\right)$, it is accepted with nonzero probability.

Hence:

$$\alpha(x, y) = 1 \Rightarrow p_{MH}(x, y) = p(x, y) \Rightarrow r_{MH}(x) = 0,$$

$$\alpha(x, y) < 1 \Rightarrow p_{MH}(x, y) < p(x, y) \Rightarrow r_{MH}(x) > 0.$$

- Let $p_1(z)$ be a multivariate continuous density:

$$p_{MH}(x, y) = p_1(x - y).$$

- Let $p_2(z)$ be symmetric:

$$p_{MH}(x, y) = p_2(x - y) = p_2(y - x), \quad \alpha(x, y) = \min\left\{\frac{f(y)}{f(x)}, 1\right\}.$$

- Let $p_3(z)$ be a multivariate continuous density:

$$p_{MH}(x, y) = p_3(y).$$

Example: Approximating $\Phi(t)$

- Use MH to approximate $F(t)$, the standard normal CDF evaluated at t.
- Generate $\{x_j\}_{j=1}^{N}$ from $N(0,1)$ using MH.

Algorithm:

1 Initialize $x_0 = 0$, choose N, and set $j = 1$.

2 Generate $x_j^* = x_{j-1} + N(0, 2.5)$.

3 Compute $\alpha(x_{j-1}, x_j^*) = \min\{\phi(x_j^*)/\phi(x_{j-1}), 1\}$.

4 Draw $u \sim \mathrm{Uniform}[0,1]$.

5 If $u \leq \alpha$, set $x_j = x_j^*$; else $x_j = x_{j-1}$.

6 If $j \leq N$, repeat from step 2.

$$F(t) \approx \frac{1}{N} \sum_{i=1}^{N} \Upsilon_{\{x_i < x_t\}}(x_i).$$

Rate of Convergence

At what speed does the chain converge? How long should it run?

- Run multiple chains with different initial values and compare within/between variation.
- Check serial correlation of draws.
- Let N grow with the serial correlation.
- Run N chains of length M with random starts; use the last value of each chain.

Acceptance Ratio

If a candidate is rejected, the current value is retained. **Acceptance rate matters:**

- For normal models, the acceptance rate should be between 23% and 45% (Roberts, Gelman, and Gilks, 1994).

M–H Example: AR(2) Model

Simulate T observations from:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t = \mathbf{w}_t' \mathbf{\Phi} + \sigma \epsilon_t,$$

where $\phi_1 = 1$, $\phi_2 = -0.5$, $\epsilon_t \sim N(0,1)$, $\mathbf{w}_t = (y_{t-1}, y_{t-2})'$ and $\mathbf{\Phi} = (\phi_1, \phi_2)'$. The stationarity conditions are:

$$\mathcal{S} = \{\mathbf{\Phi} : \phi_1 + \phi_2 < 1,\ -\phi_1 + \phi_2 < 1,\ \phi_2 > -1\}$$

and the data is

$$\mathbf{Y} = (y_3, \ldots, y_T).$$

Likelihood, Priors, and Posterior

Likelihood (conditional on y_1, y_2):

$$\ell(\mathbf{Y}|\mathbf{\Phi}, \sigma, y_1, y_2) = (\sigma^2)^{-(T-2)/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{t=3}^{T}(y_t - \mathbf{w}_t'\mathbf{\Phi})^2\right).$$

Priors:

$$\mathbf{\Phi} \in \mathcal{S} \text{ and } \sigma \in \mathbb{R}_+.$$

Posterior:

$$\pi(\mathbf{\Phi}, \sigma|\mathbf{Y}, y_1, y_2) \propto \ell(\mathbf{Y}|\mathbf{\Phi}, \sigma, y_1, y_2) 1_{\mathcal{S}}(\mathbf{\Phi}) 1_{\mathbb{R}_+}(\sigma).$$