Homework 1

Computational Math

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1 Let $\varphi(x) = e^{\sin x \cos x}$. To verify that BC1 and BC2 are satisified, we establish the stronger condition that φ is periodic:

$$\varphi(x + 2\pi) = e^{\sin(x+2\pi)}e^{\cos(x+2\pi)} = e^{\sin(x)}e^{\cos(x)} = \varphi(x)$$

Which trivially implies BC1. To check BC2 we observe:

$$\begin{aligned}
\varphi_x(x + 2\pi) &= \lim_{\delta \to 0} \frac{\varphi(x + \delta + 2\pi) - \varphi(x + 2\pi)}{\delta} \\
&= \lim_{\delta \to 0} \frac{\varphi(x + \delta) - \varphi(x)}{\delta} \\
&= \varphi_x(x)
\end{aligned}$$

And thus BC2 is satisified as well. Finally, we verify ODE via direct substitution. Taking partial derivatives we find:

$$\varphi_{xx}(x) = \Big((\cos(x) - \sin(x))^2 - (\sin(x) + \cos(x)) \Big) \varphi(x)$$

So ODE expands to:

$$\Big((\cos(x) - \sin(x))^2 - (\sin(x) + \cos(x)) + \sin(x) + \cos(x) + 2\sin(x)\cos(x) - 1 \Big) \varphi(x) = 0$$

Regrouping terms in the left subexpression gives the following elementary trig identities:

$$\begin{aligned}
(\sin(x) + \cos(x)) - (\sin(x) + \cos(x)) &= 0 \\
(\cos(x) - \sin(x))^2 + 2\sin(x)\cos(x) - 1 &= 0
\end{aligned}$$

Therefore the solution is exact.

2 First, observe that all 2π-periodic functions satisfy BC1 and BC2 (by the argument from exercise 1), and thus the approximation trivally satisifies the boundary conditions. Now we make the following observation; the Galerkin solution ...

$$\varphi(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx))$$

... may be written as a sum of complex exponentials ...

$$\varphi(x) \approx \sum_{n=-N}^{N} c_n e^{inx}$$

... by making the substitutions ...

$$\begin{aligned}
a_n &= \frac{c_n + c_{-n}}{2} \\
b_n &= i\frac{c_n - c_{-n}}{2}
\end{aligned}$$

And thus this method is isomorphic to using a Fourier series approximation. By an extension of exercise 4f to periodic functions, $\mathcal{F}\{\varphi\}(k) = ik\hat{\varphi}(k)$. Moreover, we also have the identity:

$$\begin{aligned}
\{\mathcal{F}\sin(nx)\}(k) &= \int_{-\pi}^{\pi} \left(\frac{e^{inx} - e^{-inx}}{2i} \right) e^{-ikx} dx \\
&= \frac{1}{2} i(\delta_n(k) - \delta_{-n}(k))
\end{aligned}$$

Likewise by exercise 4b, $\mathcal{F}\{\cos(nx)\}(k) = \frac{1}{2}(\delta_n(k) + \delta_{-n}(k))$. Combining this fact with the convolution theorem and the trig identity, $2\cos(x)\sin(x) = \sin(2x)$, we get the following expression for ODE:

$$-k^2\hat{\varphi}(k) + \frac{1}{2}\left(-i\delta_{-2}(k) + (1-i)\delta_{-1}(k) + (1+i)\delta_1(k) + i\delta_2(k)\right) \star \hat{\varphi}(k) = \mathcal{F}\left\{e^{\sin(x)}e^{\cos(x)}\right\}(k)$$

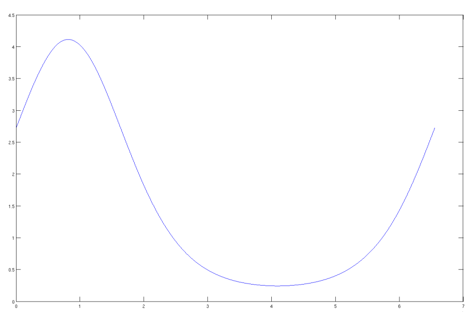
In the Galerkin approximation, this equation determines a matrix which is the sum of a Toeplitz matrix, T, with banded elements in $[-\frac{i}{2}, \frac{1}{2}(1-i), 0, \frac{1}{2}(1+i), \frac{i}{2}]$ and a diagonal matrix, D, with $D_{k,k} = -k^2$. Combined we get the following system:

$$(D + T)x = a$$

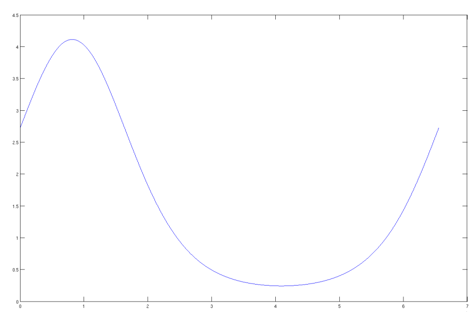
Where a is the DFT of $e^{\sin(x)}e^{\cos(x)}$ sampled at discrete intervals. Here is my MATLAB code for solving this system:

```
B = zeros(N, 5);
C = zeros(N, 1);

for k=1:N
    B(k,1) = -.5*i;
    B(k,2) = .5*(1 - i);
    B(k,3) = -(k-floor(N/2)-1)^2;
    B(k,4) = .5*(1 + i);
    B(k,5) = .5*i;
    t = (k - 1) * 2. * pi / N;
    C(k) = exp(cos(t)) * exp(sin(t));
end

A = fftshift(fft(C));
M = spdiags(B, -2:2, N, N);
x = real(ifft(ifftshift(M \ A)));
```

And here are the resulting graphs of x, C with $N = 65536$:

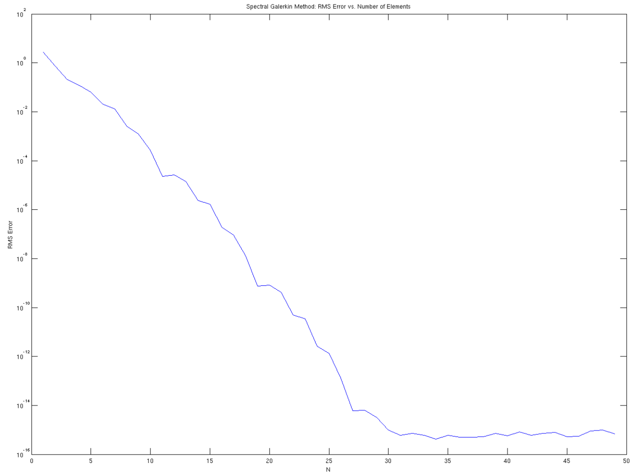


Galerkin Solution



Exact Solution

3 The following figure shows the RMS error for this method relative to the number of elements:



Up to $N \approx 30$, the accuracy of the approximation is on the order of $10^{-N/2}$, at which point the error stabilizes. This is due to the fact that at this point the precision is on the order of 10^{-15} which is the limit of what may be represented within a double. Beyond this point, any additional fluctuation is simply machine noise.

4

a

$$\begin{aligned}
\mathcal{F}\{\lambda(u+v)\}(\omega) &= \int\limits_{\mathbb{R}} (\lambda(u(x)+v(x))e^{-ixk}dx \\
&= \lambda \int\limits_{U(1)} u(x)e^{-ixk} + v(x)e^{-ixk}dx \\
&= \lambda\hat{u}(k) + \lambda\hat{v}(k)
\end{aligned}$$

b

$$\begin{aligned}
\mathcal{F}\{u(x+x_0)\}(k) &= \int\limits_{\mathbb{R}} u(x+x_0)e^{-ixk}dx \\
&= \int\limits_{\mathbb{R}} u(x)e^{-i(x+x_0)k}dx \\
&= e^{-ix_0k}\hat{u}(k)
\end{aligned}$$

c

$$\begin{aligned}
\mathcal{F}\{e^{ik_0x}u(x)\}(k) &= \int\limits_{\mathbb{R}} u(x)e^{ik_0x}e^{-ixk}dx \\
&= \int\limits_{\mathbb{R}} u(x)e^{-ix(k-k_0}dx \\
&= \hat{u}(k-k_0)
\end{aligned}$$

d

$$
\begin{aligned}
\mathcal{F}\{u(cx)\}(k) &= \int_{\mathbb{R}} u(cx)e^{-ixk}dx \\
&= \frac{1}{|c|}\int_{\mathbb{R}} u(x)e^{-i\frac{xk}{c}}dx \\
&= \frac{1}{|c|}\hat{u}(\frac{k}{c})
\end{aligned}
$$

e

$$
\begin{aligned}
\mathcal{F}\{\overline{u(x)}\}(k) &= \int_{\mathbb{R}} \overline{u(x)}e^{-ixk}dx \\
&= \overline{\int_{\mathbb{R}} u(x)e^{ixk}dx} \\
&= \overline{\hat{u}(-k)}
\end{aligned}
$$

f

$$
\begin{aligned}
\partial_x \left\{\mathcal{F}^{-1}\mathcal{F}\{u\}\right\}(x) &= \partial_x \left(\frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} u(t)e^{-itk}dt \right) e^{ixk}dk \right) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} ik \int_{\mathbb{R}} u(t)e^{-itk}dt\, e^{ixk}dk \\
&= \mathcal{F}^{-1}\{ik\hat{u}\}
\end{aligned}
$$

g Part f is applied in the last step

$$
\begin{aligned}
\mathcal{F}^{-1}\{u\}(k) &= \frac{1}{2\pi}\int_{\mathbb{R}} u(x)e^{ixk}dx \\
&= \frac{1}{2\pi}\overline{\int_{\mathbb{R}} \overline{u(x)}e^{-ixk}dx} \\
&= \frac{1}{2\pi}\overline{\hat{\bar{u}}(k)} \\
&= \frac{1}{2\pi}\hat{u}(-k)
\end{aligned}
$$

5 Let K_n be the kernel for an n-point central finite difference operator with sampling distance h. Then for $n = 2, 4$ we have:

$$
\begin{aligned}
K_2(x) &= \frac{1}{2h}\left(\delta(x-h) - \delta(x+h)\right) \\
K_4(x) &= \frac{1}{12h}\left(-\delta(x-2h) + 8\delta(x-h) - 8\delta(x+h) + \delta(x+2h)\right)
\end{aligned}
$$

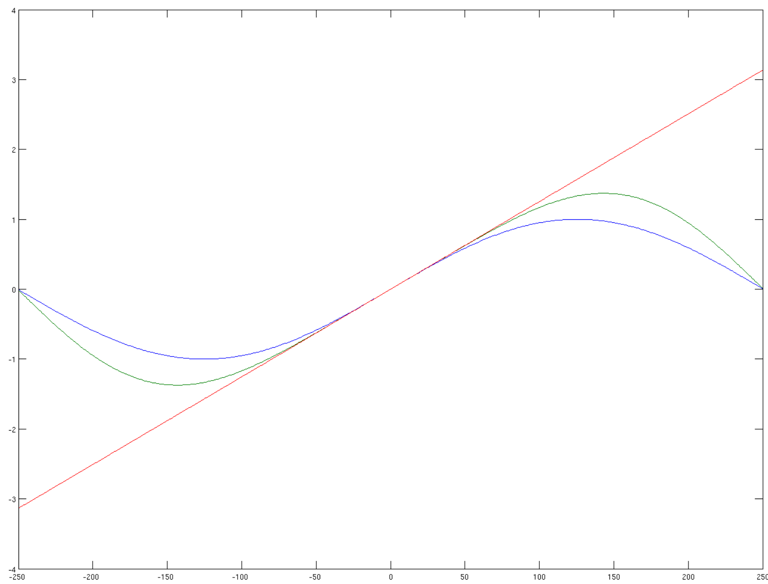
The corresponding multiplier for K_n is equivalent to taking its Fourier transform:

$$g_n = \mathcal{F}\{K_n\}$$

To compute g_n, we apply the linearity and translation identities from problem 4, along with the fact that $\mathcal{F}\{\delta\} = \frac{1}{2\pi}$ and recover the following expression for g_2, g_4:

$$\begin{aligned} g_2(k) &= \frac{1}{4\pi h}\left(e^{-ikh} - e^{ikh}\right) \\ g_4(k) &= \frac{1}{24\pi h}\left(-e^{-2ikh} + 8e^{-ikh} - 8e^{ikh} + e^{2ikh}\right) \end{aligned}$$

Plotting the imaginary component of these multipliers against g_∞ gives the following result:



g_2 is shown in blue, g_4 in green and g_∞ in red. As the order of accuracy increase, the multipliers approximate the asymptote g_∞ more closely. Generally, it appears that these operators are more accurate at lower frequencies and become progressively worse as the frequency increases.

6

a This system is a homogeneous, linear, first-order PDE. Since the coefficient on u_t is 1, the characteristic, φ curve may be parameterized as a function of t and satisifying the equation:

$$\frac{d\varphi(t)}{dt} = \frac{1}{5} + \sin^2(\varphi(t) - 1)$$

Which can be solved using Mathematica to get:

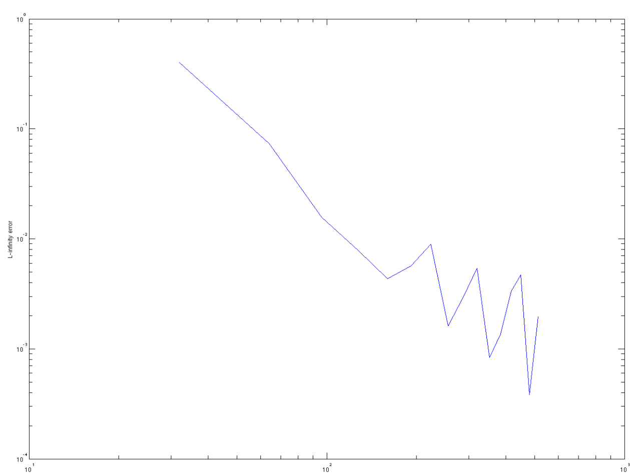
$$\varphi(t) = 1 - \tan^{-1}\left(\frac{\tan\left(\frac{\sqrt{6}}{5}(C - t)\right)}{\sqrt{6}}\right)$$

This function, when substituted into the original PDE has a period of $\frac{10\pi}{\sqrt{6}} \approx 13$.

b To solve this problem, the code from part a was modified to take a variable time / sampling step as follows:

```
function [data] = p6_param(T,N)
  h = 2*pi/N; x = h*(1:N); t = 0; dt = h/4;
  c = .2 + sin(x-1).^2;
  v = exp(-100*(x-1).^2);

  data = [v];
  vold = exp(-100*(x-.2*dt-1).^2);

  for i = 1:floor(T / dt)
      t = t+dt;
      v_hat = fft(v);
      w_hat = 1i*[0:N/2-1 0 -N/2+1:-1] .* v_hat;
      w = real(ifft(w_hat));
      vnew = vold + 2*dt*c.*w;
      vold = v;
      v = vnew;
      data(i+1,:) = v;
  end
end
```

From this, it was possible to graph the L^∞ error of the convergence, given the periodicity of the solution:



To improve this solution, one could use the exact analytic solution. This would give the correct answer for arbitrary t and boundary conditions with much less computational effort.

c As we showed at the start of problem 2, the chosen Galerkin approximation for this system is equivalent to the spectral solution up to a change of variables. Thus, the resulting solution will be the same.

7 We prove the following lemma which is sufficient to resolve most of the individual cases in theorem 3 (thereby sparing the grader the agony of parsing so much text):

Lemma If $|u| \in O(f(k)) \cap L^2(\mathbb{R})$ with u having first derivative of bounded variation; $f = \partial_k F$ where f' is positive, symmetric, monotonic and $\lim_{k\to\infty} F(k) \to 0$; and $v(j) = u(hj)$ (for $j \in \mathbb{Z}$), then for all $k \in [-\frac{\pi}{h}, \frac{\pi}{h}]$:

$$|\hat{v}(k) - \hat{u}(k)| \in O\left(\frac{1}{h}F\left(k + \frac{2\pi}{h}\right)\right)$$

Proof By theorem 2, we know that:

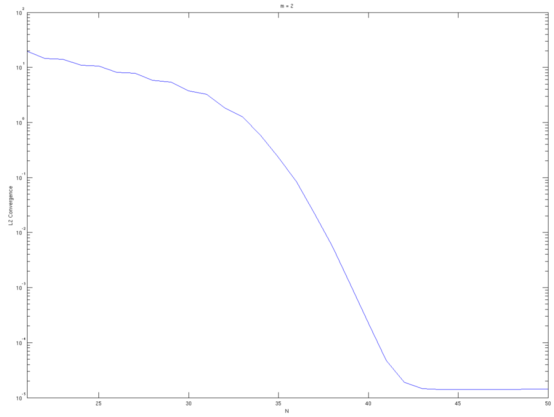
$$|\hat{v}(k)| = |\sum_{j=-\infty}^{\infty} \hat{u}(k + \frac{2\pi j}{h})|$$

Which by the hypothesis is bounded by:

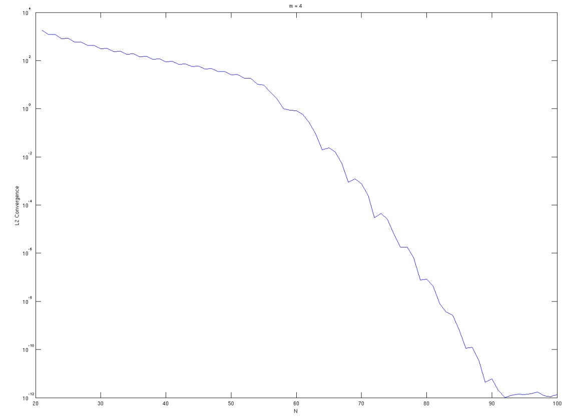
$$\begin{aligned}
|\hat{v}(k) - \hat{u}(k)| &\leq C \sum_{j=-\infty, j\neq 0}^{\infty} f\left(k + \frac{2\pi j}{h}\right) \\
&= 2C \sum_{j=1}^{\infty} f\left(k + \frac{2\pi j}{h}\right) \\
&\leq C \int\limits_{1}^{\infty} f\left(k + \frac{2\pi y}{h}\right) dy \\
&\leq \frac{C}{h} \int\limits_{k+\frac{2\pi}{h}}^{\infty} f(x)dx \\
&= \frac{CF\left(k + \frac{2\pi}{h}\right)}{h}
\end{aligned}$$

The results in the theorem 3a-c then follow naturally as corrollaries of theorems 1a-c. For theorem 3d, we take a different approach. By theorem 1d, we know that $\hat{u}$ has compact support on $[-\alpha, \alpha]$, if $h \leq \frac{\pi}{\alpha}$, then u may be extended to a function, $\hat{u'}$ with period 2π that agrees exactly with $\hat{u}$ on $[-\pi, \pi]$, such that square$(k)\hat{u'}(k) = \hat{u}(k)$. Thus, sinc$(\frac{x}{h}) \star u'(x) = u(x)$. However, convolution with sinc$(\frac{x}{h})$ fixes all points at $h\mathbb{Z}$, and thus agrees with $u'(hj) = u(hj) = v(j)$. Therefore, the transformation is exact.

8 We consider both the cases $m = 2, 4$ using the same approach. In each case, the graph shows a log-plot of the L^2 norm of the difference between subsequent eigenvalue approximations (with respect to the grid size, N). The tables consist of the Eigen values computed up to 10-digit accuracy.



L2 Convergence, $m = 2$



L2 Convergence, $m = 4$

m = 2	m = 4
1.000000000000032	1.060362090484225
2.999999999999991	3.799673029801529
4.999999999999835	7.455697937986940
6.999999999999964	11.644745511378069
9.000000000000028	16.261826018850122
10.999999999999845	21.238372918235811
13.000000000000020	26.528471183682196
14.999999999999911	32.098597710968413
16.999999999999982	37.923001027034573
18.999999999999964	43.981158097289978
20.999999999999591	50.256254516682972
23.000000000003396	56.734214055173275
24.999999999969774	63.403046986718707
27.000000000227086	70.252394628616571
28.999999998365411	77.273200481983679
31.000000010109567	84.457466274941694
32.999999939015886	91.798066808991138
35.000000313583499	99.288606660493429
36.999998392438492	106.9233073817323
39.000006910394958	114.6969173849852

To compute these results, the following modified version of the linked code was used:

```
function [ res ] = p8_param( N, m )
  L = 8;
  h = 2*pi/N; x = h*(1:N); x = L*(x-pi)/pi;
  column = [-pi^2/(3*h^2)-1/6 ...
      -.5*(-1).^(1:N-1)./sin(h*(1:N-1)/2).^2];
  D2 = (pi/L)^2*toeplitz(column);
  eigenvalues = sort(eig(-D2 + diag(x.^m)));
  res = eigenvalues(1:20);
end
```

Program wise, the main change in the code from the case where $m = 2$ to $m = 4$ is that the exponent on the diagonal matrix has to change. Otherwise, the same procedure works in both cases. In terms of the convergence, it should be noted that the $m = 4$ case converged much more slowly. Overall, it took nearly twice as many samples as the $m = 2$ case to reach a similarly stable state.