HA:

1. First check necessary condition: $\|R(p)-R(q)\|=\|p-q\|$

$R: \mathbb{R}^3 \to \mathbb{R}^3$, $p \in \mathbb{R}^3$, $q \in \mathbb{R}^3$, $R \in \mathbb{R}^{3\times 3}$

$R \in SO(3) \Rightarrow RR^T = I$, $\det(R) = 1$

The norm for this necessary condition is Euclidean norm.

$$\|R(p) - R(q)\| = \|R(p-q)\|$$

$$= [R(p-q)(p-q)^T R^T]^{1/2}$$

$$(\text{since } (p-q)(p-q)^T \in \mathbb{R})$$

$$= \|p-q\| \cdot [RR^T]^{1/2}$$

$$= \|p-q\|$$

The necessary condition holds.

Next, check the sufficient condition: $R(p \times q) = Rp \times Rq$

By definition:
$$\begin{cases} R(p \times q) = R(\|p\| \cdot \|q\| \cdot \sin\theta \cdot n) \\ Rp \times Rq = \|Rp\| \cdot \|Rq\| \cdot \sin\theta' \cdot n' \end{cases}$$

θ is the angle between p and q

n is a unit vector perpendicular to the surface that contains p and q

We first show $\|Rp\| \cdot \|Rq\| = \|p\| \cdot \|q\|$:

$$\|Rp\| = (Rpp^T R^T)^{1/2} = \|p\|$$

$$\|Rq\| = (Rqq^T R^T)^{1/2} = \|q\|$$

next, we show: $\theta = \theta'$:

By definition: $p \cdot q = \|p\| \cdot \|q\| \cdot \cos\theta$

$$Rp \cdot Rq = \|Rp\| \cdot \|Rq\| \cdot \cos\theta' = \|p\| \cdot \|q\| \cdot \cos\theta'$$

Thus, the problem reduces to show $Rp \cdot Rq = p \cdot q$

$$Rp \cdot Rq = Rp \cdot (Rq)^T = Rp \cdot q^T \cdot R^T \underset{\uparrow}{=} p \cdot q^T = p \cdot q$$

$$\begin{cases} p \cdot q^T \in \mathbb{R} \\ RR^T = I \end{cases}$$

Thus, we showed $\theta = \theta'$

We also know that: $\begin{cases} n \perp p, \ n \perp q \\ n' \perp Rp, \ n' \perp Rq \\ \|n\| = \|n'\| = 1 \end{cases}$

$$\left.\begin{array}{l} n \cdot p = n' \cdot Rp = 0 \\ n \cdot q = n' \cdot Rq = 0 \end{array}\right\} \Rightarrow \text{Assuming } p, q \neq 0, \Rightarrow n = Rn'$$

Thus, we proved $R(p \times q) = R(p) \times R(q)$

2. $v = s - r$

$$g(v) = g(s-r) = g(s) - g(r)$$
$$= p + Rs - (p + Rr)$$
$$= Rs - Rr$$
$$= R(s-r)$$
$$= Rv$$

3. a) Let $w = [w_1 \quad w_2 \quad w_3]^T$

$$\hat{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

$$\lambda I - \hat{w} = \begin{bmatrix} \lambda & -w_3 & w_2 \\ w_3 & \lambda & -w_1 \\ -w_2 & w_1 & \lambda \end{bmatrix}$$

$$\det(\lambda I - \hat{w}) = \lambda(\lambda^2 + w_1^2) + w_3(\lambda w_3 - w_1 w_2)$$
$$+ w_2(w_1 w_3 + \lambda w_2)$$
$$= \lambda^3 + \lambda w_1^2 + \lambda w_3^2 + \lambda w_2^2$$
$$= \lambda(\lambda^2 + w_1^2 + w_2^2 + w_3)^2$$
$$= 0$$

Since $\|\omega\| = 1$,

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$$

$$\lambda(\lambda^2 + 1) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = \hat{i} \quad \lambda_3 = -\hat{i}$$

$$\lambda_1 v_1 = \hat{\omega} v_1$$

$$\hat{\omega} v_1 = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = [\omega_1 \quad \omega_2 \quad \omega_3]^T$$

$$\hat{\omega} v_2 = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix} = \hat{i} \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -\omega_3 + \omega_1\omega_2 \hat{i} \\ -\hat{i}(1 - \omega_2^2) \\ \omega_1 + \omega_2\omega_3 \hat{i} \end{bmatrix}$$

v_3 is the complex conjugate of v_2

$$v_3 = \begin{bmatrix} -\omega_3 - \omega_1\omega_2 \hat{i} \\ i(1 - \omega_2^2) \\ \omega_1 - \omega_2\omega_3 \hat{i} \end{bmatrix}$$

b) $R = e^{\hat{\omega}\theta}$

From Taylor expansion:

$$R = e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^3}{3!}\hat{\omega}^3 + \cdots$$

The eigenvalue of R satisfies: $R v_i = \lambda_i v_i$

$$R v_i = e^{\hat{\omega}\theta} v_i = I v_i + \hat{\omega} v_i \theta + \frac{\theta^2}{2!}\hat{\omega}^2 v_i + \cdots$$

$$= (I v_i + \lambda_i v_i \theta + \frac{\theta^2}{2!}\hat{\omega}\lambda_i v_i + \cdots)$$

$$= (I + \lambda_i \theta + \frac{\theta^2}{2!}\hat{\omega}\lambda_i v_i + \cdots) v_i$$

$$= (I + \lambda_i \theta + \frac{\theta^2}{2!}\lambda_i^2 v_i + \cdots) v_i$$

$$= e^{\lambda_i \theta} v_i$$

Thus, $e^{\lambda_i \theta}$ is the eigenvalue of R

$\lambda_1 = e^0 = 1$, $\lambda_2 = e^{\hat{i}\theta}$, $\lambda_3 = e^{-\hat{i}\theta}$

v_1 corresponding to λ_0 is $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

Physical interpretation: the eigenvector corresponding to $\lambda = 1$ is a unit vector pointing in the direction of of ω. This means that the rotation matrix preserves the entity along the axis of rotation, which is expected.

4. a)

b) Since $v, w \in \mathbb{R}^3$

$$R(v \times w) = R\hat{v}w$$

$$Rv,\ Rw \in \mathbb{R}^3$$

$$(Rv)\times(Rw) = \widehat{Rv}\,Rw$$

From part (a), $\widehat{Rv} = R\hat{v}R^T$

$$(Rv)\times(Rw) = R\hat{v}R^TRw = R\hat{v}w = R(v \times w)$$

5. a) $Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$

$P = p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$

$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$

$p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$

① check closure under multiplication:

$$Q.P = (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\hat{i}$$
$$+ (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\hat{j} + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\hat{k}$$

$$\|Q.P\| = (q_0^2 + q_1^2 + q_2^2 + q_3^2)\cdot(p_0^2 + p_1^2 + p_2^2 + p_3^2) = 1$$

Thus, $\|Q.P\|$ is also a unit quaternion

② Check Identity under multiplication

For a quaternion e

$$e = e_0 + e_1\hat{i} + e_2\hat{j} + e_3\hat{k}$$

When $e_1 = e_2 = e_3 = 0$, $e_0 = 1$

$$Q.e = e.Q = Q$$

③ Check inverse under multiplication

$e = 1$ is the identity element.

$$Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{j}$$

$$P = p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{j}$$

Let $P = Q^{-1}$ be the inverse of Q

$$Q.P = (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\hat{i}$$
$$+ (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\hat{j} + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\hat{k}$$
$$= e$$

$$P = \frac{1}{q_0^2 + q_1^2 + q_2^2 + q_3^2}(q_0 - q_1\hat{i} - q_2\hat{j} - q_3\hat{k})$$

$$\|P\| = 1$$

Thus, $P = Q^{-1}$ is also a unit quaternion

④ Check associativity under multiplication

Let $Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$

$P = p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$

$n = n_0 + n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

It can be checked using MATLAB: $(Q.P).n = Q.(P.n)$

b) Let $x = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$

$$X = (0, x) = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$$

$$Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}, \quad \|Q\| = 1$$

$$Q\cdot X = (-\vec{q}\cdot x,\ q_0 x + \vec{q}\times\vec{x})$$

$$Q^* = q_0 - q_1\hat{i} - q_2\hat{j} - q_3\hat{k} = (q_0, -\vec{q})$$

$$\underbrace{QX}_{Q}\underbrace{Q^*}_{P} = (-\vec{q}\cdot x\, q_0 + (q_0 x + \vec{q}\times\vec{x})\cdot\vec{q},$$
$$\vec{q}\cdot x\,\vec{q} + q_0(q_0 x + \vec{q}\times\vec{x}) - (q_0 x + \vec{q}\times\vec{x})\times\vec{q})$$

$$-\vec{q}\cdot x\, q_0 + (q_0 x + \vec{q}\times\vec{x})\cdot\vec{q}$$

$$= -\vec{q}\cdot x\, q_0 + q_0 x\cdot\vec{q} + (\vec{q}\times\vec{x})\cdot\vec{q}$$

$$= -\vec{q}\cdot x\, q_0 + \vec{q}\cdot x\, q_0 + (\vec{q}\times\vec{x})\cdot\vec{q}$$

$$= (\vec{q}\times\vec{x})\cdot\vec{q}$$

$$= 0 \quad (\text{since } \vec{q}\times\vec{x} \perp \vec{q})$$

$$\vec{q}\cdot x\,\vec{q} + q_0(q_0 x + \vec{q}\times\vec{x}) + \vec{q}\times(q_0 x + \vec{q}\times\vec{x})$$

$$= (x\cdot\vec{q})\vec{q} + q_0^2 x + q_0(\vec{q}\times\vec{x}) + \vec{q}\times q_0 x + \vec{q}\times(\vec{q}\times x)$$

$$= (x\cdot\vec{q})\vec{q} + q_0^2 x + 2q_0(\vec{q}\times\vec{x}) + \vec{q}(\vec{q}\cdot x) - x(\vec{q}\cdot\vec{q})$$

$$= (q_0^2 - \vec{q}\cdot\vec{q})x + 2(q_0(\vec{q}\times\vec{x}) + (x\cdot\vec{q})\vec{q})$$

6.

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let c denotes cos, and s denotes sin

$$R_{xyz}(\psi, \theta, \phi) = R_x(\phi) R_y(\theta) R_z(\psi)$$

$$= \begin{bmatrix} C_\theta C_\psi & -C_\theta S_\psi & S_\theta \\ C_\phi S_\psi + C_\psi S_\phi S_\theta & C_\phi C_\psi - S_\phi S_\theta S_\psi & -C_\theta S_\phi \\ S_\phi S_\psi - C_\phi C_\psi S_\theta & C_\psi S_\phi + C_\phi S_\theta S_\psi & C_\phi C_\theta \end{bmatrix}$$

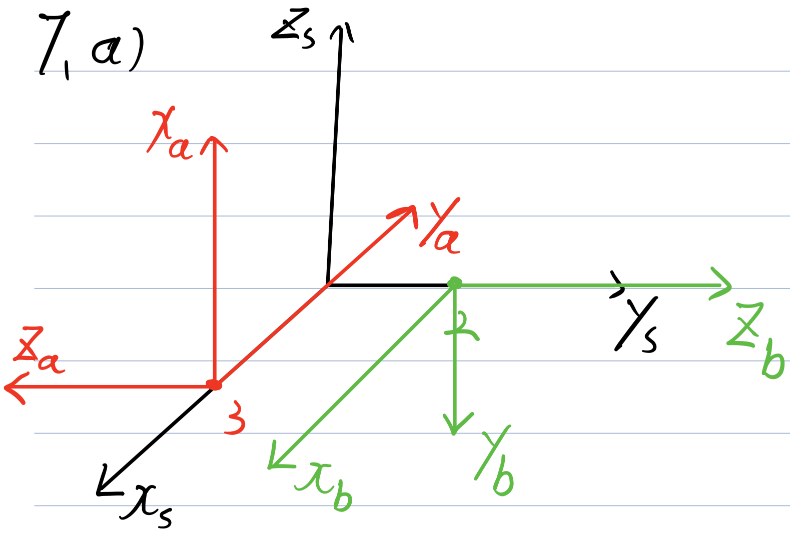
$$\triangleq \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{23} & r_{33} \end{bmatrix}$$

$$\phi = \tan^{-1}(-r_{23}, r_{33})$$

$$\theta = \tan^{-1}\left(r_{13}, \sqrt{r_{23}^2 + r_{33}^2}\right)$$

$$\psi = \tan^{-1}(-r_{12}, r_{11})$$

7, a)



b)