ME 397- ASBR

Week 4-Lecture 1

a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

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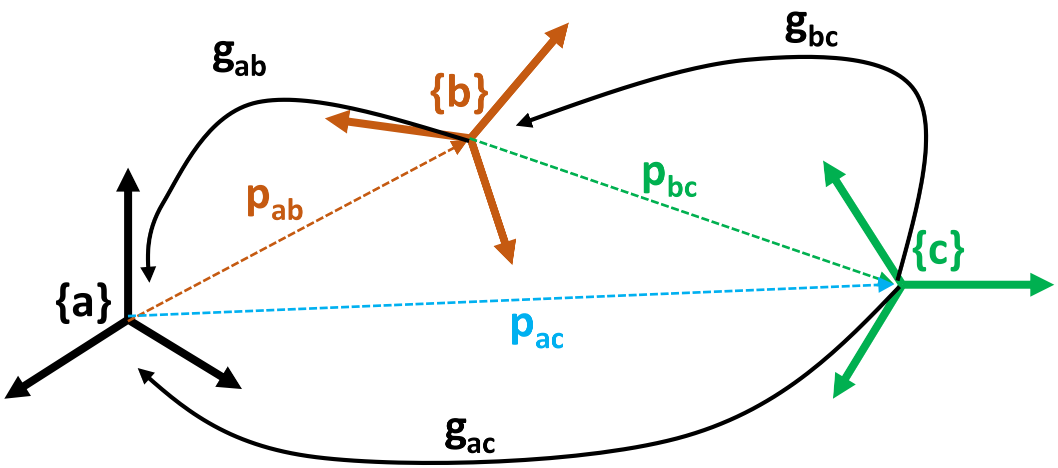
Composition of Transformations

- Rigid body transformations can be **composed** to form new rigid body transformations.
- Let $\mathbf{g_{bc}} \in \mathbf{SE(3)}$ be the configuration of **a frame C relative to a frame B**, and $\mathbf{g_{ab}}$ the configuration of **frame B relative to another frame A**. Then, using equation, the **configuration of C relative to frame A** is given by

$$\bar{g}_{ac} = \bar{g}_{ab}\,\bar{g}_{bc} = \begin{bmatrix} \underbrace{R_{ab}R_{bc}}_{\text{Rotation: } R_{ac}} & \underbrace{R_{ab}p_{bc} + p_{ab}}_{\text{Translation: } p_{ac}} \\ 0 & 1 \end{bmatrix}$$

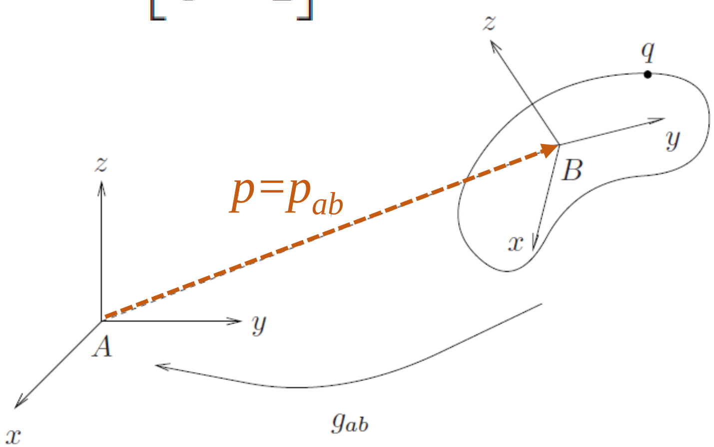


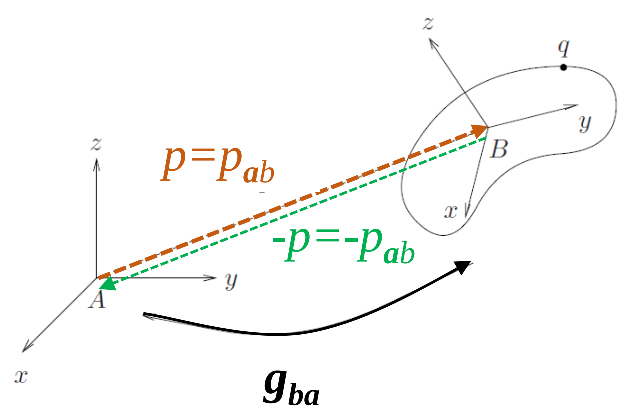
Properties of Transformation Matrices

- If $g_1, g_2 \in SE(3)$, then $g_1 g_2 \in SE(3)$.
  - The 4×4 identity element, I, is in $SE(3)$.
- The **inverse of a transformation matrix** **$g \in SO(3)$** $SE(3)$ is also a <u>transformation matrix</u>, and it has the following form:

$R_{ba} = R^{-1}_{ab}$ **P_{ba}:Origin of A defined in B**

$$\bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \qquad \bar{g}^{-1} = \begin{bmatrix} \boxed{R^T} & \boxed{-R^T p} \\ 0 & 1 \end{bmatrix} \in SE(3)$$





- The multiplication of transformation matrices **is associative**, so that $(T_1T_2)T_3 = T_1(T_2T_3)$, but generally **not commutative**: $T_1T_2 \neq T_2T_1$.
- These properties show that the set of <u>rigid transformations is a group.</u>

Uses of Transformation Matrices

- As was the case for rotation matrices, there are **three major uses** for a transformation matrix g or T:

  - ✓ (a) to **represent** the configuration (position and orientation) of a rigid body;
  - ✓ (b) to **change** the reference frame in which a vector or frame is represented;
  - ✓ (c) to **displace** a vector or frame.

- In (a) , **g or T** is thought of as **representing a frame**;

- In (b) and (c), **g or T** is thought of as an **operator** that acts on/move a **vector or frame.**

Uses of Transformation Matrices

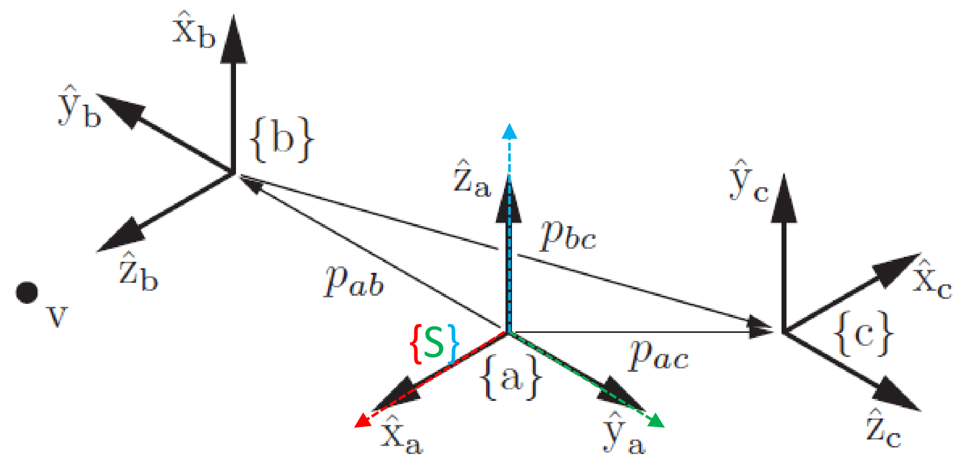
(a) To represent an orientation:

The fixed frame {s} is coincident with {a} and the frames {a}, {b}, and {c}, represented by $T_{sa} = (R_{sa}; p_{sa})$, $T_{sb} = (R_{sb}; p_{sb})$, and $T_{sc} = (R_{sc}; p_{sc})$, respectively.

$$R_{sa} = \begin{array}{c} \begin{matrix} \hat{x} & \hat{y} & \hat{z} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_{sb} = \begin{array}{c} \begin{matrix} \hat{x} & \hat{y} & \hat{z} \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \qquad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$R_{sc} = \begin{array}{c} \begin{matrix} \hat{x} & \hat{y} & \hat{z} \end{matrix} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \qquad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$



$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{bc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \qquad p_{bc} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}$$

Uses of Rotation Matrices

(b) Changing the reference frame

- By a **subscript cancellation rule** analogous to that for rotations, for any three **reference frames** {a}, {b}, and {c}, and **any vector v** expressed in {b} as v_b,

$$T_{ab}T_{bc} = T_{a\cancel{b}}T_{\cancel{b}c} = T_{ac}$$
$$T_{ab}v_b = T_{a\cancel{b}}v_{\cancel{b}} = v_a,$$

(c) Displacing (rotating and translating) a vector or a frame

- Transformation matrix T, viewed as the pair (R; p) = (Rot($\hat{\boldsymbol{\omega}}$; $\boldsymbol{\theta}$); **p**), **can act** on a frame $\mathbf{T}_{sb}$ by rotating it by $\boldsymbol{\theta}$ about an axis $\hat{\boldsymbol{\omega}}$ and translating it by **p**.
- We can extend the <u>3x3 rotation operator R</u> = Rot($\hat{\boldsymbol{\omega}}$; $\boldsymbol{\theta}$); to a 4×4 transformation matrix that <u>rotates without translating</u> i.e., $\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$
- We can similarly define a <u>**translation operator**</u> that <u>**translates without rotating**</u>,

$$\mathrm{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Uses of Rotation Matrices

- Whether we **pre-multiply** or **post-multiply** T_{sb} by T = (R; p) determines whether the $\hat{\omega}$ axis and **p** are interpreted as in the **fixed frame {s}** or in the **body frame {b}**:

$$T_{sb'} = \boxed{T}T_{sb} = \boxed{\text{Trans}(p)\,\text{Rot}(\hat{\omega},\theta)}T_{sb} \qquad \boxed{\text{(fixed frame)}}$$

$$= \boxed{\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb}+p \\ 0 & 1 \end{bmatrix}$$

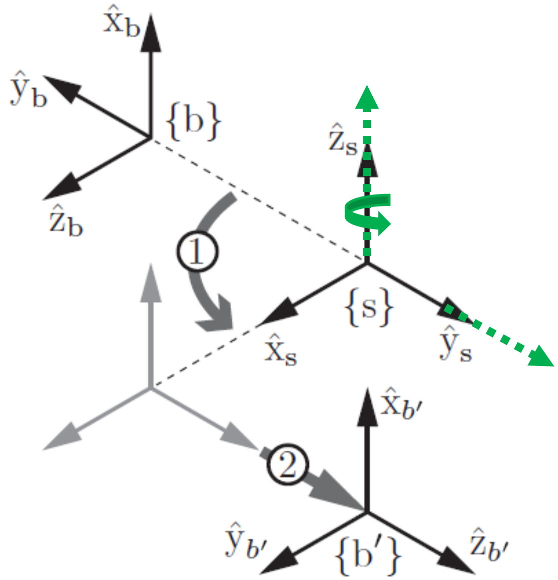
$$T_{sb''} = T_{sb}\boxed{T} = T_{sb}\boxed{\text{Trans}(p)\,\text{Rot}(\hat{\omega},\theta)} \qquad \boxed{\text{(body frame)}}$$

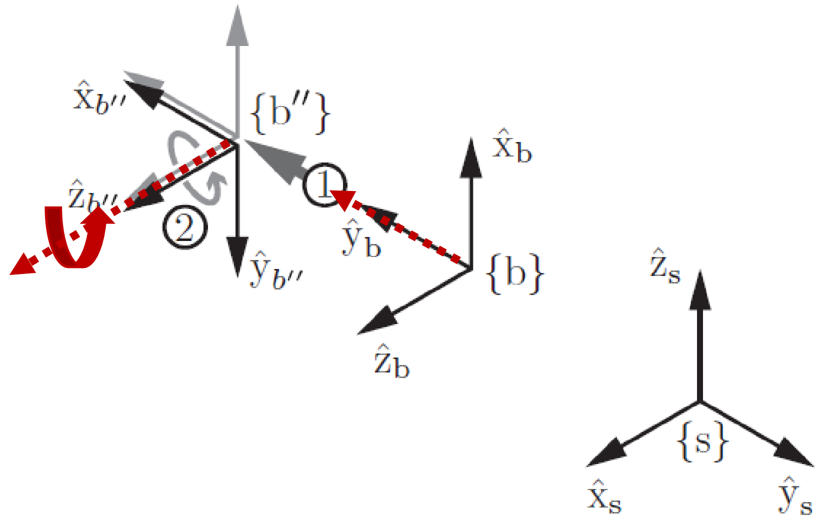
$$= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix}\boxed{\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} R_{sb}R & R_{sb}p+p_{sb} \\ 0 & 1 \end{bmatrix}$$

- The **fixed-frame transformation** can be interpreted as **first rotating the {b}** frame by **θ** about an axis $\hat{\omega}$ in the {s} frame (this rotation will cause the origin of {b} to move if it is not coincident with the origin of {s}), **then translating it by *p*** in the {s} frame to get a frame {b'}.

- The **body-frame transformation** can be interpreted as **first translating {b} by p** considered to be in the {b} frame, **then rotating about $\hat{\omega}$** in this new body frame (this does not move the origin of the frame) to get {b"}.

Uses of Rotation Matrices





$\hat{\omega}$ =(0; 0; 1), θ = 90, and p = (0; 2; 0).

(Left: Fixed-frame Transformation) The frame {b} is **first rotated** by 90 about $\hat{z}_s$ and **then translated** by two units in $\hat{y}_s$, resulting in the new frame {b'}.

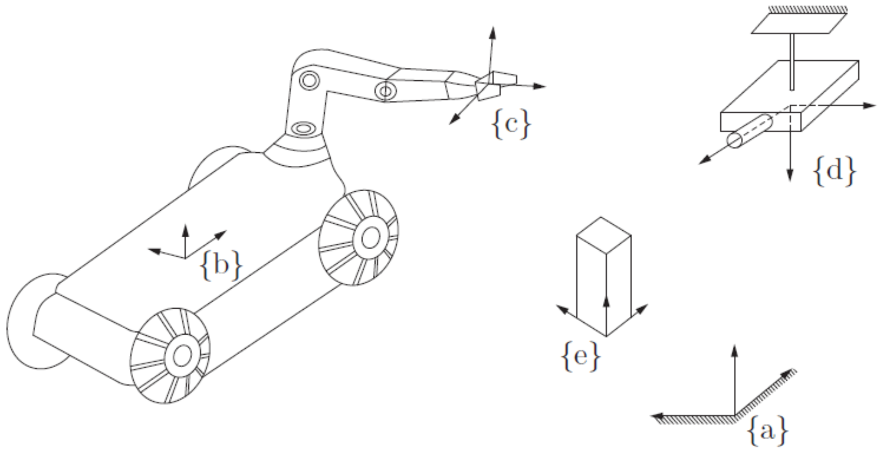
$$TT_{sb} = T_{sb'} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Right: Body-frame Transformation) The frame {b} is **first translated** by two units in $\hat{y}_b$ and **then rotated** by 90 about its $\hat{z}_b$ axis, resulting in the new frame {b"}.

$$T_{sb}T = T_{sb''} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Figure shows a robot arm mounted on a wheeled mobile platform moving in a room, and a camera fixed to the ceiling.



https://www.youtube.com/watch?v=J7Z49G443DQ

- ✓ **Frames {b} and {c}** are respectively attached to the **wheeled platform** and the **end-effector of the robot arm**, and **frame {d}** is attached to the **camera**.
- ✓ A **fixed frame {a}** has been established, and the robot must pick up an **object** with **body frame {e}**.
- ✓ Suppose that the transformations $\mathbf{T_{db}}$ **and** $\mathbf{T_{de}}$ **can be calculated** from measurements obtained with the camera.
- ✓ The transformation $\mathbf{T_{bc}}$ **can be calculated** using the arm's joint-angle measurements.
- ✓ The transformation $\mathbf{T_{ad}}$ **is assumed to be known** in advance. Suppose these calculated and known transformations are given as follows:

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

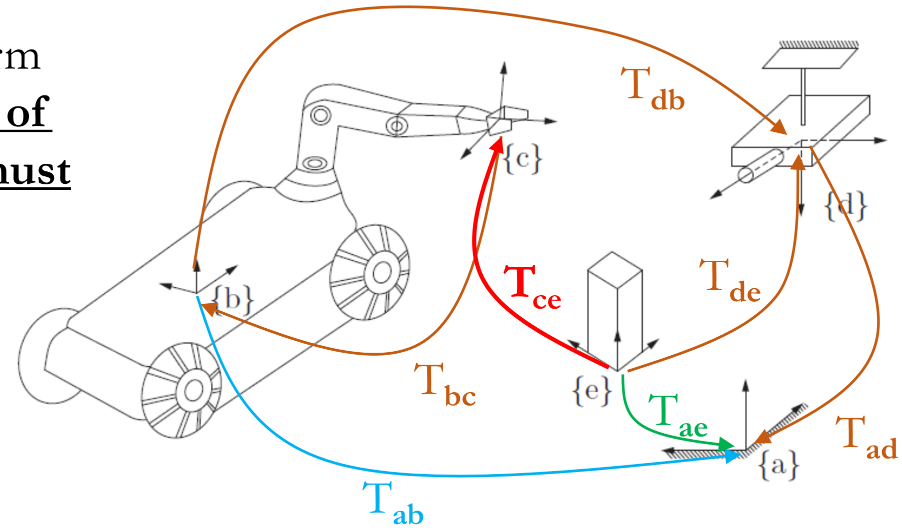
$$T_{de} = \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{ad} = \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

➢ **Calculate how to move the robot arm so as to pick up the object.**

Example

- In order to calculate how to move the robot arm so as to pick up the object, the **configuration of the object relative to the robot hand**, **T_{ce}**, **must be determined.**
- We know that: $\overbrace{\boxed{T_{ab}}\boxed{T_{bc}}\boxed{T_{ce}}}^{T_{ae}} = \overbrace{\boxed{T_{ad}T_{de}}}^{T_{ae}},$



$$\Rightarrow \quad \boxed{T_{ab}} = \boxed{T_{ad}T_{db}}$$

$$\Rightarrow \quad \boxed{T_{ce}} = (T_{ad}T_{db}T_{bc})^{-1}\,T_{ad}T_{de}$$

From the given transformations we obtain

$$T_{ad}T_{de} = \begin{bmatrix} 1 & 0 & 0 & 280 \\ 0 & 1 & 0 & -50 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{ad}T_{db}T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 230 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 160 \\ 1 & 0 & 0 & 75 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

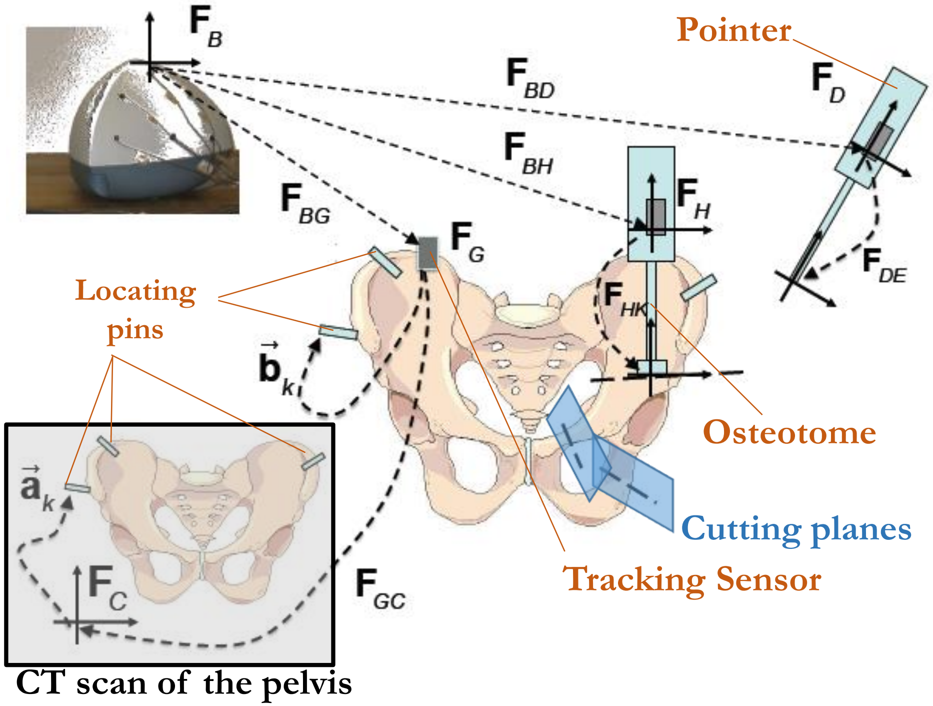
$$(T_{ad}T_{db}T_{bc})^{-1} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 70/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 390/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Rightarrow \quad T_{ce} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -260/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 130/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Computer-Assisted Osteotomy

Consider the pelvic osteotomy situation illustrated in the figure. Here we assume that a **three locating pins** have been inserted into the patient's pelvis, and that a CT scan of the pelvis with the pins inserted has been produced. The patient has been placed onto the operating table.

- A magnetic navigation system (here, the Northern Digital Aurora) is present in the room.
- **Two surgical tools** are available:
  - ✓ A probe/pointer device
  - ✓ An osteotome (essentially a fancy chisel) that will be used to cut the pelvis.



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- 6 DOF Aurora tracking sensors have been attached to the **handle of each tool** and an additional 6 DOF sensor has been **affixed rigidly to the pelvis**. The Aurora is capable of determining the position and orientation of each sensor relative to the Aurora base unit.
- Let $\mathbf{p}_{tip} = \mathbf{p}_{GE}$ be the position of the tip of the pointer tool relative to the reference marker coordinate system $\mathbf{F}_G$.
- Give a formula for computing $\mathbf{p}_{tip}$, based on the available tracking system measurements $\mathbf{F}_{Bx}$.

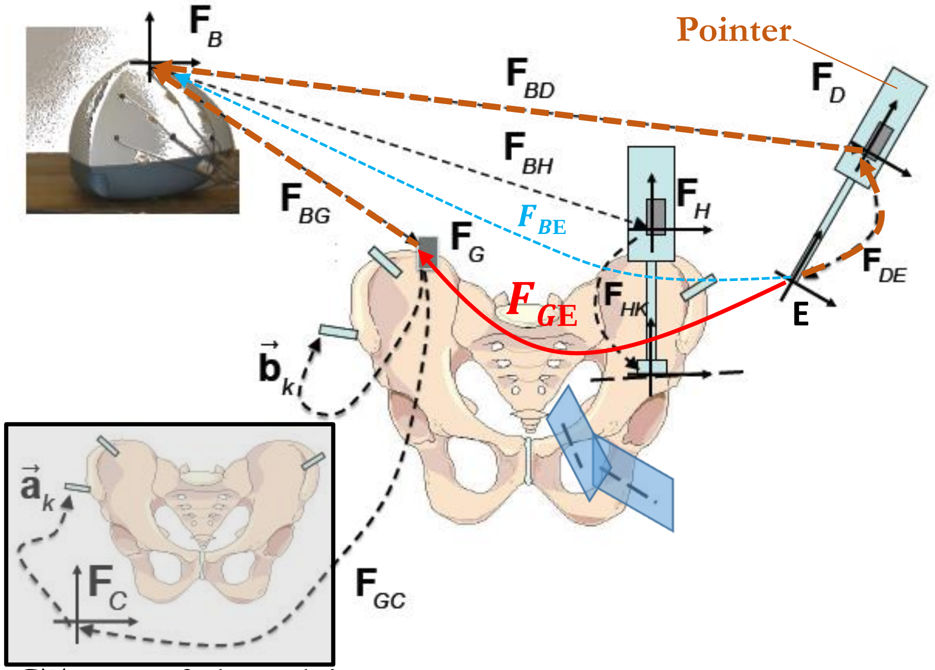
Example is from the Computer Integrated Surgery course, Russell H. Taylor, JHU

Example: Computer-Assisted Osteotomy

$$\overset{F_{BE}}{F_{BG}\,F_{GE}} = \overset{F_{BE}}{F_{BD}\,F_{DE}}$$

$$\Rightarrow \quad F_{GE} = F_{BG}^{-1}\, F_{BD}\, F_{DE}$$

p_{GE} is the last column of F_{GE}

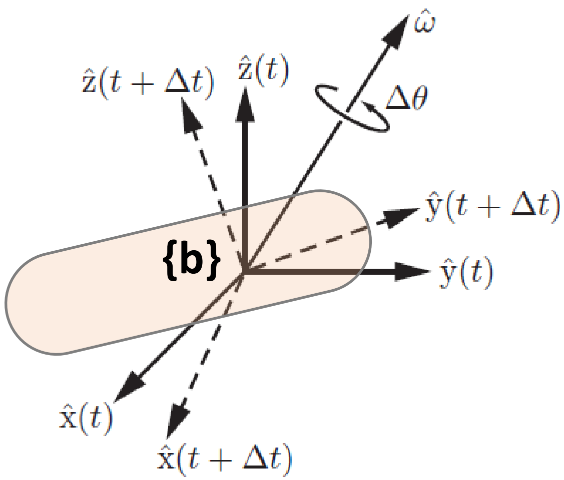


CT scan of the pelvis

Example is from the Computer Integrated Surgery course, Russell H. Taylor, JHU

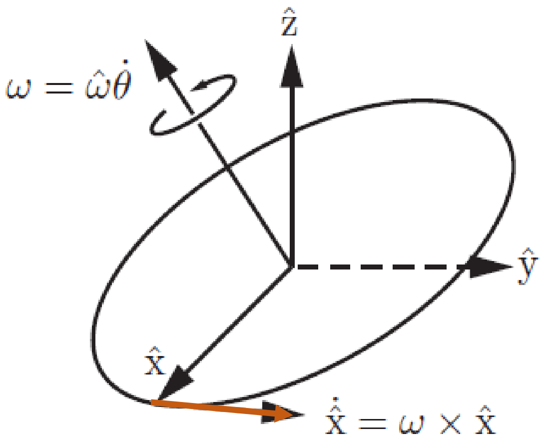
Angular Velocities

- Suppose that a **body frame** with **unit** axes $\{\hat{x}, \hat{y}, \hat{z}\}$ is attached to a **rotating body**.



- If we examine the body frame at times **t** and **t+Δt**, **the change in frame orientation** can be described as a rotation of angle $\Delta\theta$ about some unit axis $\widehat{\omega}$ passing through the origin.
- The axis $\widehat{\omega}$ is **coordinate-free**; it is not **yet** represented in any particular reference frame.
- As **t** approaches zero, the ratio $\Delta\theta/\Delta t$ becomes the **rate of rotation** $\dot{\theta}$, and $\widehat{\omega}$ is the **instantaneous axis of rotation**. Hence, angular velocity w is:

$$\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}.$$



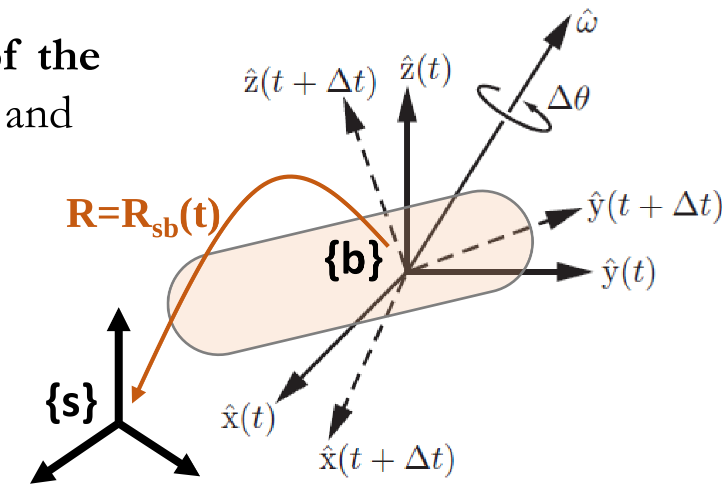
- We can then calculate **linear velocities** as:

$$\begin{aligned} \dot{\hat{\mathrm{x}}} &= \mathrm{w} \times \hat{\mathrm{x}}, \\ \dot{\hat{\mathrm{y}}} &= \mathrm{w} \times \hat{\mathrm{y}}, \\ \dot{\hat{\mathrm{z}}} &= \mathrm{w} \times \hat{\mathrm{z}}. \end{aligned}$$

- To express these equations in **coordinates**, we have to choose a reference frame in which to represent **w** typically the **fixed frame {s}** **or the body frame {b}**.

Fixed-Frame Angular Velocities

- Let R(t) be the rotation matrix describing the **orientation of the body frame with respect to the fixed frame {s}** at time t and **$\dot{R}(t)$** is its time rate of change.



- Let R(t)= $[\overset{\hat{x}}{r_1}(t); \overset{\hat{y}}{r_2}(t); \overset{\hat{z}}{r_3}(t)]$ where $\mathbf{r_i}$ is the representation of the corresponding **<u>body frame axis</u>** in the fixed frame {s}.

$$w = \hat{w}\dot{\theta}.$$

- At a specific time **t**, let $\boldsymbol{\omega}_s \in \mathbb{R}^3$ be the angular velocity $\boldsymbol{\omega}$ expressed in **fixed-frame** then we have:

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3.$$

OR $$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3]$$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$$a \times b = (a)^\wedge b.$$

- We can rewrite $\boldsymbol{\omega}_s \times \boldsymbol{r}_i$ as $[\boldsymbol{\omega}_s]R$, where $[\boldsymbol{\omega}_s]$ is a 3×3 **skewsymmetric** matrix representation of $\boldsymbol{\omega}_s \in \mathbb{R}^3$. Hence:

Skew-Symmetric angular **velocity of $\boldsymbol{\omega}$** represented in the **fixed frame**

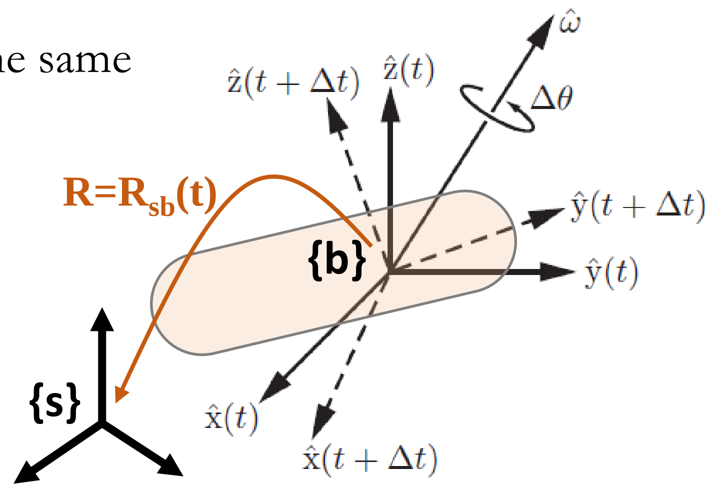
$$[\omega_s]R = \dot{R}$$

$$[\omega_s] = \dot{R}R^{-1}$$

Time rate of change of the **orientation of the body frame with respect to the fixed frame {s}**

Body-Frame Angular Velocities

- Let $\boldsymbol{\omega}_s$ **and** $\boldsymbol{\omega}_b$ be two different vector representations of the same angular velocity w expressed in the **fixed and body-frame** coordinates, respectively. Hence: $\omega_s = R_{sb}\omega_b$.



$$\omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^{\mathrm{T}}\omega_s$$

- Let us now use the skew-symmetric operator **[.]** to rewrite this equation in a matrix format:

$$\boxed{R[\omega]R^{\mathrm{T}} = [R\omega]}$$

Given any $\boldsymbol{\omega} \in \mathbb{R}^3$ and $R \in SO(3)$

$$\boxed{[\omega_s] = \dot{R}R^{-1}}$$

$$\boxed{RR^T = R^T R = I.}$$

$$\begin{aligned} [\omega_b] &= \boxed{[R^{\mathrm{T}}\omega_s]} \\ &= R^{\mathrm{T}}\boxed{[\omega_s]}R \\ &= R^{\mathrm{T}}(\dot{R}\boxed{R^{\mathrm{T}})R} \\ [\omega_b] &= R^{\mathrm{T}}\dot{R} = R^{-1}\dot{R}. \end{aligned}$$

$$[\omega_s]R = \dot{R}.$$

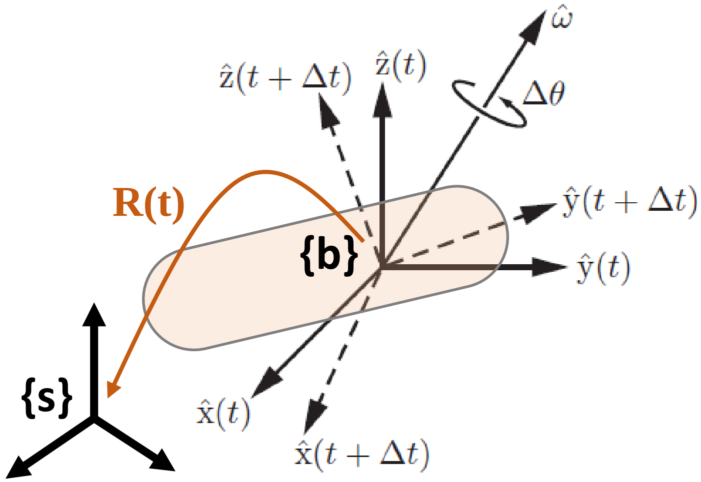
$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Skew-Symmetric angular velocity of $\boldsymbol{\omega}$ represented in the **body frame**

Time rate of change of the **orientation of the body frame with respect to the fixed frame {s}**

Angular Velocities

- Let R(t) denote the <u>orientation of the rotating frame as seen from the fixed frame</u>. Denote by **w** the angular velocity of the rotating frame. Then



$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b].$$

where $\boldsymbol{\omega_s} \in \mathbb{R}^3$ is the **fixed-frame vector** representation of **w** and $[\boldsymbol{\omega_s}] \in so(3)$ is its 3×3 matrix representation, and where $\boldsymbol{\omega_b} \in \mathbb{R}^3$ is the **body-frame vector** representation of **w** and $[\boldsymbol{\omega_b}] \in so(3)$ is its 3×3 matrix representation.

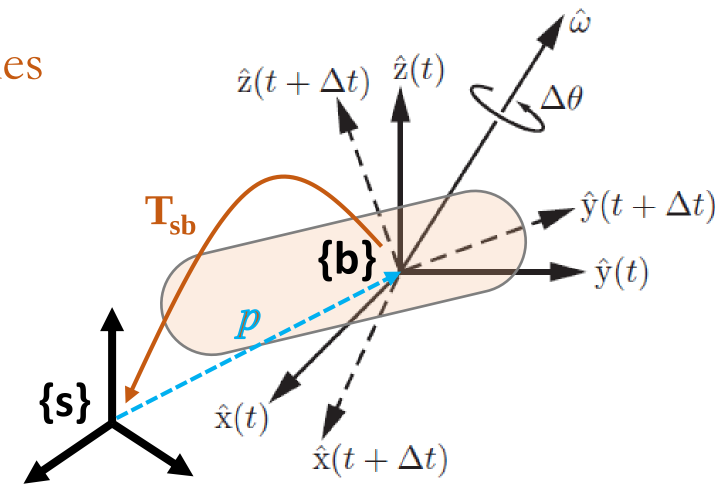
- It is important to note that the **fixed-frame angular velocity** $\boldsymbol{\omega_s}$ does **not** depend on the choice of body frame.
- **<u>Similarly</u>**, the body-frame angular velocity $\boldsymbol{\omega_b}$ does **not** depend on the choice of fixed frame.
- An angular velocity expressed in an **arbitrary frame {d}** can be represented in another **frame {c}** **<u>if we know the rotation that takes {c} to {d}:</u>**

$$\omega_c = R_{cd}\omega_d.$$

Twists

- We now consider both the linear and angular velocities of a <u>moving frame</u>. Let

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$



- Let us pre-multiply $\dot{\mathrm{T}}$ by T^{-1}:

$$\boxed{T^{-1}}\boxed{\dot{T}} = \boxed{\begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}}\boxed{\begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} \boxed{R^{\mathrm{T}}\dot{R}} & \boxed{R^{\mathrm{T}}\dot{p}} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{[\omega_b]} & \boxed{v_b} \\ 0 & 0 \end{bmatrix}.$$

$\boldsymbol{v_b} = \boldsymbol{R^T}\dot{p}$ is the **linear velocity of the origin** of {b} expressed in {b}.

- $\mathrm{T}^{-1}\dot{\mathrm{T}}$ represents the linear and angular velocities of the moving frame relative to the stationary frame {b} <u>**currently**</u> aligned with the moving frame (i.e., current body frame).

References

- Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.

- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**

- Lynch and Park, "*Modern Robotics*," Cambridge U. Press, 2017, **Chapter 3.**