3.c Consider the Lambda expression:

$$(\lambda x.\lambda y.y)((\lambda x.xxx)(\lambda x.xxx))$$

If one performs the leftmost-outermost expansion, this expression reaches the normal form, $\lambda y.y$ in one β-reduction and so it is non-divergent. Now, define $S = \lambda x.xxx$, and consider the general expression of the form:

$$(\lambda x.\lambda y.y)(S^n)$$

Where $n \geq 2$. Taking the rightmost outer most expansion leads to the following expansion

$$\begin{array}{rl} & (\lambda x.\lambda y.y)(S^n) \\ \equiv & (\lambda x.\lambda y.y)(S^{n-2}(\lambda x.xxx)(\lambda x.xxx)) \\ \vdash_\beta & (\lambda x.\lambda y.y)(S^{n-2}(\lambda x.xxx)(\lambda x.xxx)(\lambda x.xxx)) \\ \equiv & (\lambda x.\lambda y.y)(S^{n+1}) \end{array}$$

Thus each rightmost outermost β-reduction takes $(\lambda x.\lambda y.y)(S^n) \mapsto (\lambda x.\lambda y.y)(S^{n+1})$, and thus will never reach the normal form. Therefore, the rightmost-outermost expansion rule does not always find a normal form if one exists.

5.a By the hypothesis, assume there exist lambda terms, $A \neq B$ and define

$$\begin{array}{rcl} f_1 & \equiv & \lambda s.\lambda t.s \\ f_2 & \equiv & \lambda s.\lambda t.t \end{array}$$

Now suppose that $S = K$. If that were true, then it must be that:

$$Sf_2f_1AB = Kf_2f_1AB$$

And that all of their reduced normal forms are equivalent. Now reducing Sf_2f_1AB gives the following:

$$\begin{array}{rcl} Sf_2f_1AB & \equiv & (\lambda x.\lambda y.\lambda z.xz(yz))f_2f_1AB \\ & \vdash_\beta & (\lambda z.f_2z(f_1z))AB \\ & \vdash_\beta & (f_2A(f_1A))B \\ & \vdash_\beta & f_1AB \\ & \vdash_\beta & A \end{array}$$

Similarly, reducing Kf_2f_1AB gives:

$$\begin{array}{rcl} Kf_2f_1AB & \equiv & (\lambda x.\lambda y.x)f_2f_1AB \\ & \vdash_\beta & f_2AB \\ & \vdash_\beta & B \end{array}$$

But, this is a contradiction since $A \neq B$. Therefore, it must be that $S \neq K$.

6 We wish to construct a combinator, $\underline{Plus}$ such that for any Church numerals, $\underline{a}, \underline{b}$, with $a, b \in \mathbb{N}^+$:

$$\underline{Plus}\ \underline{a}\ \underline{b} \to^*_{a,b} \underline{a+b}$$

Pick:

$$\underline{Plus} \equiv \lambda a.\lambda b.\lambda f.\lambda x.(af)(bfx)$$

We now check the invariant on $\underline{Plus}$ by direct substitution:

$$\begin{aligned}
\underline{Plus}\ \underline{a}\ \underline{b} &\equiv (\lambda a.\lambda b.\lambda f.\lambda x.(af)(bfx))\,\underline{a}\,\underline{b} \\
&\vdash_\beta \lambda f.\lambda x.(\underline{a}\,f)(\underline{b}\,fx) \\
&\vdash_\beta \lambda f.\lambda x.(\lambda x'.f^a x')(f^b x) \\
&\vdash_\beta \lambda f.\lambda x.f^a(f^b x) \\
&\equiv \lambda f.\lambda x.f^{a+b}x \\
&\equiv \underline{a+b}
\end{aligned}$$

And so the definition of $\underline{Plus}$ satisifies the prescribed invariant. Now for $\underline{Times}$, we wish to find a combinator which satisfies:

$$\underline{Times}\ \underline{a}\ \underline{b} \to^*_{a,b} \underline{ab}$$

Now we select:

$$\underline{Times} \equiv \lambda a.\lambda b.\lambda f.\lambda x.(a(bf))x$$

To check the invariant, we perform a similar expansion/β-reduction:

$$\begin{aligned}
\underline{Times}\ \underline{a}\ \underline{b} &\equiv (\lambda a.\lambda b.\lambda f.\lambda x.(a(bf))x)\ \underline{a}\ \underline{b} \\
&\vdash_\beta \lambda f.\lambda x.(\underline{a}(\underline{b}\,f))x \\
&\vdash_\beta \lambda f.\lambda x.(\underline{b}\,f)^a x \\
&\vdash_\beta \lambda f.\lambda x.(\lambda x'.f^b x')^a x \\
&\vdash_\beta \lambda f.\lambda x.(\lambda x'.(f^b)^a x')x \\
&\vdash_\beta \lambda f.\lambda x.(f^b)^a x \\
&\equiv \lambda f.\lambda x.f^{ab}x \\
&\equiv \underline{ab}
\end{aligned}$$

And so we conclude that $\underline{Times}$ is indeed a proper implementation of natural number multiplication.

10.a Consider the choice:

$$W = ((\lambda w.\lambda n.n(ww))(\lambda w.\lambda n(ww)))$$

Then for any λ-term N we have:

$$\begin{aligned}
WN &= ((\lambda w.\lambda n.n(ww))(\lambda w.\lambda n(ww)))N \\
&\vdash_\beta (\lambda n.n((\lambda w.\lambda n.n(ww))(\lambda w.\lambda n(ww))))N \\
&\vdash_\beta N((\lambda w.\lambda n.n(ww))(\lambda w.\lambda n(ww))) \\
&\equiv NW
\end{aligned}$$

Thus, W has the property that for all N:

$$WN \to^*_{\alpha,\beta} NW$$

12.a If φ is a fixed point combinator, then for all lambda terms F,

$$\varphi F \to^*_{\alpha,\beta} F(\varphi F)$$

Which we check by expanding φF:

$$\begin{aligned}
\varphi F &\equiv \theta^{17}F \\
&\vdash^*_\beta (\lambda m.m(\theta^{17}m))F \\
&\vdash_\beta F(\theta^{17}F) \\
&\equiv F(\varphi F)
\end{aligned}$$

And so φ is a fixed-point combinator.

13.a We begin by expanding GY:

$$\begin{array}{rcl} GY & \equiv & (\lambda y.\lambda f.f(yf))Y \\ & \vdash_\beta & \lambda f.f(Yf) \\ & \vdash^*_{\alpha,\beta} & \lambda f.f((\lambda f'.(\lambda x.f'(xx))(\lambda x.f'(xx)))f) \\ & \vdash_\beta & \lambda f.f((\lambda x.f(xx))(\lambda x.f(xx))) \end{array}$$

Likewise, starting from Y we have:

$$\begin{array}{rcl} Y & \equiv & \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)) \\ & \vdash_\beta & \lambda f.f((\lambda x.f(xx))(\lambda x.f(xx))) \end{array}$$

Thus we have:

$$Y \to^*_{\alpha,\beta} \lambda f.f((\lambda x.f(xx))(\lambda x.f(xx))) \leftarrow^*_{\alpha,\beta} GY$$

And so $Y = GY$, which by the characterization shows that Y is a fixed point combinator.

13.b If M is a fixed point combinator, then for any F:

$$\begin{array}{rcl} GMF & \equiv & (\lambda y.\lambda f.f(yf))MF \\ & \vdash_\beta & (\lambda f.f(Mf))F \\ & \vdash_\beta & F(MF) \end{array}$$

Likewise, $MF = F(MF)$ (by the fact that M is a fixed-point combinator), and so we have that $MF = GMF$ for all F, and thus $M = GM$.

Next, if $M = GM$, then for any F once again we have:

$$\begin{array}{rcl} GMF & \equiv & (\lambda y.\lambda f.f(yf))MF \\ & \vdash^*_\beta & F(MF) \end{array}$$

Therefore, $MF = F(MF)$ and so M is a fixed-point combinator.

In conclusion, $M = GM \Leftrightarrow MF = FMF$ for all λ-terms F.

16 Consider the λ-term:

$$P_0 = \lambda z.(\lambda x.xx)((\lambda y.y)z)$$

Applying β-reduction to the left-sub expression gives:

$$P_1 = \lambda z.((\lambda y.y)z)((\lambda y.y)z)$$

Similarly right reduction results in:

$$P_2 = \lambda z.(\lambda x.xx)z$$

Yet, it would be impossible to go find a common object $P_1 \Rightarrow P_3 \Leftarrow P_2$, as both of the β reductions have a common ancestor. Thus the new definition for walk is not even weakly confluent.