Assigned: Tuesday Jan. 26, 2010
Due: Thursday Feb. 11, 2010

MATH/CS 715: Homework 1
Spring 2010

PAGE LIMIT: 20 pages (single-sided).

NOTE: Please include a cover page – this will not count toward the total page limit.

NOTE: Don't forget to include MATLAB code where appropriate – this does count toward your total pages.

Part 1: Non-Interpolating Spectral Methods

Consider the following boundary value problem with periodic boundary conditions:

$$\begin{aligned} \textbf{ODE}: &\quad \phi_{xx} + \{\sin(x) + \cos(x) + 2\sin(x)\cos(x)\}\,\phi = e^{\sin(x)} e^{\cos(x)}, \\ \textbf{BC1}: &\quad \phi(-\pi) = \phi(\pi), \\ \textbf{BC2}: &\quad \phi_x(-\pi) = \phi_x(\pi). \end{aligned}$$

1. Verify that $\phi(x) = e^{\sin(x)} e^{\cos(x)}$ is the exact solution to this BVP.
2. Devise and then implement in MATLAB a Galerkin method for approximating the solution to the above BVP. Your approximation should be of the form:
$$\phi(x) \approx a_0 + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$
3. Run your method for various values of N and plot the errors as a function of N. Explain what happens with N is taken "too large"?

Part 2: Fourier Spectral Methods

4. Let $\mathcal{F}$ denote the Fourier transform, so that $u, v \in L^2(\mathbb{R})$ have Fourier transforms $\hat{u} = \mathcal{F}\{u\}$ and $\hat{v} = \mathcal{F}\{v\}$. Verify the following properties (don't worry about rigorously justifying operations on integrals):
   (a) *Linearity.* $\mathcal{F}\{u+v\}(k) = \hat{u}(k) + \hat{v}(k)$; $\mathcal{F}\{cu\}(k) = c\hat{u}(k)$.
   (b) *Translation.* If $x_0 \in \mathbb{R}$, then $\mathcal{F}\{u(x+x_0)\}(k) = e^{ikx_0}\hat{u}(k)$.
   (c) *Modulation.* If $k_0 \in \mathbb{R}$, then $\mathcal{F}\{e^{ik_0 x} u(x)\}(k) = \hat{u}(k-k_0)$.
   (d) *Dilation.* If $c \in \mathbb{R}$ with $c \neq 0$, then $\mathcal{F}\{u(cx)\}(k) = \hat{u}(k/c)/|c|$.
   (e) *Conjugation.* $\mathcal{F}\{\bar{u}\}(k) = \overline{\hat{u}(-k)}$.
   (f) *Differentiation.* If $u_x \in L^2(\mathbb{R})$, then $\mathcal{F}\{u_x\}(k) = ik\hat{u}(k)$.
   (g) *Inversion.* $\mathcal{F}^{-1}\{u\}(k) = (2\pi)^{-1}\hat{u}(-k)$.

5. Differentiation of $u(x) = e^{ikx}$ multiplies it by $g_\infty(k) = ik$. Determine the analogous functions $g_2(k)$ and $g_4(k)$ corresponding to the second and fourth-order finite difference methods. Make a plot of $g_2(k)$, $g_4(k)$, and $g_\infty(k)$ versus k. Where in the plot do we see the order of accuracy of the finite difference formulas?

6. Download the MATLAB code for solving the variable coefficient wave equation that was discussed in lecture:

   `http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/p6.m`

   This code solves the following wave equation with periodic boundary conditions:

   $$u_t + \left(\frac{1}{5} + \sin^2(x-1)\right) u_x = 0.$$

   (a) The exact solution to the above equation is periodic in time for a certain $T \approx 13$. Determine T analytically by evaluating an appropriate integral.

   (b) Modify the downloaded program to compute $u(x,T)$ instead of $u(x,8)$. Make sure that the program stops at exactly at T. For $N = 32, 64, \ldots, 512$, determine $\max_j |u(x_j,T) - u(x_j,0)|$ and plot this error on a log-log scale as a function of N. What is the rate of convergence? How could it be improved?

   (c) Solve the BVP form Part 1 of this assignment using the Fourier spectral method. Plot the error as a function of N. Compare this error plot with the one found in problem 3.

7. In our discussion of smoothness and spectral accuracy, 4 theorems on the accuracy of spectral differentiation were presented. Show that Theorem 3 follows from Theorems 1 and 2.

8. Consider the periodic eigenvalue problem:

   $$-u_{xx} + x^m u = \lambda u, \quad x \in \mathbb{R}.$$

   Download the following the MATLAB code:

   `http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/p8.m`

   and use it to calculate for the cases $m = 2$ and $m = 4$ the first 20 eigenvalues to 10-digits of accuracy, providing good evidence that you have achieved this, and plot the results. Briefly discuss difference between the $m = 2$ and the $m = 4$ cases.