

# Likelihood Inference

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Juan Rubio-Ramírez

Emory University

- We focus on likelihood-based inference.
- Why?
  1. Likelihood principle (Berger and Wolpert, 1988).
  2. Attractive asymptotic and good small-sample behavior.
  3. Parameters for policy/welfare analysis.
  4. Simple handling of misspecification (Monfort, 1996).
  5. Enables model comparison.

1. Maximum likelihood:  $\hat{\theta} = \arg \max_{\theta} l_x(\theta)$ .
2. Bayesian:  $\pi(\theta \mid x) = \frac{l_x(\theta) \pi(\theta)}{\int l_x(u) \pi(u) \mathrm{d}u}$ .

- Data  $y_T = \{y_t\}_{t=1}^T$ .
- Model  $i \in \mathcal{M}$ ; parameter space  $\Theta_i$ .
- Likelihood  $f(y_T \mid \theta, i)$ , prior  $\pi(\theta \mid i)$ , posterior

$$\pi(\theta \mid y_T, i) = \frac{f(y_T \mid \theta, i) \pi(\theta \mid i)}{\int_{\Theta_i} f(y_T \mid u, i) \pi(u \mid i) \mathrm{d}u}.$$

Let  $y_t \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, 1)$ ,  $t = 1, \dots, n$ . Then

$$f(y_T \mid \theta) \propto \exp\left[-\frac{1}{2} \sum (y_i - \theta)^2\right] = \exp\left[-\frac{1}{2} (ns^2 + n(\theta - \bar{y})^2)\right].$$

where  $\bar{y} = \frac{1}{n} \sum y_i$  and  $s^2 = \frac{1}{n} \sum (y_i - \bar{y})^2$  are sample mean and sample variance.

$$\pi(\theta) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(\theta - \mu)^2}{2\sigma^2}\right).$$

$$\pi(\theta \mid y_T) \propto f(y_T \mid \theta) \pi(\theta) \propto \exp\left[-\frac{1}{2}(ns^2 + n(\theta - \bar{y})^2) - \frac{1}{2\sigma^2}(\theta - \mu)^2\right].$$

$$\pi(\theta \mid y_T) \propto \exp\left\{-\frac{1}{2}\left[n(\theta - \bar{y})^2 + \frac{(\theta - \mu)^2}{\sigma^2}\right]\right\}.$$

## Expand and Collect Terms

Expand the exponent:

$$-\frac{1}{2} \left[ n(\theta^2 - 2\bar{y}\theta + \bar{y}^2) + \frac{1}{\sigma^2}(\theta^2 - 2\mu\theta + \mu^2) \right].$$

Keep only terms involving  $\theta$ :

$$-\frac{1}{2} \left[ \left( n + \frac{1}{\sigma^2} \right) \theta^2 - 2 \left( n\bar{y} + \frac{\mu}{\sigma^2} \right) \theta \right].$$

Let

$$a = \frac{n\bar{y} + \mu/\sigma^2}{n + 1/\sigma^2}.$$



# Complete the Square

Rewrite inside the exponent:

$$(n + \frac{1}{\sigma^2})\theta^2 - 2(n\bar{y} + \frac{\mu}{\sigma^2})\theta = (n + \frac{1}{\sigma^2})[(\theta - a)^2 - a^2].$$

Constants that do not depend on  $\theta$  can be dropped:

$$\pi(\theta \mid y_T) \propto \exp\left[-\frac{1}{2}(n + \frac{1}{\sigma^2})(\theta - a)^2\right].$$

## Express in Compact Form

Note that

$$n + \frac{1}{\sigma^2} = \frac{\sigma^2 n + 1}{\sigma^2} = \frac{\sigma^2 + 1/n}{\sigma^2/n}, \quad a = \frac{n\bar{y} + \mu/\sigma^2}{n + 1/\sigma^2} = \frac{\bar{y}\sigma^2 + \mu/n}{\sigma^2 + 1/n}.$$

Substitute into the exponent:

$$\pi(\theta \mid y_T) \propto \exp \left[ -\frac{1}{2} \frac{\sigma^2 + 1/n}{\sigma^2/n} \left( \theta - \frac{\bar{y}\sigma^2 + \mu/n}{\sigma^2 + 1/n} \right)^2 \right].$$

- Posterior is normal:

$$\theta \mid y_T \sim \mathcal{N}\left(\frac{\bar{y} \sigma^2 + \mu/n}{\sigma^2 + 1/n}, \frac{\sigma^2/n}{\sigma^2 + 1/n}\right).$$

- As  $n \rightarrow \infty$ , posterior mean  $\rightarrow \bar{y}$  and variance  $\rightarrow 0$ .
- As  $\sigma^2 \rightarrow \infty$ , prior becomes flat  $\Rightarrow \bar{y}$  dominates.