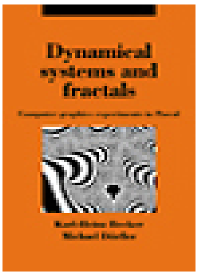

Dynamical Systems and Fractals

Computer Graphics Experiments with Pascal

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Chapter

3 - Strange Attractors pp. 55-70

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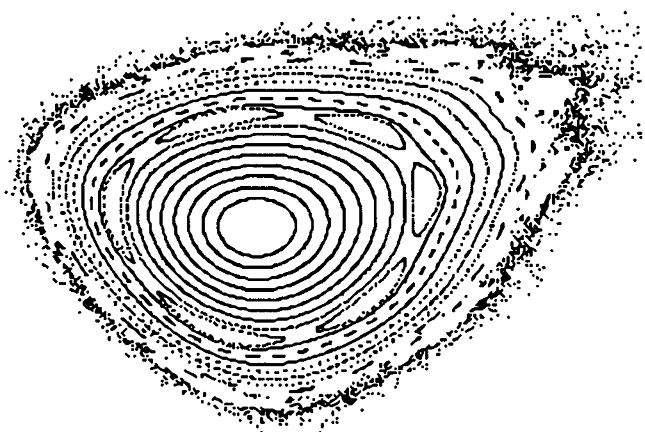
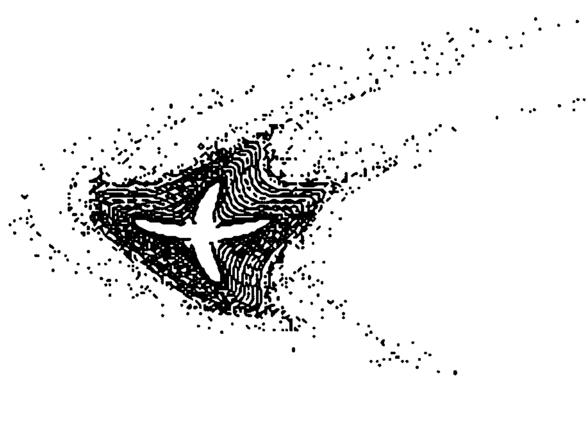
3 Strange Attractors





3.1 The Strange Attractor

Because of its aesthetic qualities, the Feigenbaum diagram has acquired the nature of a symbol. Out of allegedly dry mathematics, a fundamental form arises. It describes the connection between two concepts, which have hitherto seemed quite distinct: order and chaos, differing from each other only by the values of a parameter. Indeed the two are opposite sides of the same coin. All nonlinear systems can display this typical transition. In general we speak of the *Feigenbaum scenario* (see Chapter 9).

Indeed the fig-tree, although we have considered it from different directions, is an entirely static picture. The development in time appears only when we see the picture build up on the screen. We will now attempt to understand the development of the attractor from a rather different point of view, using the two dimensions that we can set up in a cartesian coordinate system. The feedback parameter k will no longer appear in the graphical representation, although as before it will run continuously through the range $0 \le k \le 3$. That is, we replace the independent variable k in our previous (k,p)-coordinate system by another quantity, because we want to investigate other mathematical phenomena. This trick, of playing off different parameters against each other in a coordinate system, will frequently be useful.

From the previous chapter we know that it is enough to choose k between 0 and 3. There are values between $k = 1.8$ and $k = 3$ at which we can observe the period-doubling cascade and chaos. In order to investigate the development of the Feigenbaum diagram in terms of the sequence

$$p_{n+1} = p_n + k * p_n * (1 - p_n),$$

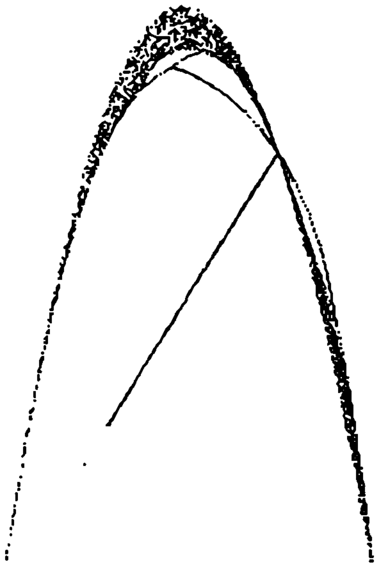
we choose as coordinate system the population values p_n and p_{n+1} which follow each other in the sequence. To the right we draw the final value p_n of the previous iteration, and we draw the result $p_{n+1} = f(p_n)$ vertically. We know this construction already from graphical iteration (see Chapter 2.1.2).

If you have already set up the program `Feigenbaum` then the modifications required are relatively easy. They relate solely to the part that does the drawing. Instead of the coordinates (k,p) we must now display $(p, f(p,k))$ on the screen. In the program fragment only the following part changes:

```
FOR i = 0 to Visible DO
BEGIN
   SetUniversalPoint (population, f(population, Feedback));
   population := f(population, Feedback);
END;
```

Nothing else need be altered.

You should make this modification to your existing program and see what happens. The final result (Figure 3.1-1) can only convey an incomplete impression of the dynamical development that occurs during its generation. You are advised to observe the gradual growth of this figure on the screen. If we choose the same scale on both axes the picture begins (for k a little larger than 0) in a less than spectacular way. The diagonal straight line, which first appears, expresses the fact that the underlying value is tending to a constant. Then $p = f(p)$. After $k = 2$ we obtain two alternating underlying values.



Data: 0, 3, 0, 1.4, 50, 50 for $0 \le k \le 3$

Figure 3.1-1 'Trace' of the parabola-attractor in the $p,f(p)$-plane.

The figure grows in two directions. Low starting values for the formula produce a higher result and then return. The curious picture here has the form of a thin curved line and runs roughly perpendicular to the original bisector. For periods 4 and 8 – when the figure grows in 4 or 8 places – it is also easy to see how the starting value p and the result $f(p)$ are connected. Thus we have built ourselves yet another measuring or observing instrument, with which we can watch the temporal development of period-doubling. As soon as we enter the chaotic region, a well-known mathematical object appears: the parabola.

If you want to draw this and similar pictures, please take a look at Exercises 3-1 and 3-2 at the end of this chapter. However, you will need a certain amount of patience, because in Figure 3.1-1 it takes some time, after the diagonal line is drawn, before points scatter on to the parabola.

In order to delve more deeply into the 'history' of the sequence, it is necessary to link together the results not just of one, but of several previous values. The investigation of the so-called Verhulst attractor (Figure 3.1-2) is especially interesting. This is the attractor corresponding to the equation

```
f(pn) = pn + 1/2*k*(3*pn*(1-pn)-pnMinus1*(1-pnMinus1))
```

which we have already encountered in §2.2.

For small k-values we soon run into a boundary, which of course lies on the diagonal of the $(p,f(p))$-coordinate system. We can observe the periods 2, 4, 8, etc. When we reach the chaos-value k=1.6 things get very exciting. At the first instant a parabola suddenly appears. Furthermore, it is not evenly filled, as we have seen already. It has an 'internal structure'.

Let us once more collect our conclusions together. The geometric form of the attractor arises because we draw the elements of a sequence in a coordinate system. To do this we represent the starting value p in one direction, and in the other the result $f(p)$ of the iteration for a fixed value of k. We first notice that the values for $f(p)$ do not leave a certain range between 0 and 1.4. Furthermore, we notice genuine chaos, revealing either no periodicity at all or a very long period. Here we know that more and more points appear in a completely unpredictable way. These points form lines or hint at their presence. Under careful observation the attractor seems to sit on a parabolic curve, defined by numerous thin lines. We want to take a closer look at that!

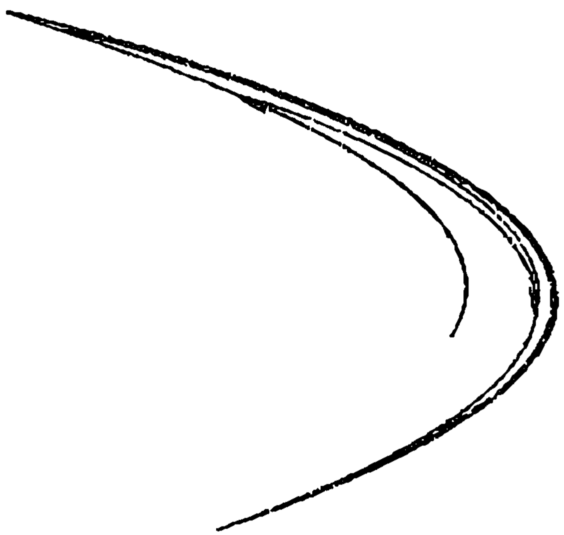


Figure 3.1-2 The Verhulst attractor for k = 1.60.

The changes in our previous program that are needed to generate the Verhulst attractor are again very simple:

Program Fragment 3.1-1

```
(* BEGIN: problem-specific procedures *)
PROCEDURE VerhulstAttractor;
  VAR i : integer;
```

```
        pn, pnMinus1, pnMinus2, oldValue: real;

    FUNCTION f(pn : real) : real;
    BEGIN
      pnMinus1 := pnMinus2;
      pnMinus2 := pn;
      f := pn + Feedback/2*(3*pn*(1-pn) -
            pnMinus1*(1-pnMinus1));
    END;

  BEGIN
    pn := 0.3; pnMinus1 := 0.3; pnMinus2 := 0.3;
    FOR i := 0 TO Invisible DO
      pn  := f(pn);
    REPEAT
      oldValue := pn;
      pn := f(pn);
      SetUniversalPoint (pn, oldValue);
    UNTIL Button;
  END;
```

The value of `Feedback` is constant during each run of the program, e.g. $k = 1.6$. In order to experiment with different k-values, the variable `Feedback` - and also `Left`, `Right`, `Bottom`, `Top` - must be input.

Using this `VerhulstAttractor` program we draw, in the first instance, the whole attractor, when p and $f(p)$ lie between 0 and 1.4 (see Exercise 3-3). If we choose a different range of values, we get different sections of it. To begin with, we look at places where there seems to be just a line, and then at the 'nodes' where the 'lines' meet or cross. These lines break up when they are magnified. In fact, they are really 'chains' into which the points arrange themselves. The picture resembles an aerial photograph of a large number of people walking in the snow along pre-defined tracks. The starting-point and destination are nowhere to be seen. By looking closely enough we can distinguish a faint track (along which the points/people lie more thickly) and parallel to it a wider one, on which the points are distributed more irregularly. A magnification of the wider track shows exactly the same structure. The same even happens if we magnify the thin track by a larger amount. If we examine the 'nodes' more carefully, we obtain something we have already encountered: a smaller version of the same attractor. This remarkable behaviour has already arisen in the Feigenbaum diagram. Many sections of the diagram produce the entire figure. This is the phenomenon of self-similarity again.

The strange Verhulst attractor exhibits a structure assembled from intricate curves

(Figure 3.1–3 ff.). By looking closely enough we encounter the same structure whenever we magnify the attractor. This structure is repeated infinitely often inside itself and occurs more often the more closely we look. The description of this 'self-similarity', and the aesthetic structure of the Verhulst attractor already referred to, are developed further in exercises at the end of the chapter.

Figure 3.1–3 Self-similarity at each step (top left projections of the attractor).

As already mentioned at the end of §2.1, we have wandered some way from our original problem ('measles in the children's home'). We will now explain the mathematical background and some possible generalisations. The equation on which the Verhulst attractor is based is well known as a numerical procedure for the solution of a differential equation. By this we mean an equation in which a function y and one of more of its derivatives occur, either linearly or nonlinearly. You remember: the first derivative y' describes how rapidly y changes. The second derivative y'' describes how y' changes, and hence the curvature of y. The simplest form for a nonlinear differential equation is

$$y' = y*(1-y) = g(y).$$

The symbol $g(y)$, which we use for $y*(1-y)$, will simplify the later description. A nonlinear differential equation is an equation in which the function y occurs quadratically, to a higher power, or for example as a trignometric expression.

Numerical methods are known for solving such equations. Starting from an initial value y_0, we try to approximate the equation in small steps. The rapidity with which we approach the limiting value (if one exists) is represented by the symbol k.

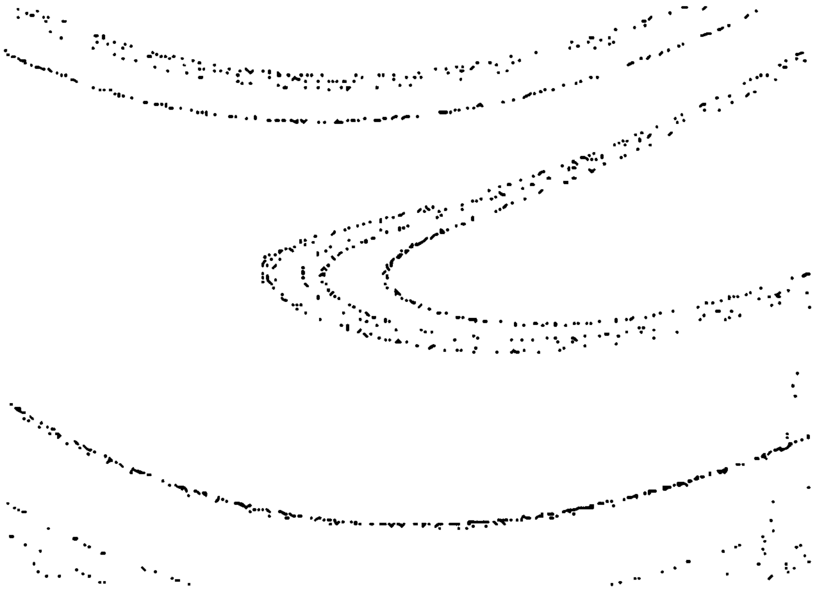


Figure 3.1-4 Self-similarity at each step (detail from the 'node' to the right of centre).

The simplest technique, known as *Euler's method*, is described e.g. in Abramowitz and Stegun (1968):

$$y_{n+1} = y_n + k * y'_n + O(k^2).$$

The final term expresses the fact that the equation is not exact, and that the error is of the order of magnitude of k^2. Since we have previously considered many iterations, the error interests us no further. The estimate is simplified if instead of $y'_n = g(y_n) = y_n*(1-y_n)$ we substitute

$$y_{n+1} = y_n + k * y'_n = y_n + k * g(y_n) = y_n + k * y_n * (1-y_n).$$

Thus we have recovered our old friend, the Feigenbaum formula, from § 2.2!

There now opens up a promising approach to interesting graphical experiments: choose a differential equation that is easy to compute, and try to approximate the solution by a numerical method. The numerical method, in the form of an iterative procedure, can then be taken as the basis of the graphical experiment.

In the same way we can derive the equation for the Verhulst attractor. The starting-point is the so-called *two-step Adams-Bashforth method*. It is somewhat more complicated than the Euler method (see Exercises 3-6 and 3-7).

3.2 The Hénon Attractor

We can find formulas for other graphically interesting attractors, without needing any special mathematical background.

Douglas Hofstadter (1981) describes the Hénon attractor. On page 7 he writes: 'It refers to a sequence of points (x_n,y_n) generated by the recursive formulas

$$x_{n+1} = y_n - a*x_n^2+1$$
$$y_{n+1} = b*x_n.$$

For the sequence illustrated the values a = 7/5 and b = 3/10 were taken; the starting values were $x_0 = 0$ and $y_0 = 0$.'

Contemplate the Hénon attractor of Figure 3.2-1.

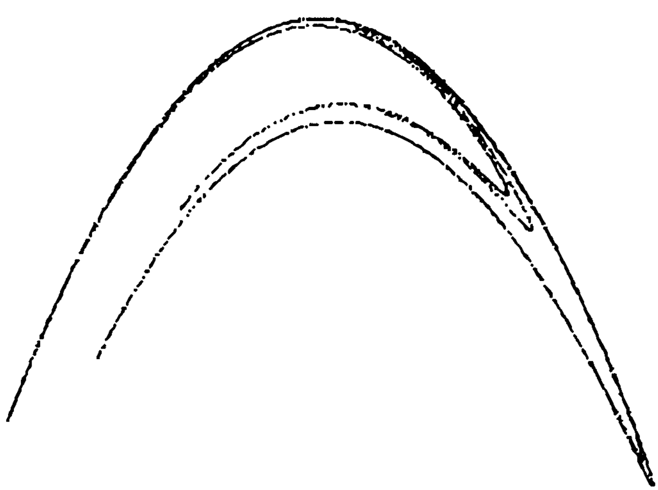


Figure 3.2-1 The Hénon attractor

Like the Feigenbaum diagram, the Hénon attractor should not be thought of as just a mathematical toy which produces remarkable computer graphics. In 1968, Michel Hénon, at the Institute for Astrophysics, Paris, proposed taking such simple quadratic mappings as models, to carry out computer-graphical simulations of dynamical systems. In particular, he was thinking of the study of the orbits of asteroids, satellites, and other heavenly bodies, or of electrically charged particles in particle accelerators.

During the period 1954-63 the mathematicians Kolmogorov, Arnold, and Moser developed a theory centred around the so-called KAM theorem. In it, they studied the behaviour of a stable dynamical system - such as, for example, a satellite circling the Earth - to clarify what happens when tiny external forces act on it. Planets or asteroids which orbit round the Sun often undergo such perturbations, so that their orbits are not truly elliptical. The KAM theorem attempts to decide whether small perturbations by external forces can lead to instability - to chaos - in the long-term behaviour. For instance an asteroid can be disturbed in its path by the gravitational force of Jupiter. Here one talks of *resonance.* Such resonances occur when the ratio of the orbital periods is a rational number. If, for example, two of Jupiter's orbits take the same time as five orbits of an asteroid, we have the case of a *2:5 resonance.*

In the title picture of Chapter 3 we see (upper figure) such a computer simulation, where in the course of the simulation ever stronger external influences are imposed. The inner curves elucidate the influence on the orbital behaviour of small external disturbances. Each point in the drawing shows the position of the asteroid after a further revolution. For small disturbances only small differences can be seen, and the system remains stable. When the influence of the external disturbance increases, we observe six 'islands'. These represent a *1:6 resonance.* A body, such as for instance an asteroid, having 1/6 the orbital period of Jupiter, would find itself in such a *resonance band.*

Further out, we see dotted regions of instability. The behaviour of an asteroid in this region is no longer predictable, because small external influences can have large effects. It is even possible for the asteroid to be catapulted out of its orbit into the 'emptiness' of the universe. Scientists believe that the gaps in the asteroid belt can be explained by this mechanism.

This brief explanation should make clear the connection between such simple formulas, and deep effects in the field of macroscopic physics. Further explanation can be found in physics texts or in Hughes (1986).

The formula to generate the title picture of Chapter 3 is as follows:

$$x_{n+1} = x_n * \cos(w) - (y_n - x_n^2) * \sin(w)$$
$$y_{n+1} = x_n * \sin(w) - (y_n - x_n^2) * \cos(w).$$

Here w is an angle in the range $0 \le w \le \pi$.

Compare the structure of this formula with the one described by Hofstadter at the start of this chapter. A program fragment for generating (other) Hénon attractors is as follows:

Program Fragment 3.2-1

```
PROCEDURE HenonAttractors;
(* x0, y0, dx0, dy0 global variables *)
  VAR
    cosA, sinA : real;
    xNew, yNew, xOld, yOld : real;
    deltaxPerPixel, deltayPerPixel : real;
    ok1, ok2 : boolean;
    i, j : integer;
BEGIN
  cosA := cos (phaseAngle); sinA := sin (phaseAngle);
  xOld := x0; yOld := y0;  {starting point of first orbit}
  deltaxPerPixel := Xscreen/(Right-Left);
  deltayPerPixel := Yscreen/(Top-Bottom);
```

```
FOR j = 1 to orbitNumber DO
BEGIN
   i := 1;
   WHILE i <= pointNumber DO
   BEGIN
      IF (abs(xOld)<= maxReal) AND (abs(yOld)<= maxReal)
      THEN
      BEGIN
         xNew := xOld*cosA - (yOld - xOld*xOld)*sinA;
         yNew := xOld*sinA + (yOld - xOld*xOld)*cosA;
         ok1 := (abs(xNew-Left) < maxInt/deltaxPerPixel);
         ok2 := (abs(Top-yNew) < MaxInt/deltayPerPixel);
         IF ok1 AND ok2 THEN
         BEGIN
            SetUniversalPoint (xNew, yNew);
         END;
         xOld := xNew;
         yOld := yNew;
      END;
      i := i+1;
   END;   {WHILE i}
   xOld := x0 + j * dx0;
   yOld := y0 + j * dy0;
 END;   {FOR j := ...}
END;
(* END : problem-specific procedures *)
```

3.3 The Lorenz Attractor

Five years before Michel Hénon began working in Paris on models for simulating dynamical systems, equally exciting things were happening elsewhere. In 1963, working in a completely different area, the American Edward N. Lorenz wrote a remarkable scientificarticle.

In his article Lorenz described a family of three particular differential equations[1] with parameters a, b, c:

$$x' = a*(y-x)$$
$$y' = b*x-y-x*z$$
$$z' = x*y-c*z.$$

Numerical analysis on a computer revealed that these equations have extremely

[1] We write the first derivatives with respect to time as x', y', z', etc. For example x' will be written in place of dx/dt.

complicated solutions. The complicated connections with and dependences upon the parameters could at first be elucidated only through computer-graphical methods.

The interpretation of the equations was exciting. In particular Lorenz sought - and found - a mathematical description which led to a rational explanation of the phenomenon of unpredictability of the weather in meteorology.

The idea of the model is as follows. The Earth is heated by the Sun. Part of the energy received at the Earth's surface is absorbed and heats the atmosphere from below. From above, the atmosphere is cooled by radiation into space. The lower, warmer layers of air want to rise upwards, and the upper, colder layers want to fall downwards. This transport problem, with oscillating layers of cold and warm air, can lead to turbulent behaviour in the atmosphere.

The picture's remarkable appearance cannot fully capture the surprising behaviour that occurs while it is being drawn on the screen. In consequence it is very important that you program and experiment for yourself.

Having said that, let us look first at some pictures of the Lorenz attractor.

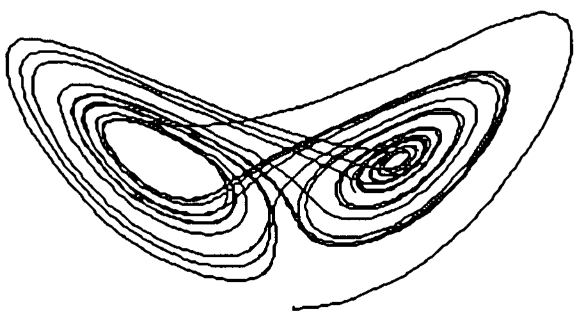


Figure 3.3-1 Lorenz attractor for a = 10, b = 28, c = 8/3, and screen dimensions -30, 30, -30, 80.

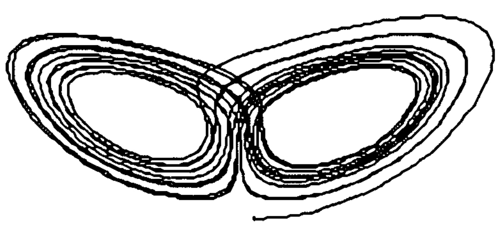


Figure 3.3-2 Lorenz attractor for a = 20, b = 20, c = 8/3, and screen dimensions -30, 30, -30, 80.

The program fragment that generates the figure is like this:

Program Fragment 3.3-1

```
(* START : Problem-specific procedures *)
PROCEDURE LorenzAttractor;
  VAR x, y, z : real;
  PROCEDURE f;
    CONST
      delta = 0.01;
    VAR
      dx, dy, dz : real;
  BEGIN
    dx := 10*(y-x);
    dy := x*(28-z)-y;
    dz := x*y - (8/3)*z;
    x := x + delta*dx;
    y := y + delta*dy;
    z := z + delta*dz;
  END;
BEGIN
  x := 1; y := 1; z := 1;
  f;
  SetUniversalPoint (x,z);
  REPEAT
    f;
    DrawUniversalLine (x, z);
  UNTIL Button;
END;
(* END : Problem-specific procedures *)
```

The behaviour which Lorenz observed on the screen in 1963 can be described in the following manner. The wandering point on the screen circles first round one of the two foci around which the two-lobed shape develops. Suddenly it changes to the other side. The point wanders on, drawing its line, until suddenly it switches back to the other side again. The behaviour of the path, in particular the change from one lobe to the other, is something that we cannot predict in the long run.

Even though this simple model is not, broadly speaking, capable of explaining the complex thermodynamic and radiative mechanisms that go on in the atmosphere, it does establish two points:

- It illustrates the basic impossibility of precise weather-prediction. Lorenz talked about this himself, saying that the 'fluttering of a butterfly's wing' can influence the weather. He called it the *butterfly effect.*

- It raises the hope that very complex behaviour might perhaps be understood through simple mathematical models.

This is a basic assumption, from which scientists in the modern theory of dynamical systems start. If this hypothesis had been wrong, and had not on occasion already proved to be correct, everybody's scientific work in this area would have been pointless.

As far as we are concerned, it remains true that certain principles on the limits to predictability exist, which cannot be overcome even with the best computer assistance.

To understand and pin down this phenomenon is the aim of scientists in the borderland between experimental mathematics, computer graphics, and other sciences.

Computer Graphics Experiments and Exercises for Chapter 3

Exercise 3-1

Modify the program `Feigenbaum` so that you can use it to draw the parabola attractor. Investigate this figure in fine detail. Look at important k-intervals, for example those with a uniform period. Look at the region near the vertex of the parabola, magnified.

Exercise 3-2

Carry out similar investigations using the number sequences described above in §2.2.

Exercise 3-3

Starting from `Feigenbaum` develop a Pascal program to draw the Verhulst attractor with the value $k = 1.6$, within the ranges $0 \le p \le 1.4$ and $0 \le f(p) \le 1.4$. Start with $p = 0.3$ and do not draw the first 20 points. Compare your result with Figure 3.1-2.

Investigate different sections from this figure, in which you define the boundaries of the drawing more closely.

Exercise 3-4

Make an animated 'movie' in which several pictures are shown one after the other. The pictures should show a section of the attractor at increasingly large magnification. We recommend a region near the 'node' with coordinates $p = 0.6$, $f(p) = 1.289$. Start with the entire picture. In this connection we offer a warning: the more extreme the magnification, the more points you must compute that lie outside the region being drawn. It can take more than an hour to put the outline of the attractor on the screen.

Exercise 3-5

Make a 'movie' in which a sequence of magnified sections of the attractor are superimposed on each other.

Exercise 3-6

The two-step Adams-Bashforth method can be written like this:

$$y_{n+1} = y_n + \tfrac{1}{2}*k*(3*g(y_n)-g(y_{n-1})).$$

If we substitute $g(y) = y*(1-y)$ we obtain

$$f(y_n) = y_n+\tfrac{1}{2}*k*(3*y_n*(1-y_n)-y_{n-1}*(1-y_{n-1})).$$

If we use the current variable p and the previous value pnMinus1, we get the familiar formula

```
f(pn) = pn+1/2*k*(3*pn*(1-pn)-pnMinus1*(1-pNminus1)).
```

Try out the above method to find solutions of differential equations, and also other variations on the method, such as:

$$y_{n+1} = y_{n-1} + 2*k*g(y_n)$$
$$y_{n+1} = y_n+ k/2*(g(y_n)+g(y_{n-1}))$$
$$y_{n+1} = y_n+k/24*(55*g(y_n)-59*g(y_{n-1})+37*g(y_{n-2})-9*g(y_{n-3})$$

or whatever else you can find in your mathematical textbooks.

Calculate Feigenbaum diagrams and draw the attractor in $(p,f(p))$-coordinates.

Exercise 3.7

We will relax the methods for solving differential equations further. The constants 3 and -1 appearing in the Adams-Bashforth method are not sacred. We simply change them.

Investigate the attractors produced by the recursion formula

```
f(p) = p + 1/2*k*(a*p*(1-p)+b*pnMinus1*(1-pnMinus1)).
```

Next work out, without drawing anything, which combinations of a and b can occur without the value $f(p)$ becoming too large. Put together a 'movie' of the changes in the attractor, occurring when the parameter a alone is varied from 2 to 3, with a = 3, b = -1 as the end point.

Exercise 3-8

Write a program to draw the Hénon attractor. Note that you do not have to use the same scale on each coordinate axis. Construct sections of this figure. Change the values of a and b in the previous exercises. Get hold of the article by Hofstadter (1981) and check his statements.

Exercise 3-9

Experiment with the system of equations for the 'planetary' Hénon attractor. Data for the title picture, for instance, are:

```
phaseAngle := 1.111; Left := -1.2, Right := 1.2;
```

```
Bottom := -1.2; Top := 1.2; x0 := 0.098; y0 := 0.061;
dx0 := 0.04; dy0 := 0.03; orbitNumber := 40;
pointNumber := 700;
```

further suitable data can be found in Hughes (1986).

Exercise 3-10

Experiment with the Lorenz attractor. Vary the parameters *a*, *b*, *c*. What influence do they have on the form of the attractor?

Exercise 3-11

Another attractor, called the *Rössler attractor*, can be obtained from the following formulas:

$$x' = -(y+z)$$
$$y' = x+(y/5)$$
$$z' = 1/5 + z*(x-5.7).$$

Experiment with this creature.