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12 December, 2016
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1 Sequences and Series

The sequence of partial sums of an infinite series either converges to a single fixed value or it diverges.

1.1 Harmonic Series

Definition 1. *The* ***harmonic series*** *is defined as:* $\sum_{n=1}^{\infty} \frac{1}{n}$

Theorem 1. *The harmonic series diverges.*

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots
\end{aligned}$$

RHS diverges $\implies$ LHS diverges. □

1.2 Geometric Series

Definition 2. *The* ***geometric series*** *is defined as:* $\sum_{n=0}^{\infty} r^n$

Theorem 2. *The geometric series* $\sum_{n=0}^{\infty} r^n$ *converges to* $\frac{1}{1-r}$ *when* $|r| < 1$.

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} r^n &= 1 + r + r^2 + r^3 \dots \\
&= \lim_{n\to\infty} (1 + r + r^2 + \dots + r^n) \\
&= \lim_{n\to\infty} \left(\frac{1 - r^{n+1}}{1 - r}\right) \\
&= \frac{1}{1-r} \quad \text{when} \quad |r| < 1
\end{aligned}$$

□