

Homework 10

$$1. \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

more eqs than variables

more variables than equations

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} R_3: R_1 - R_3 \\ R_4: R_1 - R_4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & -1 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \end{bmatrix} \begin{array}{l} R_4: R_1 - R_4 \\ R_3: R_2 + R_3 \\ R_4: R_2 - R_4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+1+1 & 1+0+2+0 & 0+0+1+1 \\ 1+0+2+0 & 1+1+4+0 & 0+1+2+0 \\ 0+0+4+1 & 0+1+2+0 & 0+1+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 2 \\ 3 & 6 & 3 \\ 2 & 3 & 3 \end{bmatrix} = A^T A$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+0+3+4 \\ 2+2+6+0 \\ 0+2+3+4 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 2 \\ 3 & 6 & 3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 9 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_2 + x_3 &= 2 \\ x_1 + 2x_2 + x_3 &= 3 \\ x_1 + x_3 &= 4 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 2 & x_1 &= x_3 \\ x_2 + x_3 &= 2 \\ x_2 &= 2 - x_1 \\ x_2 &= 2 - x_3 \end{aligned}$$

$$\begin{aligned} x_2 + x_3 &= 2 & x_3 &= 2 - x_2 \\ x_1 + x_3 &= 4 & x_3 &= 4 - x_1 \end{aligned}$$

$$\begin{aligned} x_1 &= 2 - x_2 & 2x_1 &= 4 \\ x_1 &= 4 - x_3 & x_1 &= 2 \\ 2 &= 2 - x_2 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 2 & 9 \\ 3 & 6 & 3 & 10 \\ 2 & 3 & 3 & 9 \end{bmatrix}$$

$$R_2: R_1 - R_2 \quad \begin{array}{cccc} 3 & 3 & 2 & 9 \\ -3 & -6 & -3 & -10 \\ 0 & -3 & -1 & -1 \end{array}$$

$$R_3: R_1 - \frac{3}{2}R_3$$

$$\begin{array}{cccc} 3 & 3 & 2 & 9 \\ -3 & -\frac{9}{2} & -\frac{9}{2} & -\frac{27}{2} \\ 0 & -\frac{3}{2} & -\frac{5}{2} & -\frac{9}{2} \end{array}$$

$$\begin{bmatrix} 3 & 3 & 2 & 9 \\ 0 & -3 & -1 & -1 \\ 0 & \frac{3}{2} & \frac{5}{2} & \frac{9}{2} \end{bmatrix}$$

$$R_3: R_2 - 2R_3$$

$$\begin{array}{cccc} 0 & -3 & -1 & -1 \\ 0 & 3 & 5 & 9 \\ 0 & 0 & 4 & 8 \end{array}$$

$$\begin{bmatrix} 3 & 3 & 2 & 9 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 3 & 2 & 9 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_2: R_2 - R_3$$

$$\begin{bmatrix} 0 & 3 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 3 & 0 & -1 \end{bmatrix}$$

$$R_1: R_1 - 2R_3$$

$$\begin{array}{cccc} 3 & 3 & 2 & 9 \\ 0 & 0 & -2 & -4 \\ 3 & 3 & 0 & 5 \end{array}$$

$$\begin{bmatrix} 3 & 3 & 0 & 5 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1: R_1 - R_2$$

$$\begin{array}{cccc} 3 & 3 & 0 & 5 \\ 0 & -3 & 0 & 1 \\ 3 & 0 & 0 & 6 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$x_1 = 2 \quad x_2 = -\frac{1}{3} \quad x_3 = 2$$

$$\bar{x} = \begin{bmatrix} 2 & -\frac{1}{3} & 2 \end{bmatrix}$$

$$\vec{r} = \vec{b} - A\bar{x}$$

$$A\bar{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -\frac{1}{3} \\ 2 \end{bmatrix} = \begin{bmatrix} 2 - \frac{1}{3} \\ -\frac{1}{3} + 2 \\ 2 - \frac{2}{3} + 2 \\ 2 + 2 \end{bmatrix}$$

$$\begin{array}{l} 6 - \frac{1}{3} \\ -\frac{1}{3} + 6 \\ 2 - \frac{2}{3} + 6 \\ 2 + 6 \end{array}$$

$$= \begin{bmatrix} 5 \\ 5 \\ 4 \\ 8 \end{bmatrix}$$

$$\vec{r} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -1 \\ -4 \end{bmatrix}$$

$$\|\vec{r}\|_2 = \left(\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \right)^{1/2} = \left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9} \right)^{1/2} = \left(\frac{1}{3} \right)^{1/2} = \frac{1}{\sqrt{3}}$$

2. GSGR Factorization

$$a) \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \quad A_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = y_1 \quad q_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$A \quad \|y_1\|_2 = (2^2 + 1^2 + 2^2)^{1/2} = (9)^{1/2} = 3$$

$$y_2 = A_2 - q_1 q_1^T A_2$$

$$q_1^T A_1 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \quad q_1 q_1^T A_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\frac{2}{3} - \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1$$

$$y_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ \frac{1}{3} \end{bmatrix} \quad \|y_2\|_2 = \left(\left(\frac{1}{3}\right)^2 + \left(-\frac{4}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \right)^{1/2}$$

$$= \left(\frac{1}{9} + \frac{16}{9} + \frac{1}{9} \right)^{1/2} = 2^{1/2} = \sqrt{2}$$

$$q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ \frac{1}{3} \end{bmatrix} \quad Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad R = \begin{bmatrix} 3 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$(A_1 | A_2 | A_3) = (q_1 | q_2 | q_3) \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}$$

$$b) \begin{bmatrix} 4 & 8 & 1 \\ 0 & 2 & -2 \\ 3 & 6 & 7 \end{bmatrix}$$

$$r_{ij} = \|y_j\|_2$$

$$r_{12} = q_1^T A_2$$

$$r_{13} = q_1^T A_3$$

$$r_{23} = q_2^T A_3$$

$$q_1 = \frac{y_1}{\|y_1\|_2} \quad y_1 = A_1$$

$$y_2 = A_2 - q_1 q_1^T A_2$$

$$y_3 = A_3 - q_1 q_1^T A_3 - q_2 q_2^T A_3$$

$$y_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

$$r_{11} = \|y_1\|_2 = (4^2 + 0^2 + 3^2)^{1/2} = (16 + 9)^{1/2} = 25^{1/2}$$

$$r_{11} = 5$$

$$q_1 = \frac{1}{5} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix}$$

$$q_1^T A_2 = \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} = \frac{32}{5} + \frac{18}{5} = \frac{50}{5} = 10$$

$$y_2 = \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix} (10) = \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{40}{5} \\ 0 \\ \frac{30}{5} \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$q_2 = \frac{y_2}{\|y_2\|_2} = (0^2 + 2^2 + 0^2)^{1/2} = 2$$

$$q_2 = \frac{1}{2} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

$$q_1^T A_3 = \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = \frac{4}{5} + \frac{21}{5} = 5$$

$$q_2^T A_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = 0 \cdot 1 + 0 = -2$$

$$y_3 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{13}{5} \end{bmatrix} (5) - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} (-2) = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 13 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

$$\|y_3\|_2 = (-3^2 + 0 + 4^2)^{1/2} = 5$$

$$q_3 = \frac{1}{5} \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix} \quad A = \begin{bmatrix} 4 & 8 & 1 \\ 0 & 2 & -2 \\ 3 & 6 & 7 \end{bmatrix} \quad Q = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix} \quad R = \begin{bmatrix} 5 & 10 & 5 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$3. (x, y, z) = (0, 0, 3), (0, 1, 2), (1, 1, 5), (1, 2, 6)$$

$$\text{model } z = c_0 + c_1 x + c_2 y$$

$$c_0 + c_1(0) + c_2(0) = 3$$

$$c_0 + c_1(0) + c_2(1) = 2$$

$$c_0 + c_1(1) + c_2(1) = 5$$

$$c_0 + c_1(1) + c_2(2) = 6$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 6 \end{bmatrix}$$

$$A^T A \vec{c} = A^T \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+1+1+1 & 0+0+1+1 & 0+1+1+2 \\ 0+0+1+1 & 0+0+1+1 & 0+0+1+2 \\ 0+1+1+2 & 0+1+1+4 & 0+0+1+2 \\ 0+0+1+2 & 0+1+1+4 & 0+0+1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3+2+5+6 \\ 0+0+5+6 \\ 0+2+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 11 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 11 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 4 & 16 \\ 2 & 2 & 3 & 11 \\ 4 & 3 & 6 & 19 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 4 & 16 \\ 2 & 2 & 3 & 11 \\ 4 & 3 & 6 & 19 \end{bmatrix}$$

$$R_2: R_1 - 2R_2$$

$$\begin{array}{rrrr} 4 & 2 & 4 & 16 \\ -4 & -4 & -6 & -22 \\ \hline 0 & -2 & -2 & -6 \end{array}$$

$$R_3: R_1 - R_3$$

$$\begin{array}{rrrr} 4 & 2 & 4 & 16 \\ -4 & -3 & -6 & -19 \\ \hline 0 & -1 & -2 & -3 \\ 0 & 2 & 2 & 6 \end{array}$$

$$\begin{bmatrix} 4 & 2 & 4 & 16 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$R_3: R_2 - R_3$$

$$R_3: R_2 - 2R_3$$

$$\begin{array}{rrrr} 0 & 2 & 2 & 6 \\ 0 & -2 & -4 & -6 \\ \hline 0 & 0 & -2 & 0 \end{array}$$

$$\begin{bmatrix} 4 & 2 & 4 & 16 \\ 0 & 2 & 2 & 6 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$R_2: R_2 + R_3$$

$$\begin{array}{rrrr} 0 & 2 & 2 & 6 \\ 0 & 0 & -2 & 0 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\begin{bmatrix} 4 & 2 & 0 & 16 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$R_1: R_1 + 2R_3$$

$$\begin{array}{rrrr} 4 & 2 & 4 & 16 \\ 0 & 0 & -4 & 0 \\ \hline 4 & 2 & 0 & 16 \end{array}$$

$$R_1: R_1 - R_2$$

$$\begin{bmatrix} 4 & 0 & 0 & 10 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\begin{array}{rrrr} 4 & 2 & 0 & 16 \\ 0 & -2 & 0 & -6 \\ \hline 4 & 0 & 0 & 10 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C_0 = \frac{5}{2}$$

$$C_1 = 3$$

$$C_2 = 0$$

$$\begin{bmatrix} \frac{5}{2} \\ 3 \\ 0 \end{bmatrix}$$

$$\underline{\underline{Z = \frac{5}{2} + 3X}}$$

$$\bar{C} = \begin{bmatrix} \frac{5}{2} & 3 & 0 \end{bmatrix}$$

4.

rough model of power output

fit to $f(t) = C_1 + C_2 \cos 2\pi t + C_3 \sin 2\pi t + C_4 \cos 4\pi t +$ $0 \leq t \leq 5$ write down the resulting $(1, 2726), (2, 2772), (3, 2912), (4, 2702)$
 $(5, 2682)$

$$C_1 + C_2 \cos 2\pi(1) + C_3 \sin 2\pi(1) + C_4 \cos 4\pi(1) = 2726$$

$$C_1 + C_2 \cos 2\pi(2) + C_3 \sin 2\pi(2) + C_4 \cos 4\pi(2) = 2772$$

$$C_1 + C_2 \cos 2\pi(3) + C_3 \sin 2\pi(3) + C_4 \cos 4\pi(3) = 2912$$

$$C_1 + C_2 \cos 2\pi(4) + C_3 \sin 2\pi(4) + C_4 \cos 4\pi(4) = 2702$$

$$C_1 + C_2 \cos 2\pi(5) + C_3 \sin 2\pi(5) + C_4 \cos 4\pi(5) = 2682$$

$$C_1 + C_2 \cos 2\pi(0) + C_3 \sin 2\pi(0) + C_4 \cos 4\pi(0) = 0$$