Induction & Recursion

What is induction: 1. Prove for 0 (base case).
2. Assume the statement holds for an arbitrary k, Prove it also holds for k+1.
3. Use IH (k) to prove k+1

$n \geq 0$. Prove $\sum_{i=0}^{n} (2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

Step 1 Base Case: for $n=0$ $\sum_{i=0}^{0} (2i+1)^2 = 0$ $\frac{(0+1)(2(0)+1)(2(0)+3)}{3} = 0$ $\therefore$ Holds for base case

Step 2 IH: $(n=k\geq 7)$ $\sum_{i=0}^{k} (2i+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ Use one side of this to be a stepping stone to k+1

Step 3: Prove $\sum_{i=0}^{k+1} (2i+1)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$ we must prove $\frac{(k+2)(2k+3)(2k+5)}{3} = \frac{(k+1)(2k+1)(2k+3)}{3} \cdot (2(k+1)+1)^2$

This takes the assumption and checks if adding the addition term will satisfy the equality $\left(\sum^{k+1} i = \sum^{k} i \cdot i\right)$

Step 4: Expand, Factor, prove algebraically

Example 2:

$n \geq 7$. Prove $3^n < n!$

Base case: $3^7 < 7! = 2187 < 5040$

IH: $3^k < k!$

Prove $3^{k+1} < (k+1)!$ → Use equalities to bridge IH. $3^k \cdot 3 = 3^{k+1}$

Using IH: $3^k \cdot 3 < k! \cdot 3$ → multiply both sides

$3^{k+1} < k! \cdot 3$ → If we can prove $k! \cdot 3 < (k+1)!$ this will prove it holds for k+1

$k! \cdot 3 < k! \cdot (k+1)$ → $k! \cdot (k+1) = (k+1)!$

Since $k \geq 7$: $k+1 > 7$

$3 < 7$ ✓ proven

Overall: $3^{k+1} = 3^k \cdot 3 < 3 \cdot k! < 7 \cdot k! < (k+1) \cdot k! = (k+1)!$

Recursion Example 1:

Write a recursive algorithm in pseudocode that calculates the nth member of this sequence.

$f(0) = 3$

$f(1) = 4$

$f(n) = f(n-1) - f(n-2)$

Algorithm f

Input: int n

Output: f(n)

```
if n == 0
   return 3
if n == 1
   return 4
else
  return f(n-1) - f(n-2)
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Test with $n=3$:

$f(3) = f(2) - f(1)$

$f(2) = f(1) - f(0)$

$f(3) = 4 - 3 - 4$

$= -3$

Strong Induction

Prove all $n \geq 2$ is a product of primes

Normal Induction: Prove $P(k+1)$ by assuming $P(k)$
Strong Induction: Prove $P(k+1)$ by assuming $P(2), P(3) \dots P(k)$

This is necessary when $k+1$ depends on previous cases. Instead of stepping forward one at a time, you may need to reach back for a previous value.

Base case: $n=2$. $2=2$. 2 is prime so this is a product of 1 prime
IH: Assume n can be written as a product of primes for all $2 \leq m \leq k$
Show it holds for $k+1$: $2 \leq m \leq k+1$

If $k+1$ is prime $\rightarrow$ product of 1 prime
If $k+1$ is composite: $k+1 = a \times b$ for some a, b such that $n \leq a, b \leq k+1$
Since a, b are in this range, we have assumed them to be each the product of primes
$\therefore P(k+1)$ holds