ME 397- ASBR

Week 3-Lecture 1

a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

FARSHID ALAMBEIGI, Ph.D.

Assistant Professor | Walker Department of Mechanical Engineering

Cockrell School of Engineering | The University of Texas at Austin

The University of Texas at Austin
Mechanical Engineering
Cockrell School of Engineering

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Euler Angles Representation of rotation matrix

- Rotation matrices give a **redundant description** of frame orientation.
- They are characterized by **nine elements** which are not independent but related by six constraints due to the orthogonality conditions, i.e., **column vectors r_i** are **mutually perpendicular** and have **magnitude equal to 1.**

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad r_i^T r_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

- This implies that ***three parameters* are sufficient** to describe orientation of a rigid body in space (is it correct for axis-angle representation?)
- A minimal representation of orientation can be obtained by using a set of **three angles** $\varphi = [\phi\ \vartheta\ \psi]^T$.

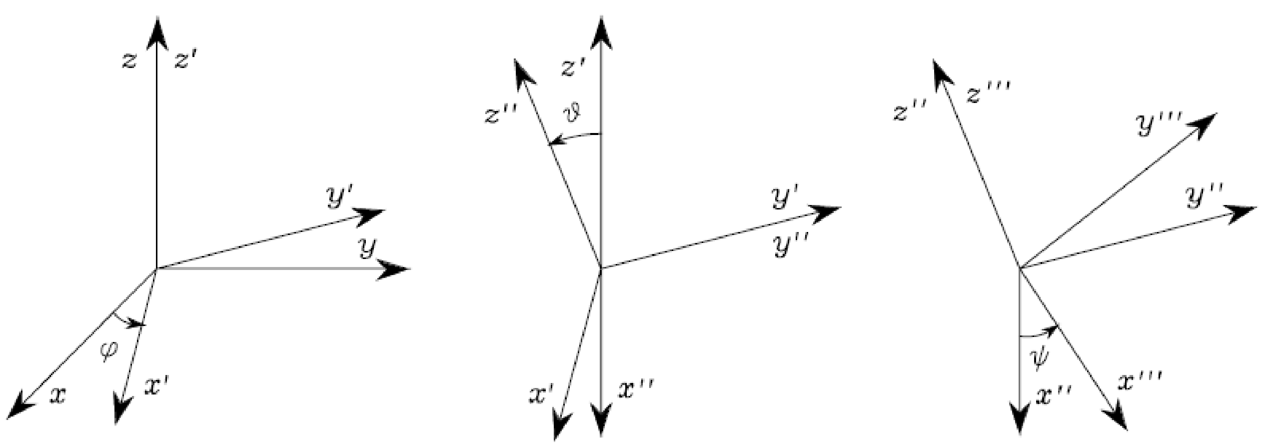
- A **generic rotation matrix** can be obtained by **composing a suitable sequence of three elementary rotations** (i.e., rotations about x, y, or z axis) while guaranteeing that **two successive rotations** are not made about parallel axes.

- This implies that **12 distinct sets of angles** are allowed out of all 27 possible combinations; each set represents **a triplet of Euler angles.**

ZYZ Euler Angles

- The rotation described by ZYZ angles is obtained as composition of the following elementary rotations:
  - ✓ First: Rotate the reference frame by the angle $\boldsymbol{\phi}$ about axis $\boldsymbol{z}$; this rotation is described by the matrix $\boldsymbol{R}_z(\phi)$
  - ✓ Second: Rotate the **current frame** by the angle $\boldsymbol{\vartheta}$ about axis **y'**; this rotation is described by the matrix $\mathbf{R}_{y'}(\boldsymbol{\vartheta})$
  - ✓ Third: Rotate the current frame by the angle ψ about axis **z''**; this rotation is described by the matrix $\mathbf{R}_{z''}(\psi)$
- The resulting frame orientation is obtained by composition of rotations via **post-multiplication**:

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$



$$R_x(\phi) := e^{\hat{x}\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

$$R_y(\beta) := e^{\hat{y}\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$R_z(\alpha) := e^{\hat{z}\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

ZYZ Euler Angles

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & \boxed{c_\varphi s_\vartheta} \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & \boxed{s_\varphi s_\vartheta} \\ \boxed{-s_\vartheta c_\psi} & \boxed{s_\vartheta s_\psi} & \boxed{c_\vartheta} \end{bmatrix}$$

➢ We are interested in solving the *inverse problem*, that is to **determine the set of Euler angles corresponding to a given rotation matrix.**

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

The function **Atan2(*y*, *x*)** computes the arctangent of the ratio ***y*/*x*** but utilizes the sign of each argument to determine which quadrant the resulting angle belongs to.

$r_{13} \neq 0$ and $r_{23} \neq 0$

$$\boxed{\varphi = \text{Atan2}(r_{23}, r_{13})}$$

Squaring and summing the elements [1, 3] and [2, 3] and using the element [3, 3]:

$$\boxed{\vartheta = \text{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right) \quad \vartheta \in (0, \pi).}$$

elements [3, 1] and [3, 2]

$$\boxed{\psi = \text{Atan2}(r_{32}, -r_{31}).}$$

ZYZ Euler Angles

- Choosing ϑ in the range $(-\pi, 0)$ leads to

$$R(\phi) = R_z(\varphi)R_{y'}(\vartheta)R_{z''}(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & \boxed{c_\varphi s_\vartheta} \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & \boxed{s_\varphi s_\vartheta} \\ \boxed{-s_\vartheta c_\psi} & \boxed{s_\vartheta s_\psi} & \boxed{c_\vartheta} \end{bmatrix}$$

$$\varphi = \text{Atan2}(-r_{23}, -r_{13})$$

$$\vartheta = \text{Atan2}\left(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \text{Atan2}(-r_{32}, r_{31}).$$

- As in the case of the exponential map, the map from **(α, β, γ) → SO(3) is surjective!**
- when $s_\vartheta = 0$; in this case, it is **possible to determine only the sum or difference of ϕ and ψ.** In fact, **if $\vartheta = 0, \pi$**, the successive rotations of ϕ and ψ are made about axes of current frames which are parallel, thus giving equivalent contributions to the rotation **(Singularity).** $R(\phi) = R_z(\varphi)R_{y'}(\vartheta)R_{z''}(\psi)$
- As in the case of the angle/axis representation, singularities in the parameterization (i.e., the lack of existence of global, smooth solutions to the inverse problem of determining the Euler angles from the rotation) occur at **R = I, the identity rotation.**

- There are **infinitely many representations** of the identity rotation in the ZYZ Euler angles parameterization in the form of **Rot(α, 0,−α) = I!**

ZYX (Roll-Pitch-Yaw) Euler Angles

- **Roll–Pitch–Yaw angles** *are a*nother set of Euler angles originates from a representation of orientation in the (aero)nautical field.
- $\phi = [\varphi\ \vartheta\ \psi]^T$ represent rotations **defined with respect to a fixed frame** attached to the center of mass of the craft.

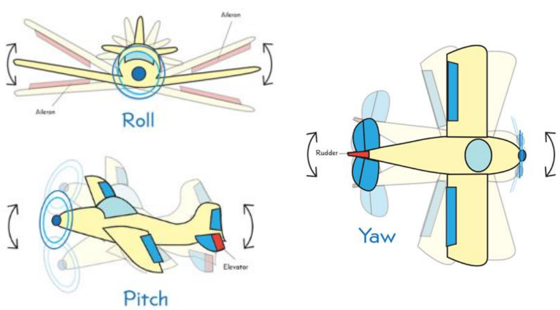
- The rotation resulting from Roll–Pitch–Yaw angles can be obtained as follows:
  - ✓ **First:** Rotate the reference frame by the **angle ψ about axis x (yaw);** this rotation is described by the matrix $\boldsymbol{R}_x(\psi)$.
  - ✓ **Second:** Rotate the reference frame by the **angle ϑ about axis y (pitch);** this rotation is described by the matrix $\mathbf{R}_y(\vartheta)$.
  - ✓ **Third:** Rotate the reference frame by the **angle ϕ about axis z (roll);** this rotation is described by the matrix $\mathbf{R}_z(\phi)$.

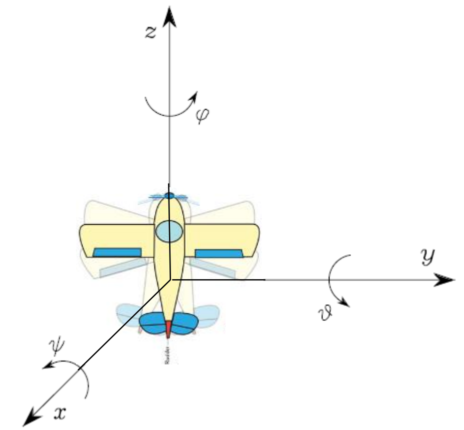
$$R_{\mathbf{x}}(\phi) := e^{\hat{\mathbf{x}}\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

$$R_{\mathbf{y}}(\beta) := e^{\hat{\mathbf{y}}\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$R_{\mathbf{z}}(\alpha) := e^{\hat{\mathbf{z}}\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi)$$





https://howthingsfly.si.edu/flight-dynamics/roll-pitch-and-yaw

ZYX (Roll-Pitch-Yaw) Euler Angles

$$
\begin{aligned}
\boldsymbol{R}(\phi) &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi) \\
&= \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}
\end{aligned}
$$

$$
\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.
$$

ϑ in the range (−π/2, π/2)

$$
\begin{aligned}
\varphi &= \mathrm{Atan2}(r_{21}, r_{11}) \\
\vartheta &= \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right) \\
\psi &= \mathrm{Atan2}(r_{32}, r_{33}).
\end{aligned}
$$

ϑ in the range (π/2, 3π/2)

$$
\begin{aligned}
\varphi &= \mathrm{Atan2}(-r_{21}, -r_{11}) \\
\vartheta &= \mathrm{Atan2}\left(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}\right) \\
\psi &= \mathrm{Atan2}(-r_{32}, -r_{33}).
\end{aligned}
$$

- ZYX Euler angles **do not** have a singularity at the identity orientation, R = I, though they **do contain** singularities when **θ = +/−π/2**.

- It is a fundamental topological fact that **singularities can never be eliminated in any 3-dimensional representation of SO(3).**

Unit Quaternions

- The unit quaternions are an alternative representation of rotations that **alleviates this singularity, but at the cost of having a fourth variable** in the representation.
  - A quaternion is a **vector quantity** of the form

$$Q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} \qquad q_i \in \mathbb{R}, i = 0, \ldots, 3,$$

where $\mathbf{q_0}$ is the **scalar component** of Q and $\vec{q} = (q_1, q_2, q_3)$ is the **vector** component and **i**, **j** and **k** are the **orthogonal complex numbers**.

- A convenient shorthand notation is $Q = (q_0, \vec{q}\,)$ with $q_0 \in R$, $\vec{q} \in R^3$
- **Quaternion Multiplication** denoted by "." is **distributive** and **associative**, but **not commutative** and satisfies the following relations:

$$a\mathrm{i} = \mathrm{i}a \qquad a\mathrm{j} = \mathrm{j}a \qquad a\mathrm{k} = \mathrm{k}a \qquad a \in \mathbb{R}$$

$$\mathrm{i}\cdot\mathrm{i} = \mathrm{j}\cdot\mathrm{j} = \mathrm{k}\cdot\mathrm{k} = \mathrm{i}\cdot\mathrm{j}\cdot\mathrm{k} = -1$$

$$\mathrm{i}\cdot\mathrm{j} = -\mathrm{j}\cdot\mathrm{i} = \mathrm{k} \qquad \mathrm{j}\cdot\mathrm{k} = -\mathrm{k}\cdot\mathrm{j} = \mathrm{i} \qquad \mathrm{k}\cdot\mathrm{i} = -\mathrm{i}\cdot\mathrm{k} = \mathrm{j}$$

Associative Law:

$(a \times b) \times c = a \times (b \times c)$

Distributive Law:

$a \times (b + c) = a \times b + a \times c$

Commutative Law

$a \times b = b \times a$

https://www.youtube.com/watch?v=d4EgbgTm0Bg&t=1365s&ab_channel=3Blue1Brown

Unit Quaternions

- The conjugate of a quaternion $Q = (q_0, \vec{q})$ is $\mathbf{Q}^* = (q_0, -\vec{q})$
- Magnitude of a quaternion satisfies:

$$\|Q\|^2 = Q \cdot Q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

- The inverse of a quaternion is: $Q^{-1} = Q^* / \|Q\|$
- Q= (1, 0) is the identity element for quaternion multiplication.

- Let $Q = (q_0, \vec{q})$ and $P = (p_0, \vec{p})$ be quaternions, where $q_0, p_0 \in R$ are the scalar parts of Q and P and $\vec{q}$, $\vec{p}$ are the vector parts. It can be shown algebraically that the product of two quaternions satisfies:

$$Q \cdot P = (q_0 p_0 - \vec{q} \cdot \vec{p},\ q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p}).$$

inner product cross product

- The **unit quaternions** are the subset of all $Q \in \boldsymbol{Q}$ such that $\|Q\|=1$.
- The unit quaternions also form a <u>group with respect to quaternion multiplication.</u>

Unit Quaternions and Axis-Angle

- **Given a rotation matrix R = exp($\hat{\omega}$θ),** we define the **associated unit quaternion *Q*** as

$$Q = \left(\cos(\theta/2),\, \omega\sin(\theta/2)\right), \qquad q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \omega\sin(\theta/2) \end{bmatrix} \in \mathbb{R}^4$$

- A detailed calculation shows that if Q_{ab} represents a **rotation between frame A and frame B**, and Q_{bc} represents a **rotation between frames B and C**, then the **rotation between A and C** is given by the quaternion:

$$Q_{ac} = Q_{ab} \cdot Q_{bc}.$$

- **Given a unit quaternion** $Q = (q_0, \vec{q})$, we can extract the corresponding axis and angle by:

$$\theta = 2\cos^{-1} q_0 \qquad \omega = \begin{cases} \dfrac{\vec{q}}{\sin(\theta/2)} & \text{if } \theta \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

- Quaternions provide an efficient representation for rotations which **do not suffer from singularities!**

Unit Quaternions

- The elements of Q can also be obtained directly <u>from the entries of a **given R**</u> as follows:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$ for any rotation $\vartheta \in [-\pi, \pi]$:

$$q_0 = \frac{1}{2}\sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

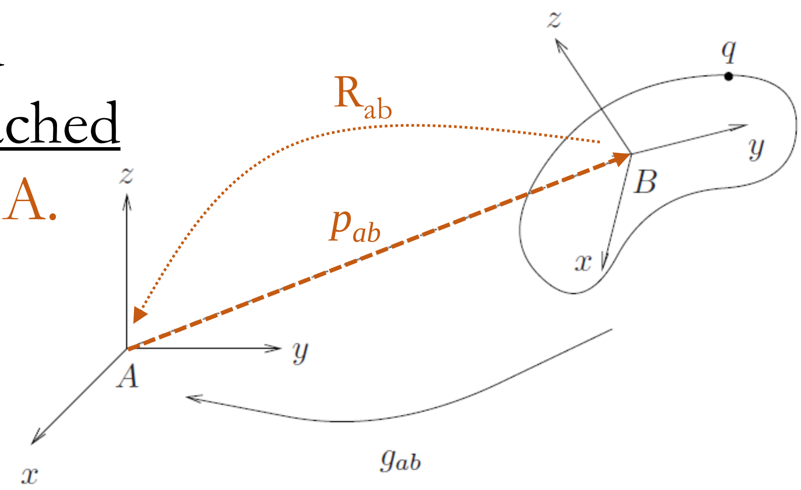
$\operatorname{sgn}(x) = 1$ for $x \geq 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$.

- **Given a unit quaternion (q_0; q_1; q_2; q_3)** the corresponding rotation matrix R is obtained as a **rotation about the unit axis**, in the **direction of (q_1; q_2; q_3)**, by an **angle $2\cos^{-1} q_0$** as:

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Rigid Motion in $\mathbb{R}^3$

- In general, rigid motions consist of **rotation** and **translation**.
- So, we need to describe the **position** and **orientation** of a coordinate frame B attached to the **body** relative to an **inertial** frame A.



- Let $\mathbf{p_{ab}} \in \mathbb{R}^3$ be the position vector of the **origin of frame B** from the **origin of frame A**, and $\mathbf{R_{ab}} \in \mathbf{SO(3)}$ the orientation of frame B, relative to frame A.
- A **configuration of the system/frame** consists of the **pair** $(\mathbf{p_{ab}}, \mathbf{R_{ab}})$.
- **The configuration space of the system** is the **product** space of $\mathbb{R}^3$ with SO(3) i.e., $\mathbb{R}^3 \times SO(3)$
- **Special Euclidean (SE) group:**

$$SE(3) = \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3).$$

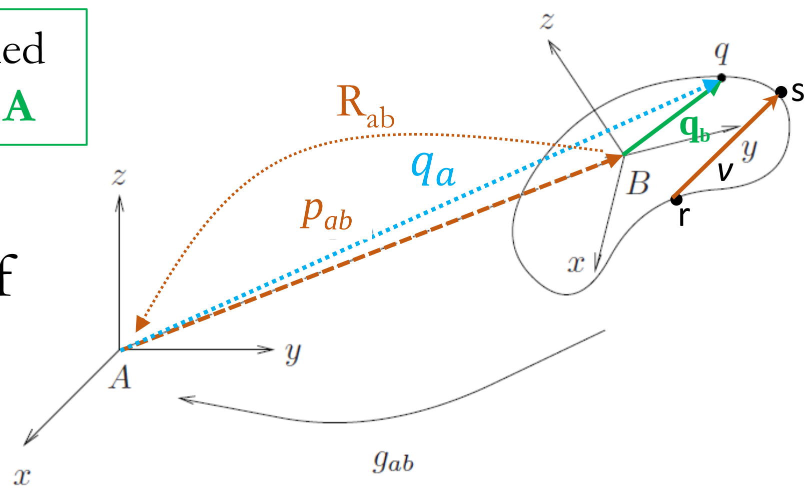
- There is a generalization to **n** dimensions,

$$SE(n) := \mathbb{R}^n \times SO(n).$$

Rigid Motion in $\mathbb{R}^3$

- Let $\mathbf{q}_a, \mathbf{q}_b \in \mathbb{R}^3$ be the coordinates of a point q relative to frames A and B, respectively.
- Given $\mathbf{q}_b$, we want to find $\mathbf{q}_a$ by a transformation of coordinates:

$$q_a = \underbrace{\boxed{p_{ab}}}_{\text{Origin of frame B wrt frame A}} + \underbrace{\boxed{R_{ab}\, q_b}}_{\text{Rotate Point q defined in frame B to frame A}}$$



- $\mathbf{g}_{ab} = (\mathbf{p}_{ab}, \mathbf{R}_{ab}) \in SE(3)$ is the **specification of the configuration** of the B frame relative to the A frame.
- **Rigid transformation g(q)** on a point q:

$$g(q) = p + Rq,$$

so that $\mathbf{q}_a = \mathbf{g}_{ab}(\mathbf{q}_b)$

- Rigid transformation $\mathbf{g} = (\mathbf{p}, \mathbf{R})$ on a vector $\mathbf{v} = \mathbf{s} - \mathbf{r}$:

$$g_*(v) := g(s) - g(r) = R(s - r) = Rv.\ \ \text{Why?!}$$

Homogeneous representation

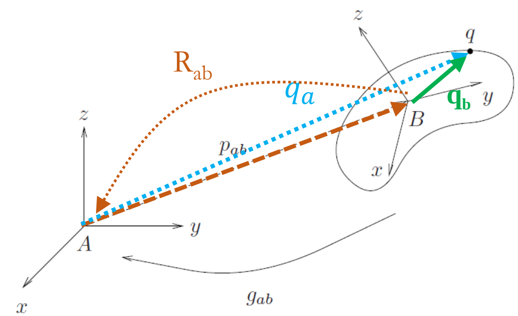
- **Homogeneous coordinates** of the **point q** in $\mathbb{R}^4$: $\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \boxed{1} \end{bmatrix}$ e.g., origin: $\bar{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boxed{1} \end{bmatrix}$

- **Vectors**, which are the difference of points, then have the form: $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \boxed{0} \end{bmatrix}$

- **$q_a = g_{ab}(q_b)$** and **$g(q) = p + Rq$** is an **affine transformation.**
  - We may represent an **affine transformation** in a **linear/homogeneous form** by:

$$\boxed{\bar{q}_a} = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}^{\bar{g}_{ab}} \begin{bmatrix} q_b \\ \boxed{1} \end{bmatrix} =: \boxed{\bar{g}_{ab}}\boxed{\bar{q}_b}.$$

Columns are basis vectors | Origin point | Point q



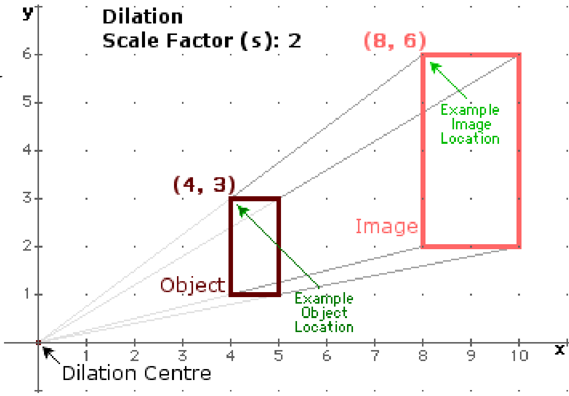
- The 4×4 matrix $\bar{g}_{ab}$ is called the **homogeneous representation** of **$g_{ab} \in SE(3)$.** In general, if $g = (p,R) \in SE(3)$, then

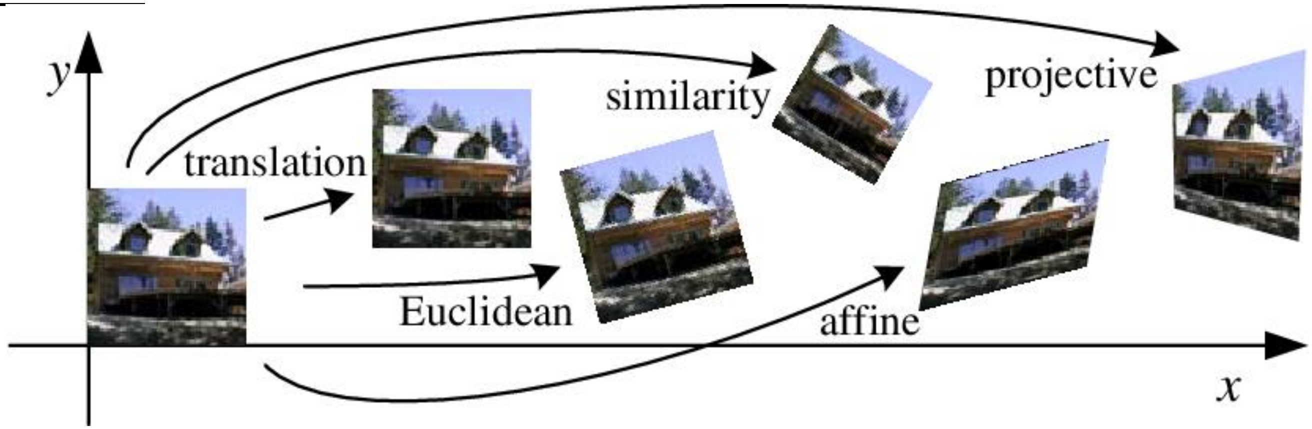
$$\bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Other types of Transformations

- In the **computer graphics/vision applications**, the number 1 in the last row is frequently **replaced by a scalar constant** which is either greater than 1 to represent dilation or less than 1 to represent contraction.
- Also, the row vector of zeros in the last row may be replaced by some other row vector to provide "**perspective transformations**."
- In both these instances, the transformation represented by the augmented matrix **no longer corresponds to a rigid displacement**.

$$\bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$



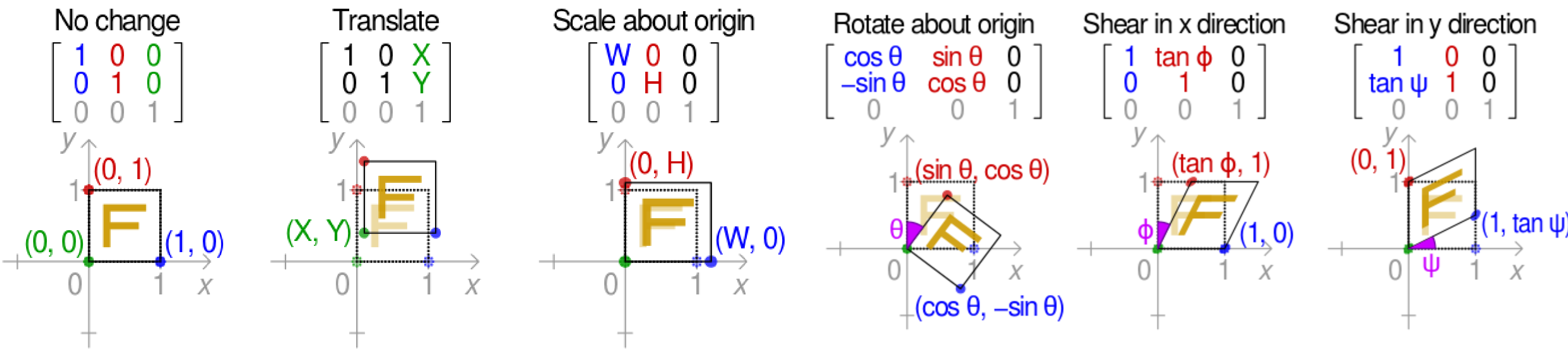


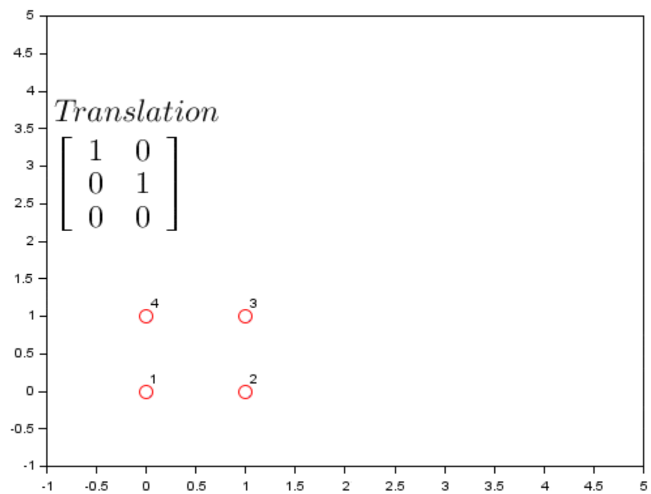
Szeliski, R., 2010. *Computer vision: algorithms and applications*. Springer Science & Business Media.

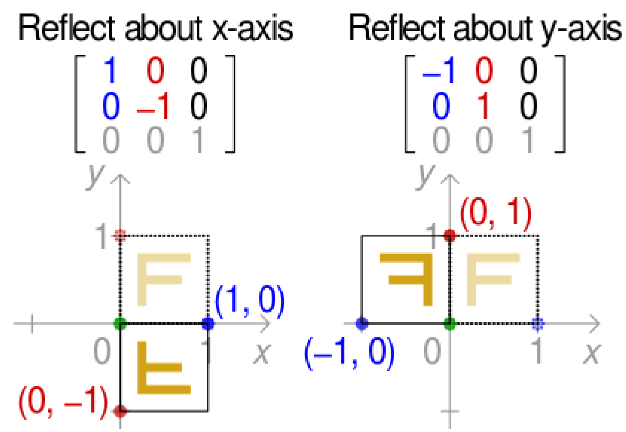
Affine Transformation: https://www.youtube.com/watch?app=desktop&v=il6Z5LCykZk&ab_channel=3Blue1Brown

perspective projection: https://wrf.ecse.rpi.edu/pmwiki/pmwiki.php/Main/HomogeneousCoords

2D affine transformation Examples







https://www.scilab.org/tutorials/computer-vision

https://ca.wikipedia.org/wiki/Fitxer:2D_affine_transformation_matrix.svg

References

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- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**
- Lynch and Park, "*Modern Robotics*," Cambridge U. Press, 2017, **Chapter 3.**