A Sample Program for the EDVAC

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(1) A $p+1$-*complex*: $X^{(p)} = (x^0, x^1, \cdots, x^p)$ consists of the *main number*: x^0, and the *satellites*: $x^1, \cdots, x^p$. Throughout what follows $p = 1, 2, \cdots$ will be fixed. A complex $X^{(p)}$ *precedes* a complex $Y^{(p)}$: $X^{(p)} \leqslant Y^{(p)}$, if their main numbers are in this order: $x^0 \leqq y^0$.

An *n-sequence* of complexes: $\left\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\right\}$.

If $0', \cdots, (n-1)'$ is permutation of $0, \cdots, (n-1)$, then the sequence $\left\{X_{0'}^{(p)}, \cdots, X_{(n-1)'}^{(p)}\right\}$ is a *permutation* of the sequence $\left\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\right\}$.

A sequence $\left\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\right\}$ is *monotone* if its elements appear in their order of precedence $X_0^{(p)} \leqslant X_1^p \leqslant \cdots \leqslant X_{n-1}^{(p)}$, i.e. $x_0^0 \leqq x_1^0 \leqq \cdots \leqq x_{n-1}^0$.

Every sequence $\left\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\right\}$ possesses a monotone permutation: $\left\{X_{0'}^{(p)}, \cdots, X_{(n-1)'}^{(p)}\right\}$ (at least one). Obtaining this monotone permutation is the operation of *sorting* the original sequence.

Given two (separately) monotone sequences $\left\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\right\}$ and $\left\{Y_0^{(p)}, \cdots, Y_{m-1}^{(p)}\right\}$, sorting the composite sequence of $\left\{X_0^{(p)}, \cdots, X_{n-1}^{(p)},\right.$ $\left.Y_0^{(p)}, \cdots, Y_{m-1}^{(p)}\right\}$ is the operation of *meshing*.

(2) We wish to formulate code instructions for sorting and for meshing and to see how much control-capacity they tie up and how much time they require.

It is convenient to consider meshing first and sorting afterwards.

(3) Consider the operation of meshing the two (separately) monotone sequences $\left\{X_0^{(p)}, \cdots, X_{n-1}^{(p)}\right\}$ and $\left\{Y_0^{(p)}, \cdots, Y_{m-1}^{(p)}\right\}$.

A natural procedure to achieve this is the following one:

> Denote the meshed sequence by $\left\{Z_0^{(p)}, \cdots, Z_{n+m-1}^{(p)}\right\}$. Assume that the ℓ first elements $Z_0^{(p)}, \cdots, Z_\ell^{(p)}$ have already been formed, $\ell = 0, 1, \cdots, n+m$. Assume that they consist

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of the n' (m') first elements of the X- (Y-) sequence: $X_0^{(p)}, \cdots, X_{n'-1}^{(p)}$ and $Y_0^{(p)}, \cdots, Y_{m'-1}^{(p)}$, with $n' = 0, 1, \cdots, n$ and $m' = 0, 1, \cdots, m$ and $n' + m' = \ell$.

Then the procedure is as follows:

(α) $n' < n, m' < m$:
Determine whether $x_{n'}^0 \leqq$ or $> y_{m'}^0$.

(α_1) $x_{n'}^0 \leqq y_{m'}^0$: $Z_\ell^{(p)} = X_{n'}^{(p)}$,
replace ℓ, n', m' by $\ell + 1, n' + 1, m'$.

(α_2) $x_{n'}^0 > y_{m'}^0$: $Z_\ell^{(p)} = Y_{m'}^{(p)}$,
replace ℓ, n', m' by $\ell + 1, n', m' + 1$.

(β) $n' < n, m' = m$: Same as (α_1).

(γ) $n' = n, m' < m$: Same as (α_2).

(δ) $n' = n, m' = m$: The process is completed.

(4) In carrying out this process, the following observations apply:

(a) The process consists of steps which are enumerated by the index, $\ell = 0, 1, \cdots, n + m$. It begins with $\ell = 0$, ends with $\ell = n + m$, and ℓ increases by 1 at every step—hence there are $n + m + 1$ steps.

(b) Each step is characterized not only by its ℓ, but also by its n', m'. Since $\ell = n' + m'$, it is preferable to characterize it by n', m' alone, and to obtain ℓ from the above formula. Thus the process begins with $(n', m') = (0, 0)$, ends with $(n', m') = (n, m)$, and at every step either n' or m' increases by 1 while the other remains constant.

(c) At the beginning of every step it is necessary to sense which of the 4 cases (α)–(δ) of (3) holds. (δ) terminates the procedure. (β), (γ) are related to (α): Indeed, (β), (γ) correspond to (α_1), (α_2). Hence in the cases (β), (γ) one may replace $x_{n'}^0, y_{m'}^0$ (when they are being inspected) by $0, 0$ or $0, -1$ and then proceed as in (α).

(d) At the end of (α) (i.e. (α_1) or (α_2), by (c) equally for (β) or (γ)) the complex $X_{n'}^{(p)}$ ($Y_{m'}^{(p)}$) must be placed in the position of the complex $Z_\ell^{(p)}$. This amounts to transferring the elements of $X_{n'}^{(p)}$ ($Y_{m'}^{(p)}$), i.e. since $x_{n'}^0$ ($y_{m'}^0$) is already available, it amounts to transferring $x_{n'}^1, \cdots, x_{n'}^p$ ($y_{m'}^1, \cdots, y_{m'}^p$). This is an unbroken sequence of p elements, followed by $x_{n'+1}^0$ ($y_{m'+1}^0$). At the next step $x_{n'}^0$ ($y_{m'}^0$) will have to be replaced (for the next inspection) by $x_{n'+1}^0$ ($y_{m'+1}^0$), hence it is simplest to transfer at the point a sequence of $p + 1$ elements, i.e. $x_{n'}^1, \cdots, x_{n'}^p, x_{n'+1}^0$ ($y_{m'}^1, \cdots, y_{m'}^p, y_{m'+1}^0$).

(e) The arrangement made at the end of (α) implies, that for $\ell \neq 0$ the quantities to be inspected for the step ℓ, i.e. $x_{n'}^0, y_{m'}^0$, are already available at the beginning of the step. In the interest of homogeneity it is therefore desirable to have the same situation at the beginning of the step $\ell = 0$, i.e. $(n', m') = (0, 0)$. Hence the step $\ell = 0$ must be preceded by a preparatory step, say step —, which makes x_0^0, y_0^0 available.

(5) The remarks of (4) define the procedure more closely. Specifically:

(f) At the beginning of a step, say step (n', m'), the following quantities must be available, i.e. placed into short tanks: $n', m', x_{n'}^0, y_{m'}^0$. Denote the short tanks containing these quantities by $\bar{1}_1, \bar{2}_1, \bar{3}_1, \bar{4}_1$. Now the first operation must turn about determining which of the the cases (α)–(δ) holds. This consists in determining which of $n' - n, m' - m$ are $\geqq 0$ or < 0. Hence n, m, too must be available, say in the short tanks $\bar{5}_1, \bar{6}_1$. According to which of the 4 cases holds, $\mathcal{C}$ must be sent to the place where its instructions begin, say the (long tank) words $1_{\alpha_1}, 1_{\beta_1}, 1_{\gamma_1}, 1_{\delta_1}$. Their numbers must be available, i.e. in short tanks, say in the short tanks $\bar{7}_1, \bar{8}_1, \bar{9}_1, \overline{10}_1$.

Finally the order which will send $\mathcal{C}$ to 1_α–1_δ must be in a short tank, say in the short tank $\overline{11}_1$.

(g) We now formulate a set of instructions to effect this 4-way decision between (α)–(δ). We state again the contents of the short tanks already assigned:

$\bar{1}_1)\ \mathcal{N}\, n'_{(-30)}$	$\bar{2}_1)\ \mathcal{N}\, m'_{(-30)}$	$\bar{3}_1)\ \mathcal{N}\, x_{n'}^0$	$\bar{4}_1)\ \mathcal{N}\, y_{m'}^0$
$\bar{5}_1)\ \mathcal{N}\, n_{(-30)}$	$\bar{6}_1)\ \mathcal{N}\, m_{(-30)}$	$\bar{7}_1)\ \mathcal{N}\, 1_{\alpha(-30)}$	$\bar{8}_1)\ \mathcal{N}\, 1_{\beta(-30)}$
$\bar{9}_1)\ \mathcal{N}\, 1_{\gamma(-30)}$	$\overline{10}_1)\ \mathcal{N}\, 1_{\delta(-30)}$	$\overline{11}_1 \ldots \to \mathcal{C}$	

Now let the instructions occupy the (long tank) words, $1_1, 2_1, \cdots$:

$1_1)\ \bar{1}_1 - \bar{5}_1$	$\sigma)\ \mathcal{N}\, n' - n_{(-30)}$	
$2_1)\ \bar{9}_1 \mathrm{\ s\ } \bar{7}_1$	$\sigma)\ \mathcal{N}\, \genfrac{}{}{0pt}{}{1_\gamma}{1_\alpha}{}_{(-30)}$	for $n' \genfrac{}{}{0pt}{}{\geqq}{<} n$
$3_1)\ \sigma \to \overline{12}_1$	$\overline{12}_1)\ \mathcal{N}\, \genfrac{}{}{0pt}{}{1_\gamma}{1_\alpha}\,(-30)$	for $n' \genfrac{}{}{0pt}{}{\geqq}{<} n$
$4_1)\ \bar{1}_1 - \bar{5}_1$	$\sigma)\ \mathcal{N}\, n' - n_{(-30)}$	
$5_1)\ \overline{10}_1 \mathrm{\ s\ } \bar{8}_1$	$\sigma)\ \mathcal{N}\, \genfrac{}{}{0pt}{}{1_\delta}{1_\beta}\,(-30)$	for $n' \genfrac{}{}{0pt}{}{\geqq}{<} n$
$6_1)\ \sigma \to \overline{13}_1$	$\overline{13}_1)\ \mathcal{N}\, \genfrac{}{}{0pt}{}{1_\delta}{1_\beta}\,(-30)$	for $n' \genfrac{}{}{0pt}{}{\geqq}{<} n$
$7_1)\ \bar{2}_1 - \bar{6}_1$	$\sigma)\ \mathcal{N}\, m' - m_{(-30)}$	
$8_1)\ \overline{13}_1 \mathrm{\ s\ } \overline{12}_1$	$\sigma)\ \mathcal{N}\, \genfrac{}{}{0pt}{}{\cdots\overline{13}_1)\cdots}{\cdots\overline{12}_1)\cdots}$	for $m' \genfrac{}{}{0pt}{}{\geqq}{<} m$
	i.e.	
	$\sigma)\ \mathcal{N}\, \genfrac{}{}{0pt}{}{1_\delta\, 1_\beta}{1_\gamma\, 1_\alpha}$	for $\genfrac{}{}{0pt}{}{m'=m, n'=n \quad m'=m, n'<n}{m'<m, n'=n \quad m'<m, n'<n}$ i.e. for $\genfrac{}{}{0pt}{}{(\delta)(\beta)}{(\gamma)(\alpha)}$, respectively.
$9_1)\ \sigma \to \overline{11}_1$	$\overline{11}_1)\ 1_\alpha, 1_\beta, 1_\gamma, 1_\delta \to \mathcal{C}$	for (α), (β), (γ), (δ), respectively.
$10_1)\ \overline{11}_1 \to \mathcal{C}$		

Now

$$\overline{11}_1)\ 1_\alpha, 1_\beta, 1_\gamma, 1_\delta \to \mathcal{C} \qquad \text{for } (\alpha),\ (\beta),\ (\gamma),\ (\delta), \text{ respectively.}$$

Thus at the end of this phase $\mathcal{C}$ is at $1_\alpha, 1_\beta, 1_\gamma, 1_\delta$, according to which case (α), (β), (γ), (δ) holds.

(h) We now pass to the case (α). This has 2 subcases (α_1) and (α_2), according to whether $x^0_{n'} \geqq$ or $< y^0_{m'}$. According to which of the 2 subcases holds, $\mathcal{C}$ must be sent to the place where its instructions begin, say the (long tank) words $1_{\alpha_1}, 1_{\alpha_2}$. Their numbers must be available, say in the short tanks $\overline{1}_2, \overline{2}_2$.

(i) We now formulate a set of instructions to effect this 2-way decision between $(\alpha_1), (\alpha_2)$. We state again the contents of the short tanks additionally assigned:

$$\overline{1}_2)\ \mathcal{N}\ 1_{\alpha_1(-30)} \qquad \overline{2}_2)\ \mathcal{N}\ 1_{\alpha_2(-30)}$$

Now the instructions follow:

$1_\alpha)\ \overline{4}_1 - \overline{3}_1$	$\sigma)\ \mathcal{N}\ y^0_{m'} - x^0_{n'}$	
$2_\alpha)\ \overline{1}_2 \mathrm{s}\ \overline{2}_2$	$\sigma)\ \mathcal{N}\ {}^{1_{\alpha_1}}_{1_{\alpha_2}}(-30)$	for $x^0_{n'} \begin{smallmatrix}\leqq\\ >\end{smallmatrix} y^0_{m'}$ i.e. for $\begin{smallmatrix}(\alpha_1)\\ (\alpha_2)\end{smallmatrix}$, respectively.
$3_\alpha)\ \sigma \to \overline{11}_1$	$\overline{11}_1)\ 1_{\alpha_1}, 1_{\alpha_2} \to \mathcal{C}$	for $(\alpha_1), (\alpha_2)$, respectively.
$4_\alpha)\ \overline{11}_1 \to \mathcal{C}$		

Now

$$\overline{11}_1)\ 1_{\alpha_1}, 1_{\alpha_2} \to \mathcal{C} \quad \text{for } (\alpha_1), (\alpha_2), \text{ respectively.}$$

Thus at the end of this phase $\mathcal{C}$ is set at $1_{\alpha_1}, 1_{\alpha_2}$, according to which case (α_1), (α_2) holds.

(j) Before turning to $(\alpha_1), (\alpha_2)$, let us dispose of the cases (β), (γ) and (δ).
According to (c), the cases $(\beta), (\gamma)$ can be handled as follows:
Additional short tanks assigned:

$$\overline{3}_2)\ \mathcal{N}\ 0 \qquad \overline{4}_2)\ \mathcal{N}\ -1$$

The instructions for (β):

$1_\beta)\ \overline{3}_2 - \overline{3}_2$	$\sigma)\ \mathcal{N}\ 0$
$2_\beta)\ 2_\alpha \to \mathcal{C}$	

and from here on like (α) with $0, 0$ for $x^0_{n'}, y^0_{m'}$.
The instructions for (γ):

$1_\gamma)\ \overline{4}_2 - \overline{3}_2$	$\sigma)\ \mathcal{N}\ -1$
$2_\gamma)\ 2_\alpha \to \mathcal{C}$	

and from here on like (α) with $0, -1$ for $x^0_{n'}, y^0_{m'}$. (For both cases of (c).)
Assuming that after the conclusion of the procedure $\mathcal{C}$ is to be sent to the (long tank) word a, the instructions for (δ) are as follows:

$$1_\delta)\ a \to \mathcal{C}$$

(k) We now pass to $(\alpha_1), (\alpha_2)$. It is necessary to state at this point, where the complexes $X_0^{(p)}, \cdots, X_{n-1}^{(p)}$ and $Y_0^{(p)}, \cdots, Y_{m-1}^{(p)}$ are stored, and where the complexes $Z_0^{(p)}, \cdots, Z_{n+m-1}^{(p)}$ are to be placed. Let the X-complexes form a sequence which begins at the (long tank) word b, also the Y-complexes a sequence beginning at c, and the Z-complexes a sequence beginning at d. Since every complex consists of $p+1$ numbers, therefore $X_{n'}^{(p)}$ begins at $b + n'(p+1)$, $Y_{m'}^{(p)}$ begins at $c + m'(p+1)$, $Z_\ell^{(p)}$ begins at $d + \ell(p+1)$. Hence $x_{n'}^u$ is at $b + n'(p+1) + u$, $y_{m'}^u$ is at $c + m'(p+1) + u$, z_ℓ^u is at $d + \ell(p+1) + u$. To conclude: The X-complexes occupy the interval from b to $b + n(p+1) - 1$, the Y-complexes occupy the interval from c to $c + m(p+1) - 1$, the Z-complexes occupy the interval from d to $d + (n+m)(p+1) - 1$.

At the beginning of the (α_1) or (α_2) phase the following further quantities must be available, i.e. placed into short tanks: $b + n'(p+1)$, $c + m'(p+1)$, $d + \ell(p+1)$. It is also convenient to have $p+1$. Denote the short tanks containing these quantities by $\overline{1}_3, \overline{2}_3, \overline{3}_3, \overline{4}_3$.

Hence these are the short tanks additionally assigned:

$\overline{1}_3$) $\mathcal{N}\ b + n'(p+1)_{(-30)}$
$\overline{2}_3$) $\mathcal{N}\ c + m'(p+1)_{(-30)}$
$\overline{3}_3$) $\mathcal{N}\ d + \ell(p+1)_{(-30)}$
$\overline{4}_3$) $\mathcal{N}\ p + 1_{(-30)}$

Finally, the transfer of the complex $X_{n'}^{(p)}$ (in (α_1)) or $Y_{m'}^{(p)}$ (in (α_2)) to the place of the complex $Z_\ell^{(p)}$ must be channeled through the short tanks. According to (d) the numbers $x_{n'}^1, \cdots, x_{n'}^p, x_{n'+1}^0$ or the numbers $y_{m'}^1, \cdots, y_{m'}^p, y_{m'+1}^0$ must be brought in (from X or Y) and the numbers $x_{n'}^0, x_{n'}^1, \cdots, x_{n'}^p$ or $y_{m'}^0, y_{m'}^1, \cdots, y_{m'}^p$ must be taken out (to Z).

Consequently the numbers $x_{n'}^0, x_{n'}^1, \cdots, x_{n'}^p, x_{n'+1}^0$ or $y_{m'}^0, y_{m'}^1, \cdots, y_{m'}^p, y_{m'+1}^0$ must be routed through the short tanks. It is clearly best to have them in the form of an unbroken sequence. The length of this sequence is $p+2$. Denote the short tanks which are designated to hold this sequence by $\overline{1}_4, \overline{2}_4, \cdots, \overline{p+1}_4, \overline{p+2}_4$.

We add: The primary function of (α_1) $[(\alpha_2)]$ is to move $x_{n'}^0, x_{n'}^1, \cdots, x_{n'}^p$ $[y_{m'}^0, y_{m'}^1, \cdots, y_{m'}^p]$ into the (long tank) words $d + \ell(p+1), d + \ell(p+1) + 1, \cdots, d + \ell(p+1) + p$.

However, there is also a secondary function: It must prepare the conditions for step $\ell + 1$. This means that it must replace the numbers $n', x_{n'}^0, b + n'(p+1), d + \ell(p+1)$ $[m', y_{m'}^0, c + m'(p+1), d + \ell(p+1)]$ in the short tanks $\overline{1}_1, \overline{3}_1, \overline{1}_3, \overline{3}_3$ $[\overline{2}_1, \overline{4}_1, \overline{2}_3, \overline{3}_3]$ by the numbers $n' + 1, x_{n'+1}^0, b + (n'+1)(p+1), d + (\ell+1)(p+1)$ $[m' + 1, y_{m'+1}^0, c + (m'+1)(p+1), d + (\ell+1)(p+1)]$.

To conclude: There are also two orders, affecting the transfers of X $[Y]$ into the short tanks, and of Z out of the short tanks, and these orders are best placed into short tanks, say $\overline{1}_5, \overline{2}_5$. They must be followed by an order returning $\mathcal{C}$ to the (α_1) or (α_2) sequence in long tanks. Hence this third order must be in $\overline{3}_5$, and it must depend on (α_1) or (α_2). I.e. it must be transferred into $\overline{3}_5$ from the (α_1) or (α_2) sequence.

(l) We now formulate 2 sets of instructions to carry out the tasks of (α_1) and (α_2), as formulated in (k):

Additional short tanks assigned:

$$
\begin{array}{ll}
\overline{1}_5)\ \ldots \nleftrightarrow \overline{1}_4 \mid p+2 & \overline{3}_5)\ \ldots \\
\overline{2}_5)\ \overline{1}_4 \nleftrightarrow \ldots \mid p+1 & \overline{4}_5)\ \mathcal{N}\ 1_{(-30)}
\end{array}
$$

The instructions for (α_1):

$$
\begin{array}{r|l}
1_{\alpha_1})\ \overline{1}_3 \to \overline{1}_5 & \overline{1}_5)\ b+n'(p+1) \nleftrightarrow \overline{1}_4 \mid p+2 \\
2_{\alpha_1})\ \overline{3}_3 \to \overline{2}_5 & \overline{2}_5)\ \overline{1}_4 \nleftrightarrow d+\ell(p+1) \mid p+1 \\
3_{\alpha_1})\ \quad \nleftrightarrow \overline{3}_5 & \overline{3}_5)\ 6_{\alpha_1} \to \mathcal{C} \\
\boxed{4_{\alpha_1})\ 6_{\alpha_1} \to \mathcal{C}} & \\
5_{\alpha_1})\ \overline{1}_5 \to \mathcal{C} & \\
\hline
\end{array}
$$

$\overline{1}_5)\ b+n'(p+1) \nleftrightarrow \overline{1}_4 \mid p+2$

$$
\begin{array}{l}
b{+}n'(p{+}1)\ )\ \mathcal{N}\ x^0_{n'} \ \text{ to } \overline{1}_4)\ \mathcal{N}\ x^0_{n'} \\
b{+}n'(p{+}1)+1)\ \mathcal{N}\ x^1_{n'} \ \text{ to } \overline{2}_4)\ \mathcal{N}\ x^1_{n'} \\
\qquad \cdots \\
b{+}n'(p{+}1)+p)\ \mathcal{N}\ x^p_{n'} \ \text{ to } \overline{(p{+}1)}_4)\ \mathcal{N}\ x^p_{n'} \\
b{+}(n'{+}1)(p{+}1))\ \mathcal{N}\ x^0_{n'+1} \ \text{ to } \overline{(p{+}2)}_4)\ \mathcal{N}\ x^0_{n'+1}
\end{array}
$$

$\overline{2}_5)\ \overline{1}_4 \nleftrightarrow d+\ell(p+1) \mid p+1$

$$
\begin{array}{l}
\overline{1}_4\ )\ \mathcal{N}\ x^0_{n'} \ \text{ to } d{+}\ell(p{+}1)\ )\ \mathcal{N}\ x^0_{n'} \\
\overline{2}_4\ )\ \mathcal{N}\ x^1_{n'} \ \text{ to } d{+}\ell(p{+}1){+}1)\ \mathcal{N}\ x^1_{n'} \\
\qquad \cdots \\
\overline{(p{+}1)}_4)\ \mathcal{N}\ x^p_{n'} \ \text{ to } d{+}\ell(p{+}1){+}p)\ \mathcal{N}\ x^p_{n'}
\end{array}
$$

$\overline{3}_5)\ 6_{\alpha_1} \to \mathcal{C}$

$$
\begin{array}{r|l}
\hline
6_{\alpha_1})\ \overline{1}_1 + \overline{4}_5 & \sigma)\ \mathcal{N}\ n'{+}1_{(-30)} \\
7_{\alpha_1})\ \sigma \to \overline{1}_1 & \overline{1}_1)\ \mathcal{N}\ n'{+}1_{(-30)} \\
8_{\alpha_1})\ \overline{(p+2)}_4 \to \overline{3}_1 & \overline{3}_1)\ \mathcal{N}\ x^0_{n'+1} \\
9_{\alpha_1})\ \overline{1}_3 + \overline{4}_3 & \sigma)\ N\ b{+}(n'{+}1)(p{+}1)_{(-30)} \\
10_{\alpha_1})\ \sigma \to \overline{1}_3 & \overline{1}_3)\ N\ b{+}(n'{+}1)(p{+}1)_{(-30)} \\
11_{\alpha_1})\ \overline{3}_3 + \overline{4}_3 & \sigma)\ N\ d{+}(\ell{+}1)(p{+}1)_{(-30)} \\
12_{\alpha_1})\ \sigma \to \overline{3}_3 & \overline{3}_3)\ N\ d{+}(\ell{+}1)(p{+}1)_{(-30)} \\
13_{\alpha_1})\ 1_1 \to \mathcal{C} & \text{To begin step } \ell+1 \text{ according to (g)}
\end{array}
$$

The instructions for (α_2):

$1_{\alpha_2})\ \overline{2}_3 \to \overline{1}_5$	$\overline{1}_5)\ c+m'(p+1) \nleftrightarrow \overline{1}_4 \mid p+2$
$2_{\alpha_2})\ \overline{3}_3 \to \overline{2}_5$	$\overline{2}_5)\ \overline{1}_4 \nleftrightarrow d+\ell(p+1) \mid p+1$
$3_{\alpha_2})\ \quad \nleftrightarrow \overline{3}_5$	$\overline{3}_5)\ 6_{\alpha_2} \to \mathcal{C}$
$\boxed{4_{\alpha_2})\ 6_{\alpha_2} \to \mathcal{C}}$	
$5_{\alpha_2})\ \overline{1}_5 \to \mathcal{C}$	
$\overline{1}_5)\ c+m'(p+1) \nleftrightarrow \overline{1}_4 \mid p+2$	
	$c{+}m'(p{+}1)\quad)\ \mathcal{N}\ y^0_{m'}$ to $\overline{1}_4)\ \mathcal{N}\ y^0_{m'}$
	$c{+}m'(p{+}1)+1)\ \mathcal{N}\ y^1_{m'}$ to $\overline{2}_4)\ \mathcal{N}\ y^1_{m'}$
	$\dots$
	$c{+}m'(p{+}1)+p)\ \mathcal{N}\ y^p_{m'}$ to $\overline{(p{+}1)}_4)\ \mathcal{N}\ y^p_{m'}$
	$c{+}(m'{+}1)(p{+}1))\ \mathcal{N}\ y^0_{m'+1}$ to $\overline{(p{+}2)}_4)\ \mathcal{N}\ y^0_{m'+1}$
$\overline{2}_5)\ \overline{1}_4 \nleftrightarrow d+\ell(p+1) \mid p+1$	
	$\overline{1}_4\quad)\ \mathcal{N}\ y^0_{m'}$ to $d{+}\ell(p{+}1)\quad)\ \mathcal{N}\ y^0_{m'}$
	$\overline{2}_4\quad)\ \mathcal{N}\ y^1_{m'}$ to $d{+}\ell(p{+}1){+}1)\ \mathcal{N}\ y^1_{m'}$
	$\dots$
	$\overline{(p{+}1)}_4)\ \mathcal{N}\ y^p_{m'}$ to $d{+}\ell(p{+}1){+}p)\ \mathcal{N}\ y^p_{m'}$
$\overline{3}_5)\ 6_{\alpha_2} \to \mathcal{C}$	
$6_{\alpha_2})\ \overline{2}_1 + \overline{4}_5$	$\sigma)\ \mathcal{N}\ m'{+}1_{(-30)}$
$7_{\alpha_2})\ \sigma \to \overline{2}_1$	$\overline{1}_1)\ \mathcal{N}\ m'{+}1_{(-30)}$
$8_{\alpha_2})\ \overline{(p+2)}_4 \to \overline{4}_1$	$\overline{4}_1)\ \mathcal{N}\ y^0_{m'+1}$
$9_{\alpha_2})\ \overline{2}_3 + \overline{4}_3$	$\sigma)\ N\ c{+}(m'{+}1)(p{+}1)_{(-30)}$
$10_{\alpha_2})\ \sigma \to \overline{2}_3$	$\overline{2}_3)\ N\ c{+}(m'{+}1)(p{+}1)_{(-30)}$
$11_{\alpha_2})\ \overline{3}_3 + \overline{4}_3$	$\sigma)\ N\ d{+}(\ell{+}1)(p{+}1)_{(-30)}$
$12_{\alpha_2})\ \sigma \to \overline{3}_3$	$\overline{3}_3)\ N\ d{+}(\ell{+}1)(p{+}1)_{(-30)}$
$13_{\alpha_2})\ 1_1 \to \mathcal{C}$	To begin step $\ell+1$ according to (g)

(6) Let us restate which short tanks are occupied at the beginning of of step ℓ, and how. This is the list:

$\overline{1}_1)\ \mathcal{N}\ n'_{(-30)}$	$\overline{2}_1)\ \mathcal{N}\ m'_{(-30)}$	$\overline{3}_1)\ \mathcal{N}\ x^0_{n'}$	$\overline{4}_1)\ \mathcal{N}\ y^0_{m'}$
$\overline{5}_1)\ \mathcal{N}\ n_{(-30)}$	$\overline{6}_1)\ \mathcal{N}\ m_{(-30)}$	$\overline{7}_1)\ \mathcal{N}\ 1_{\alpha(-30)}$	$\overline{8}_1)\ \mathcal{N}\ 1_{\beta(-30)}$
$\overline{9}_1)\ \mathcal{N}\ 1_{\gamma(-30)}$	$\overline{10}_1)\ \mathcal{N}\ 1_{\delta(-30)}$	$\overline{11}_1)\ \dots \to \mathcal{C}$	
$\overline{1}_2)\ \mathcal{N}\ 1_{\alpha_1(-30)}$	$\overline{2}_2)\ \mathcal{N}\ 1_{\alpha_2(-30)}$		
$\overline{3}_2)\ \mathcal{N}\ 0$	$\overline{4}_2)\ \mathcal{N}\ -1$		
$\overline{1}_3)\ \mathcal{N}\ b+n'(p+1)_{(-30)}$			
$\overline{2}_3)\ \mathcal{N}\ c+m'(p+1)_{(-30)}$			
$\overline{3}_3)\ \mathcal{N}\ d+\ell(p+1)_{(-30)}$			
$\overline{4}_3)\ \mathcal{N}\ p+1_{(-30)}$			
$\overline{1}_4)\ \dots$	$\overline{2}_4)\ \dots$ $\quad\cdots$	$\overline{(p+1)}_4)\ \dots$	$\overline{(p+2)}_4)\ \dots$
$\overline{1}_5)\ \dots \nleftrightarrow \overline{1}_4 \mid p+2$	$\overline{2}_5)\ \overline{1}_4 \nleftrightarrow \dots \mid p+1$	$\overline{3}_5)\ \dots$	$\overline{4}_5)\ \mathcal{N}\ 1_{(-30)}$

The first thing to note is, that this requires $11+4+4+(p+2)+4 = p+25$ short tanks. Hence, if the total number of short tanks is 32 [64], this gives the upper limit 7 [39] for p.

The second observation is, that these short tanks have the following contents when the sequence of steps $\ell = 0, 1, \cdots, n+m$ begins, i.e. at the beginning of the step $\ell = 0$. This is the list:

$\overline{1}_1)\ \mathcal{N}\,0$	$\overline{2}_1)\ \mathcal{N}\,0$	$\overline{3}_1)\ \mathcal{N}\,x_0^0$	$\overline{4}_1)\ \mathcal{N}\,y_0^0$
$\overline{5}_1)\ \mathcal{N}\,n_{(-30)}$	$\overline{6}_1)\ \mathcal{N}\,m_{(-30)}$	$\overline{7}_1)\ \mathcal{N}\,1_{\alpha(-30)}$	$\overline{8}_1)\ \mathcal{N}\,1_{\beta(-30)}$
$\overline{9}_1)\ \mathcal{N}\,1_{\gamma(-30)}$	$\overline{10}_1)\ \mathcal{N}\,1_{\delta(-30)}$	$\overline{11}_1)\ \ldots \to \mathcal{C}$	
$\overline{1}_2)\ \mathcal{N}\,1_{\alpha_1(-30)}$	$\overline{2}_2)\ \mathcal{N}\,1_{\alpha_2(-30)}$		
$\overline{3}_2)\ \mathcal{N}\,0$	$\overline{4}_2)\ \mathcal{N}\,-1$		
$\overline{1}_3)\ \mathcal{N}\,b_{(-30)}$	$\overline{2}_3)\ \mathcal{N}\,c_{(-30)}$	$\overline{3}_3)\ \mathcal{N}\,d_{(-30)}$	$\overline{4}_3)\ \mathcal{N}\,p+1_{(-30)}$
$\overline{1}_4)\ \ldots$ $\quad$ $\overline{2}_4)\ \ldots$	$\cdots$	$\overline{(p+1)}_4)\ \ldots$	$\overline{(p+2)}_4)\ \ldots$
$\overline{1}_5)\ \ldots \rightarrowtail \overline{1}_4 \mid p+2$		$\overline{2}_5)\ \overline{1}_4 \rightarrowtail \ldots \mid p+1$	
$\overline{3}_5)\ \ldots$		$\overline{4}_5)\ \mathcal{N}\,1_{(-30)}$	

Of these $p+25$ short tanks the following must form unbroken sequences: $\overline{1}_5, \overline{2}_5, \overline{3}_5$ because of their rôle in (l) (between 5_{α_1} and 6_{α_1}, and between 5_{α_2} and 6_{α_2}); $\overline{1}_4, \overline{2}_4, \cdots, \overline{(p+1)}_4, \overline{(p+2)}_4$ because of their rôle in (l) (and $\overline{1}_5$ and $\overline{2}_5$, in the two intervals mentioned above).

These are $3 + (p+2) = p+5$ short tanks. Of these $p+3$, namely $\overline{3}_5$ and $\overline{1}_4, \overline{2}_4, \cdots, \overline{(p+1)}_4, \overline{(p+2)}_4$ require no preliminary substitution; 2, namely $\overline{1}_5, \overline{2}_5$ have a fixed content.

The remaining $(p+25) - (p+5) = 20$ short tanks can be classified as follows: 12, namely $\overline{1}_1, \overline{2}_1, \overline{7}_1, \overline{8}_1, \overline{9}_1, \overline{10}_1, \overline{11}_1, \overline{1}_2, \overline{2}_2, \overline{3}_2, \overline{4}_2, \overline{4}_5$ have a fixed content; 6, namely $\overline{5}_1, \overline{6}_1, \overline{1}_3, \overline{2}_3, \overline{3}_3, \overline{4}_3$ have to be substituted from the general data of the problem (they contain $n, m, b, c, d, p+1$); 2, namely $\overline{3}_1, \overline{4}_1$ have to be substituted from the sequences X, Y (they contain x_0^0, y_0^0). It is desirable that all short tanks with a fixed content form an unbroken sequence, so that they can be substituted by one order. I.e. the 14 given here and the two given above must form an unbroken sequence. These 2 last are $\overline{1}_5, \overline{2}_5$, as noted still earlier, they must be followed by $\overline{3}_5$. This gives an unbroken sequence of $12+2+1 = 15$ short tanks.

Finally, it is desirable to have the 6 short tanks with $n, m, b, c, d, p+1$ at the beginning, and the 2 with x_0^0, y_0^0 immediately afterwards. Also to have the sequence of indefinite length $(p+2)$ at the end.

This gives the following final assignment of the p+25 short tanks used:

$$
\begin{array}{rcl}
\overline{1}) & \cdots & \overline{5}_1)\ \mathcal{N}\ n_{(-30)} \\
\overline{2}) & \cdots & \overline{6}_1)\ \mathcal{N}\ m_{(-30)} \\
\overline{3}) & \cdots & \overline{1}_3)\ \mathcal{N}\ b_{(-30)} \\
\overline{4}) & \cdots & \overline{2}_3)\ \mathcal{N}\ c_{(-30)} \\
\overline{5}) & \cdots & \overline{3}_3)\ \mathcal{N}\ d_{(-30)} \\
\overline{6}) & \cdots & \overline{4}_3)\ \mathcal{N}\ p+1_{(-30)} \\
\hline
\overline{7}) & \cdots & \overline{3}_1)\ \mathcal{N}\ x_0^0 \\
\overline{8}) & \cdots & \overline{4}_1)\ \mathcal{N}\ y_0^0
\end{array}
$$

$$
\begin{array}{rcl}
\overline{9}) & \cdots & \overline{1}_1)\ \mathcal{N}\ 0 \\
\overline{10}) & \cdots & \overline{2}_1)\ \mathcal{N}\ 0 \\
\overline{11}) & \cdots & \overline{7}_1)\ \mathcal{N}\ 1_{\alpha(-30)} \\
\overline{12}) & \cdots & \overline{8}_1)\ \mathcal{N}\ 1_{\beta(-30)} \\
\overline{13}) & \cdots & \overline{9}_1)\ \mathcal{N}\ 1_{\gamma(-30)} \\
\overline{14}) & \cdots & \overline{10}_1)\ \mathcal{N}\ 1_{\delta(-30)} \\
\overline{15}) & \cdots & \overline{11}_1)\ \ldots \to \mathcal{C} \\
\overline{16}) & \cdots & \overline{1}_2)\ \mathcal{N}\ 1_{\alpha_1(-30)} \\
\overline{17}) & \cdots & \overline{2}_2)\ \mathcal{N}\ 1_{\alpha_2(-30)} \\
\overline{18}) & \cdots & \overline{3}_2)\ \mathcal{N}\ 0 \\
\overline{19}) & \cdots & \overline{4}_2)\ \mathcal{N}\ -1 \\
\overline{20}) & \cdots & \overline{4}_5)\ \mathcal{N}\ 1_{(-30)} \\
\overline{21}) & \cdots & \overline{1}_5)\ \ldots \rightarrowtail \overline{24}\mid p+2 \\
\overline{22}) & \cdots & \overline{2}_5)\ \overline{24} \rightarrowtail \ldots \mid p+1 \\
\hline
\overline{23}) & \cdots & \overline{3}_5)\ \ldots \\
\hline
\overline{24}) & \cdots & \overline{1}_4)\ \ldots \\
\overline{25}) & \cdots & \overline{2}_4)\ \ldots \\
 & \cdots & \\
\overline{p+24}) & \cdots & \overline{(p+1)}_4)\ \ldots \\
\overline{p+25}) & \cdots & \overline{(p+2)}_4)\ \ldots
\end{array}
$$

(7) We now come to the step — mentioned in (e). We foresaw there that — would have to substitute x_0^0, y_0^0 into the proper short tanks (as we saw in (6), into $\overline{7}, \overline{8}$). We see now, however, that — has to take care of the substitution into all short tanks. More precisely: No substitutions into $\overline{23}$ and $\overline{24}, \overline{25}, \cdots, \overline{p+24}, \overline{p+25}$ are needed. And $\overline{1}, \cdots, \overline{6}$ should be substituted when the problem is set up as such. Hence the short tanks to be left for the step — are $\overline{7}, \overline{8}$ and $\overline{9}, \cdots, \overline{22}$.

We will substitute $\overline{7}, \overline{8}$ first and $\overline{9}, \cdots, \overline{22}$ afterwards. During the first operation the short tanks $\overline{1}, \cdots, \overline{6}$ are already occupied as indicated above (by $n, m, b, c, d, p+1$), while $\overline{9}, \cdots$ are still available. Hence we can use $\overline{9}, \cdots$ while carrying out the first step, the substitution of $\overline{7}, \overline{8}$, and substitute $\overline{9}, \cdots$, or more precisely $\overline{9}, \cdots, \overline{22}$, in the final form only subsequently, as a second step.

Actually it is desirable to place during the first step, the substitution of $\overline{7}, \overline{8}$, some orders into short tanks. In accordance with what was said above, we use for this $\overline{9}, \cdots$.

We now formulate the instructions which carry out all these substitutions. Let these instructions occupy the (long tank) words $1_0, 2_0, \cdots$:

$1_0)\quad \overset{t}{\to} \overline{9} \mid 3$	
$2_0)\ \ldots \overset{t}{\to} \overline{7}$	$\overline{9})\ \ldots \overset{t}{\to} \overline{7}$
$3_0)\ \ldots \overset{t}{\to} \overline{8}$	$\overline{10})\ \ldots \overset{t}{\to} \overline{8}$
$4_0)\ 8_0 \to \mathcal{C}$	$\overline{11})\ 8_0 \to \mathcal{C}$
$5_0)\ \overline{3} \to \overline{9}$	$\overline{9})\ b \overset{t}{\to} \overline{7}$
$6_0)\ \overline{4} \to \overline{10}$	$\overline{10})\ c \overset{t}{\to} \overline{8}$
$7_0)\ \overline{9} \to \mathcal{C}$	
$\overline{9})\ b \overset{t}{\to} \overline{7}$	$\overline{7})\ \mathcal{N}\ x_0^0$
$\overline{10})\ c \overset{t}{\to} \overline{8}$	$\overline{8})\ \mathcal{N}\ y_0^0$
$\overline{11})\ 8_0 \to \mathcal{C}$	
$8_0)\quad \overset{t}{\to} \overline{9} \mid 14$	
$9_0)\ \mathcal{N}\ 0$	$\overline{9})\ \mathcal{N}\ 0$
$10_0)\ \mathcal{N}\ 0$	$\overline{10})\ \mathcal{N}\ 0$
$11_0)\ \mathcal{N}\ 1_{\alpha(-30)}$	$\overline{11})\ \mathcal{N}\ 1_{\alpha(-30)}$
$12_0)\ \mathcal{N}\ 1_{\beta(-30)}$	$\overline{12})\ \mathcal{N}\ 1_{\beta(-30)}$
$13_0)\ \mathcal{N}\ 1_{\gamma(-30)}$	$\overline{13})\ \mathcal{N}\ 1_{\gamma(-30)}$
$14_0)\ \mathcal{N}\ 1_{\delta(-30)}$	$\overline{14})\ \mathcal{N}\ 1_{\delta(-30)}$
$15_0)\ \ldots \to \mathcal{C}$	$\overline{15})\ \ldots \to \mathcal{C}$
$16_0)\ \mathcal{N}\ 1_{\alpha_1(-30)}$	$\overline{16})\ \mathcal{N}\ 1_{\alpha_1(-30)}$
$17_0)\ \mathcal{N}\ 1_{\alpha_2(-30)}$	$\overline{17})\ \mathcal{N}\ 1_{\alpha_2(-30)}$
$18_0)\ \mathcal{N}\ 0$	$\overline{18})\ \mathcal{N}\ 0$
$19_0)\ \mathcal{N}\ -1$	$\overline{19})\ \mathcal{N}\ -1$
$20_0)\ \mathcal{N}\ 1_{(-30)}$	$\overline{20})\ \mathcal{N}\ 1_{(-30)}$
$21_0)\ \ldots \overset{t}{\to} \overline{24} \mid p+2$	$\overline{21})\ \ldots \overset{t}{\to} \overline{24} \mid p+2$
$22_0)\ \overline{24} \overset{t}{\to} \ldots \mid p+1$	$\overline{22})\ \overline{24} \overset{t}{\to} \ldots \mid p+1$
$23_0) 1_1 \to \mathcal{C}$	To begin step 0 according to (g)

(8) We have now a complete list of instructions:

First in short tanks $\overline{1}, \cdots, \overline{6}$, as described at the end of (6).

Second, in long tanks the words $1_0, \cdots, 23_0$ (cf. (7)); $1_1, \cdots, 10_1$ (cf. (g)); $1_\alpha, \cdots, 4_\alpha$ (cf. (i)); $1_\beta, 2_\beta$ (cf. (j)); $1_\gamma, 2_\gamma$ (cf. (j)); 1_δ (cf. (j)); $1_{\alpha_1}, \cdots, 13_{\alpha_1}$ (cf. (l)); $1_{\alpha_2}, \cdots, 13_{\alpha_2}$ (cf. (l)).

Let us consider the second category of instructions, i.e. the words in long tanks, more closely.

The first thing to note is that this requires $23 + 10 + 4 + 2 + 2 + 1 + 13 + 13 = 68$ (long tank) words.

The second observations is, that according to (j) $1_\beta, 2_\beta$ as well as $1_\gamma, 2_\gamma$ are always followed by $1_\alpha, \cdots, 4_\alpha$, and according to (i) $1_\alpha, \cdots, 4_\alpha$ are always followed by $1_{\alpha_1}, \cdots, 13_{\alpha_1}$ or by $1_{\alpha_2}, \cdots, 13_{\alpha_2}$.

Hence it is reasonable to make the final assignment of numbers to these (long tank) words in such a way that these precedences are maintained.

Actually, it is best to delay the final assignment of numbers, for reasons which will appear in (9). We make, however, a secondary assignment of numbers as follows:

$1', \cdots, 23'$	to	$1_0, \cdots, 23_0$
$24', \cdots, 33'$	to	$1_1, \cdots, 10_1$
$34', 35'$	to	$1_\beta, 2_\beta$
$36', 37'$	to	$1_\gamma, 2_\gamma$
$38', \cdots, 41'$	to	$1_\alpha, \cdots, 4_\alpha$
$42', \cdots, 54'$	to	$1_{\alpha_1}, \cdots, 13_{\alpha_1}$
$55', \cdots, 67'$	to	$1_{\alpha_2}, \cdots, 13_{\alpha_2}$
$68'$	to	1_δ

(9) The assignment of numbers at the end of (8) makes $1_\alpha, 1_\beta, 1_\gamma, 1_\delta, 1_{\alpha_1}, 1_{\alpha_2}$ equal to $38', 34', 36', 68', 42', 55'$. These numbers occur in $11_0, 12_0, 13_0, 14_0, 16_0, 17_0$, i.e. in $11', 12', 13', 14', 16', 17'$. Hence the content of these 6 (long tank) words depends implicitly on the final assignment of (long tank word) numbers to $1', \cdots, 68'$.

If this final assignment were made now, in a rigid from, then the (long tank) words $11', \cdots, 14', 16', 17'$ would be formulated accordingly, and the instructions would be completed. It is, however, preferable to have these instructions in such a form that they can begin anywhere, i.e. that their first (long tank) word can be chosen freely.

Let this first (long tank) word be e, i.e. e is $1'$. Hence $e, \cdots, e+67$ should correspond to $1', \cdots, 68'$. However it is worth while to deviate from this simple sequential correspondence for the following reasons:

(A) In $1_0, \cdots, 23_0$ (i.e. $1', \cdots, 23'$) the passage of $\mathcal{C}$ from $\overline{11}$ to 8_0 involves a delay of about one long tank, if 8_0 follows immediately upon 7_0: Indeed 7_0 is followed by $\overline{9}, \overline{10}, \overline{11}$. Hence it is better to intercalate 3 words between 7_0 and 8_0 to time correctly for $\overline{9}, \overline{10}, \overline{11}$, plus, say, 1 word for the long tank switching in $\overline{11}$. I.e., there should be 4 (empty) words between 7_0 and 8_0, i.e. $7'$ and $8'$.

(B) In $1_{\alpha_1}, \cdots, 13_{\alpha_1}$ (i.e. $42', \cdots, 54'$) there exists the same situation as in (A) between 5_{α_1} and 6_{α_1}, where $\overline{1}_5, \overline{2}_5, \overline{3}_5$ (i.e. $\overline{21}, \overline{22}, \overline{23}$) are intercalated. In order to avoid a delay of about one long tank, it is again necessary to intercalate $3+1=4$ (empty) words between 5_{α_1} and 6_{α_1}, i.e. $46'$ and $47'$.

(C) In $1_{\alpha_2}, \cdots, 13_{\alpha_2}$ (i.e. $55', \cdots, 67'$) between 5_{α_2} and 6_{α_2} the situation is exactly the same as in (B). Hence it is again advisable to intercalate 4 (empty) words between 5_{α_2} and 6_{α_2}, i.e. $59'$ and $60'$.

(D) 10_1 (i.e. $33'$) sends $\mathcal{C}$ to $\overline{11}_1$ (i.e. $\overline{15}$), and this in turn sends $\mathcal{C}$ to 1_α or 1_β or 1_γ or 1_δ (i.e. $38'$ or $34'$ or $36'$ or $68'$). In order to avoid a delay of about one long tank, it is necessary to intercalate 1 word after 10_1 to time correctly for $\overline{11}_1$, plus, say, 1 word for the long tank switching in $\overline{11}_1$. I.e., there should be 2 (empty) words after 10_1, i.e. $33'$.

Taking these matters into account the following final assignment of numbers obtains:

$e, \cdots, e+6$	to $1', \cdots, 7'$	to $1_0, \cdots, 7_0$
$e+7, \cdots, e+10$	empty (cf. (A))	
$e+11, \cdots, e+26$	to $8', \cdots, 23'$	to $8_0, \cdots, 23_0$
$e+27, \cdots, e+36$	to $24', \cdots, 33'$	to $1_1, \cdots, 10_1$
$e+37, e+38$	empty (cf. (D))	
$e+39, e+40$	to $34', 35'$	to $1_\beta, 2_\beta$
$e+41, e+42$	to $36', 37'$	to $1_\gamma, 2_\gamma$
$e+43, \cdots, e+46$	to $38', \cdots, 41'$	to $1_\alpha, \cdots, 4_\alpha$
$e+47, \cdots, e+51$	to $42', \cdots, 46'$	to $1_{\alpha_1}, \cdots, 5_{\alpha_1}$
$e+52, \cdots, e+55$	empty (cf. (B))	
$e+56, \cdots, e+63$	to $47', \cdots, 54'$	to $6_{\alpha_1}, \cdots, 13_{\alpha_1}$
$e+64, \cdots, e+68$	to $55', \cdots, 59'$	to $1_{\alpha_2}, \cdots, 5_{\alpha_2}$
$e+69, \cdots, e+72$	empty (cf. (C))	
$e+73, \cdots, e+80$	to $60', \cdots, 67'$	to $6_{\alpha_2}, \cdots, 13_{\alpha_2}$
$e+81$	to $68'$	to 1_δ

Hence the (long tank) words $11', 12', 13', 14', 16', 17'$ become

$$e+14, e+15, e+16, e+17, e+19, e+20,$$

and they contain the numbers $38', 34', 36'68', 42', 55'$, and these become

$$e+43, e+39, e+41, e+81, e+47, e+64.$$

We rewrite these (long tank) words:

$e+14)\ \mathcal{N}\ e+43_{(-30)}$
$e+15)\ \mathcal{N}\ e+39_{(-30)}$
$e+16)\ \mathcal{N}\ e+41_{(-30)}$
$e+17)\ \mathcal{N}\ e+81_{(-30)}$
$e+19)\ \mathcal{N}\ e+47_{(-30)}$
$e+20)\ \mathcal{N}\ e+64_{(-30)}$

(10) Disregarding for the time being the 6 substitutions required to produce the 6 (long tank) words enumerated at the end of (9), the total system of instructions, at the present stage, is this:

(I) The 82 (long tank) words $e, \cdots, e+81$ of (9).

(II) The 6 short tanks $\overline{1}, \cdots, \overline{6}$ of (6).

The quantities which actually determine the problem, as a function of the X, Y, are these:

(*) n, m, b, c, d, p, a, e. (For a cf. (1_δ) in (j), for e, cf. (9).)

Of these the 6 first, n, m, b, c, d, p, are given in (II), but p occurs again in (I). The others, a, e, occur in (I) only. So we must discuss how the occurrences of

(**) p, a, e

in (I) are to be taken care of.

p occurs in $20_0, 21_0$ (cf. (7)) i.e. $e+23, e+24$ (cf. (9)), a occurs in 1_δ (cf. (j)), i.e. e+81 (cf. (9)). The occurrences of e have been summarized at the end of (9).

We rewrite the (long tank) words which contain these additional substitutions:

$$\begin{aligned} &e+23) \ \ldots \rightarrowtail \overline{24} \mid p+2 \\ &e+24) \ \overline{24} \rightarrowtail \ldots \mid p+1 \end{aligned}$$

and

$$e+81) \ a \to \mathcal{C}$$

(11) The complete system of instructions, as derived in what preceded, can also be formulated as follows:

The 82 (long tank) words of (I) in (10), namely $e, \cdots, e+81$, contain only fixed symbols, except for certain occurrences of the the 3 variables of (**) in (10), namely p, a, e, in the 9 words at the end of (9) and the end of (10). Assume, that $e, \cdots, e+81$ are stated, with blanks $\ldots$ in place of these 3 variables in the 9 words in question. Call this group of 82 words $\mathcal{G}_{82}$.

Then, after $\mathcal{G}_{82}$ has been placed in the long tanks, in an unbroken sequence beginning at e, the following further steps are necessary:

First: 6 substitutions into short tanks, according to (II) in (10), and 9 substitutions into long tanks, according to the end of (9) and the end of (10). We restate these 6+9=15 substitutions:

$\overline{1}$) $\mathcal{N}\ n_{(-30)}$	$e+14$) $\mathcal{N}\ e{+}43_{(-30)}$	$e+23$) $\ldots \rightarrowtail \overline{24} \mid p+2$
$\overline{2}$) $\mathcal{N}\ m_{(-30)}$	$e+15$) $\mathcal{N}\ e{+}39_{(-30)}$	$e+24$) $\overline{24} \rightarrowtail \ldots \mid p+1$
$\overline{3}$) $\mathcal{N}\ b_{(-30)}$	$e+16$) $\mathcal{N}\ e{+}41_{(-30)}$	$e+81$) $a \to \mathcal{C}$
$\overline{4}$) $\mathcal{N}\ c_{(-30)}$	$e+17$) $\mathcal{N}\ e{+}81_{(-30)}$	
$\overline{5}$) $\mathcal{N}\ d_{(-30)}$	$e+19$) $\mathcal{N}\ e{+}47_{(-30)}$	
$\overline{6}$) $\mathcal{N}\ p+1_{(-30)}$	$e+20$) $\mathcal{N}\ e{+}64_{(-30)}$	

Denote this group by $\mathcal{S}_{15}$

After the substitutions $\mathcal{S}_{15}$ have been carried out, $\mathcal{C}$ can be sent at any time to e. This will cause the meshing to take place as desired, and after its completion send $\mathcal{C}$ to a.

The following final remark should be added: $\mathcal{G}_{82}$, as defined above, contains only fixed symbols, i.e. it is a fixed routine. With a suitable choice of $\mathcal{S}_{15}$ it will, therefore, cause any desired meshing process to take place.

Thus $\mathcal{G}_{82}$ can be stored permanently outside the machine, and it may be fed into the machine as a "sub routine," as a part of the instructions of any more extensive problem, which contains one or more meshing operations. Then $\mathcal{S}_{15}$ must be part of the "main routine" of that problem; it may be effected there in several parts if desired.

If, in particular, the problem contains several meshing operations, only those parts of $\mathcal{S}_{15}$ need to be repeated, in which those operations differ. And since $\mathcal{G}_{82}$ contains no explicit reference to its own position, i.e. to e, therefore $\mathcal{G}_{82}$ can be placed anywhere in the long tanks, it is only necessary that the "main routine" take care of the proper e (by means of its $\mathcal{S}_{15}$). This "mobility" within the long tanks is, of course, an absolute necessity for "sub routines," which are suited for use in a flexible general logical scheme of "main routines" and (possibly multiple and interchangeable) "sub routines."

(12) To conclude, we must estimate the duration of a meshing process according to the instructions which we derived.

We will not count the time in effecting $\mathcal{S}_{15}$, hence we begin when $\mathcal{C}$ reaches e. We follow the list of words $e, \cdots, e+81$ given in (9):

The process begins with the step — of (7), i.e. $1_0, \cdots, 23_0$, i.e. $e, \cdots, e+26$. Apart from 26 words $= \frac{26}{32}$ ms $= .81$ ms, there are the following delays: $\overline{9}, \overline{10}$ each averages $\frac{1}{2}$ tank $= .5$ ms; $\overline{11}$ is 1 word $= \frac{1}{32}$ ms $= .03ms$. The total is $.81 + .5 + .5 + .03 = 1.84$ ms.

Now consider a step $\ell = 0, 1, \cdots, n+m$. Its make up is as follows: It begins with $1_1, \cdots, 10_1$ of (g), i.e. $e+27, \cdots, e+38$. These are 12 words $= \frac{12}{32}$ ms $= .38$ ms, and no other delays. From here on the process splits, according to which one of the 4 cases (α), (β), (γ), (δ) obtains.

Consider (α) first. It begins with $1_\alpha, \cdots, 4_\alpha$ of (i), i.e. $e+43, \cdots, e+46$. Apart from 4 words $\frac{4}{32}$ ms $= .13$ ms, there are the following delays: At the beginning of this sequence 7 words (from $e+36$ to $e+43$) $= \frac{7}{32}$ ms $= .22$ ms; from the time of $\overline{11}_1$ (which follows upon $e+46$, and hence is $e+47$) until the beginning of (α_1) or (α_2) ($e+47$ or $e+64$), i.e. nothing or 17 words, averaging $\frac{1}{2}$ 17 words $= \frac{1}{2}\,\frac{17}{32}$ ms $= .27$ ms. This totals $.13 + .22 + .27 = .62$ ms. Next there is (α_1) [(α_2)], consisting of $1_{\alpha_1}, \cdots, 13_{\alpha_1}$ [$1_{\alpha_2}, \cdots, 13_{\alpha_2}$] of (l), i.e. $e+47, \cdots, e+63$ [$e+64, \cdots, e+80$]. Apart from 17 words $= \frac{17}{32}$ ms $= .53$ ms, there are the following delays: $\overline{1}_5$ averages $p+1$ words and $\frac{1}{2}$ tank; $\overline{2}_5$ averages $p+2$ words and $\frac{1}{2}$ tank; after 13_{α_1} [13_{α_2}] (i.e. $e+63$ [$e+80$]), there is a delay until 1_1 ($e+27$), since this delay is to be taken modulo entire tank, i.e. modulo 32 words, it amounts to 28 [11] words, i.e. an average of $\frac{1}{2}(28+11) = 19\frac{1}{2}$ words. This totals $(p+2)+(p+1)+19\frac{1}{2} = 2p+22\frac{1}{2}$ words and $\frac{1}{2}+\frac{1}{2} = 1$ tank, i.e. $\frac{2p+22\frac{1}{2}}{32}+1\,\mathrm{ms}=\frac{p}{16}+1.70\,\mathrm{ms}$. The grand total for (α) is therefore $.62+\left(\frac{p}{16}+1.70\right)\,\mathrm{ms}=\frac{p}{16}+2.32\,\mathrm{ms}$.

Consider next (β), (γ). These differ from (α) only inasmuch as they replace 1_α by $1_\beta, 2_\beta$ $[1_\gamma, 2_\gamma]$ of (j). In either case, there is, with actual operation and delays, a direct sequence from 10_1 to 2_α, i.e. from $e+36$ to $e+44$. Hence their duration is the same as (α).

Consider finally (δ). This involves the delay from 1_{10} to 1_δ of (j), from $e+36$ to $e+47$, and the word 1_δ, i.e. $e+47$, itself. This amounts to 12 words $=\frac{12}{32}\,\mathrm{ms}=.38\,\mathrm{ms}$.

Now of the $n+m+1$ steps $\ell=0,1,\cdots,n+m$ all but the last one, $n+m$, are (α), or (β), or (γ); $n+m$ is (δ). Hence there are $n+m$ lasting $.38+\left(\frac{p}{16}+2.32\right)\,\mathrm{ms}=\frac{p}{16}+2.70\,\mathrm{ms}$ and 1 lasting $.38+.38\,\mathrm{ms}=.76\,\mathrm{ms}$. The total duration of the entire meshing process is therefore this: $1.84+(n+m)\left(\frac{p}{16}+2.70\right)+.76\,\mathrm{ms}=2.60+(n+m)\left(\frac{p}{16}+2.70\right)\,\mathrm{ms}$. For $p=1$ this is $2.60+(n+m)\,2.76\,\mathrm{ms}$, for $p=7$ it is $2.60+(n+m)\,3.14\,\mathrm{ms}$, for $p=39$ it is $2.60+(n+m)\,5.14\,\mathrm{ms}$. (Concerning these p values consider the first part of (6).)