3.1

a *Want to show:* Any conic in A^2 is ismorphic to either A^1 or $A^1 - \{0\}$.

By exercise 1.1c, we know that the coordinate ring of any conic in A^2 is isomorphic to $A/\langle x^2 - y\rangle$ or $A/\langle xy - 1\rangle$; and thus each conic must be isomorphic to one of these two varieties. We now treat each case:

1.
$$A^1 \cong Y = Z(\langle x^2 - y\rangle)$$
Consider the map $\varphi : A^1 \to Y$ which takes $t \mapsto (t, t^2)$. Clearly the map is bijective and bicontinuous, and moreover regular. Therefore by theorem 3.6 it is an isomorphism of A^1 and Y.

2.
$$X = A^1 \setminus \{0\} \cong Y' = Z(\langle xy - 1\rangle)$$
Create the invertible bicontinuous map $\varphi : X \to Y'$ mapping $t \mapsto (t, \frac{1}{t})$ which is regular when $t \neq 0$. As a result, these two spaces are isomorphic by theorem 3.6.

Therefore all conics in A^2 are isomorphic to either A^1 or $A^1 \setminus \{0\}$. $\square$

b *Want to show:* A^1 is not isomorphic to any proper open subset of itself.

Take any open set $X \subset A^1$, where X is the complement of some variety cut out by a single polynomial, f. Now look at the units of $\mathcal{O}(X)$; clearly there are the constant polynomials, but moreover for each zero p of f, there also exists the family of units formed by $1/(x-p)^k$, $(x-p)^k$. These latter units are not present in $\mathcal{O}(A^1) \cong k[x]$. As a result, we conclude that $\mathcal{O}(A^1) \ncong \mathcal{O}(X)$ and so $A^1 \ncong X$. $\square$

c *Want to Show:* Any conic in P^2 is isomorphic to P^1

If Y is a conic in P^2, then Y must be a locus of the form:

$$ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fzx = 0$$

We may rewrite this as a matrix equation of the form, :

$$v^T M v = 0$$

Where:

$$M = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

and

$$v = (xyz)$$

Since M is symmetric, the spectral theorem states that M has a factorization into $U\Lambda U^*$ where Λ is a real diagonal matrix and U is orthonormal. Moreover, since the equation is a non-degenerate conic, the entries in Λ must be non-zero. Substituting and regrouping terms, we get:

$$v^T U^T \Lambda U v = 0$$

Since U is orthonormal, we may again substitute $\sqrt{\Lambda} U v \mapsto w$. Thus it is enough to find an isomorphism from P^1 onto the variety determined by

$$y^2 - xz = 0$$

But this is just the d-uple embedding given by $(s,t) \mapsto (s^2, st, t^2)$ which by problem 3.4 is an isomorphism. $\square$.

d *Want to Show:* A^2 is not homeomorphic to P^2

P^2 is homeomorphic to $A^2 \dot{\cup} A^1 \dot{\cup} A^0$. For A^2 to be homeomorphic to P^2, A^1 would have to be empty. But this is not the case. Therefore A^2 and P^2 are topologically distinct. $\square$

e *Want to Show:* If an affine variety, X, is isomorphic to a projective variety, Y, then it is a point.

According to theorem 3.4, for any projective variety $\mathcal{O}(Y) = k$. For an affine variety, $\mathcal{O}(X) = A(X)$. However $A(X) = k \implies X$ is a point. Therefore, if a projective variety is isomorphic to an affine variety it must be a point. $\square$

3.2

a Let $\varphi : A^1 \to A^2$ be defined by $t \mapsto (t^2, t^3)$.

WtS: φ defines a bijective bicontinuous map of A^1 onto the curve $y^2 = x^3$ but is not an isomorphism.

That φ is bijective can be verified by the fact that there exists a map $\varphi^{-1} : A^2 \to A^1$ taking $(x,y) \mapsto \sqrt{(x)}$ for all $y \geq 0$. Over the curve $Y = \{(x,y) | y^2 = x^3\}$, we have $\varphi \circ \varphi^{-1} = id_Y$ and likewise $\varphi^{-1} \circ \varphi = id_{A^1}$.

It is obvious that algebraic sets in Y map to algebraic sets in A^1 since φ is polynomial. Moreover, any algebraic set $Z = \{t | f(t) = 0\} \subseteq A^1$ maps to a corresponding algebraic set $Z' = \{(x,y) \in Y | f(\varphi^{-1}(x,y)) = 0\}$ (because $(x,y) \in Y \implies (x^2, y^3) \in Y$) and the corresponding algebraic sets in both varieties are just discrete points. Therefore, the map is bicontinuous.

However, φ is not a morphism. To verify this, consider the class of functions on the open set $U = Y \setminus \{(4, \pm 8)\}$. This maps under φ^{-1} to the open set $A^1 \setminus \{\pm 2\}$. Take the regular function $(y-8)$ which is well defined U and hence in $\mathcal{O}(Y)$. However the inverse image of this map is $(t^3 - 8)$ which has zeros for the points $t = \pm 2(-1)^{1/3}$ and is hence not regular on the image of U. Therefore the map is not an isomorphism. $\square$

b Let the basefield k have charateristic $p > 0$, and define $\varphi : A^1 \to A^1$ where $t \mapsto t^p$.

WtS: φ is bicontinuous and bijective, but not an isomorphism.

That φ is bicontinuous follows from the property that $(a+b)^p = a^p + b^p$ for all $a, b \in k$ by the Frobenius property, and thus it maps algebraic sets to algebraic sets.

3.5 We first must show that the d-uple embedding of a degree d hypersurface, $H \subset P^n$ becomes a hyperplane in P^N. To do this, observe that H is cut out by a homogeneous polynomial of degree d and since H is a hypersurface $I(H)$ is generated by one element of the form:

$$\sum_{d_0+d_1+\ldots d_n=d} c_{d_0 d_1 \ldots d_n} y_0^{d_0} y_1^{d_1} \ldots y_n^{d_n}$$

The d-uple embedding of this function trivially maps each coefficient to a single dimension in P^N giving the variety cut out by the set of $\binom{n+d}{n}$ equations of the form:

$$c_{d_0 d_1 \ldots d_n} = y'_{d_0 d_1 \ldots d_n}$$

This is a linear variety, and as such forms a hyperplane in P^N. Moreover, $P^N \setminus \rho_d H$ must be affine as projective space minus a hyperplane is affine. Since the image of H is completely contained in the image of P^n $\rho P^n \setminus \rho H$ is also affine. By 3.4, we know the d-uple embedding is an isomorphism and so we conclude that $P^n \setminus H$ is affine. □

3.6 *WtS:* The quasi-affine variety $X = A^2 - \{(0,0)\}$ is not affine.

First, note that $\mathcal{O}(X) \cong k[x,y]$, since all polynomials in $k[x,y]$ are non-zero on 1-dimensional subsets of A^2 and thus can not take on 0 values at only the origin. If X was affine, then by theorem 3.2 $A(X) \cong \mathcal{O}(X) \cong k[x,y] \cong A(A^2)$. But clearly $X \subset A^2$ and so $X \neq A^2$. However this contradicts theorem 3.7. Thus X is not affine. □

3.7

a *WtS:* Any two curves in P^2 have a nonempty intersection.

See part b.

b *WTS:* If $Y \subseteq P^n$ is a projective variety with dim $Y \geq 1$, then for all hypersurfaces $H \subseteq P^n$, $Y \cap H \neq \emptyset$.

Suppose $Y \cap H \neq \emptyset$. Then $Y \setminus H = Y$ and $Y \subseteq P^n \setminus H$. But $P^n \setminus H$ is affine (by problem 3.5) and therefore Y is an intersection of an algebraic set and an affine variety, Y must also be affine. But by assumption Y is also projective and so by problem 3.1e, Y must be a point. This is a contradiction if dim $Y > 1$. □

3.9 Let $X = P^1$ and Y be the 2-uple embedding of P^1 in P^2; clearly $X \cong Y$.

WTS: $S(X) \ncong S(Y)$.

First, $S(X)$ is trivial, it is just the graded ring S^1. For Y, we know from problem 3.1c that $I(Y) \cong \langle y^2 - xz \rangle$ and so $S(Y) = S/I(Y)$. But look at the units of $S(Y)$, in addition to the usual units which are of the form cx_i^n, there exist units such as xz, where $xz * xz = xz^2 \equiv_{I(Y)} -y^4$. As a result, the units of $S(Y)$ are strictly larger than those of $S(X)$ and so the two rings are non-isomorphic. $\square$

3.13 Let $Y \subseteq X$ be varieties (with Y a subvariety of X).

WTS: $\mathcal{O}_{Y,X}$ is a local ring with residue field $K(Y)$ and dimension $\dim X - \dim Y$.

That $\mathcal{O}_{Y,X}$ is a ring is obvious. Now consider the set of regular functions, $P \subset \mathcal{O}_{Y,X}$ which are zero when restricted to Y. This collection of functions forms an ideal in $\mathcal{O}_{Y,X}$ (obviously, since it is closed under addition and multiplication by another element of $\mathcal{O}_{Y,X}$). Moreover it is maximal, as the addition of any function which is non-zero on some subset of Y will be non-zero on an open subset of Y and thus by remark 3.1.1 it must extend the ideal P to include all functions in $\mathcal{O}_{Y,X}$. We also argue that P is the only maximal ideal in $\mathcal{O}_{Y,X}$; as all open sets in $\mathcal{O}_{Y,X}$ must intersect Y and thus the density argument (remark 3.1.1 again) implies that the ideal P should be included in all other ideals. Therefore, $\mathcal{O}_{Y,X}$ contains a unique maximal ideal and thus is a local ring.

To show that the residue field $\mathcal{O}_{Y,X}/P = K(Y)$, we simply observe that the functions which are non-zero on Y form a field when restricted to Y and thus by the density argument are all included in $\mathcal{O}_{Y,X}$ again.

Finally, we wish to prove that:

$$\dim K(Y) + \dim Y = \dim X$$

From 1.8A, we know that:

$$\text{height } P + \dim \mathcal{O}_{Y,X}/P = \dim \mathcal{O}_{Y,X}$$

But as we have shown, $\mathcal{O}_{Y,X}/P = K(Y)$ and as P is the space of zero-valued functions on Y, it must be that height $P = \dim Y$. Finally, $\dim O_{Y,X} = \dim X$, since the localization of $O(X)$ does not change the transcendence degree of the fractions over the base field. Therefore:

$$\dim Y + \dim K(Y) = \dim X$$

$\square$