Spectral Rigid Body Dynamics

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Overview

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An approximate model of low energy physics for stiff objects

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Pros:

- + Pretty accurate at human energy scales
- + Good for stiff materials (ie metals, plastics etc.)
- + Easy kinematic constraints (useful for mechanisms)
- + Standard animation tool (videogames!)

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- $+$ Good for stiff materials (ie metals, plastics etc.)
- $+$ Easy kinematic constraints (useful for mechanisms)
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Cons:

- \- Inaccurate at extremely large energies
- \- Bad for materials with low elastic modulus
- \- Not always solvable! (See: Painleve's paradox)

What is a Rigid Body?

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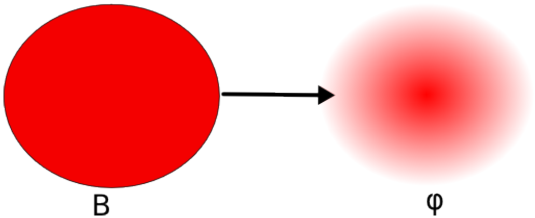

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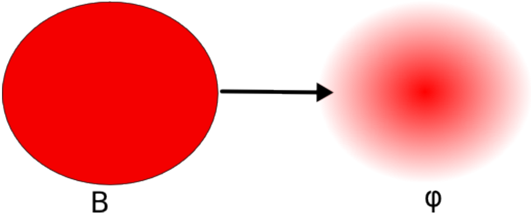

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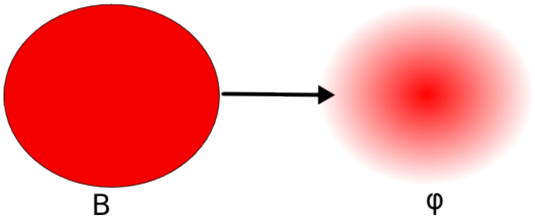

φ represents the mass distribution of B

$\varphi(x) = 0$ indicates B does not occupy the space at x

Configuration Space of a Rigid Body

Transformations of rigid mass fields must preserve distance and handedness

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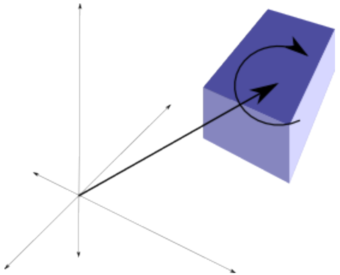

Can be parameterized by a translation t and a rotation R

Matrix: $\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$

$\binom{d+1}{2}$ degrees of freedom

Tangent space: $\mathfrak{so}(d+1)$

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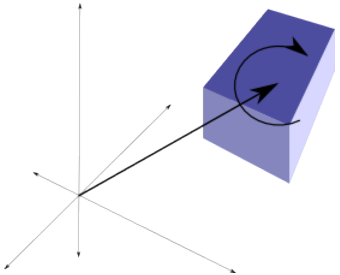

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Motions of rigid objects $\cong$ **curves** $q(t) \subset SE(d)$

Lagrangian Mechanics

Turns physics into an optimization problem.

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$$\mathcal{L}(q, \dot{q}, t) = T(\dot{q}) - U(q, t)$$

Where $T(\dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q}$ is the kinetic energy and U is a potential

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$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \frac{1}{2}\dot{q}^T M \dot{q}}{\partial \dot{q}}\right) &= \frac{\partial U(q,t)}{\partial q} \\ M\ddot{q} &= \nabla U\end{aligned}$$

Newton's equations!

Multiple Bodies

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A: Tensor sum

Let B_i, B_j be independent rigid bodies with motions q_i, q_j

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Motion	$q(t) \cong q_i(t) \oplus q_j(t)$
Lagrangian	$L(q, \dot{q}, t) = L(q_i, \dot{q}_i, t) + L(q_j, \dot{q}_j, t)$

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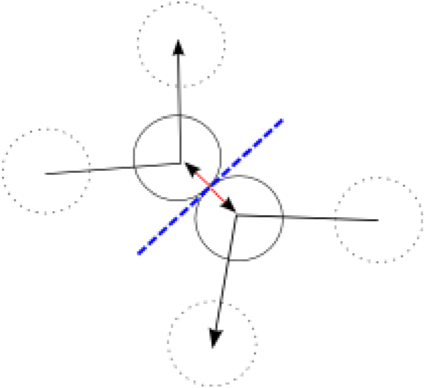
Scales to n bodies, get Lagrangian in $SE(d)^n$

Collisions

Solids can't physically occupy the same space.



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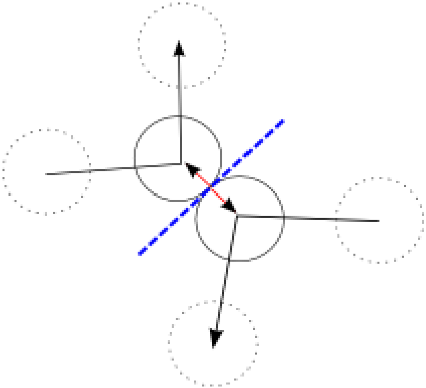

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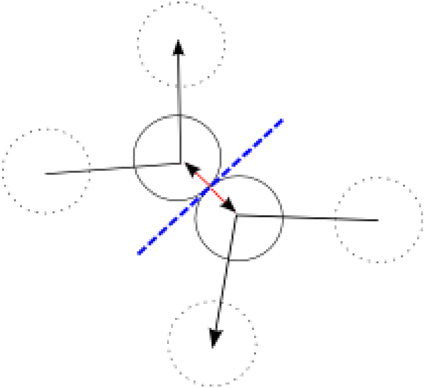

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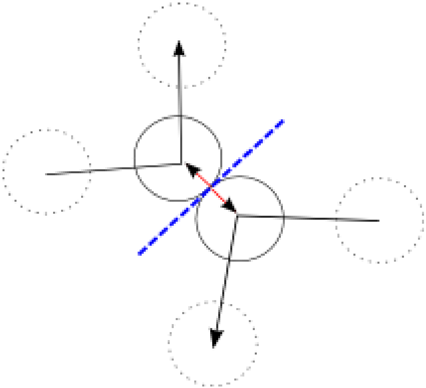

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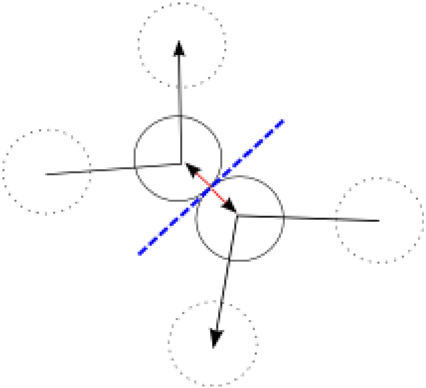

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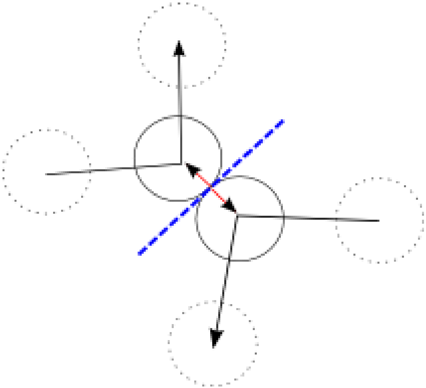

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\+ Just like high school physics

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But can be made to work with enough hacking...

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At all times no two solids intersect

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Is this really all there is to it?

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So two solids, A_i, A_j, *collide* at a configuration q_i, q_j iff:

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Define

$$C_{i,j}(q_i, q_j) \stackrel{def}{\equiv} \text{vol } q_i A_i \cap q_j A_j$$

And so we replace the impact forces with a system of differentiable holonomic inequality constraints:

$$C_{i,j} \leq 0$$

Equations of motion revisited

New problem:

$$\text{minimize} \int_{t_0}^{t_1} L(q, \dot{q}, t)dt$$

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Apply KKT conditions + Euler-Lagrange to get complementarity problem:

$$\frac{d}{dt}\left(\frac{\partial T(\dot{q}_i)}{\partial \dot{q}_i}\right) - \frac{\partial U(q, t)}{\partial q_i} + \sum_{j \neq i} \mu_{i,j} \frac{\partial C_{i,j}(q_i, q_j)}{\partial q_i} = 0$$

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Exactly elastic collision response!

Slack variables are impulse forces

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Need to compute:

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Substitute $q_j^{-1}q_i x \mapsto Rx - y$ and let $\widetilde{\mathbf{1}_{A_j}}(x) = \mathbf{1}_{A_j}(-x)$:

$$\int_{\mathbb{R}^d} \mathbf{1}_{A_i}(x)\widetilde{\mathbf{1}_{A_j}}(y - Rx)dx = \int_{\mathbb{R}^d} \mathbf{1}_{A_i}(R^{-1}x)\widetilde{\mathbf{1}_{A_j}}(y - x)dx$$

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Fourier Methods

Convolution?

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Convolution? Take a Fourier transform!

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Fix parameters $q_i = (R_i, t_i)$,

$$q_i x = R_i x + t_i$$

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Need to compute $\frac{\partial C_{i,j}(R,y)}{\partial y}$, $\frac{\partial C_{i,j}(R,y)}{\partial R}$ then use chain rule

Or by symmetry: $C_{i,j}(q_i, q_j) = C_{j,i}(q_j, q_i)$

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Where v^k denotes the k^{th} basis vector

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$$\begin{aligned}\frac{\partial C_{i,j}}{\partial y^k} &= \frac{\partial}{\partial y^k}\left(\int_{\mathbb{R}^d} \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle\omega,y\rangle}d\omega\right)\\ \ldots &= \int_{\mathbb{R}^d} 2\pi i\left\langle\omega, v^k\right\rangle \widehat{\mathbf{1}_{A_i}}(R^{-1}\omega)\overline{\widehat{\mathbf{1}_{A_j}}(\omega)}e^{2\pi i\langle\omega,y\rangle}d\omega\end{aligned}$$

Where v^k denotes the k^{th} basis vector

Conclusion: Translational gradient is just a multiplier

Rotational Gradient

Parameterize $R = \exp(\mathfrak{r})$, where $\mathfrak{r} \in \mathfrak{so}(d)$ with basis $\mathfrak{r}_{k,l}$

In otherwords $d \times d$ skew symmetric matrices, $\mathfrak{r}^T = -\mathfrak{r}$

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Not a Fourier multiplier.

But can be precalculated with $O(d)$ overhead:

$$\frac{\partial}{\partial \omega^k} \widehat{\mathbf{1}_{A_i}}(\omega) = \mathcal{F}\left(-2\pi i \left\langle x, v^k \right\rangle \mathbf{1}_{A_i}\right)(\omega)$$

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Fourier transform is neat, but what does it get us?

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Full inverse Fourier transform complexity = brute force volume

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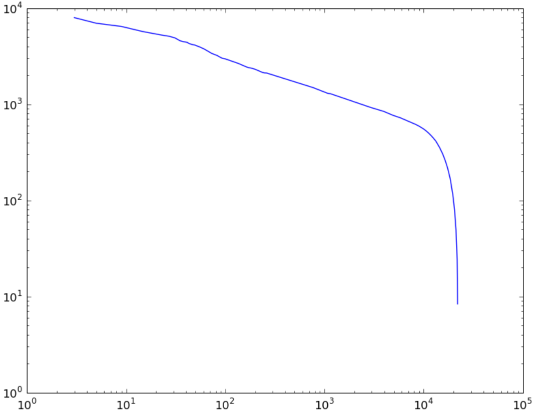
Dramatically reduces space and time complexity

Convergence

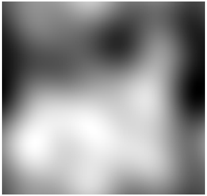
Rate of convergence, number of Fourier terms vs. L^1 error, (log-log plot)



Convergence

Qualitative effects of cutoff (full spectrum has 22185 terms)

Exact



16

32

64

256

1024

Truncated Spectra

Henceforth $d = 2$

Want: An efficient way to represent truncated spectrum

Need to deal with rotations too

Obvious answer: Polar coordinates

Pick $\omega_r = \sqrt{\omega_x^2 + \omega_y^2}$, $\omega_\theta = \tan^{-1} \frac{y}{x}$

$$\widehat{f}(\omega_x, \omega_y) \mapsto \widehat{f}^p(\omega_r, \omega_\theta)$$

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 *Problem?*

Discrete Cartesian to Polar Conversion

Can't do this exactly!

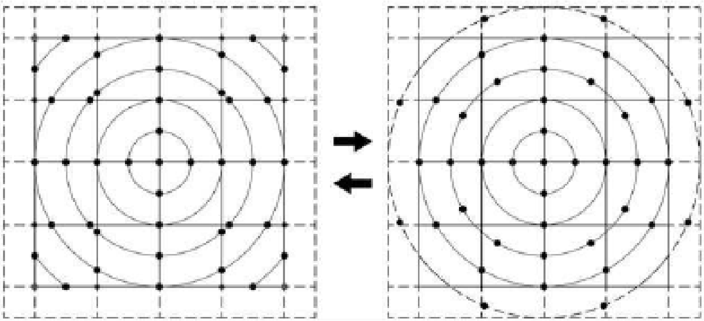

Figure from Chirikjian 2003

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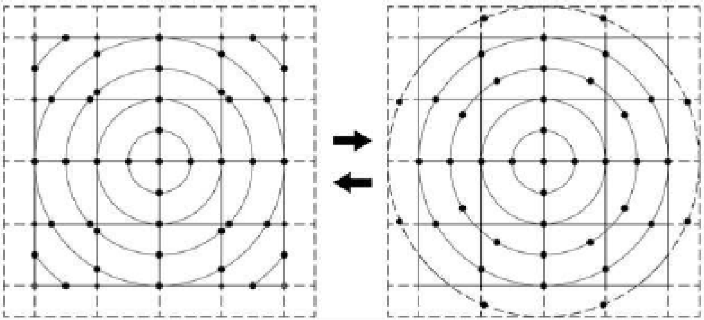

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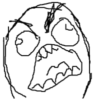
Reason: *Only nontrivial discrete subgroups of* $SE(2)$ *are 17 plane tiling groups!*



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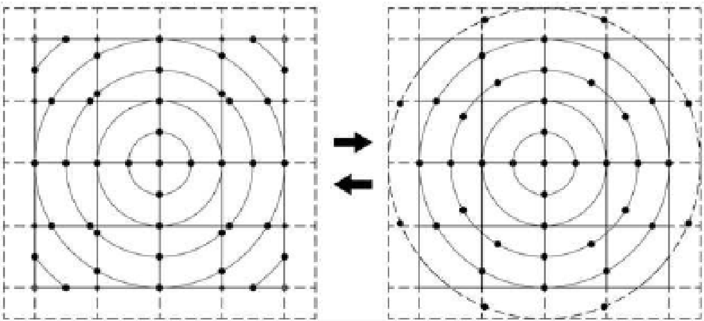

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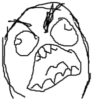

So unless you are happy with a hexagonal grid with 6 rotational samples, forget about doing this losslessly!

Bresenham Interpolation

- Approximation is a fact of life – deal with it
  - Might as well do it fast

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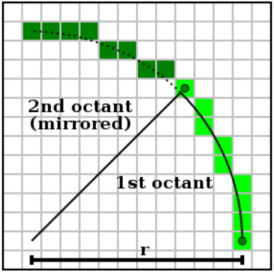
Idea: Do circle drawing in reverse, use Bresenham's algorithm



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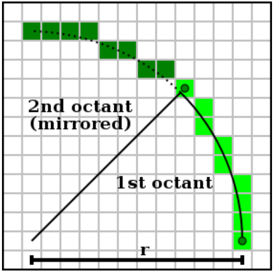

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- Sample r at uniform increments
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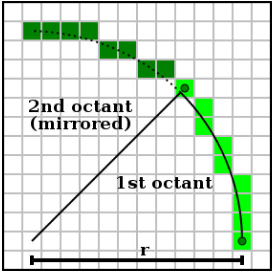

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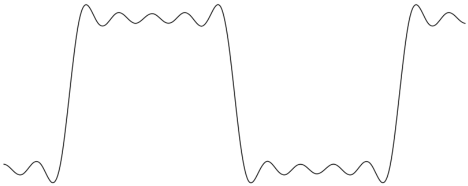
Minimal storage, uses only integer arithmetic

Gibbs Phenomenon

Truncating Fourier series causes ringing artifacts at boundaries

- Intuitively, same as low pass filtering high curvature



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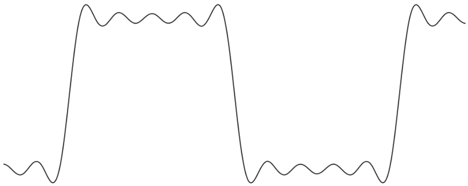

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But we don't care, just need separation between interior and exterior.

So pick some cutoff threshold; values above are in, below are out

Optimal Thresholding

Given a region $R \subset \mathbb{R}^d$ and set $A \subseteq R$ approximated by f, define quality, Q_T, of threshold T

$$Q_T = \text{vol}\,\{x \in R | f(x) > T \Leftrightarrow x \in A\}$$

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But in practice can compute T combinatorially given f, A, R