Functions

For a relation R to be a function, it must assign exactly 1 output for each input. f(x) can only produce 1 y.

Properties of functions

$f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$

determine if f is

1. Injective: Each output has only 1 input. One to one.
2. Surjective: Can produce every output in the codomain ($\mathbb{R}$). $\forall y \exists x\ f(x) = y$
3. Bijective: Both injective and surjective

$f(x) = |x|$
$f(2) = |2| = 2$
$f(-2) = |-2| = 2$
$\therefore$ Not injective

$f(x) = |x|$ cannot produce negative outputs $\therefore$ Not surjective

Not injective
Not surjective
$\therefore$ Not bijective

Function Composition

$f(x) = \sqrt{x} \quad g(x) = x^2 - 4$

$f \circ g(x) = f(g(x)) = \sqrt{x^2-4}$ Domain: $x^2 - 4 \geq 0$ → defined

$x^2 \geq 4$

$|x| \geq 2$

$(-\infty, -2] \cup [2, \infty)$

$g \circ f(x) = g(f(x)) = \sqrt{x}^2 - 4 = x - 4$ Domain: This function is defined at x such that (i) x is in domain of f
(ii) f(x) is in the domain of g

$x \geq 0$ for $\sqrt{x}$ (f(x))

$\therefore [0, \infty)$

Are they Commutative?

No, $f \circ g(x) \neq g \circ f(x)$ → function composition is never commutative

Inverse Functions

$f(x) = \frac{3x-2}{x+1}$, find $f^{-1}(x)$:

$x = \frac{3y-2}{y+1}$

$(y+1)x = 3y - 2$

$xy + x = 3y - 2$

$xy - 3y = -x - 2$

$y(x-3) = -x - 2$

$y = \frac{-x-2}{x-3}$

$f^{-1}(y) = \frac{x+2}{3-x} \quad y \neq 3$

Prove with $f(f^{-1}(y)) = y$

$$f\left(\frac{y+2}{3-y}\right) = \frac{3\cdot\left(\frac{y+2}{3-y}\right) - 2}{\left(\frac{y+2}{3-y}\right) + 1}$$

$$= \frac{\frac{3(y+2) - 2(3-y)}{3-y}}{\frac{y+2+3-y}{3-y}}$$

$$= \frac{3y+6-6+2y}{5}$$

$$= \frac{5y}{5} = y \checkmark$$

Recurrence Relations

A sequence is defined by

$a_0 = 3, \; a_{n+1} = 4a_n - 1$ for $n \geq 0$

Find first 5 terms:

$a_0 = 3$

$a_1 = 4(3) - 1 = 11$

$a_2 = 4(11) - 1 = 43$

$a_3 = 4(43) - 1 = 171$

$a_4 = 4(171) - 1 = 683$

Part 2, find closed formula

Try: $a_n = A \cdot 4^n + B$ → each term depends on the previous x4

Sub: $a_{n+1} = 4a_n - 1$ → sub into lhs and rhs

$A \cdot 4^{n+1} + B = 4(A \cdot 4^n + B) - 1$

$A \cdot 4^{n+1} + B = 4^{n+1} \cdot A + 4B - 1$

Compare coefficients: $B = 4B - 1$

$B - 4B = -1$

$-3B = -1$

$B = \frac{1}{3}$

$a_0 = 3$:

$A \cdot 4^0 + \frac{1}{3} = 3$

$A = \frac{8}{3}$

Closed Formula: $a_n = \frac{8}{3} \cdot 4^n + \frac{1}{3}$

Final step: Prove by Induction

$a_n = \frac{8}{3} \cdot 4^n + \frac{1}{3}$ for all $n \geq 0$

Base: $\frac{8}{3} \cdot 4^0 + \frac{1}{3} = 3$ and $a_0 = 3$

IH: $a_k = \frac{8}{3} \cdot 4^k + \frac{1}{3}$

Prove: $a_{k+1} = \frac{8}{3} \cdot 4^{k+1} + \frac{1}{3}$

$$= 4\left(\frac{8 \cdot 4^k + 1}{3}\right) - 1$$

$$= \frac{8 \cdot 4^{k+1} + 1}{3}$$

$a_{k+1} = 4a_k - 1$

$$= 4\left(\frac{8 \cdot 4^k + 1}{3}\right) - 1$$

Types of Sequences

Arithmetic: Constant difference between consecutive terms: $a_n = a_1 + (n-1)d$ where d = common difference
Geometric: Constant ratio between consecutive terms: $a_n = a_1 \cdot r^{n-1}$ where r = common ratio = $\frac{a_2}{a_1}$

Classify:

a) 5, 8, 11, 14, 17 → Arithmetic: $d=3$ $a_n = 5 + (n-1)3$
b) 2, 6, 18, 54, 162 → Geometric: $r=3$ $a_n = 2 \cdot 3^{n-1}$
c) 1, 4, 9, 16, 25 → Neither