Homework 5

Computational Math

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a For the test functions, choose u, v from the space of continuous functions supported on Ω; ie supp $u \subseteq \Omega$. Now for any solution u with test function v we must have:

$$\int_{\Omega} -u_{xx}(x,y)v(x,y) - u_{yy}(x,y)v(x,y)d\Omega = \int_{\Omega} f(x,y)v(x,y)d\Omega$$

Starting on the left hand side, we work term by term:

$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} -u_{xx}(x,y)v(x,y)dxdy &= \int_{-1}^{1}\left(-u_x(x,y)v(x,y)|_{-1}^{1} + \int_{-1}^{1} u_x(x,y)v_x(x,y)dx\right)dy \\
&= \int_{\Omega} u_x(x,y)v_x(x,y)d\Omega \\
&= p_0(u,v)
\end{aligned}$$

By symmetry:

$$p_1(u,v) = \int_{\Omega} u_{yy}v d\Omega = \int_{\Omega} u_y(x,y)v_y(x,y)d\Omega$$

For the right hand side, we just get:

$$b(v) = \int_{\Omega} f(x,y)v(x,y)d\Omega$$

And so the weak form of the variational problem is:

$$p_0(u,v) + p_1(u,v) = b(v)$$

b Let $(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)$ be the nodes of the element, oriented clockwise. We now solve for $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ for the node (x_1,y_1). Plugging in values, we get the following linear system:

$$\begin{aligned}
\alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1 y_1 &= 1 \\
\alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 x_1 y_2 &= 0 \\
\alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 x_1 y_3 &= 0 \\
\alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 x_4 y_4 &= 0
\end{aligned}$$

For the sake of simplicity, we rewrite the system in matrix form:

$$M\alpha = c$$

Where α is the vector of coefficients. Since c is a basis vector, the values for α at various nodes are just the corresponding rows of M^{-1}.

Now to construct the matrix equations for this system, we first consider the weak form from part a on a per element basis. Thus let φ^i, φ^j be two test functions on a quad element where

$$\varphi^i(x) = \alpha_1^i + \alpha_2^i x + \alpha_3^i y + \alpha_4^i xy$$

And:

$$\varphi_x^i(x) = \alpha_2^i + \alpha_4^i y$$

To integrate $p_0(\varphi^i, \varphi^j)$, we split the integral into two triangles, indexed by $\Delta(1,2,3)$ and $\Delta(1,3,4)$, then integrate in barycentric coordinates. We do this for the first triangle $\Delta(1,2,3)$ now. Let:

$$J = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{pmatrix}$$

And so we get the following:

$$\begin{aligned}
\int\limits_{\Delta(1,2,3)} \varphi_x^i(x,y)\varphi_x^j(x,y)dxdy &= \frac{1}{\det J}\int\limits_0^1 \int\limits_0^{1-\lambda_2} p_0(\varphi^i(J(\lambda_1,\lambda_2)), \varphi^j(J(\lambda_1,\lambda_2)))d\lambda_1 d\lambda_2 \\
&= \int\limits_0^1 \int\limits_0^{1-\lambda_2} \alpha_2^i\alpha_2^j + (\alpha_4^i\alpha_2^j + \alpha_2^i\alpha_4^j)(J_{2,1}\lambda_1 + J_{2,2}\lambda_2) + \alpha_4^i\alpha_4^j(J_{2,1}\lambda_1 + J_{2,2}\lambda_2)^2 d\lambda_1 d\lambda_2 \\
&= \frac{1}{2\det J}\left(\alpha_2^i\alpha_2^j + \frac{1}{3}(J_{2,1} + J_{2,2})(\alpha_2^i\alpha_4^j + \alpha_2^j\alpha_4^i) + \frac{1}{6}(J_{2,1} + J_{2,2})^2\alpha_4^i\alpha_4^j\right) \\
&= T_0^1
\end{aligned}$$

And by symmetry for p_1:

$$\int\limits_{\Delta(1,2,3)} \varphi_y^i(x,y)\varphi_y^j(x,y)dxdy = \frac{1}{2\det J}\left(\alpha_3^i\alpha_3^j + \frac{1}{3}(J_{1,1} + J_{1,2})(\alpha_3^i\alpha_4^j + \alpha_3^j\alpha_4^i) + \frac{1}{6}(J_{1,1} + J_{1,2})^2\alpha_4^i\alpha_4^j\right) = T_1^1$$

And let T_0^2, T_1^2 be the quantities for the second triangle (the only thing which changes is the value of J), and thus we get:

$$p_0(\varphi^i, \varphi^j) + p_1(\varphi^i, \varphi^j) = T_0^1 + T_1^1 + T_0^2 + T_1^2$$

And so the final matrix is just formed by summing over all such values. Computing the integral over f can be done using numerical quadrature.

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