An Exploration of a Bijective Mapping in the Integers

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Abstract

In this article we begin by considering the possible existence of two bijective integer valued functions where the sum of the functions is also bijective. An example of two such functions is produced. Some interesting properties of these functions, as well as the algorithm that is used to produce them, are both explored.

1 Preliminaries

Given two functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$, define a new function $(f+g) : \mathbb{Z} \to \mathbb{Z}$ by $(f+g)(n) = f(n) + g(n)$. Is it possible that f and g as well as $f+g$ could all be bijective? We will prove that it is possible by producing an example.

We begin by sequentially selecting and plotting points $(f(n), g(n))$ on a graphic grid that represents $\mathbb{Z}^2$. We will work our way through all of $\mathbb{Z}$ by considering n's in the order $0, +1, -1, +2, -2, +3, -3, \ldots$ and impose three sufficient conditions.

Selection Conditions:

1. We will select points $(f(n), g(n)) \in \mathbb{Z}^2$ such that for all $n \in \mathbb{Z}$, $f(n) + g(n) = n$. This will ensure that $(f+g)(n)$ is bijective.
2. If we are careful in our selection of points $(f(n), g(n))$ making sure that $f(n)$ never maps to the same value twice and $g(n)$ never maps to the same value twice, then both $f(n)$ and $g(n)$ will be injective.
3. And finally, if we are also careful in our selection of points $(f(n), g(n))$ making sure that $f(n)$ and $g(n)$ map to all elements of $\mathbb{Z}$, then $f(n)$ and $g(n)$ will be surjective.

In the next section we produce an example that meets these conditions.

2 A Mapping Algorithm

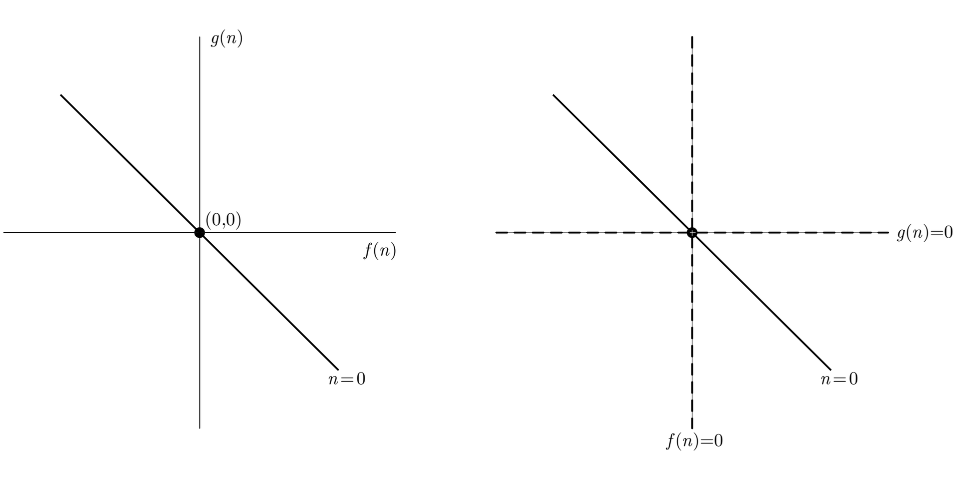

Figure 1: Beginning at the point plot of $(f(n), g(n))$ in Z^2 on the left. The first dashed *keep-off* lines are shown on the right.

Starting with $n = 0$, we note that, to meet condition 1, the point $(f(0), g(0))$ will need to be on the line labeled $n = 0$ in Figure **??**. Let's (somewhat arbitrarily) start by selecting the point $(0, 0)$. Next we draw two dashed *keep-off* lines $f(n) = 0$ and $g(n) = 0$. If we make sure that no other points that are selected lie on a keep-off line, then we will have met condition 2.

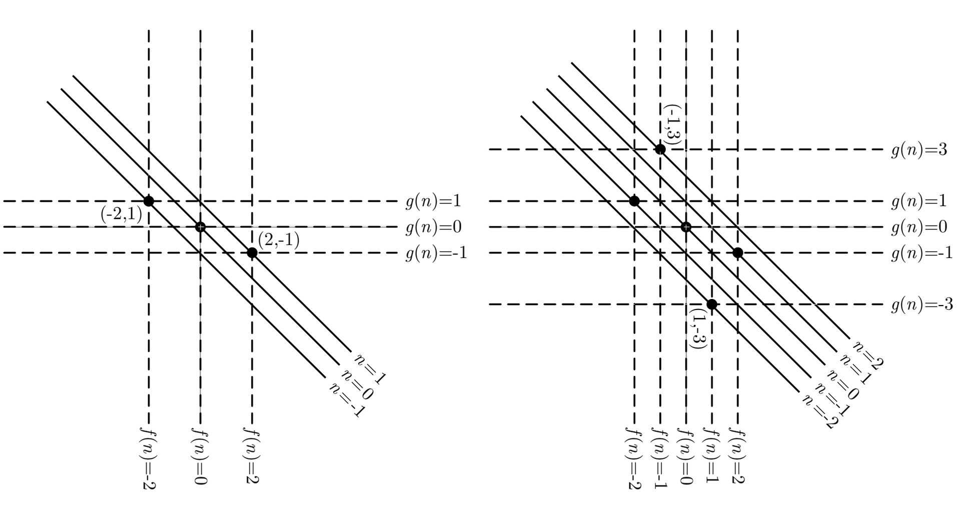

Figure 2: Evolving point selection.

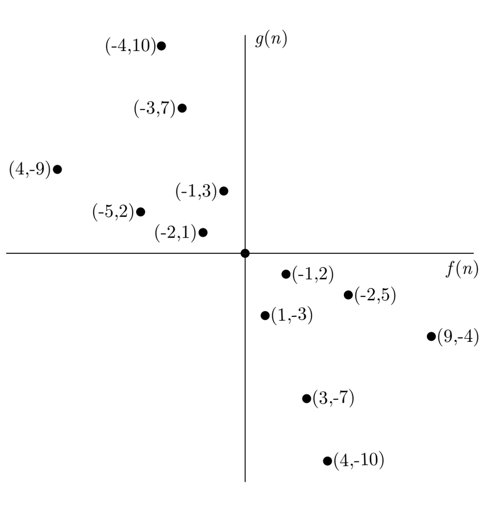

Figure 3: Plotted points up to $n = \pm 6$.

As we work our way through values of n, we note that the points we select need to lie on their respective diagonal line as shown in the evolving plot Figure **??**. In a systematic effort to meet condition 3, we will select points that are as close to $(0, 0)$ as possible. So, for $n = 1$ we select the point $(2, -1)$ and for $n = -1$ we select the point $(-2, 1)$ as shown in Figure **??**. And then, following the same rule, for $n = 2$ we select the point $(-1, 3)$ and for $n = -2$ we select the point $(1, -3)$. As we continue systematically plotting points, we make the side observation that the plotted points appear to closely fit two lines passing through zero (see Figure **??**).

n odd			n even	
$f(n)$	$g(n)$	n	$f(n)$	$g(n)$
		0	0	0
2	-1	1		
		2	-1	3
5	-2	3		
		4	-3	7
9	-4	5		
		6	-4	10
12	-5	7		
		8	-6	14
15	-6	9		
		10	-7	17
19	-8	11		
		12	-9	21
22	-9	13		
		14	-10	24
26	-11	15		
		16	-11	27
29	-12	17		

Table 1: Tabulated values for $0 \leq n \leq 17$.

In an attempt to discover an algebraic (as opposed to geometric) definition for our sequence of $f(n)$ and $g(n)$ values, we continue with a list of values as shown in Table 1. Note that because of the symmetries $f(-n) = -f(n)$ and $g(-n) = -g(n)$, we can, without loss of generality, use a table that gives values only for $n \geq 0$ as shown in Table **??**. The table has been split into columns for odd and even values of n. Note that when n is odd, $f(n)$ is positive and $g(n)$ is negative. For all even values of n greater than zero, $f(n)$ is negative and $g(n)$ is positive.

Observation of the evolving pattern leads to the following algorithm for specifying the sequence of values for $f(n)$ and $g(n)$:

Mapping Algorithm 1: (A strongly recursive definition.)

1. Initialise n equal to zero.
2. If n is even, set $f(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $f(n)$'s. Then assign $g(n)$ such that $f(n) + g(n) = n$.
3. If n is odd, set $g(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $g(n)$'s. Then assign $f(n)$ such that $f(n) + g(n) = n$.
4. Increment n by one and loop to step 2.

Of course, by the previously mentioned symmetry conditions, to complete the mapping, for all negative n, $f(n) = -f(-n)$ and $g(n) = -g(-n)$.

The algorithm guarantees that both $f(n)$ and $g(n)$ are surjective (selection condition 3) by virtue of the criteria "smallest non-negative integer value not in the set". All values of $f(n)$ and $g(n)$ will eventually get filled-in.

To show that $f(n)$ and $g(n)$ are both injective (selection condition 2), we first note that and no duplicate values will be assigned by the "smallest non-negative integer value not in the set" criteria. Thus, when n is odd, we need to show that, in algorithm step 3, $f(n)$ is always assigned a larger target value than the previous odd n and also that a smaller value hole is left unused for algorithm step 2 to subsequently fill-in.

To do this it will suffice to show that for all odd n, $f(n+2) > f(n) + 2$.

Proof.

1. Assume n is odd.
2. By Mapping Algorithm 1, step 3, $f(n) = n - g(n)$.
3. Adding 2 to n, $f(n+2) = (n+2) - g(n+2)$.
4. By Mapping Algorithm 1, step 3, $g(n+2) < g(n)$.
5. Therefore, $-g(n+2) > -g(n)$.
6. Adding $(n+2)$, $(n+2) - g(n+2) > (n+2) - g(n)$.
7. Regrouping, $(n+2) - g(n+2) > (n - g(n)) + 2$.
8. Substituting from proof steps (2) and (3), $f(n+2) > f(n) + 2$.

□

When n is even, a similar argument can be applied to $g(n)$. Thus, we have an algorithm that meets the selection conditions, thereby generating an instance of the bijective mapping that we set out to find.

3 Further Exploration

The fact that, as previously observed, the plotted points in Figure **??** appear to closely fit two lines, leads us to consider the possibility of an equivalent non-recursive definition for $f(n)$ and $g(n)$. With the assistance of a computer, we calculated the values of $f(n)$ and $g(n)$ for $0 \leq n \leq 10000001$. Some of the results are shown in Table 2.

n	$f(n)$	$g(n)$	$g(n)/f(n)$
9999	17069	-7070	-0.414201183432
10000	-7071	17071	-2.41422712488
10001	17073	-7072	-0.41422128507
99999	170709	-70710	-0.414213661846
100000	-70711	170711	-2.41420712478
100001	170712	-70711	-0.414212240499
999999	1707105	-707106	-0.414213536953
1000000	-707107	1707107	-2.41421312475
1000001	1707108	-707107	-0.414213394817
9999999	17071066	-7071067	-0.414213558778
10000000	-7071068	17071068	-2.41421352475
10000001	17071070	-7071069	-0.414213578879

Table 2: Some values generated by Mapping Algorithm 1.

The values of $f(n)$ and $g(n)$, especially when n is a power of 10, look familiar! Based on these observations, we conjecture that:

For even n,

$$\lim_{n\to\infty} \frac{f(n)}{n} = -\frac{1}{\sqrt{2}} \tag{1}$$

$$\lim_{n\to\infty} \frac{g(n)}{n} = 1 + \frac{1}{\sqrt{2}} \tag{2}$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = -\left(1 + \sqrt{2}\right) \tag{3}$$

and for odd n,

$$\lim_{n\to\infty} \frac{f(n)}{n} = 1 + \frac{1}{\sqrt{2}} \tag{4}$$

$$\lim_{n\to\infty} \frac{g(n)}{n} = -\frac{1}{\sqrt{2}} \tag{5}$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = -\left(\frac{1}{1 + \sqrt{2}}\right) \tag{6}$$

This suggests the following non-recursive equivalent to Mapping Algorithm 1:

Mapping Algorithm 2: (A non-recursive definition.)

1. If n is even,

$$f(n) = -\lceil (\tfrac{1}{\sqrt{2}}) n \rfloor$$
$$g(n) = \lceil (1 + \tfrac{1}{\sqrt{2}}) n \rfloor$$

2. If n is odd,

$$f(n) = \lceil (1 + \tfrac{1}{\sqrt{2}}) n \rfloor$$
$$g(n) = -\lceil (\tfrac{1}{\sqrt{2}}) n \rfloor$$

Note that the operator pair $\lceil\ \rfloor$ means "round to the nearest integer value".

4 A Variation

Let's leave the original problem as stated in the Introduction behind and consider a variation to Mapping Algorithm 1 in which $f(n) + g(n) = 2n$ rather than n. Of course $f(n) + g(n)$ is no longer surjective and thus not bijective.

Mapping Algorithm 3: (A strongly recursive definition.)

1. Initialise n equal to zero.
2. If n is even, set $f(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $f(n)$'s. Then assign $g(n)$ such that $f(n) + g(n) = 2n$.
3. If n is odd, set $g(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $g(n)$'s. Then assign $f(n)$ such that $f(n) + g(n) = 2n$.
4. Increment n by one and loop to step 2.

n	$f(n)$	$g(n)$	$g(n)/f(n)$
9999	26178	-6180	-0.23607609443
10000	-6180	26180	-4.23624595469
10001	26183	-6181	-0.236069205209
99999	261801	-61803	-0.236068617003
100000	-61803	261803	-4.236088863
100001	261806	-61804	-0.236067928161
999999	2618031	-618033	-0.236067869326
1000000	-618034	2618034	-4.23606791859
1000001	2618037	-618035	-0.236068092239
9999999	26180337	-6180339	-0.236067969637
10000000	-6180340	26180340	-4.23606791859
10000001	26180343	-6180341	-0.236067991928

Table 3: Some values generated by Mapping Algorithm 3.

Some calculated results are shown in Table **??**. Again the values of $f(n)$ and $g(n)$ look familiar! This time we see the golden ratio. Based on this observation, we make the conjecture that, when using Mapping Algorithm 3:

Based on these observations, we conjecture that:

For even n,

$$\lim_{n\to\infty} \frac{f(n)}{n} = -\frac{1}{\phi} \tag{7}$$

$$\lim_{n\to\infty} \frac{g(n)}{n} = 1 + \phi \tag{8}$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = -\phi^2 - \phi \tag{9}$$

and for odd n,

$$\lim_{n\to\infty} \frac{f(n)}{n} = 1 + \phi \tag{10}$$

$$\lim_{n\to\infty} \frac{g(n)}{n} = -\frac{1}{\phi} \tag{11}$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \frac{1}{-\phi^2 - \phi} \tag{12}$$

where $\phi = (1 + \sqrt{(5)})/2$.

5 A Generalisation

Consider the following parameterised generalisation, for positive integers k, in which $f(n) + g(n) = kn$. Of course $f(n) + g(n)$ is only bijective when $k = 1$.

Mapping Algorithm 4: (A strongly recursive definition parameterised by $k \in \mathbb{Z}^+$.)

1. Initialise n equal to zero.
2. If n is even, set $f(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $f(n)$'s. Then assign $g(n)$ such that $f(n) + g(n) = kn$.
3. If n is odd, set $g(n)$ equal to -1 times the smallest non-negative integer value not in the set of the absolute values of already used $g(n)$'s. Then assign $f(n)$ such that $f(n) + g(n) = kn$.
4. Increment n by one and loop to step 2.

n	$f(n)$	$g(n)$	$g(n)/f(n)$
9999	135179	-5192	-0.0384083326552
10000	-5192	135192	-26.0385208012
10001	135206	-5193	-0.038408058814
99999	1351907	-51920	-0.0384050086286
100000	-51920	1351920	-26.0385208012
100001	1351934	-51921	-0.0384049813083
999999	13519189	-519202	-0.0384048185139
1000000	-519202	13519202	-26.0384243512
1000001	13519216	-519203	-0.0384048157822

Table 4: Some values generated when $k = 13$.

Using Mapping Algorithm 4, and supported by additional calculated observations (such as shown for $k = 13$ in Table **??**), we make the conjecture that:

For all $k \in \mathbb{Z}^+$, and n even,

$$\lim_{n\to\infty} \frac{f(n)}{n} = \frac{-k}{(k-1) + \sqrt{k^2+1}} = \frac{1}{\psi_1} \tag{13}$$

$$\lim_{n\to\infty} \frac{g(n)}{n} = k - \frac{1}{\psi_1} \tag{14}$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \frac{-1}{k + \sqrt{k^2+1}} = \frac{1}{\psi_2} \tag{15}$$

and for odd n,

$$\lim_{n\to\infty} \frac{f(n)}{n} = k - \frac{1}{\psi_1} \tag{16}$$

$$\lim_{n\to\infty} \frac{g(n)}{n} = \frac{1}{\psi_1} \tag{17}$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \psi_2 \tag{18}$$

We can use ψ_1 to show a connection between the two rather special numbers $\sqrt{2}$ and the golden ratio:

When $k = 1$,

$$\sqrt{2} = -\psi_1 \tag{19}$$

and when $k = 2$,

$$\phi = -\psi_1 \tag{20}$$

Note that ψ_2 has the property that:

$$\psi_2 = -2k + \frac{1}{\psi_2} \tag{21}$$

which is equivalent to saying that the fractional part of ψ_2 is equal to $1/\psi_2$. This certainly appears to be supported by examining the $g(n)/f(n)$ column in each of the previous mapping tables.

There is also a connection between the generalised limits and solutions to certain quadratic equations. Note that we have:

$$\psi_1 = \frac{-(k-1) - \sqrt{k^2+1}}{k} \tag{22}$$

$$\psi_2 = -k - \sqrt{k^2+1} \tag{23}$$

These ψ_1 and ψ_2 are clearly solutions, respectively, to the following quadratic equations:

$$kx^2 - (2k-2)x - 2 = 0 \tag{24}$$

$$x^2 + 2kx - 1 = 0 \tag{25}$$

6 Conclusion

We have produced a parameterised algorithm that generates functions which map integers to integers. These functions give sequences, that, in the infinite limit, are conjectured to give some interesting results. The results are related specifically to two special numbers: $\sqrt{2}$ and the golden ratio. The results also generally relate a special case of a fractional part to its multiplicative inverse, as well as generally to the solution of certain quadratic equations.

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