ME 397- ASBR

Week 6-Lecture 2

a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

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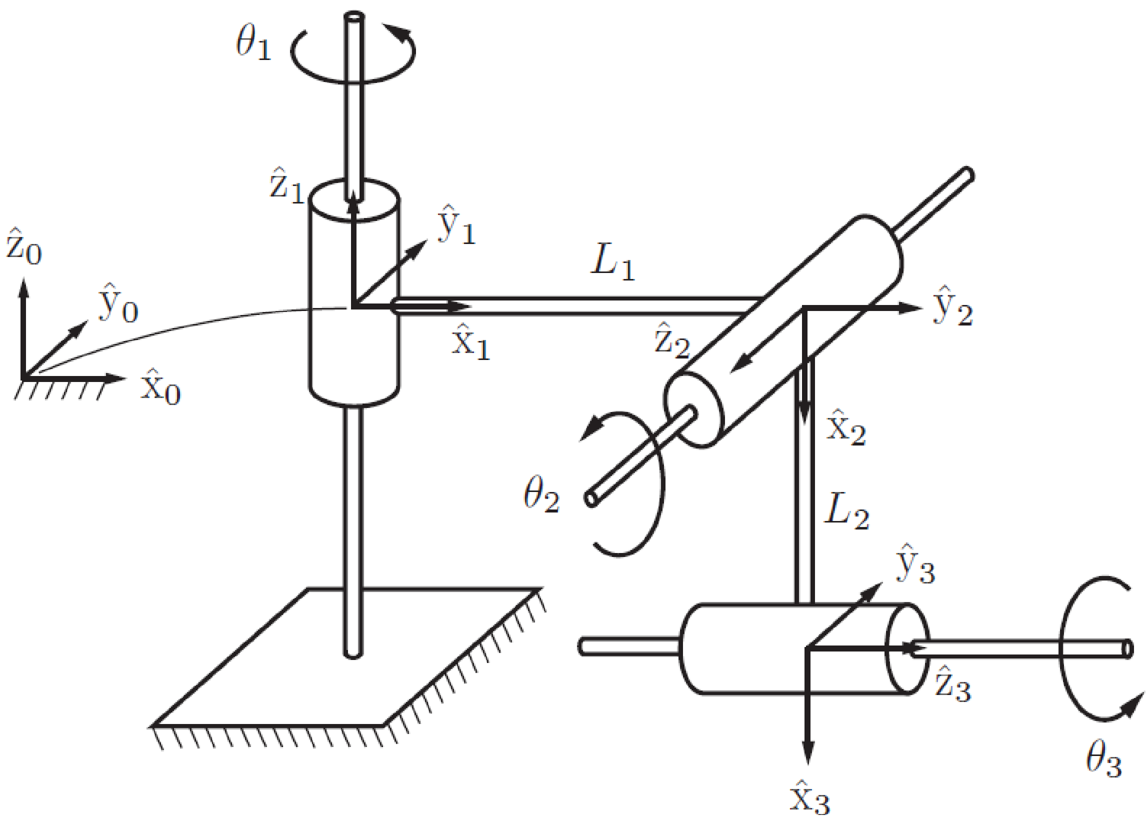
Example: 3 Revolute (R) spatial open chain

Consider the 3R open chain of the figure shown in **its home position** (all joint variables set equal to zero).

Find the forward kinematics of the robot using the **space form** of the exponential products.



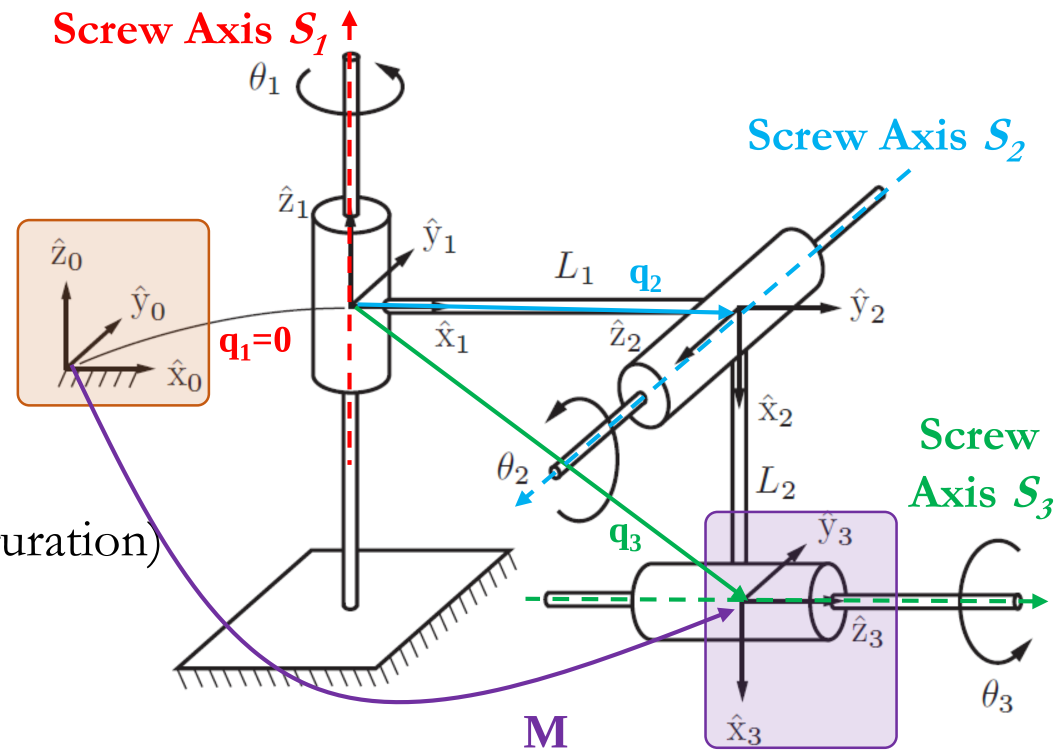
Example: 3R spatial open chain

- We should express all vectors and homogeneous transformations in terms of the **fixed frame**.

Step 1) Choose the fixed frame {0} and end-effector frame {3} as indicated in the figure.

Step 2) By inspection M (Home configuration) can be obtained as:

$$
\begin{array}{cc}
 & \begin{array}{ccc} \hat{x}_3 & \hat{y}_3 & \hat{z}_3 \end{array} \\
M = & \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$



$v = -\omega \times q$

$$v_2 = -\omega_2 \times q_2$$
$$v_3 = -\omega_3 \times q_3$$

Step 3) The screw axis $S_1 = (\omega_1; v_1)$, for joint axis 1 is $\omega_1 = (0; 0; 1)$ and $v_1 = (0; 0; 0)$

The screw axis $S_2 = (\omega_2; v_2)$, for joint axis 2 is $\omega_2 = (0; -1; 0)$ and $v_2 = (0; 0; -L_1)$

The screw axis $S_3 = (\omega_3; v_3)$, for joint axis 3 is $\omega_3 = (1; 0; 0)$ and $v_3 = (0; -L_2; 0)$

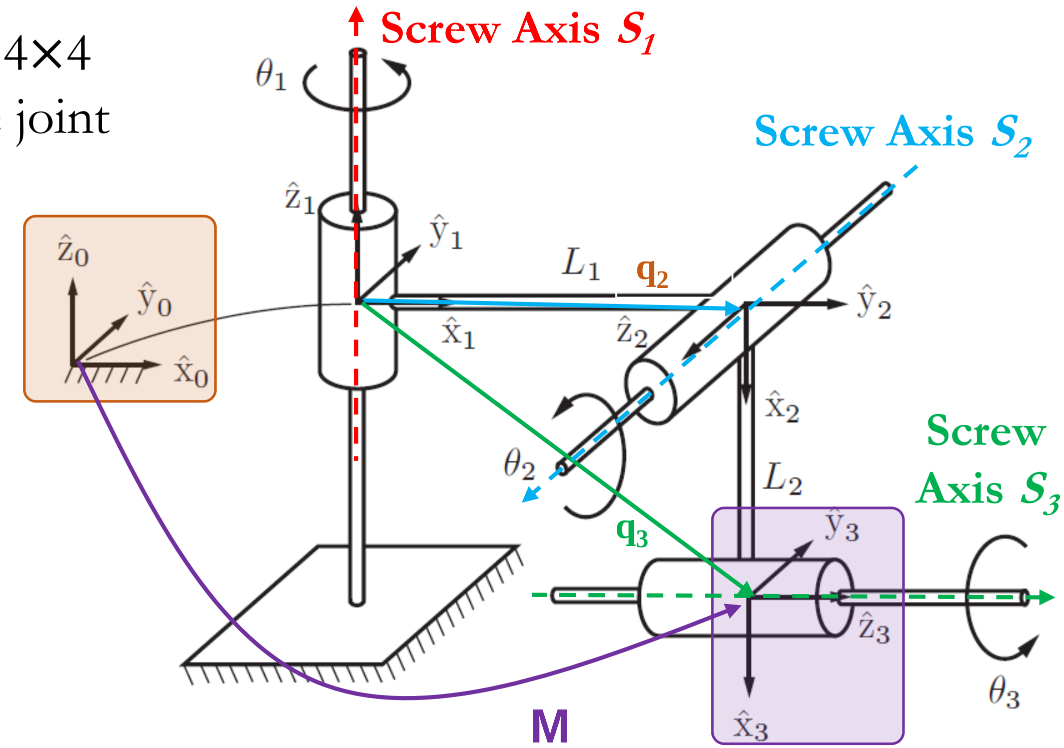
Example: 3R spatial open chain

- In summary, we have the following 4×4 matrix representations for the three joint screw axes ***S₁***, ***S₂***, and ***S₃***:

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



$$v_3 = -\omega_3 \times q_3$$
$$v_2 = -\omega_2 \times q_2$$

- The forward kinematics has the following form:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$$T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$$

where $* =$
$(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$

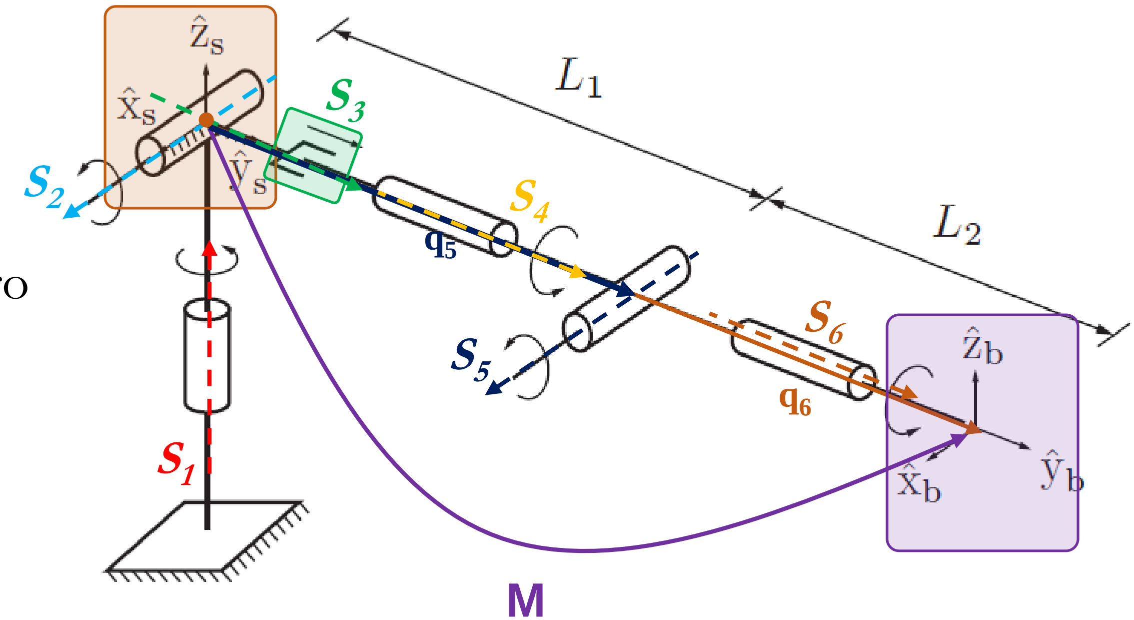
i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,-1,0)$	$(0,0,-L_1)$
3	$(1,0,0)$	$(0,L_2,0)$

Example: RRPRRR spatial open chain

Consider the six-degree-of-freedom RRPRRR spatial open chain of the Figure and find its forward kinematics.

➢ The end-effector frame in the zero position is given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



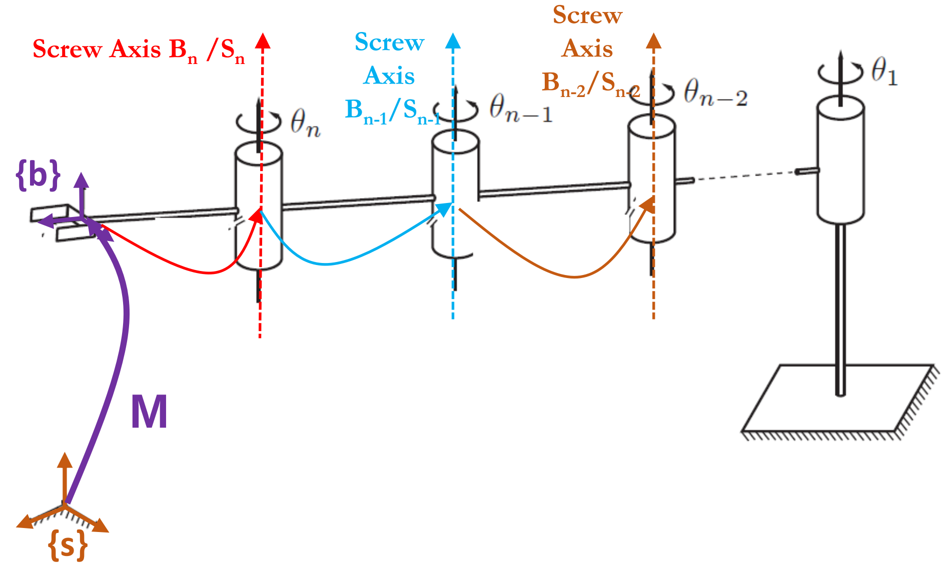
$v = -\omega \times q$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(1,0,0)$	$(0,0,0)$
3	$(0,0,0)$	$(0,1,0)$
4	$(0,1,0)$	$(0,0,0)$
5	$(1,0,0)$	$(0,0,-L_1)$
6	$(0,1,0)$	$(0,0,0)$

Note that the third joint is prismatic, so that $\boldsymbol{\omega_3} = 0$ and $\boldsymbol{v_3}$ **is a unit vector** in the direction of <u>positive translation</u>.

- An alternative form of the product of exponentials formula represents the **joint axes** as screw axes $\boldsymbol{B_i}$ in the end-effector (body) frame

- Step 1 and 2 are similar to the **Base Frame** approach.



- We define screw axes $\boldsymbol{B_i}$ **when the robot is at its zero position**, therefore:

$$T(\theta) = \boxed{M} e^{[\mathcal{B}_1]\theta_1} \cdots \boxed{e^{[\mathcal{B}_{n-1}]\theta_{n-1}}}\boxed{e^{[\mathcal{B}_n]\theta_n}}$$

where each $[B_i] = [\mathrm{Ad}_{M^{-1}}]\boldsymbol{S_i}$, **i = 1: … : n**

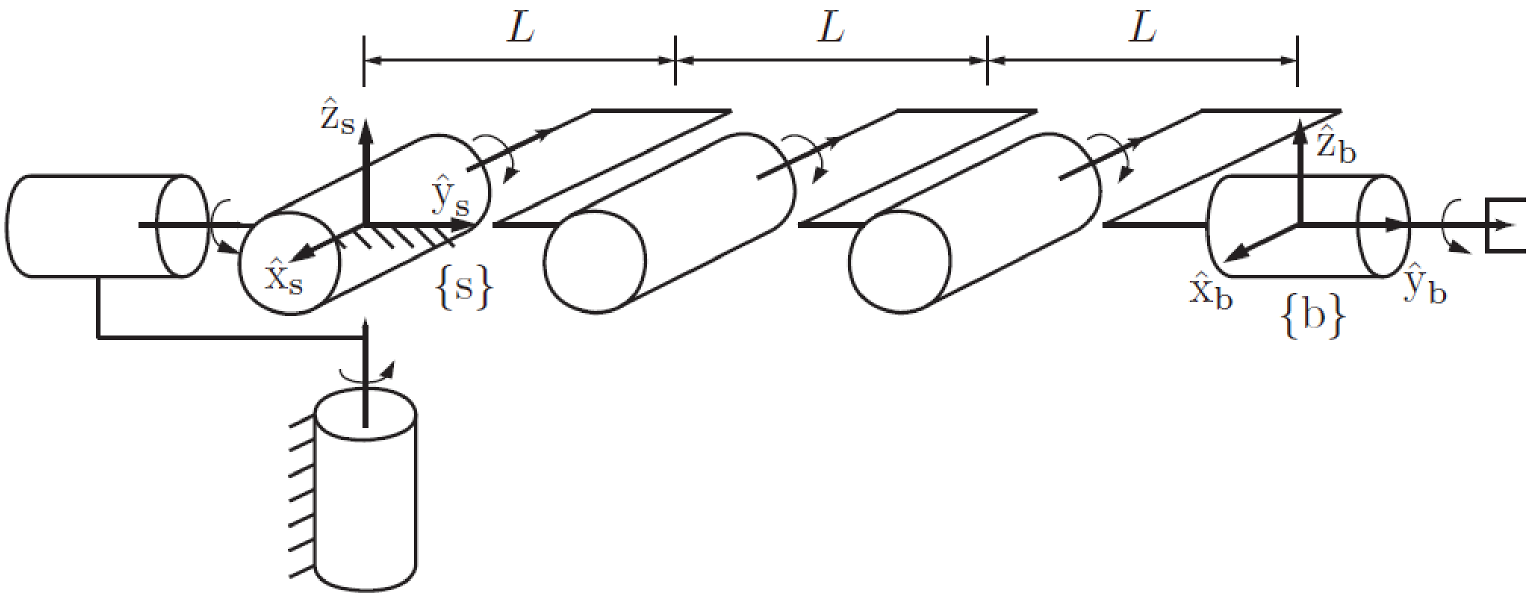
$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \boxed{\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \boxed{[\mathrm{Ad}_{T_{sb}}]}\mathcal{V}_b,$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \boxed{\begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix}} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \boxed{[\mathrm{Ad}_{T_{bs}}]}\mathcal{V}_s.$$

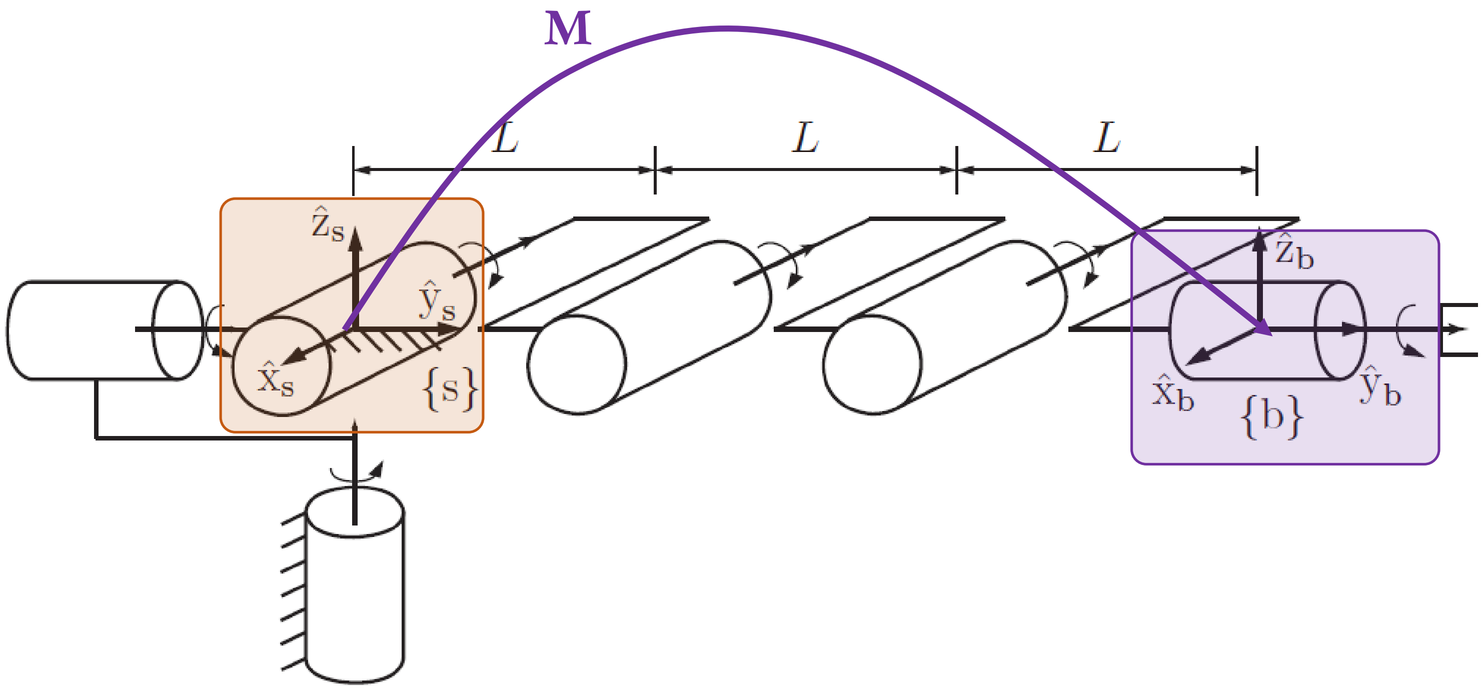
- This is the **body form** of the product of exponentials formula.

Example: 6R spatial open chain

Express **the body form** forward kinematics of the **6R open chain** shown in the Figure.



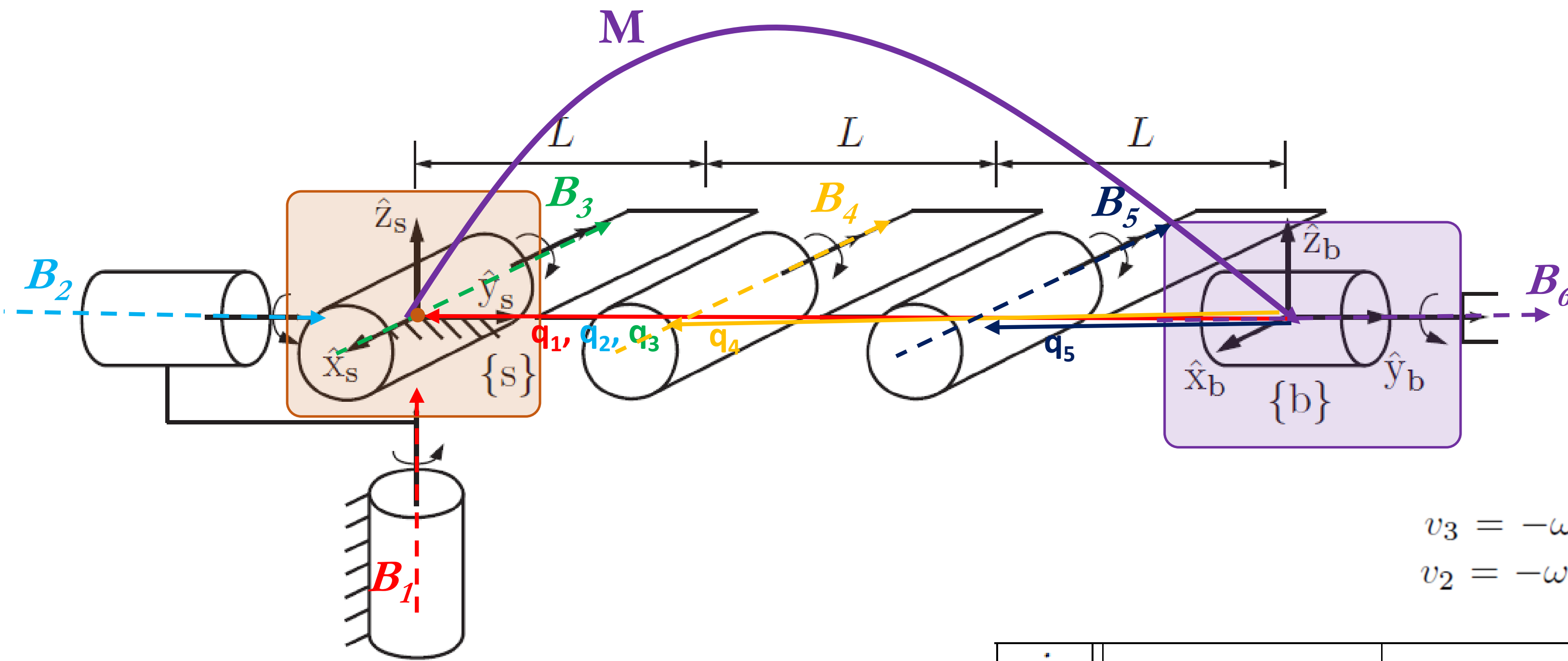
Example: 6R spatial open chain



- M **is still the same as the space form** obtained as the end-effector frame as seen from the fixed frame with the **chain in its zero position**.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: 6R spatial open chain



$$v_3 = -\omega_3 \times q_3$$
$$v_2 = -\omega_2 \times q_2$$

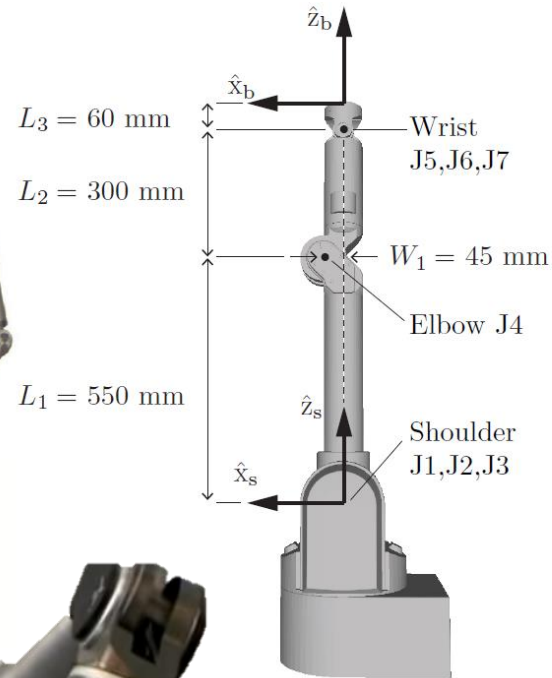
- The **screw axis** for each joint axis, expressed with respect to the end-effector frame in **<u>its zero position</u>**, is given in the table:

i	ω_i	v_i
1	$(0, 0, 1)$	$(-3L, 0, 0)$
2	$(0, 1, 0)$	$(0, 0, 0)$
3	$(-1, 0, 0)$	$(0, 0, -3L)$
4	$(-1, 0, 0)$	$(0, 0, -2L)$
5	$(-1, 0, 0)$	$(0, 0, -L)$
6	$(0, 1, 0)$	$(0, 0, 0)$

$$T(\theta) = \boxed{M}\boxed{e^{[\mathcal{B}_1]\theta_1}}\boxed{e^{[\mathcal{B}_2]\theta_2}}\cdots\boxed{e^{[\mathcal{B}_6]\theta_6}}.$$

Example: Barrett Technology's WAM 7R robot arm

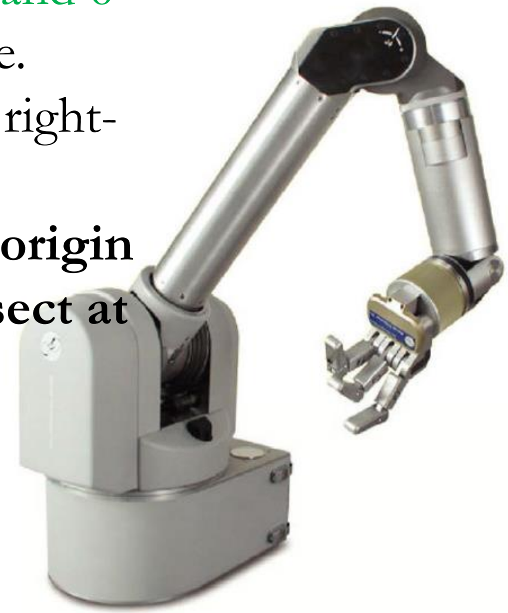
- Figure shows the redundant **Barrett Technology's WAM 7R robot** arm at its <u>zero configuration</u> (right).
- Some joints of the WAM are driven by motors placed at the base of the robot, reducing the robot's moving mass. Torques are transferred from the motors to the joints by cables winding around drums at the joints and motors.
- At the zero configuration, axes 1, 3, 5, and 7 are along $\hat{z}_s$ and axes 2, 4, and 6 are aligned with $\hat{y}_s$ out of the page. Positive rotations are given by the right-hand rule.
- Axes 1, 2, and 3 intersect at the origin of {s} and axes 5, 6, and 7 intersect at a point 60 mm from {b}.
- Express the body form forward kinematics of the robot.

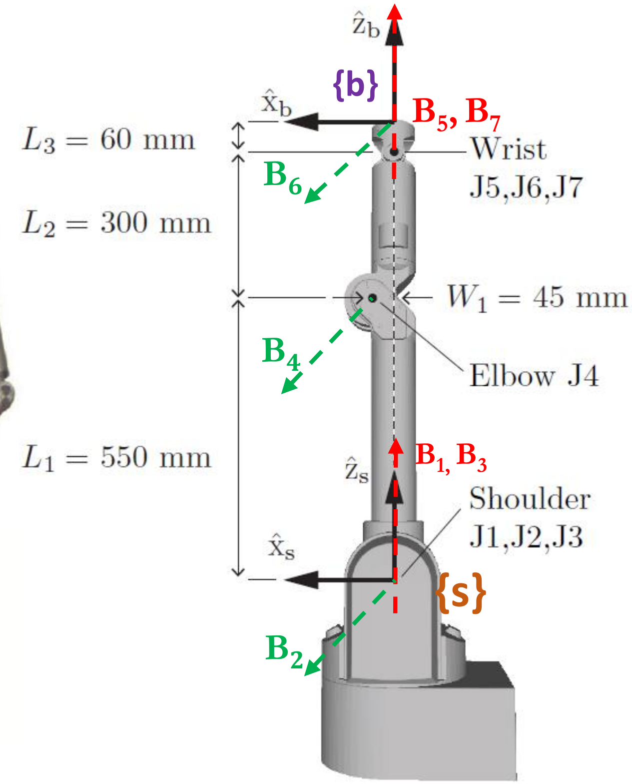


https://www.youtube.com/watch?v=oAjfjU7yxoY

Example: Barrett Technology's WAM 7R robot arm

- At the zero configuration, **axes 1, 3, 5, and 7 are along $\hat{z}_s$** and axes 2, 4, and 6 are aligned with $\hat{y}_s$ out of the page. Positive rotations are given by the right-hand rule.
- **Axes 1, 2, and 3 intersect at the origin of {s}** and **axes 5, 6, and 7 intersect at a point 60 mm from {b}.**





- The end-effector frame {b} in the zero position is given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Barrett Technology's WAM 7R robot arm

➤ The screw axes $\mathrm{B_i}$ are listed in the following table:

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(L_1+L_2+L_3,0,0)$
3	$(0,0,1)$	$(0,0,0)$
4	$(0,1,0)$	$(L_2+L_3,0,W_1)$
5	$(0,0,1)$	$(0,0,0)$
6	$(0,1,0)$	$(L_3,0,0)$
7	$(0,0,1)$	$(0,0,0)$

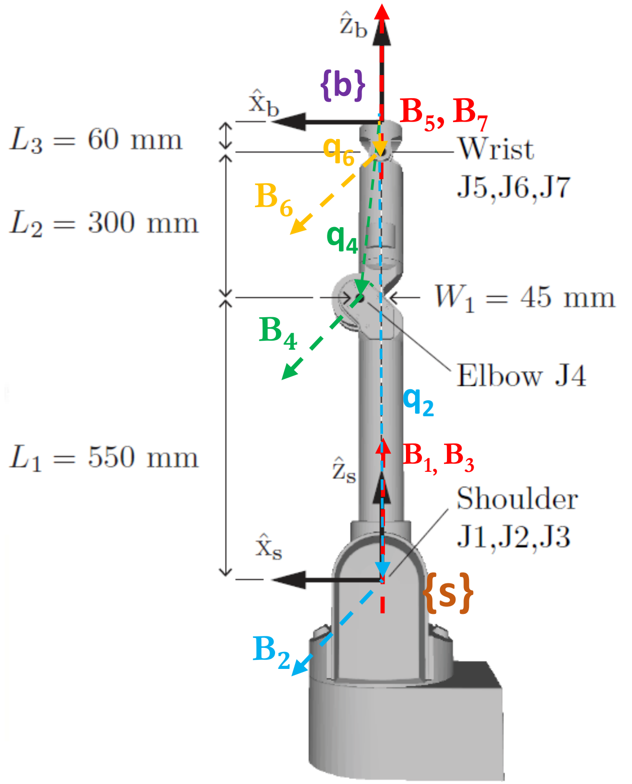


Figure shows the WAM arm with $\boldsymbol{\theta_2}$ **= 45,** $\boldsymbol{\theta_4}$ **= - 45,** $\boldsymbol{\theta_6}$ **= - 90** degrees and **all other joint angles equal to zero**, giving:

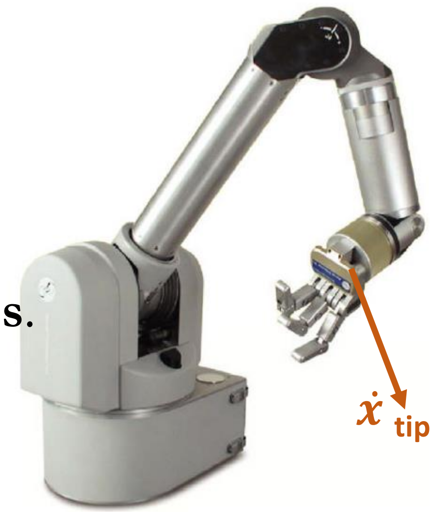
$$T(\theta) = Me^{[\mathcal{B}_2]\pi/4}e^{-[\mathcal{B}_4]\pi/4}e^{-[\mathcal{B}_6]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.3157 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.6571 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobian

➢ The forward kinematics ***f(θ)*** for a robot can be written as a nonlinear function mapping **joint-space** to the **task space**:

$$x(t) = f(\theta(t))$$

where $\boldsymbol{x} \in \mathbb{R}^m$ *is* a minimal set of coordinates defining the **end-effector configuration (pose)** and $\boldsymbol{\theta} \in \mathbb{R}^n$ is a set of **joint variables**.



➢ By the chain rule, the time derivative of ***x(t)*** at time ***t*** is

$$\dot{x} = \frac{\partial f(\theta)}{\partial \theta}\frac{d\theta(t)}{dt} = \boxed{\frac{\partial f(\theta)}{\partial \theta}}\dot{\theta}$$

$$= \boxed{J(\theta)}\dot{\theta},$$

where the velocity is given by $\dot{\boldsymbol{x}} = \frac{\boldsymbol{dx}}{\boldsymbol{dt}} \in \mathbb{R}^m$ and $\boldsymbol{J}(\boldsymbol{\theta}) \in \mathbb{R}^{m \times n}$ is called the Analytical Jacobian matrix.

➢ The Jacobian matrix $\boldsymbol{J}(\boldsymbol{\theta})$ represents the

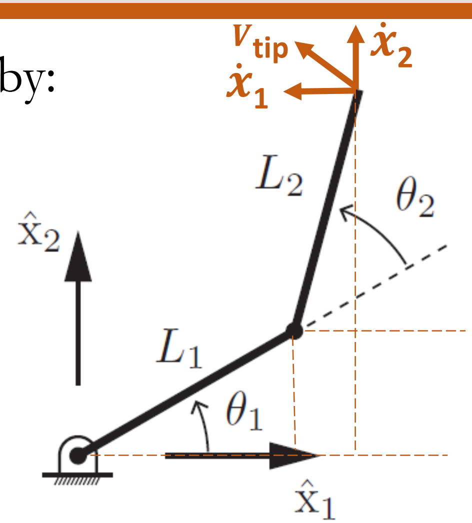
✓ **linear sensitivity** of the end-effector velocity $\dot{x}$ to the joint velocity $\dot{\boldsymbol{\theta}}(\mathbf{t})$,

✓ It is <u>**not** a constant matrix </u>and is a **function of the joint variables** $\boldsymbol{\theta}(\boldsymbol{t})$.

Jacobian

- Consider a 2R planar open chain with forward kinematics given by:

$$
\begin{aligned}
x_1 &= L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) \\
x_2 &= L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2).
\end{aligned}
$$



- Differentiating both sides with respect to time yields:

$$
\begin{aligned}
\dot{x}_1 &= -L_1\dot{\theta}_1 \sin\theta_1 - L_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) \\
\dot{x}_2 &= L_1\dot{\theta}_1 \cos\theta_1 + L_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2),
\end{aligned}
$$

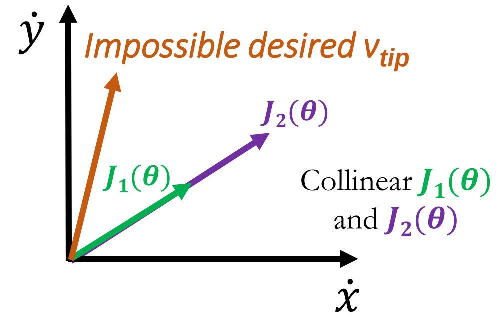
$$
\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} \underbrace{\begin{matrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \end{matrix}}_{J_1(\boldsymbol{\theta})} & \underbrace{\begin{matrix} -L_2\sin(\theta_1+\theta_2) \\ L_2\cos(\theta_1+\theta_2) \end{matrix}}_{J_2(\boldsymbol{\theta})} \end{bmatrix}}_{J(\boldsymbol{\theta})} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{\dot{\boldsymbol{\theta}}}
$$

- v_{tip} is the linear combination of $J_1(\boldsymbol{\theta})$ and $J_2(\boldsymbol{\theta})$:

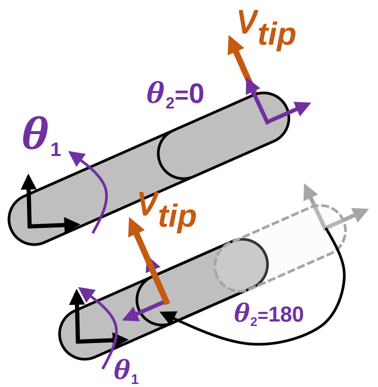
$$
v_{\text{tip}} = \boxed{J_1(\theta)}\dot{\theta}_1 + \boxed{J_2(\theta)}\dot{\theta}_2
$$

Singularity

$$v_{\text{tip}} = J(\theta)\dot{\theta} \Longrightarrow v_{\text{tip}} = \boxed{J_1(\theta)}\dot{\theta}_1 + \boxed{J_2(\theta)}\dot{\theta}_2$$



- As long as $J_1(\theta)$ and $J_2(\theta)$ are not collinear, it is <u>possible</u> to generate a tip velocity v_{tip} in <u>any arbitrary direction</u> in the x1-x2 plane by choosing <u>**appropriate joint velocities**</u>.

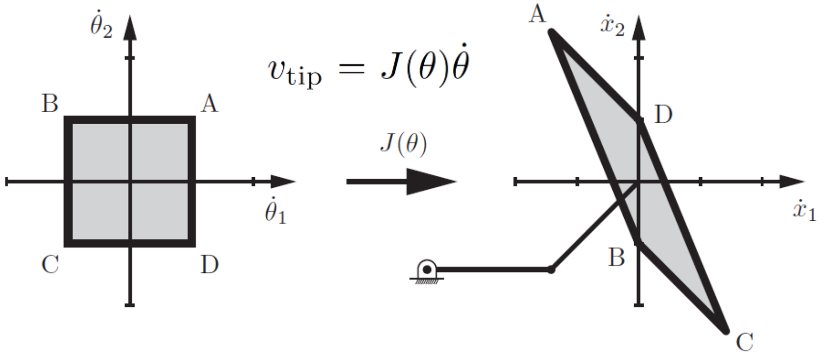


- **Singularity:** A situation where the robot tip is <u>unable to generate velocities in certain directions</u>.

- For example, for a 2 link robot, if <u>θ_2 is 0 or 180 degree</u> then, regardless of the value of θ_1, $J_1(\theta)$ and $J_2(\theta)$ will be collinear and the Jacobian $\mathbf{J(\theta)}$ becomes a singular matrix.

- In singularity the <u>ability of the tip to move in one direction is lost.</u>

- The **Jacobian** maps bounds on the rotational speed of the joints to bounds on v_{tip}.

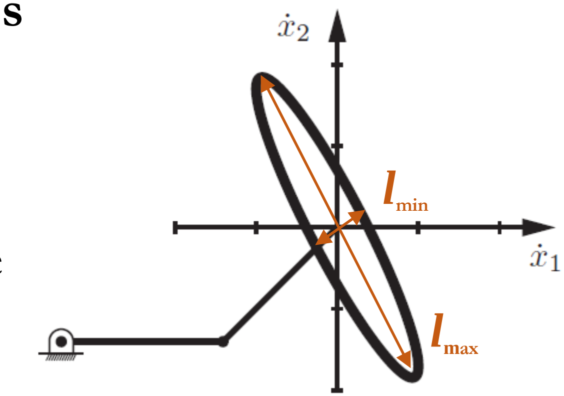


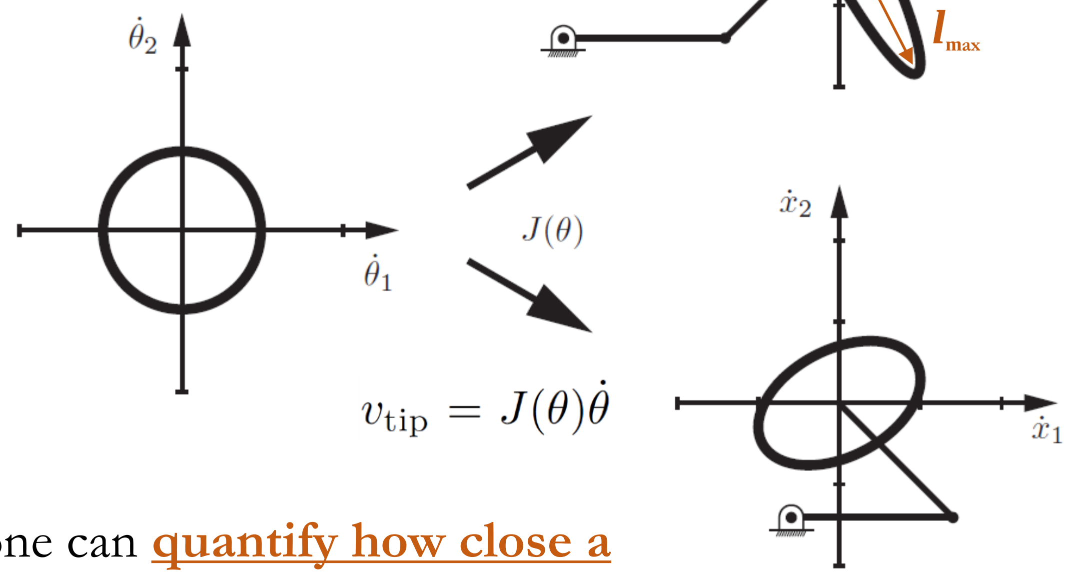
https://www.youtube.com/watch?v=UqyN7-tRS00

Manipulability Ellipsoid

- In a 2D plane, let’s consider a **unit circle of joint velocities** representing the <u>sum of squares of the joint velocities.</u>

- This **circle** maps through the Jacobian to an **ellipse in the space of tip velocities**, and this ellipse is referred to as the manipulability ellipsoid.





- As the manipulator configuration approaches a **singularity**, the <u>ellipse collapses to a line segment</u>, since the ability of the tip to move in one direction is lost.

- Using the manipulability ellipsoid one can **<u>quantify how close a given posture is to a singularity.</u>**

- The closer the ellipsoid is to a circle, i.e., the closer the **ratio $l_{max}=l_{min}$** is to 1, the <u>more easily can the tip move in arbitrary directions and thus the more removed it is from a singularity.</u>

[Manipulability demo in YouTube]

References

- Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.

- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**

- Lynch and Park, "*Modern Robotics*," Cambridge U. Press, 2017, **Chapter 3.**