

Zeta Function - Complex

Definition 1. The *zeta function* $\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}$

Converges when $\text{Re}(z) > 1$.

$$\zeta(\bar{z}) = \overline{\zeta(z)}$$
(1)

Zeta Product Formula

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}$$
(2)

Analytic Continuation

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}, \quad \text{converges when } \text{Re}(z) > 0.$$
(3)

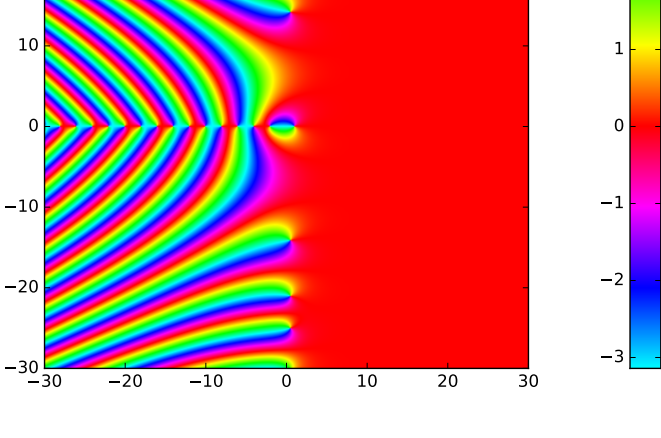
$$\zeta(z) = \frac{\eta(z)}{1 - 2^{(1-z)}}$$
(4)

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx, \quad \text{converges absolutely when } \text{Re}(z) > 0.$$
(5)

$$\Gamma(z + 1) = z\Gamma(z)$$
(6)

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z)$$
(7)

$$\zeta(1-z) = \frac{2}{(2\pi)^z} \cos\left(\frac{z\pi}{2}\right) \Gamma(z) \zeta(z)$$
(8)



Notes:

- 1. The values of $\zeta(z)$ where $\text{Re}(z) > 1$ are relatively static.
- 2. There is a single pole at $z = 1$ and zeros where $z = \{-2, -4, -6, -8, \dots\}$.

Zeta Equivalences

$$\mu(n) = \begin{cases} 1 & n \text{ square-free with an even number of prime factors} \\ -1 & n \text{ square-free with an odd number of prime factors} \\ 0 & n \text{ has a squared prime factor} \end{cases}$$
(9)

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$$
(10)

Non-trivial Zeta Zeros

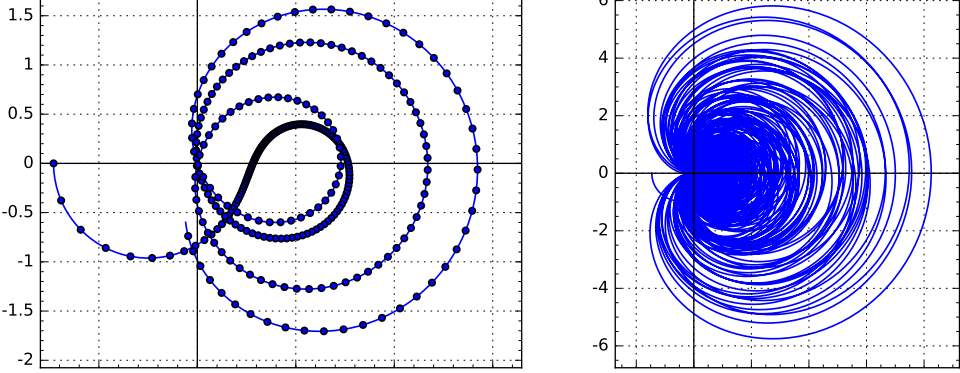
Definition 2. The *non-trivial zeta zeros* are all of the zeta zeros that are not on the real axis.

Theorem 1. The non-trivial zeta zeros are symmetric around the real axis.

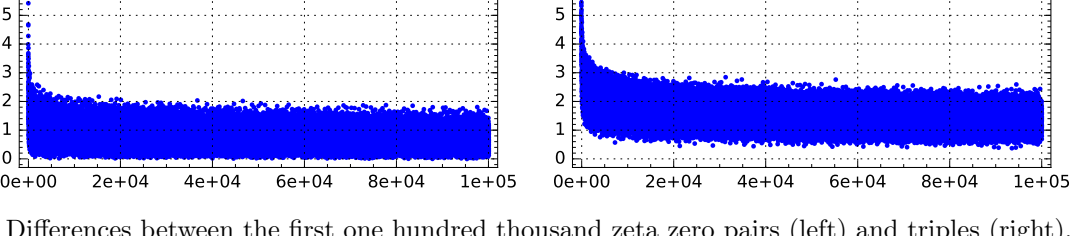
Theorem 2. The non-trivial zeta zeros are symmetric around the line where $\text{Re}(z) = \frac{1}{2}$.

Theorem 3. All of the non-trivial zeta zeros are in the *critical strip*, the region where $0 \leq \text{Re}(z) \leq 1$.

Conjecture 1. *Reimann Hypothesis*: All of the non-trivial zeros of $\zeta(s)$ are on the line $\text{Re}(z) = \frac{1}{2}$.



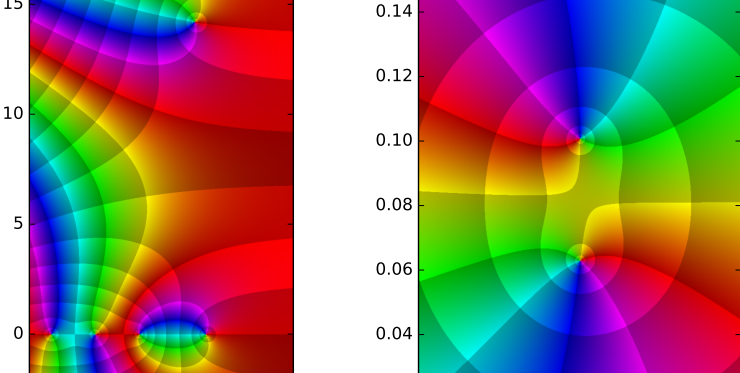
Parametric plots of $\zeta(0.5 + iv) : 0 \leq v \leq 30$ in 0.1 increments (left), $0 \leq v \leq 500$ (right).



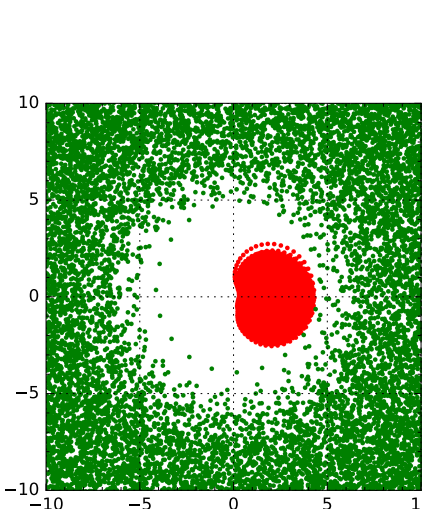
Differences between the first one hundred thousand zeta zero pairs (left) and triples (right).

Some *close-pair* zeros:

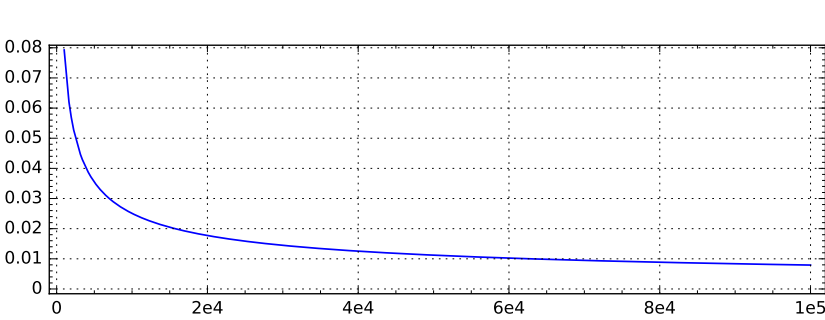
7005.062866175, 7005.100564674
71732.901207872, 71732.915909348
388858886.0022851203, 388858886.0023936899
777717772.0045702406, 777717772.0047873798



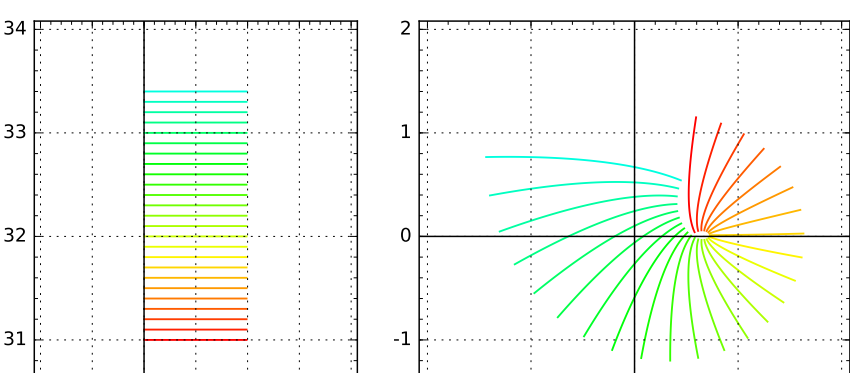
$\zeta(z)$ argument contour plot with isolines for both argument and modulus.
The first non-trivial zero at ~14.13 (left). A *close-pair* of non-trivial zeros at ~7005 (right).



$\zeta(1 + iv)$ in red and $\zeta(0 + iv)$ in green for $1000 \leq v \leq 10000$ in 0.05 increments.

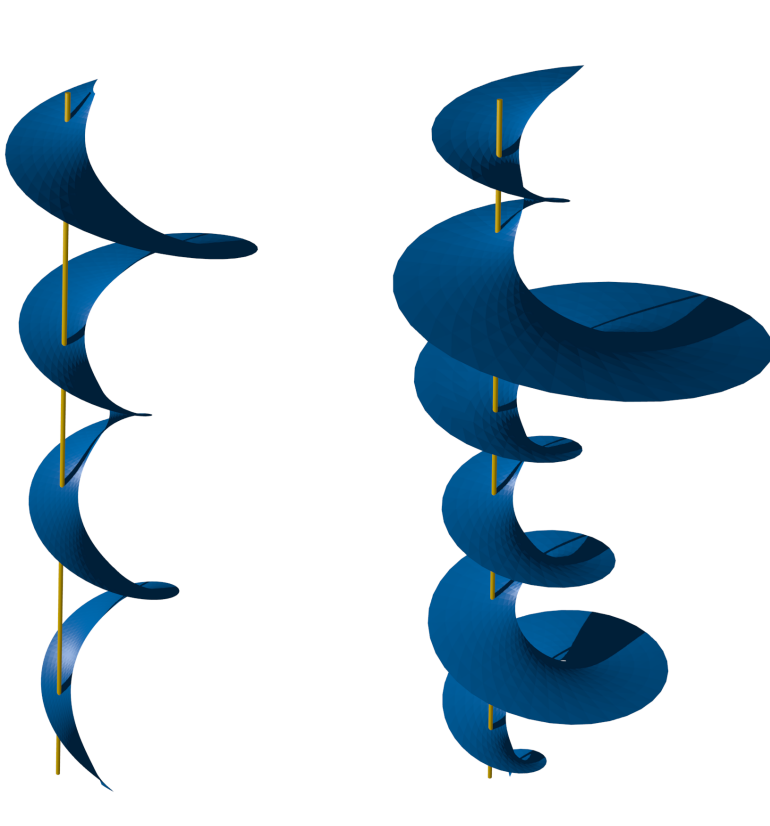


$|\zeta(1 + iv)| / |\zeta(0 + iv)|$ for $1 < v \leq 10000$



Source points $S \in \mathbb{C}$ (left) and image under mapping $\zeta(S)$ (right).

Definition 3. The zeta *critical strip spiral* is defined between constants c_1 and c_2 as the set of all points $(\text{Re}(\zeta(u + iv)), \text{Im}(\zeta(u + iv)), v)$ where $0 \leq u \leq 1$ and $c_1 \leq v \leq c_2$.



Critical strip spiral in blue and zero line $(0, 0, v)$ in yellow:
 $11 \leq v \leq 31$ (left) and $31 \leq v \leq 51$ (right).

Notes:

- 1. The critical strip spiral intersects the zero line once per spiral revolution.
- 2. As v increases, zeros on the critical strip spiral get increasingly closer together.
- 3. As v increases, the critical strip spiral gets generally wider.

Conjecture 2. Consider the $\mathbb{R} \rightarrow \mathbb{C}$ mapping given by $Z(u) = \zeta(u + iv)$ where $0 < u < 1$ and constant v is greater than one. For any v , if, for any distinct u_1 and u_2 , $Z(u_1) = Z(u_2)$ then $|u_1 - \frac{1}{2}| \neq |u_2 - \frac{1}{2}|$.

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