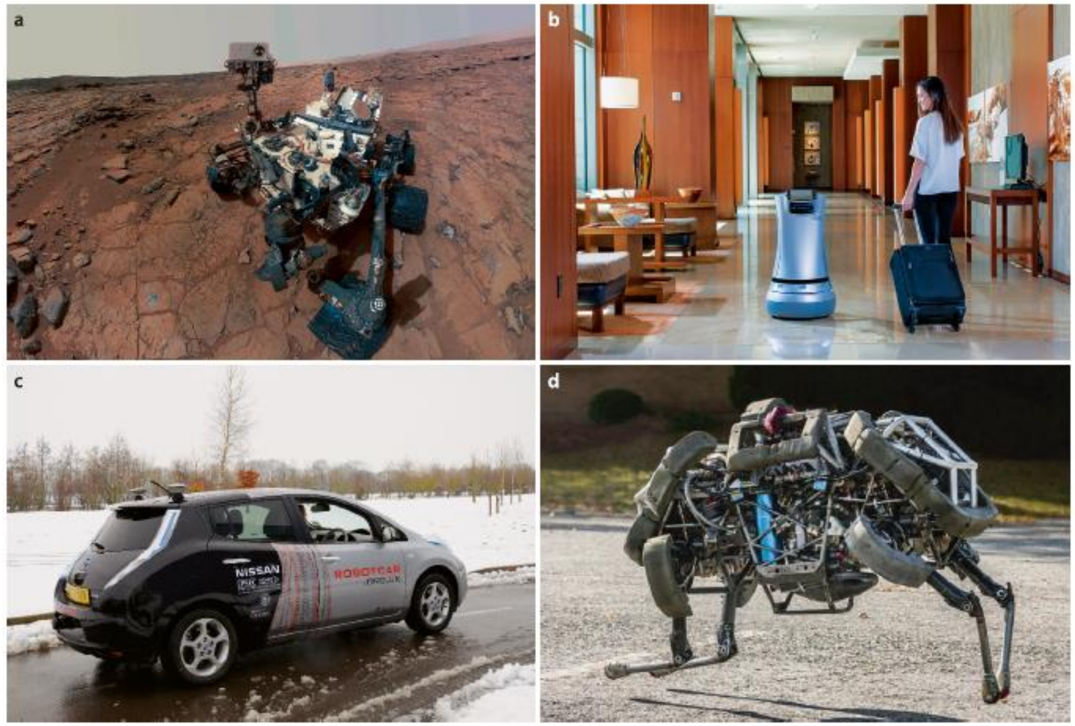
ME 397- ASBR

Week 5-Lecture 2



a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

FARSHID ALAMBEIGI, Ph.D.
Assistant Professor | Walker Department of Mechanical Engineering
Cockrell School of Engineering | The University of Texas at Austin

The University of Texas at Austin
Mechanical Engineering
Cockrell School of Engineering

Spring 2022

Matrix Logarithm of Rigid-Body Motions

- **Given** an arbitrary **transformation (R; p)∈ SE(3),** one can always find a **screw axis** $\boldsymbol{S} = (\boldsymbol{\omega}; \mathrm{v})$ and a scalar $\boldsymbol{\theta}$ such that

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} \boxed{R} & \boxed{p} \\ 0 & 1 \end{bmatrix}$$

$[\boldsymbol{S}]$ is the **4 × 4 matrix representation** of $\boldsymbol{S} = (\boldsymbol{\omega}; \mathrm{v})$

i.e., the matrix $[\boldsymbol{S}]\boldsymbol{\theta} = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix} \in se(3)$ is the matrix logarithm of T = (R; p).

- <u>**Given**</u> **(R; p)** written as T∈ SE(3), <u>find</u> a $\boldsymbol{\theta} \in [0; \pi]$ and a screw axis $\boldsymbol{S} = (\boldsymbol{\omega}; \boldsymbol{v}) \in \mathbb{R}^6$ (where <u>at least one of</u> $\|\boldsymbol{\omega}\|$ and $\|\boldsymbol{v}\|$ is **unity**) such that $e^{[\boldsymbol{S}]\boldsymbol{\theta}} = T$.

  The **vector** $\boldsymbol{S\theta} \in \mathbb{R}^6$ comprises the **exponential coordinates for T** and the **matrix** $[\boldsymbol{S}]\boldsymbol{\theta} \in se(3)$ is the **matrix logarithm of T.**

(a) If **R = I** then set $\boldsymbol{\omega} = 0$, $\boldsymbol{v} = p/\|p\|$, and $\boldsymbol{\theta} = \|p\|$.

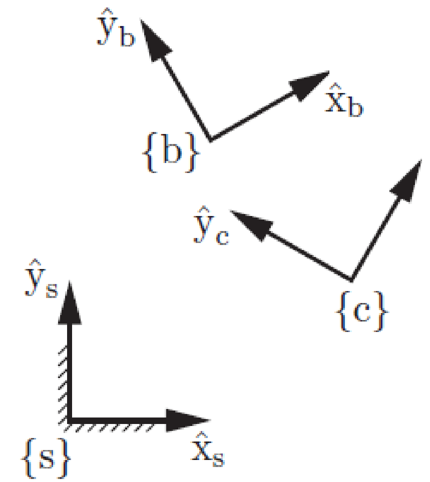
(b) Otherwise, first use the <u>**matrix logarithm on R∈ SO(3)**</u> to determine $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ for R. Next, $\boldsymbol{v}$ is calculated as:

$$\boxed{v} = G^{-1}(\theta)\boxed{p}$$

$$G^{-1}(\theta) = \frac{1}{\boxed{\theta}}I - \frac{1}{2}\boxed{[\omega]} + \left(\frac{1}{\boxed{\theta}} - \frac{1}{2}\cot\frac{\boxed{\theta}}{2}\right)\boxed{[\omega]}^2.$$

Example

The rigid-body motion is confined to the $\hat{\boldsymbol{x}}_s$-$\hat{\boldsymbol{y}}_s$ plane. The initial frame {b} and final frame {c} in the Figure can be represented by the following SE(3) matrices:



$$T_{sb} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{sc} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

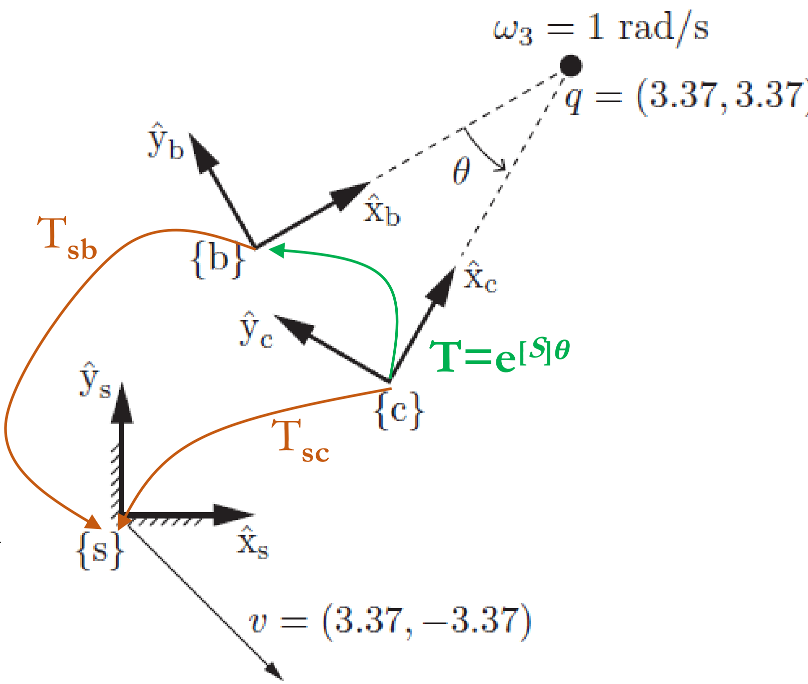
Find the **fixed frame screw motion** that displaces the frame at T_{sb} to T_{sc} .

Example

- We seek the **<u>fixed frame</u> screw motion** that displaces the frame at T_{sb} to T_{sc}:

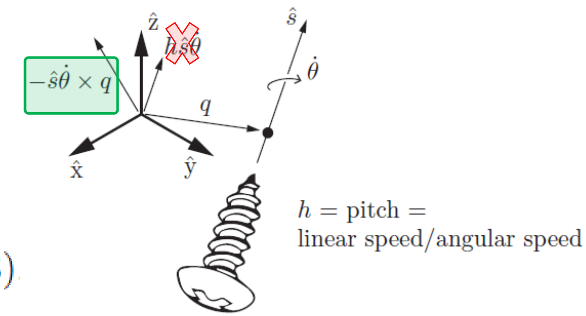
$$T_{sc} = e^{[S]\theta}\, T_{sb} \;\Longrightarrow\; T_{sc} T^{-1}_{sb} = e^{[S]\theta}$$

- Because the **<u>motion is 2D</u> and occurs in the $\hat{x}_s$-$\hat{y}_s$ plane**, the corresponding screw has an **axis of rotation** in the direction of the $\hat{z}_s$-axis and has **zero pitch** **(since it does not have a translation along the screw axis $\hat{z}_s$)**. The screw axis $S = (\omega; v)$, expressed in {s}, therefore has the following form:



$$\omega = (0, 0, \omega_3),$$
$$v = (v_1, v_2, 0).$$

$$[S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3),$$



$$[S] = \begin{bmatrix} 0 & -\omega_3 & 0 & v_1 \\ \omega_3 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$

Example

- We can apply the **matrix logarithm algorithm** directly to $T_{sc}T^{-1}_{sb}$ to obtain **[S]** (and therefore S) and $\boldsymbol{\theta}$ as follows:

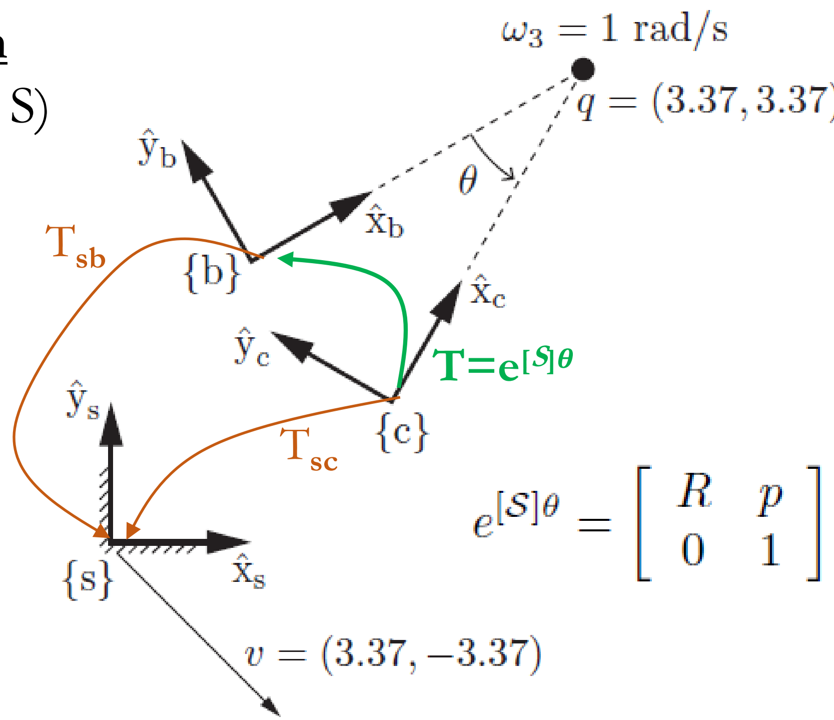
(i) We first use the matrix logarithm on **R∈ SO(3)** to determine $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ for R (Rodriguez formula: check W2-L2).

(ii) Then $\boldsymbol{v}$ is calculated as:

$$v = G^{-1}(\theta)p$$

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2.$$

(iii) The matrix $[S]\boldsymbol{\theta} = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix} \in se(3)$



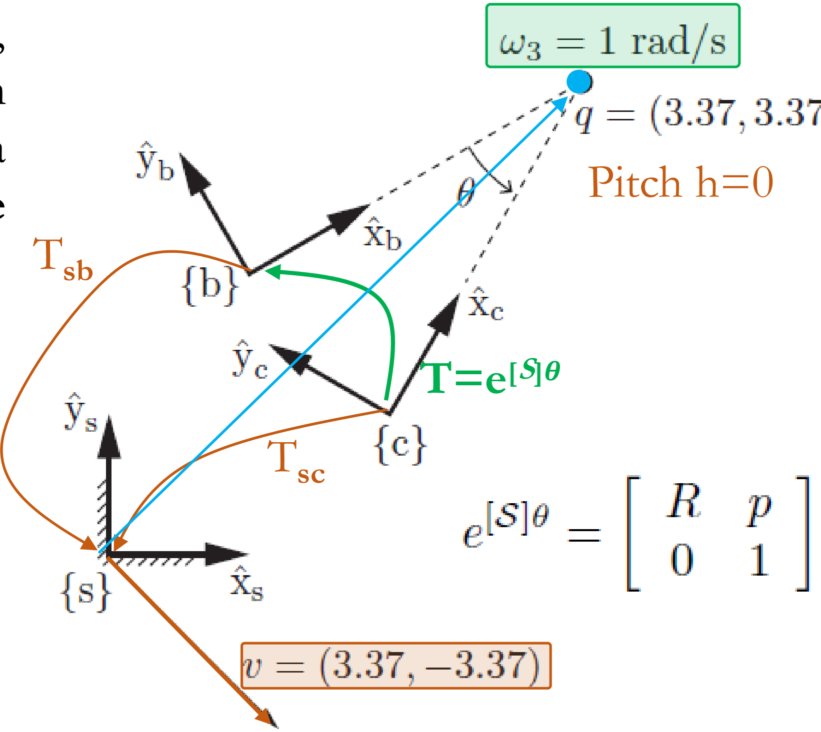
$$T_{sc}T^{-1}_{sb} = e^{[S]\theta}$$

$$[S] = \begin{bmatrix} 0 & -1 & 0 & 3.37 \\ 1 & 0 & 0 & -3.37 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3.37 \\ -3.37 \\ 0 \end{bmatrix}, \quad \boxed{\theta = \frac{\pi}{6} \text{ rad (or } 30^\circ).}$$

Example

- The value of ***S*** means that the **constant screw axis, expressed in the fixed frame {s},** is represented by an angular velocity of 1 rad/s about the $\hat{z}_s$-axis and a linear velocity of (3.37;-3.37; 0) **expressed in the frame {s}.**

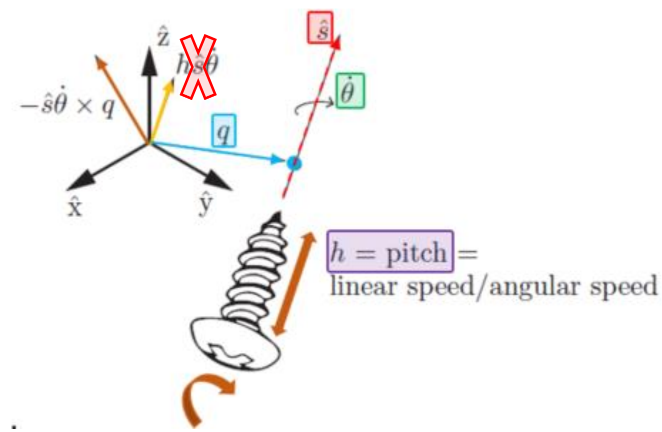
$$\mathcal{S} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3.37 \\ -3.37 \\ 0 \end{bmatrix}, \quad \theta = \frac{\pi}{6} \text{ rad (or } 30^\circ).$$



- We can also graphically determine the point $q = (q_x; q_y)$ in the $\hat{x}_s$-$\hat{y}_s$ plane through which the screw axis passes; for our example this point is given by $q = (3.37; 3.37)$.

- Screw axis can either be defined by **point q, pitch h, and axis s** OR screw axis **S**.

- Transformation T can be defined using translation and rotation about screw axis!



Summary of Rigid Body Motion

W1-L2

W2-L1

W3-L2

W3-L1

W5-L1

Rotations	Rigid-Body Motions
$R \in SO(3)$: 3×3 matrices $R^{\mathrm{T}}R = I, \det R = 1$	$T \in SE(3)$: 4×4 matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^{\mathrm{T}}$	$T^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$
change of coordinate frame: $R_{ab}R_{bc} = R_{ac}, \quad R_{ab}p_b = p_a$	change of coordinate frame: $T_{ab}T_{bc} = T_{ac}, \quad T_{ab}p_b = p_a$
rotating a frame {b}: $R = \mathrm{Rot}(\hat{\omega}, \theta)$ $R_{sb'} = RR_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ $R_{sb''} = R_{sb}R$: rotate θ about $\hat{\omega}_b = \hat{\omega}$	displacing a frame {b}: $T = \begin{bmatrix} \mathrm{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$ $T_{sb'} = TT_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ (moves {b} origin), translate p in {s} $T_{sb''} = T_{sb}T$: translate p in {b}, rotate θ about $\hat{\omega}$ in new body frame
unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$, where $\|\hat{\omega}\| = 1$	"unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$
	for a screw axis $\{q, \hat{s}, h\}$ with finite h, $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$
angular velocity is $\omega = \hat{\omega}\dot{\theta}$	twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$

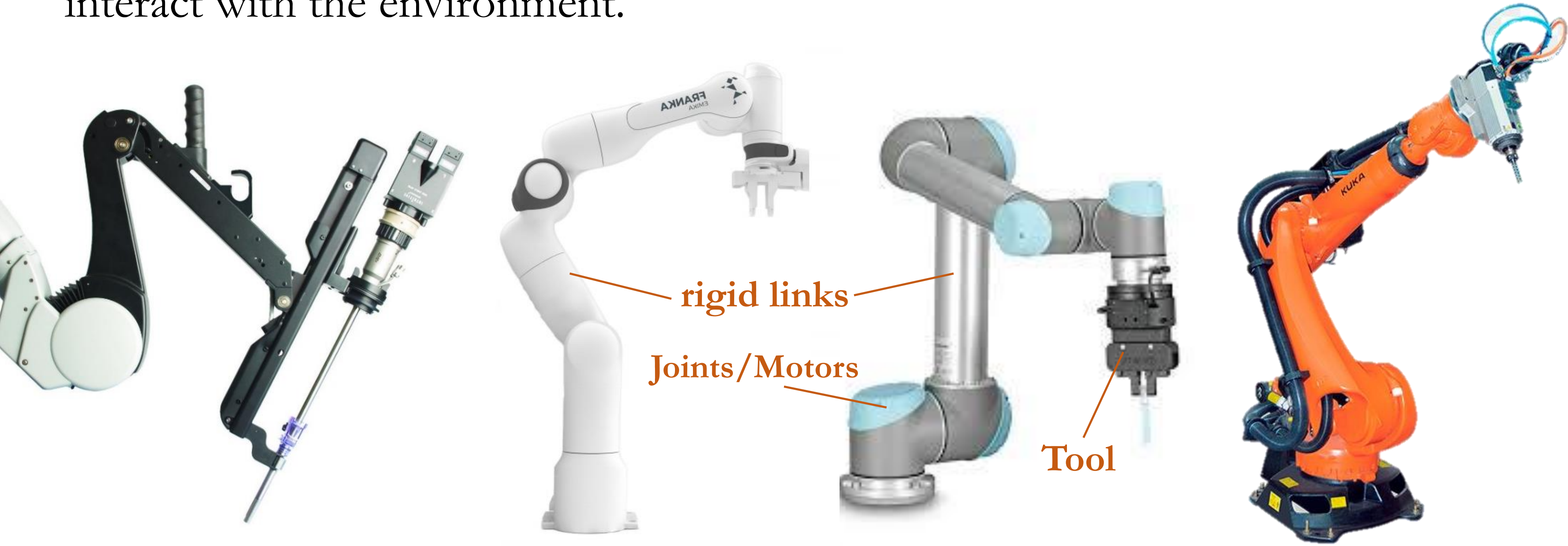
Summary of Rigid Body Motion

W4-L1 W2-L2 W4-L2 W5-L1

Rotations (cont.)	Rigid-Body Motions (cont.)
for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$ identities, $\omega, x \in \mathbb{R}^3$, $R \in SO(3)$: $[\omega] = -[\omega]^{\mathrm{T}}$, $[\omega]x = -[x]\omega$, $[\omega][x] = ([x][\omega])^{\mathrm{T}}$, $R[\omega]R^{\mathrm{T}} = [R\omega]$	for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$ (the pair (ω, v) can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context)
$\dot{R}R^{-1} = [\omega_s]$, $R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s]$, $T^{-1}\dot{T} = [\mathcal{V}_b]$
	$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$ identities: $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$, $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1 T_2}]$
change of coordinate frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b$, $\omega_a = R_{ab}\omega_b$	change of coordinate frame: $\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b$, $\mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$
exp coords for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$	exp coords for $T \in SE(3)$: $\mathcal{S}\theta \in \mathbb{R}^6$
$\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$ $$R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$$ $$I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$	$\exp : [\mathcal{S}]\theta \in se(3) \to T \in SE(3)$ $$T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$$ where $* =$ $(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$
$\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$ algorithm in Section 3.2.3.3	$\log : T \in SE(3) \to [\mathcal{S}]\theta \in se(3)$ algorithm in Section 3.3.3.2

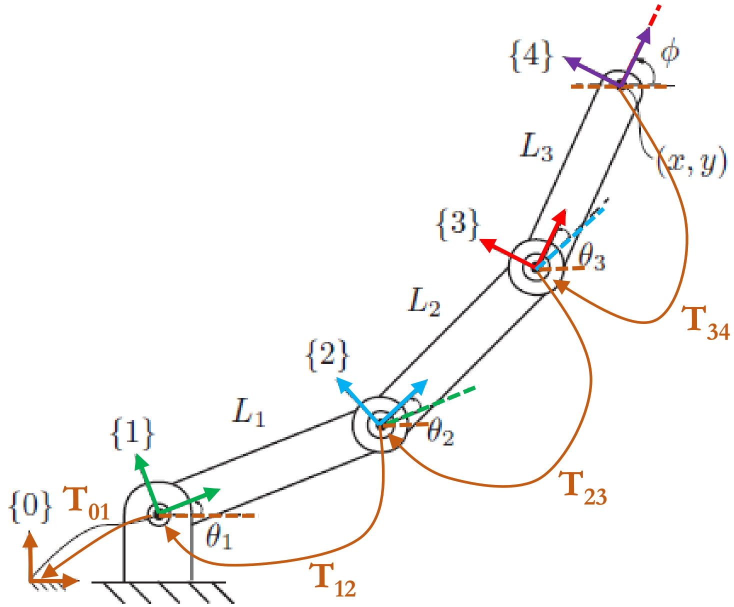
Manipulator Kinematics

- Most modern manipulators consist of a set of **rigid links** connected together by a set of joints (e.g., revolute, and prismatic) correspond to subgroups of the special Euclidean group SE(3)
- **Motors** are attached to the joints so that the overall motion of the mechanism can be controlled to perform a given task.
- A **Tool**, typically a gripper of some sort, is attached to the end of the robot to interact with the environment.



https://www.youtube.com/watch?v=bmge6R2oyL0

Forward Kinematics



- The **forward kinematics** of a robot refers to the calculation of the **position and orientation** of its end-effector frame **from its joint coordinates**.

- The Geometric forward kinematics problem for a 3R planar open chain:

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3), \\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3), \\
\phi &= \theta_1 + \theta_2 + \theta_3.
\end{aligned}
$$

- A more systematic method of deriving the forward kinematics might involve attaching reference frames {1}, {2} and {3} to each link.

- The forward kinematics can then be written as a **product of four homogeneous transformation matrices**

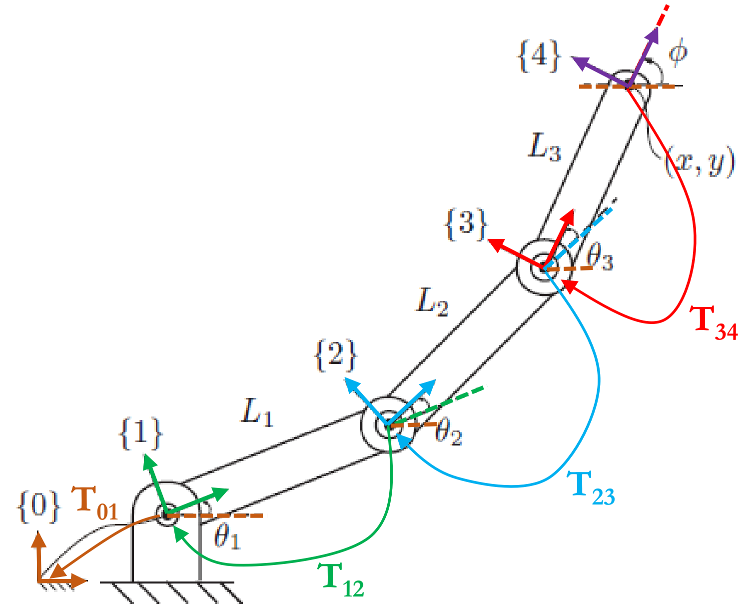
$$T_{04} = T_{01} T_{12} T_{23} T_{34}$$

Forward Kinematics

$$T_{04} = \boxed{T_{01}}\boxed{T_{12}}\boxed{T_{23}}\boxed{T_{34}}$$

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

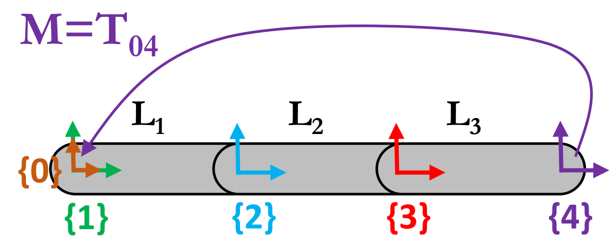


- Observe that $\mathbf{T}_{34}$ is **constant** and that each remaining $\mathbf{T}_{i-1:i}$ depends only on the joint variable $\boldsymbol{\theta}_i$.

- **D**enavit-**H**artenberg parameters (**D-H parameters**) representation of forward kinematics.

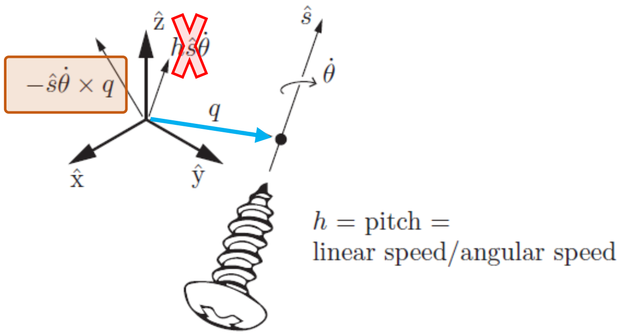
Forward Kinematics

- As an alternative approach, let us define **M** to be the **position and orientation** of **frame {4} in {0} frame** when all joint angles are set to zero (the "home or zero" position of the robot), i.e., $\mathbf{T}_{04}$. Then



$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

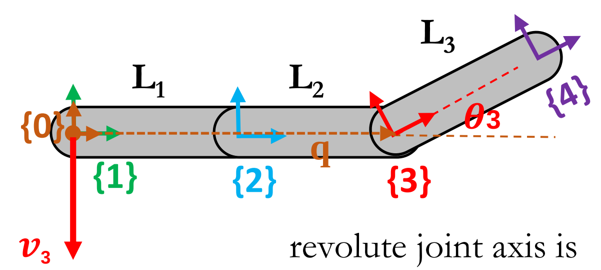
- Each **revolute joint axis** is a **zero-pitch** (2D motion) **screw axis** and its **location** is center of rotation defining **parameter q**.



- If $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are held at their **zero position** then the **screw axis** corresponding to rotating about **joint 3** can be expressed in the {0} frame as

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

$$v_3 = -\omega_3 \times q_3$$
$$q_3 = (L_1 + L_2, 0, 0)$$



revolute joint axis is out of the plane.

Forward Kinematics

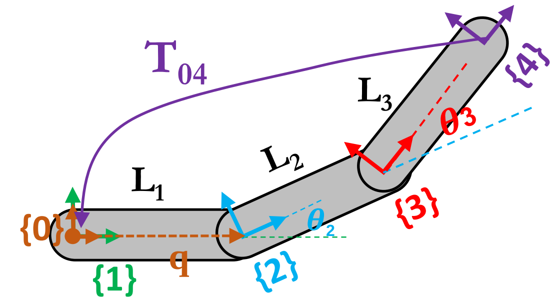
- The screw axis S_3 can be expressed in se(3) matrix form as

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ -(L_1 + L_2) \\ 0 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix}$$

- For any $\boldsymbol{\theta}_3$, the matrix exponential representation for screw motions is

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = \theta_2 = 0)$$

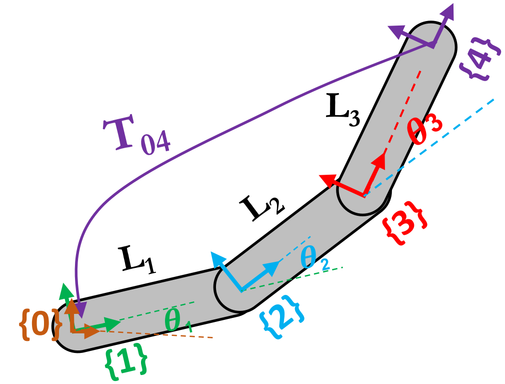
- Now, for $\boldsymbol{\theta}_1 = 0$ <u>and</u> **any <u>fixed (but arbitrary)</u>** $\boldsymbol{\theta}_3$, rotation about **joint 2** can be viewed as applying a screw motion to the rigid (link 2)/(link 3) pair, i.e.,



$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = 0)$$

$$[\mathcal{S}_2] = \begin{bmatrix} \boxed{\begin{matrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ -L_1 \\ 0 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix}$$

Forward Kinematics



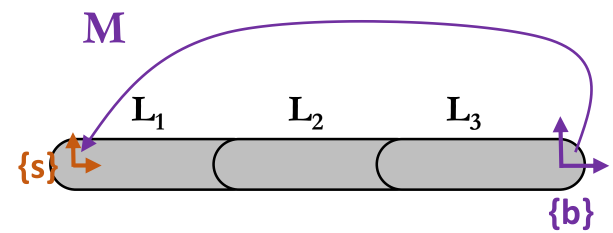
- Finally, keeping $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$ **fixed**, rotation about **joint 1** can be viewed as applying a screw motion to the entire rigid three-link assembly. We can therefore write, for arbitrary values of ($\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$):

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

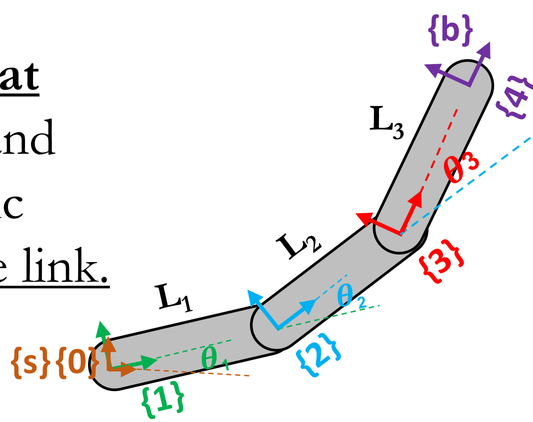
$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Thus the **forward kinematics** can be expressed as a product of matrix exponentials, each <u>corresponding to a screw motion</u>.

Product of Exponentials (PoE) Formula



- To use the PoE formula, it is **only necessary** to
  - ✓ Assign a stationary frame {s} (e.g., at the **fixed base of the robot** or anywhere else that is convenient for defining a reference frame)
  - ✓ A frame {b} at the end-effector, described by M when the robot is at its **zero position**.
- It is common to define a frame **at each link**, though, typically **at the joint axis**; these are needed for the **D-H representation** and they are useful for displaying a graphic rendering of a geometric model of the robot and for defining the mass properties of the link.



- Thus, when we are defining the kinematics of an **n-joint** robot, we may either

  (1) minimally use the frames {s} and {b} if we are only interested in the kinematics,

  (2) Or refer to {s} as frame {0}, use frames {i} for i = 1:n (the frames for links i at joints i), and use one more frame {n + 1} (corresponding to {b}) at the end-effector.
- The frame {n + 1} (i.e., {b})) is fixed relative to {n}, but it is at a more convenient location to represent the configuration of the end-effector.

Forward Kinematics: Screw Axes in the Base Frame

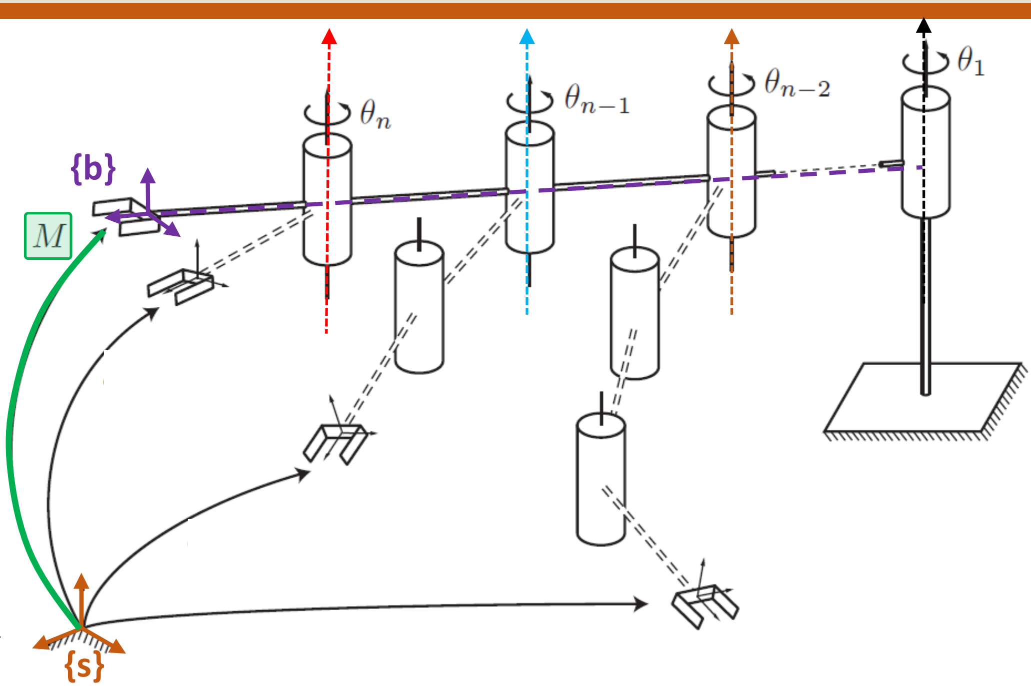
- Let's consider a general **spatial open chain** consisting of ***n*** **one-dof joints** that are **connected serially**.

- The key concept behind the PoE formula is to regard **each joint** as applying a **screw motion** to all the outward links.

- To apply the **PoE formula**, we must
  1) choose a fixed base frame {s} and an end-effector frame {b} attached to the last link.

  2) Place the robot in its zero position by setting all joint values to zero, with **assigning the direction of positive displacement** (rotation for revolute joints, translation for prismatic joints) for each joint specified.

  3) Let **M ∈ SE(3)** denote the configuration of the **end-effector frame relative to the fixed base frame** when the robot is in its **zero position.**

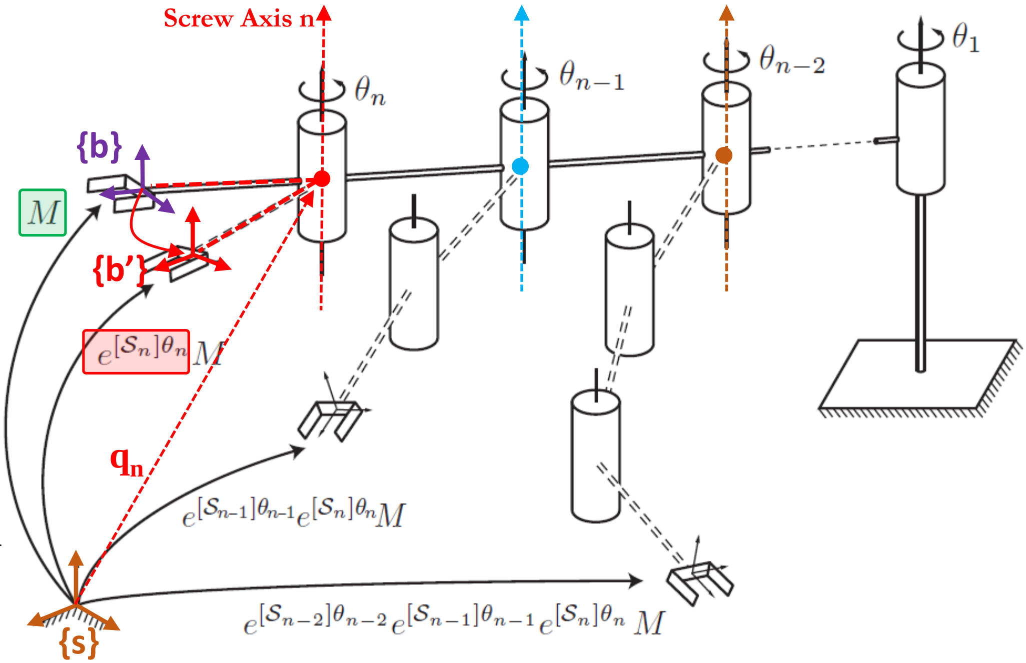


Forward Kinematics: Screw Axes in the Base Frame

4) Now suppose that joint n is displaced to some joint value $\boldsymbol{\theta}_n$. The end-effector frame M then undergoes a displacement of the form

$$T = \boxed{e^{[\mathcal{S}_n]\theta_n}} M,$$

where T ∈ SE(3) is the new configuration of the end-effector frame and $\boldsymbol{S} = (\boldsymbol{\omega}; \boldsymbol{v})$, is the screw axis of joint n as expressed in the fixed base frame.

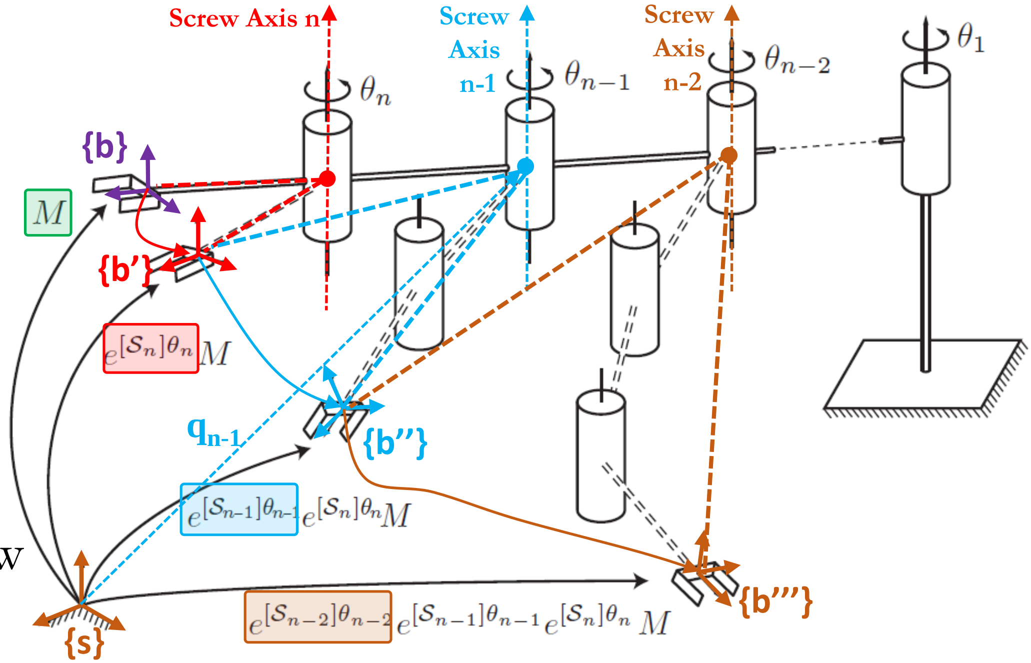


- If joint n is **revolute** (corresponding to a screw motion of **zero pitch**) then
  - ✓ $\boldsymbol{\omega}_n \in \mathbb{R}^3$ is a unit vector in the **positive direction of joint axis *n***;
  - ✓ $\mathbf{v}_n = -\boldsymbol{\omega}_n \times \mathbf{q}_n$ with $\mathbf{q}_n$ **any arbitrary point on joint axis n** as written in coordinates in the **fixed base frame**;
  - ✓ $\boldsymbol{\theta}_n$ is the joint angle.
- If joint n is **prismatic** then $\boldsymbol{\omega}_n = 0$,
  - ✓ $\mathbf{v}_n \in \mathbb{R}^3$ **is a unit vector** in the **direction of positive translation**,
  - ✓ $\boldsymbol{\theta}_n$ represents the prismatic extension/retraction.

Forward Kinematics: Screw Axes in the Base Frame

5) If we assume that joint n-1 is also allowed to vary then this has the effect of applying a screw motion to link n-1. The end-effector frame thus undergoes a displacement of the form

$$T = \boxed{e^{[\mathcal{S}_{n-1}]\theta_{n-1}}}\left(e^{[\mathcal{S}_n]\theta_n} M\right)$$



6) Continuing with this reasoning and now allowing **all the joints** (***θ***$_1$; … ; ***θ***$_n$) to vary, it follows that:

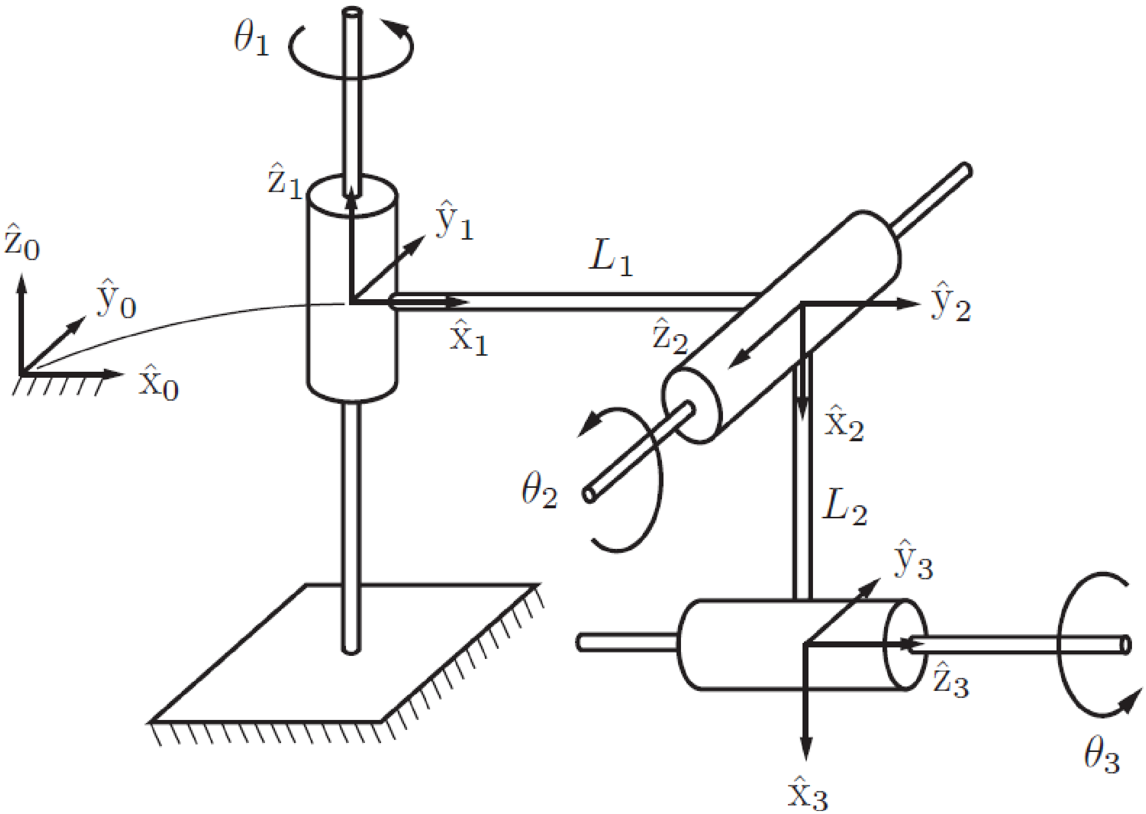
$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots \boxed{e^{[\mathcal{S}_{n-1}]\theta_{n-1}}}\boxed{e^{[\mathcal{S}_n]\theta_n}}\boxed{M}$$

- This is the **space form** (i.e., screw axes expressed in the fixed space frame **defining the order of multiplications**) **product of exponentials formula** describing the forward kinematics of an **n-dof open chain.**
- **Unlike the D-H representation,** no link reference frames need to be defined!

Example: 3 Revolute (R) spatial open chain

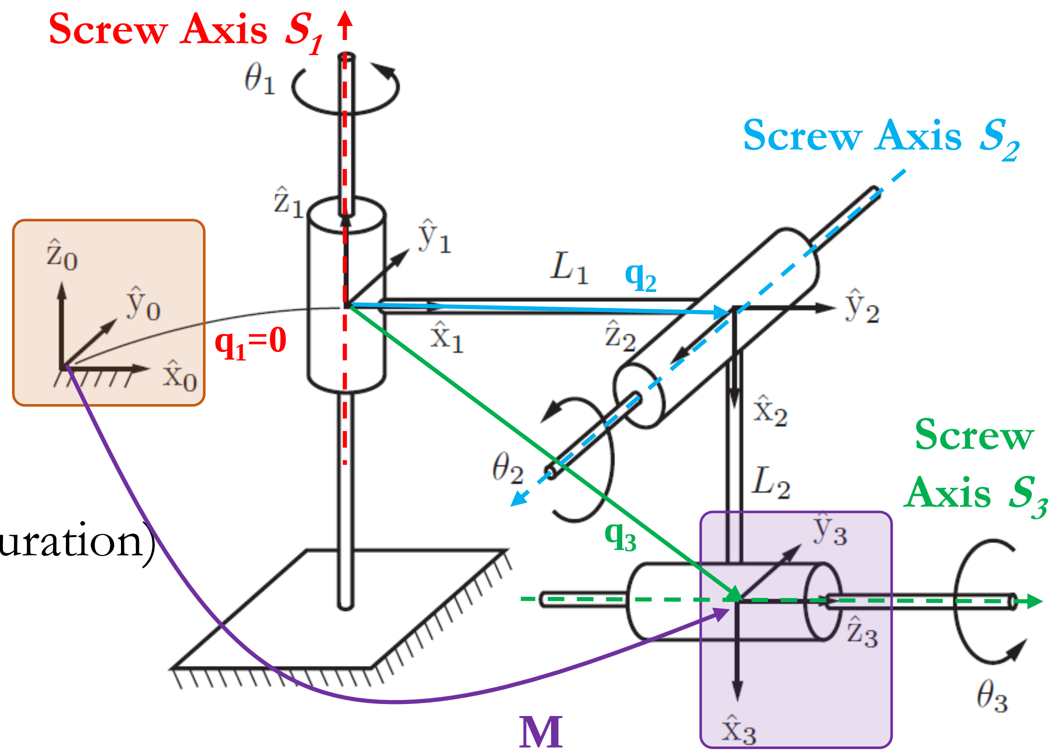
Consider the 3R open chain of the figure shown in **its home position** (all joint variables set equal to zero).

Find the forward kinematics of the robot using the **space form** of the exponential products.



Example: 3R spatial open chain

- We should express all vectors and homogeneous transformations in terms of the **fixed frame**.



Step 1) Choose the fixed frame {0} and end-effector frame {3} as indicated in the figure.

Step 2) By inspection M (Home configuration) can be obtained as:

$$M = \begin{array}{c} \begin{array}{ccc} \hat{x}_3 & \hat{y}_3 & \hat{z}_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$v = -\omega \times q$

$v_2 = -\omega_2 \times q_2$

$v_3 = -\omega_3 \times q_3$

Step 3) The screw axis $S_1 = (\omega_1; v_1)$, for joint axis 1 is $\omega_1 = (0; 0; 1)$ and $v_1 = (0; 0; 0)$

The screw axis $S_2 = (\omega_2; v_2)$, for joint axis 2 is $\omega_2 = (0; -1; 0)$ and $v_2 = (0; 0; -L_1)$

The screw axis $S_3 = (\omega_3; v_3)$, for joint axis 3 is $\omega_3 = (1; 0; 0)$ and $v_3 = (0; -L_2; 0)$

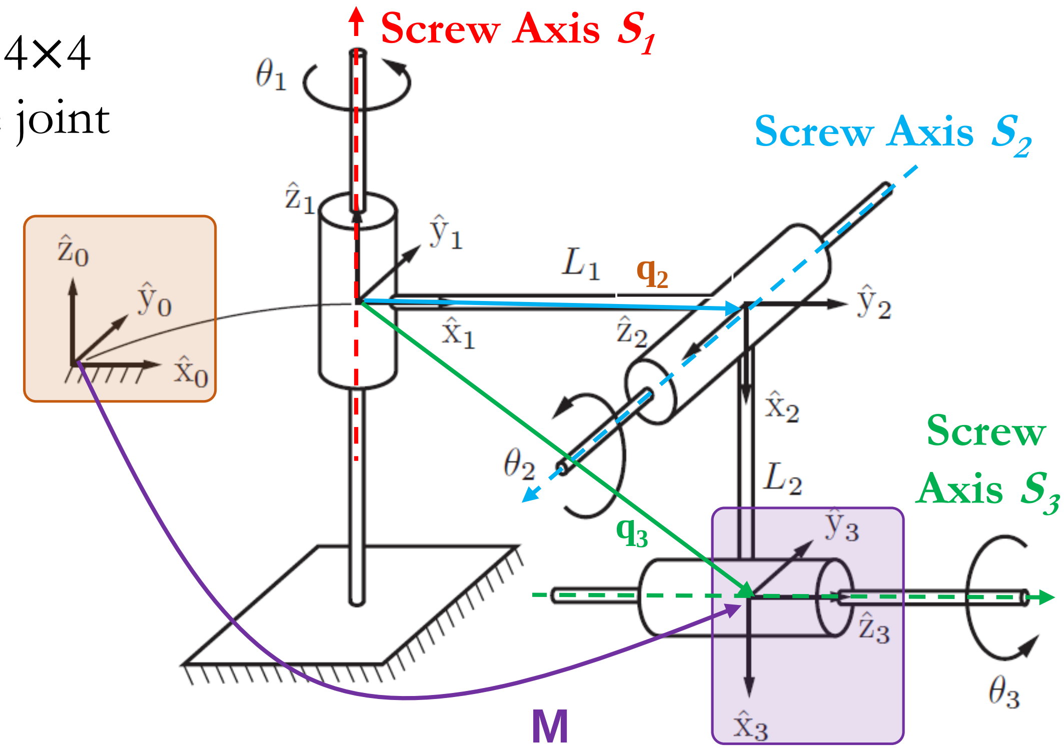
Example: 3R spatial open chain

- In summary, we have the following 4×4 matrix representations for the three joint screw axes S_1, S_2, and S_3:

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



$$v_3 = -\omega_3 \times q_3$$
$$v_2 = -\omega_2 \times q_2$$

- The forward kinematics has the following form:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$$T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$$

where $* =$
$(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, -1, 0)$	$(0, 0, -L_1)$
3	$(1, 0, 0)$	$(0, L_2, 0)$

References

- Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.

- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**

- Lynch and Park, "*Modern Robotics*," Cambridge U. Press, 2017, **Chapter 3.**