Homework 4

Computational Math

Author:
Mikola Lysenko

April 28, 2010

1

a Let u be a solution to the given ODE. Then, for any 2π periodic smooth function v it must be that:

$$\int_{-\pi}^{\pi} (u_{xx}(x) + (\sin(x) + \cos(x) + 2\sin(x)\cos(x))\, u(x))v(x)dx = \int_{-\pi}^{\pi} e^{\sin(x)}e^{\cos(x)}v(x)dx$$

By linearity, we can split the righthand side into homogeneous components and apply integration by parts. For the second order components we get:

$$\begin{aligned}
\int_{-\pi}^{\pi} u_{xx}(x)v(x)dx &= u_x(x)v(x)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u_x(x)v_x(x)dx \\
&= -\int_{-\pi}^{\pi} u_x(x)v_x(x)dx \\
&= p_2(u,v)
\end{aligned}$$

For the 0th order terms, no additional work is necessary to get a bilinear form:

$$\int_{-\pi}^{\pi} (\sin(x) + \cos(x) + 2\sin(x)\cos(x))\, u(x)v(x)dx = p_0(x)$$

And finally for the right hand side, we construct $f(v)$:

$$\int_{-\pi}^{\pi} e^{\sin(x)}e^{\cos(x)}v(x)dx = f(v)$$

By construction our space of test functions satisifies the boundary, so we have the following weak form for the system:

$$u^T(p_2 + p_0)v = f^T v$$

b To get a cG(1) finite element discretization, we restrict the space of test functions and solutions to piecewise linear maps, spanned by the basis of hat functions. Fix a collection of n ordered grid points $X = \{x_i \in [-\pi, \pi) | x_i < x_{i+1}, i \in [0, N)\}$ where $x_0 = -\pi$. Now we define the basis hat functions, $\varphi^i(x)$ as follows:

$$\varphi^i(x) = \begin{cases} \frac{x_{i+1}-x}{x_{i+1}-x_i} & \text{if } x \in [x_i, x_{i+1}) \\ \frac{x-x_{i-1}}{x_i-x_{i-1}} & \text{if } x \in [x_{i-1}, x_i) \\ 0 & \text{otherwise} \end{cases}$$

With the exception that for x_0, x_{n-1} boundary conditions are applied to make the grid cyclic. This may be done by just taking the indexes mod n and the difference terms mod 2π. Substituting this into the above equation gives the following matrix equation for the discrete Galerkin equation:

$$p_2(\varphi^i, \varphi^j) + p_0(\varphi^i, \varphi^j) = f(\varphi^j)$$

Now we expand each term separately and integrate piecewise. Starting with p_2:

$$
\begin{aligned}
p_2(\varphi^i, \varphi^j) &= -\int_{-\pi}^{\pi} \varphi_x^i(x)\varphi_x^j(x)dx \\
&= (\delta_{i,j+1} - \delta_{i,j}) \int_{x_{i-1}}^{x_i} \left(\frac{1}{x_i - x_{i-1}}\right)^2 dx + (\delta_{i,j-1} - \delta_{i,j}) \int_{x_i}^{x_{i+1}} \left(\frac{1}{x_{i+1} - x_i}\right)^2 dx \\
&= (\delta_{i,j+1} - \delta_{i,j}) \frac{1}{x_i - x_{i-1}} + (\delta_{i,j-1} - \delta_{i,j}) \frac{1}{x_{i+1} - x_i}
\end{aligned}
$$

Next we deal with p_0:

$$p_0(\varphi^i, \varphi^j) \quad =$$

And finally f:

$$f(\varphi^i) \quad =$$

Putting this together, we get the following coefficient matrix, $M_{i,j}$, such that:

$$M_{i,j} \quad = \quad p_2(\varphi^i, \varphi^j) + p_0(\varphi^i, \varphi^j)$$

And for the right hand side, we get b_i such that:

$$b_i \quad = \quad f(\varphi^i)$$

And so the final problem is to solve for some coefficients u^i such that:

$$Mu = b$$

c

2

a To start with, we modify p_0 and f from part 1, giving the following new form for M and b:

$$M_{i,j} =$$

And:

$$b_i =$$

To compute the error within the i^{th} node we do 'blah'

b

c