Proof

Square root of two is irrational

Proof.

$$p, q, k \in \mathbb{N}$$
$$\sqrt{2} \in \mathbb{R}$$

$$
\left[
\begin{array}{l}
\sqrt{2} \in \mathbb{Q} \\
\exists (p, q)\big((\sqrt{2} = p/q) \wedge (\gcd(p, q) = 1)\big) \\
(p/q)^2 = (\sqrt{2})^2 = 2 \;,\; p^2 = 2q^2 \\
2|p^2 \;,\; 2|p \\
\exists (k)(p = 2k) \\
2q^2 = 4k^2 \;,\; q^2 = 2k^2 \\
2|q^2 \;,\; 2|q \\
\big(\gcd(p, q) = 1\big) \wedge \big(2|p \wedge 2|q\big)
\end{array}
\right.
$$

$$\sqrt{2} \notin \mathbb{Q}$$
$$\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q} \quad \square$$

$\square$

Irrational power of irrational number

An irrational power of an irrational number could give a rational result.

Proof.

$$s, t \in \mathbb{R} \backslash \mathbb{Q}$$
$$\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q}$$

$$
\left[
\begin{array}{l}
\sqrt{2}^{\sqrt{2}} \in \mathbb{Q} \\
\exists (s, t)(s^t) \in \mathbb{Q}
\end{array}
\right.
$$

$$
\left[
\begin{array}{l}
\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q} \\
\sqrt{2}^{\sqrt{2}} \in \mathbb{R} \backslash \mathbb{Q} \\
\big(\sqrt{2}^{\sqrt{2}}\big)^{\sqrt{2}} = \big(\sqrt{2}\big)^2 = 2 \in \mathbb{Q} \\
\exists (s, t)(s^t) \in \mathbb{Q}
\end{array}
\right.
$$

$$\exists (s, t)(s^t) \in \mathbb{Q} \quad \square$$

$\square$

Product of sum of squares

If two integers can each be expressed as the sum of two squares, then so can their product.

Proof. Let m and m be integers such that each can be expressed as the sum of two squares:

$$m = a^2 + b^2$$
$$n = c^2 + d^2$$

Let z_m and z_n be the complex numbers given by $z_m = a + ib$ and $z_n = c + id$, so we have:

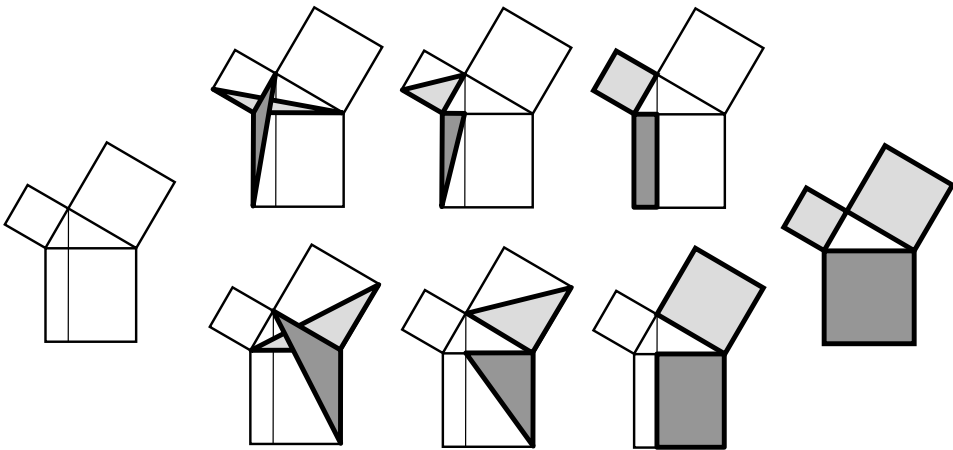$$m = |z_m|^2 = a^2 + b^2$$
$$n = |z_n|^2 = c^2 + d^2$$

Now we form the product:

$$
\begin{aligned}
mn &= (|z_m|^2)(|z_n|^2) \\
&= (|z_m||z_n|)^2 \\
&= |z_m z_n|^2 \\
&= |(a + ib)(c + id)|^2 \\
&= |(ac - bd) + i(ad + bc)|^2 \\
&= (ac - bd)^2 + (ad + bc)^2
\end{aligned}
$$

Therefore the product mn can always be expressed as the sum of two squares, namely $(ac - bd)^2$ and $(ad + bc)^2$.

$\square$

Pythagoras Theorem

Pythagoras Theorem: $c^2 = a^2 + b^2$

$\square$

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