Homework 5

Computational Math

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a For the test functions, choose u, v from the space of continuous functions supported on Ω; ie supp $u \subseteq \Omega$. Now for any solution u with test function v we must have:

$$\int_{\Omega} -u_{xx}(x,y)v(x,y) - u_{yy}(x,y)v(x,y)d\Omega = \int_{\Omega} f(x,y)v(x,y)d\Omega$$

Starting on the left hand side, we work term by term:

$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} -u_{xx}(x,y)v(x,y)dxdy &= \int_{-1}^{1}\left(-u_x(x,y)v(x,y)|_{-1}^{1} + \int_{-1}^{1} u_x(x,y)v_x(x,y)dx\right)dy \\
&= \int_{\Omega} u_x(x,y)v_x(x,y)d\Omega \\
&= p_1(u,v)
\end{aligned}$$

By symmetry:

$$p_2(u,v) = \int_{\Omega} u_{yy}v d\Omega = \int_{\Omega} u_y(x,y)v_y(x,y)d\Omega$$

For the right hand side, we just get:

$$b(v) = \int_{\Omega} f(x,y)v(x,y)d\Omega$$

And so the weak form of the variational problem is:

$$p_1(u,v) + p_2(u,v) = b(v)$$

b Let $(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$ be the nodes of the element, oriented clockwise. We now solve for $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ for the node (x_1,y_1). Plugging in values, we get the following linear system:

$$\begin{aligned}
\alpha_1 + \alpha_2x_1 + \alpha_3y_1 + \alpha_4x_1y_1 &= 1 \\
\alpha_1 + \alpha_2x_2 + \alpha_3y_2 + \alpha_4x_1y_2 &= 0 \\
\alpha_1 + \alpha_2x_3 + \alpha_3y_3 + \alpha_4x_1y_3 &= 0 \\
\alpha_1 + \alpha_2x_4 + \alpha_3y_4 + \alpha_4x_4y_4 &= 0
\end{aligned}$$

For the sake of simplicity, we rewrite the system in matrix form:

$$M\alpha = c$$

Where α is the vector of coefficients. Since c is a basis vector, the values for α at various nodes are just the corresponding rows of M^{-1}.

Now to construct the matrix equations for this system, we first consider the weak form from part a on a per element basis. Thus let φ^i, φ^j be two test functions on a quad element where

$$\varphi^i(x) = \alpha_1^i + \alpha_2^i x + \alpha_3^i y + \alpha_4^i xy$$

And:

$$\varphi_x^i(x) = \alpha_2^i + \alpha_4^i y$$

To integrate $p_1(\varphi^i, \varphi^j)$, we split the integral into two triangles, indexed by $\Delta(1,2,3)$ and $\Delta(1,3,4)$, then integrate in barycentric coordinates. We do this for the first triangle $\Delta(1,2,3)$ now. Let:

$$J = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{pmatrix}$$

And define the affine transformation:

$$\mathcal{T}(\lambda_1, \lambda_2) = J \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

And so we get the following:

$$\begin{aligned} \int\limits_{\Delta(1,2,3)} \varphi_x^i(x,y)\varphi_x^j(x,y)dxdy &= \frac{1}{\det J}\int\limits_0^1 \int\limits_0^{1-\lambda_2} \varphi_x^i(\mathcal{T}(\lambda_1,\lambda_2))\varphi_x^j(\mathcal{T}(\lambda_1,\lambda_2)))d\lambda_1 d\lambda_2 \\ &= \frac{1}{\det J}\int\limits_0^1 \int\limits_0^{1-\lambda_2} \alpha_2^i\alpha_2^j + (\alpha_4^i\alpha_2^j + \alpha_2^i\alpha_4^j)(J_{2,1}\lambda_1 + J_{2,2}\lambda_2 + y_1) \\ &\quad + \alpha_4^i\alpha_4^j(J_{2,1}\lambda_1 + J_{2,2}\lambda_2 + y_1)^2 d\lambda_1 d\lambda_2 \end{aligned}$$

To simplify the expression, make the following substitutions:

$$\begin{aligned} Q_0 &= \alpha_2^i\alpha_2^j \\ Q_1 &= \alpha_2^i\alpha_4^j + \alpha_4^i\alpha_2^j \\ Q_2 &= \alpha_4^i\alpha_4^j \end{aligned}$$

And so we get the following quantity:

$$= \frac{1}{2\det J}\left(Q_0 + y_1\left(Q_1 + y_1 Q_2\right) + \frac{J_{2,1}+J_{2,2}}{3}\left(Q_1 + \left(2y_1 + \frac{J_{2,1}+J_{2,2}}{2}\right)Q_2\right) - \frac{J_{2,1}J_{2,2}Q_2}{6}\right)$$

We shall call this quantity T_1^1, where the upper index denotes the triangle and the lower index denotes the p_1 component of the Laplacian, thus we get:

$$A(\varphi^i, \varphi^j) = p_1(\varphi^i, \varphi^j) + p_2(\varphi^i, \varphi^j) = \sum T_1^1 + T_2^1 + T_1^2 + T_2^2$$

And so the final matrix is just formed by summing over all such values. Computing f can be done approximately by sampling at the nodal values.

c Here is the code I wrote to implement the described method (in Python):

```
from numpy import *
from scipy import *
from scipy.linalg import *
from scipy.sparse import *
from scipy.linsolve import *
from pylab import *

#A quadrilateral element
class QuadElement:
    def __init__(self, ni, nx, ny):
        self.ni = ni
        self.nx = [nx[k] for k in ni]
        self.ny = [ny[k] for k in ni]
        M = matrix([ [ 1, nx[k], ny[k], nx[k] * ny[k]] for k in ni ])
        self.alpha = inv(transpose(M))

    def laplacian(self):
        res = []
        for i in range(len(self.ni)):
            for j in range(len(self.ni)):
                ali = array(self.alpha[i,1:3]).flatten()
                alj = array(self.alpha[j,1:3]).flatten()
                ahi = self.alpha[i,3]
                ahj = self.alpha[j,3]
                Q0 = ali * alj
                Q1 = ali * ahj + alj * ahi
                Q2 = ahi * ahj
                S = 0.
                for k in range(2,4):
                    J = matrix([ [self.nx[p] - self.nx[0], self.ny[p] - self.ny[0]] for p in
                        range(k-1,k+1) ])
                    X = array([J[1,0] + J[1,1], J[0,0] + J[0,1]])
                    Y = array([self.ny[0], self.nx[0]])
                    T = Q0 + Y * (Q1 + Y * Q2) + X / 3. * (Q1 + (2 * Y + X / 2.) * Q2) - Q2 *
                        array([J[1,0]*J[1,1], J[0,0]*J[0,1]]) / 12.
                    S -= sum(T) / (2. * det(J))
                res.append(((self.ni[i], self.ni[j]), S))
        return res

#Solves the finite element problem for the given mesh (which is just a list of elements)
def fe_solve(mesh, nx, ny, f, boundary, bvals):
    M = len(nx)
    A = dok_matrix((M, M))
    b = zeros((M))
    for e in mesh:
        for ((i,j), v) in e.laplacian():
            if(boundary[i]):
                continue
            A[i,j] += v
    for i in range(M):
        if(boundary[i]):
            b[i] = bvals[i]
            A[i,i] = 1
        else:
            b[i] = f(nx[i], ny[i])
    return spsolve(A, b)

#Generates a rectangular regular grid mesh over the grid G
def gen_regular_quad_mesh(grid):
    D, R, C = grid.shape
    M = R * C
    def get_index(ix, iy):
        if(ix < 0 or ix >= R or \
           iy < 0 or iy >= C):
            return -1
        idx = ix + R * iy
        return idx
    nx = grid[0,:,:].flatten()
    ny = grid[1,:,:].flatten()
    mesh = []
    for ix in range(R-1):
        for iy in range(C-1):
            mesh.append(QuadElement(     \
                [get_index(ix,    iy),   \
                 get_index(ix+1, iy),    \
                 get_index(ix+1, iy+1), \
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                  get_index(ix,    iy+1)],\
                nx, ny))
    return mesh, nx, ny, M, R, C, get_index

#Do the mesh generation
hx = 0.05
hy = 0.05
grid  = mgrid[-1:1+hx:hx,-1:1+hy:hy]
mesh, nx, ny, M, R, C, get_idx = gen_regular_quad_mesh(grid)

#Compute boundary conditions
boundary = zeros((M), 'bool')
for ix in range(C):
    boundary[get_idx(ix,0)] = True
    boundary[get_idx(ix,R-1)] = True
for iy in range(R):
    boundary[get_idx(0,iy)] = True
    boundary[get_idx(C-1,iy)] = True
bvals = zeros((M))

#Construct f
def f(x,y):
    if(sqrt(x*x + y*y) < 0.2):
        return 100.
    return 1.

#Solve problem
u = fe_solve(mesh, nx, ny, f, boundary, bvals)

#Display result
X = nx.reshape(R, C)
Y = ny.reshape(R, C)
U = u.reshape(R, C)
pcolor(X, Y, U)
savefig("prob1_result.png")
show()
```
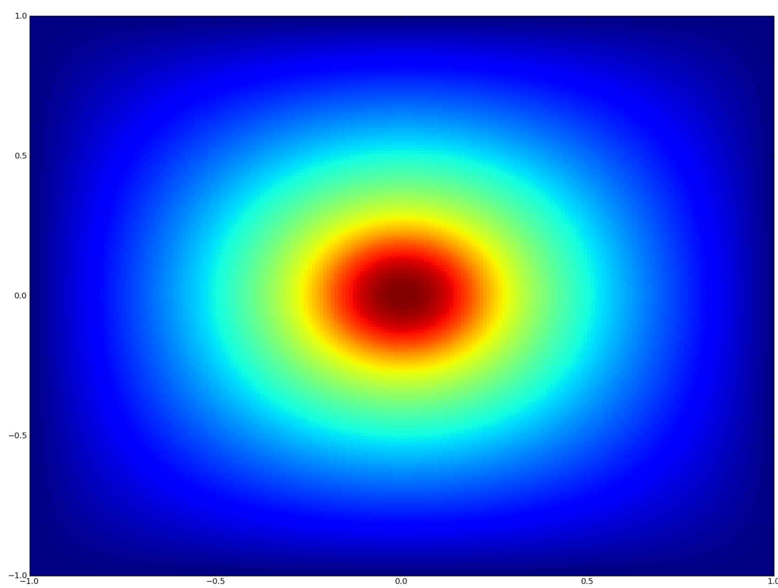

prob1.py

And here is a heatmap plot of the resulting distribution:



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