Spectral Rigid Body Dynamics

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Overview

Rigid Body Dynamics

Lagrangian Mechanics

Standard Collisions

Constraint Based Collisions

Fourier Methods

Numerical Issues

Rigid Body Dynamics

An approximate model of low energy physics for stiff objects

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Pros:

- + Pretty accurate at human energy scales
- + Good for stiff materials (ie metals, plastics etc.)
- + Easy kinematic constraints (useful for mechanisms)
- + Standard animation tool (videogames!)

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- $+$ Good for stiff materials (ie metals, plastics etc.)
- $+$ Easy kinematic constraints (useful for mechanisms)
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Cons:

- - Inaccurate at extremely large energies
- - Bad for materials with low elastic modulus
- - Not always solvable! (See: Painleve's paradox)

What is a Rigid Body?

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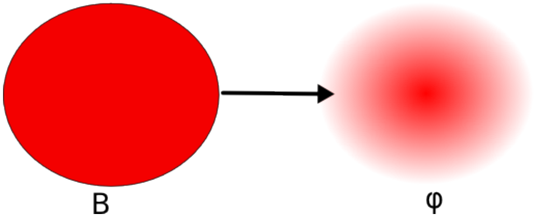

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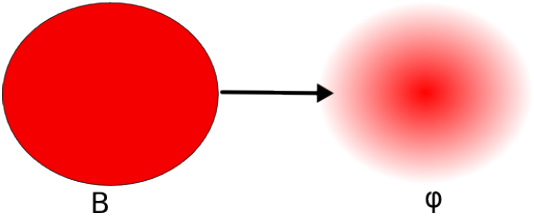

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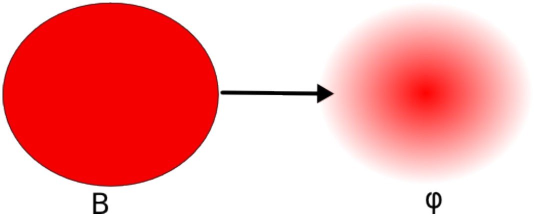

φ represents the mass distribution of B

$\varphi(x) = 0$ indicates B does not occupy the space at x

Configuration Space of a Rigid Body

Transformations of rigid mass fields must preserve distance and handedness

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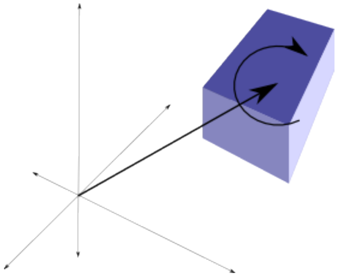

Can be parameterized by a translation t and a rotation R

Matrix: $\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$

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Tangent space: $\mathfrak{so}(d+1)$

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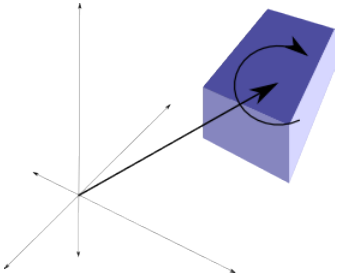

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Motions of rigid objects $\cong$ **curves** $q(t) \subset SE(d)$

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Turns physics into an optimization problem.

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$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - U(q, t)$$

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$$\underset{q:[t_0,t_1)\rightarrow SE(d)}{\operatorname{argmin}} \int_{t_0}^{t_1} L(q, \dot{q}, t)dt$$

Forward Kinematics

Solve:

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$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \frac{1}{2}\dot{q}^T M \dot{q}}{\partial \dot{q}}\right) &= \frac{\partial U(q,t)}{\partial q} \\ M\ddot{q} &= \nabla U \end{aligned}$$

Newton's equations!

Multiple Bodies

Q: How to deal with multiple independent bodies?

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A: Tensor sum

Let B_i, B_j be independent rigid bodies with motions q_i, q_j

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Motion	$q(t) \cong q_i(t) \oplus q_j(t)$
Lagrangian	$L(q, \dot{q}, t) = L(q_i, \dot{q}_i, t) + L(q_j, \dot{q}_j, t)$

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Scales to n bodies, get Lagrangian in $SE(d)^n$

Collisions

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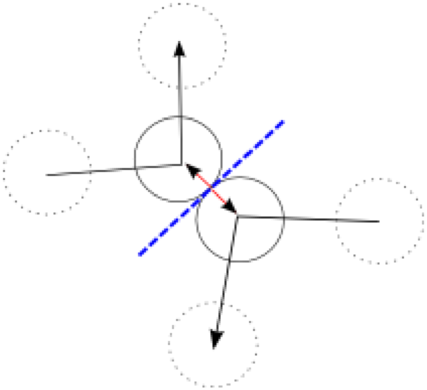

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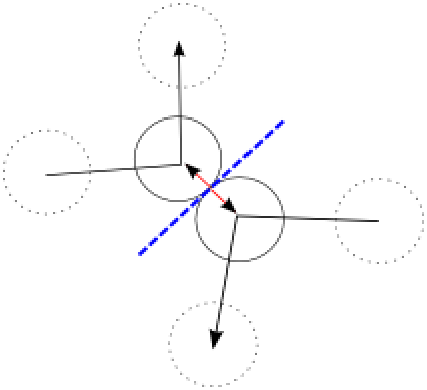

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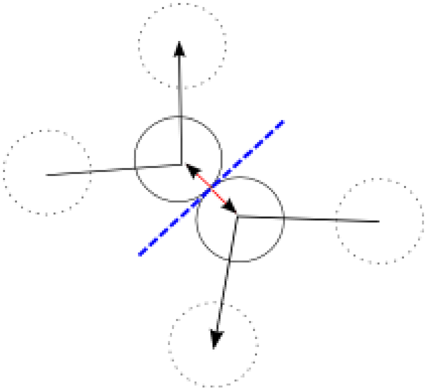

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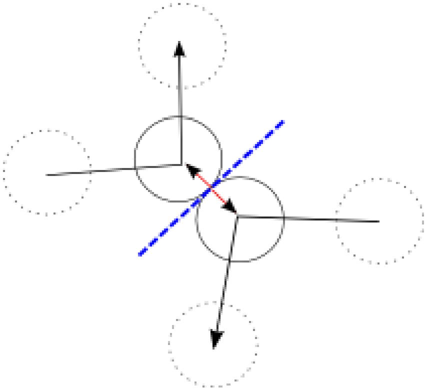

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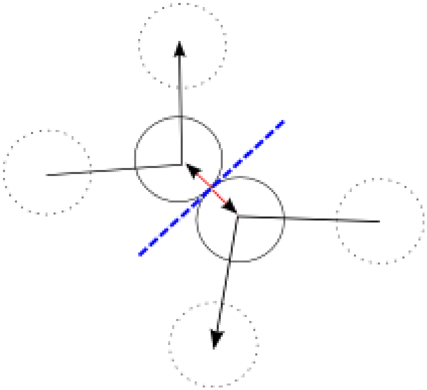

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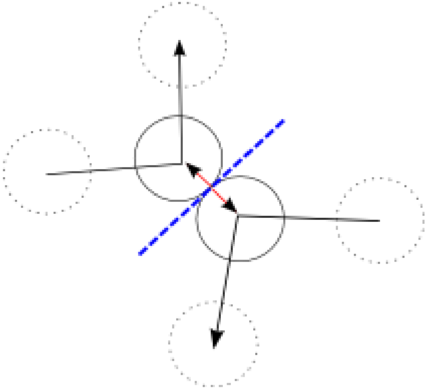

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+ Just like high school physics

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But can be made to work with enough hacking...

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At all times no two solids intersect

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Is this really all there is to it?

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Define

$$C_{i,j}(q_i, q_j) \overset{def}{\equiv} \text{vol } q_i A_i \cap q_j A_j$$

And so we replace the impact forces with a system of differentiable holonomic inequality constraints:

$$C_{i,j} \leq 0$$

Equations of motion revisited

New problem:

$$\text{minimize} \int_{t_0}^{t_1} L(q, \dot{q}, t)\,dt$$

$$\text{subject to } C_{i,j}(q_i, q_j) \leq 0 \quad \forall t \in [t_0, t_1), i \neq j$$

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Apply KKT conditions + Euler-Lagrange to get complementarity problem:

$$\frac{d}{dt}\left(\frac{\partial T(\dot{q}_i)}{\partial \dot{q}_i}\right) - \frac{\partial U(q, t)}{\partial q_i} + \sum_{j \neq i} \mu_{i,j} \frac{\partial C_{i,j}(q_i, q_j)}{\partial q_i} = 0$$

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Exactly elastic collision response!

Slack variables are impulse forces

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Need to compute:

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Observe:

$$\mathbf{1}_{A_i \cap A_j}(x) = \mathbf{1}_{A_i}(x)\mathbf{1}_{A_j}(x)$$

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So:

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Substitute $q_j^{-1}q_i x \mapsto Rx - y$ and let $\widetilde{\mathbf{1}_{A_j}}(x) = \mathbf{1}_{A_j}(-x)$:

$$\int\limits_{\mathbb{R}^d} \mathbf{1}_{A_i}(x)\widetilde{\mathbf{1}_{A_j}}(y - Rx)dx = \int\limits_{\mathbb{R}^d} \mathbf{1}_{A_i}(R^{-1}x)\widetilde{\mathbf{1}_{A_j}}(y - x)dx$$

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So,

$$C_{i,j}(q_i, q_j) = ((\mathbf{1}_{A_i} \circ R^{-1}) \star \widetilde{\mathbf{1}_{A_j}})(y)$$

Fourier Methods

Convolution?

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Convolution? Take a Fourier transform!

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Fix parameters $q_i = (R_i, t_i)$,

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Need to compute $\frac{\partial C_{i,j}(R,y)}{\partial y}$, $\frac{\partial C_{i,j}(R,y)}{\partial R}$ then use chain rule

Or by symmetry: $C_{i,j}(q_i, q_j) = C_{j,i}(q_j, q_i)$

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\ldots &= \int\limits_{\mathbb{R}^d} 2\pi i\left\langle \omega, v^k\right\rangle \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle\omega,y\rangle}d\omega
\end{aligned}$$

Where v^k denotes the k^{th} basis vector

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Start with the translational case first.

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$$\begin{aligned}\frac{\partial C_{i,j}}{\partial y^k} &= \frac{\partial}{\partial y^k}\left(\int\limits_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle\omega,y\rangle}d\omega\right)\\ \dots &= \int\limits_{\mathbb{R}^d} 2\pi i\left\langle\omega, v^k\right\rangle \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle\omega,y\rangle}d\omega\end{aligned}$$

Where v^k denotes the k^{th} basis vector

Conclusion: Translational gradient is just a multiplier

Rotational Gradient

Parameterize $R = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$ with basis $\mathfrak{r}_{k,l}$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}^T = -\mathfrak{r}$

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Not a Fourier multiplier.

But can be precalculated with $O(d)$ overhead:

$$\frac{\partial}{\partial \omega^k} \widehat{\mathbf{1}_{A_i}}(\omega) = \mathcal{F}\left(-2\pi i \left\langle x, v^k \right\rangle \mathbf{1}_{A_i} \right)(\omega)$$

Performance

Fourier transform is neat, but what does it get us?

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Full inverse Fourier transform complexity = brute force volume

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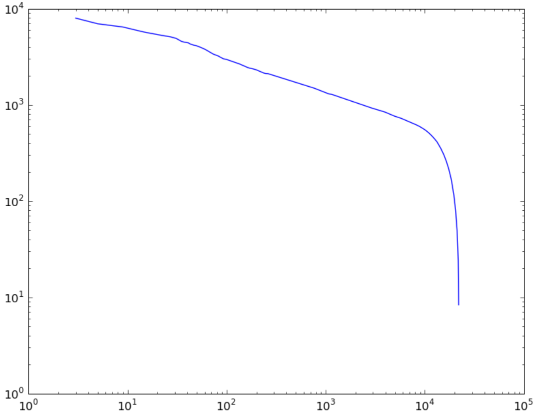
Dramatically reduces space and time complexity

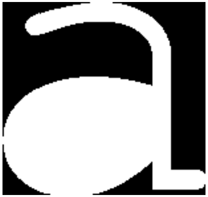
Convergence

Rate of convergence, number of Fourier terms vs. L^1 error, (log-log plot)



Convergence

Qualitative effects of cutoff (full spectrum has 22185 terms)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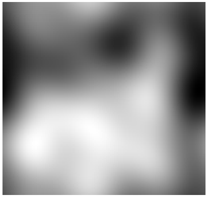
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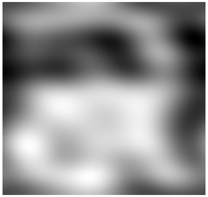
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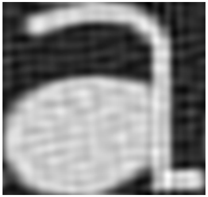
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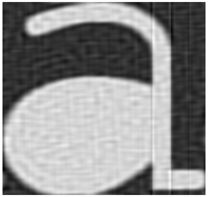
1024

Truncated Spectra

Henceforth $d = 2$

Want: An efficient way to represent truncated spectrum

Need to deal with rotations too

Obvious answer: Polar coordinates

Pick $\omega_r = \sqrt{\omega_x^2 + \omega_y^2}$, $\omega_\theta = \tan^{-1} \frac{y}{x}$

$$\widehat{f}(\omega_x, \omega_y) \mapsto \widehat{f}^p(\omega_r, \omega_\theta)$$

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Problem?

Discrete Cartesian to Polar Conversion

Can't do this exactly!

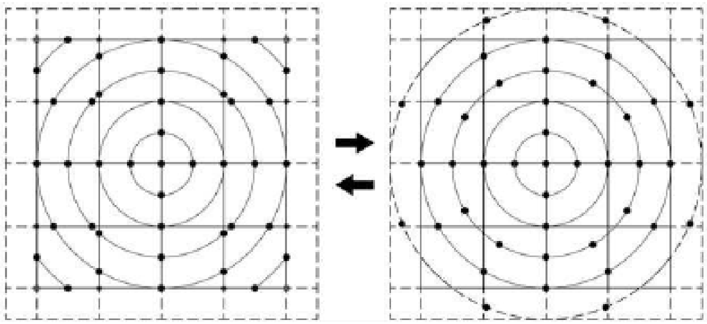

Figure from Chirikjian 2003

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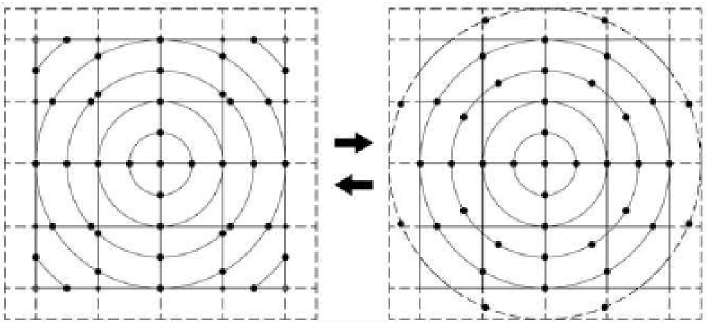
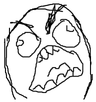


Figure from Chirikjian 2003

Reason: *Only nontrivial discrete subgroups of $SE(2)$ are 17 plane tiling groups!*



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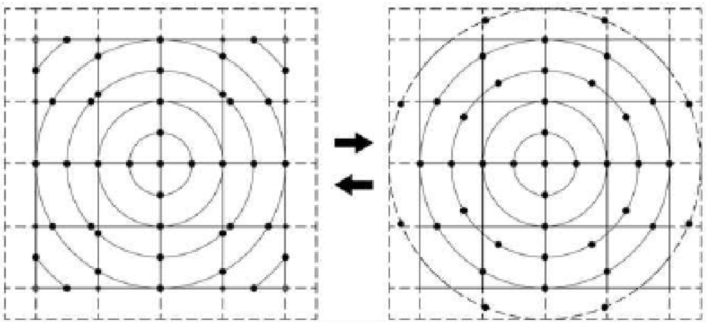

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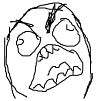

So unless you are happy with a hexagonal grid with 6 rotational samples, forget about doing this losslessly!

Bresenham Interpolation

- Approximation is a fact of life – deal with it
  - Might as well do it fast

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Idea: Do circle drawing in reverse, use Bresenham's algorithm

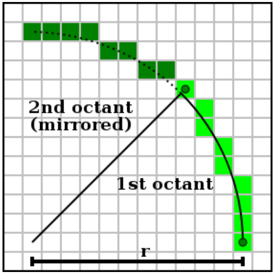

Figure from Wikipedia, 2010

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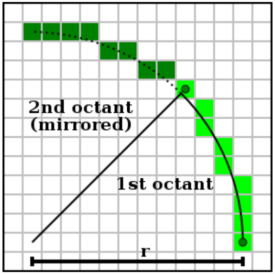

Figure from Wikipedia, 2010

- Sample r at uniform increments
- For each r, trace a circle centered at the origin
- Store $8r + 4$ radial samples in array
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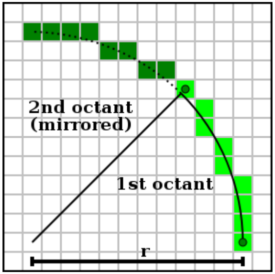

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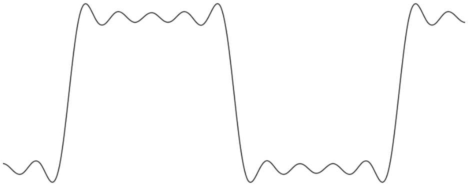
Minimal storage, uses only integer arithmetic

Gibbs Phenomenon

- Truncating Fourier series causes ringing artifacts at boundaries
  - Intuitively, same as low pass filtering high curvature



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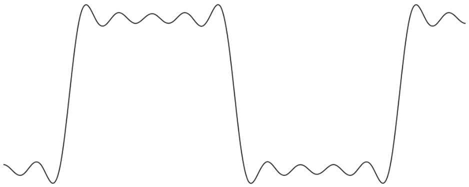

Figure from Wikipedia, 2010

But we don't care, just need separation between interior and exterior.

So pick some cutoff threshold; values above are in, below are out

Optimal Thresholding

Given a region $R \subset \mathbb{R}^d$ and set $A \subseteq R$ approximated by f, define quality, Q_T, of threshold T

$$Q_T = \text{vol}\,\{x \in R | f(x) > T \Leftrightarrow x \in A\}$$

Intuitively, number of points where threshold gives the right answer
Want to find optimal T

$$T = \operatorname*{argmax}_{T' \in \mathbb{R}} Q_{T'}$$

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But in practice can compute T combinatorially given f, A, R

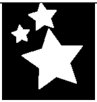
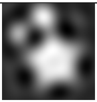
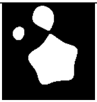
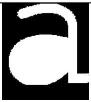
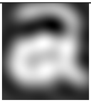
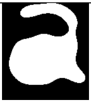
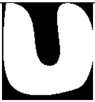
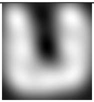
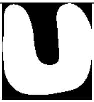
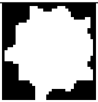
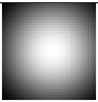
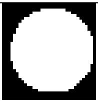
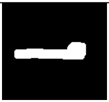
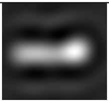
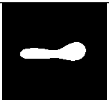
Optimal Thresholding Algorithm

```
v = []
for x in R:
    v.append( (f(x), A(x)) )
v.sort()

fv, Av = unzip(v)
Q = foldl(Av, +, 0) + foldr(Av, +, 0)

return max(zip(Q, fv)).second
```

Greedy/DP method

Takes $O(n\log(n))$ time, uses $O(n)$ space

Thresholding Results

Exact Indicator	Fourier Truncation	Optimal Cutoff
		
		
		
		
		

Radius 14 cutoff, 777 terms

Putting it together

Let $\hat{f}_A$ be the approximate Fourier series for $\mathbf{1}_A$

Then:

$$
\begin{aligned}
C_{i,j}(\rho,\theta,\phi) &= -T + \sum_{r=2}^{N_R}\sum_{t=1}^{8r+4} \widehat{f_{A_i}}\left(r, \phi\frac{dt}{d\theta} + t\right)\overline{\widehat{f_{A_j}}}(r,t) \\
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Where:

$$q_j q_i^{-1} = (R, y)$$

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Protip: Can early out using the rearrangement inequality

Prototype Physics Code

Basic implementation so far, just uses implicit Euler.
Doesn't solve for μ correctly (currently is constant)

Insert image

Features:

- OpenGL GUI
- Written 100% in Python
- Reasonable performance (though certainly not fast)
- Arbitrary shapes/mass fields supported

Download Link:

http://github.com/mikolalysenko/Collisions

To Do

- Analysis is still very incomplete
- Need to implement a proper solver for physics system
- Benchmarks, etc.
- More development/write ups needed