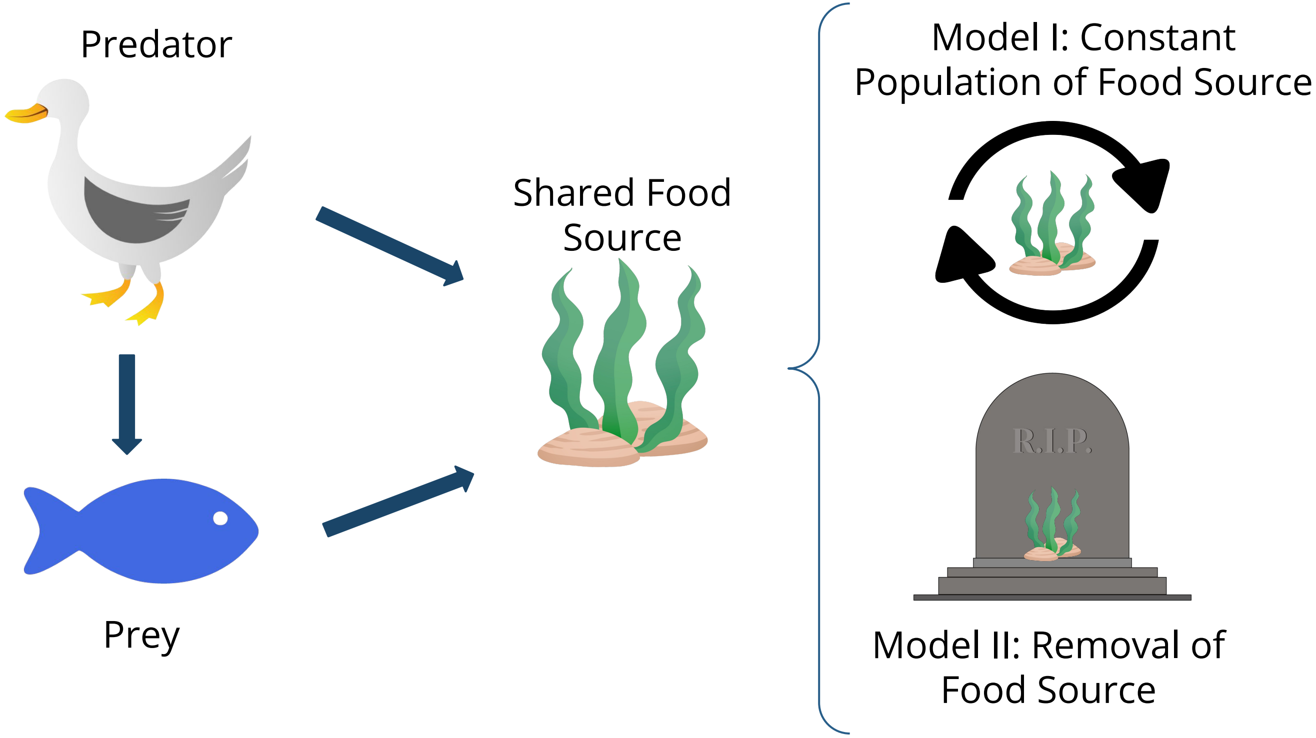
Predator-Prey Dynamics:

Removal of a Shared Food Source

By: Kevin Hefner, Jun Ryu, and Yuki Yu

Overview



Domain: Ecosystems with superpredator, mesopredator, and prey species

Our Model (Phase I)

- Lotka-Volterra based, with carrying capacity of prey and additional growth factors from shared food source
- Preserves non-negativity with positive parameters

$$\begin{cases} \dot{F} = F\left[g_F - \frac{g_F}{K_F}F + b_F P - \eta_F D\right] \\ \dot{D} = D[-d_D + b_D P + \mu_F F] \\ \dot{P} = 0; \qquad P(0) = P_0 \end{cases}$$

Parameter Recap:

g_F - unconstrained growth rate of fish population
K_F - carrying capacity of environment for fish
b_F - birth rate of fish proportional to population of plants
b_D - birth rate of ducks proportional to population of plants
η_F - death rate fish, as eaten by ducks
μ_F - birth rate of ducks due to consumption of fish as food
d_D - natural death rate of ducks

Jacobian & Equilibrium Points

- Equilibrium Points:

$$\Big(0,\ 0\Big), \quad \Big(\frac{K_F(b_F P_0 + g_F)}{g_F},\ 0\Big), \quad \Big(F^*, D^*\Big)$$

Note: See Appendix, slide 24 for (F*, D*)

- Jacobian:

$$J(F, D) = \begin{bmatrix} g_F' - 2\frac{g_F}{K_F}F - \eta_F D & -\eta_F F \\ \mu_F D & g_D' + \mu_F F \end{bmatrix}$$

- Classifications:

$$J(0,\ 0) = \begin{bmatrix} g_F' & 0 \\ 0 & g_D' \end{bmatrix} \qquad \begin{matrix} \text{unstable if} & g_D' > 0 \\ \text{saddle if} & g_D' < 0 \end{matrix}$$

$$J\Big(\frac{K_F(b_F P_0 + g_F)}{g_F}, 0\Big) = \begin{bmatrix} -g_F' & -\frac{\eta_F K_F g_F'}{g_F} \\ 0 & g_D' + \frac{\mu_F K_F g_F'}{g_F} \end{bmatrix} \qquad \begin{matrix} \text{stable if} & g_F g_D' + \mu_F K_F g_F' < 0 \\ \text{saddle if} & g_F g_D' + \mu_F K_F g_F' > 0 \end{matrix}$$

$$J(F^*, D^*) = \begin{bmatrix} \frac{g_D' g_F}{\mu_F K_F} & \frac{\eta_F g_D'}{\mu_F} \\ \mu_F D^* & 0 \end{bmatrix} \qquad \begin{matrix} \text{exists if} & g_D' \leq 0 \text{ and } g_F g_D' + \mu_F K_F g_F' \geq 0 \\ \text{stable if} & g_D' < 0 \text{ and } g_F g_D' + \mu_F K_F g_F' > 0 \end{matrix}$$

Our Model (Phase II)

- Models behavior after aquatic plants are removed from the pond

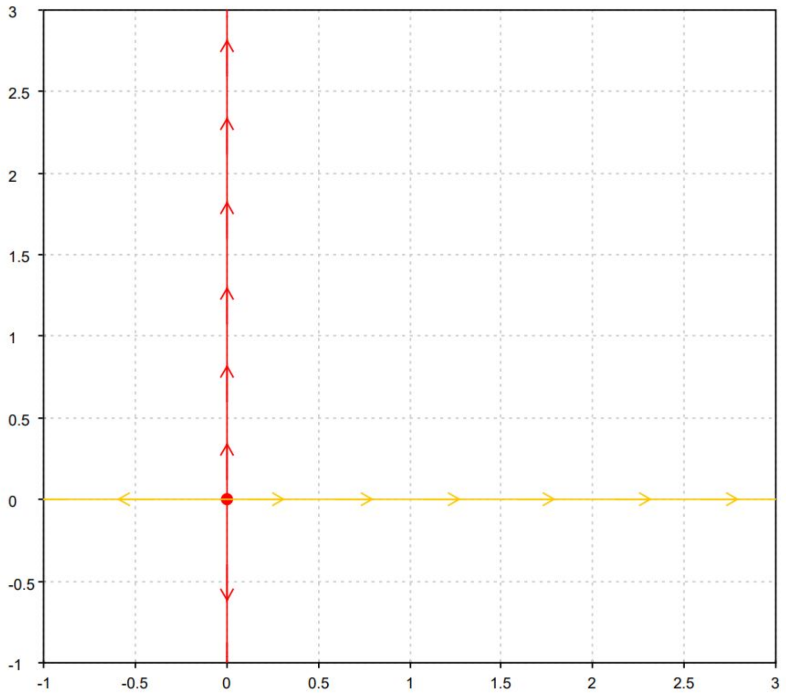
$$\begin{cases} \dot{F} = F\left[g_F - \frac{g_F}{K_F}F - \eta_F D\right] \\ \dot{D} = D[-d_D + \mu_F F] \\ \dot{P} = 0; \qquad P = 0 \end{cases}$$

Parameter Recap:

g_F - unconstrained growth rate of fish population
K_F - carrying capacity of environment for fish
η_F - death rate fish, as eaten by ducks
μ_F - birth rate of ducks due to consumption of fish as food
d_D - natural death rate of ducks

Proving Non-negativity of Both Models

1. Consider initial populations in the first quadrant.
2. Note that the vertical axis is a vertical nullcline, and the horizontal axis is a horizontal nullcline, regardless of our parameter values.
3. If any solution was to ever reach these axes, it would be unstable to escape these axes, because it can then only move along the nullcline.
4. Also, the origin is always a fixed point, so solutions cannot pass through it to the negative axes.
5. Thus, we can see that our solution beginning in the first quadrant cannot escape to another quadrant, since it cannot cross the x nor y axes.



Jacobian & Equilibrium Points

(after plants are removed)

- Equilibrium Points:

$$\Big(0,\ 0\Big), \quad \Big(K_F,\ 0\Big), \quad \Big(F_2^*, D_2^*\Big)$$

Note: See Appendix, slide 27 for (F*, D*)

- Jacobian:

$$J(F,D) = \begin{bmatrix} g_F - 2\frac{g_F}{K_F}F - \eta_F D & -\eta_F F \\ \mu_F D & -d_D + \mu_F F \end{bmatrix}$$

- Classifications:

$$J(0,\ 0) = \begin{bmatrix} g_k & 0 \\ 0 & -d_D \end{bmatrix} \qquad \text{saddle always}$$

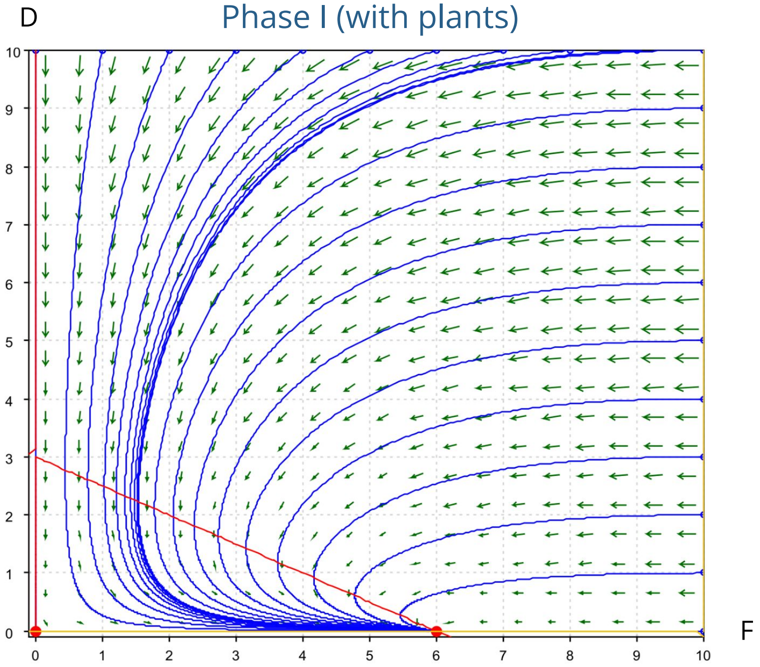
$$J(K_F,\ 0) = \begin{bmatrix} -g_F & \eta_F K_F \\ 0 & -d_D + \mu_F K_F \end{bmatrix} \qquad \begin{array}{ll} \text{saddle if} & -d_D + \mu_F K_F > 0 \\ \text{stable if} & -d_D + \mu_F K_F < 0 \end{array}$$

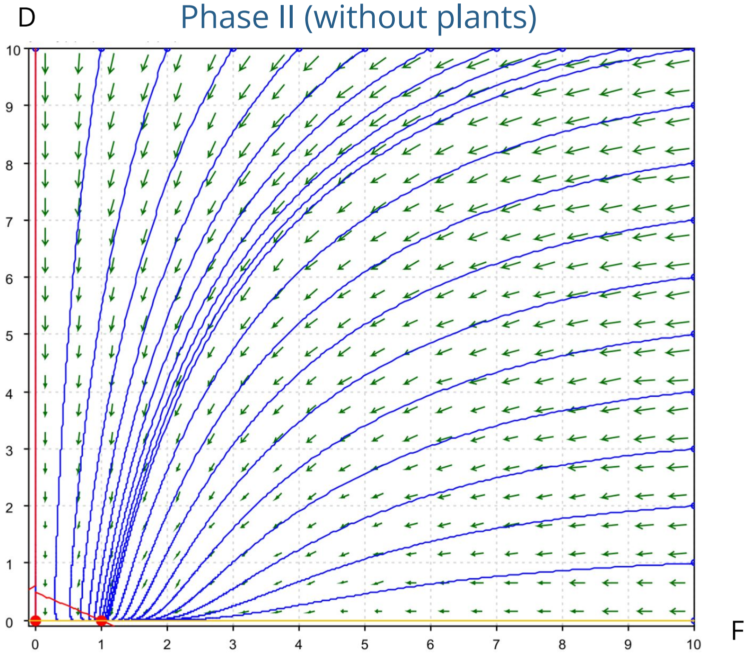
$$J(F_2^*,\ D_2^*) = \begin{bmatrix} -\frac{g_F d_D}{\mu_F K_F} & -\frac{\eta_F d_D}{\mu_F} \\ \mu_F D_2^* & 0 \end{bmatrix} \qquad \begin{array}{ll} \text{exists if} & -d_D + \mu_F K_F \geq 0 \\ \text{stable if} & -d_D + \mu_F K_F > 0 \end{array}$$

Phase Portraits

and Numerical Analysis with
Time-Series Graphs

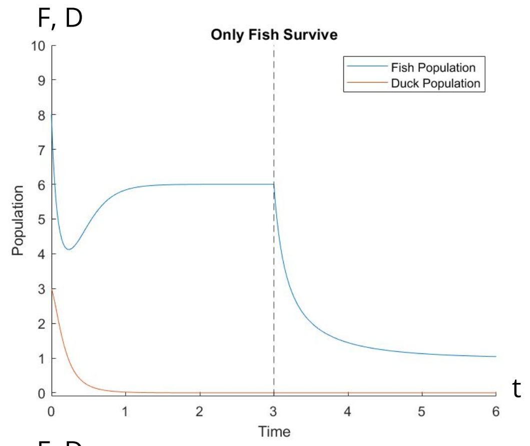
Case 1 - Predator Starvation

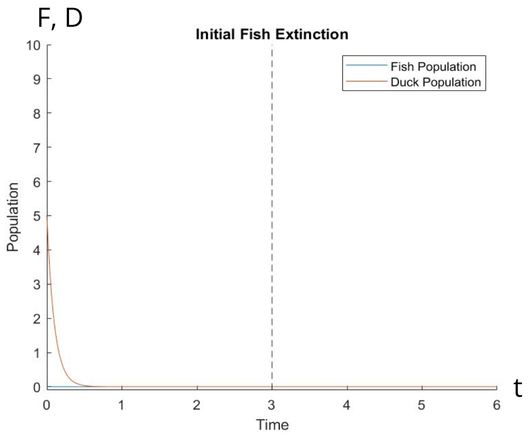


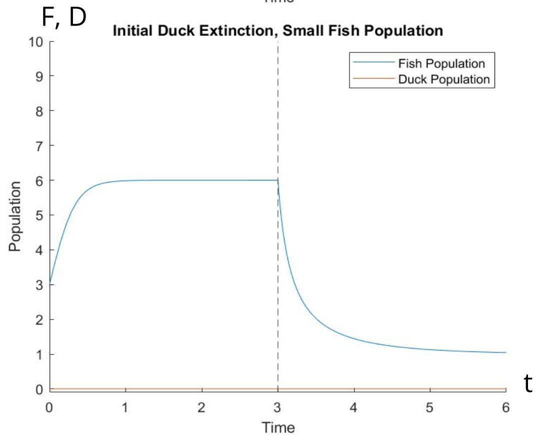


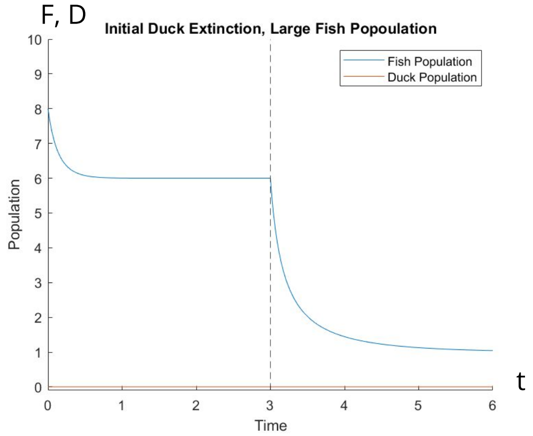
- In either case, the predator (ducks) will not be able to survive.

Time-Series Analysis (Case 1)

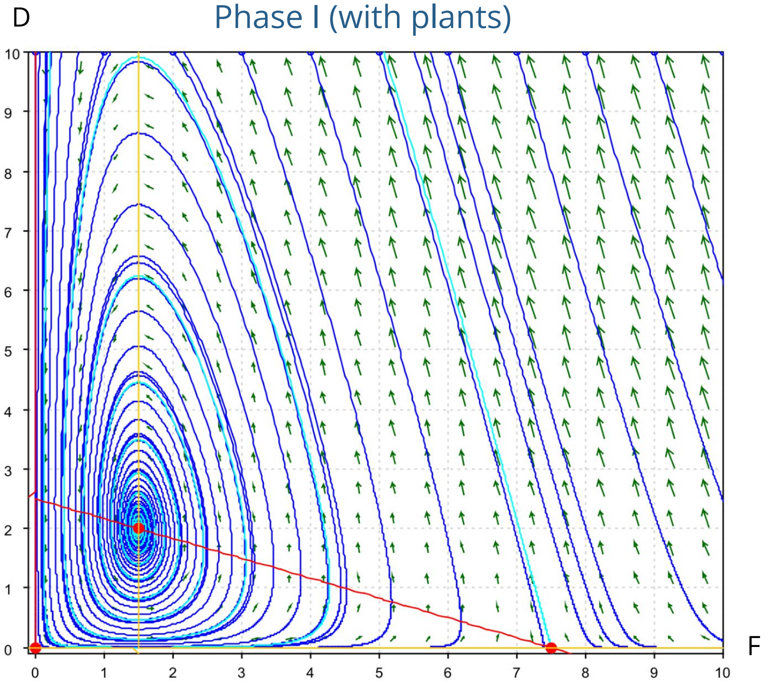


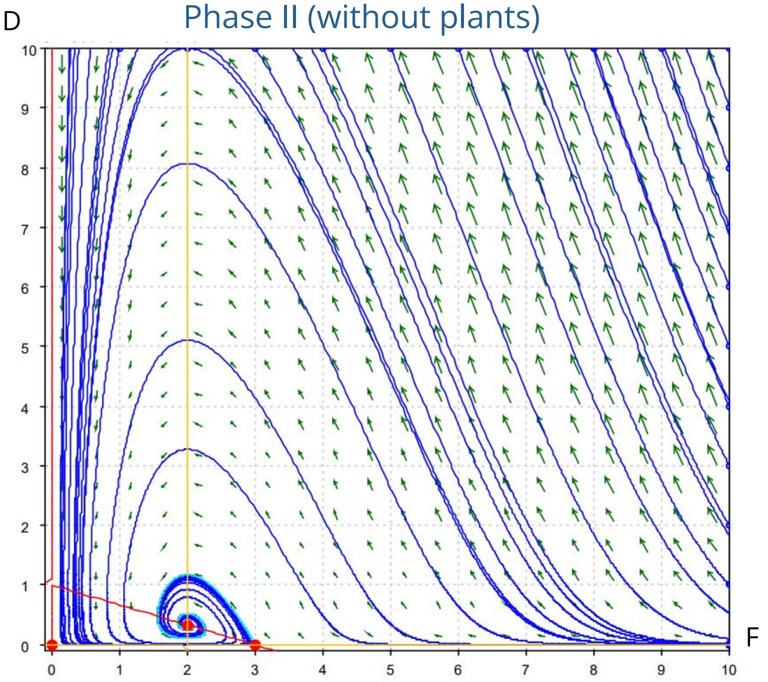






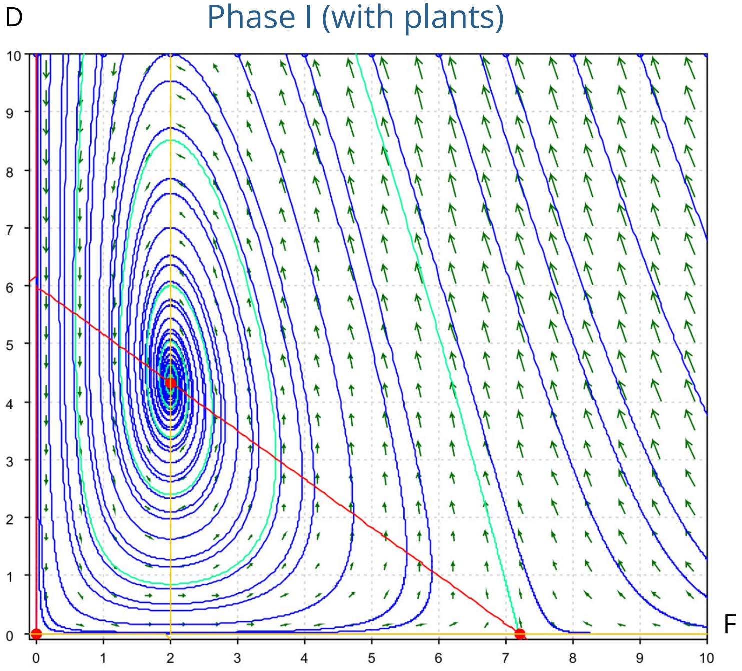
Case 2a - Inevitable Harmony

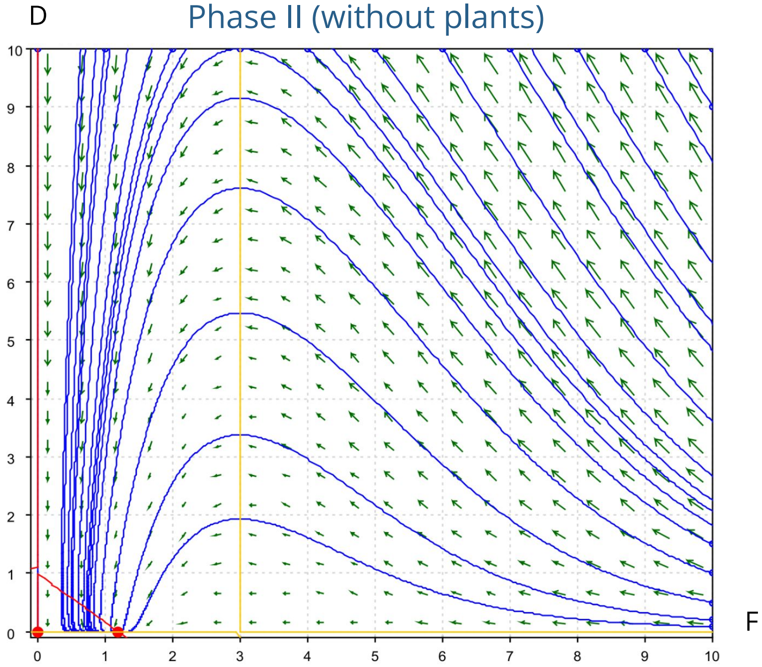




- With or without plants, both ducks and fish live in harmony

Case 2b - Conditional Harmony

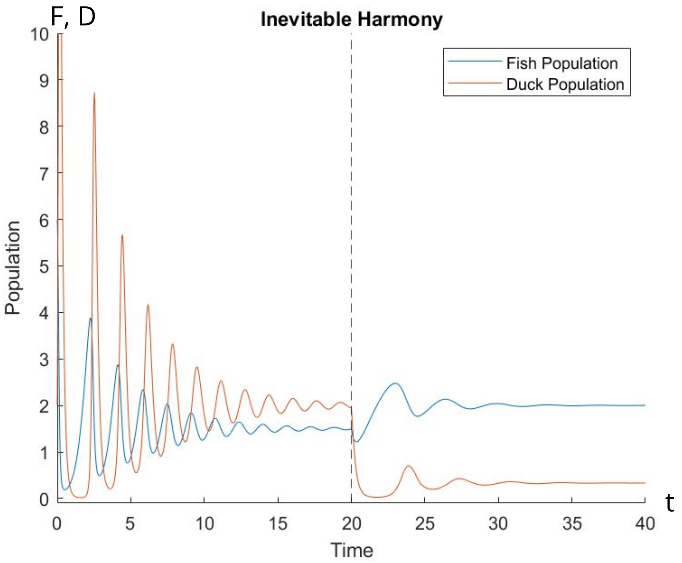




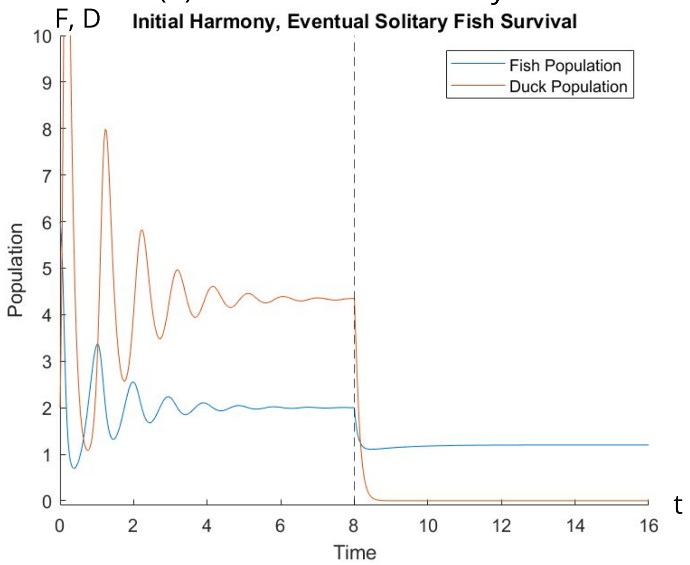
- With plants, both ducks and fish live in harmony.
- Without plants, the duck population can never survive no matter what the fish population is

Time-Series Analysis (Case 2)

Case 2(a) - Inevitable harmony

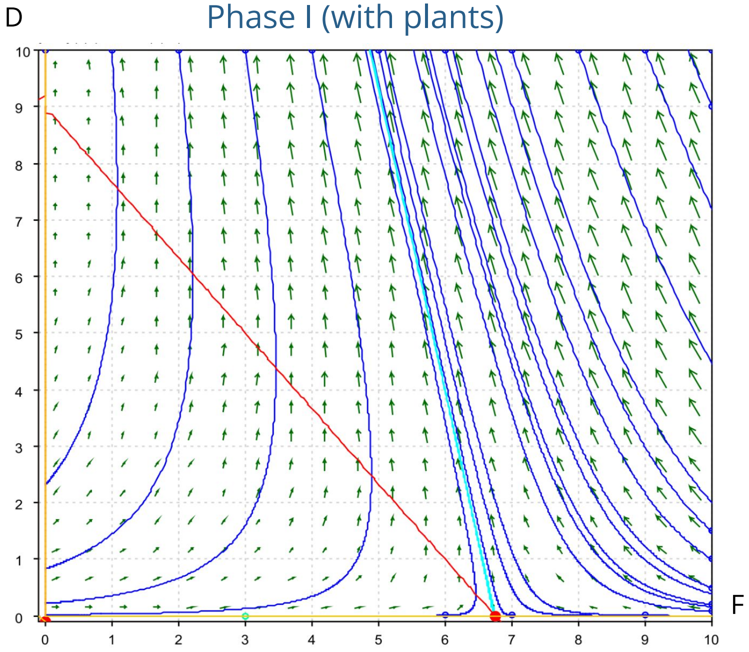


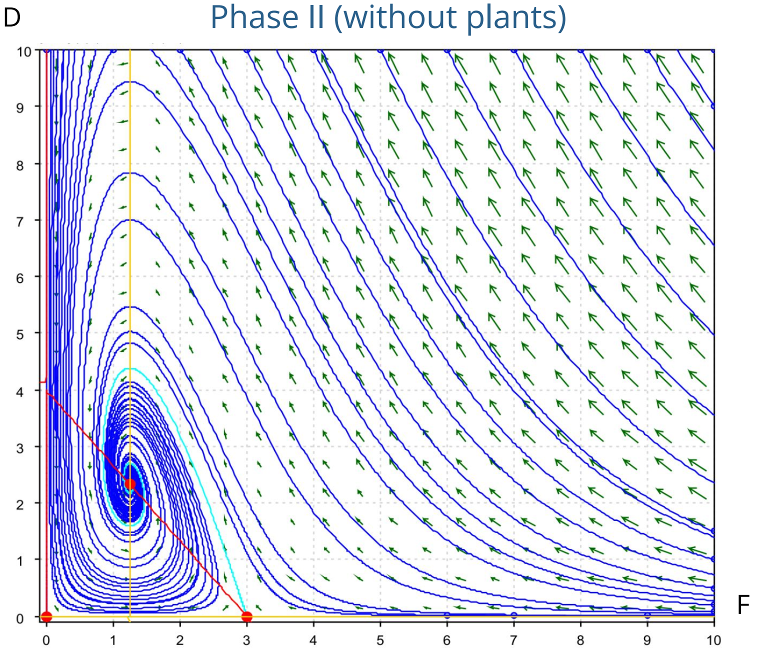
Case 2(b) - Conditional harmony



- In every case, if ducks begin as extinct, fish always approach their solitary equilibrium value, then when plants are removed, this equilibrium value decreases.
- As before, we have omitted the case when one species begins extinct, since they are congruent to case 1.

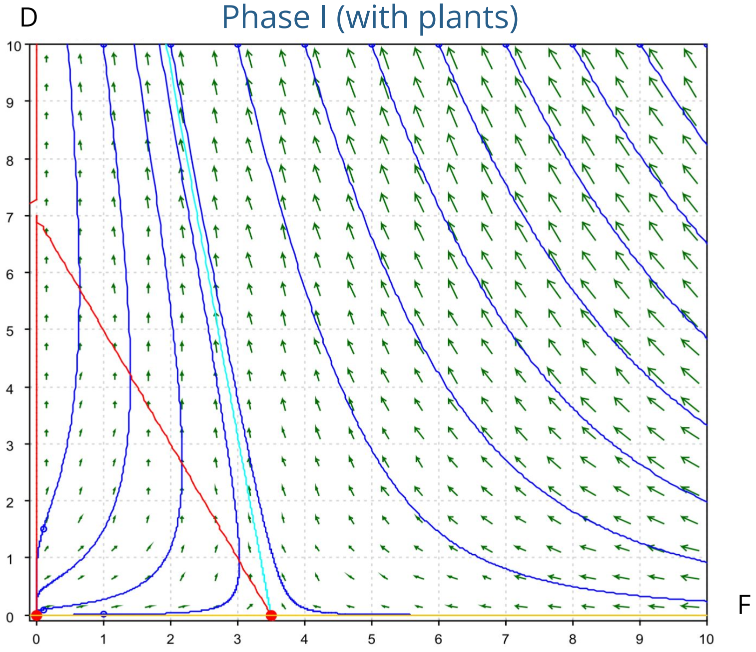
Case 3a - Strong Overfeeding

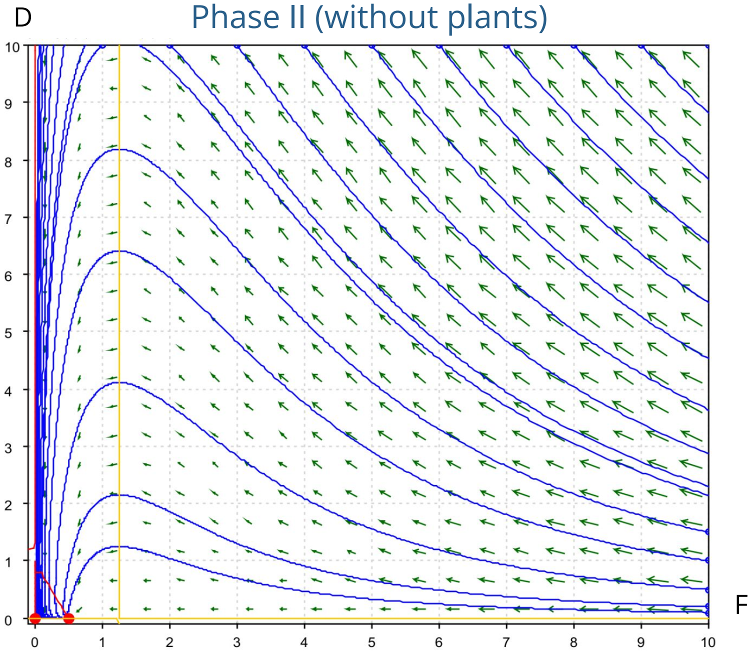




- With plants, there is enough food in the environment so that ducks can grow regardless of the fish population, growing to infinite numbers.
- When plants are removed, ducks and fish both have enough food to survive in harmony.

Case 3b - Weak Overfeeding





- When plants are removed, there is not enough food for ducks to survive, no matter the fish population.

Conclusion and Results

- There are only 5 cases when considering the two different phases of our model:
    - Predator Starvation, Inevitable/Conditional Harmony, and Strong/Weak Overfeeding
- In 3 of the 5 cases, the duck population cannot survive in the absence of plants
- In the presence of plants, the duck population has 3 different outcomes
    - Extinction, stable equilibrium, and growing without bound
- Human behavior (such as pollution) may eliminate food sources that encourage harmony between predators and prey. Without this food, predators will go extinct (as in Case 2(b)).
- In some situations, humans need not worry about over-farming a food source, because predator-prey relationship can be resilient to a loss of their food (as in Case 2(a)).

Limitations of Our Model

- Possible features that were omitted:
  - Additional predators of ducks and/or fish
  - Additional prey preferences and food sources
  - Carrying capacity of ducks
- Simplifications of plant population:
  - Direct removal of plants instead of a gradual decrease
  - Assumed to be constant
- Issues with Duck Explosion (Case 3)
  - Possibly due to the omission of the carrying capacity of ducks & direct removal of plants

Unanswered Questions

- What would happen in the presence of an apex predator (i.e. a species that can eat ducks, fish, and plants)?
- What would happen if each predator had specific preferences for the prey options?
- What would happen if plants become poisonous and the consumption of plants leads to a partial death of the predators?
- What would happen if there were a plague that affected all populations and resulted in a partial death at different rates for each respective population?

References

Courchamp, F., Langlais, M., & Sugihara, G. (1999). Cats protecting birds: Modelling the mesopredator release effect. *Journal of Animal Ecology*, *68*(2), 282–292. https://doi.org/10.1046/j.1365-2656.1999.00285.x

Haberman, R. (1998). *Mathematical models mechanical vibrations, population dynamics, and traffic flow: An introduction to applied mathematics*. Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104).

Strogatz, S. (2019). *Nonlinear Dynamics and Chaos: With applications to physics, Biology, Chemistry and Engineering*. CRC Press.

Fan, M., Kuang, Y., & Feng, Z. (2005). Cats protecting birds revisited. *Bulletin of Mathematical Biology*, *67*(5), 1081–1106. https://doi.org/10.1016/j.bulm.2004.12.002

Thank you!

Technical Contributions

Kevin Hefner - Fixed point identification, Linearization, eigenvalue considerations, Matlab coding, Creation of test cases

Jun Ryu - Unanswered questions, Conclusions, Numerical verification

Yuki Yu - Topic overview, Designing of the model, Limitations, Case & long-term behavior analysis, Mathematical verifications

Appendix - Jacobian & F.P. (I)

We can calculate the Jacobian of this model.

$$J(F, D) = \begin{bmatrix} g'_F - 2\frac{g_F}{K_F}F - \eta_F D & -\eta_F F \\ \mu_F D & g'_D + \mu_F F \end{bmatrix}$$

Note that we added some variables to clarify final results, which combine some population growth terms that are constants, with respect to F and D.

$$g'_F = g_F + b_F P$$
$$g'_D = -d_D + b_D P$$

$$\left(F^*,\ D^*\right) = \left(-\frac{g'_D}{\mu_F},\ \frac{g'_F \mu_F K_F + g_F g'_D}{\mu_F K_F \eta_F}\right)$$

Appendix - Classifications (I)

$$J(0,\,0) = \begin{bmatrix} g'_F & 0 \\ 0 & g'_D \end{bmatrix} \qquad \begin{matrix} \text{unstable if} & g'_D > 0 \\ \text{saddle if} & g'_D < 0 \end{matrix}$$

Eigenvalues are $g'_F > 0$ and g'_D since this matrix is diagonal.

$$J\left(\frac{K_F(b_F P_0 + g_F)}{g_F}, 0\right) = \begin{bmatrix} -g'_F & -\frac{\eta_F K_F g'_F}{g_F} \\ 0 & g'_D + \frac{\mu_F K_F g'_F}{g_F} \end{bmatrix} \qquad \begin{matrix} \text{stable if} & g_F g'_D + \mu_F K_F g'_F < 0 \\ \text{saddle if} & g_F g'_D + \mu_F K_F g'_F > 0 \end{matrix}$$

Eigenvalues are $-g'_F < 0$ and $g'_D + \frac{\mu_F K_F g'_F}{g_F}$ since this matrix is upper-triangular.

$$J(F^*, D^*) = \begin{bmatrix} \frac{g'_D g_F}{\mu_F K_F} & \frac{\eta_F g'_D}{\mu_F} \\ \mu_F D^* & 0 \end{bmatrix} \qquad \begin{matrix} \text{exists if} & g'_D \leq 0 \text{ and } g_F g'_D + \mu_F K_F g'_F \geq 0 \\ \text{stable if} & g'_D < 0 \text{ and } g_F g'_D + \mu_F K_F g'_F > 0 \end{matrix}$$

Similar to a matrix $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, when the fixed point exists with $g'_D < 0$, we have

$$a = \frac{g'_D g_F}{\mu_F K_F} < 0, \qquad b = \frac{\eta_F g'_D}{\mu_F} < 0, \qquad c = \mu_F D^* > 0$$

Eigenvalues are $\frac{a+\sqrt{a^2+4bc}}{2} < 0$ and $\frac{a-\sqrt{a^2+4bc}}{2} < 0$.

Appendix - Cases (I)

We can consider every combination of the cases that determine stability and existence above at each fixed to find that the only possible configurations are:

- saddle/stable/DNE
- saddle/saddle/stable
- unstable/saddle/DNE

These have been labeled as Case 1, Case 2, and Case 3 respectively.

Appendix - Jacobian & F.P. (II)

We can calculate the Jacobian of the second model that represents the dynamics after plants are eliminated.

$$J(F, D) = \begin{bmatrix} g_F - 2\frac{g_F}{K_F}F - \eta_F D & -\eta_F F \\ \mu_F D & -d_D + \mu_F F \end{bmatrix}$$

Note the similarity to the previous Jacobian.

The third fixed point is quite long in it's exact form, and has been represented in the main presentation with 2 introduced variables.

$$\left(F_2^*,\ D_2^*\right) = \left(\frac{d_D}{\mu_F},\ \frac{g_F \mu_F K_F - g_F d_D}{\mu_F K_F \eta_F}\right)$$

Appendix - Classifications (II)

$$J(0,\,0) = \begin{bmatrix} g_k & 0 \\ 0 & -d_D \end{bmatrix} \qquad \text{saddle always}$$

Eigenvalues are $g_k > 0$ and $-d_D < 0$ since this matrix is diagonal.

$$J(K_F,\,0) = \begin{bmatrix} -g_F & \eta_F K_F \\ 0 & -d_D + \mu_F K_F \end{bmatrix} \qquad \begin{array}{rl} \text{saddle if} & -d_D + \mu_F K_F > 0 \\ \text{stable if} & -d_D + \mu_F K_F < 0 \end{array}$$

Eigenvalues are $-g_F < 0$ and $-d_D + \mu_F K_F$ since this matrix is upper triangular.

$$J(F_2^*,\,D_2^*) = \begin{bmatrix} -\frac{g_F d_D}{\mu_F K_F} & -\frac{\eta_F d_D}{\mu_F} \\ \mu_F D_2^* & 0 \end{bmatrix} \qquad \begin{array}{rl} \text{exists if} & -d_D + \mu_F K_F \geq 0 \\ \text{stable if} & -d_D + \mu_F K_F > 0 \end{array}$$

Similar to a matrix $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, when the fixed point exists with $-d_D + \mu_F K_F > 0$, we have

$$a = -\frac{g_F d_D}{\mu_F K_F} < 0, \qquad b = -\frac{\eta_F d_D}{\mu_F} < 0, \qquad c = \mu_F D_2^* > 0$$

Eigenvalues are $\frac{a+\sqrt{a^2+4bc}}{2} < 0$ and $\frac{a-\sqrt{a^2+4bc}}{2}$.

Appendix - Cases (II)

Again, consider every combination of the conditions that determine stability and existence at each fixed point, to determine that the only possible configurations are:

- saddle/saddle/stable
- saddle/stable/DNE

These cases have been labeled as sub-cases of the previous, specifically sub-case (a) and sub-case (b).

Appendix - Combining Cases

In order to determine what combinations of the previous cases (in the first and second phase of the model) are possible, we consider combinations of the conditions that give rise to each case above.

Case 1(a)

$$g_F(-d_D + b_D P_0) + \mu_F K_F(g_F + b_F P_0) < 0$$

$$g_F(-d_D + \mu_F K_F) + g_F b_D P_0 + \mu_F K_F B_F P_0 < 0 \qquad \Rightarrow \qquad -d_D + \mu_F K_F < 0$$

Cases 2 & 3

$$g_F(-d_D + b_D P_0) + \mu_F K_F(g_F + b_F P_0) > 0$$

$$g_F(-d_D + \mu_F K_F) + g_F b_D P_0 + \mu_F K_F B_F P_0 > 0$$

either sign of $-d_D + \mu_F K_F$ is possible.

Appendix - Complete Cases

This yields a final, complete list of every possible case of the various stabilities and existences of our fixed points, after combining the two phases of our model.

Case 1: saddle/stable/DNE

  (a) saddle/stable/DNE

Case 2: saddle/saddle/stable

  (a) saddle/saddle/stable
  (b) saddle/stable/DNE

Case 3: unstable/saddle/DNE

  (a) saddle/saddle/stable
  (b) saddle/stable/DNE

Appendix - Matlab Code

```
% constants
g_F = 1;
K_F = 1;
b_F = 1;
b_D = 1;
eta_F = 2;
mu_F = 1;
d_D = 15;
P_0 = 5;
% initial conditions and time-scale
F_0 = 8;
D_0 = 3;
initial_cond_title = "Only Fish Survive";
y_0 = [F_0 D_0];
t_0 = 0;
t_1 = 3;
t_2 = 6;
```

Appendix - Matlab Code

```
%% numerical approximation (Phase I)
P = P_0;
dot_F = @(t, y) y(1)*( g_F - (g_F/K_F)*y(1) + b_F*P -
eta_F*y(2) );
dot_D = @(t, y) y(2)*( -d_D + b_D*P + mu_F*y(1) );
dot_y = @(t, y) [dot_F(t, y); dot_D(t, y)];
tspan_I = [t_0 t_1];
[t_I, y_I] = ode45(dot_y, tspan_I, y_0);
%% numerical approximation (Phase II)
P = 0;
dot_F = @(t, y) y(1)*( g_F - (g_F/K_F)*y(1) + b_F*P -
eta_F*y(2) );
dot_D = @(t, y) y(2)*( -d_D + b_D*P + mu_F*y(1) );
dot_y = @(t, y) [dot_F(t, y); dot_D(t, y)];
tspan_II = [t_1 t_2];
[t_II, y_II] = ode45(dot_y, tspan_II, y_I(length(y_I),:));
```

Appendix - Matlab Code

```
%% plotting
t_full = [t_I; t_II];
F_numer = [y_I(:,1); y_II(:,1)];
D_numer = [y_I(:,2); y_II(:,2)];
clf;
figure(1);
hold on;
xlim([t_0 t_2]);
ylim([-0.1 10]);
plot(t_full, F_numer);
plot(t_full, D_numer);
xline(t_1, 'k--');
legend("Fish Population", "Duck Population");
title(initial_cond_title);
xlabel('Time');
ylabel('Population')
hold off;
```