2.4

a *WTS:* There is a 1-1 inclusion reversing correspondence between algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals of S not equal to $S_+ = \bigoplus_{d>0} S_d$ given by:

$$Y \subseteq \mathbb{P}^n \mapsto I(Y) = \{f \in S | f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in Y\}$$

$$\mathfrak{a} \in S \mapsto Z(\mathfrak{a}) = \{P \in \mathbb{P}^n | f(P) = 0 \forall f \in \mathfrak{a}\}$$

WTS: $Y = Z(I(Y))$

By definition $Y \subseteq Z(I(Y))$. Because Y is algebraic, there exists some $f \in S$ such that $f(P) = 0 \Leftrightarrow P \in Y$. As a result, $f \in I(Y)$ and so $Z(I(Y)) \subseteq Y$. Therefore we have an equality.

WTS: $\mathfrak{a} = I(Z(\mathfrak{a}))$

Once again, it is obvious that $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$. Moreover, because $\mathfrak{a}$ is radical, by the Nullstellensatz $I(Z(\mathfrak{a})) \subseteq \mathfrak{a}$ and so the correspondence is 1-1.

WTS: For all varieties $Y_1, Y_2 \subseteq \mathbb{P}^n$:

$$Y_1 \subset Y_2 \implies I(Y_1) \supset I(Y_2)$$

This follows because if $Y_1 \subset Y_2$, then $f(P) = 0$ for all $P \in Y_2$, then $f(Q) = 0$ for all $Q \in Y_1$ and so $f \in I(Y_1)$.

WTS: For all radical ideals $\mathfrak{a}_1, \mathfrak{a}_2 \in S \setminus \{S_+\}$:

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \implies Z(\mathfrak{a}_1) \supset Z(\mathfrak{a}_2)$$

If $f \in \mathfrak{a}_2$, $f(Q) = 0 \implies Q \in Z(\mathfrak{a}_1)$ and so $Q \in Z(\mathfrak{a}_1) \implies Q \in Z(\mathfrak{a}_2)$. Therefore the correspondence is inclusion reversing. □

b An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

WTS: Y irreducible $\implies I(Y)$ is prime.

If $Y = \emptyset$, then $I(Y)$ contains only constant functions and so it is prime.

$Y \neq \emptyset$ is irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper closed subsets.

Suppose $I(Y)$ is not prime; then there exists a pair $f, g \in S \setminus I(Y)$ such that $fg \in I(Y)$, then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$, thus

$$Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$$

both of which are closed. But since Y is irreducible, either $Y \subseteq Z(f)$ or $Y \subseteq Z(g)$ and so one of them must be in $I(Y)$. But this is a contradiction and so we must conclude that $I(Y)$ is prime.

WTS: $\mathfrak{a}$ prime $\implies Z(\mathfrak{a})$ irreducible.

Suppose that $Z(\mathfrak{a}) = Y_1 \cup Y_2$; then $I(Z(\mathfrak{a})) = \mathfrak{a} = I(Y_1) \cap I(Y_2)$. But $\mathfrak{a}$ is prime so either $\mathfrak{a} = I(Y_1)$ or $\mathfrak{a} = I(Y_2)$. Therefore $Z(\mathfrak{a}) = Y_1$ or Y_2, and hence it is irreducible.

□

c $\mathbb{P}^n$ is irreducible.

By part b, it is enough to show that $I(\mathbb{P}^n)$ is prime. However, $I(\mathbb{P}^n) = \{0\}$, which is trivially prime. Therefore $\mathbb{P}^n$ is irreducible. $\square$

2.8 *WTS:* A projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree.

From 2.6, we know that $\dim S(Y) = \dim Y + 1$ and by 2.7 we know that $\dim \mathbb{P}^n = n$. Since $S(Y)$ is homogeneous and $I(Y)$ is prime (given that Y is a variety), proposition 1.7 implies that

$$\begin{aligned} \dim Y &= \dim \mathbb{P}^n - \text{height } I(Y) \\ &= n - 1 \end{aligned}$$

$\square$

2.9 If $Y \subseteq \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an openset $U_0 = \mathbb{P}^n \setminus \{x_0 = 0\}$ by the homeomorphism $\varphi_0 : U_0 \to \mathbb{A}^n$ where

$$\varphi_0(x_0, ..., x_n) = \left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0} \right)$$

Then define the projective closure $\bar{Y}$ such that:

$$\bar{Y} = \bigcap \{Y' \subseteq \mathbb{P}^n | Y' \text{ is algebraic and } Y \subseteq \varphi_0(Y')\}$$

a *WTS:* $I(\bar{Y})$ is the ideal generated by $\beta(I(Y))$, where

$$\beta(g) = x_0^e g \left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0} \right)$$

and $e = \deg(g)$. Recall:

$$I(\bar{Y}) = \{f \in S | f(p) = 0 \text{ for all } p \in U_0 \wedge \varphi_0(p) \in Y\}$$

$$\beta(I(Y)) = \{\beta(g) | g \in A \wedge g(x) = 0 \text{ for all } x \in Y\}$$

For all $g \in I(Y)$, there exists a $f \in S$ such that $f \circ \varphi_0 = \beta \circ g$; and so taking the closure of $Z(\beta(I(Y)))$ corresponds to finding the ideal generated by $\beta(I(Y))$. As a result $I(\bar{(Y)})$ is generated by $\beta(I(Y))$. $\square$

b Let $Y = \{(t, t^2, t^3) | t \in k\}$.

From homework 1, $I(Y)$ is generated by $\{y - x^2, z - x^3\}$.

However, $I(\bar{Y}) = \langle yw - x^2, xz - y^2, xw - yz \rangle$, which is generated by elements homogeneous of degree 2. As a result, applying β to the generators of $I(Y)$ does not always yield generators for $I(\bar{Y})$. $\square$

2.12 For given $n, d > 0$ let $M_0, M_1, ...M_N$ be all the monomials of degree d in $n+1$ variables $x_0, .., x_n$ where $N = \binom{n+d}{n} - 1$. Define a mapping $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ by sending $P = (a_0, ..., a_n)$ to the point $\rho_d(P) = (M_0(a), M_1(a), ..., M_N(a))$ obtained by substituting a_i in the monomials M_j.

a Let $\theta : k[y_0, ...y_N] \to k[x_0, ..., x_n]$ be the homomorphism defined by sending y_i to M_i and let $\mathfrak{a}$ be the kernel of θ.

WTS: $\mathfrak{a}$ is a homogeneous prime ideal.

The fact that $\mathfrak{a}$ is an ideal follows from the fact that it is a kernel. Moreover, Im θ is a polynomial ring over an algebraically closed field and so $k[x_0, ..., x_n]/\mathfrak{a}$ is entire and thus $\mathfrak{a}$ is prime.

That $\mathfrak{a}$ is homogeneous follows from the fact that each M_i is a homogeneous monomial of degree d, and thus substituting variables of homogeneous degree does not change the homogeneity of a polynomial.

The fact that $Z(\mathfrak{a})$ is a variety follows from 2.4(b) and the above predicate. $\square$

b *WTS:* $Z(\mathfrak{a}) \subseteq \rho_d(\mathbb{P}^n)$

WTS: $Z(\mathfrak{a}) \supseteq \rho_d(\mathbb{P}^n)$

c *WTS:* ρ_d is a homeomorphism of $\mathbb{P}^n$ onto $Z(A)$.

From part b, we know ρ_d is onto. Moreover, since ρ_d is a polynomial it sends closed sets to closed sets; so all that remains is to check that:

WTS: ρ_d is 1-1.

Suppose there exist $p, q \in \mathbb{P}^n$ such that $p \neq q$ and $\rho_d(p) = \rho_d(q)$. However, since $p \neq q$, there must exist some term x_j such that $p_j \neq q_j$. Now consider the monomial term $M_i = x_j^d$. Clearly $\rho_d(p)_i \neq \rho_d(q)_i$ and so we have a contradiction. $\square$

d Pick $n = 1, d = 3$, then $N = \binom{n+d}{n} - 1 = 3$ and

$$\rho_3(x_0, x_1) = (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$$

$$I(\rho_3(\mathbb{P}^{\nVdash})) \cong \langle yw - x^2, xz - y^2, xw - yz \rangle$$

$\square$

2.14 Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be defined by sending the ordered pair $(a_0, ..., a_r) \times (b_0, ..., b_s)$ to $(..., a_i b_j, ...)$ lexicographically, where $N = rs + r + s$. It is evident that ψ is well-defined and injective.

WTS: Im ψ is a subvariety of $\mathbb{P}^N$.

Consider the ring homomorphism $\phi : k[z_{i,j}] \to k[x_0, ..., x_r, y_0, ..., y_s]$ such that $\phi(z_{i,j}) = x_i y_j$. The kernel of this map is clearly a prime ideal (since its image is a polynomial ring). Likewise ϕ is the dual of ψ acting on homomorphism, and so Ker $\psi \cong I(\text{Im } \phi)$ or $Z(\text{Ker } \phi) = \text{Im } \psi$. $\square$

2.15 Define $Q \subseteq \mathbb{P}^3$ by the equation $xy - zw = 0$.

a *WTS:* $Q = \psi(\mathbb{P}^1 \times \mathbb{P}^1)$ where

$$\psi(s,t,u,v) = (su, tv, tu, sv)$$

Picking coordinates x, y, z, w gives the equations:

$$\begin{aligned} x &= su \\ y &= tv \\ z &= tu \\ w &= sv \end{aligned}$$

Solving for s, t, u, v in terms of x, y, z, w gives:

$$\begin{aligned} s &= \frac{x}{u} \\ u &= \frac{z}{t} \\ t &= \frac{y}{v} \\ v &= \frac{w}{s} \end{aligned}$$

Substituting into s:

$$s = \frac{x}{u} = \frac{xt}{z} = \frac{xy}{zv} = \frac{xys}{zw}$$

Cancelling s gives $1 = \frac{xy}{zw}$ or $xy - zw = 0$ which is exactly Q. $\square$

b *WTS:* Q contains two families of lines, $\{L_t\}, \{M_t\}$ parameterized by $t \in \mathbb{P}^1$ such that for all $t, u \in \mathbb{P}^1$; $L_t \neq L_u \implies L_t \cap L_u = \emptyset$; $M_t \neq M_u \implies M_t \cap M_u = \emptyset$ and $L_t \cap M_u =$ one point.

Because $\mathbb{P}^1$ is projective, we split the parameter u into a pair (s,t) modulo the relation $(s,t) \cong (\lambda s, \lambda t)$. Pick $L_u = \{(x,y,z,w) \in Q | sx - tz\}$ and $M_u = \{(x,y,z,w) \in Q | sy - tw\}$; which clearly satisfy the projective equivalence relation.

For any pair of points $p, q \in \mathbb{P}^1$, where $p = (s,t), q = (u,v)$; $p \neq q \implies L_p \neq L_q$. Moreover the intersection term is given by:

$$\begin{aligned} sx - tz &= 0 \\ ux - vz &= 0 \\ xy - zw &= 0 \end{aligned}$$

However, this system is overdetermined if $u/v \neq s/t$, and so the only possible choice for xy is 0. But this is not in $\mathbb{P}^3$ and therefore $L_p \cap L_q = \emptyset$. Symmetrically

$M_p \neq M_q \implies M_p \cap M_q = \emptyset$. Now consider $L_p \cap M_q$. This gives the system of equations:

$$\begin{aligned} sx - tz &= 0 \\ uy - vw &= 0 \\ xy - zw &= 0 \end{aligned}$$

The solution to this system is the set given by the set $(x, y, z, w) = (t, w, s, u)$ which is a point $\mathbb{P}^3$ and so $Ł_p \cap M_q$ is in fact a point. $\square$

c *WTS:* Q contains curves not contained in $M_t \cup L_t$: ie the twisted cubic:

$$c = \{(x, y, z, w) \in \mathbb{P}^3 | yw - x^2 = 0, xz - w^2 = 0, xy - zw = 0\}$$

However the only curves contained in $\mathbb{P}^1 \times \mathbb{P}^1$ are the families L_p and M_q as described above. Yet, c intersects each curve in L_p and M_q such that the region $c \cap L_p$ given by:

$$\begin{aligned} sx - tz &= 0 \\ yw - x^2 &= 0 \\ xz - w^2 &= 0 \\ xy - zw &= 0 \end{aligned}$$

Substituting $x = tz/s$:

$$\begin{aligned} yw - \frac{t^2}{s^2} z^2 &= 0 \\ \frac{t}{s} z^2 - w^2 &= 0 \\ \frac{t}{s} zy - zw &= 0 \end{aligned}$$

And so $w = \pm\sqrt{\frac{t}{s}} z$:

$$\begin{aligned} \pm\sqrt{\frac{t}{s}} yz - \frac{t^2}{s^2} z^2 &= 0 \\ \frac{t}{s} yz \mp \sqrt{\frac{t}{s}} z^2 &= 0 \end{aligned}$$

Which has a solution for $\frac{t}{s} = 0, 1$. However by part b, c can not be in either $\{L_p\}$ or $\{M_p\}$ since it intersects two curves in both sets. Therefore $c \notin \mathbb{P}^1 \times \mathbb{P}^1$ and so we must conclude that the Zariski topology on Q is distinct from $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$