ME 397- ASBR

Week 2-Lecture 1

a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

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Spring 2022

Rigid Body Transformation

A **mapping** $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a **rigid body transformation** if it satisfies the following properties:

1. (**Necessary condition**) Length is preserved for all points p and q:

$$\|g(p) - g(q)\| = \|p - q\|$$

2. (**Sufficient Condition**) The cross product (orientation) is preserved for all vectors ***v*** and ***w***

$$g_*(v \times w) = g_*(v) \times g_*(w)$$

- We may define the **space of rotation matrices** in $R^{n\times n}$ by

$$SO(n) = \{R \in \mathbb{R}^{n\times n} : RR^T = I, \det R = +1\}$$

- $R^{-1} = R^T$

Rotations are rigid body transformations

A rotation **R ∈ SO(3)** is a rigid body transformation; that is,

1. **R** preserves distance:

$$||Rq - Rp|| = ||q - p|| \;\; for\ all\ q, p \in \mathbb{R}^3$$

2. **R** preserves orientation:

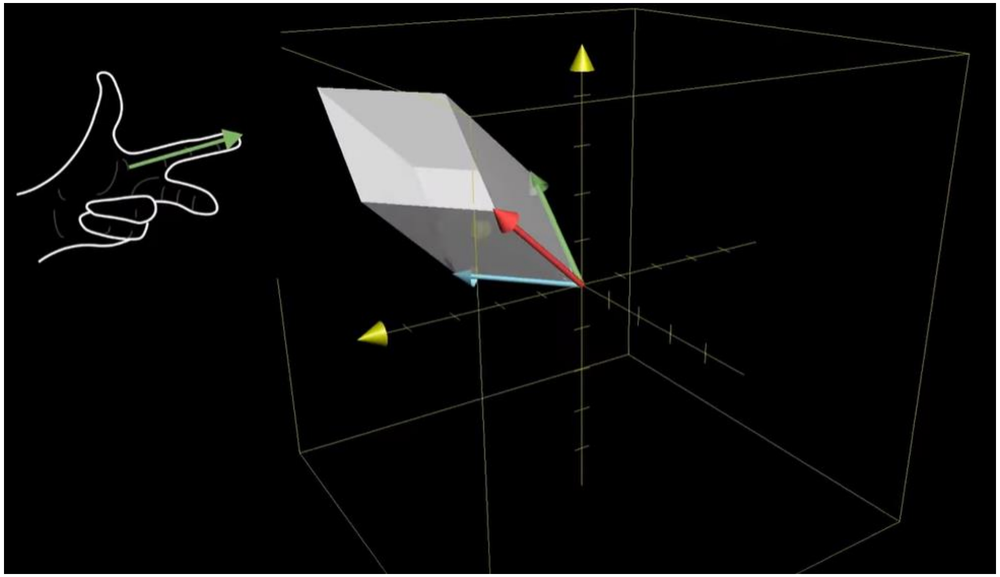
$$R(v \times w) = Rv \times Rw \;\; for\ all\ v, w \in \mathbb{R}^3$$

i.e., the rotation of the cross product of two vectors is the cross product of the rotation of each of the vectors by **R**.

❖ Proof in your **THA1**!

Physical Interpretation of Rotation Matrices

- Geometrically, determinant can be viewed as the volume scaling factor of the linear transformation described by the matrix.
- The determinant is **positive or negative** according to whether the linear mapping **preserves or reverses the orientation of *n*-space**.
- det(R)=+1 means that it's a **rigid body transformation** that does not **change the length** and **orientation**!



https://www.youtube.com/watch?v=Ip3X9LOh2dk

Uses of Rotation Matrices

- There are **three** major uses for a rotation matrix **R**:

  (a) To represent an orientation;
  (b) To change the reference frame in which a vector or a frame is represented (Solved example);
  (c) To rotate a vector or a frame.

- In (a) , **R** is thought of as **representing a frame**;

- In (b) and (c), **R** is thought of as an **operator** that acts on a **vector or frame** (changing its reference frame or rotating it).

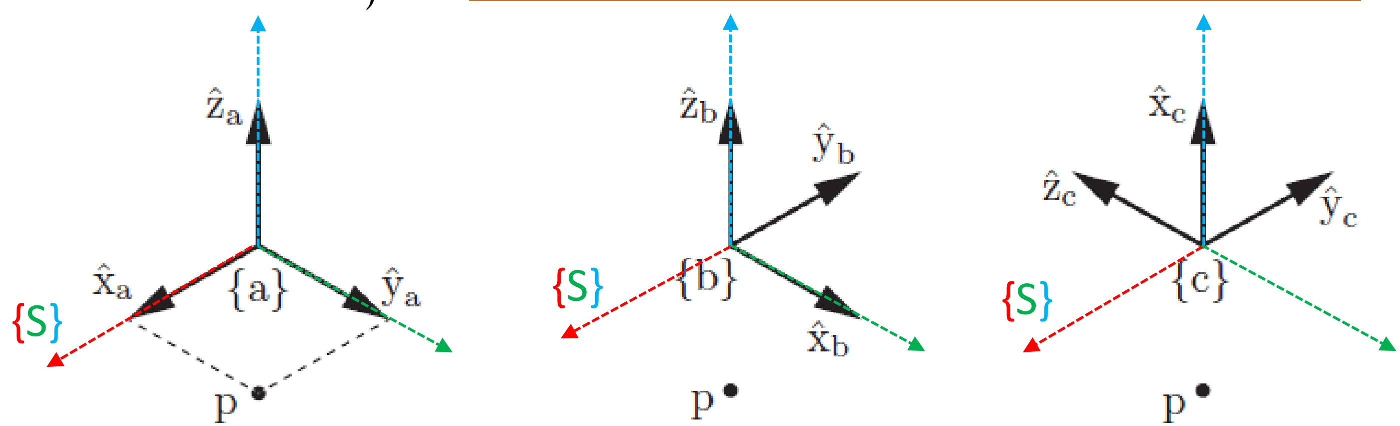
Uses of Rotation Matrices

(a) To represent an orientation:

- ✓ Frames {a}, {b}, and {c} representing the same space with the same origin.
- ✓ RGB color frame is a **fixed space frame {s}**, which is aligned with frame {a}.
- ✓ The orientations of the three frames **relative to {s}** can be written as $\mathbf{R}_f$, which implicitly referring to the orientation of frame $\{f\}$ relative to the fixed frame {s}, i.e., $\mathbf{R}_{sf}$

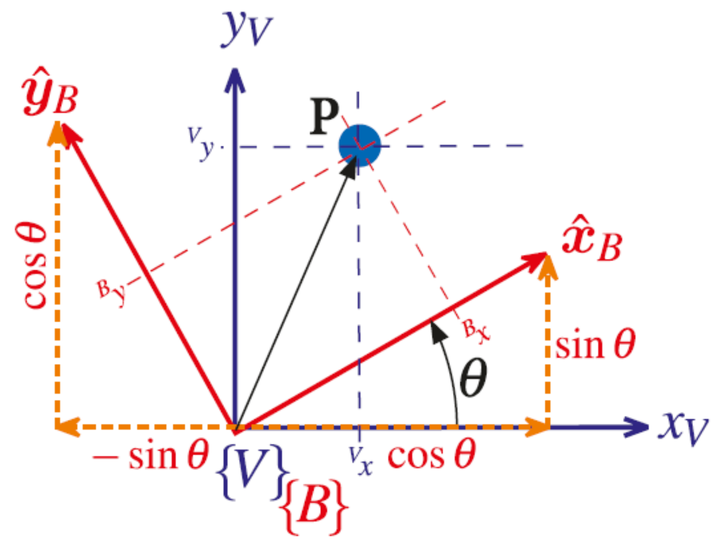
$$R_a = \begin{array}{c} \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \quad R_b = \begin{array}{c} \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \\ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \quad R_c = \begin{array}{c} \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \end{array} \\ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

- ➢ A rotation matrix is just **a collection of three unit vectors!**



2D Rotation Matrix

- **Goal:** Given known position **P** in the rotated frame {B}, find its position wrt **fixed frame {V}**.
- The point **P** can be considered with respect to the 2D red (Rotated) or blue coordinate frame **with the same origin:** i.e., $^{V}\mathbf{p} = {}^{B}\mathbf{p}$



$$\begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = \boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix} \qquad \begin{pmatrix} {}^{V}x \\ {}^{V}y \end{pmatrix} = \boxed{{}^{V}R_{B}} \begin{pmatrix} {}^{B}x \\ {}^{B}y \end{pmatrix}$$

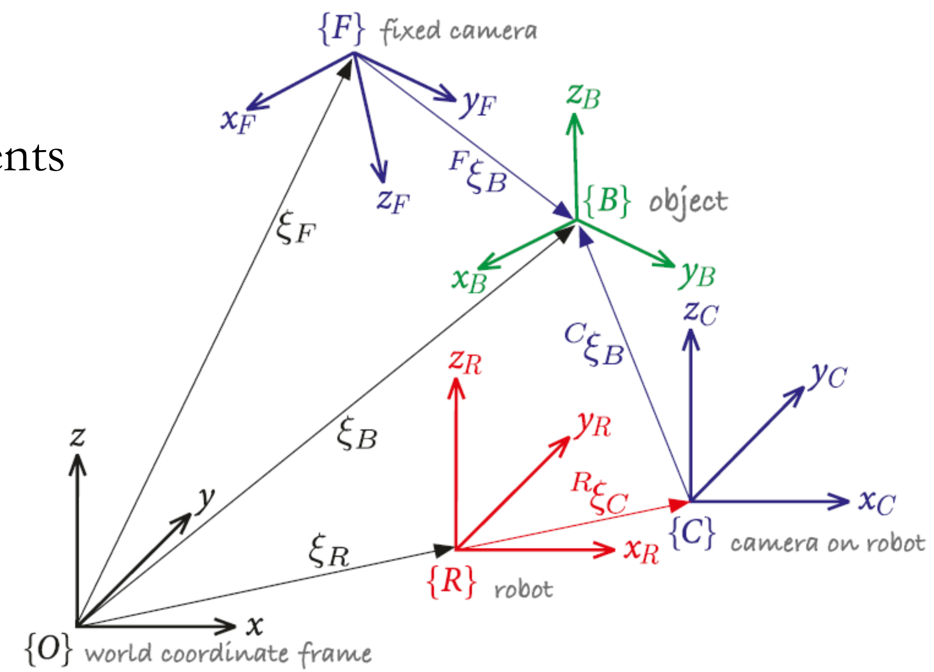
Uses of Rotation Matrices

(b) Changing the reference frame

- If the rotation matrix $\mathbf{R}_{ab}$ represents **the orientation of {b} in {a}** and $\mathbf{R}_{bc}$ represents the **orientation of {c} in {b},** then we have:

$$R_{ac} = R_{ab} R_{bc}.$$

- Notation: $\mathbf{R}_{ab} = {}^{a}\mathbf{R}_{b}$ **both** represents the **orientation of {b} in {a}.**



Uses of Rotation Matrices

(b) Changing the reference frame

- If the rotation matrix $\mathbf{R}_{ab}$ represents the orientation of {b} in {a} and $\mathbf{R}_{bc}$ represents the orientation of {c} in {b}, then we have:

$$R_{ac} = R_{ab} R_{bc}.$$

- $\mathbf{R}_{bc}$ can be viewed as a **representation of the orientation** of {c}.
- While $\mathbf{R}_{ab}$ can be viewed as a **mathematical operator** that changes the reference frame from {b} to {a}.
- A **subscript cancellation rule** helps us to remember this property:

$$R_{ab} R_{bc} = R_{a\cancel{b}} R_{\cancel{b}c} = R_{ac}$$

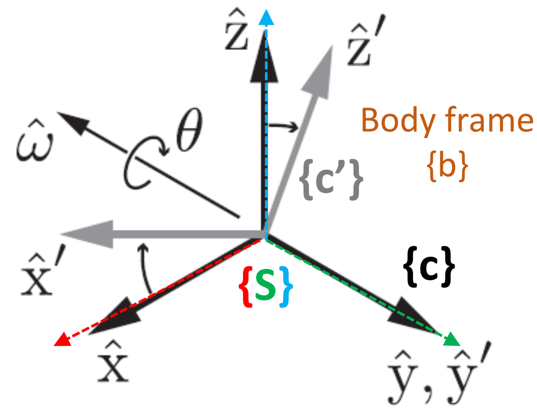
- The **reference frame of a vector** can also be changed by a rotation matrix:

$$R_{ab} p_b = R_{a\cancel{b}} p_{\cancel{b}} = p_a.$$

Uses of Rotation Matrices

(c) Rotating a vector or a frame

- We rotate the frame **{c}** about a unit axis $\hat{\boldsymbol{\omega}}$ by an amount θ, the new frame is $\{c'\}$ and can define it by $\mathbf{R}_{Sc'}$.
- We can also see **R** as a **rotation operator**, instead of as an orientation, i.e., $R = \mathrm{Rot}(\hat{\omega}, \theta)$



$$\mathrm{Rot}(\hat{x}, \theta) = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ \cos\theta \\ \sin\theta \end{matrix} & \begin{matrix} 0 \\ -\sin\theta \\ \cos\theta \end{matrix} \end{bmatrix}, \quad \mathrm{Rot}(\hat{y}, \theta) = \begin{bmatrix} \begin{matrix} \cos\theta \\ 0 \\ -\sin\theta \end{matrix} & \boxed{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}} & \begin{matrix} \sin\theta \\ 0 \\ \cos\theta \end{matrix} \end{bmatrix}$$

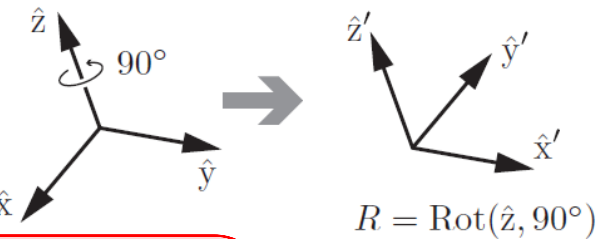
$$\mathrm{Rot}(\hat{z}, \theta) = \begin{bmatrix} \begin{matrix} \cos\theta \\ \sin\theta \\ 0 \end{matrix} & \begin{matrix} -\sin\theta \\ \cos\theta \\ 0 \end{matrix} & \boxed{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}} \end{bmatrix}.$$

- ***Rot*** is an <u>**independent-to-frame operation**</u> that rotates the orientation <u>**represented by the Identity matrix to the orientation represented by R.**</u>
- We have to specify whether the axis of rotation $\hat{\boldsymbol{\omega}}$ is expressed in **{S}** or **body frame {b},** (e.g., {c} in the figure).

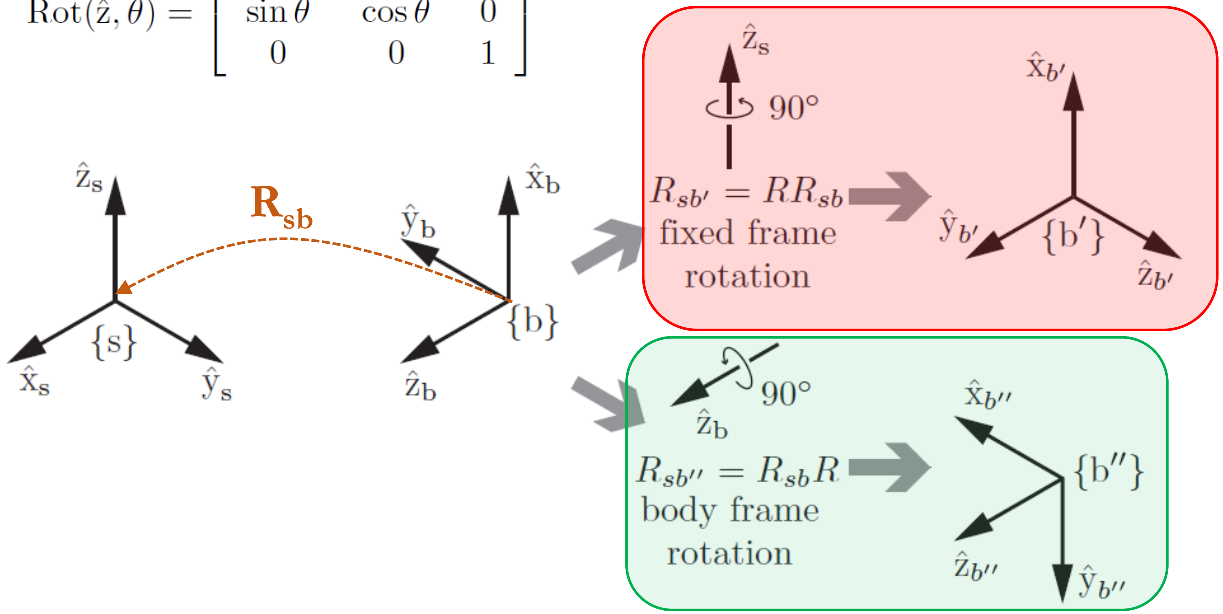
- Depending on our choice, <u>the same numerical $\hat{\boldsymbol{\omega}}$</u> (and therefore the same numerical R) <u>corresponds to **different rotation axes** in the underlying space</u>!!!

Uses of Rotation Matrices

- The **independent rotation operator** **R = Rot(z^; 90)** gives the orientation of the right-hand frame in the left-hand frame.



$$\mathrm{Rot}(\hat{z},\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

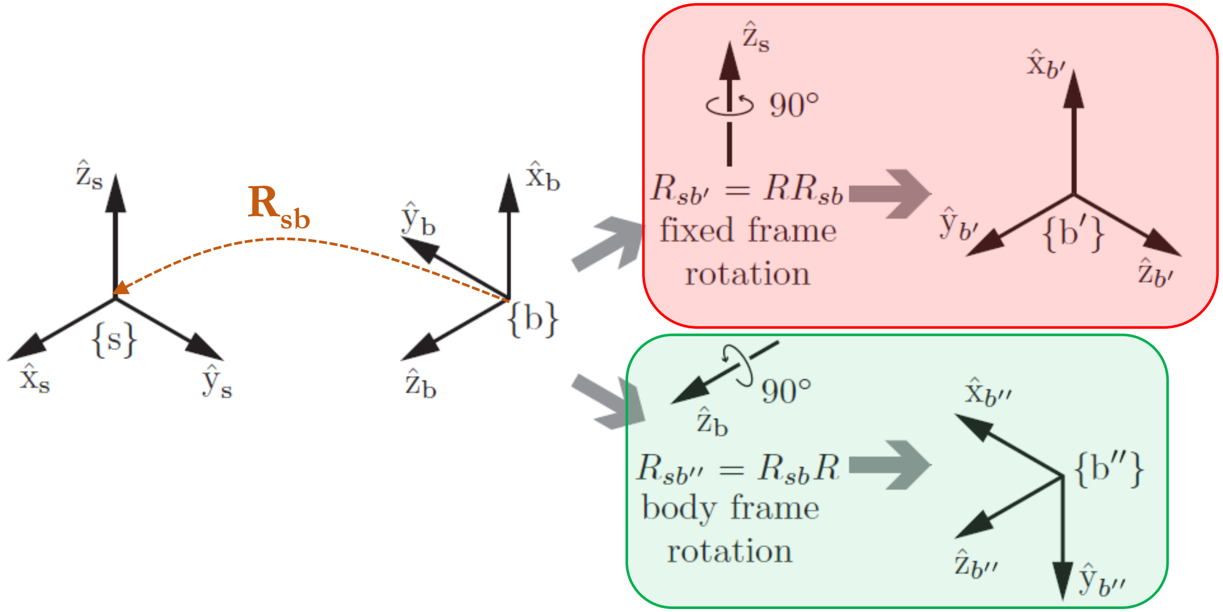


- The quantity RR_{sb} rotates {b} by 90 degree about the **fixed-frame axis** $\mathbf{\hat{z}_s}$ to **{b'}.**
- The quantity $R_{sb}R$ rotates {b} by 90 degree about the **body-frame axis** $\mathbf{\hat{z}_b}$ to **{b"}.**

$$R_{sb'} = \text{rotate\_by\_}R\text{\_in\_\{s\}\_frame}\;(R_{sb}) = \boxed{R}R_{sb}$$

$$R_{sb''} = \text{rotate\_by\_}R\text{\_in\_\{b\}\_frame}\;(R_{sb}) = R_{sb}\boxed{R}.$$

Uses of Rotation Matrices



- In other words, **pre-multiplying** **by R = Rot($\hat{\omega}$; θ)** yields a rotation about an axis $\hat{\omega}$ considered to be in the **fixed frame**, and **post-multiplying by R** yields a rotation about $\hat{\omega}$ considered as being in the **body frame**.

- The quantity RR_{sb} rotates {b} by 90 degree about the **fixed-frame axis $\hat{z}_s$** to **{b'}.**
- The quantity $R_{sb}R$ rotates {b} by 90 degree about the **body-frame axis $\hat{z}_b$** to **{b''}.**

$$R_{sb'} = \text{rotate\_by\_}R\text{\_in\_}\{s\}\text{\_frame}\ (R_{sb}) = RR_{sb}$$

$$R_{sb''} = \text{rotate\_by\_}R\text{\_in\_}\{b\}\text{\_frame}\ (R_{sb}) = R_{sb}R.$$

Skew Symmetric matrices

- Given a vector $x = [x_1\ x_2\ x_3]^T \in R^3$, define: $[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$
- The matrix **[x] or also sometime shown by $\hat{x}$** is a **3×3 skew-symmetric matrix** representation of vector x; that is $[x] = -[x]^T$
- The set of all **3×3** **<u>real</u>** **skew-symmetric matrices** is called **so(3)**.
- The set of skew-symmetric matrices **so(3)** is called the **Lie algebra** of the **Lie group SO(3)**.
- The space of <u>n×n skew-symmetric matrices *S*</u> is: $so(n) = \{S \in \mathbb{R}^{n\times n} : S^T = -S\}$

- We can write the **cross product of two vectors** as: $a \times b = (a)^\wedge b.$
- The **<u>sum</u>** of two skew-symmetric matrices is skew-symmetric $(v+w)^\wedge = \widehat{v} + \widehat{w}.$
- The **<u>scalar multiple</u>** of any element of so(3) is an element of so(3)

- The <u>elements on the diagonal</u> of a skew-symmetric matrix are **<u>zero</u>**, and therefore its **trace equals zero**.
- If A is a **real skew-symmetric matrix** and λ **<u>is a real eigenvalue, then λ=0</u>** i.e. the **<u>nonzero eigenvalues</u>** of a skew-symmetric matrix are **<u>purely imaginary</u>**.
- **<u>Determinant</u>** of an n×n skew-symmetric matrix is **zero** if **<u>n is odd.</u>**

Vector linear differential equation

The linear differential equation $\boxed{\dot{x}(t) = Ax(t)}$ *with initial condition* $x(0) = x_0$, *where* $\boxed{A \in \mathbb{R}^{n\times n} \textit{ is constant}}$ *and* $x(t) \in \mathbb{R}^n$, *has solution*

$$x(t) = e^{At}x_0$$

where

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots .$$

The matrix exponential e^{At} *further satisifies the following properties:*

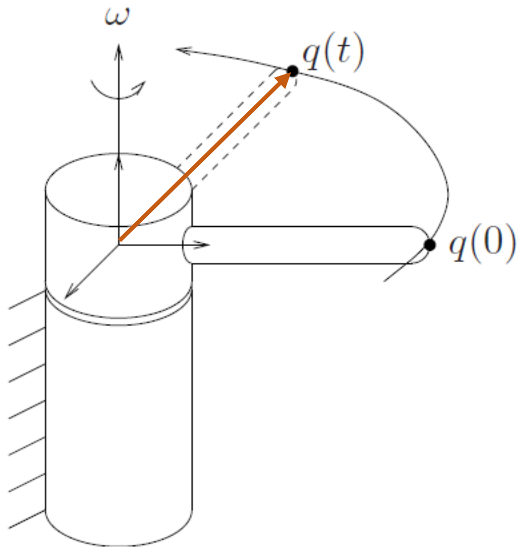
(a) $d(e^{At})/dt = Ae^{At} = e^{At}A$.

(b) *If* $A = PDP^{-1}$ *for some* $D \in \mathbb{R}^{n\times n}$ *and invertible* $P \in \mathbb{R}^{n\times n}$ *then* $e^{At} = Pe^{Dt}P^{-1}$.

(c) *If* $AB = BA$ *then* $e^Ae^B = e^{A+B}$.

(d) $(e^A)^{-1} = e^{-A}$.

Exponential coordinates for rotation



- Let's rotate the rigid body at **constant unit velocity about the axis ω**, the velocity of the point, $\dot{q}$, may be written as the following **linear differential equation**:

$$\dot{q}(t) = \omega \times q(t) = \widehat{\omega} q(t) \quad \Rightarrow \quad q(t) = e^{\widehat{\omega} t} q(0)$$

where q(0) is the initial (t = 0) position of the point and $e^{\widehat{\omega} t}$ is the matrix exponential:

$$e^{\widehat{\omega} t} = I + \widehat{\omega} t + \frac{(\widehat{\omega} t)^2}{2!} + \frac{(\widehat{\omega} t)^3}{3!} + \cdots$$

- It follows that if we **rotate** about the **axis ω at unit velocity** for **θ units of time (t = θ)**, then the **net rotation** is given by: $R(\omega, \theta) = e^{\widehat{\omega} \theta}$

- Given a matrix $\widehat{\omega} \in so(3)$, $\|\omega\| = 1$, and a real number θ ∈ R, we write the exponential $\widehat{\omega}\theta$ as

$$\exp(\widehat{\omega}\theta) = e^{\widehat{\omega}\theta} = I + \theta\widehat{\omega} + \frac{\theta^2}{2!}\widehat{\omega}^2 + \frac{\theta^3}{3!}\widehat{\omega}^3 + \ldots$$

- This is an **infinite series** and, hence, not useful from a computational standpoint.
- It can be shown that if **the matrix is constant** and **finite** then this series is always guaranteed **to converge to a finite limit.**

Exponential coordinates for rotation

- Given $\widehat{a} \in so(3)$, the following relations hold:

$$\widehat{a}^2 = aa^T - \|a\|^2 I$$
$$\widehat{a}^3 = -\|a\|^2 \widehat{a}$$

and higher powers can be calculated recursively.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

- Utilizing this with $\boxed{a = \omega\theta,\ \|\omega\| = 1}$, we have:

$$\exp(\widehat{\omega}\theta) = e^{\widehat{\omega}\theta} = I + \theta\widehat{\omega} + \frac{\theta^2}{2!}\widehat{\omega}^2 + \frac{\theta^3}{3!}\widehat{\omega}^3 + \ldots$$

$$\Rightarrow \quad e^{\widehat{\omega}\theta} = I + \boxed{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}\widehat{\omega} + \boxed{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}\widehat{\omega}^2$$

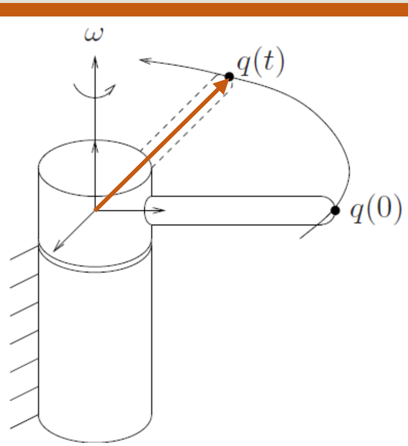
$$\Rightarrow \quad \mathrm{Rot}(\hat{\omega}, \theta) = e^{\widehat{\omega}\theta} = I + \widehat{\omega}\boxed{\sin\theta} + \widehat{\omega}^2\boxed{(1 - \cos\theta)} \in SO(3)$$ **Rodrigues' formula**

- $\boxed{\text{When } \|\omega\| \neq 1,}$ it may be verified:

$$e^{\widehat{\omega}\theta} = I + \frac{\widehat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\widehat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\theta))$$

Exponential coordinates for rotation

- The quantity $e^{\widehat{\omega}\theta}\boldsymbol{q}$ has the effect of rotating $\boldsymbol{q}$ about the fixed-frame axis **ω** by an angle **θ**.



- Similarly, considering that a rotation matrix **R** consists of three column vectors, the rotation matrix $R' = \mathrm{Rot}(\hat{\omega}, \theta)R$ is the orientation achieved by rotating R by an angle **θ** about **axis ω** **in the fixed frame.**

- Reversing the order of matrix multiplication, $R'' = R\,\mathrm{Rot}(\hat{\omega}, \theta)$ is the orientation achieved by **rotating R in the body frame.**

$$R'' = Re^{[\hat{\omega}_2]\theta_2} \neq R' = e^{[\hat{\omega}_2]\theta_2} R.$$

- Exponentials of skew symmetric matrices are orthogonal!
- Given a skew-symmetric matrix $\widehat{\omega} \in so(3)$ *and* $\theta \in \mathbb{R}$ ➡ $e^{\widehat{\omega}\theta} \in SO(3).$

- **Geometrically**, the **skew symmetric matrix corresponds to an axis of rotation** and the **exponential map generates the rotation** corresponding to rotation about the axis by a specified amount θ.

References

- Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.

- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**

- Lynch and Park, "*Modern Robotics*," Cambridge U. Press, 2017, **Chapter 3.**