

An Exploration of a Bijective Mapping in the Integers

John Rugis*

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Abstract

In this article we begin by considering the possible existence of two bijective integer valued functions where the sum of the functions is also bijective. An example of two such functions is produced. Some interesting properties of these functions, as well as the algorithm that is used to produce them, are both explored.

1 Preliminaries

Given two functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$, define a new function $(f+g) : \mathbb{Z} \rightarrow \mathbb{Z}$ by $(f+g)(n) = f(n) + g(n)$. Is it possible that f and g as well as $f+g$ could all be bijective? We will prove that it is possible by producing an example.

We begin by sequentially selecting and plotting points $(f(n), g(n))$ on a graphic grid that represents \mathbb{Z}^2 . We will work our way through all of \mathbb{Z} by considering n 's in the order $0, +1, -1, +2, -2, +3, -3, \dots$ and impose three sufficient conditions.

Selection Conditions:

1. We will select points $(f(n), g(n)) \in \mathbb{Z}^2$ such that for all $n \in \mathbb{Z}$, $f(n) + g(n) = n$. This will ensure that $(f+g)(n)$ is bijective.
2. If we are careful in our selection of points $(f(n), g(n))$ making sure that $f(n)$ never maps to the same value twice and $g(n)$ never maps to the same value twice, then both $f(n)$ and $g(n)$ will be injective.
3. And finally, if we are also careful in our selection of points $(f(n), g(n))$ making sure that $f(n)$ and $g(n)$ map to all elements of \mathbb{Z} , then $f(n)$ and $g(n)$ will be surjective.

In the next section we produce an example that meets these conditions.

2 A Mapping Algorithm

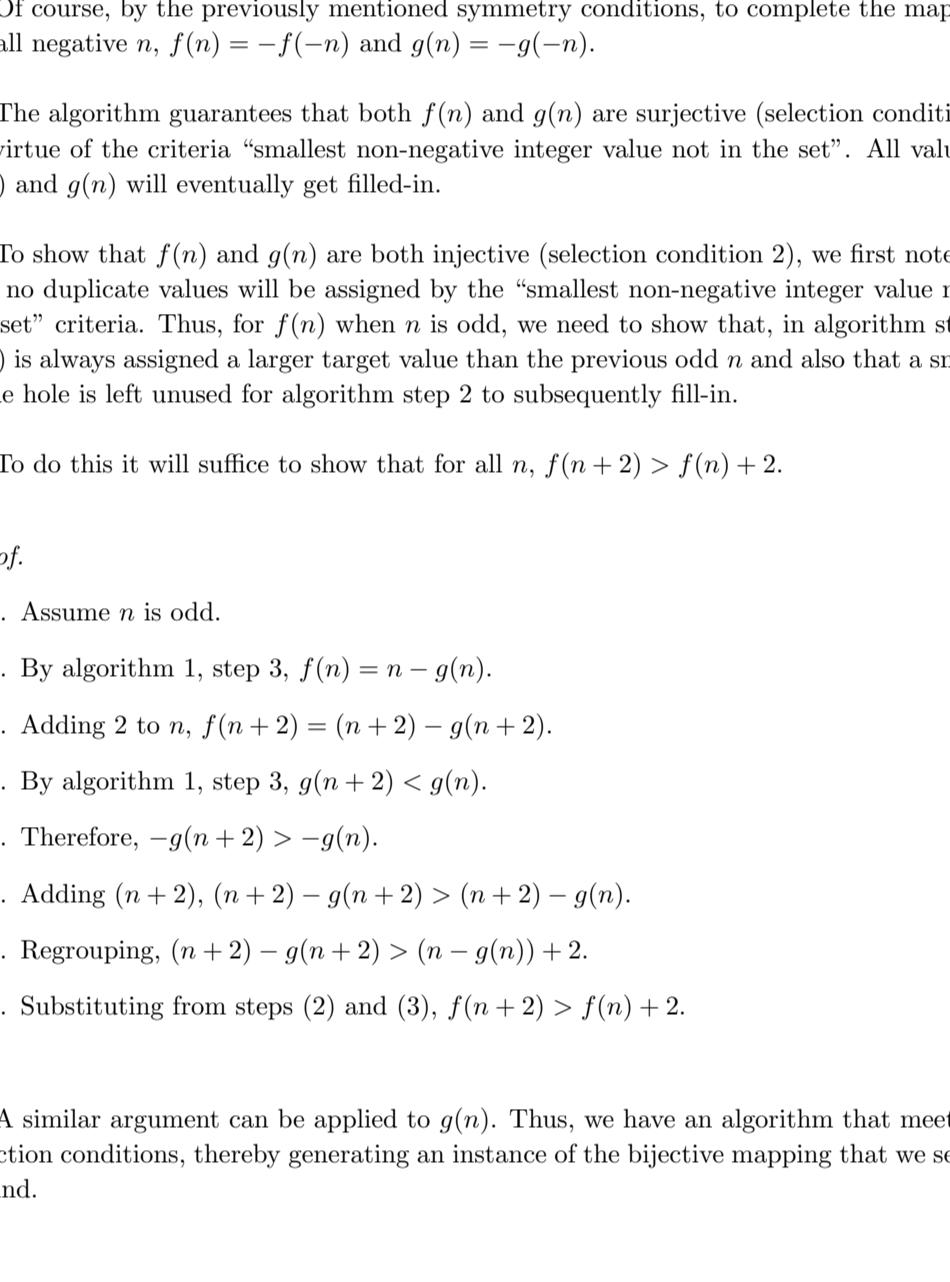


Figure 1: Beginning at the point plot of $(f(n), g(n))$ in \mathbb{Z}^2 on the left. The first dashed *keep-off* lines are shown on the right.

Starting with $n = 0$, we note that, to meet condition 1, the point $(f(0), g(0))$ will need to be on the line labeled $n = 0$ in Figure 1. Let's (somewhat arbitrarily) start by selecting the point $(0, 0)$. Next we draw two dashed *keep-off* lines $f(n) = 0$ and $g(n) = 0$. If we make sure that no other points that are selected lie on a *keep-off* line, then we will have met condition 2.

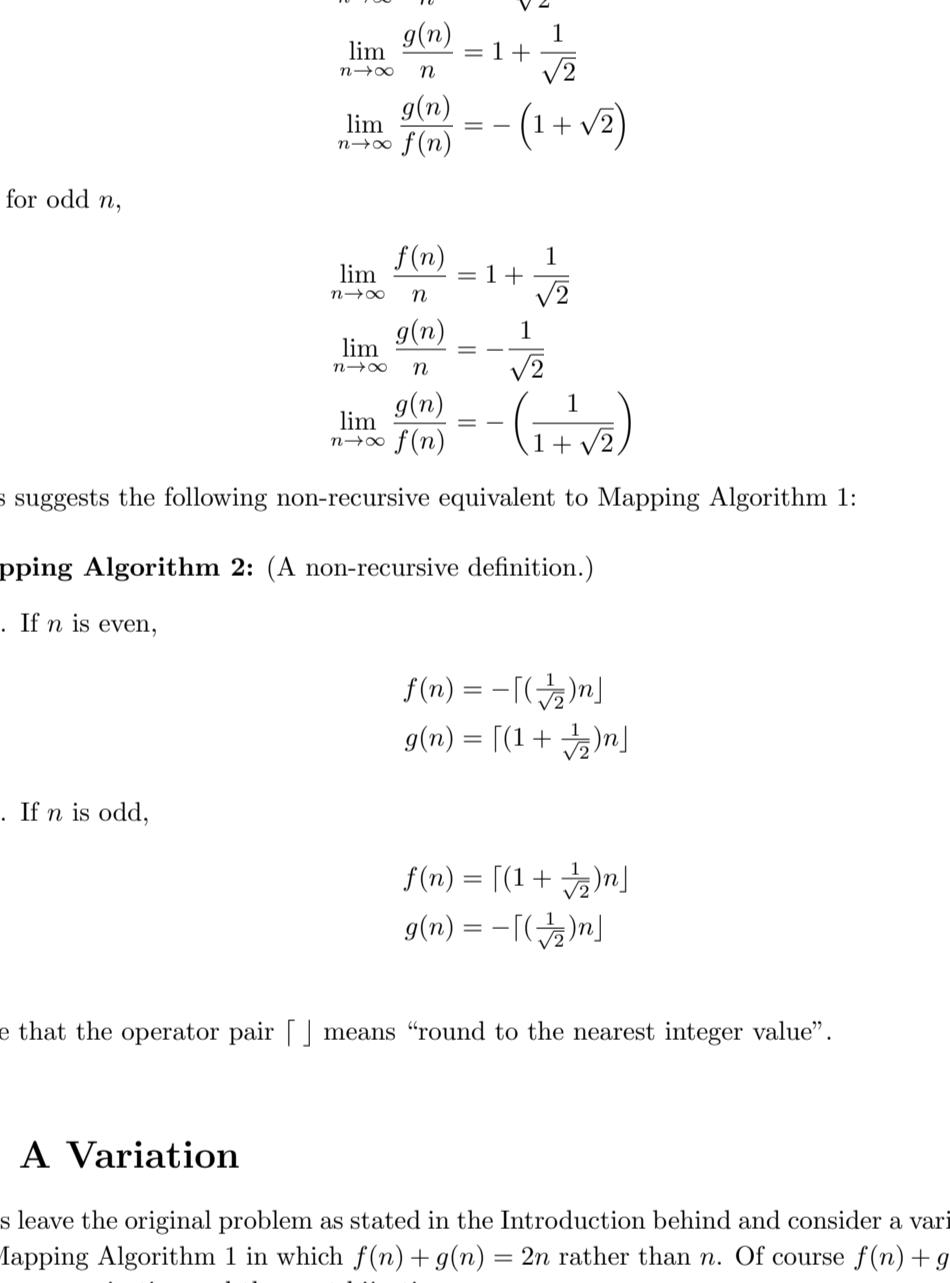


Figure 2: Evolving point selection.

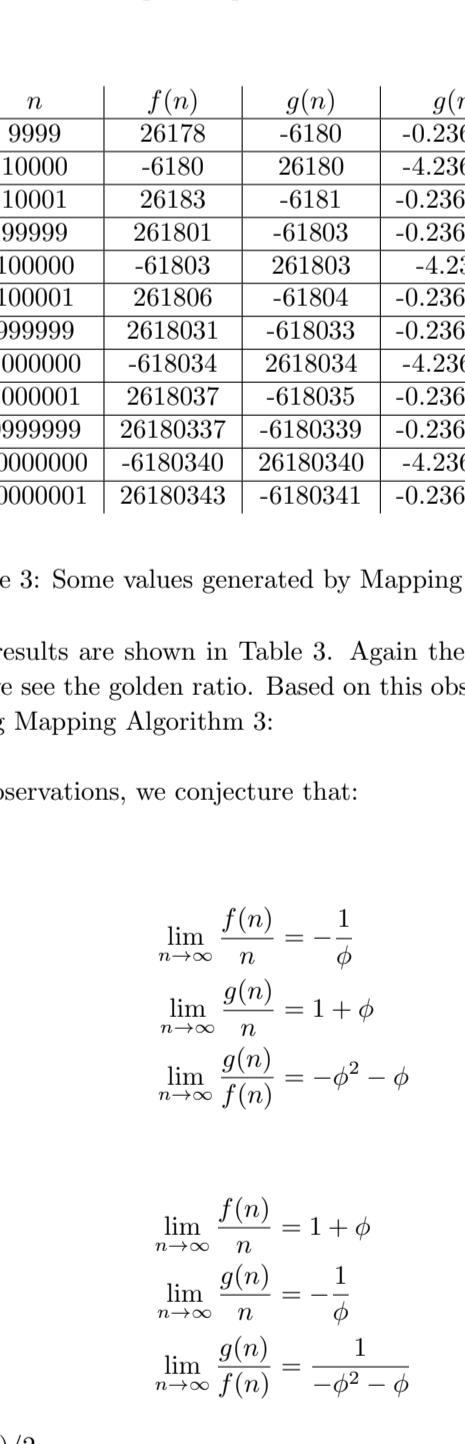


Figure 3: Plotted points up to $n = \pm 6$.

As we work our way through values of n , we note that the points we select need to lie on their respective diagonal line as shown in the evolving plot Figure 2. In a systematic effort to meet condition 3, we will select points that are as close to $(0, 0)$ as possible. So, for $n = 1$ we select the point $(2, -1)$ and for $n = -1$ we select the point $(-2, 1)$ as shown in Figure 2. And then, following the same rule, for $n = 2$ we select the point $(-1, 3)$ and for $n = -2$ we select the point $(1, -3)$. As we continue systematically plotting points, we make the side observation that the plotted points appear to closely fit two lines passing through zero (see Figure 3).

Table 1: Tabulated values for $0 \leq n \leq 17$.

In an attempt to discover an algebraic (as opposed to geometric) definition for our sequences of $f(n)$ and $g(n)$ values, we continue with a list of values as shown in Table 1. Note that because of the symmetries $f(-n) = -f(n)$ and $g(-n) = -g(n)$, we can, without loss of generality, use a table that gives values only for $n \geq 0$ as shown in Table 1. The table has been split into columns for odd and even values of n . Note that when n is odd, $f(n)$ is positive and $g(n)$ is negative. For all even values of n greater than zero, $f(n)$ is negative and $g(n)$ is positive.

Observation of the evolving pattern leads to the following algorithm for specifying the sequence of values for $f(n)$ and $g(n)$:

Mapping Algorithm 1: (A strongly recursive definition.)

1. Initialise n equal to zero.
2. If n is even, set $f(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $f(n)$'s. Then assign $g(n)$ such that $f(n) + g(n) = n$.
3. If n is odd, set $g(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $g(n)$'s. Then assign $f(n)$ such that $f(n) + g(n) = n$.
4. Increment n by one and loop to step 2.

Of course, by the previously mentioned symmetry conditions, to complete the mapping, for all negative n , $f(n) = -f(-n)$ and $g(n) = -g(-n)$.

The algorithm guarantees that both $f(n)$ and $g(n)$ are surjective (selection condition 3) by virtue of the criteria "smallest non-negative integer value not in the set". All values of $f(n)$ and $g(n)$ will eventually get filled-in.

To do this it will suffice to show that for all n , $f(n+2) > f(n)+2$.

Proof.

1. Assume n is odd.
2. By algorithm 1, step 3, $f(n) = n - g(n)$.
3. Adding 2 to n , $f(n+2) = (n+2) - g(n+2)$.
4. By algorithm 1, step 3, $g(n+2) < g(n)$.

5. Therefore, $-g(n+2) > -g(n)$.

6. Adding $(n+2)$, $(n+2) - g(n+2) > (n+2) - g(n)$.

7. Regrouping, $(n+2) - g(n+2) > (n - g(n)) + 2$.

8. Substituting from steps (2) and (3), $f(n+2) > f(n) + 2$.

□

A similar argument can be applied to $g(n)$. Thus, we have an algorithm that meets the selection conditions, thereby generating an instance of the bijective mapping that we set out to find.

3 Further Exploration

The fact that, as previously observed, the plotted points in Figure 3 appear to closely fit two lines, leads us to consider the possibility of an equivalent non-recursive definition for $f(n)$ and $g(n)$. With the assistance of a computer, we calculated the values of $f(n)$ and $g(n)$ for $0 \leq n \leq 10000000$. Some of the results are shown in Table 2.

Table 2: Some values generated by Mapping Algorithm 1.

The values of $f(n)$ and $g(n)$, especially when n is a power of 10, look familiar! Based on these observations, we conjecture that:

Mapping Algorithm 2: (A non-recursive definition.)

1. Initialise n equal to zero.
2. If n is even, set $f(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $f(n)$'s. Then assign $g(n)$ such that $f(n) + g(n) = n$.
3. If n is odd, set $g(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $g(n)$'s. Then assign $f(n)$ such that $f(n) + g(n) = n$.
4. Increment n by one and loop to step 2.

Based on these observations, we conjecture that:

For even n ,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{n} = -\frac{1}{\sqrt{2}} \quad (1)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{n} = 1 + \frac{1}{\sqrt{2}} \quad (2)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = -(1 + \sqrt{2}) \quad (3)$$

and for odd n ,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{n} = 1 + \frac{1}{\sqrt{2}} \quad (4)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{n} = -\frac{1}{\sqrt{2}} \quad (5)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \left(\frac{1}{1 + \sqrt{2}} \right) \quad (6)$$

where $\phi = (1 + \sqrt{5})/2$.

5 A Generalisation

Consider the following parameterised generalisation, for positive integers k , in which $f(n) + g(n) = kn$. Of course $f(n) + g(n)$ is only bijective when $k = 1$.

Mapping Algorithm 4: (A strongly recursive definition parameterised by $k \in \mathbb{Z}^+$.)

1. Initialise n equal to zero.
2. If n is even, set $f(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $f(n)$'s. Then assign $g(n)$ such that $f(n) + g(n) = kn$.
3. If n is odd, set $g(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $g(n)$'s. Then assign $f(n)$ such that $f(n) + g(n) = kn$.
4. Increment n by one and loop to step 2.

Table 3: Some values generated by Mapping Algorithm 3.

Some calculated results are shown in Table 3. Again the values of $f(n)$ and $g(n)$ look familiar! This time we see the golden ratio. Based on this observation, we make the conjecture that, when using Mapping Algorithm 3:

Based on these observations, we conjecture that:

For even n ,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{n} = -\frac{1}{\phi} \quad (7)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{n} = 1 + \frac{1}{\phi} \quad (8)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = -\phi^2 - \phi \quad (9)$$

and for odd n ,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{n} = 1 + \frac{1}{\phi} \quad (10)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{n} = -\frac{1}{\phi} \quad (11)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \frac{1}{-\phi^2 - \phi} \quad (12)$$

where $\phi = (1 + \sqrt{5})/2$.

4 A Variation

Let's leave the original problem as stated in the Introduction behind and consider a variation to Mapping Algorithm 1 in which $f(n) + g(n) = 2n$ rather than n . Of course $f(n) + g(n)$ is no longer surjective and thus not bijective.

Mapping Algorithm 3: (A strongly recursive definition.)

1. Initialise n equal to zero.
2. If n is even, set $f(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $f(n)$'s. Then assign $g(n)$ such that $f(n) + g(n) = 2n$.
3. If n is odd, set $g(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $g(n)$'s. Then assign $f(n)$ such that $f(n) + g(n) = 2n$.
4. Increment n by one and loop to step 2.

Table 4: Some values generated when $k = 13$.

Using Mapping Algorithm 4, and supported by additional calculated observations (such as shown for $k = 10$ in Table 4), we make the conjecture that:

For even n ,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{n} = -\frac{1}{\sqrt{2}} \quad (13)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{n} = 1 + \frac{1}{\sqrt{2}} \quad (14)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \frac{1}{(1 + \sqrt{2})^2} \quad (15)$$

and for odd n ,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{n} = 1 + \frac{1}{\sqrt{2}} \quad (16)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{n} = -\frac{1}{\sqrt{2}} \quad (17)$$

$$\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \frac{1}{(1 - \sqrt{2})^2} \quad (18)$$

We can use ψ_1 to show a connection between the two rather special numbers $\sqrt{2}$ and the golden ratio:

$$\psi_1 = -\lceil (\frac{1}{\sqrt{2}})n \rceil \quad (19)$$

and when $k = 2$,

$$\phi = -\psi_1 \quad (20)$$

Note that ψ_2 has the property that:

$$\psi_2 = -2k + \frac{1}{\psi_1} \quad (21)$$

which is equivalent to saying that the fractional part of ψ_2 is equal to $1/\psi_1$. This certainly appears to be supported by examining the $g(n)/f(n)$ column in each of the previous mapping tables.

There is also a connection between the generalised limits and solutions to certain quadratic equations. Note that we have:

$$\psi_1 = \frac{-(k-1) - \sqrt{k^2 + 1}}{k} \quad (22)$$

$$\psi_2 = -k - \sqrt{k^2 + 1} \quad (23)$$

These ψ_1 and ψ_2 are clearly solutions, respectively, to the following quadratic equations:

$$2x^2 - (2k-2)x - 2 = 0 \quad (24)$$