Homework 1

Computational Math

Author:
Mikola Lysenko

January 28, 2010

1 Let $\varphi(x) = e^{\sin x \cos x}$. To verify that BC1 and BC2 are satisified, we establish the stronger condition that φ is periodic:

$$\varphi(x+2\pi) = e^{\sin(x+2\pi)}e^{\cos(x+2\pi)} = e^{\sin(x)}e^{\cos(x)} = \varphi(x)$$

Which trivially implies BC1. To check BC2 we observe:

$$\begin{aligned}
\varphi_x(x+2\pi) &= \lim_{\delta\to 0} \frac{\varphi(x+\delta+2\pi) - \varphi(x+2\pi)}{\delta} \\
&= \lim_{\delta\to 0} \frac{\varphi(x+\delta) - \varphi(x)}{\delta} \\
&= \varphi_x(x)
\end{aligned}$$

And thus BC2 is satisified as well. Finally, we verify ODE via direct substitution. Taking partial derivatives we find:

$$\varphi_{xx}(x) = \Big((\cos(x) - \sin(x))^2 - (\sin(x) + \cos(x)) \Big) \varphi(x)$$

So ODE expands to:

$$\Big((\cos(x) - \sin(x))^2 - (\sin(x) + \cos(x)) + \sin(x) + \cos(x) + 2\sin(x)\cos(x) - 1 \Big) \varphi(x) = 0$$

Regrouping terms in the left subexpression gives the following elementary trig identities:

$$\begin{aligned}
(\sin(x) + \cos(x)) - (\sin(x) + \cos(x)) &= 0 \\
(\cos(x) - \sin(x))^2 + 2\sin(x)\cos(x) - 1 &= 0
\end{aligned}$$

Therefore the solution is exact.

2 First, observe that all periodic functions satisfy BC1 and BC2 (by the argument from exercise 1), and so we consider only the solutions which are periodic on the interval $[-\pi, \pi)$. By an extension of exercise 4f to periodic functions, $\mathcal{F}\{\varphi\}(k) = ik\hat{\varphi}(k)$. Moreover, we also have the identity:

$$\begin{aligned}
\{\mathcal{F}\sin(nx)\}(k) &= \int_{-\pi}^{\pi} \left(\frac{e^{inx} - e^{-inx}}{2i} \right) e^{-ikx} dx \\
&= \frac{1}{2} i(\delta_n(k) - \delta_{-n}(k))
\end{aligned}$$

Likewise by exercise 4b, $\mathcal{F}\{\cos(nx)\}(k) = \frac{1}{2}(\delta_n(k) + \delta_{-n}(k))$. Combining this fact with the convolution theorem and the trig identity, $2\cos(x)\sin(x) = \sin(2x)$, we get the following expression for ODE:

$$-k^2\hat{\varphi}(k) + \frac{1}{2}\left(-i\delta_{-2}(k) + (1-i)\delta_{-1}(k) + (1+i)\delta_1(k) + i\delta_2(k)\right) \star \hat{\varphi}(k) = \mathcal{F}\left\{ e^{\sin(x)}e^{\cos(x)} \right\}(k)$$

In the Galerkin approximation, this equation determines a matrix which is the sum of a Toeplitz matrix, T, with banded elements in $[-\frac{i}{2}, \frac{1}{2}(1-i), 0, \frac{1}{2}(1+i), \frac{i}{2}]$ and a diagonal matrix, D, with $D_{k,k} = -k^2$. Combined we get the following system:

$$(D+T)x = a$$

Where a is the DFT of $e^{\sin(x)}e^{\cos(x)}$ sampled at discrete intervals. Here is my MATLAB code for solving this system:

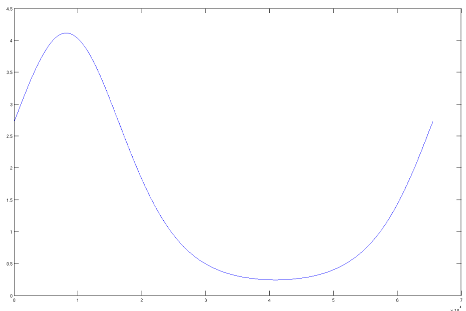
```
B = zeros(N, 5);
C = zeros(N, 1);

for k=1:N
    B(k,1) = -.5*i;
```

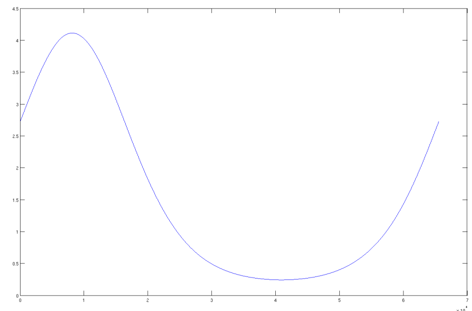
```
    B(k,2) = .5*(1 - i);
    B(k,3) = -(k-floor(N/2)-1)^2;
    B(k,4) = .5*(1 + i);
    B(k,5) = .5*i;
    t = (k - 1) * 2. * pi / N;
    C(k) = exp(cos(t)) * exp(sin(t));
end

A = fftshift(fft(C));
M = spdiags(B, -2:2, N, N);
x = real(ifft(ifftshift(M \ A)));
```

And here are the resulting graphs of x, C with $N = 65536$:

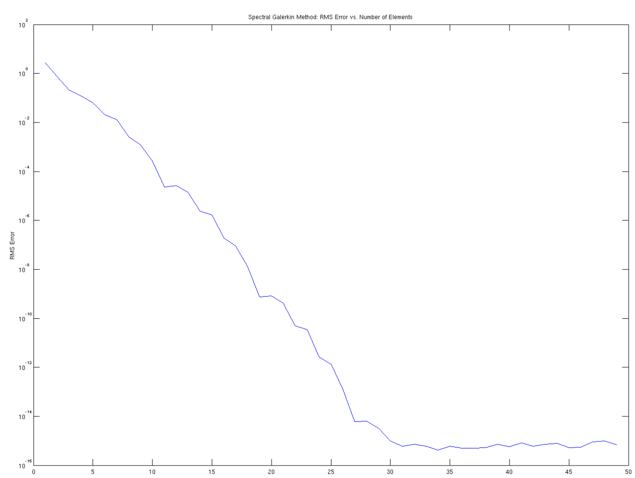


Galerkin Solution



Exact Solution

3 The following figure shows the RMS error for this method relative to the number of elements:



Up to $N \approx 30$, the accuracy of the approximation is on the order of $10^{-N/2}$, at which point the error stabilizes. This is due to the fact that at this point the precision is on the order of 10^{-15} which is the limit of what may be represented within a double. Beyond this point, any additional fluctuation is simply machine noise.

4

a

$$\begin{aligned}\mathcal{F}\{\lambda(u+v)\}(\omega) &= \int_{\mathbb{R}} (\lambda(u(x)+v(x))e^{-ixk}dx \\ &= \lambda \int_{U(1)} u(x)e^{-ixk} + v(x)e^{-ixk}dx \\ &= \lambda\hat{u}(k) + \lambda\hat{v}(k)\end{aligned}$$

b

$$\begin{aligned}\mathcal{F}\{u(x+x_0)\}(k) &= \int_{\mathbb{R}} u(x+x_0)e^{-ixk}dx \\ &= \int_{\mathbb{R}} u(x)e^{-i(x+x_0)k}dx \\ &= e^{-ix_0k}\hat{u}(k)\end{aligned}$$

c

$$\begin{aligned}\mathcal{F}\{e^{ik_0x}u(x)\}(k) &= \int_{\mathbb{R}} u(x)e^{ik_0x}e^{-ixk}dx \\ &= \int_{\mathbb{R}} u(x)e^{-ix(k-k_0}dx \\ &= \hat{u}(k-k_0)\end{aligned}$$

d

$$\begin{aligned}\mathcal{F}\{u(cx)\}(k) &= \int_{\mathbb{R}} u(cx)e^{-ixk}dx \\ &= \frac{1}{|c|}\int_{\mathbb{R}} u(x)e^{-i\frac{xk}{c}}dx \\ &= \frac{1}{|c|}\hat{u}(\frac{k}{c})\end{aligned}$$

e

$$\begin{aligned}\mathcal{F}\{\overline{u(x)}\}(k) &= \int_{\mathbb{R}} \overline{u(x)}e^{-ixk}dx \\ &= \overline{\int_{\mathbb{R}} u(x)e^{ixk}dx} \\ &= \overline{\hat{u}(-k)}\end{aligned}$$

f

$$\begin{aligned}
\partial_x \left\{\mathcal{F}^{-1}\mathcal{F}\{u\}\right\}(x) &= \partial_x \left(\frac{1}{2\pi} \int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}} u(t) e^{-itk} dt \right) e^{ixk} dk \right) \\
&= \frac{1}{2\pi} \int\limits_{\mathbb{R}} ik \int\limits_{\mathbb{R}} u(t) e^{-itk} dt \, e^{ixk} dk \\
&= \mathcal{F}^{-1}\{ik\hat{u}\}
\end{aligned}$$

g Part f is applied in the last step

$$\begin{aligned}
\mathcal{F}^{-1}\{u\}(k) &= \frac{1}{2\pi} \int\limits_{\mathbb{R}} u(x) e^{ixk} dx \\
&= \frac{1}{2\pi} \overline{\int\limits_{\mathbb{R}} \overline{u(x)} e^{-ixk} dx} \\
&= \frac{1}{2\pi} \overline{\hat{\bar{u}}(k)} \\
&= \frac{1}{2\pi} \hat{u}(-k)
\end{aligned}$$

5 The n^{th} order central finite difference kernel with step width h is given by the following formula:

$$\Delta_{n,h}(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \sqcap \left(\left(x + \frac{n}{2} - j \right) h \right)$$

Where $\sqcap$ denotes the rectangle function:

$$\sqcap(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2} \\ \frac{1}{2} & \text{if } |x| = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Fixing h discretizes the kernel, and so we may replace $\sqcap$ with δ giving the following expression:

$$\Delta_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \delta \left(x + \frac{n}{2} - j \right)$$

We now shall compute the discrete Fourier transform of Δ_n. Following the notation in the problem, define the multiplier g_n as:

$$g_n(k) = \mathcal{F}\{\Delta_n\}(k)$$

To compute g_n, we apply the linearity and translation identities from problem 4, along with the fact that $\mathcal{F}\{\delta\} = 1$ and recover the following formula:

$$\begin{aligned}
g_n(k) &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} e^{-ik\left(\frac{n}{2} - j\right)} \\
&= e^{-\frac{ikn}{2}} \sum_{j=0}^{n} e^{ij(k+\pi)} \binom{n}{j}
\end{aligned}$$

But this is just a binomial expansion:

$$g_n(k) = e^{-\frac{ikn}{2}} \left(1 - e^{ik}\right)^n$$