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Regularized deconvolution on the 2D-Euclidean motion group

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Abstract
This paper considers a class of inverse problems involving deconvolution density estimation over the Euclidean motion group. Group representations of the Euclidean motion group are used to break apart convolution products, followed by compression and Tikhonov regularization to invert the distortion operator which is assumed known and compact. The integrated mean-squared error for the deconvolution density estimator is calculated whereby polynomial bounds on the recovery are obtained.

1. Introduction

In this paper, we consider a class of inverse problems involving deconvolution density estimation over a non-commutative non-compact Lie group. In particular we will address the situation of the Euclidean motion group. The idea behind this class of inverse problems is to use non-commutative Fourier methods to break apart convolution products in a manner analogous to the classical commutative deconvolution case. Some papers covering the classical case can be found in, for example, [1, 6, 8, 13, 17]. Recently, there has been interest in the non-commutative case including the compact situation, see, for example, [9, 11, 12, 14, 16], as well as the non-compact situation, see, for example, [3, 5, 20–23].

We will approach this problem in terms of observations consisting of distorted measurements over the Euclidean motion group. The main difference with the classical commutative case [17] as well as the non-commutative compact case [12] is that the distortion operator is now a non-invertible infinite-dimensional matrix and is therefore ill-posed. A finite-dimensional approximation is needed which will be carried out by invoking compression and regularization. Regularization will involve the method of Tikhonov which was first suggested for the Euclidean motion group by Chirikjian [3]. Although some discussion of Tikhonov regularization for the Euclidean motion group is carried out in the latter reference, a detailed theoretical analysis is not hitherto available and will be thoroughly examined for the first time in this paper, where precise integrated mean-squared error calculations are obtained. We note that these calculations depend on the spectral properties of the known distortion operator. If the spectrum of the compression of the distortion operator is bounded polynomially, then the regularized deconvolution density estimator has polynomial convergence in the integrated mean-squared error. We note that Tikhonov regularization is very popular in the classical commutative case, see, for example, [2, 7, 15].

In this paper, we do not explicitly consider applications; however, applications are abundant in situations involving motion data on the plane. Examples of such include biomechanics where one may study the planar motion of say, a knee joint, or robotics data, where one is interested in manipulating planar robotic motions. We refer the interested reader to Chirikjian and Kyatkin [4] for an abundance of applications.

This paper is organized as follows. Section 2 reviews the Euclidean motion group and Fourier analysis on that group. Section 3 then states the Euclidean motion inverse problem along with a proposal for recovering the deconvolution density using compression and Tikhonov regularization. Following this, in section 4, we state the main results with respect to bounds on the integrated mean-squared error. It is shown that the regularized estimator can achieve the same rate of convergence as the ordinary estimator with the proofs collected in section 6. Section 5 provides a discussion on details of the distortion operator and some technical details are collected in an appendix.

2. The Euclidean motion group and Fourier analysis

Consider the three-dimensional Euclidean motion group $\mathbb{SE}(2)$, which can be realized as a semi-direct product of the 2×2 rotation group $\mathbb{SO}(2)$, and the additive group $\mathbb{R}^2$. An element $g \in \mathbb{SE}(2)$ will be written as $g = (R_\theta, \mathbf{r})$ where $\mathbf{r} \in \mathbb{R}^2$, $R_\theta \in \mathbb{SO}(2)$ and $\theta \in [0, 2\pi)$. In this form, $g^{-1} = (R'_\theta, -R'_\theta \mathbf{r})$ where superscript $'$ denotes transpose. The identity element of the group is $(\mathbf{I}_2, \mathbf{0})$, where $\mathbf{I}_\mu$ is generically the $\mu \times \mu$ identity matrix and $\mathbf{0} = (0, 0)'$. The group operation can be written $(R_\theta, \mathbf{r})(R_\phi, \mathbf{s}) = (R_\theta R_\phi, \mathbf{r} + R_\theta \mathbf{s})$, where $\theta, \phi \in [0, 2\pi)$.

A natural embedding of $\mathbb{SE}(2)$ is to represent each group element as a 3×3 matrix with the corresponding group action and inversion being matrix multiplication and inversion, respectively. In particular, each element $g \in \mathbb{SE}(2)$ can be parameterized in rectangular coordinates as

$$g(\theta, r_1, r_2) = \begin{pmatrix} \cos\theta & -\sin\theta & r_1 \\ \sin\theta & \cos\theta & r_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where $\theta \in [0, 2\pi)$ and $\mathbf{r} = (r_1, r_2)' \in \mathbb{R}^2$, or in polar coordinates as

$$g(\theta, \phi, r) = \begin{pmatrix} \cos\theta & -\sin\theta & r\cos\phi \\ \sin\theta & \cos\theta & r\sin\phi \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where $\mathbf{r} = (r\cos\phi, r\sin\phi)' \in \mathbb{R}^2$, $r \geqslant 0$ and $\theta, \phi \in [0, 2\pi)$.

The irreducible unitary representations (see [19] for information on irreducible unitary representations) for $\mathbb{SE}(2)$ will be defined over $L^2(\mathbb{S}^1)$ the vector space of square integrable functions on the unit circle by

$$U(g, p)\varphi(\mathbf{x}) \equiv \mathrm{e}^{-\mathrm{i}p(\mathbf{r}\cdot\mathbf{x})}\varphi(R'_\theta \mathbf{x}), \tag{3}$$

where $g \in \mathbb{SE}(2)$, $\mathrm{i}^2 = -1$, $p \in [0, \infty)$, $\mathbf{x} \in \mathbb{S}^1$ and $\varphi \in L^2(\mathbb{S}^1)$. That these are the irreducible representations verified in [4, 18]. We note that this group representation observes the group homomorphism property, $U(g_1g_2, p) = U(g_1, p)U(g_2, p)$, and $U(g, p)^* = U(g^{-1}, p)$ for $g, g_1, g_2 \in \mathbb{SE}(2)$, $p \in [0, \infty)$ and superscript * denotes adjoint.

In general, representations can be expressed as unitary operators in a basis for the underlying vector space. In order to represent $U(g, p)$ as a matrix we note that any function in $L^2(\mathbb{S}^1)$ can be expressed as a Fourier series of orthonormal basis functions. Hence the matrix elements of the operator $U(g, p)$, denoted by $u_{\ell m}(g, p)$, will be represented with respect to the basis functions $\{\mathrm{e}^{-\mathrm{i}\ell\psi} : \ell \in \mathbb{Z}, \psi \in [0, 2\pi)\}$ as

$$\begin{aligned} u_{\ell m}(g, p) &= \langle \mathrm{e}^{\mathrm{i}\ell\psi}, U(g, p)\,\mathrm{e}^{\mathrm{i}m\psi} \rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\ell\psi}\,\mathrm{e}^{-\mathrm{i}(r_1 p\cos\psi + r_2 p\sin\psi)}\,\mathrm{e}^{\mathrm{i}m(\psi-\theta)}\,\mathrm{d}\psi, \end{aligned} \tag{4}$$

for all $\ell, m \in \mathbb{Z}$, where the inner product is $\langle \varphi_1, \varphi_2 \rangle = \int_0^{2\pi} \varphi_1(\psi)\overline{\varphi_2(\psi)}\,\mathrm{d}\psi$ with overline denoting conjugation for $\varphi_1, \varphi_2 \in L^2(\mathbb{S}^1)$. Henceforth, no distinction will be made between the operator $U(g, p)$ and the corresponding infinite-dimensional matrix with elements $u_{\ell m}(g, p)$, $\ell, m \in \mathbb{Z}$. Finally, note that the collection

$$\{u_{\ell m}(\cdot, p) | \ell, m \in \mathbb{Z}, p \in [0, \infty)\} \tag{5}$$

forms a complete orthonormal basis for $L^2(\mathbb{SE}(2))$.

The Fourier transform on $\mathbb{SE}(2)$ can now be defined relative to the irreducible unitary representations (3). In the following, the measure $\mathrm{d}g$ is the $\mathbb{R}^2$ invariant, $\mathbb{SO}(2)$ normalized Haar measure on $\mathbb{SE}(2)$. Furthermore, we will denote by $L^q(\mathbb{SE}(2))$ the space of functions $f : \mathbb{SE}(2) \to \mathbb{R}$ so that $\int_{\mathbb{SE}(2)} |f(g)|^q\,\mathrm{d}g < \infty$ for $q \geqslant 1$.

A function, $f : \mathbb{SE}(2) \to \mathbb{R}$, is said to be rapidly decreasing if

$$\lim_{r\to\infty} r^m f(g(\theta, \phi, r)) = 0$$

for all $m \in \mathbb{Z}_+$ where $g = g(r, \phi, \theta) \in \mathbb{SE}(2)$ is the polar coordinate representation (2). The Fourier transform of $f \in L^1(\mathbb{SE}(2)) \cap L^2(\mathbb{SE}(2))$, a rapidly decreasing function is

$$\widehat{f}(p) = \int_{\mathbb{SE}(2)} f(g)U(g^{-1}, p)\,\mathrm{d}g, \qquad p \in [0, \infty)$$

and its inverse transform is

$$f(g) = \int_0^\infty \mathrm{tr}(\widehat{f}(p)U(g, p))\,p\,\mathrm{d}p, \qquad g \in \mathbb{SE}(2),$$

where 'tr' stands for the trace of the object in question.

For convenience, $\widehat{f}(p)$ will often be represented as an infinite-dimensional matrix where the matrix elements will use the matrix elements of $U(g, p)$ as defined in (4) giving

$$\widehat{f}_{\ell m}(p) = \langle \mathrm{e}^{\mathrm{i}\ell\psi}, \widehat{f}(p)\,\mathrm{e}^{\mathrm{i}m\psi} \rangle = \int_{\mathbb{SE}(2)} f(g)u_{\ell m}(g^{-1}, p)\,\mathrm{d}g \quad \forall \ell, m \in \mathbb{Z}.$$

Likewise, the inversion can be written in terms of the matrix elements as

$$f(g) = \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_0^\infty \widehat{f}_{\ell m}(p)u_{m\ell}(g, p)\,p\,\mathrm{d}p. \tag{6}$$

Some properties of Fourier transform are, for $f, h \in L^1(\mathbb{SE}(2)) \cap L^2(\mathbb{SE}(2))$, define convolution by

$$f * h(g) = \int_{\mathbb{SE}(2)} f(g)h(g^{-1}y)\,\mathrm{d}y, \qquad g \in \mathbb{SE}(2). \tag{7}$$

Then

$$\widehat{(f * h)}_{\ell m}(p) = \sum_j \widehat{h}_{\ell j}(p)\widehat{f}_{jm}(p), \qquad p \in [0, \infty). \tag{8}$$

A very important property is the Plancherel formula

$$\|f\|_2^2 = \int_0^\infty \|\widehat{f}(p)\|_{\mathrm{tr}}^2 \, p \, \mathrm{d}p, \tag{9}$$

where for some operator A, $\|A\|_{\mathrm{tr}}^2 = \mathrm{tr}(AA^*)$ is the square of the Hilbert–Schmidt norm and $\|f\|_2^2 = \int_{\mathbb{SE}(2)} |f(g)|^2 \, \mathrm{d}g$ is the $L^2(\mathbb{SE}(2))$-norm. We note that further details regarding Fourier analysis on the Euclidean motion group can be found in [4, 15].

3. Deconvolution density estimation

We can use the Riemannian structure to define the Laplacian on $\mathbb{SE}(2)$, i.e., the elliptic second-order partial differential operator associated with the usual Riemannian metric on $\mathbb{SE}(2)$. In particular, using differential operators $\mathfrak{X}_i$ the general Laplacian can be written as

$$\sum_{i,j} a_{ij}\mathfrak{X}_i\mathfrak{X}_j + \sum_j b_j\mathfrak{X}_j$$

giving the whole class of functions. The choice made here is to let $b_j = 0$ for all j and let $a_{ij} = \delta_{ij}$. Let $\mathfrak{se}(2)$ be the Lie algebra of $\mathbb{SE}(2)$ so that we have $\mathfrak{se}(2) = \mathbb{R}^2 + \mathfrak{so}(2)$, a vector space sum, where $\mathfrak{so}(2)$ is the Lie algebra of $\mathbb{SO}(2)$. We choose the following basis as it is the natural choice for $\mathbb{SE}(2)$; however, we could use any basis associated with the eigenvectors of any elliptic second-order partial differential operator. We can take the following matrices as a basis for $\mathfrak{se}(2)$:

$$\mathfrak{X}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{X}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{X}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

For $\mathfrak{X} \in \mathfrak{se}(2)$, consider the one-parameter subgroup $\exp(t\mathfrak{X})$ of $\mathfrak{se}(2)$, where $\exp : \mathfrak{se}(2) \to \mathbb{SE}(2)$ is the exponential map. The left-invariant vector field on $\mathfrak{se}(2)$ can now be defined by

$$\tilde{\mathfrak{X}} f = \frac{\mathrm{d}}{\mathrm{d}t} f(g \exp(t\mathfrak{X}))|_{t=0},$$

where $f : \mathbb{SE}(2) \to \mathbb{R}$. Thus, with respect to the basis (10), we have

$$\exp(t\mathfrak{X}_1) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\exp(t\mathfrak{X}_2) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\exp(t\mathfrak{X}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix},$$

for $t \in \mathbb{R}$.

In polar coordinates (2), the left-invariant vector fields are

$$\tilde{\mathfrak{X}}_1 = \frac{\partial}{\partial\theta},$$
$$\tilde{\mathfrak{X}}_2 = \cos(\theta-\phi)\frac{\partial}{\partial r} + \frac{\sin(\theta-\phi)}{r}\frac{\partial}{\partial\phi},$$
$$\tilde{\mathfrak{X}}_3 = -\sin(\theta-\phi)\frac{\partial}{\partial r} + \frac{\cos(\theta-\phi)}{r}\frac{\partial}{\partial\phi}.$$

In rectangular coordinates (1), the left-invariant vector fields are

$$\tilde{\mathfrak{X}}_1 = \frac{\partial}{\partial\theta},$$
$$\tilde{\mathfrak{X}}_2 = \cos(\theta)\frac{\partial}{\partial r_1} - \sin(\theta)\frac{\partial}{\partial r_2},$$
$$\tilde{\mathfrak{X}}_3 = \sin(\theta)\frac{\partial}{\partial r_1} + \cos(\theta)\frac{\partial}{\partial r_2}.$$

The Laplacian on $\mathbb{SE}(2)$ is

$$\Delta = -\tilde{\mathfrak{X}}_1^2 - \tilde{\mathfrak{X}}_2^2 - \tilde{\mathfrak{X}}_3^2$$
$$= -\frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial r_1^2} - \frac{\partial^2}{\partial r_2^2}.$$

The eigenfunctions of the Laplacian are given by $u_{\ell m}(g,p)$ in (5) which have eigenvalues (m^2+p^2), $m\in\mathbb{Z}$, $p\in[0,\infty)$. Define the adjoint of the Laplace operator, Δ^*, by

$$\Delta^* u_{\ell m}(g,p) = \overline{\Delta u_{m\ell}(g,p)},$$

for $g\in\mathbb{SE}(2)$, $p\in[0,\infty)$ and $\ell,m\in\mathbb{Z}$. These are easily determined by looking at the left-invariant vector fields of $\mathbb{SE}(2)$ where details can be found in [18].

The Sobolev condition with respect to the operator $1+\Delta^*+\Delta$, specifying a radius $Q>0$ is

$$\Theta(s,Q) = \left\{ f \,\middle|\, \int_0^\infty (1+\ell^2+m^2+2p^2)^s \|\widehat{f}(p)\|_{\mathrm{tr}}^2 p\,\mathrm{d}p < Q \right\}, \tag{11}$$

where $s>3/2$ and superscript $*$ here is in reference to the formal adjoint of Δ as an operator. We note that Sobolev functions are rapidly decreasing functions.

To set up the statistical problem, consider random $\mathbb{SE}(2)$ elements X, Y and Z with densities f, h and k, respectively, and with X and Z assumed independent. Then if we observe X indirectly via

$$Y = ZX \tag{12}$$

the relationship among the densities is given by convolution $h = k \star f$ as defined in (7). To formalize this we are interested in estimating the unknown density f when f is related to the observation density h by the model

$$h(g) = (k\star f)(g) \qquad g\in\mathbb{SE}(2), \tag{13}$$

and where k is assumed to be a known function.

Taking the Fourier transform of (13) and using the convolution property (8) result in operators that can be represented as infinite matrices

$$\widehat{h}(p) = \widehat{f}(p)\widehat{k}(p)$$

with $p \in [0, \infty)$. The matrix elements of $\widehat{h}(p)$ can be written as

$$\widehat{h}_{\ell m}(p) = \sum_{q=-\infty}^{\infty} \widehat{f}_{\ell q}(p)\widehat{k}_{qm}(p)$$

with $p \in [0, \infty)$. We call $\widehat{k}$ the known distortion operator and assume that it is a compact operator.

Formally,

$$\widehat{f}(p) = \widehat{h}(p)\widehat{k}^{-1}(p) \tag{14}$$

assuming invertibility of $\widehat{k}(p)$, where $\widehat{k}^{-1}(p)$ is the (formal) inverse of $\widehat{k}(p)$. In coordinates the Fourier coefficients for f is

$$\widehat{f}_{\ell m}(p) = \sum_{q=-\infty}^{\infty} \widehat{h}_{\ell q}(p)\widehat{k}_{qm}^{-1}(p)$$

with $p \in [0, \infty)$.

Two major considerations for implementing this procedure are, first, reducing the infinite-dimensional problem down to a finite-dimensional approximation, and second, addressing the ill-posed condition of non-invertibility of the known compact distortion operator. Both of these considerations can be addressed in the following way. In particular, for an operator A acting on a countable Hilbert space so that in some basis, if $A = (a_{ij})_{i,j\in\mathbb{Z}}$ then the compression for some $T > 0$ is denote by $A_T = (a_{ij})_{|i|,|j|\leqslant T}$. We then consider the application of this compression to the operators $\widehat{h}$ and $\widehat{k}$, along with assumptions of invertibility of the latter for each compression.

Assuming that the compression of the distortion operator $\widehat{k}_T(p)$ is invertible for any $T > 0$ and $p \in [0, \infty)$, the finite-dimensional version of (14) is

$$\widehat{h}_T(p)\widehat{k}_T^{-1}(p), \qquad p \in [0, \infty). \tag{15}$$

Even with the invertibility condition imposed, it is possible that the inversion is unstable due to small eigenvalues; consequently, for efficient numerical implementation, regularization is needed. As discussed in [3], a Tikhonov regularized version would use some $\nu > 0$ and

$$\widehat{h}_T(p)\widehat{k}_T^*(p)(\widehat{k}_T(p)\widehat{k}_T^*(p) + \nu\mathbf{I}_{2T+1})^{-1}, \qquad p \in [0, \infty). \tag{16}$$

It is obvious to see that (15) follows as a special case of (16) when $\nu = 0$.

The coefficients $\widehat{h}_{\ell m}$ are unknown since $\widehat{f}_{\ell m}$ are unknown, but, assuming a random sample $Y_1, \ldots, Y_n \in \mathbb{SE}(2)$ is available from h, we can write the empirical Fourier transform coefficients as

$$\widehat{h}^n_{\ell m}(p) = \frac{1}{n}\sum_{j=1}^{n} u_{\ell m}\big(Y_j^{-1}, p\big) \qquad |\ell|, |m| \leqslant T$$

for some $T > 0$. This in turn allows us to write

$$\widehat{f}^n_{\ell m}(p) = \sum_{|q|\leqslant T} \widehat{h}^n_{\ell q}(p)\widehat{k}^{-1}_{qm}(p), \qquad |\ell|, |m| \leqslant T$$

for (15). For (16) define

$$\widehat{f}^{n\nu}_{\ell m}(p) = \sum_{|q|\leqslant T} \widehat{h}^n_{\ell q}(p)(k_T^*(p)(\widehat{k}_T(p)\widehat{k}_T^*(p) + \nu\mathbf{I}_{2T+1})^{-1})_{qm}, \qquad |\ell|, |m| \leqslant T$$

and note that $\widehat{f}^{n0}_{\ell m}(p) = \widehat{f}^n_{\ell m}(p)$.

Finally, define

$$f^n(g) = \int_0^T \sum_{|\ell|,|m|\leqslant T} \widehat{f}^n_{\ell m}(p) u_{m\ell}(g,p)p\,\mathrm{d}p \tag{17}$$

and

$$f^{n\nu}(g) = \int_0^T \sum_{|\ell|,|m|\leqslant T} \widehat{f}^{n\nu}_{\ell m}(p) u_{m\ell}(g,p)p\,\mathrm{d}p, \tag{18}$$

and note that $f^{n0} = f^n$.

4. Integrated mean-squared error bounds

In this section, we will assess the performance of the estimators (17) and (18) in terms of the integrated mean-squared error.

The following notations will be used. Let $\{a_n\}$ and $\{b_n\}$ denote two real sequences of numbers. We write $a_n \ll b_n$ to mean $a_n \leqslant Cb_n$ for some $C > 0$, as $n \to \infty$, the Vinogradov notation. The notation $a_n = o(b_n)$ will mean $a_n/b_n \to 0$, as $n \to \infty$, consequently, the expression $o(1)$ would mean a sequence converging to 0. Furthermore, $a_n \asymp b_n$ whenever $a_n \ll b_n$ and $b_n \ll a_n$, as $n \to \infty$. Finally, for some operator A on a separable (countably infinite-dimensional) Hilbert space denote the spectrum of it by $\Lambda(A)$.

For the compression $\widehat{f}_T(p), 0 \leqslant p \leqslant T, T > 0$, spectral conditions can be formulated as, for $p, T > 0$ and $\beta \geqslant 0$, there exists $0 < \gamma_0 \leqslant \gamma_1 < \infty$ such that

$$\Lambda(\widehat{f}_T(p)) \subset [\gamma_0(T^2+p^2)^{-\beta}, \gamma_1(T^2+p^2)^{-\beta}] \tag{19}$$

as $T \to \infty$, and β is a smoothness parameter dictating a polynomial rate of decay.

We will also need the following decay condition on the restriction as in for $p, T > 0$ and $\beta \geqslant 0$:

$$\int_0^T \sum_{|q|\leqslant T, |q'|>T} |\widehat{k}_{q'q}(p)|^2 p\,\mathrm{d}p = o(T^{-2s-4\beta-1}) \tag{20}$$

as $T \to \infty$.

We have the following where we will denote the $L^2(\mathbb{SE}(2))$ norm by $\|\cdot\|_2$. Note also that $\mathbb{E}$ denotes expectation with respect to the density h in all following appearances.

Theorem 4.1. *Suppose for $p, T > 0$ and $\beta \geqslant 0$, there exists $0 < \gamma_0 \leqslant \gamma_1 < \infty$ that satisfy (19) and (20). If $\nu \geqslant 0$ then*

$$\mathbb{E}\|f^{n\nu} - f\|_2^2 \ll \frac{T^3}{n} \frac{2^{-\beta}\gamma_1 T^{-2\beta} + n\nu^2}{(\gamma_0 T^{-2\beta} + \nu)^2} + T^{-2s}$$

as $T, n \to \infty$ for $f \in \Theta(s, Q)$, $s > 3/2$.

In the special case of (14) and hence (17), we have the following.

Corollary 4.2. *If $\nu = 0$, then*

$$\mathbb{E}\|f^n - f\|_2^2 \ll \frac{T^{2\beta+3}}{n} + T^{-2s}$$

as $T, n \to \infty$ for $f \in \Theta(s, Q)$, $s > 3/2$.

The following best upper bound rate can be obtained for the special case of (14) and hence (17).

Corollary 4.3. *If* $\nu = 0$ *and* $T \asymp n^{1/(2s+2\beta+3)}$ *then*

$$\mathbb{E}\|f^n - f\|_2^2 \ll n^{-\frac{2s}{2s+2\beta+3}}$$

as $n \to \infty$ *for* $f \in \Theta(s, Q)$, $s > 3/2$.

We would also like to point out that in the case where the X is observed directly, i.e., $Z = (\mathbf{I}_2, \mathbf{0})'$ in (12), then this would correspond to $\beta = 0$ and we would be performing density estimation directly.

For this we have the following.

Corollary 4.4. *If* $\nu = 0$ *and* $T \asymp n^{1/(2s+3)}$, *then*

$$\mathbb{E}\|f^n - f\|_2^2 \ll n^{-\frac{2s}{2s+3}}$$

as $n \to \infty$ *for* $f \in \Theta(s, Q)$, $s > 3/2$.

The last two results indicate an upper bound rate of convergence that is compatible with lower bound rates of convergence in the statistical literature over compact manifolds, hence Lie groups, see, for example, [11]. These are the first such results for non-compact non-commutative Lie groups.

As a result of corollary 4.3, we are interested in how the choice of the regularization parameter ν affects integrated mean-squared error. We have the following results which provide sufficient conditions that give us the same $L^2(\mathbb{SE}(2))$-bound.

Theorem 4.5. *Suppose for* $p, T > 0$ *and* $\beta \geqslant 0$, *there exists* $0 < \gamma_0 \leqslant \gamma_1 < \infty$ *that satisfy (19) and (20). If* $T^{2\beta} = o(n)$ *and*

$$0 \leqslant \nu \leqslant \frac{\xi\gamma_0 + \sqrt{\xi^2\gamma_0^2 + (n - \xi T^{2\beta})(2^{-\beta}\gamma_1 - \xi\gamma_0)T^{-2\beta}}}{n - \xi T^{2\beta}}$$

then

$$\mathbb{E}\|f^{n\nu} - f\|_2^2 \ll \frac{T^{2\beta+3}}{n} + T^{-2s},$$

as $T, n \to \infty$ *for* $f \in \Theta(s, Q)$, $s > 3/2$.

Consequently, we obtain the following.

Corollary 4.6. *If* $T \asymp n^{1/(2s+2\beta+3)}$ *and* $0 \leqslant \nu \leqslant \frac{\gamma_0\xi}{n}(1 + o(1))$ *then*

$$\mathbb{E}\|f^{n\nu} - f\|_2^2 \ll n^{-\frac{2s}{2s+2\beta+3}}$$

as $n \to \infty$ *for* $f \in \Theta(s, Q)$, $s > 3/2$.

Again, with respect to direct density estimation as in corollary 4.4, we have the following:

Corollary 4.7. *If* $T \asymp n^{1/(2s+3)}$ *and* $0 \leqslant \nu \leqslant \frac{\gamma_0\xi}{n}(1 + o(1))$ *then*

$$\mathbb{E}\|f^{n\nu} - f\|_2^2 \ll n^{-\frac{2s}{2s+3}}$$

as $n \to \infty$ *for* $f \in \Theta(s, Q)$, $s > 3/2$.

5. Discussion

We would like to make some remarks on the conditions in theorems 4.1 and 4.5. In particular, we would like to verify that they are non-vacuous and that they are relatively easy to check.

Starting with the first condition (19), one can adopt the definition of the $\mathbb{SO}(3)$-Laplace distribution as specified in [9] where

$$\widehat{k}_{\ell m}(p) = \frac{1}{1+\sigma^2(\ell^2+m^2+2p^2)}\delta_{\ell m},$$

$\sigma^2 \geqslant 1/2$, $\delta_{\ell m}$ denotes Kronecker delta and $\ell, m \in \mathbb{Z}$. This would correspond to the situation where $\gamma = 1/(2\sigma^2)$ and $\beta = 1$.

By Fourier inversion (6), the corresponding function would be

$$k(g) = \int_0^\infty \sum_{\ell=-\infty}^{\infty} \frac{1}{1+2\sigma^2(\ell^2+p^2)} u_{\ell\ell}(g,p)\,p\,\mathrm{d}p,$$

for $g \in \mathbb{SE}(2)$. One can verify that this is a probability density function and will be referred to as the $\mathbb{SE}(2)$-Laplace distribution.

As for condition (20), we would like to remark that the condition on the distortion operator $\widehat{k}$ in many cases will be trivially satisfied, or can at least be easily checked. In particular, if we assume that $k \in L^2(\mathbb{SE}(2))$, then through the Plancherel formula (9),

$$\int_{\mathbb{SE}(2)} |k(g)|^2\,\mathrm{d}g = \int_0^\infty \sum_{q=\infty}^{\infty}\sum_{q'=\infty}^{\infty} |\widehat{k}_{q'q}(p)|^2\,p\,\mathrm{d}p$$

is finite, hence the condition

$$\int_0^T \sum_{|q|\leqslant T, |q'|>T} |\widehat{k}_{q'q}(p)|^2\,p\,\mathrm{d}p = o(T^{-2s-4\beta-1})$$

as $T \to \infty$ is stating the manner in which an infinite part of the summation vanishes.

As examples if $\widehat{k}(p)$ is diagonal as in the $\mathbb{SE}(2)$-Laplace distribution, or is band-limited for all $p \in [0,\infty)$, then the condition is trivial in the former since $\widehat{k}_{q'q}(p) = 0$ for $|q'| > T$ and $|q| \leqslant T$, while in the latter, $\widehat{k}(p) = 0$ for $p > T$ at some finite value. See [3, 20] for more examples.

More generally, if $\widehat{k}(p)$ is a banded matrix involving nonzero sub-diagonal and super-diagonal terms, then the above condition becomes

$$\int_0^T (|\widehat{k}_{-T-1,T}(p)|^2 + |\widehat{k}_{T+1,-T}(p)|^2)\,p\,\mathrm{d}p = o(T^{-2s-4\beta-1})$$

as $T \to \infty$. In such cases, this condition describes a joint decay condition on the sub- and super-diagonal terms.

6. Proofs

The method of proof is to take the integrated mean-squared error and separate the latter into the integrated variance and integrated bias components as

$$\mathbb{E}\|f^{n\nu} - f\|_2^2 = \mathbb{E}\|f^{n\nu} - \mathbb{E}f^{n\nu}\|_2^2 + \|\mathbb{E}f^{n\nu} - f\|_2^2.$$

The object then is to calculate each component separately.

Condition (19) involves the spectral structure of $\widehat{k}(p)$. In particular, for $\widehat{k}_T(p)$ a $(2T+1)\times(2T+1)$ matrix for each $T>0$ and $p\in[0,\infty)$, let

$$\widehat{k}_T(p)\widehat{k}_T^*(p) = V_T(p)D_T(p)V_T^*(p) \tag{21}$$

denote the spectral decomposition, where $V_T(p)V_T^*(p) = V_T^*(p)V_T(p) = \mathbf{I}_{2T+1}$, $D_T(p)$ is a diagonal matrix with diagonal entries $d_j(p)$, $|j|\leqslant T$ and $p\in[0,\infty)$ and superscript $*$ means conjugate transpose. Consequently, (19) is equivalent to the statement, for $p, T>0$ and $\beta\geqslant 0$, there exists $0<\gamma_0\leqslant\gamma_1<\infty$ such that

$$\gamma_0(T^2+p^2)^{-\beta}\leqslant d_j(p)\leqslant\gamma_1(T^2+p^2)^{-\beta} \tag{22}$$

for $|j|\leqslant T$.

The integrated bias term is presented below with the details of the calculation in appendix A. In particular

$$\|f-\mathbb{E}f^{n\nu}\|_2^2 \ll \sum_{j=-T}^{T}\int_0^T\left(\frac{\nu}{d_j(p)+\nu}\right)^2 p\,\mathrm{d}p + T^{-2s}(1+o(1)) \tag{23}$$

as $T\to\infty$ and $f\in\Theta(Q,s)$, $s>3/2$.

The integrated variance term is presented below with the details of the calculation in appendix B. In particular

$$\mathbb{E}\|f^{n\nu}-\mathbb{E}f^{n\nu}\|^2 \ll \frac{1}{n}\int_0^T\sum_{|j|\leqslant T}\frac{d_j(p)}{(d_j(p)+\nu)^2}p\,\mathrm{d}p \tag{24}$$

as $T\to\infty$.

Finally, by combining (23) and (24), we can determine the integrated mean-squared error for the regularized case

$$\mathbb{E}\|f^{n\nu}-f\|^2 \ll \frac{1}{n}\int_0^T\sum_{j=-T}^{T}\frac{d_j(p)+n\nu^2}{(d_j(p)+\nu)^2}p\,\mathrm{d}p + T^{-2s} \tag{25}$$

as $n, T\to\infty$.

6.1. Proof of theorem 4.1

By (22) we have

$$\begin{aligned}
\int_0^T\sum_{j=-T}^{T}\frac{d_j+n\nu^2}{(d_j+\nu)^2}p\,\mathrm{d}p &\leqslant \int_0^T\sum_{j=-T}^{T}\frac{\gamma_1(T^2+p^2)^{-\beta}+n\nu^2}{(\gamma_0(T^2+p^2)^{-\beta}+\nu)^2}p\,\mathrm{d}p\\
&\leqslant (2T+1)\frac{2^{-\beta}\gamma_1T^{-2\beta}+n\nu^2}{(\gamma_0T^{-2\beta}+\nu)^2}\int_0^T p\,\mathrm{d}p\\
&\ll T^3\frac{2^{-\beta}\gamma_1T^{-2\beta}+n\nu^2}{(\gamma_0T^{-2\beta}+\nu)^2},
\end{aligned}$$

as $n, T\to\infty$.

Using this inequality in (25) gives us

$$\mathbb{E}\|f^{n\nu}-f\|^2 \ll \frac{T^3}{n}\frac{2^{-\beta}\gamma_1T^{-2\beta}+n\nu^2}{(\gamma_0T^{-2\beta}+\nu)^2}+T^{-2s}$$

as $n, T\to\infty$.

6.2. Proof of theorem 4.5

The approach here is to obtain the inequality

$$\frac{2^{-\beta}\gamma_1 T^{-2\beta} + n\nu^2}{(\gamma_0 T^{-2\beta} + \nu)^2} \leqslant \xi T^{2\beta}$$

for some $\xi > 0$.

One notes that by assuming $T^{2\beta} = o(n)$, by solving for ν one obtains the dominant root as

$$\frac{\xi\gamma_0 + \sqrt{\xi^2\gamma_0^2 + (n - \xi T^{2\beta})(2^{-\beta}\gamma_1 - \xi\gamma_0)T^{-2\beta}}}{n - \xi T^{2\beta}}$$

thus providing a sufficient condition on ν.

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Appendix A

In this appendix, we will provide the technical details of the bias (23) and variance (24) calculations. For convenience we will adopt the following notation:

$$\kappa^\nu(p) = \widehat{k}_T(p)^*(\widehat{k}_T(p)\widehat{k}_T(p)^* + \nu\mathbf{I}_{2T+1})^{-1}.$$

A.1. Bias

First consider evaluating

$$\begin{aligned} f(g) - \mathbb{E}f^{n\nu}(g) &= \int_0^\infty \sum_{|\ell|>T}\sum_{|m|>T} \widehat{f}_{\ell m}(p)u_{m\ell}(g,p)p\,\mathrm{d}p \\ &\quad + \int_0^\infty \sum_{|\ell|\leqslant T}\sum_{|m|>T} \widehat{f}_{\ell m}(p)u_{m\ell}(g,p)p\,\mathrm{d}p \\ &\quad + \int_0^\infty \sum_{|\ell|>T}\sum_{|m|\leqslant T} \widehat{f}_{\ell m}(p)u_{m\ell}(g,p)p\,\mathrm{d}p \\ &\quad + \int_T^\infty \sum_{|\ell|\leqslant T}\sum_{|m|\leqslant T} \widehat{f}_{\ell m}(p)u_{m\ell}(g,p)p\,\mathrm{d}p \\ &\quad + \int_0^T \sum_{|\ell|\leqslant T}\sum_{|m|\leqslant T} (\widehat{f}_{\ell m}(p) - \mathbb{E}f^{n\nu}_{\ell m}(p))u_{m\ell}(g,p)p\,\mathrm{d}p. \end{aligned}$$

The first four terms will simplify primarily due to the Sobolev condition (11). The interim step is to multiply by the factor $(1+\ell^2+m^2+2p^2)^{-s}(1+\ell^2+m^2+2p^2)^s$. This gives a factor of T^{-2s} due to the compression and leaves the first four terms following the above equality (multiplied by $(1+\ell^2+m^2+2p^2)^s$) bounded by the Sobolev condition (11).

In particular, let us calculate one of the terms. The rest will follow similarly,

$$\begin{aligned}
\int_0^\infty \sum_{|\ell|,|m|>T} |\widehat{f}_{\ell m}(p)|^2 p \,\mathrm{d}p
&\leqslant \int_0^\infty \sum_{|\ell|,|m|>T} (1+\ell^2+m^2+2p^2)^{-s}(1+\ell^2+m^2+2p^2)^{s}|\widehat{f}_{\ell m}(p)|^2 p\,\mathrm{d}p \\
&\ll T^{-s}\int_0^\infty \sum_{|\ell|,|m|>T} (1+\ell^2+m^2+2p^2)^{s}|\widehat{f}_{\ell m}(p)|^2 p\,\mathrm{d}p \\
&\leqslant QT^{-s},
\end{aligned}$$

since $f \in \Theta(s, Q)$, $s > 3/2$.

Now, consider the final term of the bias decomposition without the integration. In particular, we have

$$\begin{aligned}
\sum_{|\ell|,|m|\leqslant T} \left|\widehat{f}_{\ell m}(p) - \mathbb{E}\widehat{f}^{n\nu}_{\ell m}(p)\right|^2 &= \sum_{|\ell|,|m|\leqslant T}\left|\widehat{f}_{\ell m}(p) - \sum_{|q|\leqslant T}\mathbb{E}\widehat{h}^n_{\ell q}(p)\kappa^\nu_{qm}(p)\right|^2 \\
&= \sum_{|\ell|,|m|\leqslant T}\left|\widehat{f}_{\ell m}(p) - \sum_{|q|\leqslant T}\widehat{h}_{\ell q}(p)\kappa^\nu_{qm}(p)\right|^2 \\
&= \sum_{|\ell|,|m|\leqslant T}\left|\widehat{f}_{\ell m}(p) - \sum_{|q|\leqslant T}\sum_{q'=-\infty}^{\infty}\widehat{f}_{\ell q'}(p)\widehat{k}_{q'q}(p)\kappa^\nu_{qm}(p)\right|^2 \\
&= \sum_{|\ell|,|m|\leqslant T}\left|\widehat{f}_{\ell m}(p) - \sum_{q'=-\infty}^{\infty}\widehat{f}_{\ell q'}(p)\sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^\nu_{qm}(p)\right|^2 \\
&= \sum_{|\ell|,|m|\leqslant T}\left|\sum_{q'=-\infty}^{\infty}\widehat{f}_{\ell q'}(p)\left(\delta_{q'm} - \sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^\nu_{qm}(p)\right)\right|^2 \\
&\leqslant \sum_{|\ell|,|m|\leqslant T}\sum_{q'=-\infty}^{\infty}|\widehat{f}_{\ell q'}(p)|^2\sum_{q'=-\infty}^{\infty}\left|\delta_{q'm} - \sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^\nu_{qm}(p)\right|^2 \\
&\leqslant \sum_{|\ell|\leqslant T}\sum_{q'=-\infty}^{\infty}|\widehat{f}_{\ell q'}(p)|^2\sum_{|m|\leqslant T}\sum_{q'=-\infty}^{\infty}\left|\delta_{q'm} - \sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^\nu_{qm}(p)\right|^2 .
\end{aligned}$$

We note that we can bound

$$\sum_{|\ell|\leqslant T}\sum_{q'=-\infty}^{\infty}|\widehat{f}_{\ell q'}(p)|^2 \leqslant \int_{\mathbb{SE}(2)}|f(g)|^2\,\mathrm{d}g,$$

and furthermore,

$$\begin{aligned}
\sum_{|m|\leqslant T}\sum_{q'=-\infty}^{\infty}\left|\delta_{q'm} - \sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^\nu_{qm}(p)\right|^2 &= \sum_{|m|\leqslant T}\sum_{|q'|<T}\left|\delta_{q'm} - \sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^\nu_{qm}(p)\right|^2 \\
&\quad + \sum_{|m|\leqslant T}\sum_{|q'|>T}\left|\sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^\nu_{qm}(p)\right|^2 . \qquad \text{(A.1)}
\end{aligned}$$

The first term has the following calculation:

$$\sum_{|m|\leqslant T}\sum_{|q'|<T}\left|\delta_{q'm}-\sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^{\nu}_{qm}(p)\right|^2$$

$$\begin{aligned}
&= \|\mathbf{I}_{2T+1}-\widehat{k}_T(p)\widehat{k}^*_T(p)(\widehat{k}_T(p)\widehat{k}^*_T(p)+\nu\mathbf{I}_{2T+1})^{-1}\|^2_{\mathrm{tr}}\\
&= \|\mathbf{I}_{2T+1}-V_T(p)D_T(p)(D_T(p)+\nu\mathbf{I}_{2T+1})^{-1}V_T(p)^*\|^2_{\mathrm{tr}}\\
&= \|\mathbf{I}_{2T+1}-D_T(p)\,(D_T(p)+\nu\mathbf{I}_{2T+1})^{-1}\,\|^2_{\mathrm{tr}}\\
&= \sum_{j=-T}^{T}\left(\frac{\nu}{d_j(p)+\nu}\right)^2.
\end{aligned}$$

Now consider the second term in (A.1) along with integration. We note that

$$\int_0^T\sum_{|m|\leqslant T}\sum_{|q'|>T}\left|\sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)\kappa^{\nu}_{qm}(p)\right|^2 p\,\mathrm{d}p$$

$$\begin{aligned}
&= \int_0^T\sum_{|m|\leqslant T}\sum_{|q'|>T}\left|\sum_{|q|\leqslant T}\widehat{k}_{q'q}(p)((\widehat{k}_T(p)\widehat{k}_T(p)^*+\nu\mathbf{I}_{2T+1})^{-1})_{qm}(p)\right|^2 p\,\mathrm{d}p\\
&\leqslant \int_0^T\sum_{|q'|>T}\sum_{|q|\leqslant T}|\widehat{k}_{q'q}(p)|^2\sum_{|m|\leqslant T}\sum_{|q|\leqslant T}|((\widehat{k}_T(p)\widehat{k}_T(p)^*+\nu\mathbf{I}_{2T+1})^{-1})_{qm}|^2 p\,\mathrm{d}p\\
&= \int_0^T\sum_{|j|\leqslant T}\frac{1}{(d_j(p)+\nu)^2}\sum_{|q'|>T}\sum_{|q|\leqslant T}|\widehat{k}_{q'q}(p)|^2 p\,\mathrm{d}p\\
&\leqslant (2T+1)\int_0^T\sup_{|j|\leqslant T}\frac{1}{(d_j(p)+\nu)^2}\sum_{|q'|>T}\sum_{|q|\leqslant T}|\widehat{k}_{q'q}(p)|^2 p\,\mathrm{d}p\\
&\ll T\int_0^T\sup_{|j|\leqslant T}\frac{1}{d_j(p)^2}\sum_{|q'|>T}\sum_{|q|\leqslant T}|\widehat{k}_{q'q}(p)|^2 p\,\mathrm{d}p\\
&\ll T^{4\beta+1}\int_0^T\sum_{|q'|>T}\sum_{|q|\leqslant T}|\widehat{k}_{q'q}(p)|^2 p\,\mathrm{d}p,
\end{aligned}$$

as $T\to\infty$.

Thus by using this we get

$$\begin{aligned}
\int_0^T\sum_{|\ell|\leqslant T}\sum_{|m|\leqslant T}\left|\widehat{f}_{\ell m}(p)-\mathbb{E}\widehat{f}^{n\nu}_{\ell m}(p)\right|^2 p\,\mathrm{d}p &\ll \int_0^T\sum_{j=-T}^{T}\left(\frac{\nu}{d_j(p)+\nu}\right)^2 p\,\mathrm{d}p\\
&\quad +T^{-2s}\left(1+T^{2s+4\beta+1}\int_0^T\sum_{|q'|>T}\sum_{|q|\leqslant T}|\widehat{k}_{q'q}(p)|^2 p\,\mathrm{d}p\right),
\end{aligned}$$

as $T\to\infty$.

Appendix B. Variance

We note that in the general case

$$\mathbb{E}\|f^{n\nu}-\mathbb{E}f^{n\nu}\|^2=\mathbb{E}\int_0^T\sum_{|\ell|,|m|\leqslant T}\left|\widehat{f}^{n\nu}_{\ell m}(p)-\mathbb{E}\widehat{f}^{n\nu}_{\ell m}(p)\right|^2 p\,\mathrm{d}p$$
$$=\mathbb{E}\int_0^T\sum_{|\ell|,|m|\leqslant T}\left|\sum_{|q|\leqslant T}\left(\widehat{h}^n_{\ell q}(p)-\mathbb{E}\widehat{h}^n_{\ell,q}(p)\right)\kappa^\nu_{qm}(p)\right|^2 p\,\mathrm{d}p. \tag{B.1}$$

Now apply expectation to just the summand and let Y denote a generic random $\mathbb{SE}(2)$ quantity

$$\sum_{|\ell|,|m|\leqslant T}\mathbb{E}\left|\sum_{|q|\leqslant T}\left(\widehat{h}^n_{\ell q}(p)-\mathbb{E}\widehat{h}^n_{\ell,q}(p)\right)\kappa^\nu_{qm}(p)\right|^2$$
$$\leqslant\sum_{|\ell|<\infty}\sum_{|m|\leqslant T}\mathbb{E}\left|\sum_{|q|\leqslant T}\left(\widehat{h}^n_{\ell q}(p)-\mathbb{E}\widehat{h}^n_{\ell q}(p)\right)\kappa^\nu_{qm}(p)\right|^2$$
$$=\sum_{|\ell|<\infty}\sum_{|m|\leqslant T}\frac{1}{n}\mathbb{E}\left|\sum_{|q|\leqslant T}\left(u_{\ell q}(Y^{-1},p)-\widehat{h}_{\ell q}(p)\right)\kappa^\nu_{qm}(p)\right|^2$$
$$\ll\sum_{|\ell|<\infty}\sum_{|m|\leqslant T}\frac{1}{n}\mathbb{E}\sum_{|q'|,|q|\leqslant T}u_{\ell q}(Y^{-1},p)\overline{u_{\ell q'}(Y^{-1},p)}\kappa^\nu_{qm}(p)\overline{\kappa^\nu_{q'm}(p)}$$
$$=\sum_{|m|\leqslant T}\frac{1}{n}\mathbb{E}\sum_{|q'|,|q|\leqslant T}\left\{\sum_{|\ell|<\infty}u_{\ell q}(Y^{-1},p)\overline{u_{\ell q'}(Y^{-1},p)}\right\}\kappa^\nu_{qm}(p)\overline{\kappa^\nu_{q'm}(p)}$$
$$=\sum_{|m|\leqslant T}\frac{1}{n}\mathbb{E}\sum_{|q'|,|q|\leqslant T}\left\{\sum_{|\ell|<\infty}\overline{u_{q\ell}(Y,p)}\overline{u_{\ell q'}(Y^{-1},p)}\right\}\kappa^\nu_{qm}(p)\overline{\kappa^\nu_{q'm}(p)}$$
$$=\sum_{|m|\leqslant T}\frac{1}{n}\mathbb{E}\sum_{|q'|,|q|\leqslant T}\{\overline{u_{qq'}(YY^{-1},p)}\}\kappa^\nu_{qm}(p)\overline{\kappa^\nu_{q'm}(p)}$$
$$=\sum_{|m|\leqslant T}\frac{1}{n}\mathbb{E}\sum_{|q'|,|q|\leqslant T}u_{qq'}(e,p)\kappa^\nu_{qm}(p)\overline{\kappa^\nu_{q'm}(p)}$$
$$=\sum_{|m|\leqslant T}\frac{1}{n}\sum_{|q'|,|q|\leqslant T}\delta_{qq'}\kappa^\nu_{qm}(p)\overline{\kappa^\nu_{q'm}(p)}$$
$$=\frac{1}{n}\sum_{|m|\leqslant T}\sum_{|q|\leqslant T}\left|\kappa^\nu_{qm}(p)\right|^2. \tag{B.2}$$

Here we used the fact that $U(g,p)^*=U(g^{-1},p)$ as well as the homomorphism property $U(g,p)U(g^{-1},p)=U(gg^{-1},p)=U(e,p)=(\delta_{\ell m})$ where $e=(\mathbf{I}_2,\mathbf{0})$, the unit element of $\mathbb{SE}(2)$ and $\delta_{\ell m}$ is the Kronecker delta.

We note that

$$\sum_{|q|,|m|\leqslant T} \left|\kappa^{\nu}_{qm}(p)\right|^2 = \left\|\kappa^{\nu}_{qm}(p)\right\|^2_{\mathrm{tr}}$$
$$= \mathrm{tr}\{\widehat{k}_T(p)\widehat{k}_T(p)^*(\widehat{k}_T(p)\widehat{k}_T(p)^* + \nu\mathbf{I}_{2T+1})^{-2}\}$$
$$= \sum_{|j|\leqslant T} \frac{d_j(p)}{(d_j(p)+\nu)^2}. \tag{B.3}$$

Consequently, by substituting (B.3) and (B.2) into (B.1), we obtain the formula for the variance (24).

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