

## Zeta Function

**Definition 1.** The zeta function  $\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}$

Converges when  $\operatorname{Re}(z) > 1$ .

$$\zeta(\bar{z}) = \overline{\zeta(z)} \quad (1)$$

## Zeta Product Formula

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}, \quad \text{converges when } \operatorname{Re}(z) > 1. \quad (2)$$

## Analytic Continuation

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx, \quad \text{converges absolutely when } \operatorname{Re}(z) > 0. \quad (3)$$

$$\Gamma(z+1) = z\Gamma(z) \quad (4)$$

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z) \quad (5)$$

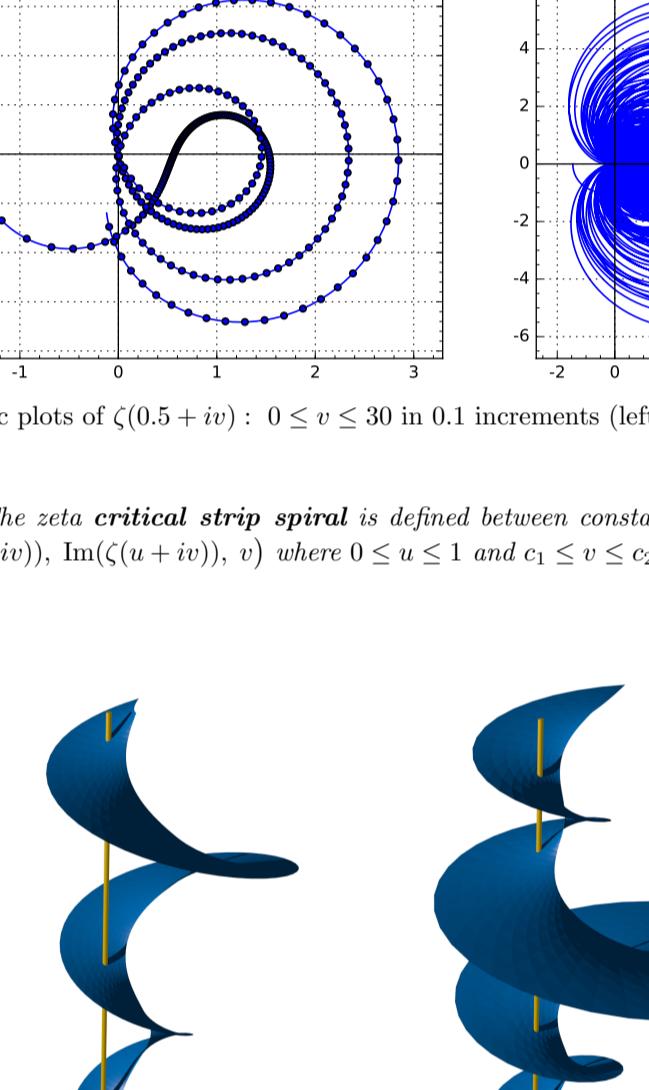
$$\zeta(1-z) = \frac{2}{(2\pi)^z} \cos\left(\frac{z\pi}{2}\right) \Gamma(z) \zeta(z) \quad (6)$$

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z}, \quad \text{converges when } \operatorname{Re}(z) > 0. \quad (7)$$

$$\zeta(z) = \frac{\eta(z)}{(1 - 2^{1-z})} \quad (8)$$

$$\mu(n) = \begin{cases} 1 & n \text{ square-free with an even number of prime factors} \\ -1 & n \text{ square-free with an odd number of prime factors} \\ 0 & n \text{ has a squared prime factor} \end{cases} \quad (9)$$

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} \quad (10)$$



Notes:

1. The values of  $\zeta(z)$  where  $\operatorname{Re}(z) > 1$  are relatively static.

2. There is a single pole at  $z = 1$  and zeros where  $z = \{-2, -4, -6, -8, \dots\}$ .

## Reimann Function

Alternative formulation:

$$R(x) = 1 + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n n! \zeta(n+1)} \quad (11)$$

## Non-trivial Zeta Zeros

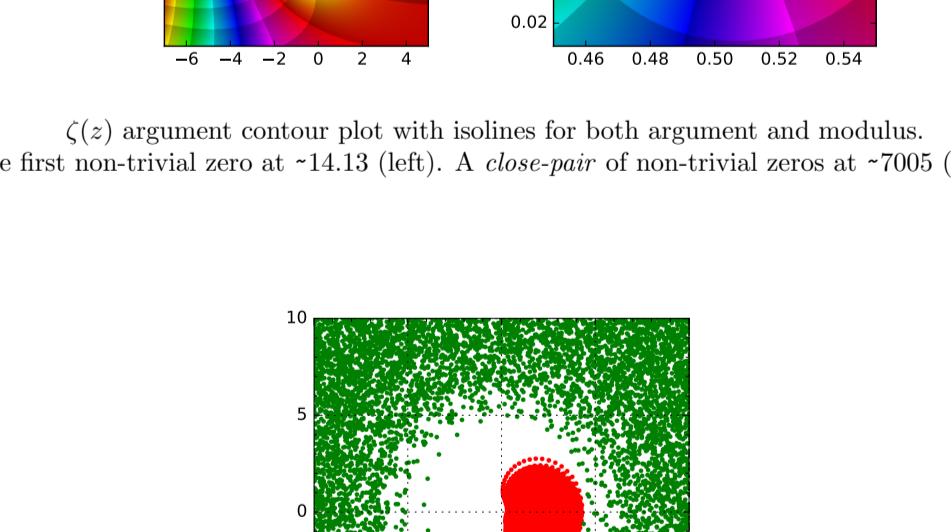
**Definition 2.** The non-trivial zeta zeros are all of the zeta zeros that are not on the real axis.

**Theorem 1.** The non-trivial zeta zeros are symmetric around the real axis.

**Theorem 2.** The non-trivial zeta zeros are symmetric around the line where  $\operatorname{Re}(z) = \frac{1}{2}$ .

**Theorem 3.** All of the non-trivial zeta zeros are in the critical strip, the region where  $0 \leq \operatorname{Re}(z) \leq 1$ .

**Conjecture 1. Reimann Hypothesis:** All of the non-trivial zeros of  $\zeta(s)$  are on the line  $\operatorname{Re}(z) = \frac{1}{2}$ .



**Definition 3.** The zeta critical strip spiral is defined between constants  $c_1$  and  $c_2$  as the set of all points  $(\operatorname{Re}(\zeta(u+iv)), \operatorname{Im}(\zeta(u+iv)), v)$  where  $0 \leq u \leq 1$  and  $c_1 \leq v \leq c_2$ .

Critical strip spiral in blue and zero line  $(0, 0, v)$  in yellow:  $11 \leq v \leq 31$  (left) and  $31 \leq v \leq 51$  (right).

Notes:

1. The critical strip spiral intersects the zero line once per spiral revolution.

2. As  $v$  increases, zeros on the critical strip spiral get increasingly closer together.

3. As  $v$  increases, the critical strip spiral gets generally wider.



## Some close-pair zeros:

7005.062866175, 7005.100564674

71732.901207872, 71732.915909348

388858886.0022851203, 388858886.0023936899

777717772.0045702406, 777717772.0047873798



$\zeta(z)$  argument contour plot with isolines for both argument and modulus.

The first non-trivial zero at  $\sim 14.13$  (left). A close-pair of non-trivial zeros at  $\sim 7005$  (right).



$\zeta(1+iv)$  in red and  $\zeta(0+iv)$  in green for  $1000 < v < 10000$  in 0.05 increments.



$|\zeta(1+iv)| / |\zeta(0+iv)|$  for  $1000 < v < 10000$

**Conjecture 2.** Consider the  $\mathbb{R} \rightarrow \mathbb{C}$  mapping given by  $s(u) = \zeta(u+iv)$  where  $0 < u < 1$  and constant  $v$  is greater than one. For any  $v$ , if, for any distinct  $u_1$  and  $u_2$ ,  $s(u_1) = s(u_2)$  then  $|u_1 - \frac{1}{2}| \neq |u_2 - \frac{1}{2}|$ .

**Conjecture 3.** Consider the  $\mathbb{R} \rightarrow \mathbb{C}$  mapping given by  $s(u) = \zeta(u+iv)$  where  $0 < u < 1$  and constant  $v$  is greater than one. For any  $v$ , if, for any distinct  $u_1$  and  $u_2$ ,  $s(u_1) = s(u_2)$  then  $|s(1) - s(u_1)| < |s(1)|$  and thus neither  $s(u_1)$  nor  $s(u_2)$  is equal to zero.

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