# Gibbs Sampling

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## Gibbs Sampling — Systematic Scan

Let $x^{(1)} = (x_1^{(1)}, \ldots, x_d^{(1)})$ be initial. For $t = 2, 3, \ldots$

$$
\begin{aligned}
&\text{Sample } X_1^{(t)} \sim \pi\big(x_1 \mid X_2^{(t-1)}, \ldots, X_d^{(t-1)}\big),\\
&\qquad\vdots\\
&\text{Sample } X_j^{(t)} \sim \pi\big(x_j \mid X_{1:j-1}^{(t)}, X_{j+1:d}^{(t-1)}\big),\\
&\qquad\vdots\\
&\text{Sample } X_d^{(t)} \sim \pi\big(x_d \mid X_{1:d-1}^{(t)}\big).
\end{aligned}
$$

Questions: Do conditionals define a unique joint $\pi$? Is $\pi$ invariant? Does the chain converge?

## Invariance of the Gibbs Sampler

For $d = 2$, the Gibbs kernel is

$$K(x^{(t-1)}, x^{(t)}) = \pi(x_1^{(t)} \mid x_2^{(t-1)})\, \pi(x_2^{(t)} \mid x_1^{(t)}).$$

**Proposition.** Systematic-scan Gibbs admits $\pi$ as invariant distribution.

## Proof for $d = 2$

We have

$$\int K(x,y)\pi(x)\,dx = \int \pi(y_2 \mid y_1)\,\pi(y_1 \mid x_2)\,\pi(x_1,x_2)\,dx_1 dx_2$$

$$= \pi(y_2 \mid y_1)\int \pi(y_1 \mid x_2)\pi(x_2)\,dx_2$$

$$= \pi(y_2 \mid y_1)\pi(y_1) = \pi(y_1,y_2) = \pi(y).$$

# Example: Bivariate Normal

Let $X = (X_1, X_2) \sim \mathcal{N}(\mu, \Sigma)$, $\mu = (\mu_1, \mu_2)$,

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{pmatrix}.$$

Gibbs updates:

$$X_1^{(t)} \sim \mathcal{N}\Big(\mu_1 + \tfrac{\rho}{\sigma_2^2}(X_2^{(t-1)} - \mu_2),\ \sigma_1^2 - \tfrac{\rho^2}{\sigma_2^2}\Big),$$
$$X_2^{(t)} \sim \mathcal{N}\Big(\mu_2 + \tfrac{\rho}{\sigma_1^2}(X_1^{(t)} - \mu_1),\ \sigma_2^2 - \tfrac{\rho^2}{\sigma_1^2}\Big).$$

Large $|\rho| \Rightarrow$ slow mixing.