Formal Deduction

Proving arguments in a purely syntactic way using formal rules for deduction. The goal is to define a relation called formal deducibility ($\vdash$) to mechanically/syntactically check the truth of a proof. These proofs will not rely on any semantic properties which is the main distinction

The Main Idea

Formal Deducibility is a relation between a set of formulas Σ (premises) and formula A (conclusion).

This is written a: $\Sigma \vdash A$ → A is formally deductible (provable) from Σ

$$\left.\begin{array}{l} \Sigma \cup A = \Sigma, A \\ \Sigma \cup \Sigma' = \Sigma, \Sigma' \end{array}\right\} \text{common rewrites}$$

The 11 Rules of Formal Deduction

1. $A \vdash A$ → Reflexivity

2. if: $\Sigma \vdash A$
   then: $\Sigma, \Sigma' \vdash A$ → Addition of premises (+)

3. if: $\Sigma, \neg A \vdash B$
   and: $\Sigma, \neg A \vdash \neg B$ → $\neg$ elimination
   then: $\Sigma \vdash A$

4. if: $\Sigma \vdash A \to B$
   and: $\Sigma \vdash A$ → $\to$ elimination
   then: $\Sigma \vdash B$

5. if: $\Sigma, A \vdash B$
   then: $\Sigma \vdash A \to B$ → $\to$ introduction

6. if: $\Sigma \vdash A \wedge B$
   then: $\Sigma \vdash A$ → $\wedge$ elimination
   and: $\Sigma \vdash B$

7. if: $\Sigma \vdash A$
   and: $\Sigma \vdash B$ → $\wedge$ introduction
   then: $\Sigma \vdash A \wedge B$

8. if: $\Sigma, A \vdash C$
   and: $\Sigma, B \vdash C$ → $\vee$ elimination
   then: $\Sigma, A \vee B \vdash C$

9. if: $\Sigma \vdash A$
   then: $\Sigma \vdash A \vee B$ → $\vee$ introduction
   and: $\Sigma \vdash B \vee A$

10. if: $\Sigma \vdash A \leftrightarrow B$
    and: $\Sigma \vdash A$ → $\leftrightarrow$ elimination
    then: $\Sigma \vdash B$

11. if: $\Sigma, A \vdash B$
    and: $\Sigma, B \vdash A$ → $\leftrightarrow$ introduction
    then: $\Sigma \vdash A \leftrightarrow B$

Each of these deductibility formulas can be formally expressed as ______ "is a theorum" like $\Sigma \vdash A$ is a theorum, basically meaning A is provable by the premises of Σ!. The meaning of theorum is that this statement can be proved valid by this logic system

$\vdash$ vs $\vDash$:

The distinction between deductability and tautological equivalence is crucial

$\Sigma \vdash A$: A is provable from Σ

$\Sigma \vDash A$: A is true in all models Σ is true (equivalent)

Example 1:

($\in$) if $A \in \Sigma$
then $\Sigma \vdash A$ (membership rule) → if A is a premise A can be derived from Σ

Proof: Suppose $A \in \Sigma$ and $\Sigma' = \Sigma - \{A\}$ (Σ is $A \cup \Sigma'$)

1) $A \vdash A$ (by ref) → generated by reflexivity, $A \vdash A$ is always a valid starting point
2) $A, \Sigma' \vdash A$ (by +, (1)) → generated by + rule applied to step 1.

These steps constitute a formal proof of $\Sigma \vdash A$, ($\in$) is now formally proven and is a new theorum

This proof is basically saying if $A \vdash A$ and $A \in \Sigma$ then $\Sigma \vdash A$ by breaking Σ into Σ' and A.

Proving Hypothetical Syllogism with Formal Deduction

$(\in)\ A \to B, B \to C \vdash A \to C$

(1) $A \to B, B \to C, A \vdash A \to B$ (by $(\in)$) → from premises
(2) $A \to B, B \to C, A \vdash A$ (by $(\in)$) → from Premises
(3) $A \to B, B \to C, A \vdash B$ (by ($\to$ elim), (1),(2)) → $A \to B$ and $A \therefore B$
(4) $A \to B, B \to C, B \vdash C$ (by $(\in)$)
(5) $A \to B, B \to C, A \vdash C$ (by ($\to$ elim), (3),(4)) → $B \to C$ and $B \therefore C$
(6) $A \to B, B \to C \vdash A \to C$ (by ($\to$ intro), from (5) discharging A) → A and $C \therefore A \to C$
↳ This discharges the temporary assumption that A is true.