Spectral Rigid Body Dynamics

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Overview

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Pros:

- $+$ Pretty accurate at human energy scales
- $+$ Good for stiff materials (ie metals, plastics etc.)
- $+$ Easy kinematic constraints (useful for mechanisms)
- $+$ Standard animation tool (videogames!)

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- $+$ Good for stiff materials (ie metals, plastics etc.)
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Cons:

- - Inaccurate at extremely large energies
- - Bad for materials with low elastic modulus
- - Not always solvable! (See: Painleve's paradox)

What is a Rigid Body?

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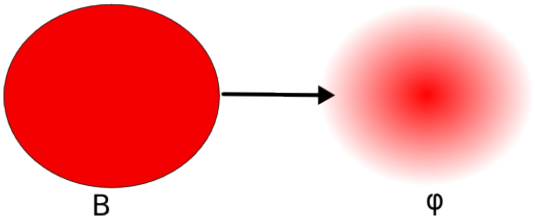

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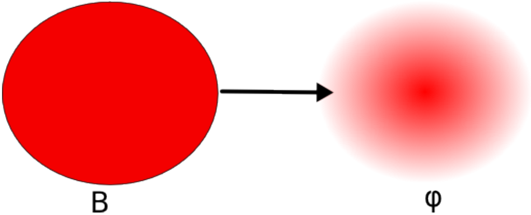

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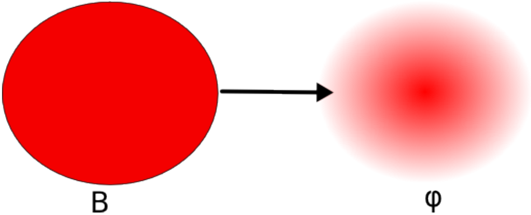

φ represents the mass distribution of B

$\varphi(x) = 0$ indicates B does not occupy the space at x

Configuration Space of a Rigid Body

Transformations of rigid mass fields must preserve distance and handedness

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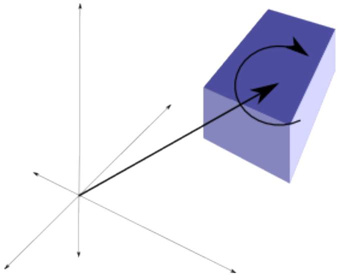

Can be parameterized by a translation t and a rotation R

Matrix: $\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$

$\binom{d+1}{2}$ degrees of freedom

Tangent space: $\mathfrak{so}(d+1)$

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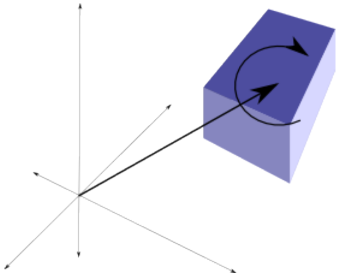

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Motions of rigid objects $\cong$ **curves** $q(t) \subset SE(d)$

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$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \frac{1}{2}\dot{q}^T M \dot{q}}{\partial \dot{q}}\right) &= \frac{\partial U(q,t)}{\partial q} \\ M\ddot{q} &= \nabla U \end{aligned}$$

Newton's equations!

Multiple Bodies

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A: Tensor sum

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Motion	$q(t) \cong q_i(t) \oplus q_j(t)$
Lagrangian	$L(q, \dot{q}, t) = L(q_i, \dot{q}_i, t) + L(q_j, \dot{q}_j, t)$

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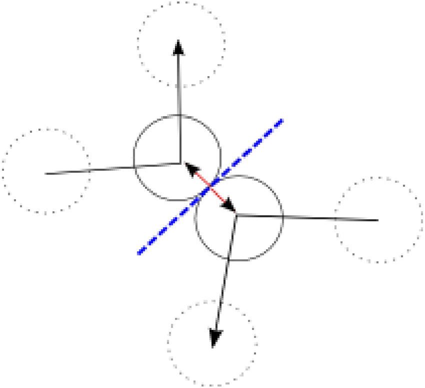
Scales to n bodies, get Lagrangian in $SE(d)^n$

Collisions

Solids can't physically occupy the same space.



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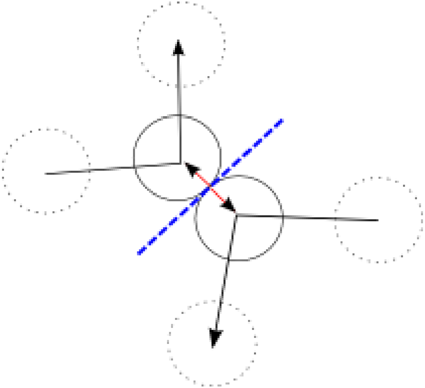

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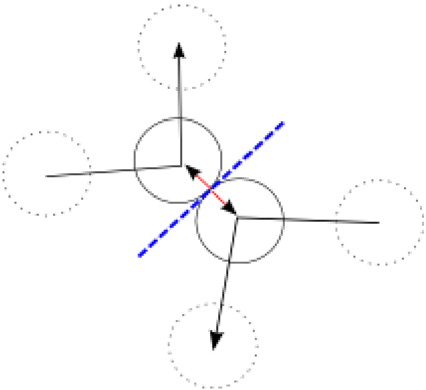

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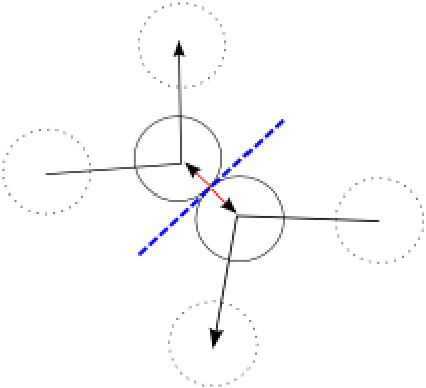

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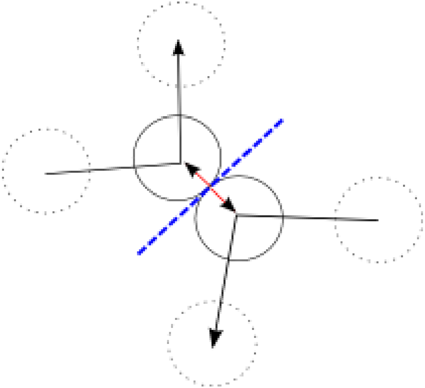

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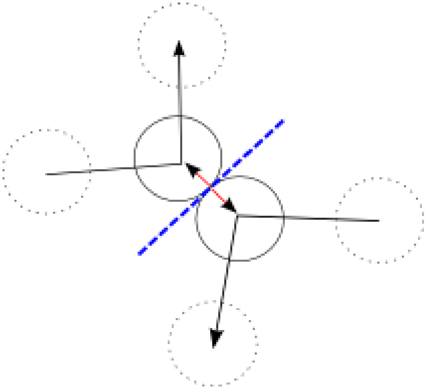

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\+ Just like high school physics

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But can be made to work with enough hacking...

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At all times no two solids intersect

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Is this really all there is to it?

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Define

$$C_{i,j}(q_i, q_j) \overset{def}{\equiv} \text{vol } q_i A_i \cap q_j A_j$$

And so we replace the impact forces with a system of differentiable holonomic inequality constraints:

$$C_{i,j} \leq 0$$

Equations of motion revisited

New problem:

$$\text{minimize } \int_{t_0}^{t_1} L(q, \dot{q}, t)dt$$

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Apply KKT conditions + Euler-Lagrange to get complementarity problem:

$$\frac{d}{dt}\left(\frac{\partial T(\dot{q}_i)}{\partial \dot{q}_i}\right) - \frac{\partial U(q,t)}{\partial q_i} + \sum_{j \neq i} \mu_{i,j} \frac{\partial C_{i,j}(q_i, q_j)}{\partial q_i} = 0$$

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Exactly elastic collision response!

Slack variables are impulse forces

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Need to compute:

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Substitute $q_j^{-1}q_i x \mapsto R(x-y)$ and let $\widetilde{\mathbf{1}_{A_j}}(x) = \mathbf{1}_{A_j}(-x)$:

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Fourier Methods

Convolution?

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Fix parameters $q_i = (R_i, t_i)$,

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Solve for R, y,

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Need to compute $\frac{\partial C_{i,j}(R_i, t_i, R_j, t_j)}{\partial R_i}$, $\frac{\partial C_{i,j}(R_i, t_i, R_j, t_j)}{\partial t_i}$

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Where v^k denotes the k^{th} basis vector

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$$\begin{aligned}
\frac{\partial C_{i,j}}{\partial t_i^k} &= \frac{\partial}{\partial t_i^j}\left(\int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega)\overline{\widehat{\mathbf{1}_{A_j}}(R_j R_i^{-1}\omega)}e^{2\pi i\left\langle\omega, R_i R_j^{-1}t_j - t_i\right\rangle}d\omega\right) \\
&= \int_{\mathbb{R}^d} 2\pi i\left\langle\omega, v^k\right\rangle \widehat{\mathbf{1}_{A_i}}(\omega)\overline{\widehat{\mathbf{1}_{A_j}}(R_j R_i^{-1}\omega)}e^{2\pi i\left\langle\omega, R_i R_j^{-1}t_j - t_i\right\rangle}d\omega
\end{aligned}$$

Where v^k denotes the k^{th} basis vector

Conclusion: Translational gradient is just a multiplier

Rotational Gradient

Parameterize $R_i = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}_{k,l} = -\mathfrak{r}_{l,k}$

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$$\frac{\partial C_{i,j}}{\partial \mathfrak{r}_{k,l}} = \frac{\partial}{\partial \mathfrak{r}_{k,l}} \left(\int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) \overline{\widehat{\mathbf{1}_{A_j}}(R_j \exp(-\mathfrak{r})\omega)} e^{2\pi i \left\langle \omega, \exp(\mathfrak{r}) R_j^{-1} t_j - t_i \right\rangle} d\omega \right)$$

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$$\begin{aligned}
\frac{\partial C_{i,j}}{\partial \mathfrak{r}_{k,l}} &= \frac{\partial}{\partial \mathfrak{r}_{k,l}} \left(\int\limits_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) \overline{\widehat{\mathbf{1}_{A_j}}(R_j \exp(-\mathfrak{r})\omega)} e^{2\pi i \left\langle \omega, \exp(\mathfrak{r}) R_j^{-1} t_j - t_i \right\rangle} d\omega \right) \\
\ldots &= \int\limits_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) e^{2\pi i \left\langle \omega, \exp(\mathfrak{r}) R_j^{-1} t_j - t_i \right\rangle} \left(- \left\langle \overline{\nabla \widehat{\mathbf{1}_{A_j}}(R\omega)}, \mathrm{ad}_{\mathfrak{r}_{k,l}} R_j \omega \right\rangle \right. \\
&+ \left. \overline{\widehat{\mathbf{1}_{A_j}}(R_j \exp(-\mathfrak{r})\omega)} 2\pi i \left\langle R_j \mathrm{ad}_{\mathfrak{r}_{k,l}} \omega, t_j \right\rangle \right) d\omega
\end{aligned}$$

Rotational Gradient

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$$\begin{aligned}\frac{\partial C_{i,j}}{\partial \mathfrak{r}_{k,l}} &= \frac{\partial}{\partial \mathfrak{r}_{k,l}} \left(\int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) \overline{\widehat{\mathbf{1}_{A_j}}(R_j \exp(-\mathfrak{r})\omega)} e^{2\pi i \left\langle \omega, \exp(\mathfrak{r}) R_j^{-1} t_j - t_i \right\rangle} d\omega \right) \\ \dots &= \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) e^{2\pi i \left\langle \omega, \exp(\mathfrak{r}) R_j^{-1} t_j - t_i \right\rangle} \left(- \left\langle \overline{\nabla \widehat{\mathbf{1}_{A_j}}(R\omega)}, \mathrm{ad}_{\mathfrak{r}_{k,l}} R_j \omega \right\rangle \right. \\ &+ \left. \overline{\widehat{\mathbf{1}_{A_j}}(R_j \exp(-\mathfrak{r})\omega)} 2\pi i \left\langle R_j \mathrm{ad}_{\mathfrak{r}_{k,l}} \omega, t_j \right\rangle \right) d\omega\end{aligned}$$

Get two terms:

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\dots &= \int\limits_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) e^{2\pi i \left\langle \omega, \exp(\mathfrak{r}) R_j^{-1} t_j - t_i \right\rangle} \left(- \left\langle \overline{\nabla \widehat{\mathbf{1}_{A_j}}(R\omega)}, \mathrm{ad}_{\mathfrak{r}_{k,l}} R_j \omega \right\rangle \right. \\
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\end{aligned}$$

Get two terms: a multiplier (easy),

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\end{aligned}$$

Get two terms: a multiplier (easy), a gradient (can be precomputed).

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&+ \left. \overline{\widehat{\mathbf{1}_{A_j}}(R_j \exp(-\mathfrak{r})\omega)} 2\pi i \left\langle R_j \mathrm{ad}_{\mathfrak{r}_{k,l}} \omega, t_j \right\rangle \right) d\omega
\end{aligned}$$

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Computationally not too bad, but still pretty messy in d-dimensional space.