# Spectral Rigid Body Dynamics

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# Overview

Rigid Body Dynamics

Lagrangian Mechanics

Standard Collisions

Constraint Based Collisions

Fourier Methods

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An approximate model of low energy physics for stiff objects

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Pros:

- + Pretty accurate at human energy scales
- + Good for stiff materials (ie metals, plastics etc.)
- + Easy kinematic constraints (useful for mechanisms)
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- $+$ Good for stiff materials (ie metals, plastics etc.)
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Cons:

- - Inaccurate at extremely large energies
- - Bad for materials with low elastic modulus
- - Not always solvable! (See: Painleve's paradox)

# What is a Rigid Body?

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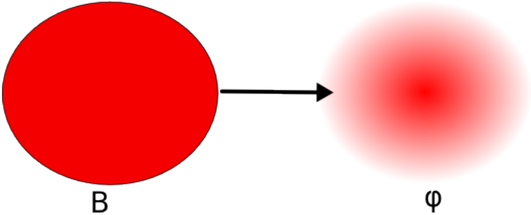

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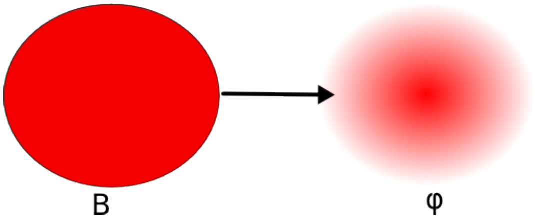


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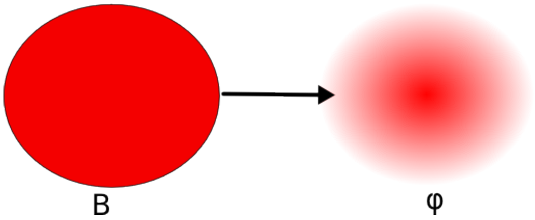


$\varphi$ represents the mass distribution of $B$

$\varphi(x) = 0$ indicates $B$ does not occupy the space at $x$

# Configuration Space of a Rigid Body

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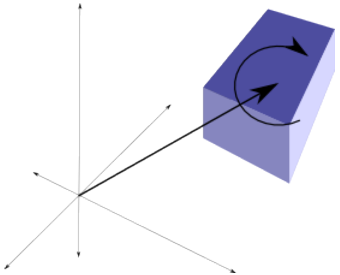

Can be parameterized by a translation $t$ and a rotation $R$

Matrix: $\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$

$\binom{d+1}{2}$ degrees of freedom

Tangent space: $\mathfrak{so}(d+1)$

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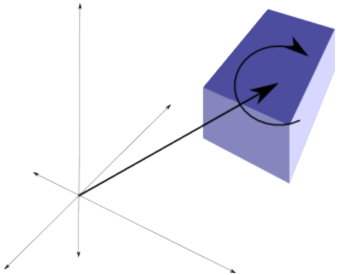

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**Motions of rigid objects** $\cong$ **curves** $q(t) \subset SE(d)$

# Lagrangian Mechanics

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$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - U(q, t)$$

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In other words:

$$\operatorname*{argmin}_{q:[t_0,t_1)\to SE(d)} \int_{t_0}^{t_1} L(q, \dot{q}, t)dt$$

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Newton's equations!

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A: Tensor sum

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| Motion | $q(t) \cong q_i(t) \oplus q_j(t)$ |
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Scales to $n$ bodies, get Lagrangian in $SE(d)^n$

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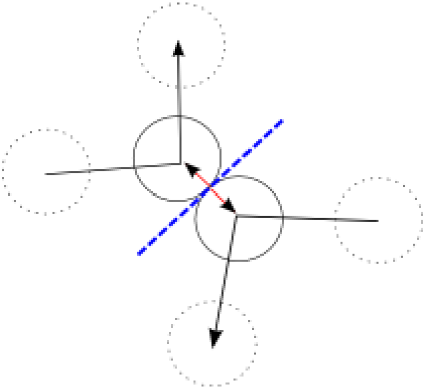

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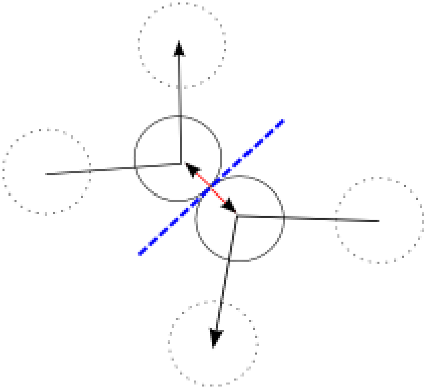

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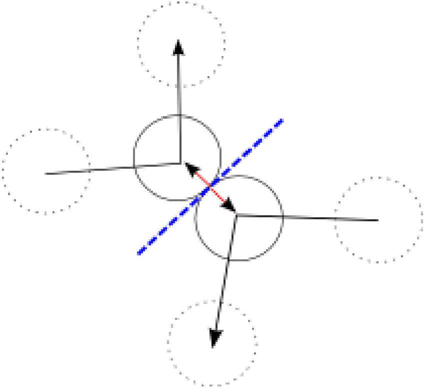

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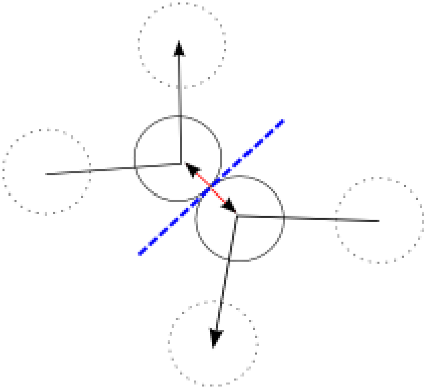

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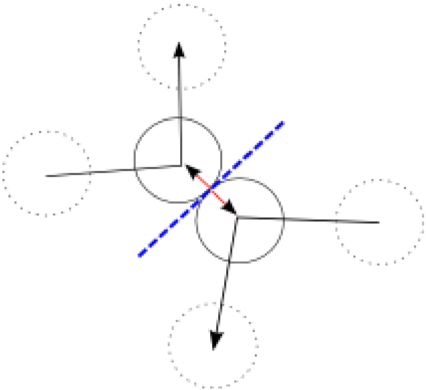

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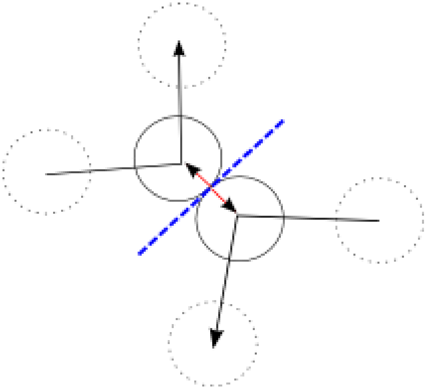

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\+ Just like high school physics

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But can be made to work with enough hacking...

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Is this really all there is to it?

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Define

$$C_{i,j}(q_i, q_j) \stackrel{def}{\equiv} \text{vol } q_i A_i \cap q_j A_j$$

And so we replace the impact forces with a system of differentiable holonomic inequality constraints:

$$C_{i,j} \leq 0$$

# Equations of motion revisited

New problem:

$$\text{minimize} \int_{t_0}^{t_1} L(q, \dot{q}, t)dt$$

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Apply KKT conditions + Euler-Lagrange to get complementarity problem:

$$\frac{d}{dt}\left(\frac{\partial T(\dot{q}_i)}{\partial \dot{q}_i}\right) - \frac{\partial U(q,t)}{\partial q_i} + \sum_{j \neq i} \mu_{i,j} \frac{\partial C_{i,j}(q_i, q_j)}{\partial q_i} = 0$$

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Exactly elastic collision response!

Slack variables are impulse forces

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Observe:

$$\mathbf{1}_{A_i \cap A_j}(x) = \mathbf{1}_{A_i}(x)\mathbf{1}_{A_j}(x)$$

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So:

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Substitute $q_j^{-1}q_i x \mapsto Rx - y$ and let $\widetilde{\mathbf{1}_{A_j}}(x) = \mathbf{1}_{A_j}(-x)$:

$$\int\limits_{\mathbb{R}^d} \mathbf{1}_{A_i}(x)\widetilde{\mathbf{1}_{A_j}}(y - Rx)dx = \int\limits_{\mathbb{R}^d} \mathbf{1}_{A_i}(R^{-1}x)\widetilde{\mathbf{1}_{A_j}}(y - x)dx$$

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$$C_{i,j}(q_i, q_j) = ((\mathbf{1}_{A_i} \circ R^{-1}) \star \widetilde{\mathbf{1}_{A_j}})(y)$$

# Fourier Methods

Convolution?

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Convolution? Take a Fourier transform!

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What about gradients? (ie $\frac{\partial C_{i,j}(q_i,q_j)}{\partial q_i}$)

Fix parameters $q_i = (R_i, t_i)$,

$$q_i x = R_i x + t_i$$

$$q_i^{-1} = (R_i^{-1}, -R_i^{-1} t_i)$$

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Need to compute $\frac{\partial C_{i,j}(R_i, t_i, R_j, t_j)}{\partial R_i}$, $\frac{\partial C_{i,j}(R_i, t_i, R_j, t_j)}{\partial t_i}$

Or by symmetry: $C_{i,j}(q_i, q_j) = C_{j,i}(q_j, q_i)$

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$$\frac{\partial C_{i,j}}{\partial t_j^k} = \frac{\partial}{\partial t_i^j}\left(\int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R_i R_j^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\left\langle \omega, t_j - R_j R_i^{-1} t_i\right\rangle} d\omega\right)$$

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$$
\begin{aligned}
\frac{\partial C_{i,j}}{\partial t_j^k} &= \frac{\partial}{\partial t_i^j}\left( \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R_i R_j^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i \left\langle \omega, t_j - R_j R_i^{-1} t_i \right\rangle} d\omega \right) \\
&= \int_{\mathbb{R}^d} 2\pi i \left\langle \omega, v^k \right\rangle \widehat{\mathbf{1}_{A_i}}(R_i R_j^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i \left\langle \omega, t_j - R_j R_i^{-1} t_i \right\rangle} d\omega
\end{aligned}
$$

Where $v^k$ denotes the $k^{th}$ basis vector

# Translational Gradient

Start with the translational case first.

Let $t_j^k$ denote the $k^{th}$ component of $t_j$,then

$$\begin{aligned}\frac{\partial C_{i,j}}{\partial t_j^k} &= \frac{\partial}{\partial t_i^j}\left(\int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R_i R_j^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i\left\langle \omega, t_j - R_j R_i^{-1} t_i\right\rangle} d\omega\right) \\ &= \int_{\mathbb{R}^d} 2\pi i \left\langle \omega, v^k \right\rangle \widehat{\mathbf{1}_{A_i}}(R_i R_j^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i\left\langle \omega, t_j - R_j R_i^{-1} t_i\right\rangle} d\omega\end{aligned}$$

Where $v^k$ denotes the $k^{th}$ basis vector

Conclusion: Translational gradient is just a multiplier

# Rotational Gradient

Parameterize $R_j = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}_{k,l} = -\mathfrak{r}_{l,k}$

# Rotational Gradient

Parameterize $R_j = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}_{k,l} = -\mathfrak{r}_{l,k}$

$$\frac{\partial C_{i,j}}{\partial \mathfrak{r}_{k,l}} = \frac{\partial}{\partial \mathfrak{r}_{k,l}} \left( \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R_i \exp(-\mathfrak{r})\omega) \overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i \left\langle \omega, t_j - \exp(\mathfrak{r}) R_i^{-1} t_i \right\rangle} d\omega \right)$$

# Rotational Gradient

Parameterize $R_j = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}_{k,l} = -\mathfrak{r}_{l,k}$

$$
\begin{aligned}
\frac{\partial C_{i,j}}{\partial \mathfrak{r}_{k,l}} &= \frac{\partial}{\partial \mathfrak{r}_{k,l}} \left( \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R_i \exp(-\mathfrak{r})\omega) \overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i \left\langle \omega, t_j - \exp(\mathfrak{r}) R_i^{-1} t_i \right\rangle} \, d\omega \right) \\
\ldots &= \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) e^{2\pi i \langle \omega, y \rangle} \left( - \left\langle \overline{\nabla \widehat{\mathbf{1}_{A_j}}(R\omega)}, R \mathfrak{r}_{k,l} \omega \right\rangle \right. \\
&+ \left. \overline{\widehat{\mathbf{1}_{A_j}}(R\omega)} 2\pi i \left\langle R \mathfrak{r}_{k,l} \omega, t_j \right\rangle \right) d\omega
\end{aligned}
$$

# Rotational Gradient

Parameterize $R_j = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$

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$$
\begin{aligned}
\frac{\partial C_{i,j}}{\partial \mathfrak{r}_{k,l}} &= \frac{\partial}{\partial \mathfrak{r}_{k,l}} \left( \int\limits_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R_i \exp(-\mathfrak{r})\omega) \overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i \left\langle \omega, t_j - \exp(\mathfrak{r}) R_i^{-1} t_i \right\rangle} d\omega \right) \\
\ldots &= \int\limits_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) e^{2\pi i \langle \omega, y \rangle} \left( - \left\langle \overline{\nabla \widehat{\mathbf{1}_{A_j}}(R\omega)}, R\mathfrak{r}_{k,l}\omega \right\rangle \right. \\
&+ \left. \overline{\widehat{\mathbf{1}_{A_j}}(R\omega)} 2\pi i \left\langle R\mathfrak{r}_{k,l}\omega, t_j \right\rangle \right) d\omega
\end{aligned}
$$

Get two terms:

## Rotational Gradient

Parameterize $R_j = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$

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$$\ldots = \int\limits_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) e^{2\pi i \langle \omega, y \rangle} \Big( - \left\langle \overline{\nabla \widehat{\mathbf{1}_{A_j}}(R\omega)}, R \mathfrak{r}_{k,l} \omega \right\rangle$$

$$+ \overline{\widehat{\mathbf{1}_{A_j}}(R\omega)} 2\pi i \left\langle R \mathfrak{r}_{k,l} \omega, t_j \right\rangle \Big) d\omega$$

Get two terms: a multiplier (easy),

# Rotational Gradient

Parameterize $R_j = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}_{k,l} = -\mathfrak{r}_{l,k}$

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\frac{\partial C_{i,j}}{\partial \mathfrak{r}_{k,l}} &= \frac{\partial}{\partial \mathfrak{r}_{k,l}} \left( \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R_i \exp(-\mathfrak{r})\omega) \overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i \left\langle \omega, t_j - \exp(\mathfrak{r}) R_i^{-1} t_i \right\rangle} d\omega \right) \\
\ldots &= \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) e^{2\pi i \langle \omega, y \rangle} \left( - \left\langle \overline{\nabla \widehat{\mathbf{1}_{A_j}}(R\omega)}, R\mathfrak{r}_{k,l}\omega \right\rangle \right. \\
&+ \left. \overline{\widehat{\mathbf{1}_{A_j}}(R\omega)} 2\pi i \left\langle R\mathfrak{r}_{k,l}\omega, t_j \right\rangle \right) d\omega
\end{aligned}$$

Get two terms: a multiplier (easy), a gradient (can be precomputed).

# Rotational Gradient

Parameterize $R_j = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}_{k,l} = -\mathfrak{r}_{l,k}$

$$\begin{aligned}
\frac{\partial C_{i,j}}{\partial \mathfrak{r}_{k,l}} &= \frac{\partial}{\partial \mathfrak{r}_{k,l}} \left( \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R_i \exp(-\mathfrak{r})\omega) \overline{\widehat{\mathbf{1}_{A_j}}(\omega)} e^{2\pi i \left\langle \omega, t_j - \exp(\mathfrak{r}) R_i^{-1} t_i \right\rangle} d\omega \right) \\
\ldots &= \int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(\omega) e^{2\pi i \langle \omega, y \rangle} \left( - \left\langle \overline{\nabla \widehat{\mathbf{1}_{A_j}}(R\omega)}, R \mathfrak{r}_{k,l} \omega \right\rangle \right. \\
&+ \left. \overline{\widehat{\mathbf{1}_{A_j}}(R\omega)} 2\pi i \left\langle R \mathfrak{r}_{k,l} \omega, t_j \right\rangle \right) d\omega
\end{aligned}$$

Get two terms: a multiplier (easy), a gradient (can be precomputed).

Computationally not too bad, but still pretty messy in $d$-dimensional space.