Homework 9

1. $\int_0^1 \frac{dx}{x^2+1}$

a) $I = \int_0^1 \frac{1}{x^2+1}\,dx \quad a=1 \quad u=x \quad du=dx$

$I = \arctan(x)\Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

b) Calculate I using composite midpoint.

$\int_a^b f(x)\,dx = h\sum_{i=1}^{m} f(w_i) + \frac{(b-a)}{24} h^2 f''(c)$

$h = \frac{1}{1} = 1 \quad w_1 = \frac{1}{2} \qquad \sum f(w_1) = f\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^2 + \frac{4}{4}} = \frac{4}{5}$

$\int_0^1 \frac{dx}{x^2+1} = 1\left(\frac{4}{5}\right) + \frac{1}{24}(1^2)(-2)$

$= \frac{4}{5} \mp \frac{1}{12}$

$m=2$

$h = \frac{1}{2} \quad w_1 = \frac{1}{4} \quad w_2 = \frac{3}{4} \quad f(w_1) = f\left(\frac{1}{4}\right) = \frac{16}{17}$

$f(w_2) = f\left(\frac{3}{4}\right) = \frac{16}{25}$

$\int_0^1 \frac{dx}{x^2+1} = \frac{1}{2}\sum^{2} f(w_i) + \frac{1}{24}\left(\frac{1}{2}\right)^2(-2)$

$= \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right] - \frac{1}{48}$

$= \frac{1}{2}\left[\frac{16}{17} + \frac{16}{25}\right] - \frac{1}{48}$

$= \left[\frac{8}{17} + \frac{8}{25}\right] - \frac{1}{48}$

$= \left[\frac{200}{425} + \frac{136}{425}\right] - \frac{1}{48}$

$= \frac{336}{425} \mp \frac{1}{48}$

$m=4. \quad h = \frac{1}{4} \quad w_1 = \frac{1}{8} \; w_2 = \frac{3}{8} \; w_3 = \frac{5}{8} \; w_4 = \frac{7}{8}$

$\int_0^1 \frac{dx}{x^2+1} = \frac{1}{4}\sum^{4} f(w_i) + \frac{1}{24}\left(\frac{1}{4}\right)^2(-2)$

$= \frac{1}{4}\left[\frac{64}{65} + \frac{64}{73} + \frac{64}{89} + \frac{64}{113}\right] - \frac{2}{384}$

$= \left[\frac{16}{65} + \frac{16}{73} + \frac{16}{89} + \frac{16}{113}\right] - \frac{1}{192}$

c) calculate I using composite trapezoid rule

$$\int_a^b f(x)dx = \frac{h}{2}\left(y_0 + y_m + 2\sum_{i=1}^{m-1} y_i\right) - \frac{(b-a)}{12}h^2 f''(c)$$

$m=1 \quad h = \frac{1}{1} = 1 \qquad y_0 = f(0) = 1$

$y_1 = f(1) = \frac{1}{2}$

$$\int_0^1 \frac{dx}{x^2+1} = \frac{1}{2}\left(1 + \frac{1}{2} + 2\sum_{i=1}^{0} y_i\right) - \frac{1}{12}(1)(-2)$$

$$= \frac{1}{2}\left(\frac{3}{2}\right) \pm \frac{1}{6}$$

$$= \frac{3}{4} \pm \frac{1}{6}$$

$m=2 \quad h = \frac{1}{2} \qquad y_0 = 1 \quad y_2 = \frac{1}{2}$

$$f\left(\frac{1}{4}\right) = \frac{1}{\frac{1}{16}+\frac{16}{16}} = \frac{16}{17}$$

$$\int_0^1 \frac{dx}{x^2+1} = \frac{\frac{1}{2}}{2}\left(1 + \frac{1}{2} + 2\sum_{i=1}^{1} y_i\right) - \frac{1}{12}\left(\frac{1}{2}\right)^2(-2)$$

$$= \frac{1}{4}\left(\frac{3}{2} + 2\left(\frac{16}{17}\right)\right) - \frac{2}{48}$$

$$= \left(\frac{3}{8} + \frac{8}{17}\right) \pm \frac{1}{24}$$

$m=4 \quad h = \frac{1}{4} \qquad y_0 = 1 \quad y_4 = \frac{1}{2}$

$x_1 = \frac{1}{4} \quad x_2 = \frac{1}{2} \quad x_3 = \frac{3}{4}$

$y_1 = \frac{16}{17} \quad y_2 = \frac{4}{5} \quad y_3 = \frac{16}{25}$

$$\int_0^1 \frac{dx}{x^2+1} = \frac{\frac{1}{4}}{2}\left(1 + \frac{1}{2} + 2\sum_{i=1}^{3} y_i\right) - \frac{1}{12}\left(\frac{1}{4}\right)^2(-2)$$

$$= \frac{1}{8}\left(\frac{3}{2} + 2\left(\frac{16}{17} + \frac{4}{5} + \frac{16}{25}\right)\right) - \frac{-2}{12}\left(\frac{1}{16}\right)$$

$$= \left(\frac{3}{16} + \frac{4}{17} + \frac{1}{5} + \frac{4}{25}\right) \pm \frac{1}{96}$$

d. Calculate I using composite Simpson's rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + y_m + 4\sum_{i=1}^{m} y_{2i-1} + 2\sum_{i=1}^{m-1} y_{2i}\right] - \frac{(b-a)}{180} h^4 f^{(4)}(c)$$

$m = 1 \quad h = x_2 - x_1 = x_1 - x_0$

$h = 1 - x_1 = x_1 - 0 \quad x_1 = \frac{1}{2} \quad h = \frac{1}{2}$

$$\int_0^1 f(x)\,dx = \frac{\frac{1}{2}}{3}\left[1 + \frac{1}{2} + 4\sum_{i=1}^{1} y_{2i-1} + 2\sum_{i=1}^{0} y_{2i}\right] - \frac{1}{180}\left(\frac{1}{2}\right)^4 (24$$

$= \frac{1}{6}\left[\frac{3}{2} + 4(y_1) + 0\right] - \frac{24}{(180)(16)}$

$= \frac{1}{6}\left[\frac{3}{2} + 4\left(\frac{4}{5}\right)\right] - \frac{1}{120}$

$= \frac{1}{6}\left[\frac{3}{2} + \frac{16}{5}\right] - \frac{1}{120}$

$= \left[\frac{1}{4} + \frac{8}{15}\right] \mp \frac{1}{120}$

$m = 2.$ $\quad y_0 = 1 \quad y_4 = \frac{1}{2}$

$y_1 = \frac{16}{17} \quad y_2 = \frac{4}{5} \quad y_3 = \frac{16}{25}$

$h = \frac{1 - x_3}{x_3 - x_2} = \frac{x_2 - x_1}{x_1 - 0} \quad \frac{1 - \frac{3}{4}}{\frac{3}{4} - \frac{1}{2}} \quad \frac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{4} - 0} \quad h = \frac{1}{4}$

$$\int_0^1 \frac{dx}{x^2+1} = \frac{\frac{1}{4}}{3}\left[1 + \frac{1}{2} + 4\sum_{i=1}^{2} y_{2i-1} + 2\sum_{i=1}^{1} y_{2i}\right] - \frac{1}{180}\left(\frac{1}{4}\right)^4 (24)$$

$= \frac{1}{12}\left[\frac{3}{2} + 4\left[\frac{16}{25} + \frac{16}{17}\right] + 2\left[\frac{4}{5}\right]\right] - \frac{24}{180(256)}$

$= \frac{1}{12}\left[\frac{3}{2} + \frac{64}{25} + \frac{64}{17} + \frac{8}{5}\right] - \frac{1}{1920}$

$= \left[\frac{1}{8} + \frac{16}{75} + \frac{16}{51} + \frac{2}{15}\right] - \frac{1}{1920}$

2. $\int_0^3 x e^{-x^2}\,dx$

$\int_0^3 x e^{-x^2}\,dx \quad u = -x^2 \quad du = -2x\,dx$

$-\frac{1}{2}\int_0^3 e^u\,du = -\frac{1}{2}\left[e^u\right]_0^3 = -\frac{1}{2} e^{-x^2}\Big|_0^3 = -\frac{1}{2}e^{-9} + \frac{1}{2}e^0$

$= -\frac{1}{2}e^{-9} + \frac{1}{2}$

the slope based on the theoretical error term:

$\left(\frac{(b-a)}{2^n}\right)^2$

3. $F = \frac{-1}{2} e^{-9} + \frac{1}{2}$

slope from the theoretical error term:

$\left(\frac{b-a}{2^n}\right)^4$

4. $\int_0^1 x^{1/2}\,dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$

slope from trapezoid theoretical error term:

$\left(\frac{b-a}{2^n}\right)^2$

slope from Simpson's theoretical error term:

$\left(\frac{b-a}{2^n}\right)^4$

I am not sure why the slope in problem 4 is is greater than the slope from the theoretical error while the slope from problems 2 and 3 seem equal.

5. Prove the second order formula for the third derivative.

$$f'''(x) = \frac{f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)}{2h^3} + O(h^2)$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2} f''(x) + \frac{(2h)^3}{6} f'''(x) + O(h^4)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4)$$

$$f(x-h) = f(x) - hf'(x) + \frac{(-h)^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + O(h^4)$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{(-2h)^2}{2} f''(x) - \frac{(2h)^3}{6} f'''(x) + O(h^4)$$

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(x) + \frac{8h^3}{6}f'''(x) + O(h^4)$$

$$-2f(x+h) = -2f(x) - 2hf'(x) - h^2f''(x) - \frac{h^3}{3}f'''(x) - 2O(h^4)$$

$$+2f(x-h) = 2f(x) - 2hf'(x) + h^2f''(x) - \frac{h^3}{3}f'''(x) + 2O(h^4)$$

$$-f(x-2h) = -f(x) + 2hf'(x) - 2h^2f''(x) + \frac{8h^3}{6}f'''(x) - O(h^4)$$

$$0 + 0 + 0 + 2h^3f'''(x) + 0$$

$$f'''(x) = \frac{2h^3f'''(x)}{2h^3} + O(h^2)$$

$$f'''(x) = f'''(x) + O(h^2)$$