# FOL Semantics

Syntax tells us how to define and write formulas
Semantics tell us what they mean

To evaluate a formula like: $\forall x(P(x) \to Q(x))$
We need to know: 1. The domain of $x$
2. Definitions of the predicates
3. What any constant refers to
4. The values of any free variables

} All this together is called a valuation

## Valuations

A valuation $v$ consists of

1: A domain $D$ - A non empty set of objects ($\mathbb{N}, \mathbb{Z}, \mathbb{R}$, students at western)

2: An interpretation of non-logical symbols.

| Example | → | Under valuation $v$ |
|---|---|---|
| Constant $a$ | → | an element $a^v \in D$ |
| Free var $x$ | → | a value $x^v \in D$ |
| n-ary relation $F(x,y)$ | → | a set $F^v \subseteq D^n$ |
| m-ary function $f(x)$ | → | a function $f^v: D^m \to D$ |

3. Assignment: Free variables get specific values ($y^v = 7$), bound variables are controlled by quantifiers ($\forall x(F(x))$)

## Value of a Formula

Formulas evaluate to either $0$ = False or $1$ = True

### Atomic formulas

$F(t_1, \dots t_n) = 1 \equiv (t_1^v, \dots t_n^v) \in F^v$

If $F^v = \{1, 3, 7\}$
$t^v = \{3\}$
$\to F(t)^v = 1$ true $t^v$ in $F$

### Equality

$(t_1 = t_2)^v = \begin{cases} 1 \text{ if } t_1^v = t_2^v \\ 0 \text{ otherwise} \end{cases}$

### Connectives

Same as propositional logic

### Quantifiers

To evaluate $\forall x A(x)$ → try every $x \in D$
To evaluate $\exists x A(x)$ → try at least 1 $x \in D$

$x$ may appear alongside other free variables so overwrite $x$'s value temporarily using $v(x/d)$ = valuation equal to $v$ except $x^v = d$

$(\forall x A(x))^v = 1 \equiv A^{v(x/d)} = 1 \quad \forall d \in D$

$(\exists x A(x))^v = 1 \equiv A^{v(x/d)} = 1 \quad$ for some $d \in D$

$v(x/d)$ means temporarily assign $x$ to a new value $d \in D$ to test the quantifier under a new valuation

Example Because this is Vague

$\forall x \exists y (F(y) \wedge G(x,y))$

Valuation 1: Domain = Ships
F = on fire
G = is larger than

Every ship is smaller than a ship thats on fire

Valuation 2: Domain = $\mathbb{N}$
F = is prime
G = $<$

Every natural number is less than a prime

Key: Same syntax, meaning changes based on valuation