# Linear Algebra

$$\text{unitary matrix:} \quad U^{-1} = U^{\dagger} \tag{1}$$

$$\text{takes unit vectors to a unit vectors} \tag{2}$$

$$\begin{cases} \text{all cols length 1} \\ \text{any two cols orthogonal} \end{cases} \tag{3}$$

$$\begin{cases} \text{all rows length 1} \\ \text{any two rows orthogonal} \end{cases} \tag{4}$$

$$\text{hermitian matrix:} \quad H = H^{\dagger} \tag{5}$$

Hermitian matrices have real diagonal entries and have real eigenvalues.

Hermitian matrices are unitarily diagonalizable: $H = UDU^{\dagger}$

Two hermitian matrices are simultaneously diagonalizable if and only if they commute.

$$(H_a H_b = H_b H_a) \Leftrightarrow ((H_a = U D_a U^{\dagger}) \text{ and } (H_b = U D_b U^{\dagger})) \tag{6}$$

# Qubits

$$\text{state } |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{7}$$

$$\text{probabilities } |\alpha|^2 \text{ and } |\beta|^2 \tag{8}$$

$$|\alpha|^2 + |\beta|^2 = 1 \tag{9}$$

$$\text{rotate } \textit{global phase} \text{ so that } \alpha \in \mathbb{R}^+ \tag{10}$$

$$\text{leaves two degrees of freedom} \tag{11}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{12}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{13}$$

$$|+\rangle = \tfrac{1}{\sqrt{2}} |0\rangle + \tfrac{1}{\sqrt{2}} |1\rangle \tag{14}$$

$$|-\rangle = \tfrac{1}{\sqrt{2}} |0\rangle - \tfrac{1}{\sqrt{2}} |1\rangle \tag{15}$$

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{16}$$

$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \tag{17}$$

$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \tag{18}$$

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{19}$$

## B-Sphere

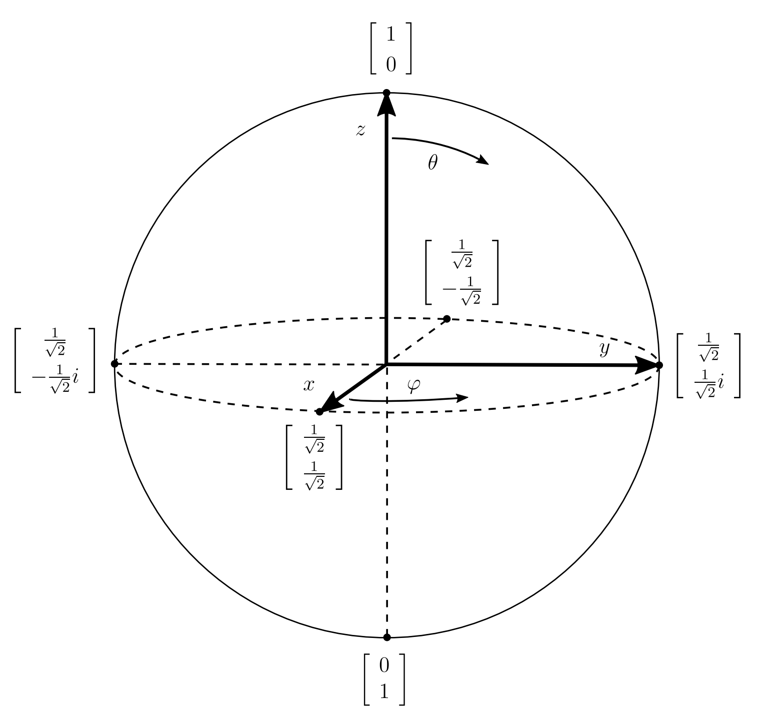



Figure 1: State vectors mapped to surface of unit radius sphere.

For an arbitrary state vector $|\psi\rangle$,

$$|\psi\rangle = \cos\left(\tfrac{\theta}{2}\right) |0\rangle + e^{i\varphi} \sin\left(\tfrac{\theta}{2}\right) |1\rangle = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{bmatrix} \tag{20}$$

$$x = \sin\theta \cos\varphi \tag{21}$$

$$y = \sin\theta \sin\varphi \tag{22}$$

$$z = \cos\theta \tag{23}$$

## Gates

| | | |
|---|---|---|
| $X$ | $-\boxed{X}-$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |
| $Y$ | $-\boxed{Y}-$ | $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ |
| $Z$ | $-\boxed{Z}-$ | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ |
| $H$ | $-\boxed{H}-$ | $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ |
| $S$ | $-\boxed{S}-$ | $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ |
| $T$ | $-\boxed{T}-$ | $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$ |

Table 1: 1-qubit gates

$$XY = -YX = iZ \tag{24}$$

$$YZ = -ZY = iX \tag{25}$$

$$ZX = -XZ = iY \tag{26}$$

$$SS = Z \tag{27}$$

$$TT = S \tag{28}$$

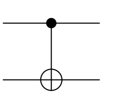
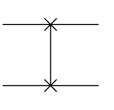
| | | |
|---|---|---|
| *CNOT* |  | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ |
| *SWAP* |  | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |

Table 2: 2-qubit gates

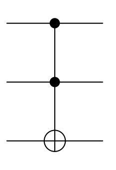
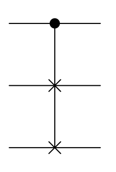
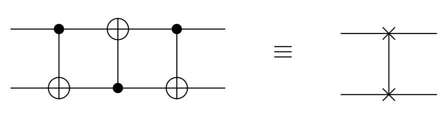
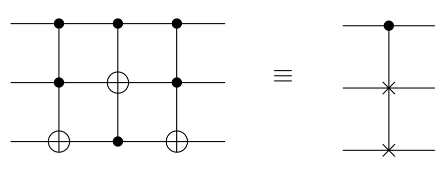
| | | |
|---|---|---|
| *TOFF* |  | $\left(\begin{array}{cccc\|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$ |
| *FRED* |  | $\left(\begin{array}{cccc\|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$ |

Table 3: 3-qubit gates

$FRED$ is *conservative* (i.e. preserves total number of ones) and thus can be used to construct a *billiard ball* model of logic computation.

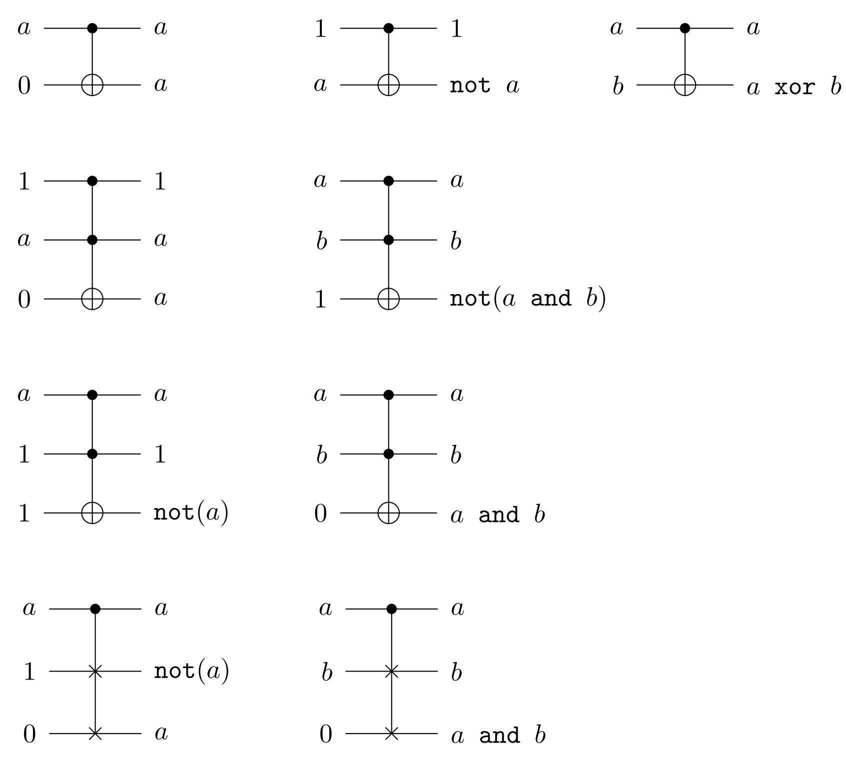
Equivalent circuits:





# Quantum Logic Gates

Number of distinct $n$-input 1-output logic circuits: $2^{2^n}$




# Quantum States

$$n = \text{ number of qubits} \tag{29}$$

$$\text{state vector dimension:} \quad d = 2^n \tag{30}$$

$$\text{state space:} \quad \mathbb{C}^d \tag{31}$$

$$\text{state vector:} \quad |\psi\rangle \equiv \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ a_{d-1} \end{bmatrix} \in \mathbb{C}^d \tag{32}$$

$$\text{set global phase such that:} \quad a_0 \in \mathbb{R}^+ \tag{33}$$

$$\sum_{n=0}^{d-1} |a_n|^2 = 1 \tag{34}$$

$$\langle\psi| \equiv |\psi\rangle^{\dagger} \tag{35}$$

$$\langle\psi| = \left[\ a_0^*, a_1^*, \cdots, a_{d-1}^*\ \right] \tag{36}$$

$$\text{inner product:} \quad \langle\psi|\psi\rangle = 1 \tag{37}$$

# Quantum Circuits and Algebra

$U_A \otimes U_B \quad \equiv$