

Calculus

Function Limit

$f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$

$$\lim_{x \rightarrow c} f(x) = l$$
$$\equiv \forall(\epsilon > 0) \exists(\delta > 0) \forall x ((0 < |x - c| < \delta) \Rightarrow (|f(x) - l| < \epsilon))$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^1$

$$\lim_{x \rightarrow x_0} f(x) = l$$
$$\equiv \forall(\epsilon > 0) \exists(\delta > 0) \forall x ((0 < \|x - x_0\| < \delta) \Rightarrow (|f(x) - l| < \epsilon))$$

Limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (1)$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \quad (2)$$

Differentiation

$f : \mathbb{R}^1 \rightarrow \mathbb{R}^1, f'(x) \in \mathbb{R}^1$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

$$\sin' = \cos \quad (3)$$

Differentiation Formulae

$$(fg)' = f'g + fg' \quad (4)$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad (5)$$

$$(f(g))' = f(g)'g' \quad (6)$$

Integration

Change of Variables

$$\int_a^b f(g)g' dx = \int_{g(a)}^{g(b)} f(u) du \quad (7)$$

$$\iint_D f(x, y) dx dy = \iint_{D'} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \quad (8)$$
$$\text{where } \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Integration by Parts

$$\int_a^b f(x) dx = \int_{v(a)}^{v(b)} u dv = uv - \int_{u(a)}^{u(b)} v du \quad (9)$$

Natural Logarithm

$$\ln x \equiv \int_1^x \frac{1}{t} dt$$

$$\int_1^e \frac{1}{t} dt \equiv 1$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow 0} (1+x)^{1/x} \quad (10)$$

Partial Derivative

$f : \mathbb{R}^3 \rightarrow \mathbb{R}^1, \frac{\partial f}{\partial x} \in \mathbb{R}^1$

$$\frac{\partial f}{\partial x}(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

Gradient

scalar field $f : \mathbb{R}^3 \rightarrow \mathbb{R}^1, \nabla f \in \mathbb{R}^3$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl

vector field $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \operatorname{curl} \mathbf{F} \in \mathbb{R}^3$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}$$

Line Integral

$$\iint_P \mathbf{F} \cdot d\mathbf{r} = \int_P F_1 dx + F_2 dy + F_3 dz = \int_{P'} \left(F_1 \frac{\partial x}{\partial t} + F_2 \frac{\partial y}{\partial t} + F_3 \frac{\partial z}{\partial t} \right) dt$$

if $\mathbf{F} = \nabla g$,

$$\int_P \mathbf{F} \cdot d\mathbf{r} = \int_{P'} dg$$

Surface Integral

$$\iint_S dS = \iint_{S'} |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

$$= \iint_R \sqrt{1 + f_x^2 + f_y^2} dx dy$$

Green's Theorem

in the plane

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

Stoke's Theorem

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{F} dV$$

Conservative Field

1. Vector field \mathbf{F} is conservative.

$$\int \mathbf{F} \cdot d\mathbf{r} \text{ is path independent, } \oint \mathbf{F} \cdot d\mathbf{r} = 0$$

2. A scalar function φ exists such that $\mathbf{F} = \nabla \varphi$.

3. $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

4. $\mathbf{F} \cdot d\mathbf{r}$ is an exact differential.

$$\mathbf{F} \cdot d\mathbf{r} = \nabla \varphi \cdot d\mathbf{r} = dg$$

Directional Derivative

$f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ at \mathbf{c} in the direction of $\mathbf{u}, f'(\mathbf{c}; \mathbf{u}) \in \mathbb{R}^1$

$$f'(\mathbf{c}; \mathbf{u}) = \lim_{h \rightarrow 0} \frac{\|\mathbf{f}(\mathbf{c} + h\mathbf{u}) - \mathbf{f}(\mathbf{c})\|}{h}$$
$$= \nabla f(\mathbf{c}) \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} \quad \text{if } \mathbf{f} \text{ is differentiable}$$

Total Derivative

The total derivative of $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ at \mathbf{c} is $\mathbf{T}_c(\mathbf{v})$ where

$$\mathbf{f}(\mathbf{c} + \mathbf{v}) = \mathbf{f}(\mathbf{c}) + \mathbf{T}_c(\mathbf{v}) + \|\mathbf{v}\| \mathbf{E}_c(\mathbf{v})$$
$$\text{and } \mathbf{E}_c(\mathbf{v}) \rightarrow \mathbf{0} \text{ as } \mathbf{v} \rightarrow \mathbf{0}$$

f differentiable at \mathbf{c} implies that f is continuous at \mathbf{c} and the directional derivative exists in all directions.

Notes

The total derivative of $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ at (x, y) is $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$ where

$$f(x+h, y+k) = f(x, y) + \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} + \|(h, k)\| E(h, k)$$

and $E(h, k) \rightarrow 0$ as $(h, k) \rightarrow (0, 0)$

Note: $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \nabla f(x, y) \cdot (h, k)$