Foundations of Arithmetic and Algebra

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Contents

Preface

Forget everything you know, or at least think you know, about math except how to count to nine and the single digit numbers: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. After the course is over, or if you can compartmentalize, separately, you can think about how the content of this book connects with things you have learned before, but it is important at this stage to be clear, I intend to develop all concepts from the ground up, and you should only consult the ideas in this book to contend with problems in this book. The book is written to be self-contained, and has a rather idiosyncratic development. I hope that by engaging with this book, you will be better connected to the core concepts of math usually taught at this level, and as result, you will find yourself better prepared to cope with future math courses in a more traditional style, or, indeed, to understand and make use of math in your non-academic lives. I hope that is not too ambitious a goal!

Justin Young

Part 1

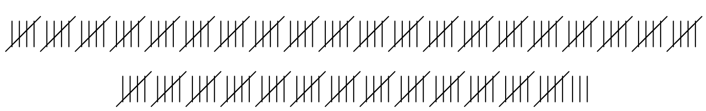
Numbers, Decimals, Fractions

CHAPTER 1

Numbers

None of us were there, but I'm pretty sure that the origins of what we call "mathematics" (or simply, "math", here in the USA) lie in the concept of counting. There were concrete reasons to count objects, and this immediately gave rise to some interesting properties and puzzles. The first concept that is difficult to understand about numbers is that $1, 2, 3, 4, \ldots$ are not numbers but merely *a particular way of referring to numbers.* This notation for numbers has not existed for very long compared to the breadth of human history on Earth, and there have been many other ways of referring to numbers in writing, and in language.

The simplest way to write counting numbers is to simply make a slash mark for each additional object we want to count. So if we had, for example, 15 sheep to count, we might scratch $|||||||||||||||$ into something to remember that amount, and we can easily verify if the number of sheep matches the marks as needed. The next step is to make larger numbers of marks easier to count. Imagine, for example, trying to count 168 apples using only slash marks. It would be very difficult to look at the slash marks and figure out exactly how many apples there are without literally counting them again. So, we might group together 5 slash marks by drawing 4 slash marks and then making a slash through them. Then, at a glance we can see the number of groups of 5 and we could use those to count up to 165, then we would have 3 left over slash marks.



This idea is more workable, but still a bit inefficient.

We do not have time to delve deeply into the topic of notation for numbers in this course, but I hope this gives you an idea of the scope of the problem, and how long it took for humans to arrive at a workable system, which is now used in most parts of the world: place value, and base-10.

1.1. Counting, Base-10, and place value

The central idea of base-10 is that we count objects by grouping them in 10s, and whenever we have 10 of a group, we group those together. We use place value to keep track of how many objects are in a group. In this way, we can represent any counting number, and all we need are 10 symbols. Let's get into the details.

The standard symbols we use for base-10 are: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. We use these to start from zero (this is an important development in its own right) represented by 0, then we get $|$ represented by 1, $||$ is 2, $|||$ is 3, $||||$ is 4, $|||||$ is 5, $||||||$ is 6, $|||||||$ is 7, $||||||||$ is 8, $|||||||||$ is 9. In this way, we can represent the counting numbers for counting up to | | | | | | | | | objects. What happens if we have one more object? This is where place value and the base-10 system go to work: we group together the ten objects (| | | | | | | | | |) and we shift one place to the left so 10 means 1 group of (| | | | | | | | | |) objects and 0 single objects. Next, we have one group of ten objects and one single object (| | | | | | | | | |) |, this is represented in base-10 by 11. Then, we have one group of ten objects and two single objects (| | | | | | | | | |) | |, this is represented by 12. We continue in this way until we reach one group of ten objects and nine single objects: (| | | | | | | | | |) | | | | | | | | | which is represented by 19. What happens if we add one more object?

In this case, as always in base-10, we group together anything we have ten of, so we get two groups of ten (| | | | | | | | | |)(| | | | | | | | | |) and zero single objects, which is written 20. From here we continue with 21, 22, 23, etc. Notice that, in a two-digit number, from left to right, the first digit tells us how many *groups of ten objects* we have, and the second digit tells us how many *single objects* we have. For this reason, we call the leftmost digit in a two-digit number the **tens** place, and the rightmost digit is the **ones** place.

For example, the number 47 means: four groups of ten objects, and 7 single objects. We could draw the grouped slash marks to illustrate this as follows: (| | | | | | | | | |)(| | | | | | | | | |)(| | | | | | | | | |)(| | | | | | | | | |) | | | | | | |. If you count the slash marks you should see 47, but the important point here is to see how the notation 47 connects to the way we are grouping the objects. One trick that can save some time to illustrate numbers in this way is to write an abbreviation for a group of ten objects, for example we could let $X = (|\,|\,|\,|\,|\,|\,|\,|\,|\,|)$. Using this *object abbreviation*, we could write[1] $XXXX\,|\,|\,|\,|\,|\,|\,|$ for 47. It should help to get this visual connection between the picture and the notation. Understanding in a clear way the method we use in our base-10 notation is essential for grasping more advanced concepts and calculations that appear later in the course.

The next obvious question is: what do we do when we have ten groups of ten objects? We do the same thing as before: we group together the ten groups of ten, and move one place to the left, this new place is called the **hundreds** place and it will count the number of ten groups of ten. If you are familiar with basic math, you probably know that ten groups of ten objects is a hundred objects. Once we reach nine groups of ten and nine single objects, and then we add one more object, we now have ten groups of ten, or $(XXXXXXXXXX)$, we can also abbreviate this $C = (XXXXXXXXXX)$. Using these abbreviations, we could express the number of objects described by the base-10 number 347 as $CCCXXXX\,|\,|\,|\,|\,|\,|\,|$ meaning: three groups of a hundred (ten groups of ten), four groups of ten, and seven single objects. This gives a clear meaning to each place, and connects the way we write the number with the number of objects we are describing.

Exercises

Exercise 1.1.1. Consider the following two digit number in base-10: 68. Draw slash marks grouped in tens as above to illustrate the number of objects described by this notation. Do not use object abbreviations like X for ten.

[1] This is vaguely similar to Roman numerals but NOT the same, note that in Roman numerals 47 is written $XLVII$. Roman numerals do not use base-10 or place value.

Exercise 1.1.2. Use the object abbreviations $|$ for one object, X for ten objects, and C for a hundred objects. Regroup and abbreviate the following objects according to the base-10 system, then write the corresponding base-10 notation for the number of objects:

(a)
$$X\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|$$

(b)
$$|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|$$

(c)
$$CXXXXXXXXXXX\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|$$

Exercise 1.1.3. If we use the abbreviation M for ten hundreds:
$$M = (CCCCCCCCCC),$$
then M is an abbreviation for a thousand objects. For example,
$$MMCCCXXXX\,|\,|\,|\,|\,|$$
would represent 2345 objects in base-10.

(a) Use the object abbreviations ($|, X, C,$ and M) to write notation for 5678 objects.

(b) Use the object abbreviations to write notation for 9001 objects.

(c) Use the object abbreviations to write notation for 9999 objects.

(d) Using the idea of grouping in tens (as we do in base-10), how might we regroup and abbreviate **one more than**
$$9999 = MMMMMMMMMCCCCCCCCCXXXXXXXXX\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,?$$

Exercise 1.1.4. Write the numbers from 1 to 30 using object abbreviations, notice where regrouping and abbreviation is required.

1.2. Decimals

In the last section, we saw how we count in base-10 by grouping in tens and moving to the left in place value. If we now reverse our thinking, we see that, as we go *from left to right* each place value is one-tenth of the size of the place before. For example, if we divide one hundred $C = (XXXXXXXXXXX)$ into ten equal parts, we get ten $X = (|\,|\,|\,|\,|\,|\,|\,|\,|\,|)$. Similarly, if we divide ten $X = (|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|)$ into ten equal parts we get one single object $|$. What if we continue this line of thinking?

Imagine dividing one single object into ten equal parts, then we could conceivably decrease the size of our place value and go to the **right** of the ones place. In this case, we would get *tenths*, and we can use the abbreviation $| = $ (**********).

Our convention is to put a period or decimal point to the right of the ones place to denote the passage from whole object to partial objects. Thus, we could write two whole objects and six tenths of an object as 2.6, or in object abbreviation form:

$$||******.$$

Now, we can continue the pattern and further divide a tenth into ten equal parts: $* = (++++++++++)$ called *hundredths* of an object. Using the above abbreviations we can write:

$$23.45 = XX|||****+++++.$$

Note that, just as in base-10 whole numbers, as we go right to left, the value of the next place is 10 times the previous place, and as we go left to right, the value of the next place is *one-tenth* of the previous place. For example, the hundreds place is worth ten times the tens place, and the hundreds place is worth one-tenth of the thousands place. In this way, we can continue to the right in our base-10 decimal numbers to get the *thousandths* place, this place has value one-tenth of the hundredths place. So, ten thousandths should be equal to one-hundredth, if we use, for example δ to represent a thousandth, then we must have $+ = (\delta\delta\delta\delta\delta\delta\delta\delta\delta\delta)$. To write the base-10 decimal number 2.783 with our object abbreviations, for example, we would write $2.783 = ||*******++++++++\delta\delta\delta$.

Exercises

Exercise 1.2.1. Most of you will be familiar with a real world context in which we divide a whole object into tenths: currency. If we consider a dollar to be a whole object, then what common form of currency represents a *tenth* of a dollar?

Exercise 1.2.2. What common from of currency represents a *hundredth* of a dollar?

Exercise 1.2.3. Consider the amount of money given by \$34.26. We could abbreviate this decimal by $XXX||||**++++++$. For each object abbreviation: X, $|$, $*$, and $+$, describe what it represents in common US currency. Then, explain the decimal value 34.26 in terms of this currency in a way that fits with the base-10 system.

Exercise 1.2.4. Note that, for example, although we have a quarter coin having a value of \$0.25, this coin can not be used as an object abbreviation in base-10. Rather, we would describe 0.25 in base-10 as $**+++++$ (two tenths of an object, and five hundredths of an object), or in common US currency terms: two dimes and five pennies. For each of the examples below, describe how we can use our base-10 object abbreviations, and common US currency to represent the following currency (none of the examples below can be used as an object abbreviation in base-10).

(a) a nickel
(b) a five-dollar bill
(c) a twenty-dollar bill
(d) a fifty-dollar bill

Exercise 1.2.5. Consider the base-10 decimal 59.237. What is the value of the 7 in terms of the base-10 object $+ = 0.01$, or in terms of a penny (viewing the number as a dollar amount)?

Exercise 1.2.6. Fill in the blanks only by reasoning about the base-10 system and US currency, do not calculate:

(a) A penny is one ________ of a dollar.
(b) A dime is one ________ of a dollar.
(c) A dollar is one ________ of \$10.
(d) A dollar is ________ times a penny.
(e) A dollar is ________ times a dime.
(f) \$1000 is ________ times \$100.

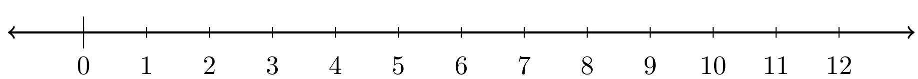
Exercise 1.2.7. Be sure you are careful to distinguish fractional decimal values (to the right of the decimal point), and whole base-10 values (to the left of the decimal point). For example the *hundreds* place is NOT the same as the *hundredths* place. To illustrate this, describe 5 hundreds and 5 hundredths in terms of dollar amounts. Then, write each value using our object abbreviations.

1.3. Order

Our first and simplest concept of the order of numbers is simply in terms of size. If you have \$9 and I have \$7, then, since you have more money than I do, we say that 9 is larger than 7 as a number. In this way, we can easily order the counting numbers according to size. We will use the abbreviations $a < b$ to mean that a is smaller than b, and $a > b$ to mean that a is larger than b, where a and b are any counting numbers. In terms of the above example, we would say that $9 > 7$ (and we could also say that $7 < 9$). In order to refine and expand our concept for comparing numbers, a very convenient way of visualizing numbers comes from thinking of them as *lengths*, this leads to the concept of number lines.

1.3.1. Number Lines, Negative Numbers.

We start by imagining a ruler, say in inches. Note that the ruler starts from zero, and typically measures up to 12 inches. If we abstract this idea, and think of the distance between each pair of counting numbers (including zero and one) as having a fixed length, we get something like the number line below.



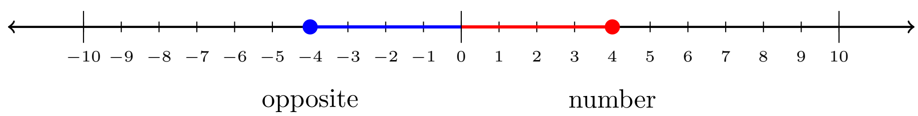
Of course, the ruler doesn't need to stop at 12, and if we let the numbers keep going forever, we have the idea of the number line for the usual counting numbers. You may notice, however, that there is an arrow pointing to the left of zero as well. We already have some common real-world interpretations of this, the most common being degrees below zero in temperature (Celsius or Fahrenheit), but we can also think of elevation below sea level, or owing money as negative amounts. These count the same way as positive counting numbers but they go up to the left, imagine a thermometer on its side to get the idea:

Notice the essential symmetry of the number line, the negative numbers are like mirror images of the positive ones. From the idea of temperature, for example, we know that -3 degrees is warmer than -9 degrees. In that way, we can say that $-3 > -9$. If we translate this into number line terms, we can now say that $a > b$ means a is to the *right* of b on the number line and $a < b$ means a is to the *left* of b on the number line, *for any numbers*, positive or negative. Once we introduce the concept of negative numbers in this way, we no longer want to think in terms of size, but rather in terms of position on the number line. Consequently, we change our language slightly and say that $a > b$ really means a is *greater than* b, and $a < b$ really means a is *less than* b. To use this vocabulary, we could say that 3 is less than 7, but 3 is greater than -4, or in symbolic terms $3 < 7$ and $3 > -4$.

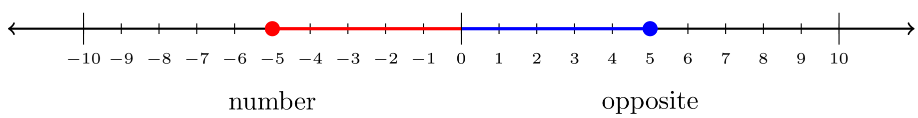
We will now define the opposite of any number on the number line.

Definition 1.3.1 (opposite). Let x be any nonzero number on the number line. Then the *opposite* of x, denoted $-x$ is defined to be the number on the number line that is the same distance from 0 as x, but not equal to x.

Since this definition is a bit oblique, let's spell it out with examples. First, the easiest case, if we have a positive number like 4 for example, then the opposite of 4 is simply -4. See the number line below showing that both numbers are the same distance from zero (four number line units), but are not the same!

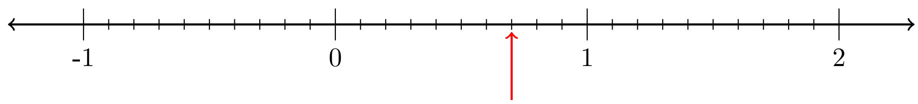


We should now talk about the opposite of a negative number. For example, consider -5, what is the opposite of this number? First, observe that this number is 5 units away from zero. What other number is 5 units away from zero? You guessed it, the number 5.

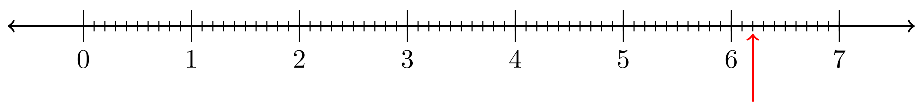


Thus, the opposite of -5 is 5, and therefore, using our notation, this implies (by definition alone!) that $-(-5) = 5$.

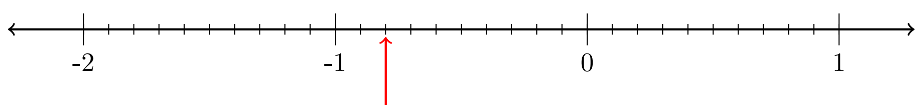
1.3.2. Decimals on the Number Line. Recall that our definition of decimals involves dividing objects into ten equal parts. If we now think of numbers as *lengths from zero* on the number line, we can divide up the space between the whole number values to obtain decimal values. For example, we could view the number 0.7 as dividing the space between zero and one on the number line into ten equal parts, and then using the first seven of them:



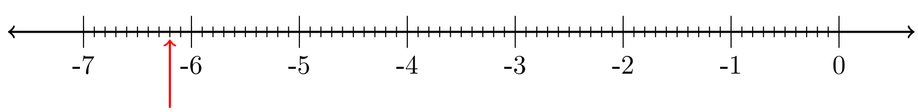
In this way, the segment from zero to the arrow above represents the number 0.7. If we want a larger decimal value we go out to the ones place first, then move the appropriate number of tenths. For example, the number 6.2 is located via the arrow below.



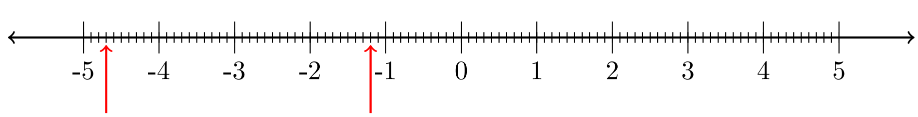
Negative decimals work in the same way, dividing the segments on the negative side of the number line into ten equal parts. Notice carefully, to locate a negative value starting from zero, we move to the *left*, the opposite direction that we move for positive values. For example, the number -0.8 is labelled below.



Let's look at one more negative value on the number line -6.2 notice carefully that the value is two tenths to the *left* of -6. Compare to the number line showing positive 6.2 above. This is a key point to make sure you understand about negative numbers on the number line.



At this point, we are ready to start comparing positive and negative decimals. For example, observe the locations of -4.7 and -1.2 on the number line below.



From this picture, we can easily see that -4.7 is to the *left* of -1.2 on the number line, and therefore:

$$-4.7 < -1.2$$

We can *also* see that -1.2 is to the *right* of -4.7 on the number line (switching perspectives), which means therefore:

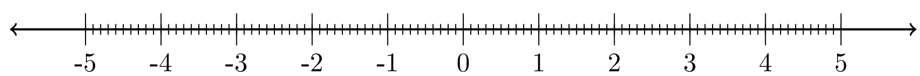
$$-1.2 > -4.7$$

Note that these two statements are completely equivalent: if a is to the left of b, then b is to the right of a, so we can always phrase any inequality of real numbers as a 'greater than' OR as a 'less than'. Developing the flexibility to see this relationship from either point of view is essential for learning to interpret and use inequalities. We record this result formally in the theorem below.

THEOREM 1.3.2. *The inequality $a < b$ is equivalent to the inequality $b > a$ (both inequalities are true or both are false).*

Exercises

EXERCISE 1.3.1. Find the following values on the number line: $1.2, 3.6, -1.7, -2.9, -0.3$.



EXERCISE 1.3.2. Use a number line sketch to illustrate the following inequalities:

(a) $4.2 < 4.7$

(b) $-4.2 > -4.7$

(c) $-5 < 0$

(d) $1.9 > -6.2$.

EXERCISE 1.3.3. Fill in the blanks:

(a) $4.2 < 4.7$ because ________ is to the ________ of ________ on the number line.

(b) $-4.2 > -4.7$ because ________ is to the ________ of ________ on the number line.

(c) $-5 < 0$ because ________ is to the ________ of ________ on the number line.

(d) $1.9 > -6.2$ because ________ is to the ________ of ________ on the number line.

EXERCISE 1.3.4. Using the number line definition of the order of numbers, explain why any positive number is always greater than any negative number. Then, explain why any negative number is always less than any positive number. (Note that the explanations are slightly different, even though the two statements are equivalent.)

EXERCISE 1.3.5. The symbols $\leq$ and $\geq$ are used for less than *or equal to* and greater than *or equal to* respectively. One way to read $a \leq b$ in terms of the number line is to say a is either to the left of b or is the same as b. For example, if we know $2.6 < 5.4$, for example, then we automatically have $2.6 \leq 5.4$ as well. This adds no new information, and is in fact a weaker statement, but sometimes it is convenient to include equality as a possibility in an inequality (I know that sounds contradictory!). Note that $2.6 \leq 2.6$ is true, as well as $2.6 \geq 2.6$. It may seem silly to phrase it this way, since obviously $2.6 = 2.6$, but in practice we often don't know the specific value of a number, or we want to represent a range of numbers, and thus it is useful to be a bit more flexible with our inequalities, so flexible that we allow equality! For the following inequalities, answer true or false.

(a) $4.2 \leq 4.7$

(b) $-4.2 \leq -4.7$

(c) $-5 \leq 0$

(d) $1.9 \geq -6.2$

(e) $2 < 2$

(f) $3 \geq 3$

(g) $0 \geq -9$

Exercise 1.3.6. Find the opposite of each number below.

(a) number $= 7$, opposite $= ?$

(b) number $= -8$, opposite $= ?$

(c) number $= 0$, opposite $= ?$

Exercise 1.3.7. The absolute value of a number x, denoted $|x|$ is defined to be the (positive or zero) distance from x to 0 on the number line.

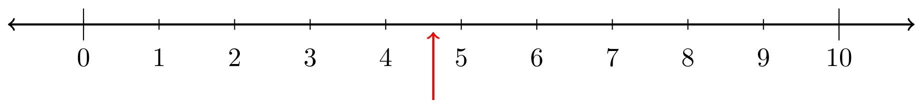
(a) Find $|7|$.

(b) Find $|-8|$.

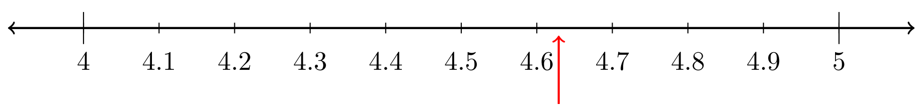
(c) Use the definitions to explain why $|x| = |-x|$ (a number and its opposite have the same absolute value).

1.4. Order of decimals and zooming in

We would now like to expand beyond one decimal place and compare decimal numbers with two, three, four, or even ten decimal places. A good way to practice this is to visualize zooming in to a portion of the number line. For example, if we look at the decimal number 4.629, it would be located as shown below on the number line.



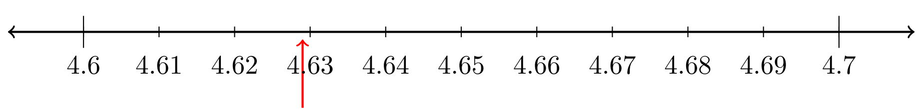
Note that the larger tick marks are tracking the ones place (in this case the ones place of 4.629 is a 4). The smaller tick marks are tracking the tenths place (in this case the tenths place of 4.629 is a 6). We will always follow this convention with our Base-10 number lines: only large and small tick marks (no other sizes), and they always track adjacent decimal places in order of size. Now, if we *zoom in* to the little interval between whole numbers (in this case 4 and 5) that contains our given value 4.629, we can see more precisely where the decimal 4.629 is located.



On this number line the large tick marks are tracking tenths.

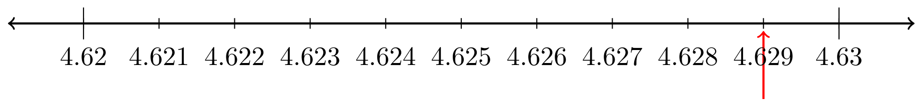
Note carefully that the tenths place of the left- and right-most values on the number line above is zero: $4 = 4.0$ and $5 = 5.0$, so the distance between two adjacent large tick marks is always a tenth. However, be careful, just because two numbers are a tenth apart doesn't mean they can be used as large tick marks tracking tenths. For example, we could **not** have a number line with 2.37 and 2.47 on adjacent large tick marks, this would not fit with the base-10 or decimal system because, although they are a tenth apart, they have nonzero decimal values to the right of the tenths place!

We see our given value 4.629 is now between the large tick mark values 4.6 and 4.7. In this case, the small tick marks are tracking the next smallest decimal value: hundredths (a tenth of a tenth). You should now try to visualize what it would look like to zoom in on the interval between 4.6 and 4.7, what would the large and small tick marks be tracking? Where would our given value be located? The zoomed in number line would be as follows:



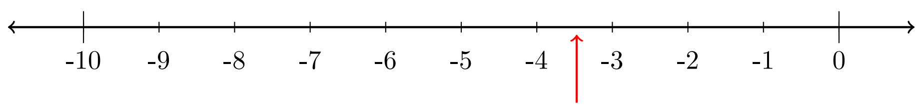
This time note that the large tick marks are tracking hundredths.

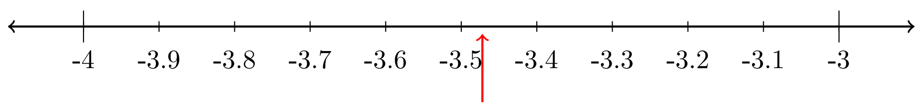
The small tick marks are tracking the next smallest decimal place now: thousandths (a tenth of a hundredth). We now see that our given value 4.629 is located between the values 4.62 and 4.63, much closer to 4.63 since the thousandths place is a 9, so we have located our number sitting directly on a small tick mark, the ninth one between 4.62 and 4.63. We could zoom in again, but this would put our value on a large tick mark, as follows:

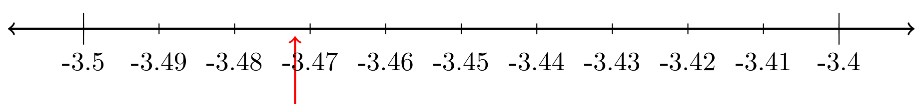


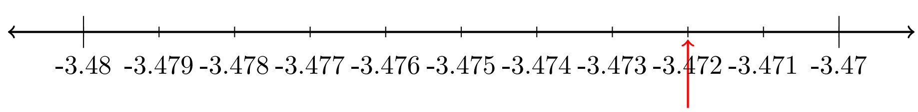
On this number line the large tick marks are tracking thousandths, and the small tick marks are tracking ten-thousandths (a tenth of a thousandth). We will not typically zoom in this far, as the previous number line located our given value on a small tick mark, and the extra precision provided by the ten-thousandths does not alter our understanding of the location of 4.629 on the number line.

We can, of course, also zoom in on the location of a negative decimal. To give an example, the following sequence of number lines will zoom in on -3.472.







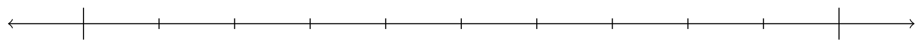


Pay close attention to the fact that, just as for whole numbers, negative decimals count up to the *left*. For example, -2.9 is to the left of -2.1 on the number line. If we now recall our interpretation of order in terms of left and right on the number line, we are now equipped to visualize more complicated decimals in order. We can see that, $-3.14 > -3.3$, for example.

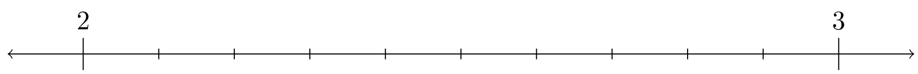
Exercises

Exercise 1.4.1. What is the hundredths place of 4.6? What is the hundredths place of 4.7? What is the hundredths place of 4.629?

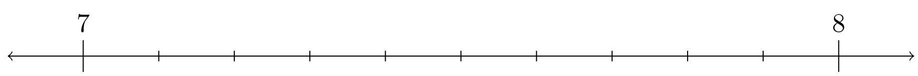
Exercise 1.4.2. Draw a sequence of number lines to zoom in on the following values, use the template below for each number line, starting with the given one for each part. Your final number line should have the given value on a small tick mark. Clearly indicate the approximate position of the given value on each number line, and label enough tick marks so that I can follow your thinking.



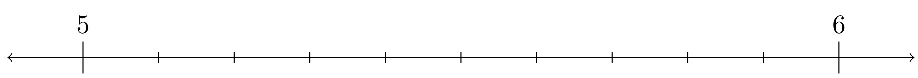
(a) 2.345
Starting number line:



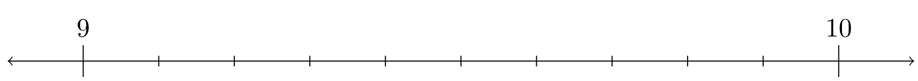
(b) 7.238
Starting number line:



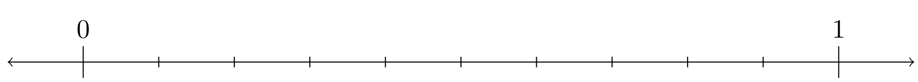
(c) 5.6
Starting number line:



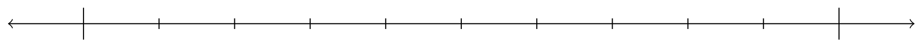
(d) 9.1234
Starting number line:



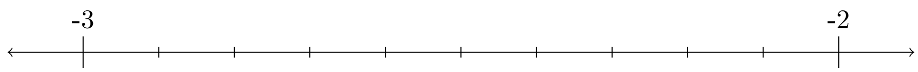
(e) 0.53
Starting number line:



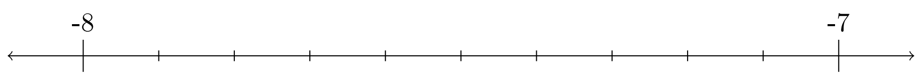
EXERCISE 1.4.3. Draw a sequence of number lines to zoom in on the following values, starting with the number line below. Clearly indicate the approximate position of the given value on each number line.



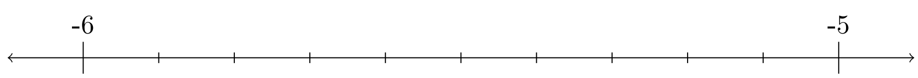
(a) -2.345
Starting number line:



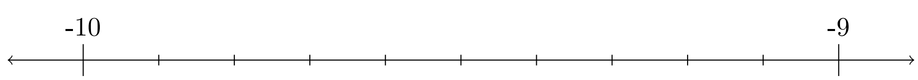
(b) -7.238
Starting number line:



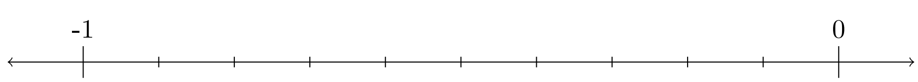
(c) -5.6
Starting number line:



(d) -9.1234
Starting number line:



(e) -0.53
Starting number line:

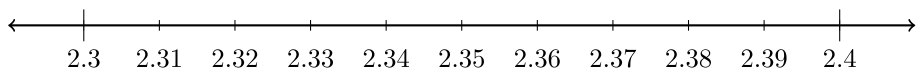


Exercise 1.4.4. Explain why $3.2 > 3.19$ (even though $2 < 19$) by drawing a number line zoomed in enough to show both values on a tick mark.

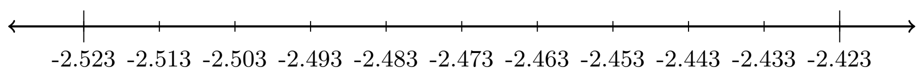
Exercise 1.4.5. Now explain why $-3.2 < -3.19$ by drawing a number line zoomed in enough to show both values on a tick mark.

Exercise 1.4.6. Indicate the number lines that do **NOT** fit with the structure of the Base-10 system, and briefly explain why in each case.

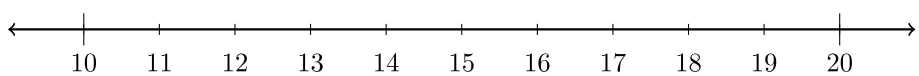
(a) Number line:



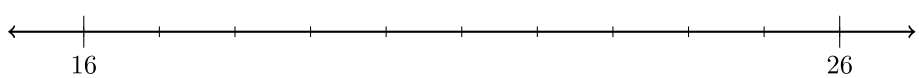
(b) Number line:



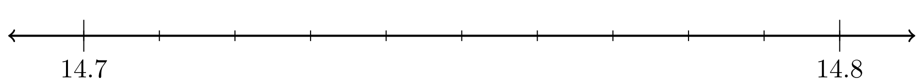
(c) Number line:



(d) Number line:



(e) Number line:



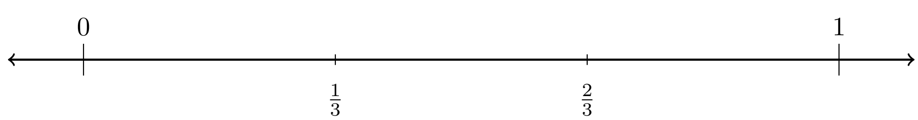
CHAPTER 2

Fractions

2.1. Definition of fractions and the number line

There is no more feared and despised topic in elementary math than fractions. Luckily, if we have understood the development until now, it will not be a big leap forward to understand fractions. We have already discussed decimals, in which we divide a whole into ten equal parts, each equal to a tenth of the whole. In decimal form, we write a tenth as 0.1, but now we want to allow other equal divisions, so we will use the alternative fraction notation $0.1 = \frac{1}{10}$.

We can now, for example, divide the whole number 1 into *three* equal parts, each one of size $\frac{1}{3}$ of the whole amount 1. One of these parts is called one-third, and written $\frac{1}{3}$. Two of these parts is called two-thirds, and written $\frac{2}{3}$. Three of these parts is called three-thirds, and written $\frac{3}{3}$. Note that, if we divide the whole amount into *three* equal parts and we then take three of those parts, together they make the whole amount again. Thus, we see that $\frac{3}{3} = 1$. We can also subdivide the interval from 0 to 1 on the number line into three equal parts, and discover the above fractions on the number line in a natural way.



We now arrive at our fundamental definition of fraction.

Definition 2.1.1. For positive whole numbers m and n, we define $\frac{m}{n}$ to mean m parts of size $\frac{1}{n}$ of the whole amount. The whole amount must always be specified, but by default the whole amount is the number 1. The whole number m is referred to as the *numerator* and the whole number n is referred to as the *denominator* of the fraction.

By convention, we can easily allow m (the numerator) to equal zero and the above definition still makes sense. In general, zero parts of any size will equal zero, so we have $\frac{0}{n} = 0$ for any $n > 0$. For example, $\frac{0}{3} = 0$, $\frac{0}{4} = 0$, $\frac{0}{5} = 0$, etc. We will discuss negative fractions in a later section.

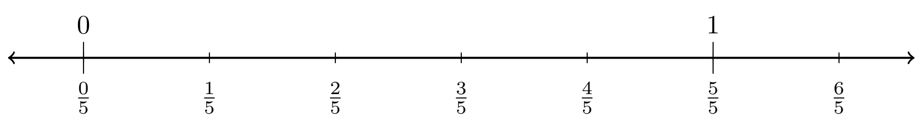
Let's now discuss the whole amount, which is an essential component to a clear understanding of fractions, but often under-emphasized due to the default status of the number 1. Imagine that you complete a 9 mile run. Let your running distance (9 miles) be the whole amount this time, what is one-third of your running distance? If we let the symbol m stand for a mile, then we can represent 9 miles as $mmmmmmmmm$. How can we divide this into three equal parts? Easily, in threes as follows

$$9 \text{ miles } = mmmmmmmmm = (mmm)(mmm)(mmm).$$

Thus, you can see that we can divide the whole amount 9 miles into three equal parts, each equal to 3 miles. Thus, $\frac{1}{3}$ of your running distance is 3 miles, $\frac{2}{3}$ of your running distance is 6 miles (*two* parts of size $\frac{1}{3}$ of the running distance), and $\frac{3}{3}$ of your running distance is, of course, the full 9 miles.

Note carefully in the above example that $\frac{3}{3} \neq 9$, but rather $\frac{3}{3}$ *of the running distance* is equal to 9 miles (the whole running distance). A key clue word for keeping track of the whole amount is the word **of**. In more practical problems we often want the whole amount to be something other than the default of the real number 1, developing the flexibility to change whole amounts will make fractions easier to understand, and will also show you their usefulness!

We can easily understand so-called improper fractions (a fraction where $m > n$) from the point of view of our definition. For example, $\frac{6}{5}$ simply means six parts of size $\frac{1}{5}$ of the whole amount. If we use the default whole amount 1 we can visualize this on the number line below.



Now, let's go back to the running distance example earlier: let the whole amount be your total running distance of 9 miles. We have already established that $\frac{1}{3}$ of your running distance is 3 miles, $\frac{2}{3}$ of your running distance is 6 miles, and $\frac{3}{3} = 1$ of your running distance is 9 miles (this last one should be obvious once we have understood the definition of fraction, as 1 of the whole amount is always the whole amount). Now, what distance would be $\frac{4}{3}$ of your running distance? This should be easy to visualize now. We have already seen the 9 miles grouped in thirds as below:

$$(mmm)(mmm)(mmm).$$

All we need to do now is add one more third to the distance to obtain:

$$(mmm)(mmm)(mmm)(mmm).$$

Now you can count for yourself and see that $\frac{4}{3}$ of your running distance would be 12 miles.

As you can see, it is fairly easy to understand thirds of a total amount that is easily divided into 3 equal groups. What about fourths? What is $\frac{1}{4}$ of your total running distance of 9 miles? In order to solve this problem, we need to *subdivide* the miles into fractional parts themselves, in order to obtain 4 equal groups giving us the total distance of 9 miles. This is where whole amounts become extremely important. Now, for reasons that will hopefully become clear as the book progresses, we should divide each mile into 4 equal parts, I will visualize these parts using the symbol $*$. So, in our visualization we have $m = ****$.

Let's pause here and note that the symbol $*$ stands for one equal part of four, which together make one mile. So, by definition of fraction, if we use one mile as the whole amount, we see that $*$ is equal to $\frac{1}{4}$ *of a mile*. Note that $\frac{1}{4}$ of a mile is different from our goal which is $\frac{1}{4}$ of your total running distance (of 9 miles). This is where keeping track of whole amounts, and clearly understanding the whole amount corresponding to all fractions matter, it might be helpful to label all fractions you use with a whole amount, at least to start with, to make sure you don't get confused or lose track of which fractions refer to which whole amounts.

Back to the task at hand, we can now visualize our total running distance of 9 miles:

$$mmmmmmmmm$$

in terms of our subdivisions $m = ****$ as follows:

$$mmmmmmmmm = (****)(****)(****)(****)(****)(****)(****)(****)(****).$$

Now that we have subdivided the total running distance in this way, it is now possible to put your total running distance of 9 miles into 4 equal groups, each equal to $\frac{1}{4}$ of your total running distance as follows:

$$mmmmmmmmm = (*********)(*********)(*********)(*********).$$

In this way, we can see, from our definition of fraction, that $\frac{1}{4}$ of your total running distance of 9 miles can be visualized as: $(*********)$. If we now recall that the distance corresponding to $*$ is equal to $\frac{1}{4}$ of a mile, we now see that $\frac{1}{4}$ of your total running distance is equal to nine parts of size $*$ which is equal to $\frac{1}{4}$ of mile, and thus, $\frac{1}{4}$ of your total running distance of 9 miles, is equal to $\frac{9}{4}$ *of a mile*. From here we can easily determine that $\frac{2}{4}$ of the total running distance is $(*********)(*********)$ which is equal to $\frac{18}{4}$ of a mile. Similarly, $\frac{3}{4}$ of the total running distance is $(*********)(*********)(*********)$ which is equal to $\frac{27}{4}$ of a mile. You can count the $*$ to be sure, by regrouping in tens to determine how many $*$ we have in our decimal system:

$$(*********)(*********)(*********) = (**********)(**********)*******$$

Thus, we have two group of ten $*$ and seven single $*$ equalling 27 $*$, and since each $*$ is equal to $\frac{1}{4}$ of a mile, we see that the above distance is equal to $\frac{27}{4}$ of a mile, which is in turn $\frac{3}{4}$ of your total running distance of 9 miles. Do you see why whole amounts are important? Please take the time to digest this example, as it draws together many of the concepts of this section.

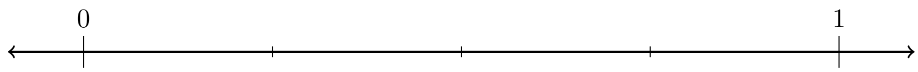
If you do the relevant exercise below, you should find that $\frac{4}{4}$ of the running distance is equal to $\frac{36}{4}$ miles, four groups of nine $*$, which you can then regroup into three groups of ten $*$ and six single $*$. Recall, however, that $\frac{4}{4} = 1$ of the running distance, so it must now be true that

$$\frac{36}{4} = 9 \text{ miles.}$$

This is a first example of *simplifying fractions* which we will discuss more in later chapters.

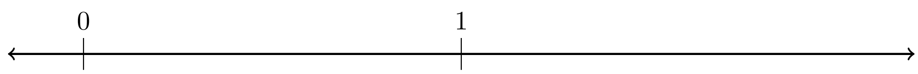
Exercises

EXERCISE 2.1.1. We could also divide the whole number 1, and the interval from 0 to 1 into four equal parts, obtaining fourths, written $\frac{1}{4}$. Complete the number line below:



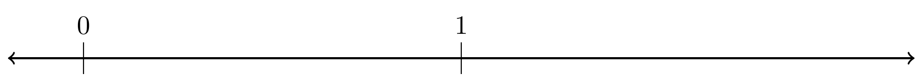
Why is it true that $\frac{4}{4} = 1$?

Exercise 2.1.2. Subdivide the number line below using the default whole amount to plot the improper fraction $\frac{7}{4}$ as accurately as you can by hand.



Exercise 2.1.3. Given the whole amount of your running distance of 9 miles as discussed in the example above, use the definition of fraction and determine $\frac{4}{4}$ of your total running distance, and then determine $\frac{5}{4}$ of your total running distance. Draw pictures as above, and show the groupings both in fourths of the running distance, and in tens to count the number of $*$ required. In this way, it should be easier for you to visualize the fractions involved.

Exercise 2.1.4. Subdivide the number line below using the default whole amount to plot the fractions below as accurately as you can by hand.



(a) Plot $\frac{5}{6}$.

(b) Plot $\frac{6}{6}$. Can you simplify this fraction?

(c) Plot $\frac{7}{6}$.

(d) Plot $\frac{11}{6}$. Label the segments on the number line to illustrate the numerator 11.

Exercise 2.1.5. You are making cookies for a large party, and the recipe calls for 14 cups of flour. Use the whole amount of 14 cups of flour to complete the parts below. Use the visualization 14 cups is equal to *cccccccccccccc*, and subdivide as needed to complete the parts below.

(a) Oh no! Half of the invited guests said they won't be attending the party! Find $\frac{1}{2}$ of the flour in the recipe, state your answer in cups or fractions of a cup.

(b) Oh no! Four-sevenths of the invited guests said they won't be attending the party! Find $\frac{3}{7}$ of the flour in the recipe, state your answer in cups or fractions of a cup.

(c) Oh no! Two-thirds of the invited guests said they won't be attending the party! Find $\frac{1}{3}$ of the flour in the recipe, state your answer in cups or fractions of a cup. (Hint: subdivide a cup into three equal parts. Regroup your subdivisions into tens to explain your numerator.)

(d) Oh no! One-third of the invited guests said they won't be attending the party! Find $\frac{2}{3}$ of the flour in the recipe, state your answer in cups or fractions of a cup. Regroup your subdivisions into tens to explain your numerator.

(e) Find $\frac{3}{3}$ of the flour in the recipe, state your answer in cups or fractions of a cup. Regroup your subdivisions into tens to explain your numerator. Can you simplify the fraction in your answer by reasoning about fractions?

Exercise 2.1.6. T'Challa runs a distance of 30 miles. Natasha runs $\frac{3}{4}$ of T'Challa's running distance. Use reasoning about fractions, and draw a visualization in symbols or on the number line to determine how far Natasha runs.

Exercise 2.1.7. A recipe for pizza calls for 1 cup of cheese and 4 cups of flour. You check the refrigerator and find that you only have $\frac{2}{3}$ of a cup of cheese. Find how much flour you will need for the reduced recipe by reasoning about fractions and drawing a visualization.

Exercise 2.1.8. Let's try to think backwards. Steve runs 16 miles, but that is only $\frac{3}{5}$ of the distance that Wanda runs. Use reasoning about fractions and a visualization to find how far Wanda runs in miles. Be sure to keep track of the whole amounts for every fraction involved.

Exercise 2.1.9. Use our definition of fraction to explain why $\frac{0}{3} = 0$, $\frac{0}{4} = 0$, and indeed, that $\frac{0}{n} = 0$ for any positive denominator n.

2.2. Simplifying fractions, equivalent fractions

Many of you already know, or at least have already heard about such equations as $\frac{1}{2} = \frac{2}{4}$, or $\frac{6}{8} = \frac{3}{4}$. We are now equipped with a clear idea of what fractions mean, and can therefore grasp the reason for these equations in terms of our definition. For example, if we say that $\frac{1}{2} = \frac{2}{4}$, we are saying that if you divide a whole amount into two equal parts and take one of them (this is $\frac{1}{2}$ by definition), this will give you the same value as dividing that same whole amount into four equal parts and taking two of them (this is $\frac{2}{4}$ by definition). Let's see this in a particular example.

Tony is 72 inches tall, while Peter is only $\frac{1}{2}$ as tall as Tony. How tall is Peter? In order to see this we need to divide 72 inches into two equal groups. This will be easier to see if we go back to our object abbreviations for base-10. In terms of those symbols, we have $72 = XXXXXXX\,|\,|$. If we want to put this amount into two equal groups, we can start by putting the tens into two equal groups, as large as possible: $XXXXXXX = (XXX)(XXX)X$. As you see, we obtain two groups of three tens (symbol X), and one ten left over. We now subdivide the leftover ten $X = |\,|\,|\,|\,|\,|\,|\,|\,|\,|$, and together with the remaining two inches, we now have

$| | | | | | | | | | | |$ inches left to put into two equal groups: $(| | | | | |)(| | | | | |)$, in this case, two groups of six works. Now we put these two groups together with the two groups of tens we already found, and we obtain:

$$XXXXXXX\,| | = [(XXX)\,| | | | | |][(XXX)\,| | | | | |]$$

where we have used square brackets to outline the two groups. Note that the number of miles is exactly the same on both sides, we have just regrouped them into two equal groups. This method is much quicker for larger numbers than writing out seventy-two single objects and trying to put them in two equal groups by hand. From the above, we now see that Peter's height, which is $\frac{1}{2}$ of Tony's height (the whole amount), is equal to $(XXX)\,| | | | | |$ or 36 inches.

Continuing the above example. Shuri's height is $\frac{2}{4}$ of Tony's height. Let's now try to group the whole amount 72 inches into four equal groups to determine Shuri's height. Similarly, we use the object abbreviations, starting with the tens: $XXXXXXX = (X)(X)(X)(X)XXX$, we are only able to put one group of ten in each of the four groups, and we have three tens leftover, now we need to subdivide what remains into single inches and put them in four equal groups: $XXX\,| | = (| | | | | | | | | |)(| | | | | | | | | |)(| | | | | | | | | |)\,| |$. This may require some thought, but we can regroup the objects as follows:

$$(| | | | | | | | | |)(| | | | | | | | | |)(| | | | | | | | | |)\,| | =$$
$$(| | | | | | | |)(| | | | | | | |)(| | | | | | | |)(| | | | | | | |).$$

Be sure you agree that we have the same number of objects (representing inches) on each side of the equation above. Thus, we put the 72 inches (or whole amount) into four equal groups of $X\,| | | | | | | |$:

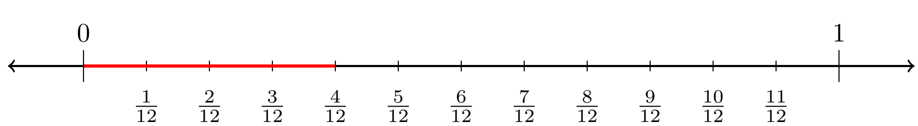
$$XXXXXXX\,| | = (X\,| | | | | | | |)(X\,| | | | | | | |)(X\,| | | | | | | |)(X\,| | | | | | | |).$$

To find Shuri's height, we take two of these groups to obtain $\frac{2}{4}$ of the whole amount (Tony's height):

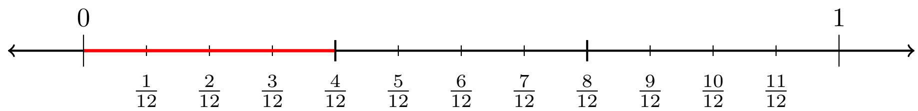
$$(X\,| | | | | | | |)(X\,| | | | | | | |) = XX(| | | | | | | | | |)\,| | | | | | = XXX\,| | | | | |$$

where we regroup according to the base-10 system to find that Shuri's height is 36 inches as well! Note that since we obtained the same height for Peter and Shuri, it must be true that $\frac{1}{2}$ of Tony's height is the same as $\frac{2}{4}$ of Tony's height, and therefore we see why $\frac{1}{2} = \frac{2}{4}$ explicitly in terms of our definition!

We may also view equivalent fractions from the point of view of regrouping objects. For example, consider the fraction $\frac{4}{12}$, that is, four parts of size $\frac{1}{12}$ of the whole amount. If we let the line segment from zero to one on the number line, or the number 1, be our whole amount, then we can visualize the fraction $\frac{4}{12}$ by dividing the segment/number into twelve equal parts and taking the first four of them.



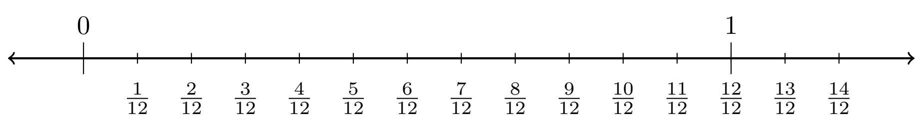
Now, if we group the segment from zero to one into groups of the length of the $\frac{4}{12}$ segment above, we obtain the following:



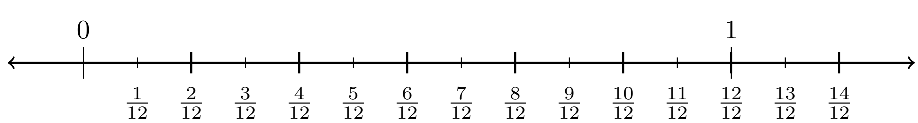
Observe that this divides the segment from zero to one into *three* equal segments of length $\frac{4}{12}$, and therefore by definition, each one has length $\frac{1}{3}$, and finally we see that $\frac{4}{12} = \frac{1}{3}$, since $\frac{4}{12}$ accounts for the first of these segments. As a bonus, we also see that $\frac{8}{12} = \frac{2}{3}$, and $\frac{12}{12} = \frac{3}{3}$, though this last is clear from the fact that $\frac{12}{12} = 1 = \frac{3}{3}$.

We could also see this by taking D to stand for our whole amount, and letting d be $\frac{1}{12}$ of D, so $D = dddddddddddd$. In this way, we can group the d into three equal groups as follows: $(dddd)(dddd)(dddd) = D$. Thus, we see that $(dddd)$ is equal to $\frac{1}{3}$ of the whole amount D, and since it also stands for four parts of size $\frac{1}{12}$ of the whole amount, we obtain the same result $\frac{4}{12} = \frac{1}{3}$. It is good to be able to visualize and interpret fractions both with objects and with the number line. A more common way to visualize fractions is with pieces of pie, I leave that to the reader!

We interpret the equation $\frac{4}{12} = \frac{1}{3}$ as a simplification of the fraction $\frac{4}{12}$, since making the denominator smaller gives us fewer pieces of the whole amount, and therefore a more compact expression. How do we know when we are able to simplify a fraction? Perhaps it would be good to do another example before we make any general statements, let us consider an improper fraction $\frac{14}{12}$. Recall that this means 14 parts of size $\frac{1}{12}$ of the whole amount. We may visualize this on the number line as follows:



Now, if we group the segments above in twos, we obtain *both* a complete and even subdivision of the interval from zero to one (our whole amount) *and* a subdivision of the interval from zero to $\frac{14}{12}$ (our given fraction). Any time this is possible, we should be able to simplify the fraction. We will illustrate the subdivision below with tick marks.



You should see that now the segment from zero to one (our whole amount) is divided by the larger tick marks into *six* equal parts (this shows, for example, that $\frac{2}{12} = \frac{1}{6}$). Furthermore, the segment from zero to $\frac{14}{12}$ is equal to seven of these larger segments, and therefore $\frac{14}{12} = \frac{7}{6}$. The takeaway here is that to simplify a fraction $\frac{m}{n}$ we need to be able to simultaneously regroup the n parts making up the whole amount, and the m of those parts making up the fraction $\frac{m}{n}$ using the same regrouping. We can also visualize this with objects as follows: $D = dddddddddddd$ is our whole amount, so $\frac{1}{12}$ of the whole amount is represented by d, and $\frac{14}{12}$ of the whole amount is represented by $\frac{14}{12}$ of $D = dddddddddddddd$. We can then regroup in twos, to obtain, simultaneously: $D = (dd)(dd)(dd)(dd)(dd)(dd)$ and $\frac{14}{12}$ of $D = (dd)(dd)(dd)(dd)(dd)(dd)(dd)$. This is another way to see that (dd) is equal to $\frac{1}{6}$ of the whole amount D, and that $\frac{14}{12}$ of D is also equal to $\frac{7}{6}$ of D.

Exercises

Exercise 2.2.1. Continuing the above example, Maria's height is $\frac{2}{3}$ of Tony's height of 72 inches, while Monica's height is $\frac{4}{6}$. Show explicitly using object abbreviations and our definition of fraction that Maria and Monica are the same height, and use that to conclude that $\frac{2}{3} = \frac{4}{6}$.

Exercise 2.2.2. Agatha has \$60. Scott has $\frac{3}{4}$ of the amount that Agatha has, while Hope has $\frac{9}{12}$ of that same amount. Use base-10 object abbreviations to illustrate why Scott and Hope have the same amount of money, and conclude that $\frac{3}{4} = \frac{9}{12}$.

Exercise 2.2.3. Write your own word problem, and write its solution, to explain why $\frac{2}{5} = \frac{4}{10}$. Going back to decimals, this means that $0.4 = \frac{2}{5}$ as numbers.

Exercise 2.2.4. We can also view the content of this section as a way to *complicate* (the opposite of simplify) fractions, or to make the denominator *larger*. For example, if we look at $\frac{1}{2}$ we can expand the two equal parts into five subparts, so if W is the whole amount, then $W = hh$ where h is $\frac{1}{2}$ of the whole amount, then we let $h = ttttt$, and so finally $W = hh = (ttttt)(ttttt)$. In this way we see that t is $\frac{1}{10}$ of W, and also that h is $\frac{5}{10}$ of W, therefore $\frac{1}{2} = \frac{5}{10} = 0.5$. Use a similar idea to explain why $\frac{1}{5} = \frac{2}{10} = 0.2$.

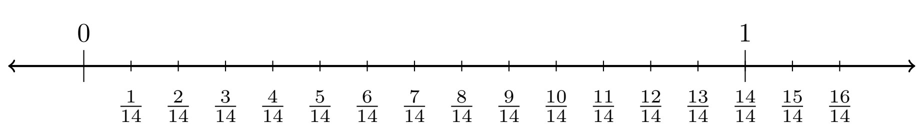
Exercise 2.2.5. We have already observed in previous sections that $0.1 = 0.10$, illustrate this in fraction form $\frac{1}{10} = \frac{10}{100}$ using object abbreviations.

Exercise 2.2.6. For a bit more of a challenge, try subdividing into very small pieces to explain why $\frac{1}{4} = \frac{25}{100} = 0.25$.

Exercise 2.2.7. Consider the improper fraction $\frac{16}{14}$.

(a) Use simple regrouping of object abbreviations to show that $\frac{2}{14} = \frac{1}{7}$.

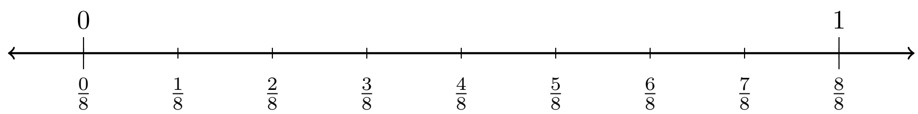
(b) Regroup the subdivisions on the number line below to show why $\frac{16}{14} = \frac{8}{7}$.



(c) Now, continuing the above number line, can you simplify $\frac{18}{14}$?

2.3. Order of fractions, common denominators, other comparison methods

We would now like to come up with some techniques for comparing fractions in order on the number line. Note that if two fractions have the same denominator, such as $\frac{6}{8}$ and $\frac{7}{8}$, we can easily see that $\frac{6}{8} < \frac{7}{8}$ since they are both made up of pieces of the same size, but 6 pieces of size $\frac{1}{8}$ of the whole amount is smaller than 7 pieces of size $\frac{1}{8}$ of the whole amount. So, essentially, we can use the fact that $6 < 7$, since all we did was change what 6 and 7 were counting (whole ones versus parts of size $\frac{1}{8}$ of the whole amount), but 6 things is always less than 7 things. We can also see this on the number line below.

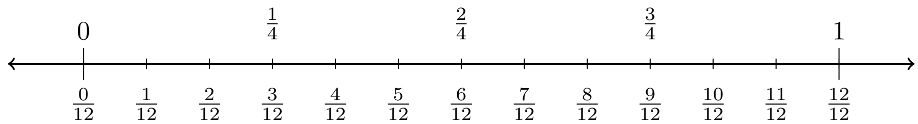


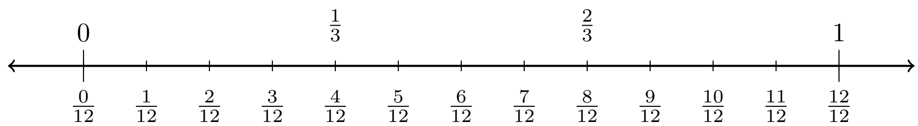
Observe that $\frac{6}{8}$ is to the left of $\frac{7}{8}$ on the number line, so $\frac{6}{8} < \frac{7}{8}$ also fits with our earlier definition of order in terms of the number line.

Given the above exercise, it should also be true that $\frac{3}{4} < \frac{7}{8}$, this one is not so obvious from the definitions, as without some work or explanation, we don't know exactly how pieces of size $\frac{1}{4}$ relate to pieces of size $\frac{1}{8}$. This shows immediately why common denominators (having denominators in common - be sure you don't go on autopilot with this phrase, it just means their respective parts are the same size) are useful in comparing fractions, once the parts are the same size, we only have to compare how many of them we have, which is just a comparison of whole numbers. This is fairly easy to do using methods already mentioned.

How do we obtain common denominators? Let's do an example: say we want to compare $\frac{3}{4}$ and $\frac{2}{3}$. We can accomplish this by finding a common subdivision for each of the two fractions as follows. Let W be the whole amount, so $W = ffff$ where f stands for one part of size $\frac{1}{4}$ of the whole amount, and $W = ttt$ where t stands for one part of size $\frac{1}{3}$ of the whole amount. Here is a nice trick to create a common subdivision (the switch trick): we subdivide the fourths $f = ppp$ into three equal parts, and we subdivide the thirds $t = qqqq$ into four equal parts. For reasons that will become clear later in the book, these give us pieces of the same size (i.e. $p = q$) which we can see if we just look at each subdivision in terms of the whole amount: $W = ffff = (ppp)(ppp)(ppp)(ppp)$ and $W = ttt = (qqqq)(qqqq)(qqqq)$, simply by counting (use base-10 reorganization to show how to count the parts in base-10: $(pppppppppp)pp = (qqqqqqqqqq)qq$ we have that p and q both represent equal parts of size $\frac{1}{12}$ of the whole amount (it takes 12 of p, or 12 of q to equal the whole amount W). Now that we have this common subdivision, which we will simply call p from now on to get rid of excess letters, we can interpret our two given fractions in terms of this subdivision, and compare them. We see that $\frac{3}{4}$ of W is $fff = (ppp)(ppp)(ppp)$ which is $\frac{9}{12}$ of W. We also see that $\frac{2}{3}$ of W is $tt = (pppp)(pppp)$ which is $\frac{8}{12}$ of W. It is now clear by simple counting that $\frac{9}{12} > \frac{8}{12}$, and therefore $\frac{3}{4} > \frac{2}{3}$.

The number lines below show a different way of visualizing the same idea, we simply divide segments on the number line instead of using object abbreviations.



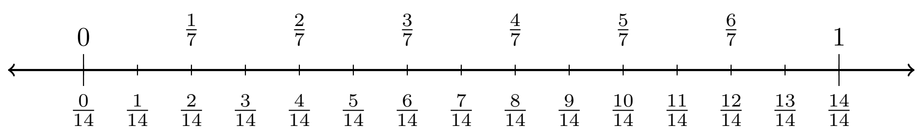


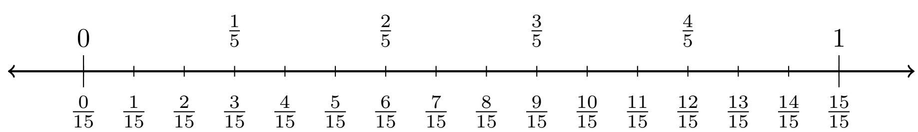
The number lines clearly show that $\frac{3}{4}$ is to the right of $\frac{2}{3}$, but we can also see an illustration of another important idea for comparing fractions: the larger the denominator, the smaller the parts. For example, if we want to compare $\frac{1}{4}$ and $\frac{1}{3}$, we can simply observe, continuing the above object abbreviations: $W = ffff = ttt$, so it simply must be true that f represents a smaller amount than t, because it takes four fs to get W, whereas it only takes three ts to get W. Thus, $\frac{1}{4} < \frac{1}{3}$, which you can also see using the number lines above.

Thus, if we have a common *numerator*, we can also compare fractions, since in this case the number of parts is equal, so the fraction with the larger parts must be larger. For example, we know right away that $\frac{10}{4} < \frac{10}{3}$, since $\frac{1}{4} < \frac{1}{3}$, and therefore ten parts of size $\frac{1}{4}$ of the whole amount must be smaller than ten parts of size $\frac{1}{3}$ of the whole amount. We can also see this on a number line, as we count up to $\frac{10}{4}$ by segments of length $\frac{1}{4}$, which is smaller than segments of length $\frac{1}{3}$, used to count up to $\frac{10}{3}$, and thus $\frac{10}{4}$ must be to the left of $\frac{10}{3}$ on the number line, and therefore $\frac{10}{4} < \frac{10}{3}$.

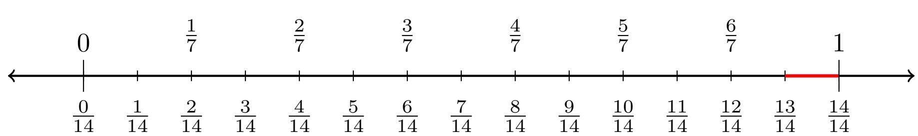
Let's compare the fractions $\frac{2}{5}$ and $\frac{3}{7}$ using common numerators. We can start with objects, let W be the whole amount, and so $W = fffff$ where f is one part of size $\frac{1}{5}$ of W. Similarly, $W = sssssss$ where s is one part of size $\frac{1}{7}$ of W. We now employ the switch trick, but we use the *numerators* 2 and 3 rather than the denominators as we did above. So, we subdivide the f into three equal parts $f = ttt$, and we subdivide the s into two equal parts $s = hh$. Thus, we can express our two given fractions as follows: $\frac{2}{5}$ of W is given by $ff = (ttt)(ttt)$, and $\frac{3}{7}$ of W is given by $sss = (hh)(hh)(hh)$. We see that each fraction is given by *six* parts, but not necessarily of the same size, we need to determine the size of t and of h. We do this by comparing each of them to the whole amount W as follows: $W = fffff = (ttt)(ttt)(ttt)(ttt)(ttt)$ and $W = sssssss = (hh)(hh)(hh)(hh)(hh)(hh)(hh)$. By counting, you can see that 15 of t gives us W, and therefore t represents $\frac{1}{15}$ of the whole amount, whereas it takes 14 of h to give us the whole amount W, so h represents $\frac{1}{14}$ of the whole amount. Finally, we now have that $\frac{2}{5} = \frac{6}{15}$, and $\frac{3}{7} = \frac{6}{14}$, and since $\frac{1}{14} > \frac{1}{15}$, we now conclude that $\frac{6}{14} > \frac{6}{15}$, and in turn: $\frac{3}{7} > \frac{2}{5}$. Observe that this method is equally useful for comparing fractions, it always works! The primary reason it is rarely taught is because common denominators have other uses in arithmetic that common numerators do not. The more methods you have for understanding fractions and how to compare them, the better!

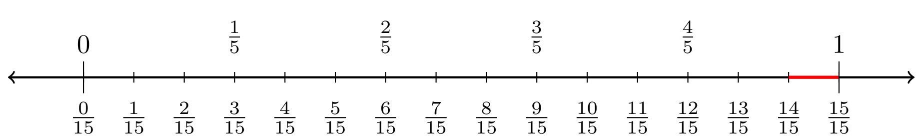
To illustrate common numerators on the number line, we do essentially the same thing as we did with objects.





The segment from zero to $\frac{6}{15}$ is clearly shorter than the segment from zero to $\frac{6}{14}$. These number lines can also help us understand a final way of comparing fractions: benchmarking. Note the fractions $\frac{13}{14}$ and $\frac{14}{15}$ in the above number lines. You see that $\frac{13}{14}$ is exactly $\frac{1}{14}$ distance to the *left* of 1, and that $\frac{14}{15}$ is exactly $\frac{1}{15}$ distance to the *left* of 1. Since it is the same number 1 in both cases, and since the distance $\frac{1}{14}$ is larger than the distance $\frac{1}{15}$, we see that $\frac{14}{15}$ is closer to 1 from the left side than $\frac{13}{14}$ is.





By this common comparison to the number 1, we see that $\frac{13}{14}$ must be to the left of $\frac{14}{15}$ on the number line, and so by our previous definition: $\frac{13}{14} < \frac{14}{15}$. Using similar reasoning, we can use the same number lines above to conclude that $\frac{6}{7} > \frac{4}{5}$. Observe that if we continue to the *right* of 1, we can also compare improper fractions.

Exercises

EXERCISE 2.3.1. Use the number line to explain why $\frac{6}{8} = \frac{3}{4}$.

EXERCISE 2.3.2. Compare $\frac{2}{5}$ and $\frac{3}{7}$ using common denominators (illustrate with objects or number lines), and make sure you get the same result as we did above.

EXERCISE 2.3.3. Use the definition of fraction to explain why $\frac{12}{17} > \frac{10}{17}$.

EXERCISE 2.3.4. Use the switch trick and object abbreviations to compare the fractions $\frac{3}{4}$ and $\frac{4}{5}$ using common denominators.

EXERCISE 2.3.5. Use the switch trick and object abbreviations to compare the fractions $\frac{4}{6}$ and $\frac{5}{9}$ using common numerators.

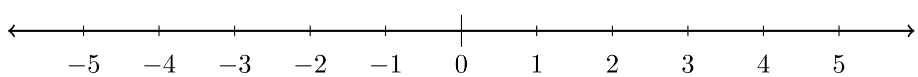
EXERCISE 2.3.6. Use a number line and benchmarking to explain why $\frac{15}{14} > \frac{16}{15}$. Pay careful attention to how each fraction relates to the number 1 on the number line. These fractions will be to the *right* of 1 on the number line.

EXERCISE 2.3.7. What is wrong with the following reasoning? *Since* $3 < 5$, *and* $4 < 6$, *we have* $\frac{3}{4} < \frac{5}{6}$. (Do not just compare the fractions, use the definitions of fraction and order to explain why this is not a valid way to compare fractions.)

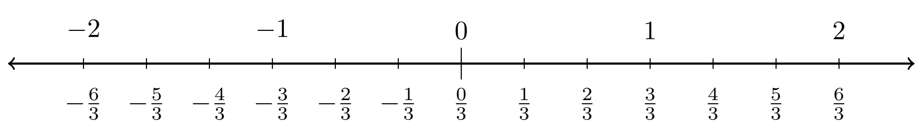
EXERCISE 2.3.8. Now compare the fractions $\frac{3}{4}$ and $\frac{5}{6}$ using a method from this section: common denominators, common numerators, or benchmarking (or all three!).

2.4. Negative fractions

So far we have only talked about positive fractions (to the right of zero on the number line). Luckily, if you understand how negative whole numbers are represented on the number line, this will be a natural extension of that. Recall that negative values on the number line count up to the left as below.



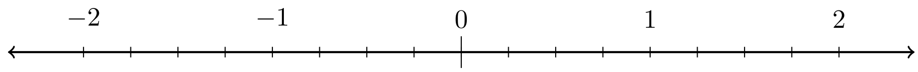
Similarly, given a denominator, such as 3, we divide up the interval from 0 to -1 into three equal parts as well, but we count negative thirds up to the left, so we get the following.



As you can see, this number line arrangement means that, for example, $-\frac{5}{6} < -\frac{1}{6}$, because $-\frac{5}{6}$ is to the left of $-\frac{1}{6}$ on the number line. This can be counterintuitive, but no more so than the related fact that $-5 < -1$.

Exercises

EXERCISE 2.4.1. Fill in the number line below with positive and negative *fourths* (use denominator 4).



EXERCISE 2.4.2. Fill in the blanks, and draw a number line to illustrate each statement:

(a) $-\frac{2}{3} < -\frac{1}{3}$ because ________ is to the ________ of ________ on the number line.

(b) $-\frac{5}{4} < -1$ because ________ is to the ________ of ________ on the number line.

(c) $-\frac{1}{5} > -\frac{1}{3}$ because ________ is to the ________ of ________ on the number line.

(d) $-2 > -\frac{13}{6}$ because ________ is to the ________ of ________ on the number line.

(e) $-2 < \frac{1}{3}$ because ________ is to the ________ of ________ on the number line.

Part 2

Operations

CHAPTER 3

Addition

3.1. Addition and subtraction of positive whole numbers

Once we have understood the basic structure of whole numbers, decimals, and fractions, it is time to combine them in various ways. Operations are essentially ways of combining two (or more) numbers and producing a new number. The simplest common operation is addition. We can start by thinking about what happens when we need a total. For example, say you start with \$5 and then you find \$6 on the ground, how much do you have now? The answer is, by definition of addition, you have $5 + 6$ dollars (try to avoid the need to calculate this sum until we have a reason to, the purpose here is the meaning of addition, we will talk about calculation shortly).

Definition 3.1.1. Given *positive whole* numbers m and n, we define $m + n$ to be the number of objects you have if you *start* with m objects, *and then* get n more objects. The numbers m and n are called **terms**, and the number $m + n$ is called the **sum** of m and n.

If we want to calculate the result of a simple sum, the most intuitive thing to do is write down object abbreviations for both terms, then combine them and regroup according to the Base-10 system. For example, the sum $5 + 6$ can be calculated using $5 = |\,|\,|\,|\,|$ and $6 = |\,|\,|\,|\,|\,|$, so together we have $5+6 = |\,|\,|\,|\,| + |\,|\,|\,|\,|\,| = (|\,|\,|\,|\,|\,|\,|\,|\,|\,|)\,| = X\,| = 11$.

We can define subtraction for positive whole numbers in a similar way. Say you have \$15 and you spend \$8 on gas for your car, how much money do you have left? The answer is, by definition of subtraction, you have $15 - 8$ dollars.

Definition 3.1.2. Given *positive whole* numbers m and n, such that $m > n$, then $m - n$ is defined to be the number of objects you have if you *start* with m objects, *and then* take away n objects. The numbers m and n are again called **terms**, and the number $m - n$ is called the **difference** between m and n.

Note carefully that, for this definition, the difference $m - n$ only makes sense if $m > n$. We will expand our definitions later to allow for other possibilities. We can calculate simple differences by the method above, using object abbreviations, but this time we remove objects. For example, to calculate $15 - 8$ we use $15 = X\,|\,|\,|\,|\,|$, and $8 = |\,|\,|\,|\,|\,|\,|\,|$. First we expand $15 = X\,|\,|\,|\,|\,| = |\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|$, then we want to take away $8 = |\,|\,|\,|\,|\,|\,|\,|$, so we get $15 - 8 = |\,|\,|\,|\,|\,|\,|\,\cancel{|}\,\cancel{|}\,\cancel{|}\,\cancel{|}\,\cancel{|}\,\cancel{|}\,\cancel{|}\,\cancel{|} = |\,|\,|\,|\,|\,|\,| = 7$. Observe that, by definition, we also see from this calculation that $7 + 8 = 15$. This reflects a general relationship between addition and subtraction that is important to clarify.

Theorem 3.1.3. *If a, b, and c are positive whole numbers and $a + b = c$, then $c - a = b$ and $c - b = a$.*

Similarly, if a*,* b*, and* c *are positive whole numbers so that* $a > b$ *and* $a - b = c$*, then* $b + c = a$ *and* $c + b = a$*.*

Since the theorem is phrased abstractly, I want to point out how it applies to our examples above. We already calculated $5 + 6 = 11$ above, so if you have 5 objects, and then get 6 more objects, you now have 11 objects. It follows easily that if you start with 11 objects, and take away 6 of them, you now have 5 objects: $11 - 6 = 5$. Similarly, if you start with 11 objects and take away 5 of them, you must have 6 objects left: $11 - 5 = 6$. This illustrates the first statement in the theorem.

Continuing with our other example, we calculated that $15-8 = 7$, so if you have 15 objects and take away 8 of them, you now have 7 objects. It stands to reason that if you start with 7 objects and then get 8 objects, you now have 15 of them: $7 + 8 = 15$. Similarly, with a bit more thought, it should be clear that if you start with 8 objects and then get 7 of them, you now have 15 objects: $8+7 = 15$. We can illustrate this using colors: $15 = |\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,| = |\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,| = 7+8$ and $15 = |\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,| = |\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,| = 8+7$. Note carefully that if you view the red objects as the objects removed, we can also interpret these object drawings as illustrating the subtractions $15 - 8 = 7$ and $15 - 7 = 8$, so we come full circle. It is extremely helpful going forward in this book if you can flexibly switch point of view between addition and subtraction as needed.

Consider the following word problem: *Loki has* 8 *apples. He gets* 5 *more apples. How many does he have now?* We can represent this word problem as an equation involving addition, using the definition above $8 + 5 =?$. Note that the unknown quantity is represented by a question mark. Now consider the slightly different word problem: *Loki had* 8 *apples. He gets some more apples, and now he has* 13 *apples. How many did he get?* We can represent this word problem using the following equation $8+? = 13$. In this case, observe that we can use our flexibility developed above to change this into a subtraction problem: $13 - 8 =?$. Here we are not concerned about the answer to the problem, but about how we can first translate it into an equation using the definitions, then about how we can rewrite that equation to make the problem easier to solve.

We can also write word problems that naturally translate into subtraction equations: *Loki has* 8 *apples. He eats* 5 *of them. How many does he have now?* The equation representing this word problem is $8-5 =?$. Changing the problem slightly: *Loki has* 8 *apples. He eats some of them, and now he has* 3 *apples. How many did he eat?* The equation representing this word problem is $8-? = 3$. From our discussion above, we can rewrite this equation in addition form: $3+? = 8$, and then we can rewrite it again as a simple subtraction: $8-3 =?$, making it straightforward to solve. Note again that we are **not interested in solving the equation**, it is just as important, if not more so, to understand how to convert a statement in words into some sort of mathematical equation or expression, as it is to actually solve that equation. The solving of equations is sometimes over-emphasized leading to a severe gap in translation skills, and a widespread fear of word problems.

Word problems are the most common way humans encounter problems that can be solved using math, going back to the origins of counting. There would be no math at all if it were not for word problems, so learn to appreciate them! The abstract math is easier because it is a distillation of concepts arising from practical situations.

Exercises

EXERCISE 3.1.1. Use object abbreviations and regrouping in Base-10 to calculate the sum $6 + 7$.

EXERCISE 3.1.2. Consider this word problem: *Loki had some apples. He gets* 5 *more apples, and now he has* 13 *apples. How many apples did he start with?* Translate this into an equation, then rewrite the equation using subtraction to make it easier to solve. Do not solve the equation.

EXERCISE 3.1.3. Consider this word problem: *Loki had some apples. He eats* 5 *of them, and now he has* 3 *apples. How many did he start with?* Translate this into an equation, then rewrite the equation to make it easier to solve. Do not solve the equation.

EXERCISE 3.1.4. Use object abbreviations and regroup/expand as needed to calculate the sums and differences in Base-10.

(a) $9 + 4$

(b) $9 - 4$

(c) $14 - 8$

(d) $14 - 6$

(e) $13 + 12$

EXERCISE 3.1.5. Consider the equation: $9 + 6 = 15$. Rewrite the equation in two different ways using subtraction. Illustrate using object abbreviations.

EXERCISE 3.1.6. Consider the equation: $17 - 8 = 9$. Rewrite the equation using addition. Then, rewrite the equation using a different subtraction. Illustrate using object abbreviations.

EXERCISE 3.1.7. Consider the word problems below, translate each into an equation using the definitions of addition and subtraction. Then, if needed, rewrite the equations to make them easier to solve. Do not solve the equations.

(1) Nebula has 13 apps on her smartphone. She installs 6 more apps. How many apps does she have now?

(2) Nebula has 13 apps on her smartphone. She installs some more apps and now she has 19 apps. How many apps did she install?

(3) Nebula had some apps on her smartphone. She installs 6 more apps, and now she has 19 apps. How many apps did she start with?

(4) Nebula has 13 apps on her smartphone. She deletes 6 apps. How many apps does she have now?

(5) Nebula has 13 apps on her smartphone. She deletes some apps, and now she has 7 apps. How many apps did she delete?

(6) Nebula has some apps on her smartphone. She deletes 6 apps, and now she has 7 apps. How many apps did she start with?

Exercise 3.1.8. Write a word problem that naturally translates into the equations below using the definitions of addition and subtraction. Try to use the same situation, but change the wording of the problem to produce the variations.

(a) $7 + 4 =?$

(b) $7+? = 11$

(c) $? + 4 = 11$

(d) $7 - 4 =?$

(e) $7-? = 3$

(f) $? - 4 = 3$

3.2. Commutative and Associative Properties, the web of addition

We are now ready to develop some basic properties of addition, commonly used and intuitively clear, and along the way we will encounter ideas about learning single digit addition facts, such as $5 + 6 = 11$ as a part of a web of facts, rather than as individual bits of information to be memorized. The purpose of this is to minimize the amount of information you need to commit to memory to be able to do simple addition problems in your head reliably. As a bonus, we will also have a deeper understanding of addition!

3.2.1. The Associative Property of Addition. Since this property seems intuitively obvious to most students, I want to explain why it has content. Recall that we have defined $m + n$ as the amount of objects you have if you start with m objects, and then get n more objects. Note that the expression $m + n + p$ does not have a single meaning under this definition, there are two ways we can read the expression: $(m + n) + p$, which means we start with m objects, then get n more objects, then finally get p further objects, or $m+(n+p)$ which means we start with m objects, and then get $n + p$ more objects, and in turn, $n + p$ objects means we (separately) start with n objects and then get p more objects.

All this abstraction is much more convoluted than the underlying idea, let's look at an example: *Drax has* \$2*, he first gets* \$3 *from Groot, and then gets* \$4 *from Bucky. How much money does Drax have?* Again, resist the urge to calculate the answer, the point here is understanding the definition of addition. What sum does this problem give us if we follow the definition? In this case it would be:

$$(2 + 3) + 4$$

dollars. Now, consider the slightly different example: *Drax has* \$2*. Groot starts with* \$3*, then gets* \$4 *from Bucky and finally gives all he has to Drax. How much money does Drax have?* First note that, before he gives what he has to Drax, Groot has $3+4$ dollars by definition. After Drax receives Groot's money, he will have:

$$2+(3+4)$$

dollars. These two scenarios are clearly different, and hopefully they more clearly illustrate the difference than the abstract description given above. You may want to reread that now, and perhaps it will be clearer what the point is.

At this stage, we are now ready to explain the associative property, it says in the context of the example word problems above that *Drax has the same amount of money in the end of both word problems*:

$$(2+3)+4=2+(3+4).$$

We can easily see why this is true using colors and object abbreviations:

$$(|\,|\,{\color{red}|\,|\,|})\,{\color{blue}|\,|\,|\,|} = |\,|\,({\color{red}|\,|\,|}\,{\color{blue}|\,|\,|\,|}).$$

This shows that the two scenarios simply give a different grouping of the same amount of money. If we actually calculate the additions in parentheses, by counting the objects in parentheses for example, we see that the equation becomes:

$$5+4=2+7$$

which clearly shows that the associative property has content, as this equation should not be obvious without calculation, and yet the associative property implies it! We are now ready to state the associative property abstractly.

THEOREM 3.2.1 (Associative Property of Addition). *Let m, n, and p be positive whole numbers, then $(m+n)+p=m+(n+p)$.*

3.2.2. The Commutative Property of Addition. This property seems even more obvious than the previous one, so I will again try to illustrate the content of the property using two slightly different word problems. *Peggy has* 3 *bananas. Then, Hela gives Peggy* 4 *bananas. How many bananas does Peggy have now?* By definition of addition, the answer is $3+4$ bananas (do not calculate!). Now consider the word problem: *Peggy has* 4 *bananas. Then, Hela gives Peggy* 3 *bananas. How many bananas does Peggy have now?* Again by definition, the answer is $4+3$ bananas. Note that these word problems, though similar, are definitely not the same! The commutative property says in this context that Peggy has the same number of bananas at the end of both problems:

$$3+{\color{red}4}={\color{red}4}+3.$$

We can again illustrate this with objects:

$$|\,|\,|\,{\color{red}|\,|\,|\,|} = {\color{red}|\,|\,|\,|}\,|\,|\,|.$$

Note that this property, unlike the associative property, only involves a single addition, and is about the order in which we do the addition. In general, if we start with m objects and then get n more objects, we have the same number of objects as if we start with n objects and then get m more objects, or, to put it more succinctly:

THEOREM 3.2.2 (Commutative Property of Addition). *Let m and n be positive whole numbers, then $m+n=n+m$.*

3.2.3. Application: the web of single digit addition. An important and useful set of facts to know in order to become more fluent with the basic operations of math is the result of adding any two single digit whole numbers. For example, what is $7 + 6$? If you do not know immediately, don't worry, we will now outline a method for calculating such sums, and how to relate them to others. Once you practice this several times, you will find that, not only can you more easily recall all the single digit sums, but you have a much deeper understanding of how they are related. It is extremely important that you **do not use a calculator** for anything in this section, or indeed in this book. A calculator is a tool to save you time doing long calculations, but it should not be a way to avoid learning basic facts.

We will display the additions we want to learn in the array below.

$$\begin{array}{ccccccccc}
1+1 & 1+2 & 1+3 & 1+4 & 1+5 & 1+6 & 1+7 & 1+8 & 1+9 \\
2+1 & 2+2 & 2+3 & 2+4 & 2+5 & 2+6 & 2+7 & 2+8 & 2+9 \\
3+1 & 3+2 & 3+3 & 3+4 & 3+5 & 3+6 & 3+7 & 3+8 & 3+9 \\
4+1 & 4+2 & 4+3 & 4+4 & 4+5 & 4+6 & 4+7 & 4+8 & 4+9 \\
5+1 & 5+2 & 5+3 & 5+4 & 5+5 & 5+6 & 5+7 & 5+8 & 5+9 \\
6+1 & 6+2 & 6+3 & 6+4 & 6+5 & 6+6 & 6+7 & 6+8 & 6+9 \\
7+1 & 7+2 & 7+3 & 7+4 & 7+5 & 7+6 & 7+7 & 7+8 & 7+9 \\
8+1 & 8+2 & 8+3 & 8+4 & 8+5 & 8+6 & 8+7 & 8+8 & 8+9 \\
9+1 & 9+2 & 9+3 & 9+4 & 9+5 & 9+6 & 9+7 & 9+8 & 9+9
\end{array}$$

This may seem intimidating at first (there are 81 additions in the array), but using the properties we have now learned, we can reduce our workload significantly. First, observe that, if you remove the doubles along the diagonal, i.e. $1 + 1$, $2 + 2$, $3 + 3$, $4 + 4$, etc, the table is symmetric, every other has a corresponding entry in the opposite order. For example, we have $6 + 3$ below the diagonal, and we have $3 + 6$ above the diagonal.

$$\begin{array}{ccccccccc}
\color{blue}{1+1} & 1+2 & 1+3 & 1+4 & 1+5 & 1+6 & 1+7 & 1+8 & 1+9 \\
2+1 & \color{blue}{2+2} & 2+3 & 2+4 & 2+5 & 2+6 & 2+7 & 2+8 & 2+9 \\
3+1 & 3+2 & \color{blue}{3+3} & 3+4 & 3+5 & \color{red}{3+6} & 3+7 & 3+8 & 3+9 \\
4+1 & 4+2 & 4+3 & \color{blue}{4+4} & 4+5 & 4+6 & 4+7 & 4+8 & 4+9 \\
5+1 & 5+2 & 5+3 & 5+4 & \color{blue}{5+5} & 5+6 & 5+7 & 5+8 & 5+9 \\
6+1 & 6+2 & \color{red}{6+3} & 6+4 & 6+5 & \color{blue}{6+6} & 6+7 & 6+8 & 6+9 \\
7+1 & 7+2 & 7+3 & 7+4 & 7+5 & 7+6 & \color{blue}{7+7} & 7+8 & 7+9 \\
8+1 & 8+2 & 8+3 & 8+4 & 8+5 & 8+6 & 8+7 & \color{blue}{8+8} & 8+9 \\
9+1 & 9+2 & 9+3 & 9+4 & 9+5 & 9+6 & 9+7 & 9+8 & \color{blue}{9+9}
\end{array}$$

By the commutative property we know that $6 + 3 = 3 + 6$, so if we can just remember one of these, for example $6 + 3 = 9$, then we should automatically know that $3 + 6 = 6 + 3 = 9$ as well. In this way, we cut our workload almost in half, as we can just learn the doubles on the diagonal, and all sums $m + n$ where $m > n$.

$1+1$								
$2+1$	$2+2$							
$3+1$	$3+2$	$3+3$						
$4+1$	$4+2$	$4+3$	$4+4$					
$5+1$	$5+2$	$5+3$	$5+4$	$5+5$				
$6+1$	$6+2$	$6+3$	$6+4$	$6+5$	$6+6$			
$7+1$	$7+2$	$7+3$	$7+4$	$7+5$	$7+6$	$7+7$		
$8+1$	$8+2$	$8+3$	$8+4$	$8+5$	$8+6$	$8+7$	$8+8$	
$9+1$	$9+2$	$9+3$	$9+4$	$9+5$	$9+6$	$9+7$	$9+8$	$9+9$

We have now reduced our workload to 45 additions from 81! The first step is now to learn all sums that add up to a number less than ten:

$1+1$								
$2+1$	$2+2$							
$3+1$	$3+2$	$3+3$						
$4+1$	$4+2$	$4+3$	$4+4$					
$5+1$	$5+2$	$5+3$	$5+4$	$5+5$				
$6+1$	$6+2$	$6+3$	$6+4$	$6+5$	$6+6$			
$7+1$	$7+2$	$7+3$	$7+4$	$7+5$	$7+6$	$7+7$		
$8+1$	$8+2$	$8+3$	$8+4$	$8+5$	$8+6$	$8+7$	$8+8$	
$9+1$	$9+2$	$9+3$	$9+4$	$9+5$	$9+6$	$9+7$	$9+8$	$9+9$

This amounts to the sums (in both orders, of course) shown in magenta above. These are easy to count if you ever forget, for example $5+3 = |\,|\,|\,|\,|\,|\,|\,| = 8$. I do not recommend trying to memorize them directly, simply use them in as many ways as you can (in your daily lives as well!) and you will find that they stick in your head eventually. You may use this array as a tool as well, note that, if you go diagonally up to the right, you get the same sum along this diagonal, for example: $5+1 = 4+2 = 3+3 = 6$. You can also see that these sums are the same by using the associative property. For example, say you know that $5+1=6$, and you want to calculate $4+2$, which you forgot, simply observe that $2 = 1+1$ (I think all of you know this one) and then apply the associative property:

$$4+2 = 4+(1+1) = (4+1)+1 = 5+1 = 6.$$

In general, if two distinct sums have the same value, you should be able to relate them either using the associative property as we just demonstrated, or by using the commutative property as we did above to switch the order of the terms in the sum.

The next step is to learn the main anti-diagonal of sums that 'make-a-ten':

$1+1$								
$2+1$	$2+2$							
$3+1$	$3+2$	$3+3$						
$4+1$	$4+2$	$4+3$	$4+4$					
$5+1$	$5+2$	$5+3$	$5+4$	$5+5$				
$6+1$	$6+2$	$6+3$	$6+4$	$6+5$	$6+6$			
$7+1$	$7+2$	$7+3$	$7+4$	$7+5$	$7+6$	$7+7$		
$8+1$	$8+2$	$8+3$	$8+4$	$8+5$	$8+6$	$8+7$	$8+8$	
$9+1$	$9+2$	$9+3$	$9+4$	$9+5$	$9+6$	$9+7$	$9+8$	$9+9$

These sums are shown in olive above $9+1=8+2=7+3=6+4=5+5=10$. These sums are crucial for learning the remaining sums below the anti-diagonal, which have values greater than ten, and take the longest to memorize, typically. Once you have learned these sums (and the corresponding sums in the opposite order, i.e. $3+7=7+3=10$ as well), you will now use these to help you understand and eventually memorize the remaining entries in the array. We will apply the associative property (and the commutative property if needed), to figure out the remaining sums using those we already know, and always 'making a ten' first.

Let's consider an example: $8+5$. We have already calculated this sum earlier in the text directly, but what we want to do now is relate it to sums we have already learned. First, we look at the term on the left, and we think to ourselves $8+?=10$. Then we remember our entry from the array: $8+2=10$. So, now we want to decompose $5=2+?$, so that we can apply the associative property:

$$8+5=8+(2+?)=(8+2)+?=10+?$$

Note carefully that, in the base-10 system, $10+$ (any single digit number) is very easy, as we simply put the single digit number in the ones place (we only alter values to left of the ones place if we have ten or more, which would require more than one digit!). To illustrate the point: $10+7=17$, $10+4=14$, etc. Or, in Base-10 object abbreviation terms: $10+7=X\,|\,|\,|\,|\,|\,|\,|=17$. All of this shows, that if we can just solve the problem $5=2+?$, then we will immediately know that $8+5=1?$.

To solve the problem $5=2+?$, you can either remember the fact $5=3+2=2+3$ using previous knowledge and the commutative property, or, you can use our previous discussion of the relationship between addition and subtraction to calculate: $5=2+?$ means $5-2=?$. So, if we have 5 objects and remove 2 of them, how many do we have left? $|\,|\,|\,|\,|=|\,|\,|=3$.

Putting it all together, we are now in a position to calculate $8+5$ using the make-a-ten method:

$$8+5=8+(2+3)=(8+2)+3=10+3=13$$

Observe that if we start with an entry above the main diagonal (eliminated from the array using the commutative property) we can always return to the more familiar part of the array using the commutative property. For example, $6+9=9+6$. Then, we recall that $9+1=10$, so we want to decompose $6=1+?$, we find that $6=1+5$, and finally we obtain:

$$6+9=9+6=9+(1+5)=(9+1)+5=10+5=15$$

This last example puts all the properties together! We have only scratched the surface of how these single-digit addition facts are related, but hopefully this gives you enough of a sampling to feel empowered to use these properties to help you learn these facts. Even if you already know how to add single digit numbers in your head, this gives you a glimpse into the connections between the facts you know. Ultimately it is just as important, if not more so, to know how facts are related as it is to know the individual facts themselves.

Exercises

Exercise 3.2.1. Calculate $7 + 5$ using the make-a-ten method. What about $5 + 7$? Can you apply the properties above to calculate this sum as well?

Exercise 3.2.2. Write a word problem whose solution is the sum $(5 + 6) + 7$, according to the definition of addition. Then, write a slightly different word problem whose solution is $5 + (6 + 7)$, according to the definition of addition. Finally, use object abbreviations to show that these two sums are equal $(5+6)+7 = 5+(6+7)$. What property does this illustrate?

Exercise 3.2.3. Write a word problem whose solution is the sum $5 + 4$, according to the definition of addition. Then, write a slightly different word problem whose solution is $4 + 5$, according to the definition of addition. Finally, use object abbreviations to show that these two sums are equal $5 + 4 = 4 + 5$. What property does this illustrate?

Exercise 3.2.4. Explain why $7 + 6 = 6 + 7$ using a property of numbers.

Exercise 3.2.5. Explain why $7 + 6 = 8 + 5$ using a property of numbers.

Exercise 3.2.6. Use the make-a-ten method to calculate $8+6$. Show a sequence of equations and indicate where you use any properties.

Exercise 3.2.7. Use the make-a-ten method to calculate $7+9$. Show a sequence of equations and indicate where you use any properties.

Exercise 3.2.8. Does subtraction satisfy the associative property? For example, is it true that $(15 - 8) - 6 = 15 - (8 - 6)$?

Exercise 3.2.9. Observe that a single digit addition fact like $9 + 8 = 17$ can also give us two subtraction facts: $17 - 9 = 8$ and $17 - 8 = 9$. Write down the two subtraction facts we can derive from each addition fact below.

(a) $4 + 6 = 10$

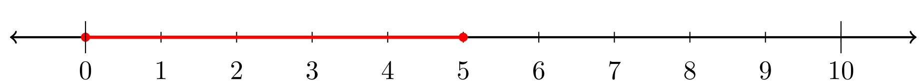
(b) $7 + 5 = 12$

(c) $8 + 7 = 15$

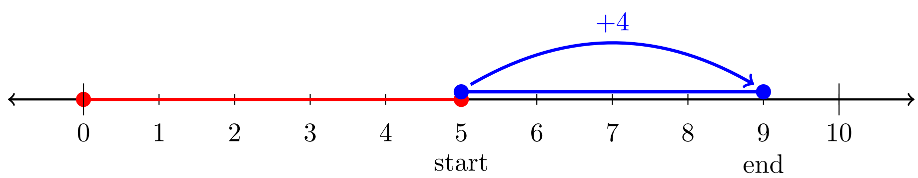
(d) $9 + 9 = 18$ (Do we actually get two subtraction facts from this one? Why or why not?)

3.3. Addition on the number line and negative numbers

We will now show a number line interpretation of addition for positive whole numbers, and use that idea as inspiration for the definition of adding (and subtracting) negative numbers. The idea behind this interpretation is that we can view a positive whole number as that same number of concurrent segments of length 1 on the number line. Consider the following example: $5 + 4$, by our previous definition, this means we start with 5 objects, in this case 5 intervals of length one, to the right of zero as below:

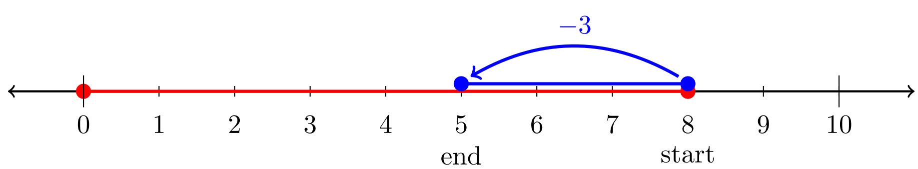


Next, starting at 5 on the number line we move 4 intervals to the right from there as follows:



The place where you arrive on the number line now represents $5 + 4$ segments of length 1 to the right of 0 on the number line, and therefore we see that $5+4 = 9$ using this interpretation of the definition.

We can also interpret subtraction as moving to the *left* on the number line. For example, we want to calculate $8 - 3$ using the number line interpretation. Then, we start at 8, with 8 segments of length 1 to the right or zero on the number line, then we take away 3 of them, effectively moving to the left 3 units.

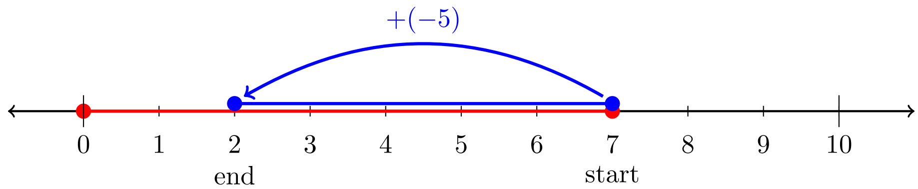


The end point of this process can now be read as $8 - 3$ segments of length 1 to the right of 0 on the number line, and therefore we see that $8 - 3 = 5$ using this interpretation.

Since the number line goes on forever in *both* directions, and since the above interpretation fits and gives us the same answers as our original definition of addition and subtraction, we can now expand the definition of addition to negative numbers as follows.

Definition 3.3.1 (addition). Let m and n be any whole numbers, positive or negative, then $m + n$ is defined to be the place on the number line obtained by starting at m and moving n units from there on the number line. Here, if n is positive, moving n units means moving to the *right*, and if n is negative, moving n units means moving to the *left*.

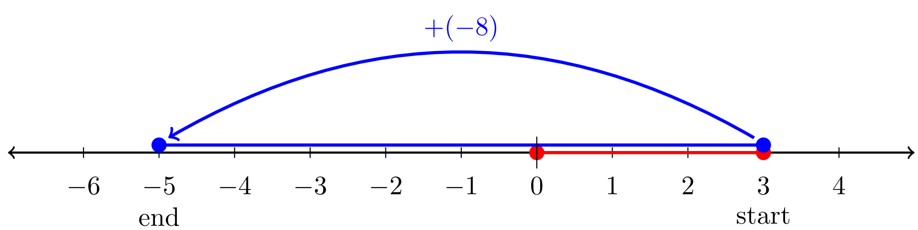
To illustrate the above definition with a concrete example, $7 + (-5)$ means to start at 7 and to move -5 units on the number line, which according to the definition means move 5 units to the *left* from there.



Thus, according to this new definition $7 + (-5) = 2$. Observe that this could also be interpreted, according to our description of subtraction above, as showing that $7 - 5 = 2$. We will use this observation to **eliminate subtraction** as a separate operation going forward, and we will systematically reinterpret every subtraction as addition of the opposite.

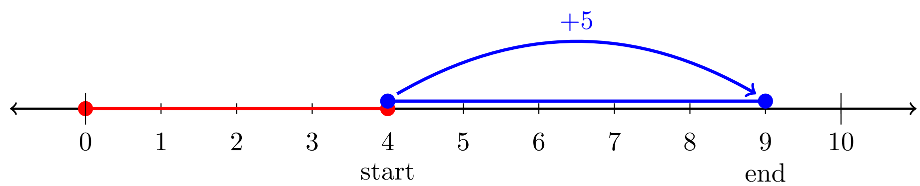
Definition 3.3.2 (Subtraction). For any numbers m and n, we define $m - n = m + (-n)$.

We will return to our reinterpretation of subtraction shortly, but it is important to show at this stage what we have gained just in terms of addition. We are now prepared to add any two numbers, whether positive or negative. So, let's look at $3 + (-8)$. According to our definition, this means we start at 3, and then move -8 units on the number line (this means 8 units to the *left*). The number we arrive at will be the value $3 + (-8)$.



Therefore, using the above number line, we see that, by definition, $3 + (-8) = -5$. We can also read this in terms of subtraction: $3 - 8 = 3 + (-8) = -5$. Now, we are equipped to subtract numbers $m - n$, even if $m \leq n$.

Let's think through the consequences. What is $4 - (-5)$, for example? We need to be very careful in how we read this expression: it means, by definition, $4 +$ the opposite of (-5). What is the opposite of -5? As we saw in a previous section, it's the number that is five units from zero, but not equal to -5 itself: simply 5. Thus, just following the definitions, we must now say that $4-(-5) = 4+5$. So, we can calculate $4 - (-5) = 4 + 5$ using the number line by starting at 4 and moving 5 units (to the right) on the number line.



Thus, we see that $4 + 5 = 9$ (fairly easy), but more interestingly: $4 - (-5) = 9$ as well!

Exercises

Exercise 3.3.1. Show how to use the definition (draw a number line for each one) to calculate the following.

(a) $2 + 6$

(b) $6 + 2$

(c) $-3 + (-4)$ (Hint: start at -3 and move -4 units on the number line)

(d) $-3 + 4$

(e) $-5 + 5$

(f) $4 + (-2)$

(g) $-2 + 4$

Exercise 3.3.2. Rewrite each subtraction as addition of the opposite, then use the definition (draw a number line) to calculate the following.

(a) $2 - 6$

(b) $6 - 2$

(c) $-3 - (-4)$

(d) $-3 - 4$

(e) $-5 - 5$

(f) $4 - (-2)$

(g) $5 - (-5)$

(h) $5 - 5$

Exercise 3.3.3. In the evening, it is 13° Fahrenheit outside. At night, the temperature drops by 20°, what is the outside temperature at night? First interpret this as an addition or subtraction problem, then use a number line (much like a thermometer on its side!) to indicate the solution using the definition.

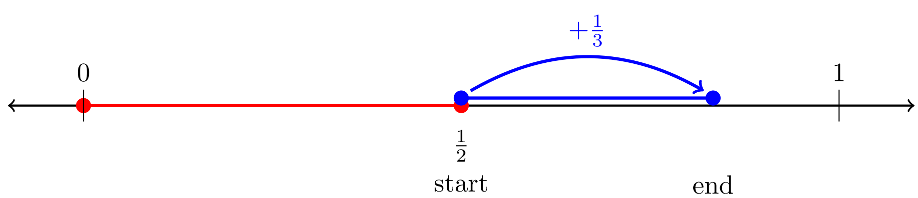
EXERCISE 3.3.4. We can interpret a sum with three terms $(4+5)+6$ using the number line as: start at 4, then move 5 units (arriving at $4+5$), then move 6 units. We can also interpret $4+(5+6)$ as: start at 4, then move $5+6$ units. Note that we can move $5+6$ units by first moving 5 units, then moving 6 units. Therefore, we have $(4+5)+6 = 4+(5+6)$ (the associative property) using our number line definition as well.

(a) Explain why moving $5+(-3) = 2$ units on the number line is the same as first moving 5 units, then moving -3 units. Draw a number line to illustrate your point.

(b) Now show that $(-2+(-4))+(-3) = -2+((-4)+(-3))$ using our number line definition of addition, and draw a number line to illustrate.

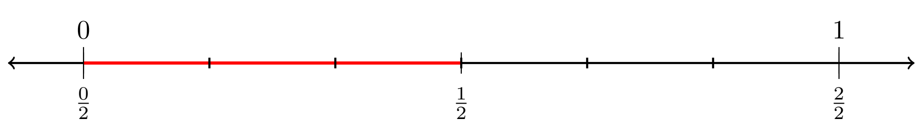
3.4. Adding fractions

Since we understand fractions on the number line, and we have our definition of addition (and subtraction) on the number line, we are now able to discuss adding fractions. The primary difficulty with adding fractions is the fact that they are typically stated in terms of different sized parts. For example, what is $\frac{1}{2}+\frac{1}{3}$? From the point of view of our number line definition, this is not so difficult: we start at $\frac{1}{2}$ and then we move $\frac{1}{3}$ on the number line:

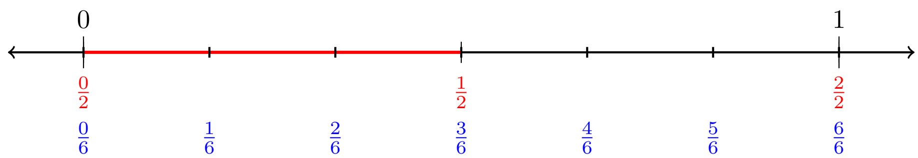


How do we determine the end point on the number line above? We need a way to relate the lengths given by $\frac{1}{2}$ and $\frac{1}{3}$, and as it stands that relationship is not obvious.

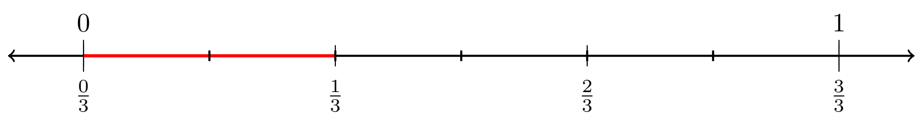
Recall that, by subdividing, we can find a common denominator to compare any two fractions. We can use the exact same idea to figure out where the end point is. We will employ the switch trick, we divide the parts of size $\frac{1}{2}$ of the whole amount (given by the interval from zero to one) into *three* equal parts.



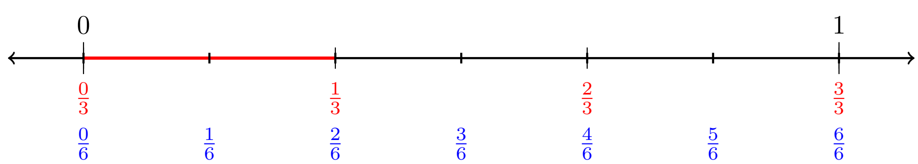
Observe that this divides the interval from zero to one into *six* equal parts, which therefore each have size $\frac{1}{6}$ of the whole amount by definition. We now see the values of the new tick marks are given by:



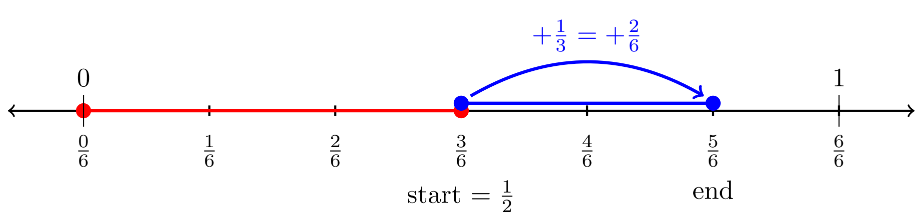
Thus, we obtain the equivalent fraction $\frac{1}{2} = \frac{3}{6}$. Now, we finish the switch trick by subdividing the parts of size $\frac{1}{3}$ of the whole amount into *two* equal parts.



Observe that this subdivision divides the interval from zero to one into *six* equal parts as well (the magic of the switch trick: splitting two parts into three, and splitting three parts into two both give six smaller parts). Thus, we see our equivalent fractions:



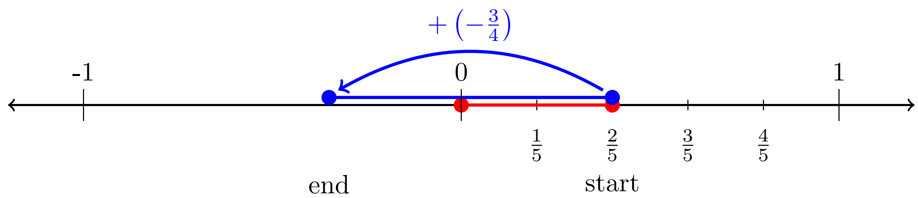
In particular, we observe that $\frac{1}{3} = \frac{2}{6}$. Now, we can easily calculate the addition $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6}$ using the number line:



We now see that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Let's note here that, once we find a common denominator, we can complete the addition simply by adding the whole number numerators. Since $\frac{1}{2} = \frac{3}{6}$ is 3 parts of size $\frac{1}{6}$ of the whole amount, and $\frac{1}{3} = \frac{2}{6}$ is 2 parts of size $\frac{1}{6}$ of the whole amount, the sum $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6}$ must be $3 + 2$ parts of size $\frac{1}{6}$ of the whole amount, which we can calculate (or simply count) to be

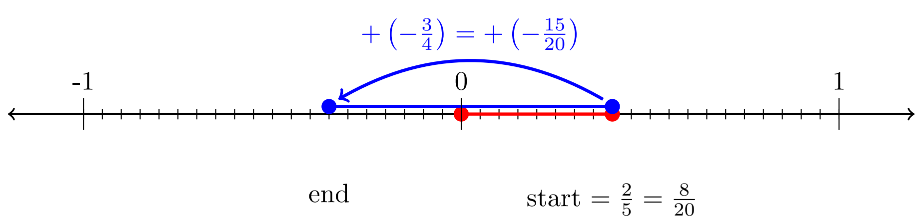
$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}.$$

Let's note that once we understand this method, using the number line interpretation, we can now easily understand subtracting fractions, or adding negative fractions as well. For example, consider the subtraction $\frac{2}{5} - \frac{3}{4}$. First, we reinterpret this as addition of the opposite: $\frac{2}{5} - \frac{3}{4} = \frac{2}{5} + \left(-\frac{3}{4}\right)$. Now, we should clearly understand that we can find this value by starting at $\frac{2}{5}$ and moving $-\frac{3}{4}$ units on the number line. This will mean moving to the *left* in this case. Since $\frac{3}{4} > \frac{2}{5}$ (why?) we must end up to the left of zero:



We now employ the switch trick: subdivide the fifths into four equal parts, and subdivide the fourths into five equal parts. Let's simply do this with object abbreviations this time. Let $W = fffff$ be the whole amount, so $f = \frac{1}{5}$ of W, then we subdivide $f = tttt$, and obtain $W = fffff = (tttt)(tttt)(tttt)(tttt)(tttt)$, so $t = \frac{1}{20}$ of W. Then, we see that $\frac{2}{5} = (tttt)(tttt) = \frac{8}{20}$ of W. Similarly, we start with $W = qqqq$, so $q = \frac{1}{4}$ of W, and then we subdivide $q = ttttt$ (we know in advance that we get the same subdivision, so we use the single letter t to reduce clutter), and we obtain $W = qqqq = (ttttt)(ttttt)(ttttt)(ttttt)$, so we see indeed that $t = \frac{1}{20}$ of W in this case as well. Finally, we get that $\frac{3}{4} = qqq = (ttttt)(ttttt)(ttttt) = \frac{15}{20}$ of W.

Therefore, we want to start at $\frac{2}{5} = \frac{8}{20}$, and then move $-\frac{3}{4} = -\frac{15}{20}$, which means $\frac{15}{20}$ to the *left* on the number line.



Here we have suppressed the labels of all the tick marks to avoid clutter, but you should be able to see that we end up at $-\frac{7}{20}$, so finally we have our result! Note that, even with negative fractions, we can still calculate this result by adding the whole number numerators, we treat a negative fraction as a fraction with a negative numerator: $-\frac{3}{4} = -\frac{15}{20} = \frac{-15}{20}$. Since we want to move 15 small tick marks to the *left* of $\frac{2}{5} = \frac{8}{20}$ (zoom in with your device and count the tick marks to make sure my description is accurate!), we can accomplish this as follows:

$$\frac{2}{5} - \frac{3}{4} = \frac{2}{5} + \left(-\frac{3}{4}\right) = \frac{8}{20} + \left(-\frac{15}{20}\right) = \frac{8 + (-15)}{20} = \frac{-7}{20} = -\frac{7}{20}.$$

We have now developed tools for adding and subtracting fractions using the number line or object abbreviations to find a common denominator, always using the number line to visualize the direction we move, and using the number line (either with the fractions, or just to calculate the whole number numerator) to find the final value. Hopefully this demystifies the process of adding fractions a bit.

Exercises

EXERCISE 3.4.1. Without any calculation, use only the definitions of fraction and addition (on the number line) to explain how we know that $\frac{1}{2} + \frac{1}{3} < 1$.

EXERCISE 3.4.2. Show how to express the two fractions with a common denominator in *two* ways: using a number line, and using object abbreviations. You may use the switch trick for all of these, but is there a simpler common denominator for any of them?

(a) $\frac{1}{4}$ and $\frac{1}{3}$

(b) $\frac{3}{5}$ and $\frac{2}{3}$

(c) $\frac{7}{4}$ and $\frac{5}{6}$

(d) $\frac{3}{8}$ and $\frac{3}{4}$

EXERCISE 3.4.3. Calculate the additions below using a number line to illustrate your result.

(a) $\frac{1}{4} + \frac{1}{3}$

(b) $\frac{3}{5} + \left(-\frac{2}{3}\right)$

(c) $-\frac{7}{4} + \frac{5}{6}$

(d) $\frac{3}{8} + \frac{3}{4}$

EXERCISE 3.4.4. Calculate the subtractions below by first rewriting each as an addition, then using a number line to illustrate your result.

(a) $\frac{1}{4} - \frac{1}{3}$

(b) $\frac{3}{5} - \left(-\frac{2}{3}\right)$

(c) $-\frac{7}{4} - \frac{5}{6}$

(d) $\frac{3}{8} - \frac{3}{4}$

Exercise 3.4.5. Show why adding fractions with a common denominator is easy by calculating the following and illustrating your result with a number line: $\frac{4}{5} + \frac{2}{5}$.

Exercise 3.4.6. Clint has one bag of fertilizer that is $\frac{2}{5}$ full, and another bag of fertilizer (of the same size) that is $\frac{2}{7}$ full. He wants to combine the two into a single bag of fertilizer, how full will it be? (Clearly state the whole amount in this problem, and use a number line to illustrate your result.)

Exercise 3.4.7. Explain how you know that Clint's bag of fertilizer from the previous problem will not overflow. In other words, use the definition of fraction and addition alone (without calculating, or changing denominators) to explain why $\frac{2}{5} + \frac{2}{7} < 1$.

3.5. Addition algorithm

We will now discuss the standard algorithm[1] for adding numbers in Base-10. Counting objects or moving on the number line are great ways to understand the meaning of addition, and also work well for calculation involving relatively small values. However, sometimes it is necessary to add larger numbers such as $384 + 57$. This can also be done using object abbreviations, and we will illustrate how it works, but there is a more efficient way to calculate this value using Base-10 and adding place by place, or column by column.

First, let's look at object abbreviations for these values:

$$384 = CCCXXXXXXXX\,||||$$

and

$$57 = XXXXX\,|||||||.$$

It is straightforward now, to see that

$$384 + 57 = CCCXXXXXXXXXXXXX\,|||||||||||.$$

The only issue is we do not immediately see how to write this number in Base-10 because we have more than ten tens (Xs) and we have more than ten ones ($|$s). Recall from earlier that, in this scenario, we start from the smallest place value (ones in this case) and we work our way up, regrouping in tens along the way.

To start, we look at the ones:

$$||||||||||| = (||||||||||)\,| = X\,|$$

(*from here we know our ones place is* $|$).

Then, we take that new X and we combine with the others:

$$XXXXXXXXXXXXXX = (XXXXXXXXXX)XXXX = CXXXX$$

(recall that ten tens is one hundred (C), *from here our tens place is* $XXXX$).

Now, we add this new C to the given hundreds:

$$CCCC,$$

here we do not have ten or more objects, and this is the largest place value, so we are done (*from here our hundreds place is* $CCCC$)!

[1]An algorithm is a step-by-step procedure to perform a calculation or solve a problem

We just need to put together what we obtained above:

$$CCCCXXXX\,| = 441.$$

Congratulations! You just performed a multi-column addition! If that doesn't seem obvious to you, let me point out the key idea: every time we group in tens such as $|\,|\,|\,|\,|\,|\,|\,|\,|\,| = X$, we are doing what is traditionally called 'carrying the one', but, as you see, what is actually happening is that we are grouping a smaller place value into ten, and obtaining 'one' of the next largest place value. We then combine that with the others (this is the 'carry' part) and regroup again until we are done. Let's illustrate what this looks like when we use our traditional Base-10 notation instead of objects.

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 384 \\ +57 \\ \hline 441 \end{array}$$

This is a very compact and efficient way to express the exact method we used above with the objects. To compare, we start from the right hand column: $4+7 = 11$ (note how our single digit addition facts are used routinely). This corresponds to our ones calculation obtaining $X \ = 11$ in the end. Now, we take the ten $X = 10$, and we keep track of it by putting an additional one above the tens place, this one actually stands for ten, but since we put it in the tens column, we don't need to write the zero! We write the leftover one below the line in the ones column. So far, what we have is:

$$\begin{array}{r} {\scriptstyle 1} \\ 384 \\ +57 \\ \hline 1 \end{array}$$

Next, we add down the tens column: $1 + 8 + 5 = 14$ (we omit the parentheses since we know by the Associative Property that either grouping leads to the same result). Note carefully that this means 14 *tens* (in other words, the value is 140). This means the 'one' in 14 is actually ten tens, or 100. Thus, we write a one above the hundreds column, and we write the leftover tens (there are 4 of them) below the line in the tens column.

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 384 \\ +57 \\ \hline 41 \end{array}$$

To finish the calculation, we now add down the hundreds column: $1 + 3 = 4$, thus we get four hundreds and nothing to carry (fewer than ten hundreds), and we write a 4 below the line in the hundreds column. There is nothing left to calculate, so this is the end!

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 384 \\ +57 \\ \hline 441 \end{array}$$

You should now see how, if you proceed from left to right, step by step, we are doing the exact same calculations and regroupings that we did with the object abbreviations, we are simply using place value in Base-10 to write everything in a more efficient way.

Note that, to add decimals in this way, we only need to use object abbreviations for the fractional (decimal) values. For example, if we let | = ********** (so * represents 0.1), and * = ++++++++++ (so + represents 0.01 - note the ambiguity!). Then, using the same idea, but starting with the smallest object abbreviation, we can regroup our way up the place values and we essentially obtain the same method for adding, for example,

4.39 + 1.67 = (||||***+++++++++) + (|******+++++++)
= |||||*********++++++++++++++++.

Now, we regroup the hundredths

++++++++++++++++ = (++++++++++)++++++ = *++++++.

Thus we obtain ++++++ = 0.06 for the hundredths place. We carry the new * and combine with the others to obtain ********** = (**********) = |, this time we get ten tenths or a one, with no tenths left over, so we carry the one, and our tenths place will be a zero (0.0)! Now we combine the new one with the others: |||||| = 6 for the ones place, and since this is the largest place value we now know that $4.39 + 1.67 = 6.06$. In multi-column format this looks like:

$$\begin{array}{r} {\scriptstyle 1\ 1} \\ 4.39 \\ +1.67 \\ \hline 6.06 \end{array}$$

Exercises

Exercise 3.5.1. Show the multi-column additions step by step and explain how they are related to the object abbreviations above.

$$\begin{array}{r} {\scriptstyle 1\ 1} \\ 4.39 \\ +1.67 \\ \hline 6.06 \end{array}$$

Exercise 3.5.2. Calculate the sum in *two* ways: using object abbreviations and regrouping, and then using multi-column addition, as shown above.

(a) $473 + 89$

(b) $403 + 89$

(c) $473 + 189$

(d) $473 + 180$

(e) $1473 + 789$

(f) $1073 + 789$

(g) $732 + 105$

(h) $4.73 + 0.89$

(i) $47.3 + 18.9$

(j) $14.73 + 7.89$

(k) $107.3 + 78.9$

EXERCISE 3.5.3. In a previous section we discussed the 'make a ten' method for learning single digit addition facts. There is a related idea called 'make a hundred' that can also be useful for larger sums and differences. The idea is that you try to learn how to make a hundred from a one or (usually) two digit number in your head: $26+? = 100$. This can be broken down into two steps, if you have $ab+? = 100$, where ab is a two-digit number (we allow $a = 0$ for a one digit number), then you solve $a+? = 9$ and $b + \overline{?} = 10$, and the solution to the original problem is $ab+?\overline{?} = 100$. For example, back to our example $26+? = 100$: we solve $2+? = 9$ and $6 + \overline{?} = 10$ and we get $? = 7$ and $\overline{?} = 4$, so therefore $26 + 74 = 100$. Check that this works, and explain why, using our addition algorithm or object abbreviations, if $a+? = 9$ and $b + \overline{?} = 10$, then $ab+?\overline{?} = 100$.

3.6. Subtraction algorithm

We now present the standard algorithm for subtracting numbers in Base-10. Recall that, for addition, we would regroup in tens and add one (or more if necessary) to the next largest place value. For subtraction, we need to do the opposite, break apart larger place values into tens and move them to the next smallest place value. This is usually called *borrowing*. We will follow a similar presentation to the previous section, first we will illustrate the idea with object abbreviations, then we will show how to make the process efficient with columns.

The easy case is when each digit of the larger number is larger than each corresponding digit of the smaller number, for example, $956 - 123$. In this case we simply cross off digit-by-digit. So we write

$$956 = CCCCCCCCCXXXXX\,|\,|\,|\,|\,|\,|$$

and

$$123 = CXX\,|\,|\,|\,.$$

We start with

$$956 = CCCCCCCCCXXXXX\,|\,|\,|\,|\,|\,|$$

and we cross off $3 = |\,|\,|$ ones (we will put them in red as well):

$$956 - 3 = CCCCCCCCCXXXXX\,|\,|\,|\,\cancel{|\,|\,|},$$

then we cross off 2 tens $= XX$:

$$956 - 23 = CCCCCCCCCXXX\cancel{XX}\,|\,|\,|\,\cancel{|\,|\,|},$$

and finally we cross off 1 hundred $= C$:

$$956 - 123 = CCCCCCCC\cancel{C}XXX\cancel{XX}\,|\,|\,|\,\cancel{|\,|\,|} = CCCCCCCCXXX\,|\,|\,| = 833.$$

The slightly harder case is when one or more of the digits of the larger number is smaller than the corresponding digits of the smaller number. Let's do an example in detail to illustrate: $624 - 198$. First, we write our numbers using object abbreviations as usual:

$$624 = CCCCCCXX\,|\,|\,|\,|$$

and

$$198 = CXXXXXXXXX\,|\,|\,|\,|\,|\,|\,|\,|.$$

Notice that we can not simply cross of 8 ones since we only have 4 available in the number 624. The simple solution is to break apart one of the larger place values to get some more ones available to cross off. We take one of the tens and replace it by ten ones: $X = |\,|\,|\,|\,|\,|\,|\,|\,|\,|$ to obtain:

$$624 = CCCCCCXX\,|\,|\,|\,| = CCCCCCX\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|.$$

Now we have enough ones to cross off $8 = |\,|\,|\,|\,|\,|\,|\,|$ of them:

$$624 - 8 = CCCCCCX\,|\,|\,|\,|\,|\,|\,\cancel{|\,|\,|\,|\,|\,|\,|\,|}.$$

We now want to cross off 9 tens $= XXXXXXXXX$, but, again, we clearly do not have enough available. Luckily, we have already seen how to deal with this issue: we break apart one of the larger place values to get more tens to cross off. In this case we take one of the hundreds: $C = XXXXXXXXXX$ to obtain first:

$$624 - 8 = CCCCCCX\,|\,|\,|\,|\,|\,|\,\cancel{|\,|\,|\,|\,|\,|\,|\,|} =$$
$$CCCCCXXXXXXXXXXX\,|\,|\,|\,|\,|\,|\,\cancel{|\,|\,|\,|\,|\,|\,|\,|},$$

and now we have enough tens to cross off 9 of them:

$$624 - 98 = CCCCCXX\cancel{XXXXXXXXX}\,|\,|\,|\,|\,|\,|\,\cancel{|\,|\,|\,|\,|\,|\,|\,|}.$$

The last step is straightforward (no borrowing required), we just need to cross off 1 hundred $= C$ to obtain the result:

$$624 - 198 = CCCC\cancel{C}XX\cancel{XXXXXXXXX}\,|\,|\,|\,|\,|\,|\,\cancel{|\,|\,|\,|\,|\,|\,|\,|} =$$
$$CCCCXX\,|\,|\,|\,|\,|\,| = 426.$$

Recall that the subtraction $624 - 198 = 426$ can also be read as the addition: $624 = 426 + 198$, so it is possible to check your work using the addition algorithm from the previous section.

Let's look at the neat, abbreviated version of this using place value and columns to keep track of the borrowing:

$$\begin{array}{cccc}
 & 5 & 11 & \\
 & & \cancel{1} & 14 \\
 & \cancel{6} & \cancel{2} & \cancel{4} \\
- & 1 & 9 & 8 \\
\hline
 & 4 & 2 & 6
\end{array}$$

We will now go through this step-by-step, you should compare each step here with the steps above using the object abbreviations. First, we start with the ones column:

$$\begin{array}{cccc}
 & 6 & 2 & 4 \\
- & 1 & 9 & 8 \\
\hline
\end{array}$$

We would like to subtract $4 - 8$, but we can not do it without using negative numbers (see exercises for a method involving negative numbers). So, we borrow ten ones from the tens column, and reduce the tens place by one to obtain:

$$\begin{array}{rrrr} & & 1 & 14 \\ & 6 & \cancel{2} & \cancel{4} \\ - & 1 & 9 & 8 \\ \hline \end{array}$$

This allows us to subtract down the ones column $14 - 8 = 6$ (note that our single digit addition facts are also involved here: $6 + 8 = 14$).

$$\begin{array}{rrrr} & & 1 & 14 \\ & 6 & \cancel{2} & \cancel{4} \\ - & 1 & 9 & 8 \\ \hline & & & 6 \end{array}$$

Next, we would like to subtract down the tens column $1 - 9$, but, again, we can not do it without using negative numbers, so we borrow ten tens from the hundreds column, and reduce the hundreds place by one to obtain:

$$\begin{array}{rrrr} & 5 & 11 & \\ & & \cancel{1} & 14 \\ & \cancel{6} & \cancel{2} & \cancel{4} \\ - & 1 & 9 & 8 \\ \hline & & & 6 \end{array}$$

We are now able to subtract $11 - 9 = 2$ (single digit addition fact $2 + 9 = 11$) and we now have:

$$\begin{array}{rrrr} & 5 & 11 & \\ & & \cancel{1} & 14 \\ & \cancel{6} & \cancel{2} & \cancel{4} \\ - & 1 & 9 & 8 \\ \hline & & 2 & 6 \end{array}$$

Finally, we now subtract down the hundreds column: $5 - 1 = 4$ and we have the final result.

$$\begin{array}{rrrr} & 5 & 11 & \\ & & \cancel{1} & 14 \\ & \cancel{6} & \cancel{2} & \cancel{4} \\ - & 1 & 9 & 8 \\ \hline & 4 & 2 & 6 \end{array}$$

Subtracting decimals is substantially similar, we simply use smaller subdivisions, the only issue that comes up sometimes is it could be necessary to add some zeros to the numbers to allow both values to align column-wise. We will show a completed subtraction involving decimals and leave the reader to work out the details using object abbreviations and then going column by column borrowing as needed, we will do the subtraction: $5.3 - 1.79 = 5.30 - 1.79 = 3.51$.

$$\begin{array}{rrrr} & 4 & 12 & \\ & & \cancel{2} & 10 \\ & \cancel{5}. & \cancel{3} & \cancel{0} \\ - & 1. & 7 & 9 \\ \hline & 3. & 5 & 1 \end{array}$$

Exercises

Exercise 3.6.1. Use object abbreviations to illustrate the calculation. Then, complete the problem from the beginning, column by column, comparing each step with the object abbreviations. Note the importance of adding the zero $5.3 = 5.30$ when we use the algorithm.

$$\begin{array}{rrrr} & 4 & 12 & \\ & & \cancel{2} & 10 \\ & \cancel{5}. & \cancel{3} & \cancel{0} \\ - & 1. & 7 & 9 \\ \hline & 3. & 5 & 1 \end{array}$$

Exercise 3.6.2. Calculate the difference in *two* ways: using object abbreviations and regrouping, and then using multi-column subtraction, as shown above.

(a) $473 - 189$

(b) $403 - 289$

(c) $570 - 189$

(d) $473 - 180$

(e) $1473 - 789$

(f) $1073 - 789$

(g) $732 - 105$

(h) $4.73 - 0.89$

(i) $47 - 18.9$

(j) $14.7 - 7.89$

(k) $107.3 - 78$

Exercise 3.6.3. Consider the following subtraction method:

$$\begin{array}{rrrr} & 6 & 2 & 4 \\ - & 1 & 9 & 8 \\ \hline & 5 & 0 & 0 \\ - & & 7 & 4 \\ \hline & 4 & 2 & 6 \end{array}$$

Illustrate this method with object abbreviations (use another color or marking to indicate 'negative objects'). Note the usefulness of knowing how to 'make a hundred': $100 = 74 + 26$. Finally, complete the following subtraction using this method: $473 - 189$.

CHAPTER 4

Multiplication

4.1. The definition of multiplication

We will now discuss multiplication, and we will define it in a fairly concrete and clear way that will hopefully always be in the back of your mind when using multiplication. Let's go through an example first before presenting the abstract definition. Let's say you have 3 bags and you want to put 2 lollipops into each bag, then, as we will see, we can take groups = bags, and units/objects = lollipops, and then by definition, the total number of lollipops to be put in bags is $= 3 \cdot 2$ (I will generally use the symbol $\cdot$ for multiplication, but other books may use $\times$ or other symbols). Note carefully that this definition has nothing to do with *calculating* the product[1]. Rather, we are interested at this stage in the *structure* of multiplication. In other words, the question is: what kinds of situations call for a multiplication to calculate? As usual, we start with positive whole numbers, subsequent sections will develop multiplication for other types of numbers.

DEFINITION 4.1.1 (multiplication). Let m and n be positive whole numbers. If there are m groups with n objects or units in one group, then $m \cdot n$ is defined to be the total number of objects or units in the m groups.

This definition is phrased vaguely or abstractly to make it more flexible, so it applies to as many situations as possible. Let's consider further examples. One nice way to apply the definition is for unit conversions. There are 12 inches in 1 foot. How many inches are in 3 feet? To answer this question we can take groups = feet, and units = inches. We then see that there are 12 inches (units) in 1 foot (group), so therefore, by the above definition, there are $3 \cdot 12$ inches (units) in 3 feet (groups).

A classroom has desks arranged into 4 rows with 5 desks in each row. We can display this arrangement as an *array*:

$$\begin{matrix} d & d & d & d & d \\ d & d & d & d & d \\ d & d & d & d & d \\ d & d & d & d & d \end{matrix}$$

In this case, we can take groups = rows and objects = desks. We see that there are 5 desks (objects) in 1 row (group), with 4 rows (groups) in all. Therefore, by definition, there are $4 \cdot 5$ desks in the classroom.

Exercises

EXERCISE 4.1.1. For each of the following situations, write a multiplication (**do not calculate it**) that describes the requested amount, and state in words

[1] A *product* is the result of a multiplication.

what your result means. Use the definition of multiplication: describe the groups and units/objects.

(a) Thanos wants to hand out boxes of cookies to all his minions. He puts 4 cookies in each box, and each of his 13 minions will receive a box of cookies. How many cookies does he need in all?

(b) Maria is measuring the dimensions of her fighter plane using meters, and she finds that it is 14 meters long. There are 100 centimeters in one meter. How long is her plane in centimeters?

Exercise 4.1.2. An auditorium has chairs arranged into 8 rows with 9 chairs in each row.

(a) Write a multiplication (**do not calculate it**) that describes the total number of chairs in the auditorium. Use the definition of multiplication: describe the groups and units/objects.

(b) Draw an array to illustrate the arrangement of chairs.

(c) Now, using your array, you can organize the chairs by *columns*. Write a multiplication (**do not calculate it**) that describes the total number of chairs in the auditorium. Use columns as your groups and chairs as your objects. How does this change the multiplication from above? What can you conclude about the two multiplications?

Exercise 4.1.3. Write a word problem whose solution is given by the multiplication $5 \cdot 6$. Describe the groups and units/objects. Try to come up with a situation not already used in the book.

4.2. Commutative and Associative Properties, the web of multiplication

In this section we will discuss two basic properties of multiplication, and then we will apply them to assist in learning the basic single digit multiplication facts, just as we did for addition.

4.2.1. The Associative Property of Multiplication. We start by explaining why this property has content, namely, what is the difference between $m \cdot (n \cdot p)$ and $(m \cdot n) \cdot p$. If you have completely understood the definition, you may already see, but to be completely clear: each of these expressions has *two* multiplications. The expression $m \cdot (n \cdot p)$ describes an amount with m groups of $n \cdot p$ objects in each group (the objects in each group in this case are further organized into n (sub)groups with p objects in each (sub)group). On the other hand, the expression $(m \cdot n) \cdot p$ describes an amount with $m \cdot n$ groups of p objects in each group (the *groups* in this case are further organized into m (super)groups with n *groups* (playing the role of 'objects' here) in each (super)group). This second multiplication is a bit more difficult because the groups themselves are organized into groups. This idea should become clearer by doing an example.

Here is our example: *Ajak works in a grocery store and needs to count the inventory of boxes of granola bars. He has arranged them into stacks with* 5 *boxes* *in each stack. The stacks themselves are arranged into an array with* 3 *rows and* 4 *columns. How many boxes of granola bars are there?* There are, in fact, many more than two ways to count this using multiplication, but we will choose two ways that illustrate the Associative property. First, we will let groups = stacks and objects = boxes. We already know there are 5 boxes in each stack, so the number of boxes is (total # of stacks)$\cdot$5. Interestingly, in this case, the stacks themselves are arranged into an array:

$$\begin{array}{cccc} s & s & s & s \\ s & s & s & s \\ s & s & s & s \end{array}$$

Therefore, we an count the number of stacks using multiplication as well, with groups = rows and objects = stacks (note that the objects of this multiplication are the groups of the next multiplication - this is allowed!). So we get that there are a total of $3 \cdot 4$ stacks of boxes in all (3 rows (groups) of 4 stacks (objects)). Finally, we put it together with the above, and we find that there are $(3 \cdot 4) \cdot 5$ boxes of granola bars. Again, resist the urge to calculate, think structure not results!

Now, we will reorganize our thinking about the above example, and count the boxes in a slightly different way. We start with groups = rows, but this time we take objects = *boxes*. Note that the boxes in each row are organized into 4 stacks of 5 boxes, so we can count the number of boxes in each row by taking groups = stacks (in one row) and objects = boxes. There are 4 stacks in one row, and 5 boxes in each stack, so by definition, there are $4 \cdot 5$ boxes in each row. Finally, we see that with this organization, we have, by definition of multiplication, $3 \cdot (4 \cdot 5)$ boxes of granola bars.

Note in each case that if we have two multiplications, then we need two distinct groups and objects set-ups to describe the expression using the definition (which only allows for one multiplication at a time). More importantly, note that we counted the *same boxes of granola bars* in *two different ways*. The number of boxes of granola bars remains the same regardless of how we count them, so we now know that $(3 \cdot 4) \cdot 5 = 3 \cdot (4 \cdot 5)$. Observe that we do **not** conclude that these values are equal by calculating! I have never mentioned up to this point what the final value of this multiplication is! We are concluding that the values are equal because they are both ways of counting the same number of objects. This is a useful way to explain why two expressions are equal without doing any calculation.

The equation $(3 \cdot 4) \cdot 5 = 3 \cdot (4 \cdot 5)$ is an example of the Associative Property of Multiplication, and more importantly, the idea behind the equation explains why the property is true. We can always count objects organized into an array of stacks in these two (and many more) different ways!

THEOREM 4.2.1 (Associative Property of Multiplication). *Let m, n, and p be positive whole numbers, then $(m \cdot n) \cdot p = m \cdot (n \cdot p)$.*

4.2.2. The Commutative Property of Multiplication. This property is significantly easier to explain than the previous one. First, as usual, we explain why it has content. The expressions $m \cdot n$ and $n \cdot m$ do **not** mean the same thing according to our definition of multiplication. The expression $m \cdot n$ refers to m groups with n objects/units in each group, whereas the expression $n \cdot m$ refers to n groups with m objects/units in each group. The fact that the result of these two multiplications is the same is the fundamental content of the Commutative Property of Multiplication.

We can fairly easily explain this property using the example of an array. Recall the example: A classroom has desks arranged into 4 rows with 5 desks in each row. We can display this arrangement as an *array*:

$$\begin{array}{ccccc} d & d & d & d & d \\ \hline d & d & d & d & d \\ \hline d & d & d & d & d \\ \hline d & d & d & d & d \end{array}$$

In this case, we can take groups = rows and objects = desks. We see that there are 5 desks (objects) in 1 row (group), with 4 rows (groups) in all. Therefore, by definition, there are $4 \cdot 5$ desks in the classroom. So far, this is just a repeat of what we did before. Now, though, we observe that we can organize the array in a different way, we can take groups = *columns* and retain objects = desks. This organization looks something like:

$$\begin{array}{c|c|c|c|c} d & d & d & d & d \\ d & d & d & d & d \\ d & d & d & d & d \\ d & d & d & d & d \end{array}$$

We now have 5 columns with 4 desks in each column, so the total number of desks in the array is $5 \cdot 4$ by definition of multiplication.

Although the organization is different, we counted the same array of desks in both cases, so therefore, we must have $4 \cdot 5 = 5 \cdot 4$. This is an example of the Commutative Property of Multiplication, and it illustrates why it's true as well! We now state the property.

THEOREM 4.2.2 (Commutative Property of Multiplication). *Let m and n be positive whole numbers, then $m \cdot n = n \cdot m$.*

4.2.3. Application: the web of single digit multiplication. Another important and useful set of facts to know in order to become more fluent with the basic operations of math is the result of multiplying any two single digit whole numbers. For example, what is $7 \cdot 6$? If you do not know immediately, don't worry, we will now outline a method for calculating such products, and how to relate them to others. Once you practice this several times, you will find that, not only can you more easily recall all the single digit products, but you have a much deeper understanding of how they are related. Reminder: it is extremely important that you **do not use a calculator** for anything in this section.

We will display the multiplications we want to learn in the array below.

$$\begin{array}{ccccccccc}
1\cdot 1 & 1\cdot 2 & 1\cdot 3 & 1\cdot 4 & 1\cdot 5 & 1\cdot 6 & 1\cdot 7 & 1\cdot 8 & 1\cdot 9 \\
2\cdot 1 & 2\cdot 2 & 2\cdot 3 & 2\cdot 4 & 2\cdot 5 & 2\cdot 6 & 2\cdot 7 & 2\cdot 8 & 2\cdot 9 \\
3\cdot 1 & 3\cdot 2 & 3\cdot 3 & 3\cdot 4 & 3\cdot 5 & 3\cdot 6 & 3\cdot 7 & 3\cdot 8 & 3\cdot 9 \\
4\cdot 1 & 4\cdot 2 & 4\cdot 3 & 4\cdot 4 & 4\cdot 5 & 4\cdot 6 & 4\cdot 7 & 4\cdot 8 & 4\cdot 9 \\
5\cdot 1 & 5\cdot 2 & 5\cdot 3 & 5\cdot 4 & 5\cdot 5 & 5\cdot 6 & 5\cdot 7 & 5\cdot 8 & 5\cdot 9 \\
6\cdot 1 & 6\cdot 2 & 6\cdot 3 & 6\cdot 4 & 6\cdot 5 & 6\cdot 6 & 6\cdot 7 & 6\cdot 8 & 6\cdot 9 \\
7\cdot 1 & 7\cdot 2 & 7\cdot 3 & 7\cdot 4 & 7\cdot 5 & 7\cdot 6 & 7\cdot 7 & 7\cdot 8 & 7\cdot 9 \\
8\cdot 1 & 8\cdot 2 & 8\cdot 3 & 8\cdot 4 & 8\cdot 5 & 8\cdot 6 & 8\cdot 7 & 8\cdot 8 & 8\cdot 9 \\
9\cdot 1 & 9\cdot 2 & 9\cdot 3 & 9\cdot 4 & 9\cdot 5 & 9\cdot 6 & 9\cdot 7 & 9\cdot 8 & 9\cdot 9
\end{array}$$

This may seem intimidating at first (there are 81 multiplications in the array), but using the properties we have now learned, we can reduce our workload significantly. First, observe that, if you remove the squares along the diagonal, i.e. $1 \cdot 1$, $2 \cdot 2$, $3 \cdot 3$, $4 \cdot 4$, etc, the table is symmetric, every other has a corresponding entry in the opposite order. For example, we have $6 \cdot 3$ below the diagonal, and we have $3 \cdot 6$ above the diagonal.

$1 \cdot 1$	$1 \cdot 2$	$1 \cdot 3$	$1 \cdot 4$	$1 \cdot 5$	$1 \cdot 6$	$1 \cdot 7$	$1 \cdot 8$	$1 \cdot 9$
$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$	$2 \cdot 6$	$2 \cdot 7$	$2 \cdot 8$	$2 \cdot 9$
$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 3$	$3 \cdot 4$	$3 \cdot 5$	$3 \cdot 6$	$3 \cdot 7$	$3 \cdot 8$	$3 \cdot 9$
$4 \cdot 1$	$4 \cdot 2$	$4 \cdot 3$	$4 \cdot 4$	$4 \cdot 5$	$4 \cdot 6$	$4 \cdot 7$	$4 \cdot 8$	$4 \cdot 9$
$5 \cdot 1$	$5 \cdot 2$	$5 \cdot 3$	$5 \cdot 4$	$5 \cdot 5$	$5 \cdot 6$	$5 \cdot 7$	$5 \cdot 8$	$5 \cdot 9$
$6 \cdot 1$	$6 \cdot 2$	$6 \cdot 3$	$6 \cdot 4$	$6 \cdot 5$	$6 \cdot 6$	$6 \cdot 7$	$6 \cdot 8$	$6 \cdot 9$
$7 \cdot 1$	$7 \cdot 2$	$7 \cdot 3$	$7 \cdot 4$	$7 \cdot 5$	$7 \cdot 6$	$7 \cdot 7$	$7 \cdot 8$	$7 \cdot 9$
$8 \cdot 1$	$8 \cdot 2$	$8 \cdot 3$	$8 \cdot 4$	$8 \cdot 5$	$8 \cdot 6$	$8 \cdot 7$	$8 \cdot 8$	$8 \cdot 9$
$9 \cdot 1$	$9 \cdot 2$	$9 \cdot 3$	$9 \cdot 4$	$9 \cdot 5$	$9 \cdot 6$	$9 \cdot 7$	$9 \cdot 8$	$9 \cdot 9$

By the commutative property we know that $6 \cdot 3 = 3 \cdot 6$, so if we can just remember one of these, for example $3 \cdot 6 = 18$, then we should automatically know that $6 \cdot 3 = 3 \cdot 6 = 18$ as well. In this way, we cut our workload almost in half, as we can just learn the squares on the diagonal, and all products $m \cdot n$ where $m < n$. It is actually easier to understand the upper part of the array in the case of multiplication, since multiplication is inherently less symmetric than addition, and keeping the number of groups smaller seems somehow simpler.

$1 \cdot 1$	$1 \cdot 2$	$1 \cdot 3$	$1 \cdot 4$	$1 \cdot 5$	$1 \cdot 6$	$1 \cdot 7$	$1 \cdot 8$	$1 \cdot 9$
	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$	$2 \cdot 6$	$2 \cdot 7$	$2 \cdot 8$	$2 \cdot 9$
		$3 \cdot 3$	$3 \cdot 4$	$3 \cdot 5$	$3 \cdot 6$	$3 \cdot 7$	$3 \cdot 8$	$3 \cdot 9$
			$4 \cdot 4$	$4 \cdot 5$	$4 \cdot 6$	$4 \cdot 7$	$4 \cdot 8$	$4 \cdot 9$
				$5 \cdot 5$	$5 \cdot 6$	$5 \cdot 7$	$5 \cdot 8$	$5 \cdot 9$
					$6 \cdot 6$	$6 \cdot 7$	$6 \cdot 8$	$6 \cdot 9$
						$7 \cdot 7$	$7 \cdot 8$	$7 \cdot 9$
							$8 \cdot 8$	$8 \cdot 9$
								$9 \cdot 9$

We have now reduced our workload to 45 multiplications from 81! The top row is fairly easy, for example, one group of 6 objects is clearly 6 objects, so $1 \cdot 6 = 6$, and similarly for the other entries. Next, you can work on the doubles, and in this case, we can connect this with our addition facts, as, for example 2 groups of 7 objects, is the same number as '7 objects, then 7 more objects': $2 \cdot 7 = 7 + 7 = 14$. If we need to, we can use the connection with addition to calculate any of these sums, though it may require the addition algorithm if you have trouble doing it in your head. For example: $3 \cdot 8$ is three groups of 8 objects, so therefore $3 \cdot 8 = 8 + 8 + 8$ (we omit parentheses since, by the associative property of addition, we know that we get the same result either way). Now, we can calculate $8+8+8 = (8+8)+8 = 16+8 = 24$ using the algorithm for the last addition if necessary. Thus, we obtain $3 \cdot 8 = 24$.

We can also employ a 'bootstrapping' technique, if we know one multiplication, we should be able to get the next by adding one more group. For example, we observed above that $3 \cdot 8 = 24$, so what is $4 \cdot 8$? Well, it is 4 groups of 8 objects, which is clearly the same as 3 groups of eight objects and one more group of 8 objects, in equation form: $4 \cdot 8 = 3 \cdot 8 + 8 = 24 + 8$. So, we can use the addition algorithm again, if needed, to find that $4 \cdot 8 = 24 + 8 = 32$.

We can use the Associative Property for some of the larger products. For example, if we know that $2 \cdot 7 = 14$, then we get, for example that $6 \cdot 7 = (3 \cdot 2) \cdot 7 = 3 \cdot (2 \cdot 7) = 3 \cdot 14 = 14 + 14 + 14 = 28 + 14 = 42$. Note that we used all the previous ideas here as well as the Associative Property: converting to addition, using the addition algorithm. If we happen to know that $3 \cdot 7 = 21$, then we can use the associative property in a slightly different way: $6 \cdot 7 = (2 \cdot 3) \cdot 7 = 2 \cdot (3 \cdot 7) = 2 \cdot 21 = 21 + 21 = 42$.

Sometimes the larger products are harder to remember, so you can break up the groups and combine them using addition as needed, for example, if you want to find $9 \cdot 9$, which is 9 groups of 9 objects, you can break this up into 4 groups of 9 objects ($4 \cdot 9 = 36$) and then 5 groups of 9 objects ($5 \cdot 9 = 45$) - note carefully that this involves knowing the single digit addition fact $4 + 5 = 9$ for counting the groups. Then, you put it together using addition: $9 \cdot 9 = 4 \cdot 9 + 5 \cdot 9 = 36 + 45 = 81$.

The important point here is that the properties and definitions give you tools to relate these facts to other facts that you may already know. So, don't treat this as a table of things to memorize, but try to connect any fact you don't know to other facts that you do know! In this way, you build a web of facts that eventually renders memorization unnecessary! Furthermore, understanding how facts are related is just as important, if not more so, as knowing the facts themselves (yes, I repeat myself!).

Exercises

Exercise 4.2.1. A classroom has students sitting at desks arranged in an array with 6 rows and 7 columns. Assume all desks are occupied.

(a) Each student receives a package of 8 markers to use on a class assignment. Use the definition of multiplication to count the markers in **two different ways** to illustrate why $(6 \cdot 7) \cdot 8 = 6 \cdot (7 \cdot 8)$. Be sure you assign groups and objects for each multiplication (there should be a total of *four* set-ups of the form groups = something, and objects = something). Do not calculate anything.

(b) Now, count the desks in *two* different ways to illustrate why $6 \cdot 7 = 7 \cdot 6$. Be sure you assign groups and objects for each multiplication (there should be a total of *two* set-ups of the form groups = something, and objects = something). Do not calculate anything.

Exercise 4.2.2. Invent your own scenario that can be used to explain why $(9 \cdot 10) \cdot 11 = 9 \cdot (10 \cdot 11)$ by counting objects in the real world. Explain why your example works using the definition of multiplication.

Exercise 4.2.3. Invent your own scenario that can be used to explain why $12 \cdot 13 = 13 \cdot 12$ by counting objects in the real world. Explain why your example works using the definition of multiplication.

Exercise 4.2.4. Assume as given that $3 \cdot 9 = 27$. Use the associative property of multiplication and the definition of multiplication to explain why $6 \cdot 9 = 27 + 27$. Then, calculate the sum $27 + 27$ using the addition algorithm.

EXERCISE 4.2.5. Explain the distinction between $8 \cdot 9$ and $9 \cdot 8$ in terms of the definition of multiplication. Then, use the definition of multiplication to explain why, nonetheless, $8 \cdot 9 = 9 \cdot 8$.

EXERCISE 4.2.6. Given that $7 \cdot 9 = 63$, find $8 \cdot 9$ using bootstrapping.

EXERCISE 4.2.7. Given that $3 \cdot 6 = 18$ and $4 \cdot 6 = 24$, find $7 \cdot 6$ using these two facts, the definition of multiplication, and the addition algorithm.

4.3. The Distributive Property and negative number multiplication

4.3.1. The Distributive Property. So far, all of the properties of numbers we have discussed have involved only one operation: addition or multiplication. The distributive property is unique in that it involves *both* addition and multiplication, and explains how they interact. Since this property is more complicated, I don't feel the need to convince you that it has content, as this one is not usually immediately obvious to students.

Let's look at a simple example of an array: at the post office, P.O. (redundant!) boxes are arranged into an array with 12 rows and 13 columns.

$$\begin{array}{ccccccccccccc}
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b
\end{array}$$

Previously, our standard way of counting the boxes would be to take groups = rows and objects = P.O. boxes, and we find that there are $12 \cdot 13$ boxes in all (12 rows of 13 boxes in each row), using the definition of multiplication. However, this is not the only way to count the boxes! We have already seen some other ways, such as using columns, but here I want to focus on two new ways (already implicit in the previous section): subdividing the objects before and after multiplication. To clarify, instead of viewing the rows as containing 13 boxes, we can imagine there is a dividing wall splitting up the rows as follows:

$$
\begin{array}{ccccccc|cccccc}
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b
\end{array}
$$

This allows us to count the boxes in the following slightly different way. Using the dividing wall, we can view each row as containing 7 boxes, and then 6 more boxes, for a total of $7 + 6$ boxes, using the definition of addition. You might say that $7 + 6 = 13$, and I would say "You are right, but the point here is structural, we are not interested in calculation, the point is that we *can* view the number of boxes as $7 + 6$." So, just bear with me for now. If we view the number of boxes in each row as $7 + 6$, then, still using groups = rows and objects = P.O. boxes, we get that the total number of boxes is $12 \cdot (7 + 6)$. See the footnote[2] if you are concerned about the parentheses.

Alternatively, we can count the boxes by using the wall to split the boxes into two separate arrays, one with 7 columns and one with 6 columns, and then adding the numbers of boxes in each array together. In this way, again using groups and objects as above. We get that there are $12 \cdot 7$ boxes in the left array, and $12 \cdot 6$ boxes in the right array. By definition of addition, then, there are a total of $12 \cdot 7 + 12 \cdot 6$ boxes altogether. Since we have counted the same array of P.O. Boxes in two different ways, we know the resulting values must be equal!

$$12 \cdot (7 + 6) = 12 \cdot 7 + 12 \cdot 6$$

We have now seen an example of the distributive property, and an explanation for why it's true. Let's go ahead and state it.

THEOREM 4.3.1 (The Distributive Property). *Let m, n, and p be positive whole numbers. Then, $m \cdot (n + p) = m \cdot n + m \cdot p$.*

There are many variations on this property, for example, we can divide the columns into three sections to get $m \cdot (n+p+q) = m \cdot n + m \cdot p + m \cdot q$. We can also subdivide the groups instead of the objects. For example, say there is a horizontal

[2]Here, we must write parentheses around $7+6$, the expression $12 \cdot 7+6$ means to first multiply $12 \cdot 7$, and then add 6, which is clearly not the number of P.O. boxes! We will clarify this further when we discuss the order of operations.

wall trim dividing the P.O. Boxes from above like this:

$$\begin{array}{ccccccccccccc}
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
\hline
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b \\
b & b & b & b & b & b & b & b & b & b & b & b & b
\end{array}$$

Again, using the exact same groups and objects as above, we can see the rows (groups) as subdivided into 7 rows and then 5 more rows, for a total of $7 + 5$ rows. Each row has 13 boxes (objects), so by definition again, we see that there are $(7 + 5) \cdot 13$ boxes in the array.

Finally, we can view the above as two distinct arrays of boxes, the top array has 7 rows of 13 boxes, which is $7 \cdot 13$ boxes, and the bottom array has 5 rows of 13 boxes, which is $5 \cdot 13$ boxes, for a toal of $7 \cdot 13 + 5 \cdot 13$ boxes in all. We have once again counted the boxes in two different ways and so we obtain:

$$(7 + 5) \cdot 13 = 7 \cdot 13 + 5 \cdot 13$$

We could also have derived this by applying the commutative property of multiplication a few times, and using the earlier version of the distributive property:

$$(7 + 5) \cdot 13 = 13 \cdot (7 + 5) = 13 \cdot 7 + 13 \cdot 5 = 7 \cdot 13 + 5 \cdot 13$$

This is one of many possible examples of building a web of relationships implied by all these properties. Learning to view the same fact from multiple perspectives is a key step in the process of understanding the structure of arithmetic (and math in general).

We collect the above variations on the Distributive Property below, we will simply say 'Distributive Property' to describe any of these.

THEOREM 4.3.2 (Distributive Property Variations). *Let m, n, p and q be positive whole numbers. Then,*

$$m \cdot (n + p + q) = m \cdot n + m \cdot p + m \cdot q$$

and

$$(m + n) \cdot p = m \cdot p + n \cdot p$$

4.3.2. Negative number multiplication. Our definition of multiplication works well for positive whole numbers, and thus far, we have used such numbers almost exclusively in our development of the properties of numbers and operations. It is possible, using the idea of owing money, for example, to come up with interpretations of the definition of multiplication involving negative numbers. However, we will instead use a theoretical approach common in mathematics: definition by insisting on certain properties. We would like to define multiplication of negative numbers in such a way that preserves the Associative Property of Multiplication, the Commutative Property of Multiplication, and the Distributive Property. These properties together completely determine the definition.

First of all, what about multiplication by zero? Well, it is fairly easy to argue that *any number of groups of zero objects is still zero objects*, so it works well with our definition to say that $m \cdot 0 = 0$ for any positive number m. By the commutative property, we must also have $0 \cdot m = m \cdot 0 = 0$ for any positive number m. Finally zero groups of zero objects should also be zero objects (I know that sounds strange), so it make sense to define $0 \cdot 0 = 0$. We should be able to see easily from here, that the three properties mentioned above are still valid if we expand multiplication to include zero in this way. For example, $(m \cdot n) \cdot p = m \cdot (n \cdot p)$ since, if any one of m, n or p is zero, then both sides must equal zero as well, by our definitions.

Next, we will try do define an expression like $m \cdot n$ where n is negative, and m is positive. We can appeal to the definition in this case by thinking about owing money. Say you owe \$6 to 4 different friends, how much do you owe in all? We can think of this as groups = friends, and units= dollars. In this case you owe \$6 (units) to each friend (group), so we will say that each friend 'has' $-\$6$ dollars. In this way, we get that you owe $4 \cdot (-6)$ dollars in all by the usual definition of multiplication. It should be clear that the total amount your 4 friends 'have', by calculating in any of the ways mentioned previously, is $-(4 \cdot 6) = -24$ dollars.

If the above argument seemed a bit unconvincing, here's another one that may or may not be more convincing. We already know that $4 \cdot 6 = 24$ by calculation using any technique from above. We also know that $6 + (-6) = 0$ by thinking in terms of the number line: start at 6 and then move -6 units on the number line. Therefore, if we have the distributive property, we must have:

$$4 \cdot 6 + 4 \cdot (-6) = 4 \cdot (6 + (-6)) = 4 \cdot 0 = 0$$

using our rules for multiplication by zero developed in this section. Now, we know that the value $4 \cdot (-6) =?$ must satisfy the equation $24+? = 0$ by the above calculation. So, if you start at 24, how many units do you move to end up at zero? The answer is clearly -24, i.e. 24 units to the left. It follows from here that we must have $4 \cdot (-6) = -24$ if we want the distributive property to continue to hold! By the commutative property, we must also have $(-6) \cdot 4 = -24$. It follows that:

Lemma 4.3.3. *If m and n are positive whole numbers, then $m \cdot n$ and $m \cdot (-n)$ are opposites, and $m \cdot n$ and $(-m) \cdot n$ are opposites:*

$$m \cdot (-n) = -(m \cdot n) = (-m) \cdot n$$

We now have many paths open to us to define the only remaining product of whole numbers not yet covered by the above: the product of two negative numbers. We can argue in terms of the distributive property: we know that $4 \cdot (-6) = -24$ by above, so $(-4) \cdot (-6) =?$ is determined by the distributive property (variation) as follows:

$$? + (-24) = (-4) \cdot (-6) + 4 \cdot (-6) = ((-4) + 4)) \cdot 6 = 0 \cdot 6 = 0.$$

Therefore, $(-4) \cdot (-6)$ must be the *opposite* of $4 \cdot (-6) = -24$. As we discussed earlier, the opposite of -24 is 24, so therefore $(-4) \cdot (-6) = 24$.

I hope that our careful development of the properties (associative, commutative, distributive) for positive whole numbers have convinced you that they are useful, and so you will agree that it is helpful to define multiplication in such a way that we can continue to have these properties available to us. I would suggest reviewing the earlier sections on the properties of multiplication and the distributive property to remind you how helpful and intuitive those properties are! I generally find that students are unconvinced by arguments such as the one in this section, but here I will flip the question around: how would *you* prefer to define something like $4\cdot(-6)$? Do you have a simpler or more reasonable way of defining this value? If so, please let me know! Ultimately this section is both defining the value of a product of negative numbers, *and* making an argument that the definition is good because it preserves some of the nice properties we had before for positive numbers.

We can now extend the above lemma to any whole numbers, and in this way we also have a definition of the product of any whole numbers, positive, negative, or zero.

LEMMA 4.3.4. *If m and n are* any *whole numbers, then $m \cdot n$ and $m \cdot (-n)$ are opposites, and $m \cdot n$ and $(-m) \cdot n$ are opposites:*

$$m \cdot (-n) = -(m \cdot n) = (-m) \cdot n$$

If you want to parse this for a product of two negative numbers it could look like: $(-4)\cdot(-6) = -(4\cdot(-6)) = -(-(4\cdot 6)) = 4\cdot 6 = 24$. Note we apply the lemma twice, once for each of the negative factors.

Exercises

EXERCISE 4.3.1. Invent a scenario involving an array, and interpret it in different ways to explain the equation $2 \cdot (3 + 4) = 2 \cdot 3 + 2 \cdot 4$. Use the definition of multiplication, describe the groups and objects/units for each multiplication. Do not calculate the values of either side of the equation.

EXERCISE 4.3.2. Invent a scenario involving an array, and interpret it in different ways to explain the equation $(2 + 3) \cdot 4 = 2 \cdot 4 + 3 \cdot 4$. Use the definition of multiplication, describe the groups and objects/units for each multiplication. Do not calculate the values of either side of the equation.

EXERCISE 4.3.3. Use the idea of owing money to explain the equation $5\cdot(-3) = -15$.

EXERCISE 4.3.4. Assume that $5 \cdot 3 = 15$ and use the distributive property to explain the equation $5 \cdot (-3) = -15$.

EXERCISE 4.3.5. Assume that $5\cdot(-3) = -15$, explain why $(-3)\cdot 5 = -15$ using a property of numbers from this book.

EXERCISE 4.3.6. Assume that $5\cdot(-3) = -15$ and use the distributive property to explain the equation $(-5) \cdot (-3) = 15$.

EXERCISE 4.3.7. Argue in terms of the number line that if $m+? = 0$, then ? must be the opposite of m, i.e. $? = -m$. (This applies whether m is positive or negative, but feel free to split up the cases to make your argument, if you prefer).

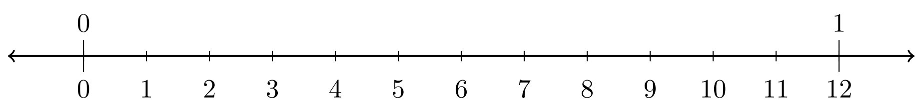
EXERCISE 4.3.8. Argue in terms of the number line that if $0+? = 0$, then ? must also be zero, i.e. $? = 0$.

EXERCISE 4.3.9. Now, use the fact that $0 + 0 = 0$, the previous problem, and the distributive property to explain why $m \cdot 0 = 0$ for any whole number m.

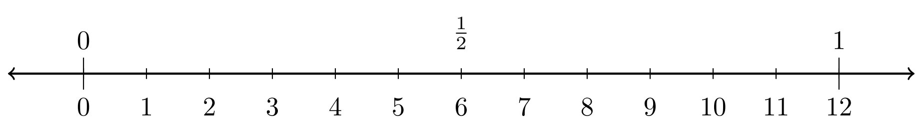
EXERCISE 4.3.10. Use the rules for multiplication by negative numbers to explain why $-1 \cdot n = n \cdot (-1) = -n$ for any whole number n.

4.4. Fraction multiplication

We will now extend the definition of multiplication to fractions. First, we will focus on positive fractions. It is relatively easy to tweak, or even repeat, some of the scenarios we have already used to allow for both fractional group amounts, and fractional unit/object amounts. For example, we can look at a simple unit conversion problem. Let groups = feet and units = inches. We know that there are 12 inches in 1 foot, so how many inches are in $\frac{1}{2}$ of a foot? By definition of multiplication, then, there are $\frac{1}{2} \cdot 12$ inches in $\frac{1}{2}$ of a foot. We can calculate this value by using a visualization on the number line:



Here we have indicated feet above the number line, and inches below the number line. Here we can easily see that half of a foot must appear in the following position:



Thus, half of a foot must be 6 inches, and so by above, we have $\frac{1}{2} \cdot 12 = 6$.

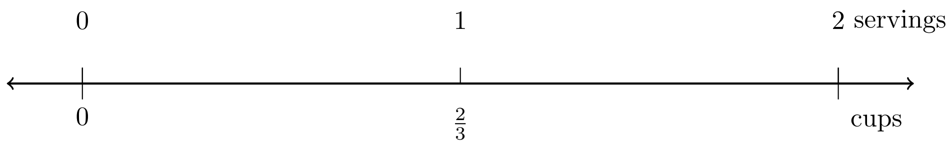
We can also visualize this with object abbreviations $f = iiiiiiiiiiii$, which can be divided into two equal groups as follows: $f = (iiiiii)(iiiiii)$, and thus $\frac{1}{2}$ of f (the whole amount) is $(iiiiii) = 6$ inches. Again by definition of multiplication, this implies that $\frac{1}{2} \cdot 12 = 6$.

One nice practical example to illustrate fraction multiplication is serving size: *One serving of orange juice is $\frac{2}{3}$ cup. How any cups in $\frac{3}{4}$ of a serving?* Note very carefully that there are *two* different whole amounts involved in this problem. The fraction $\frac{2}{3}$ is of the whole amount 1 cup, whereas the fraction $\frac{3}{4}$ is of a serving of orange juice. It is very important to keep the whole amounts straight to understand this problem.

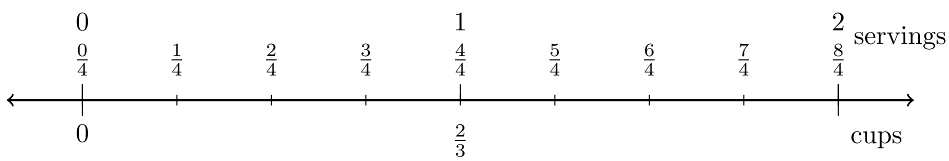
Here is a key moment to help you set this up to use the definition of multiplication, observe the wording here: **One serving** of orange juice is $\frac{2}{3}$ **cup**. The most natural way to read this in terms of groups and units is to take groups = servings of orange juice, and units = cups. Then, we have $\frac{2}{3}$ units (cup) in one group (serving). We are then able to find the number of units (cup) in $\frac{3}{4}$ of a serving by doing the multiplication (by definition):

$$\frac{3}{4} \cdot \frac{2}{3}.$$

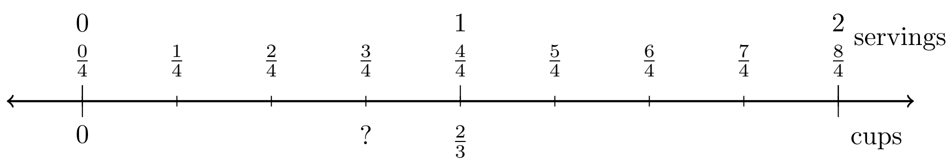
We can calculate this value by using double-labeled number lines, as we did above. The numbers above the line are servings of orange juice, and the numbers below the line are cups. We therefore have 1 (serving) on top lined up with $\frac{2}{3}$ (cup or cups[3]) on the bottom.



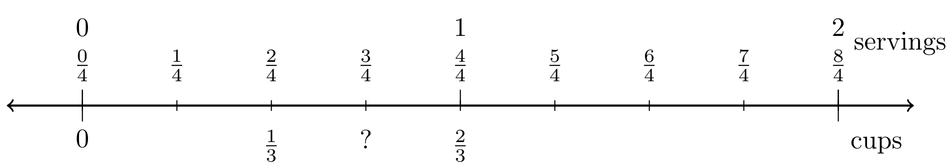
Note that we do not give a numerical value for cups on the right side of the number line, to line up with 2 servings, this is for two reasons: first, we don't know what that value is at the moment, and second, we do not need to know to complete the problem, so we leave it unlabeled as a placeholder. Next, we want to divide servings into 4 equal parts, each of size $\frac{1}{4}$ of a serving.



Ultimately, we want to find the value for cups that goes underneath $\frac{3}{4}$ servings, we use a '?' below to indicate the value we seek.

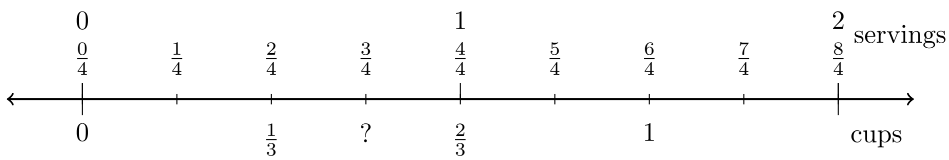


Let's proceed slowly as this is the trickiest part. The key point to understand about $\frac{2}{3}$ cups is that it is 2 parts of size $\frac{1}{3}$ cup. So, it is itself divided into **two** equal parts! Thus, if we find the **halfway** point between 0 and $\frac{2}{3}$ cups, that must represent **one** part of size $\frac{1}{3}$ cup, i.e., $\frac{1}{3}$ cup.

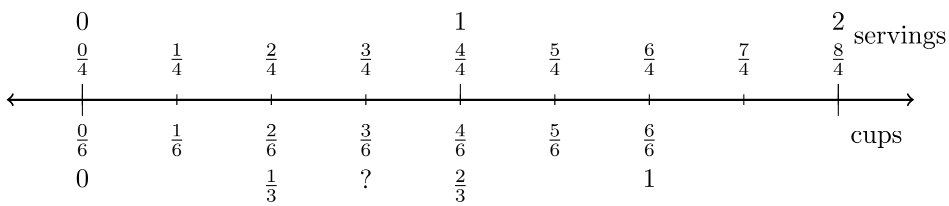


In this way, we can now see that $\frac{2}{4} = \frac{1}{2}$ of a serving of juice is equal to $\frac{1}{3}$ cup. This is not what we need just yet, but note the progress, we have already learned something interesting about the relationship between these two whole amounts. Observe now, that we can find the location of 1 cup below the number line simply by going to $\frac{3}{3} = 1$ cup, now that we know the equivalent of $\frac{1}{3}$ cup in servings:

[3]For fractional units it is acceptable to say $\frac{2}{3}$ cup, $\frac{2}{3}$ of a cup, or $\frac{2}{3}$ cups



To complete the problem, we now flip our point of view and think about the whole amount 1 cup on the bottom of the number line. The tick marks above divide 1 cup into *six* equal parts, therefore the distance between tick marks represents $\frac{1}{6}$ cup. If we simply count tick marks we can easily find the value we were looking for:

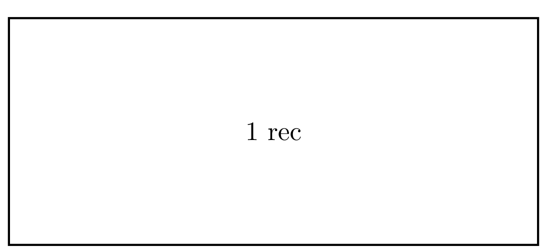


Now we can read the value we wanted from below the number line: $\frac{3}{4}$ of a serving is equal to $\frac{3}{6}$ cups. This implies, using the definition of multiplication that

$$\frac{3}{4} \cdot \frac{2}{3} = \frac{3}{6} \text{ cups.}$$

We could simplify the fraction using techniques from earlier in the book to get $\frac{3}{6} = \frac{1}{2}$ cups of juice in $\frac{3}{4}$ serving, but that is not necessary here. The point is to use the definitions of fraction and multiplication, together with a double-labeled number line to calculate a multiplication of fractions. This type of exercise, while not at all efficient, forces you to understand the concepts and is therefore well worth the effort of going through a few times.

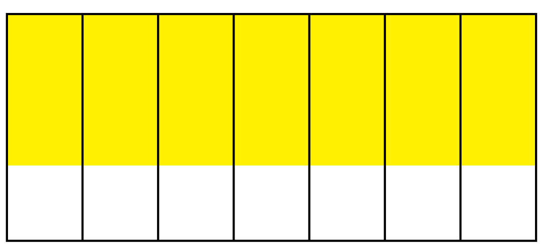
We will now discuss a different way of calculating products of fractions using the idea of rectangles to represent units. To start with, we will make our unit amount the rectangle below.



To avoid ambiguity, we will refer to the unit above as a *rec*. The above picture represents 1 rec. Now, we subdivide the rectangle into thirds, each of size $\frac{1}{3}$ rec.

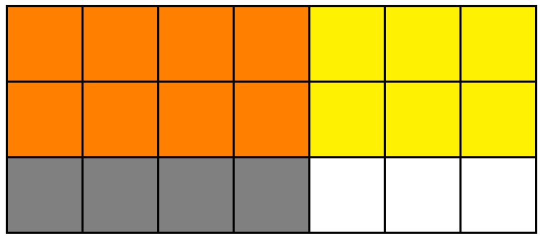
In this way, we can let groups = the yellow rectangle below formed by the *two of the horizontal strips* above and we let units = recs, so 1 group is equal to $\frac{2}{3}$ rec.

This may seem an odd sort of group, and it is rather artificial, but the goal here is to be able to do a multiplication of the form $? \cdot \frac{2}{3}$, so we need to form groups and units so that 1 group has $\frac{2}{3}$ units, this is a simple visual way to illustrate that idea. The definition of multiplication can now be applied to find a fraction of a group! What if we want to find $\frac{4}{7}$ of a group in terms of recs (fractions of the whole amount given by the original rectangle)? First of all, observe that by definition, that will be equal to $\frac{4}{7} \cdot \frac{2}{3}$ recs. Thus, if we can figure out what fraction of the rectangle that takes up, we can calculate the product. If you've made it this far, this is the easy part, now we simply divide the group into seven equal parts *horizontally* as follows:



Now, we look at *four* of those horizontal sections:

Then, we look at how these four sections overlap with our group given in yellow above to show what $\frac{4}{7}$ of a group (shown in orange) looks like as a fraction of the rec(tangle).



Now, the last part is we want to calculate the orange region as a fraction of the rec(tangle) given at the start. We will do this in a systematic way that does not in any way depend on the particular values taken by our fractions, but will work to calculate any product. First, observe that the original rec(tangle) is decomposed into an array of squares, and that our orange region is also decomposed into an array of squares. This means, we can calculate all the values we need by letting squares = units, and using multiplication to count how many squares we have in the full rec(tangle) and in the orange region. First, observe that the rec(tangle) is made up of 3 rows with 7 squares in each row. If we let groups = rows, and units = squares, we are then able to see that we have a total of $3 \cdot 7$ squares in the rectangle (if you wish you can calculate $3 \cdot 7 = 21$, but that is NOT a good idea at this stage, as we are trying to learn why we multiply). From this, it follows by the *definition* of fraction that each square represents $\frac{1}{3 \cdot 7}$ of the original rec(tangle), and is therefore equal to $\frac{1}{3 \cdot 7}$ rec.

Now that we know the size of a square in terms of recs (our given unit from above), we need to find out how many of these squares give us the orange region, and then we will know the value of the orange region in recs, and we will have calculated the product above. Again, we see that the orange region is made up of 2 rows with 4 squares in each row, by definition then, the orange region is made up of a total of $2 \cdot 4$ squares. Each square has size $\frac{1}{3 \cdot 7}$ rec, and therefore the orange region is equal to $\frac{2 \cdot 4}{3 \cdot 7}$ rec, by definition of fraction!

We can now conclude from the above that:

$$\frac{4}{7} \cdot \frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 7} = \frac{4 \cdot 2}{7 \cdot 3} \left(= \frac{8}{21} \right) \text{ rec}$$

We have reversed the factors: $\frac{4 \cdot 2}{7 \cdot 3}$ to make it easier to see the general rule here: we multiply across. We could have organized groups and units a bit differently to obtain this order directly, but we have the Commutative Property, so there's no particular reason to prefer $2 \cdot 4$ or $4 \cdot 2$ for example, as they both have the same value, and we are just calculating here. Note, however, that to set up the original multiplication, we had to use a certain order dictated by the definition of multiplication. This subdivided rectangle approach is the easiest way to visualize multiplying fractions and provides a relatively easy way to calculate them is well, so this is worth spending some more effort!

THEOREM 4.4.1 (Fraction Multiplication Rule). *Let* $\frac{m}{n}$ *and* $\frac{p}{q}$ *be positive fractions, then*

$$\frac{m}{n} \cdot \frac{p}{q} = \frac{m \cdot p}{n \cdot q}.$$

Note carefully that calculating fraction multiplication **does not require a common denominator**! Though the concepts and the visualization can be trickier, the actual calculation of multiplication of fractions is straightforward and if you are comfortable with whole number multiplication, you can do it quickly and easily.

Exercises

EXERCISE 4.4.1. There are 12 inches in a foot. Use the definition of multiplication to write an expression for the number of inches in $\frac{2}{3}$ of a foot. Calculate the result by reasoning about fractions: use object abbreviations, pictures, or a number line. Do not apply the fraction multiplication rule.

EXERCISE 4.4.2. There are 100 centimeters in a meter. Use the definition of multiplication to write an expression that represents $\frac{1}{3}$ of a meter in centimeters. Do not calculate.

EXERCISE 4.4.3. On a package of cheddar cheese it says that there are $\frac{4}{5}$ ounces in 1 serving of cheese.

(a) How many ounces in 3 servings of cheddar cheese? Use the definition of multiplication to write an expression, and calculate it by reasoning about fractions.

(b) How many ounces in $\frac{1}{4}$ servings of cheddar cheese? Use the definition of multiplication to write an expression, and calculate it by reasoning about fractions: use object abbreviations, pictures, or a number line.

(c) How many ounces in $\frac{2}{3}$ servings of cheddar cheese? Use the definition of multiplication to write an expression, and calculate it using a rectangle as your unit and subdivide it as an array. Do not apply the fraction multiplication rule.

EXERCISE 4.4.4. Above we used reasoning about fractions to find that

$$\frac{3}{4} \cdot \frac{2}{3} = \frac{3}{6},$$

but according to the fraction multiplication rule:

$$\frac{3}{4} \cdot \frac{2}{3} = \frac{3 \cdot 2}{4 \cdot 3} = \frac{6}{12}.$$

Explain why these answers are equivalent:

$$\frac{3}{6} = \frac{6}{12}$$

by reasoning about fractions with object abbreviations, pictures, or a number line.

EXERCISE 4.4.5. Follow the steps below.

(a) Explain why $\frac{n}{1} = n$ for any positive whole number n using the definition of fraction.

(b) Now use the rule for multiplication of fractions to explain why $\frac{1}{m} \cdot n = \frac{n}{m}$.

4.5. Extending the rules to fractions, negative fractions

We will now declare that the previous properties we had for whole numbers, also apply to fractions.

THEOREM 4.5.1. *All previous properties: commutative, associative, and distributive, apply to addition and multiplication of fractions.*

As a result, we have the previous lemma, now applied to fractions as well as whole numbers.

LEMMA 4.5.2. *If m and n are* any *numbers (including fractions), then $m \cdot n$ and $m \cdot (-n)$ are opposites, and $m \cdot n$ and $(-m) \cdot n$ are opposites:*

$$m \cdot (-n) = -(m \cdot n) = (-m) \cdot n$$

To be completely clear, we should justify these statements using the rules we developed above. However, the explanations would be substantially similar to ones we've already given, so we will omit them and ask the reader to either believe the rules apply to fractions, or work out an explanation by doing some examples.

Let's briefly illustrate how we multiply negative fractions using the rule above. Say we want to calculate $\frac{3}{7} \cdot \left(-\frac{4}{3}\right)$. Then, we simply apply the lemma above to write:

$$\frac{3}{7} \cdot \left(-\frac{4}{3}\right) = -\left(\frac{3}{7} \cdot \frac{4}{3}\right) = -\left(\frac{3 \cdot 4}{7 \cdot 3}\right) = -\frac{12}{21}$$

using our fraction multiplication rule at the end. Our general strategy will be like this, we deal with any opposites and then, in the end, we always calculate positive products, applying the opposite at the end if needed.

Note that we also have the distributive property. *A cake recipe calls for $\frac{1}{3}$ cup of sugar for the icing and $\frac{1}{2}$ cup of sugar for the cake batter. Write an expression for the number of cups of sugar used in $\frac{2}{3}$ of the recipe.* Here we can take groups = cake recipes, and units = cups of sugar. Then, we are told that one cake recipe has $\frac{1}{3} + \frac{1}{2}$ cups of sugar in all. Thus, if we want the number of cups of sugar in $\frac{2}{3}$ of the recipe, we have the following amount:

$$\frac{2}{3} \cdot \left(\frac{1}{3} + \frac{1}{2}\right).$$

We used a number line earlier to calculate $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$. So we have:

$$\frac{2}{3} \cdot \left(\frac{1}{3} + \frac{1}{2}\right) = \frac{2}{3} \cdot \left(\frac{5}{6}\right) = \frac{2 \cdot 5}{3 \cdot 6} = \frac{10}{18} \text{ cups}$$

using the fraction multiplication rule. On the other hand, the distributive property says that:

$$\frac{2}{3} \cdot \left(\frac{1}{3} + \frac{1}{2}\right) = \frac{2}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{9} + \frac{2}{6}$$

Exercises

EXERCISE 4.5.1. Use the lemma to explain why $\left(-\frac{3}{7}\right) \cdot \left(-\frac{4}{3}\right) = \frac{12}{21}$.

EXERCISE 4.5.2. Use reasoning about fractions, object abbreviations, pictures, or a number line, to explain why the two fractional expressions are equal:

$$\frac{2}{9} + \frac{2}{6} = \frac{10}{18}.$$

EXERCISE 4.5.3. Use the properties of numbers and the fraction multiplication rule to calculate the following values. Use reasoning about fractions to simplify your answers if you can.

(a) $(-3) \cdot \frac{2}{3}$.

(b) $\frac{1}{6} \cdot \left(-\frac{5}{2}\right)$

(c) $\left(-\frac{4}{5}\right) \cdot \left(-\frac{3}{4}\right)$.

EXERCISE 4.5.4. A pizza recipe calls for $\frac{2}{3}$ tablespoon of salt for the dough and $\frac{3}{4}$ tablespoon of salt for the sauce. Altogether, $\frac{2}{3} + \frac{3}{4}$ tablespoons of salt are needed per pizza recipe. Let groups = pizza recipes, and units = tablespoons of salt.

(a) Calculate the amount of salt in tablespoons needed for $\frac{1}{2}$ of the pizza recipe by first adding the fractions $\frac{2}{3} + \frac{3}{4} =?$ using a common denominator as in a previous chapter. Use objects, pictures, or a number line to explain your result. Then, multiply $\frac{1}{2} \cdot ?$ using the fraction multiplication rule.

(b) Now, apply the distributive property:

$$\frac{1}{2} \cdot \left(\frac{2}{3} + \frac{3}{4}\right) = \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{3}{4} = \frac{2}{6} + \frac{3}{8}$$

and use reasoning about fractions to explain why this answer is equivalent to your answer from the previous part.

4.6. More on subtraction, fractions

We have already seen subtraction presented as the missing term in an addition problem. For example, if $5+? = 13$, then we necessarily have $13 - 5 =?$. When this was introduced, it only applied to positive whole numbers, but we can observe here that it also works with negative values: $-5+? = -13$ means $-13 - (-5) =?$, and if we recall our opposite rules, we know that $-13 - (-5) = -13 + 5 = -8$ by thinking in terms of the number line. Note that, by the distributive property, we have $-13 + 5 = -1 \cdot (13 - 5) = -1 \cdot 8 = -8$. Thus, if we need to calculate with larger negative numbers such as $-145 + 37$, we can instead calculate $145 - 37$ and then take the opposite.

This idea also works with fractions, if we have $\frac{1}{3}+? = \frac{2}{5}$, then we must have $\frac{2}{5} - \frac{1}{3} =?$. We can calculate this by using common denominators as before. Recall the switch trick, we subdivide the fifths into threes $f = ttt$, and then two fifths is $ff = (ttt)(ttt)$, but five fifths (or one whole amount) is $fffff = (ttt)(ttt)(ttt)(ttt)(ttt)$, so we see therefore that $t = \frac{1}{15}$ of the whole amount and $\frac{2}{5} = \frac{6}{15}$. Note that, given our rule for multiplying fractions, we can also subdivide into threes by multiplying by one whole divided into three equal parts, i.e., $\frac{3}{3}$. We know that for any number n, $1 \cdot n = n$, essentially by definition of multiplication: one group of n objects is n objects. So we must have $\frac{3}{3} \cdot \frac{2}{5} = \frac{2}{5}$, but on the other hand, by our fraction multiplication rule

$$\frac{3}{3} \cdot \frac{2}{5} = \frac{3 \cdot 2}{3 \cdot 5} = \frac{6}{15}.$$

Thus, we can see that $\frac{2}{5} = \frac{6}{15}$ via multiplication as well. This is a useful shortcut once you have understood the concept using abbreviations or pictures. We want to express $\frac{1}{3}$ using the same denominator 15, so by the switch trick, we want subdivide the thirds into *five* equal parts, which means we multiply by one whole divided into five equal parts, i.e., we multiply by $\frac{5}{5} = 1$. Thus, we obtain

$$\frac{5}{5} \cdot \frac{1}{3} = \frac{5}{15},$$

and we are able to calculate the subtraction:

$$\frac{2}{5} - \frac{1}{3} = \frac{6}{15} - \frac{5}{15} = \frac{1}{15}$$

and in terms of the original addition problem: $\frac{1}{3} + \frac{1}{15} = \frac{2}{5}$.

Exercises

Exercise 4.6.1. Find the missing term in each addition below by rewriting the problem as a subtraction and then using methods and ideas from this book to calculate the difference.

(a)
$$-7 + ? = -15$$

(b)
$$154 + ? = 73$$

(Hint: $73 - 154 = 73 + (-154) = -154 + 73 = -(154 - 73)$).

(c)
$$\frac{2}{3} + ? = \frac{3}{4}$$

(d)
$$\frac{3}{4} + ? = \frac{2}{3}$$

(e)
$$\frac{11}{6} + ? = \frac{4}{3}$$

Exercise 4.6.2. A chocolate chip cookie recipe calls for $\frac{3}{4}$ cup of sugar. Jessica accidentally added $\frac{2}{3}$ cup of sugar instead. How much more sugar does Jessica need to add to follow the recipe? Write an addition problem with a missing term, then solve the problem using subtraction.

4.7. Multiplication algorithm, multiplying decimals

In this section we develop an algorithm for calculating any product of numbers written in Base-10 (including decimals) using only single digit facts and some observations about multiplication by ten in base-10.

4.7.1. What's so special about multiplying by ten? Let's look at a simple example. *There are* 3 *feet in a yard. Use the definition of multiplication to write an expression for the number of feet in* 10 *yards.* We've seen many examples like this, we take groups = yards and units = feet, and so we get $10 \cdot 3$ feet in ten yards. If we draw ten groups of three objects in an array:

$$\begin{array}{ccc}
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & |
\end{array}$$

We want to count the objects in the array and express that number in Base-10, this is particularly easy in this case as we have exactly ten objects in three columns:

$$\begin{array}{|c|c|c|}
\hline
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
| & | & | \\
\hline
\end{array}$$

which is equivalent to XXX, or three groups of ten objects, and zero single objects. That is, we see that $10 \cdot 3 = 30$. Note that we get exactly one group of ten for every single object in $3 = |\,|\,|$.

What about decimals? *A quarter is worth* \$0.25*. How much is* 10 *quarters worth in dollars?* In this case we take groups = quarters, and units = dollars. Then, we get a total of $10 \cdot 0.25$ dollars. Using our object abbreviations above, we can write $0.25 = **{+}{+}{+}{+}{+}$. Then, we can display ten of these in an array.

$$\begin{array}{ccccccc}
* & * & + & + & + & + & + \\
* & * & + & + & + & + & + \\
* & * & + & + & + & + & + \\
* & * & + & + & + & + & + \\
* & * & + & + & + & + & + \\
* & * & + & + & + & + & + \\
* & * & + & + & + & + & + \\
* & * & + & + & + & + & + \\
* & * & + & + & + & + & + \\
* & * & + & + & + & + & +
\end{array}$$

Observe that the columns naturally divide this array into groups of ten, and, we have $^{**********} = |$ and $++++++++++ = *$. So, the above array is equivalent to $|\,|^{*****} = 2.5$ dollars, or as we would usually write: $\$2.50 = 10 \cdot 0.25$. See the pattern? We shift the whole number or decimal places to the *left* by one. The above arrays illustrate why this happens, we get a group of ten for each whole number or decimal place. Note that this applies to $10 \cdot 3 = 30$ as well, we just need to put a zero in the ones place since 3 has no tenths!

Since we have the commutative property, this also applies to the other order: $3 \cdot 10 = 30$ and $0.25 \cdot 10 = 2.5$.

Theorem 4.7.1 (multiplication by ten). *Let d be any number. Then, the base-10 representation of $10 \cdot d = d \cdot 10$ is the same as the base-10 representation of d, but shifted to the left one place.*

Observe that we can also apply the above to multiplication by one-tenth:

$$0.1 \cdot a = \frac{1}{10} \cdot a = b$$

means that b is one-tenth of a, so ten equal parts of size b is equal to a, so, in other words, $10 \cdot b = a$. This means that the base-10 representation of a is the same as the base-10 representation of b, but shifted to the *left* one place. If we reverse our point of view that means that the base-10 representation of b is the base-10 representation of a but shifted to the *right* one place. In this way, we can quickly calculate the product of any two single-nonzero-digit numbers in base-10 by finding the product of the single nonzero digits, and then working out where to shift the decimal point. Let's do a few examples. The single digit multiplication is shown in blue, the placement of the decimal point is determined by the place values of the factors.

$$\begin{aligned}
40 \cdot 3 &= \quad 120 \\
600 \cdot 70 &= \quad 42,000 \\
200 \cdot 0.4 &= \quad 80 \\
90 \cdot 0.07 &= \quad 6.3 \\
0.3 \cdot 0.08 &= \quad 0.024
\end{aligned}$$

Let's break down the second one. First of all $600 = 6 \cdot 100$ and $70 = 7 \cdot 10$. Therefore,

$$600 \cdot 70 = (6 \cdot 100) \cdot (7 \cdot 10) = (6 \cdot 7) \cdot (100 \cdot 10)$$

by the associative and commutative properties of multiplication (we have the same four factors, in a different order and with different parentheses, we omit the details of applying the properties as they are a bit tedious to show since we have three multiplications not two). Now, we have $6 \cdot 7 = 42$, and we have $100 \cdot 10 = (10 \cdot 10) \cdot 10$. For each factor of 10 we shift to the left one place, therefore, we will get our result by shifting 42 to the left *three* places:

$$(6 \cdot 7) \cdot (100 \cdot 10) = 42 \cdot (10 \cdot 10 \cdot 10) = 42,000$$

Let's look at a decimal example, we have $90 = 9 \cdot 10$ and $0.07 = 7 \cdot 0.01$. So we break down the product as before:

$$90 \cdot 0.07 = (9 \cdot 10) \cdot (7 \cdot 0.01) = (9 \cdot 7) \cdot (10 \cdot 0.01).$$

We now apply the single digit multiplication fact $9 \cdot 7 = 63$. We just need to work out $10 \cdot 0.01$ as a product of tens and tenths. Observe that a tenth of a tenth is a hundredth, or in other words $0.1 \cdot 0.1 = 0.01$ (you can also get this from the multiplication by ten rule as outlined above, or using fractions). Therefore, $10 \cdot 0.01 = 10 \cdot (0.1 \cdot 0.1)$, and so to multiply 63 by this number we shift *left* one time and we shift *right* two times. Shifting to the left one time and then to the right two times is the same as shifting to the right one time, or in terms of the number line: $-1 + 2 = 1$. Another way to see this is to apply the rule above to get $10 \cdot 0.01 = 0.1$. Thus, we shift the place values of 63 to the right one place in base-10 and we obtain:

$$90 \cdot 0.07 = (9 \cdot 10) \cdot (7 \cdot 0.01) = (9 \cdot 7) \cdot (10 \cdot 0.01) = 63 \cdot (10 \cdot 0.1 \cdot 0.1) = 6.3$$

4.7.2. The algorithm. We will now outline a method for multiplying any two numbers in base-10 by multiplying them digit-by-digit, then working out the decimal placement using the multiplication by ten rules from the previous section. We will show you an example first, then break down the steps so you understand how to do it. We will first do $562 \cdot 47$.

$$\begin{array}{cccccc}
 & & & 5 & 6 & 2 \\
 & & & \cdot & 4 & 7 \\
\hline
 & & \color{red}{1} & \color{red}{1} & & \\
 & & & & 1 & 4 \\
 & & & 4 & 2 & 0 \\
 & & 3 & 5 & 0 & 0 \\
 & & & & 8 & 0 \\
 & & 2 & 4 & 0 & 0 \\
+ & 2 & 0 & 0 & 0 & 0 \\
\hline
 & 2 & 6 & 4 & 1 & 4
\end{array}$$

We have shown the carries from the addition in red for clarity. There are now two questions we want to answer: why does it work like this, and how do you do it? First let's look at why it works like this as a lead in. We can always break down a base-10 number into a sum of single-nonzero-digit numbers as follows: $562 = 2 + 60 + 500$ and $47 = 7 + 40$ (we write from smallest to largest for convenience). According to the distributive property, then, we can calculate the product as follows:

$$562 \cdot 47 = (2 + 60 + 500) \cdot (7 + 40) = 2 \cdot 7 + 60 \cdot 7 + 500 \cdot 7 + 2 \cdot 40 + 60 \cdot 40 + 500 \cdot 40$$

As you see, we just calculate the products of all the single-nonzero-digit parts and then add the results together. This explains why we add column-wise at the end.

Now, let's take it step by step, we write the problem, lining up columns as follows:

$$\begin{array}{rrrrr} & & 5 & 6 & 2 \\ & & \cdot & 4 & 7 \\ \hline \end{array}$$

We do the first product which is $2 \cdot 7 = 14$ and we write that first:

$$\begin{array}{rrrrr} & & 5 & 6 & 2 \\ & & \cdot & 4 & 7 \\ \hline & & & 1 & 4 \end{array}$$

Next is the product $60 \cdot 7 = 420$:

$$\begin{array}{rrrrr} & & 5 & 6 & 2 \\ & & \cdot & 4 & 7 \\ \hline & & & 1 & 4 \\ & & 4 & 2 & 0 \end{array}$$

Followed by $500 \cdot 7 = 3500$:

$$\begin{array}{rrrrr} & & 5 & 6 & 2 \\ & & \cdot & 4 & 7 \\ \hline & & & 1 & 4 \\ & & 4 & 2 & 0 \\ & 3 & 5 & 0 & 0 \end{array}$$

Now we proceed to $40 \cdot 2 = 80$ (notice these are exact same products from the distributive property calculation above):

$$\begin{array}{rrrrr} & & 5 & 6 & 2 \\ & & \cdot & 4 & 7 \\ \hline & & & 1 & 4 \\ & & 4 & 2 & 0 \\ & 3 & 5 & 0 & 0 \\ & & & 8 & 0 \end{array}$$

Then, $40 \cdot 60 = 2400$.

$$\begin{array}{rrrrr} & & 5 & 6 & 2 \\ & & \cdot & 4 & 7 \\ \hline & & & 1 & 4 \\ & & 4 & 2 & 0 \\ & 3 & 5 & 0 & 0 \\ & & & 8 & 0 \\ & 2 & 4 & 0 & 0 \end{array}$$

Finally, $40 \cdot 500 = 20000$ (watch your zeroes carefully on this one, $5 \cdot 4 = 20$ so there may be an extra zero you are not expecting):

$$\begin{array}{rrrrr} & & 5 & 6 & 2 \\ & & \cdot & 4 & 7 \\ \hline & & & 1 & 4 \\ & & 4 & 2 & 0 \\ & 3 & 5 & 0 & 0 \\ & & & 8 & 0 \\ & 2 & 4 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{array}$$

Now, we know we need to add the values up to get the product, so we do our addition algorithm column by column:

$$\begin{array}{cccccc}
 & & & 5 & 6 & 2 \\
 & & & \cdot & 4 & 7 \\ \hline
 & & & & 1 & 4 \\
 & & & 4 & 2 & 0 \\
 & & 3 & 5 & 0 & 0 \\
 & & & & 8 & 0 \\
 & & 2 & 4 & 0 & 0 \\
+ & 2 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & 4
\end{array}$$

We carry a one if we need to:

$$\begin{array}{cccccc}
 & & & 5 & 6 & 2 \\
 & & & \cdot & 4 & 7 \\ \hline
 & & & 1 & & \\
 & & & & 1 & 4 \\
 & & & 4 & 2 & 0 \\
 & & 3 & 5 & 0 & 0 \\
 & & & & 8 & 0 \\
 & & 2 & 4 & 0 & 0 \\
+ & 2 & 0 & 0 & 0 & 0 \\ \hline
 & & & & 1 & 4
\end{array}$$

And so on (we've seen this movie before):

$$\begin{array}{cccccc}
 & & & 5 & 6 & 2 \\
 & & & \cdot & 4 & 7 \\ \hline
 & & 1 & 1 & & \\
 & & & & 1 & 4 \\
 & & & 4 & 2 & 0 \\
 & & 3 & 5 & 0 & 0 \\
 & & & & 8 & 0 \\
 & & 2 & 4 & 0 & 0 \\
+ & 2 & 0 & 0 & 0 & 0 \\ \hline
 & & & 4 & 1 & 4
\end{array}$$

followed by

$$\begin{array}{cccccc}
 & & & 5 & 6 & 2 \\
 & & & \cdot & 4 & 7 \\ \hline
 & & 1 & 1 & & \\
 & & & & 1 & 4 \\
 & & & 4 & 2 & 0 \\
 & & 3 & 5 & 0 & 0 \\
 & & & & 8 & 0 \\
 & & 2 & 4 & 0 & 0 \\
+ & 2 & 0 & 0 & 0 & 0 \\ \hline
 & & 6 & 4 & 1 & 4
\end{array}$$

followed by

$$
\begin{array}{cccccc}
 & & & 5 & 6 & 2 \\
 & & & \cdot & 4 & 7 \\
\hline
 & & {\color{red}1} & {\color{red}1} & & \\
 & & & & 1 & 4 \\
 & & & 4 & 2 & 0 \\
 & & 3 & 5 & 0 & 0 \\
 & & & & 8 & 0 \\
 & & 2 & 4 & 0 & 0 \\
+ & 2 & 0 & 0 & 0 & 0 \\
\hline
 & 2 & 6 & 4 & 1 & 4
\end{array}
$$

We arrive at the end, and we see that $562 \cdot 47 = 26414$.

We would like to do an example with decimals, but to illustrate the concept, we will take the same two numbers and shift the digits, to show you that the algorithm is the same no matter where you put the decimal point. We will calculate $56.2 \cdot 0.47$. First, observe the single digit breakdown: $56.2 = 0.2+6+50$ and $0.47 = 0.07+0.04$. So we calculate:

$$56.2\cdot0.47 = (0.2+6+50)\cdot(0.07+0.4) = 0.2\cdot0.07+6\cdot0.07+50\cdot0.07+0.2\cdot0.4+6\cdot0.4+50\cdot0.4$$

The main issue is deciding where to put the decimal point for each product, we already know the digits from our calculation above, as the single digit products are all the same! Note that, for example, $0.2 \cdot 0.07$ is 14 shifted to the right three places, so $0.2 \cdot 0.07 = 0.014$. We will use excessive spacing for the decimal point to clearly illustrate how everything lines up, particularly below the line, we now calculate all the products listed above:

$$
\begin{array}{ccccccc}
 & & & 5 & 6 & . & 2 \\
 & & \cdot & 0 & . & 4 & 7 \\
\hline
 & & 0 & . & 0 & 1 & 4 \\
 & & 0 & . & 4 & 2 & 0 \\
 & & 3 & . & 5 & 0 & 0 \\
 & & 0 & . & 0 & 8 & 0 \\
 & & 2 & . & 4 & 0 & 0 \\
+ & 2 & 0 & . & 0 & 0 & 0
\end{array}
$$

Now, we add up the values, carrying as needed, just like before:

$$
\begin{array}{ccccccc}
 & & & 5 & 6 & . & 2 \\
 & & \cdot & 0 & . & 4 & 7 \\
\hline
 & & {\color{red}1} & & {\color{red}1} & & \\
 & & 0 & . & 0 & 1 & 4 \\
 & & 0 & . & 4 & 2 & 0 \\
 & & 3 & . & 5 & 0 & 0 \\
 & & 0 & . & 0 & 8 & 0 \\
 & & 2 & . & 4 & 0 & 0 \\
+ & 2 & 0 & . & 0 & 0 & 0 \\
\hline
 & 2 & 6 & . & 4 & 1 & 4
\end{array}
$$

Thus, we obtain $56.2 \cdot 0.47 = 26.414$. Note that these are exact same digits as $562 \cdot 47 = 26414$ but shifted to the right three places.

Exercises

Exercise 4.7.1. Show using an array of objects in base-10 that $10 \cdot 4.2 = 42$.

Exercise 4.7.2. Calculate the following single-nonzero-digit products by doing a single-digit multiplication, and then shifting to the appropriate place.

(a) $6 \cdot 80$

(b) $60 \cdot 8$

(c) $60 \cdot 0.8$

(d) $0.60 \cdot 8$

(e) $60 \cdot 50$

(f) $0.6 \cdot 0.05$

Exercise 4.7.3. Decompose the numbers into a sum of single-nonzero-digit terms from smallest to largest, as shown in the section.

(a) 9684

(b) 9.684

(c) 1.05

Exercise 4.7.4. Carry out the algorithm to multiply the numbers.

(a) $753 \cdot 29$

(b) $7.53 \cdot 2.9$

(c) $509 \cdot 48$

(d) $5.09 \cdot 0.48$

Part 3

Algebra

CHAPTER 5

Division

5.1. Division: how many groups

We are going to approach division similar to the way we approached subtraction, as a solution to a *multiplication* problem with a missing factor. Let's state the basic rules abstractly, and then we will discuss the meaning.

THEOREM 5.1.1. *Let m and n be any numbers (positive, negative, fractions, decimals). Then, the multiplication problems $m \cdot ? = n$* ***and*** *$? \cdot m = n$ are* ***both*** *equivalent to the division problem $n \div m =?$.*

We will take this as essentially the definition of division. Note that although the division $n \div m =?$ can only be written one way, the multiplication equations that are equivalent can be written in *two* different ways: $m \cdot ? = n$ and $? \cdot m = n$. Since our definition of multiplication is asymmetric (each factor plays a different role, the left factor is the number of groups and the right factor is the number of units/objects in one group), we get two different interpretations of division. If the missing factor is on the left $? \cdot m = n$, the we call this a '*how many groups*' division problem for, more or less, obvious reasons, the solution to the division problem tells us how many groups we need.

Here is a simple example: *Yondu has* 35 *candies and wants to put them in bags with* 5 *candies in each bag. How many bags does he need?* We can set this up as a *how many groups* division problem by taking groups = bags and objects = candies. Then, the number of bags Yondu needs is ? (unknown), and we have 5 candies in each bag, making a total of 35 candies. Thus, we get the equation:

$$? \cdot 5 = 35$$

By our interpretation of division given at the beginning of this section this means $? = 35 \div 5$. The solution to this is fairly easy to visualize, one bag looks like $b = (ccccc)$, how many of these bags does Yondu need to use 35 candies? Well,

$$bbbbbbb = (ccccc)(ccccc)(ccccc)(ccccc)(ccccc)(ccccc)(ccccc)$$

makes 35 candies, so Yondu needs $? = 7$ bags. If we remember the single digit multiplication facts, we can see the solution is 7 just by remembering $7 \cdot 5 = 35$. We used our knowledge of multiplication to calculate $35 \div 5 = 7$.

Another example: *Nick has a rope that is* 42 *meters long. He wants to cut the rope into pieces that are each* 7 *meters long. How many pieces of rope will he have?* We assign groups = pieces of rope and units = meters. We see that there are 7 units (meters) in one group (piece of rope). So, we are led to the equation $? \cdot 7 = 42$ where ? represents the number of pieces of rope he will have. By definition, we see that $? = 42 \div 7$. If we also recall the fact $6 \cdot 7 = 42$ then we immediately have: $42 \div 7 = 6$ pieces of rope, since $6 \cdot 7 = 42$.

Exercises

Exercise 5.1.1. For each problem below, set up the solution as a *how many groups* division problem by using the definition of multiplication. Then, solve the problem using your knowledge of single digit multiplication facts.

(a) Valkyrie wants to buy some swords. Each sword costs \$9 and she has \$72 to spend. How many swords can she buy?

(b) Rocket wants to put cookies into packages. He has a total of 45 cookies and he puts 5 cookies in each package. How many packages will he have?

(c) Sam is giving out dog treats in gift bags. He puts 3 treats in each bag and he has 18 treats to put in bags. How many gift bags will he have?

Exercise 5.1.2. Write a *how many groups* word problem for $30 \div 6 =?$. Solve the problem using single digit multiplication facts, and interpret your answer in context.

5.2. Division: how many units/objects in one group

Returning to the observation from the previous section, if we have instead the missing factor on the right: $m \cdot ? = n$, then we call this a '*how many units/objects in one group*' division problem. In this case, the result will tell us, evidently, how many units or objects should be in one group. We will now go through the same abstract division problems as the previous section, but rewrite the word problems so that they are now *how many units*[1] division problems.

Our first example: *Yondu has* 35 *candies and wants to divide them equally among* 5 *bags. How many candies should he put in each bag?* We use the same set up as before: groups = bags and objects = candies. In this case, however, the number 5 represents the number of groups, so the multiplication problem now has the form: $5 \cdot ? = 35$. Again, in this case, we have $5 \cdot 7 = 35$, so therefore $? = 35 \div 5 = 7$, but this time, it context, it means 7 candies in each bag. Note carefully that, although both problems involve the abstract division fact $35 \div 5 = 7$, the interpretation of this value is different in each case.

We can also use abbreviations to visualize this solution. In this case we have $bbbbb = (ccccccc)(ccccccc)(ccccccc)(ccccccc)(ccccccc)$, a different arrangement than in the previous section. Note how, in this case 5 is the number of bags, whereas in the previous problem, 5 was the number of candies in each bag.

Next example: *Nick has a rope that is* 42 *meters long. He wants to cut the rope into* 7 *equal pieces. How long will each piece of group be?* Again, we have the same set up with groups = pieces of rope and units = meters. This time, though, the multiplication problem has the missing factor on the right: $7 \cdot ? = 42$. Again, the result abstractly follows from $7 \cdot 6 = 42$, so we get $? = 42 \div 7 = 6$. This time, though, the number 6 means each piece of rope is 6 meters long.

Exercises

[1] We will sometimes simply say *how many units* for *how many units/objects in one group* just to keep the text from getting cluttered, but if it helps, you should mentally add the last part.

Exercise 5.2.1. For each problem below, set up the solution as a *how many units/objects in one group* division problem by using the definition of multiplication. Then, solve the problem using your knowledge of single digit facts.

(a) Valkyrie buys 9 swords. She spent \$72. How much did each sword cost?

(b) Rocket wants to put cookies into packages. He has a total of 45 cookies and he has 5 packages. How many cookies will go in each package?

(c) Sam is giving out dog treats in gift bags to 3 dogs. He puts an equal number of treats in each bag and he has 18 treats in all. How many treats will go in each bag?

Exercise 5.2.2. Write a *how many units in one group* word problem for $30 \div 6 = ?$. Solve the problem using single digit multiplication facts, and interpret your answer in context.

Exercise 5.2.3. What if there are zero objects in one group? In that case, we could present a *how many groups* division problem such as $? \cdot 0 = 37$ as describing the situation of having 37 objects to put into groups of zero objects, how many groups do we need? Clearly this question makes no sense, as we can never get 37 objects, no matter how many groups we have, if each group has 0 objects in it! Thus, the division $37 \div 0 = ?$ is **undefined** for this reason. Give a similar explanation in terms of a *how many units in one group* division problem to explain why $48 \div 0$ is also undefined by our definition.

5.3. Fraction division

There are many interesting connections between fractions and division. Let's first make an observation. Consider the division: $35 \div 5 = 7$. One way we visualized this was $bbbbb = (ccccccc)(ccccccc)(ccccccc)(ccccccc)(ccccccc)$, i.e., 5 bags with 7 candies in each bag. In this way, you can see the 35 candies are divided into 5 equal parts, so it should also be true, according to the definition of fraction, that $\frac{1}{5}$ of the 35 candies is 7 candies. We could set this up, a bit artificially, but validly, as groups = all the candies, and objects = candies, so that there are 35 objects (candies) in each group (all the candies). In this case, we see that $\frac{1}{5}$ of a group (all of the candies) is equal to 7 candies, and so, in terms of multiplication, we have:

$$\frac{1}{5} \cdot 35 = 7$$

Recall from our multiplication rule that $\frac{1}{5} \cdot 35 = \frac{35}{5} = ?$ as well.

This idea works well in general, if we set up $n \div m = ?$ as a *how many units* division problem: $m \cdot ? = n$, we have n units divided up into m equal groups, and therefore, in the same way, we have $\frac{1}{m} \cdot n = ?$. Thus, we get the following:

Theorem 5.3.1. *For any numbers n and m, we have $n \div m = \frac{n}{m}$.*

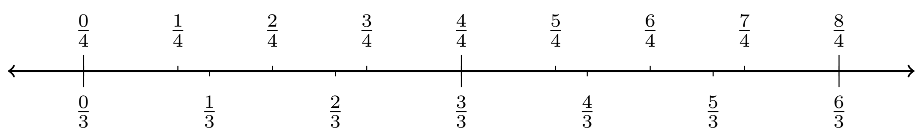
For this reason, we will sometimes use the fraction bar to indicate a division. The preceding discussion explains why this fits with our definitions so far. Technically, we need to attend to the sign rules, but the sign rules for division follow from those for multiplication, so they are perfectly compatible. We also need to justify why this works for non-whole numbers, but that too follows from essentially the same reasoning, we could make the candies in the above example into $\frac{1}{3}$ candies to get division with a common denominator, and any two fractions can be expressed with a common denominator.

Let's look at another example: *A cake recipe calls for* $\frac{2}{3}$ *tablespoon of baking powder. How many cakes can you make with* $\frac{7}{4}$ *tablespoons of baking powder?* Hopefully it is clear that the answer will be some sort of fraction. Let's set this up in terms of multiplication first: we take groups = cake recipes and units = tablespoons. Then, the multiplication equation becomes

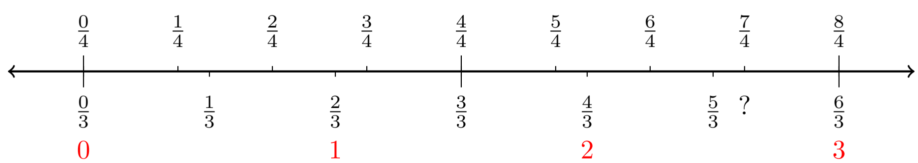
$$? \cdot \frac{2}{3} = \frac{7}{4}.$$

This is therefore a *how many groups* division problem. The easiest way to see this is that in our set-up, we took groups to be cake recipes, and the question said 'how many cakes can you make'. Learning to judge which type of division we need is a skill to develop, but one place to start is to write down groups and units and see how that fits with the problem statement.

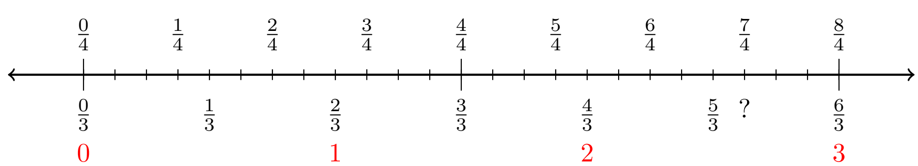
According to the definition, we have $? = \frac{7}{4} \div \frac{2}{3}$ cakes, in our above example. What is this value, though? We want to find how many groups of size $\frac{2}{3}$ we can break up $\frac{7}{4}$ into. Let's use a number line.



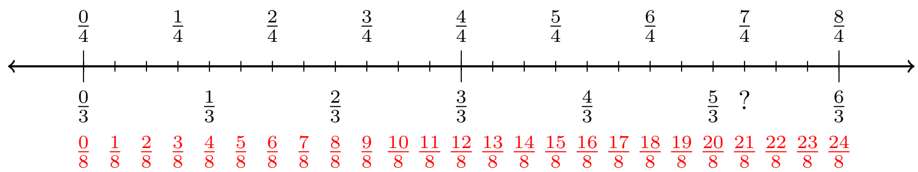
Now, since $\frac{2}{3}$ tablespoons represents 1 cake (group) in the above number line, we will show the number of cakes (in red) corresponding to the number of tablespoons below the number line:



So the question is now, what value is represented by the '?' in the above number line? The easiest way to see this is to subdivide as before to find a common denominator for thirds and fourths. We will use the switch trick which means we subdivide the fourths into three equal parts, and the thirds into four equal parts.



Now that we've found this common subdivision, we can see that the distance between two small tick marks above represents *both* $\frac{1}{12}$ of a tablespoon **and** $\frac{1}{8}$ of a cake. Observe that the tick marks divide the interval from zero to one tablespoon (expressed as $\frac{4}{4}$) into 12 equal parts, and at the same time, they divide the interval between zero and one cake (in red) into 8 equal parts. Thus, we can count cakes in *eighths* as follows (shown in red):



Thus, we see that the value we were looking for is $? = \frac{21}{8}$ cakes. Note that thinking in different whole amounts (tablespoons vs. cakes) is very important for understanding this method. If we want to derive a general rule for dividing fractions from this method, one simple way is by thinking in common denominators. First of all, recall the original multiplication equation:

$$? \cdot \frac{2}{3} = \frac{7}{4}.$$

If we express the two fractions involved with a common denominator, using the switch trick, then, as you can see in the number line above, we get $\frac{2}{3} = \frac{8}{12}$ and $\frac{7}{4} = \frac{21}{12}$. So, the equation becomes:

$$? \cdot \frac{8}{12} = \frac{21}{12}.$$

Since the fractional parts are now the same size, the question is, how many groups of 8 (twelfths) can we fit in 21 (twelfths)? Note that these values are a bit abstract, but in these terms, we have the equivalent division problem:

$$? \cdot 8 = 21$$

and therefore by definition $? = \frac{21}{8}$. This is an interesting trick, we can build the fraction into the units to show why the two division problems are equivalent. In the division problem

$$? \cdot \frac{8}{12} = \frac{21}{12}$$

the groups = cakes and units = tablespoons. If we keep the groups the same but make units = $\frac{1}{12}$ tablespoons, then we get the division problem

$$? \cdot 8 = 21$$

where '?' stands for the exact same number, and the two division problems are equivalent!

In order to derive the rule from this, we need to pay close attention to where each value came from. First of all, we wrote

$$\frac{2}{3} = \frac{4 \cdot 2}{4 \cdot 3} = \frac{8}{12}$$

and note carefully that the 4 is used here because of the 4 in the denominator of $\frac{7}{4}$ and we are doing the switch trick (which always works). Then, on the other side, we have

$$\frac{7}{4} = \frac{3 \cdot 7}{3 \cdot 4} = \frac{21}{12}$$

and in this case we use 3 here from the 3 in the denominator of $\frac{2}{3}$ (switch trick again). When we decompose the final result, we see that the answer we sought was:

$$? = \frac{21}{8} = \frac{3 \cdot 7}{2 \cdot 4},$$

here we use the color blue to show the values that come from $\frac{2}{3}$, and now we see using our rule for multiplying fractions that:

$$? = \frac{21}{8} = \frac{3 \cdot 7}{2 \cdot 4} = \frac{3}{2} \cdot \frac{7}{4}.$$

To rephrase this in terms of division, we have

$$\frac{7}{4} \div \frac{2}{3} = \frac{3}{2} \cdot \frac{7}{4}.$$

Note that the *divisor*[2] $\left(\frac{2}{3}\right)$ is turned upside down $\left(\frac{3}{2}\right)$ to perform the multiplication on the right side of the equation. This calls for a definition.

DEFINITION 5.3.2. Given a fraction $\frac{p}{q}$ the *reciprocal* of this fraction is $\frac{q}{p}$.

Using this vocabulary, we see that the rule is: we divide fractions by multiplying the dividend by the reciprocal of the divisor.

THEOREM 5.3.3 (Fraction Division Rule). *Let $\frac{n}{m}$ and $\frac{p}{q}$ be fractions. Then,*

$$\frac{n}{m} \div \frac{p}{q} = \frac{q}{p} \cdot \frac{n}{m}.$$

We wrote the multiplication on the left, and used the color blue, to try to make sure you pay close attention to which fraction requires a reciprocal (the divisor), and that there is *no longer any division* on the right hand side of the equation (the whole point of the rule). These are two of the most common errors in applying this rule. To avoid these mistakes, and to develop your sense for why the rule works, you will calculate some divisions without using the rule in the exercises. If you want to check your work, you have the option of using the switch trick to get a common denominator, and the correct value is fairly easy to see from there.

Exercises

EXERCISE 5.3.1. *Sersi has* 36 *muffins and wants to divide them equally among* 9 *packages. How many muffins should she put in each package?*

(a) First, use the definition of multiplication to set this up as a *how many units in one group* division problem: $9 \cdot ? = 36$.

(b) Then, use the definition of multiplication in a different way to set this up as a multiplication problem: $\frac{1}{9} \cdot 36 = ?$.

(c) Combine the previous two steps to explain why $36 \div 9 = \frac{36}{9}$.

(d) Now rewrite the above problem to make it a *how many groups* division problem and interpret the result $36 \div 9 = ?$ in context of your new problem.

[2] In a division expression $n \div m$, the value n is called the *dividend* and the value m is called the *divisor*

EXERCISE 5.3.2. *One serving of pretzels is* 6 *ounces. How many servings are there in a 72-ounce bag of pretzels?*

(a) First, use the definition of multiplication to set this up as a *how many groups* division problem: $? \cdot 6 = 72$.

(b) Then, use the definition of multiplication in a different way to set this up as a multiplication problem: $72 \cdot \frac{1}{6} =?$ (*Hint: Reverse groups and units.*)

(c) Combine the previous two steps to explain why $72 \div 6 = \frac{72}{6}$.

(d) Use a method or idea from the book to explain why $36 \cdot 2 = 72$. Then use this to calculate the answer to the problem.

(e) Now rewrite the above problem to make it a *how many units in one group* division problem and interpret the result $72 \div 6 =?$ in context of your new problem.

EXERCISE 5.3.3. *A recipe for brownies calls for* $\frac{3}{4}$ *cup of flour. How many batches of the recipe can be made with* $\frac{7}{6}$ *cups of flour?*

(a) Use the definition of multiplication to set this up as a division problem: $\frac{7}{6} \div \frac{3}{4} =?$. Which type of division problem is it?

(b) Use a number line to calculate the result of this division as shown in the text.

(c) Now, use the fraction division rule to calculate the result.

EXERCISE 5.3.4. *Assume that* $\frac{1}{3}$ *of a pancake recipe calls for* $\frac{1}{2}$ *teaspoon of salt. How much salt is needed for the full recipe?*

(a) Use the definition of multiplication to set this up as a division problem: $\frac{1}{2} \div \frac{1}{3} =?$. Which type of division problem is it?

(b) Use a number line to calculate the result of this division as shown in the text.

(c) Now, use the fraction division rule to calculate the result.

EXERCISE 5.3.5. *Ikaris runs* $\frac{4}{5}$ *of a mile, but that is only* $\frac{2}{5}$ *of the distance that Kingo runs. How far did Kingo run?*

(a) Use the definition of multiplication to set this up as a division problem: $\frac{4}{5} \div \frac{2}{5} =?$. Which type of division problem is it?

(b) Use a number line to calculate the result of this division as shown in the text.

(c) Now, use the fraction division rule to calculate the result.

5.4. Division algorithm, decimal division

We will now discuss a standard algorithm for carrying out a division by hand. Let's start with $344 \div 9$. We will show the full example first, then break down how it works step-by-step.

$$
\begin{array}{r r l}
 & 8 & \text{(1s of 9s)} \\
 & 30 & \text{(10s of 9s)} \\
\cline{2-2}
9 \big) & 344 & \\
- & 270 & \leftarrow 30 \cdot 9 \\
\cline{2-2}
 & 74 & \\
- & 72 & \leftarrow 8 \cdot 9 \\
\cline{2-2}
 & 2 &
\end{array}
$$

This result means that $344 \div 9 = 38$ remainder 2. We get the quotient by adding the numbers at the top: $38 = 8 + 30$. This result means that $344 = 38 \cdot 9 + 2$ (you can check this).

Now, let's talk through the process. First of all, we think of $344 \div 9 =?$ as a *how many groups* division problem: $? \cdot 9 = 344$. So, we want to put 344 objects into groups of 9 objects, and count how many groups we get. We might, of course, have some objects left over at the end (the remainder). We carry out the algorithm by thinking in tens of groups. So, we start with the largest group of tens *of nines* that we can have without going over 344 objects. So, for example, starting with 100 groups (or ten tens) $100 \cdot 9 = 900 > 344$ is too many objects, so then we look at 10 groups, $10 \cdot 9 = 90 < 344$ is not too large. Now, we look for multiples of 10 groups, necessarily fewer than ten (since we already saw that 100 is too large). So, we look at $20 \cdot 9 = 180 < 344$, which is not too large, so we proceed to $30 \cdot 9 = 270 < 344$, and then to $40 \cdot 9 = 360 > 344$ which is too large. Therefore, we stop at $30 \cdot 9 = 270$. The idea is that 30 groups of 9 objects is 270 objects, so we record the 30 groups at the top, and then subtract $344 - 270$ and proceed to group the remaining objects.

$$
\begin{array}{r r l}
 & 30 & \text{(10s of 9s)} \\
\cline{2-2}
9 \big) & 344 & \\
- & 270 & \leftarrow 30 \cdot 9 \\
\cline{2-2}
 & 74 &
\end{array}
$$

We have omitted the borrowing required to carry out the subtraction to avoid clutter, but feel free to write it if it helps!

Now, we want to put the remaining 74 objects into groups of 9. We start with the next place value in Base-10, which is the ones place in this case. So we take 'ones' of nines: $1 \cdot 9 = 9 < 74$, $2 \cdot 9 = 18 < 74$, $3 \cdot 9 = 37 < 74$, and we keep going like this until we get to $8 \cdot 9 = 72 < 74$ and $9 \cdot 9 = 81 > 74$. Thus, we take 8 more groups of 9, which gives us $8 \cdot 9 = 72$ objects. So, we record 8 more groups above and then we subtract $74 - 72$ and try to group the remaining objects in nines as well.

$$\begin{array}{rl} 8 & \text{(1s of 9s)} \\ 30 & \text{(10s of 9s)} \\ 9\overline{)\ 344} & \\ -\ \underline{270} & \leftarrow 30\cdot 9 \\ 74 & \\ -\ \underline{72} & \leftarrow 8\cdot 9 \\ 2 & \end{array}$$

Now, $2 < 9$, so we can not make any more groups of 9, therefore, 2 is the *remainder* of the division, the number of objects left over, not put into groups. The total number of groups given by $30 + 8 = 38$ is referred to as the *quotient* (in this case the whole number quotient - see next paragraph).

We can present the result of a division in numerous ways. First, we can simply say, as we did at the start, that $344 \div 9 = 38$ remainder 2. If you recall our work on fractions, however, if we want to describe the result as a number of groups, with no objects left over, we can see how much of a group of 9 we get from 2 objects. According to the definition of multiplication, the number of groups (?) of 9 objects given by 2 objects is determined by $? \cdot 9 = 2$, a *how many groups* division problem, so the result is $? = 2 \div 9 = \frac{2}{9}$, by our work on fractions. Thus, we could present the result as $344 \div 9 = 38 + \frac{2}{9}$ groups of 9 (equal to 344 objects - this is the fractional quotient).

Note carefully that, when we present 2 as a remainder, it refers to 2 objects left over, which we do *not* put in groups of 9. When we present the result as $38 + \frac{2}{9}$, the fraction $\frac{2}{9}$ is a number of *groups*. The two ideas are very closely related, but you need to think carefully about which answer makes the most sense depending on the context of the problem, sometimes a fractional number of groups makes sense, and sometimes left over objects works better.

We can also obtain decimal results using the algorithm, we just need to continue past the decimal point, and continue with smaller base-10 place values. Instead of stopping at ones, we proceed to tenths, hundredths, and so on. Let's do a simple example: $57 \div 4$. To start with we look at 100s of 4s, which is $100 \cdot 4 = 400 > 57$, and so too large. Then, we look at 10s of 4s which is $10 \cdot 4 = 40 < 57$, so we start here. The next number of tens of fours is, $20 \cdot 4 = 80 > 57$ which is again too large. So, we record 10 groups above the line, and then subtract $10 \cdot 4 = 40$ objects from the 57 objects we started with.

$$\begin{array}{rl} 10 & \text{(10s of 4s)} \\ 4\overline{)\ 57} & \\ -\underline{40} & \leftarrow 10\cdot 4 \\ 17 & \end{array}$$

Now, we want to group the remaining 17 objects, we look at 'ones' of 4s, clearly $1 \cdot 4 = 4 < 17$ is not too large. So, we look at $2 \cdot 4 = 8 < 17$, then $3 \cdot 4 = 12 < 17$, and finally $4 \cdot 4 = 16 < 17$. Since $5 \cdot 4 = 20 > 17$, we stop at 4 groups of 4, which is 16 objects. We record the 4 groups above the line, and subtract the 16 objects from the remaining 17 objects.

$$\begin{array}{rl}
4 & \text{(1s of 4s)} \\
10 & \text{(10s of 4s)} \\
4\,\overline{)\,57} & \\
-40 & \leftarrow 10\cdot 4 \\
\hline
17 & \\
-16 & \leftarrow 4\cdot 4 \\
\hline
1 &
\end{array}$$

We have 1 as the remainder of left over objects and $10 + 4 = 14$ as the whole number quotient. So, $57 \div 4 = 14$ remainder 1, or we can present the quotient as a fractional number of groups $57 \cdot 4 = 14 + \frac{1}{4}$, since 1 object is $\frac{1}{4}$ of a group of 4.

Now, however, we can continue the process by adding a decimal point and a zero to everything, and proceeding to look at groups of 0.1 (tenths) of 4s.

$$\begin{array}{rl}
4.0 & \text{(1s of 4s)} \\
10.0 & \text{(10s of 4s)} \\
4\,\overline{)\,57.0} & \\
-40.0 & \leftarrow 10\cdot 4 \\
\hline
17.0 & \\
-16.0 & \leftarrow 4\cdot 4 \\
\hline
1.0 &
\end{array}$$

The question now is: how many 0.1 (tenths) of 4s can we take from 1.0 without going over? We start with $0.1 \cdot 4 = 0.4 < 1.0$, then proceed to $0.2 \cdot 4 = 0.8 < 1.0$, and finally we stop when we pass 1.0 with $0.3 \cdot 4 = 1.2 > 1.0$. We record 0.2 of a group of 4 above the line, and subtract $0.2 \cdot 4 = 0.8$ from 1.0 below as before.

$$\begin{array}{rl}
0.2 & \text{(0.1s of 4s)} \\
4.0 & \text{(1s of 4s)} \\
10.0 & \text{(10s of 4s)} \\
4\,\overline{)\,57.0} & \\
-\ 40.0 & \leftarrow 10\cdot 4 \\
\hline
17.0 & \\
-\ 16.0 & \leftarrow 4\cdot 4 \\
\hline
1.0 & \\
-0.8 & \leftarrow 0.2\cdot 4 \\
\hline
0.2 &
\end{array}$$

Note that the decimal subtraction algorithm can be used to carry out the last subtraction. We are now left with 0.2 objects, and we proceed with the next decimal place 0.01s (hundredths) of 4s. First we look at $0.01 \cdot 4 = 0.04 < 0.2$ (careful here, $0.2 = 0.20$ the decimal place matters). Then, we have $0.02 \cdot 4 = 0.08 < 0.2$, $0.03 \cdot 4 = 0.12 < 0.2$, $0.04 \cdot 4 = 0.16 < 0.2$, and finally $0.05 \cdot 4 = 0.20 = 0.2$! Thus, if we take 0.05 groups of 4 we will have no (fractional) objects remaining, so the division will be over! We record the number of groups 0.05 above, and we subtract $0.05 \cdot 4 = 0.20$ from the remaining 0.2 objects to obtain zero objects left over.

$$
\begin{array}{rrr}
 & 0.05 & (0.01\text{s of }4\text{s}) \\
 & 0.20 & (0.1\text{s of }4\text{s}) \\
 & 4.00 & (1\text{s of }4\text{s}) \\
 & 10.00 & (10\text{s of }4\text{s}) \\
\cline{2-2}
4 \big) & 57.00 & \\
 & -\underline{40.00} & \leftarrow 10\cdot 4 \\
 & 17.00 & \\
 & -\underline{16.00} & \leftarrow 4\cdot 4 \\
 & 1.00 & \\
 & -\underline{0.80} & \leftarrow 0.2\cdot 4 \\
 & 0.20 & \\
 & -\underline{0.20} & \leftarrow 0.05\cdot 4 \\
 & 0.00 &
\end{array}
$$

Note that we add another zero after the decimal point to make room for hundredths. We get the decimal quotient by adding the number of groups above the line $10 + 4 + 0.2 + 0.05 = 14.25$ groups of 4 is equal to 57 objects, or in other words: $14.25 \cdot 4 = 57$, or $57 \div 4 = 14.25$. We can also view this as a fraction to decimal conversion idea, if we recall that $\frac{57}{4} = 57 \div 4 = 14.25$. Recall the fractional quotient $14 + \frac{1}{4}$ and comparing to the decimal quotient $14.25 = 14 + 0.25$, we see that $\frac{1}{4} = 0.25$ as a decimal (you could also derive this by doing decimal division on $1 \div 4$). Thus, we also have a method for expressing a fraction as a decimal!

We should also observe that if you just move the decimal point to the right two places the above calculation shows: $5700 \div 4 = 1425$, as you can verify. We can see this using the rules and properties we have for multiplication. First, we know for sure, by definition of division, that $14.25 \cdot 4 = 57$. Now multiply both sides by 100:

$$100 \cdot (14.25 \cdot 4) = 100 \cdot 57$$

and apply the associative property and the rule for multiplication by 10 (or in this case $10 \cdot 10 = 100$) and we get:

$$1425 \cdot 4 = (100 \cdot 14.25) \cdot 4 = 100 \cdot (14.25 \cdot 4) = 100 \cdot 57 = 5700.$$

So by definition of division, we have $5700 \div 4 = 1425$. This is another way of looking at the above, we move the decimal point over until we get something evenly divisible by 4, then we move it back to get the decimal quotient.

Some fractional values do not convert so cleanly into decimal form. For example, if we look at $\frac{1}{3} = 1 \div 3$. We can try to find the decimal version of this value as before. In this case we have to start with tenths of 3s because one group of 3 objects is more than 1 object!

$$\begin{array}{rrr}
 & 0\,.\,0\,0\,0\,3 & (0.0001\text{s of }3\text{s}) \\
 & 0\,.\,0\,0\,3\,0 & (0.001\text{s of }3\text{s}) \\
 & 0\,.\,0\,3\,0\,0 & (0.01\text{s of }3\text{s}) \\
 & \underline{0\,.\,3\,0\,0\,0} & (0.1\text{s of }3\text{s}) \\
3\,\big) & 1\,.\,0\,0\,0\,0 & \\
 & -\underline{0\,.\,9\,0\,0\,0} & \leftarrow 0.3\cdot 3 \\
 & 0\,.\,1\,0\,0\,0 & \\
 & -\underline{0\,.\,0\,9\,0\,0} & \leftarrow 0.03\cdot 3 \\
 & 0\,.\,0\,1\,0\,0 & \\
 & -\underline{0\,.\,0\,0\,9\,0} & \leftarrow 0.003\cdot 3 \\
 & 0\,.\,0\,0\,1\,0 & \\
 & -\underline{0\,.\,0\,0\,0\,9} & \leftarrow 0.0003\cdot 3 \\
 & 0\,.\,0\,0\,0\,1 &
\end{array}$$

This shows three steps in the process of doing the decimal division to find $1 \div 3$. Notice some patterns in the above. First of all, in the quotient part above the line, we get a single digit of 3 each time, that moves over one decimal place as we go up (0.3 then 0.03 then 0.003, and so on). Also observe that each time we subtract, we obtain a single digit of 1 each time which also moves over one decimal place as we go down (we start with 1.000, then subtract to get 0.100, then subtract to get 0.010, then subtract to get 0.001, and so on). Since $3 \cdot 3 = 9$, and $10 - 9 = 1$, we just have the same pattern over and over again, shifting one decimal place to the right each time. Thus, we see that we should have $1 \div 3 = 0.333333\ldots$ (repeating 3 forever to the right of the decimal point). This is the first example we've seen in this book of an infinite decimal!

The infinite decimal form then, of $\frac{1}{3}$ is

$$\frac{1}{3} = 0.333333\ldots.$$

If we multiply each side by three, we see that $3 \cdot \frac{1}{3} = \frac{3}{3} = 1$, but on the right we get $3 \cdot 0.33333\ldots = 0.9999999\ldots$, so we have:

$$1 = 0.999999\ldots.$$

Thus, we get a bizarre infinite decimal expansion for the number 1!

The above is interesting, and useful for understanding some more complicated issues that come up with decimals. However, this is the reason I prefer to avoid decimals for the most part. If you use a calculator to calculate something with $\frac{1}{3}$, the calculator will not be able to store the decimal value, since it has finite storage, it will cut off and/or round the decimal somewhere, say 0.333333333. Though this value is fairly close, it is simply **not true** that

$$0.3333333333 = \frac{1}{3},$$

the '...' indicate that the 3s must continue forever to the right if we want the precise value of $\frac{1}{3}$. Thus, for hand calculations, and for precision, it is preferable to use the fractional value $\frac{1}{3}$ to avoid loss of information due to finite storage!

Exercises

EXERCISE 5.4.1. Cosmo has 387 dog treats to put into gift bags with 8 treats in each bag. How many bags does he need?

(a) Set up the solution as a division problem using the definition of multiplication. Which type of division problem is it?

(b) Carry out the division using the algorithm.

(c) Present your solution as a whole number quotient with remainder, and interpret it in context of the problem.

EXERCISE 5.4.2. At a pizza party, Elaine observes that there are 217 slices of pizza left. Each pizza has 7 slices. How many pizzas are left?

(a) Set up the solution as a division problem using the definition of multiplication. Which type of division problem is it?

(b) Carry out the division using the algorithm.

(c) Present your solution as a whole number quotient with remainder, and interpret it in context of the problem.

(d) Now present your solution as a fractional quotient, and interpret it in context of the problem.

EXERCISE 5.4.3. Consider the division problem $67 \div 5$.

(a) Carry out the division using the algorithm.

(b) Present your solution as a whole number quotient with remainder.

(c) Now present your solution as a fractional quotient.

(d) Continue your division algorithm to express the quotient as a decimal.

EXERCISE 5.4.4. Consider the division problem $91 \div 8$.

(a) Carry out the division using the algorithm.

(b) Present your solution as a whole number quotient with remainder.

(c) Now present your solution as a fractional quotient.

(d) Continue your division algorithm to express the quotient as a decimal.

EXERCISE 5.4.5. Write the fraction $\frac{1}{8}$ as a decimal using the division algorithm.

EXERCISE 5.4.6. Write the fraction $\frac{2}{3}$ as a decimal using the division algorithm.

EXERCISE 5.4.7. Given that $7900 \div 4 = 1975$, what is $79 \div 4 =$? Explain your answer.

5.5. Negative number division

The good news is: there isn't much new here, I just want to briefly spell out the rules for dividing negative numbers. Ultimately the rules for dividing negative numbers come from the rules for multiplying negative numbers, since we define division in terms of multiplication.

Let's look at some examples: say we want to divide $-24 \div 8 =?$. We can rephrase this in two ways, and if we choose, for example, *how many groups* then this becomes

$$? \cdot 8 = -24.$$

If you recall our rules for multiplying negative numbers, you know that ? must be negative, since the product $? \cdot 8 = -24$ is negative. You also know that if you solve $? \cdot 8 = 24$ first, all you need to do is take the opposite to solve the same equation with -24 on the right hand side. So, since $3 \cdot 8 = 24$, we know that $(-3) \cdot 8 = -24$, and thus $-24 \div 8 = -3$.

A similar argument can be made about $25 \div (-8) =?$ in this case, we still have that ? is a negative number, but since $25 \div 8$ is not a whole number, we can just express the result in fraction form $? = -\frac{25}{8}$, which can also be written as a quotient with fractional remainder: $? = -\left(3 + \frac{1}{8}\right)$ (note carefully that we use parentheses above because of the order of operations). We could also calculate the decimal value using the algorithm and get $? = -3.125$. We will generally avoid using the whole number quotient with remainder format when dividing negative numbers.

The final case is something like $(-30) \div (-6) =?$, in this case if you think about the associated multiplication equation $? \cdot (-6) = -30$ it must be the case here that ? is a *positive* number, as the product of two negative numbers can never be -30. Thus, we get $? = 30 \div 6$ (in a sense, the two opposite signs 'cancel' each other out), and so $? = 5$. It is useful to verify your results like this: $5 \cdot (-6) = -30$ to avoid common sign errors. I will now present the rules in abstract form.

LEMMA 5.5.1. *If m and n are* any *numbers, then $m \div n$ and $m \div (-n)$ are opposites, and $m \div n$ and $(-m) \div n$ are opposites:*

$$m \div (-n) = -(m \div n) = (-m) \div n$$

Exercises

EXERCISE 5.5.1. Calculate the following divisions, use the algorithm as needed. Use fractional quotients.

(a) $(-56) \div 8 =?$.

(b) $(-48) \div (-6) =?$.

(c) $42 \div (-6) =?$.

(d) $43 \div (-6) =?$.

(e) $(-48) \div (-7) =?$.

EXERCISE 5.5.2. Express the fraction as a decimal using the algorithm and the rules for dividing negative numbers.

(a) $\frac{-9}{4}$.

(b) $\frac{9}{-5}$.

(c) $\frac{-23}{-8}$.

(d) $\frac{-8}{3}$.

CHAPTER 6

Algebraic concepts

6.1. Order of Operations

I would rather not write this chapter. Honestly, this is the worst chapter in the book. I apologize in advance for this chapter.

However, let me explain why I'm putting this chapter in. First of all, although grammar and spelling can be quite boring parts of learning language and reading, they are extremely important for communication, arbitrary though they may be. Second of all, though the presentation of this book is a bit unorthodox, I want you to come away from this book with concepts and skills that you can use in other math courses, and in your lives (though, in this latter case, probably less so for this chapter). Since we are leading to a discussion of algebra in this part of the book, we are going to need to resolve some ambiguities in written math. There is a standard way to resolve these ambiguities, not quite universal around the world, but widely agreed-upon, and any deviation will generally be met with confusion and misunderstanding. The difficult part is that this is not about math facts or concepts or ideas, but merely about convention. There is an alternate universe where different conventions are used and they get along just fine! The conventions I will present to you are not laws, or even important ideas, they are just like the spelling of *night* versus the spelling of *knight*, fairly arbitrary and not interesting in their own right, but necessary to know to comprehend what you see in math books. Ok, on with the show.

The standard order of operations that we will follow is as follows:

(1) Grouping symbols (including parentheses, brackets, absolute value bars, and fraction bars).

(2) *Exponents* (not covered in the book so far, included here for completeness, see special topics at the end of the book).

(3) Multiplication and division - read from left to right.

(4) Addition and subtraction - read from left to right.

I will now note a few things on the above list. We always read left to right for everything, this is convenient for us since English is also written that way, but not all languages are! There is a common mnemonic PEMDAS (parentheses, exponents, multiplication, division, addition, subtraction) for remembering this list. I prefer to include more than parentheses as first on the list, as other grouping symbols are often used and serve the same function, so adjust the mnemonic to say *GEMDAS* instead, if you please. If we ignore exponents for the moment, since we don't use them for the most part in this book, then the order of operations comes down to this: multiplication before addition. Recall that subtraction is derived from addition and division is derived from multiplication. So, really all of this boils down to multiplication before addition. We have already seen examples of this in the book that I didn't comment on to avoid over-crowding the presentation, but consider the following expression.

$$2 \cdot 3 + 5$$

There are two natural ways to read that expression, multiplication first, or addition first. If we do multiplication first we get:

$$2 \cdot 3 + 5 = 6 + 5 = 11.$$

If we do addition first we get:

$$2 \cdot 3 + 5 = 2 \cdot 8 = 16 \ (\textit{This is incorrect!})$$

This last is presented in red because it violates the order of operations! (Sometimes it is helpful to show incorrect math, but I want to make sure this is not misunderstood.) The main point of the order of operations is that we read this expression the first way, so $2 \cdot 3 + 5 = 6 + 5 = 11$. The main point of the top priority 'grouping symbols', and in this case we primarily mean parentheses, is to have a way around the order of operations. So, if we want to have the expression $2 \cdot 3 + 5$ but with the addition first, then we simply put parentheses around the addition part: $2 \cdot (3 + 5)$. Now, according to the order of operations, it is correct to calculate:

$$2 \cdot (3 + 5) = 2 \cdot 8 = 16.$$

The final point I want to make is that the associative property and the distributive property show ways of calculating that can violate the order of operations, and the point of having the properties in those cases is to let you know it's ok to ignore the order of operations in those cases. This is an important point about the order of operations: it serves to resolve ambiguities, but sometimes the ambiguities don't matter! For example, if we have an addition like $2 + 3 + 4$, then according to the order of operations we add from left to right:

$$2 + 3 + 4 = 5 + 4 = 9.$$

However, given the associative property, we know that we can also add from right to left:

$$(2 + 3) + 4 = 2 + (3 + 4) = 2 + 7 = 9.$$

Since we will often suppress parentheses when they are not necessary, it is important to understand that we are only allowed to leave out parentheses if the order does not affect the result.

The distributive property can also be used to calculate in a different order from what the order of operations prescribes. For example, recall the above calculation:

$$2 \cdot (3 + 5) = 2 \cdot 8 = 16.$$

If we apply the distributive property here (which does not fit with the order of operations), we would calculate the left hand side as follows,

$$2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5 = 6 + 10 = 16.$$

Thank you, good night!

Exercises

EXERCISE 6.1.1. Consider the expression $3 \cdot 5 + 7$.

(a) Calculate the result of the expression using the order of operations.

(b) Now rewrite the expression according to the order of operations so that we should do the addition first and calculate the result.

EXERCISE 6.1.2. Consider the expression $3 + 5 \cdot 7$.

(a) Calculate the result of the expression using the order of operations.

(b) Now rewrite the expression according to the order of operations so that we should do the addition first and calculate the result.

EXERCISE 6.1.3. Consider the expression $3 + 35 \div 7$.

(a) Calculate the result of the expression using the order of operations.

(b) Now rewrite the expression according to the order of operations so that we should do the addition first and calculate the result. Present your result as a whole number quotient with remainder.

EXERCISE 6.1.4. The distributive property says that

$$3 \cdot (4 + 5) = 3 \cdot 4 + 3 \cdot 5 = 12 + 15 = 27.$$

Explain why

$$3 \cdot 4 + 5 \neq 27.$$

6.2. Variables

We will now introduce variables.

DEFINITION 6.2.1. A *variable* is a letter or other symbol that stands for a number within a specified or understood set of numbers.

We have already seen many examples of variables in this book, simply not named as such until now. For example, when we stated the commutative property, we said 'Let m and n be positive whole numbers, then $m+n = n+m$'. In this case, n and m are both variables, standing for numbers within the set of positive whole numbers. Here, the set of numbers is specified, but other times it is implied. The point is that n and m are allowed to take *any* values within the specified (in this case) set of positive whole numbers. As we eventually discovered, the commutative property remains true if we allow n and m to be chosen from the larger set of all real numbers, and in that case we have changed the meaning of the variables m and n by allowing them to be chosen from a larger set of numbers. Thus, the meaning of a variable is not simply the letter or symbol used, but primarily what the specified or understood set of numbers is that determines the possible values of the variable.

Another common way we introduced variables that comes up frequently in algebra is as the unknown symbol ? in an equation. For example, we presented a subtraction such as $9 - 5$ as the solution to the equation $5+? = 9$. Here ? is functioning as a variable (we can use *any* symbol for a variable, not only letters), and the understood set of numbers that ? is allowed to stand for is the *implied* set of all numbers that satisfy the equation $5+? = 9$. In this case, there is **only one** such number, namely $? = 9 - 5 = 4$. This is an important special case, we are allowed to use a variable to stand for a single number, whose value is determined by an equation, but which we have not yet calculated. In this situation, the understood set of numbers is simply the set consisting of the single number 4, in the end, but we don't 'know' that until we solve the equation. Before doing any work, we can dodge this issue of specifying the set of numbers by using the equation $5+? = 9$ and we simply let '?' belong to the set of all numbers that satisfy this equation. Thus, solving the equation amounts to specifying or determining the set of numbers that ? could possibly be in this context. Solving equations in algebra frequently amounts to this, and therefore it is important to understand how this relates to our definition of variable.

We used the question mark in equations to avoid any potential confusion that comes from introducing variables, but now that we have the concept, we can restate the above example as follows. Let x stand for any number in the set of all numbers that satisfy the equation $5 + x = 9$. Then, we must have $x = 9 - 5 = 4$ by our theorem on subtraction. This is a more traditional way of presenting the same idea. Indeed, one would rarely see it stated so elaborately, but rather one would more likely see something like, simply: solve for x

$$5 + x = 9,$$

and one must understand that x here is a variable and the understood set of numbers is the set of all numbers that satisfy the equation. My apologies for slipping into the indefinite pronoun 'one'.

In any case, we see that we have already developed the tool of subtraction (or rather, addition of the opposite) for solving algebraic equations. Another such example is division. *Let y stand for any number that satisfies the equation*

$$7y = 35,$$

determine all possible values of y. Here, we need to be clear that $7y$ stands for $7 \cdot y$, we frequently do not include the multiplication symbol $\cdot$, and simply use concatenation (writing next to) to indicate multiplication. We could also write $y7$ or $y \cdot 7$, but traditionally it is standard to always write specified numbers like 7 on the left and variables like y on the right. Since we have the commutative property: $7 \cdot y = y \cdot 7$, we can always do this if needed. All of this understood, we see that $7y = 35$ is really a *how many units in one group* division problem, so there is only one possible value of y (and only one solution to the equation), which is $y = 35 \div 7 = \frac{35}{7} = 5$.

6.2.1. Expressions. It is important here to make a distinction that always causes some confusion. Any string of mathematical symbols that makes sense is **not automatically called an equation.** We have a name for a meaningful string of mathematical symbols that is *not* an equation, such a string is called an expression.

DEFINITION 6.2.2. An *expression* is a meaningful string of mathematical operations (addition, subtraction, multiplication, division, exponents, etc.) and numbers, variables, or both. In particular, an expression does **not** include the equals sign: $=$.

The above is such a vague concept as to be nearly useless, but we introduce it primarily to have a name for strings of math symbols that do not represent an equation. It is important conceptually to break the habit you may have of referring to an expression like $7x+17$ as an 'equation'. Since there is no equals sign involved, $7x + 17$ is properly called an expression. Of course, if we write $7x + 17$, we do not determine the possible values of the variable x by an equation, so it is up to us to specify the set of possible values that x can be chosen from. Generally, when the set of possible values is not specified, the default is to take the largest possible set of values that makes sense for that expression. In this case, we can let x be any type of number we have referred to so far: positive, negative, fraction, decimal, etc.

For other expressions, we may need to constrain the variable more. For example, recall from earlier that division by zero is undefined, so if we write $18 \div z$, then the variable z can stand for any number *except zero*! Thus, the implied set of values for a variable in an expression can be limited according to the limitations of any operations in the given expression.

6.2.2. Evaluating expressions. One useful procedure that we will need to understand is *evaluating an expression.* The basic idea of evaluating an expression is replacing any variables in the expression by given specific values, and then determining what number the resulting expression is equal to by carrying out the operations or simplifying. A numerical expression can be evaluated just by completing the last mentioned step.

For example, given the expression $7x + 17$, we can evaluate this expression for $x = 3$ by replacing x by the number 3: $7 \cdot 3 + 17$ (write the multiplication symbol here $73 \neq 7 \cdot 3$!), and then we calculate the value of this numerical expression.

$$7 \cdot 3 + 17 = 21 + 17 = 38$$

Note that we used the order of operations here by doing the multiplication first. We could also simply evaluate a numerical expression like $3 - 7 + 8$ as follows:

$$3 - 7 + 8 = -4 + 8 = 4.$$

Recall that the order of operations tells us to always read from left to right, so we do the subtraction $3 - 7 = -4$ first, then add 8.

Definition 6.2.3. Two expressions are said to be *equivalent* if they evaluate to the same number for all values within the set of values allowed by the variables involved. We use the notation '=' to indicate that two expression are equivalent.

The above definition is a bit abstract, so to illustrate a specific example: let x represent any positive number, then $x + x$ is equivalent to $2 \cdot x$, since 2 groups of x objects is equal to $2x$, or $x + x$. Therefore we write $x + x = 2x$ (here the '=' sign means the left hand side is equivalent to the right hand side) since this is true for any positive number x. It turns out that $x + x = 2x$ for negative values as well, but we need to argue in a slightly different way, we use the distributive property and the equation $1 + 1 = 2$:

$$x + x = 1 \cdot x + 1 \cdot x = (1 + 1) \cdot x = 2 \cdot x = 2x.$$

This works since the distributive property holds for any numbers, whereas our definition of multiplication works best for positive numbers. It is useful to understand the same idea from different points of view.

Another class of equivalent expressions is given by the properties of arithmetic we have discussed, for example, the commutative property of addition says:

$$n + m = m + n$$

where n and m can stand for any numbers. This is also an example of equivalent expressions. So you see, we've been using this idea in the book already!

Generally speaking, we will use properties (associative, commutative, distributive), together with facts about numbers ($2+3=5$, $4 \cdot 6 = 24$, etc), to determine if two expressions are equivalent, or, more commonly, to *simplify* an expression by finding an equivalent expression that has fewer terms or operations, or just seems simpler in some way. This is necessarily a subjective concept (what is simple to you may not seem so to me), but you should be able to make a concrete argument (in terms of number of terms/factors, etc.) any time you simplify an expression as to why the final equivalent expression is truly simpler than the one you started with. For a silly example:

$$x + 1 = -7 + 3 \cdot \left(x + \frac{1}{3}\right) + 7$$

these expressions are equivalent (verify this!), but the one on the left has two terms and one operation, whereas the one on the right has 3 terms and 4 operations!

Exercises

EXERCISE 6.2.1. Okoye puts x erasers into each of 9 bags, and then has 4 erasers left over. An expression for the total number of erasers Okoye has, in terms of x, is $9x + 4$.

(a) What are the possible values for the variable x?

(b) Evaluate the expression $9x + 4$ for $x = 3$.

(c) Evaluate the expression $9x + 4$ for $x = 5$.

EXERCISE 6.2.2. Christine has a podcast app that charges \$5 per month (prorated, so you pay \$2.50 for half a month, for example) to use, and a \$2 one-time fee to install on your phone.

(a) Write an expression for the total cost of using the app for m months.

(b) What are the possible values for the variable m?

(c) Evaluate the expression for $m = 3$.

(d) Evaluate the expression for $m = 5.5$.

(e) Evaluate the expression for $m = \frac{3}{4}$.

EXERCISE 6.2.3. Karun has c candy bars. He gives away $\frac{1}{5}$ of his candy bars. Then, he gets 7 more candy bars.

(a) Write an expression for the number of candy bars Karun has.

(b) Write an equivalent, but different expression for the number of candy bars Karun has. Either simplify your expression above, or write a new one from scratch.

(c) What are the possible values for the variable c?

(d) Evaluate the two expressions for $c = 3$ and verify that you get the same result.

Exercise 6.2.4. Consider the expression $3y + 4$.

(a) Write a scenario for the expression, explain the meaning of the variable.

(b) What are the possible values for the variable y?

Exercise 6.2.5. Consider the expression $(y + 4) \cdot 3$.

(a) Write a scenario for the expression, explain the meaning of the variable.

(b) What are the possible values for the variable y?

Exercise 6.2.6. Verify that the expressions $3y + 4$ and $(y + 4) \cdot 3$ are NOT equivalent by evaluating each of them for $y = 2$.

Exercise 6.2.7. Explain why the expressions $3(y + 4)$ and $(4 + y) \cdot 3$ are equivalent using properties of arithmetic.

Exercise 6.2.8. Explain why the expressions $3(y+4)$ and $3y+12$ are equivalent using properties of arithmetic.

6.3. Equations

Let's begin with the definition.

Definition 6.3.1. An *equation* is the mathematical statement that two expressions are equal.

This definition is set up to be flexible enough to apply to any situation that we want to describe as an equation. As a practical matter, most equations we will be solving are much simpler than this definition suggests. We have, in fact, already discussed tools for solving certain equations in previous chapters, we just didn't mention it at the time!

For example, recall that an addition statement such as $4 + 3 = 7$ can also be expressed in terms of subtraction in two ways: $7 - 4 = 3$ and $7 - 3 = 4$. Thus, if we have an equation such as $x + 3 = 7$, then we automatically know that $x = 7 - 3$. Note carefully that if we write an equation involving variable expressions, the default assumption is that the variables can take any value that makes the equation true. This means that often the set of possible values for a variable will simply be determined by the possible solutions to an equation. In those cases, we typically do not explicitly mention the set of possible values, it should be understood in this way. For the equation $x + 3 = 7$, there is only one possible solution to the equation, given by $7 - 3 = x$, and therefore $x = 7 - 3 = 4$ is the only possible value for the variable x here. For the equations we will focus on in this book, it will usually be true that an equation has only one solution, but do not assume this! As you progress in your math courses you will find that equations can have two, three, or even infinitely many solutions!

The above discussion shows that the operation of subtraction (or, really, adding the opposite) is a useful tool for solving equations. A different way to phrase the same idea discussed above is to start with the equation $x+3=7$ and then, instead of rewriting it in terms of subtraction right away, we add the opposite of 3 to both sides:

$$(x+3)+(-3)=7+(-3).$$

This should make sense: if two expressions are equal and we add -3 to each of them, they should still be equal! Now, we apply the associative property to the left side of the equation: $(x+3)+(-3)=x+(3+(-3))$. We can add $3+(-3)=0$ (as we saw earlier in the book), so we get $x+0=0$ on the left hand side. On the other side we can add: $7+(-3)=4$. Thus, the original equation becomes:

$$x=4.$$

This equation clearly has only one solution, namely $x=4$, or the set of all possible values for the variable x is the one-element set containing only the number 4. The idea of doing the same operation to both sides of an equation to attempt to put it in a simpler form is an extremely useful idea. It is not efficient to go back to the definitions of all the operations every time. Further, the use of the associative property will usually go unmentioned at this stage, if we see three numbers added in a row, we know implicitly that we can add them in either way, so we will simply write $x+3+(-3)=x+0=x$, for example.

The other operation we have secretly used to solve equations is division. Recall that we had two forms of division: *how many groups* and *how many units in one group*, depending on where the question mark belonged in the multiplication equation. We can now see that, if we use a more standard variable name like x, then a how many groups division problem could look like $x\cdot 8=32$, and a how many units in one group division problem for the same values would be $8\cdot x=32$. In the first example, x represents the number of groups, and in the second example, x represents the number of units in one group. Thus, the interpretation of x changes depending on the type of division problem, however, numerically, in each case we have $x=32\div 8=4$.

Note that, you will not typically see an equation such as the above written like $x\cdot 8=32$, or even $8\cdot x=32$. As we stated earlier, when variables are involved in an expression, we usually write any coefficients[1] on the left, and we generally do not write the multiplication symbol: '$\cdot$'. We would instead express the above equation as $8x=32$. However, just because it is not commonly done, doesn't mean it is incorrect! Whenever the interpretation of division matters, in a concrete situation, not in some abstract calculation, it is important to place the unknown in the correct place, and this sometimes means breaking conventions. I wanted to name the conventions, and I will generally follow them to prepare you for future math courses, but keep in the back of your mind that if it makes sense to write an expression unconventionally, it's ok to do so!

We now have the main tools we need to solve a variety of equations: adding the opposite to both sides, and division. We can now solve more complex equations such as $4x+7=15$ with multiple steps. We start by adding -7 to both sides:

$$4x+7+(-7)=15+(-7).$$

[1]For our purposes, a *coefficient* is a number which is multiplied by a variable expression.

Observe that we put (-7) in parentheses above to clearly separate the opposite from the addition, if you write $+-7$ it can easily be misread. Then, we simplify:

$$4x + 0 = 8$$

and finally obtain

$$4x = 8$$

Now, we can solve for x using division: $x = 8 \div 4 = 2$.

We do not have space to cover every type of equation that might come up, but hopefully I can point out the most common issues. Ultimately, you will learn how to solve equations by doing it. First, what if the final division has a remainder? For example, say we end up with the equation: $3y = 7$. Then, the solution should be $y = 7 \div 3$, which is not a whole number, as you can see. In this case, the presentation of the solution will depend on the problem, sometimes it may make sense to present it as a quotient with remainder: $y = 2$ (remainder: 1), if the result can only be a whole number of groups or units. In other cases, we may want to present a fractional remainder $y = 2 + \frac{1}{3}$, when that makes sense. Recall, as well, that we have the following equality $7 \div 3 = \frac{7}{3}$, so we can also simply say $y = \frac{7}{3}$ if that makes sense. Of course the fractional values: $\frac{7}{3} = 2 + \frac{1}{3}$ are equal so this last is a matter of choice for what makes the most sense or is the most useful[2]. We could also use the decimal form $x = 2.333333\ldots$.

Another important point is to pay attention to the order of operations: $4x + 7$ is calculated in a different order from $4(x + 7)$, and therefore, the equation

$$4(x + 7) = 15$$

may have a different solution than the one we solved above. How do we solve it? We have options! The easiest way to solve it is to apply the distributive property to the left hand side. We should not add anything to both sides, as the order of operations says the $x + 7$ in parentheses should be done before the multiplication $4 \cdot (x + 7)$, and therefore $4(x + 7) + (-7) \neq 4x$. Thus, we do the following:

$$4(x + 7) = 4x + 4 \cdot 7 = 4x + 28.$$

Thus, the equation becomes:

$$4x + 28 = 15.$$

Now, we see that the addition comes first on the left side, so we add the opposite -28 to both sides:

$$4x + 28 + (-28) = 15 + (-28).$$

The left hand side now easily simplifies to $4x + 0 = 4x$:

$$4x = 15 + (-28).$$

To calculate the right hand side $15 + (-28)$ we recall our rules for addition of negative numbers, we start at 15 and move -28 on the number line (28 units to the left), by the commutative property this is the same is $(-28) + 15$, start at -28 and move 15 units to the right. Finally, you can see that this is just the *opposite* of starting at 28 and moving 15 units to the left, so putting it all together:

$$15 + (-28) = (-28) + 15 = -(28 - 15) = -13.$$

[2]I do not use mixed numbers such as $2\frac{1}{3}$ in this book, instead I write such numbers using addition: $2 + \frac{1}{3}$.

We get the final result by subtracting $28-15=13$ using the algorithm or any other method from the book, and then taking the opposite. This technique for reinterpreting addition of negative numbers in terms of subtraction of positive numbers and taking the opposite is an important trick for avoiding calculator dependence. I encourage you to think through the example.

Back to the equation

$$4x = 15 + (-28),$$

we have now calculated the right hand side so the equation becomes:

$$4x = -13.$$

We now apply our rules for dividing negative numbers and we get $x = (-13) \div 4 = -\frac{13}{4}$. Feel free to write the fraction as a whole number with fractional remainder or as a decimal, but the point here is how this result is different from the solution to the equation $4x + 7 = 15$. So pay attention to the order of operations!

Exercises

EXERCISE 6.3.1. Solve each equation, express the answer as a whole number if possible, otherwise, you may leave it in fraction form. Do not use decimals.

(a) $x + 9 = 17$

(b) $y + (-9) = 17$

(c) $7z = 28$

(d) $-7p = 28$

(e) $3t + 9 = 31$

(f) $3(n + 9) = 31$

(g) $3m + 9 = -31$

(h) $3(x - 9) = 31$

(i) $-3(y + 9) = 31$

6.4. Combining like terms, negative coefficients, fractions

Sometimes an equation will have more than one variable term in it, for example $3x + 4x = 17$. In this case, we can simplify the left hand side by thinking about it from the definition. If we have 3 groups with x units in each group, and then 4 more groups with x units in each group, how many groups of x units do we have now? By definition of addition, we have $3 + 4$ such groups. Thus,

$$3x + 4x = (3 + 4)x = 7x.$$

Note that the first equation above is just the distributive property, and this reasoning is essentially one way we justified that property. This process of combining variable terms like this is referred to as *combining like terms*. Traditionally, this terminology is used to remind you not to combine unlike terms in an expression like $3x + 4y$, for example. But, since we proceed from a clear definition of multiplication, it should be clear that $3x + 4y$ does not simplify since the number of units in each group is different in each term! Now that we have simplified the equation $3x + 4x = 17$ to $7x = 17$, you should be able to solve it from there.

We apply the same sort of reasoning when we have negative coefficients, except that a negative number of groups doesn't usually make sense, so we just use the rule formally. For example, if we want to solve $5y - 3y = 13$, then we simply think of it as $5y + (-3y) = 13$, and we combine like terms on the left hand side to get $(5 + (-3))y = 13$, which becomes $2y = 13$.

More interestingly, what happens if we have variable terms on both sides of the equation? For example, $6z = 8z + 19$. In this case, we first apply the idea we used for moving numbers from one side to the other: adding the opposite. Given that we want to solve for z and isolate the number 19, it makes sense to move the $8z$ to the left hand side, so we add $-8z$ to both sides as follows.

$$(-8z) + 6z = (-8z) + 8z + 19$$

On the right hand side we have $(-8z) + 8z + 19 = ((-8) + 8)z + 19 = 0z + 19 = 0 + 19 = 19$, since, zero groups of any number of units is 0 units! Note that we do not combine the 19 with the z terms as 19 is not given as a multiplication. On the left hand side we have $(-8z) + 6z = ((-8) + 6)z = -2z$. So, finally we obtain:

$$-2z = 19$$

and the solution is

$$z = 19 \div (-2) = -(19 \div 2) = -\frac{19}{2} = -\left(9 + \frac{1}{2}\right) = -9.5$$

where we have written the answer in several different forms to show you the possibilities. The first step $z = 19 \div (-2)$ is just from the definition. The second $19 \div (-2) = -(19 \div 2)$ comes from the sign rules for division. After that, we can simplify $19 \div 2$ in many ways by writing it as a fraction, by doing long division with fractional quotient, or by doing long division to get a decimal quotient. Depending on the context, one of these ways of presenting the result might make more sense than the others, in the abstract, it does not matter at all, as all the above solutions are equivalent values.

We may also have fractions as coefficients, as regular numbers or both in a given equation. Let's look at an example:

$$\frac{2}{3}p = \frac{2}{5}.$$

In this case, we simply solve by the definition: $p = \frac{2}{5} \div \frac{2}{3}$, and then apply the rule for dividing fractions,

$$p = \frac{2}{5} \div \frac{2}{3} = \frac{2}{5} \cdot \frac{3}{2} = \frac{2 \cdot 3}{5 \cdot 2} = \frac{6}{10} = 0.6 = \frac{3}{5}.$$

Here, we have written the result in various forms again to show you the possibilities.

We can also look at an example where adding fractions is required:

$$\frac{4}{5}t + \frac{2}{6} = 17.$$

First of all, we know that we should add the opposite $-\frac{2}{6}$ to both sides of the equation,

$$\frac{4}{5}t + \frac{2}{6} + \left(-\frac{2}{6}\right) = 17 + \left(-\frac{2}{6}\right).$$

On the left hand side, we get the usual cancellation involving the opposite (the whole point of adding the opposite), $\frac{4}{5}t + \left(\frac{2}{6} + \left(-\frac{2}{6}\right)\right) = \frac{4}{5}t + 0 = \frac{4}{5}t$. On the right hand side we need to add $17 + \left(-\frac{2}{6}\right)$. How do we do this? Essentially the only way is to find a common denominator. It is easy to find a common denominator with a whole number, how many parts of size $\frac{1}{6}$ is 17 whole amounts? Well, 1 whole amount is $\frac{6}{6}$, 2 whole amounts is $\frac{12}{6}$, 3 whole amounts is $\frac{18}{6}$, and continuing in this way, we get that 17 whole amounts is $\frac{17\cdot 6}{6}$. You can calculate $17 \cdot 6 = (10+7)\cdot 6 = 10\cdot 6 + 7 \cdot 6 = 60 + 42 = 102$, which is essentially the multiplication algorithm. Thus, we get $17 = \frac{17\cdot 6}{6} = \frac{102}{6}$. Now, we are ready to add the fractions on the right hand side:

$$17 + \left(-\frac{2}{6}\right) = \frac{102}{6} + \left(-\frac{2}{6}\right) = \frac{102+(-2)}{6} = \frac{100}{6}.$$

We can simplify this fraction a bit if we notice that we can pair up the objects, so 2 parts of size $\frac{1}{6}$ is the same is $\frac{1}{3}$, so for every $\frac{2}{6}$ we get $\frac{1}{3}$, i.e., half as many thirds as sixths, and therefore $\frac{100}{6} = \frac{2\cdot 50}{2\cdot 3} = \frac{50}{3}$. Another way to see this is to think of the rule for multiplying fractions in reverse:

$$\frac{50}{3} = 1 \cdot \frac{50}{3} = \frac{2}{2}\cdot\frac{50}{3} = \frac{2\cdot 50}{2\cdot 3} = \frac{100}{6}.$$

The order here is we read from right to left. If we think in division terms, we have $\frac{100}{6} = 100 \div 6$, and if we read this as a *how many groups* division problem, we have 100 objects to put in groups of 6. Note that if we combine two objects into one 'double object' , each group of 6 single objects is the same as 3 'double objects', and at the same time, 100 objects is the same as 50 double objects. Thus, we see that 100 objects in groups of 6 is the same number of groups as 50 double objects in groups of 3, and thus $\frac{100}{6} = \frac{50}{3}$. It is important to reason in terms of the definition and to be able to apply the rules formally.

In case the above is a bit obscure, I will illustrate with object abbreviations. So, a group of 6 is given by $||||||$, this is equal to one whole amount, so $| = \frac{1}{6}$ of the whole amount. Now, if we group in pairs (double objects), we get that one whole amount is $1 = |||||| = (||)(||)(||)$, so it is equal to 3 objects of the form $(||)$, and thus $(||) = \frac{1}{3}$ of the whole amount. Since we are putting the $|$ objects into groups of two, the number of $|$ objects is the number of $(||)$ objects, times 2. So if there are 100 $|$ objects, then there are $100 \div 2$ $(||)$ objects, and if we calculate 100 of $|$ is equal to $100 \div 2 = 50$ of $(||)$. Therefore, $\frac{100}{6} = \frac{50}{3}$. If we wish we can also view this in terms of multiplication as follows:

$$\frac{100}{6} = \frac{2\cdot 50}{2\cdot 3} = \frac{2}{2}\cdot\frac{50}{3} = 1\cdot\frac{50}{3} = \frac{50}{3}.$$

Note that we have simply reversed the order of the above equations to emphasize this interpretation.

Exercises

EXERCISE 6.4.1. Solve each equation, express the answer as a whole number if possible, otherwise, you may leave it in fraction form. If the solution is a fraction, try to simplify it.

(a) $x + 9 = 17x$

(b) $y + (-9) = -17y$

(c) $7z = \frac{28}{9}$

(d) $-7p + \frac{1}{3} = 28$

(e) $3t - \frac{2}{3} = \frac{31}{2}$

(f) $\frac{4}{5}(n + 9) = 31$

(g) $3m + 9 = -\frac{31}{4}$

(h) $3\left(x - \frac{9}{5}\right) = 31$

(i) $-\frac{1}{3}\left(y + \frac{2}{3}\right) = \frac{31}{4}$

Part 4

Special Topics

CHAPTER 7

Percent

Percentages are a very special way of presenting fractions to make them more relatable and understandable to a wider audience in sports, politics, advertisements, etc. The essential point to understand about percents is that the name literally tells you what they are: 'per' means 'out of' and 'cent' means 'one hundred'. So, essentially, something like 60% really means 60 out of 100, or, in terms we've used in this book, 60 parts of size $\frac{1}{100}$:

$$60\% = \frac{60}{100}.$$

Thus, you should understand the percent sign % to be a stylized way of writing $\frac{1}{100}$, it is the only fractional amount that has a special symbol.

Percent bears a special relationship to Base-10 decimals as well, recall that $\frac{1}{100} = 0.01$, so we also have:

$$60\% = \frac{60}{100} = 0.60 = 0.6 = \frac{6}{10} = \frac{3}{5}.$$

Here we see a variety of ways to express this percentage as a fraction and as a decimal. It is important to be able to think about percent from a variety of points of view to be able to solve different types of problems.

We can also convert any decimal or fraction into percent form. For decimals this is very easy, for example consider 0.45. We want to write $0.45 = \frac{?}{100} = ? \div 100$, which means by definition, $? = 0.45 \cdot 100 = 45$. Therefore, $0.45 = \frac{45}{100} = 45\%$. Note that all we need to do is shift the decimal point. Observe that a number larger than 1, such as 2.34 will give an 'improper percentage' larger than 100, so $2.34 = \frac{234}{100} = 234\%$.

For fractions things can get a bit trickier as all fractions don't play well with Base-10. First we consider a friendly fraction like $\frac{3}{4}$. We want to write $\frac{3}{4} = \frac{?}{100}$, this means that $? = \frac{3}{4} \cdot 100 = 300 \div 4 = 75$, as you can calculate with the algorithm. So, therefore, $\frac{3}{4} = \frac{75}{100} = 75\%$.

If we pick a different fraction like $\frac{2}{3}$ we can proceed in the same way: $\frac{2}{3} = \frac{?}{100}$, therefore $? = \frac{2}{3} \cdot 100 = 200 \div 3$. If we do the division algorithm here, we get the following.

$$
\begin{array}{rrr}
 & 0.06 & (0.01\text{s of 3s}) \\
 & 0.60 & (0.1\text{s of 3s}) \\
 & 6.00 & (1\text{s of 3s}) \\
 & 60.00 & (10\text{s of 3s}) \\
\cline{2-2}
3 \,\big) & 200.00 & \\
 & -180.00 & \leftarrow 60 \cdot 3 \\
\cline{2-2}
 & 20.00 & \\
- & 18.00 & \leftarrow 6 \cdot 3 \\
\cline{2-2}
 & 2.00 & \\
- & 1.80 & \leftarrow 0.6 \cdot 3 \\
\cline{2-2}
 & 0.20 & \\
- & 0.18 & \leftarrow 0.06 \cdot 3 \\
\cline{2-2}
 & 0.02 &
\end{array}
$$

If you study this division for a bit, you can see the pattern that will continue forever, we get a remainder of 2 shifting to the right, and we get a quotient of 6 shifting to the right forever, thus $200 \div 3 = 60+6+0.6+0.06+0.006+\ldots = 66.666\ldots$ where the sixes continue forever to the right. If you type this in a calculator, it will probably round up at some point and say 66.6666666667, but you can see from the above calculation that the 7 is inaccurate, and it only occurs because the calculator does not have infinite storage. Generally, I prefer to avoid infinite decimals and use the fraction form $\frac{2}{3}$ instead, but since we are specifically trying to write this as a percentage, we will allow an exception in this case. Thus, we get $\frac{2}{3} = \frac{66.666\ldots}{100} = 66.666\ldots\%$.

Since percentages are inherently fractions, they always reference a whole amount which may be understood or stated explicitly. A simple example is discounts. *An office supply store sells notebooks for* 20% *off the regular price. The regular price is* \$3.00. *What is the discount amount, and what is the sale price?* In this case, the whole amount is the regular price of \$3.00. The discount amount is 20% of \$3.00. How do we find this discount amount? We apply the definition of multiplication, let groups = notebooks at regular price, and units = dollars. Then, the cost of 1 notebook is \$3.00, so what is the cost of 20% of that? Well, we simply multiply $20\% \cdot \$3.00 =?$ and this will be the cost of '20%' of a notebook. How do we carry out the multiplication?

At this point you have a variety of options. You could write

$$20\% = \frac{20}{100} = \frac{20 \cdot 1}{20 \cdot 5} = \frac{20}{20} \cdot \frac{1}{5} = 1 \cdot \frac{1}{5} = \frac{1}{5}$$

in fraction form, and then multiply

$$? = 20\% \cdot 3 = \frac{1}{5} \cdot 3 = \frac{3}{5}$$

Now, we can do a long division for $3 \div 5$ and we find that $\frac{3}{5} = 0.6$. So therefore, the discount amount is \$0.60.

You could also use the decimal form $20\% = 0.20 = 0.2$, and then multiply $? = 20\% \cdot 3 = 0.2 \cdot 3 = 0.6$ using the algorithm or just the rule for single nonzero digit numbers in Base-10. Another method for calculation is to use a percent table, starting with

$$\$3.00 \to 100\%,$$

we now use the rule for dividing by 10 to get:

$$\$0.30 \to 10\%,$$

and finally we double this amount to get:

$$\$0.60 \to 20\%.$$

You want to be able to approach this calculation in all of these ways to deepen your understanding of percentages, fractions and decimals.

Finally, we can figure out the sale price by subtracting the discount amount \$0.60 from the regular price \$3.00:

$$\$3.00 - \$0.60 = 3 - 0.6 = (0.4 + 2.6) - 0.6 = 0.4 + (2.6 - 0.6) = 0.4 + 2 = 2.4 = \$2.40.$$

Here, I used the associative property, you could also use the algorithm to perform the subtraction. In any case, the sale price is \$2.40.

We can also flip the problem around. *A shirt is on sale for* \$15.60 *from a regular price of* \$20. *What is the discount rate as a percentage of the regular price?* In this case we want to find the discount amount first: $20 - 15.60 = \$4.40$. Then, we want to find what percentage this is of \$20, the regular price. So, essentially, we are looking at the following[1]: $\frac{4.4}{20} = \frac{?}{100}$. Here we could proceed as above and do long division, but since $100 = 5 \cdot 20$ there is a simpler way, we apply the rule for multiplying fractions.

$$\frac{4.4}{20} = 1 \cdot \frac{4.4}{20} = \frac{5}{5} \cdot \frac{4.4}{20} = \frac{5 \cdot 4.4}{5 \cdot 20}.$$

We now need to calculate $5 \cdot 4.4 = 22$ (you can use the algorithm here). Finally, we have

$$\frac{5 \cdot 4.4}{5 \cdot 20} = \frac{22}{100} = 22\%$$

and so the discount rate is 22%.

Exercises

EXERCISE 7.0.1. Write each percentage as simplified fraction, then write it as a decimal.

(a) 30%

(b) 40%

(c) 50%

(d) 85%

(e) 100%

(f) 200%

[1]We allow a decimal numerator here, if you prefer you can think of this as a division $\frac{4.4}{20} = 4.4 \div 20$.

(g) 150%

EXERCISE 7.0.2. Write each decimal or fraction as a percentage, use long division to find infinite decimals, if needed.

(a) 0.6

(b) 0.05

(c) 1.2

(d) $\frac{3}{5}$

(e) $\frac{1}{2}$

(f) $\frac{9}{8}$

(g) $\frac{4}{9}$

EXERCISE 7.0.3. A retail store is selling shirts for 30% off the regular price. The regular price for a shirt is \$15.00.

(a) Find the discount amount using the definition of multiplication.

(b) Find the sale price.

EXERCISE 7.0.4. The town of Townsville has 1040 voting residents. Candidate A and Candidate B run in an election for mayor. Candidate A receives 65% of the votes. Assume all voting residents vote for one of the two candidates.

(a) Find how many votes Candidate A received using the definition of multiplication.

(b) Find how many votes Candidate B received.

(c) What percent of the vote did Candidate B receive? Calculate this in two ways.

EXERCISE 7.0.5. A retail store is selling socks for \$7.30 discounted from the regular price of \$10.

(a) Find the discount amount using subtraction.

(b) Find the discount rate as a percentage of the regular price.

CHAPTER 8

Exponents

The basic idea of exponents is to have a shorthand for repeated multiplication. Let's give the definition and then look at some examples.

DEFINITION 8.0.1. Let $n > 0$ be a whole number, and let b be any number. Then, the expression b^n stands for $b \cdot b \cdot b \cdots b$ or b multiplied by itself n times. The number n is called the *exponent* and the number b is called the *base*.

Two simple examples to help you understand the roles of exponent and base. First we have 2^3 which means the number 2 multiplied by itself 3 times:

$$2^3 = 2 \cdot 2 \cdot 2 = 4 \cdot 2 = 8.$$

Then, we switch roles and we have 3^2, or 3 multiplied by itself 2 times:

$$3^2 = 3 \cdot 3 = 9.$$

Recall that exponents come before everything but grouping symbols in the order of operations. So an expression such as:

$$4 + 5^2 + 3 \cdot 4$$

should be simplified by doing $5^2 = 5 \cdot 5 = 25$ first

$$4 + 25 + 3 \cdot 4,$$

then the multiplication $3 \cdot 4 = 12$

$$4 + 25 + 12,$$

then addition from left to right (by the associative property we can also go right to left here)

$$4 + 25 + 12 = 29 + 12 = 41.$$

One important point of confusion to clarify with respect to exponential expressions is the role of negative numbers in the base, and the use of parentheses. For a simple example to illustrate the point:

$$(-2)^4 \neq -2^4.$$

If it looks like those values should be the same to you, write down what you think the expression evaluates to and compare it with the following discussion. The key to distinguishing the two expressions is to understand how to parse the two exponential expressions in terms of exponent and base. Note that, for any number n we have $-n = -1 \cdot n$ (e.g. $-6 = -1 \cdot 6$), so we should consider the opposite to come after the exponent in the order of operations. To be fair, this point is left ambiguous by the order of operations list, and I think it is because the opposite is an operation on a single number and many people think of it as simply a shorthand for multiplying by -1 (which is equivalent). This is all just to let you know that opposites should be done after exponents but before anything else in the order of operations, though clarifying parentheses are preferred to avoid hard-to-read expressions.

To return to the above example, because of the parentheses, the expression $(-2)^4$ has base $= -2$ and exponent $= 4$. Thus, $(-2)^4$ is -2 multiplied by itself 4 times,

$$(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 4 \cdot (-2) \cdot (-2) = -8 \cdot (-2) = 16.$$

On the other hand, the expression -2^4 has base $= 2$, exponent $= 4$, and then we take the opposite of that. Thus the expression should be simplified as follows:

$$-2^4 = -2 \cdot 2 \cdot 2 \cdot 2 = -4 \cdot 2 \cdot 2 = -8 \cdot 2 = -16.$$

Now we see that since $16 \neq -16$, we also have

$$(-2)^4 \neq -2^4.$$

To add to the confusion, if the exponent is odd, then the above distinction actually doesn't affect the result. So, the following exponential expressions actually *are* equal:

$$(-2)^3 = -2^3.$$

The reasoning is the same as we used above, we should still parse them differently, it just turns out we get the same result. The expression $(-2)^3$ has base $= -2$ and exponent $= 3$. Thus, $(-2)^3$ is -2 multiplied by itself 3 times,

$$(-2)^3 = (-2) \cdot (-2) \cdot (-2) = 4 \cdot (-2) = -8.$$

On the other hand, the expression -2^3 has base $= 2$, exponent $= 3$, and then we take the opposite of that. Thus the expression should be simplified as follows:

$$-2^3 = -2 \cdot 2 \cdot 2 = -4 \cdot 2 = -8.$$

Now, since $-8 = -8$ we see why

$$(-2)^3 = -2^3.$$

Note carefully, though, that the way we deconstructed the expressions to evaluate them was the same in both examples. The issue is that if you multiply a negative number by itself 3 times you get a negative result, but 4 times gives a positive result. Since the only difference in these expressions is the sign of the result, we can only illustrate the difference clearly when the base is negative and the exponent is even. Nonetheless, the expressions $(-2)^3$ and -2^3 are still different expressions, and are evaluated differently, even though they lead to the same result.

Since the base b is allowed to be any number, we can also have a fractional base, which is simply evaluated according to the usual rules for multiplying fractions:

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 3} = \frac{4}{9}.$$

In order to effectively do algebraic manipulations with variable expressions involving exponents, it is helpful to develop some rules for multiplying exponential expressions with the same base, the same exponent, and for simplifying multiple exponents. Let's first look at multiplying exponential expressions. We start with numerical examples. First of all, if the base is not the same, and the exponents are different, there is not much we can do but keep the exponentials separate, for example $2^3 \cdot 5^2 = 8 \cdot 25 = 200$. If, however, the base is the same, then we can simplify, $2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2)$, here we write the parentheses to show you the two exponential expressions expanded into multiplication form. According to the associative property, that expression is simply equal to 2 multiplied by itself 5 times *in any order*, thus, we get,

$$2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2) = 2^5.$$

Notice that the number of factors of 2 that we get is 5, with 3 coming from 2^3 and 2 coming from 2^2. Ultimately, we can think of this as simply adding the exponents $3 + 2 = 5$ to get the total number of factors of 2 in the expression $2^3 \cdot 2^2$. This is the essential idea in the rule below.

THEOREM 8.0.2. *Let m and n be positive whole numbers, and let b be any number. Then, $b^n \cdot b^m = b^{n+m}$.*

We will now extend the above rule to allow m and n to be any whole numbers (we will leave the topic of fractional or decimal exponents for another book). First, what should the exponent zero do? Consider 2^0, if we assume that the above rule holds, we can let $m = 0$ and $n = 1$, and then we have:

$$2^0 \cdot 2 = 2^0 \cdot 2^1 = 2^{0+1} = 2^1 = 2.$$

By the definition of division, this implies that $2^0 = 2 \div 2 = 1$. This argument works for any base, so we get the following result.

THEOREM 8.0.3. *Let b be any number*[1]*, then $b^0 = 1$.*

Now that we have a clear value for the exponent zero, we are ready to define negative exponents. Let's start with an example to illustrate the point, what is $x = 2^{-3}$, for example? If we apply the rules above, we get the following equation:

$$x \cdot 2^3 = 2^{-3} \cdot 2^3 = 2^{-3+3} = 2^0 = 1$$

so by the definition of division, and writing the result in fraction form, we get: $x = 1 \div 2^3 = \frac{1}{2^3}$. Thus, $2^{-3} = \frac{1}{2^3}$ (in this particular case we can further simplify $\frac{1}{2^3} = \frac{1}{8}$, but we focus on the general rule for now). This argument works for any negative exponent, or indeed, for the opposite of any exponent, positive or negative. We state the result below.

THEOREM 8.0.4. *Let n be any whole number, and let b be any number, then*

$$b^{-n} = 1 \div (b^n) = \frac{1}{b^n}.$$

The last rule to discuss is the double exponent rule, for an example let's look at $\left(2^3\right)^4$. If we unpack this expression one step at a time, we have the outer exponent 4, with base 2^3, so we multiply 2^3 by itself 4 times and we obtain

$$\left(2^3\right)^4 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 = 2^{3+3+3+3} = 2^{4 \cdot 3}.$$

Here we applied the rule for multiplying exponential expressions with the same base, and we used the definition of multiplication $4 \cdot 3 = 3 + 3 + 3 + 3$ (4 groups of 3). This shows the general pattern. We now state the rule.

THEOREM 8.0.5. *Let m and n be any whole numbers, and let b be any number. Then, $\left(b^n\right)^m = b^{m \cdot n}$.*

---

[1]This includes $b = 0$, we have $0^0 = 1$, which causes no trouble outside of calculus.

One common point of confusion comes from distinguishing this rule from the rule: $b^n \cdot b^m = b^{n+m}$. For this purpose, I suggest you keep the following example in mind

$$2^3 \cdot 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$$

compared to

$$\left(2^3\right)^4 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{12}.$$

If you actually write out the individual factors, it is much more difficult to make the mistake of confusing the two rules.

Exercises

Exercise 8.0.1. For each exponential, state the base and exponent, and then evaluate the exponential expression.

(a) 4^3

(b) 3^4

(c) 7^0

(d) -4^3

(e) $(-4)^3$

(f) -3^4

(g) $(-3)^4$

(h) $\left(\frac{4}{3}\right)^3$

Exercise 8.0.2. For each exponential, state the base and exponent, and then evaluate the exponential expression.

(a) 4^{-3}

(b) 3^{-4}

(c) -4^{-3}

(d) $(-4)^{-3}$

(e) -3^{-4}

(f) $(-3)^{-4}$

(g) $\left(\frac{4}{3}\right)^{-3}$

Exercise 8.0.3. Show all of the factors to explain the difference between $3^2 \cdot 3^4$ and $\left(3^2\right)^4$.

Exercise 8.0.4. Consider the expression $3 \cdot 5^2 + 7$.

(a) Calculate the result of the expression using the order of operations.

(b) Now rewrite the expression according to the order of operations so that we should do the multiplication first and calculate the result.

Exercise 8.0.5. Using ideas from the text, what can we say about the following base $b = 0$ exponential expressions?

(a) 0^1

(b) 0^2

(c) 0^0

(d) 0^{-1}

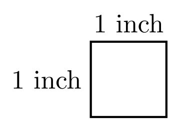
(e) 0^{-2}

CHAPTER 9

Basic Geometry

We will now go over some basic concepts in geometry. First of all, we start with length measurements. There are many units used to measure length from our US customary units like inches, feet, yards, miles, etc., to metric units like centimeters, meters, kilometers. We can convert from unit to unit using the definition of multiplication.

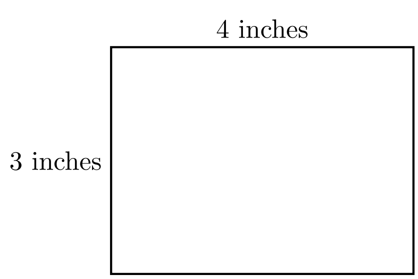
Once we have a unit of length, such as inches, we then form squares that are 1 inch by 1 inch.



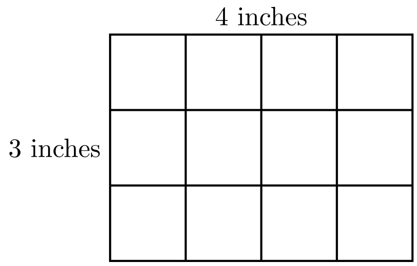
This is referred to as a *square inch*, sometimes we write in^2 to abbreviate. When we want to speak about area generally, without referencing a specific unit, we will simply call the units of length *units*, and therefore the units of area will be called *square units* or abbreviated units^2.

Definition 9.0.1. Given units of length, we define the *area* of a shape to be the number of square units it takes to cover the shape without gaps or overlaps, the square units are allowed to be cut in pieces and rearranged.

This definition is a bit vague, but for our purposes it will be sufficient. Let's look at the area of a rectangle from this point of view. Say we have a rectangle that is 4 inches wide and 3 inches long.



In order to find the area of this shape, we simply form a grid of square inches by dividing horizontally and vertically as follows.



Now, we can count the squares manually, or, more usefully, we can count them using the definition of multiplication. We take groups = rows, and units = square inches. Then, we see that there are 3 rows, with 4 square inches in each row in the rectangle, so the rectangle is covered by a total of $3 \cdot 4$ square inches, and therefore the area of the rectangle is $3 \cdot 4$ square inches. We could calculate the product $3 \cdot 4 = 12$ square inches, but the point here is to see how the length and width measurements correspond to a way of counting the number of square inches in the rectangle. Note that the 3 in $3 \cdot 4$ refers to the *number of rows* and NOT to inches. Similarly, the 4 in $3 \cdot 4$ refers to *square inches* and NOT to inches. These numbers correspond, but they are not the same, so be careful when you set up multiplication to count area that you are NOT thinking 3 in. $\cdot$ 4 in.. Nonetheless, we have the following formula.

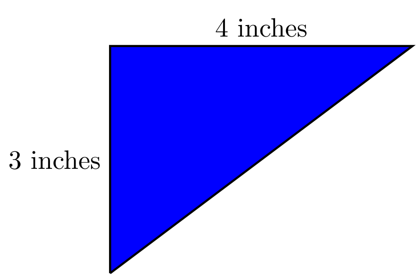
THEOREM 9.0.2. *Let R be a rectangle with length L units and width W units, then the area of R is $L \cdot W$ square units.*

In order to calculate the area of more complicated shapes than rectangles, we need to formalize some consequences of the definition of area.

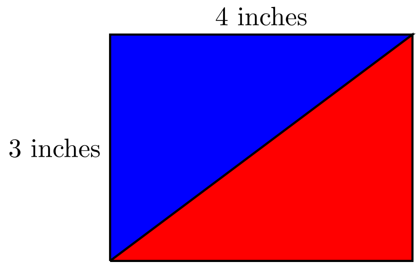
THEOREM 9.0.3 (Moving Principle). *If a shape is moved from one place to another, it still has the same area.*

THEOREM 9.0.4 (Additivity Principle). *If a shape is decomposed into pieces, without gaps or overlaps, then the area of the shape is the sum of the areas of the pieces.*

Together, these principles will allows us to cut and rearrange a shape in various ways to make it easier to calculate the area. The simplest example of this is a triangle. Let's take a right triangle (a triangle with a $90°$ angle) with base 4 inches, and height 3 inches.



If we double the triangle and arrange it as below



we get a rectangle 4 inches by 3 inches, which we know to have area 12 in.^{2}. By the moving and additivity principles, the area of the rectangle is equal to the sum of the areas of the two triangles

$$A_{\text{rectangle}} = A_{triangle} + A_{triangle}$$

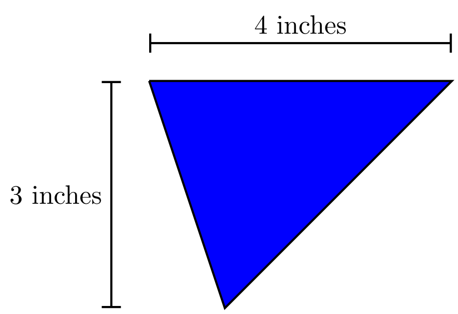
Since the two triangles are congruent[1], they have the same area, call that common area $A_t = A_{triangle} = A_{triangle}$. Then, we have,

$$A_{\text{rectangle}} = A_{triangle} + A_{triangle} = A_t + A_t = 2 \cdot A_t$$

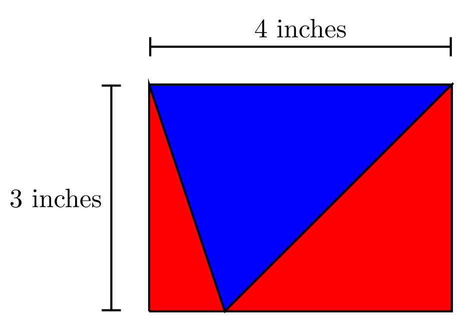
and so by the definition of division, we have:

$$A_t = A_{\text{rectangle}} \div 2 = 12 \div 2 = 6 \text{ in.}^2.$$

We can make a similar argument if the triangle is not a right triangle but has the same base and height.

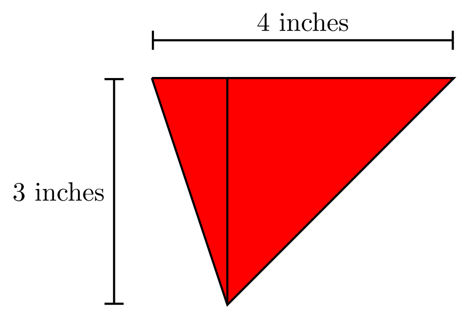


We fill in the rectangle as below:



[1]For our purposes, two shapes are *congruent* if you can move them so that they will perfectly overlap each other.

If we take the two red triangles and rearrange them as follows



we get a triangle congruent to the original blue triangle.

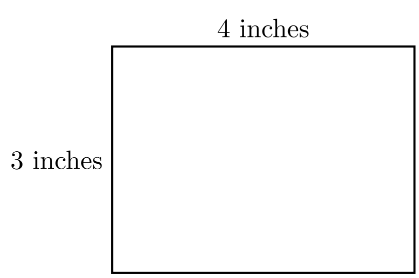
Since we again obtained a 3 inch by 4 inch rectangle made up of two copies of our triangle, we again obtain an area of 6 in.^{2} for the blue triangle. This argument can be made with any triangle, and so we obtain the classic formula.

THEOREM 9.0.5. *Let T be a triangle with base b units and height h units, then the area of T is $b \cdot h \div 2$ square units.*

Our final concept is perimeter.

DEFINITION 9.0.6. The *perimeter* of a shape is the length of its outer boundary (measured in units of length).

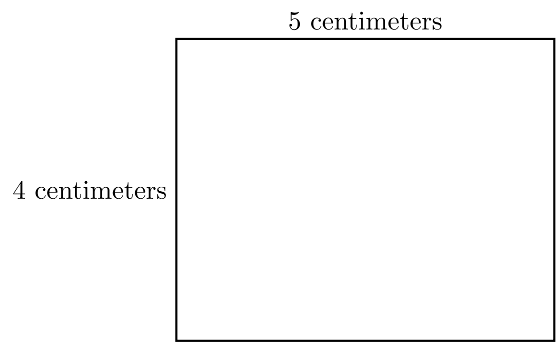
To find the perimeter of simple shapes like rectangles and triangles, we simply add up the lengths of the outer edges. For example, the rectangle



has perimeter equal to $4 + 3 + 3 + 4$ inches, which is $4 + 3 + 3 + 4 = 14$ in. if you calculate it. Note the difference between the perimeter and area, not just that we have different values, but the units of measurement are different, perimeter is a *length*, not an area.

Exercises

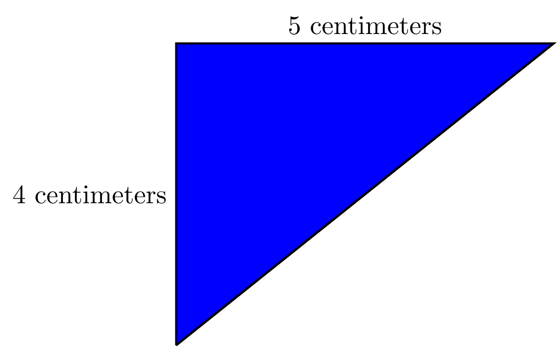
EXERCISE 9.0.1. Consider the rectangle below.



(a) Use the definition of multiplication (not the formula) to calculate the area of this rectangle and clearly state your units.

(b) Find the perimeter of this rectangle, and clearly state your units.

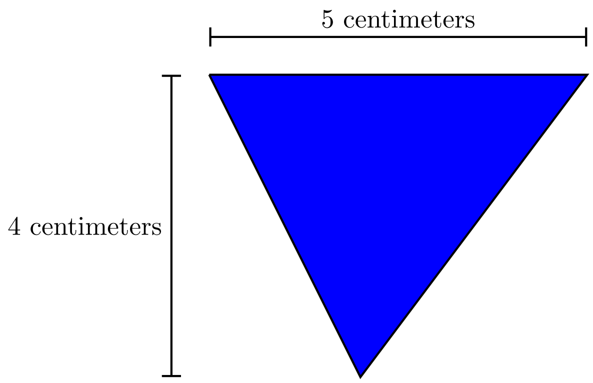
EXERCISE 9.0.2. Consider the triangle below.



(a) Use the moving and additivity principles (not the formula) to calculate the area of this triangle and clearly state your units.

(b) Call the length of the unlabeled side of the triangle h (measured in centimeters). Write an expression for the perimeter of the triangle in terms of h. Clearly state your units.

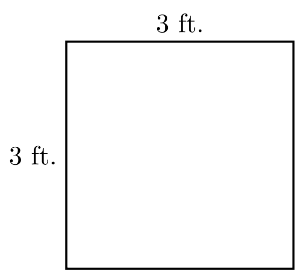
EXERCISE 9.0.3. Consider the triangle below.



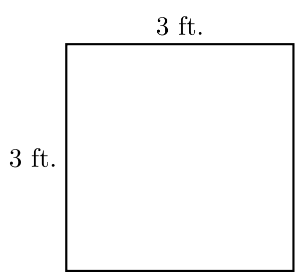
(a) Use the moving and additivity principles (not the formula) to calculate the area of this triangle and clearly state your units.

(b) Define two variables and write an expression for the perimeter of this triangle in terms of your two variables. Clearly state your units.

Exercise 9.0.4. Explain why the area of the square below is NOT 3 square feet. Do NOT calculate the area of the square in your explanation. Use only the definition of area.



Exercise 9.0.5. Use the definition of multiplication (not the formula) to calculate the area of the square below and clearly state your units.



CHAPTER 10

Prime numbers

In this chapter we will focus on positive whole numbers and discover a way of building them up from certain basic kinds of numbers called prime numbers. Let's start with a definition.

DEFINITION 10.0.1. Let N, a, and b be positive whole numbers and assume that $N = a \cdot b$. We say that this is a *proper factorization of* N if we have $a < N$ and $b < N$.

For example, if we look at $6 = 2 \cdot 3$, this is a proper factorization of 6 because $2 < 6$ and $3 < 6$. On the other hand, if we want to factor 5 as a product of whole numbers, there are only two ways to do it: $5 = 5 \cdot 1 = 1 \cdot 5$, and in each case we do NOT have $5 < 5$, and therefore there is NO proper factorization of 5. We therefore say that 5 is a prime number, this illustrates the following definition.

DEFINITION 10.0.2. Let $N > 1$ be a whole number[1]. If N has no proper factorization, then we say that N is a *prime number*.

The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$. If we think about the proper factorization $6 = 2 \cdot 3$, we see that 2 and 3 are both prime numbers, and so we call this the prime factorization of 6. Think about the following process: starting with a number N find a proper factorization, if there are none, then N is prime by definition, if there is one, look at the factors, do they have a proper factorization? If so, factor them, and look at the new factors, do they have a proper factorization? If we continue in this way, we see that the factors are decreasing in size each time, and therefore at some point we will obtain only prime numbers as factors, and if we multiply all the factors we get in this way we have a factorization of the number N as a product of only prime numbers. This argument is the idea behind the following result.

THEOREM 10.0.3 (Fundamental Theorem of Arithmetic). *Let $N > 1$ be a whole number, then, up to order, there is a unique way to write N as a product of only prime numbers.*

Let's illustrate the argument and the theorem using a larger number with more steps involved. Consider $N = 90$, can you find a proper factorization of this number? Indeed, we have, for example, $90 = 10 \cdot 9$. Now, we try to find a proper factorization of $10 = 5 \cdot 2$ and $9 = 3 \cdot 3$. In both cases, we see that the factors 2, 3, and 5 are all prime numbers. So, we have found the prime factorization of $N = 90$ as we show below:

$$90 = 10 \cdot 9 = (5 \cdot 2) \cdot (3 \cdot 3).$$

[1] We exclude $N = 1$ as a prime number for technical reasons. If we allowed 1 to be prime, we would lose uniqueness in the fundamental theorem of arithmetic, for example $14 = 2 \cdot 7 \cdot 1 = 2 \cdot 7 \cdot 1 \cdot 1 \cdot 1$.

What if we start with a different proper factorization? For example, say we find instead $90 = 3 \cdot 30$. In this case, 3 is already prime, so we look at $30 = 6 \cdot 5$ where the factor 5 is prime, but 6 is not, so we continue to factor: $6 = 2 \cdot 3$. Putting it all back together, we get the prime factorization:

$$90 = 3 \cdot 30 = 3 \cdot (6 \cdot 5) = 3 \cdot ((2 \cdot 3) \cdot 5).$$

Note that if we change parentheses (associative property) and reorder the factors (commutative property) we get the exact same prime factors as before, one factor of 2, one factor of 5 and two factors of 3. This illustrates the uniqueness of prime factorization, we don't necessarily get the same parentheses or the same order, but we always get the same list of prime factors.

Once we have this idea of unique prime factorization, we immediately see that the list of prime numbers must continue forever: *there are infinitely many prime numbers*. We can argue this as follows, let $2, 3, 5, 7, \ldots, p$ be the list of all prime numbers up to p. Consider the number

$$N = (2 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot p) + 1.$$

By definition, N is not divisible by any of the prime numbers $2, 3, 5, 7, \ldots, p$, since the equation shows it has remainder 1 if we divide by any of them. Therefore, either N itself is a larger prime number than any in the list, or N has a prime factor larger than any in the list, in either case, the list $2, 3, 5, 7, \ldots, p$ does NOT contain all prime numbers.

Exercises

Exercise 10.0.1. Find a prime factorization of each number below.

(a) 8

(b) 70

(c) 158

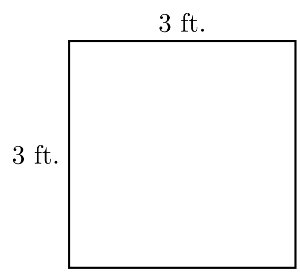
(d) 36

Exercise 10.0.2. Find the next 3 prime numbers greater than 29. Justify your answer.

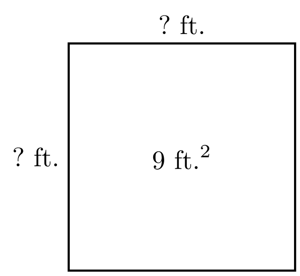
CHAPTER 11

Irrational numbers

In the previous section, we learned how to calculate the area of the square below



if we just use the formula, we get $3 \cdot 3 = 9$ ft.^{2}. What if we reverse the problem? Say we start with a square that has area 9 ft.^{2} and ask, what is the length of a side of the square?



We can use a variable $x =?$ feet for the side length, and the above problem translates into the equation $x \cdot x = 9$, or, using exponents:

$$x^2 = 9.$$

We already know a solution to this equation from above, of course, we have $x = 3$ ft. This leads to the following definition.

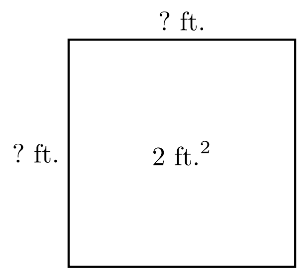
Definition 11.0.1. Let A be a positive number, then we say that b is a *square root* of A if $b^2 = A$. If, in addition, b is positive[1], then we use the notation $b = \sqrt{A}$.

From this definition, we can see that 3 is a square root of 9, since $3^2 = 9$, and furthermore, 3 is positive, so we also have $3 = \sqrt{9}$. In general, if $A > 0$ is a positive

[1]Observe that $(-3)^2 = 9$, so by definition -3 is a square root of 9. We specifically use 'radical' notation for the positive square root.

number, then the positive square root of A, given by the notation $\sqrt{A}$ is the length of a side of a square with area A ft.^{2}. This is essentially the reason why we call it a *square* root.

Now, consider the following problem, say the square below has area 2 square feet.



What is the length of a side of the square? We know from the definition that $? = \sqrt{2}$ ft.. This follows essentially from the definition, but we would like to know more about this number. It is not a whole number for sure, as we can see that $1^2 = 1 < 2 < 4 = 2^2$, so we should have $1 < \sqrt{2} < 2$. Is it a rational number? In other words, is there an improper fraction $\frac{p}{q}$ so that $\frac{p}{q} = \sqrt{2}$? The answer is NO! As a result we say that $\sqrt{2}$ is an *irrational number.*

THEOREM 11.0.2. *The number $\sqrt{2}$ is irrational.*

There are many ways to explain this, we will use prime factorization ideas. First of all, observe that if you square a number: 3^2, $6^2 = 36 = 2^2 \cdot 3^2$, $10^2 = 100 = 2^2 \cdot 5^2$, $12^2 = 2^4 \cdot 3^2$, etc., each prime factor can always be put in groups of two with none left over. To illustrate,

$$12^2 = 2^4 \cdot 3^2 = (2 \cdot 2 \cdot 2 \cdot 2) \cdot (3 \cdot 3) = (2 \cdot 2) \cdot (2 \cdot 2) \cdot (3 \cdot 3).$$

The reason for this is that when you square a number n, you automatically get two copies of each prime factor of n giving the prime factorization of n^2. In other words, this means that each prime factor of n^2 occurs an *even* number of times, i.e., the number of times is divisible by two.

Now, let's look at the number $x = \sqrt{2}$. All we know about it is from the definition: $x > 0$ and $x^2 = 2$. What if we have $x = \frac{p}{q}$ where p and q are positive whole numbers? In that case we would have

$$2 = \left(\frac{p}{q}\right)^2 = \left(\frac{p}{q}\right) \cdot \left(\frac{p}{q}\right) = \frac{p^2}{q^2}$$

using the rule for multiplying fractions. Next we recall that $\frac{p^2}{q^2} = p^2 \div q^2$, and so by definition of division, we must have $2 = p^2 \div q^2$, and therefore in terms of multiplication:

$$q^2 \cdot 2 = p^2.$$

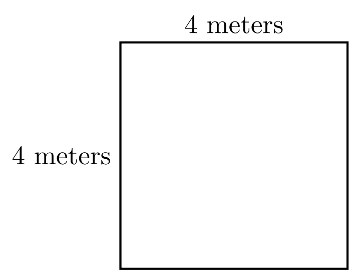
From above, we know that the prime factors of p^2 and q^2 can be put into groups of two with none left over. However, there is a problem: on the left side of the above equation, we have $q^2 \cdot 2$, and 2 is prime, so the number of factors of the prime 2 in $q^2 \cdot 2$ can NOT be put into groups of two with none left over, there will be ONE left over! This is a problem because $q^2 \cdot 2 = p^2$ and the right hand side p^2 must have an even number of factors of 2, i.e., they can be put into groups of two with none left over. This is a contradiction to the uniqueness of prime factorizations (Fundamental Theorem of Arithmetic), and so our assumption $x = \frac{p}{q}$ is necessarily incorrect, and therefore $x = \sqrt{2}$ must be irrational!

If you type $\sqrt{2}$ into a calculator, it will give you a partial decimal expansion such as 1.41421356237, but any such partial expansion is rational, and furthermore, the decimal expansion of any rational number is finite or repeating (see $\frac{1}{3} = 0.3333333\ldots$ for example), therefore the decimal expansion of $\sqrt{2}$ is both infinite, and non-repeating. This means that not only will a calculator fail to calculate it perfectly, but the entire decimal expansion of $\sqrt{2}$ is, in fact, unknown[2] since that would require knowing an infinite non-repeating sequence of single digit numbers!

There are many other irrational numbers, often given special names, such as the number e, the number π and the golden ratio (which is not rational!). Since we need significantly more effort to explain these numbers, we will stop our exploration of irrational numbers here.

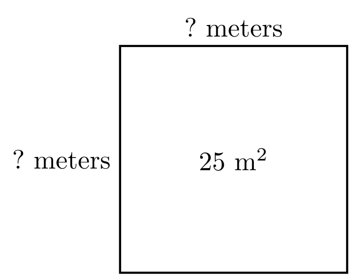
Exercises

Exercise 11.0.1. Consider the square below.



Find the area of the square, clearly state your units.

Exercise 11.0.2. Consider the square below.



Find the length of a side of the square, notated '?' above, clearly state your units.

Exercise 11.0.3. Use radical notation (i.e. $\sqrt{\text{something}} = \text{something else}$) to express the equation $4^2 = 16$.

[2]As of January 2022, the record for computation of the decimal expansion of $\sqrt{2}$ out to 10,000,000,001,000 digits was completed by a computer in about 18.5 days, see `http://www.numberworld.org/y-cruncher/records.html`.

Exercise 11.0.4. Use prime factorization to explain why each value below is irrational.

(a) $\sqrt{3}$

(b) $\sqrt{6}$

(c) $\sqrt{12}$

(d) $\sqrt{102}$

(e) $\sqrt{p}$ where p is any prime number

Bibliography

[B] Beckmann, S., *Mathematics for Elementary Teachers with Activities, 5th ediction*, Pearson (2017).

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