Metropolis–Hastings Homework: Bivariate Normal Sampling

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Objective

In this homework, you will implement a Metropolis–Hastings (MH) algorithm to draw from a **bivariate normal** distribution

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \mathbf{\Sigma} \right),$$

where μ_1, μ_2 and $\mathbf{\Sigma}$ are known parameters. You will use a **flat prior** on $\mathbf{\Sigma}$ restricted to the set of **symmetric positive definite** matrices.

Model Setup

- Let the true parameters be:

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}.$$

- The target density is:

$$f(\mathbf{x}) \propto \exp\bigl(-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\bigr)\, 1_{\mathbb{S}(2)}(\mathbf{\Sigma}).$$

  where $\mathbb{S}(2)$ is the set of definite positive matrices of dimension two.

- You will construct a Metropolis–Hastings chain $\{\mathbf{x}_t\}_{t=1}^{N}$ whose stationary distribution is this $f(\mathbf{x})$.

Flat Prior on Covariance Matrix

Although the target $\mathbf{\Sigma}$ is known for this exercise, assume in your setup that the prior on $\mathbf{\Sigma}$ is **flat** over the set:

$$\mathcal{S} = \left\{ \mathbf{\Sigma} \in \mathbb{R}^{2\times 2} : \mathbf{\Sigma} = \mathbf{\Sigma}',\ \mathbf{\Sigma} \succ 0 \right\}.$$

This means you only accept covariance matrices that are symmetric positive definite (SPD). You may check SPD by verifying that $\mathbf{\Sigma}$ has positive eigenvalues.

Algorithm

1. Initialize the chain at an arbitrary point $\mathbf{x}_0$.
2. At each iteration t:
   (a) Propose a new draw:
   $$\mathbf{x}^* = \mathbf{x}_{t-1} + \boldsymbol{\eta}, \quad \boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, c^2 I_2),$$
   where $c > 0$ controls the proposal variance.

   (b) Compute the acceptance probability:
   $$\alpha(\mathbf{x}_{t-1}, \mathbf{x}^*) = \min\left\{1, \frac{f(\mathbf{x}^*)}{f(\mathbf{x}_{t-1})}\right\}.$$
   (c) Draw $u \sim \text{Uniform}[0, 1]$.

   (d) If $u \leq \alpha(\mathbf{x}_{t-1}, \mathbf{x}^*)$, accept the candidate and set $\mathbf{x}_t = \mathbf{x}^*$; otherwise set $\mathbf{x}_t = \mathbf{x}_{t-1}$.
3. Repeat steps 2(a)–2(d) for N iterations.

Questions

1. Write the mathematical expression for the log target density $\log f(\mathbf{x})$ (up to a constant).
2. Write the Metropolis–Hastings acceptance probability explicitly in terms of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$.
3. Implement the MH algorithm in `MATLAB` to draw $N = 10{,}000$ samples from $f(\mathbf{x})$.
4. Plot the trajectory of the chain and the histogram of each marginal distribution.
5. Compute the empirical mean and covariance of your draws, and compare them with the true $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$.
6. Experiment with different proposal variances c^2 (e.g., $c = 0.1, 0.5, 1, 2$). How does it affect the acceptance rate and mixing?