Homework 8

1. calculate the following integrals using the mid point rule. Give an estimate.

a) $\int_0^1 \ln(x+1)\,dx$ $\quad h=1-0=1 \quad w=0+\frac{1}{2}=\frac{1}{2}$

$f(x)=\ln(x+1) \quad f(w)=f(\frac{1}{2})=\ln(\frac{3}{2})$

$f''(0)=-1 \quad f''(1)=\frac{-1}{4}$

$\int_0^1 \ln(x+1)\,dx=(1)(\ln(\frac{3}{2}))+\frac{1^3}{24}(-1)$

$=\ln(\frac{3}{2})\pm\frac{1}{24}$

b) $\int_{-0.5}^{0.5} x\ln(x+1)\,dx$ $\quad h=0.5-(-0.5)=1 \quad w=-0.5+\frac{1}{2}=0$

$f''(\frac{-1}{2})=6 \quad f''(\frac{1}{2})=\frac{14}{9} \quad f(0)=0$

$\int_{-0.5}^{0.5} x\ln(x+1)\,dx=(1)(0)+\frac{1^3}{24}(6)$

$=0\pm\frac{1}{4}$

c) $\int_{-0.2}^{0.2}\cos^2x\,dx$ $\quad h=0.2-(-0.2)=0.4=\frac{2}{5} \quad w=-0.2+\frac{0.4}{2}=-0.2+0.2=0$

$f(x)=\cos^2x \quad f(w)=f(0)=1$

$f''(0)=-2$

$\int_{-0.2}^{0.2}\cos^2x\,dx=(\frac{2}{5})(1)+\frac{(\frac{2}{5})^3}{24}(-2)=\frac{2}{5}+\frac{\frac{8}{125}}{24}(-2)=\frac{2}{5}+\frac{\frac{-16}{125}}{24}$

$=\frac{2}{5}\pm\frac{16}{3000}$

d) $\int_0^{\pi/2} e^{\sin x}\,dx$ $\quad h=\frac{\pi}{2}-0=\frac{\pi}{2} \quad w=0+\frac{\frac{\pi}{2}}{2}=\frac{\pi}{4}$

$f(x)=e^{\sin x} \quad f(w)=f(\frac{\pi}{4})=e^{\sin\frac{\pi}{4}}=e^{\frac{\sqrt{2}}{2}}$

$f''(0)=1 \quad f''(\frac{\pi}{2})=-e$

$\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx=(\frac{\pi}{2})(e^{\frac{\sqrt{2}}{2}})+\frac{(\frac{\pi}{2})^3}{24}(-e)=\frac{\pi e^{\sqrt{2}/2}}{2}+\frac{\frac{\pi^3}{8}}{24}(-e)$

$=\frac{\pi e^{\sqrt{2}/2}}{2}\pm\frac{\pi^3 e}{192}$

2. Calculate the following integrals with the trapazoid rule. Give an estimate.

a) $\int_0^1 \ln(x+1)\,dx$

$h = 1 - 0 = 1$

$y_0 = \ln 1 = 0 \quad y_1 = \ln 2$

$f''(0) = -1 \quad f''(1) = \frac{-1}{4}$

$\int_0^1 \ln(x+1)\,dx = \frac{1}{2}(0 + \ln 2) - \frac{1^3}{12}(-1)$

$= \ln\sqrt{2} \pm \frac{1}{12}$

b) $\int_{-0.5}^{0.5} x\ln(x+1)\,dx$

$h = 0.5 - (-0.5) = 1$

$y_0 = -\frac{1}{2}\ln(\frac{1}{2}) = \ln\sqrt{2}$

$y_1 = \frac{1}{2}\ln(\frac{3}{2}) = \ln(\frac{\sqrt{3}}{\sqrt{2}})$

$f''(\frac{-1}{2}) = 6 \quad f''(\frac{1}{2}) = \frac{14}{9}$

$\int_{-0.5}^{0.5} x\ln(x+1)\,dx = \frac{1}{2}\left(\ln\sqrt{2} + \ln\left(\frac{\sqrt{3}}{\sqrt{2}}\right)\right) - \frac{1^3}{12}(6)$

$= \frac{1}{2}(\ln\sqrt{3}) \pm \frac{6}{12} = \ln(3^{1/4}) \pm \frac{1}{2}$

c) $\int_{-0.2}^{0.2} \cos^2 x\,dx$

$h = 0.2 - (-0.2) = 0.4$

$y_0 = \cos^2(-0.2) = \cos^2(0.2)$

$y_1 = \cos^2(0.2) = \cos^2(0.2)$

$f''(0) = -2$

$\int_{-0.2}^{0.2} \cos^2 x\,dx = \frac{\frac{2}{5}}{2}\left(\cos^2(0.2) + \cos^2(0.2)\right) - \frac{(\frac{2}{5})^3}{12}(-2)$

$= \frac{1}{5}(2\cos^2(0.2)) \pm \frac{16}{1500}$

$= \frac{2}{5}(\cos^2(0.2)) \pm \frac{16}{1500}$

d) $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$

$h = \frac{\pi}{2} \quad y_0 = 1 \quad y_1 = e$

$f''(0) = 1 \quad f''(\frac{\pi}{2}) = -e$

$\int_0^{\pi/2} e^{\sin x}\,dx = \frac{\frac{\pi}{2}}{2}(1 + e) - \frac{(\frac{\pi}{2})^3}{12}(-e) = \frac{\pi}{4}(1+e) \pm \frac{\pi^3 e}{96}$

$= \frac{\pi}{4}(1+e) \pm \frac{\pi^3 e}{96}$

3. Calculate the following integrals using the Simpson rule. Give an estimate.

a) $\int_0^1 \ln(x+1)dx$ $\quad h = 1 - x_1 \quad h = x_1 - 0 \quad h = \frac{1}{2}$

$1 - x_1 = x_1 - 0 \quad x_1 = \frac{1}{2}$

$y_0 = \ln 1 = 0 \quad y_1 = \ln\frac{3}{2}$

$y_2 = \ln 2$

$f^{(4)}(0) = -6 \quad f^{(4)}(1) = \frac{-3}{8}$

$\int_0^1 \ln(x+1)dx = \frac{\frac{1}{2}}{3}\left(0 + 4\ln\frac{3}{2} + \ln 2\right) - \frac{\left(\frac{1}{2}\right)^5}{90}(-6)$

$= \frac{1}{6}\left(\ln\left(\frac{3}{2}\right)^4(2)\right) \pm \frac{6}{2880} = \frac{1}{6}\left(\ln\frac{81}{16}(2)\right) \pm \frac{3}{1440}$

$= \frac{1}{6}\left(\ln\frac{81}{8}\right) \pm \frac{3}{1440}$

b) $\int_{-0.5}^{0.5} x\ln(x+1)dx$ $\quad h = 0.5 - x_1 \quad h = x_1 - (-0.5) \quad h = 0.5 = \frac{1}{2}$

$0.5 - x_1 = x_1 + 0.5 \quad 0 = 2x_1 \quad x_1 = 0$

$y_0 = \frac{-1}{2}\ln\frac{1}{2}$

$y_1 = 0 \quad y_2 = \frac{1}{2}\ln\frac{3}{2}$

$f^{(4)}\left(\frac{-1}{2}\right) = 112 \quad f^{(4)}\left(\frac{1}{2}\right) = \frac{176}{27}$

$\int_{-0.5}^{0.5} x\ln(x+1)dx = \frac{\frac{1}{2}}{3}\left(\frac{-1}{2}\ln\frac{1}{2} + 4(0) + \frac{1}{2}\ln\frac{3}{2}\right) - \frac{\left(\frac{1}{2}\right)^5}{90}(112)$

$= \frac{1}{6}\left(\ln\sqrt{2} + \ln\frac{\sqrt{3}}{\sqrt{2}}\right) - \frac{112}{2880} = \frac{1}{6}\left(\ln\left(\frac{\sqrt{3}}{\sqrt{2}}\right)\left(\sqrt{2}\right)\right) - \frac{7}{180}$

$= \ln 3^{1/12} \pm \frac{7}{180}$

c) $\int_{-0.2}^{0.2} \cos^2 x\,dx$ $\quad h = 0.2 - x_1 \quad h = x_1 + 0.2 \quad h = 0.2 = \frac{1}{5}$

$0.2 - x_1 = x_1 + 0.2 \quad 0 = 2x_1 \quad x_1 = 0$

$y_0 = \cos^2(0.2)$

$y_1 = 1 \quad y_2 = \cos^2(0.2)$

$f^{(4)}(0) = 8$ $\quad \frac{\frac{8}{3125}}{90}$

$\int_{-0.2}^{0.2} \cos^2 x\,dx = \frac{\frac{1}{5}}{3}\left(\cos^2(0.2) + 4(1) + \cos^2(0.2)\right) - \frac{\left(\frac{1}{5}\right)^5}{90}(8)$

$= \frac{1}{15}\left(2\cos(0.2) + 4\right) - \frac{8}{281250}$

$= \frac{2}{15}\left(\cos^2(0.2) + 2\right) \pm \frac{4}{140625}$

d) $\int_0^{\frac{\pi}{2}} e^{\sin x}dx$ $h=\frac{\pi}{2}-x_1$ $h=x_1-0$ $\frac{\pi}{2}-x_1=x_1$ $\frac{\pi}{2}=2x_1$ $x_1=\frac{\pi}{4}$

$y_0=1$ $y_1=e^{\sqrt{2}/2}$ $y_2=e$

$f^{(4)}(0)=-3$ $f^{(4)}(\frac{\pi}{2})=4e$ $\frac{\pi^5 \cancel{4}e}{\frac{4^{\cancel{5}4}}{90}}$

$$\int_0^{\frac{\pi}{2}} e^{\sin x}dx = \frac{\frac{\pi}{4}}{3}\left(1+4e^{\sqrt{2}/2}+e\right)-\frac{\left(\frac{\pi}{4}\right)^5}{90}(4e)$$

$$=\frac{\pi}{12}\left(1+4e^{\sqrt{2}/2}+e\right)-\frac{\pi^5 e}{4^4(90)}$$

$$=\frac{\pi}{12}\left(1+4e^{\sqrt{2}/2}+e\right)\pm\frac{\pi^5 e}{23040}$$

4. Formulate a trapazoid rule for the integration of parametric function $x=x(t)$ $y=y(t)$ $t\in[a,b]$

$\int_a^b y(t)x'(t)dt$ $f(t)=y(t)x'(t)$ $h=b-a$

$y_0=y(a)x'(a)$ $y_1=y(b)x'(b)$

$$\int_a^b y(t)x'(t)dt=\frac{b-a}{2}\left(y(a)x'(a)+y(b)x'(b)\right)-\frac{(b-a)^3}{12}f''(c)$$

for $a\le c\le b$

5. Find the degree of approx.

$$\int_{-1}^1 f(x)dx \approx f\left(\frac{-1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$$

I could only get the answer from THEOREM 5.6 in Sauer. $n=2 \Rightarrow DOP=3$

6. Find the error term for Simpson's $\frac{3}{8}$ rule

for $f(x)=x^3$: $f'(x)=3x^2$ $f''(x)=6x$ $f'''(x)=6$ $f^{(4)}(x)=0$

for $f(x)=x^4$: $f'(x)=4x^3$ $f''(x)=12x^2$ $f'''(x)=24x$ $f^{(4)}(x)=24$

error term: $c\frac{f^{(4)}(x)}{4!}(b-a)h^4=\frac{c(b-a)^5}{4!\,3^4}f^{(4)}(c)$

for $f(x)=x^3$: $\frac{c(b-a)^5}{4!\,3^4}(0)=0$

for $f(x)=x^4$: $\frac{c(b-a)^5}{4!\,3^4}(24)=\frac{c(b-a)^5}{81}$