ME 397- ASBR

Week 2-Lecture 1

a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

FARSHID ALAMBEIGI, Ph.D.
Assistant Professor | Walker Department of Mechanical Engineering
Cockrell School of Engineering | The University of Texas at Austin

The University of Texas at Austin
Mechanical Engineering
Cockrell School of Engineering

Spring 2022

Skew Symmetric matrices

- Given a vector $x = [x_1\ x_2\ x_3]^T \in R^3$, define: $[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$

- The space of <u>n×n skew-symmetric matrices *S*</u> is: $so(n) = \{S \in \mathbb{R}^{n\times n} : S^T = -S\}$

- We can write the **cross product of two** **vectors** as: $a \times b = (a)^{\wedge} b.$

Vector linear differential equation

The linear differential equation $\boxed{\dot{x}(t) = Ax(t)}$ *with initial condition* $x(0) = x_0$, *where* $\boxed{A \in \mathbb{R}^{n\times n} \text{ is constant}}$ *and* $x(t) \in \mathbb{R}^n$, *has solution*

$$x(t) = e^{At}x_0$$

where

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots .$$

The matrix exponential e^{At} *further satisifies the following properties:*

(a) $d(e^{At})/dt = Ae^{At} = e^{At}A$.

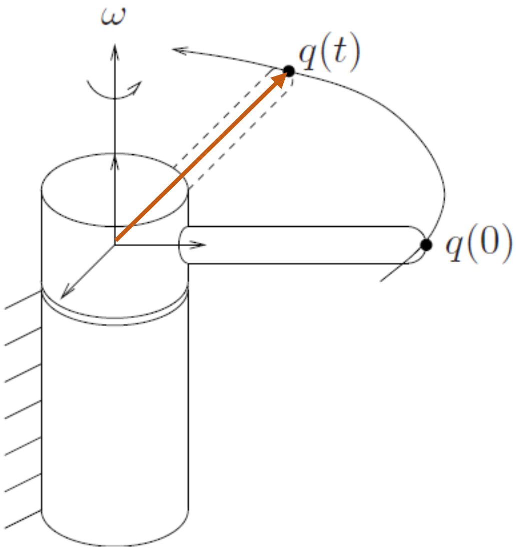
(b) *If* $A = PDP^{-1}$ *for some* $D \in \mathbb{R}^{n\times n}$ *and invertible* $P \in \mathbb{R}^{n\times n}$ *then* $e^{At} = Pe^{Dt}P^{-1}$.

(c) *If* $AB = BA$ *then* $e^A e^B = e^{A+B}$.

(d) $(e^A)^{-1} = e^{-A}$.

Exponential coordinates for rotation

- Let's rotate the rigid body at **constant unit velocity about the axis ω**, the velocity of the point, $\dot{q}$, may be written as the following **linear differential equation**:



$$\dot{q}(t) = \omega \times q(t) = \widehat{\omega} q(t) \quad \Rightarrow \quad q(t) = e^{\widehat{\omega} t} q(0)$$

where q(0) is the initial (t = 0) position of the point and $e^{\widehat{\omega} t}$ is the matrix exponential:

$$e^{\widehat{\omega} t} = I + \widehat{\omega} t + \frac{(\widehat{\omega} t)^2}{2!} + \frac{(\widehat{\omega} t)^3}{3!} + \cdots$$

- It follows that if we **rotate** about the **axis ω at unit velocity** for **θ units of time (t = θ)**, then the **net rotation** is given by: $R(\omega, \theta) = e^{\widehat{\omega} \theta}$
- Given a matrix $\widehat{\omega} \in so(3)$, $\|\omega\| = 1$, and a real number θ ∈ R, we write the exponential $\widehat{\omega}\theta$ as

$$\exp(\widehat{\omega}\theta) = e^{\widehat{\omega}\theta} = I + \theta\widehat{\omega} + \frac{\theta^2}{2!}\widehat{\omega}^2 + \frac{\theta^3}{3!}\widehat{\omega}^3 + \ldots$$

- This is an **infinite series** and, hence, not useful from a computational standpoint.
- It can be shown that if **the matrix is constant** and **finite** then this series is always guaranteed **to converge to a finite limit**.

Exponential coordinates for rotation

- Given $\widehat{a} \in so(3)$, the following relations hold:

$$\widehat{a}^2 = aa^T - \|a\|^2 I$$
$$\widehat{a}^3 = -\|a\|^2 \widehat{a}$$

and higher powers can be calculated recursively.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

- Utilizing this with $\boxed{a = \omega\theta, \ \|\omega\| = 1}$, we have:

$$\exp(\widehat{\omega}\theta) = e^{\widehat{\omega}\theta} = I + \theta\widehat{\omega} + \frac{\theta^2}{2!}\widehat{\omega}^2 + \frac{\theta^3}{3!}\widehat{\omega}^3 + \ldots$$

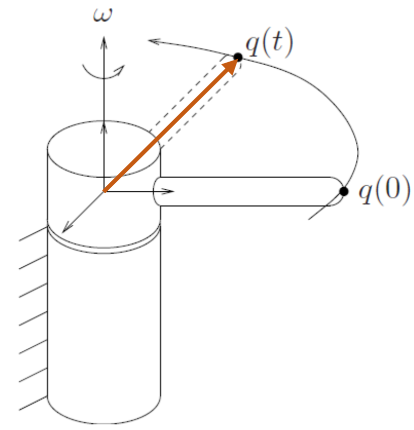
$$\Rightarrow \quad e^{\widehat{\omega}\theta} = I + \boxed{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}\widehat{\omega} + \boxed{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}\widehat{\omega}^2$$

$$\Rightarrow \quad \mathrm{Rot}(\hat{\omega}, \theta) = e^{\widehat{\omega}\theta} = I + \widehat{\omega}\boxed{\sin\theta} + \widehat{\omega}^2\boxed{(1 - \cos\theta)} \in SO(3)$$ **Rodrigues' formula**

- $\boxed{\text{When } \|\omega\| \neq 1,}$ it may be verified:

$$e^{\widehat{\omega}\theta} = I + \frac{\widehat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\widehat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\theta))$$

Exponential coordinates for rotation



- The quantity $e^{\widehat{\omega}\theta}\boldsymbol{q}$ has the effect of rotating $\boldsymbol{q}$ about the fixed-frame axis **ω** by an angle **θ**.
- Similarly, considering that a rotation matrix **R** consists of three column vectors, the rotation matrix $R' = \mathrm{Rot}(\hat{\omega}, \theta)R$ is the orientation achieved by rotating R by an angle **θ** about **axis ω** **in the fixed frame.**
- Reversing the order of matrix multiplication, $R'' = R\,\mathrm{Rot}(\hat{\omega}, \theta)$ is the orientation achieved by **rotating R in the body frame.**

$$R'' = Re^{[\hat{\omega}_2]\theta_2} \neq R' = e^{[\hat{\omega}_2]\theta_2}R.$$

- Exponentials of skew symmetric matrices are orthogonal!
- Given a skew-symmetric matrix $\widehat{\omega} \in so(3)$ *and* $\theta \in \mathbb{R}$ ➡ $e^{\widehat{\omega}\theta} \in SO(3)$.

- **Geometrically**, the **skew symmetric matrix corresponds to an axis of rotation** and the **exponential map generates the rotation** corresponding to rotation about the axis by a specified amount θ.

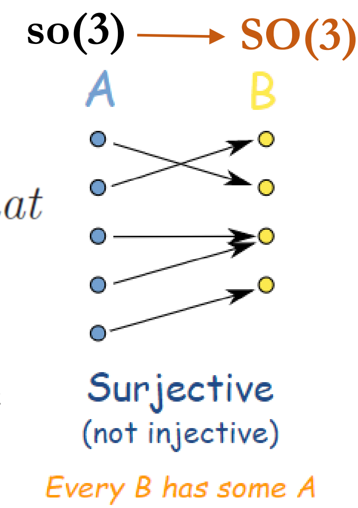
Matrix Logarithm of Rotations

- The **matrix logarithm** is the inverse of the matrix exponential.
- Just as the matrix exponential **integrates** the **matrix representation of an angular velocity** for one second to give an orientation $R \in SO(3)$, the matrix logarithm **differentiates** an $R \in SO(3)$ to find the **matrix representation of a constant angular velocity** which, if integrated for one second, rotates a frame from I to R.

$$\begin{array}{llcl} \exp: & [\hat{\omega}]\theta \in so(3) & \rightarrow & R \in SO(3), \\ \log: & R \in SO(3) & \rightarrow & [\hat{\omega}]\theta \in so(3). \end{array}$$

- The exponential map is **surjective** onto **SO(3)** i.e.,

Given $R \in SO(3)$, *there exists* $\omega \in \mathbb{R}^3$, $\|\omega\| = 1$ *and* $\theta \in \mathbb{R}$ *such that* $R = \exp(\widehat{\omega}\theta)$.



- **Surjective** means that every "B" has **at least one** matching "A" (maybe more than one). There won't be a "B" left out.

Matrix Logarithm of Rotations

<u>Given an R</u>, we equate terms of **R** and **exp($\hat{\omega}$ θ)** and solve the corresponding equations.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad v_\theta = 1 - \cos\theta, c_\theta = \cos\theta, \text{ and } s_\theta = \sin\theta.$$

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta) \\ &= \begin{bmatrix} 1 - v_\theta(\omega_2^2+\omega_3^2) & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & 1 - v_\theta(\omega_1^2+\omega_3^2) & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & 1 - v_\theta(\omega_1^2+\omega_2^2) \end{bmatrix} \\ &= \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix}. \end{aligned}$$

$$\text{tr } R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta.$$

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} = \frac{1}{2\boxed{\sin\theta}}(R - R^{\mathrm{T}})$$

WHAT ?

- Because of the **<u>sin θ term</u>** in the denominator, **[$\hat{\omega}$] <u>is not well defined if θ is an integer multiple of π</u>** **<u>(Singularity).</u>**

Matrix Logarithm of Rotations

Algorithm: Given R ϵ SO(3), find a **θ ϵ [0; π]** and a unit rotation axis $\omega \in \mathbb{R}^3, \|\omega\| = 1,$ such that $e^{\hat{\omega}\theta} = R$.

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} = \frac{1}{\boxed{2\sin\theta}}(R - R^{\mathrm{T}})$$

(a) If **R = I** then **θ = 0** and $\hat{\boldsymbol{\omega}}$ **is undefined**.

$$\operatorname{tr} R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta.$$

(b) If **tr (R) = -1** then **θ** = **π**. Set $\hat{\boldsymbol{\omega}}$ equal to any of the following three vectors:

$$R = e^{[\hat{\omega}]\pi} = I + 2[\hat{\omega}]^2.$$

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix} \quad \hat{\omega} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix} \text{ or } \quad \hat{\omega} = \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

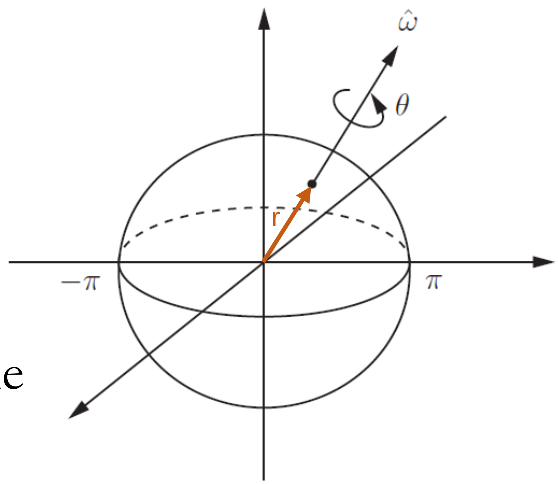
(c) Otherwise,

$$\theta = \cos^{-1}\left(\tfrac{1}{2}(\operatorname{tr} R - 1)\right) \in [0, \pi)$$

$$\hat{\omega} = \frac{1}{2\sin\theta}(R - R^{\mathrm{T}})$$

- Since every Rϵ SO(3), satisfies one of the three cases in the algorithm, **for every R** there exists a matrix logarithm $\hat{\boldsymbol{\omega}}$**θ** and therefore a set of exponential coordinates **ωθ (Surjective** onto **SO(3)).**

Matrix Logarithm of Rotations



- Because the matrix logarithm calculates exponential coordinates ωθ satisfying $||\hat{\omega}\theta|| \leq \pi$, we can picture the **rotation group SO(3) as a solid ball of radius π**.

  - Given a point r ϵ R^3 in this solid ball,

    let $\omega = r/||r||$ be the **unit axis** in the direction from the origin to the point r and

    let $\theta = ||r||$ be the **distance from the origin to r**, so that r = **ωθ**.

  - The **rotation matrix** corresponding to **r** can then be regarded as a rotation about the axis **ω** by an angle **θ**.

- For any R ϵ SO(3) such that **tr R ≠-1**, there exists a unique r in the interior of the solid ball such that $e^{[r]} = R$.

- In the event that **tr R = -1**, **log R** is given by two antipodal points **on the surface** of this solid ball.

Exponential coordinates and Matrix Logarithm

$$\begin{array}{llcl} \exp: & [\hat{\omega}]\theta \in so(3) & \rightarrow & R \in SO(3), \\ \log: & R \in SO(3) & \rightarrow & [\hat{\omega}]\theta \in so(3). \end{array}$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{\widehat{\omega}\theta} = I + \widehat{\omega}\sin\theta + \widehat{\omega}^2(1-\cos\theta) \in SO(3)$$ **Rodrigues' formula**

- When $\|\omega\| \neq 1$, it may be verified:

forward

$$e^{\widehat{\omega}\theta} = I + \frac{\widehat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\widehat{\omega}^2}{\|\omega\|^2}(1-\cos(\|\omega\|\theta)).$$

inverse

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

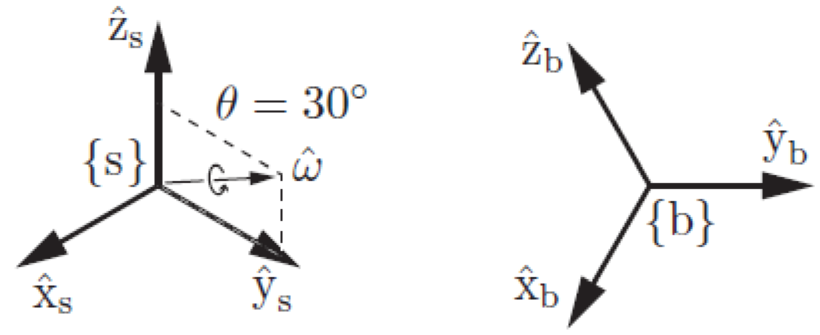
$$\theta = \cos^{-1}\left(\tfrac{1}{2}(\mathrm{tr}\, R - 1)\right) \in [0, \pi)$$

$$\hat{\omega} = \frac{1}{2\sin\theta}(R - R^{\mathrm{T}})$$

- Since every Rϵ SO(3), satisfies one of the three cases in the algorithm, **for every R** there exists a matrix logarithm **$\widehat{\omega}\theta$** and therefore a set of exponential coordinates **ωθ (Surjective** onto **SO(3)).**

Example

(a) The frame {b} is obtained by rotation from an initial orientation aligned with the frame {s} about a **unit axis** $\boldsymbol{\omega}$ =(0; 0.866; 0.5) by an **angle** $\boldsymbol{\theta}$ = 30° = 0.524 rad.
Find the rotation matrix representation of {b}.

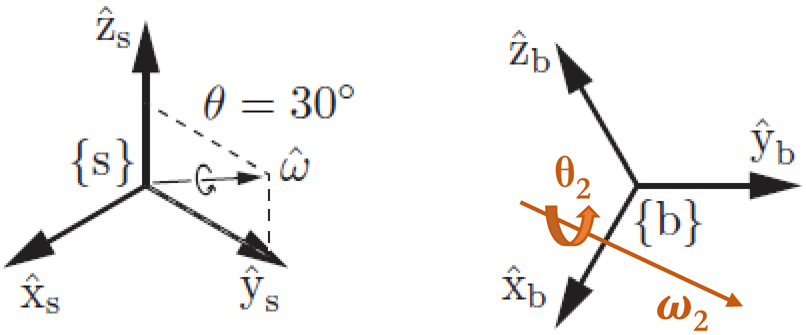


$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{\widehat{\omega}\theta} = I + \widehat{\omega}\sin\theta + \widehat{\omega}^2(1 - \cos\theta)$$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Example

(b) If {b} is then rotated by θ_2 about a **fixed-frame axis** $\boldsymbol{\omega}_2 \neq \boldsymbol{\omega}_1$ or a **body frame axis,** what will be the final rotation in each case?



$$\text{Rot}(\hat{\omega}, \theta) = e^{\widehat{\omega}\theta} = I + \widehat{\omega}\sin\theta + \widehat{\omega}^2(1 - \cos\theta)$$

Euler Angles Representation of rotation matrix

- Rotation matrices give a **redundant description** of frame orientation.
- They are characterized by **nine elements** which are not independent but related by six constraints due to the orthogonality conditions, i.e., **column vectors r_i** are **mutually perpendicular** and have **magnitude equal to 1.**

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad r_i^T r_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

- This implies that ***three parameters* are sufficient** to describe orientation of a rigid body in space (is it correct for axis-angle representation?)
- A minimal representation of orientation can be obtained by using a set of **three angles** $\varphi = [\phi\ \vartheta\ \psi]^T$.

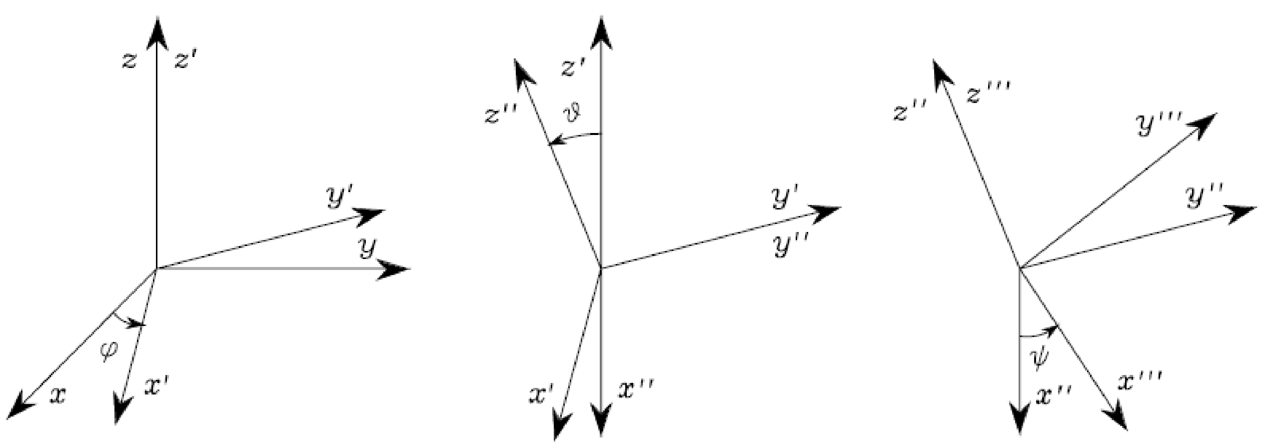
- A **generic rotation matrix** can be obtained by **composing a suitable sequence of three elementary rotations** (i.e., rotations about x, y, or z axis) while guaranteeing that **two successive rotations** are not made about parallel axes.

- This implies that **12 distinct sets of angles** are allowed out of all 27 possible combinations; each set represents **a triplet of Euler angles.**

ZYZ Euler Angles

- The rotation described by ZYZ angles is obtained as composition of the following elementary rotations:
  - ✓ First: Rotate the reference frame by the angle ϕ about axis z; this rotation is described by the matrix $R_z(\phi)$
  - ✓ Second: Rotate the **current frame** by the angle ϑ about axis **y'**; this rotation is described by the matrix $\mathbf{R}_{y'}(\vartheta)$
  - ✓ Third: Rotate the current frame by the angle ψ about axis **z''**; this rotation is described by the matrix $\mathbf{R}_{z''}(\psi)$
- The resulting frame orientation is obtained by composition of rotations via **post-multiplication**:

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$



$$R_x(\phi) := e^{\widehat{x}\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

$$R_y(\beta) := e^{\widehat{y}\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$R_z(\alpha) := e^{\widehat{z}\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

ZYZ Euler Angles

$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & \boxed{c_\varphi s_\vartheta} \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & \boxed{s_\varphi s_\vartheta} \\ \boxed{-s_\vartheta c_\psi} & \boxed{s_\vartheta s_\psi} & \boxed{c_\vartheta} \end{bmatrix}$$

- We are interested in solving the *inverse problem*, that is to **determine the set of Euler angles corresponding to a given rotation matrix.**

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

The function **Atan2(*y*, *x*)** computes the arctangent of the ratio ***y/x*** but utilizes the sign of each argument to determine which quadrant the resulting angle belongs to.

$r_{13} \neq 0$ and $r_{23} \neq 0$

$$\varphi = \text{Atan2}(r_{23}, r_{13})$$

Squaring and summing the elements [1, 3] and [2, 3] and using the element [3, 3]:

$$\vartheta = \text{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right) \quad \vartheta \in (0, \pi).$$

elements [3, 1] and [3, 2]

$$\psi = \text{Atan2}(r_{32}, -r_{31}).$$

ZYZ Euler Angles

- Choosing ϑ in the range $(-\pi, 0)$ leads to

$$\varphi = \text{Atan2}(-r_{23}, -r_{13})$$
$$\vartheta = \text{Atan2}\left(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$
$$\psi = \text{Atan2}(-r_{32}, r_{31}).$$

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$
$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & \boxed{c_\varphi s_\vartheta} \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & \boxed{s_\varphi s_\vartheta} \\ \boxed{-s_\vartheta c_\psi} & \boxed{s_\vartheta s_\psi} & \boxed{c_\vartheta} \end{bmatrix}$$

- As in the case of the exponential map, the map from **(α, β, γ) → SO(3) is surjective!**
- when $\boldsymbol{s_\vartheta = 0}$; in this case, it is **possible to determine only the sum or difference of $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$.** In fact, **if $\boldsymbol{\vartheta = 0, \pi}$**, the successive rotations of ϕ and ψ are made about axes of current frames which are parallel, thus giving equivalent contributions to the rotation **(Singularity).** $\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$
- As in the case of the angle/axis representation, singularities in the parameterization (i.e., the lack of existence of global, smooth solutions to the inverse problem of determining the Euler angles from the rotation) occur at **R = I, the identity rotation.**

- There are **infinitely many representations** of the identity rotation in the ZYZ Euler angles parameterization in the form of **Rot(α, 0,−α) = I!**

ZYX (Roll-Pitch-Yaw) Euler Angles

- **Roll–Pitch–Yaw angles** *are a*nother set of Euler angles originates from a representation of orientation in the (aero)nautical field.
- $\phi = [\varphi\ \vartheta\ \psi]^T$ represent rotations **defined with respect to a fixed frame** attached to the center of mass of the craft.

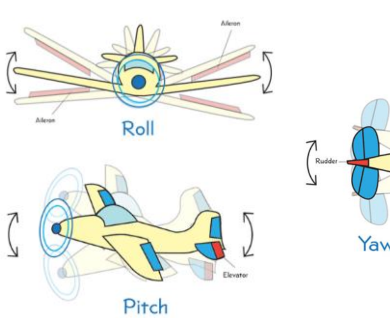
- The rotation resulting from Roll–Pitch–Yaw angles can be obtained as follows:
  - ✓ **First:** Rotate the reference frame by the **angle ψ about axis x (yaw);** this rotation is described by the matrix $\boldsymbol{R}_x(\psi)$.
  - ✓ **Second:** Rotate the reference frame by the **angle ϑ about axis y (pitch);** this rotation is described by the matrix $\mathbf{R}_y(\boldsymbol{\vartheta})$.
  - ✓ **Third:** Rotate the reference frame by the **angle ϕ about axis z (roll);** this rotation is described by the matrix $\mathbf{R}_z(\boldsymbol{\phi})$.

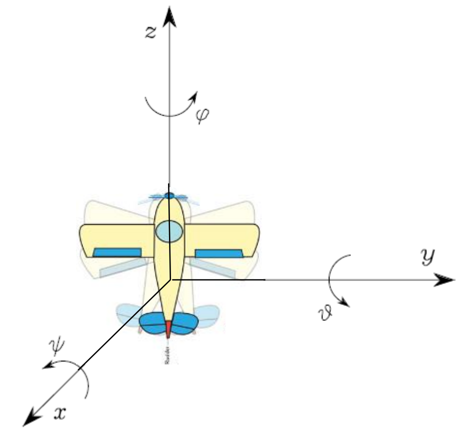
$$R_{\mathbf{x}}(\phi) := e^{\hat{\mathbf{x}}\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

$$R_{\mathbf{y}}(\beta) := e^{\hat{\mathbf{y}}\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$R_{\mathbf{z}}(\alpha) := e^{\hat{\mathbf{z}}\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi)$$





https://howthingsfly.si.edu/flight-dynamics/roll-pitch-and-yaw

ZYX (Roll-Pitch-Yaw) Euler Angles

$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi)$$
$$= \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

ϑ in the range $(-\pi/2, \pi/2)$

$$\varphi = \mathrm{Atan2}(r_{21}, r_{11})$$
$$\vartheta = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right)$$
$$\psi = \mathrm{Atan2}(r_{32}, r_{33}).$$

ϑ in the range $(\pi/2, 3\pi/2)$

$$\varphi = \mathrm{Atan2}(-r_{21}, -r_{11})$$
$$\vartheta = \mathrm{Atan2}\left(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}\right)$$
$$\psi = \mathrm{Atan2}(-r_{32}, -r_{33}).$$

- ZYX Euler angles **do not** have a singularity at the identity orientation, R = I, though they **do contain** singularities when $\theta = +/-\pi/2$.

- It is a fundamental topological fact that **singularities can never be eliminated in any 3-dimensional representation of SO(3).**

References

- Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.
- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**
- Lynch and Park, "*Modern Robotics,*" Cambridge U. Press, 2017, **Chapter 3.**