APPM 5600: Homework #1
Due Wednesday, September 3

1 (10 points; 5 each) How would you perform the following calculations to avoid cancellation?

i. Evaluate $\frac{1-\cos(x)}{\sin(x)}$ for $|x| \ll 1$.

ii. Evaluate $(x+1)^{1/3} - 1$ for $|x| \ll 1$.

2 (20 points) Consider the polynomial $p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$.

i. (5 pts) Plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ (i.e. $x = [1.920 : 0.001 : 2.080];$) evaluating p via its coefficients.

ii. (5 pts) Produce the same plot again, now evaluating p via the expression $(x-2)^9$. (Put plots i and ii on the same axes)

iii. (5 pts) Produce the same plot again, now evaluating p via the expression $(x-2)^9 = ((((((((x-18)x+144)x-672)x+2016)x-4032)x+5376)x-4608)x+2304)x-512$. (Put plots i, ii, and iii on the same axes)

iv. (5 pts) Explain why (ii) is more accurate than (iii) which is more accurate than (i).

3 (10 points) Prove that the condition number of a nonsingular matrix is greater than or equal to 1. Your proof should work for any matrix operator norm, not just the 2-norm.

4 (10 points; 5 each) Atkinson chapter 7, #7. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. The function

$$q(\boldsymbol{x}) = \boldsymbol{x}^T \mathbf{A} \boldsymbol{x}$$

is called the quadratic form determined by $\mathbf{A}$. It is a quadratic polynomial in n variables $x_1, \ldots, x_n$ and it occurs when considering the maximization or minimization of a function of n variables.

(a) Prove that if $\mathbf{A}$ is skew-symmetric then $q(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$.

(b) For a general square matrix, define $\mathbf{A}_1 = (\mathbf{A}+\mathbf{A}^T)/2$ and $\mathbf{A}_2 = (\mathbf{A}-\mathbf{A}^T)/2$ and note that $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$. (No work required yet.) Show that $\mathbf{A}_1$ is symmetric and $q(\boldsymbol{x}) = \boldsymbol{x}^T \mathbf{A}_1 \boldsymbol{x}$ for all $\boldsymbol{x}$. This shows that the coefficient matrix $\mathbf{A}$ for a quadratic form can always be assumed to be symmetric, without any loss of generality.

5 (10 points) Suppose that the $n \times n$ matrix $\mathbf{A}$ is full (a.k.a. dense, i.e. all nonzero entries), and that you've computed the LU factorization at $\mathcal{O}(n^3)$ cost. How much would it cost (i.e. operation count) to use the Sherman-Morrison identity

$$\left(\mathbf{A} + \boldsymbol{u}\boldsymbol{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\boldsymbol{u}\boldsymbol{v}^T\mathbf{A}^{-1}}{1 + \boldsymbol{v}^T\mathbf{A}^{-1}\boldsymbol{u}}$$

with the already-computed LU factorization of $\mathbf{A}$ to solve the system $(\mathbf{A} + \boldsymbol{u}\boldsymbol{v}^T)\boldsymbol{x} = \boldsymbol{b}$? Is this more expensive, less expensive, or the same cost as solving the system by performing Gaussian elimination on the new system? Give your answer in terms of floating point operations; e.g. 'both methods would cost $\mathcal{O}(n^{100})$ floating point operations'.

6 (5 points) Atkinson chapter 8, #6. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be matrices of size $m \times n$, $n \times p$, and $p \times q$, respectively. Do an operations count for computing $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$. Give examples of when one order of computation is preferable over the other. Extra notes: You don't need an exact count, only leading order. You only need to provide one example where one order is preferable. Your example cannot have one of the matrices be scalar.

7 (20 points) Let $\mathbf{A}$ be an $n \times n$ matrix with 4 on the diagonal, -1 immediately above and below the diagonal, and -1 in the $(1, n)$ and $(n, 1)$ entries. This matrix is symmetric positive definite & strictly diagonally-dominant, so no pivoting is required when performing Gaussian elimination.

- (10 points) What is the operation count to reduce this matrix to upper triangular form using Gaussian Elimination? (You may give your result as $\mathcal{O}(n^p)$, but you have to justify your answer.)
- (10 points) Let $\mathbf{A} = \mathbf{LU}$ be the LU factorization of $\mathbf{A}$. Where are the nonzero entries in the $\mathbf{U}$?

8 (10 points) Problem #2 from Atkinson, chapter 8. Consider the linear system

$$\begin{aligned} 6x + 2y + 2z &= -2 \\ 2x + \frac{2}{3}y + \frac{1}{3}z &= 1 \\ x + 2y - z &= 0 \end{aligned}$$

which has exact solution $x = 2.6$, $y = -3.8$, $z = -5$.

(a) Solve the system using Gaussian elimination without pivoting, rounding each arithmetic operation to 4 digits (round to nearest, rounding up from 0.00005).

(b) Repeat part (a) using partial pivoting.

Note: Throughout, only keep 4 decimal digits total (not 4 after the decimal point) and round after *every* arithmetic operation.