Assignmen- 2 MCS-1

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Q-1

$$p(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^{x}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

(a) To check weather a given random variable x is discrete,

prove that $\sum_{x=0}^{\infty} p(x) = 1$

$$\Rightarrow \frac{3}{4}\left(\frac{1}{4}\right)^{0} + \frac{3}{4}\left(\frac{1}{4}\right)^{1} + \cdots$$

$$a = \frac{3}{4}, \quad r = \frac{1}{4}$$

$$S = \frac{a}{1-r} = \frac{3}{4\left(1-\frac{1}{4}\right)} = \frac{3 \times 4}{4 \times 3} = 1$$

Hence, $\boxed{\sum_{x=0}^{\infty} p(x) = S = 1}$ proved

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Q-1

(b) $P(X=2)$

$$P(X=2) = \frac{3}{4}\left(\frac{1}{4}\right)^2 = \frac{3}{4^3} = \frac{3}{64}$$

(c) $P(X \le 2)$

$$P(X \le 2) = \sum_{x=0}^{2} p(x)$$

$$= \frac{3}{4}\left(\frac{1}{4}\right)^0 + \frac{3}{4}\left(\frac{1}{4}\right)^1 + \frac{3}{4}\left(\frac{1}{4}\right)^2$$

$$= \frac{16\times3 + 4\times3 + 3}{64}$$

$$= \frac{48+12+3}{64} = \frac{63}{64}$$

$$\boxed{P(X \le 2) = \frac{63}{64}}$$

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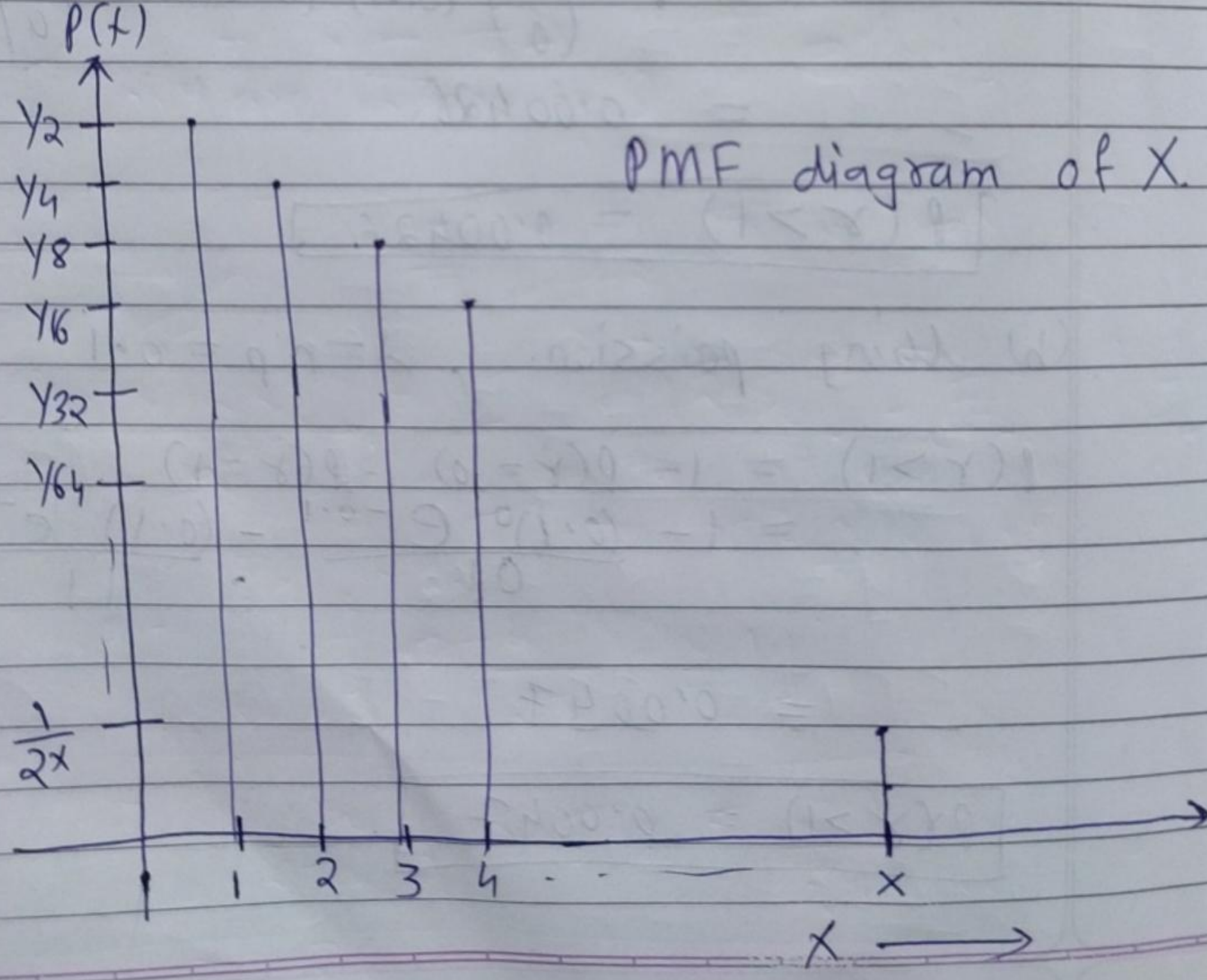
Q-2

(a) $p(H) = p(T) = \frac{1}{2}$

X = Number of tosses for first head

$$\text{PMF of } X = \begin{cases} \frac{1}{2^x}, & x = 1, 2, 3, \ldots \\ 0, & \text{otherwise.} \end{cases}$$



PMF diagram of X.

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(b) CDF of X.

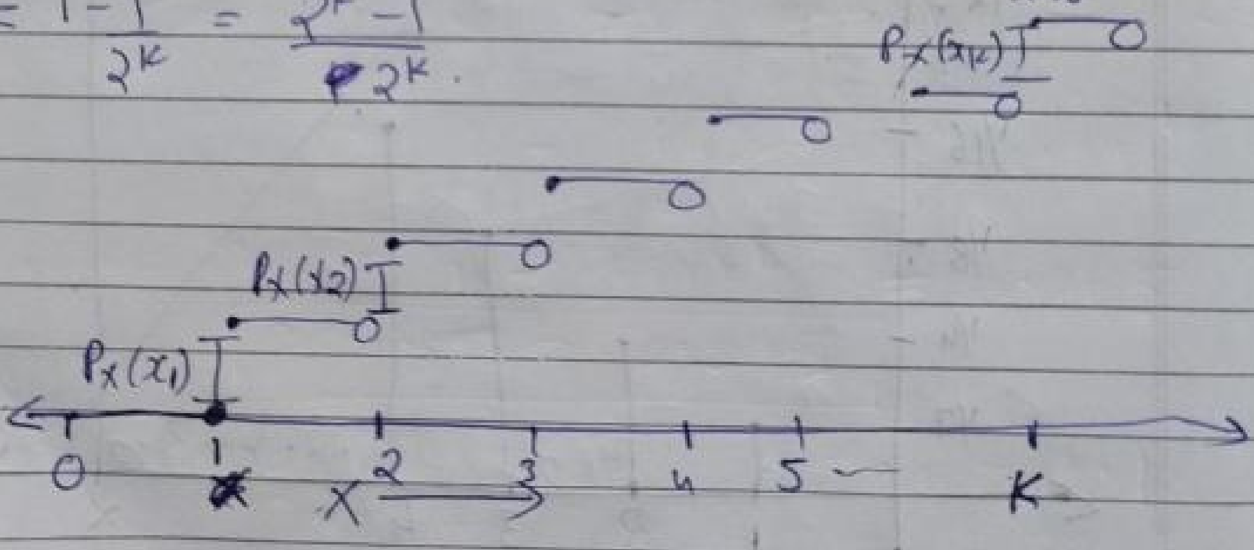
x	1	2	3	4	5	. —	k
$F_X(x)$	$1/2$	$\frac{1}{2}+\frac{1}{2^2}$	$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}$				

$$F_X(x) = \sum_{x_k \le x} P_X(x_k)$$

$$F_X(x) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^k}$$

$$= 1 - \frac{1}{2^k} = \frac{2^k - 1}{2^k}.$$

$$\lim_{x \to \infty} F_X(x) = 1$$



CDF of Random Variable X

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(c) $P(1 < x \leq 4)$

$P(1 < x \leq 4) = P(2) + P(3) + P(4)$

$= \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$

$= \frac{4+2+1}{16} = \frac{7}{16}$

$$\boxed{P(1 < x \leq 4) = \frac{7}{16}}$$

Q-3

$p(err) = 0.01$

$n = 10, \quad r > 1$

(a) Using binomial

$P(X=r) = {}^nC_r \, p^r q^{n-r}$

$$P(r>1) = 1 - P(r=0) - P(r=1)$$
$$= 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{0}(0.01)^1(0.99)^9$$
$$= 0.00426$$

$$\boxed{P(r>1) = 0.00426}$$

(b) Using poission, $\lambda = np = 0.1$

$$P(r>1) = 1 - P(r=0) - P(r=1)$$
$$= 1 - \frac{(0.1)^0 e^{-0.1}}{0!} - \frac{(0.1)^1 e^{-0.1}}{1!}$$

$$= 0.0047$$

$$\boxed{P(r>1) = 0.0047}$$

Q-4

$X \sim N(0, \sigma^2)$

$$F_{X|X>0}(x) = \frac{F_X(x)}{P(X>0)} = \frac{F_X(x)}{1-F_X(0)} = \frac{F_X(x)}{1-1/e}$$

$$= 2F_X(x)$$

1. $E[X|X>0] = \int_0^\infty x F_{X|X>0}(x)\,dx$

$$= \int_0^\infty x \cdot 2\, 1/\sqrt{2\pi\sigma^2}\; e^{-x^2/2\sigma^2}$$

$$= 2\int_0^\infty x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}.$$

let $\frac{x^2}{2\sigma^2} = y$

$$\frac{2x\,dx}{2\sigma^2} = dy$$

$$x\,dx = \sigma^2 dy$$

So, $E[X|X>0] = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^\infty e^{-y} \sigma^2 dy$

$$= \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= \sigma\sqrt{\frac{2}{\pi}}\,[0-(-1)]$$

$$E[X|X>0] = \sigma\sqrt{\frac{2}{\pi}}$$

Q4

2.

$$E[x^2 | x>0] = \int_{-\infty}^{\infty} x^2 f_{x|x>0}(x)\, dx$$

$$= \int_{0}^{\infty} x^2 \frac{2}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx$$

let $\frac{x^2}{2\sigma^2} = y$, $x\,dx = \sigma^2 dy$

$$= \frac{2\sigma^2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} \sqrt{2\sigma^2 y} \cdot e^{-y} dy$$

$$= \frac{2\sigma(\sqrt{2}\sigma)}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{y}\, e^{-y} dy$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \times \frac{1}{2}\sqrt{\pi} = \sigma^2$$

$$Var[x|x>0] = E[x^2|x>0] - \left[E[x|x>0]\right]^2$$

$$= \sigma^2 - \left(\sigma^2 \frac{2}{\pi}\right)$$

$$= \sigma^2\left(1 - \frac{2}{\pi}\right)$$

$$\boxed{Var[x|x>0] = \sigma^2\left(1-\frac{2}{\pi}\right)}$$

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Q-5

Let us assume random variable x is memoryless.

$$P(X > x+\theta \mid X > \theta) = P(X > x)$$

$$\frac{P(X > x+\theta)}{P(X > \theta)} = P(X > x)$$

$$1 - F_x(X = x+\theta)$$
$$= (1 - F_x(X=\theta))(1 - F_x(X=x))$$

$$G_x(X = x+\theta) = G_x(X=\theta)\, G_x(X=x)$$

Putting $\theta = x$, $G_x(X = 2x) = G_x^2(X=x)$

$\theta = 2x$, $G_x(X = 3x) = G_x^3(X=x)$

$\theta = (\lambda-1)x$, $G_x(X = \lambda x) = G_x^{\lambda}(X=x)$

$= K^{\lambda}$ as $G_x(X=x)$ is constant.

$= e^{\lambda \ln K}$

Q - 6

given: $f_X(x) = Ke^{-\lambda|x|}, \lambda > 0, -\infty < x < \infty$

1. Find K.

$$\int_{-\infty}^{\infty} K e^{-\lambda|x|}\, dx = 1$$

$$\int_{-\infty}^{0} K e^{+\lambda x}\, dx + \int_{0}^{\infty} K e^{-\lambda x}\, dx = 1$$

$$\frac{K}{\lambda}\left[e^{\lambda x}\right]_{-\infty}^{0} + \frac{-K}{\lambda}\left[e^{-\lambda x}\right]_{0}^{\infty} = 1$$

$$\frac{K}{\lambda}[1-0] + -\frac{K}{\lambda}[0-1] = 1$$

$$\frac{K}{\lambda} + \frac{K}{\lambda} = 1 \Rightarrow \boxed{K = \frac{\lambda}{2}}$$

$$\boxed{K = \frac{\lambda}{2}}$$

Q - 6

3. Find CDF of X

$$F_X(X \le x) \Rightarrow \int_{-\infty}^{x} K e^{\lambda x} \, dx$$
$-\infty < x < 0$

$$\Rightarrow \frac{K}{\lambda} [e^{\lambda x}]_{-\infty}^{x}$$

$$\Rightarrow \frac{1}{2} e^{\lambda x} \quad (\because K = \lambda/2)$$

$$\Rightarrow \boxed{\frac{e^{\lambda x}}{2}}, \quad -\infty < x < 0$$

$$F_X(X \le x) = \int_{-\infty}^{x} K e^{-\lambda|x|} \, dx$$
$-\infty < x < \infty$

$$\Rightarrow \int_{-\infty}^{0} K e^{\lambda x} \, dx + \int_{0}^{x} K e^{-\lambda x} \, dx$$

$$\Rightarrow \frac{K}{\lambda} \cdot [e^{\lambda x}]_{-\infty}^{0} + \frac{-K}{\lambda} [e^{-\lambda x}]_{0}^{x}$$

$$\Rightarrow \frac{1}{2}[1-0] + \frac{-1}{2}[e^{-\lambda x} - 1]$$

$$\Rightarrow \frac{2 - e^{-\lambda x}}{2}$$

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$$F_X(x) = \begin{cases} \dfrac{e^{\lambda x}}{2} & , -\infty < x < 0 \\ \dfrac{2 - e^{-\lambda x}}{2} & , 0 \le x < \infty \end{cases}$$

3.

$$E(X=x) = \int_{-\infty}^{\infty} x \cdot \frac{\lambda}{2} e^{-\lambda|x|} \, dx$$

$$= \frac{\lambda}{2} \int_{-\infty}^{0} x e^{\lambda x} \, dx + \frac{\lambda}{2} \int_{0}^{\infty} x e^{-\lambda x} \, dx$$

Let $\lambda x = t$ | Let $\lambda x = -t_1$

$dx = \frac{1}{\lambda} dt$ | $dx = -\frac{1}{\lambda} dt_1$

$$\frac{\lambda}{2} \cdot \frac{1}{\lambda} \int_{-\infty}^{0} (-t_1) e^{-t_1} \left(-\frac{1}{\lambda}\right) dt_1 + \frac{\lambda}{2} \cdot \frac{1}{\lambda} \int_{0}^{\infty} t e^{-t} \frac{1}{\lambda} dt$$

$$= \frac{1}{2\lambda} \int_{-\infty}^{0} t_1 e^{-t_1} \, dt_1 + \frac{1}{2\lambda} \int_{0}^{\infty} t e^{-t} \, dt$$

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$$F_X(x) = \begin{cases} \dfrac{e^{\lambda x}}{2} & ; -\infty < x < 0 \\ \dfrac{2 - e^{-\lambda x}}{2} & ; 0 \le x < \infty \end{cases}$$

3.

$$E(X=x) = \int_{-\infty}^{\infty} x \cdot \frac{\lambda}{2} e^{-\lambda|x|} dx$$

$$= \frac{\lambda}{2} \int_{-\infty}^{0} x e^{\lambda x} dx + \frac{\lambda}{2} \int_{0}^{\infty} x e^{-\lambda x} dx$$

Let $\lambda x = t$ $dx = \frac{1}{\lambda} dt$	Let $\lambda x = -t_1$ $dx = -\frac{1}{\lambda} dt_1$

$$\frac{\lambda}{2} \cdot \frac{1}{\lambda} \int_{-\infty}^{0} (-t_1) e^{-t_1} \left(-\frac{1}{\lambda}\right) dt_1 + \frac{\lambda}{2} \cdot \frac{1}{\lambda} \int_{0}^{\infty} t e^{-t} \frac{1}{\lambda} dt$$

$$= \frac{1}{2\lambda} \int_{-\infty}^{0} t_1 e^{-t_1} dt_1 + \frac{1}{2\lambda} \int_{0}^{\infty} t e^{-t} dt$$

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$$= \frac{-1}{2\lambda}\int_0^\infty t_1 e^{-t_1} dt_1 + \frac{1}{2\lambda}\int_0^\infty t e^{-t} dt$$

$$= 0$$

even ~~odd~~ function

$$\boxed{E(\lambda = x) = 0}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|} dx$$

$$= \frac{2\lambda}{2}\int_0^\infty x^2 e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-t} \frac{t^2}{\lambda^2} \frac{dt}{\lambda}$$

$\lambda x = t$
$x = dt/\lambda$

$$= \frac{1}{\lambda^2}\int_0^\infty e^{-t} t^2 dt$$

$$= \frac{2}{\lambda^2}$$

$$Var(\lambda) = E[x^2] - E[x=\lambda]$$

$$= \frac{2}{\lambda^2} - 0 = \frac{2}{\lambda^2} \text{ u.}$$

Q-7

$X \Rightarrow$ 4 - sided die

$R_X = \{1, 2, 3, 4\}$

$Y = $ 6 sided die

Total pairs $= 4 \times 6$ $= 24$

$R_Y = \{1, 2, 3, 4, 5, 6\}$

$Z = (X+Y)/2$

$R_Z = \{1, 2, 3/2, 5/2, 3, 7/2, 4, 9/2, 5\}$

1.

X	1	2	3	4
P(X)	1/4	1/4	1/4	1/4

$E(X) = \frac{1}{4}(1+2+3+4)$

$= 5/2$

$E(X^2) = \frac{1}{4}(1^2+2^2+3^2+4^2)$

$= \frac{15}{2}$

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$$Var(X) = E[X^2] - [E[X]]^2$$

$$= \frac{15}{2} - \frac{25}{4}$$

$$\boxed{Var(X) = 5/4}$$

$$E(Y) = \frac{1}{6}*1 + \frac{1}{6}*2 + \frac{1}{6}*3 + \frac{1}{6}*4 + \frac{1}{6}*5 + \frac{1}{6}*6$$

$$= \frac{1}{6}[1+2+3+4+5+6] = 7/2$$

$$E(Y^2) = \frac{1}{6}[1^2+2^2+3^2+4^2+5^2+6^2]$$

$$= \frac{91}{6}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{91}{6} - \frac{49}{4}$$

$$= \frac{182 - 147}{12} = 35/12$$

$$\boxed{Var(Y) = 35/12}$$

Q-07

$$E(z) = E\left(\frac{x+y}{2}\right) = \frac{1}{2}E(x+y)$$

$$= \frac{1}{2}[E(x) + E(y)] = \frac{1}{2}\left[\frac{5}{2} + \frac{7}{2}\right]$$

$$\boxed{E(z) = 3}$$

$$Var(z) = \frac{1}{4}Var(x) + \frac{1}{4}Var(y)$$

$$= \frac{1}{4}\left[\frac{5}{4} + \frac{35}{12}\right] = \frac{25}{24}$$

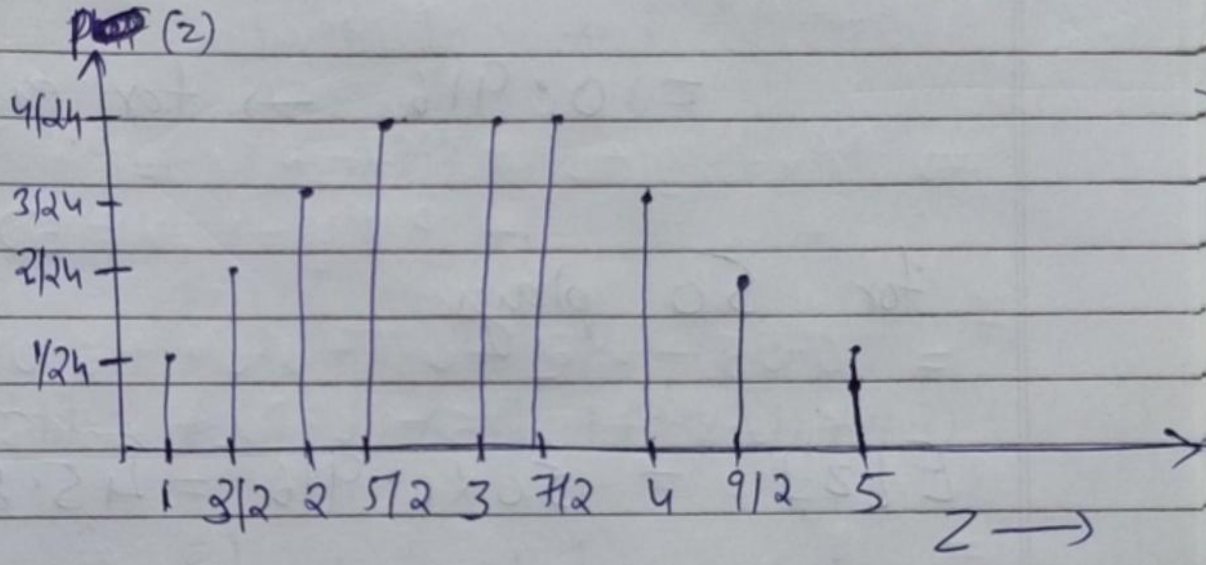
$$\boxed{Var(z) = \frac{25}{24}}$$

Q-07

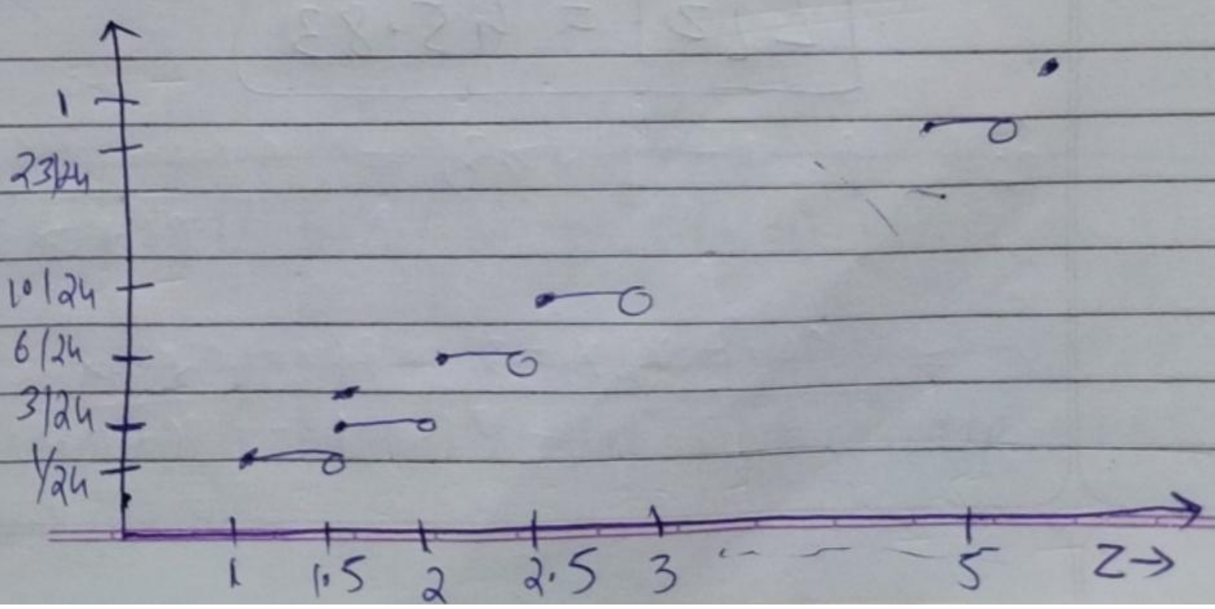
2] $Z \Rightarrow$

Z	1	3/2	2	5/2	3	7/2	4	9/2	5
P(z)	1/24	2/24	3/24	4/24	4/24	4/24	3/24	2/24	1/24

PMF of Z



CDF of Z



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Q-7

3. Let z be event when $x > y$

$z = \{4, 6, 8, -1\}$ $\quad x = \{1, 2, 3, 4\}$

$y : \{1, 2, 3, 4, 5, 6\}$

$$E[z] = 4\left(\frac{1}{24}\right) + 6\left(\frac{2}{24}\right) + 8\left(\frac{3}{24}\right) - 1\left(\frac{18}{24}\right)$$

$= 0 \cdot 916 \rightarrow$ for each time

for 50 plays

$E[z] = 50 \times 0 \cdot 916 = 45 \cdot 83$

$$\boxed{E[z] = 45 \cdot 83}$$

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Q-8

Are X and Y independent random variables?

$\Rightarrow$

$$P(X=1) = \cancel{P(X=1)}$$

$$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$$

$$= \frac{1}{18} + \frac{1}{9} + \frac{1}{6}$$

$$= \frac{1+2+3}{18}$$

$$= \frac{1}{3}$$

$$P(Y=1) = P(Y=1, X=1) + P(Y=1, X=2) + P(Y=1, X=3)$$

$$= \frac{1}{18} + \frac{1}{9} + \frac{1}{6}$$

$$= \frac{1+2+3}{18} = \frac{1}{3}$$

$$\boxed{P(X=1, Y=1) = \frac{1}{18} \neq P(X=1)\cdot P(Y=1) \Rightarrow \frac{1}{9}.}$$

Hence, X and Y are not independent variables

Q-9

1. Justify that this is a valid PMF

for valid PMF in joint probability.

$$\sum_{\substack{x_i \in X, \\ y_j \in Y}} P(X = x_i, Y = y_j) = 1$$

$\Rightarrow P(x=1, y=0) + P(x=2, y=0) + P(x=3, y=0)$
$+ \; \sim \; + P(y=1, x=5).$

$= 0.05 + 0.2 + 0.1 + 0.04 + 0.01 + 0.01$
$+ 0.09 + 0.15 + 0.20 + 0.15$

$= 1$

Hence, proved that given PMF is valid.

2. $P(Y=1 \mid X \geq 3) = \dfrac{P(Y=1, X \geq 3)}{P(X \geq 3)}$

$$= \frac{P(Y=1, X=3) + P(Y=1, X=4) + P(Y=1, X=5)}{1 - P(X=1) - P(X=2)}$$

Q-9

$P(X=1) = 0.05 + 0.01 = 0.06$
$P(X=2) = 0.20 + 0.09 = 0.29$

$$P(Y=1 \mid X \geq 3) = \frac{0.15 + 0.20 + 0.15}{1 - 0.06 - 0.29}$$

$$= \frac{0.5}{0.65} = \frac{50}{65} = \frac{10}{13}$$

$$\boxed{P(Y=1 \mid X \geq 3) = \frac{10}{13}}$$

$$P(Y=0, X \geq 3) = P(Y=0, X=3) + P(Y=0, X=4) + P(Y=0, X=5)$$

$$\cancel{= 0.15 + 0.20 +}$$

$$= 0.1 + 0.04 + 0.01$$

$$= 0.15$$

3. Marginal PMF of X

$$P(X=1) = P(X=1, Y=0) + P(X=1, Y=1)$$

$$= 0.05 + 0.01$$

$$= 0.06$$

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$P(X=2) = P(X=2, Y=0) + P(X=2, Y=1)$
$= 0.2 + 0.09$
$= 0.29$

$P(X=3) = P(X=3, Y=0) + P(X=3, Y=1)$
$= 0.1 + 0.15$
$= 0.25$

$P(X=4) = P(X=4, Y=0) + P(X=4, Y=1)$
$= 0.04 + 0.20$
$= 0.24$

$P(X=5) = P(X=5, Y=0) + P(X=5, Y=1)$
$= 0.01 + 0.15$
$= 0.16$

4. Marginal PMF of Y :-

$P(Y=0) = P(Y=0, X=1) + P(Y=0, X=2) + P(Y=0, X=3)$
$+ P(Y=0, X=4) + P(Y=0, X=5) + P(Y=0, X=6)$
$= 0.05 + 0.2 + 0.1 + 0.04 + 0.01$
$= 0.4$

$P(Y=1) = P(Y=1, X=1) + P(Y=1, X=2) + P(Y=1, X=3)$
$+ P(Y=1, X=4) + P(Y=1, X=5) + P(Y=1, X=6)$

$= 0.01 + 0.09 + 0.15 + 0.20 + 0.15$
$= 0.6$

Q-9

5. $E[X \mid Y=1]$

$$= \sum_{\substack{x_i \in X \\ Y=1}} x_i \, P(X=x_i, Y=1) / P(Y=1)$$

$$= \cancel{1 * 0.05 + 2 * 0.2}$$

$$= (1 * 0.01 + 2 * 0.09 + 3 * 0.15 + 4 * 0.20 + 5 * 0.15) / 0.6$$

$$= ((0.01 + 0.18 + 0.45 + 0.80 + 0.75)) / 0.6$$

$$= (2.19) / 0.6$$

$$\boxed{E(X \mid Y=1) = 3.65}$$

$$P(X=x) = \begin{cases} 0.06, & x=1 \\ 0.29, & x=2 \\ 0.25, & x=3 \\ 0.24, & x=4 \\ 0.16, & x=5 \\ 0, & \text{otherwise} \end{cases}$$

$$P(Y=y) = \begin{cases} 0.4, & y=0 \\ 0.6, & y=1 \\ 0, & \text{otherwise} \end{cases}$$

Q-10

1.

As x and y are independent

$$E(XY) = E(X) \cdot E(Y)$$

$$E[XY] = \iint x \left(\text{pmf of } x\right) \cdot y \left(\text{pmf of } y\right) dx\,dy$$

$$= \iint xy * 1 \; dx\,dy$$

$$= \int_0^1 \int_0^1 xy \; dx\,dy$$

$$= \int_0^1 y \left[\frac{x^2}{2}\right]_0^1 dy = \int_0^1 y\left(\frac{1}{2} - 0\right) dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}.$$

$$\boxed{E(XY) = 1/4}$$

2. $E(e^{X+Y}) = \iint e^{X+Y} * 1 \; dx\,dy$

$$= \int_0^1 \int_0^1 e^{X} \cdot e^{Y} \; dx\,dy$$

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$$= \int_0^1 e^y [e^x]_0^1 \, dy$$

$$= \int_0^1 e^y (e-1) \, dy$$

$$= e-1 \, (e^y)_0^1$$

$$= (e-1)(e-1) = (e-1)^2$$

$$\cancel{E[e^{x+y}] = (e-1)^2}$$

$$\boxed{E[e^{x+y}] = (e-1)^2}$$

3. $E[x^2+y^2+xy]$

$$= E(x^2) + E(y^2) + E(xy)$$

$$E(x^2) = \int_0^1 x^2 (1) \, dx = (x^3/3)_0^1 = 1/3$$

$$E(y^2) = \int_0^1 y^2 (1) \, dy = (y^3/3)_0^1 = 1/3$$

$$E(xy) = 1/4$$

$$E(x^2+y^2+xy) = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{11}{12}$$ //.

Q-11

X and Y are independent.

$N(0,1)$, $Z = 7 + X + Y$
$W = 1 + Y.$

$$F_{xy}(x,y) = \frac{1}{2\pi} e^{-1/2(x^2+y^2)}$$

$X = Z - W - 6$, $Y = W - 1$,
$h_1(Z,W)$, $h_2(Z,W)$

$$|J| = \begin{vmatrix} \frac{\partial h_1}{\partial Z} & \frac{\partial h_1}{\partial W} \\ \frac{\partial h_2}{\partial Z} & \frac{\partial h_2}{\partial W} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$F_{WZ}(w,z) = F_{xy}(z-w-6, w-1)\,|J|$$
$$= \frac{1}{2\pi} e^{-1/2[(z-w-6)^2+(w-1)^2]} \times 1$$

$$F_{WZ}(w,z) = \frac{1}{2\pi} e^{-\frac{1}{2}[(z-w-6)^2+(w-1)^2]}$$

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$$f_{XY}(x,y) = \begin{cases} x^2 + \frac{1}{3}y, & -1 \le x \le 1, \; 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

1. $f_{X|Y}(x|y)$,

$$\text{PDF} = \frac{f(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-1}^{1} x^2 + \frac{1}{3}y \, dx$$

$$= \left[\frac{x^3}{3}\right]_{-1}^{1} + \frac{1}{3}y\,[x]_{-1}^{1}$$

$$= \frac{1}{3} - \left(\frac{-1}{3}\right) + \frac{1}{3}y\,(2)$$

$$= \frac{2}{3} + \frac{2y}{3} = \frac{2}{3}(1+y)$$

$$\text{PDF} = \frac{3x^2+y}{\cancel{3} \cdot \frac{2}{\cancel{3}}(1+y)} = \frac{3x^2+y}{2(1+y)}$$

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2. $P(x > 0 \mid y = y)$.

$$= \frac{P(x > 0, y = y)}{P(y = y)}$$

$$= \frac{\int_0^1 x^2 + \frac{1}{3}y \, dx}{\frac{2}{3}(1+y)}$$

$$\cancel{= \frac{x^2[\,]>0] + \frac{1}{3}[y^2]_0^1}{\frac{2}{3}(1+y)}}$$

$$\cancel{= \frac{3x^2 + y\,1}{2(1+y)}} = \frac{\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{3}y[1-0]}{\frac{2}{3}(1+y)}$$

$$= \frac{1+y}{3 \div \frac{2}{3}(1+y)} = \frac{1}{2}$$

$$\boxed{P(x > 0 \mid y = y) = \frac{1}{2}}$$

Q - 12

3.

X and Y are said to be independent if

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \text{ for all } x,y.$$

$f_X(x)$ PDF of only X $= \int_0^1 x^2 + \frac{y}{3}\, dy$

$$= x^2 [y]_0^1 + \left[\frac{y^2}{2 \cdot 3}\right]_0^1$$

$$= x^2 y + \frac{1}{6} = \frac{6x^2 y + 1}{6}$$

$$f_Y(y) = \int_{-1}^{1} x^2 + \frac{y}{3}\, dx$$

$$= \left[\frac{x^3}{3}\right]_{-1}^{1} + \frac{y}{3}[x]_{-1}^{1}$$

$$= \frac{1}{3}[1-(-1)] + \frac{y}{3}[1-(-1)]$$

$$= \frac{2}{3} + \frac{2y}{3} = \frac{2}{3}(1+y).$$

Q-12

$$f_x(x) \cdot f_y(y) = \frac{(6x^2y+1)}{\cancel{6}\,3} \cdot \frac{\cancel{2}(1+y)}{\cancel{6}\,3}$$

$$= \frac{6x^2y + 6x^2y^2 + 1 + y}{9}$$

$$\neq f_{(x,y)}(x,y) \neq x^2 + \frac{1}{3}y$$

So, X and Y are dependent.

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$\Gamma(\alpha+1) = \alpha \Gamma \alpha.$

1. $\Gamma(7/2)$ $\qquad \Gamma(1/2) = \sqrt{\pi}$

$= \Gamma(5/2+1)$

$= \frac{5}{2} \Gamma(5/2)$

$= \frac{5}{2} * \frac{3}{2} \Gamma(3/2)$

$= \frac{5}{2} * \frac{3}{2} * \frac{1}{2} \Gamma(1/2)$

$= \frac{15}{8} \sqrt{\pi}$

2.

$\int_0^\infty x^7 e^{-5x} dx = \left[\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha} \right]$

$\Rightarrow \alpha - 1 = 7 \Rightarrow \alpha = 8$

$\Rightarrow \lambda = 5.$

$\int_0^\infty x^7 e^{-5x} dx = \frac{\Gamma(8)}{5^8} = \frac{7!}{5^8}$

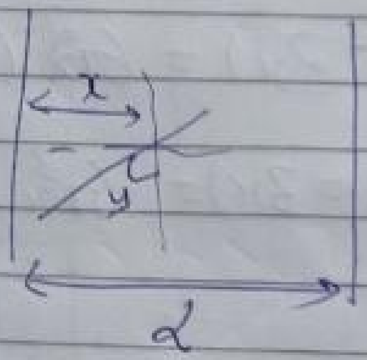
Q-14

$X \sim U(0, \alpha/2)$, $Y = U(0, \pi/2)$

$P(X = x) = \dfrac{1}{\frac{\alpha}{2} - 0} = 2/\alpha$ $\quad P(Y = y) = \dfrac{1}{\frac{\pi}{2} - 0} = 2/\pi$

$P(X = x, Y = y) = \dfrac{2}{\alpha} * \dfrac{2}{\pi} = \dfrac{4}{\alpha\pi}$

[$\because$ X, Y are independent]



From the fig we observe that

$x - \frac{l}{2} \sin y \le 0$ if needle has to intersect any line

i.e. $x \le l/2 \sin y$

$$P = \int_0^{\pi/2} \int_0^{l/2 \sin y} \frac{4}{\alpha\pi} \, dx \, dy$$

$$= \int_0^{\pi/2} \frac{4}{\alpha\pi} \frac{l}{2} \sin y \, dy = \frac{2l}{\alpha\pi} \int_0^{\pi/2} \sin y \, dy$$

$$= \frac{2l}{\alpha\pi}$$

Q-15

$X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$

$\mu_{x+y} = \mu_x + \mu_y$

$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 + 2\,\mathrm{cov}(x, y)$

$= \sigma_x^2 + \sigma_y^2$ ($\because$ independent)

$(X+Y) \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

Q-16

$X_1 \sim \text{Normal}(2, 3) \Rightarrow \mu = 2,\ \sigma^2 = 3$
$X_2 \sim \text{Normal}(1, 4) \Rightarrow \mu = 1,\ \sigma^2 = 4$

1. $Y = 2X_1 + 3X_2$

$$
\begin{aligned}
E[Y] &= 2E(X_1) + 3E(X_2) \\
&= 2(2) + 3(1) \\
&= 4 + 3 \\
&= 7
\end{aligned}
$$

$\boxed{E[Y] = 7}$

$$
\begin{aligned}
Var[Y] &= Var[2X_1 + 3X_2] \\
&= 4\,Var[X_1] + 9\,Var[X_2] \\
&= 4(3) + 9(4) \\
&= 12 + 36
\end{aligned}
$$

$\boxed{Var[Y] = 48}$ $\quad Y \sim \text{Normal}(7, 48)$

2. $Y = X_1 - X_2$

$$
\begin{aligned}
E[Y] &= E[X_1] - E[X_2] \\
&= 2 - 1 \\
&= 1
\end{aligned}
$$

$\boxed{E[Y] = 1}$

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$$\begin{aligned} Var[Y] &= Var[X_1 - X_2] \\ &= 1 * Var[X_1] + 1 * Var[X_2] \\ &= 3 \times 1 + 4 \times 1 \\ &= 7 \end{aligned}$$

$$\boxed{Var[Y] = 7}$$

$Y \sim Normal(1, 7)$