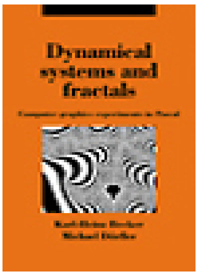

Dynamical Systems and Fractals

Computer Graphics Experiments with Pascal

Karl-Heinz Becker, Michael Dörfler, Translated by I. Stewart

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Chapter

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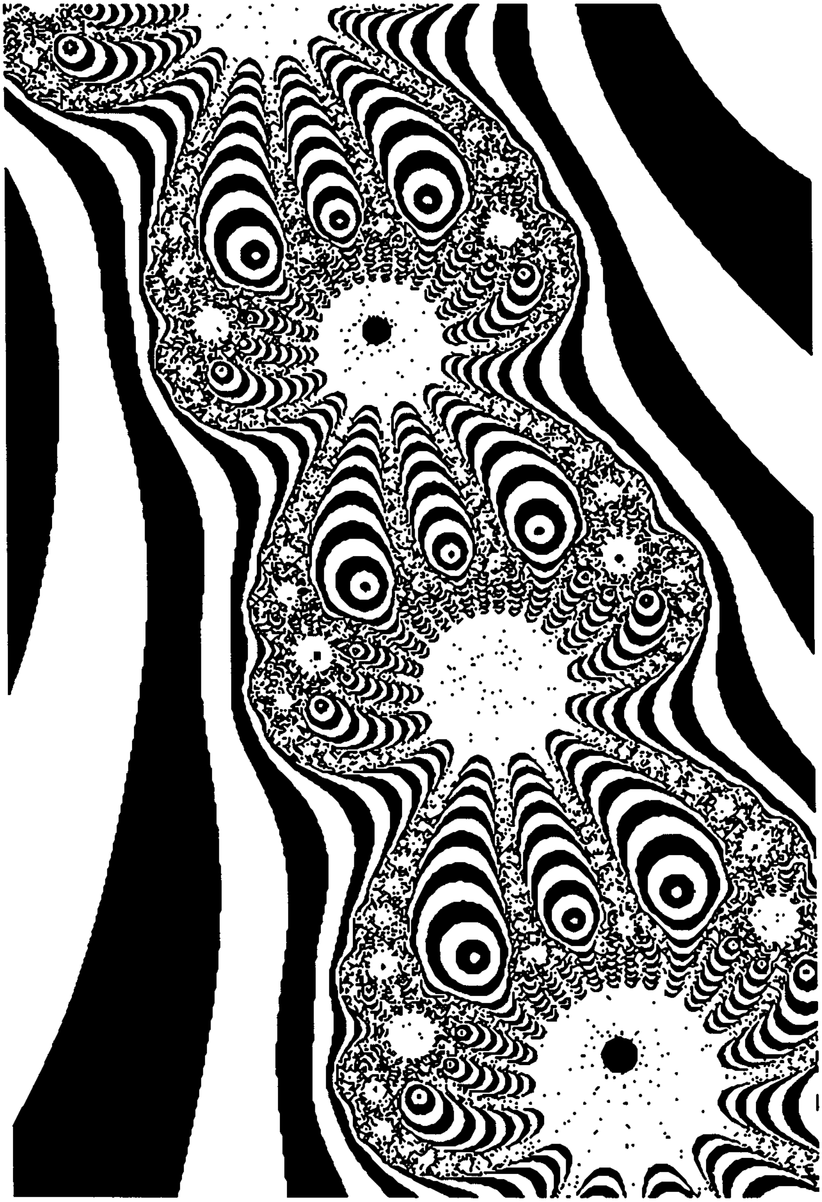
4 Greetings from Sir Isaac



In the previous chapters we saw what the 140-year-old Verhulst formula is capable of when we approach it with modern computers. Now we pursue the central ideas of self-similarity and chaos in connection with two further mathematical classics. These are *Newton's method* for calculating zeros, and the *Gaussian plane* for representing complex numbers.

In both cases we are dealing with long-established methods of applied mathematics. In school mathematics both are given perfunctory attention from time to time, but perhaps these considerations will stimulate something to change that.

4.1 Newton's Method

A simple mathematical example will demonstrate that chaos can be just around the next corner.

Our starting point is an equation of the third degree, the cubic polynomial

$$y = f(x) = (x+1)*x*(x-1) = x^3-x.$$

This polynomial has zeros at $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$ (Figure 4.1-1).

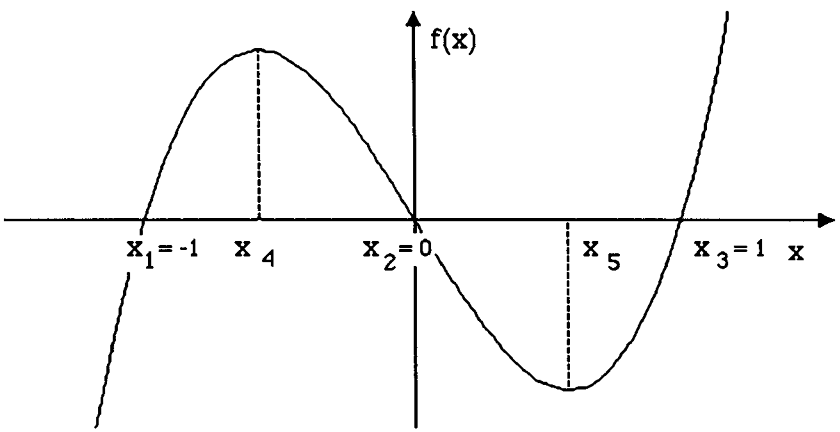


Figure 4.1-1 The graph of the function $f(x) = (x+1)*x*(x-1)$.

In order to introduce chaos into the safe world of this simple mathematical equation, we will apply the apparently harmless Newton method to this function.

Sir Isaac Newton was thinking about a widely encountered problem in mathematics: to find the zeros of a function, for which only the formula is known. For equations of the first and second degree we learn simple methods of solution at school, and complicated and tedious methods are known for polynomials of degree 3 or 4. For degree 5 there is no simple expression for the solution in closed form. However, complicated equations like these, and others containing trigonometric or other functions, are of interest in many applications.

Newton's point of departure was simple: find the zeros by trial and error. We start with an arbitrary value, which we will call x_n. From this the function value $f(x_n)$ is calculated. In general we will not have found a zero, that is, $f(x_n) \neq 0$. But from here we can 'take aim' at the zero, by constructing the tangent to the curve. This can be seen in Figure 4.1-2. When constructing the tangent we need to know the slope of the curve. This quantity is given by the derivative $f'(x_n)$, which can often be found easily.[1]

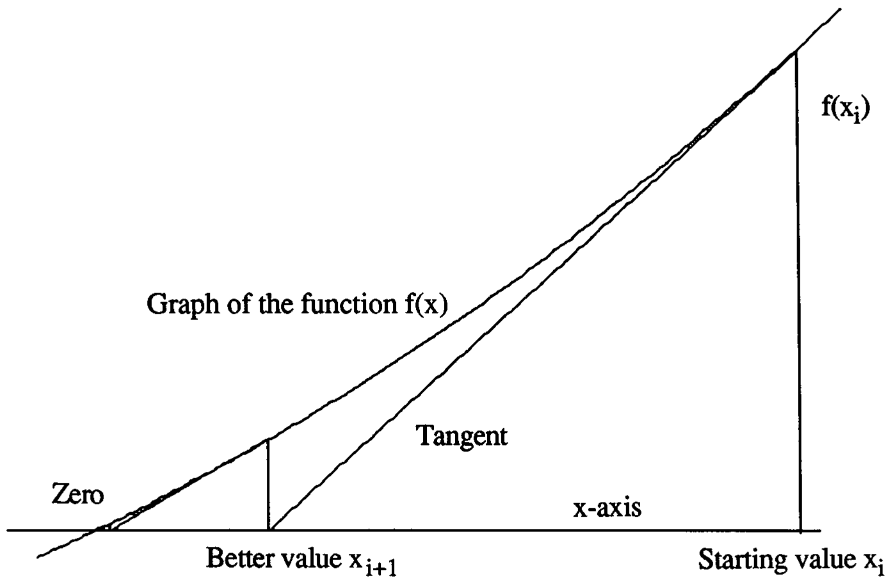


Figure 4.1-2 How Newton's method leads to a zero

A further problem should be mentioned. If $f'(x_n) = 0$, we find ourselves at a minimum, maximum, or inflexion point. Then we must carry out the analysis at the point $x_n + dx$.

The right-angled triangle in Figure 4.1-2 represents a slope of height $f(x_n)$ and width $f(x_n)/f'(x_n)$. From this last expression we can correct our approximate zero x_n, to get a better value for the zero:

$$x_{n+1} = x_n - f(x_n)/f'(x_n).$$

An even better approximation arises if we carry out this calculation again with x_{n+1} as input.

Basically Newton's method is just a feedback scheme for computing zeros.

[1] Even when $f'(x_n)$ is not known as an explicit function we can approximate the differential quotient closely, by $f(x_n) = (f(x_n+dx)-f(x_n-dx))/(2*dx)$, where dx is a small number, e.g. 10^{-6}.

When we get close enough to the zero for our purposes, we stop the calculation. A criterion for this might be, for example, that f(x) is close enough to zero ($|f(x_n)| \le 10^{-6}$) or that x does not change very much ($|x_n - x_{n+1}| \le 10^{-6}$).

For further investigation we return to the above cubic equation

$$f(x) = x^3 - x$$

for which

$$f'(x) = 3x^2 - 1.$$

Then we compute the improved value x_{n+1} from the initial value x_n using

$$x_{n+1} = x_n - (x_n^3 - x_n)/(3x_n^2 - 1).$$

If we exclude[2] the two irrational critical points $x_4 = -\sqrt{(1/3)}$ and $x_5 = \sqrt{(1/3)}$, then from any starting value we approach one of the three zeros x_1, x_2, or x_3. These zeros thus have the nature of attractors, because every possible sequence of iterations tends towards one of the zeros. This observation leads to a further interesting question: given the starting value, which zero do we approach? More generally: what are the basins of attraction of the three attractors x_1, x_2, and x_3?

The first results of this method will be shown in three simple sketches.

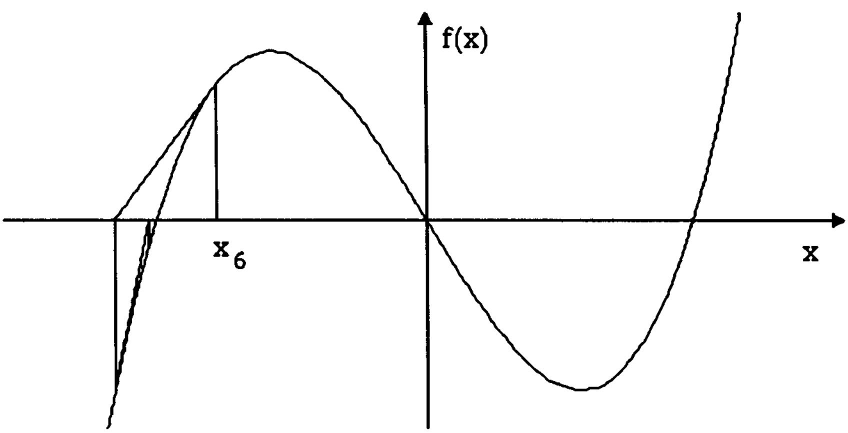


Figure 4.1-3 Initial value x_6 leads to attractor x_1.

We know the position of the axes and the graph of the function. At each iteration, we draw a vertical line and construct the tangent to the curve at that point. The result of Figures 4.1-3 to 4.1-5 is not particularly surprising: if we begin with values x_6, x_7, or x_8 close to an attractor, the the iteration converges towards that same attractor.

By further investigation we can establish:

---

[2]These numbers cannot be represented exactly in the computer. The Newton method fails here because the first derivative $f'(x_n) = 0$. Graphically, this follows because the tangents at these points are horizontal, and obviously cannot cut the x-axis, because they run parallel to it.

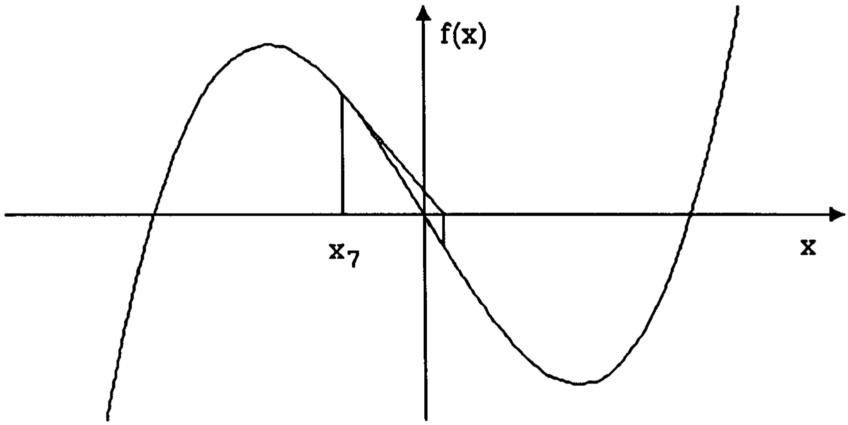


Figure 4.1–4 Initial value x_7 leads to attractor x_2.

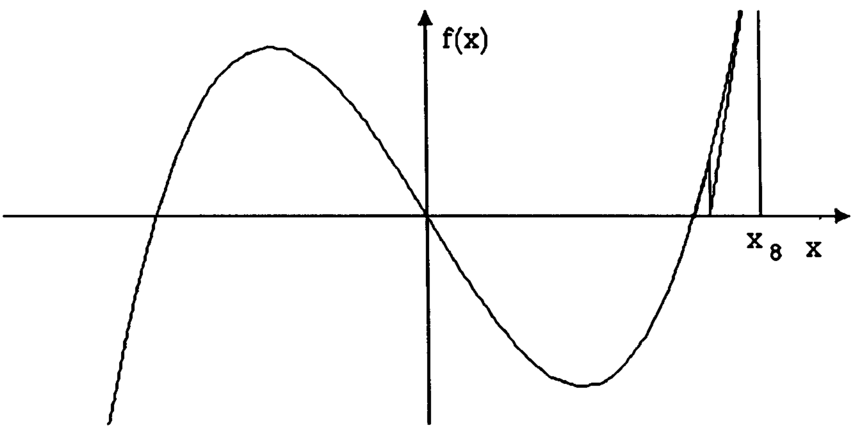


Figure 4.1–5 Initial value x_8 leads to attractor x_3.

- The basin of attraction of the attractor x_1 includes the region
  $-\infty < x < x_4 = -\sqrt{(1/3)}$.
- The basin of attraction of the attractor x_3 includes the region
  $x_5 = \sqrt{(1/3)} < x < \infty$.
  In particular, this region is symmetrically placed relative to the basin of attraction of x_1.
- The numbers near the origin belong to the basin of attraction of x_2.
- If we have found the attractor for a given initial value, then nearby initial values lead to the same attractor.

We expect exceptions only where the graph of the function has a maximum or a minimum. But we have already excluded these points from our investigations.

If we now take a glance at Figures 4.1-6 to 4.1-8, we realise that not everything is at simple as it appears from the above. In the next sequence of pictures we begin from three very close initial values, namely

$x_9 = 0.447\ 20$,

$x_{10} = 0.447\ 25$, and

$x_{11} = 0.447\ 30$.

We found these values by trial.

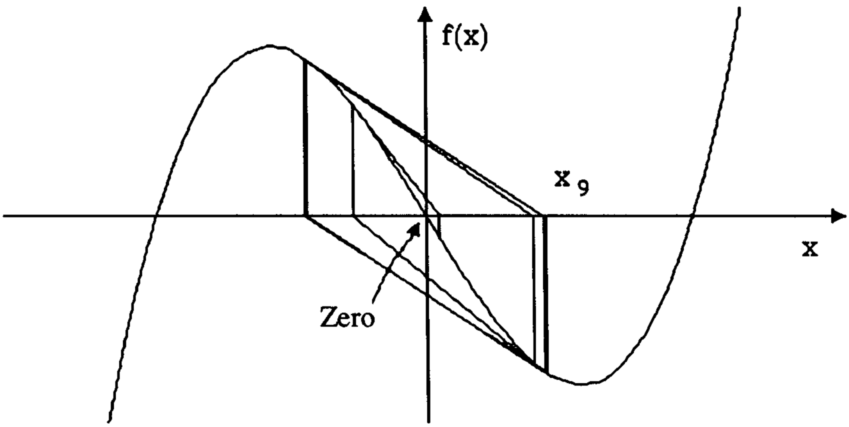


Figure 4.1-6 Initial value x_9 leads to attractor x_2.

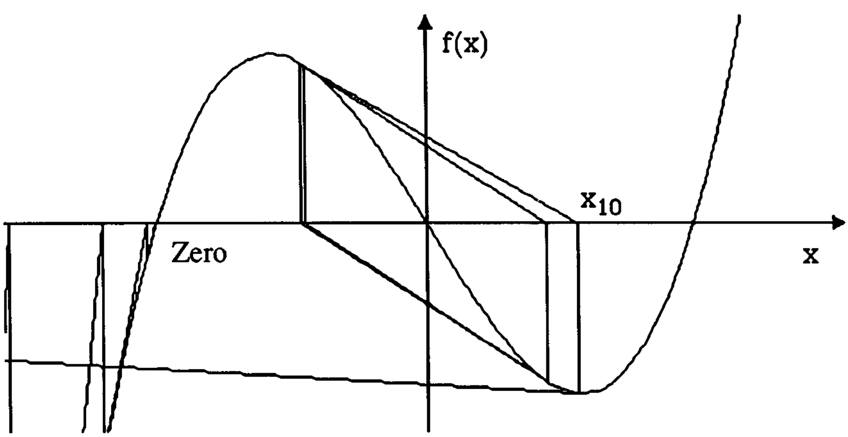


Figure 4.1-7 Initial value x_{10} leads to attractor x_1.

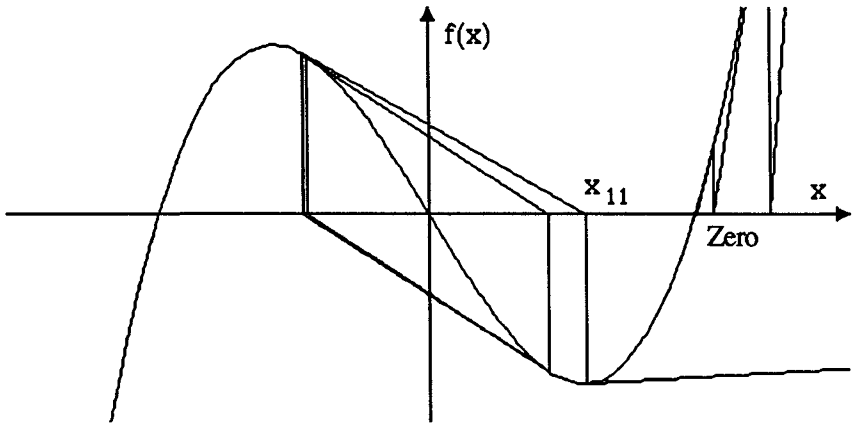


Figure 4.1-8 Initial value x_{11} leads to attractor x_3.

Despite their closeness, and despite the smooth and 'harmless' form of the graph of the function, the Newton method leads to the three different attractors. A sensible prediction seems not to be possible here. We refer to this 'breakdown of predictability' when we speak below of chaos.

In all areas of daily life, and also in physics and mathematics, we make use of a great number of unspoken assumptions, when we describe things or processes. One of the basic principles of physics is the causality principle.[3] Recall that this states that *the same causes* lead to *the same effects*. If this rule did not hold, there would be no technical apparatus upon which one could rely. Interestingly, this precept is often handled in a very cavalier fashion. We formulate this generalisation as the strong causality principle: *similar causes* lead to *similar effects*.

That this statement does not hold in general is obvious every Saturday in Germany when the lottery numbers are called - the result is not similar, even though the 49 balls begin each time in the same (or at least a similar) arrangement. Our definition of chaos is no more than this:

A chaotic system is one in which the strong causality principle is broken.

In the next step - and indeed in the whole book - we will show that such chaos is not totally arbitrary, but that at least in some regions an order, at first hard to penetrate, lies behind it.

In order to explain this order, in Figure 4.1-9 we have drawn the basins of attraction for the three attractors of the function $f(x)$ in different shades of grey and as rectangles of different heights. Everywhere we find a short, medium-grey rectangle, the iteration tends towards the attractor x_1. The basin of attraction of x_2 can be recognised as the light-grey, medium-height rectangles; and all points which tend towards x_3 can

[3]Causality: logical consequence.

be seen as high dark-grey rectangles.

Can you reconstruct the results collected together after Figure 4.1-5 from Figure 4.1-9?

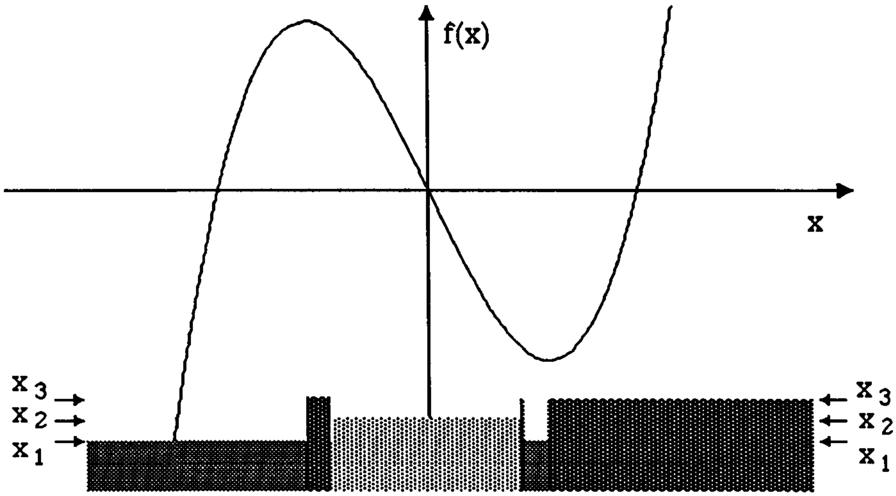


Figure 4.1-9 Graphical representation of the basins of attraction.

Especially interesting are the *chaotic zones*, in which there is a rapid interchange between the basins (of attraction). We show a magnified version of the left-hand region (for x-values in the range $-0.6 < x < -0.4$) in Figure 4.1-10. The section from Figure 4.9-9 is stretched along the x-axis by a factor of 40. The graph of the function in this range is barely distinguishable from a straight line.

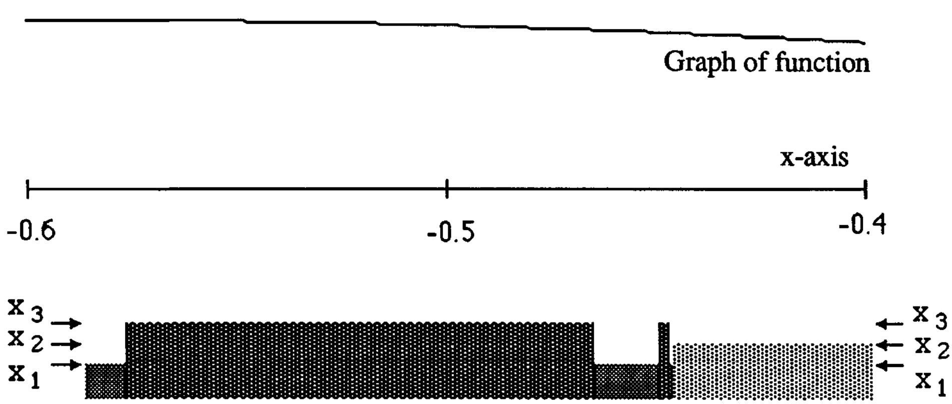


Figure 4.1-10 Basins of attraction (detail of Figure 4.1-9).

If we now investigate the basins shown by the grey areas we observe the following:

- On the outside we find the basins of x_1 and x_2.
- A large region from the basin of x_3 has sneaked between them.
- Between the regions for x_3 and x_2 there is another region for x_1.
- Between the regions for x_1 and x_2 there is another region for x_3.
- Between the regions for x_3 and x_2 there is another region for x_1,

and so on.

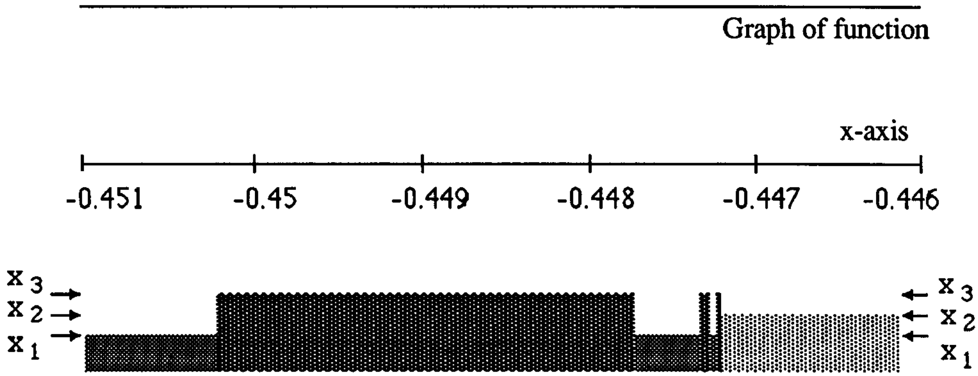
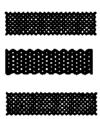


Figure 4.1-11 Basins of attraction (detail of Figure 4.1-10).

A further magnification by a factor of 40 in Figure 4.1-11 shows the same scheme again, but on a finer scale. We have already met the scientific description of this phenomenon: self-similarity.

The apparent chaos reveals itself as a strongly ordered zone.

In a further investigation we will now set to work, calculating as closely as possible the points that separate the basins from each other. The corresponding program will not be described further here: it is left as an exercise. These boundary points will be denoted g_i. Only the results are shown, in Table 4-1.



- The first value is given by $g_1 = -0.577\ 35\ldots$.
- If $x < g_1$ then x belongs to the basin of x_1.
- If $g_1 < x < g_2$ then x belongs to the basin of x_3.
- If $g_2 < x < g_3$ then x belongs to the basin of x_1, and so on.

Using Table 4-1 we have discovered a simple mathematical connection between the g_i-values. Namely, the quotient tends to a constant value. In fact

$$\lim_{n\to\infty} \frac{g_n - g_{n-1}}{g_{n+1} - g_n} = 6.0 = q \ .$$

Index n	g_n	$(g_n-g_{n-1})/(g_{n+1}-g_n)$
1	-0.577 350 269 189 626	-
2	-0.465 600 621 433 678	7.256 874 166 975 182
3	-0.450 201 477 782 476	6.179 501 149 801 554
4	-0.447 709 505 812 910	6.029 219 709 825 583
5	-0.477 296 189 979 436	6.004 851 109 839 370
6	-0.447 227 359 657 766	6.000 807 997 292 021
7	-0.447 215 889 482 132	6.000 134 651 772 122
8	-0.447 213 977 829 095	6.000 022 441 303 783
9	-0.447 213 659 221 447	6.000 003 740 154 308
10	-0.447 213 606 120 205	6.000 000 623 270 044
11	-0.447 213 597 269 999	6.000 000 039 232 505
12	-0.447 213 595 794 965	-

Table 4-1 The basin boundaries

For x-values greater than zero the same result holds but now with positive g_i-values. The resulting quotient is the same.

A few words to explain these numbers are perhaps in order.

- The number g_1 has the value $g_1 = \sqrt{(1/3)} = 3^{-1/2}$.

This is worked out by applying school methods to investigate the curve

$$f(x) = x^3 - x.$$

At $x = g_1$ the first derivative

$$f'(x) = 3x^2 - 1$$

takes the value $f'(g_1) = 0$.

The function $f(x)$ has an extreme value there, at which the derivative changes sign, so an increasing function becomes decreasing. Because the slope (first derivative) plays a special role in Newton's method, this leads us to conclude that the points on the right and left of an extreme value belong to distinct basins of attraction.

- For the limiting value of the g_i we have:

$$\lim_{n\to\infty} g_n = \sqrt{\frac{1}{5}} = x_g \ .$$

This value too can be expressed analytically.[4]

[4]Analytic behaviour is here intended in comparison with the numerical behaviour found previously.

For this purpose we use Figures 4.1-4 to 4.1-6. In all three cases the iteration runs several times almost symmetrically round the origin, before it decides on its broader course. In the extreme case we can convince ourselves, that there must exists a point x_g, such that the iteration can no longer escape, and indeed that each iteration only changes the sign. After two steps the original value recurs.

For x_g we must have

$$x_g = \frac{f(x_g)}{f'(x_g)} = -x_g \quad \text{or} \quad x_g = \frac{x_g^3 - x_g}{3x_g^2-1} = -x_g \;.$$

Simplifying the equation, we get

$$5x_g^2 = 1$$

whence the above expression.

- As regards q:

Why this quotient always has the value $q = 6$, we cannot explain here. It puzzled us too.

Further experiments make it possible to show that q is always 6 if we investigate a cubic function, whose zeros lie on the real line, and are equal distances apart. In other cases we instead find a value $q_A > 6$ and a value $q_B < 6$.

Computer Graphics Experiments and Exercises for §4.1

The only experiments for this chapter are, exceptionally, rather short. You can of course try to work out Table 4-1, or similar tables for other functions.

The next section will certainly be more interesting graphically.

4.2 Complex Is Not Complicated

In previous chapters we have formulated the two basic principles of graphical phenomena that concern us: self-similarity and boundaries. The first concept we have encountered many times, despite the different types of representation that are possible in a cartesian coordinate system.

In previous figures the boundaries between basins of attraction have not always been clearly distinguishable. In order to investigate these boundaries more carefully, we will change our previous methods of graphical representation and switch to the two-dimensional world of surfaces. We thus encounter a very ingenious and elegant style of graphics, by which we can also show the development of the boundary in two dimensions on a surface.

No one would claim that what we have discussed so far is entirely simple. But now it becomes 'complex' in a double sense. Firstly, what we are about to consider is really complicated, unpredictable and not at all easy to describe. Secondly, we will be dealing with mathematical methods that have come to be called 'calculations with complex numbers'. It is not entirely necessary to understand this type of calculation in order to generate the pictures: you can also use the specific formulas given. For that reason we will write out the important equations in full detail. In some motivation and generalisations a knowledge of complex numbers will prove useful. The theory of complex numbers also plays a role in physics and engineering, because of its numerous applications. This is undeniable: just look at a mathematics textbook on the subject. Independently we have collected together the basic ideas below.

The complex numbers are an extension of the set of real numbers. Recall that the real numbers comprise all positive and negative whole numbers, all fractions, and all decimal numbers. In particular all numbers that are solutions of a mathematical equation belong to this set. These might, for example, be the solutions of the quadratic equation

$$x^2 = 2,$$

namely $\sqrt{2}$, the 'square root of 2', or the famous number π, which gives the connection between the circumference and diameter of a circle. There is only one restriction. You cannot take the square root of a negative number. So an equation like

$$x^2 = -1$$

has no solutions in real numbers. Such restrictions are very interesting to mathematicians. New research areas always appear when you break previously rigid rules.

The idea of introducing imaginary numbers was made popular by Carl Friedrich Gauss (1777–1855), but it goes back far earlier. In 1545 Girolamo Cardano used imaginary numbers to find solutions to the equations $x+y = 10$, $xy = 40$. Around 1550, Raphael Bombelli used them to find real roots to cubic equations.

They are imaginary in the sense that they have no position on the number line and exist only in the imagination. The basic imaginary number is known as i and its properties are defined thus:

$$i * i = -1.$$

The problem of the equation

$$x^2 = -1$$

is thus solved at a stroke. The solutions are

$$x_1 = i \text{ and } x_2 = -i.$$

If you remember this, the rules of calculation are very straightforward.

A few examples of calculation with imaginary numbers should clarify the computational rules:[5]

- $2i * 3i = -6$.
- $\sqrt{(-16)} = \pm\, i*4 = \pm 4i$.
- The equation $x^4 = 1$ has four solutions 1, −1, i, and −i: they are all 'fourth roots' of 1.
- $6i-2i = 4i$.

Imaginary numbers can be combined with real numbers, so that something new appears. These numbers are called *complex numbers.* Examples of complex numbers are 2+3i or 3.141592 - 1.4142*i.

[5] You will find further examples at the end of the chapter.

A whole series of mathematical and physical procedures can be carried out especially elegantly and completely using complex numbers. Examples include damped oscillations, and the electrical behaviour of circuits that contain capacitors and resistors. In addition, deep mathematical theories (such as function theory) can be constructed using complex numbers, a fact that is not apparent when we discuss just the basics.

All equations for the basic rules, which are important for the respresentation of boundary behaviour, can be expressed using elementary mathematics.

We begin with the rule

$$i*i = -1$$

and the notation

$$z = a+i*b$$

for complex numbers.

Two numbers z_1 and z_2, which we wish to combine, are

$$z_1 = a+i*b \text{ and } z_2 = c+i*d.$$

Then the following basic rules of calculation hold:

Addition

$$z_1+z_2 = (a+i*b)+(c+i*d) = (a+c) + i*(b+d).$$

Subtraction

$$z_1-z_2 = (a+i*b)-(c+i*d) = (a-c) + i*(b-d).$$

Multiplication

$$z_1*z_2 = (a+i*b)*(c+i*d) = (a*c-b*d) + i*(a*d+b*c).$$

The *square* is a special case of multiplication:

$$z_1^2 = z_1*z_1 = (a+i*b)^2 = (a^2-b^2) + 2*i*a*b.$$

Division

Here a small problem develops: all expressions that appear must be manipulated so that only real numbers appear in the denominator. For

$$\frac{1}{z_2} = \frac{1}{c+i*d}$$

this can be achieved by multiplying by $(c-i*d)$, the *complex conjugate* of the denominator:

$$\frac{1}{c+i*d} = \frac{c-i*d}{(c+i*d)*(c-i*d)} = \frac{c-i*d}{c^2+d^2} .$$

From this we get the rule for division:

$$\frac{z_1}{z_2} = \frac{a+i*b}{c+i*d} = \frac{a*c+b*d}{c^2+d^2} + i*\frac{b*c-a*d}{c^2+d^2} .$$

Furthermore, there is also a geometrical representation for all complex numbers and all of the mathematical operations. The two axes of an (x,y)-coordinate system are identified with the real and imaginary numbers respectively. We measure the real numbers along the x-axis and the imaginary numbers along the y-axis. Each point

P = (x,y) of this *Gaussian plane* [6] represents a complex number

$$z = x + i*y$$

(Figure 4.2-1).

Multiplication can be understood better graphically than by way of the above equation. To do this we consider not, as before, the real and imaginary parts of the complex number, but the distance of the corresponding point from the origin and the direction of the line that joins them (Figure 4.2-1). Instead of cartesian coordinates (x,y) we use *polar coordinates* (r,φ). In this polar coordinate system, multiplication of two complex numbers is carried out by the following rule:

If $z_1*z_2 = z_3$, then $r_1*r_2 = r_3$ and $\varphi_1+\varphi_2 = \varphi_3$.

We multiply the distances from the origin and add the polar angles. The distance r is described as the *modulus* of the number z and written $r = |z|$.

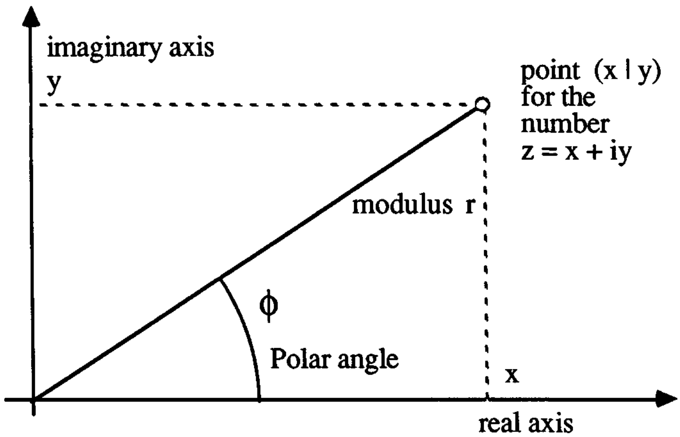


Figure 4.2-1 A point in the Gaussian plane and its polar coordinates.

What connection does the complex plane have with our mathematical experiments, with chaos, with computer graphics?

The answer is simple. Until now the region in which we have calculated, and which we have drawn, has been a section of the real-number axis, from which the parameter k was chosen. We carried out our calculations, and drew the result, for each point of that section - as long as it remained within the limits of the screen. Now we let the parameter become complex. As a result the equations for calculating chaotic systems

[6] *Translator's note*: Gauss represented complex numbers on a plane in about 1811. In many countries the Gaussian plane is known as the ***Argand diagram***, after Jean-Robert Argand who published it in 1806. A Danish surveyor, Caspar Wessell, has a greater claim than either, having independently described the idea in 1797. The above is the conventionally recognised trio: for some reason everyone seems to ignore the fact that John Wallis used a plane to represent a complex number geometrically in his *Algebra* of 1673.

are especially simple. In particular we can represent the results directly in the complex plane. Depending on the shape of the computer screen we use a rectangular or square section of the plane. For each point in the section - as long as it remains within the limits of the screen - we carry out our calculations. The complex number corresponding to this point represents the current parameter value. After iteration we obtain the value $f(z)$ from the result of our calculation, which tells us how the corresponding screen point is coloured.

Complex numbers aren't so hard, are they? Or aren't they?

Computer Graphics Experiments and Exercises for §4.2

Exercise 4.2-1

Draw on millimetre graph paper a section of the complex plane. Using a scale 1 unit = 1 cm, draw the points which correspond to the complex numbers

$z_1 = 2-i*2$, $z_2 = -0.5+1*1.5$, and $z_3 = 2-i*4$.

Join these points to the origin. Do the same for the points

$z_4 = z_1+z_2$ and $z_5 = z_3-z_1$.

Do you recognise an analogy with the addition and subtraction of vectors?

Exercise 4.2-2

The following connection holds between cartesian coordinates (x,y) and polar coordinates with distance r and polar angle φ:

$r^2 = x^2+y^2$ and $\tan \varphi = y/x$.

If $x = 0$ and $y > 0$, then $\varphi = 90°$.

If $x = 0$ and $y < 0$, then $\varphi = 270°$.

If $x = 0$ and also $y = 0$, then $r = 0$ and the angle φ is not defined.

Recall that for multiplication the following then holds: if $z_1*z_2 = z_3$ then $r_1*r_2 = r_3$ and $\varphi_1+\varphi_2 = \varphi_3$. Express this result in colloquial terms.

Investigate whether both methods of multiplication lead to the same result, using the numbers in Exercise 4.2-1.

Exercise 4.2-3

In the complex plane, what is the connection between:

- A number and its complex conjugate?
- A number and its square?
- A number and its square root?

Exercise 4.2-4

If all the previous exercise have been too easy for you, try to find a formula for powers. How can you calculate the number

$$z = (a+i*b)^p$$

when p is an arbitrary positive real number?

Exercise 4.2-5

Formulate all of the algorithms (rules of calculation) in this section in a programming language.

4.3 Carl Friedrich Gauss meets Isaac Newton

Of course, these two scientific geniuses never actually met each other. When Gauss was born, Newton was already fifty years dead. But that will not prevent us from arranging a meeting between their respective ideas and mathematical knowledge.

We transform Newton's method into a search for zeros in the complex plane. The iterative equations derived above will be applied to complex numbers instead of reals. This is a trick that has been used in innumerable mathematical, physical, and technical problems. The advantage is that many important equations can be completely solved, and the graphical representations are clearer. The normally important real solutions are considered as a special case of the complex.

Our starting point (§4.1) was

$$f(x) = x^3-x.$$

For this the Newton method takes the form

$$x_{n+1} = x_n - (x_n^3-x_n)/(3x_n^2-1).$$

For complex numbers it is very similar:

$$z_{n+1} = z_n - (z_n^3-z_n)/(3z_n^2-1).$$

Recalling that $z_n = x_n+i*y_n$, this becomes

$$z_{n+1} = \frac{2*(x_n^3-3x_ny_n^2+i*(3x_n^2y_n-y_n^3))}{3x_n^2-3y_n^2-1+i*6x_ny_n} .$$

Further calculations, in particular complex division, can be carried out more easily on a computer.

The calculation has thus become a bit more complicated. But that is not the only problem that faces us here. Now it is no longer enough to study a segment of the real line. Instead, our pictures basically lie in a section of the complex plane. This two-dimensional rectangular area must be investigated point by point. The iteration must therefore be carried out for each of 400 points in each of 300 lines.[7]

We know the mathematical result already from the previous chapter: one of the three

[7]These data can vary from program to program and computer to computer. Most of our pictures use a screen of 400 × 300 pixels.

zeros x_1, x_2, x_3 on the real axis will be reached.[8] This remains true even when the iteration starts with a complex number.

The graphical result is nevertheless new. To show how the three basins of attraction fit together, we have shaded them in grey in Figure 4.3-1, just as we did in §4.1.

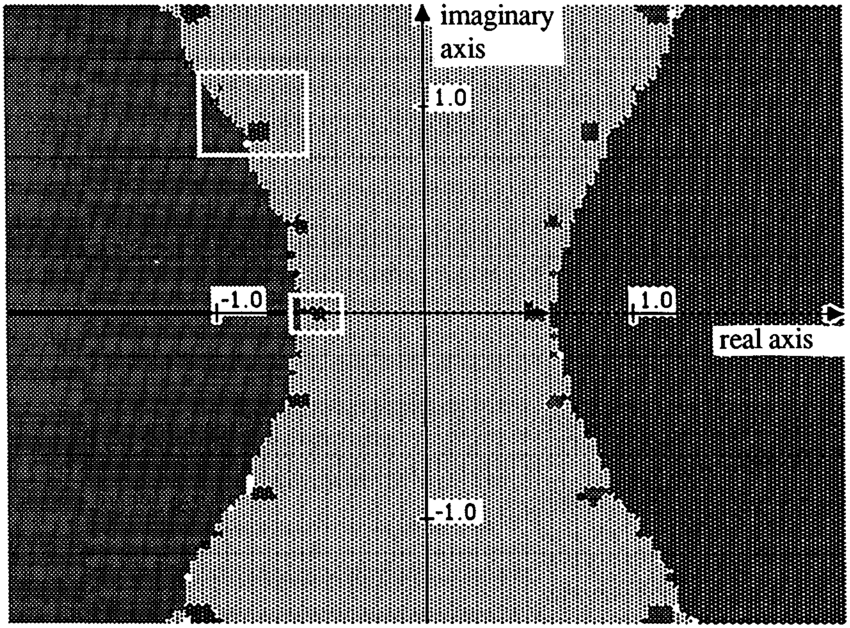


Figure 4.3-1 The basins of attraction in the complex plane.[9]

Thus the basin of x_1 is medium grey, that of x_2 is light grey, and that of x_3 is dark grey. All points for which it cannot be decided, after 15 iterations, towards which attractor they are tending, are left white.

Along the real axis, we again find the various regions that were identified in §4.1. 'Chaos', as it first appeared to us in Figure 4.1-9, we recognise in the small multi-coloured regions. We have defined chaos as the 'breakdown of predictability'. The graphical consequence of this uncertainty is fine structures 'less than one pixel in resolution'. We can investigate their form only by magnification.

The interesting region, which we investigated on the real axis in Figures 4.1-9 to 4.1-11, is shown in Figure 4.3-2 on a large scale. Again, self-similarity and regularity of the structure can be seen.

[8]We retain the names from §4.1, even though we are working with complex numbers z_1 etc. This is permissible, because the imaginary parts are zero.

[9]The two outlined regions on the real axis and above it will be explored in more detail in the following pictures.

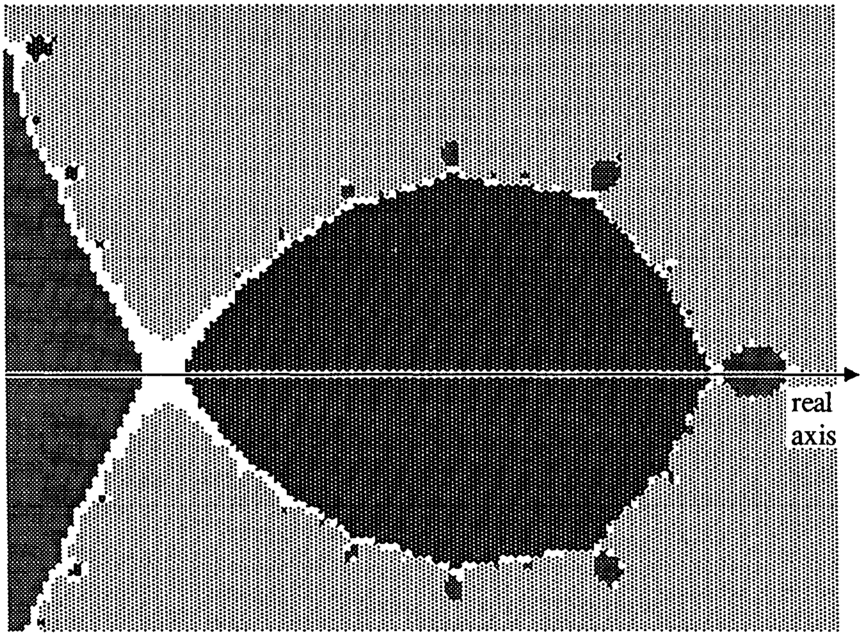


Figure 4.3-2 Section from Figure 4.3-1 left of centre.

Compared with the appearance on the real axis, which we have seen already, Figure 4.3-1 reveals something new. In many different places there appear 'grapelike' structures like Figure 4.3-2. An example appears magnified in Figure 4.3-3.

Self-similarity does not just occur on the real axis in these graphical experiments. In general, where a boundary between two basins of attraction occurs, similar figures are observed, sprinkled ever more thickly along the border. The same section as in Figure 4.3-3 leads to the next picture, in a different experiment. In this drawing only points are shown for which it cannot be decided, after 12 iterations, to which basin they belong. Thus the white areas correspond to those which in the previous pictures are shown in grey. Their structure is somewhat reminiscent of various sizes of 'blister' attached to a surface.

Further magnified sections reveal a similar scheme. The basins of attraction sprout ever smaller offshoots. One of the first mathematicians to understand and investigate such recursive structures was the Frenchman Gaston Julia. After him we call a complex boundary with self-similar elements a *Julia set*.

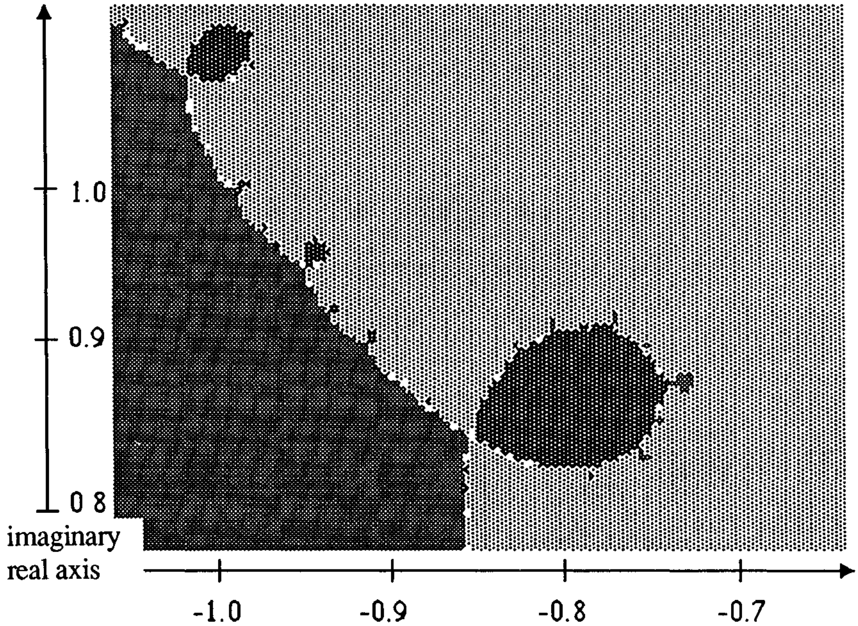


Figure 4.3-3 At the boundary between two basins of attraction.

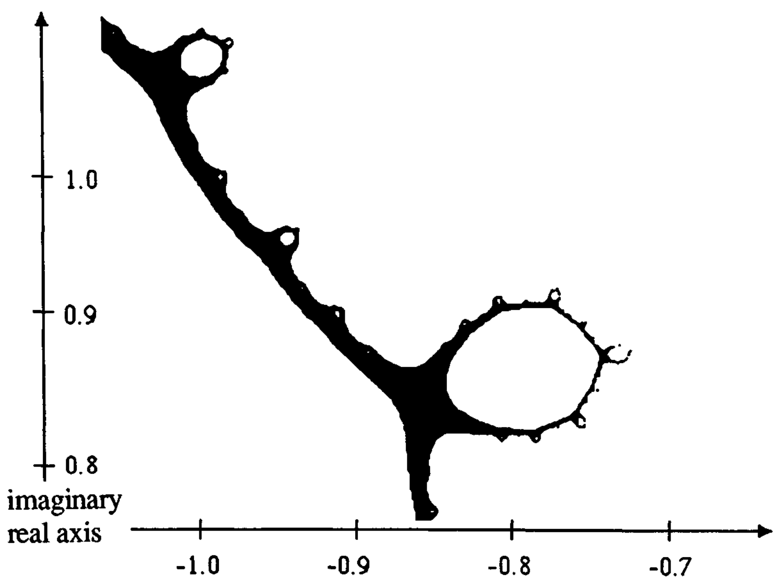


Figure 4.3-4 The boundary between two basins of attraction.

To close this section, which began with a simple cubic equation and immediately led into 'complex chaos', we illustrate a further possibility, visible in Figure 4.3-1, in graphical form. Instructions for the production of these pictures are to be found in the next chapter.

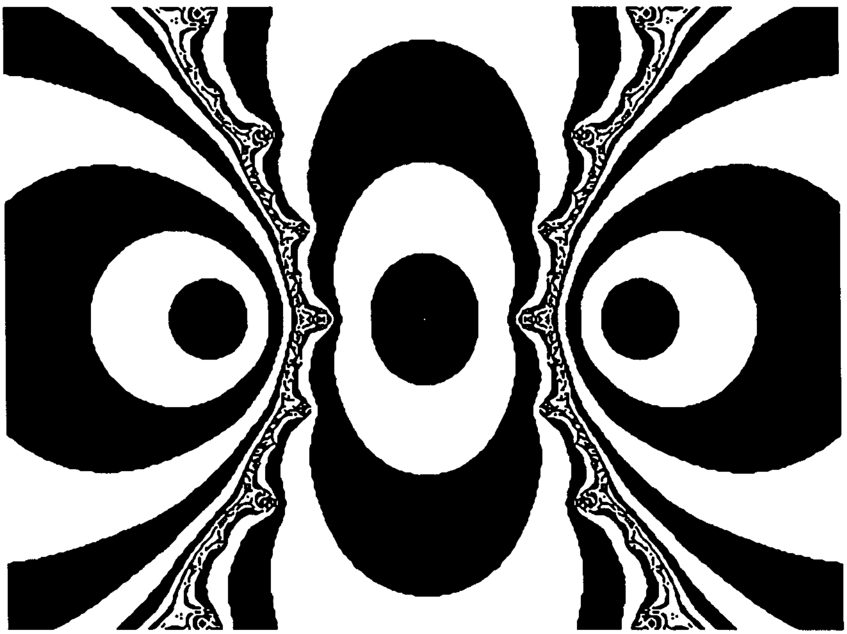


Figure 4.3-5 Stripes approaching the boundary.