Probability and Statistics

Assignment 2 (Due: 10/09)

1 Discrete Random Variables

1. (3 points) Let the function $p(x)$ be defined as follows

$$p(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^x, & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

   1. Check that $p(x)$ is a PMF. of discrete random variable.
   2. Find $P(X = 2)$.
   3. Find $P(X \leq 2)$.

2. (3 points) A fair coin is tossed repeatedly until first head appears. Let X denote the R.V. for the number of tosses required until the first head. Answer the following:

   1. Calculate PMF of X and sketch the PMF.
   2. Calculate the CDF. of X and sketch the CDF.
   3. Find $P(1 < X \leq 4)$.

3. (2 points) (Binomial and Poisson Distribution) The probability of error in transmission of a bit through a noisy channel is $p = 0.01$.

   1. What is the probability that out of the 10 received digits, there are more than one bits with error? Show using Binomial.
   2. Show above using Poisson approximation.

4. (2 points) Let $X = N(0, \sigma^2)$. Find $E[X \mid X > 0]$ and $\mathrm{Var}(X \mid X > 0)$.

5. (2 points) (Exponential Distribution) We know that exponential R.V. is memoryless. Show the converse, that is, show that any random variable that is memoryless must be an exponential R.V.

6. (3 points) (Laplace Distribution) A random variable X defined as follows

$$f_X(x) = ke^{-\lambda|x|}, \quad \lambda > 0, \quad -\infty < x < \infty,$$

where k is a constant is called a Laplace R.V.

1. What is k?
2. Find the CDF of X.
3. Find the mean and variance of X.

7. (2 points) There are two dice: 4-sided and 6-sided. Let X denote the outcome of rolling 4 sided die and Y denote the outcome of rolling 6-sided die. Let $Z = (X+Y)/2$. Answer the following:

1. Find the variance of X, Y, Z.
2. Plot the graph of PMF and CDF of Z
3. Consider the game that if $X > Y$, you get $2X$ rupees and otherwise you loose 1 rupee. Find the expected total profit if you play 50 times.

2 Two Random Variables

8. (2 points) If the joint PMF of X and Y is given by the following table Are X and Y

X/Y	1	2	3
1	1/18	1/9	1/6
2	1/9	1/6	1/18
3	1/6	1/18	1/9

Table 1: Joint PMF of X and Y.

independent random variables?

	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$
$y=0$	0.05	0.2	0.1	0.04	0.01
$y=1$	0.01	0.09	0.15	0.20	0.15

Table 2: Joint PMF of X and Y.

9. (5 points) Air India was recently acquired by TATA group. To understand the customer satisfaction for the services offered, the airline decides to do a survey. The passengers are asked to rate the quality on a scale of 1 to 5. They are also asked to rate on the same scale whether the air hostesses were attentive. Their ratings are changed to a scale between 0 and 1. The quality is denoted by the random variable X and the quality of service by air hostesses by random variable Y. The joint PMF is given in Table 2. Answer the following:

   1. Justify that this is a valid PMF.
   2. Find $P(Y = 1 \mid X \geq 3)$ and $P(Y = 0, X \geq 3)$.
   3. Find the marginal PMF of X.
   4. Find the marginal PMF of Y.
   5. Find $E[X|Y = 1]$.

10. (2 points) Let X and Y be two independent Uniform$(0, 1)$ random variables. Find

    1. $E[XY]$.
    2. $E[e^{X+Y}]$.
    3. $E[X^2 + Y^2 + XY.]$

11. (2 points) (Method of Transformation) Let X and Y be two independent $N(0, 1)$. Let

$$Z = 7 + X + Y$$
$$W = 1 + Y$$

    Find the joint PDF of Z and W.

12. (2 points) The joint PDF of two random variables is given as follows

$$f_{XY}(x, y) = \begin{cases} x^2 + \frac{1}{3}y & -1 \leq x \leq 1, 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Calculate the following:

1. $f_{X|Y}(x \mid y)$, the conditional PDF.
2. $P(X > 0 \mid Y = y)$. Does this depend on y?
3. Are X and Y independent?

13. (2 points) (Gamma Function) Find the value of I:

1. $\mathrm{I} = \Gamma(7/2)$
2. $\mathrm{I} = \int_0^\infty x^7 e^{-5x}\, dx$

14. (4 points) (Joint Probability) A surface has infinite parallel lines, equally spaced and d distance apart from each other. Suppose we have a needle of length l which we throw randomly on the surface. What is the probability that this needle intersects a line on the surface? Assume $l < d$ and that the needle and the surface lie in the same cartesian plane.

15. (4 points) (Normal Distribution) If $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ are independent, prove that:

$$X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

16. (2 points) (Normal Distribution) Let X_1 be a normal random variable with $\mu = 2$ and $\sigma^2 = 3$ and let X_2 be a normal random variable with $\mu = 1$ and $\sigma^2 = 4$. Assuming that X_1 and X_2 are independent, What is the distribution of the linear combination:

1. $Y = 2X_1 + 3X_2$?
2. $Y = X_1 - X_2$?