Perturbation Methods

Juan F. Rubio-Ramírez

Emory University

Presentation

We want to solve functional equations of the form

$$F(d(x)) = 0.$$

where the unknown is $d(x)$. Perturbation solves the problem by using Taylor's theorem and specifying an nth-order approximation

$$d_n(x, \Phi) = \sum_{i=0}^{n} \Phi_i \left(x - x_0\right)^i.$$

We need to find the coefficients $\{\Phi_i\}_i$. How?

Motivation

- Many complicated mathematical problems have either
  1. a particular case, or
  2. a related problem,

  that is easy to solve.
- Often, we can use the solution of the simpler problem as a building block of the general solution.
- This approach has been very successful in physics.

Applications to Economics

- Judd and Guu (1993) showed how to apply perturbation methods to economic problems.
- Recently, these methods have gained popularity; second-order approximations are easy to compute and markedly improve accuracy.
- Perturbation theory generalizes the well-known linearization strategy.
- Sometimes referred to as *asymptotic methods*.

A Baby Example: A Basic RBC

$$\max_{\{c_t, k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} k_t,$$

$$z_{t+1} = \rho z_t + \sigma \varepsilon_{t+1},\ \varepsilon_t \sim \mathcal{N}(0,1).$$

FOC of this problem

$$\frac{1}{c_t} = \beta \, \mathbb{E}_t \left[\frac{1}{c_{t+1}} \alpha e^{z_{t+1}} k_{t+1}^{\alpha - 1} \right]$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} k_t,$$

$$z_{t+1} = \rho z_t + \sigma \varepsilon_{t+1}, \; \varepsilon_t \sim \mathcal{N}(0, 1).$$

Main Idea

- Transform the problem by introducing a small perturbation parameter.
- Solve the transformed problem for a particular choice (typically setting the parameter to zero).
- Use that solution to approximate the solution of the original problem.

A Perturbation Approach

- Choose the standard deviation σ as the perturbation parameter.
- Set $\sigma = 0 \Rightarrow$ deterministic model: $z_t = 0$ and $e^{z_t} = 1$.
- The deterministic steady state is straightforward to compute.

A Re-parametrized Policy Function

We search for policy functions

$$c_t = c(k_t, z_t; \sigma).$$

$$k_{t+1} = k(k_t, z_t; \sigma).$$

$$z_{t+1} = \rho z_t + \sigma \varepsilon_{t+1}.$$

- Note the appearance of the parameter σ.
- We build a local approximation around $\sigma = 0$.

Taylor's Theorem (assume $\delta = 1$)

Equilibrium conditions:

$$
\begin{aligned}
0 &= \mathbb{E}_t\left[\frac{1}{c(k_t, z_t; \sigma)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}}, k(k_t, z_t; \sigma)^{\alpha - 1}}{c(k(k_t, z_t; \sigma), \rho z_t + \sigma \varepsilon_{t+1}; \sigma)}\right], \\
0 &= c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) - e^{z_t} k_t^{\alpha} \\
0 &= z_{t+1} - \rho z_t - \sigma \varepsilon_{t+1}.
\end{aligned}
$$

We will take derivatives w.r.t. k_t, z_t, and σ and apply Taylor's theorem around the deterministic steady state. (How to proceed in logs?)

Important Question

Is $c(k, 0; 0)$ different from $c(k, 0; \sigma)$?

Asymptotic Expansion for c_t

Let k be the steady state capital in the deterministic model. Then

$$\begin{aligned} c_t = {} & c(k,0;0) + c_k(k,0;0)(k_t - k) + c_z(k,0;0)z_t + c_\sigma(k,0;0)\sigma \\ & + \tfrac{1}{2}c_{kk}(k_t - k)^2 + \tfrac{1}{2}c_{kz}(k_t - k)z_t + \tfrac{1}{2}c_{k\sigma}(k_t - k)\sigma \\ & + \tfrac{1}{2}c_{zk}z_t(k_t - k) + \tfrac{1}{2}c_{zz}z_t^2 + \tfrac{1}{2}c_{z\sigma}z_t\sigma \\ & + \tfrac{1}{2}c_{\sigma k}\sigma(k_t - k) + \tfrac{1}{2}c_{\sigma z}\sigma z_t + \tfrac{1}{2}c_{\sigma\sigma}\sigma^2 + \cdots . \end{aligned}$$

Asymptotic Expansion for k_{t+1}

$$
\begin{aligned}
k_{t+1} = {} & k(k,0;0) + k_k(k,0;0)(k_t - k) + k_z(k,0;0)z_t + k_\sigma(k,0;0)\sigma \\
& + \tfrac{1}{2}k_{kk}(k_t - k)^2 + \tfrac{1}{2}k_{kz}(k_t - k)z_t + \tfrac{1}{2}k_{k\sigma}(k_t - k)\sigma \\
& + \tfrac{1}{2}k_{zk}z_t(k_t - k) + \tfrac{1}{2}k_{zz}z_t^2 + \tfrac{1}{2}k_{z\sigma}z_t\sigma \\
& + \tfrac{1}{2}k_{\sigma k}\sigma(k_t - k) + \tfrac{1}{2}k_{\sigma z}\sigma z_t + \tfrac{1}{2}k_{\sigma\sigma}\sigma^2 + \cdots .
\end{aligned}
$$

Notation

Define the vector of equilibrium conditions as

$$F(k_t, z_t; \sigma) = \mathbb{E}_t \begin{bmatrix} \dfrac{1}{c(k_t, z_t; \sigma)} - \beta \dfrac{\alpha e^{z_{t+1}} k(k_t, z_t; \sigma)^{\alpha-1}}{c(k(k_t, z_t; \sigma), z_{t+1}; \sigma)} \\ c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) - e^{z_t} k_t^{\alpha} \\ z_{t+1} - \rho z_t - \sigma \varepsilon_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Equivalently, write $F(k_t, z_t; \sigma) = \mathbb{E}_t H(c_t, c_{t+1}, k_t, k_{t+1}, z_t, z_{t+1}; \sigma)$, where subscripts H_i denote derivatives w.r.t. the ith argument (evaluated at steady state when used below).

Zero-Order Approximation

Evaluate at $(k_t, z_t, \sigma) = (k, 0, 0)$:

$$F(k, 0; 0) = 0.$$

Steady state (for $\delta = 1$):

$$\frac{1}{c} = \beta\, \alpha (k)^{\alpha - 1} \quad \Leftrightarrow \quad 1 = \alpha\beta (k)^{\alpha - 1}.$$

Thus

$$k = (\alpha\beta)^{\frac{1}{1-\alpha}}, \qquad c = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}. \quad \text{(or using the resource constraint)}$$

How good is this approximation?

First-Order Approximation

Take derivatives of $F(k_t, z_t; \sigma)$ and evaluate at $(k, 0, 0)$:

$$F_k(k, 0; 0) = 0, \qquad F_z(k, 0; 0) = 0, \qquad F_\sigma(k, 0; 0) = 0.$$

Solving the System I

Recall

$$F(k_t, z_t; \sigma) = \mathbb{E}_t H\big(c(k_t, z_t; \sigma), c(k(k_t, z_t; \sigma), \rho z_t + \sigma\varepsilon_{t+1}), k_t, k(k_t, z_t; \sigma), z_t, \rho z_t + \sigma\varepsilon_{t+1}\big) = 0.$$

Since F is identically zero, its derivatives of any order are also zero. Therefore (at the steady state):

$$\begin{aligned}
F_k &= H_1 c_k + H_2(c_k k_k) + H_3 + H_4 k_k = 0, \\
F_z &= H_1 c_z + H_2(c_k k_z + c_z \rho) + H_4 k_z + H_5 + H_6 \rho = 0, \\
F_\sigma &= H_1 c_\sigma + H_2(c_k k_\sigma + c_z \mathbb{E}\varepsilon_{t+1} + c_\sigma) + H_4 k_\sigma + H_6 \mathbb{E}\varepsilon_{t+1} = 0.
\end{aligned}$$

Solving the System II

Where

$$H_1 = \begin{bmatrix} -1 \\ \frac{}{c_{ss}^2} \\ 1 \\ 0 \end{bmatrix}, H_2 = \begin{bmatrix} \frac{1}{c_{ss}^2} \\ 0 \\ 0 \end{bmatrix}, H_3 = \begin{bmatrix} 0 \\ -\alpha k_{ss}^{\alpha-1} \\ 0 \end{bmatrix}, H_4 = \begin{bmatrix} \frac{\alpha-1}{k_{ss}c_{ss}} \\ 1 \\ 0 \end{bmatrix}, H_5 = \begin{bmatrix} 0 \\ -k_{ss}^{\alpha} \\ -\rho \end{bmatrix}, H_6 = \begin{bmatrix} \frac{1}{c_{ss}} \\ 0 \\ 1 \end{bmatrix}$$

Unknowns at first order: $c_k, c_z, k_k, k_z, c_\sigma, k_\sigma$. We have a system to determine them. How to solve it efficiently?

Solving the System II

Note that the pair

$$F_k = 0, \quad F_z = 0$$

constitutes a quadratic system in the four unknowns (c_k, c_z, k_k, k_z) and does not depend on (c_σ, k_σ). Procedures to solve such systems: Blanchard and Kahn (1980), Uhlig (1999), Sims (2000), Klein (2000) — all equivalent. Why quadratic? Stable vs. unstable manifolds.

How to do it?

Rewrite as

$$\begin{bmatrix} H_4 & H_6 & H_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{bmatrix} \begin{bmatrix} k_k \\ 0 \end{bmatrix} = - \begin{bmatrix} H_3 & H_5 & H_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ c_k \end{bmatrix},$$

$$\begin{bmatrix} H_4 & H_6 & H_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{bmatrix} \begin{bmatrix} k_z \\ \rho \end{bmatrix} = - \begin{bmatrix} H_3 & H_5 & H_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ c_z \end{bmatrix}.$$

How to do it?

Or compactly

$$\underbrace{\begin{bmatrix} H_4 & H_6 & H_2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} k_k & k_z \\ 0 & \rho \end{bmatrix}}_{h_x} = - \underbrace{\begin{bmatrix} H_3 & H_5 & H_1 \end{bmatrix}}_{B} M.$$

Let $h_x = \begin{bmatrix} k_k & k_z \\ 0 & \rho \end{bmatrix}$ and unknown vector $\begin{bmatrix} c_k & c_z \end{bmatrix}^\top$.

How to do it?

Let Λ and P be the eigenvalue and eigenvector matrices of h_x:

$$h_x P = P\Lambda.$$

Then

$$AMP\Lambda = -BMP.$$

How to do it?

Define

$$Z := MP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ c_k & c_z \end{bmatrix} P, \qquad A := \begin{bmatrix} H_4 & H_6 & H_2 \end{bmatrix}, \qquad B := -\begin{bmatrix} H_3 & H_5 & H_1 \end{bmatrix}.$$

Then the generalized eigenvalue problem reads

$$AZ\Lambda = BZ.$$

Generalized Eigenvalue Problem

For given A and B, there exists a matrix $V = [V_1 \; V_2]$ and a block-diagonal $D = \text{diag}(D_{11}, D_{22})$ such that

$$A \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} = B \begin{bmatrix} V_1 & V_2 \end{bmatrix}.$$

D_{11} collects roots with $|\lambda| < 1$.

Stable Roots and Recovery

Assume the number of eigenvalues with $|\lambda| < 1$ equals the number of states. Then set $\Lambda = D_{11}$ and $Z = V_1$:

$$MP = V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} \quad \Rightarrow \quad P = V_{11}, \qquad \begin{bmatrix} c_k & c_z \end{bmatrix} = V_{21} P^{-1}.$$

Finally,

$$h_x = P D_{11} P^{-1}.$$

Importantly, we know that some of the entries of h_x are related to the exogenous process.

Solving the System III

With c_k, c_z, k_k, k_z in hand, determine (c_σ, k_σ) from the linear homogeneous system

$$F_\sigma = H_1 c_\sigma + H_2(c_k k_\sigma + c_\sigma) + H_4 k_\sigma = 0 \quad \Longrightarrow \quad \begin{bmatrix} H_1 + H_2 & H_2 c_k + H_4 \end{bmatrix} \begin{bmatrix} c_\sigma \\ k_\sigma \end{bmatrix} = 0.$$

Hence

$$c_\sigma = k_\sigma = 0.$$

Interpretation

- Since $c_\sigma = k_\sigma = 0$, the first-order solution exhibits **certainty equivalence** — no precautionary behavior.
- Distinguish *risk aversion* (second derivative of utility) from the *precautionary motive* (third derivative of utility): Leland (1968), Kimball (1990).
- Ignoring numerical error, the *policy* functions have no approximation error at first order; the *value function* still does.

Comparison with Linearization

- After Kydland and Prescott (1982), LQ methods/linearization became popular to solve macro models.
- Properly implemented, linearization, LQ, and first-order perturbation are equivalent.
- Advantages of perturbation: access to higher-order terms with clear theorems.

Second-Order Approximation

Take second derivatives at $(k, 0, 0)$:

$$F_{kk} = 0, \quad F_{kz} = 0, \quad F_{k\sigma} = 0, \quad F_{zz} = 0, \quad F_{z\sigma} = 0, \quad F_{\sigma\sigma} = 0.$$

(Remember Young's theorem.)

Solving the Second-Order System

- Substitute known first-order coefficients.
- Obtain a linear system of 12 equations in 12 unknowns.
- Cross terms in $k\sigma$ and $z\sigma$ are zero.
- Conjecture: all terms with odd powers of σ vanish.

Correction for Risk

- A term in σ^2 appears at second order.
- Captures the precautionary motive.
- Certainty equivalence no longer holds.
- Key advantage of going to second order.

Higher-Order Terms

- The recursion can be continued to arbitrarily high order.
- Major advantage: the procedure is *recursive*.
- Often, a few iterations suffice.
- Accuracy depends on the objective (see Fernández-Villaverde, Rubio-Ramírez, and Santos, 2006).