Real Analysis

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Contents

Chapter 1

The Real Number System

1.1 Ordered Field

Definition 1.1. *A field F is said to be an ordered field with respect to a particular subset $P \subseteq F$ if*

O1. $\forall x, y \in P, x + y \in P$ *and* $x \cdot y \in P$.

O2. $\forall x \in F$, *one and only one of the following statements hold:*

$$x \in P \vee -x \in P \vee x = 0$$

.

Definition 1.2. *If $x \in P$, we say that x is **positive** and if $-x \in P$, then we say x is **negative**.*

Definition 1.3. *Given $x, y \in F$, we say that x and y have the **same sign** if $x, y \in P \vee -x, -y \in P$ (either both positive or both negative). We say that x and y have **opposite signs** if $-x, y \in P$, or $x, -y \in P$ (one positive and the other one negative).*

Definition 1.4. *We define the symbols $<, \leq, >$ and $\geq$ in an ordered field as follow*

- $x < y \iff y - x \in P$.
- $x > y \iff y < x \iff x - y \in P$.
- $x \leq y \iff x < y \vee x = y \iff y - x \in P \vee x = y$.
- $x \geq y \iff x > y \vee x = y \iff x - y \in P \vee x = y$.

Remark. *Since* $0 = 0 \implies 0 \notin P$ *(hence* 0 *is not positive) and* $-0 \notin P$ *(hence* 0 *is not negative).*

Theorem 1.5. (a) x *is positive* $\iff x > 0$.

(b) x *is negative* $\iff x < 0$.

(c) $x > 0 \iff -x < 0$.

(d) $x < y \iff y - x > 0 \iff x - y < 0$.

(e) $x > y \iff x - y > 0 \iff y - x < 0$.

Theorem 1.6. *Let* $x, y \in F$ *an ordered field. Then one and only one of the following statements hold:* $x < y, x > y \vee x = y$.

Corollary 1.6.1. *Let* $x \in F$. *Then one and only one of the following statements hold:* $x < 0, x > 0 \vee x = 0$.

Theorem 1.7. *Let* $x, y \in F$. *Then*

(a) $x \leq y \iff x \not> y$.

(b) $x \geq y \iff x \not< y$.

(c) *If* $x \leq y$ *and* $y \leq x$ *then* $x = y$.

(d) $\forall x, y \in \mathbb{R}, x \leq y$ *or* $y \leq x$.

Theorem 1.8 (Combination of Positive and Negative Elements)**.** *Let* F *be an ordered field. Then*

(a) $\forall x, y \in F, x > 0 \wedge y > 0 \implies x + y > 0 \wedge x \cdot y > 0$.

(b) $\forall x, y \in F, x < 0 \wedge y < 0 \implies x + y < 0 \wedge x \cdot y > 0$.

(c) $\forall x \in F, x \neq 0 \implies x^2 > 0$.

(d) $\forall x \in F, x^2 \geq 0$.

(e) $\forall x, y \in F, x > 0 \wedge y < 0 \implies x \cdot y < 0$.

(f) $\forall x, y \in F, x \cdot y > 0 \implies x$ *and* y *have same signs.*

(g) $\forall x, y \in F, x \cdot y < 0 \implies x$ *and* y *have opposite signs.*

Theorem 1.9. $1 > 0$.

Corollary 1.9.1. $-1 < 0$.

Theorem 1.10. *Let $x, y, z \in F$ and ordered field. Then*

(a) $(x < y) \wedge (y < z) \implies x < z$.

(b) $(x \leq y) \wedge (y \leq z) \implies x \leq z$

(c) $x < y \iff x + z < y + z$.

(d) $(x < y) \wedge (z > 0) \implies x \cdot z < y \cdot z$

(e) $(x \leq y) \wedge (z > 0) \implies x \cdot z \leq y \cdot z$.

(f) $(x \leq y) \wedge (z \geq 0) \implies x \cdot z \leq y \cdot z$.

(g) $(x < y) \wedge (z < 0) \implies x \cdot z > y \cdot z$.

Theorem 1.11. (a) *If $x > 0$ then $\dfrac{1}{x} > 0$.*

(b) *If $x < 0$ then $\dfrac{1}{x} < 0$.*

(c) *If $x < y$ and $z > 0$ then $\dfrac{x}{z} < \dfrac{y}{z}$.*

(d) *If $x < y$ and $z < 0$ then $\dfrac{x}{z} > \dfrac{y}{z}$.*

Theorem 1.12. (a) $0 < x < y \iff 0 < \dfrac{1}{y} < \dfrac{1}{x}$.

(b) $x < y < 0 \iff \dfrac{1}{y} < \dfrac{1}{x} < 0$.

(c) $(x < y) \wedge (u < v) \implies x + u < y + v$.

(d) $(0 < x < y) \wedge (0 < u < v) \implies (x \cdot u < y \cdot v) \wedge \left(\dfrac{x}{v} < \dfrac{y}{u}\right)$.

(e) $(0 \leq x \leq y) \wedge (0 \leq u \leq v) \implies (0 \leq x \cdot u \leq y \cdot v)$

(f) $x < y \implies x < \dfrac{x+y}{2} < y$.

Theorem 1.13. (a) $(x < y) \wedge (w < z) \implies x + w < y + z$.

(b) $(x < y) \wedge (w \leq z) \implies x + w < y + z$.

1.2 Natural Numbers and The Principle of Mathematical Induction

Definition 1.14. *A set $A \subseteq \mathbb{R}$ is said to be* ***inductive*** *if*

1. $1 \in A$, *and*
2. $\forall x \in \mathbb{R}, x \in A \implies x + 1 \in A$.

Theorem 1.15. *The intersection of any collection of inductive sets is inductive.*

Definition 1.16. *The set of* ***natural numbers*** *is the intersection of all the inductive subsets of $\mathbb{R}$. In symbols,*

$$\mathbb{N} = \cap S,$$

where S denotes the collection of all inductive subsets of $\mathbb{R}$.

Theorem 1.17. *The set of natural numbers is the smallest inductive subset of $\mathbb{R}$, in the sense that if A is an inductive subset of R then $\mathbb{N} \subseteq A$.*

Theorem 1.18. (a) *All natural numbers are positive.*

(b) *1 is the smallest natural number. That is, $\forall n \in \mathbb{N}, n \geq 1$.*

(c) *If n is a natural number other than 1, then $n - 1$ is also a natural number. That is $\forall n \in \mathbb{N}$, if $n > 1$, then $n - 1 \in \mathbb{N}$.*

Theorem 1.19 (The Principle of Mathematical Induction). *Let $P(n)$ be a statement concerning natural numbers. Then*

$$P(1) \text{ and } (\forall k \in \mathbb{N}) \, [P(k) \implies P(k+1)] \implies \forall n \in \mathbb{N}, P(n).$$

Theorem 1.20. *Let $P(n)$ be a statement concerning natural numbers, then following statements are equivalent:*

1. ***The Principle of Mathematical Induction***

$$P(1) \text{ and } (\forall k \in \mathbb{N}) \, [P(k) \implies P(k+1)] \implies \forall n \in \mathbb{N}, P(n).$$

2. ***The Principle of Strong Mathematical Induction***

$$P(1) \text{ and } (\forall k \in \mathbb{N}) \, [(\forall j \leq k, P(j)) \implies P(k+1)] \implies \forall n \in \mathbb{N}, P(n).$$

where j ranges over natural numbers in this statement.

3. ***Well Ordering Principle***
   Every nonempty set of natural numbers contains a least element.

Remark. *Alternate definition of The Principle of Strong Mathematical Induction.*

$$(\forall k \in \mathbb{N}) \left[(\forall j < k, P(j)) \implies P(k) \right] \implies \forall n \in \mathbb{N}, P(n).$$

where j ranges over natural numbers in this statement.

Theorem 1.21. *Show that:*

(a) $\forall m, n \in \mathbb{N}, m < n \implies n - m \in \mathbb{N}$.

(b) $\forall n \in \mathbb{N}$, *there is no natural number between n and $n + 1$.*

(c) $\mathbb{N}$ *is closed under addition.*

(d) $\mathbb{N}$ *is closed under multiplication.*

(e) $\mathbb{N}$ *is not closed under subtraction or division.*

1.3 Integer Powers of Real Numbers

Definition 1.22. *The set of integers is the set*

$$\mathbb{Z} = \{x \in \mathbb{R} : x \in \mathbb{N} \textit{ or } -x \in \mathbb{N} \textit{ or } x = 0\}$$

Definition 1.23. *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$a^n = \begin{cases} a \textit{ if } n = 1 \\ a^{n-1} \cdot a \textit{ if } n \geq 2 \end{cases}$$

We define $a^0 = 1$ for all $a \in \mathbb{R} - \{0\}$.

Definition 1.24. *Let $a \in \mathbb{R} - \{0\}$ and $n \in \mathbb{N}$. Then*

$$a^{-n} = \frac{1}{a^n}$$

Theorem 1.25. *Let $a \in \mathbb{R} - \{0\}$, then for every $n \in \mathbb{N}$,*

$$a^{-n} \overset{def}{=} \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$$

Theorem 1.26. *Show that:*

(a) $\forall x \in \mathbb{R}_{\geq 0}, \forall n \in \mathbb{N}$, $x^n \geq 0$.

(b) $\forall x \in \mathbb{R}_{>0}, \forall n \in \mathbb{N}$, x^{-n}.

Theorem 1.27. *Let x and y be real numbers, and let $m, n \in \mathbb{N}$. Show that*

(a) $(xy)^n = x^n y^n$,

(b) $x^{n+m} = x^n x^m$,

(c) $(x^n)^m = x^{nm}$.

Theorem 1.28. *Let x and y be nonzero real numbers and $m, n \in \mathbb{N}$. Show that*

(a) $(xy)^{-n} = x^{-n} y^{-n}$,

(b) $x^{-n-m} = x^{-n} x^{-m}$,

(c) $(x^{-n})^{-m} = x^{nm}$.

Theorem 1.29. *Show that*

(a)
$$(x^n - y^n) = (x - y) \sum_{k=1}^{n} x^{n-k} y^{k-1}$$
is valid for all $x, y \in \mathbb{R}$ and every $n \in \mathbb{N}$.

(b) *Suppose that x and y are postive real numbers. If $x^n < y^n$ for some $n \in \mathbb{N}$, prove that $x < y$.*

(c) *Suppose that x and y are nonnegative real numbers. If $x^n = y^n$ for some $n \in \mathbb{N}$, prove that $x = y$.*

(d) *Suppose that x and y are nonnegative real numbers. If $x^n \leq y^n$ for some $n \in \mathbb{N}$, prove that $x \leq y$.*

1.4 Rational Powers of Real Numbers

Theorem 1.30 (The Existence of nth root)**.** *For every positive real number a and every natural number n there exists a unique positive number b such that $b^n = a$.*
Proved later after introducing completeness.

Definition 1.31. *The set of **rational numbers** is the set*

$$\mathbb{Q} = \left\{ x \in \mathbb{R} : \exists\, m \in \mathbb{Z}, \exists\, n \in \mathbb{N} \text{ such that } n \neq 0, \text{ and } x = \frac{m}{n} \right\}$$

If x is positive real number and $r = \frac{m}{n}$, we define x^r by

$$x^r := (x^m)^{\frac{1}{n}}$$

Remark. *Show that this definition is well defined.*

If $x < 0$ and n is even, $x^{\frac{1}{n}}$ has no meaning. Since if n is even ($n = 2k$ for some k $\in \mathbb{N}$) then

$$x = (x^{\frac{1}{n}})^n = (x^{\frac{1}{n}})^{2k} = ((x^{\frac{1}{n}})^k)^2 \geq 0$$

If n is odd, then we define $x^{\frac{1}{n}}$ by

$$x^{\frac{1}{n}} := -\left((-x)^{\frac{1}{n}} \right)$$

Remark. *nth root of 0 is 0 for $\forall n \in \mathbb{N}$.*

1.5 Additional Properties of Inequalities

Theorem 1.32. *Show that:*

(a) *Let $x, y \in \mathbb{R}_{>0}$. Then $\forall n \in \mathbb{N}$, $0 < x < y \iff x^n < y^n$.*

(b) *Let $x, y \in \mathbb{R}_{\geq 0}$. Then $\forall n \in \mathbb{N}$, $x \leq y \iff x^n \leq y^n$.*

(c) *Let $x, y \in \mathbb{R}_{>0}$. Then $\forall n \in \mathbb{N}$, $0 < x < y \iff x^{\frac{1}{n}} < y^{\frac{1}{n}}$.*

(d) *Let $x, y \in \mathbb{R}_{\geq 0}$. Then $\forall n \in \mathbb{N}$, $x \leq y \iff x^{\frac{1}{n}} \leq y^{\frac{1}{n}}$.*

Theorem 1.33 (Forcing Principle)**.** *Let $x, y \in \mathbb{R}$. Then*

(a) $\forall \epsilon > 0$, $x \leq \epsilon \implies x \leq 0$.

(b) $\forall \epsilon > 0$, $x \leq y + \epsilon \implies x \leq y$.

(c) $\forall \epsilon > 0$, $|x| \leq \epsilon \implies x = 0$.

(d) $\forall \epsilon > 0$, $|x - y| \leq \epsilon \implies x = y$.

1.6 Intervals

Definition 1.34. *$\forall a, b \in \mathbb{R}$, we define the $[a, b]$ to be the set*

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

Note that $[a, a] = \{a\}$ and $[2, 1] = \emptyset$ since $2 \leq x \leq 1$ is false, $\forall x \in \mathbb{R}$.

An ***interval*** *in $\mathbb{R}$ is any subset $I \subseteq \mathbb{R}$ such that $\forall x, y \in I, x < y \implies [x, y] \subseteq I$.*

Theorem 1.35. *Show that following sets are intervals:*

(a) $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$*;*

(b) $(a, b) = \{x \in \mathbb{R} : a < x < b\}$*;*

(c) $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$*;*

(d) $(a, \infty) = \{x \in \mathbb{R} : a < x\}$*;*

1.7 Absolute Value

Definition 1.36. *Let $x \in \mathbb{R}$. Then*

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Theorem 1.37. $\forall x \in \mathbb{R}, |x|^2 = |x^2| = x^2$.

Theorem 1.38. $\forall x \in \mathbb{R}, |x| = \sqrt{x^2}$.

Theorem 1.39. $\forall x \in \mathbb{R}, |x| \geq 0$.

Theorem 1.40. $\forall x \in \mathbb{R}, |-x| = |x|$.

Theorem 1.41. $\forall x \in \mathbb{R}, -|x| \leq x \leq |x|$.

Theorem 1.42. $\forall x \in \mathbb{R}, |x| = 0 \iff x = 0$.

Theorem 1.43. $\forall x, y \in \mathbb{R}, |x| = y \implies x = y$ *or* $x = -y$.

Theorem 1.44. *Let $a \in \mathbb{R}$ and suppose $a > 0$. Then*

$$\forall x \in \mathbb{R}, \Big[|x| < a \iff -a < x < a \Big]$$

.

Theorem 1.45. *Let $a \in \mathbb{R}$ and suppose $a \geq 0$. Then*

$$\forall x \in \mathbb{R}, \Big[|x| \leq a \iff -a \leq x \leq a \Big]$$

.

Theorem 1.46. *Let $a \in \mathbb{R}$ and suppose $a > 0$. Then*

$$\forall x \in \mathbb{R}, \Big[|x| > a \iff x < -a \vee x > a \Big]$$

.

Theorem 1.47. *Let $a \in \mathbb{R}$ and suppose $a \geq 0$. Then*

$$\forall x \in \mathbb{R}, \Big[|x| \geq a \iff x \leq -a \vee x \geq a \Big]$$

.

Theorem 1.48. $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$.

Theorem 1.49. $\forall x, y, z \in \mathbb{R}, |x - z| \leq |x - y| + |y - z|$.

Theorem 1.50. $\forall x, y \in \mathbb{R}, \big||x| - |y|\big| \leq |x - y| \iff |x| - |y| \leq |x - y| \wedge |y| - |x| \leq |x - y|$.

Theorem 1.51. $\forall x, y \in \mathbb{R}, \big||x| - |y|\big| \leq |x + y| \iff |x| - |y| \leq |x + y| \wedge |y| - |x| \leq |x + y|$.

Theorem 1.52. $\forall x, y \in \mathbb{R}, |xy| = |x||y|$.

Theorem 1.53. $\forall x, y \in \mathbb{R}$ *with* $y \neq 0, \left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$.

Theorem 1.54. $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}, |x^n| = |x|^n$.

1.8 Some Useful Inequalities and Identities

Theorem 1.55 (Bernoulli's inequality)**.** *Prove that for every $n \in \mathbb{N}$ and every real number $x \geq -1$,*

$$(1 + x)^n \geq 1 + nx$$

Theorem 1.56. *Let $x \in \mathbb{R} - \{0\}$. Prove that for every $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

Lemma 1.57. *Let n and k be natural numbers and let $1 \leq k \leq n$. Then*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Theorem 1.58 (Binomial Theorem)**.** *If x and y are arbitrary real numbers and $n \in \mathbb{N}$, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Theorem 1.59. *If x and y are arbitrary real numbers then*

$$|xy| \leq \frac{1}{2}(x^2 + y^2).$$

Theorem 1.60. *Let $x_1, \ldots, x_n$ be nonnegative real numbers, and let $m \in \mathbb{N}$. Prove that*

$$\sqrt[m]{x_1 + \cdots + x_n} \leq \sqrt[m]{x_1} + \cdots + \sqrt[m]{x_n}$$

Theorem 1.61 (The Cauchy-Schwartz Inequality)**.** *If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are arbitrary real numbers, then*

$$\left(\sum_{i=1}^{n} x_i y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right)$$

Theorem 1.62 (Minokowski's Inequality)**.** *If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are arbitrary real numbers, then*

$$\left(\sum_{i=1}^{n} (x_i + y_i)^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} y_i^2 \right)^{\frac{1}{2}}$$

Chapter 2

Cardinality

Definition 2.1. *Let A and B be sets. Then we say that A and B are* ***equinumerous*** *($A \approx B$) provided that there exists a bijection from the set A to the set B.*

Theorem 2.2. *Let A, B, and C be sets.*

(a) *For each set $A, A \approx A$.*

(b) *For all sets A and B, if $A \approx B$, then $B \approx A$.*

(c) *For all sets A, B and C, if $A \approx B$ and $B \approx C$, then $A \approx C$.*

Theorem 2.3. *Let E be the set of all even natural numbers and let D be the set of all odd natural numbers. Prove that,*

(a) $\mathbb{N} \approx E$.

(b) $\mathbb{N} \approx D$.

(c) $\mathbb{N} \approx \mathbb{Z}$.

(d) $\mathbb{R}_{>0} \approx \mathbb{R}$.

Theorem 2.4. *Suppose $A \approx B$ and $C \approx D$. Then:*

(a) $A \times C \approx B \times D$.

(b) *If A and C are disjoint and B and D are disjoint, then $A \cup C \approx B \cup D$.*

2.1 Finite Sets

Theorem 2.5. *Let $m, n \in \mathbb{N}$. If $\mathbb{N}_n \approx \mathbb{N}_m$, then $n = m$.*

Corollary 2.5.1. *If $A \approx \mathbb{N}_m$ and $A \approx \mathbb{N}_n$ then $m = n$.*

Definition 2.6. *We define $\mathbb{N}_k = \{1, 2, ..., k\}$. A set A is a* ***finite set*** *provided that $A = \emptyset$ or $A \approx \mathbb{N}_k$ for some natural number k.*
If A is finite, we say that the ***cardinality*** *of A, denoted as $|A|$, is 0 if $A = \emptyset$, or k if $A \approx N_k$.*
A set is an ***infinite set*** *provided that it is not a finite set. So a set A is infinite $\iff$ $A \neq \emptyset$ and there is not bijection between A and $\mathbb{N}_k$, $\forall k \in \mathbb{N}$.*

Lemma 2.7. *If $A \approx \emptyset$ then $A = \emptyset$.*

Theorem 2.8. *Any set equivalent to a finite set A is a finite set and has the same cardinality as A.*

Theorem 2.9. *Suppose A and B are finite sets. Then*

$$A \approx B \iff |A| = |B|.$$

Theorem 2.10. *Suppose A and B are finite sets. Then*

(a) *If $A \cap B = \emptyset$ then $A \cup B$ is finite and $|A \cup B| = |A| + |B|$.*

(b) *$A \cup B$ is finite, and $|A \cup B| = |A| + |B| - |A \cap B|$.*

(c) *$A \times B$ is finite, and $|A \times B| = |A| \times |B|$.*

Lemma 2.11. *If A is a finite set and $x \notin A$, then $A \cup \{x\}$ is a finite set and $|A \cup \{x\}| = |A| + 1$.*

Lemma 2.12. *For each natural number m, if $A \subseteq \mathbb{N}_m$, then A is a finite set and $|A| \leq m$.*

Theorem 2.13. *If S is a finite set and A is a subset of S, then A is a finite set and $|A| \leq |S|$.*

Corollary 2.13.1. *If A is a finite set and $x \in A$, then $A - \{x\}$ is a finite set and $|A - \{x\}| = |A| - 1$.*

Theorem 2.14 (The Pigeonhole Principle)**.** *Let A and B be finite sets. If $|A| > |B|$, then any function $f : A \to B$ is not an injection.*

Theorem 2.15. *For every finite set A, $|\mathcal{P}(A)| = 2^{|A|}$.*

Theorem 2.16. *A finite set is not equivalent to any of its proper subsets.*

2.2 Countably Infinite and Countable Sets

Theorem 2.17. *If a set A is equivalent to any of its proper subset, then A is infinite.*

Corollary 2.17.1. (a) *The set of natural numbers is infinite.*

(b) *The set of real numbers is infinite.*

Corollary 2.17.2. *If A is an infinite set and B is a finite set, then A and B are not equivalent.*

Theorem 2.18. *Let A and B be sets.*

(a) *If A is infinite and $A \approx B$, then B is infinite.*

(b) *If A is infinite and $A \subseteq B$, then B is infinite.*

Definition 2.19. *A set A is* ***countably infinite*** *provided that $A \approx \mathbb{N}$.*
So A is countably infinite $\iff$ there exists a bijection between A and $\mathbb{N}$. A set is ***countable*** *provided that it is finite or countably infinite. An infinite set that is not countably infinite is called an* ***uncountable set***. *So a set A if uncountable $\iff$ A is infinite and there is no bijection between A and $\mathbb{N}$.*
Since $\mathbb{N} \approx \mathbb{Z}$, the set $\mathbb{Z}$ of integers is countably infinite.

Remark. *So if A is countably infinite ($A \approx \mathbb{N}$) then A is an infinite set since $\mathbb{N}$ is infinite (Theorem 2.18). So a finite set can not be countably infinite.*
If A is countably infinite ($A \approx \mathbb{N}$) then the elements of A can be enumerated in an interminable list as $A = \{a_1, a_2, a_3, \ldots\}$ where $a_i = f(i), \forall i \in \mathbb{N}$ where f is a bijection from $\mathbb{N}$ and A. Since f is onto and one-to-one, each element of A appears in this list one and only one times.

Theorem 2.20. *Show that:*

(a) *The set of positive rational numbers is countably infinite.*

(b) *The set of negative rational numbers is countably infinite.*

Theorem 2.21. *If A is a countably infinite set, then $A \cup \{x\}$ is a countably infinite set.*

Theorem 2.22. *If A is a countably infinite set and B is a finite set, then $A \cup B$ is a countably infinite set.*

Theorem 2.23. *If A and B are disjoint countably infinite set, then $A \cup B$ is a countably infinite set.*

Theorem 2.24. *The set $\mathbb{Q}$ of all rational numbers is countably infinite.*

Theorem 2.25. *If A and B are both countably infinite, then so is $A \times B$.*

Theorem 2.26. *The set $\mathbb{N} \times \mathbb{N}$ is countably infinite.*

Theorem 2.27. *Any set equivalent to a countable set is countable.*

Corollary 2.27.1. *If A is a countable set and B is an uncountable set, then A and B are not equivalent.*

Theorem 2.28. *Every subset of the natural numbers is countable.*

Corollary 2.28.1. *Every subset of a countable set is countable.*

Corollary 2.28.2. *If A and B are both countably infinite, then so is $A \cup B$.*

Theorem 2.29. *Let A be a nonempty set. The following statements are equivalent:*

(a) *A is countable.*

(b) *There is a function $f : \mathbb{N} \to A$ that is onto.*

(c) *There is a function $f : A \to \mathbb{N}$ that is one-to-one.*

Theorem 2.30. *Suppose A and B are countable sets. Then:*

(a) *$A \times B$ is countable.*

(b) *$A \cup B$ is countable.*

Theorem 2.31. *The union of countably many countable sets in countable.*

2.3 Uncountable Sets

Theorem 2.32. *The open interval (0, 1) of real numbers in uncountable.*

Theorem 2.33. *Let A and B be sets.*

(a) *If A is uncountable and $A \approx B$ then B is uncountable.*

(b) *If A is uncountable and $A \subseteq B$ then B is uncountable*

Corollary 2.33.1. *The sets of real numbers* $\mathbb{R}$ *is uncountable.*

Corollary 2.33.2. *If* A *is a countable set and* B *is an uncountable set then* A *and* B *are not equivalent.*

Theorem 2.34. *For* $a, b \in \mathbb{R}$, *with* $a < b$, $(a, b) \approx (0, 1)$.

Theorem 2.35. *For* $a, b \in \mathbb{R}$, *with* $a < b$, $(a, b) \approx \mathbb{R}$.

Theorem 2.36. *The set of irrational numbers is uncountable.*

Definition 2.37. *If* A *and* B *are sets, then we will say that* B *dominates* A*, and write* $A \preceq B$*, if there is a function* $f : A \to B$ *that is one-to-one. We write* $A \prec B$ *if and only if* $A \preceq B$ *and* $A \not\approx B$.

Theorem 2.38. *For every nonempty set* A*, the sets* $\mathcal{P}(A)$ *and* 2^A *are equivalent.*

Theorem 2.39 (Cantor's Theorem)**.** *For every set* A*,* $A \prec \mathcal{P}(A)$.

Corollary 2.39.1. $\mathcal{P}(\mathbb{N})$ *is an uncountable set.*

Theorem 2.40 (The Schröder-Bernstein Theorem)**.** *If* A *and* B *are sets such that* $A \preceq B$ *and* $B \preceq A$*, then* $A \approx B$.

Theorem 2.41. *The sets* $\mathcal{P}(\mathbb{N})$ *and* $\mathbb{R}$ *are equivalent.*

Corollary 2.41.1. *The sets* 2^N *and* $\mathbb{R}$ *are equivalent.*

Remark (The Continuum Hypothesis)**.** *There exists no set* S *such that*

$$\mathbb{N} \prec S \prec \mathbb{R}.$$

Chapter 3

The Completeness Property and its Applications

3.1 Bounded Sets

Definition 3.1. *Let $A \subseteq \mathbb{R}$ and $u \in \mathbb{R}$. We say that:*

(a) *u is an **upper bound** for A if $\forall x \in A, x \leq u$.*

(b) *u is a **lower bound** for A if $\forall x \in A, u \leq x$.*

(c) *u is a **maximum element** of A if $u \in A$ and $\forall x \in A, x \leq u$.*

(d) *u is a **minimum element** of A if $u \in A$ and $\forall x \in A, u \leq x$.*

*If A has an upper bound we say that A is **bounded above**; if A has a lower bound we say that A is **bounded below**. If A is bounded above and below, we say that A is **bounded**.*

Theorem 3.2. *Let $A \subseteq \mathbb{R}$. Then A is bounded $\iff \exists M > 0$ such that $|x| \leq M$ for all $x \in A$.*

Theorem 3.3. (a) *A set cannot have more than one maximum or mroe than one minimum element.*

(b) *Every nonempty finite set has both a maximum element and a minimum element.*

3.2 Suprema and Infima

Definition 3.4. *Suppose that F is an ordered field and $A \subseteq F$. We say that an element $u \in F$ is*

(a) *a **least upper bound (supremum)** of A if u is an upper bound for A and for all upper bounds v for $A, u \leq v$. The notation we use is $u = sup(A)$.*

(b) *a **greatest lower bound (infimum)** of A if u is a lower bound for A and for all lower bounds v for $A, u \geq v$. The notation we use is $u = inf(A)$.*

Theorem 3.5. (a) *A set cannot have more than one greatest lower bound.*

(b) *A set cannot have more than one least upper bound.*

(c) *If a set has the minimum (or maximum) element, then that element is the greatest lower bound (or least upper bound) of A.*

(d) *If a set contains the greatest lower bound (or least upper bound) then that element is the minimum (or maximum) element of A.*

3.3 The Completeness Axiom

<u>**Completeness Axiom**</u>: Every nonempty set of real numbers that is bounded above has the least upper bound.

Theorem 3.6. *Every nonempty $S \subseteq \mathbb{R}$ that is bounded below has the greatest lower bound.*

Theorem 3.7. *Let $a < b$ in an ordered field F. Then $a = inf(a, b)$ and $b = sup(a, b)$.*

Theorem 3.8. *Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded. Let $A \subseteq S$ be nonempty. Prove that A is bounded. Then prove that $sup(A) \leq sup(S)$ and $inf(S) \leq inf(A)$.*

Theorem 3.9. *Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded above. Let $\beta = sup(S)$. Prove that $\forall \epsilon > 0, \exists x \in S$ such that $\beta - \epsilon < x$.*

Theorem 3.10. *Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded below. Let $\alpha = inf(S)$. Prove that $\forall \epsilon > 0, \exists x \in S$ such that $x < \alpha + \epsilon$.*

Theorem 3.11. *Let $S \subseteq \mathbb{R}$ be nonempty and bounded and let $k \in \mathbb{R}$. Define the set $k + S = \{k + x : x \in S\}$. Prove that:*

(a) $sup(k + S) = k + sup(S)$.

(b) $inf(k + S) = k + inf(S)$.

Theorem 3.12. *Let $S \subseteq \mathbb{R}$ be nonempty and bounded and let $k \in \mathbb{R}$. Define the set $kS = \{kx : x \in S\}$. Then the set kS is bounded and*

(a) *if $k \geq 0$, then $sup(kS) = k\ sup(S)$ and $inf(kS) = k\ inf(S)$.*

(b) *if $k < 0$, then $sup(kS) = k\ inf(S)$ and $inf(kS) = k\ sup(S)$.*

Theorem 3.13. *Given nonempty subsets A and B of $\mathbb{R}$, let C denote the set $C = \{x + y : x \in A$ and $y \in B\}$. If A and B have suprema then C has a supremum and $sup(C) = sup(A) + sup(B)$.*

Definition 3.14. *A function $f : D \to \mathbb{R}$ is bounded if the set $f(D) = \{f(x) : x \in D\}$ is bounded.*

Remark. *$f : D \to \mathbb{R}$. f is bounded.*
$\iff \exists\, a, b \in \mathbb{R}$ *such that* $a \leq f(x) \leq b,\ \forall x \in D$.
$\iff \exists M > 0$ *such that* $|f(x)| \leq M,\ \forall x \in D$.

Theorem 3.15. *Let D be a nonempty subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be bounded. Define $f + g : D \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x),\ \forall x \in D$. Then $f + g$ is bounded i.e., $(f + g)(D)$ is bounded and*

(a) $sup((f + g)(D)) \leq sup(f(D)) + sup(g(D))$.

(b) $inf(f(D)) + inf(g(D)) \leq inf((f + g)(D))$.

Theorem 3.16. *Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ by nonempty. Suppose $\forall x \in A, \forall y \in B, x \leq y$. Then A is bounded above and B is bounded below and $sup(A) \leq inf(B)$.*

Corollary 3.16.1. *Let $A \subseteq \mathbb{R}$ by nonempty and let $c \in \mathbb{R}$. Suppose $\forall x, y \in A$, $x - y < c$. Then A is bounded and $sup(A) - inf(A) \leq c$.*

Theorem 3.17. *Suppose that D is a nonempty set and $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. If $\forall x, y \in D, f(x) \leq g(y)$ then $f(D)$ is bounded above adn $g(D)$ is bounded below and $sup(f(D) \leq inf(g(D)$.*

Theorem 3.18 (Archimedean Property of $\mathbb{R}$)**.** *The set of natural numbers is not bounded above in $\mathbb{R}$.*

Theorem 3.19. *The following statements are equivalent:*

(a) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ *such that* $n > x$.

(b) *(Teaspoon and The Sea) (Every journey begins with a single step)* $\forall x, y \in \mathbb{R}$ *with* $x > 0, \exists n \in \mathbb{N}$ *such that* $nx > y$.

(c) $\forall x > 0, \exists n \in \mathbb{N}$ *such that* $0 < \dfrac{1}{n} < x$.

Theorem 3.20. *If* $x < y + \dfrac{1}{n}$ *for every natural number* n *then* $x \leq y$.

Theorem 3.21. *Let* A *be a nonempty subset of* $\mathbb{R}$, *and let* a *be a real number such that* $\forall n \in \mathbb{N}, a + \frac{1}{n}$ *is an upper bound for* A *and* $a - \frac{1}{n}$ *is not an upper bound for this set. Then prove that* a *is the supremum of* A.

Theorem 3.22. *Let* p *be a prime. Then there is no element in* $\mathbb{Q}$ *whose square is* p.

Theorem 3.23. *Let* p *be a prime number. Then there exists a unique positive real number* x *such that* $x^2 = p$.

Theorem 3.24 (The Existence of nth root; extending previous theorem)**.** *For every positive real number* a *and every natural number* n *there exists a unique positive number* b *such that* $b^n = a$.

Theorem 3.25. *The ordered field* $\mathbb{Q}$ *of rational numbers is not complete.*

Theorem 3.26. $\forall x \in \mathbb{R}, \exists!\, m \in \mathbb{Z}$ *such that* $m - 1 \leq x < m$.

Corollary 3.26.1. *Let* $x, y \in \mathbb{R}$. *If* $y - x > 1$ *then* $\exists\, m \in \mathbb{Z}$ *such that* $x < m < y$.

Definition 3.27. *Let* $D \subseteq \mathbb{R}$. *We say* D *is dense in* $\mathbb{R}$ *if* $\forall x, y \in \mathbb{R}$, *if* $x < y$, *then* $\exists d \in D$ *such that* $x < d < y$.

Theorem 3.28. *The set* $\mathbb{Q}$ *of rational numbers is dense in* $\mathbb{R}$.

Lemma 3.29. *Let* $x \in \mathbb{Q}$ *be nonzero and* $y \in \mathbb{R}$ *be irrational. Then* xy *is irrational.*

Theorem 3.30. *The set of irrational numbers is dense in* $\mathbb{R}$.

Theorem 3.31 (Nested Interval Theorem)**.** *Let* $\{I_n : n \in \mathbb{N}\}$ *be a set of nonempty closed intervals* $I_n = [a_n, b_n]$ *such that* $I_{n+1} \subseteq I_n, \forall \in \mathbb{N}$. *Then*

(a) $\bigcap_{n=1}^{\infty} I_n$ *is nonempty closed interval.*

(b) *if* $\lim_{n\to\infty}(b_n - a_n) = 0$ *then* $\bigcap_{n=1}^{\infty} I_n$ *consists of only one point.*

Theorem 3.32 (Bolzano-Weierstrass Theorem for Sets)**.** *Every infinite bounded set of real numbers has an accumulation point.*

Chapter 4

Topology

4.1 Neighborhoods

Definition 4.1. *Let* $a \in \mathbb{R}$. *Then* $N_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\} = (a-\epsilon, a+\epsilon)$. *Therefore*

$$\begin{aligned} x \in N_\epsilon(a) &\iff |x-a| < \epsilon \\ &\iff -\epsilon < x-a < \epsilon \\ &\iff a-\epsilon < x < a+\epsilon \\ &\iff x \in (a-\epsilon, a+\epsilon) \end{aligned}$$

Definition 4.2. *Let* $a \in \mathbb{R}$. *Let* $\epsilon > 0$. *Then* $N^*_\epsilon = \{x \in \mathbb{R} : x \neq a \wedge |x-a| < \epsilon\} = N_\epsilon(x) - \{a\} = (a-\epsilon, a) \cup (a, a+\epsilon)$.

Theorem 4.3. *Let* $a \in \mathbb{R}$. *Let* $0 < \epsilon_1 < \epsilon_2$. *Then*

1. $N_{\epsilon_1}(a) \subseteq N_{\epsilon_2}(a)$.
2. $N^*_{\epsilon_1}(a) \subseteq N^*_{\epsilon_2}(a)$.

Corollary 4.3.1. *Let* $a \in \mathbb{R}$. *Let* $0 < \epsilon_1 \leq \epsilon_2$. *Then* $N_{\epsilon_1}(a) \subseteq N_{\epsilon_2}(a)$ *and* $N^*_{\epsilon_1}(a) \subseteq N^*_{\epsilon_2}(a)$

Corollary 4.3.2. *Let* $a \in \mathbb{R}$.
If $\epsilon_1 > 0$ *and* $\epsilon_2 > 0$ *then* $N_{\epsilon_1}(a) \cap N_{\epsilon_2}(a) = N_\epsilon(a)$ *where* $\epsilon = min\{\epsilon_1, \epsilon_2\}$.

Theorem 4.4. *Let* $a, b \in \mathbb{R}$ *and* $\epsilon > 0$.
Then if $a < b$ *then* $(a,b) = N_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$.

4.2 Interior, Exterior, And Boundary

Definition 4.5. *Let $A \subseteq \mathbb{R}$. The interior of A is given by $int(A) = \{x \in \mathbb{R} : \exists\, \epsilon > 0,\ \text{such that } N_\epsilon(x) \subseteq A\}$.*

Theorem 4.6. *Let $A \subseteq \mathbb{R}$. Then $int(A) \subseteq A$.*

Theorem 4.7. *Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Let $\epsilon > 0$, then $N_\epsilon(x) \subseteq A \iff N_\epsilon(x) \cap A^c = \emptyset$.*

Remark.

- $x \in int(A) \iff \exists\, \epsilon > 0$ *such that* $N_\epsilon(x) \subseteq A \iff \exists\, \epsilon > 0$ *such that* $N_\epsilon(x) \cap A^c = \emptyset$.
- $x \notin int(A) \iff \forall \epsilon > 0, N_\epsilon(x) \nsubseteq A \iff \forall \epsilon > 0, N_\epsilon(x) \cap A^c \neq \emptyset$.

Definition 4.8. *Let $A \subseteq \mathbb{R}$. The exterior of A is given by*

$$ext(A) = \{x \in \mathbb{R} : \exists\, \epsilon > 0 \text{ such that } N_\epsilon(x) \subseteq A^c\}.$$

Remark.

- $x \in ext(A) \iff \exists\, \epsilon > 0$ *such that* $N_\epsilon(x) \subseteq A^c \iff \exists\, \epsilon > 0$ *such that* $N_\epsilon(x) \cap A = \emptyset$.
- $x \notin ext(A) \iff \forall\, \epsilon > 0, N_\epsilon(x) \nsubseteq A^c \iff \forall\, \epsilon > 0, N_\epsilon(x) \cap A \neq \emptyset$.

Definition 4.9. *Let $A \subseteq \mathbb{R}$. The **boundary** of A is given by $\partial(A) = \{x \in \mathbb{R} : \forall \epsilon > 0, N_\epsilon(x) \cap A^c \neq \emptyset \text{ and } N_\epsilon(x) \cap A \neq \emptyset\}$*

Remark.

$$\begin{aligned}
x \in \partial(A) &\iff \forall \epsilon > 0, N_\epsilon(x) \cap A^c \neq \emptyset \,\wedge\, N_\epsilon(x) \cap A \neq \emptyset \\
&\iff \forall \epsilon > 0, N_\epsilon(x) \cap A^c \neq \emptyset \,\wedge\, \forall \epsilon > 0, N_\epsilon(x) \cap A \neq \emptyset \\
&\iff x \notin int(A) \,\wedge\, x \notin ext(A)
\end{aligned}$$

$$\begin{aligned}
x \notin \partial(A) &\iff \exists \epsilon > 0, N_\epsilon(x) \cap A^c = \emptyset \,\vee\, \exists \epsilon > 0, N_\epsilon(x) \cap A = \emptyset \\
&\iff x \in int(A) \,\vee\, x \in ext(A)
\end{aligned}$$

Theorem 4.10 (The first partition theorem)**.** *For any $A \subseteq \mathbb{R}$ we have*

(a) $int(A) \cup \partial(A) \cup ext(A) = \mathbb{R}$.

(b) (i) $int(A) \cap \partial(A) = \emptyset$.
  (ii) $ext(A) \cap \partial(A) = \emptyset$.
  (iii) $int(A) \cap ext(A) = \emptyset$.

4.3 Isolated, Accumulation, And Closure

Definition 4.11. *Let $A \subseteq \mathbb{R}$. The set of* ***isolated*** *(discrete) points of A is given by $A^{\circ} = \{x \in \mathbb{R} : \exists\, \epsilon > 0,\ \text{such that } N_{\epsilon}(x) \cap A = \{x\}\}$*

Theorem 4.12. *Let $A \subseteq \mathbb{R}$. Then $x \in A^{\circ} \iff x \in A \wedge \exists\epsilon > 0$ such that $N^{*}_{\epsilon}(x) \cap A = \emptyset$.*

Remark.

$$\begin{aligned} x \in A^{\circ} &\iff \exists\, \epsilon > 0,\ \text{such that } N_{\epsilon}(x) \cap A = \{x\} \\ &\iff x \in A \wedge \exists\, \epsilon > 0 \text{ such that } N^{*}_{\epsilon}(x) \cap A = \emptyset \end{aligned}$$

Definition 4.13. *Let $A \subseteq \mathbb{R}$. The set of* ***accumulation*** *point of A is given by $acc(A) = \{x \in \mathbb{R} : \forall\epsilon > 0, N^{*}_{\epsilon}(x) \cap A \neq \emptyset\}$.*

Remark.

- $x \in acc(A) \iff \forall\epsilon > 0, N^{*}_{\epsilon}(x) \cap A \neq \emptyset$
- $x \notin acc(A) \iff \exists\epsilon > 0, N^{*}_{\epsilon}(x) \cap A = \emptyset$

Remark.

Theorem 4.14 (The second partition theorem)**.** *For any $A \subseteq \mathbb{R}$ we have*

(a) $A^{\circ} \cup acc(A) \cup ext(A) = \mathbb{R}$.

(b) (i) $A^{\circ} \cap acc(A) = \emptyset$.

(ii) $A^{\circ} \cap ext(A) = \emptyset$.

(iii) $acc(A) \cap ext(A) = \emptyset$.

Definition 4.15. *Let $A \subset \mathbb{R}$. The* ***closure*** *of A is given by, $cl(A) = int(A) \cup \partial(A)$.*

Theorem 4.16. *Let $A \subseteq \mathbb{R}$. Then*

$$\begin{aligned} cl(A) &= int(A) \cup \partial(A) = (ext(A))^{c} \\ &= A \cup \partial(A) \\ &= A^{\circ} \cup acc(A) \\ &= A \cup acc(A) \end{aligned}$$

Theorem 4.17. $int(int(A)) = int(A)$.

Theorem 4.18. $int(ext(A)) = ext(A)$.

Theorem 4.19. $int(A) \subseteq ext(ext(A))$.

Theorem 4.20. *$ext(A) \subseteq ext(int(A))$.*

Theorem 4.21. *$\partial(\partial(A)) \subseteq \partial(A)$.*

Theorem 4.22. *$\partial(int(A)) \subseteq \partial(A)$.*

Theorem 4.23. *$\partial(A) = \partial(A^c)$.*

Theorem 4.24. *$\partial(ext(A)) \subseteq \partial(A)$.*

Theorem 4.25. *$int(A) \cup int(B) \subseteq int(A \cup B)$.*

Theorem 4.26. *$ext(A \cup B) \subseteq ext(A) \cap ext(B)$.*

Theorem 4.27. *$int(A \cap B) = int(A) \cap int(B)$.*

Theorem 4.28. *$ext(A \cup B) = ext(A) \cap ext(B)$.*

Theorem 4.29. *$\partial(A \cup B) \subseteq \partial(A) \cup \partial(B)$.*

Theorem 4.30. *If $A \subseteq B$ then $int(A) \subseteq int(B)$.*

Theorem 4.31. *If $A \subseteq B$ then $ext(B) \subseteq ext(A)$.*

Theorem 4.32. *$int(A) \subseteq acc(A)$.*

Theorem 4.33. *$A^\circ \subseteq \partial(A)$.*

Theorem 4.34. (a) *$x \in \partial(A) \wedge x \notin A \implies x \in acc(A)$.*

(b) *$x \in acc(A) \wedge x \notin A \implies x \in \partial(A)$.*

4.4 Open and Closed Sets

Definition 4.35. *Let $A \subseteq \mathbb{R}$. We say A is* ***open*** *$\iff \forall x \in A, \exists\, \epsilon > 0$ such that $N_\epsilon(x) \subseteq A \iff \forall x \in A, x \in int(A) \iff A \subseteq int(A)$.*
We say A is ***closed*** *when $\partial(A) \subseteq A$.*

Theorem 4.36. *Let $A \subseteq \mathbb{R}$. Then the following statements are equivalent:*

(a) *A is an open set.*

(b) *$A = int(A)$.*

(c) *$A \cap \partial(A) = \emptyset$.*

(d) *A^c is a closed set.*

Theorem 4.37. *Let $A \subseteq \mathbb{R}$. Then the following statements are equivalent:*

(a) *A is a closed set.*

(b) *$A = int(A) \cup \partial(A)$.*

(c) *$acc(A) \subseteq A$.*

(d) *A^c is an open set.*

Theorem 4.38 (Finite Sets)**.** (a) *Every point of a finite set A is an isolated point of A i.e., $A \subseteq A^\circ$.*

(b) *Every point of a finite set A is a boundary point of A i.e., $A \subseteq \partial(A)$.*

(c) *Finite sets have no interior points i.e., A is finite $\implies int(A) = \emptyset$.*

(d) *Finite sets have no accumulation points i.e., A is finite $\implies acc(A) = \emptyset$.*

(e) *A finite set is a closed set.*

Theorem 4.39. *A nonempty open set must be an infinite set.*

Theorem 4.40. *The union of any collection of open sets is open.*

Theorem 4.41. *The intersection of any finite number of open sets is open.*

Theorem 4.42. *The intersection of any collection of closed sets is closed.*

Theorem 4.43. *The union of any finite number of closed sets is closed.*

Theorem 4.44. *Let $(a, b) \subseteq \mathbb{R}$. Then (a, b) is an open set.*

Corollary 4.44.1. $\forall x \in \mathbb{R}$ *and* $\forall \epsilon > 0$

(a) *$N_\epsilon(x)$ is an open set.*

(b) *$N_\epsilon^*(x)$ is an open set.*

Theorem 4.45. (a) *$A = (-\infty, a)$ is an open set.*

(b) *$B = (a, \infty)$ is an open set.*

Theorem 4.46. *$[a, b]$ is a closed set.*

Theorem 4.47. *$(a, b]$ is neither open nor closed.*

Theorem 4.48. *Let A be a set of real numbers. Then,*

(a) $int(A) = \bigcup\{$*all open subsets of* $A\}$

(b) $int(A)$ *is the largest open subset of* A*, in the sense that if* B *is an open subset of* A *then* $B \subseteq int(A)$*.*

Theorem 4.49. *Let* $A \subseteq \mathbb{R}$*. I*

Theorem 4.50. $int(A)$ *is open.*

Theorem 4.51. $ext(A)$ *is open.*

Theorem 4.52. $cl(A)$ *is closed.*

Theorem 4.53. $\partial(A)$ *is closed.*

Theorem 4.54. $acc(A)$ *is closed.*

Theorem 4.55. (a) $\emptyset$ *and* $\mathbb{R}$ *are open.*

(b) $\emptyset$ *and* $\mathbb{R}$ *are closed.*

Theorem 4.56. *Let* $A \subseteq \mathbb{R}$*.*

(a) *If* A *has an infimum then* $inf(A) \in \partial(A)$*.*

(b) *If* A *has a supremum then* $sup(A) \in \partial(A)$*.*

Theorem 4.57. *Suppose* $A \subseteq \mathbb{R}$ *and* $x \in \mathbb{R}$*. Then* x *is an accumulation point of the set* $A \iff$ *every neighborhood of* x *contains infinitely many points of* A *i.e.,* $\forall \epsilon > 0, N_\epsilon(x) \cap A$ *is infinite.*

Theorem 4.58 (Sequential Criterion for Accumulation Points)**.**

(a) $x \in acc(A) \implies \exists$ *sequence* $\langle a_n \rangle$ *of points of* A *other than* x*, such that* $a_n \to x$*.*

(b) *Let* $\langle a_n \rangle$ *be a sequence of points of* A *other than* x *such that* $a_n \to x$*. Then* $x \in acc(A)$*.*

Theorem 4.59 (Sequential Criterion for Closed Sets)**.** *A set* A *is closed* $\iff \forall$ *convergent sequences* $\langle a_n \rangle$ *of points of* A*,* $lim_{n\to\infty} a_n \in A$*.*

4.5 Compact Sets

Definition 4.60. *A set A is said to be compact if whenever it is contained in the union of a family $\mathcal{F} = \{O_i : i \in I\}$ of open sets (O_i is an open set for all $i \in I$) then it is contained in the union of some finite number of the sets in $\mathcal{F}$.*
So A is compact if whenever $A \subseteq \bigcup_{i \in I} O_i$ where O_i is an open set for all $i \in I$ then $\exists J \subseteq I$ such that J is finite and $A \subseteq \bigcup_{i \in J} O_i$.
If $\mathcal{F} = \{O_i : i \in I\}$ is a family of open sets such that $A \subseteq \bigcup_{i \in I} O_i$ then $\mathcal{F}$ is called an open cover to A. Given an open cover $\mathcal{F} = \{O_i : i \in I\}$ of A, $\mathcal{G} = \{O_i : i \in J\}$ is called an open subcover of A if $J \subseteq I$ and $A \subseteq \bigcup_{i \in J} O_i$.
Thus A is compact if and only if every open cover contains a finite subcover.

Theorem 4.61. *Show that $A = (0, 2)$ is not compact.*

Theorem 4.62. *Any finite set is compact.*

Theorem 4.63. *If A is a nonempty closed bounded subset of $\mathbb{R}$ then A has a maximum and a minimum.*

Theorem 4.64. *Every compact set is closed and bounded.*

Theorem 4.65. *Every closed, bounded interval of real numbers is compact.*

Theorem 4.66. *A closed subset of a compact set is compact.*

Theorem 4.67 (Heine-Borel)**.** *Let $A \subseteq \mathbb{R}$. Then A is compact $\iff$ A is closed and bounded.*

Theorem 4.68 (Sequential Criterion for Compactness)**.** *Let $A \subseteq \mathbb{R}$. Then A is compact $\iff$ every sequence of points of A has a subsequence that converges to a point of A.*

4.6 The Cantor Set

Chapter 5

Sequences

Definition 5.1. *Let $k \in \mathbb{Z}$. Define $D_k = \{m \in \mathbb{Z} : m \geq k\}$. So $D_{-2} = \{-2, -1, 0, 1, 2, \ldots\}$.*
*A **sequence** is a function $a : D_k \to \mathbb{R}$. We normally denote the value $a(n)$ by a_n where a_n is called the nth term of the sequence.*
A sequence a is denoted as $\langle a_n \rangle_{n=1}^{\infty}$.

Definition 5.2. *A sequence $\langle a_n \rangle$ is said to **converge** to the real number L provided that*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N},\, n > N \implies |a_n - L| < \epsilon.$$

If $\langle a_n \rangle$ converges to L then L is called the limit of the sequence and we write $\lim_{n\to\infty} a_n = L$, $\lim a_n = L$ or $a_n \mapsto L$.
*If a sequence $\langle a_n \rangle$ does not converge, then we say $\langle a_n \rangle$ **diverges**. So a sequence $\langle a_n \rangle$ diverges provided*

$$\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N} \text{ such that } n > N \wedge |a_n - L| \geq \epsilon.$$

5.1 Algebra of Limits

Theorem 5.3. *Let $\langle a_n \rangle$ be a sequence and $L \in \mathbb{R}$. Then*

(a) $\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0.$

(b) $\lim_{n\to\infty} a_n = L \iff \lim_{n\to\infty} a_n - L = 0.$

(c) $\lim_{n\to\infty} a_n = L \iff \lim_{n\to\infty} |a_n - L| = 0.$

(d) $\lim_{n\to\infty} a_n = L \implies \lim_{n\to\infty} |a_n| = |L|$.

Theorem 5.4. $\lim_{n\to\infty} \frac{1}{n} = 0$.

Theorem 5.5. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Theorem 5.6. *Let $\langle a_n \rangle$ be a constant sequence i.e., $a_n = c$ for all $n \in \mathbb{N}$, where $c \in \mathbb{R}$, is a constant. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = c$.*

Theorem 5.7. *If $\lim_{n\to\infty} a_n = L, L > 0$ and $a_n \geq 0$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.*

Theorem 5.8. *Let $\langle s_n \rangle, \langle a_n \rangle$ be sequences and let $L \in \mathbb{R}$. If*

(i) $|s_n - L| \leq k\,|a_n|$ *for all $n \geq m$, where $k > 0$ and $m \in \mathbb{N}$.*

(ii) $\lim_{n\to\infty} a_n = 0$

then $\lim_{n\to\infty} s_n = L$.

Corollary 5.8.1. *Let $x \in \mathbb{R}$ be such that $|x| < 1$. Then $\lim_{n\to\infty} x^n = 0$.*

Corollary 5.8.2. *Suppose that $\lim_{n\to\infty} a_n = L$. let $r \in \mathbb{R}$ and $\langle u_n \rangle$ be such that $|u_n - r| \leq k|a_n - L|$ for all $n \geq m$ for some $k > 0$ and $m \in \mathbb{N}$. Then $\lim_{n\to\infty} u_n = r$.*

Corollary 5.8.3. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Theorem 5.9. *If $\lim_{n\to\infty} a_n = L, L \geq 0$ and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.*

Theorem 5.10 (Uniqueness of limit)**.** *A sequence cannot converge to more than one real number.*

Theorem 5.11 (Alternate definition of limit)**.** $\lim_{n\to\infty} a_n = L \iff \forall \epsilon > 0$, *all but finitely many terms of the sequence $\langle a_n \rangle$ are in the interval $(L-\epsilon, L+\epsilon)$.*

Theorem 5.12. *Let $D \subseteq \mathbb{R}$ be dense in $\mathbb{R}$. Let x be any real number. Then there is a sequence $\langle d_n \rangle$ that converges to x where $d_n \in D$ for all $n \in \mathbb{N}$.*

Definition 5.13 (Bounded Sequence)**.** *A sequence $\langle a_n \rangle$ is **bounded** if the set $S = \{a_n : n \in \mathbb{N}\}$ is bounded. So $\langle a_n \rangle$ is bounded*
$\iff \exists a, b \in \mathbb{R}$ *such that* $a \leq a_n \leq b$ *for all* $n \in \mathbb{N}$
$\iff \exists M > 0$ *such that* $|a_n| \leq M$ *for all* $n \in \mathbb{N}$.

Theorem 5.14. *Every convergent sequence is bounded*

Theorem 5.15 (Limit Algebra)**.** *Suppose that the sequence $\langle a_n \rangle$ and $\langle b_n \rangle$ converge to limits L and M respectively and $c \in \mathbb{R}$ is a constant. Then*

(a) $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L \pm M.$

(b) $\lim\limits_{n\to\infty} ca_n = c \cdot \lim\limits_{n\to\infty} a_n = cL.$

(c) $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = LM.$

(d) $\lim\limits_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{M}$ *provided $b_n \neq 0$ for all $n \in \mathbb{N}$ and $M \neq 0$.*

(e) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{M}$ *provided $b_n \neq 0$ for all $n \in \mathbb{N}$ and $M \neq 0$.*

Theorem 5.16 (Only the 'Tail' matters)**.** *Given a sequence $\langle s_n \rangle$ and $k \in \mathbb{N}$, let $\langle t_n \rangle$ be the sequence defined by $t_n = s_{n+k}$ (first k terms skipped). Show that $\langle t_n \rangle$ converges if and only if $\langle s_n \rangle$ converges and if they converge then $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n$.*

Theorem 5.17 (The Squeeze Theorem)**.** *Let $\langle s_n \rangle$ and $\langle t_n \rangle$ be convergent sequences such that $\lim\limits_{n\to\infty} s_n = L = \lim\limits_{n\to\infty} t_n$. If $\langle y_n \rangle$ is a sequence satisfying $s_n \leq y_n \leq t_n$ for all $n \geq m$ where $m \in \mathbb{N}$, then $\lim\limits_{n\to\infty} y_n = L$.*

Theorem 5.18. *For any fixed $c > 0$, $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$*

Theorem 5.19. (a) *If $\lim\limits_{n\to\infty} a_n = L$ and $L > 0$, then $a_n > 0$ for all sufficiently large n;*

(b) *If $\lim\limits_{n\to\infty} a_n = L$ and $L < 0$, then $a_n < 0$ for all sufficiently large n.*

Theorem 5.20. *If $\lim\limits_{n\to\infty} a_n = L$ where $a_n \geq 0$ for all $n \in \mathbb{N}$, then $L \geq 0$.*

Theorem 5.21. *If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ where $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $L \leq M$.*

Theorem 5.22. *Suppose that $\lim\limits_{n\to\infty} a_n = L$. Let a and b be real numbers.*

(a) *If $a_n \leq b$ for all $n \in \mathbb{N}$ then $L \leq b$.*

(b) *If $a \leq a_n$ for all $n \in \mathbb{N}$ then $a \leq L$.*

Corollary 5.22.1. *Let $\langle a_n \rangle$ be a sequence whose terms are all in $[a, b]$. If $\lim_{n\to\infty} a_n = L$, then $L \in [a, b]$.*

Corollary 5.22.2. *Let $\langle a_n \rangle$ be a sequence such that $\lim_{n\to\infty} a_n = L$ and let $M \geq 0$. If $|a_n| \leq M$ for all $n \in \mathbb{N}$ then $|L| \leq M$.*

5.2 Monotone Sequences

Definition 5.23. *A sequence $\langle a_n \rangle$ is said to be*

(a) ***monotone increasing*** *if $\forall n \in \mathbb{N}, a_n \leq a_{n+1}$; that is*

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \cdots$$

(b) ***monotone decreasing*** *if $\forall n \in \mathbb{N}, a_n \geq a_{n+1}$; that is*

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \cdots$$

(c) ***strictly increasing*** *if $\forall n \in \mathbb{N}, a_n < a_{n+1}$; that is*

$$a_1 < a_2 < \cdots < a_n < a_{n+1} \cdots$$

(d) ***strictly decreasing*** *if $\forall n \in \mathbb{N}, a_n > a_{n+1}$; that is*

$$a_1 > a_2 > \cdots > a_n > a_{n+1} \cdots$$

(e) ***monotone*** *if it is monotone increasing or monotone decreasing.*

Theorem 5.24 (Monotone Convergence Theorem)**.** *Every bounded monotone sequence converges. More precisely,*

(a) *if $\langle a_n \rangle$ is a monotone increasing sequence that is bounded above, then $\lim_{n\to\infty} a_n = sup\{a_n : n \in \mathbb{N}\}$;*

(b) *if $\langle a_n \rangle$ is a monotone decreasing sequence that is bounded below, then $\lim_{n\to\infty} a_n = inf\{a_n : n \in \mathbb{N}\}$;*

5.3 Subsequences

Definition 5.25. *Suppose $\langle a_n \rangle$ is a sequence. if $\langle n_k \rangle$ is a strictly increasing sequence of natural numbers (i.e., $n_1 < n_2 < \cdots < n_k < \cdots$) then the sequence $\langle a_{n_k} \rangle$ is said to be the* ***subsequence*** *of $\langle a_n \rangle$.*

Lemma 5.26. *If $\langle n_k \rangle$ is a strictly increasing sequence of natural numbers, then $\forall k \in \mathbb{N}, n_k \geq k$.*

Theorem 5.27. *A sequence $\langle a_n \rangle$ converges to a real number $L \iff$ every subsequence of $\langle a_n \rangle$ converges to L.*

Theorem 5.28. *Every sequence has a monotone subsequence.*

Theorem 5.29 (Bolzano-Weierstrass Theorem for Sequences)**.** *Every bounded sequence has a convergent subsequence.*

Definition 5.30. *A real number L is a* ***cluster point*** *of a sequence $\langle a_n \rangle$ if every neighborhood of L contains infinitely many term of the sequence $\langle a_n \rangle$ i.e., $\forall \epsilon > 0, a_n \in (L - \epsilon, L + \epsilon)$ for infinitely many values of $n \iff \forall \epsilon > 0, \{n \in \mathbb{N} : |a_n - L| < \epsilon\}$ is infinite.*

Theorem 5.31. *Let $\langle a_n \rangle$ be a sequence. If $a_n \to L$ then L is a cluster point of $\langle a_n \rangle$.*

Theorem 5.32. *Let $\langle a_n \rangle$ be a sequence and let $L \in \mathbb{R}$. Then the following statements are equivalent;*

(a) *L is a cluster point of $\langle a_n \rangle$;*

(b) *$\forall \epsilon > 0, \forall m \in \mathbb{N}, \exists n > m$ such that $|a_n - L| < \epsilon$;*

(c) *$\exists \langle a_{n_k} \rangle$ a subsequence of $\langle a_n \rangle$ converging to L.*

Theorem 5.33. *Let $\langle a_n \rangle$ be a sequence such that $\lim\limits_{n \to \infty} a_n = L$. Then L is the only cluster point of $\langle a_n \rangle$.*

Theorem 5.34. *Let $\langle a_n \rangle$ be a bounded sequence. Then $\langle a_n \rangle$ has atleast one cluster point.*

Theorem 5.35. *Let $\langle a_n \rangle$ be a bounded sequence such that it has one and only one cluster point L. Then $\langle a_n \rangle$ converges to L.*

5.4 Cauchy Sequences

Definition 5.36. *A sequence $\langle a_n \rangle$ is a* ***Cauchy sequence*** *if it satisfies the following criteria;*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n \in \mathbb{N}, m, n > N \implies |a_m - a_n| < \epsilon.$$

Theorem 5.37. *Every convergent sequence is a Cauchy sequence.*

Theorem 5.38. *Every Cauchy sequence is bounded.*

Theorem 5.39 (Cauchy Convergence Criterion)**.** *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Theorem 5.40. *If some subsequence of a Cauchy sequence converges to a real number L, then the sequence itself also converges to L.*

Chapter 6

Limits of Functions

Definition 6.1. *If $f : D_f \to \mathbb{R}$ and a is an accumulation point of D_f then*

$$\lim_{x \to a} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in D_f, 0 < |x-a| < \delta \implies |f(x)-L| < \epsilon.$$

Negation of $\lim_{x\to a} f(x) = L$: *f does not have limit L at a* $\iff$

$\exists \epsilon > 0$ *such that* $\forall \delta > 0, \exists x \in D_f$ *such that* $0 < |x-a| < \delta \wedge |f(x) - L| \geq \epsilon$.

Remark. *We require $a \in acc(D_f)$ since we want to talk about the value of $f(x)$ as the value of x gets closer and closer to a.*

Theorem 6.2 (Uniqueness of Limits)**.** *A function cannot have more than one limit as $x \to a$.*

Theorem 6.3 (Sequential Criterion for Limits of Functions)**.**

$$\lim_{x \to a} f(x) = L \iff \forall \text{ sequences } \langle x_n \rangle \text{ in } D_f - \{a\}, \text{ if } x_n \to a \text{ then } f(x_n) \to L.$$

Negation : *f does not have limit L at a if and only if $\exists$ sequence $\langle x_n \rangle$ in $D_f - \{a\}$ such that $x_n \to a$ but the sequence $\langle f(x_n) \rangle$ does not converge to L.*

Theorem 6.4. *If $\exists$ sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in $D_f - \{a\}$ which both converge to a, but the sequence $\langle f(x_n) \rangle$ and $\langle f(y_n) \rangle$ do not both converge to the same number, then $\lim_{x\to a} f(x)$ does not exist.*

Example 6.5. *Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.*

Theorem 6.6 (Absolute Value and Limits)**.**

(a) $\lim_{x\to a} f(x) = 0 \iff \lim_{x\to a} |f(x)| = 0$.

(b) $\lim_{x\to a} f(x) = L \iff \lim_{x\to a} |f(x) - L| = 0$.

(c) $\lim_{x\to a} f(x) = L \implies \lim_{x\to a} |f(x)| = |L|$.

Theorem 6.7. *Let $f : D_f \to \mathbb{R}$ and a is an accumulation point of D_f. Then $\lim_{x\to a} f(x) = L \iff$ for each neighborhood V of L there exists a deleted neighborhood U of a such that $f(U \cap D_f) \subseteq V$ (or $U \cap D_f \subseteq f^{-1}(V)$).*

Theorem 6.8. *Let $f : D_f \to \mathbb{R}$ and $a \in acc(D_f)$. If f is constant, say $f(x) = c$, on some deleted neighborhood of a, then $\lim_{x\to a} f(x) = c$.*

Theorem 6.9. *If $\lim_{x\to a} f(x) = L \in \mathbb{R}$, then there is some deleted neighborhood U of a such that f is bounded on $U \cap D_f$.*

Theorem 6.10 (Fundamental Limit)**.** *For every $a \in \mathbb{R}, \lim_{x\to a} x = a$*

Theorem 6.11. *Suppose $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = M,$ and $a \in \mathbb{R}$. Then*

(a) $\displaystyle\lim_{x\to a} cf(x) = c\,L$.

(b) $\displaystyle\lim_{x\to a} (f(x) \pm g(x)) = L \pm M$.

(c) $\displaystyle\lim_{x\to a} (f(x) \cdot g(x)) = LM$.

(d) $\displaystyle\lim_{x\to a} \left(\frac{1}{g(x)}\right) = \frac{1}{M} \qquad$ (*if* $M \neq 0$).

(e) $\displaystyle\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M} \qquad$ (*if* $M \neq 0$).

In (b), (c), and (e) we assume that a is an accumulation point of $D_f \cap D_g$.

Theorem 6.12. *If $\lim_{x\to a} f(x) = L$ and $f(x) \geq 0$ for all x in some deleted neighborhood of a, then $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$.*

Definition 6.13. *A* ***polynomial*** *(in one variable) is a function of the form*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \cdots, a_n$ are (constant) real numbers.

Theorem 6.14 (Limits of polynomials)**.** *For any polynomial $p(x)$ and any $a \in \mathbb{R}$*

$$\lim_{x \to a} p(x) = p(a).$$

Definition 6.15. *A **rational function** (of one variable) is any function of the form*

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

Theorem 6.16 (Limits of Rational Functions)**.** *For any rational function $r(x) = \dfrac{p(x)}{q(x)}$ and any $a \in \mathbb{R}, \lim_{x \to a} r(x) = r(a)$ provided that $q(a) \neq 0$.*

Theorem 6.17 (Only What Happens in a Deleted Neighborhood of c Matters)**.** *Suppose $\lim_{x \to c} f(x) = L$, and $f(x) = g(x)$ for all x in some deleted neighborhood of c. Then $lim_{x \to c} g(x) = L$.*

Theorem 6.18 (The *Squeeze* Principle for Functions)**.**

(a) ***The First Squeeze Principle:*** *Suppose $f(x) \leq g(x) \leq h(x)$ for all x in some deleted neighborhood of c and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$. Then $\lim_{x \to c} g(x) = L$.*

(b) ***The Second Squeeze Principle:*** *Suppose $\lim_{x \to c} g(x) = 0$. If $|f(x) - L| \leq |g(x)|$, for all x in some deleted neighborhood of c, then $\lim_{x \to c} f(x) = L$.*

Example 6.19. *Use the squeeze principle to prove that*

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Theorem 6.20 (Limits Preserve Inequalities)**.**

(a) *If $\lim_{x \to a} f(x) = L$ and $f(x) \leq K$ for all x in some deleted neighborhood of a, then $L \leq K$.*

(b) *If $\lim_{x \to a} f(x) = L$ and $f(x) \geq K$ for all x in some deleted neighborhood of a, then $L \geq K$*

(c) *Let $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$ and $a \in acc(D_f \cap D_g)$. If $\lim_{x \to a} f(x)$ and $\lim_{x \to c} g(x)$ exist, and $f(x) \leq g(x)$ for all x in some deleted neighborhood of a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.*

Theorem 6.21 (Change of Variables in Limits)**.** *Suppose* $\lim_{h\to c} g(h) = a$ *and* $\lim_{x\to a} f(x) = L$ *where* c *and* a *are accumulation points of* D_g *and* D_f *respectively and* $g(h) \in D_f - \{a\}$ *for all* $h \in D_g$ *in some deleted neighborhood of* c*. Then*

$$\lim_{h\to c} f(g(h)) = \lim_{x\to a} f(x) = L$$

.

Theorem 6.22. *Suppose* $f : D_f \to \mathbb{R}$*. Then*

$$\lim_{x\to a} f(x) = L \iff \lim_{h\to 0} f(a+h) = L$$

6.1 One-Sided Limits

Definition 6.23 (Limit from the Left)**.** *Suppose* $f : D_f \to \mathbb{R}$ *and* $a \in acc(D_f \cap (-\infty, a))$*. Then we say* f *has limit* L *as* x *approaches* a *from the left, written* $\lim_{x\to a^-} f(x) = L$*, if*

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that, } \forall x \in D_f, a - \delta < x < a \implies |f(x) - L| < \epsilon$$

.

Example 6.24. *Prove that*

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = -1$$

Definition 6.25 (Limit from the Right)**.** *Suppose* $f : D_f \to \mathbb{R}$ *and* $a \in acc(D_f \cap (a, +\infty))$*. Then we say* f *has limit* L *as* x *approaches* a *from the right, written* $\lim_{x\to a^+} f(x) = L$*, if*

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that, } \forall x \in D_f, a < x < a + \delta \implies |f(x) - L| < \epsilon$$

.

Example 6.26. *Prove that*

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = 1$$

Theorem 6.27. *Suppose* $f : D_f \to \mathbb{R}$*.*

(a) *If* $\lim_{x\to a^-} f(x) = L$ *then* $\exists \delta > 0$ *such that* f *is bounded on* $(a-\delta, a) \cap D_f$*.*

(b) *If* $\lim_{x\to a^+} f(x) = L$ *then* $\exists\delta > 0$ *such that* f *is bounded on* $(a, a+\delta) \cap D_f$.

Theorem 6.28 (Limits from the Left Preserve Inequalities). (a) *If* $\lim_{x\to a^-} f(x) = L$ *and* $\exists\delta_1 > 0$ *such that* $f(x) \leq K$ *for all* $x \in (a-\delta_1, a) \cap D_f$, *then* $L \leq K$.

(b) *If* $\lim_{x\to a^-} f(x) = L$ *and* $\exists\delta_1 > 0$ *such that* $f(x) \geq K$ *for all* $x \in (a-\delta_1, a) \cap D_f$, *then* $L \geq K$.

(c) *Let* $f : D_f \to \mathbb{R}$ *and* $g : D_g \to \mathbb{R}$ *and suppose* $a \in acc(D_f \cap D_g \cap (-\infty, a))$. *If* $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^-} g(x)$ *exist and* $\exists\delta > 0 \ni \forall x \in (a-\delta, a) \cap (D_f \cap D_g), f(x) \leq g(x)$ *then* $\lim_{x\to a^-} f(x) \leq \lim_{x\to a^-} g(x)$.

Theorem 6.29. *If* a *is an accumulation point of* $D_f \cap (-\infty, a)$ *and* a *is an accumulation point of* $D_f \cap (a, +\infty)$ *then* $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$.

Example 6.30.

$$\textit{Prove that } \lim_{x\to 2} \frac{|x-2|}{x-2} \textit{ does not exist.}$$

6.2 Infinity in Limits

6.2.1 Infinity as a Limit

Definition 6.31. *Suppose* $f : D_f \to \mathbb{R}$ *and* $a \in acc(D_f)$. *Then*

(a) $\lim_{x\to a} f(x) = +\infty$ *if*

$$\forall M > 0, \exists\delta > 0 \ni \forall x \in D_f, 0 < |x-a| < \delta \implies f(x) > M.$$

(b) $\lim_{x\to a} f(x) = -\infty$ *if*

$$\forall M > 0, \exists\delta > 0 \ni \forall x \in D_f, 0 < |x-a| < \delta \implies f(x) < -M.$$

Theorem 6.32. *Suppose* $f : D_f \to \mathbb{R}$. *Then* $\lim\limits_{x\to a} f(x) = +\infty \iff f(x) > 0$ *for all* x *in some deleted neighborhood of* a *and* $\lim\limits_{x\to a} \dfrac{1}{f(x)} = 0$

Example 6.33. *Prove that* $\lim\limits_{x\to 2} \dfrac{3x-5}{(x-2)^2} = +\infty$.

Theorem 6.34. *Suppose* $\lim_{x\to a} f(x) = +\infty, \lim_{x\to a} g(x) = +\infty, \lim_{x\to a} h(x) = -\infty$ *and* $\lim_{x\to a} k(x) = -\infty$*. Then*

(a) $\lim_{x\to a}(f(x) + g(x)) = +\infty$*;*

(b) $\lim_{x\to a}(f(x)g(x)) = +\infty$*;*

(c) $\lim_{x\to a}(h(x) + k(x)) = -\infty$*;*

(d) $\lim_{x\to a}(h(x)k(x)) = +\infty$*;*

(e) $\lim_{x\to a}(f(x)h(x)) = -\infty$*.*

In (a) and (b), $a \in acc(D_f \cap D_g)$*. In (c) and (d),* $a \in acc(D_h \cap D_k)$*. In (e)* $a \in acc(D_f \cap D_h)$*.*

Theorem 6.35 (Comparison Test)**.** *Suppose that* $f(x) \leq g(x)$ *for all* x *in some deleted neighborhood of* a*.*

(a) *If* $\lim_{x\to a} f(x) = +\infty$*, then* $\lim_{x\to a} g(x) = +\infty$*;*

(b) *If* $\lim_{x\to a} g(x) = -\infty$*, then* $\lim_{x\to a} f(x) = -\infty$

6.2.2 Limit at Infinity

Definition 6.36. $\lim\limits_{x\to+\infty} f(x) = L \iff D_f$ *is unbounded above, and*

$$\forall \epsilon > 0, \exists N > 0 \ni \forall x \in D_f, x > N \implies |f(x) - L| < \epsilon.$$

Definition 6.37. $\lim\limits_{x\to-\infty} f(x) = L \iff D_f$ *is unbounded below, and*

$$\forall \epsilon > 0, \exists N > 0 \ni \forall x \in D_f, x < -N \implies |f(x) - L| < \epsilon.$$

Definition 6.38. $\lim\limits_{x\to+\infty} f(x) = +\infty \iff D_f$ *is unbounded above, and*

$$\forall M > 0, \exists N > 0 \ni \forall x \in D_f, x > N \implies f(x) > M.$$

Definition 6.39. $\lim\limits_{x\to+\infty} f(x) = -\infty \iff D_f$ *is unbounded above, and*

$$\forall M > 0, \exists N > 0 \ni \forall x \in D_f, x > N \implies f(x) < -M.$$

Definition 6.40. $\lim\limits_{x\to-\infty} f(x) = +\infty \iff D_f$ *is unbounded below, and*

$$\forall M > 0, \exists N > 0 \ni \forall x \in D_f, x < -N \implies f(x) > M.$$

Definition 6.41. $\lim_{x\to-\infty} f(x) = -\infty \iff D_f$ *is unbounded below, and*

$$\forall M > 0, \exists N > 0 \ni \forall x \in D_f, x < -N \implies f(x) < -M.$$

Example 6.42. *Prove that* $lim_{x\to+\infty}(5-4x) = -\infty$.

Theorem 6.43. (a) $\forall n \in \mathbb{N}, \lim_{x\to+\infty} x^n = +\infty$;

(b) $\forall n \in \mathbb{N}$, *if n is even, then* $\lim_{x\to-\infty} x^n = +\infty$;

(c) $\forall n \in \mathbb{N}$, *if n is odd, then* $\lim_{x\to-\infty} x^n = -\infty$.

Theorem 6.44. (a) *Let* $f : D_f \to \mathbb{R}$ *and suppose* $(0, +\infty) \subseteq D_f$. *Then*

$$\lim_{x\to0^+} f(x) = L \iff \lim_{x\to+\infty} f\left(\frac{1}{x}\right) = L;$$

(b) *Let* $f : D_f \to \mathbb{R}$ *and suppose* $(-\infty, 0) \subseteq D_f$. *Then* $\lim_{x\to0^-} f(x) = L \iff \lim_{x\to-\infty} f\left(\frac{1}{x}\right) = L$;

Theorem 6.45. *Let* $f : D_f \to \mathbb{R}$ *and a be a real number.*

(a) *Suppose* $(a, +\infty) \subseteq D_f$ *and* $\forall x > a \implies f(x) > 0$. *Then* $\lim_{x\to+\infty} f(x) = +\infty \iff \lim_{x\to+\infty} \frac{1}{f(x)} = 0$.

(b) *Suppose* $(a, +\infty) \subseteq D_f$ *and* $\forall x > a \implies f(x) < 0$. *Then* $\lim_{x\to+\infty} f(x) = -\infty \iff \lim_{x\to+\infty} \frac{1}{f(x)} = 0$.

(c) *Suppose* $(-\infty, a) \subseteq D_f$ *and* $\forall x < a \implies f(x) > 0$. *Then* $\lim_{x\to-\infty} f(x) = +\infty \iff \lim_{x\to-\infty} \frac{1}{f(x)} = 0$.

(d) *Suppose* $(-\infty, a) \subseteq D_f$ *and* $\forall x < a \implies f(x) < 0$. *Then* $\lim_{x\to-\infty} f(x) = -\infty \iff \lim_{x\to-\infty} \frac{1}{f(x)} = 0$.

Chapter 7

Continuous Functions

7.1 Continuity of a Function at a Point

Definition 7.1. *Suppose $f : D_f \to \mathbb{R}$ and $a \in D_f$. Then f is **continuous at** a if*

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in D_f, |x - a| \implies |f(x) - f(a)| < \epsilon.$$

Example 7.2. *Prove that the function $f(x) = 3x^2 - 2x - 1$ is continuous at $a = 2$.*

Theorem 7.3. *Suppose $f : D_f \to \mathbb{R}$ and $a \in acc(D_f)$. Then f is continuous at $a \iff \lim_{x\to a} f(x) = f(a)$.*

Theorem 7.4 (Sequential Criterion for Continuity of f at a)**.** *A function $f : D_f \to \mathbb{R}$ is continuous at a point $a \in D_f \iff \forall$ sequences $\langle x_n \rangle$ in D_f, if $x_n \to a$ then $f(x_n) \to f(a)$.*
Negation : *A function $f : D_f \to \mathbb{R}$ is discontinuous at a point $a \in D_f$ if and only if $\exists$ sequence $\langle x_n \rangle$ in D_f such that $x_n \to a$ but the sequence $\langle f(x_n) \rangle$ does not converge to $f(a)$.*

Example 7.5. *The **signum function, sgn**$(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is discontinuous at $a = 0$.*

Theorem 7.6. *Polynomial functions are continuous everywhere.*

Theorem 7.7. *A rational function $r(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, is continuous everywhere on its domain i.e., at every real number x for which $q(x) \neq 0$.*

Example 7.8. (a) *The absolute value function $f(x) = |x|$ is continuous everywhere.*

(b) *The square root function $f(x) = \sqrt{x}$ is continuous everywhere on its domain $[0, +\infty)$.*

Example 7.9 (A function That is Continuous Nowhere). *The **Dirchlet function** $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ is discontinuous everywhere.*

Example 7.10. *The **Thomae's function** $T(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \neq 0, \text{ where } m \in \mathbb{Z}, n \in \mathbb{N}, \text{ and hav} \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$*

Theorem 7.11 (Algebra of Continuous Function). *Suppose f and g are continuous at a point a and let $c \in \mathbb{R}$. Then,*

(a) *cf is continuous at a;*

(b) *$f \pm g$ is continuous at a;*

(c) *$f \cdot g$ is continuous at a;*

(d) *$\dfrac{1}{g}$ is continuous at a, if $g(a) \neq 0$.*

(e) *$\dfrac{f}{g}$ is continuous at a, if $g(a) \neq 0$.*

Theorem 7.12 (Composite Functions). (a) *Suppose f is continuous at a and g is continuous at $f(a)$. Then the composite function $g \circ f$ is continuous at a.*

(b) *Suppose $lim_{x \to a} f(x) = b \in D_g$ and g is continuous at b. Then*

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right) = g(b)$$

.

7.2 Monotonic Functions

Definition 7.13. *A function f is*

(a) ***monotone increasing*** *on a set* $A \subseteq D_f$ *if* $\forall x_1, x_2$ *in* A*,*

$$x_1 < x_2 \implies f(x_1) \leq f(x_2);$$

(b) ***monotone decreasing*** *on a set* $A \subseteq D_f$ *if* $\forall x_1, x_2$ *in* A*,*

$$x_1 < x_2 \implies f(x_1) \geq f(x_2);$$

(c) ***strictly increasing*** *on a set* $A \subseteq D_f$ *if* $\forall x_1, x_2$ *in* A*,*

$$x_1 < x_2 \implies f(x_1) < f(x_2);$$

(d) ***strictly decreasing*** *on a set* $A \subseteq D_f$ *if* $\forall x_1, x_2$ *in* A*,*

$$x_1 < x_2 \implies f(x_1) > f(x_2);$$

(e) ***monotone*** *on a set* $A \subseteq D_f$ *if it is monotone increasing or monotone decreasing.*

7.3 Continuity on Compact Sets and Intervals

Theorem 7.14 (Continuous functions *preserve* compactness)**.** *If* A *is a compact set and* $f : A \to \mathbb{R}$ *is continuous, then* $f(A)$ *is compact.*

Corollary 7.14.1 (Extreme Value Theorem)**.** *If* A *is a nonempty compact set and* $f : A \to \mathbb{R}$ *is continuous then* f *has the extreme value property on* A*:*

(a) $\exists u = min\, f(A) = min\{f(x) : x \in A\}$*, and*

(b) $\exists v = max\, f(A) = max\{f(x) : x \in A\}$.

That is, a continuous function assumes a maximum and a minimum value on any nonempty compact set.

Theorem 7.15 (Continuous functions *preserve* intervals)**.** *Suppose* I *is an interval and* $f : I \to \mathbb{R}$ *is continuous. Then* $f(I)$ *is an interval.*

Corollary 7.15.1 (Intermediate Value Theorem)**.** *Suppose* $a < b$*. Any continuous* $f : [a, b] \to \mathbb{R}$ *must satisfy the* ***intermediate value property*** *on* $[a, b]$*:*

$$\forall y \text{ between } f(a) \text{ and } f(b), \exists c \in [a, b] \ni f(c) = y.$$

7.4 Uniform Continuity

Definition 7.16. *A function $f : D_f \to \mathbb{R}$ is* ***continuous*** *on a set $A \subseteq D_f$ if*

$$\forall a \in A, \forall \epsilon > 0, \exists \delta > 0 \ni \forall x \in D_f, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

Definition 7.17. *A function $f : D_f \to \mathbb{R}$ is* ***uniformly continuous*** *on a set $A \subseteq D_f$ if*

$$\forall \epsilon > 0, \exists \delta > 0 \ni \forall x, y \in A, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Example 7.18. *Prove that the function $f(x) = 2x$ is uniformly continuous on $\mathbb{R}$.*

Example 7.19. *Prove that the function $f(x) = 3x^2 - 2x - 1$ is uniformly continuous on the interval $[-1, 5]$.*

Theorem 7.20. *If $f : D_f \to \mathbb{R}$ is uniformly continuous on the set $A \subseteq D_f$, then $f|_A$ is continuous on A.*

Corollary 7.20.1. *If $f : D \to \mathbb{R}$ is uniformly continuous on D, then f is continuous on D.*

Example 7.21. *The converse of the previous theorem is not true. The function $f(x) = \dfrac{1}{x}$ is continuous on $(0, 1)$ but is not uniformly continuous there.*

Theorem 7.22. *If f is uniformly continuous on a bounded set A then f is bounded on A.*

Theorem 7.23. *If $f : A \to \mathbb{R}$ is continuous on a compact set A, then f is uniformly continuous on A.*