Assigned: Thursday April 8, 2010
Due: Tuesday April 27, 2010

MATH/CS 715: Homework 5
Spring 2010

PAGE LIMIT: 20 pages (single-sided).

NOTE: Please include a cover page – this will not count toward the total page limit.

NOTE: Don't forget to include MATLAB code where appropriate – this does count toward your total pages.

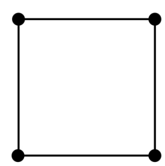
2D Structured FEM:

1. (20 points) Consider the 2D Poisson equation:

$$\begin{aligned} \textbf{PDE}: &\quad -u_{xx} - u_{yy} = f(x,y) \quad \text{in} \quad \Omega = [-1,1] \times [-1,1], \\ \textbf{BC}: &\quad u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

   (a) Recast this problem as a variational problem. Clearly state the test and trial function spaces.

   (b) Develop a finite element method for this problem based on square (not triangular) elements of the form:



   On each element, the solution on the four corners is interpolated with the following interpolant:

$$s(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy.$$

   Be sure to explicit write out the linear system that arises from your FEM.

   (c) Use this method to solve the 2D Poisson eqution with

$$f(x,y) = \begin{cases} 100 & \text{if } \sqrt{x^2+y^2} < 0.2 \\ 1 & \text{otherwise} \end{cases}$$

   HINT: use the `pcg` command in MATLAB to invert the linear system.

2D Unstructured FEM:

2. Dowload the 2D mesh generation code from the course website:

   `http://www.math.wisc.edu/~rossmani/math713/Meshgen.tar.gz`

   Make sure that you can get the provided example to work in MATLAB.

   Next, use the MATLAB code to generate a mesh for the following geometry: a star-shaped region with a circular hole. The boundaries of this region are zero contours of the following functions:

$$f_{\text{outer}}(r,\theta) = r - 0.75 - 0.25\sin(5\theta),$$
$$f_{\text{inner}}(r,\theta) = (0.25)^2 - x^2 - y^2.$$

   NOTE: in order to get a good mesh, you may have to do apply one or both of the following tricks:

   (a) You may need to fix a few points on the boundary (especially points near "corners"). You can do this by adding values to the `fix` vector.

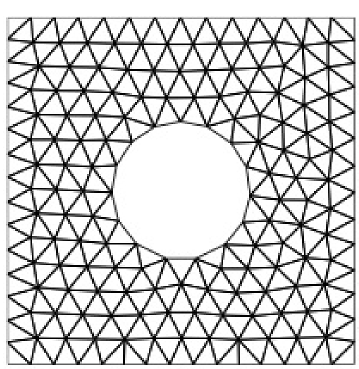
   (b) You may need to use a slightly non-uniform grid. At the momment the code uses `huniform`, but you can replace it with a function `fh.m` that takes as input the vector `[x; y]` and returns a number. The grid spacing will be reduced in regions where `fh.m` is small and increased where `fh.m` is large. Below is an example of a square with a circular hole.

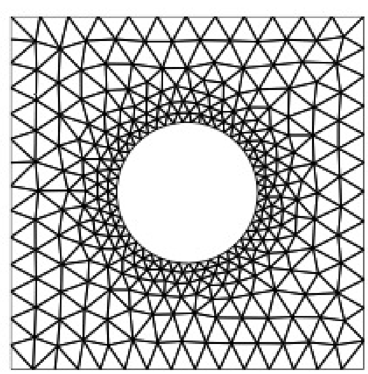
   Example #1 (uniform grid spacing):

```
>> fd=inline(ddiff(drectangle(p,-1,1,-1,1),dcircle(p,0,0,0.4)),p);
>> pfix=[-1,-1;-1,1;1,-1;1,1];
>> [p,t]=distmesh2d(fd,@huniform,0.05,[-1,-1;1,1],pfix);
```

   Example #2 (non-uniform grid spacing):

```
>> fd=inline(ddiff(drectangle(p,-1,1,-1,1),dcircle(p,0,0,0.4)),p);
>> pfix=[-1,-1;-1,1;1,-1;1,1];
>> fh=inline(min(4*sqrt(sum(p.^2,2))-1,2),p);
>> [p,t]=distmesh2d(fd,fh,0.05,[-1,-1;1,1],pfix);
```





3. Consider the following Laplace equation on the domain created in Problem #2:

$$\textbf{PDE}: \quad u_{xx} + u_{yy} = 0 \quad \text{in interior,}$$
$$\textbf{BCs}: \quad u = \begin{cases} \sin(5\theta) & \text{if} \quad f_{\text{inner}}(r,\theta) = 0, \\ 0 & \text{if} \quad f_{\text{outer}}(r,\theta) = 0. \end{cases}$$

   Derive a finite element method for this problem (be careful with the boundary conditions). Construct a sufficiently resolved grid and solve the finite element variational problem.

4. Consider the following eigenvalue problem in the domain created in Problem #2:

$$\textbf{PDE}: \quad \phi_{xx} + \phi_{yy} = -\lambda\phi \quad \text{in interior,}$$
$$\textbf{BCs}: \quad \phi = 0 \quad \text{on the boundary.}$$

   Derive a finite element method for this problem. Construct a sufficiently resolved grid and approximately compute the first four eigenvalues and corresponding eigenfunctions. Produce color (or at least grayscale) plots of the eigenfunctions to show how each one varies in space.