# Spectral Rigid Body Dynamics

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May 6, 2010

# Overview

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Pros:

- + Pretty accurate at human energy scales
- + Good for stiff materials (ie metals, plastics etc.)
- + Easy kinematic constraints (useful for mechanisms)
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- + Pretty accurate at human energy scales
- + Good for stiff materials (ie metals, plastics etc.)
- + Easy kinematic constraints (useful for mechanisms)
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Cons:

- - Inaccurate at extremely large energies
- - Bad for materials with low elastic modulus
- - Not always solvable! (See: Painleve's paradox)

# What is a Rigid Body?

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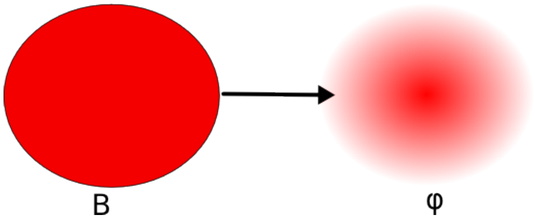

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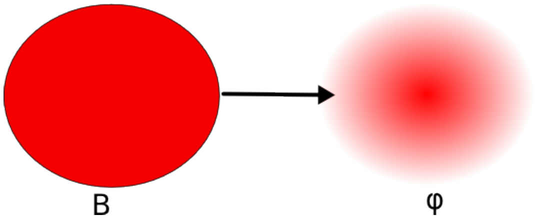


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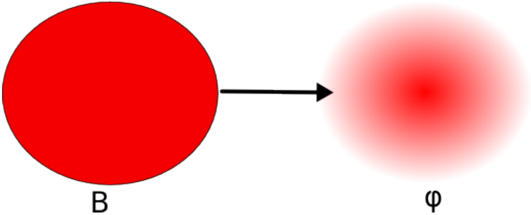


$\varphi$ represents the mass distribution of $B$

$\varphi(x) = 0$ indicates $B$ does not occupy the space at $x$

# Configuration Space of a Rigid Body

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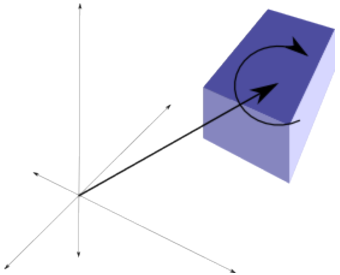

Can be parametrized by a translation $t$ and a rotation $R$

Matrix: $\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$

$\binom{d+1}{2}$ degrees of freedom

Tangent space: $\mathfrak{so}(d+1)$

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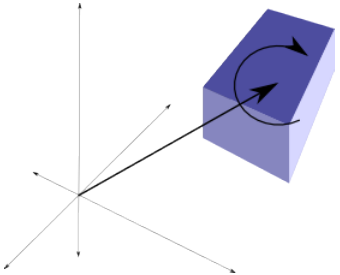

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**Motions of rigid objects** $\cong$ **curves** $q(t) \subset SE(d)$

# Lagrangian Mechanics

Turns physics into an optimization problem.

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Given a configuration curve $q$ at time $t$, define the *Lagrangian*

$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - U(q, t)$$

Where $T(\dot{q}) = \operatorname{tr} \frac{1}{2}\dot{q}^T M\dot{q}$ is the kinetic energy and $U$ is a potential

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In other words:

$$\operatorname*{argmin}_{q:[t_0,t_1)\to SE(d)} \int_{t_0}^{t_1} L(q, \dot{q}, t)dt$$

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Solve:

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Unpack definitions and simplify:

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Newton's equations!

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A: Tensor sum

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| Motion | $q(t) \cong q_i(t) \oplus q_j(t)$ |
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Scales to $n$ bodies, get Lagrangian in $SE(d)^n$

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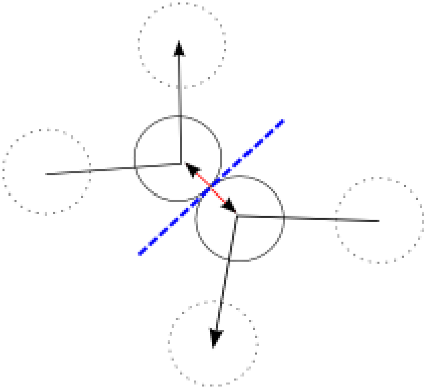

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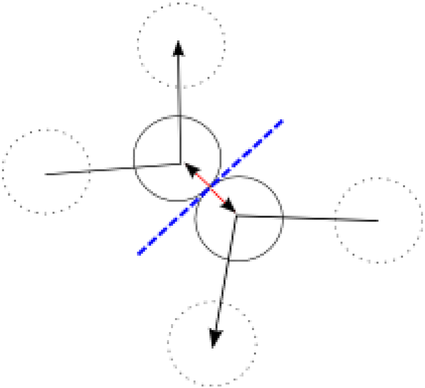

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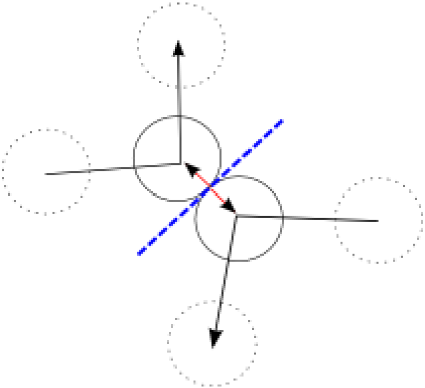

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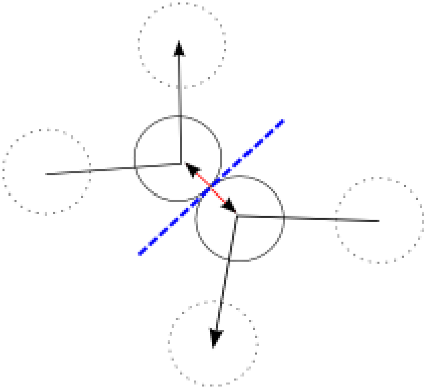

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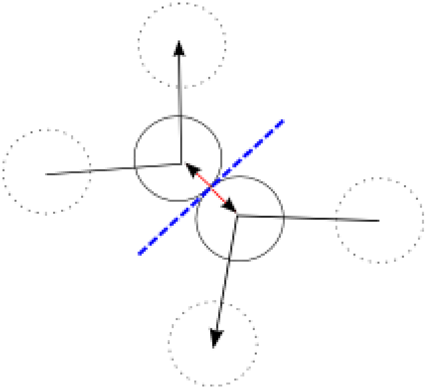

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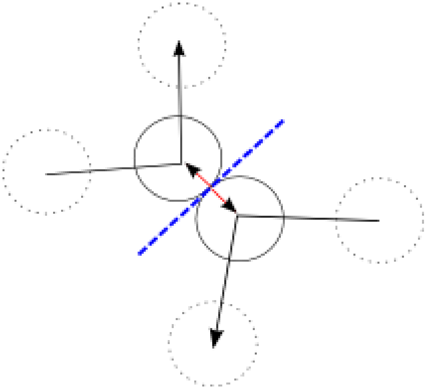

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\+ Just like high school physics

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- \- Penalty methods don't guarantee separation
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But can be made to work with enough hacking...

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Is this really all there is to it?

# Constraint Based Impacts

For any body $B_i$ with mass field $\varphi_i$, define the set:

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So two solids, $A_i, A_j$, *collide* at a configuration $q_i, q_j$ iff:

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Define

$$C_{i,j}(q_i, q_j) \overset{def}{\equiv} \text{vol } q_i A_i \cap q_j A_j$$

And so we replace the impact forces with a system of differentiable holonomic inequality constraints:

$$C_{i,j} \leq 0$$

# Equations of motion revisited

New problem:

$$\text{minimize} \int_{t_0}^{t_1} L(q, \dot{q}, t)\,dt$$

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Apply KKT conditions + Euler-Lagrange to get complementarity problem:

$$\frac{d}{dt}\left(\frac{\partial T(\dot{q}_i)}{\partial \dot{q}_i}\right) - \frac{\partial U(q,t)}{\partial q_i} + \sum_{j \neq i} \mu_{i,j} \frac{\partial C_{i,j}(q_i, q_j)}{\partial q_i} = 0$$

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Exactly elastic collision response!

Slack variables are impulse forces

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Observe:

$$\mathbf{1}_{A_i \cap A_j}(x) = \mathbf{1}_{A_i}(x)\mathbf{1}_{A_j}(x) \qquad \text{vol } A_i = \int_{\mathbb{R}^d} \mathbf{1}_{A_i}(x)dx$$

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So:

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Substitute $q_j^{-1}q_i x \mapsto Rx - y$ and let $\widetilde{\mathbf{1}_{A_j}}(x) = \mathbf{1}_{A_j}(-x)$:

$$\int_{\mathbb{R}^d} \mathbf{1}_{A_i}(x)\widetilde{\mathbf{1}_{A_j}}(y - Rx)dx = \int_{\mathbb{R}^d} \mathbf{1}_{A_i}(R^{-1}x)\widetilde{\mathbf{1}_{A_j}}(y - x)dx$$

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So,

$$C_{i,j}(q_i, q_j) = ((\mathbf{1}_{A_i} \circ R^{-1}) \star \widetilde{\mathbf{1}_{A_j}})(y)$$

# Fourier Methods

Convolution?

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Convolution? Take a Fourier transform!

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What about gradients? (ie $\frac{\partial C_{i,j}(q_i, q_j)}{\partial q_i}$)

Fix parameters $q_i = (R_i, t_i)$,

$$q_i x = R_i x + t_i$$

$$q_i^{-1} = (R_i^{-1}, -R_i^{-1} t_i)$$

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Need to compute $\frac{\partial C_{i,j}(R,y)}{\partial y}$, $\frac{\partial C_{i,j}(R,y)}{\partial R}$ then use chain rule

Or by symmetry: $C_{i,j}(q_i, q_j) = C_{j,i}(q_j, q_i)$

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\ldots &= \int\limits_{\mathbb{R}^d} 2\pi i\left\langle\omega, v^k\right\rangle \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle\omega,y\rangle}d\omega
\end{aligned}
$$

Where $v^k$ denotes the $k^{th}$ basis vector

# Translational Gradient

Start with the translational case first.

Let $y^k$ denote the $k^{th}$ component of $y$,then

$$\begin{aligned}\frac{\partial C_{i,j}}{\partial y^k} &= \frac{\partial}{\partial y^k}\left(\int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle\omega,y\rangle}d\omega\right)\\ \dots &= \int_{\mathbb{R}^d} 2\pi i\left\langle\omega, v^k\right\rangle \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle\omega,y\rangle}d\omega\end{aligned}$$

Where $v^k$ denotes the $k^{th}$ basis vector

Conclusion: Translational gradient is just a multiplier

# Rotational Gradient

Parametrize $R = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$ with basis $\mathfrak{r}_{k,l}$

In other words $d \times d$ skew symmetric matrices, $\mathfrak{r}^T = -\mathfrak{r}$

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\ldots &= \int\limits_{\mathbb{R}^d} - \left\langle \nabla \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega), R^{-1}\mathfrak{r}_{k,l}\omega \right\rangle \overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i \langle \omega, y \rangle} d\omega
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Not a Fourier multiplier.

But can be precalculated with $O(d)$ overhead:

$$\frac{\partial}{\partial \omega^k} \widehat{\mathbf{1}_{A_i}}(\omega) = \mathcal{F}\left(-2\pi i \left\langle x, v^k \right\rangle \mathbf{1}_{A_i}\right)(\omega)$$

# Performance

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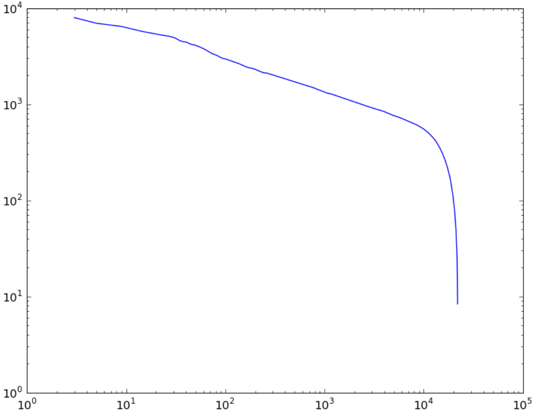
Dramatically reduces space and time complexity

# Convergence

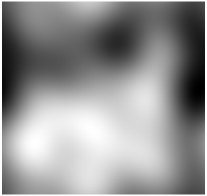
Rate of convergence, number of Fourier terms vs. $L^1$ error, (log-log plot)



# Convergence

Qualitative effects of cutoff (full spectrum has 22185 terms)

Exact



16

32

64

256

1024

# Truncated Spectra

Henceforth $d = 2$

Want: An efficient way to represent truncated spectrum

Need to deal with rotations too

Obvious answer: Polar coordinates

Pick $\omega_r = \sqrt{\omega_x^2 + \omega_y^2}$, $\omega_\theta = \tan^{-1} \frac{y}{x}$

$$\widehat{f}(\omega_x, \omega_y) \mapsto \widehat{f}^p(\omega_r, \omega_\theta)$$

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 *Problem?*

# Discrete Cartesian to Polar Conversion

Can't do this exactly!

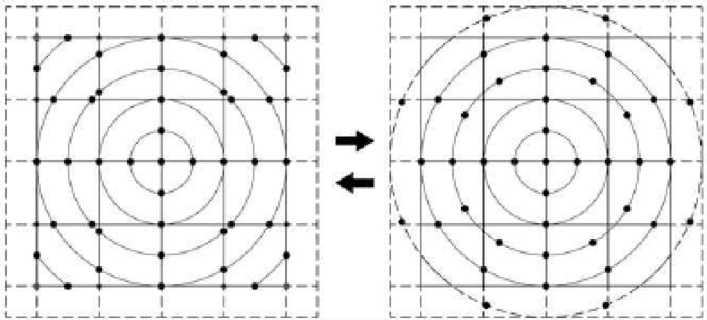

Figure from Chirikjian 2003

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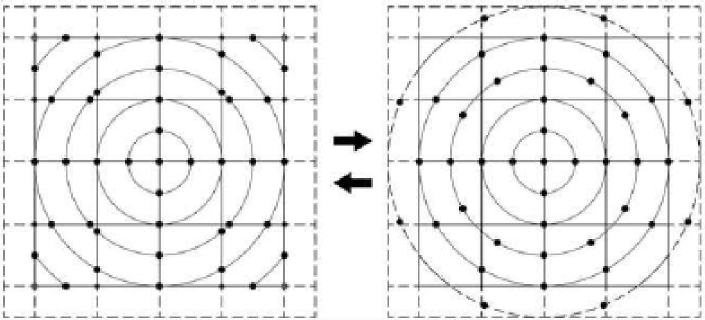

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Reason: *Only nontrivial discrete subgroups of* $SE(2)$ *are 17 plane tiling groups!*

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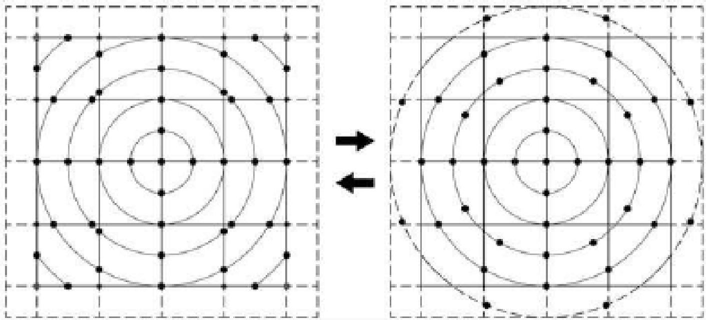

Figure from Chirikjian 2003

Reason: *Only nontrivial discrete subgroups of $SE(2)$ are 17 plane tiling groups!*
So unless you are happy with a hexagonal grid with 6 rotational samples, forget about doing this losslessly!

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- Approximation is a fact of life – deal with it
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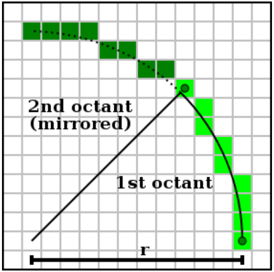


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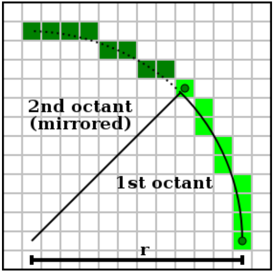


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- Sample $r$ at uniform increments
- For each $r$, trace a circle centered at the origin
- Store $8r + 4$ radial samples in array
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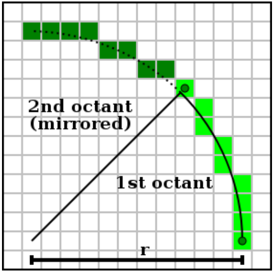


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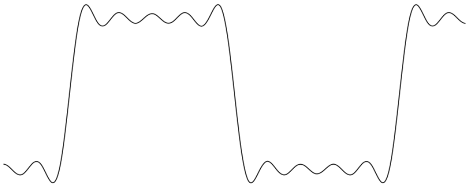
Minimal storage, uses only integer arithmetic

# Gibbs Phenomenon

Truncating Fourier series causes ringing artifacts at boundaries

- Intuitively, same as low pass filtering high curvature



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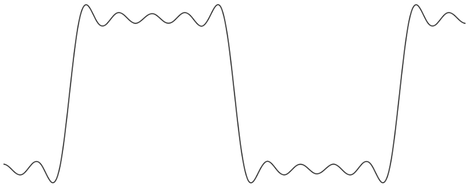

Figure from Wikipedia, 2010

But we don't care, just need separation between interior and exterior.

So pick some cutoff threshold; values above are in, below are out

# Optimal Thresholding

Given a region $R \subset \mathbb{R}^d$ and set $A \subseteq R$ approximated by $f$, define quality, $Q_T$, of threshold $T$

$$Q_T = \text{vol}\,\{x \in R | f(x) > T \Leftrightarrow x \in A\}$$

*Intuitively, number of points where threshold gives the right answer*
Want to find optimal $T$

$$T = \underset{T' \in \mathbb{R}}{\text{argmax}}\, Q_{T'}$$

Seems hard to estimate $T$ analytically

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But in practice can compute $T$ combinatorially given $f, A, R$

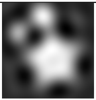
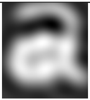
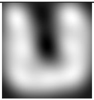
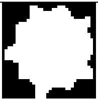
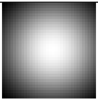
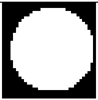
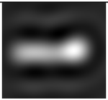
# Optimal Thresholding Algorithm

```
v = []
for x in R:
    v.append( (f(x), A(x)) )
v.sort()

fv, Av = unzip(v)
Q = foldr(Av, +, 0) - foldl(Av, +, 0)

return max(zip(Q, fv)).second
```

Greedy/DP method

Takes $O(n\log(n))$ time, uses $O(n)$ space

# Low Pass Filter/Thresholding Results

| Exact Indicator | Fourier Truncation | Optimal Cutoff |
|---|---|---|
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

Radius 14 cutoff, 777 terms

# Putting it together

Let $\hat{f}_A$ be the approximate Fourier series for $\mathbf{1}_A$

Then:

$$
\begin{aligned}
C_{i,j}(\rho,\theta,\phi) &= -T + \sum_{r=2}^{N_R} \sum_{t=1}^{8r+4} \widehat{f_{A_i}}\left(r, \phi\frac{dt}{d\theta} + t\right) \overline{\widehat{f_{A_j}}}(r,t) \\
\ldots &\quad \exp\left(\frac{2\pi i}{N}\rho r \cos\left(t\frac{d\theta}{dt} + \theta\right)\right) r\frac{dt}{d\theta}
\end{aligned}
$$

Where:

$$q_j q_i^{-1} = (R, y)$$

$$y = \rho(\cos(\theta)v^x + \sin(\theta)v^y)$$

$$R = \exp(\phi\mathfrak{r}_{0,1})$$

$$dt = 8r + 4$$

$$d\theta = 2\pi$$

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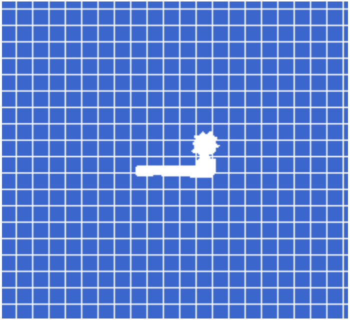
*Protip: Can early out using the rearrangement inequality*

# Prototype Physics Code

Basic implementation so far, just uses implicit Euler.
Doesn't solve for $\mu$ correctly (currently set to constant)



Features:

- OpenGL GUI
- Written 100% in Python
- Reasonable performance (though certainly not fast)
- Arbitrary shapes/mass fields supported

Download Link:

http://github.com/mikolalysenko/Collisions

# To Do

- Analysis is still very incomplete
- Need to implement a proper solver for physics system
- Benchmarks, etc.
- More development/write ups needed

# References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