Assigned: Thursday March 18, 2010
Due: Thursday April 8, 2010

MATH/CS 715: Homework 4
Spring 2010

PAGE LIMIT: 20 pages (single-sided).

NOTE: Please include a cover page – this will not count toward the total page limit.

NOTE: Don't forget to include MATLAB code where appropriate – this does count toward your total pages.

1D Finite Element Methods

1. Consider the following boundary value problem with periodic boundary conditions:

$$\begin{aligned}
\textbf{ODE}: &\quad \phi_{xx} + \{\sin(x) + \cos(x) + 2\sin(x)\cos(x)\}\,\phi = e^{\sin(x)}e^{\cos(x)},\\
\textbf{BC1}: &\quad \phi(-\pi) = \phi(\pi),\\
\textbf{BC2}: &\quad \phi_x(-\pi) = \phi_x(\pi).
\end{aligned}$$

   (a) Recast this equation into a variational problem, stating the trial and test function spaces.
   (b) Develop a cG(1) finite element method for this problem. Explicitly write out the coefficient matrix and use the trapezoidal rule to evaluate the necessary integrals. Write out the method for a general non-uniform grid spacing (you will need this in Problem 2(c).)
   (c) Numerically determine the convergence rate using various grid spacings on a uniformly spaced grid (the exact solution is $\phi(x) = e^{\sin(x)}e^{\cos(x)}$). Briefly comment on this result compared to HW #1 results (i.e., Galerkin method based on trigonometric polynomials and the Fourier pseudo-spectral method).

2. Consider the following boundary value problem with periodic boundary conditions:

$$\begin{aligned}
\textbf{ODE}: &\quad \phi_{xx} + 50\,\phi = 2450\sin^{48}(x)\cos^2(x),\\
\textbf{BC1}: &\quad \phi(-\pi) = \phi(\pi),\\
\textbf{BC2}: &\quad \phi_x(-\pi) = \phi_x(\pi),\\
\textbf{Solution}: &\quad \phi(x) = \sin^{50}(x).
\end{aligned}$$

   (a) Determine an *a posteriori* error bound for this ODE using your cG(1) method from Problem 1.
   (b) **Global refinement**. Modify your code from Problem 1 so that you iteratively change the number of grid points used in solving the ODE until your *a posteriori* error bound is below a user-prescribed tolerance. In this GLOBAL REFINEMENT approach, you should always use equally spaced points. Start the iteration process by solving the ODE with very small number of grid points (e.g., 4 or 5). Test your algorithm with a variety of tolerance and report your findings. Compare tolerance values to actual errors.

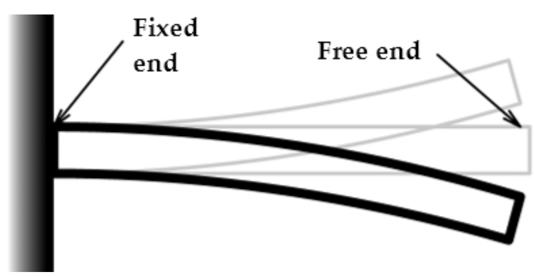
(c) **Local refinement**. Modify your code so that you iteratively modify the number of grid points used in solving the ODE until your *a posteriori* error bound is below a user-prescribed tolerance. In this LOCAL REFINEMENT approach, you will only introduce additional grid points in regions where the error bound is violated. The algorithm should do the following:

- Start with a uniform grid with a relatively small number of points. Let the grid spacing be h.
- Solve equation using the FEM and calculate an error estimate in each element.
- In those elements where the error estimate is above the tolerance, add extra grid points in order to ensure that the current error estimate is below the given tolerance.
- Solve the equation on this new grid and repeat until error is below tolerance in each element. Test your algorithm with a variety of tolerances and report your findings. Compare tolerance values to actual errors.

Beam equation

3. Consider the Beam equation from mechanics with boundary conditions that model a *cantilever* beam:

$$\begin{aligned} \textbf{PDE}: &\quad u^{(iv)} = f(x) \quad x \in (0,1)\,, \\ \textbf{BC}: &\quad u(0) = 0, \quad u'(0) = 0, \quad u''(1) = 0, \quad u'''(1) = 0. \end{aligned}$$



(a) Recast this problem as a variational problem. Clearly state the test and trial function spaces.

(b) Develop a finite element method for this problem.

(c) Prove an *a priori* error estimate for this method in the *energy norm*:

$$\|e\|_E = \left\{ \int_0^1 \left(e''\right)^2 \, dx \right\}^{1/2}.$$

(d) Prove an *a priori* error estimate for this method in the L_2 norm:

$$\|e\|_{L_2} = \left\{ \int_0^1 (e)^2 \, dx \right\}^{1/2}.$$