Minimum Multicut

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Multicuts and Multiflows Continuing on applications of primal dual schema we now consider a generalization of the mincut problem known as *minimum multicut*. Given a graph $G = (V, E)$, some capacity function $c : E \to \Re^+$ associated with each edge and a collection of pairs of vertices $\{(s_1, t_1), ..., (s_k, t_k)\}$ with $s_i, t_i \in V$ such that $s_i \neq t_i$, a minimum multicut, $D \subseteq E$ is the collection of edges with smallest total cost whose removal separates each (s_i, t_i) pair. This problem may be expressed as an integer program:

$$\text{minimize} \sum_{e \in E} c(e) d_e$$

$$\text{subject to } \forall 1 \leq i \leq k, P = \{s_i, p_1, ..., t_i\} : \sum_{e \in P} d_e \geq 1$$

$$d_e \in \{0, 1\}$$

Where P is some path from s_i to t_i. In general, the number of constraints in this system is exponential in k. A special case of this problem is the *minimum multiway cut*, where instead of being given some collection of pairs of vertices to separate, we are given a set $\{s_1, ...s_k\} \subseteq V$ and try to separate each (s_i, s_j) for all $i \neq j$. This version is NP-hard for $k \geq 3$.

Dual to the multicut problem is the parallel generalization of network flow: *maximum integer multiflow*. Briefly, the integer program for multiflow is the following:

$$\text{maximize} \sum_{i=1}^{k} f_i$$

$$\text{subject to } \forall e \in E : \sum_{i | P_i \ni e} f_i \leq c(e)$$

$$f_i \in \mathbb{Z}^+$$

Where P_i denotes some path from s_i to t_i and f_i the flow between the respective source/sink. For general graphs, the best known approximation of the minimum multicut is $O(\log k)$. No good approximation algorithm is known for integer maximum multiflow.

Approximate Multiflows/Multicuts on Trees In the case of trees, there is a factor 2 approximation to the minimum multicut problem which we will construct then analyze using approximate primal/dual complementary slackness. In trees, there is at most one vertex disjoint path between any pair of vertices. We use this property to simplify the mincut program:

$$\text{minimize} \sum_{e \in E} c(e) d_e$$

$$\text{subject to } \forall 1 \leq i \leq k : \sum_{e \in P_i} d_e \geq 1$$

$$d_e \in \{0,1\}$$

Where P_i is uniquely determined. Before giving an approximation algorithm to this problem, we introduce to some terms: let the *root* of G be any vertex r, the *depth* of a vertex v is the length of the shortest path from r to v and let $lca(u,v)$ be the lowest common ancestor of some vertices u, v, that is:

$$lca(u,v) = \min_{t \in P_{uv}} depth(t)$$

If on some path from v to r, edge e_1 occurs before e_2 we say that e_1 is *deeper*. With this notation we now introduce the following approximation algorithm for minimum multicut. Let f be a multiflow and D be a multicut:

$f \leftarrow 0, D \leftarrow \emptyset$
for each $v \in V$ in nonincreasing order of depth
    do for each pair (s_i, t_i) such that $lca(s_i, t_i) = v$
        do Route maximum integral flow from s_i to t_i through f
            Add to D all edges saturated by this flow in order
Let $e_1, e_2, \ldots e_l$ the ordered list of edges in D
for $j = l$ down **to** 1
    do if $D - \{e_j\}$ is a multicut
        then $D \leftarrow D - \{e_j\}$

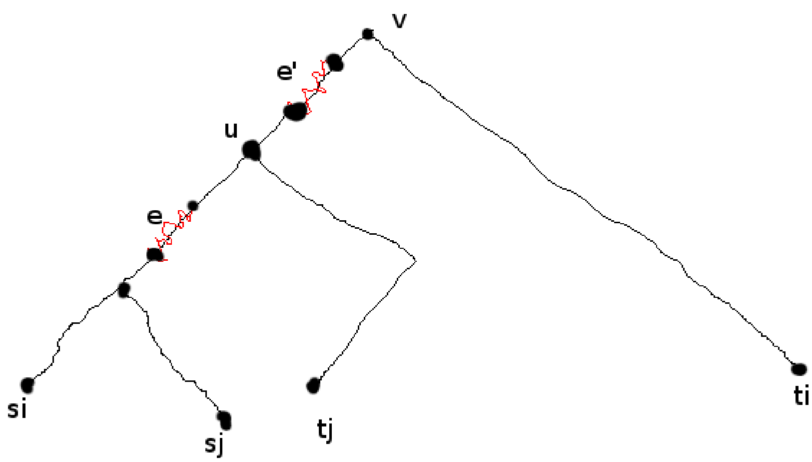
To establish the performance of this approximation algorithm, we prove the following lemma:

Lemma Let $f_i \neq 0$ for some (s_i, t_i) path where $lca(s_i, t_i) = v$. Then at most one edge is picked in the multicut from s_i to v and one edge from v to t_i.

Proof First, consider the path from s_i to v. Suppose two edges e, e' on the path from s_i to v are both picked for the multicut and without loss of generality assume e is deeper. Both e, e' stayed in D through the second for loop. Consider what happened when e was tested; since it was not discarded there must exist some (s_j, t_j) such that e is the *unique* edge of D on the $s_j \ldots t_j$ path at that time. Now let $u = lca(s_j, t_j)$. Since e is unique on the s_j, t_j path, e' is not on it we know that:

$$depth(u) < depth(e') < depth(v)$$

And so we conclude that u must have been processed before v. Between processing u and v, D must contain some edge e'' on the path s_j, t_j. By hypothesis, e can not be added before v is processed. Since v is an ancestor of u, e is added to D after e'' and so the order in D must be such that $e'' \prec e$ and therefore in the final for loop e is tested before e''. However this is a contradiction, when e was tested it was not discarded since it was the unique edge cutting the path from s_j to t_j, and yet the path was also cut by e''. This argument is illustrated in the following diagram:



Theorem The algorithm gives a factor-2 approximation.

Proof To show this, we apply approximate primal-dual complementary slackness conditions with $\alpha = 1, \beta = 2$. Our goal is to show:

$$\forall e \in E: \quad d_e \neq 0 \implies \sum_{i: P_i \ni e} f_i = c(e)$$
$$\forall 1 \leq i \leq k: \quad f_i \neq 0 \implies \sum_{e \in P_i} d_e \leq 2$$

The first condition is trivially satisified, since we never over saturate any flow. For the second condition, the above lemma proves that if there is some flow between s_i, t_i, then we never picked more than one cut edge per the path v to s_i and v to t_i (with $v = lca(s_i, t_i)$ and so the total number of cut edges from s_i to t_i is less than 2. As a result, we obtain a factor $\alpha\beta = 2$ approximation.