CS787 Homework 2

Mikola Lysenko

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1.

i. $\emptyset \subseteq V$ is vacuously covered by any matching, so $\emptyset \in \mathcal{I}$.

ii. If $I \in \mathcal{I}$ is covered by a matching, then the same matching must cover all subsets of I, so $\mathcal{I}$ satisifies downward closure.

iii. Suppose $I, J \in \mathcal{I}$ and $|I| < |J|$. By definition there exist matchings M_I, M_J covering the vertices in both I, J. If there are any vertices in $J - I$ which are incident to M_I, then we may add them immediately to I, so without loss of generality assume that the vertices $J - I \subseteq M_J - M_I$. With respect to $M_I \cup M_J$ each vertex, $c \in M_J - M_I$ has degree exactly 1 and is thus the start of a unique maximal path with edges alternating in M_J, M_I. By partitioning, there are only 3 places where the path can end:

a. $M_J - M_I$

b. $M_I - M_J - I$

c. $I - M_J$

Now it is known by hypothesis that $|I - M_J| \leq |I - J| < |J - I| \leq |M_J - M_I|$, so there always exists some path which does not end in case c. In either case a or b, take the symmetric difference of the path with M_I to create a matching containing c and covering I. By induction, this eventually reduces $|M_J - M_I| = |J - I|$ thereby forcing the exchange of some element in $J - I$ to I.

Thus, we have shown that the matroid axioms hold for $(S, \mathcal{I})$.

2. Let $G = (Q \cup S, E)$ be a bipartite graph with components S, Q and edges $E = \{(q_i, e_i) | q_i \in Q, e_j \in q_i\}$. Then any matching $M = \{(q_{i(1)}, e_{j(1)}), ..., (q_{i(t)}, e_{j(t)})\}$ of G has distinct $e_{j(k)}$ with each $e_{j(k)}$ associated to a unique $q_{i(k)}$, and is thus is a transversal. Additionally, in any transversal $T = \{e_{j(1)}, ... e_{j(t)}\}$ each $e_{j(k)}$ and $q_{i(k)}$ are distinct and form an edge within G. Therefore partial transversals of (S, Q) are equivalent to matchings on G.

To build the transversal matroid, take the construction for G given above and compose it with the definition from problem 1's matching matroid.

3. [1] Let $S = B_1 \cup ... \cup B_m$ as described and pick $Q = \{B_{i,k} | 1 \leq k \leq d_i, 1 \leq i \leq m\}$ where $B_{i,j} = B_i$. Then any partial transversal $T = \{e_{j(k)}\}$ of (S, Q) also

[1] This proof is slightly more general than required as it does not make use of the fact that each B_i is disjoint.

satisfies $T \subseteq S$ and that for all B_i, $B_i \cap T$ contains no more than d_i members of S, so $|T \cap B_i| \leq d_i$. Next, take any $I \subseteq S$ such that for all B_i, $|I \cap B_i| \leq d_i$, and match each element of $e \in I \cap B_i$ to some distinct $B_{i,j}$ (which is always possible by cardinality) giving some partial transversal of (S, Q). Therefore, each I is equivalent to a unique partial transversal up to permutation, and the partition matroid is therefore a special case of the transversal matroid.

4. Let $G = (V, E)$ be a graph. For each vertex, $v_i \in V$, let $B_i \subseteq E$ be the edges incident to v_i. Then any partition $S = B_1 \cup ... \cup B_m$ with limits $d_i = 1$ satisfies the property that no edge is incident to no more than one vertex. Moreover, for any collection of edges in G satisfying the property are also a partition of S, d_i (as each edge appears in no B_i more than once and each $d_i = 1$). So, we obtain a matroid by invoking problem 3. The from case is equivalent to solving the problem on the transpose of G.

If a subset, I, of edges is independent in both matroids, then for each vertex v the both the number of edges in I incident to or from any v is no more than 1. To generate these sets, we modify the above construction by adding the additional sets B'_i of all edges incident from v along with the limits $d'_i = 1$.

5. The only constraint on subsets of W is that "the same row can have at most one element"; therefore, each row may be considered independently. Within a single row, we are only allowed to choose one element so the maximum weight must equal the weight of the maximum element. This gives the following algorithm:

```
MAX-WEIGHT(W)
  S ← ∅
  for i ← 1 to m
        do S ← S ∪ min({w_ij | j ∈ [1, n]})
  return S
```

To express this problem as a matroid, we partition W row-wise into sets $q_i = \{w_{i,1}, ...w_{i,j}\}$ and apply the transversal matroid from problem 2 (weighting the elements according to W.) A valid transversal, $T = \{w_{1,j(1)}, ..., w_{m,j(m)}\}$, satisifies the constraint that no two elements are in the same row and moreover, a maximum weight transversal also

6. First observe that the total amount of time needed to complete n jobs on the machine is no greater than n, as each job has a unit cost, so without loss of generality assume that each $d_i \leq n$. Let $S = \{e_1, ..., e_n\}$ be the collection of jobs, each with deadline d_j and penalty p_n as described and $Q = \{q_1, ...q_n\}$ with $q_t = \{e_j | 1 \leq j \leq n, d_j \leq t\}$. A partial transversal of (S, Q) is also a valid schedule for the machine, as each job, $e_{j(t)}$ associated with some particular q_t could be executed at the time t and by symmetry each schedule with a job, $e_{j(t)}$

at time t is a partial transversal. Therefore, the set of valid schedules containing no empty time slots forms a matroid.

Now, assign weights to jobs within the transversal matroid equal to their penalty. As a result, a maximum weight schedule minimizes the total penalties of the jobs which were not executed. So, we conclude that the optimal schedule can be found greedily.