

Spectral Rigid Body Dynamics

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Overview

Rigid Body Dynamics

Lagrangian Mechanics

Standard Collisions

Constraint Based Collisions

Fourier Methods

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An approximate model of low energy physics for stiff objects

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Pros:

- ▶ + Pretty accurate at human energy scales
- ▶ + Good for stiff materials (ie metals, plastics etc.)
- ▶ + Easy kinematic constraints (useful for mechanisms)
- ▶ + Standard animation tool (videogames!)

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Cons:

- ▶ - Inaccurate at extremely large energies
- ▶ - Bad for materials with low elastic modulus
- ▶ - Not always solvable! (See: Painleve's paradox)

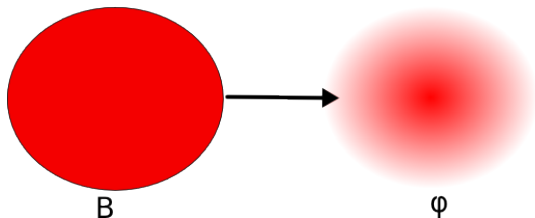
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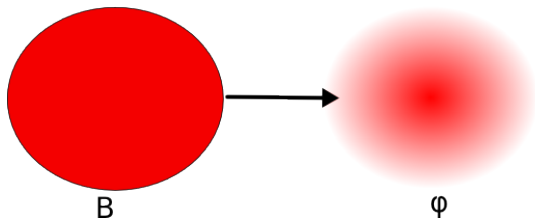
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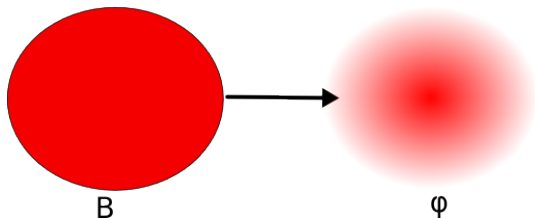


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$\varphi(x) = 0$ indicates B does not occupy the space at x

Configuration Space of a Rigid Body

Transformations of rigid mass fields must preserve distance and handedness

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In other words, must be a direct Euclidean isometry

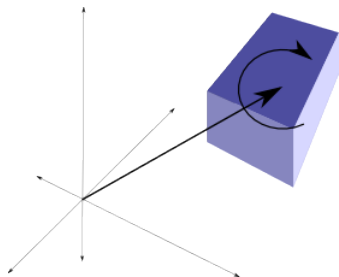
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$$\text{Matrix: } \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$$

$\binom{d+1}{2}$ degrees of freedom

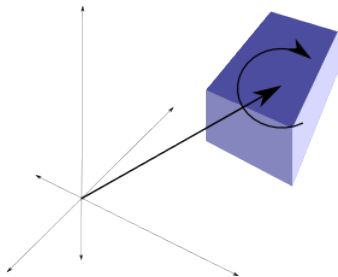
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Motions of rigid objects \cong curves $q(t) \subset SE(d)$

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Turns physics into an optimization problem.

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$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - U(q, t)$$

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$$\operatorname{argmin}_{q: [t_0, t_1] \rightarrow SE(d)} \int_{t_0}^{t_1} L(q, \dot{q}, t) dt$$

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Newton's equations!

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Configuration space $SE(d)^2 \cong SE(d) \oplus SE(d)$

Motion $q(t) \cong q_i(t) \oplus q_j(t)$

Lagrangian $L(q, \dot{q}, t) = L(q_i, \dot{q}_i, t) + L(q_j, \dot{q}_j, t)$

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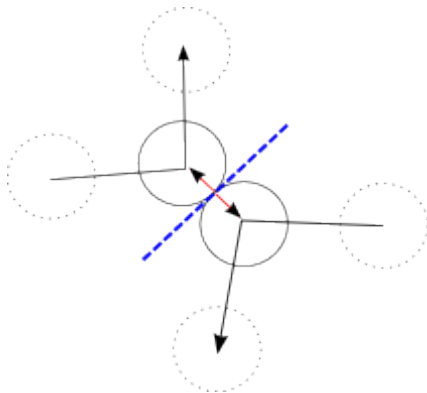
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Scales to n bodies, get Lagrangian in $SE(d)^n$

Collisions

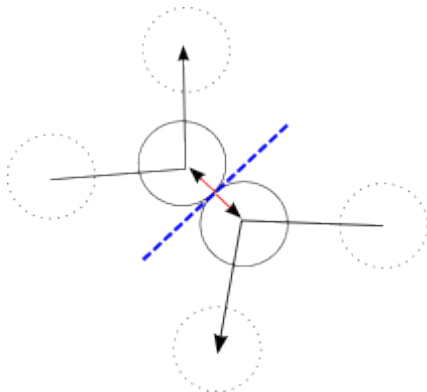
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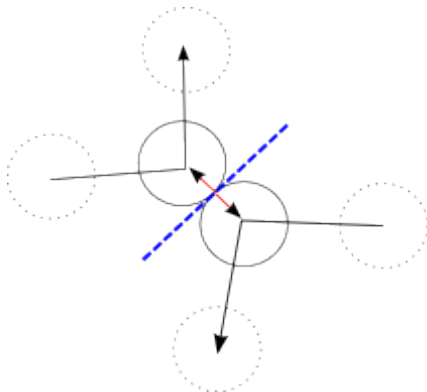
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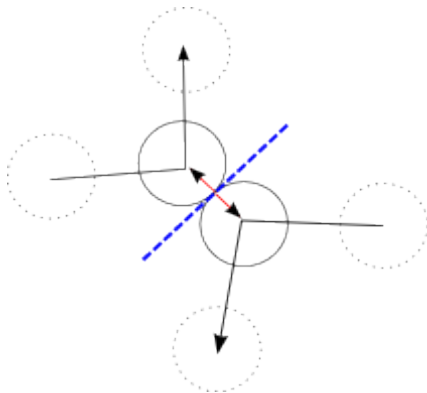
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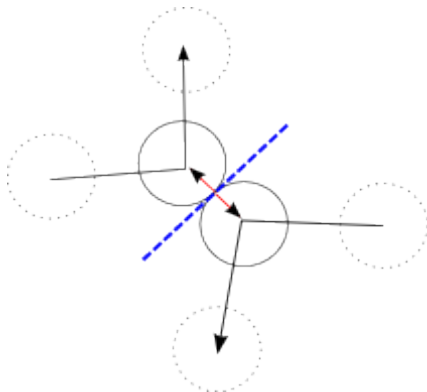
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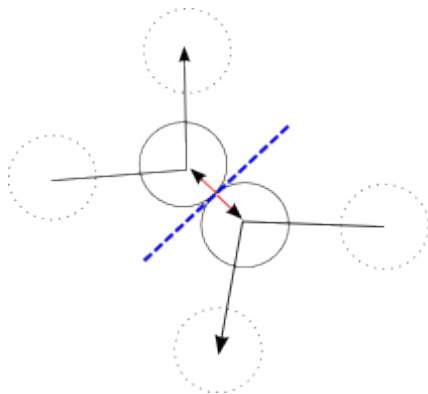
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+ Just like high school physics

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But can be made to work with enough hacking...

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At all times no two solids intersect

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Is this really all there is to it?

Constraint Based Impacts

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Define

$$C_{i,j}(q_i, q_j) \stackrel{\text{def}}{=} \operatorname{vol} q_i A_i \cap q_j A_j$$

And so we replace the impact forces with a system of differentiable holonomic inequality constraints:

$$C_{i,j} \leq 0$$

Equations of motion revisited

New problem:

$$\text{minimize } \int_{t_0}^{t_1} L(q, \dot{q}, t) dt$$

subject to $C_{i,j}(q_i, q_j) \leq 0 \quad \forall t \in [t_0, t_1), i \neq j$

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Apply KKT conditions + Euler-Lagrange to get complementarity problem:

$$\frac{d}{dt} \left(\frac{\partial T(\dot{q}_i)}{\partial \dot{q}_i} \right) - \frac{\partial U(q, t)}{\partial q_i} + \sum_{j \neq i} \mu_{i,j} \frac{\partial C_{i,j}(q_i, q_j)}{\partial q_i} = 0$$

$$0 \leq \mu_{i,j} \perp -C_{i,j} \geq 0$$

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Exactly elastic collision response!

Slack variables are impulse forces

Calculating $C_{i,j}$

Need to compute:

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Observe:

$$\mathbf{1}_{A_i \cap A_j}(x) = \mathbf{1}_{A_i}(x) \mathbf{1}_{A_j}(x)$$

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So:

$$C_{i,j}(q_i, q_j) = \int_{\mathbb{R}^d} \mathbf{1}_{A_i}(q_i^{-1}x) \mathbf{1}_{A_j}(q_j^{-1}x) dx$$

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Substitute $q_j^{-1}q_i x \mapsto Rx - y$ and let $\widetilde{\mathbf{1}}_{A_j}(x) = \mathbf{1}_{A_j}(-x)$:

$$\int_{\mathbb{R}^d} \mathbf{1}_{A_i}(x) \widetilde{\mathbf{1}}_{A_j}(y - Rx) dx = \int_{\mathbb{R}^d} \mathbf{1}_{A_i}(R^{-1}x) \widetilde{\mathbf{1}}_{A_j}(y - x) dx$$

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$$C_{i,j}(q_i, q_j) = ((\mathbf{1}_{A_i} \circ R^{-1}) \star \widetilde{\mathbf{1}}_{A_j})(y)$$

Fourier Methods

Convolution?

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Convolution? Take a Fourier transform!

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Fix parameters $q_i = (R_i, t_i)$,

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Need to compute $\frac{\partial C_{i,j}(R, y)}{\partial y}$, $\frac{\partial C_{i,j}(R, y)}{\partial R}$

Or by symmetry: $C_{i,j}(q_i, q_j) = C_{j,i}(q_j, q_i)$

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Conclusion: Translational gradient is just a multiplier

Rotational Gradient

Parameterize $R = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$ with basis $\mathfrak{r}_{k,l}$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}^T = -\mathfrak{r}$

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Unfortunately not a Fourier multiplier.

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Can be precalculated with $O(d)$ overhead.

In summary...

Unified method for dealing