Spectral Rigid Body Dynamics

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Overview

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An approximate model of low energy physics for stiff objects

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Pros:

- + Pretty accurate at human energy scales
- + Good for stiff materials (ie metals, plastics etc.)
- + Easy kinematic constraints (useful for mechanisms)
- + Standard animation tool (videogames!)

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- $+$ Good for stiff materials (ie metals, plastics etc.)
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Cons:

- - Inaccurate at extremely large energies
- - Bad for materials with low elastic modulus
- - Not always solvable! (See: Painleve's paradox)

What is a Rigid Body?

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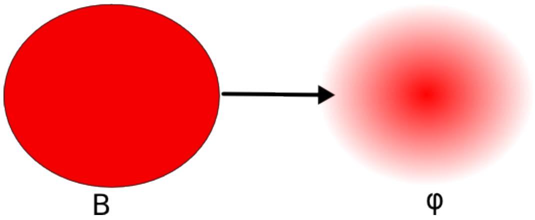

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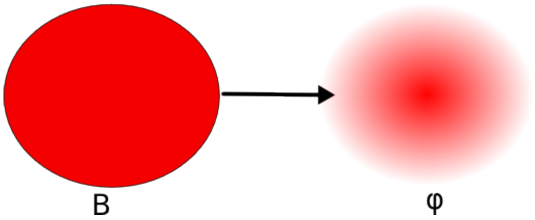

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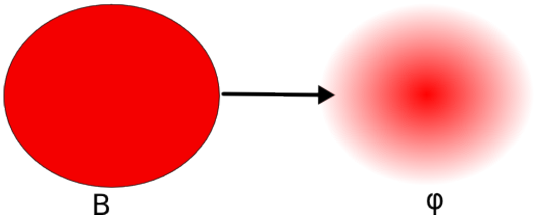

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$\varphi(x) = 0$ indicates B does not occupy the space at x

Configuration Space of a Rigid Body

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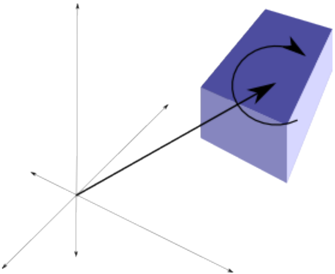

Can be parameterized by a translation t and a rotation R

Matrix: $\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$

$\binom{d+1}{2}$ degrees of freedom

Tangent space: $\mathfrak{so}(d+1)$

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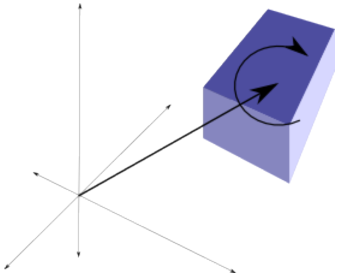

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Motions of rigid objects $\cong$ **curves** $q(t) \subset SE(d)$

Lagrangian Mechanics

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$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - U(q, t)$$

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$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \frac{1}{2}\dot{q}^T M \dot{q}}{\partial \dot{q}}\right) &= \frac{\partial U(q, t)}{\partial q} \\ M\ddot{q} &= \nabla U \end{aligned}$$

Newton's equations!

Multiple Bodies

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A: Tensor sum

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Lagrangian	$L(q, \dot{q}, t) = L(q_i, \dot{q}_i, t) + L(q_j, \dot{q}_j, t)$

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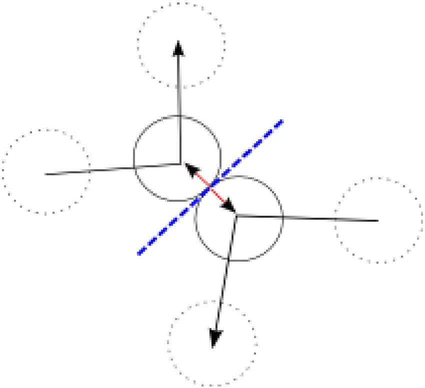
Scales to n bodies, get Lagrangian in $SE(d)^n$

Collisions

Solids can't physically occupy the same space.



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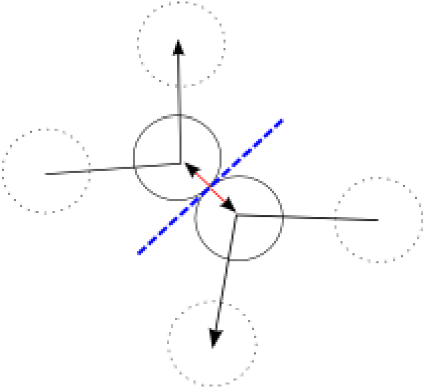

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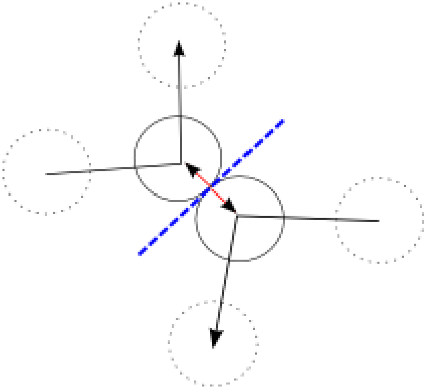

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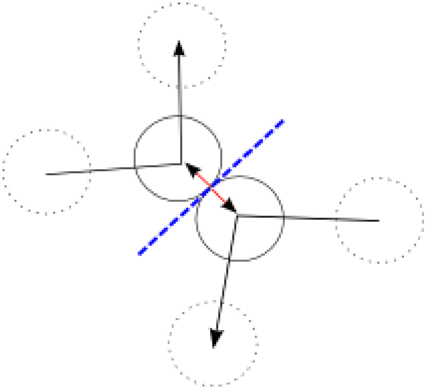

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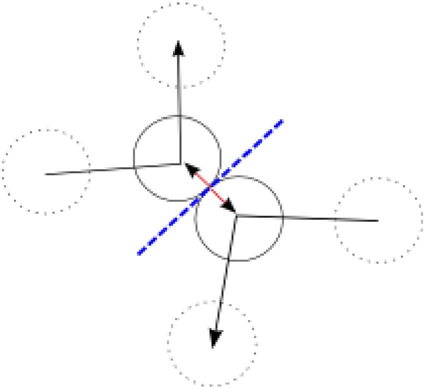

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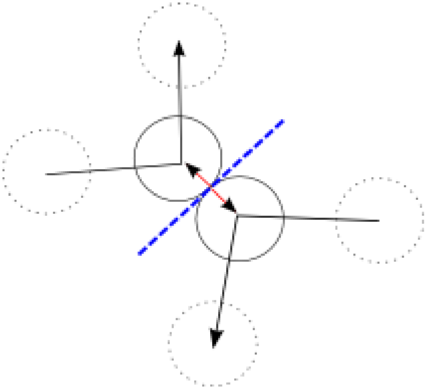

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\+ Just like high school physics

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But can be made to work with enough hacking...

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At all times no two solids intersect

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Is this really all there is to it?

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Define

$$C_{i,j}(q_i, q_j) \stackrel{def}{\equiv} \text{vol } q_i A_i \cap q_j A_j$$

And so we replace the impact forces with a system of differentiable holonomic inequality constraints:

$$C_{i,j} \leq 0$$

Equations of motion revisited

New problem:

$$\text{minimize} \int_{t_0}^{t_1} L(q, \dot{q}, t)dt$$

$$\text{subject to } C_{i,j}(q_i, q_j) \leq 0 \quad \forall t \in [t_0, t_1), i \neq j$$

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Apply KKT conditions + Euler-Lagrange to get complementarity problem:

$$\frac{d}{dt}\left(\frac{\partial T(\dot{q}_i)}{\partial \dot{q}_i}\right) - \frac{\partial U(q, t)}{\partial q_i} + \sum_{j \neq i} \mu_{i,j} \frac{\partial C_{i,j}(q_i, q_j)}{\partial q_i} = 0$$

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Exactly elastic collision response!

Slack variables are impulse forces

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Need to compute:

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Substitute $q_j^{-1}q_i x \mapsto Rx - y$ and let $\widetilde{\mathbf{1}_{A_j}}(x) = \mathbf{1}_{A_j}(-x)$:

$$\int_{\mathbb{R}^d} \mathbf{1}_{A_i}(x)\widetilde{\mathbf{1}_{A_j}}(y - Rx)dx = \int_{\mathbb{R}^d} \mathbf{1}_{A_i}(R^{-1}x)\widetilde{\mathbf{1}_{A_j}}(y - x)dx$$

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Fourier Methods

Convolution?

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Fix parameters $q_i = (R_i, t_i)$,

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$$q_i^{-1} = (R_i^{-1}, -R_i^{-1} t_i)$$
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Need to compute $\frac{\partial C_{i,j}(R,y)}{\partial y}$, $\frac{\partial C_{i,j}(R,y)}{\partial R}$ then use chain rule

Or by symmetry: $C_{i,j}(q_i, q_j) = C_{j,i}(q_j, q_i)$

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Where v^k denotes the k^{th} basis vector

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Start with the translational case first.

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$$\begin{aligned} \frac{\partial C_{i,j}}{\partial y^k} &= \frac{\partial}{\partial y^k}\left(\int\limits_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle \omega, y\rangle} d\omega \right) \\ \dots &= \int\limits_{\mathbb{R}^d} 2\pi i \left\langle \omega, v^k \right\rangle \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle \omega, y\rangle} d\omega \end{aligned}$$

Where v^k denotes the k^{th} basis vector

Conclusion: Translational gradient is just a multiplier

Rotational Gradient

Parameterize $R = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$ with basis $\mathfrak{r}_{k,l}$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}^T = -\mathfrak{r}$

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Not a Fourier multiplier.

But can be precalculated with $O(d)$ overhead:

$$\frac{\partial}{\partial \omega^k} \widehat{\mathbf{1}_{A_i}}(\omega) = \mathcal{F}\left(-2\pi i \left\langle x, v^k \right\rangle \mathbf{1}_{A_i} \right)(\omega)$$

Performance

Fourier transform is neat, but what does it get us?

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Full inverse Fourier transform complexity = brute force volume

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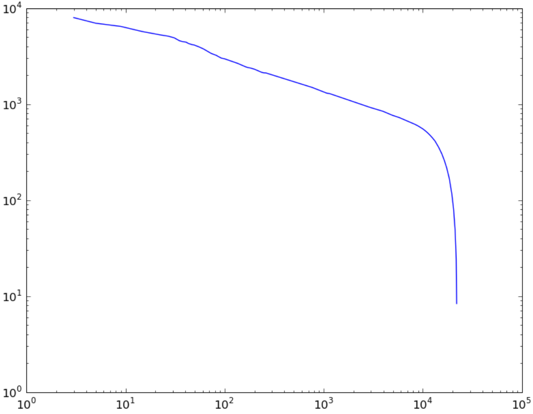
Dramatically reduces space and time complexity

Convergence

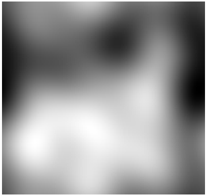
Rate of convergence, number of Fourier terms vs. L^1 error, (log-log plot)



Convergence

Qualitative effects of cutoff (full spectrum has 22185 terms)

Exact



16

32

64

256

1024

Truncated Spectra

Henceforth $d = 2$

Want: An efficient way to represent truncated spectrum

Need to deal with rotations too

Obvious answer: Polar coordinates

Pick $\omega_r = \sqrt{\omega_x^2 + \omega_y^2}$, $\omega_\theta = \tan^{-1} \frac{y}{x}$

$$\widehat{f}(\omega_x, \omega_y) \mapsto \widehat{f}^p(\omega_r, \omega_\theta)$$

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Problem?

Discrete Cartesian to Polar Conversion

Can't do this exactly!

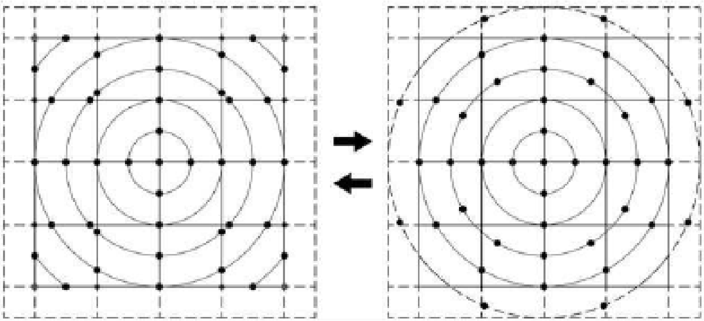

Figure from Chirikjian 2003

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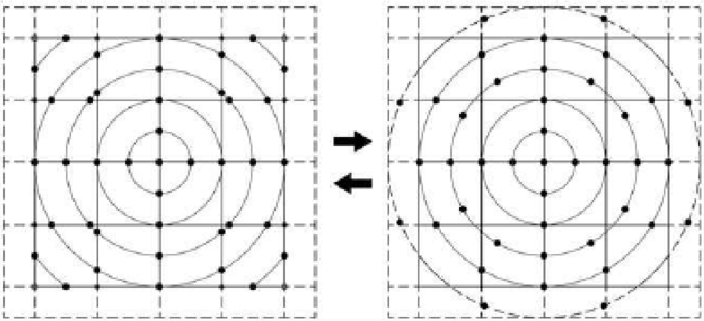

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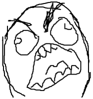
Reason: *Only nontrivial discrete subgroups of $SE(2)$ are 17 plane tiling groups!*



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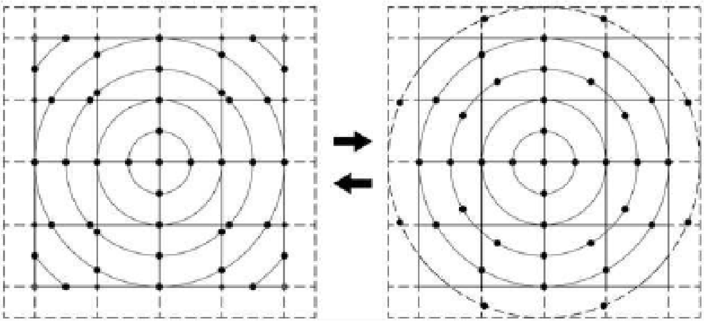

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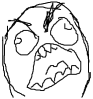

So unless you are happy with a hexagonal grid with 6 rotational samples, forget about doing this losslessly!

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- Approximation is a fact of life – deal with it
  - Might as well do it fast

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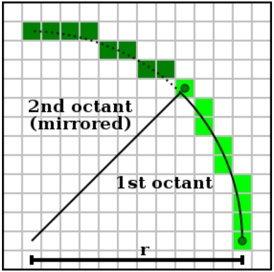

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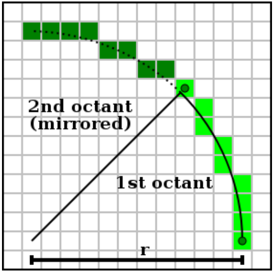

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- Sample r at uniform increments
- For each r, trace a circle centered at the origin
- Store $8r + 4$ radial samples in array
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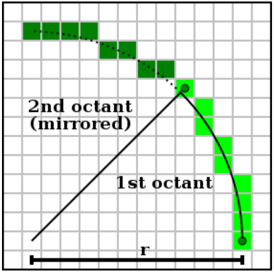

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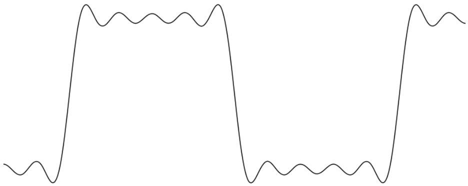
Minimal storage, uses only integer arithmetic

Gibbs Phenomenon

Truncating Fourier series causes ringing artifacts at boundaries

- Intuitively, same as low pass filtering high curvature



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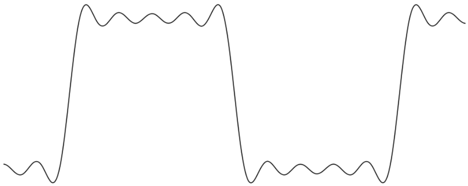

Figure from Wikipedia, 2010

But we don't care, just need separation between interior and exterior.

So pick some cutoff threshold; values above are in, below are out

Optimal Thresholding

Given a region $R \subset \mathbb{R}^d$ and set $A \subseteq R$ approximated by f, define quality, Q_T, of threshold T

$$Q_T = \text{vol}\left\{x \in R | f(x) > T \Leftrightarrow x \in A\right\}$$

Intuitively, number of points where threshold gives the right answer
Want to find optimal T

$$T = \underset{T' \in \mathbb{R}}{\text{argmax}}\, Q_{T'}$$

Seems hard to estimate T analytically

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But in practice can compute T combinatorially given f, A, R

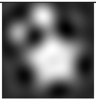
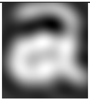
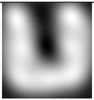
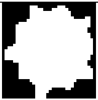
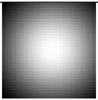
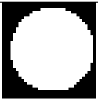
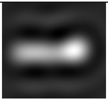
Optimal Thresholding Algorithm

```
v = []
for x in R:
    v.append( (f(x), A(x)) )
v.sort()

fv, Av = unzip(v)
Q = foldl(Av, +, 0) + foldr(Av, +, 0)

return max(zip(Q, fv)).second
```

Greedy/DP method

Takes $O(n \log(n))$ time, uses $O(n)$ space

Low Pass Filter/Thresholding Results

Exact Indicator	Fourier Truncation	Optimal Cutoff
		
		
		
		
		

Radius 14 cutoff, 777 terms

Putting it together

Let $\hat{f}_A$ be the approximate Fourier series for $\mathbf{1}_A$

Then:

$$\begin{aligned} C_{i,j}(\rho,\theta,\phi) &= -T + \sum_{r=2}^{N_R} \sum_{t=1}^{8r+4} \widehat{f_{A_i}}\left(r, \phi\frac{dt}{d\theta} + t\right) \overline{\widehat{f_{A_j}}}(r,t) \\ \ldots & \exp\left(\frac{2\pi i}{N}\rho r \cos\left(t\frac{d\theta}{dt} + \theta\right)\right) r\frac{dt}{d\theta} \end{aligned}$$

Where:

$$q_j q_i^{-1} = (R, y)$$

$$y = \rho(\cos(\theta)v^x + \sin(\theta)v^y)$$

$$R = \exp(\phi\mathfrak{r}_{0,1})$$

$$dt = 8r + 4$$

$$d\theta = 2\pi$$

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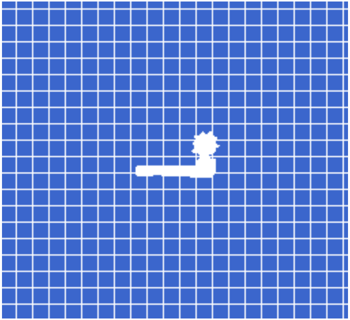
Protip: Can early out using the rearrangement inequality

Prototype Physics Code

Basic implementation so far, just uses implicit Euler.

Doesn't solve for μ correctly (currently set to constant)



Features:

- OpenGL GUI
- Written 100% in Python
- Reasonable performance (though certainly not fast)
- Arbitrary shapes/mass fields supported

Download Link:

http://github.com/mikolalysenko/Collisions

To Do

- Analysis is still very incomplete
- Need to implement a proper solver for physics system
- Benchmarks, etc.
- More development/write ups needed

References

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