Assigned: Thursday Feb. 11, 2010
Due: Thursday Feb. 25, 2010

MATH/CS 715: Homework 2
Spring 2010

PAGE LIMIT: 20 pages (single-sided).

NOTE: Please include a cover page – this will not count toward the total page limit.

NOTE: Don't forget to include MATLAB code where appropriate – this does count toward your total pages.

Fourier Spectral Methods

1. Consider the following thrid-order boundary value problem with periodic boundary conditions:

$$u_{xxx} + q_1(x)\, u_{xx} + q_2(x)\, u = 3\, e^{\sin(x)} e^{\cos(x)},$$
$$q_1(x) = \sin(x)\cos(x),$$
$$q_2(x) = -2\cos^4(x) + \cos^3(x) + (8 + 3\sin(x))\cos^2(x) - (1 + \sin(x))\cos(x).$$

   (a) Verify that $u(x) = e^{\cos(x)} e^{\sin(x)}$ is the exact solution to this BVP.

   (b) Derive the third-order Fourier spectral differentiation matrix $D_N^{(3)}$ by carefully differentiating the periodic Sinc function.

   (c) Numerically solve this BVP through the use of the appropriate Fourier spectral differentiation matrices. Provide plots of both the solution and the error for various N.

Chebyshev Spectral Methods

2. Consider the following boundary value problem

$$\textbf{ODE}: \quad u_{xx} + 4u_x + e^x u = \sin(8x),$$
$$\textbf{BCs}: \quad u(-1) = u(1) = 0.$$

   Solve this problem using a Chebyshev spectral method. To 8 digits of accuracy, what is u(0) (8 digits of accuracy means relative error is $< 5 \times 10^{-9}$)? How large did you have to make N in order to achieve this?

3. The solution to the Sturm-Liouville eigenvalue problem

$$\textbf{ODE}: \quad \phi_{xx} + \lambda\sigma(x)\phi = 0$$
$$\textbf{BCs}: \quad \phi(0) = \phi(1) = 0$$

   can be approximated by an asymptotic formula in the limit as $\lambda \to \infty$. To leading order, one can show that

$$\phi(x) \approx \sigma^{-1/4} \sin\left[\sqrt{\lambda} \int_0^x \sqrt{\sigma(\xi)}\, d\xi\right].$$

(a) Use the boundary conditions to determine an asymptotic formula for the eigenvalues.

(b) Numerically solve this problem with $\sigma(x) = 1+x$. Make a table of the first seven eigenvalues, comparing the asymptotic to the numerically converged solutions. Plot the first four eigenvectors, again comparing the asymptotic and numerical solutions. Comment on your results.

4. Download the MATLAB code for solving the nonlinear BVP discussed in lecture:

   `http://www.comlab.ox.ac.uk/nick.trefethen/p14.m`

   (a) In this program the error norm is observed to be reduced by a factor of about 0.2943 in each iteration. This explains why 30 steps are required to reduce the error to 10^{-14}. Add one or two lines of code to compute the eigenvalues of an appropriate matrix to show where the number 0.2943 comes from.

   (b) Devise an alternative to `p14.m` based on Newton iteration rather than fixed-point iteration. Do you observe quadratic convergence?

5. Consider the nonlinear initial boundary value problem

$$\begin{aligned} \textbf{PDE}: &\quad u_t = u_{xx} + e^u, \\ \textbf{BCs}: &\quad u(-1,t) = u(1,t) = 0, \\ \textbf{IC}: &\quad u(x,0) = 0. \end{aligned}$$

Solve this problem using a Chebyshev spectral in space. If one were to use Forward Euler method in time, what would the time-step restriction be? Instead of Forward Euler, use an operator split scheme:

$$\begin{aligned} v^\star &= \log\left(\frac{2\exp(v^n)}{2-\exp(v^n)\Delta t}\right) && \text{(solve ODE part exactly on } \Delta t/2) \\ v^{\star\star} &= v^\star + \frac{\Delta t}{2}\left(v^\star_{xx} + v^{\star\star}_{xx}\right) && \text{(full time step on diffusion part)} \\ v^{n+1} &= \log\left(\frac{2\exp(v^{\star\star})}{2-\exp(v^{\star\star})\Delta t}\right) && \text{(solve ODE part exactly on } \Delta t/2). \end{aligned}$$

To at least four digits of accuracy, what is $u(0, 3.5)$, and what is the time t_5 such that $u(0, t_5) = 5$?