Homework 2

Computational Math

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1

a I checked the exactness condition for the given u using sympy and a Python script:

```
from sympy import *;

def uexact(x):
return exp(sin(x)) * exp(cos(x));

def q1(x):
return sin(x) * cos(x);

def q2(x):
return -2 * cos(x)**4 + cos(x)**3 + (8 + 3 * sin(x)) * cos(x)**2 - (1 + sin(x)) * cos(x);

u = uexact(x);
ux = diff(u, x);
uxx = diff(ux, x);
uxxx = diff(uxx, x);
print trigsimp(simplify(uxxx + q1(x) * uxx + q2(x) * u(x) - 3 * exp(sin(x)) * exp(cos(x)))) == 0;
```

By inspection, it should be clear that if uexact is an exact solution for the ode, then it ought to print out True; which is exactly what happens, and so the exactness condition is satisified.

b Based on class discussion, pick

$$u(x) = \sum_{j=0}^{N} w_j C_j(x)$$

Where

$$C_j(x) = \frac{1}{N} \sin\left[\frac{N(x - x_j)}{2}\right] \cot\left[\frac{(x - x_j)}{2}\right]$$

In the interval $[-\pi, \pi)$. Now observe that C_j has a nice, finite Fourier series expansion, (assuming N is divisible by 4):

$$C_j(x) = \frac{1}{N}\left(-\frac{1}{2}e^{\pm iN/2(x-x_j)} + \sum_{k=-N/2}^{N/2} e^{ik(\pi+x-x_j)}\right)$$

Differentiating C_j, d-times is equivalent to applying the Fourier multiplier $(ik)^d$, and so:

$$\partial_x^d C_j(x) = \frac{1}{N}\left(-\frac{(iN)^d}{2^{d+1}}e^{\pm iN/2(x-x_j)} + \sum_{k=-N/2}^{N/2} (ik)^d e^{ik(\pi+x-x_j)}\right)$$

To compute the weights for the differentiation matrix, D^d, we need need to compute:

$$D^d_{p,q} = \langle \partial_x^d C_p, C_q \rangle$$

But due to the cancellative properties of sinc interpolation, this reduces to:

$$D^d_{p,q} = \partial_x^d C_p(x_q)$$

Pick $d = 3$ and we are done.

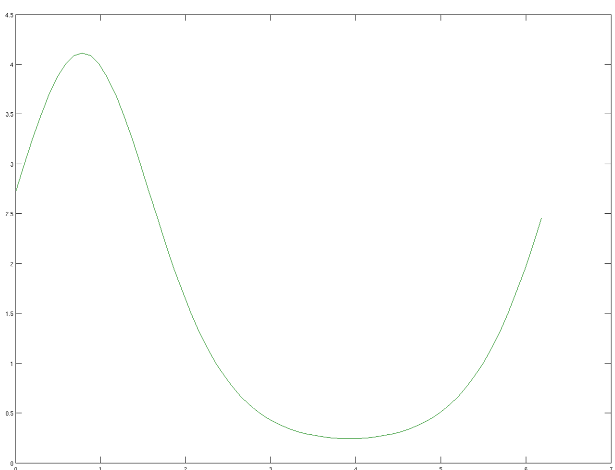
c We compute the differentiation matrix numerically using the fast fourier transform. Here is some MATLAB code I wrote to do this:

```
function [ D ] = sinc_d( N, d )
  k = [0:(N/2) (1-N/2):-1];
  ch = ones(1,N);
  chd = (i * k).^d .* ch;
  dc = real(ifft(chd));
  D = toeplitz(dc, dc([1 N:-1:2]));
```

Once we have the matrix computed, we just solve for the solution using the vanilla Galerkin method for sinc functions:

```
x = (0:N-1) * 2 * pi / N;
D2 = sinc_d(N, 2);
D3 = sinc_d(N, 3);
s = sin(x);
c = cos(x);
q1 = s .* c;
q2 = -2.*c.^4 + c.^3 + (8. + 3. * s) .* c.^2 - (1. + s) .* c;
f = 3. * exp(s) .* exp(c);
L = D3 + diag(q1) * D2 + diag(q2);
u = L \ f';
```

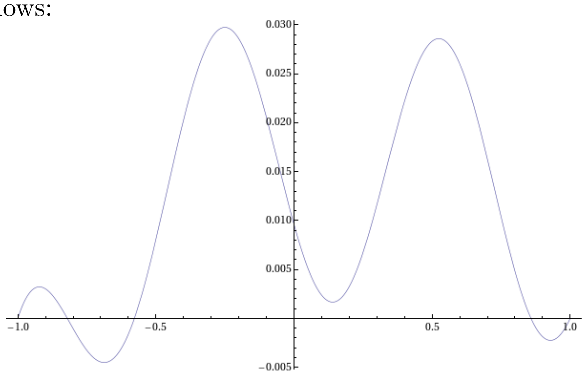
Here is a plot of the exact solution and the Galerkin method, and as you can see the plots are nearly identical:



2 First, observe that the exact solution to this equation is given by:

$$
\begin{aligned}
u(x) &= \frac{e^{-2x}}{J_4\left(2\sqrt{e}\right)Y_4\left(\frac{2}{\sqrt{e}}\right) - J_4\left(\frac{2}{\sqrt{e}}\right)Y_4\left(2\sqrt{e}\right)} \times \\
& \left(J_4\left(\frac{2}{\sqrt{e}}\right)\left(24Y_4\left(2\sqrt{e}\right)J_4\left(2\sqrt{e^x}\right) - e^2(;5;-e)Y_4\left(2\sqrt{e^x}\right)\right)\int_1^{-1} -\frac{1}{24}\pi e^{2t}\sin(8t)Y_4\left(2\sqrt{e^t}\right)\,dt \right. \\
&+ J_4\left(2\sqrt{e^x}\right)\left(e^2(;5;-e)Y_4\left(\frac{2}{\sqrt{e}}\right) - 24J_4\left(\frac{2}{\sqrt{e}}\right)Y_4\left(2\sqrt{e}\right)\right)\int_1^{x} -\frac{1}{24}\pi e^{2t}\sin(8t)Y_4\left(2\sqrt{e^t}\right)\,dt \\
&+ 2Y_4\left(\frac{2}{\sqrt{e}}\right)\left(Y_4\left(2\sqrt{e}\right)J_4\left(2\sqrt{e^x}\right) - J_4\left(2\sqrt{e}\right)Y_4\left(2\sqrt{e^x}\right)\right)\int_1^{-1}\frac{1}{2}\pi e^{4s}\,(;5;-e^s)\sin(8s)\,ds \\
&+ \left. 2Y_4\left(2\sqrt{e^x}\right)\left(J_4\left(2\sqrt{e}\right)Y_4\left(\frac{2}{\sqrt{e}}\right) - J_4\left(\frac{2}{\sqrt{e}}\right)Y_4\left(2\sqrt{e}\right)\right)\int_1^{x}\frac{1}{2}\pi e^{4s}\,(;5;-e^s)\sin(8s)\,ds \right)
\end{aligned}
$$

This was computed using mathematica, and is suitably hideous. The symbols Y, K are Bessel functions and $(;;)$ is the hypergeometric function of the second kind. A plot of this exact solution in mathematica is as follows:

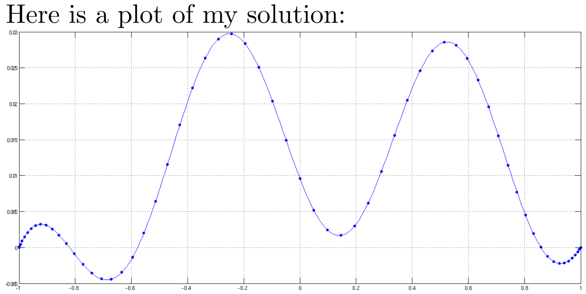


To solve the system with the Chebyshev method, I used cheb.m, obtained from Nick Trefethen's page and constructed the following solution procedure:

```
[D,x] = cheb(N);
D0 = diag(exp(x(2:N)));
D1 = 4. * D(2:N,2:N);
D2 = D^2;
D2 = D2(2:N,2:N);
f = sin(8. * x(2:N));

L = D2 + D1 + D0;
u = L\f;
u = [0;u;0];
```

Here is a plot of my solution:



I found that L^∞ error stabilized to 5e-10 around $N \approx 32$.

3

a Assume that $\sigma(x) > 0$ for all $x \in [0, 1]$, (otherwise the solution could be potentially imaginary or undetermined, and so it would be impossible to directly apply Sturm-Liouville theory). From the boundary condition, we know that $\varphi(1) = 0$. Combined with the asymptotic solution for the eigenmodes, we get the following equation for λ:

$$0 = \sigma(1)^{-\frac{1}{4}} \sin\left(\sqrt{\lambda}\int_0^1 \sqrt{\sigma(\xi)}d\xi\right)$$

Which is true if and only if $\sin(...) = 0$, and so it must be that $... = 0, \pi, 2\pi, ...k\pi$ for all $k \in \mathbb{Z}$. Thus:

$$\sqrt{\lambda}\int_0^1 \sqrt{\sigma(\xi)}d\xi = k\pi$$

And so

$$\lambda \approx \left(\frac{k\pi}{\int_0^1 \sqrt{\sigma(\xi)}d\xi}\right)^2$$

b This equation may be rewritten as the eigenvalue problem:

$$\frac{1}{\sigma(x)}\varphi_{xx} = \lambda\varphi$$

To solve this, we again use the cheb.m code from Nick Trefethen. Here is my MATLAB code:

```
[D,x] = cheb(N);
D = 2. * D;
x = .5 * (x + 1);
sigma = 1. + x;
D2 = D^2;
D2 = D2(2:N,2:N);
L = diag( 1 ./ sigma(2:N) ) * D2;
[V,Lam] = eig(L);
```

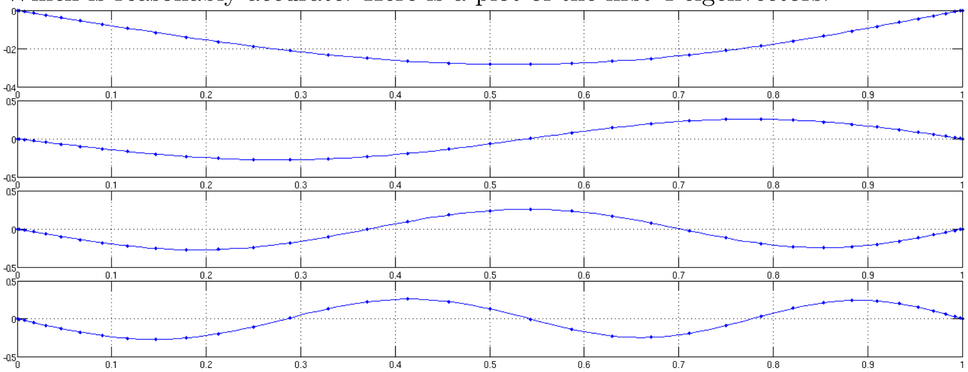
Substituting $\sigma(x) = 1 + x$, we get $\int_0^1 \sqrt{1 + x}dx = \frac{2}{3}(2\sqrt{2} - 1)$ and so:

$$\lambda \approx \frac{9\pi^2 k^2}{4\left(2\sqrt{2} - 1\right)^2}$$

The first 7 eigen values vs. asymptotic approximation:

Computed	Estimate
6.5484	6.6424
26.4649	26.5697
59.6742	59.7818
106.1700	106.2788
165.9513	166.0607
239.0177	239.1275
325.3691	325.4790

Which is reasonably accurate. Here is a plot of the first 4 eigenvectors:



4

a Given that this program solves the BVP given by:

$$u_{xx} = e^u$$

Subject to the BCs $u(-1) = u(1) = 0$ via the numerical method:

$$D2 * v_{k+1} = \exp(v_k)$$

(Where $D2$ is the second order Chebyshev differentiation matrix). Thus, the L^∞ difference between successive terms is:

$$|v_{k+1} - v_k|_\infty = |D2^{-1}\exp(v_k) - v_k|_\infty$$

Let λ_{max} be the magnitude of the largest eigenvalue of $D2^{-1}$, and V be its eigen vector. Then this quantity is maximized by the choice $v_k = \log(V)$, which gives the asymptotic estimate:

$$|v_{k+1} - v_k|_\infty \approx \lambda_{max} - \log(\lambda_{max})$$

The ratio of convergence is given by:

$$\Delta = \frac{|v_{k+2} - v_{k+1}|_\infty}{|v_{k+1} - v_k|_\infty} = 2\frac{|D2^{-1}\exp(D2^{-1}\exp(v_k) - v_k) - D2^{-1}\exp(v_k) + v_k|_\infty}{|v_{k+1} - v_k|_\infty}$$

Again, we take the eigenvalue expansion and obtain our final approximation for Δ:

$$\Delta \approx 2\frac{\lambda_{max}\exp(\lambda_{max} - \log(\lambda_{max})) - \lambda_{max} + \log(\lambda_{max})}{\lambda_{max} - \log(\lambda_{max})}$$

We now evaluate our estimate for Δ numerically using the following MATLAB code:

```
[v,lam] = eigs(inv(D2))
lmax = abs(lam(1))
delta =  2 * (lmax * exp(lmax - log(lmax)) - lmax + log(lmax)) / (lmax - log(lmax))
```

Which gives the value $\Delta \approx .2924$.

b We begin by rewriting $u_{xx} = e^u$ as a functional equation:

$$f(u) = u_{xx} - e^u$$

Which is discretized using the Chebyshev method to give:

$$f(v) = D2v - \exp(v)$$

Which has the first variation:

$$f'(v) = D2 - \Lambda_{\exp(v)}$$

Where $\Lambda_{\exp(v)}$ is a diagonal matrix with values $\exp(v)$. Thus the Newton's method solution for k^{th} iterate is:

$$v_{k+1} = v_k - \frac{f(v_k)}{f'(v_k)}$$

Which we compute using the following MATLAB snippet:

```
f = D2 * u - exp(u);
df = D2  - diag(exp(u));
unew = u - df \ f;
```

Unlike the fixed point method, this converges in 5 iterations, which is consistent with the observation that it ought to be quadratic convergence.

5 The forward Euler method is stable for values of $\Delta t \approx |\lambda|^{-1}$ where λ is the largest eigenvalue of the system. In this case, since we are dealing with a second-order dispersive system, the largest eigenvalue should be in $O(N^2)$ where N is the number of samples. So to ensure stability, the timestep should be in $\Delta t \in O(N^{-2})$.

The split timestep integration method is implemented as described, though some work was necessary to figure out the parametric form of v^{**}. From the equation, we have:

$$\begin{aligned} v^{**} &= v^* + \frac{\Delta t}{2} D2(v^* + v^{**}) \\ (I - \frac{\Delta t}{2} D2)v^{**} &= (I + \frac{\Delta t}{2} D2)v^* \\ v^{**} &= (I - \frac{\Delta t}{2} D2)^{-1}(I + \frac{\Delta t}{2} D2)v^* \end{aligned}$$

I implemented the split timestep method as described using the following MATLAB code:

```
[D,x] = cheb(N);
D2 = D^2;
D2 = D2(2:N,2:N);
M = inv(eye(N-1) - .5 * delta_t * D2) * (eye(N-1) + .5 * delta_t * D2);
v = zeros(N-1,1);
for t = 0:delta_t:t_max
  vs  = M * log( 2. * exp(v) ./ (2. - exp(v) * delta_t) );
 v = log( 2. * exp(vs) ./ (2. - exp(vs) * delta_t));
end
```

From this, I numerically computed $u(0, 3.5) \approx 3.5205$ and that $u(0, 3.5364) \approx 5.0000$.