

Sequences and Series

The sequence of partial sums of an infinite series either converges to a single fixed value or it diverges.

Harmonic Series

Definition 1. The *harmonic series* is defined as: $\sum_{n=1}^{\infty} \frac{1}{n}$

Theorem 1. The harmonic series diverges.

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots\end{aligned}$$

RHS diverges \implies LHS diverges. □

Geometric Series

Definition 2. The *geometric series* is defined as: $\sum_{n=0}^{\infty} r^n$

Theorem 2. The geometric series $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ when $|r| < 1$.

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} r^n &= 1 + r + r^2 + r^3 \dots \\ &= \lim_{n \rightarrow \infty} (1 + r + r^2 + \dots + r^n) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1 - r^{n+1}}{1 - r} \right) \\ &= \frac{1}{1 - r} \quad \text{when } |r| < 1\end{aligned}$$
□