1.1

a Let $I(Y) = \langle x^2 - y \rangle$ and $f, g \in A^2$. Construct a mapping $\phi : A^2/I(Y) \to k[t]$ such that:

$$\phi(f(x,y)) = f(t,t^2)$$

ϕ is a ring homomorphism since $\phi((fg)(x,y)) = (fg)(t,t^2) = f(t,t^2)g(t,t^2)$ and $\phi((f+g)(x,y)) = f(t,t^2) + g(t,t^2)$. ϕ is also injective:

$$\begin{aligned}\phi(f(x,y) + g(x,y)(x^2 - y)) &= \phi(f(x,y)) + g(t,t^2)(t^2 - t^2) \\ &= \phi(f(x,y))\end{aligned}$$

And so $\phi(f) = \phi(g)$ iff $f = g$. Finally, ϕ is surjective since for all $p \in k[t]$, there exists some $f(x,y) = p(x) \in A^2$ such that $\phi(f) = p$. Therefore ϕ is an isomorphism and $A(Y) \cong k[t]$.

b The equivalence classes of $A(Z)$ are isomorphic to $A^1 \dot{\cup} A^1$. Topologically, this consists of two disconnected components and so it cannot be isomorphic to A^1

c Take any quadratic $ax^2 + bxy + cy^2 \in A$ (it suffices to consider this general form, since the linear component could be removed via translation/scaling of the curve). Factor the expression via the quadratic formula to get an expression for x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} y$$

If $b^2 - 4ac = 0$, then the expression is single valued, and so via the substitution trick from part a we conclude that the coordinate ring for the conic is identically $k[x]$. Otherwise, the expression is two-valued and so looks like $A(Z)$

1.2 First $Y = \{y - x^2 = 0\} \cap \{z - x^3 = 0\}$, which follows from placing x in bijection with t. Consequently, Y is the intersection of two varieties; $\{y - x^2 = 0\}$ and $\{z - x^3 = 0\}$ and so $I(Y) = \langle y - x^2, z - x^3 \rangle$. Because $I(Y)$ is principle and has 2 generators, height $I(Y) = 2$ and so:

$$\dim Y = \dim A^3 - \text{height } I(Y) = 1$$

Finally picking $\phi(f(x,y,z)) = f(t,t^2,t^3)$ gives an isomorphism by an argument symmetric to 1.1a.

1.3 Consider the ideals $I(Y_1) = \langle z - 1, x^2 - yz \rangle, I(Y_2) = \langle x, y \rangle, I(Y_3) = \langle x, z \rangle$. Direct calculation shows that $I(Y) = \langle x^2 - yz, xz - x \rangle = I(Y_1) \cap I(Y_2) \cap I(Y_3)$. Moreover, since each $I(Y_i)$ is generated by irreducible polynomials, the varieties Y_1, Y_2, Y_3 are irreducible.

1.7

a That $ii \Rightarrow i$ is obvious (since a sequence of sets is also a family). To show $i \Rightarrow ii$, let S be a family of closed subsets of X. Because X is Noetherian, any sequence $s_i \supseteq s_{i+1} \supseteq ...$ of closed subsets in X has a minimal element. Since S forms a partial ordering by inclusion, Zorn's lemma states that S has a minimal element.

Now, to show that $i \Rightarrow iii$, consider any sequence of open sets $t_i \subseteq t_{i+1} \subseteq ...$ with $t_i \subseteq X$. Taking the complement of each t_i gives a collection of closed sets $\bar{t_i}$ such that $\bar{t_i} \supseteq \bar{t_{i+1}} \supseteq ...$, which by i contains a minimal element, $\bar{t_n}$. Therefore, t_n is the maximal element of $t_i, t_{i+1}, ...$

A symmetric argument applies to show that $iii \Leftrightarrow iv$ and $iii \Rightarrow i$ (replacing all instances of open with closed, and $\supset$ with $\subset$).

c Since the closed sets in a subset of X are contained in X; a closed sequence of such subsets is alos a closed sequence in X and so it must have a minimal element and thus all subsets of X are also Noetherian.

1.8 Split H into irreducible components and consider each individually. If we take the union of a height $n - r$ ideal with a height 1 prime ideal, the resulting ideal is height $n - r - 1$ unless the height 1 ideal is contained in the $n - r$ ideal. Therefore, the dimension of each irreducible component of the intersection must be $r - 1$ (unless the intersection is strictly contained in H).

1.9 If a is prime, then height $a = r$ and so the equality is satisified. Otherwise, height $a < r$. Therefore;

$$\text{dim } A = \text{dim } A^n - \text{height } a \geq n - r$$

1.11 $I(Y) = \{x^4 - y^3, y^5 - z^4, x^5 - z^3\}$