Zeta Function

Definition 1. *The **zeta function*** $\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}$

Converges when $\mathrm{Re}(z) > 1$.

$$\zeta(\overline{z}) = \overline{\zeta(z)} \tag{1}$$

Zeta Product Formula

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}\ , \quad \text{converges when } \mathrm{Re}(z) > 1. \tag{2}$$

Analytic Continuation

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}\ , \quad \text{converges when } \mathrm{Re}(z) > 0. \tag{3}$$

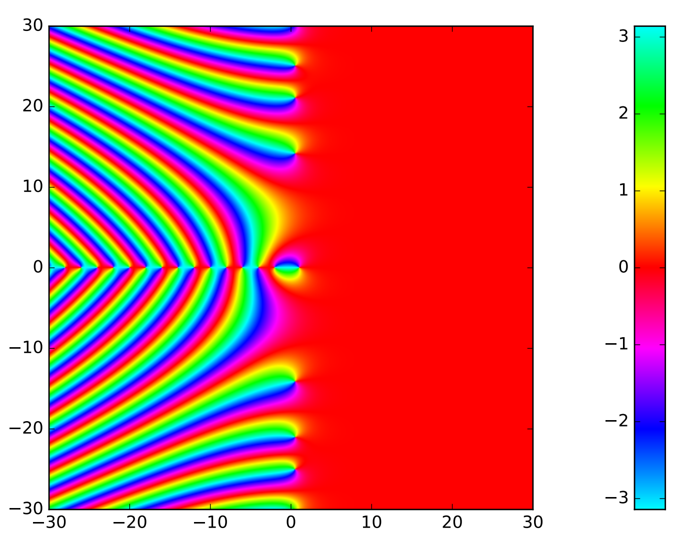
$$\zeta(z) = \frac{\eta(z)}{1 - 2^{(1-z)}} \tag{4}$$

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx\ , \quad \text{converges absolutely when } \mathrm{Re}(z) > 0. \tag{5}$$

$$\Gamma(z+1) = z\Gamma(z) \tag{6}$$

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \tag{7}$$

$$\zeta(1-z) = \frac{2}{(2\pi)^z} \cos\left(\frac{z\pi}{2}\right) \Gamma(z)\zeta(z) \tag{8}$$



Contour plot: $\arg\zeta(z)$

Notes:

1. The values of $\zeta(z)$ where $\mathrm{Re}(z) > 1$ are relatively static.
2. There is a single pole at $z = 1$ and zeros where $z = \{-2, -4, -6, -8, \dots\}$.

Zeta Equivalences

$$\mu(n) = \begin{cases} 1 & n \text{ square-free with an even number of prime factors} \\ -1 & n \text{ square-free with an odd number of prime factors} \\ 0 & n \text{ has a squared prime factor} \end{cases} \tag{9}$$

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} \tag{10}$$

Reimann Function

Alternative formulation:

$$\mathrm{R}(x) = 1 + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n\, n!\, \zeta(n+1)} \tag{11}$$

Non-trivial Zeta Zeros

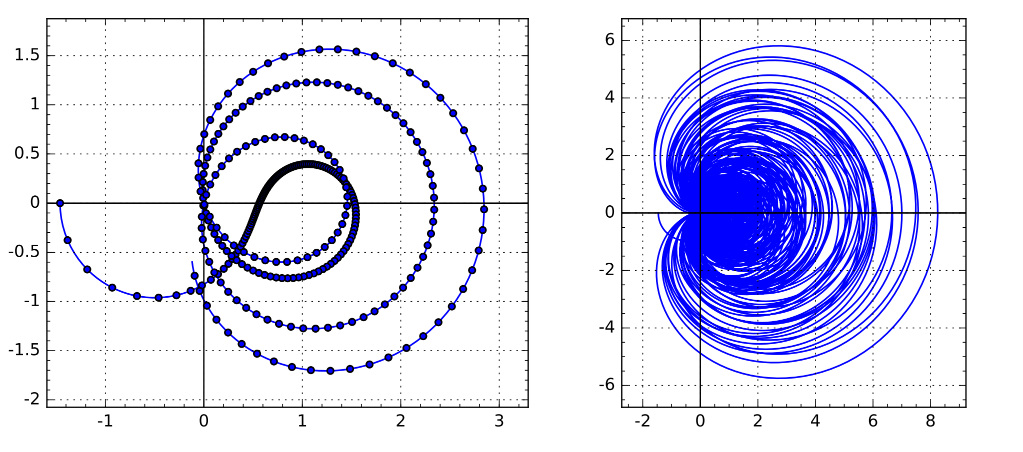
Definition 2. *The **non-trivial zeta zeros** are all of the zeta zeros that are not on the real axis.*

Theorem 1. *The non-trivial zeta zeros are symmetric around the real axis.*

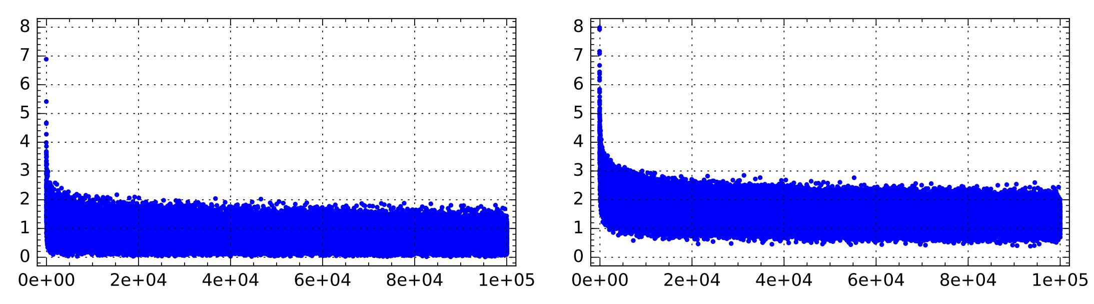
Theorem 2. *The non-trivial zeta zeros are symmetric around the line where* $\mathrm{Re}(z) = \frac{1}{2}$.

Theorem 3. *All of the non-trivial zeta zeros are in the **critical strip**, the region where* $0 \le \mathrm{Re}(z) \le 1$.

Conjecture 1. ***Reimann Hypothesis:*** *All of the non-trivial zeros of* $\zeta(s)$ *are on the line* $\mathrm{Re}(z) = \frac{1}{2}$.

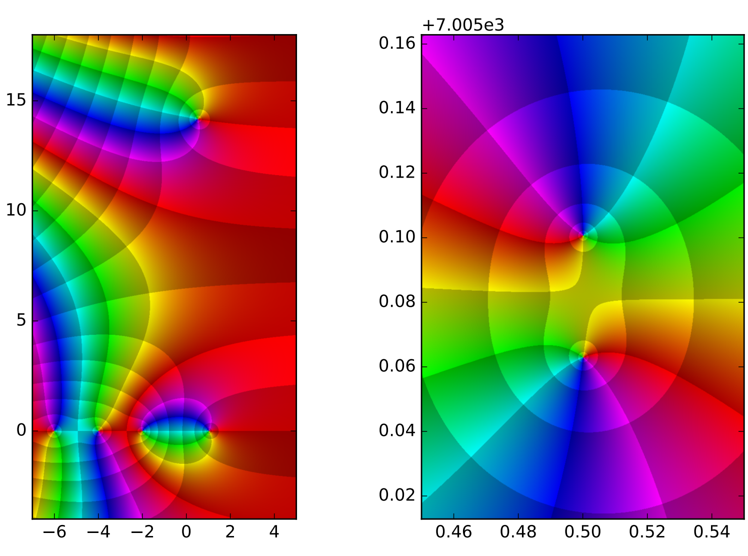


Parametric plots of $\zeta(0.5 + iv):\ 0 \le v \le 30$ in 0.1 increments (left), $0 \le v \le 500$ (right).

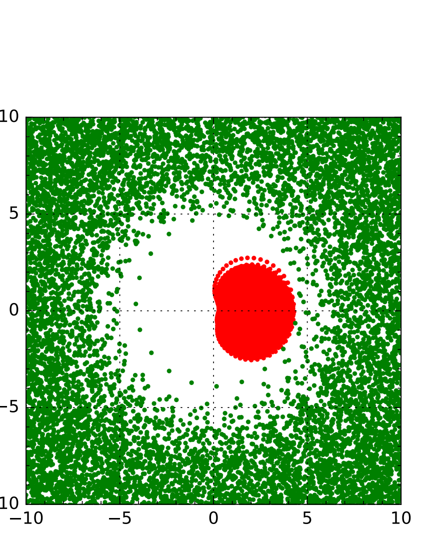


Differences between the first one hundred thousand zeta zero pairs (left) and triples (right).

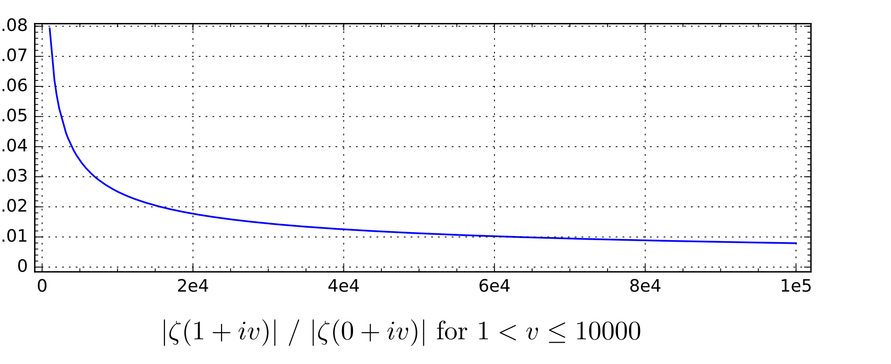
Some *close-pair* zeros:
7005.062866175, 7005.100564674
71732.901207872, 71732.915909348
388858886.0022851203, 388858886.0023936899
777717772.0045702406, 777717772.0047873798



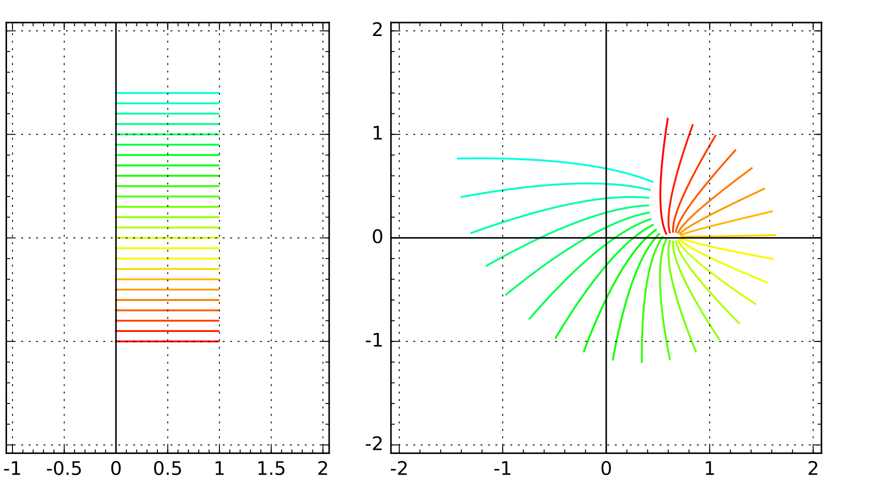
$\zeta(z)$ argument contour plot with isolines for both argument and modulus.
The first non-trivial zero at ~14.13 (left). A *close-pair* of non-trivial zeros at ~7005 (right).



$\zeta(1+iv)$ in red and $\zeta(0+iv)$ in green for $1000 \le v \le 10000$ in 0.05 increments.

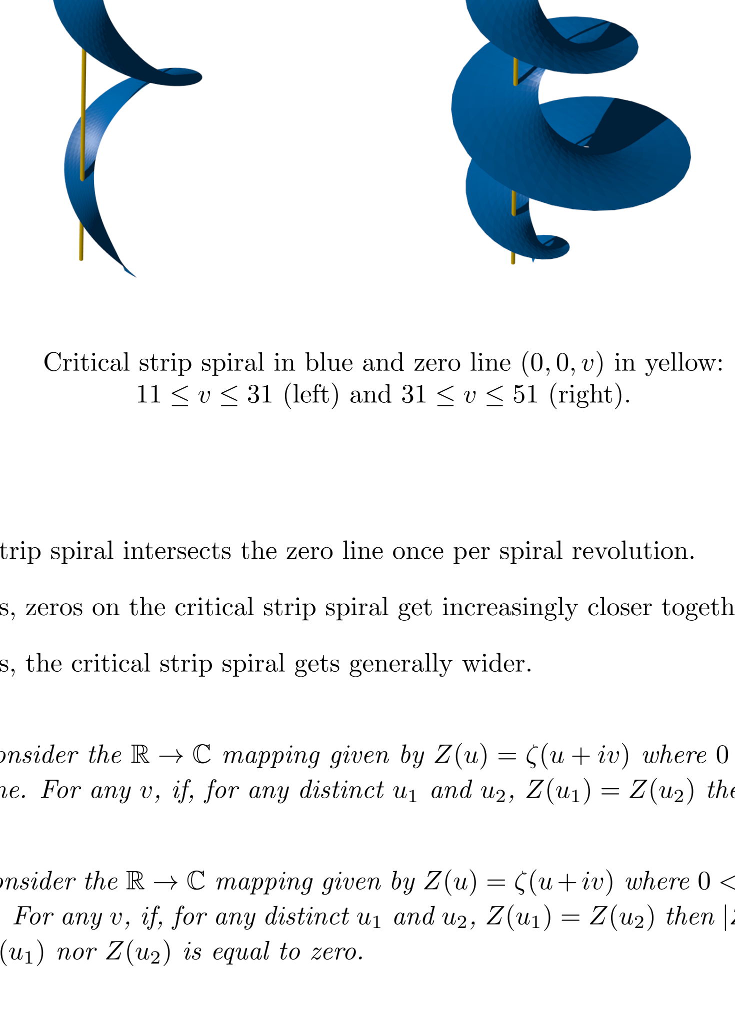


$|\zeta(1+iv)|\ /\ |\zeta(0+iv)|$ for $1 < v \le 10000$



Source points $S \in \mathbb{C}$ (left) and image under mapping $\zeta(S)$ (right).

Definition 3. *The zeta **critical strip spiral** is defined between constants* c_1 *and* c_2 *as the set of all points* $\big(\mathrm{Re}(\zeta(u+iv)),\ \mathrm{Im}(\zeta(u+iv)),\ v\big)$ *where* $0 \le u \le 1$ *and* $c_1 \le v \le c_2$.



Critical strip spiral in blue and zero line $(0, 0, v)$ in yellow:
$11 \le v \le 31$ (left) and $31 \le v \le 51$ (right).

Notes:

1. The critical strip spiral intersects the zero line once per spiral revolution.
2. As v increases, zeros on the critical strip spiral get increasingly closer together.
3. As v increases, the critical strip spiral gets generally wider.

Conjecture 2. *Consider the* $\mathbb{R} \to \mathbb{C}$ *mapping given by* $Z(u) = \zeta(u+iv)$ *where* $0 < u < 1$ *and constant* v *is greater than one. For any* v*, if, for any distinct* u_1 *and* u_2, $Z(u_1) = Z(u_2)$ *then* $|u_1 - \frac{1}{2}| \neq |u_2 - \frac{1}{2}|$.

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