Zeta Function

Definition 1. *The **zeta function*** $\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}$

The zeta function converges when $\mathrm{Re}(z) > 1$.

Zeta Product Formula

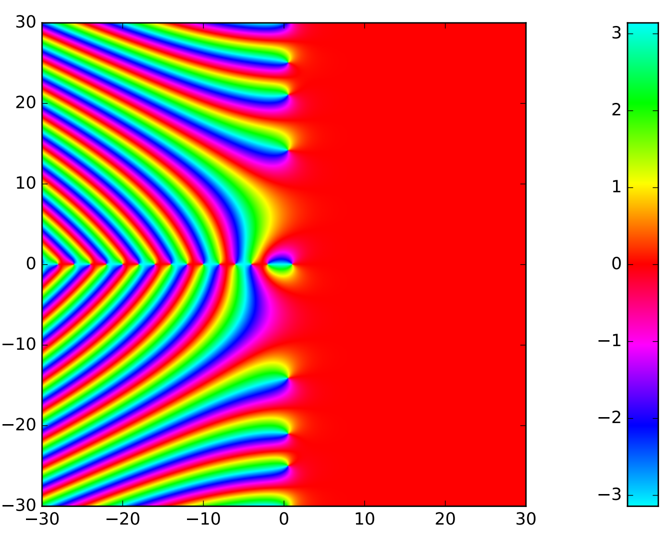
$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}} \quad \text{converges when } \mathrm{Re}(z) > 1. \tag{1}$$

Reimann Function

Alternative formulation:

$$\mathrm{R}(x) = 1 + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n\, n!\, \zeta(n+1)} \tag{2}$$

Analytic Continuation, Zeta Function Computation



Contour plot: $\arg \zeta(z)$

Notes: (at least in the range shown above)

1. The values of $\zeta(z)$ where $\mathrm{re}(z) > 1$ are relatively static.

2. There is a single pole at $z = 1$ and zeros where $z = \{-2, -4, -6, -8, \dots\}$.

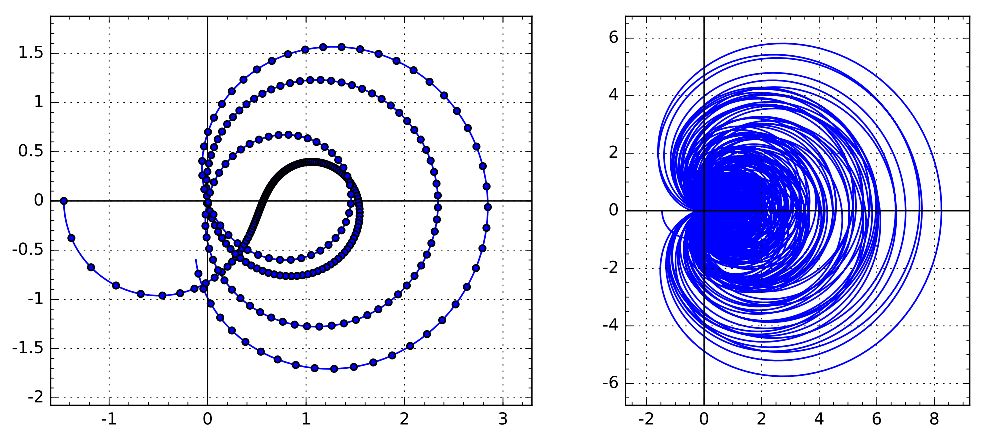
3. Zeros are either on the real axis or in the *critical strip*, the region where $0 \leq \mathrm{re}(z) \leq 1$.

4. Zeros are symmetric around the real axis.

Non-trivial Zeta Zeros

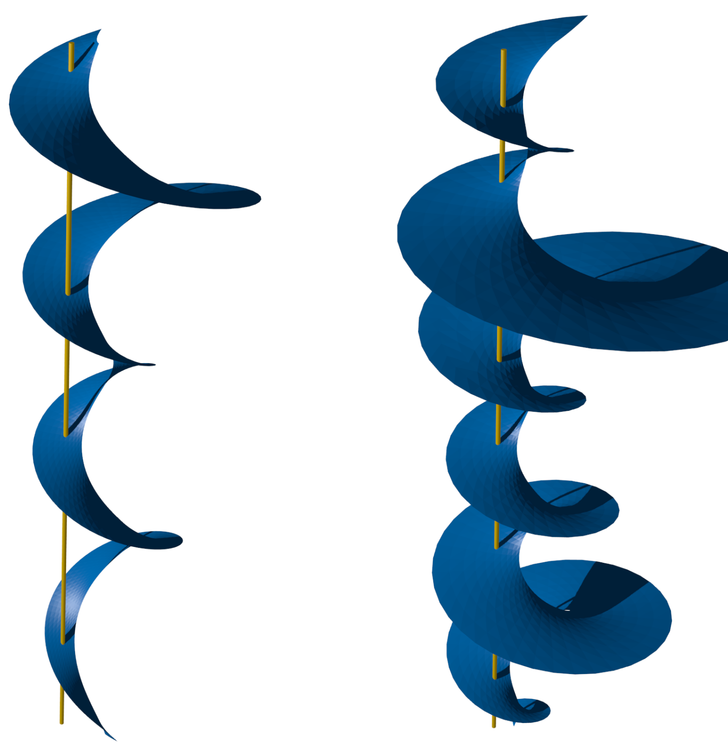
Definition 2. *The **non-trivial zeta zeros** are all of the zeta zeros that are not on the real axis.*

Conjecture 1. ***Reimann Hypothesis:*** *All of the non-trivial zeros of $\zeta(s)$ are on the line $re(z) = \frac{1}{2}$.*



Parametric plots of $\zeta(0.5 + iv)$: $0 \leq v \leq 30$ in 0.1 increments (left), $0 \leq v \leq 500$ (right).

Definition 3. *The zeta **critical strip spiral** is defined between constants c_1 and c_2 as the set of all points $\big(\mathrm{re}(\zeta(u+iv)),\ \mathrm{im}(\zeta(u+iv)),\ v\big)$ where $0 \leq u \leq 1$ and $c_1 \leq v \leq c_2$.*



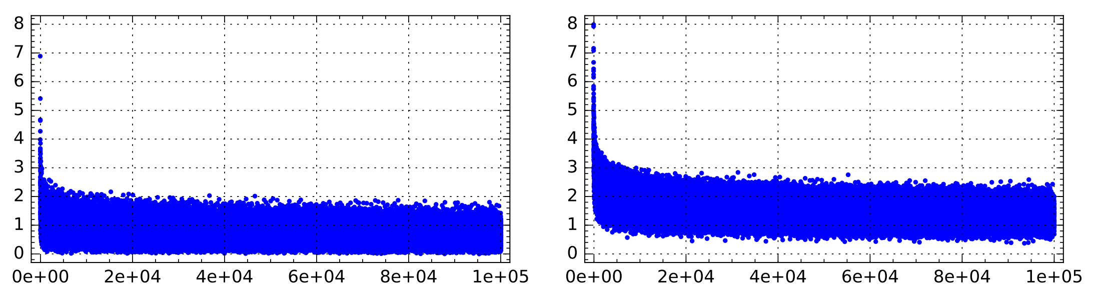
Critical strip spiral in blue and zero line $(0, 0, v)$ in yellow:
$11 \leq v \leq 31$ (left) and $31 \leq v \leq 51$ (right).

Notes: (at least in the range shown above)

1. The critical strip spiral intersects the zero line once per spiral revolution.

2. As v increases, zeros on the critical strip spiral get increasingly closer together.

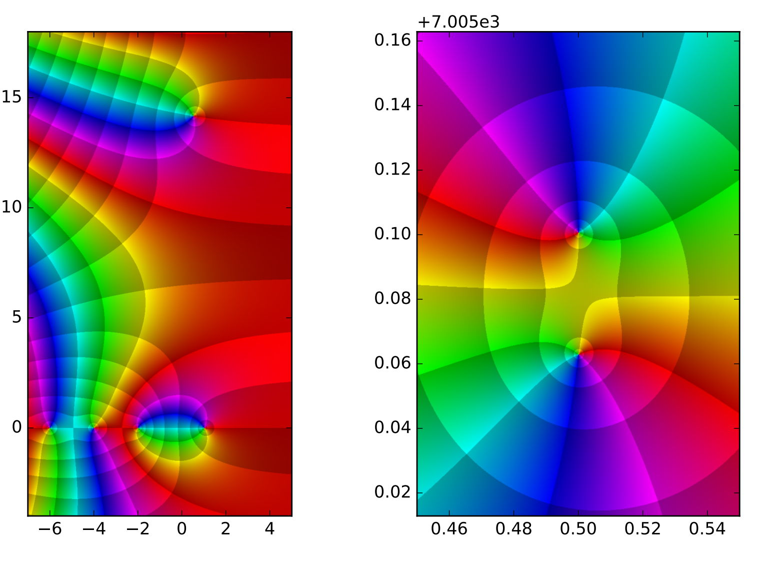
3. As v increases, the critical strip spiral gets generally wider.



Differences between the first one hundred thousand zeta zero pairs (left) and triples (right).

Some *close-pair* zeros:

7005.062866175	71732.901207872	388858886.0022851203	777717772.0045702406
7005.100564674	71732.915909348	388858886.0023936899	777717772.0047873798



$\zeta(z)$ argument contour plot with isolines for both argument and modulus.
The first non-trivial zero at ~14.13 (left). A *close-pair* of non-trivial zeros at ~7005 (right).

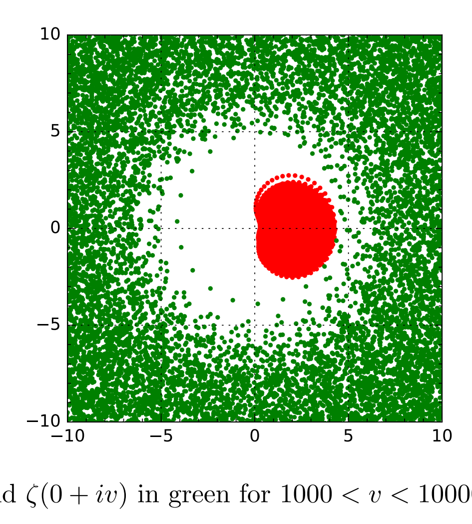
$$\zeta(1-z) = \frac{2}{(2\pi)^z} \cos\left(\frac{z\pi}{2}\right) \gamma(z)\zeta(z) \tag{3}$$

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} \tag{4}$$

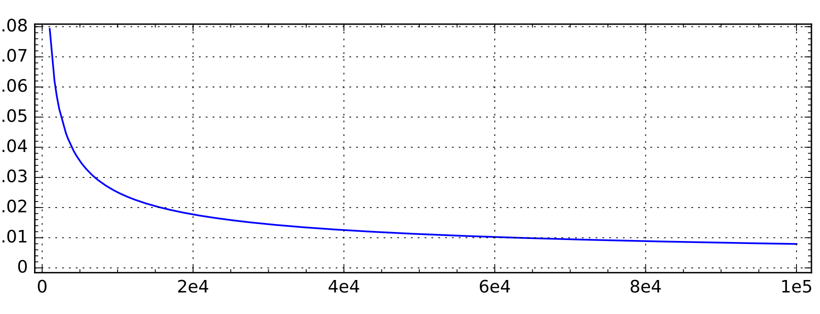
Theorem 1. *The non-trivial zeta zeros are symmetric around the real axis.*

Theorem 2. *All of the non-trivial zeta zeros are in the critical strip.*

Theorem 3. *The non-trivial zeta zeros are symmetric around the line where $re(z) = \frac{1}{2}$.*



$\zeta(1 + iv)$ in red and $\zeta(0 + iv)$ in green for $1000 < v < 10000$ in 0.05 increments.



$|\zeta(1 + iv)|\ /\ |\zeta(0 + iv)|$ for $1000 < v < 10000$

Conjecture 2. *Consider the $\mathbb{R} \to \mathbb{C}$ mapping given by $s(u) = \zeta(u + iv)$ where $0 < u < 1$ and v is any real constant greater than one. If $s(u_1) = s(u_2)$ then $|u_1 - \frac{1}{2}| \neq |u_2 - \frac{1}{2}|$.*

Conjecture 3. *Consider the $\mathbb{R} \to \mathbb{C}$ mapping given by $s(u) = \zeta(u + iv)$ where $0 < u < 1$ and v is any real constant greater than one. Whenever $s(u_1) = s(u_2)$, $|\zeta(1 + iv) - \zeta(\frac{1}{2} + iv)| < |\zeta(1 + iv)|$ and thus neither $s(u_1)$ nor $s(u_2)$ is equal to zero.*

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