Calculus

Function Limit

$f : \mathbb{R}^1 \to \mathbb{R}^1$

$$\begin{aligned} \lim_{x \to c} f(x) &= l \\ &\equiv \forall(\epsilon > 0)\ \exists(\delta > 0)\ \forall x\big((0 < |x - c| < \delta) \Rightarrow (|f(x) - l| < \epsilon)\big) \end{aligned}$$

$f : \mathbb{R}^n \to \mathbb{R}^1$

$$\begin{aligned} \lim_{\boldsymbol{x} \to \boldsymbol{x_0}} f(\boldsymbol{x}) &= l \\ &\equiv \forall(\epsilon > 0)\ \exists(\delta > 0)\ \forall x\big((0 < \|\boldsymbol{x} - \boldsymbol{x_0}\| < \delta) \Rightarrow (|f(\boldsymbol{x}) - l| < \epsilon)\big) \end{aligned}$$

Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{1}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \tag{2}$$

Differentiation

$f : \mathbb{R}^1 \to \mathbb{R}^1$, $f'(x) \in \mathbb{R}^1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

$$\sin' = \cos \tag{3}$$

Differentiation Formulae

$$(fg)' = f'g + fg' \tag{4}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \tag{5}$$

$$\big(f(g)\big)' = f(g)'g' \tag{6}$$

Integration

Change of Variables

$$\int_a^b f(g)g'\,dx \ = \ \int_{g(a)}^{g(b)} f(u)\,du \tag{7}$$

$$\begin{aligned} \iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D'} f\left(g(u,v), h(u,v)\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ \text{where} \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \end{aligned} \tag{8}$$

Integration by Parts

$$\int_a^b f(x)\,dx \ = \ \int_{v(a)}^{v(b)} u\,dv \ = \ uv - \int_{u(a)}^{u(b)} v\,du \tag{9}$$

Natural Logarithm

$$\ln x \quad \equiv \quad \int_1^x \frac{1}{t}\,dt$$

$$\int_1^{\mathrm{e}} \frac{1}{t}\,dt \equiv 1$$

$$\mathrm{e} = \sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to 0} (1+x)^{1/x} \tag{10}$$

Partial Derivative

$f : \mathbb{R}^3 \to \mathbb{R}^1$, $\frac{\partial f}{\partial x} \in \mathbb{R}^1$

$$\frac{\partial f}{\partial x}(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

Gradient

scaler field $f : \mathbb{R}^3 \to \mathbb{R}^1$, $\nabla f \in \mathbb{R}^3$

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

maximum rate of change $= \|\nabla f\|$

Divergence

vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{div} \boldsymbol{F} \in \mathbb{R}^1$

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl

vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl} \boldsymbol{F} \in \mathbb{R}^3$

$$\begin{aligned} \operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \end{aligned}$$

Line Integral

$$\int\limits_P \boldsymbol{F} \cdot d\mathbf{r} = \int\limits_P F_1\,dx + F_2\,dy + F_3\,dz = \int\limits_{P'} \left(F_1 \frac{\partial x}{\partial t} + F_2 \frac{\partial y}{\partial t} + F_3 \frac{\partial z}{\partial t}\right) dt$$

if $\boldsymbol{F} = \nabla g$,

$$\int\limits_P \boldsymbol{F} \cdot d\mathbf{r} = \int\limits_{P'} dg$$

Surface Integral

$$\begin{aligned} \iint\limits_S dS &= \iint\limits_{S'} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv \\ &= \iint\limits_R \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy \end{aligned}$$

Green's Theorem

in the plane

$$\oint \boldsymbol{F} \cdot d\mathbf{r} = \oint F_1\,dx + F_2\,dy = \iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Stoke's Theorem

$$\oint \boldsymbol{F} \cdot d\mathbf{r} = \iint\limits_S (\nabla \times \boldsymbol{F}) \cdot \mathbf{n}\,dS$$

Divergence Theorem

$$\iint\limits_S \boldsymbol{F} \cdot \mathbf{n}\,dS = \iiint\limits_V \nabla \cdot \boldsymbol{F}\,dV$$

Conservative Field

1. Vector field $\boldsymbol{F}$ is conservative.

$$\int \boldsymbol{F} \cdot d\boldsymbol{r} \text{ is path independent, } \oint \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

2. A scaler function φ exists such that $\boldsymbol{F} = \nabla g$.
3. Curl $\boldsymbol{F} = \mathbf{0}$.
4. $\boldsymbol{F} \cdot d\boldsymbol{r}$ is an exact differential.

$$\boldsymbol{F} \cdot d\boldsymbol{r} = \nabla g \cdot d\boldsymbol{r} = dg$$

Directional Derivative

$\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^n$ at $\boldsymbol{c}$ in the direction of $\boldsymbol{u}$, $\boldsymbol{f}'(\boldsymbol{c}\,;\boldsymbol{u}) \in \mathbb{R}^1$

$$\begin{aligned} \boldsymbol{f}'(\boldsymbol{c}\,;\boldsymbol{u}) &= \lim_{h \to 0} \frac{\|\boldsymbol{f}(\boldsymbol{c} + h\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{c})\|}{h} \\ &= \nabla f(\boldsymbol{c}) \cdot \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \qquad \text{if } \boldsymbol{f} \text{ is differentiable} \end{aligned}$$

Total Derivative

The total derivative of $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ at $\boldsymbol{c}$ is $\boldsymbol{T_c}(\boldsymbol{v})$ where

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{c} + \boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{c}) + \boldsymbol{T_c}(\boldsymbol{v}) + \|\boldsymbol{v}\|\boldsymbol{E_c}(\boldsymbol{v}) \\ \text{and } \boldsymbol{E_c}(\boldsymbol{v}) \to \mathbf{0} \text{ as } \boldsymbol{v} \to \mathbf{0} \end{aligned}$$

$\boldsymbol{f}$ differentaible at $\boldsymbol{c}$ implies that $\boldsymbol{f}$ is continuous at $\boldsymbol{c}$ and the directional derivative exists in all directions.

The total derivative of $f : \mathbb{R}^2 \to \mathbb{R}^1$ at (x,y) is $\left[\frac{\partial f}{\partial x}\,\frac{\partial f}{\partial y}\right]$ where

$$\begin{aligned} f(x+h, y+k) = f(x,y) + \left[\tfrac{\partial f}{\partial x}\,\tfrac{\partial f}{\partial y}\right] \begin{bmatrix} h \\ k \end{bmatrix} + \|(h,k)\|\ E(h,k) \\ \text{and } E(h,k) \to 0 \text{ as } (h,k) \to (0,0) \end{aligned} \tag{11}$$

Note: $\left[\frac{\partial f}{\partial x}\,\frac{\partial f}{\partial y}\right] \begin{bmatrix} h \\ k \end{bmatrix} = \nabla f(x,y) \cdot (h,k)$

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