

# 1 Infinite Series

The sequence of partial sums of an infinite series either converges to a single fixed value or it diverges.

## 1.1 Harmonic Series

**Definition 1.** The **harmonic series** is defined as:  $\sum_{n=1}^{\infty} \frac{1}{n}$

**Theorem 1.** The harmonic series diverges.

*Proof.*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots\end{aligned}$$

RHS diverges  $\implies$  LHS diverges. □

## 1.2 Geometric Series

**Definition 2.** The **geometric series** is defined as:  $\sum_{n=0}^{\infty} r^n$

**Theorem 2.** The geometric series  $\sum_{n=0}^{\infty} r^n$  converges to  $\frac{1}{1-r}$  when  $|r| < 1$ .

*Proof.*

$$\begin{aligned}\sum_{n=0}^{\infty} r^n &= 1 + r + r^2 + r^3 \dots \\ &= \lim_{n \rightarrow \infty} (1 + r + r^2 + \dots + r^n) \\ &= \lim_{n \rightarrow \infty} \left( \frac{1 - r^{n+1}}{1 - r} \right) \\ &= \frac{1}{1 - r} \text{ when } |r| < 1\end{aligned}$$

□