

Pedestal Structure and Stability in High-Performance Plasmas on Alcator C-Mod

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Thank you to...

- The thesis committee: JW Hughes, DG Whyte, AE White, JP Freidberg
- The I-mode crew: AE Hubbard, JL Terry, I Cziegler, A Dominguez, SG Baek, C Theiler, RM Churchill, ML Reinke, JE Rice...
- Physops: R Granetz, S Shiraiwa, S Wolfe, S Wukitch...
- C-Mod operations, engineering, researchers and techs
- PSFC grad students, past and present
- Family and friends
- the audience!

Outline

■ Context & Motivation

- ▶ High-performance regimes
- ▶ Pedestal physics
- ▶ Introduction to I-mode

■ Pedestal Modeling & Theory:

- ▶ Peeling-balloonning MHD stability
- ▶ Kinetic-balloonning mode turbulence

■ ELMy H-Mode Physics¹

- ▶ EPED Modeling on C-Mod

¹JR Walk *et al.*, Nuclear Fusion 52 (2012)

Outline

■ I-Mode Pedestals & Global Performance^{2,3}

- ▶ Pedestal response to fueling, heating power
- ▶ Pedestal widths and gradients
- ▶ Global performance and confinement scalings

■ I-Mode Pedestal Stability

- ▶ P-B MHD, KBM modeling
- ▶ ELM characterization

■ Summary, Future Work, & Questions

²JR Walk *et al.*, *Physics of Plasmas* **21** (2014)

³Invited talk, APS-DPP Nov. 2013

The problem...

By default (“L-mode”), rapid transport of energy and particles from plasma driven by turbulence

- and energy transport gets *worse* with more heating power!
- need very strong magnetic field and/or large machine size to overcome poor plasma performance

L-mode likely not suitable for (economical) power plant development.

The solution?

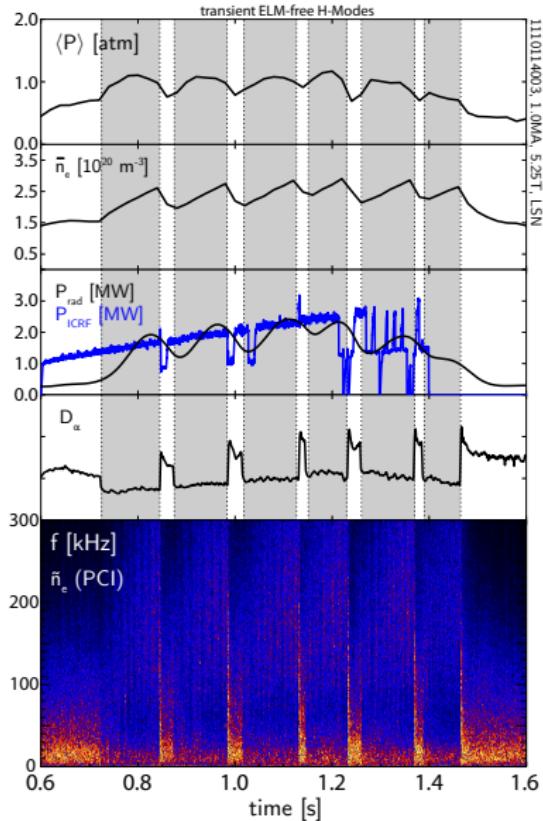
Under right conditions, plasma forms “transport barrier” in edge, with steep gradients in density and temperature – the *pedestal*
→ plasma transitions to “high-confinement” or H-mode

- immediate factor of ~ 2 increase in energy confinement
- pedestal supports higher core pressures = fusion power density

pedestal height sets strong constraint on global performance

...But this has problems of its own

- increased particle confinement
= plasma retains impurities as well as fuel ions
- radiated power ($\sim Z^2$ for a given impurity species) increases, overcomes heating power \rightarrow plasma drops back into L-mode
- inherently transient state

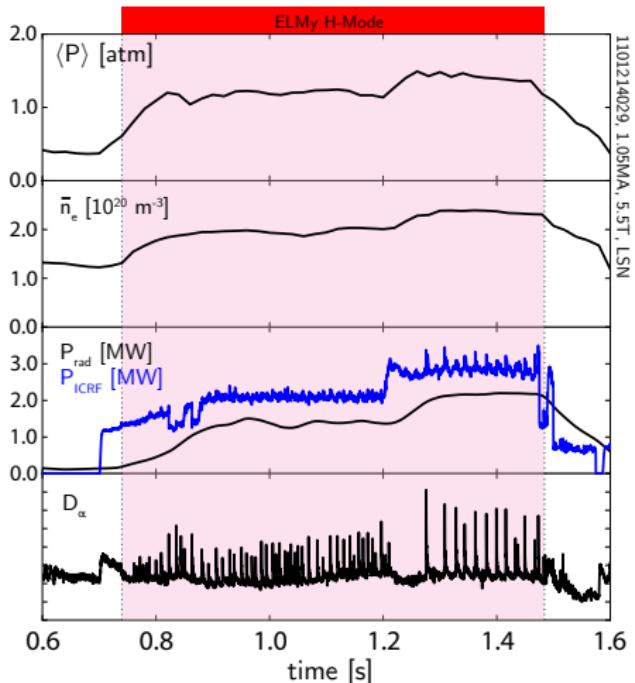


so, we need:

- high energy confinement
- low particle confinement (low enough, at least)
- ... and that's it, right?

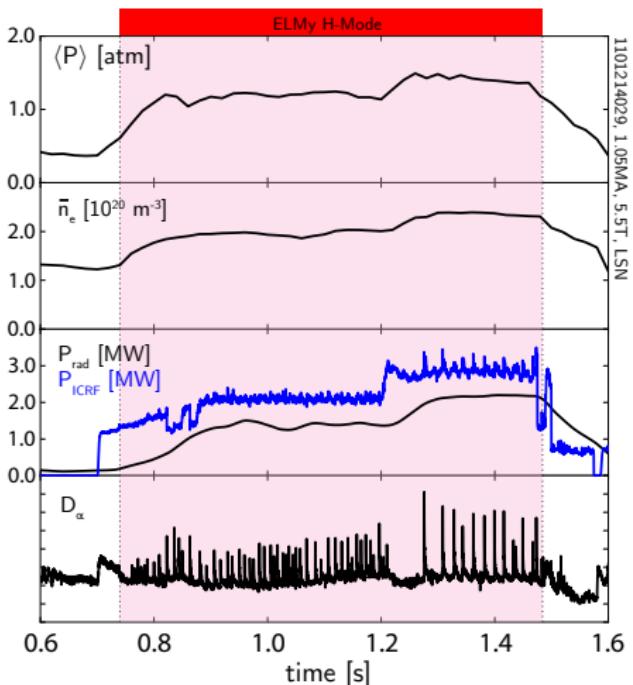
The solution? (part II)

- Edge-Localized Modes (ELMs)
 - instabilities that relax the pedestal, drive bursts of energy, particle transport, enough to prevent impurity accumulation



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- Edge-Localized Modes (ELMs)
 - instabilities that relax the pedestal, drive bursts of energy, particle transport, enough to prevent impurity accumulation
- large ELMs drive pulsed heat loads in excess of plasma-facing material tolerances



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- avoid, mitigate, or suppress large ELMs



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 - ▶ engineering solutions:
pellet pacing, resonant magnetic perturbations

so, we need:

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 - ▶ engineering solutions:
pellet pacing, resonant magnetic perturbations
 - ▶ physics solutions:
pedestal regulation by fluctuations below ELM limit

The solution? (part III)

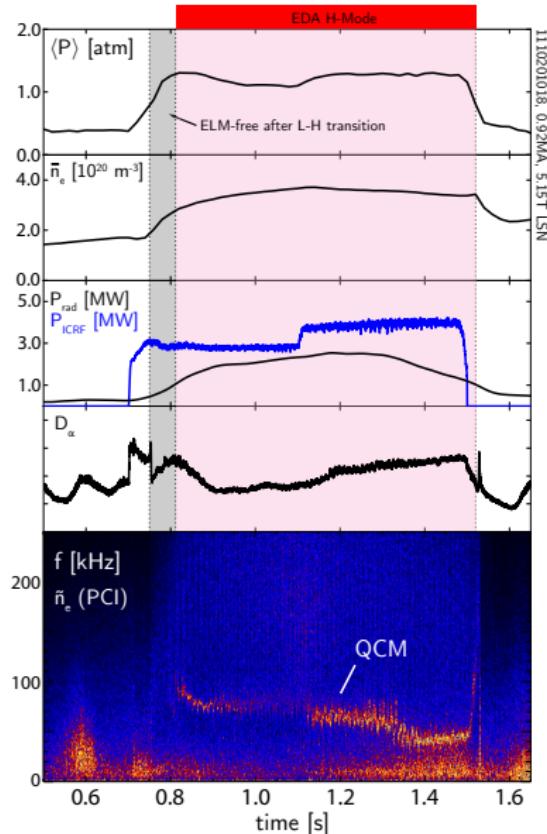
A number of fluctuation-regulated regimes have been observed:

- EDA H-mode
Quasi-Coherent Mode (QCM) – C-Mod, AUG(?)
- Quiescent H-mode
Edge Harmonic Oscillator (EHO) – DIII-D, JET, AUG
- High-Recycling Steady H-mode
edge MHD fluctuations – JFT-2M
- type-II, -III ELMy H-modes
small, rapid ELMs – various devices

generally capable of stationary high performance without large ELMs

EDA H-mode (on C-Mod and elsewhere)

- pedestal regulated by continuous edge fluctuation (QCM), rather than bursts of ELM transport
- steady density, $P_{rad} \rightarrow$ stationary operation possible with good performance

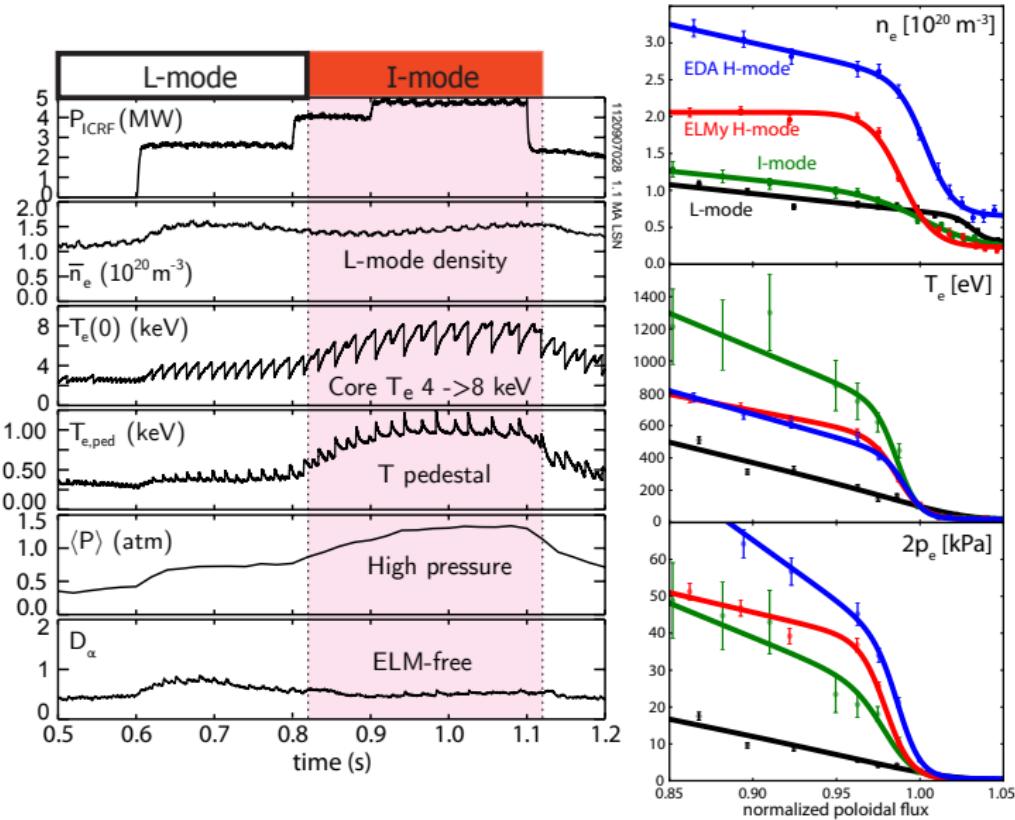


But each of these has drawbacks

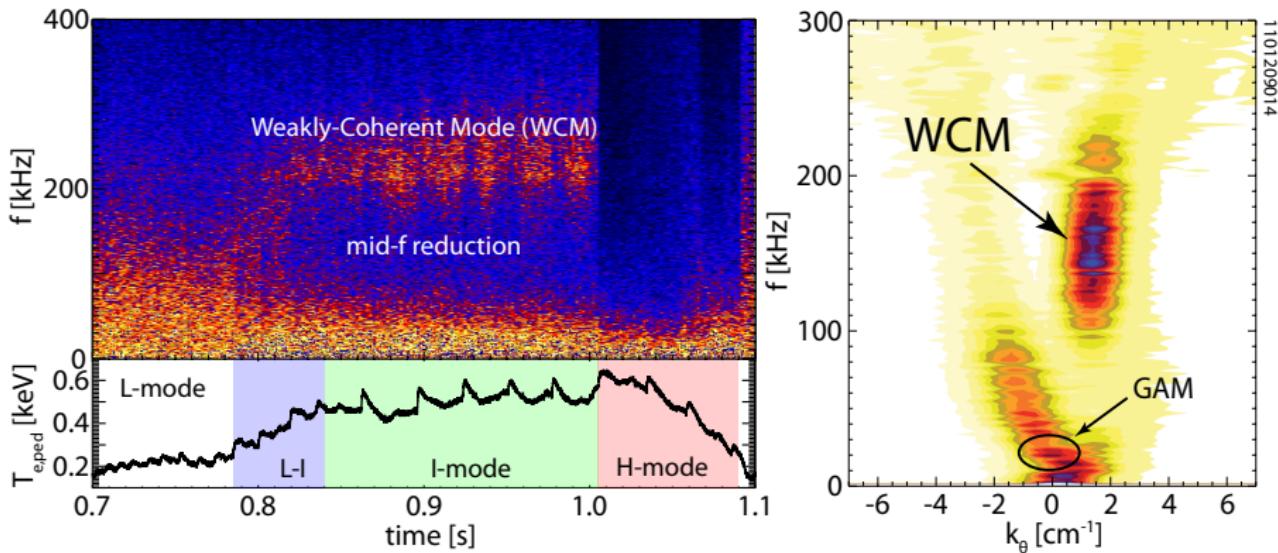
- engineering requirements, e.g. high beam torque for QH-mode
- access limits: high collisionality for EDA, shaping for type-II ELMs

Can we do better?

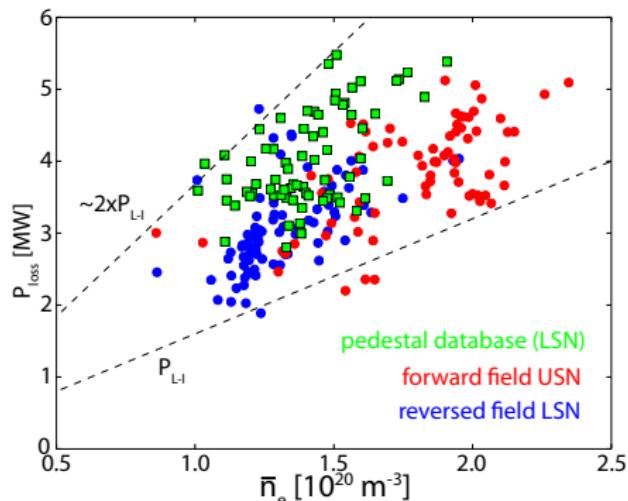
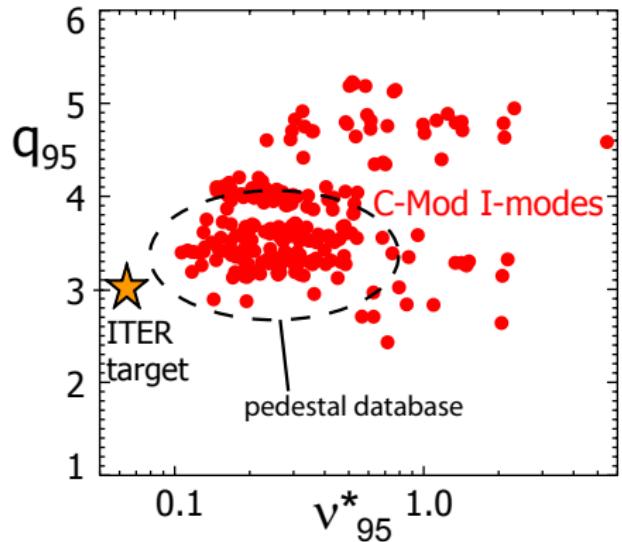
A challenger appears: the I-mode



I-mode pedestal regulated by Weakly-Coherent Mode (WCM)



Robust I-mode access on C-Mod



- I-mode accessed over range of edge current profiles, low-mid collisionalities
- “Unfavorable” ∇B orientation (ion ∇B drift away from primary X-point) – forward-field upper-null or reversed-field lower-null operation
- Sustain mode with heating power up to $\sim 2\times$ above L-I threshold

last intro slide here

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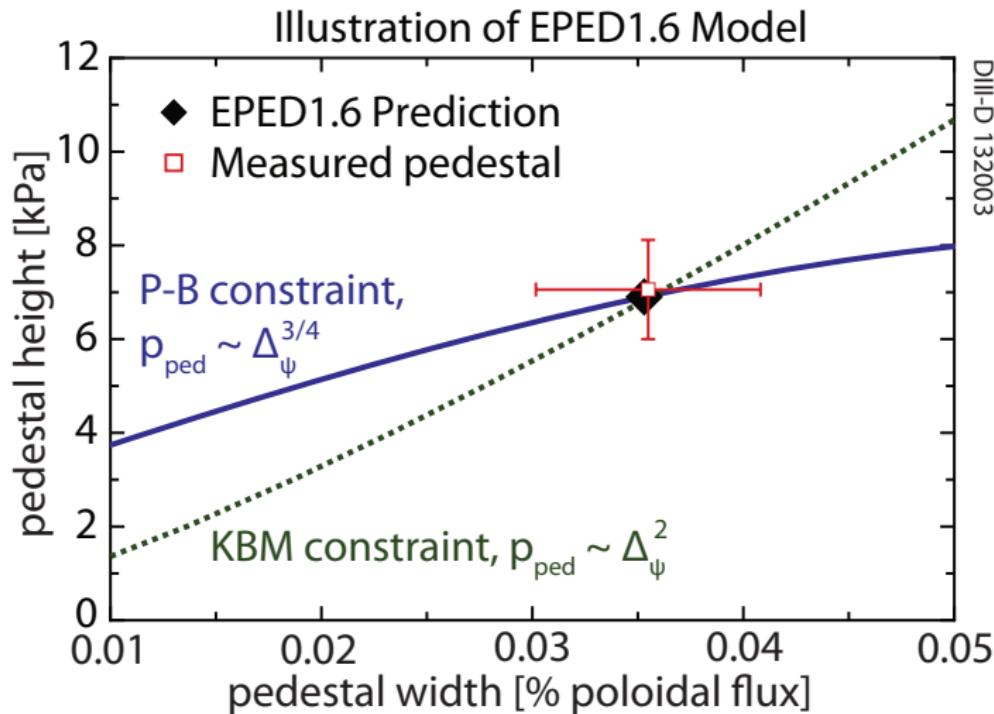
■ ELMy H-Mode Physics¹

- ▶ EPED Modeling on C-Mod

¹JR Walk *et al.*, Nuclear Fusion 52 (2012)

how much detail here for modeling, vs in extra slides?

Predictive Model for ELMy H-modes – EPED⁴



⁴PB Snyder *et al.*, Nuclear Fusion **51** (2011)

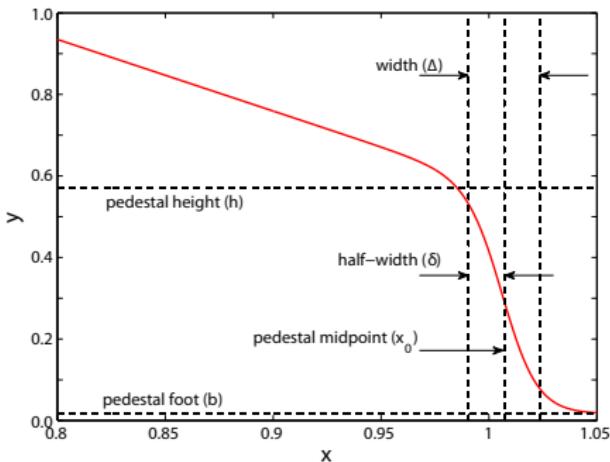
Pedestal Structure Definitions

pedestals fitted by

$$z = \frac{x_0 - x}{\delta}$$

$$mtanh(\alpha, z) = \frac{(1 + \alpha z)e^z - e^{-z}}{e^z + e^{-z}}$$

$$y = \frac{h + b}{2} + \frac{h - b}{2} mtanh(\alpha, z)$$



rigorous definition for pedestal width $\Delta = 2\delta$, continuous and differentiable throughout pedestal profile

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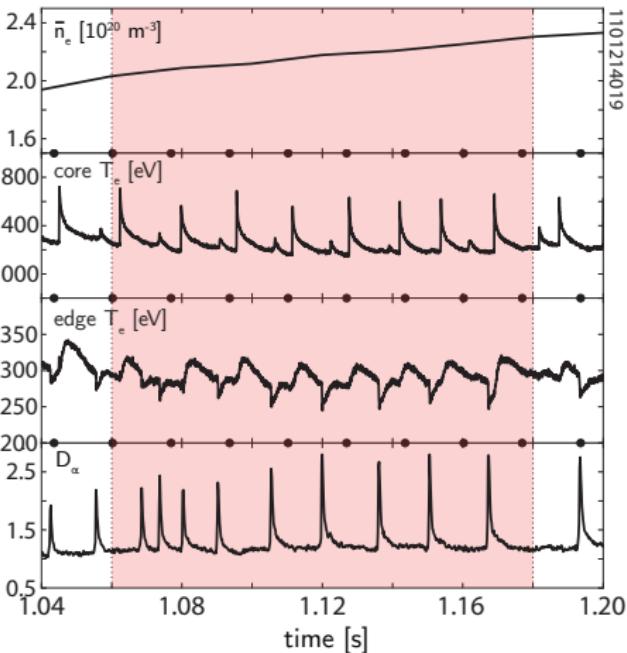
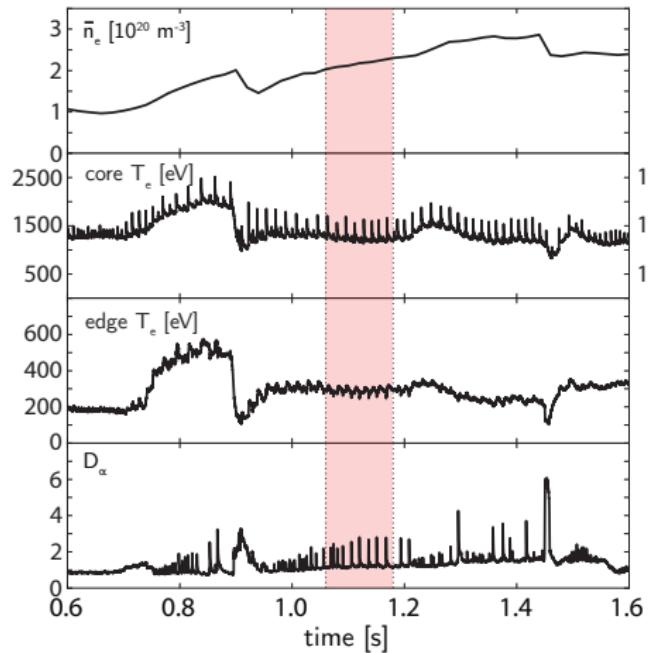
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■ ELM My H-Mode Physics¹

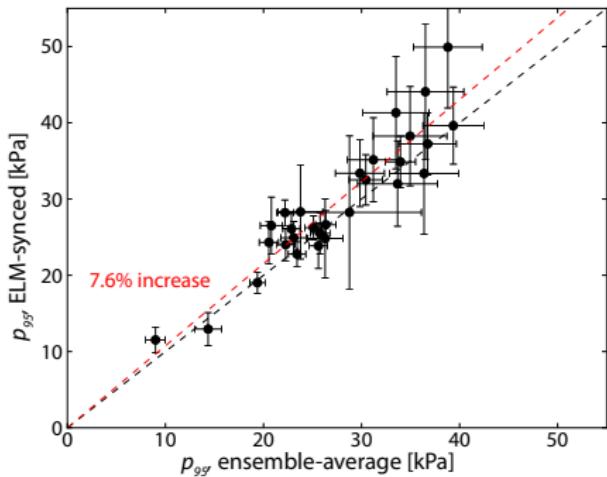
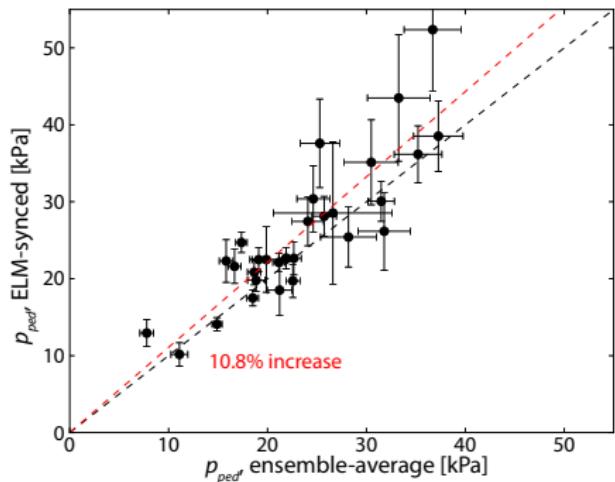
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Target steady ELMy phases for study

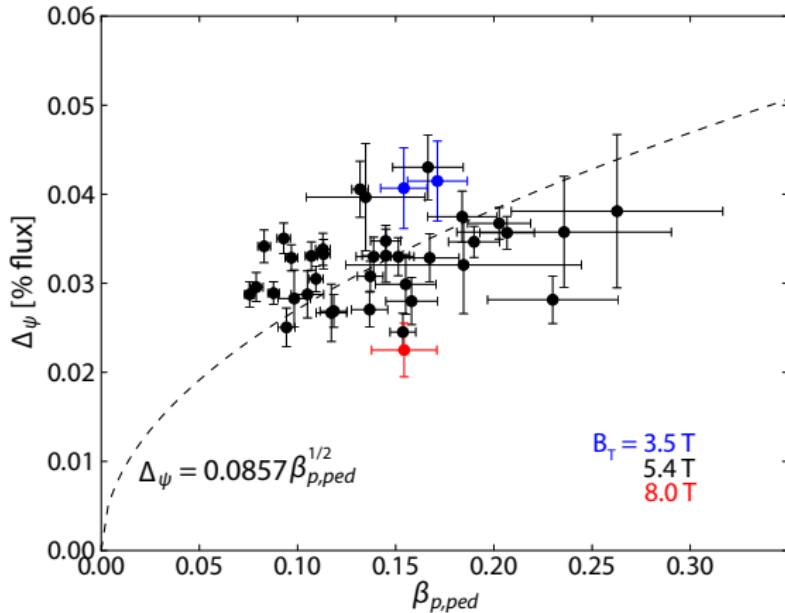


ELM cycle binning necessary to capture pedestal limit



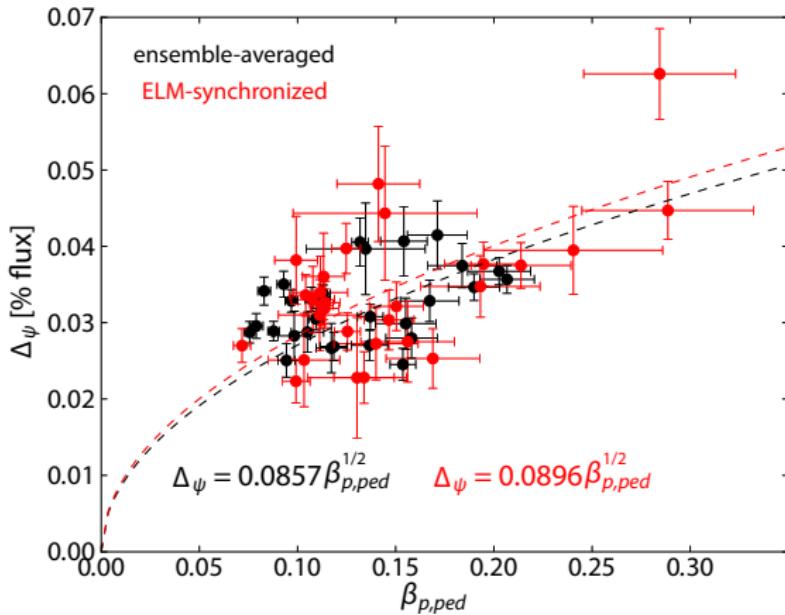
Take profile data immediately preceding ELM crash (typically last 20% of ELM cycle) for pedestal structure at point of instability – necessary, but difficult given ELM frequency on C-Mod (subset of data prepared thus).

Pedestal width described well by KBM limit



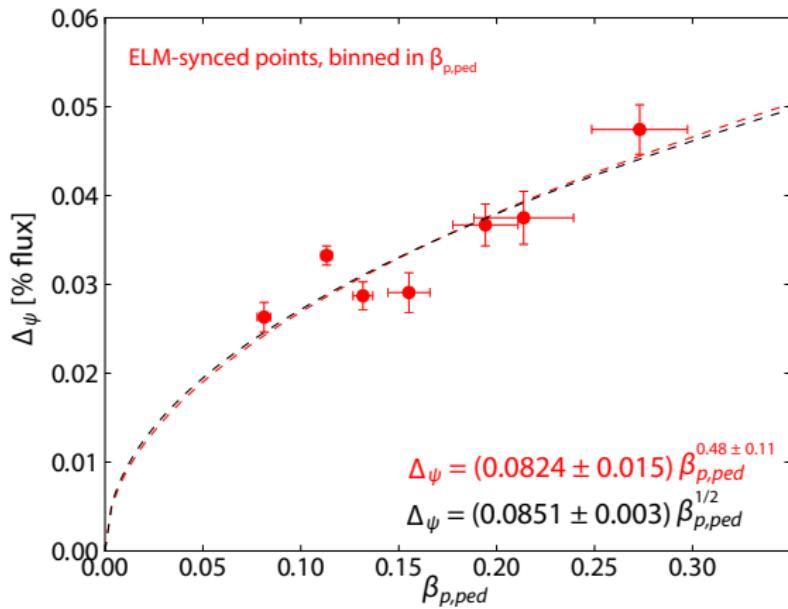
KBM limit predicts width $\Delta_\psi = G(\nu^*, \varepsilon, \dots) \beta_{p,ped}^{1/2}$

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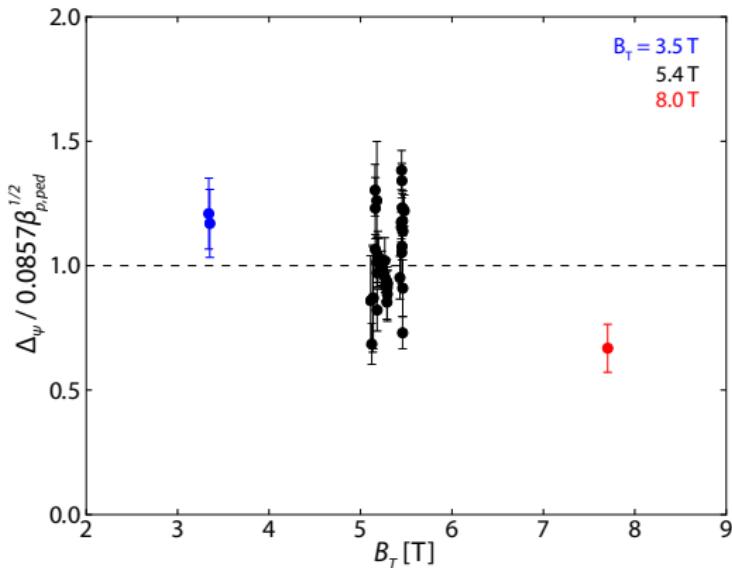
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Minimal dependence of width on other parameters

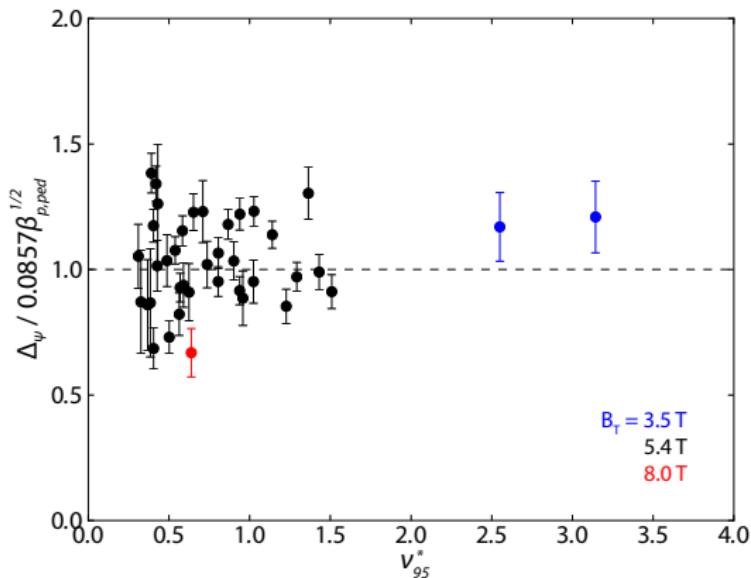
Secondary dependences via pedestal width normalized to $\beta_{p,ped}^{1/2}$ scaling



- broader pedestal at low field qualitatively \sim some modeling efforts, but within spread at standard field, strong covariance with current
- no variation with collisionality, despite effects on edge current \rightarrow magnetic shear
- no variation with gyroradius (expected as dependence in G)

Minimal dependence of width on other parameters

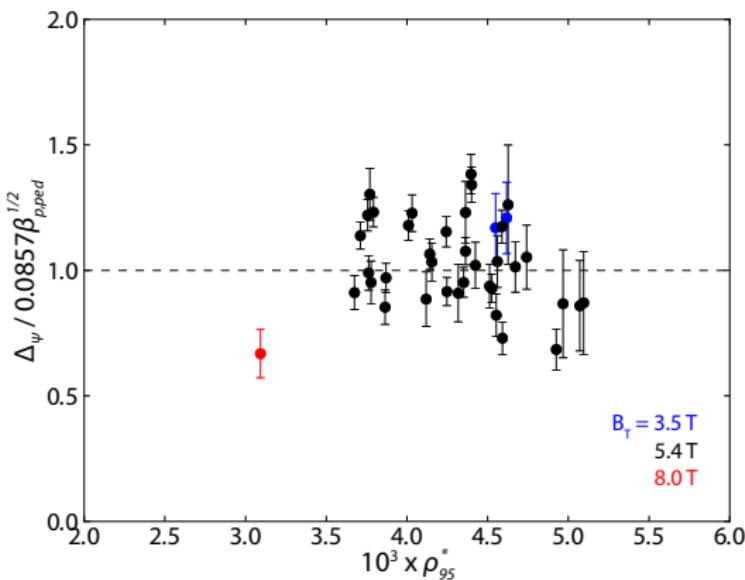
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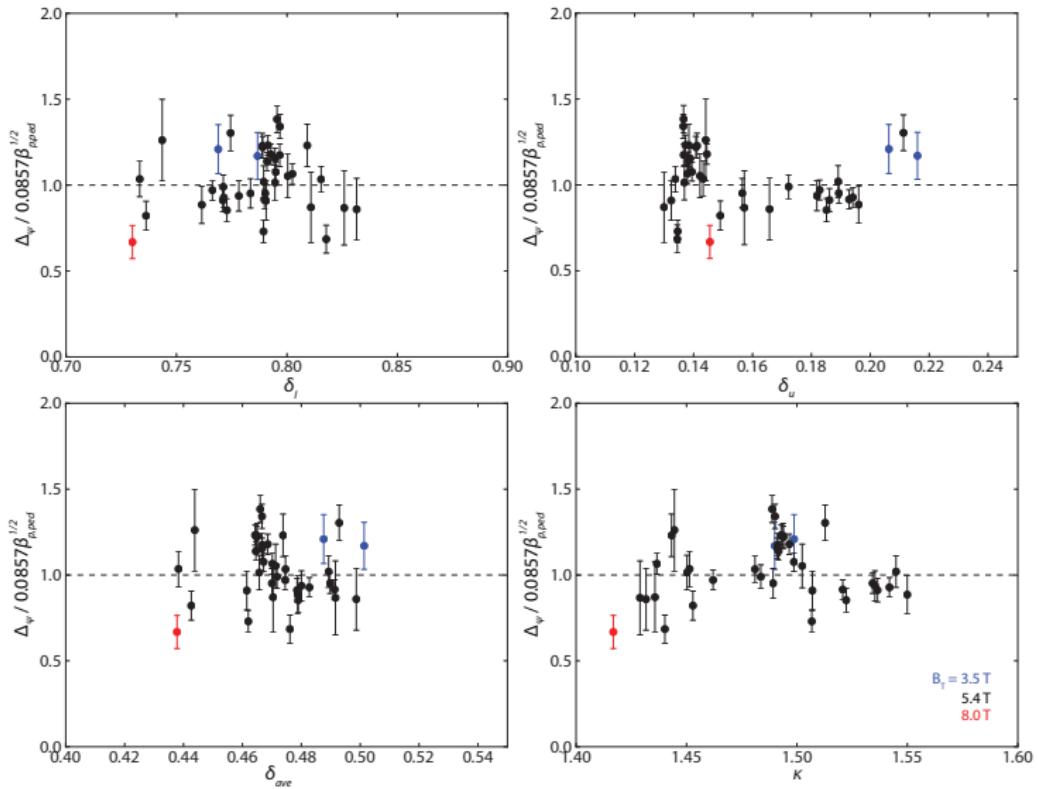
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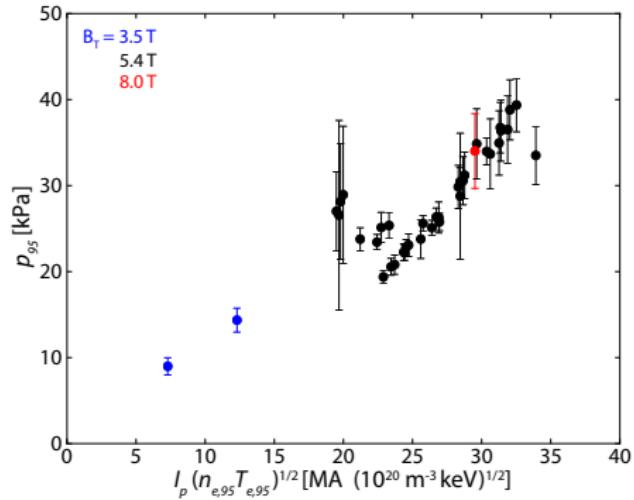
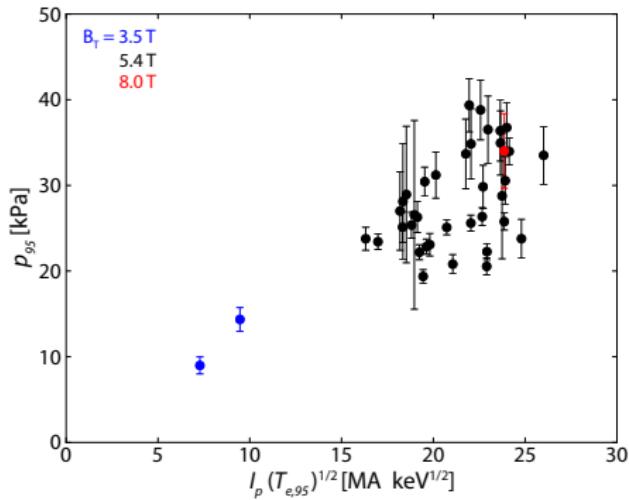


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Minimal dependence of normalized width on shaping

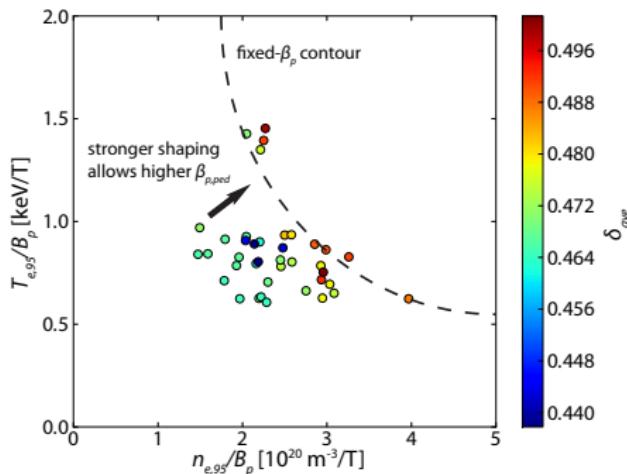
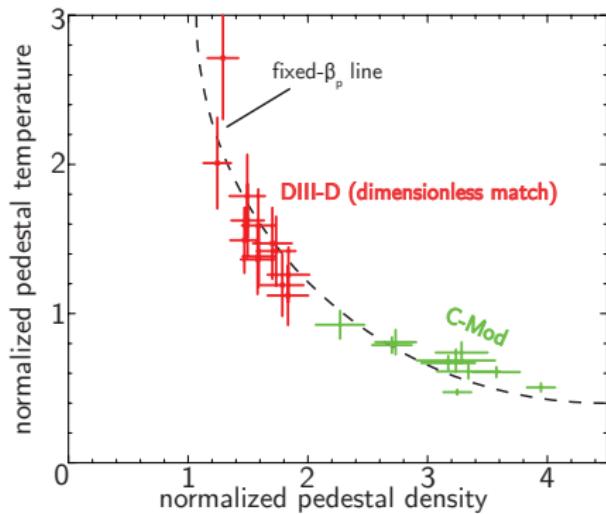


Pedestal height predicted by ballooning ∇p limit

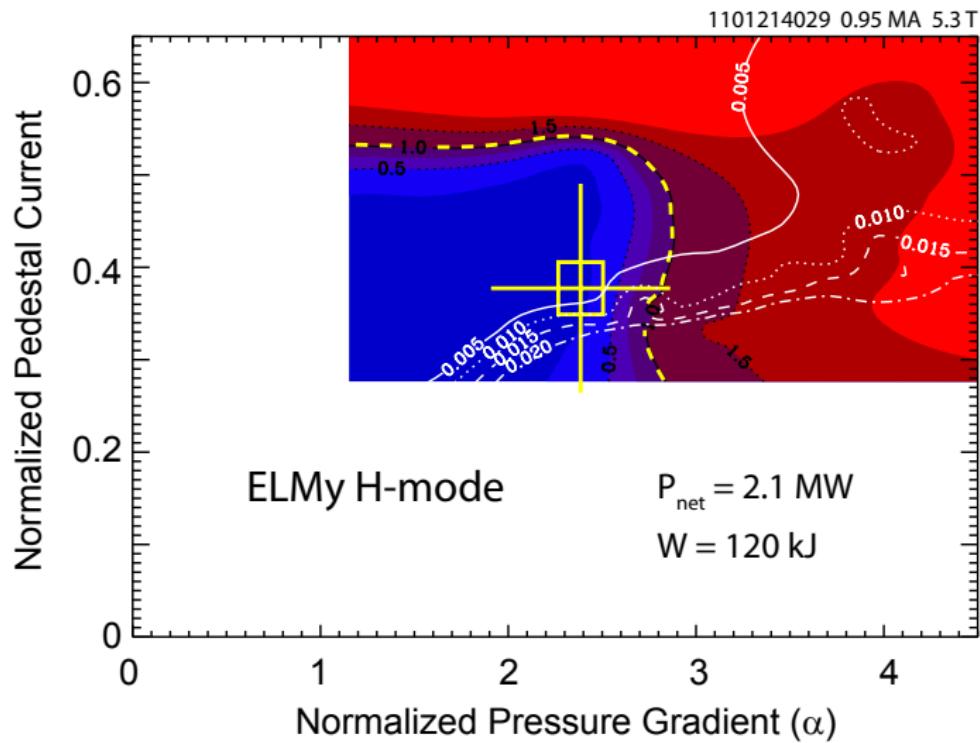


Pedestal height $p_{ped} \sim \nabla p \times \Delta_p \rightarrow \sim I_p^2 \Delta_p$ from ballooning MHD
predicted well by $\Delta_p \sim \sqrt{\beta_{p,ped}}$, less so by $\Delta_p \sim \rho_{i,pol}$

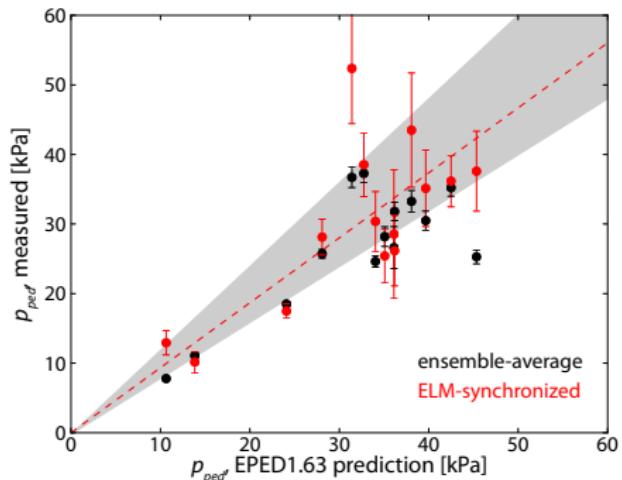
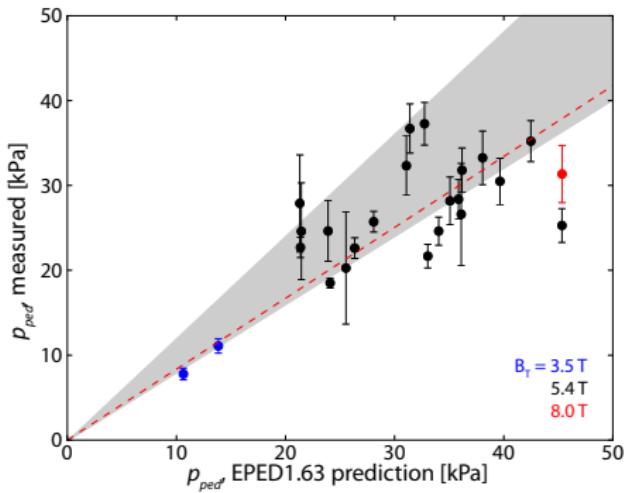
Robust width, gradient limit = attainable $\beta_{p,ped}$ limited in ELM My H-mode



Computational modeling of P-B MHD, KBM captures ELMy pedestal

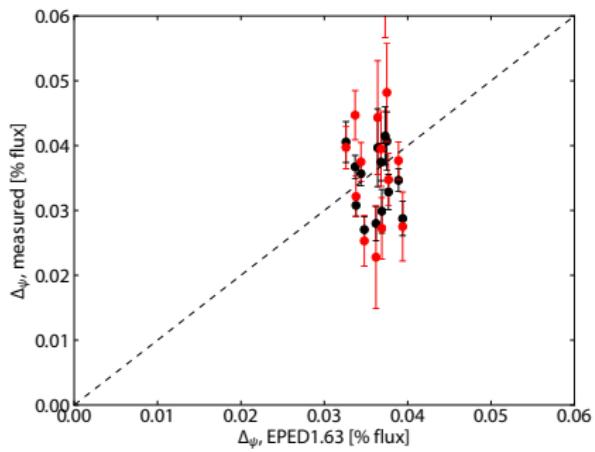
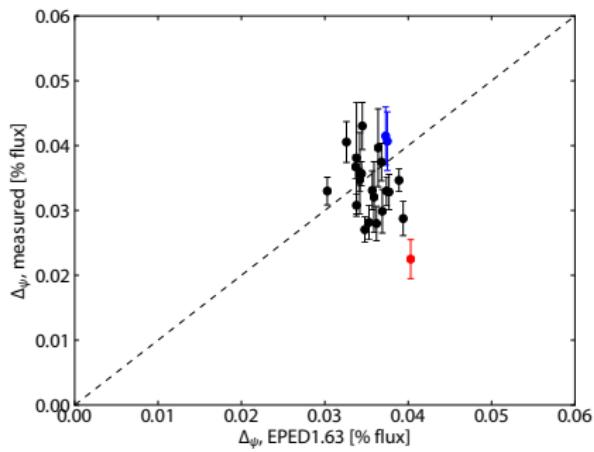


EPED predicts pedestal height for ELM-binned pedestals



measured to predicted ratio of 0.835 ± 0.036 for ensemble-averaged data, 0.934 ± 0.066 for ELM-synced pedestals, well within expected $\pm 20\%$ accuracy for EPED predictions

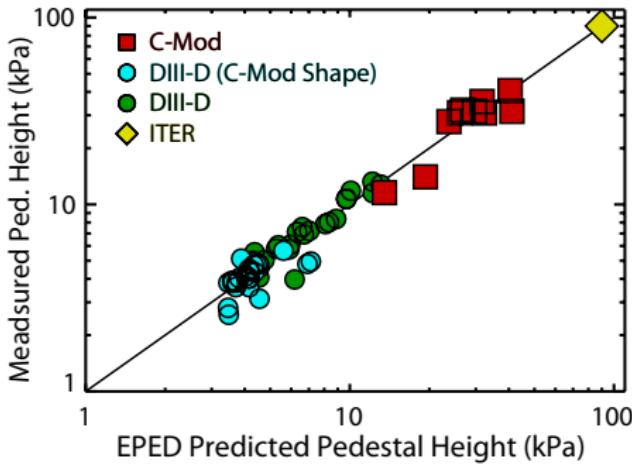
Width varies over narrow range, hard to predict



Pedestal width varies little over range of 3 – 5% of poloidal flux, difficult to extract trend – EPED reproduces robust width to within $\pm 20\%$ uncertainty

Experiments expand parameter space tested in EPED⁵

- reach highest field (8 T), highest thermal pressure, within factor of ~ 2 of ITER pedestal target
- C-Mod contribution to multi-machine Joint Research Target
- reliable physics-based understanding of H-mode pedestal limits



⁵RJ Groebner et al., Nuclear Fusion 53 (2013)

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■ I-Mode Pedestal Stability

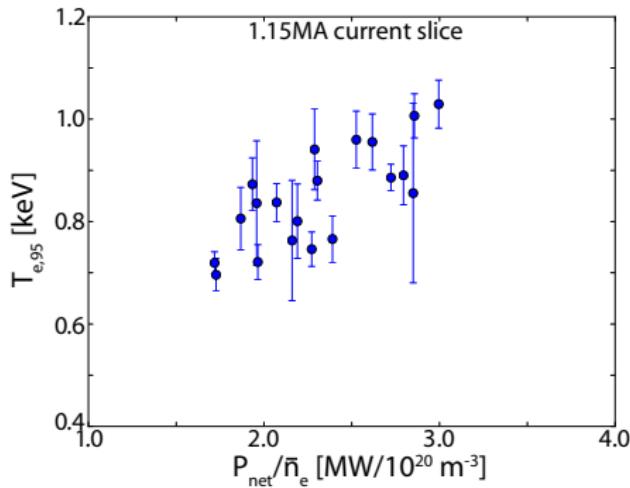
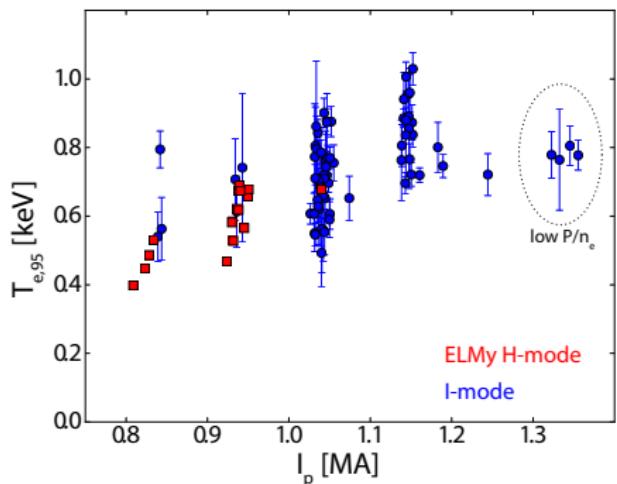
- ▶ P-B MHD, KBM modeling
- ▶ ELM characterization

■ Summary, Future Work, & Questions

²JR Walk *et al.*, *Physics of Plasmas* 21 (2014)

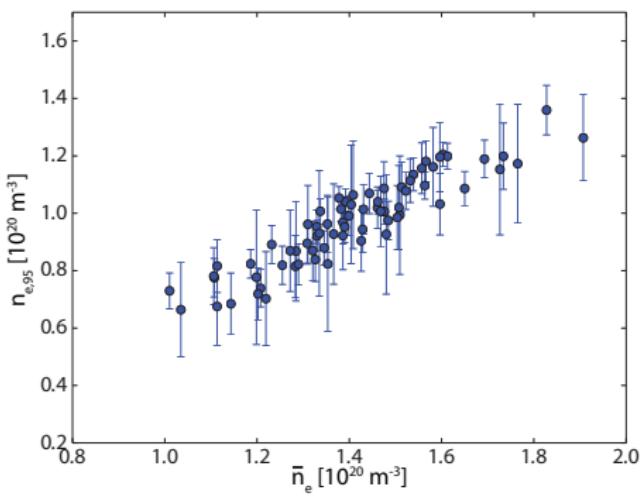
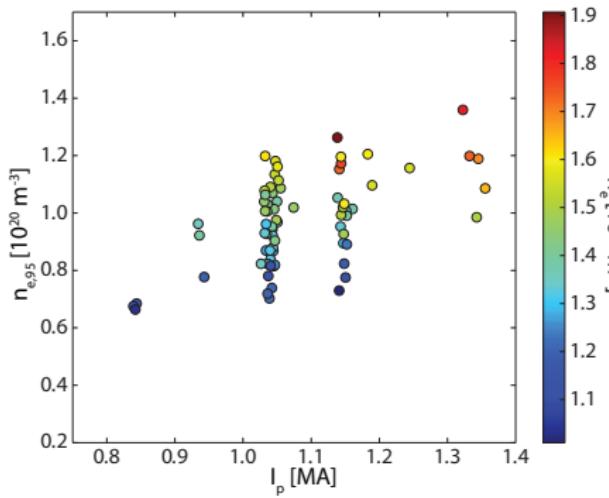
³Invited talk, APS-DPP Nov. 2013

Temperature pedestal H-mode-like, set by plasma current, heating power

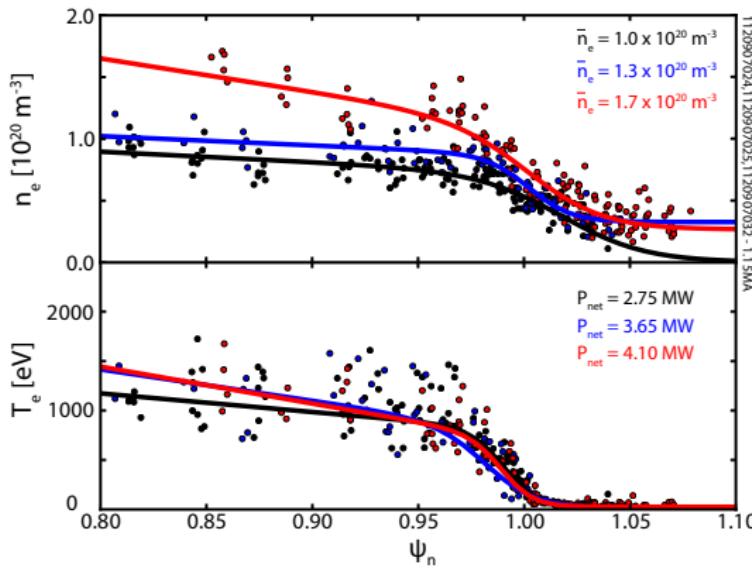


- pedestal T_e shows positive trending $T_e \sim I_p$, spread at given current due to heating power
- input power strongly affects pedestal temperature as with EDA H-mode – more properly, **power per particle sets pedestal temperature** at fixed current

In contrast, density set by operator fueling,
with L-mode-like profile

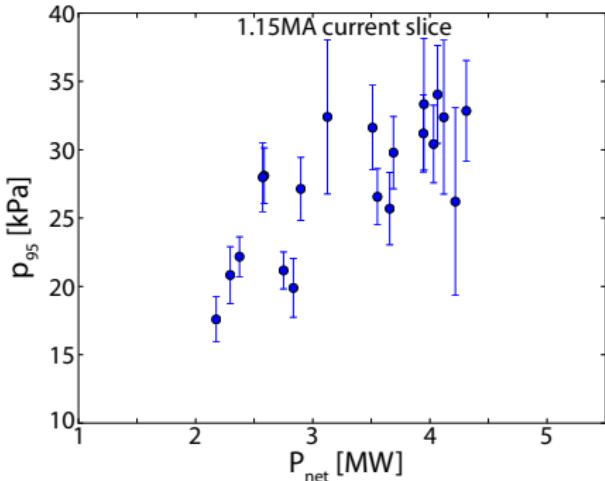
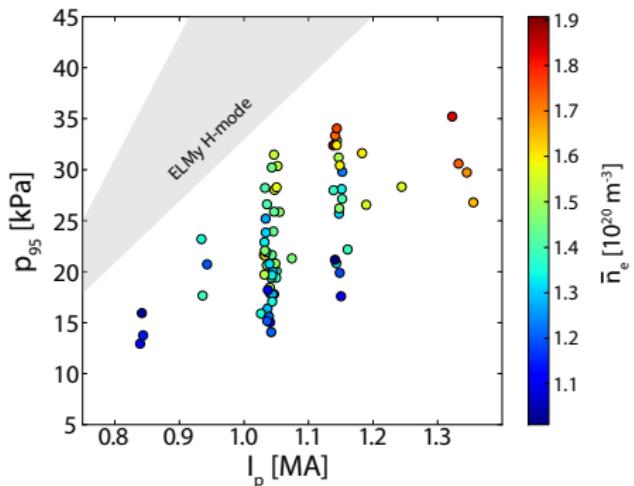


Pedestal density separately controlled from temperature, independent of MHD limits



- with sufficient power to maintain P_{net}/\bar{n}_e , temperature pedestal matched across range of fueling
- Contrasts to MHD-limited pedestals (fixed $\beta_{p,ped} \rightarrow$ limit on $n_e T_e$) – path to strongly increase pedestal beta

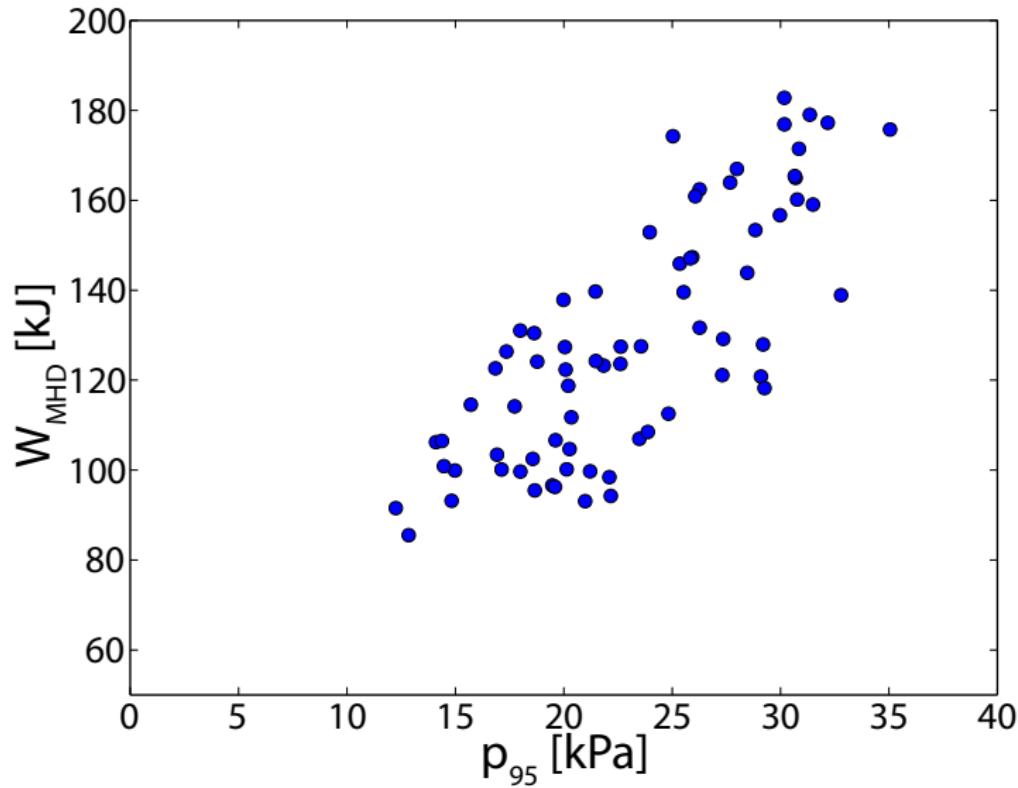
I-mode pedestal pressure scales with current, heating power, fueling, competitive to H-mode



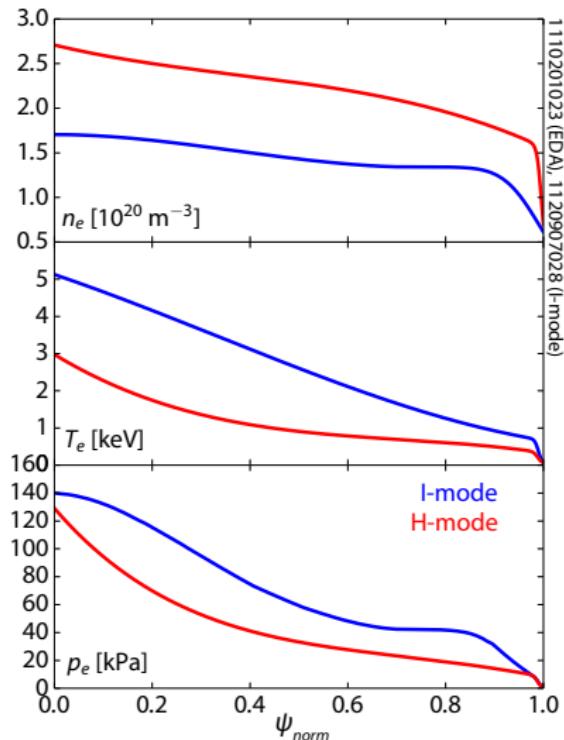
- Pedestal pressure increases at least as $p_{ped} \sim I_p$, due to increased $T_e \sim I_p$ and more fueling (fixed f_{Gr}) at higher current
- Pedestal pressure at fixed current $\sim P_{net}$ (consistent with $T_e \sim P/n_e$), corresponds to favorable scaling of energy confinement with heating power
- Fueling (with sufficient power to maintain temperature pedestal) strongly increases pedestal pressure

What does this get us?

Pedestal impacts core, global performance

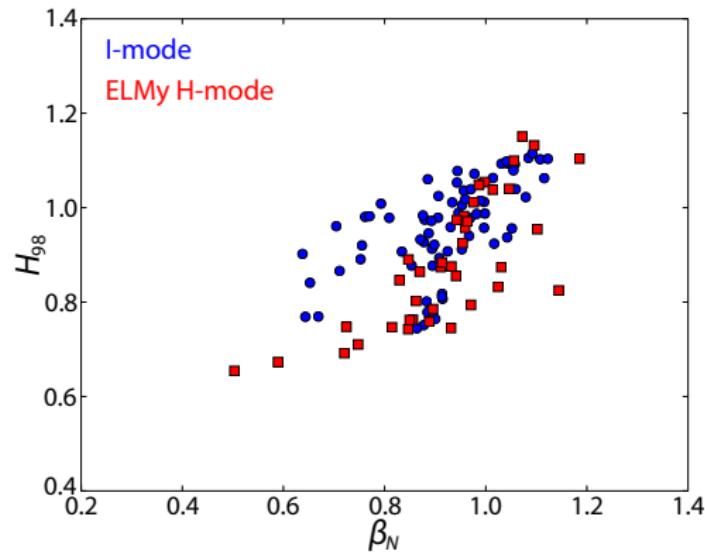
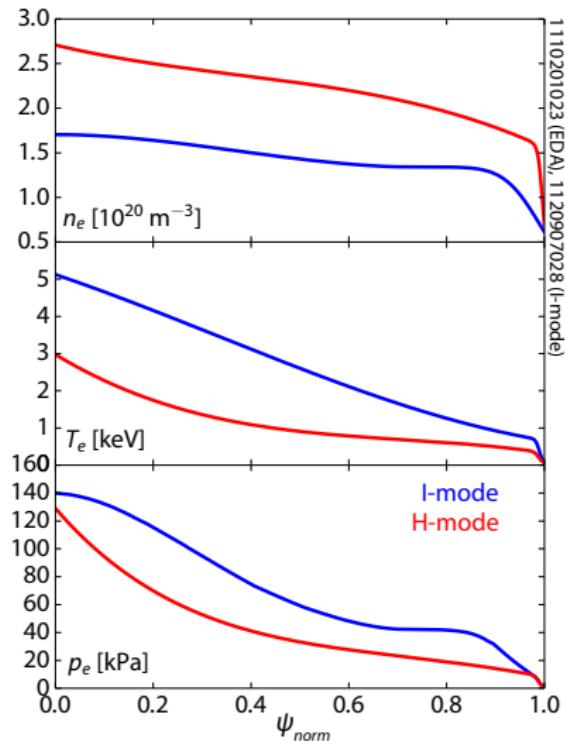


Strong temperature pedestal supports high core temperature, pressure



- stiff ($R/L_{T_e} \sim \text{fixed}$) temperature profiles \rightarrow higher T_{ped} supports greatly increased core temperatures
- provided moderated density peaking ($n_{e,0}/\langle n_e \rangle \sim 1.1 - 1.3$ in I-mode), reaches comparable core, vol-average pressure despite relaxed p_{ped}
- fusion-reactive plasma where $T_e > 4 \text{ keV}$, high T_{ped} maximizes fusing volume

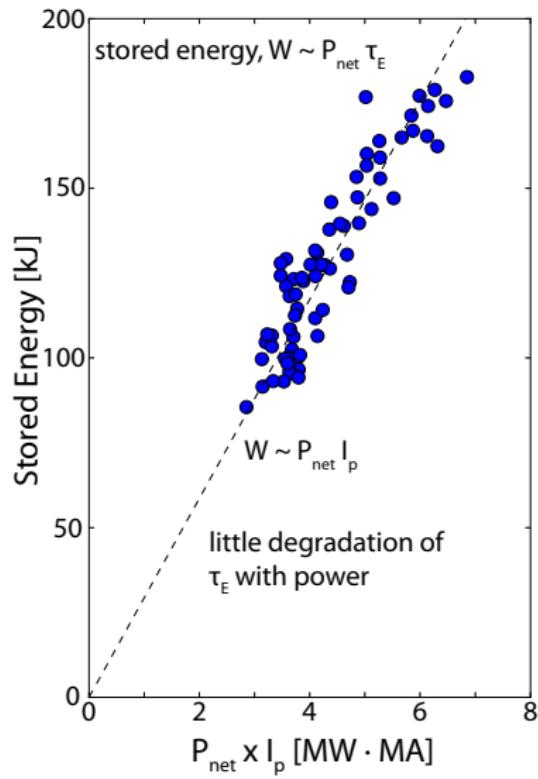
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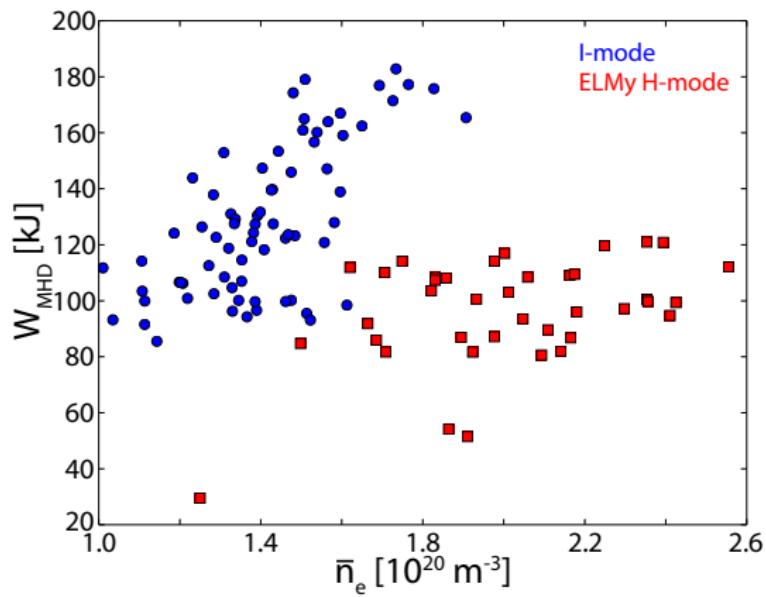
→ same $\langle \beta_N \rangle$, normalized confinement to ELMy H-mode

Energy confinement lacks degradation with heating power

- Stored energy $W \sim P\tau_E$
- for H-mode, expect $\tau_E \sim I_p$, with power degradation $\tau_E \sim P^{-0.7}$ thus $W \sim I_p P^{0.3}$
- I-mode stored energy $W \sim I_p P_{net} \rightarrow$ little/no degradation of τ_E with heating power
- reflects lack of MHD limit on pedestal



...As well as with fueling



- I-mode stored energy set by pedestal pressure, responding strongly to fueling
- ELMy H-mode stored energy set by pedestal β_p , limited by MHD constraint – mainly increase pedestal β_p with stronger shaping, flat relation with fueling

First pass at an I-mode confinement scaling

Following practice in ITER89,ITER98 scalings, express I-mode energy confinement as a power law of the form

$$\tau_E = C I_p^{\alpha_{Ip}} B_T^{\alpha_{BT}} \bar{n}_e^{\alpha_{ne}} R^{\alpha_R} \varepsilon^{\alpha_\varepsilon} \kappa^{\alpha_\kappa} P_{loss}^{\alpha_P}$$

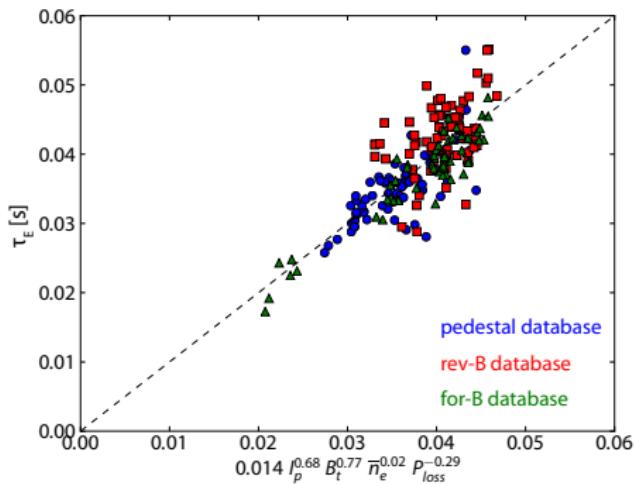
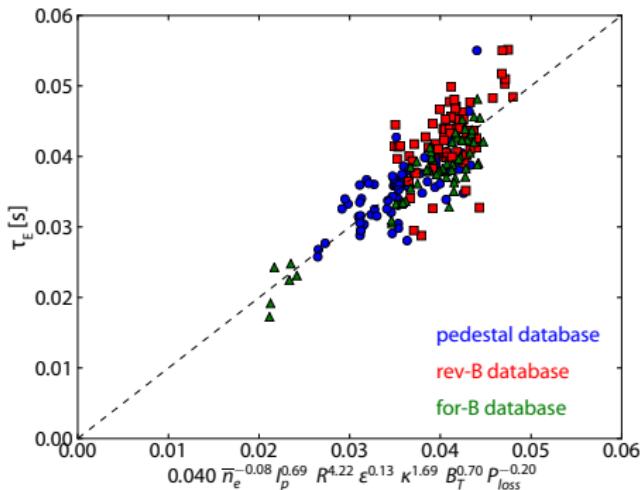
Using high-res pedestal database plus older forward- and reversed-field datasets for expanded parameter range

Reduced fitting parameter set captures I-mode physics

	(a)	(b)	(c)
C	0.040 ± 0.066	0.014 ± 0.002	0.056 ± 0.008
I_p	0.686 ± 0.074	0.685 ± 0.076	0.676 ± 0.077
B_T	0.698 ± 0.075	0.768 ± 0.072	0.767 ± 0.072
\bar{n}_e	-0.077 ± 0.055	0.017 ± 0.048	0.006 ± 0.048
R	4.219 ± 4.623		2^*
ε	0.127 ± 1.144		0.5^*
κ	1.686 ± 0.398		
P_{loss}	-0.197 ± 0.048	-0.286 ± 0.042	-0.275 ± 0.042
r^2	0.713	0.685	0.683

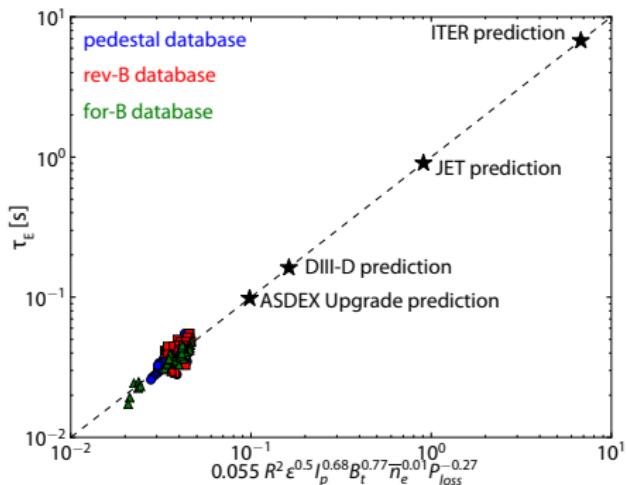
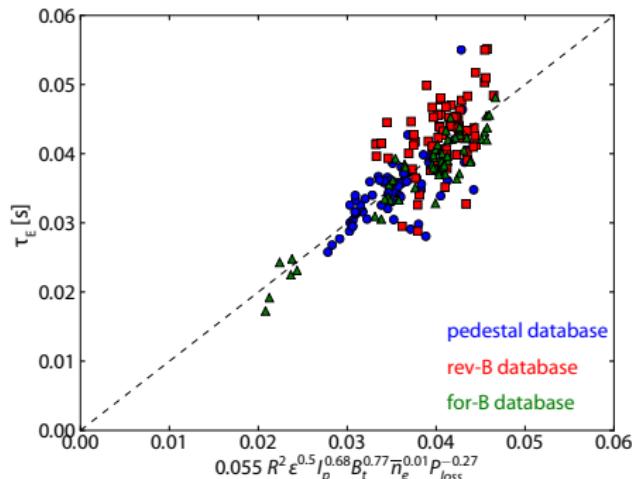
poor fitting in R , ε , κ due to restricted range in dataset

Reduced fitting parameter set captures I-Mode physics



Both fits capture weak degradation of τ_E with heating power, strong response to current, field

Thought experiment: apply ITER98-like size dependence to I-mode scaling, extrapolate to larger machines?



Apply fixed $R^2 \sqrt{\epsilon}$ size scaling – does not impact C-Mod data fit.
Weak degradation $\tau_E \sim P_{loss}^{-0.27}$ extrapolates highly favorably to large devices – $\tau_E \sim 8$ s for ITER!

I-mode behavior beneficial for global performance, confinement

- high temperature pedestal supports steep core ∇T , comparable pressure and confinement despite relaxed pedestal
- stored energy responds strongly to heating power, fueling, consistent with pedestal response
 - ▶ good news: in burning plasma, alpha heating power $\sim n_e^2$

First-pass confinement scaling laws consistent with observed behaviors

- single-machine scaling captures strong response to current, field, weak degradation with heating power ($\tau_E \sim P^{-\alpha}$, $\alpha < 0.3$)
 - ▶ consistent with assumptions in ITER simulations leading to $Q = 10$ scenario⁶
- extrapolates to $\tau_E \sim 8\text{ s}$ for ITER(!)

⁶DG Whyte, APS-DPP Nov. 2011

Outline

■ I-Mode Pedestals & Global Performance^{2,3}

- ▶ Pedestal response to fueling, heating power
- ▶ Pedestal widths and gradients
- ▶ Global performance and confinement scalings

■ I-Mode Pedestal Stability

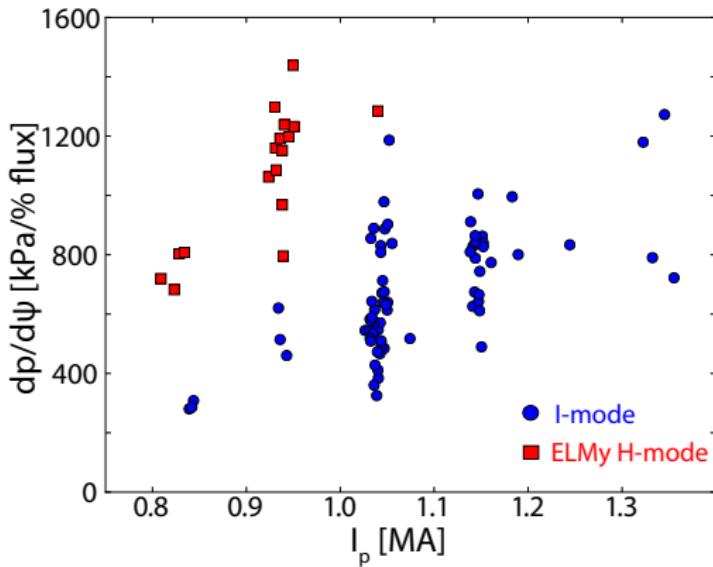
- ▶ P-B MHD, KBM modeling
- ▶ ELM characterization

■ Summary, Future Work, & Questions

²JR Walk *et al.*, *Physics of Plasmas* 21 (2014)

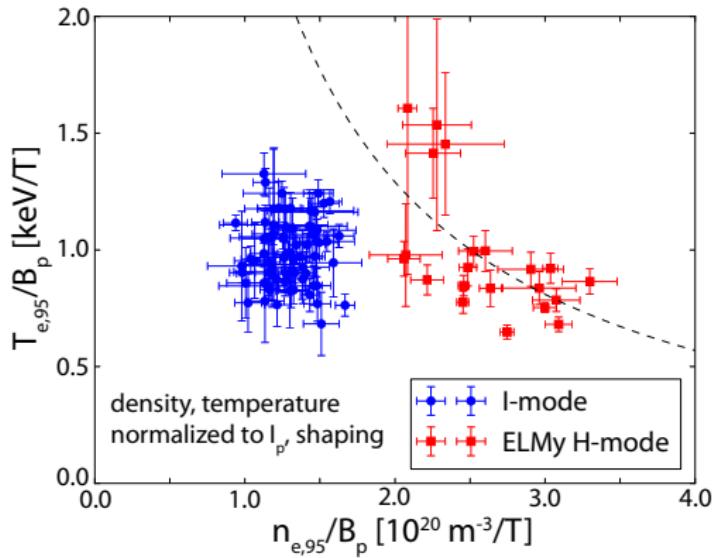
³Invited talk, APS-DPP Nov. 2013

Pedestal pressure gradient suggests MHD stability, headroom for performance improvement



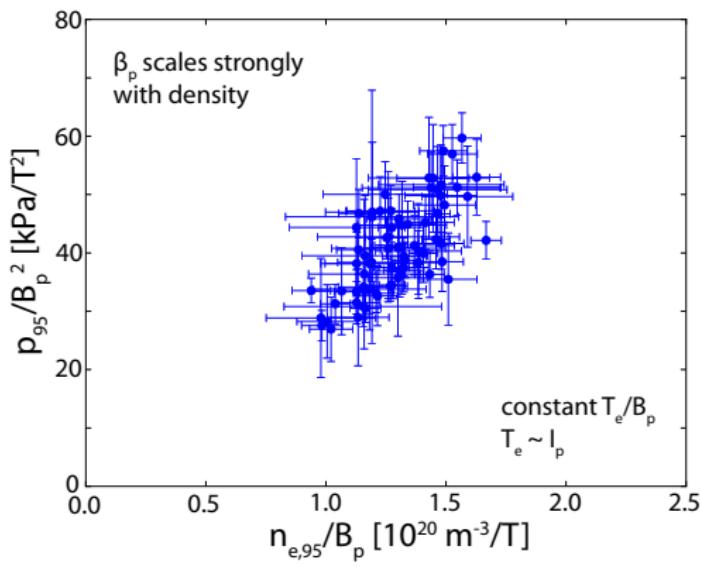
- Pedestal ∇p shallower at given I_p than ELMy H-mode due to lack of density pedestal
- Gradients scale more weakly than $\nabla p \propto I_p^2$ from ballooning MHD (critical-gradient) stability boundary

I-mode pedestal scalings consistent with stability against peeling-balloonning MHD



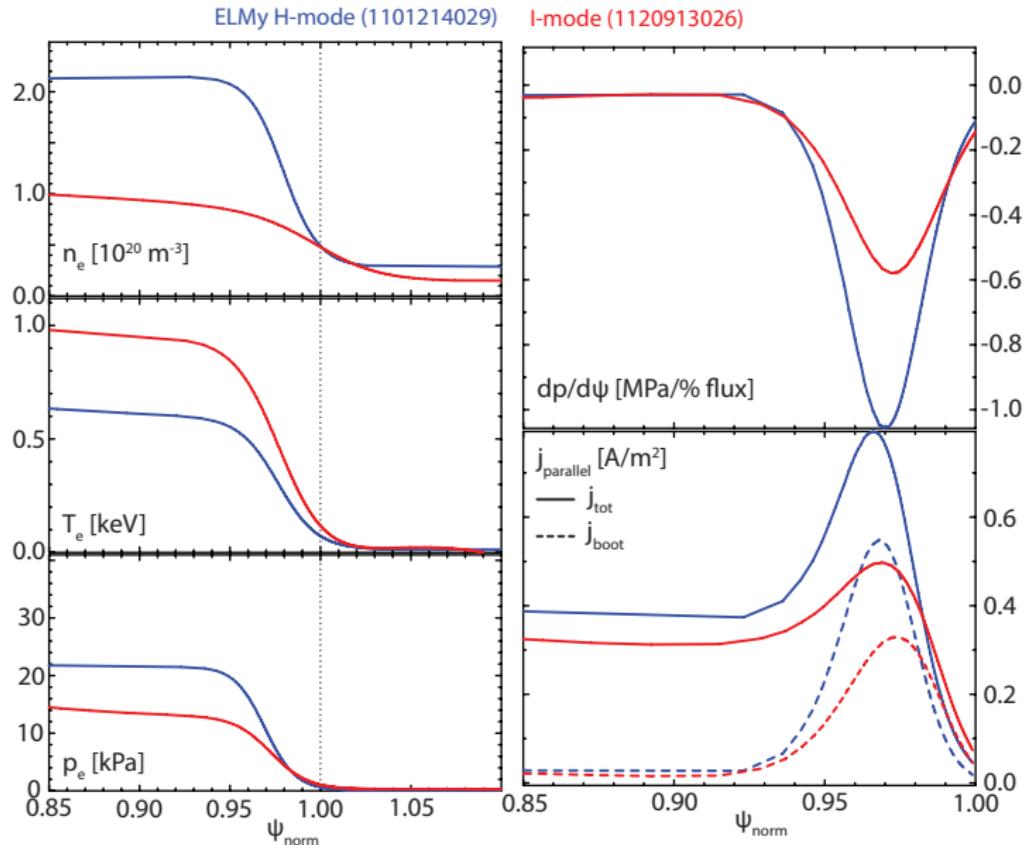
- ballooning stability, to lowest order, limits pedestal β_p in ELMy H-mode; I-mode n_e , T_e independent, rather than fixed $n_e T_e$
- Pedestal β_p scaling with density consistent with constant $T_{e,95}/B_p \rightarrow T_e \sim I_p$, rather than $T_e \sim 1/n_e$

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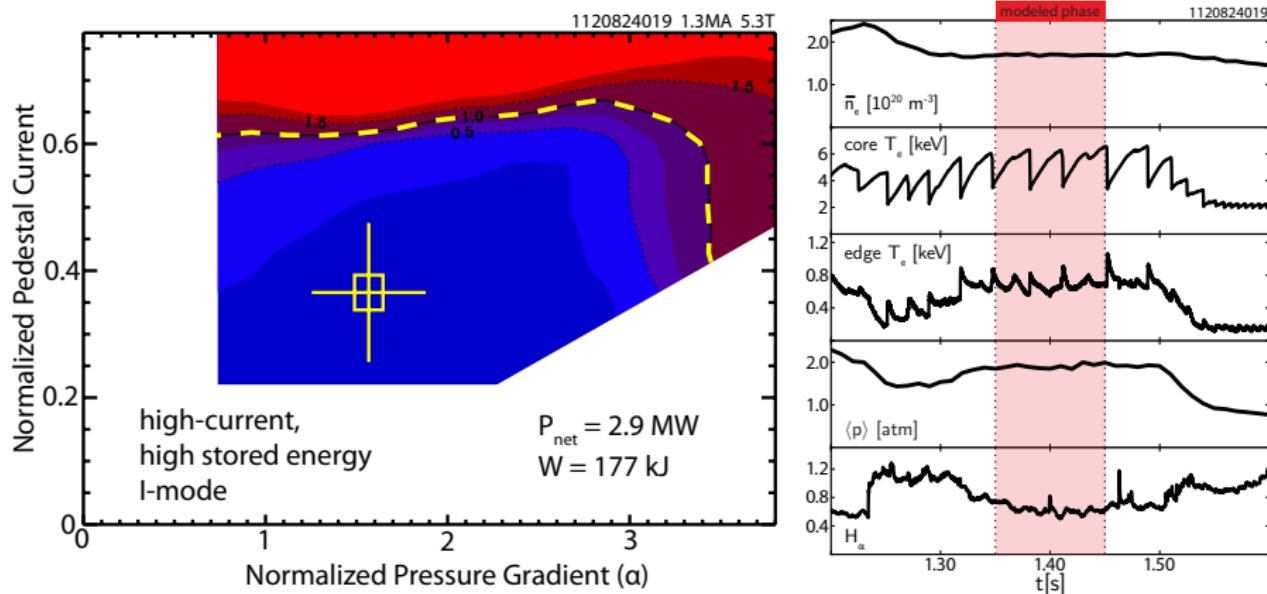


- ballooning stability, to lowest order, limits pedestal β_p in ELM My H-mode; I-mode n_e , T_e independent, rather than fixed $n_e T_e$
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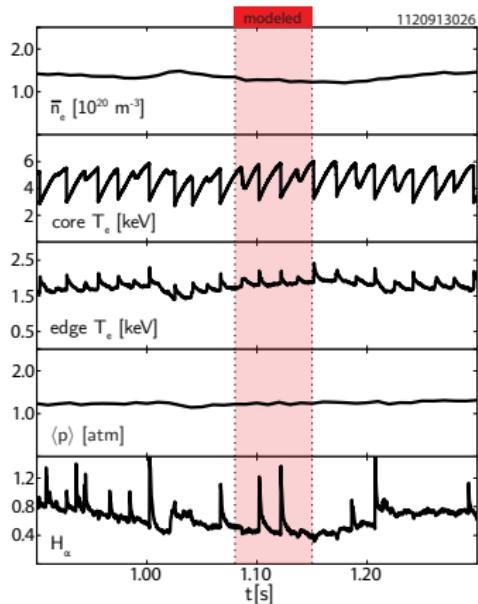
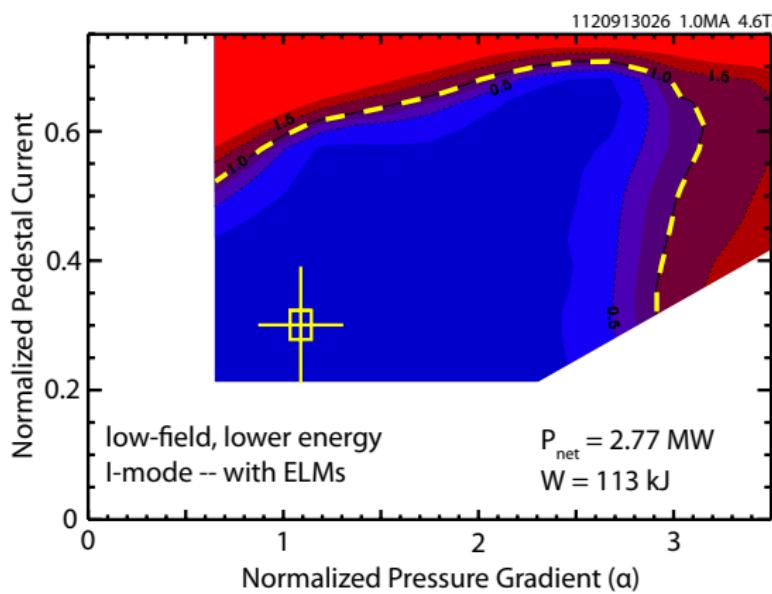
Stability is self-enforcing for I-mode pedestals



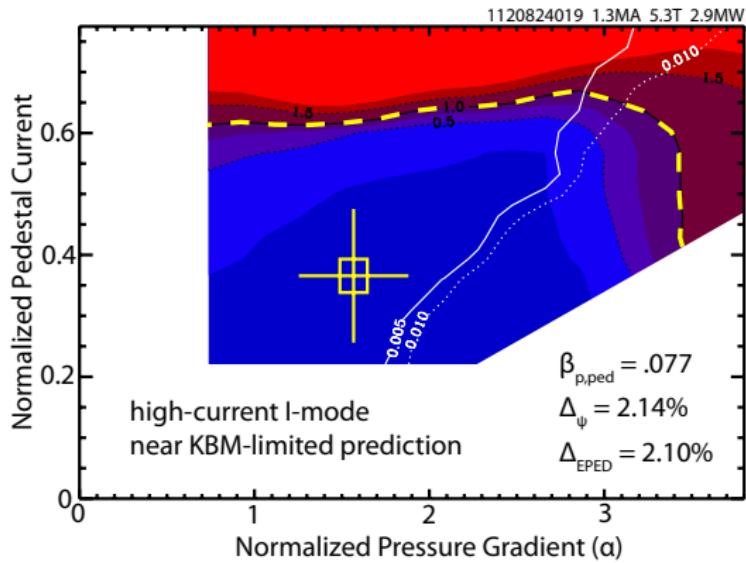
I-mode pedestal strongly stable against peeling-balloonning MHD, KBM turbulence



Including in low-field, low-energy cases exhibiting apparent ELMs

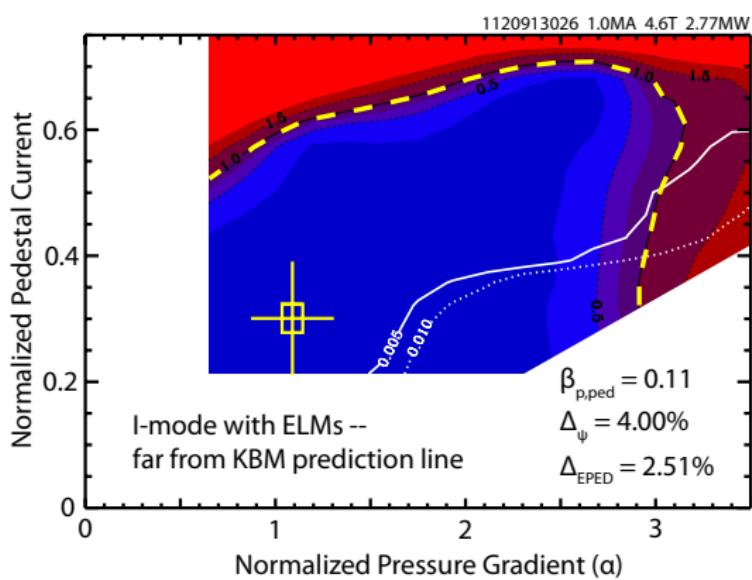


I-mode pedestal modeled to be below KBM threshold as well, including cases with ELMs



- KBM-limited (EPED1) prediction for width, $\Delta_{EPED} = 0.076\beta_{p,ped}^{1/2}$
- BALOO calculates width of pedestal locally beyond threshold – mode onset when half of pedestal is unstable
- I-mode cases spanning width range modeled below threshold

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EPED physics assumptions alone are not capturing all observed edge behavior in I-mode

I-mode pedestal is stable against identified ELM triggers – however, in minority of cases (12 time windows / 10 unique shots, out of dataset of 72 time windows / 52 shots), particularly at low field (~ 4.6 T) exhibit small, intermittent events that appear to be ELMs.

→ need to more carefully characterize these events!

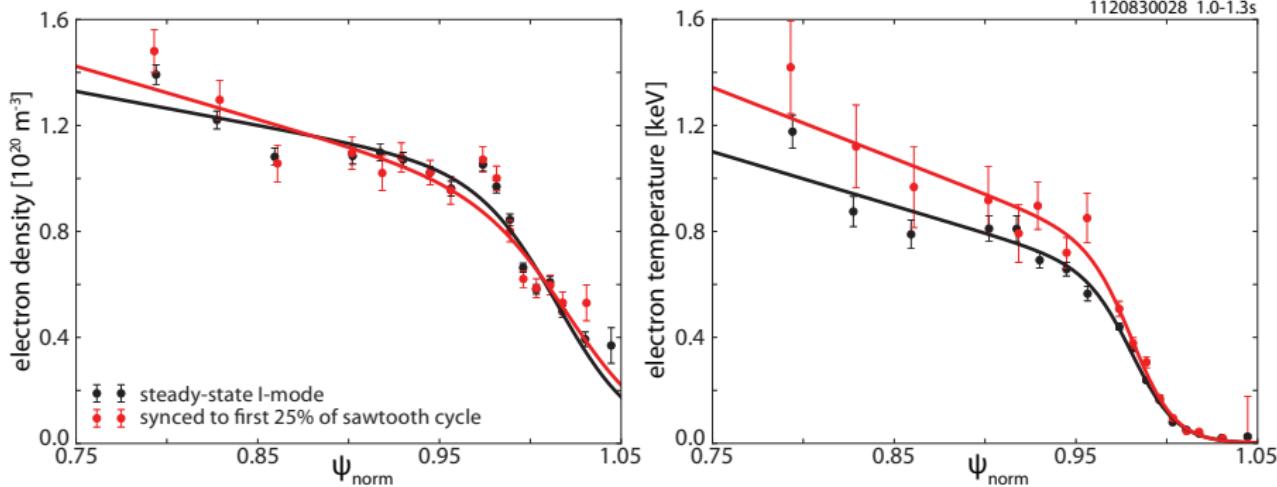
What is an ELM?

ELMs denoted by:

- burst of ionization in edge → spike in H_{α} light
- “explosive” crash in pedestal, both temperature and pressure
- unstable fluctuation leading up to ELM (P-B MHD instability, magnetic precursors, turbulent fluctuations...)

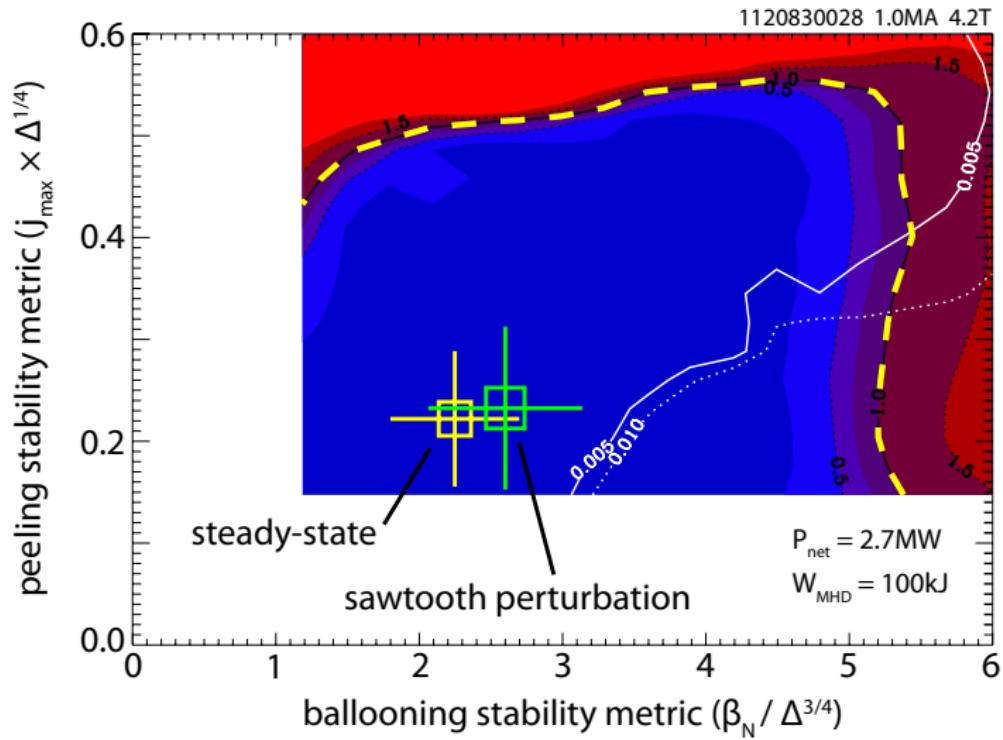
observations of ELMs in I-mode based on H_{α} spikes, in most cases ($\sim 70\%$) tied to the sawtooth heat pulse reaching the edge – good place to begin!

Sawtooth heat pulse modifies temperature pedestal, little impact on density profile

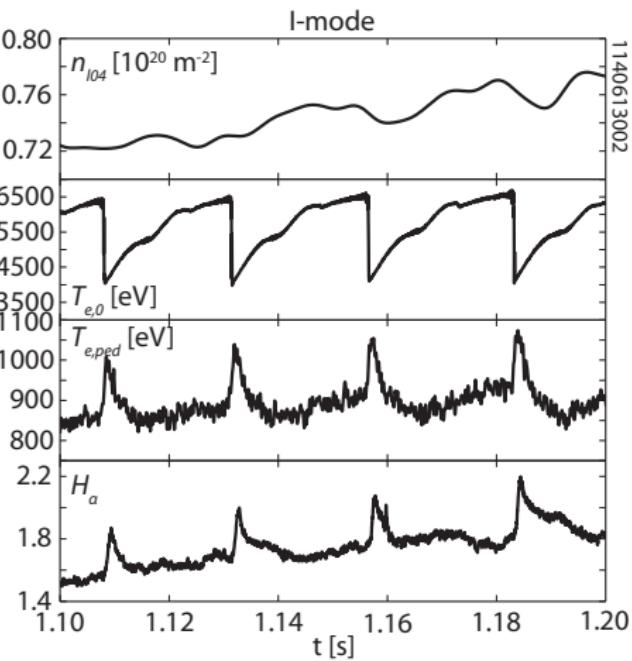
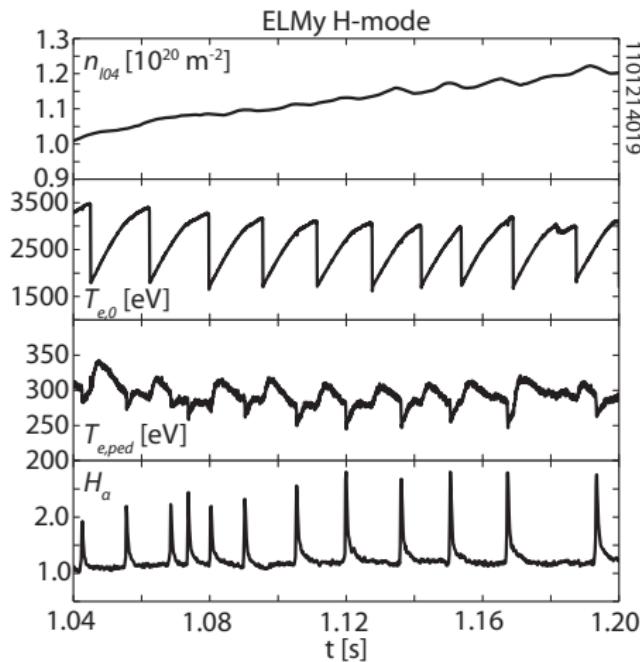


prepare data by masking TS frames to first 25% of sawtooth cycle
(immediately following heat pulse reaching edge) – similar to
ELM-binning technique used for ELMy H-mode

Sawtooth measurably perturbs pedestal in stability space, but insufficient to reach ELM threshold



I-mode sawtooth H_α events (mostly) do not exhibit characteristic temperature crash expected for ELMs



These events do not appear to be ELMs

Given the lack of a temperature pedestal crash and the computed stability against known ELM triggers, these events appear to not be instability-driven ELMs at all – best described simply as “sawtooth-driven H_α bursts.”

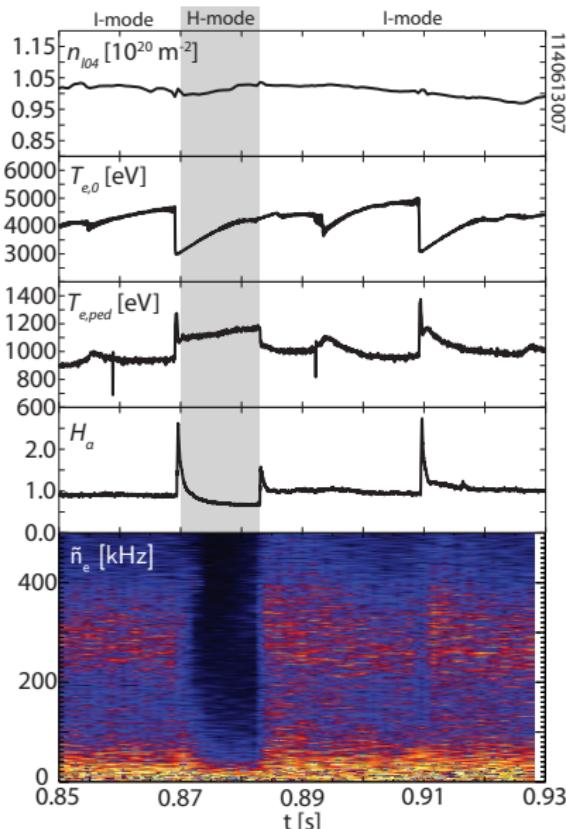
possible explanations:

- ionization front impacting neutrals in SOL?
- density transport effect from sawtooth interaction with WCM?

which raise questions:

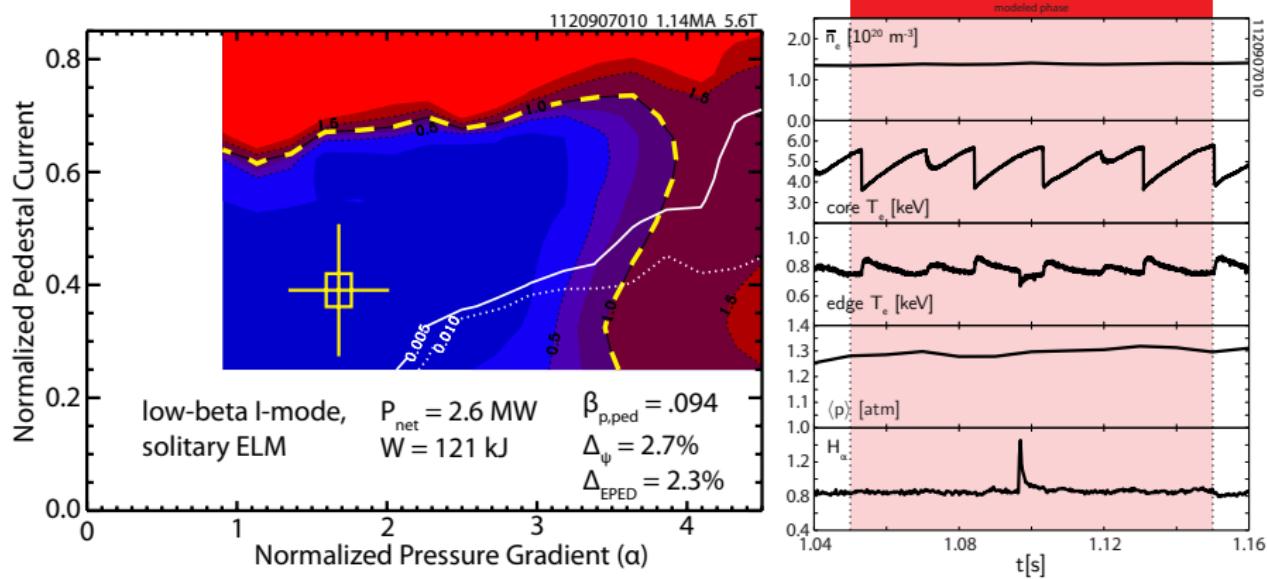
- why are these sometimes triggered – field, shaping, density effects?
- under comparable conditions, why are H_α spikes inconsistently triggered by similarly-sized sawteeth?

Minority of H_α spikes do appear to be ELMs, however



- Few events (10 out of 37 total events) do exhibit observable drop in edge T_e with spike, are not necessarily triggered by sawteeth
- suggests these are “true” ELMs – but perturbation is still small, < 1% stored energy drop, T_e crash on sawtooth-triggered ELMs insufficient to overcome T_e increase from heat pulse

Stationary pedestal structure around intermittent ELMs still stable to P-B MHD, KBM threshold



- H_{α} spikes (putative ELMs) in minority of I-mode cases (12 out of 72 time windows)
- Majority ($\sim 70\%$, 27/37) of identified events do not appear to be ELMs at all, consistent with computed stability and observed pedestal behavior – rather, are benign H_{α} pulses triggered by sawteeth
- a few do exhibit ELM behavior, not necessarily triggered on sawteeth – but these are rare, small
 - ▶ stationary pedestal structure still stable – transient modifications to hit stability boundary?

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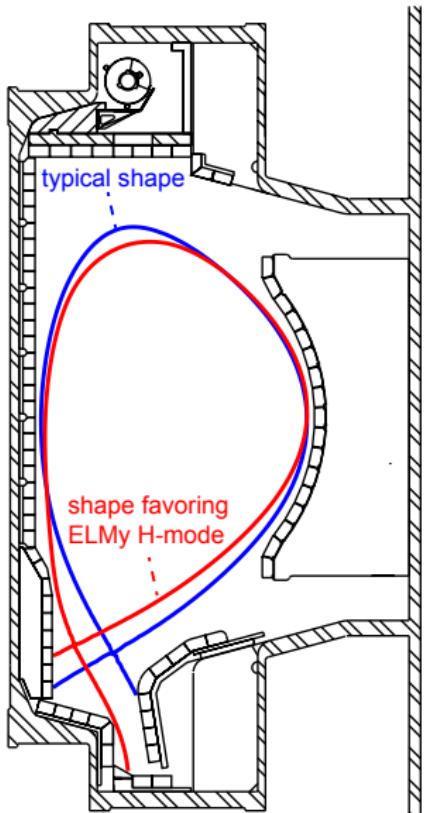
²JR Walk *et al.*, *Physics of Plasmas* 21 (2014)

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Supplemental Slides



Plasma shaping in C-Mod operation

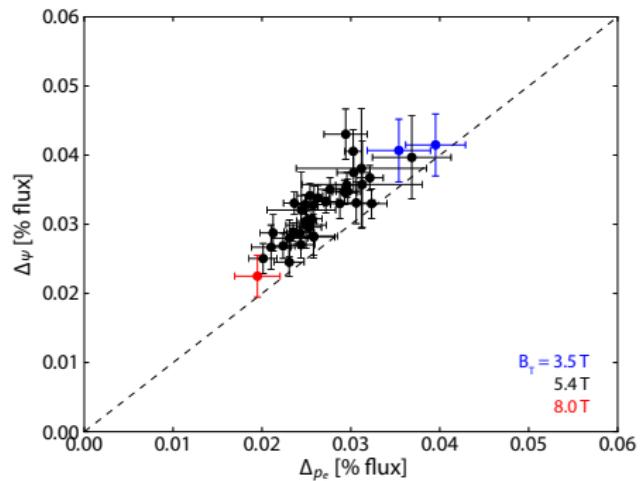
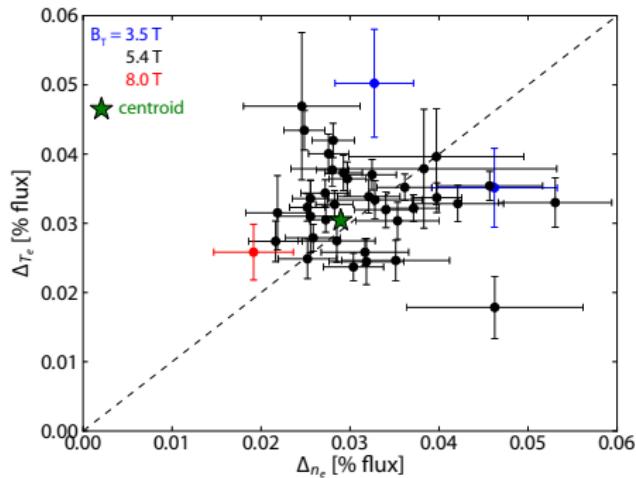


- I-mode operates at typical shaping for C-Mod plasmas (with reversed I_p , B_T for unfavorable ∇B drift)
- ELM My H-mode on C-Mod requires special shaping with low elongation, upper triangularity, high lower triangularity – in normal shaping in forward field, reach ELM-free H-mode (low ν^*) or EDA H-mode (high ν^*)

ELMy H-mode

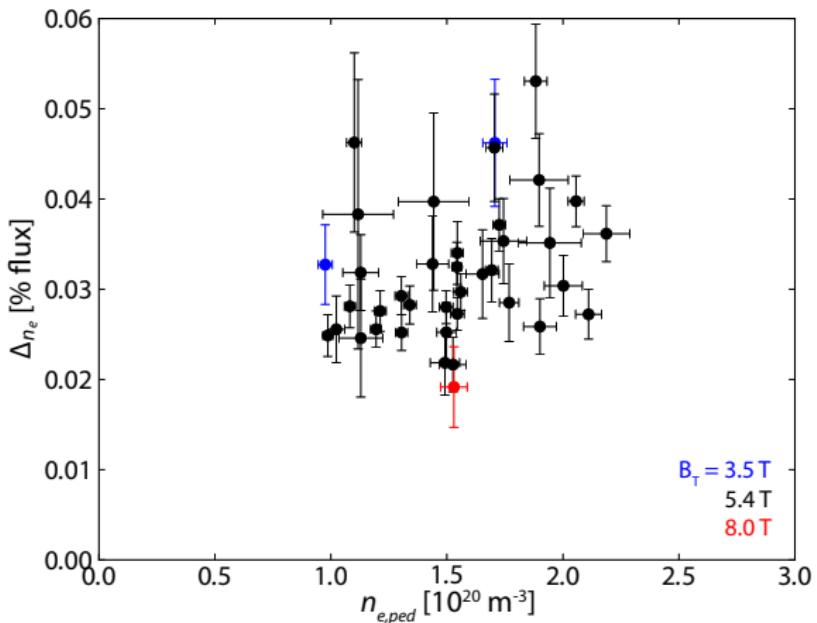


Density, temperature, and pressure widths in ELMy H-mode



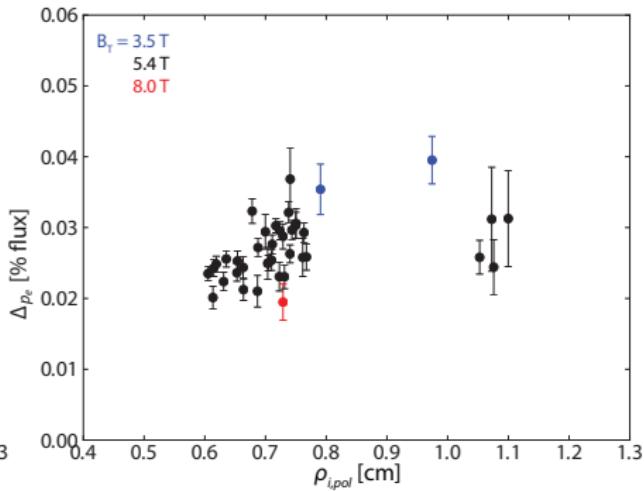
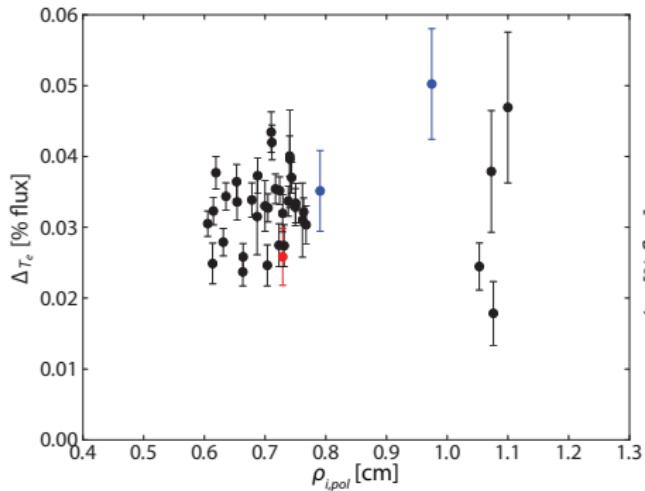
$$\Delta_\psi = (\Delta n_e + \Delta T_e)/2, \text{ tracks with directly-measured } \Delta p_e$$

Density pedestal width in ELM My H-mode inconsistent with neutral-penetration model



as expected from high-density, neutral-opaque SOL on C-Mod

Temperature, pressure pedestal widths not well-described by gyroradius scaling



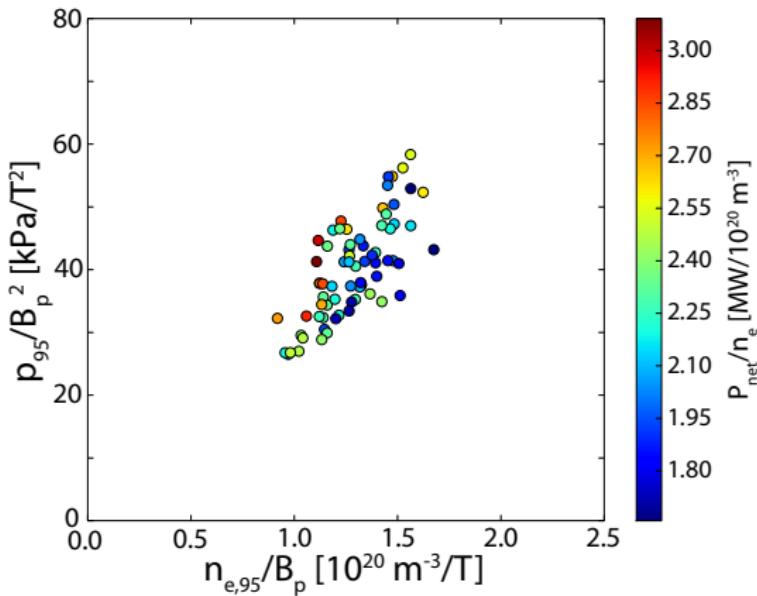
T_e pedestal width uncorrelated; p_e pedestal width trend due to covariance between $\rho_{i,pol} \sim \sqrt{T_{e,ped}}/I_p$ and $\sqrt{\beta_{p,ped}} \sim \sqrt{n_{e,ped} T_{e,ped}}/I_p$

I-mode pedestals



$\beta_{p,95}$ scaling with norm. density in addition to heating-power response

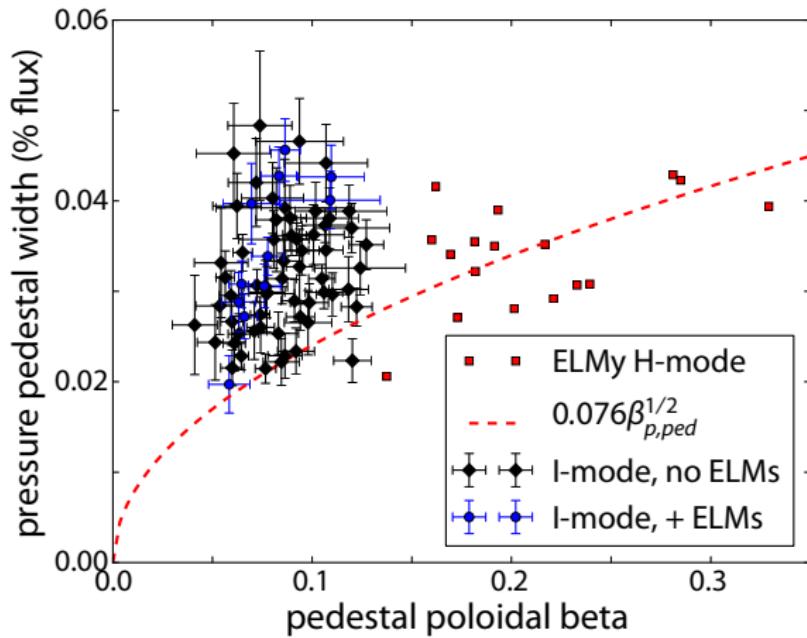
- P_{net}/\bar{n}_e sets slope of T_e/B_p line – response of pedestal temperature at fixed current
- pressure responds to fueling, provided sufficient power to maintain the pedestal



I-mode stability & ELM characterization



I-modes exhibiting edge H_α spikes / ELMs typically at lower β_p range



MHD stability analysis



$$\vec{Q} = \nabla \times (\vec{\xi} \times \vec{B})$$

$$\delta W = \delta W_F + \delta W_S + \delta W_V$$

$$\begin{aligned}\delta W_F = & \frac{1}{2} \int_P d^3\vec{r} \left[\frac{|\vec{Q}|^2}{\mu_0} + \frac{B^2}{\mu_0} \left| \nabla \cdot \vec{\xi}_\perp + 2\vec{\xi}_\perp \cdot \vec{\kappa} \right|^2 + \gamma p \left| \nabla \cdot \vec{\xi} \right|^2 \right. \\ & \left. - 2 \left(\vec{\xi}_\perp \cdot \nabla p \right) \left(\vec{\kappa} \cdot \vec{\xi}_\perp^* \right) - j_{||} \left(\vec{\xi}_\perp^* \times \vec{b} \right) \cdot \vec{Q}_\perp \right]\end{aligned}$$

$$\delta W_S = \frac{1}{2} \int_S dS \left| \hat{n} \cdot \vec{\xi}_\perp \right|^2 \hat{n} \cdot \left[\nabla \left(p + \frac{B^2}{2\mu_0} \right) \right]$$

$$\delta W_V = \frac{1}{2} \int_V d^3\vec{r} \frac{|B_1|^2}{\mu_0}$$

$$X = RB_p \xi_\psi$$

$$ik_{\parallel} = \frac{1}{JB} \left(\frac{\partial}{\partial \chi} + i n \nu \right) \quad \quad \nu = JB_T / R$$

$$P = \sigma X + \frac{B_p^2}{\nu B^2} \frac{F}{n} \frac{\partial}{\partial \psi} (JBk_{\parallel} X)$$

$$Q = \frac{X}{B^2} \frac{dp}{d\psi} + \frac{F^2}{\nu R^2 B^2} \frac{1}{n} \frac{\partial}{\partial \psi} (JBk_{\parallel} X)$$

$$\sigma = - \frac{F}{B^2} \frac{dp}{d\psi} - \frac{dF}{d\psi} = - \frac{j_{\parallel}}{B}$$

$$\delta W = \pi \iint d\psi d\chi \left\{ \frac{JB^2}{R^2 B_p^2} |k_{\parallel} X|^2 + \frac{R^2 B_p^2}{JB^2} \left| \frac{1}{n} \frac{\partial}{\partial \psi} (JBk_{\parallel} X) \right|^2 \right. \\ - \frac{2J}{B^2} \frac{dp}{d\psi} \left[|X|^2 \frac{\partial}{\partial \psi} \left(p + \frac{B^2}{2} \right) - \frac{iF}{JB^2} \frac{\partial}{\partial \chi} \left(\frac{B^2}{2} \right) \frac{X^* \partial X}{n \partial \psi} \right] \\ - \frac{X^*}{n} JBk_{\parallel} \left(X \frac{d\sigma}{d\psi} \right) + \frac{1}{n} [PJBk_{\parallel}^* Q^* + P^* JBk_{\parallel} Q] \\ \left. + \frac{\partial}{\partial \psi} \left[\frac{\sigma}{n} X^* JBk_{\parallel} X \right] \right\}$$

magnetic line bending: stabilizing

$$\delta W = \pi \iint d\psi d\chi \left\{ \frac{JB^2}{R^2 B_p^2} |k_{\parallel} X|^2 + \frac{R^2 B_p^2}{JB^2} \left| \frac{1}{n} \frac{\partial}{\partial \psi} (JB k_{\parallel} X) \right|^2 \right\}$$

ballooning
drive

$$- \frac{2J}{B^2} \frac{dp}{d\psi} \left[|X|^2 \frac{\partial}{\partial \psi} \left(p + \frac{B^2}{2} \right) - \frac{iF}{JB^2} \frac{\partial}{\partial \chi} \left(\frac{B^2}{2} \right) \frac{X^* \partial X}{n \partial \psi} \right]$$

kink
drive

$$- \frac{X^*}{n} JB k_{\parallel} \left(X \frac{d\sigma}{d\psi} \right) + \frac{1}{n} [P J B k_{\parallel}^* Q^* + P^* J B k_{\parallel} Q] + \frac{\partial}{\partial \psi} \left[\frac{\sigma}{n} X^* J B k_{\parallel} X \right] \} \quad \begin{array}{l} \text{magnetic curvature: stabilizing inboard,} \\ \text{destabilizing outboard} \end{array}$$

surface term: peeling drive

ELITE solves this for given n by encoding the poloidal angle χ in a straight-line coordinate via the “ballooning transform,”

$$\omega = \frac{1}{q} \int^{\chi} \nu \, d\chi$$

(note $2\pi q = \oint \nu \, d\chi$) and decomposing the displacement X into poloidal harmonics

$$X = \sum_m u_m(\psi) e^{-im\omega}$$

centered on the (m, n) rational surface. Define a “fast” radial variable

$$x = m_0 - nq$$
$$m_0 = \text{Int}(nq_a) + 1$$

readily convertible between x and ψ .

Euler-Lagrange equation minimizing the energy may be expressed in this harmonic expansion by a set of coupled equations⁷

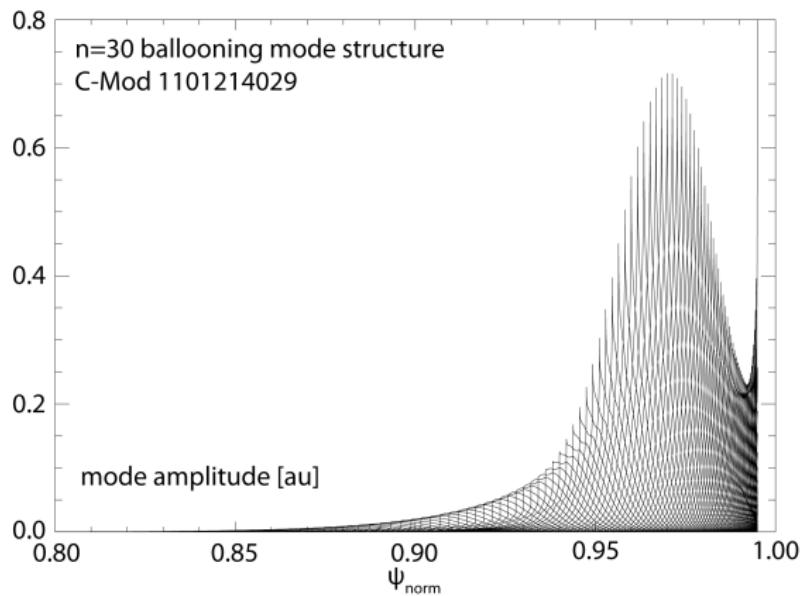
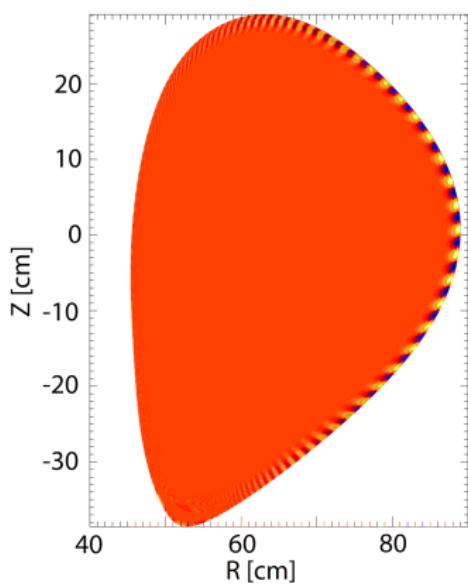
$$A_{m,m'}^{(2)} \frac{d^2 u_m}{d\psi^2} + A_{m,m'}^{(1)} \frac{du_m}{d\psi} + A_{m,m'}^{(0)} u_m = 0$$

describing coupling between harmonics m, m' , with matrix elements A calculated from the equilibrium describing the mode amplitudes. Features:

- u_m varies rapidly ($\sim x$, comparable to spacing between rational surfaces), needs fine mesh; A set by equilibrium parameters, varies more slowly, can be calculated on coarse mesh for numerical efficiency
- modes only couple to few nearest neighbors, so most $A_{m,m'}$ may be ignored
- can quickly calculate “fictitious eigenvalue” for stable/unstable determination, or true eigenvalue γ^2 for mode growth rate when inertial terms included (modifications to matrix elements A)

⁷HR Wilson et al., *Physics of Plasmas* **9** (2002)

ELITE radial, poloidal mode structure for $n = 30$ ballooning mode



BALOO solves simplified version of ballooning energy formulation, in the $n \rightarrow \infty$ limit: reduces to 1-D eigenvalue equation,

$$(L_0 + \omega^2 M_0)f = 0$$

$$X = f(\psi, y)e^{-in \int \nu dy}$$

for displacement X and ballooning-transformed angle y , given by

$$L_0 f = \frac{\partial}{\partial y} \left\{ \frac{1}{JR^2 B_p^2} \left[1 + \left(\frac{R^2 B_p^2}{B} \int^y \frac{d\nu}{d\psi} dy \right)^2 \right] \frac{\partial f}{\partial y} \right\}$$

$$+ f \left\{ \frac{2J}{B^2} \frac{dp}{d\psi} \frac{\partial}{\partial \psi} \left(p + \frac{B^2}{2} \right) - \frac{F}{B^4} \frac{dp}{d\psi} \left(\int^y \frac{d\nu}{d\psi} dy \right) \frac{\partial B^2}{\partial y} \right\}$$

$$M_0 f = \frac{J}{R^2 B_p^2} \left[1 + \left(\frac{R^2 B_p^2}{B} \int^y \frac{d\nu}{d\psi} dy \right)^2 \right] f$$