Master's Colloquium

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Technical University of Munich
Department of Informatics
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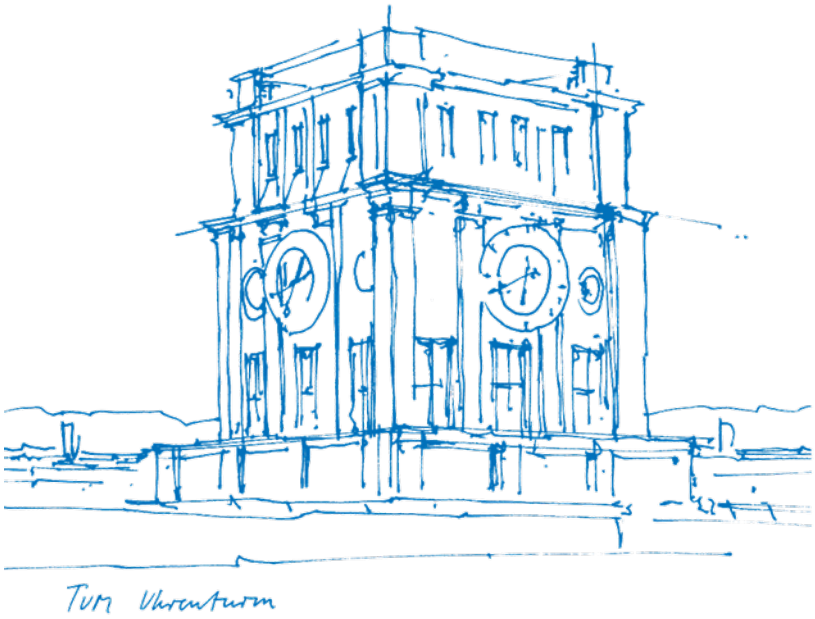

Master's Colloquium

An Experimental Evaluation of Heuristics for Approximate Maxflow Algorithms

Maxflows: Runtimes

- Directed Graph: $\mathcal{O}(m \cdot \min(m^{1/2}, n^{1/3}))$ (Goldberg, Rao)

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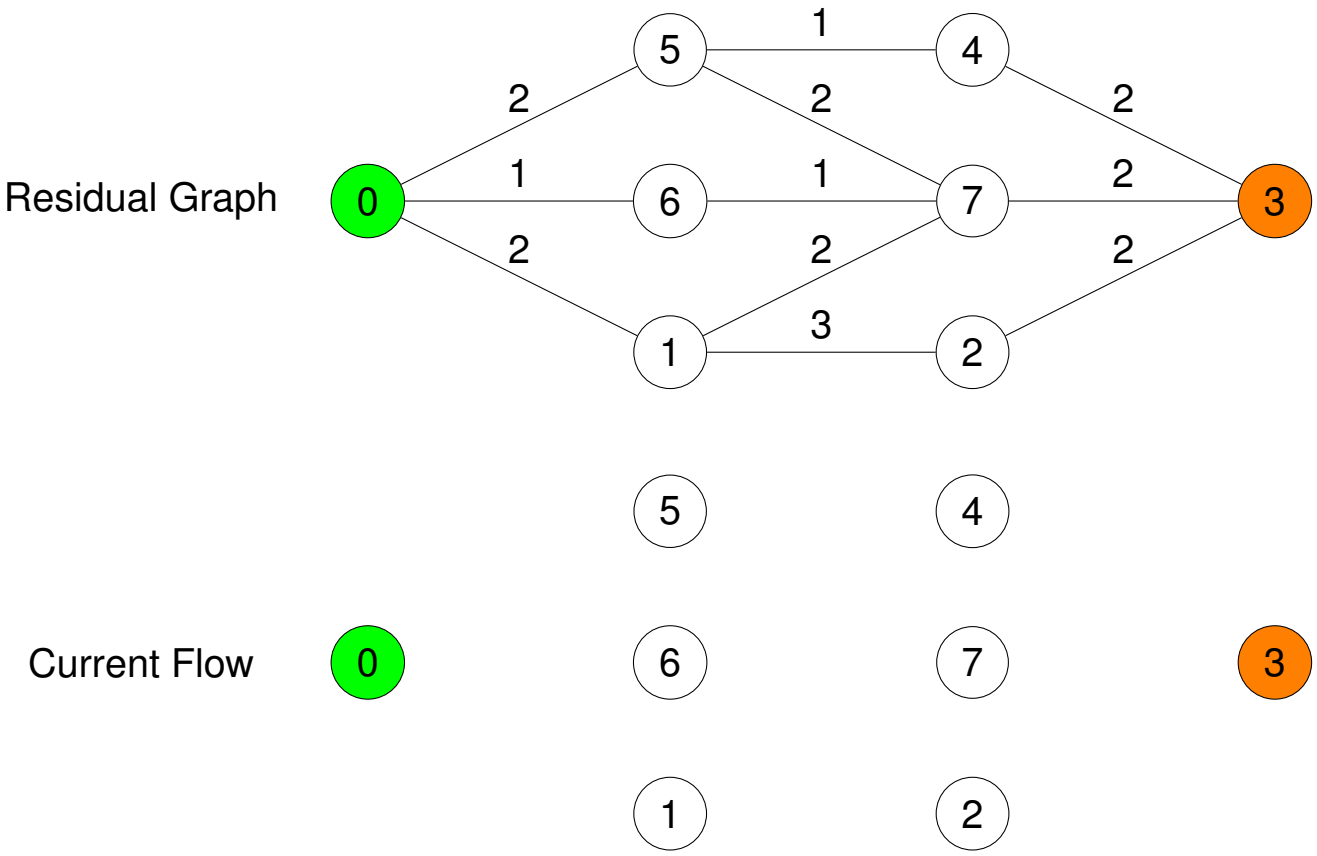
- Sherman: Single-commodity flows (undirected graph, approximative): $(1+\varepsilon)$-optimal in $\mathcal{O}(m^{1+o(1)}\varepsilon^{-2})$

Maxflows


Residual Graph
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Current Flow
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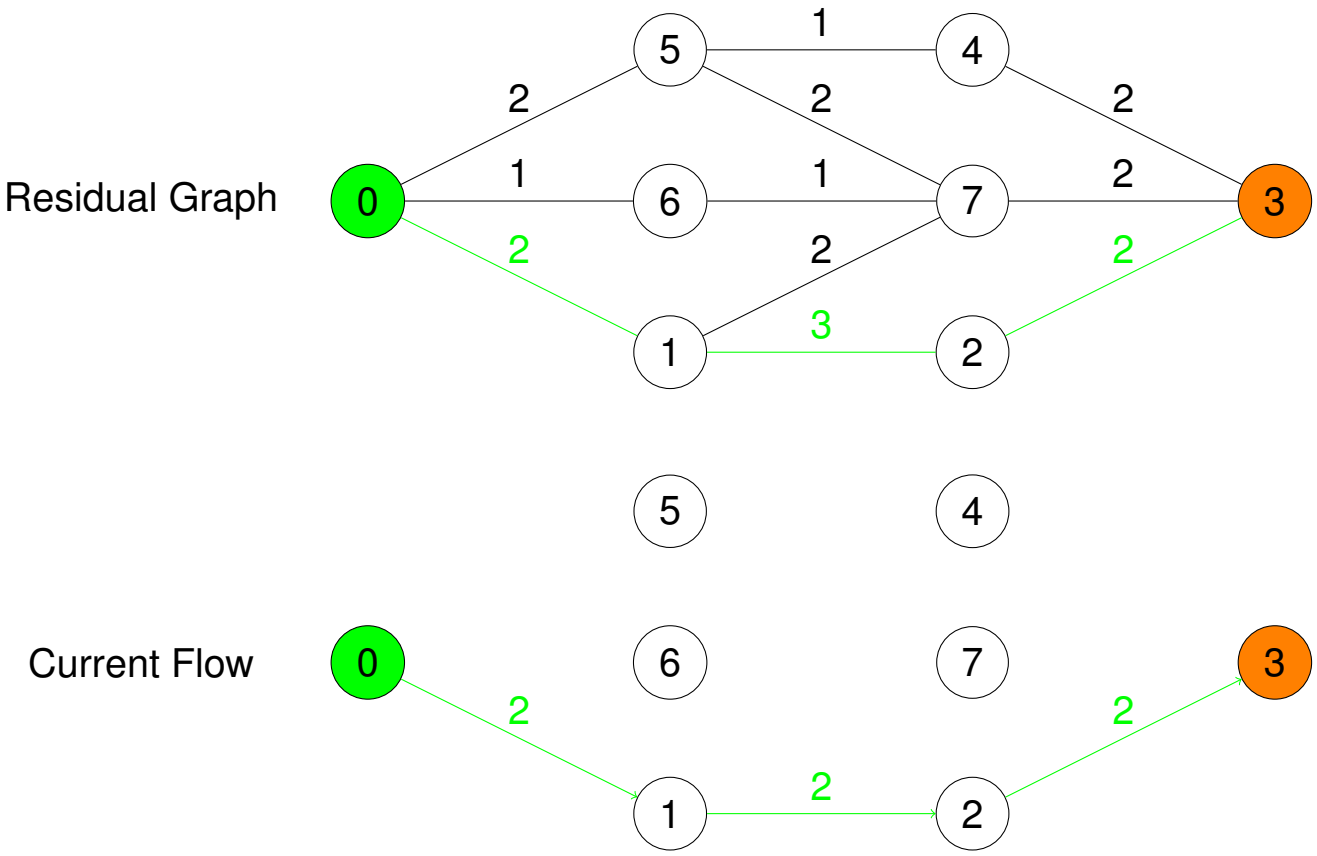
Maxflows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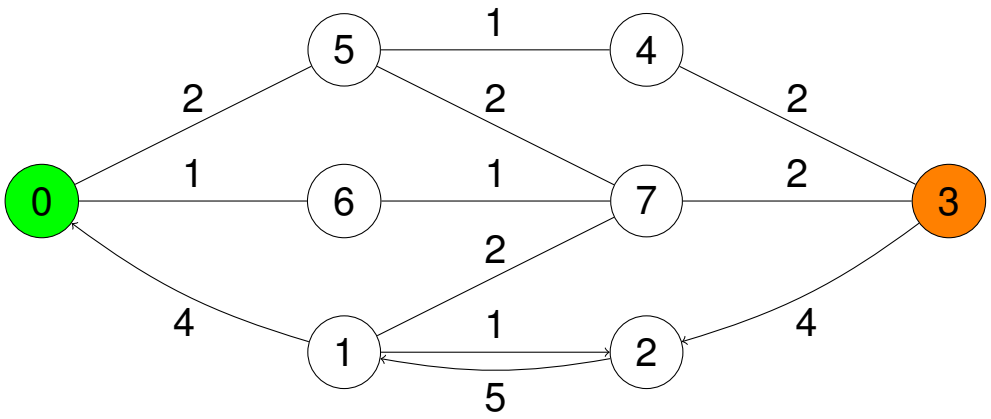
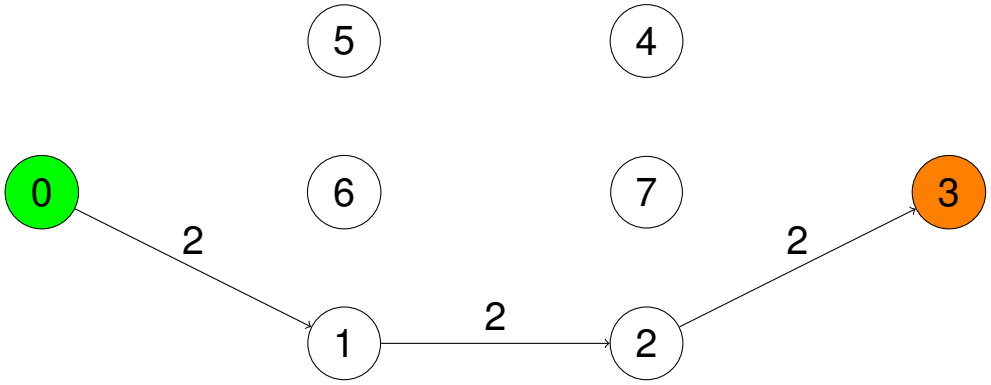
Maxflows

Residual Graph

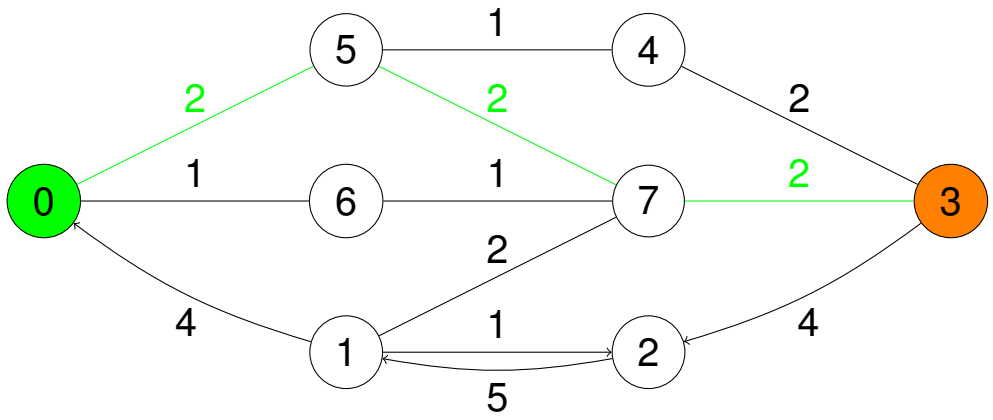


Current Flow

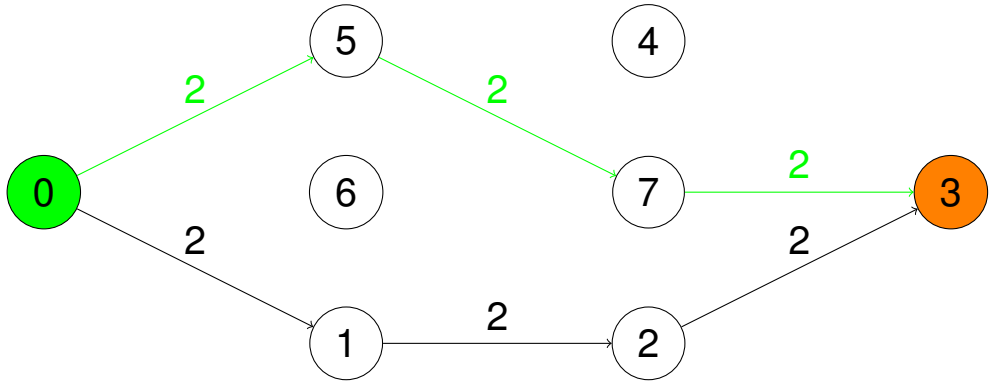


Maxflows

Residual Graph

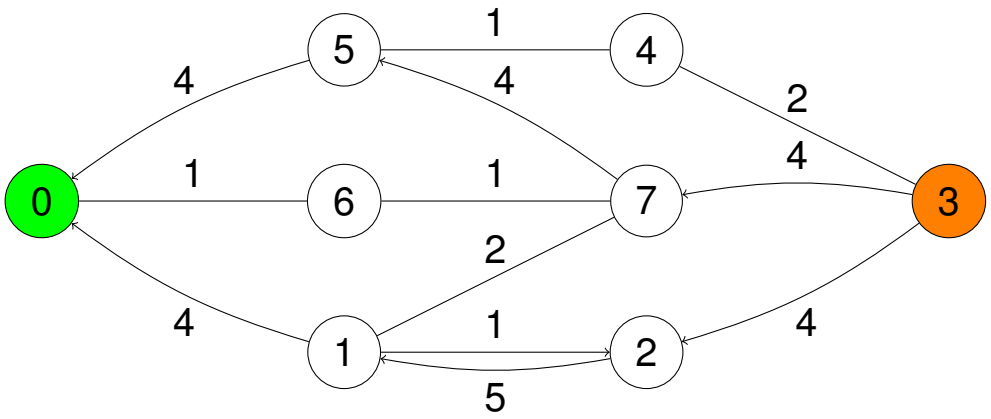


Current Flow

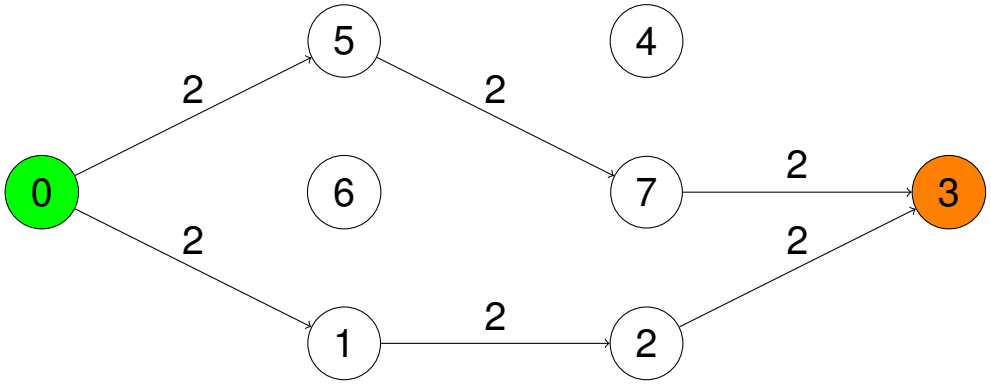


Maxflows

Residual Graph

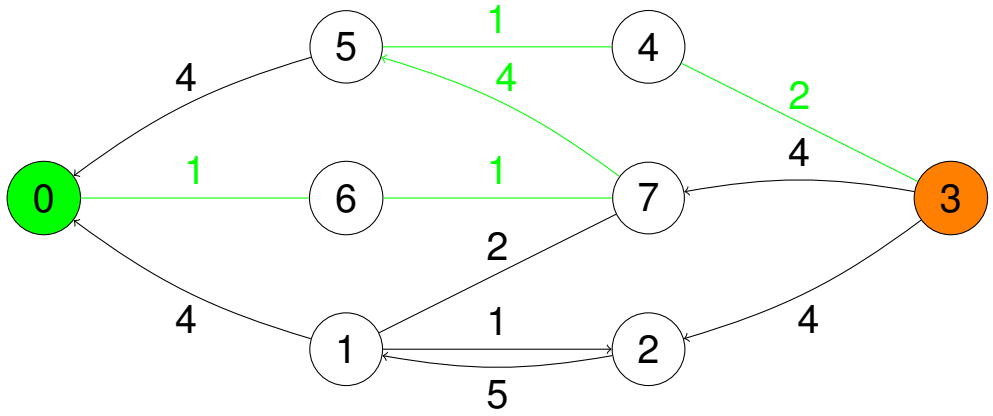


Current Flow



Maxflows

Residual Graph



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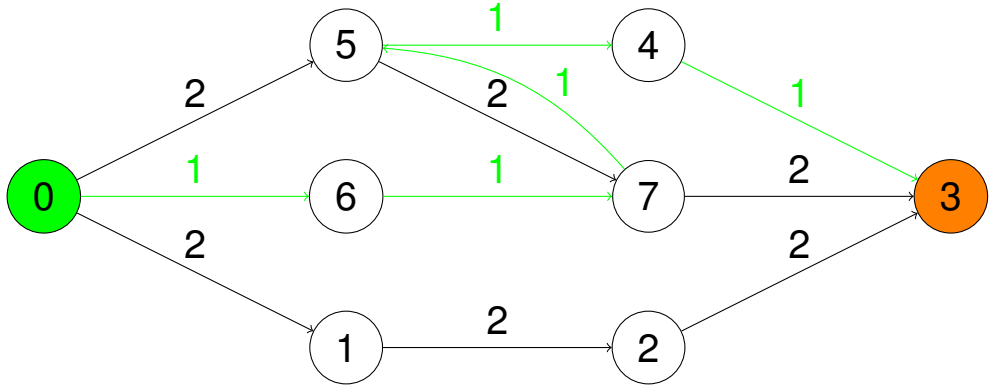

Maxflows

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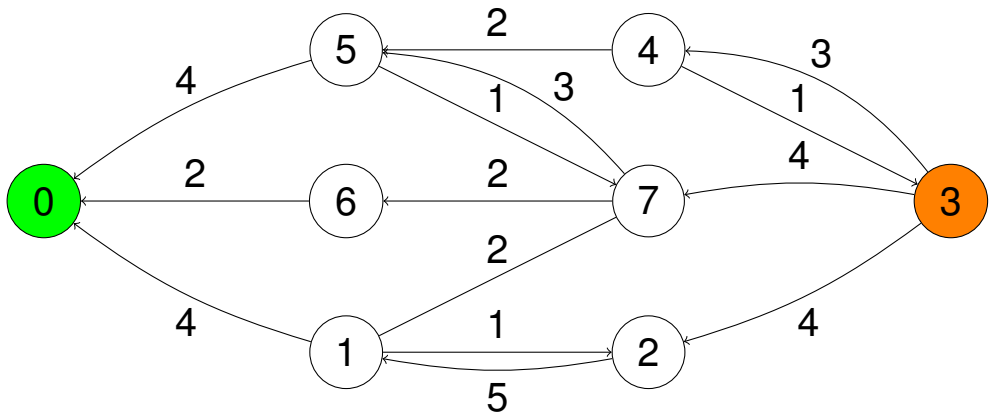

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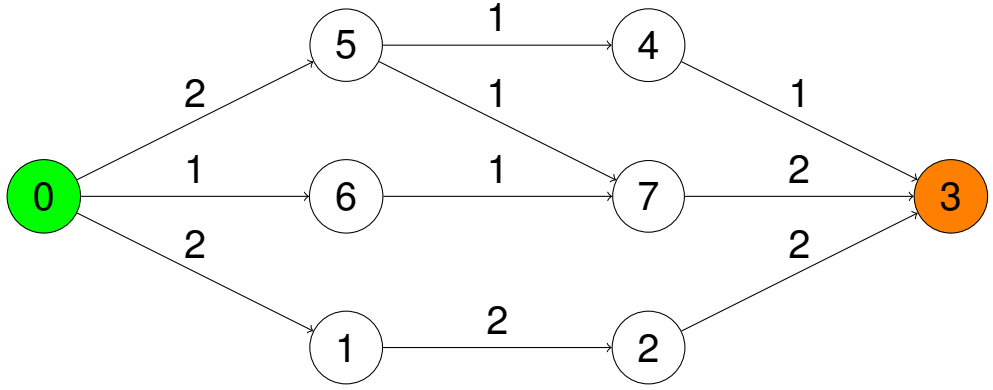

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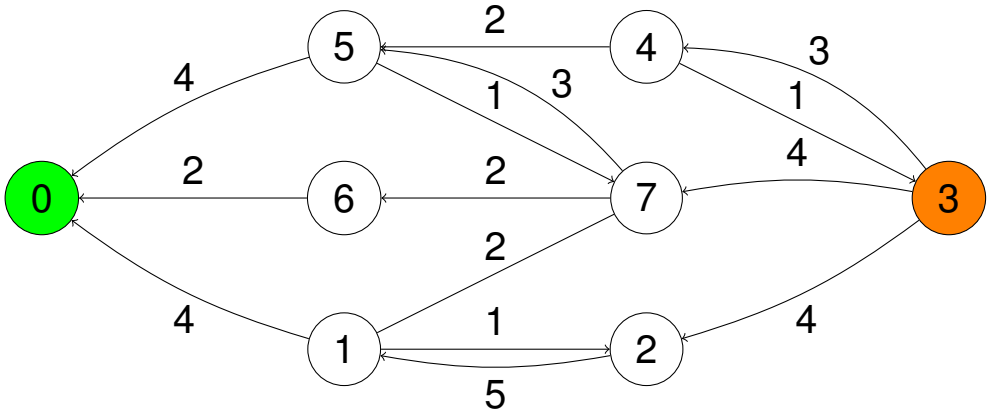

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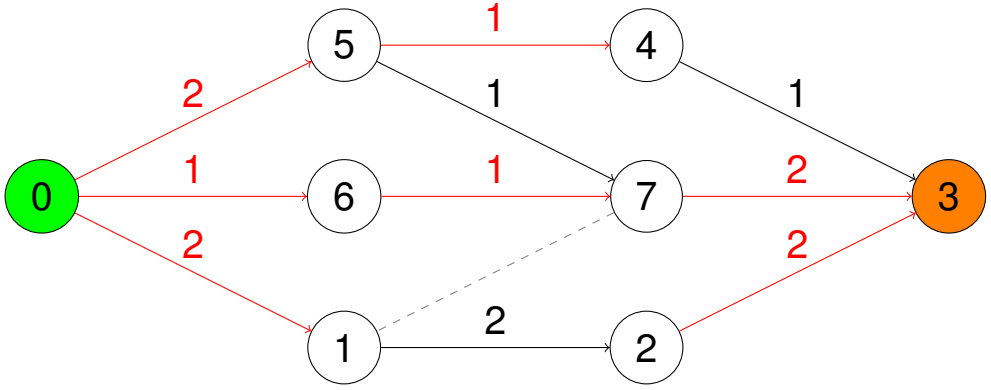

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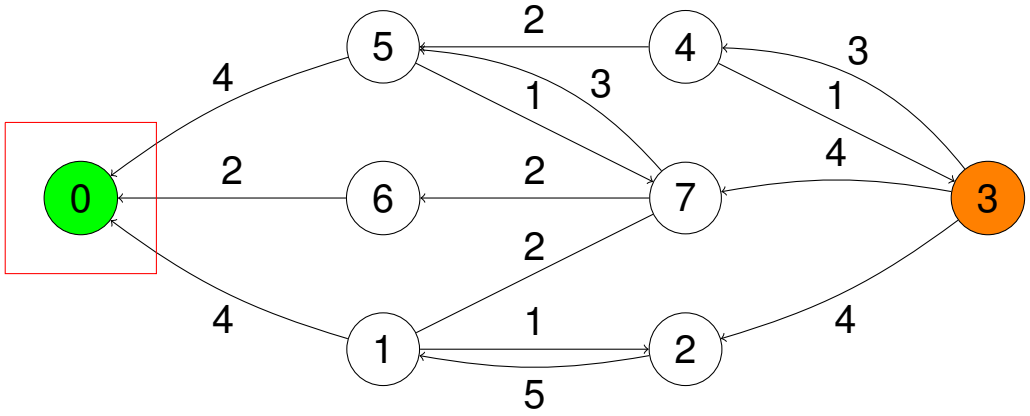

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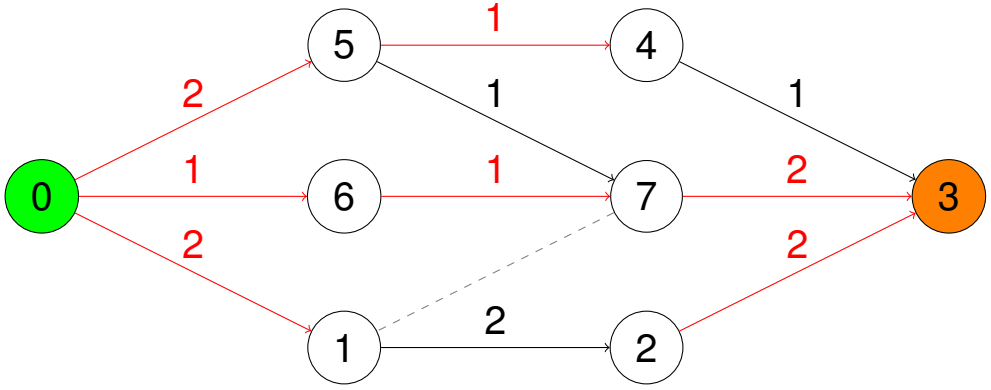

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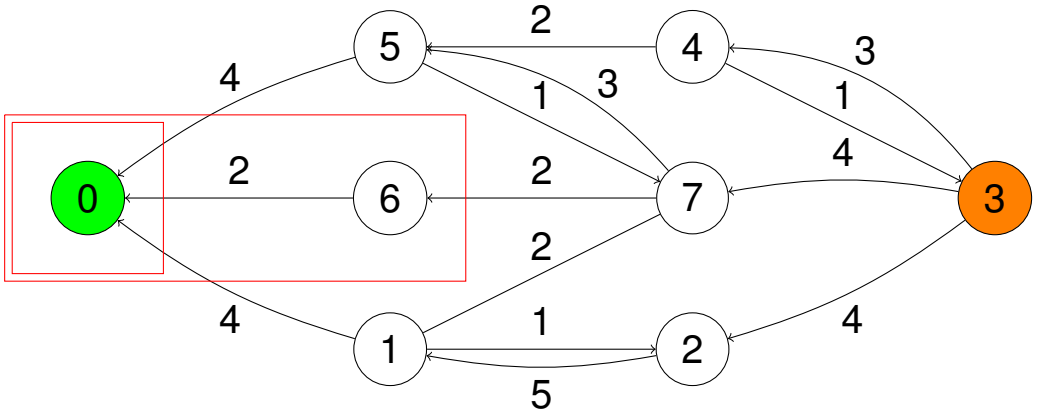

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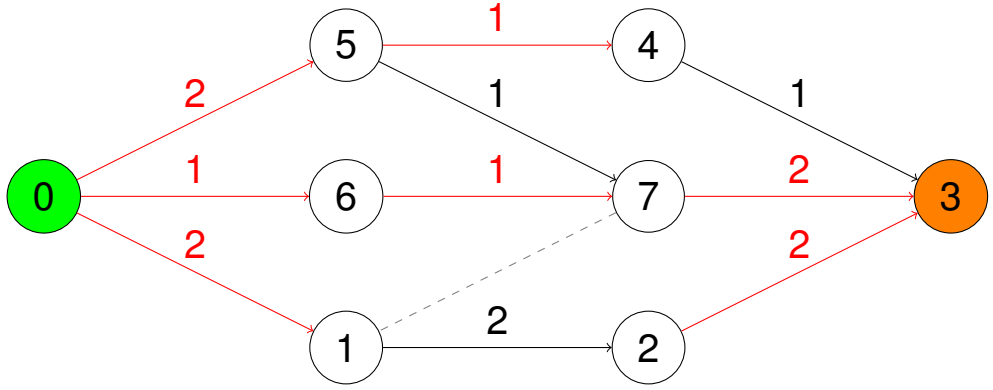
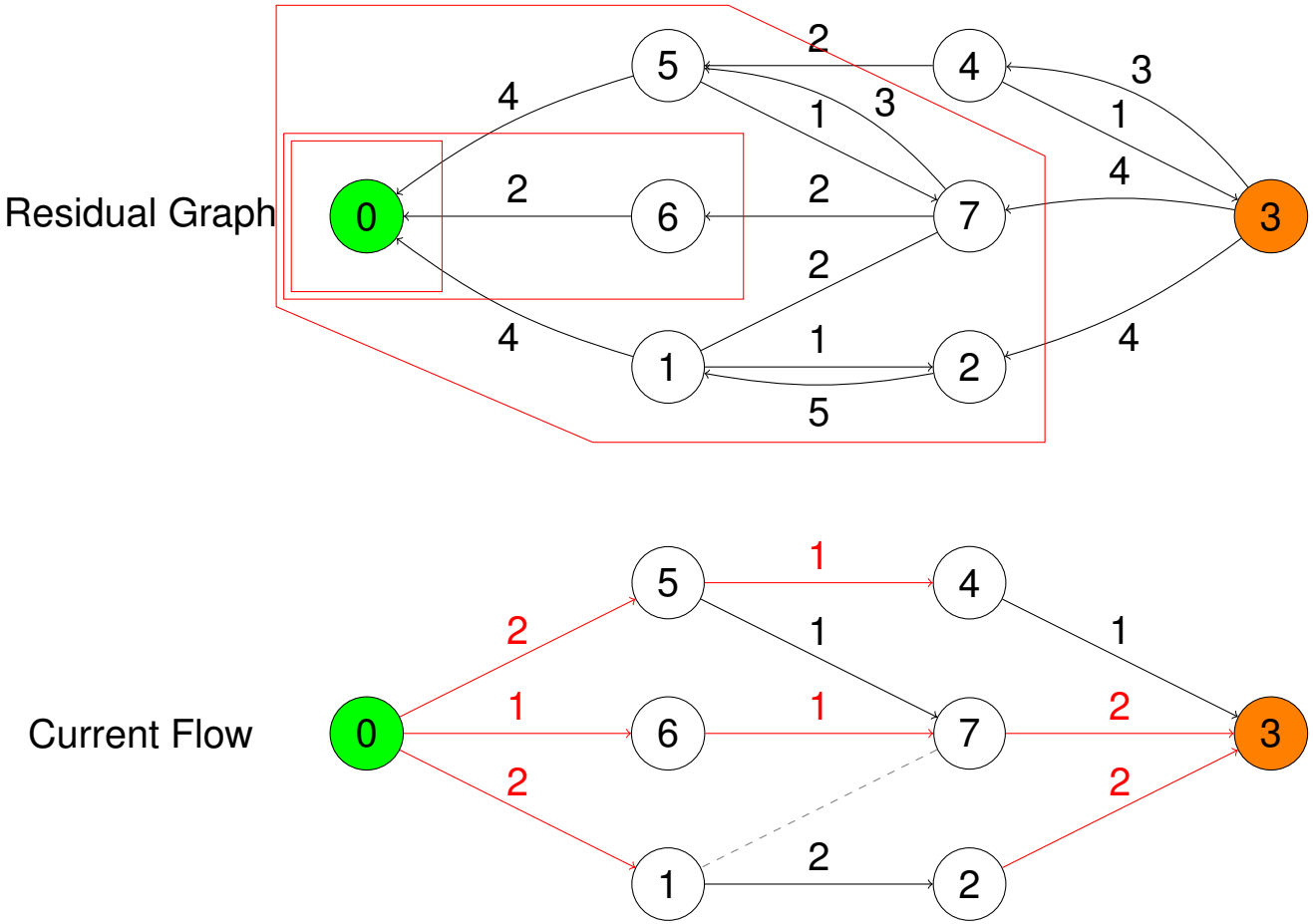


Maxflows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Minimum Congestion Flow

- Maximum Flow Problem:

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  - s-t-flow of maximal value

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  - maximum flow $= \frac{\text{minimum congestion flow } f}{\text{maximum congestion of } f}$

Congestion Approximator

- Optimal congestion = maximal congested cut

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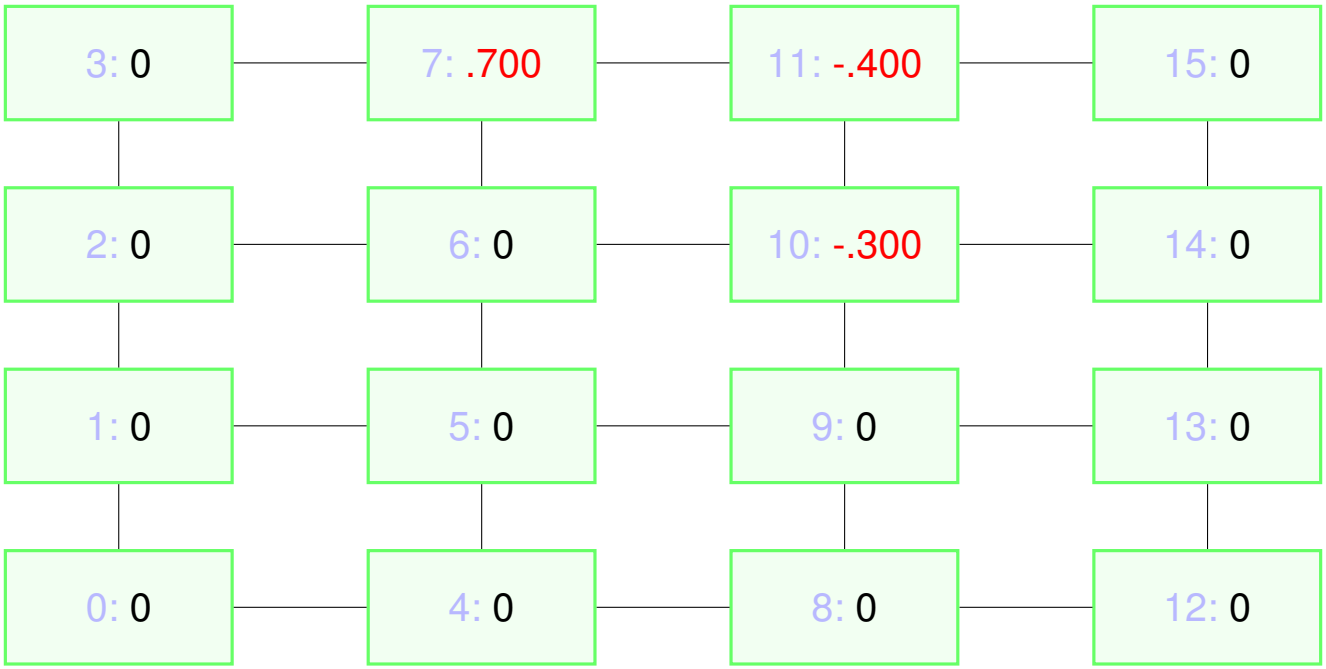
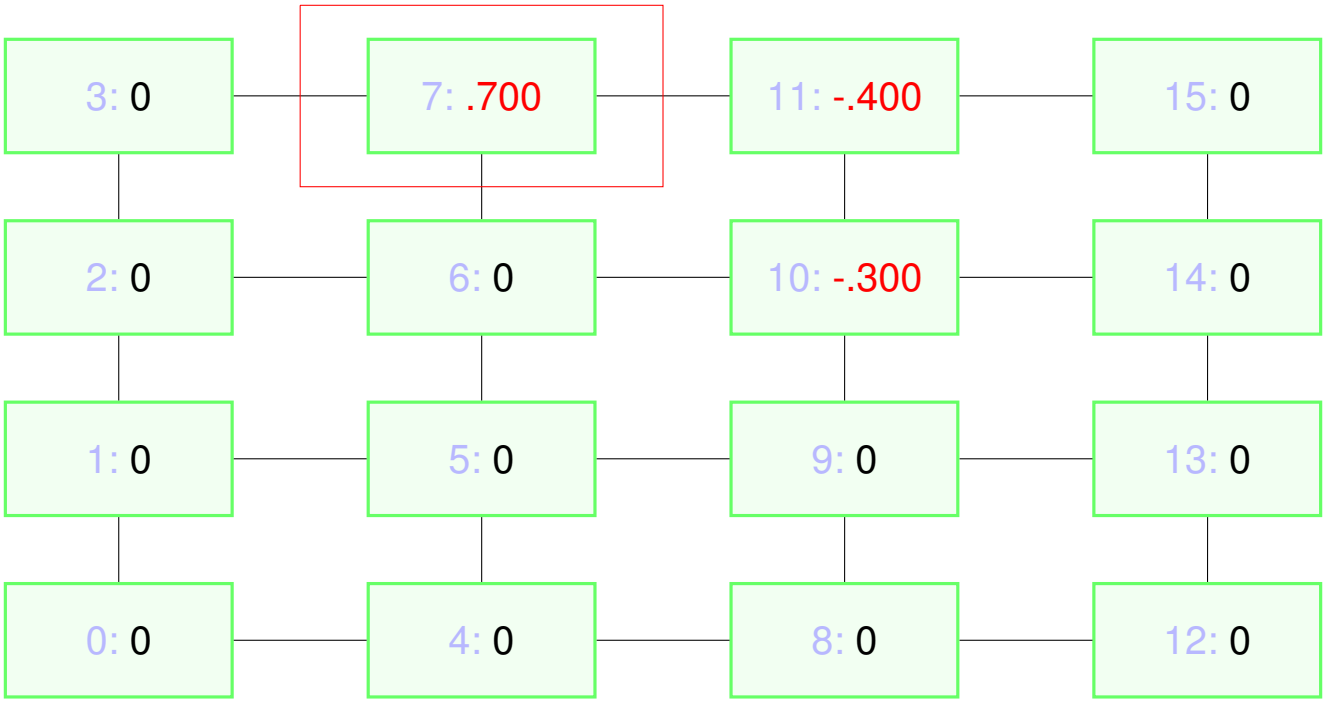
- Optimizing goal: minimizing maximum congestion via its approximation

Congestion Approximator

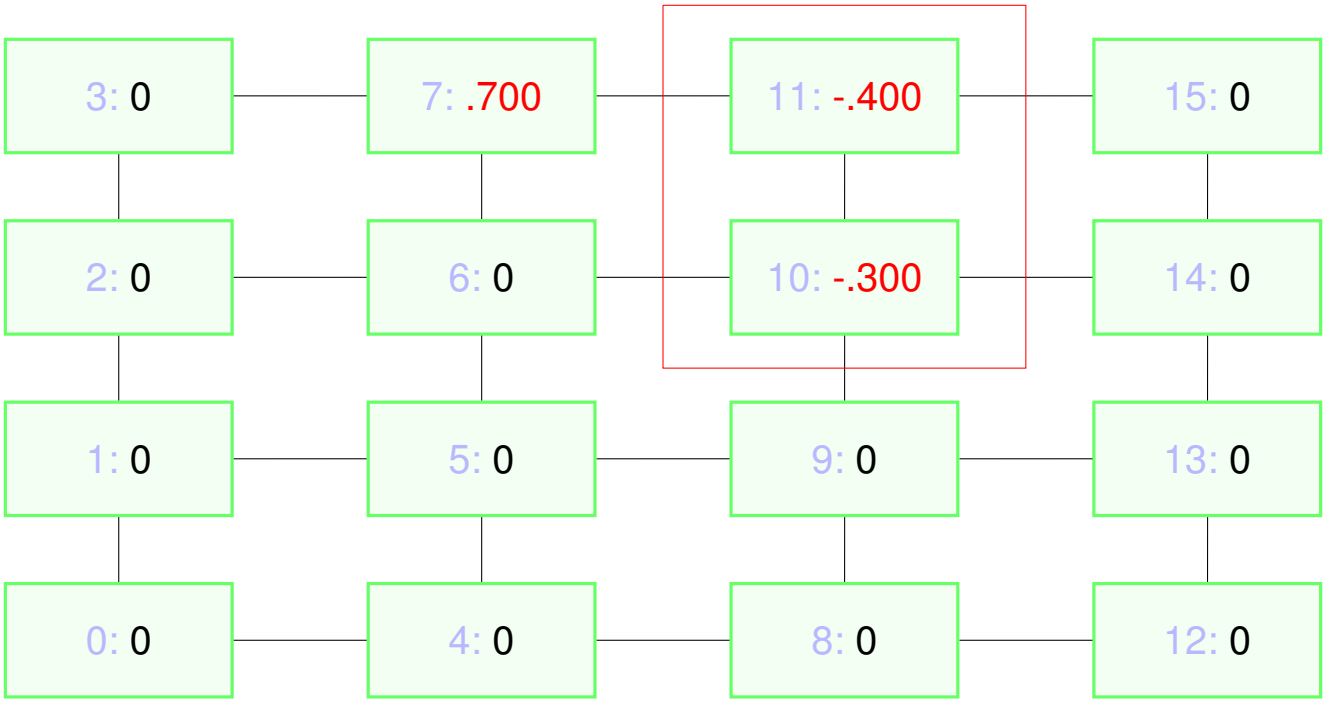


Congestion Approximator



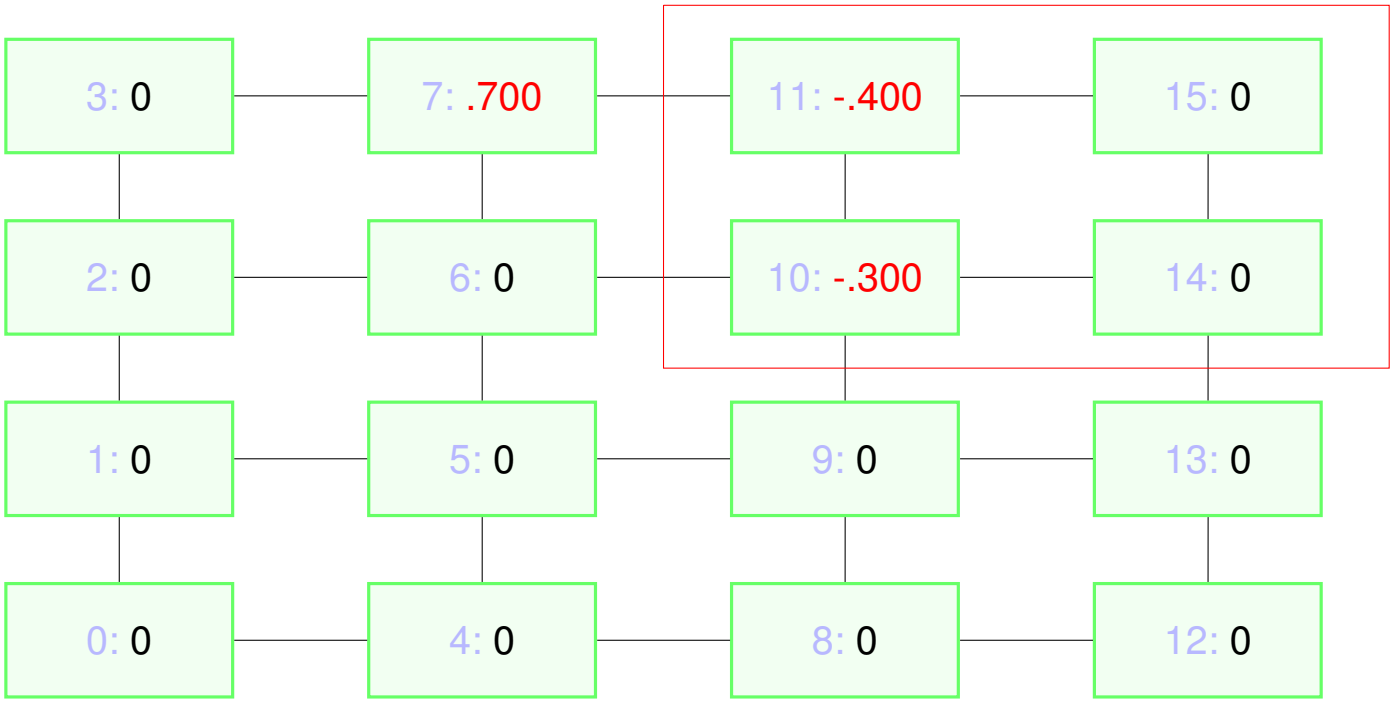
Congestion $\geq 0.7/3$

Congestion Approximator



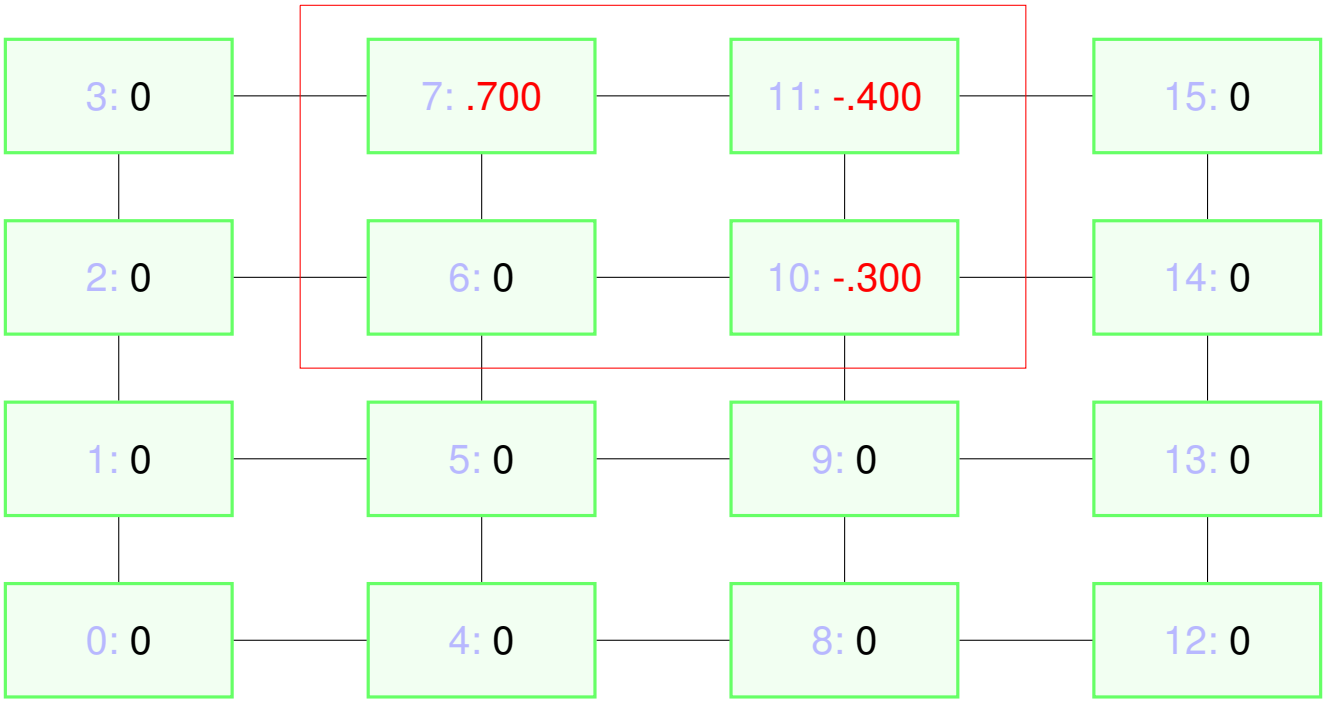
Congestion $\geq 0.7/5$

Congestion Approximator



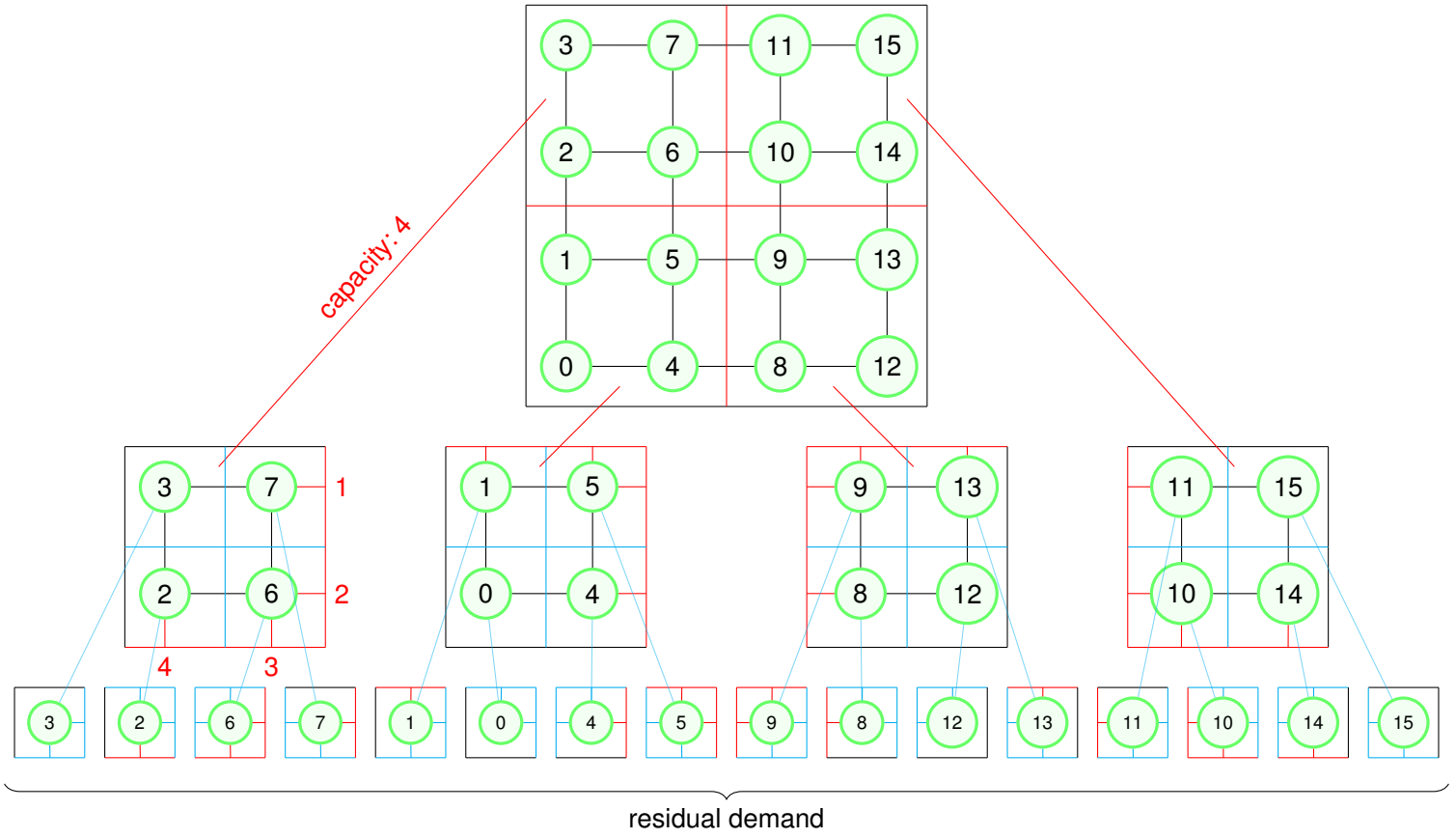
Congestion $\geq 0.7/4$

Congestion Approximator



Congestion ≥ 0

Approximator Scheme



α-Congestion Approximator

Definition:

$$\|Rb\|_\infty \leq \text{opt}(b) \leq \alpha \|Rb\|_\infty, \quad \text{for all demands } b$$

opt(b): Minimum Congestion for b

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opt(b): Minimum Congestion for b

Equivalent iff $Rb \neq 0$:

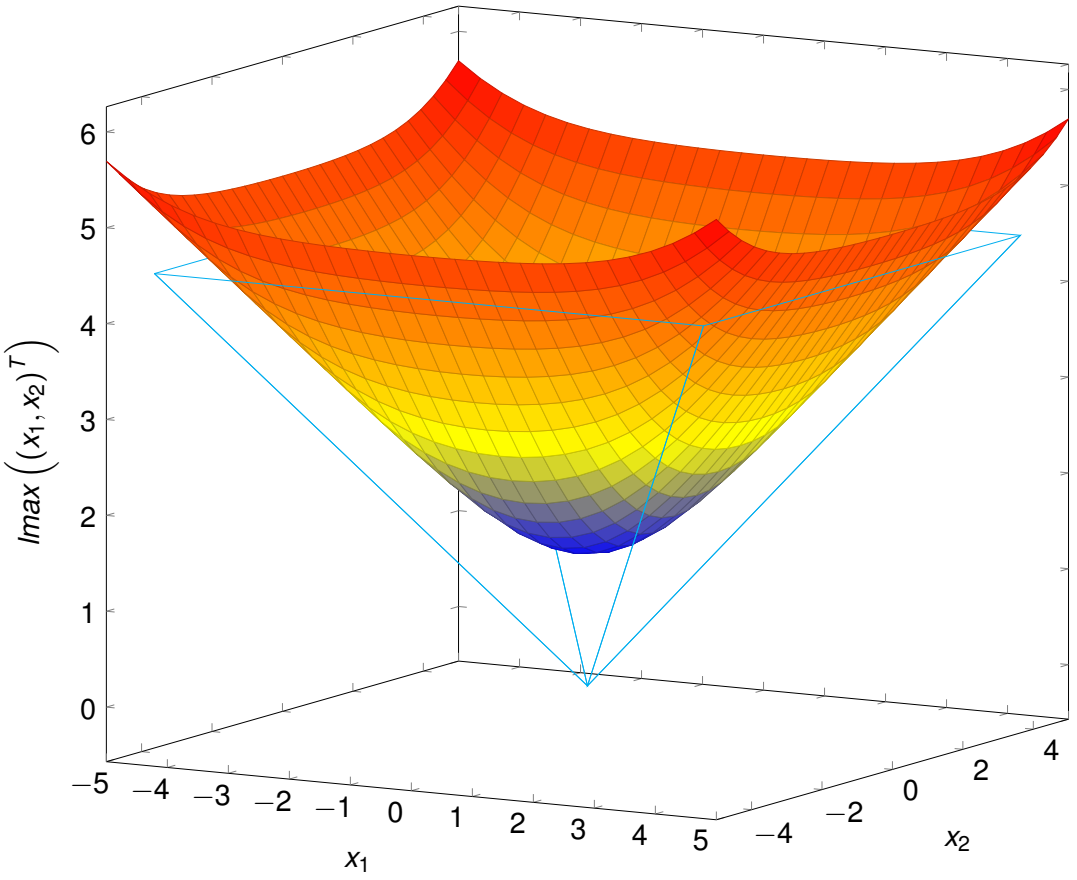
$$1 \leq \frac{\text{opt}(b)}{\|Rb\|_\infty} \leq \alpha, \quad \text{for all demands } b$$

Potential Function

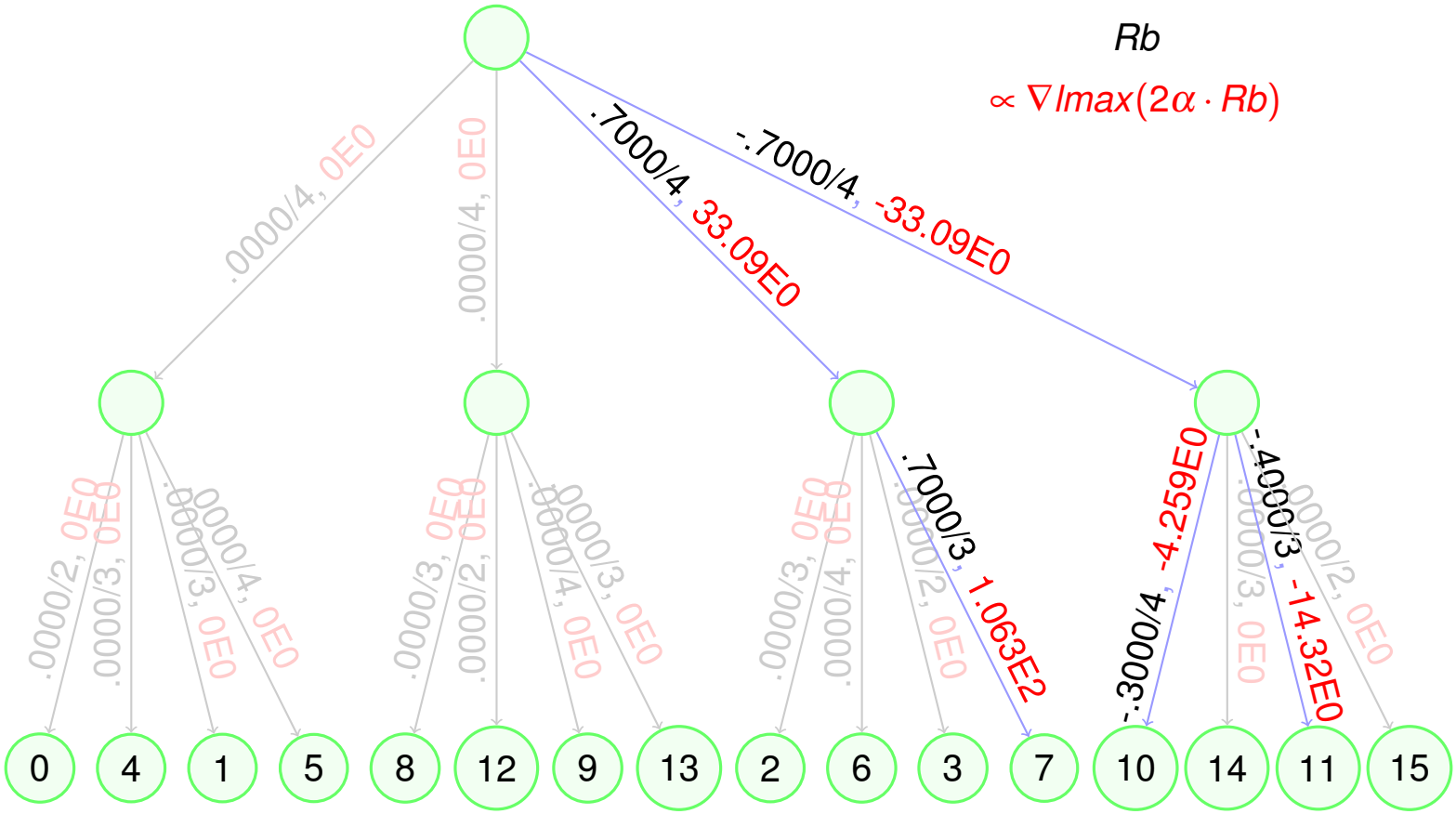
$$\Phi(f) := \left\| C^{-1} f \right\|_\infty + 2\alpha \left\| R(b - Bf) \right\|_\infty$$
$$\phi(f) := lmax\left(C^{-1} f\right) + lmax\left(2\alpha \cdot R(b - Bf)\right)$$
$$\nabla\phi(f) = \left(C^{-1}\right)^T \cdot \nabla lmax\left(C^{-1} f\right) - 2\alpha \cdot B^T R^T \cdot \nabla lmax(2\alpha \cdot R(b - Bf))$$

$$lmax(\vec{x}) := ln\left(\sum_i e^{x_i} + e^{-x_i}\right)$$
$$\left(\nabla lmax(\vec{x})\right)_j = \frac{e^{x_j} - e^{-x_j}}{\sum_i e^{x_i} + e^{-x_i}}$$

Potential Function



Congestion Approximator



Grid Graphs

Grid Graphs

- Undirected

Grid Graphs

- Undirected
- Unit-Capacity

Grid Graphs

- Undirected
- Unit-Capacity
- Multi-Dimensional

Grid Graphs

- Undirected
- Unit-Capacity
- Multi-Dimensional
- $\rightarrow$ can be defined via dimensional node count vector $(n_1, ..., n_d)$

Grid Graphs

Node count:

$$n = \prod_{i=1}^{d} n_i$$

Edge count:

$$m = \left(\sum_{i=1}^{d} \frac{n_i - 1}{n_i} \right) \cdot n$$

$$\rightarrow m \in \Theta(d \cdot n)$$

Grid Graphs

Capacity of hypercube cut $([a_1, b_1], \ldots, [a_d, b_d])$:

$$\sum_{i=1}^{d} \left((\overline{\delta}_{a_i,0} + \overline{\delta}_{b_i,n_i-1}) \cdot \prod_{j=1, j\neq i}^{d} (b_j - a_j + 1) \right)$$

with $\overline{\delta}_{x,y} = 1 - \delta_{x,y}$ (Kronecker-Delta)

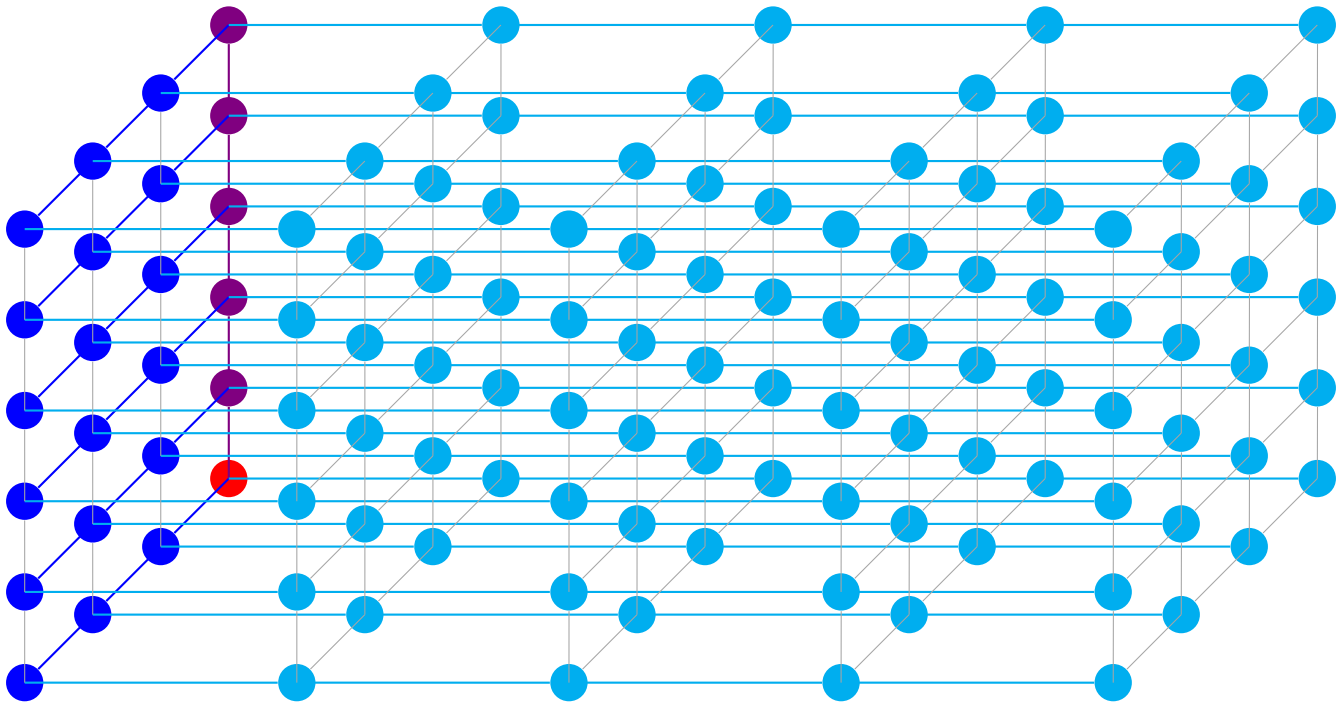
Algorithm

1: **procedure** CompleteRoute(b, ε)
2: $r \leftarrow b$
3: $(f_0, S_0) \leftarrow \texttt{AlmostRoute}(r, \varepsilon)$
4: **for** $i \leftarrow 1, \log_2(2m)$ **do**
5: $r \leftarrow r - Bf_{i-1}$
6: $(f_i, S_i) \leftarrow \texttt{AlmostRoute}(r, 1/2)$
7: **end for**
8: $r \leftarrow r - Bf_i$ ▷ Assuming $i = log(2m)$ after the loop.
9: $f_{i+1} \leftarrow \texttt{GridMST.route}(r)$
10: **return** $\sum_{j=0}^{i+1} f_j$
11: **end procedure**

Algorithm

1: **procedure** ALMOSTROUTE(b, ε)
2: **repeat**
3: **while** $\phi(f) < 16\varepsilon^{-1}\log(n)$ **do**
4: $f \leftarrow \frac{17}{16} \cdot f$
5: $b \leftarrow \frac{17}{16} \cdot b$
6: **end while**
7: $\delta \leftarrow \|C\nabla\phi(f)\|_1$
8: **if** $\delta \geq \varepsilon/4$ **then**
9: $f_e \leftarrow f_e - \frac{\delta}{1+4\alpha^2}\text{sgn}(\nabla\phi(f)_e)c_e$
10: **end if**
11: **until** $\delta < \varepsilon/4$
12: **return** f, $\nabla\phi(f)$
13: **end procedure**

Maximal Spanning Tree



MST Routing

Algorithm 1 Routing Demand b through a (Maximal) Spanning Tree of Grid Graph G

```
1: procedure RouteMST(GridGraph G, GridDemand b)
2:     f ← 0
3:     r ← b                                                    ▷ Residual demand
4:     (n_1, ..., n_d) ← G.nodesPerDimension
5:     for i ← 1, d do
6:         for j ← 1, i − 1 do                                  ▷ Set v_start and v_end properly
7:             (v_start)_j ← 0
8:             (v_end)_j ← 0
9:         end for
10:        (v_start)_i ← n_i − 1
11:        (v_end)_i ← 1
12:        for j ← i + 1, d do
13:            (v_start)_j ← n_j − 1
14:            (v_end)_j ← 0
15:        end for
16:        for v ← v_start, v_end do
17:            u ← v
18:            u_i ← v_i − 1
19:            e ← (u, v)
20:            f_e ← r_v                                        ▷ r_v: Residual demand of leaf v. Every iterated v is a leaf iff the iteration is valid.
21:            r_u ← r_u + r_v
22:            r_v ← 0
23:        end for
24:    end for
25:    return f
26: end procedure
```

MST Routing

Algorithm 2 Simplified Version

```
1:  procedure RouteMST2(GridGraph G, GridDemand b)
2:      f ← 0
3:      r ← b                                          ▷ Residual demand
4:      i ← 1
5:      for k ← n − 1, 1 do
6:          v⃗ ← toVector(k)
7:          while v⃗_i = 0 do
8:              i ← i + 1
9:          end while
10:         v⃗_i ← v⃗_i − 1
11:         u ← toIndex(v⃗)
12:         e ← (u, k)
13:         f_e ← r_k
14:         r_u ← r_u + r_k
15:         r_k ← 0
16:     end for
17:     return f
18: end procedure
```

Implementation

Implementation

- Basic DS & Algorithm(s)

Implementation

- Basic DS & Algorithm(s)
- Step Size Optimization

Implementation

- Basic DS & Algorithm(s)
- Step Size Optimization
- Empiric Search for α

Step Size Optimization

Step Size Optimization

- Default: $f_i \leftarrow f_i - s_i$

Step Size Optimization

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- Approach: $f_i \leftarrow f_i - h \cdot s_i$

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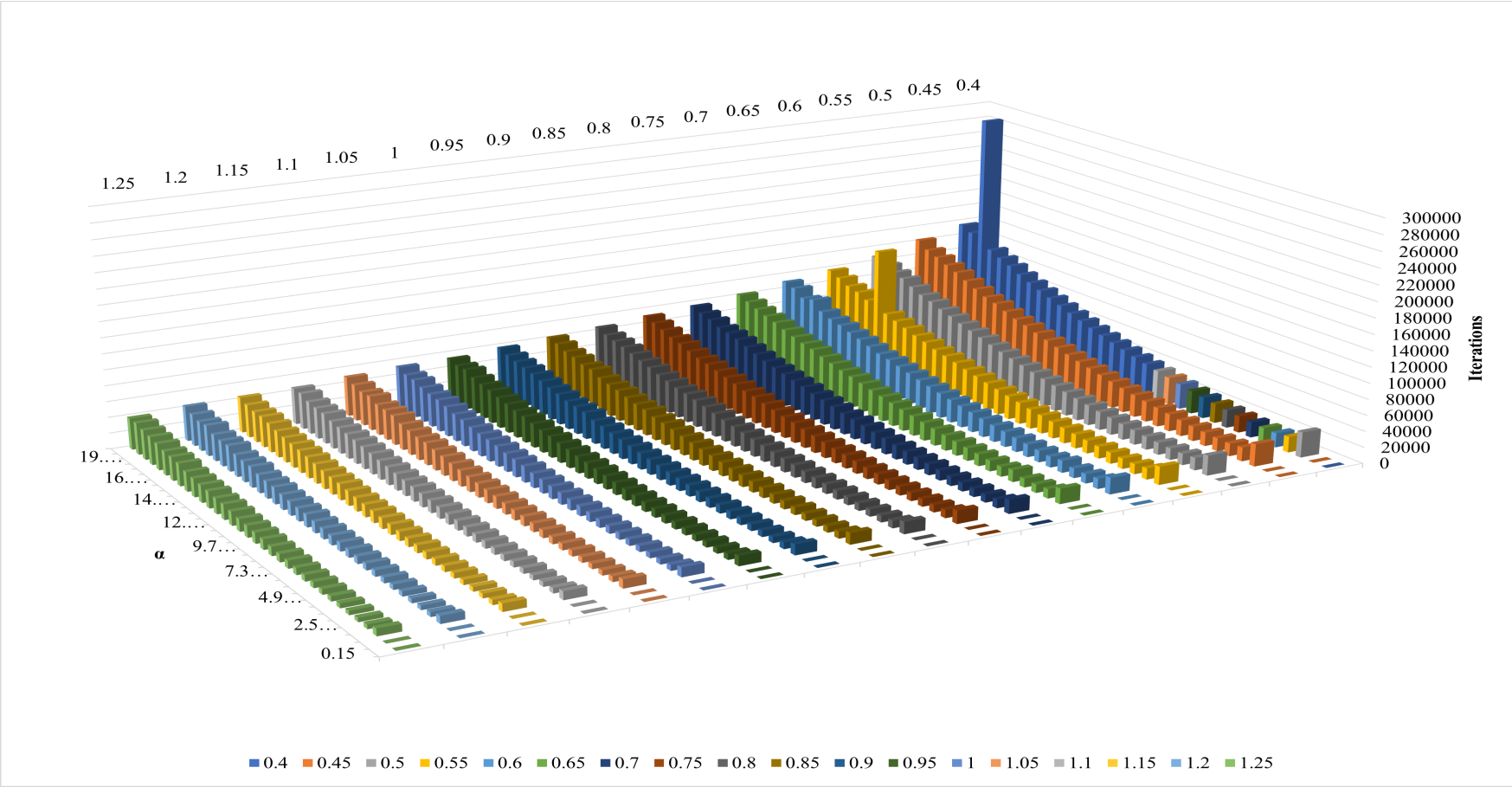
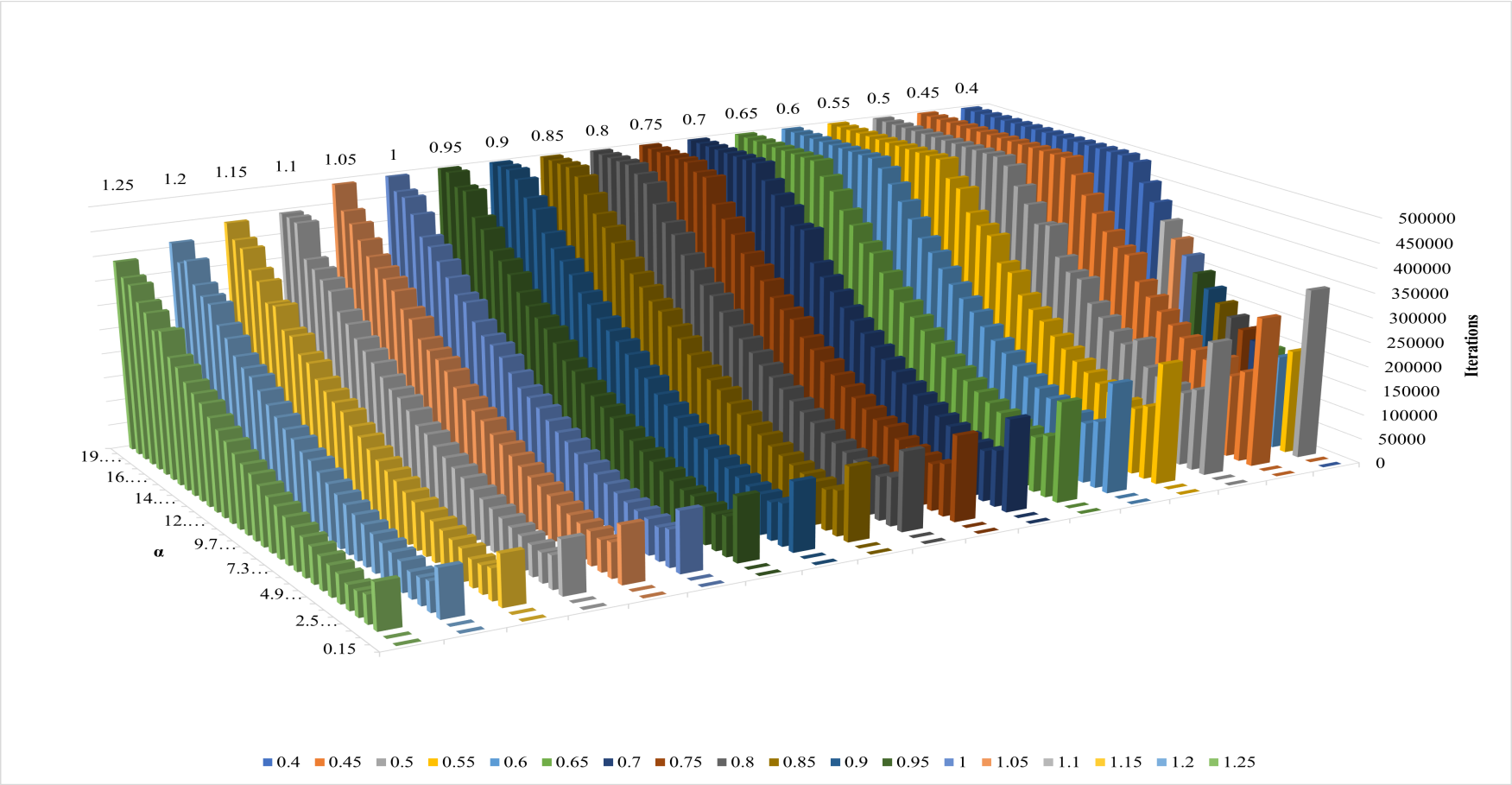
$$\frac{\vartheta}{\vartheta h}\phi(f - h \cdot s) = (-C^{-1}s)^T \cdot \nabla lmax\left(C^{-1} \cdot (f - h \cdot s)\right) + (2\alpha RBs)^T \cdot \nabla lmax\left(2\alpha R(b - B(f - h \cdot s))\right)$$

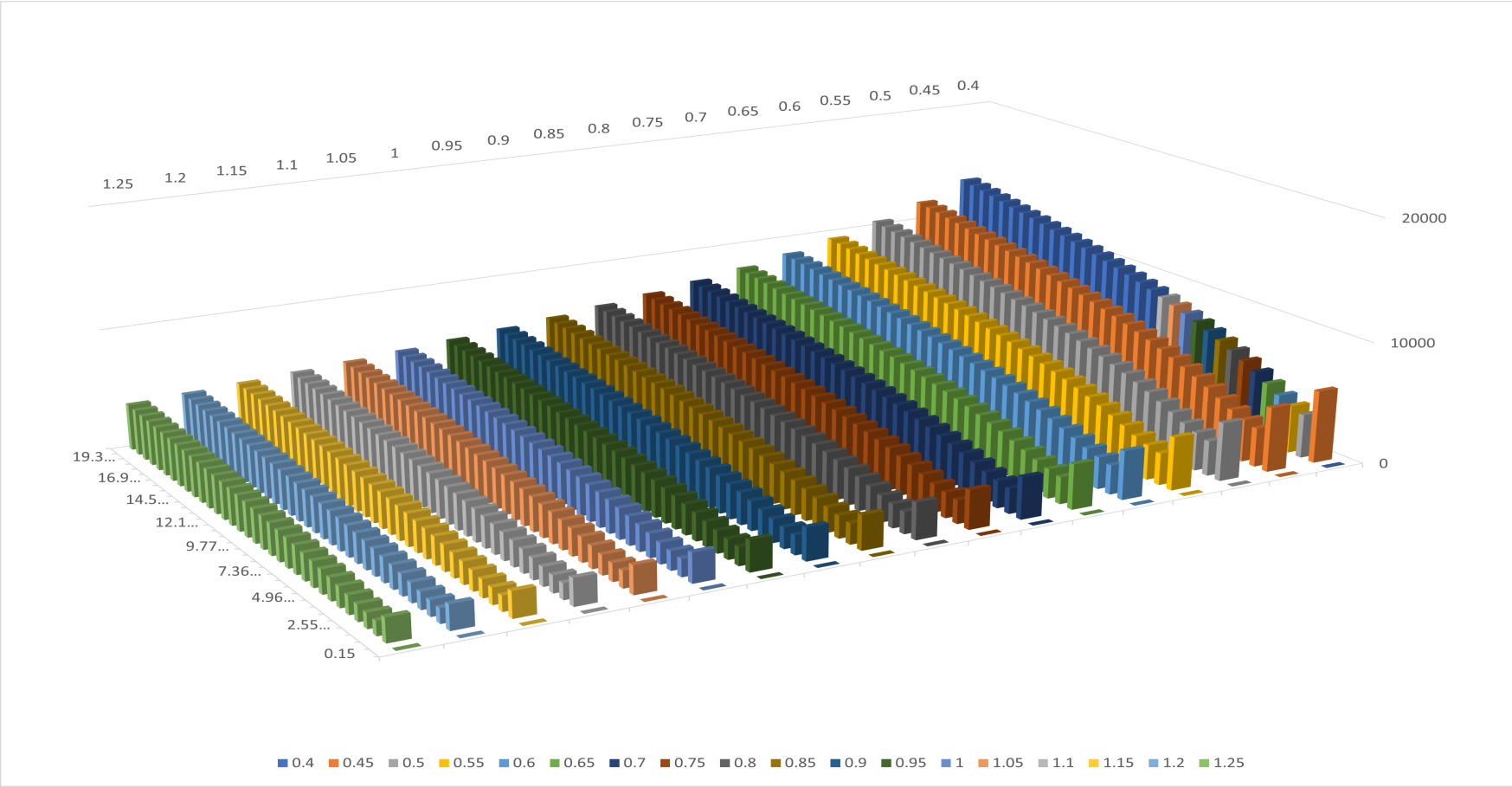
Static Optimization



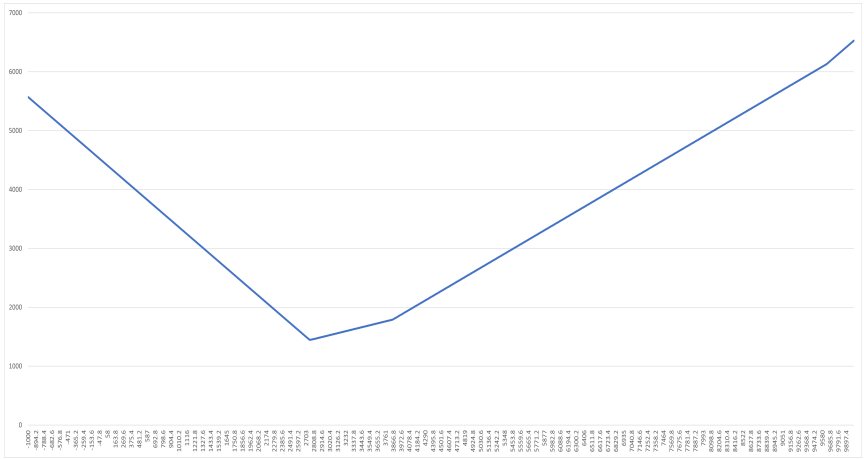
Static Optimization

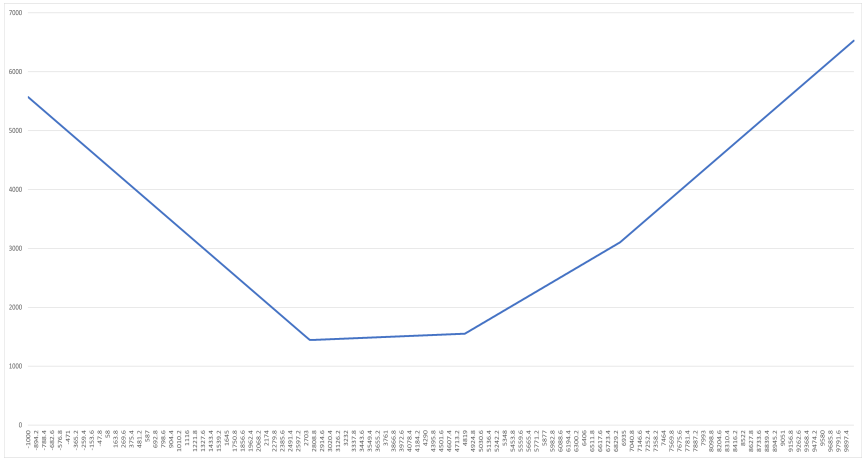


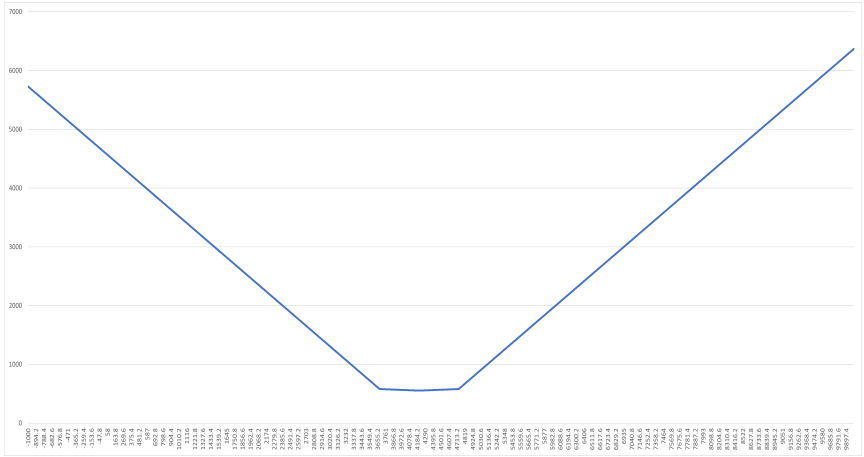
Static Optimization



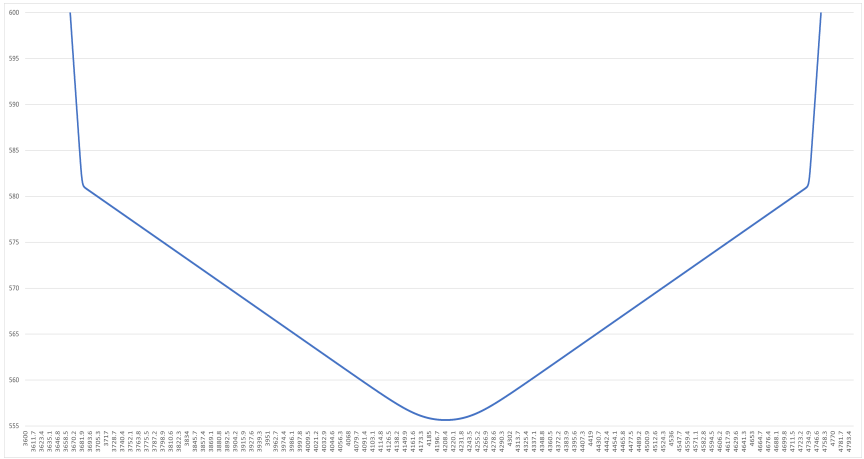
$$\phi(f - h \cdot s)$$







$$\phi(f - h \cdot s)$$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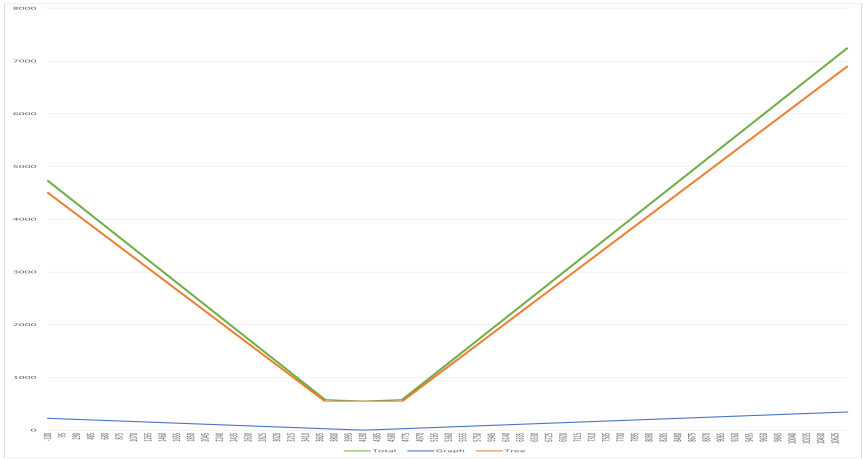
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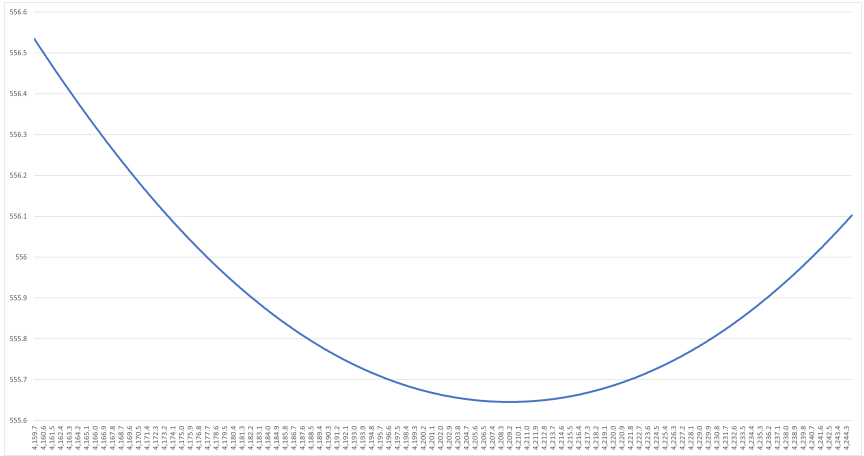
5000
4000
3000
2000
1000
0
Total
Graph
Tree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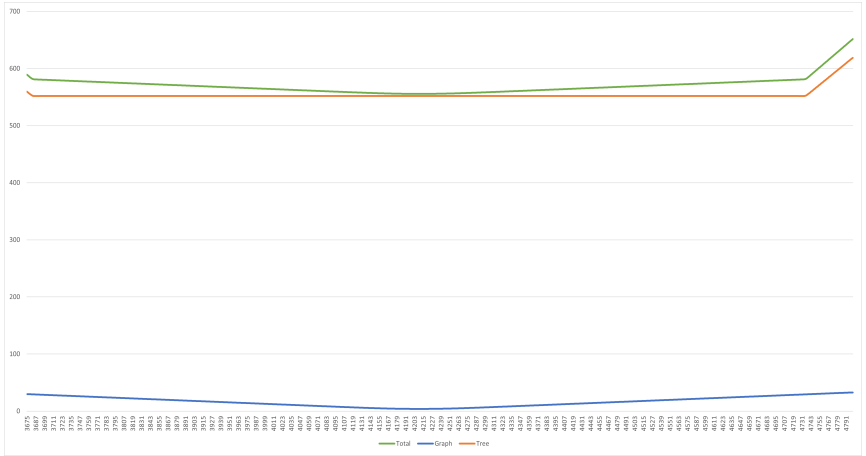
555.6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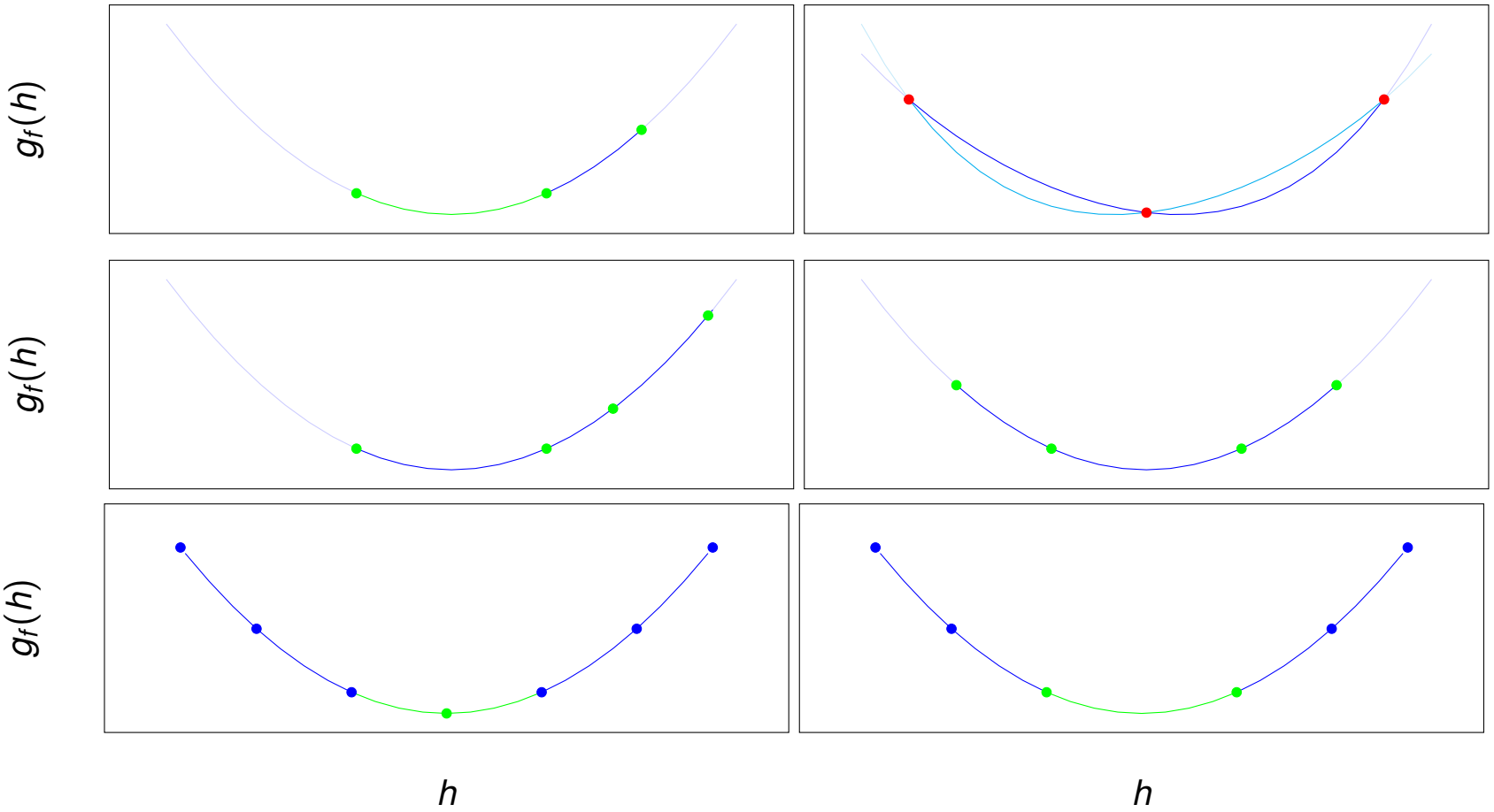
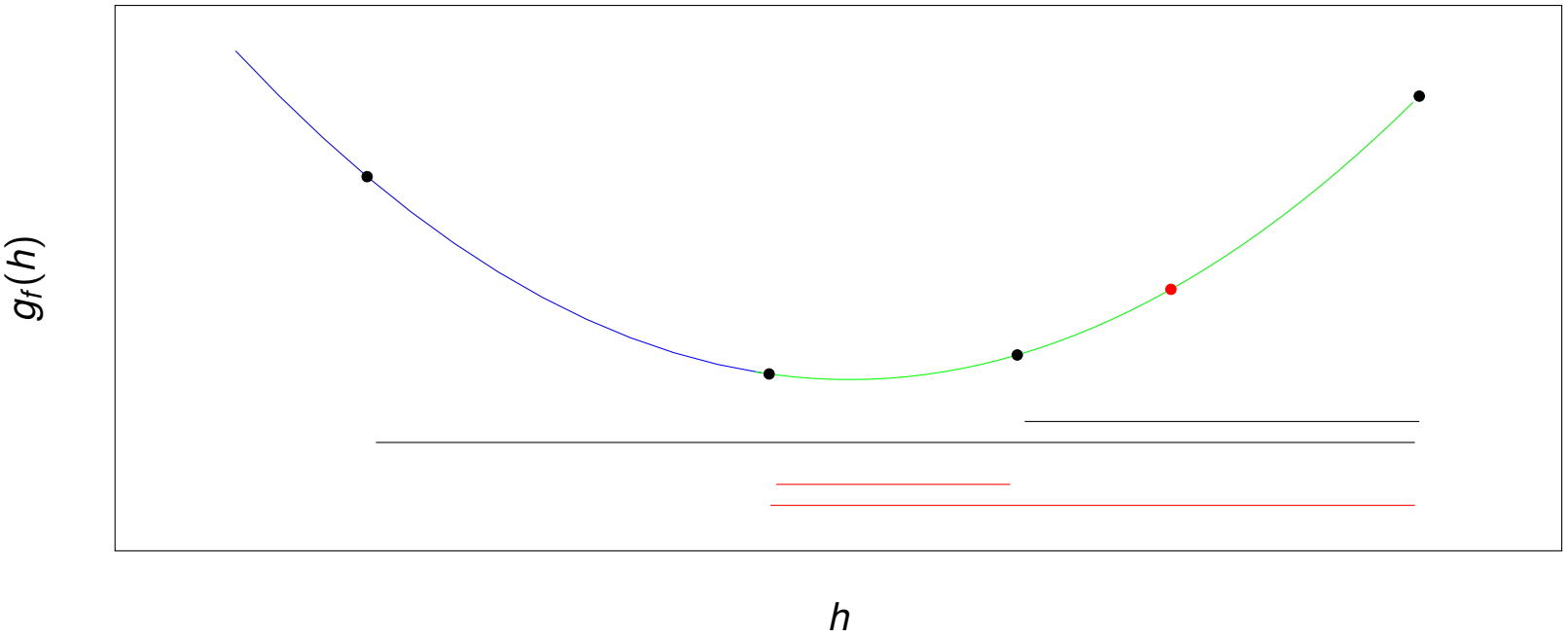
600
500
400
300
200
100
0
Total
Graph
Tree

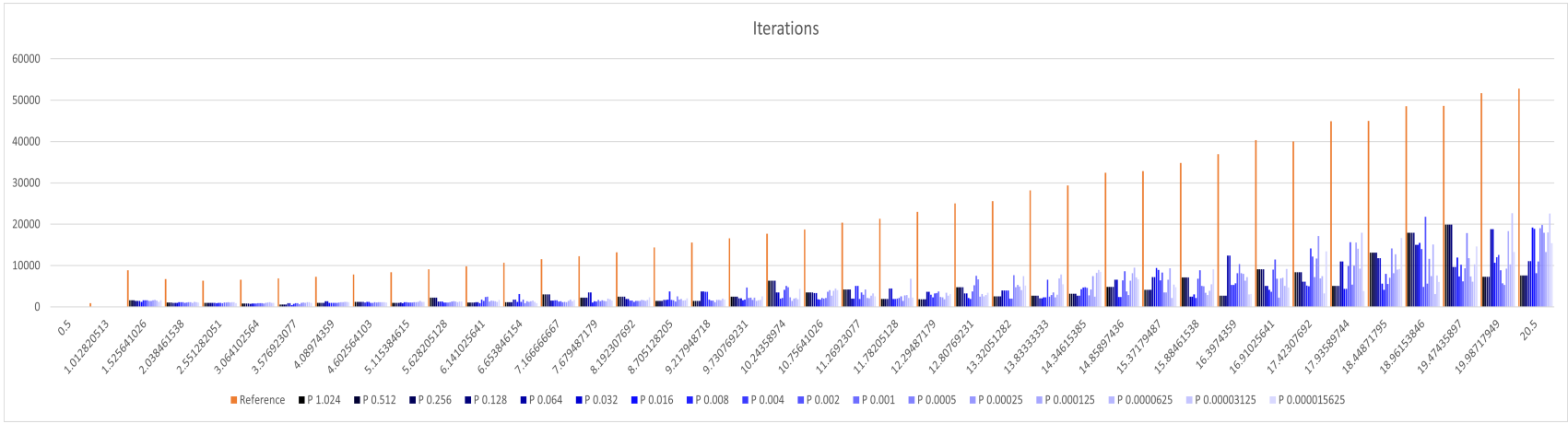
Line Search

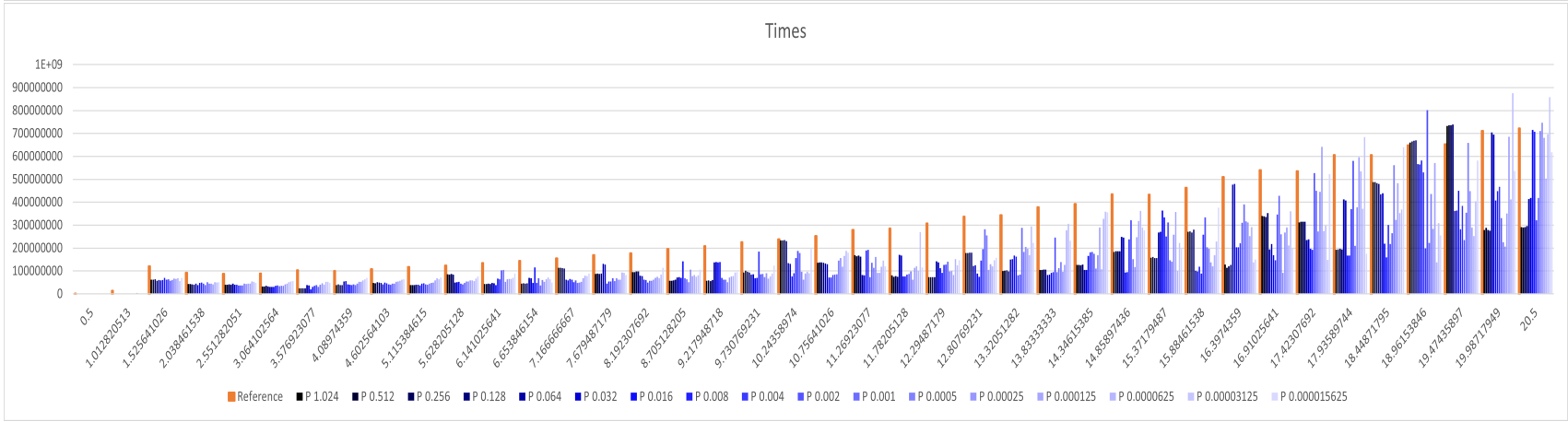


Golden Section Search

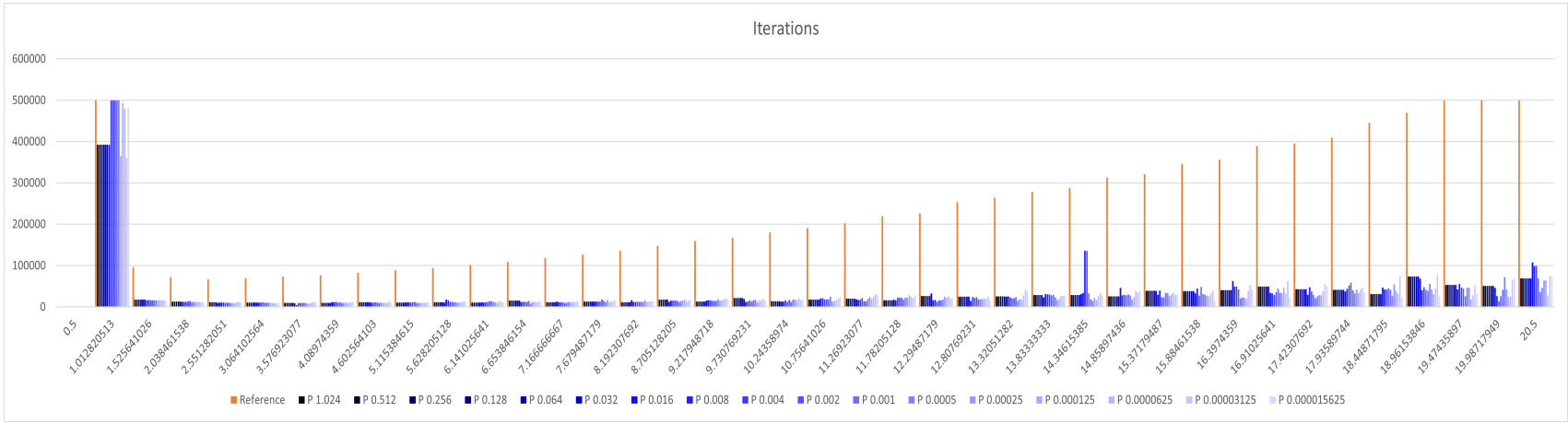


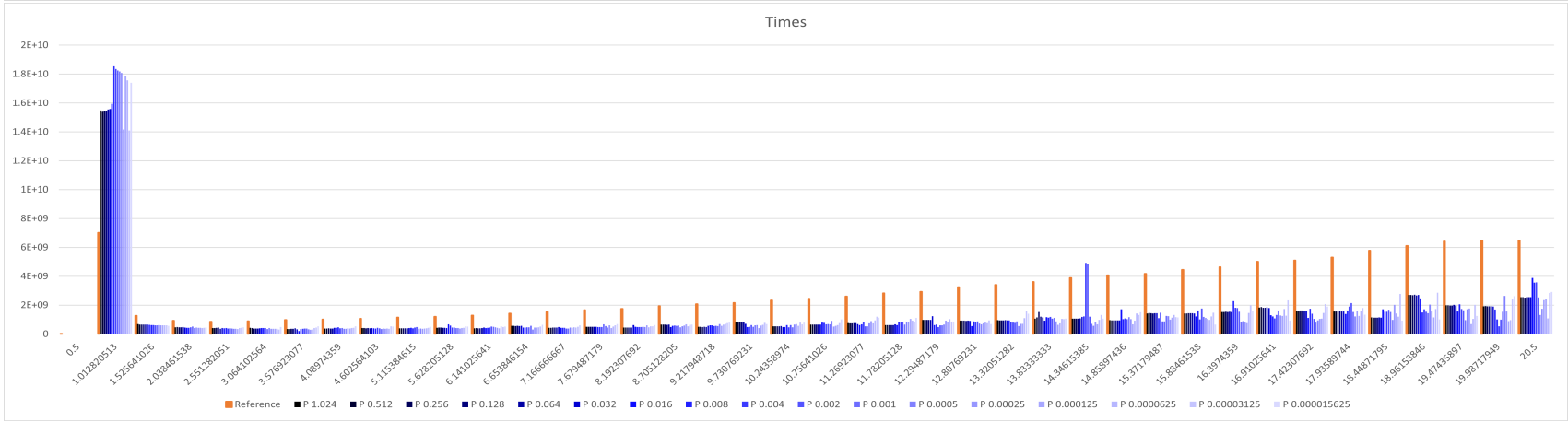
Dynamic Optimization Results



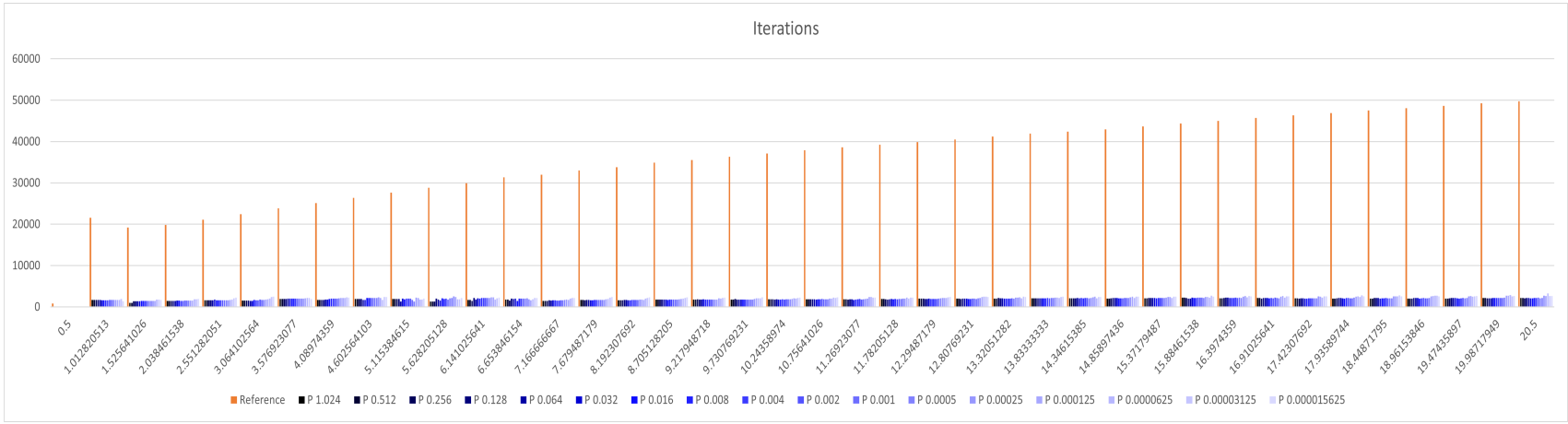


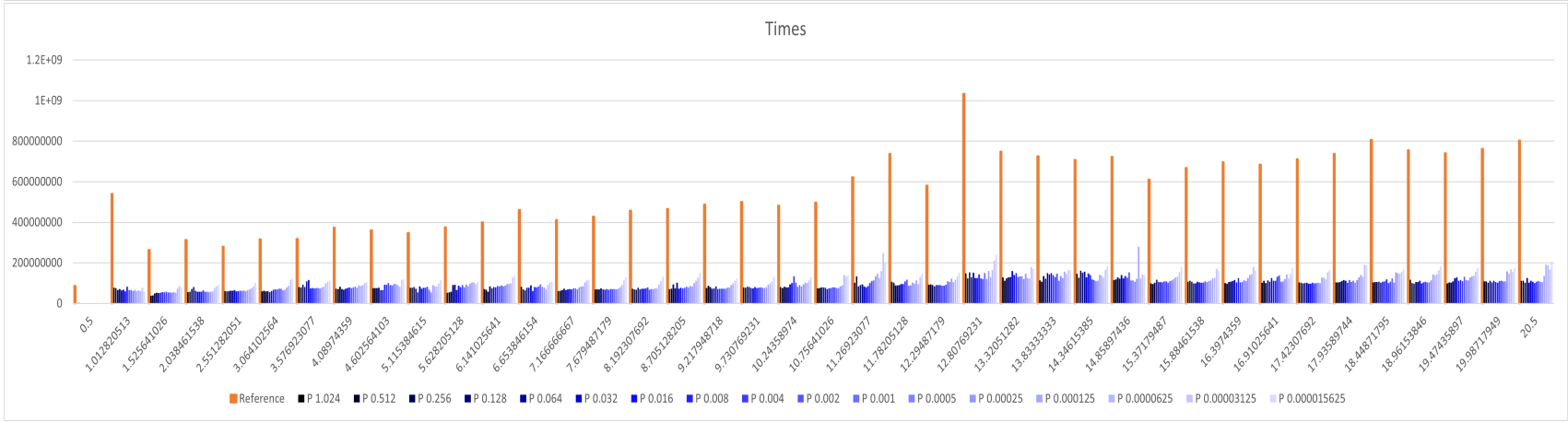
Dynamic Optimization Results





Dynamic Optimization Results





Empiric Search for α

Empiric Search for α

- Random Sampling for 1D grid graphs

Empiric Search for α

- Random Sampling for 1D grid graphs
- 1D-sampling usable in higher dimensions

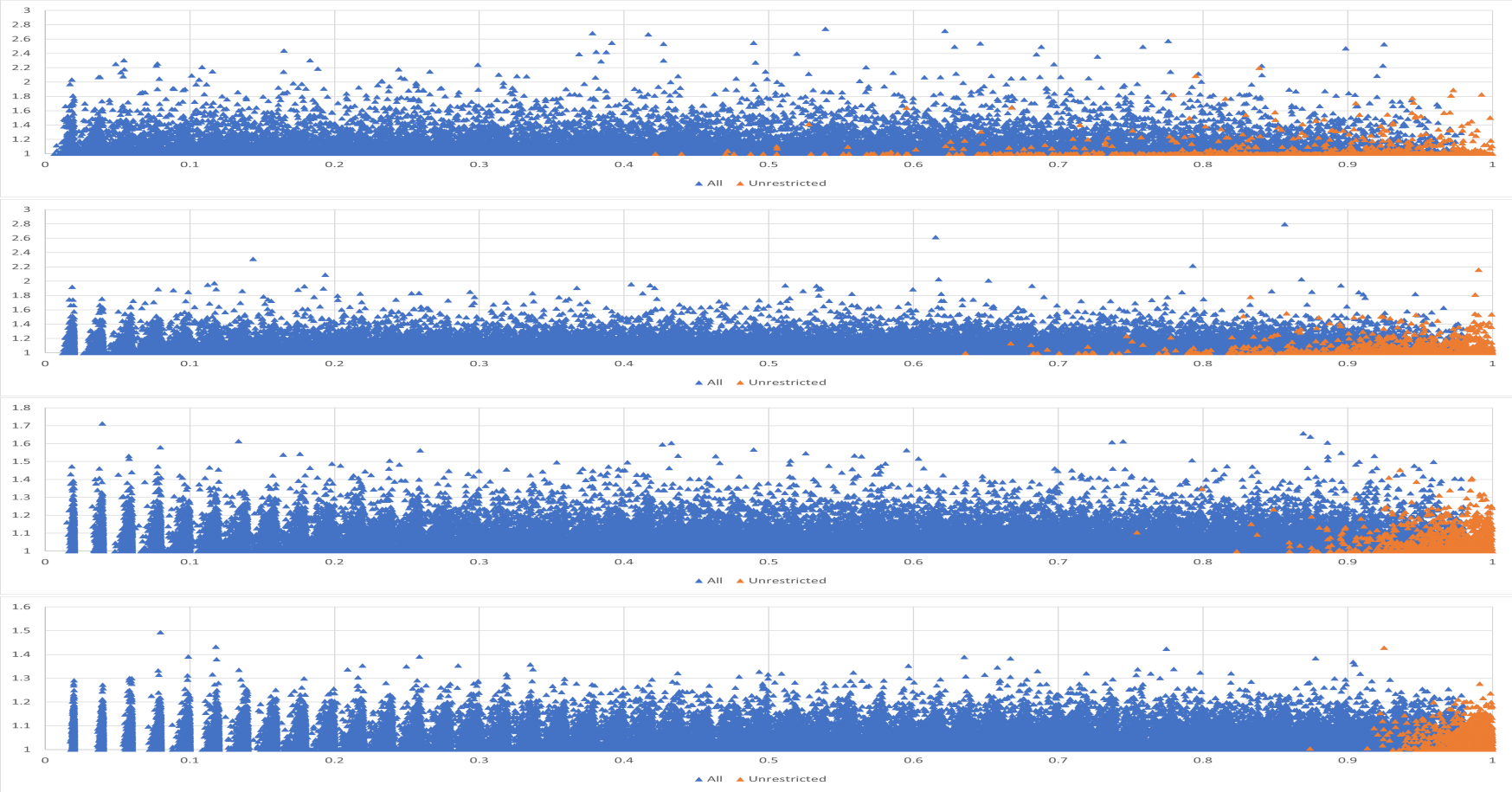
Empiric Search for α

- Random Sampling for 1D grid graphs
- 1D-sampling usable in higher dimensions
- Randomized Cut with Maximum Congestion

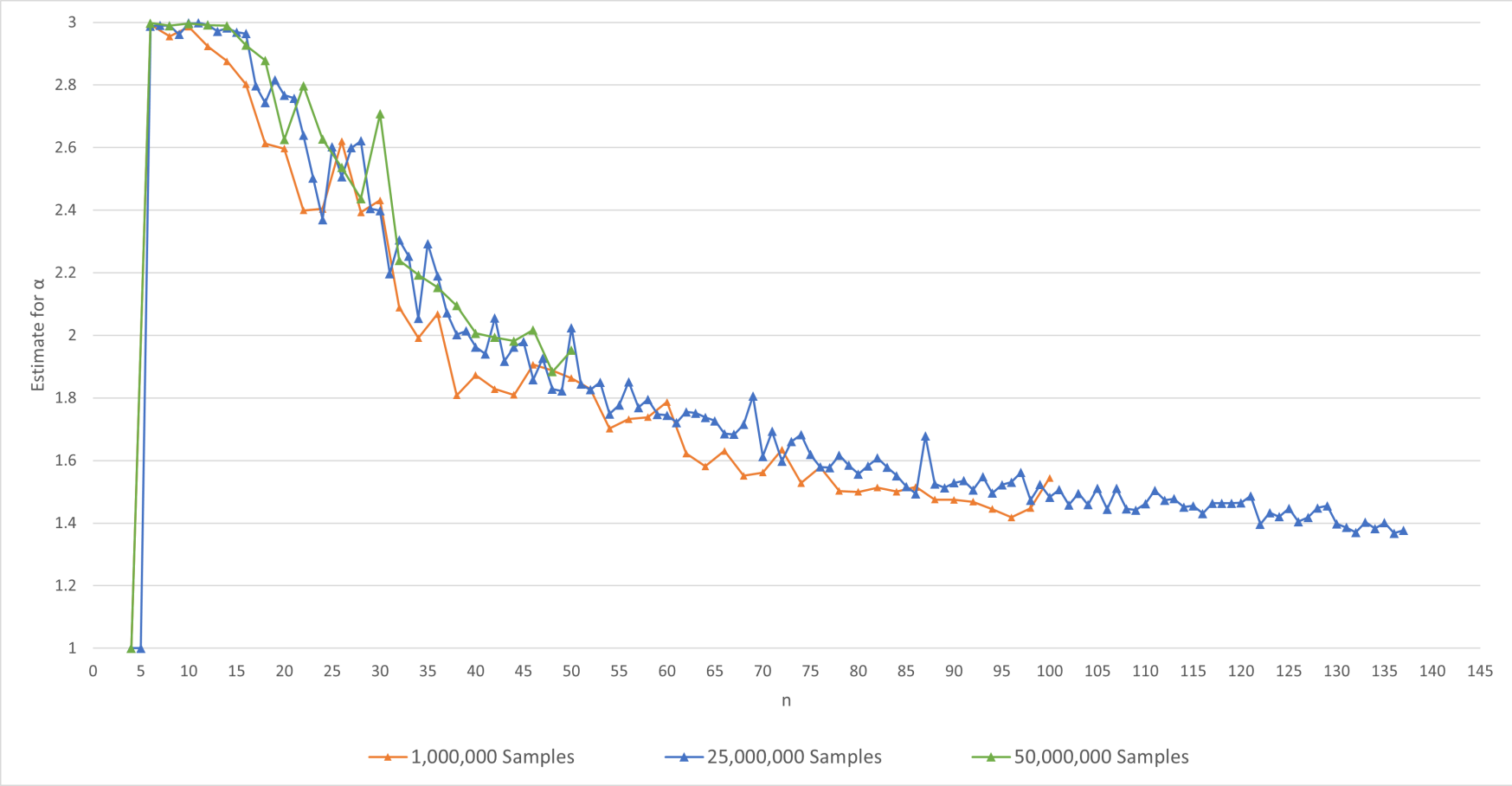
Empiric Search for α

- Random Sampling for 1D grid graphs
- 1D-sampling usable in higher dimensions
- Randomized Cut with Maximum Congestion
- Pre-Calculating Optimal Maximum Congestion

1D-Sampling for α



1D-Sampling for α



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- Master's Thesis: `https://github.com/js97/approximate-flows/blob/main/Thesis-Paper/Paper.pdf`
- Sherman's Approximative Maxflow Algorithm [She13]: Jonah Sherman. "Nearly Maximum Flows in Nearly Linear Time". In: *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*. 2013, pp. 263-269. DOI: 10.1109/FOCS.2013.36.