

Zeta Function

Definition 1. The *zeta function*  $\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}$

Converges when  $\text{Re}(z) > 1$ .

$$\zeta(\bar{z}) = \overline{\zeta(z)}$$
(1)

Zeta Product Formula

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}} \text{ , } \text{converges when } \text{Re}(z) > 1.$$
(2)

Analytic Continuation

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx \text{ , } \text{converges absolutely when } \text{Re}(z) > 0.$$
(3)

$$\Gamma(z + 1) = z\Gamma(z)$$
(4)

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1 - z) \zeta(1 - z)$$
(5)

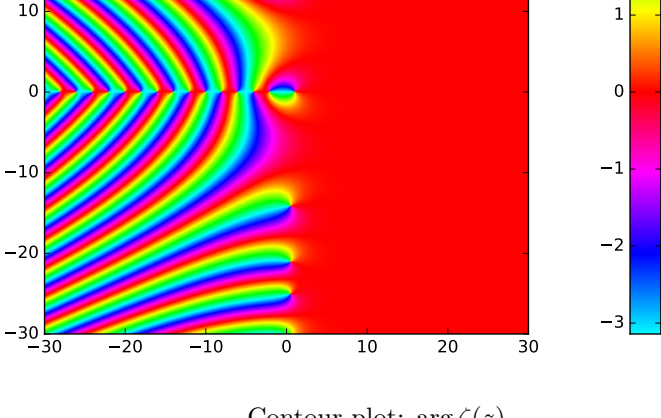
$$\zeta(1 - z) = \frac{2}{(2\pi)^z} \cos\left(\frac{z\pi}{2}\right) \Gamma(z) \zeta(z)$$
(6)

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \text{ , } \text{converges when } \text{Re}(z) > 0.$$
(7)

$$\zeta(z) = \frac{\eta(z)}{1 - 2^{(1-z)}}$$
(8)

$$\mu(n) = \begin{cases} 1 & n \text{ square-free with an even number of prime factors} \\ -1 & n \text{ square-free with an odd number of prime factors} \\ 0 & n \text{ has a squared prime factor} \end{cases}$$
(9)

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$$
(10)



Notes:

- 1. The values of  $\zeta(z)$  where  $\text{Re}(z) > 1$  are relatively static.
- 2. There is a single pole at  $z = 1$  and zeros where  $z = \{-2, -4, -6, -8, \dots\}$ .

Reimann Function

Alternative formulation:

$$R(x) = 1 + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n n! \zeta(n+1)}$$
(11)

Non-trivial Zeta Zeros

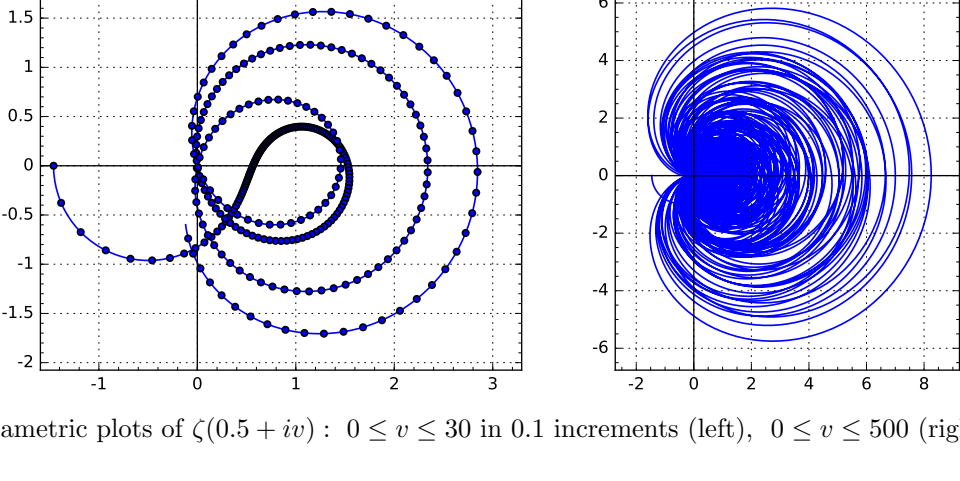
Definition 2. The *non-trivial zeta zeros* are all of the zeta zeros that are not on the real axis.

Theorem 1. The non-trivial zeta zeros are symmetric around the real axis.

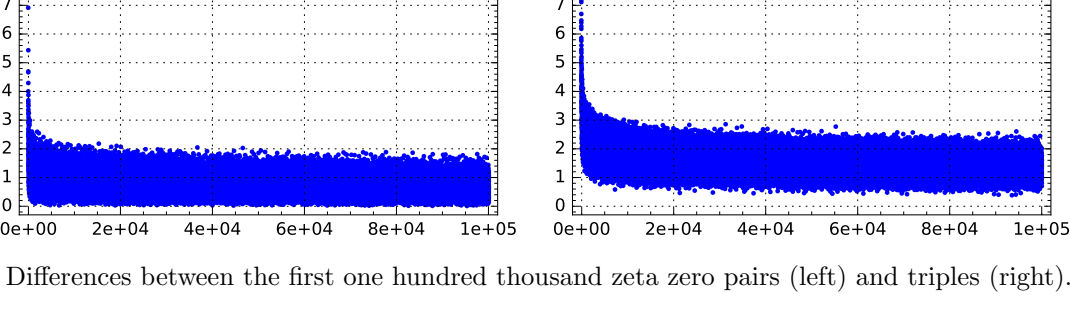
Theorem 2. The non-trivial zeta zeros are symmetric around the line where  $\text{Re}(z) = \frac{1}{2}$ .

Theorem 3. All of the non-trivial zeta zeros are in the *critical strip*, the region where  $0 \leq \text{Re}(z) \leq 1$ .

Conjecture 1. *Reimann Hypothesis:* All of the non-trivial zeros of  $\zeta(s)$  are on the line  $\text{Re}(z) = \frac{1}{2}$ .



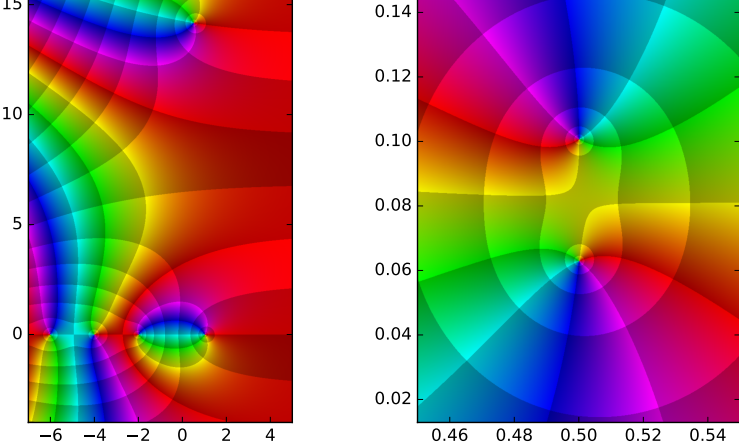
Parametric plots of  $\zeta(0.5 + iv) : 0 \leq v \leq 30$  in 0.1 increments (left),  $0 \leq v \leq 500$  (right).



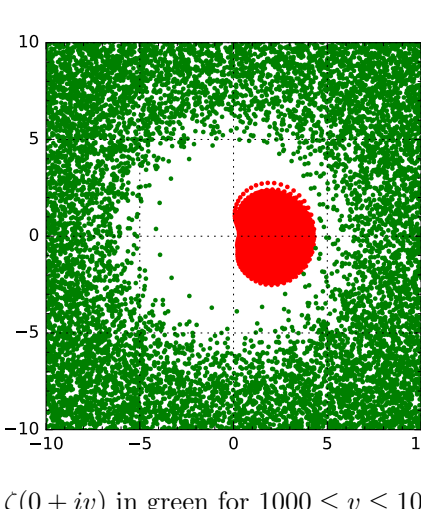
Differences between the first one hundred thousand zeta zero pairs (left) and triples (right).

Some *close-pair* zeros:

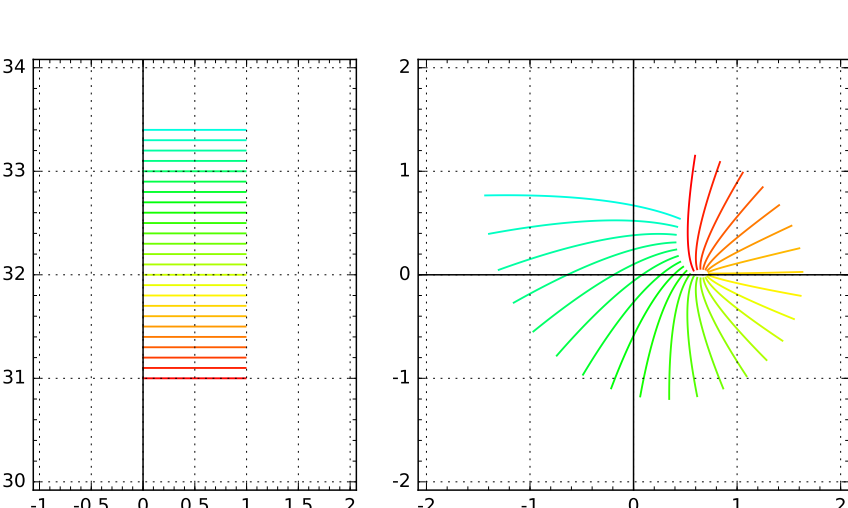
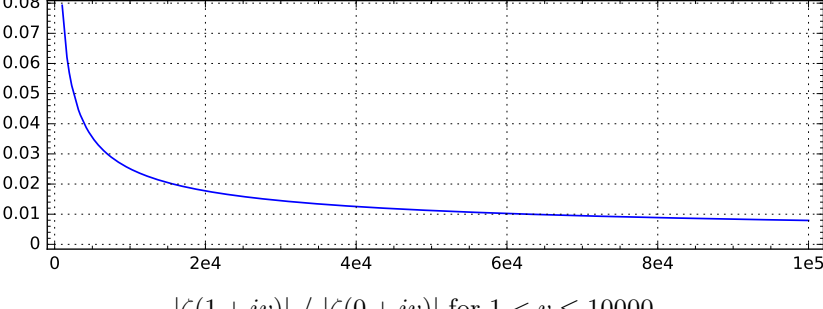
7005.062866175, 7005.100564674  
71732.901207872, 71732.915909348  
388858886.0022851203, 388858886.0023936899  
777717772.0045702406, 777717772.0047873798



$\zeta(z)$  argument contour plot with isolines for both argument and modulus.  
The first non-trivial zero at  $\sim 14.13$  (left). A *close-pair* of non-trivial zeros at  $\sim 7005$  (right).

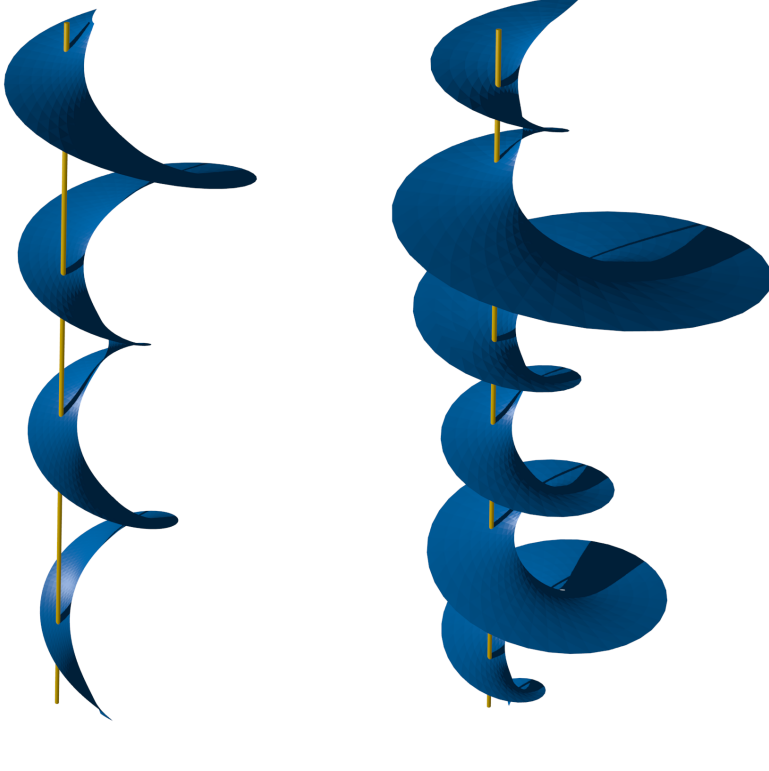


$\zeta(1 + iv)$  in red and  $\zeta(0 + iv)$  in green for  $1000 \leq v \leq 10000$  in 0.05 increments.



Source points  $S \in \mathbb{C}$  (left) and image under mapping  $\zeta(S)$  (right).

Definition 3. The zeta *critical strip spiral* is defined between constants  $c_1$  and  $c_2$  as the set of all points  $(\text{Re}(\zeta(u + iv)), \text{Im}(\zeta(u + iv)), v)$  where  $0 \leq u \leq 1$  and  $c_1 \leq v \leq c_2$ .



Critical strip spiral in blue and zero line  $(0, 0, v)$  in yellow:  
 $11 \leq v \leq 31$  (left) and  $31 \leq v \leq 51$  (right).

Notes:

- 1. The critical strip spiral intersects the zero line once per spiral revolution.
- 2. As  $v$  increases, zeros on the critical strip spiral get increasingly closer together.
- 3. As  $v$  increases, the critical strip spiral gets generally wider.

Conjecture 2. Consider the  $\mathbb{R} \rightarrow \mathbb{C}$  mapping given by  $Z(u) = \zeta(u + iv)$  where  $0 < u < 1$  and constant  $v$  is greater than one. For any  $v$ , if, for any distinct  $u_1$  and  $u_2$ ,  $Z(u_1) = Z(u_2)$  then  $|u_1 - \frac{1}{2}| \neq |u_2 - \frac{1}{2}|$ .

Conjecture 3. Consider the  $\mathbb{R} \rightarrow \mathbb{C}$  mapping given by  $Z(u) = \zeta(u + iv)$  where  $0 < u < 1$  and constant  $v$  is greater than one. For any  $v$ , if, for any distinct  $u_1$  and  $u_2$ ,  $Z(u_1) = Z(u_2)$  then  $|Z(1) - Z(u_1)| < |Z(1) - Z(u_2)|$  and thus neither  $Z(u_1)$  nor  $Z(u_2)$  is equal to zero.