4.1 $U \cap V$ is open since,

$$\begin{aligned} U &= X/\bar{U} \\ V &= X/\bar{V} \\ U \cap V &= X/\bar{U}/\bar{V} \end{aligned}$$

Therefore, remark 3.1.1 implies $f = g$ over $U \cup V$ is $U \cap V = 0$. If $U \cap V$ is empty, then gluing U and V gives a regular function trivially. That there exists a maximal U on which f is defined follows from homework exercise 1.7a(iv) with the above result.

4.2 This result is symmetric to the above.

4.3

a f is defined when $x_0 \neq 0$ so the set is $P^2/\{x_0 = 0\}$

b Begin by embedding $A^1_{y'}$ into P^1 via $y' \mapsto (y', 1)$. So the induced map $\varphi : P^2 \to P^1$ is the mapping:

$$\varphi(x_0, x_1, x_2) = (\frac{x_1}{x_0}, 1)$$

However, this map may be rationalized to give:

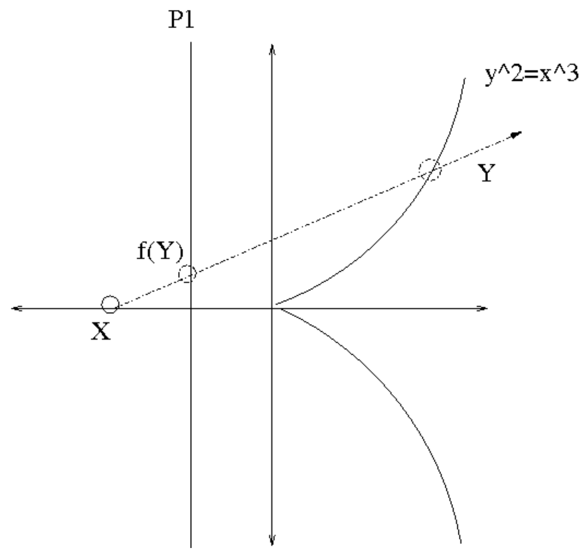
$$\varphi(x_0, x_1, x_2) = (x_1, x_0)$$

Which is defined for all P^2.

4.4

a By exercise 3.1c, all conics in P^2 are isomorphic to P^1 and since isomorphism implies birational equivalence they are all rational curves.

b Take the following projection:



Where the line P^1 is embedded as $x = 0$ into A^2. Fixing a point $X = (-1, 0)$ and taking the intersection of lines with P^1 gives a birational map between P^1 and the cuspidal cubic. Therefore, the two sets are birationally equivalent.

c This is identical to the above procedure.

5.1

Problem	Picture	f	$\partial_x f$	$\partial_y f$	Singular Points
a	Tacnode	$-x^2 + x^4 + y^4$	$-2x + 4x^3$	$4y^3$	$(0, 0)$
b	Node	$x^6 - xy + y^6$	$-2x + 4x^3$	$4y^3$	$(0, 0)$
c	Cusp	$-x^3 + x^4 + y^2 + y^4$	$-3x^2 + 4x^3$	$2y + 4y^3$	$(0, 0)$
d	Triple Point	$x^4 - x^2y - xy^2 + y^4$	$-3x^2 + 4x^3$	$2y + 4y^3$	$(0, 0)$

5.2

Problem	Picture	f	$\partial_x f$	$\partial_y f$	$\partial_z f$	Singular Points
a	Pinch Point	$xy^2 - z^2$	y^2	$2xy$	$-2z$	$\{y = 0, z = 0\}$
b	Conical Double Point	$x^2 + y^2 - z^2$	$2x$	$2y$	$-2z$	$\{(0, 0, 0)\}$
c	Double Line	$x^3 + xy + y^3$	$3x^2 + y$	$x + 3y^2$	0	$\{x = 0, y = 0\}$

5.6

a We handle both cases separately. For the cusp we have the generators:

$$
\begin{aligned}
f_1 &= -x^3 + x^4 + y^2 + y^4 \\
f_2 &= xu - yt
\end{aligned}
$$

With the Jacobian matrix:

$$
\begin{pmatrix}
-3x^2 + 4x^3 & 2y + 4y^3 & 0 & 0 \\
u & -t & x & -y
\end{pmatrix}
$$

Which is non-singular subject to $f_1 = 0, f_2 = 0$.
For the node, we get:

$$
\begin{aligned}
f_1 &= x^6 - xy + y^6 \\
f_2 &= xu - yt
\end{aligned}
$$

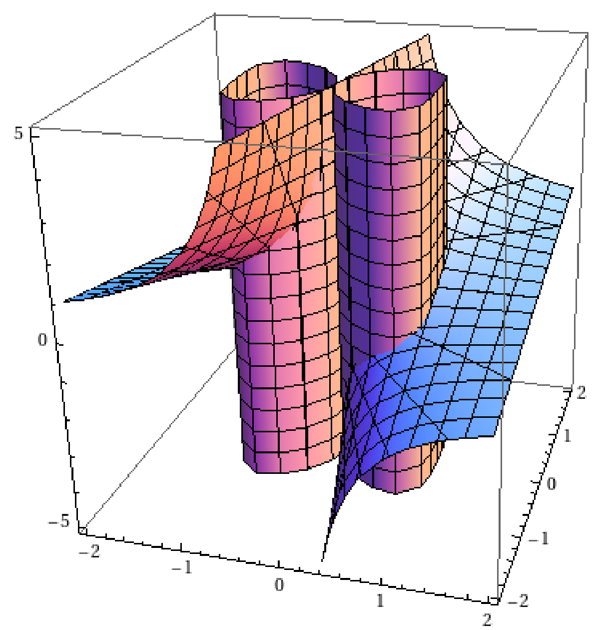
With the Jacobian matrix:

$$
\begin{pmatrix}
x^5 - y & y^5 - x & 0 & 0 \\
u & -t & x & -y
\end{pmatrix}
$$

b If there are two distinct tangents to P, then there must be two distinct lines through the point at P. Intersecting with the blow-up variety then gives two distinct intersections. As a result the intersection curve is no longer singular.

c The generators of the blow up of 5.1a are given as follows:

$$
\begin{aligned}
f_1 &= x^4 + y^4 - x^2 \\
f_2 &= xu - yt
\end{aligned}
$$



To compute the singular points; we now construct the Jacobian matrix of the generators of $\tilde{Y}$ for the two affine subspaces cut out by $t = 1$ and $u = 1$:

For $t = 1$:

$$\begin{pmatrix} -2x + 4x^3 & 4y^3 & 0 \\ u & -1 & x \end{pmatrix}$$

For $u = 1$:

$$\begin{pmatrix} -2x + 4x^3 & 4y^3 & 0 \\ 1 & -t & y \end{pmatrix}$$

In both cases, the rank of the matrix is 2, since the bottom row vectors $(u, -1)$ and $(1, -t)$ do not contain any x, y terms and the first row does not contain any constant or u, t terms. Since these two sets cover $A^2 \times P^1$, the local ring at all points in the variety is regular and thus $\tilde{Y}$ is nonsingular.

7.1

a The d-uple embedding of P^n in P^N is given by the intersection of n hypersurfaces of degree d. Since this embedding is an isomorphism (exercise 3.4) and projective space is singly connected, there is exactly one intersection component in the image of the d-uple embedding with multiplicity 1. Therefore, by theorem 7.7, the degree of the d-uple embedding is the product of the degree of each component and so it must be d^n.

b We showed that the Segre mapping was an isomorphism in problem 3.16. Moreover, the Hilbert polynomial of P^r is $\varphi_{P^r}(t) = \binom{r+t}{r}$ (using the argument on p.52) and so the Hilbert polynomial of the embedding, Q, is given by:

$$\begin{aligned} \varphi_Q(t) &= \varphi_{P^r}(t)\varphi_{P^s}(t) \\ &= \binom{r+t}{r}\binom{s+t}{s} \end{aligned}$$

Looking at the leading coefficient for $\varphi_Q(t)$, we get $\frac{1}{r!s!}t^{r+s}$. To solve for the degree of Q, d, we apply a second result from p.52 to see that:

$$\frac{d}{(r+s)!} = \frac{1}{r!s!}$$

and thus

$$d = \binom{r+s}{r}$$

7.5

a In dimension 1, for any point on the curve Y there is some line (call it H) which intersects that point. The intersection multiplicity of this line and that point is identical to the self-intersection multiplicity (by the definition from page 53). So for each point that H intersects Y we have by theorem 7.7:

$$\sum_{x \in Y \cap H} i(Y, H; x) = \text{deg } Y \text{deg } H$$

However, deg $Y = d$ and deg $H = 1$ so the self intersection multiplicity for all points along H is given by:

$$\sum_{x \in Y \cap H} i(Y, H; x) = d$$

And therefore for any point the intersection multiplicity of that point must be strictly less than d.

b Take a line tangent to that point, which then intersects the curve exactly once. Because the line intersects the curve once, we may project the rest of Y onto this tangent line using the method described in 4.4.

7.8 The number of subspaces in P^n is finite so there exists a minimum subspace containing Y^r of dimensions $r + 1$.