

Zeta Function - Complex

Definition 1. The zeta function $\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}$

Converges when $\operatorname{Re}(z) > 1$.

$$\zeta(\bar{z}) = \overline{\zeta(z)} \quad (1)$$

Zeta Product Formula

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}, \quad \text{converges when } \operatorname{Re}(z) > 1. \quad (2)$$

Analytic Continuation

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}, \quad \text{converges when } \operatorname{Re}(z) > 0. \quad (3)$$

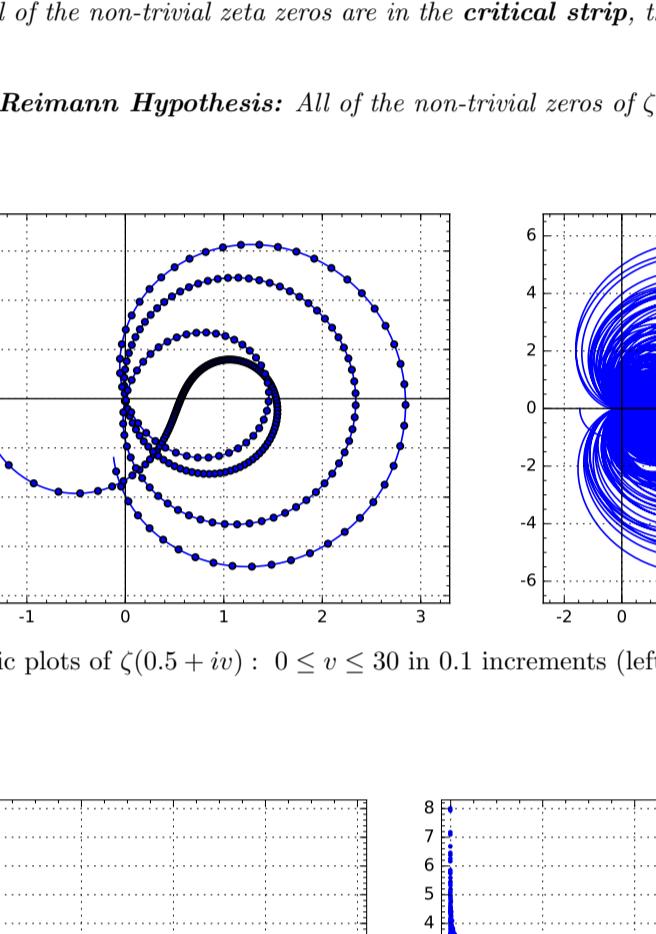
$$\zeta(z) = \frac{\eta(z)}{1 - 2^{(1-z)}} \quad (4)$$

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx, \quad \text{converges absolutely when } \operatorname{Re}(z) > 0. \quad (5)$$

$$\Gamma(z+1) = z\Gamma(z) \quad (6)$$

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad (7)$$

$$\zeta(1-z) = \frac{2}{(2\pi)^z} \cos\left(\frac{z\pi}{2}\right) \Gamma(z)\zeta(z) \quad (8)$$



Notes:

1. The values of $\zeta(z)$ where $\operatorname{Re}(z) > 1$ are relatively static.
2. There is a single pole at $z = 1$ and zeros where $z = \{-2, -4, -6, -8, \dots\}$.

Zeta Equivalences

$$\mu(n) = \begin{cases} 1 & n \text{ square-free with an even number of prime factors} \\ -1 & n \text{ square-free with an odd number of prime factors} \\ 0 & n \text{ has a squared prime factor} \end{cases} \quad (9)$$

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} \quad (10)$$

Non-trivial Zeta Zeros

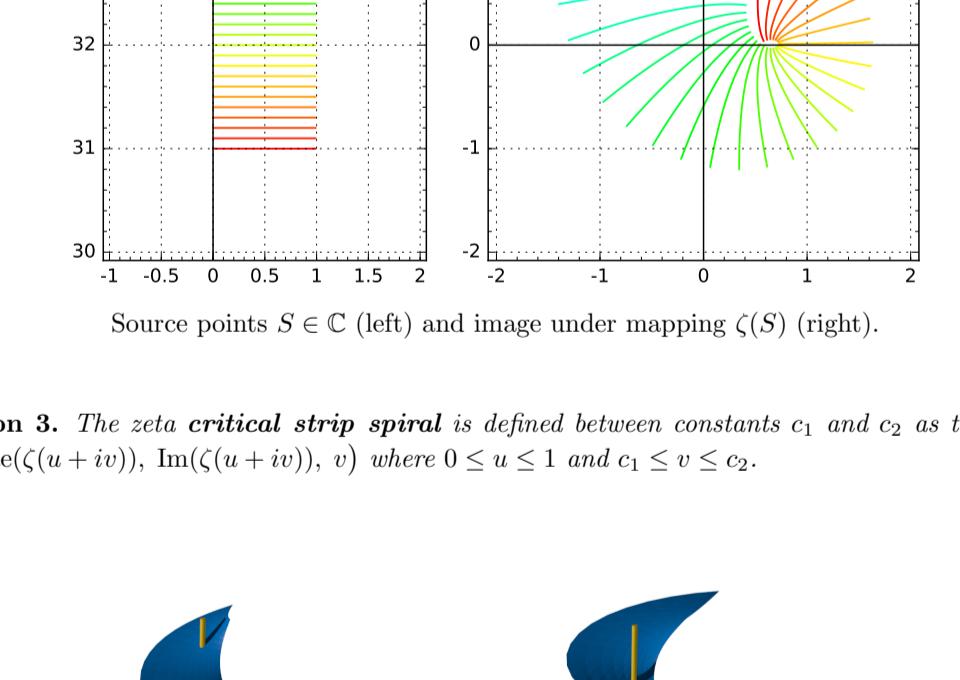
Definition 2. The **non-trivial zeta zeros** are all of the zeta zeros that are not on the real axis.

Theorem 1. The non-trivial zeta zeros are symmetric around the real axis.

Theorem 2. The non-trivial zeta zeros are symmetric around the line where $\operatorname{Re}(z) = \frac{1}{2}$.

Theorem 3. All of the non-trivial zeta zeros are in the **critical strip**, the region where $0 \leq \operatorname{Re}(z) \leq \frac{1}{2}$.

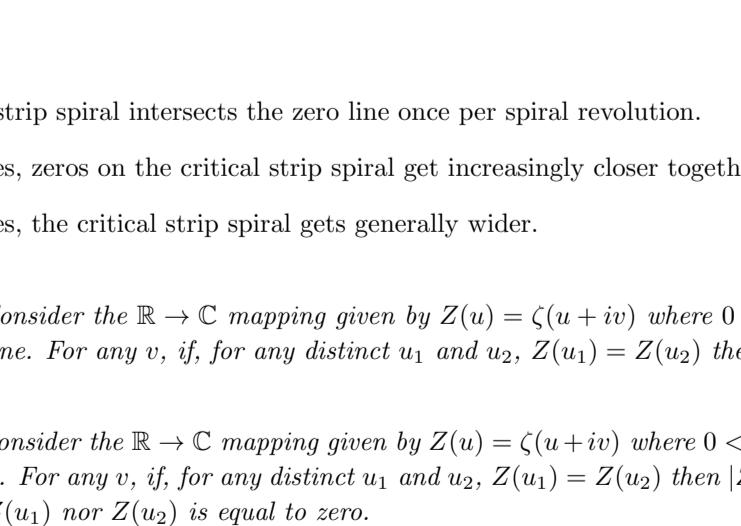
Conjecture 1. Riemann Hypothesis: All of the non-trivial zeros of $\zeta(s)$ are on the line $\operatorname{Re}(z) = \frac{1}{2}$.



Parametric plots of $\zeta(0.5 + iv)$: $0 \leq v \leq 30$ in 0.1 increments (left), $0 \leq v \leq 500$ (right).

Some close-pair zeros:

7005.062866175, 7005.100564674
71732.901207872, 71732.915909348
388858886.0022851203, 388858886.0023936899
777717772.0045702406, 777717772.0047873798



$\zeta(z)$ argument contour plot with isolines for both argument and modulus. The first non-trivial zero at ~ 14.13 (left). A close-pair of non-trivial zeros at ~ 7005 (right).



$\zeta(1+iv)$ in red and $\zeta(0+iv)$ in green for $1000 \leq v \leq 10000$ in 0.05 increments.



Critical strip spiral in blue and zero line $(0, 0, v)$ in yellow: $11 \leq v \leq 31$ (left) and $31 \leq v \leq 51$ (right).

Notes:

1. The critical strip spiral intersects the zero line once per spiral revolution.
 2. As v increases, zeros on the critical strip spiral get increasingly closer together.
 3. As v increases, the critical strip spiral gets generally wider.
- Conjecture 2.** Consider the $\mathbb{R} \rightarrow \mathbb{C}$ mapping given by $Z(u) = \zeta(u + iv)$ where $0 < u < 1$ and constant v is greater than one. For any v , if, for any distinct u_1 and u_2 , $Z(u_1) = Z(u_2)$ then $|u_1 - \frac{1}{2}| \neq |u_2 - \frac{1}{2}|$.
- Conjecture 3.** Consider the $\mathbb{R} \rightarrow \mathbb{C}$ mapping given by $Z(u) = \zeta(u + iv)$ where $0 < u < 1$ and constant v is greater than one. For any v , if, for any distinct u_1 and u_2 , $Z(u_1) = Z(u_2)$ then $|Z(1) - Z(u_1)| < |Z(1)|$ and thus neither $Z(u_1)$ nor $Z(u_2)$ is equal to zero.