Math 170S HW5 Jun Ryu

1) a) Posterior pdf of θ:

Since Y is the sum of n observations,

Y is a Poisson distr. with $\lambda = n\theta$

Given $Y=y$: $g(y|\theta) = \dfrac{n^y \theta^y e^{-n\theta}}{y!}$

The prior pdf is gamma distr. with params α & β:

$$h(\theta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma\alpha \cdot \beta^\alpha}$$

So, our posterior pdf $k(\theta|y)$ is $\dfrac{g(y|\theta)\,h(\theta)}{\int_{-\infty}^{\infty} g(y|\theta)h(\theta)d\theta}$

We evaluate the integral first:

$$\int_{-\infty}^{\infty} \frac{n^y \theta^y e^{-n\theta}}{y!} \cdot \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma\alpha \cdot \beta^\alpha} d\theta$$

Take out all constants then simplify:

$$\frac{n^y}{y!\,\Gamma\alpha\,\beta^\alpha} \int_0^\infty \theta^y e^{-n\theta} \theta^{\alpha-1} e^{-\theta/\beta} d\theta = \frac{n^y}{y!\,\Gamma\alpha\,\beta^\alpha} \int_0^\infty \theta^{y+\alpha-1} e^{-\theta(n+\frac{1}{\beta})} d\theta$$

Turn the integrand into a gamma distr:

$$= \frac{n^y \Gamma_{y+\alpha} \left(\frac{1}{n+\frac{1}{\beta}}\right)^{y+\alpha}}{y!\, \Gamma\alpha\, \beta^\alpha} \underbrace{\int_0^\infty \frac{\theta^{y+\alpha-1} e^{-\theta(n+\frac{1}{\beta})}}{\Gamma_{y+\alpha} \left(\frac{1}{n+\frac{1}{\beta}}\right)^{y+\alpha}}\, d\theta}_{\text{gamma}\left(\alpha+y,\, \frac{1}{n+\frac{1}{\beta}}\right)}$$

so this resolves to 1

$$= \frac{n^y \Gamma_{y+\alpha}\, \beta^{y+\alpha}}{y!\, \Gamma\alpha\, \beta^\alpha (n\beta+1)^{y+\alpha}} = \frac{n^y \Gamma_{y+\alpha}\, \beta^y}{y!\, \Gamma\alpha\, (n\beta+1)^{y+\alpha}}$$

Now, the numerator:

$$g(y|\theta)h(\theta) = \frac{n^y \theta^y e^{-n\theta}}{y!} \cdot \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\Gamma\alpha\, \beta^\alpha}$$

Combining:

$$\frac{\frac{\cancel{n^y} \theta^y e^{-n\theta}}{\cancel{y!}} \cdot \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\cancel{\Gamma\alpha}\, \beta^\alpha}}{\frac{\cancel{n^y} \Gamma_{y+\alpha}\, \beta^y}{\cancel{y!}\, \cancel{\Gamma\alpha}\, (n\beta+1)^{y+\alpha}}} = \frac{\theta^{y+\alpha-1} e^{-n\theta-\theta/\beta} (n\beta+1)^{y+\alpha}}{\Gamma_{y+\alpha}\, \beta^{\alpha+y}}$$

$$= \frac{\theta^{y+\alpha-1} e^{-n\theta-\theta/\beta}}{\Gamma_{y+\alpha} \left(\frac{1}{n+\frac{1}{\beta}}\right)^{y+\alpha}}$$

So, this gives us a $\boxed{\text{gamma distribution with parameters } \left(\alpha+y,\ \frac{1}{n+\frac{1}{\beta}}\right)}$

b) The loss function is minimized when $w(y)$ is the mean of our posterior pdf. The mean of a gamma distribution is simply the product of the 2 parameters. So,

$$w(y) = E(Y|\theta) = \boxed{\frac{\alpha+y}{n+\frac{1}{\beta}}}$$

c) Now, the above Bayes' Estimator can be rewritten as:

$$\frac{\alpha}{n+\frac{1}{\beta}} + \frac{y}{n+\frac{1}{\beta}} = \frac{\frac{1}{\beta}}{n+\frac{1}{\beta}} \cdot \alpha\beta + \frac{n}{n+\frac{1}{\beta}} \cdot \frac{y}{n}$$

Thus, this shows $w(y)$ is a weighted avg. of MLE $\frac{y}{n}$ and prior mean $\alpha\beta$, with respective weights as circled above.

2) a) $X_1 \ldots X_n$ has a gamma distr. with params α Known and $\theta = \frac{1}{\tau}$

We have $g(x_i|\tau) = \dfrac{x_i^{\alpha-1} e^{-\tau x_i}}{\Gamma\alpha \, (\frac{1}{\tau})^{\alpha}}$

$$g(x_1 \ldots x_n|\tau) = g(x_1|\tau)\cdot g(x_2|\tau) \cdots g(x_n|\tau)$$

$$= \frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\tau \sum_{i=1}^{n} x_i}}{\left(\Gamma\alpha \, (\frac{1}{\tau})^{\alpha}\right)^{n}}$$

Our prior pdf $h(\tau)$ is a gamma distr. with params α_0 & θ_0 $\Rightarrow$ $h(\tau) = \dfrac{\tau^{\alpha_0-1} e^{-\tau/\theta_0}}{\Gamma\alpha_0 \, \theta_0^{\alpha_0}}$

Similar to Q1, we will calculate the integral:

$$\int_{-\infty}^{\infty} g(x|\tau)\, h(\tau)\, d\tau = \int_{0}^{\infty} \frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\tau \sum_{i=1}^{n} x_i}}{\left(\Gamma\alpha \, (\frac{1}{\tau})^{\alpha}\right)^{n}} \cdot \frac{\tau^{\alpha_0-1} e^{-\tau/\theta_0}}{\Gamma\alpha_0 \, \theta_0^{\alpha_0}}\, d\tau$$

Take out constants:

$$\frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}{\Gamma\alpha^{n}\, \Gamma\alpha_0 \, \theta_0^{\alpha_0}} \int_{0}^{\infty} \tau^{\alpha n} e^{-\tau \sum_{i=1}^{n} x_i} \tau^{\alpha_0 -1} e^{-\tau/\theta_0}\, d\tau$$

Simplify:

$$\frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}{\Gamma\alpha^n \, \Gamma\alpha_0 \, \theta_0^{\alpha_0}} \int_0^\infty \tau^{\alpha n+\alpha_0-1} \, e^{-\tau\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)} \, d\tau$$

We again use the technique of turning the integrand into a gamma distr. and we are left with:

$$\frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \Gamma(n\alpha+\alpha_0) \left(\dfrac{1}{\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}}\right)^{n\alpha+\alpha_0}}{\Gamma\alpha^n \, \Gamma\alpha_0 \, \theta_0^{\alpha_0}}$$

$$= \frac{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \Gamma(n\alpha+\alpha_0) \, \theta_0^{n\alpha}}{\Gamma\alpha^n \, \Gamma\alpha_0 \left(\sum_{i=1}^{n} \theta_0 x_i + 1\right)^{n\alpha+\alpha_0}}$$

Combining all:

$$\frac{\dfrac{\cancel{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}} \, e^{-\tau \sum_{i=1}^{n} x_i}}{\left(\cancel{\Gamma\alpha} \left(\frac{1}{\tau}\right)^{\alpha}\right)^n} \cdot \dfrac{\tau^{\alpha_0-1} e^{-\tau/\theta_0}}{\cancel{\Gamma\alpha_0} \, \theta_0^{\alpha_0}}}{\dfrac{\cancel{\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}} \, \Gamma(n\alpha+\alpha_0) \, \theta_0^{n\alpha}}{\cancel{\Gamma\alpha^n} \, \cancel{\Gamma\alpha_0} \left(\sum_{i=1}^{n} \theta_0 x_i + 1\right)^{n\alpha+\alpha_0}}}$$

...

$$= \frac{\tau^{\alpha n} \cdot \tau^{\alpha_0 - 1} \cdot e^{-\tau \sum_{i=1}^{n} x_i} \cdot e^{-\tau/\theta_0} \cdot \left(\sum_{i=1}^{n} \theta_0 x_i + 1\right)^{n\alpha + \alpha_0}}{\Gamma(n\alpha+\alpha_0) \cdot \theta_0^{n\alpha} \cdot \theta_0^{\alpha_0}}$$

$$= \frac{\tau^{\alpha n + \alpha_0 - 1} \, e^{-\tau\left(\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}\right)}}{\Gamma(n\alpha+\alpha_0) \cdot \left(\frac{1}{\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}}\right)^{n\alpha+\alpha_0}}$$

This posterior pdf is a $\boxed{\text{gamma distribution with parameters } \alpha n + \alpha_0 \text{ and } \frac{1}{\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}}}$

b) The mean of the above gamma distribution is the product of the 2 parameters:

$$\frac{\alpha n + \alpha_0}{\sum_{i=1}^{n} x_i + \frac{1}{\theta_0}} = \frac{\alpha n + \alpha_0}{n\bar{X} + \frac{1}{\theta_0}} = \boxed{\frac{\alpha n \theta_0 + \alpha_0 \theta_0}{n\theta_0 \bar{X} + 1}}$$

The boxed mean is written in terms of sample mean $\bar{X}$ and prior mean $\alpha_0 \theta_0$ ✓

3) We realize that through Q1 and Q2 that we don't actually need to calculate $K_1(y)$ (the integral) because a lot of things end up canceling out in the end. For this question, we will use the concept of proportionality ($\propto$) to simplify our problem:

So, we have from given that our prior distr:

$$\pi(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta}, \quad \theta > 0$$

and $y \sim Exp(\theta)$ with $y = \{y_1 \dots y_n\}$ iid

so $g(y_i|\theta) \propto \theta e^{-y_i\theta}$

$$\Rightarrow g(y|\theta) = g(y_1|\theta) \cdot g(y_2|\theta) \cdots g(y_n|\theta)$$

$$\propto \theta^n e^{-(y_1+y_2 \cdots y_n)\theta}$$

Our posterior pdf:

$$k(\theta|y) \propto \theta^{\alpha-1} e^{-\beta\theta} \cdot \theta^n e^{-(y_1+y_2+\cdots y_n)\theta}$$

$$\propto \theta^{\alpha+n-1} e^{-\theta(\beta+n\bar{y})}$$

Based on this proportion, our posterior pdf is a gamma distribution with parameters $(\alpha+n, \beta+n\bar{y})$