# Complex Numbers

The imaginary number $i \equiv \sqrt{-1}$.

Complex numbers, quaternions and octonions are composed of real and imaginary numbers:
$\mathbb{C}$ - Complex numbers: (1x) real, (1x) imaginary, a field.
$\mathbb{H}$ - Quaternions: (1x) real, (3x) imaginary, multiplication not commutative.
$\mathbb{O}$ - Octonions: (1x) real, (7x) imaginary, multiplication neither commutative nor associative.

## Complex Number Definitions

$z \in \mathbb{C}$ and $x, y, r, \theta \in \mathbb{R}$.
Note: $-\pi < \theta \leq \pi$ gives principle value.

| | |
|---|---|
| Cartesian Representation | $z = x + iy$ |
| Complex Conjugate | $\overline{z} = x - iy$ |
| Modulus | $r = \lvert z\rvert = \sqrt{x^2 + y^2}$ |
| Argument | $\theta = \arctan(y/x)$ |
| Real Part | $\mathrm{re}(z) = x = r\cos\theta$ |
| Imaginary Part | $\mathrm{im}(z) = y = r\sin\theta$ |
| Polar Representation | $z = re^{i\theta}$ |

## Complex Number Theorems

| | |
|---|---|
| Euler's Formula | $e^{i\theta} = \cos\theta + i\sin\theta = \mathrm{cis}\,\theta$ |
| Demoivre's Theorem | $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ |
| Triangle Inequality | $\lvert z_1 \pm z_2\rvert \leq \lvert z_1\rvert + \lvert z_2\rvert$ |

## Complex Identities

$$e^{i\pi} = -1 \tag{1}$$

$$z = r(\cos\theta + i\sin\theta) \tag{2}$$

$$\mathrm{re}(z) = (z + \overline{z})/2 \tag{3}$$

$$\mathrm{im}(z) = (z - \overline{z})/2i \tag{4}$$

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2 \tag{5}$$

$$\sin\theta = (e^{i\theta} - e^{-i\theta})/2i \tag{6}$$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \tag{7}$$

$$|z| = |\overline{z}| \tag{8}$$

$$|z_1||z_2| = |z_1 z_2| = |z_1\overline{z_2}| \tag{9}$$

$$|z|^2 = |z^2| = |z\overline{z}| = z\overline{z} \tag{10}$$

$$||z_1| - |z_2|| \leq |z_1| - |z_2| \leq |z_1 \pm z_2| \leq |z_1| + |z_2| \tag{11}$$

$$z_0 z_1 = r_0 r_1 \big(\cos(\theta_0 + \theta_1) + \sin(\theta_0 + \theta_1)\big) \tag{12}$$

$$z_0{}^n = r_0{}^n(\cos n\theta_0 + i\sin n\theta_0) \tag{13}$$

$$\begin{gathered} z_0{}^{n/m} = \sqrt[m]{r_0{}^n}\Big(\cos\big(\tfrac{n}{m}(\theta_0 + 2k\pi)\big) + i\sin\big(\tfrac{n}{m}(\theta_0 + 2k\pi)\big)\Big) \\ \text{where } k = 0, 1, \ldots, m-1 \end{gathered} \tag{14}$$

Let $z = \cos\theta + i\sin\theta$, then

$$z^n = \cos(n\theta) + i\sin(n\theta) \qquad \frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta) \tag{15}$$

$$\cos(n\theta) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \qquad \sin(n\theta) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right) \tag{16}$$

$$\cos^n\theta = \frac{1}{2^n}\left(z + \frac{1}{z}\right)^n \qquad \sin^n\theta = \frac{1}{(2i)^n}\left(z - \frac{1}{z}\right)^n \tag{17}$$

## Complex Function Definitions

$$\ln(z) = \ln|z| + i\arg(z) \tag{18}$$

$$e^z = e^{x+iy} = e^x e^{iy} \tag{19}$$

$$z^c = e^{c\ln z} \tag{20}$$

$$\cos(z) = (e^{iz} + e^{-iz})/2 \tag{21}$$

$$\sin(z) = (e^{iz} - e^{-iz})/2i \tag{22}$$

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