CS787 Homework 4

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March 26, 2009

1. Let A, B be the two components of the bipartite graph $K_{n,n}$ each containing n elements. Because each vertex in A is connected to all vertices in B by a single edge and likewise back to all vertices in A by a path of length 2, it would be impossible to disconnect A unless all vertices in B were completely removed. A symmetric argument holds in the case of B. Therefore, the only way to disconnect $K_{n,n}$ is to completely remove one of the connected components which would take at least n vertices.

2. Consider G, a $k \times k$ square grid embedded in the plane with unit length edges. The planar separator theorem assures us that we have some separator $S \subset G$ which partitions G into two disconnected components, where each component, $C, C' \subset G$, contains at least $k^2/3 - 4k$ vertices. Note that due to the embedding of G, $S \cup C$ forms a polygon whose area may be expressed using Pick's theorem:

$$A = I + B - 1$$

Where A is the area of $S \cup C$, I is the number of interior points in $S \cup C$ and B is the number of points on the boundary, $\partial(S \cup C)$. Because $I + B = |S \cup C| \geq |C|$:

$$A \geq k^2/3 - 4k - 1$$

Now suppose that there was some circle with radius r and area A, then:

$$\begin{aligned} \pi r^2 &= A \\ \pi r^2 &\geq k^2/3 - 4k - 1 \\ r &\geq \frac{1}{\sqrt{\pi}}\sqrt{k^2/3 - 4k - 1} \end{aligned}$$

The circumference of this circle, $c = 2\pi r$, satisifies

$$c \geq 2\sqrt{\pi(k^2/3 - 4k - 1)} > (2 + \epsilon)k$$

for all $0 < \epsilon < 2\sqrt{\pi/3} - 2$ and sufficiently large k. Now recall a well-known theorem from the calculus of variations: a circle has maximal area for a given circumference. Combined with the fact that each edge in G is unit length, this implies that the number of edges in $\partial(S \cup C)$ is strictly larger than $(2 + \epsilon)k$. A symmetric argument holds for the other connected component C', and so:

$$|\partial(S \cup C')| + |\partial(S \cup C)| > (4 + 2\epsilon)k$$

Because $|\partial G| = 4k$, $G = S \cup C \cup C'$ and $C \cap C' = \emptyset$, S must contain at least $2\epsilon k$ vertices. Fixing $n = k^2$, we conclude that all $1/3 - 2/3$ separators for this family of graphs contain $\Omega(\sqrt{n})$ vertices. [1]

[1] Good grief. I hope there was an easier way to prove this lower bound.

3. Without loss of generality, assume that $G = (V, E, \theta, -)$ is connected and triangulated, $|V| = n$, $|E| = m$ and $v_0 \in V$.

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PLANAR-SEPARATOR(V, E, θ, −)
rank[...] ← −inf
count[...] ← 0
parent[...] ← ∅
treeedge[...] ← false
tovisit ← MAKE-QUEUE(v0)
rank[v0] ← 0
while NOT-EMPTY(tovisit)
      do v ← POP(tovisit)
         for each v′ ∈ NEIGHBORS(v) such that rank[v′] > 0
               do rank[v′] ← rank[v] + 1
                  count[rank[v′]] ← count[rank[v′]] + 1
                  parent[v′] ← v
                  treeedge[(v, v′)] ← true
                  PUSH(tovisit, v′)
t1 ← 0
s1 ← 0
for i ← 1 to max_v rank
      do s′ ← s1 + count[i]
         if s1 ≤ n/2 and s′ > n/2
           then t1 ← i
                break
         s1 ← s′
t0 ← 0
s0 ← 0
for i ← 1 to max_v rank
      do s′ ← s0 + count[i]
         if s0 + 2(count[t1] − count[t0]) ≤ 2√s1
           then t1 ← i
         s0 ← s′
t2 ← 0
for i ← max_v rank to 1
      do s′ ← s0 − count[i]
         if s0 + 2(count[t2] − count[t1] − 1) ≤ 2√(n − s1)
           then t2 ← i
         s0 ← s′
A ← ∅
B ← ∅
C ← ∅
for each v ∈ V
      do if rank[v] < t0
           then A ← A ∪ {v}
                REMOVE-VERTEX(G, v)
         elseif rank[v] = t0
           then rank[v] ← 0
                parent[v] ← x
                ADD-EDGE(G, (x, v))
         elseif rank[v] > t2
           then C ← C ∪ {v}
                REMOVE-VERTEX(G, v)
           else
                rank[v] ← rank[v] − t0
B ← LOCATE-CYCLE-WITH-COST<2/3-IN-DUAL(G)
G ← G − B
A ← A ∪ CONNECTED-TO-X(G)
C ← C ∪ (G − A)
go to sleep[a]
return (A, B, C)
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[a] I give up. It's 2am and this is taking forever.

4. Recall that the cycles in G are defined as the orbits of the group generated by $\{\theta, \theta^*, -\}$ acting on E, aka $|E/\{\theta, \theta^*, -\}|$. However, since $\theta^* = \theta \circ -$, this is the same as the group generated by $\{\theta, -\}$. From class, we know that $|E/\{\sigma_i, -\}| = |E/\{\sigma_{i+1}, -\}|$, so by definition the number of cycles under σ_i is the same as those for σ_{i+1}.

5. Given invertible automorphisms $\theta : E \to E, - : E \to E$, construct the permutation arrays $P = (p_1, p_2, ...p_n), Q = (q_1, q_2, ...q_n)$ such that $e_{p_i} = \theta(e_i)$ and $e_{q_i} = -(e_i)$ for all edges $e_x \in E, 1 \leq x \leq |E| = n$. By definition, $G^* = (E, \theta^*, -)$ where $\theta^* = \theta \circ -$, which is equivalent to the permutation:

$$P^* = (p_{q_1}, p_{q_2}, ...p_{q_n})$$

Computing P^* takes $O(n)$ time where $n = |E|$, therefore the cost of constructing G^* from G using an array representation is in $O(|E|)$.