Final

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Question 1: Use the MH Truncated Multivariate Normal in a Half-Space

Let $X = (X_1, X_2, X_3)^\top \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ with given $\boldsymbol{\mu} \in \mathbb{R}^3$ and positive definite $\Sigma \in \mathbb{R}^{3\times 3}$. Consider the half-space truncation

$$\mathcal{A} \;=\; \{x \in \mathbb{R}^3 : \; \mathbf{1}^\top x = x_1 + x_2 + x_3 > 0\}.$$

Your task is to generate draws from the *truncated* distribution $X \mid \{\mathbf{1}^\top X > 0\}$ and to compare two approaches.

Target density (up to a constant):

$$f(x) \;\propto\; \phi_3(x; \boldsymbol{\mu}, \Sigma)\, \mathbf{1}\{\mathbf{1}^\top x > 0\},$$

where $\phi_3(\cdot; \boldsymbol{\mu}, \Sigma)$ is the $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ density on $\mathbb{R}^3$.

What to do

(A) Direct MH for the truncated normal.

Implement a Metropolis–Hastings (MH) sampler that targets $\pi_{\text{trunc}}(x)$ directly using a *symmetric random-walk Gaussian* proposal

$$x' = x + s\,\varepsilon, \qquad \varepsilon \sim \mathcal{N}(\mathbf{0}, I_3),$$

where $s > 0$ is a step-size tuning parameter.

- If $x' \notin \mathcal{A}$, reject automatically.
- Otherwise, accept with probability $\min\{1, \phi_3(x'; \boldsymbol{\mu}, \Sigma)/\phi_3(x; \boldsymbol{\mu}, \Sigma)\}$ (you should implement this on the log scale).

Report: acceptance rate, trace plots of $\mathbf{1}^\top X^{(t)}$, and an estimate of $\mathbb{E}[X \mid \mathbf{1}^\top X > 0]$ with Monte Carlo standard errors.

(B) Untruncated MH + Accept/Reject (A/R).

First, run an MH sampler that targets the *untruncated* $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ (same random-walk proposal). Then apply a simple A/R filter that *keeps* only the draws with $\mathbf{1}^\top X > 0$.

Report: the empirical keep-rate (i.e., $\Pr(\mathbf{1}^\top X > 0)$ estimated from the chain), how many total MH iterations you needed to collect the same number of truncated draws as in (A), and the resulting estimate of $\mathbb{E}[X \mid \mathbf{1}^\top X > 0]$ with Monte Carlo standard errors.

Deliverables

- Your MATLAB code (clearly organized and commented).
- A short writeup with: (i) a description of your proposal and tuning choice s, (ii) acceptance rates, (iii) effective sample sizes (ESS) for each component, (iv) computational cost comparison between (A) and (B), and (v) a brief discussion of which approach you would prefer here and why.

Default values for testing

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0.6 & 0.2 \\ 0.6 & 1 & 0.4 \\ 0.2 & 0.4 & 1 \end{bmatrix}, \quad \text{iterations} = 50{,}000.$$

Question 2: Solve RBC model with demand shock

We consider a standard infinite-horizon RBC model with the following features, except that the stochastic shock now affects the **utility function** rather than the production function.

- Infinite horizon representative agent.
- Preferences are affected by a stochastic demand (preference) shock:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{d_t} \frac{C_t^{1-\gamma}}{1-\gamma}.$$

- Budget Constraint:

$$C_t + K_{t+1} = K_t^{\alpha}.$$

- Stochastic *demand (preference)* shock:

$$d_{t+1} = \sigma \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathcal{N}(0,1).$$

Use $\alpha = 0.4$, $\gamma = 2$, $\sigma = 1$, and $\beta = 0.9$.

Things to do

1. Write the first-order conditions of the model. Remember there must be three equations since we are looking for policy functions for consumption, capital, and the demand shock.

   (i) Euler equation, (ii) Resource constraint, (iii) Law of motion for d_t.

2. Find the steady-state analytically (set $d_t = 0$ and all variables constant over time).
3. Write the system of equilibrium conditions in the form

$$H(c_t, c_{t+1}, k_t, k_{t+1}, d_t, d_{t+1}) = 0.$$

4. Write the policy functions we are looking for as functions of (k_t, d_t, σ):

$$c_t = g_c(k_t, d_t, \sigma), \qquad k_{t+1} = g_k(k_t, d_t, \sigma).$$

5. Write H_i for $i = 1, \ldots, 6$, that is, each of the six scalar equilibrium conditions that compose H.
6. Write H_i for $i = 1, \ldots, 6$ evaluated at the steady-state.
7. Compute F_k, F_d, and F_σ.
8. Construct the matrices $\mathbf{A}$ and $\mathbf{B}$ from the linearized system.
9. Solve for the policy derivatives $\mathbf{h}_x$ and $\mathbf{c}_x$ both using dynare and the method we devolped in class.

Question 3: A Gibbs Sampler for bayesian regression

Model and likelihood. Let $x_1, \ldots, x_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with unknown (μ, σ^2). Write $\tau \equiv \sigma^{-2}$, $\bar{x} = \frac{1}{n}\sum_i x_i$, and $S = \sum_{i=1}^n (x_i - \bar{x})^2$. The (kernel of the) likelihood is

$$L(\mathbf{x}; \mu, \tau) \propto \tau^{n/2} \exp\Big(-\tfrac{\tau}{2}[n(\mu-\bar{x})^2 + S]\Big) = \underbrace{\tau^{1/2} \exp\Big(-\tfrac{n\tau}{2}(\mu-\bar{x})^2\Big)}_{\mathcal{N}\left(\theta=\bar{x},\ h=\frac{1}{\tau n}\right)} \underbrace{\tau^{(n-1)/2} \exp\Big(-\tfrac{S}{2}\tau\Big)}_{\mathrm{Ga}\left(a=\frac{n+1}{2},\ \ b=\frac{S}{2}\right)}.$$

where $\mathbf{x} = x_1, \ldots, x_n$, remeber that density of normal with mean θ and precision matrix h is $h^{1/2}e^{\left(-\frac{h}{2}(x-\theta)^2\right)}$ and the density of gamma is proportional to $\tau^{a-1}e^{-b\tau}$.

Conditionals

$$\boxed{\mu \mid \tau, x \ \sim\ \mathcal{N}\left(\theta = \bar{x},\ h = \frac{1}{\tau n}\right)}$$

$$\boxed{\tau \mid x \ \sim\ \mathrm{Ga}\left(a = \frac{n+1}{2},\quad b = \frac{S}{2}\right)}$$

so, the normal draw depends on τ but the draw of τ does not.

Gibbs (independent τ draw):

$$\tau^{(m+1)} \sim \mathrm{Ga}\left(\tfrac{n+1}{2},\ \tfrac{S}{2}\right), \qquad \mu^{(m+1)} \sim \mathcal{N}\big(\bar{x},\ 1/(\tau^{(m+1)}n)\big).$$

Tasks.

1. Simulate data with $\mu^\star = 1.0$, $\sigma^\star = 2.0$, $n = 200$.
2. Implement Gibbs and compare histograms of draws for μ and σ with the true values.