

Complex Analysis

Complex Analysis Definitions

$z \in \mathbb{C}$, $\epsilon, \delta \in \mathbb{R}$, $S \subseteq \mathbb{C}$ and $f : S \rightarrow \mathbb{C}$.

ϵ -neighborhood: The ϵ -neighborhood of z_0 , $N_\epsilon(z_0) \equiv \{z : |z - z_0| < \epsilon\}$.

Interior Point: z is an interior point of S means that $\exists(\epsilon > 0)(N_\epsilon(z) \subseteq S)$.

Open Set: S is open iff every element z of S has some ϵ -neighborhood that is a subset of S .

$$S \text{ is open} \equiv \forall(z \in S) \exists(\epsilon > 0)(N_\epsilon(z) \subseteq S)$$

Alternatively, S is open iff every point of S is an interior point.

Relatively Open Set: $V \subseteq S$ is relatively open in S means that $\forall(z \in V) \exists(\epsilon > 0)((N_\epsilon(z) \cap S) \subseteq V)$.

Set Complement: The complement of a set S , $\mathbb{C} \setminus S \equiv \{z : z \notin S\}$.

Closed Set: A set S is closed iff its complement is open. Alternatively, a set S is closed iff it contains all its limit points. Note that S could be neither open nor closed.

Limit Point: z_0 is a limit point of a set S iff every ϵ -neighborhood of z_0 contains a point of S other than z_0 .

$$z_0 \text{ is a limit point of } S \equiv \forall(\epsilon > 0) \exists(z \in S)(z \in N_\epsilon(z_0) \wedge z \neq z_0)$$

Isolated Point: Any point in S that is not a limit point is said to be an isolated point.

Function Limit: Given z_0 a limit point of S ,

$$\lim_{z \rightarrow z_0} f(z) = l \equiv \forall(\epsilon > 0) \exists(\delta > 0) \forall(z \in S)((0 < |z - z_0| < \delta) \Rightarrow (|f(z) - l| < \epsilon))$$

Function Continuity: The function f is continuous at z_0 iff

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) &= f(z_0) \quad \text{when } z_0 \text{ is a limit point,} \\ \forall(\epsilon > 0) \exists(\delta > 0)((N_\delta(z_0) \cap S) &\subseteq N_\epsilon(f(z_0))) \quad \text{when } z_0 \text{ is an isolated point.} \end{aligned}$$

Infinite Sequence Limit: The infinite sequence $\{z_n\}$ converges to $l \in \mathbb{C}$ by definition iff for every ϵ -neighborhood of l , there is an $m \in \mathbb{N}$ such that if $n > m$ then z_n is in the given neighborhood.

$$\lim_{n \rightarrow \infty} f(z_n) = l \equiv \forall(\epsilon > 0) \exists(m) \Big((n > m) \Rightarrow (|z_n - l| < \epsilon) \Big)$$

Proper Convergence: The infinite sequence $\{z_n\}$ converges properly to $l \in \mathbb{C}$ iff $\{z_n\}$ converges to l and $z_n \neq l$ ever.

Complex Derivative

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Complex Analysis Examples

Given the sequence $z_n = \frac{i}{n} \Big(1 + (-1)^n \Big) = \left\{ 0, \frac{i}{1}, 0, \frac{i}{2}, 0, \frac{i}{2}, \dots \right\}$,

$$\lim_{n \rightarrow \infty} z_n = 0$$

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Complex Analysis Theorems

Closed Sets: A set is closed iff it contains all its limit points.

Function Continuity and Open Sets: A complex function $f : S \rightarrow \mathbb{C}$ is continuous iff for every open set U , the inverse image $f^{-1}(U)$ is relatively open in S .

Complex Sequence Convergence: Given a sequence $\{z_n\}$ where $z_n = x_n + iy_n$,

$$\lim_{n \rightarrow \infty} z_n = z \quad \Leftrightarrow \quad \left(\lim_{n \rightarrow \infty} x_n = x \wedge \lim_{n \rightarrow \infty} y_n = y \right)$$