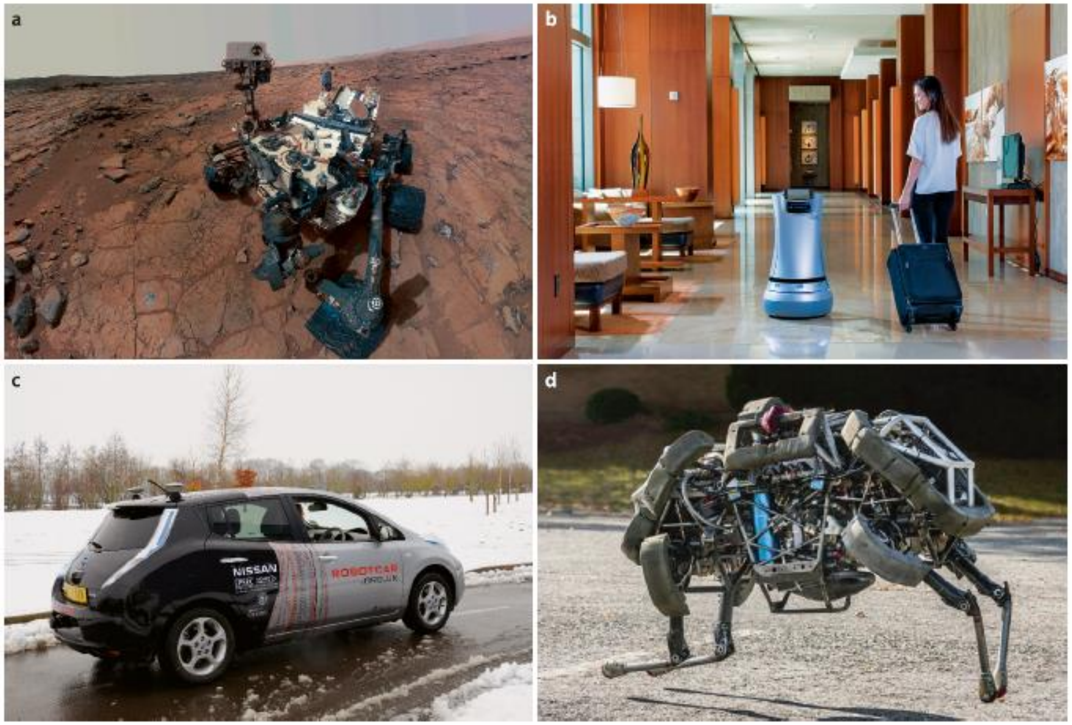
ME 397- ASBR

Week 1-Lecture 2



a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

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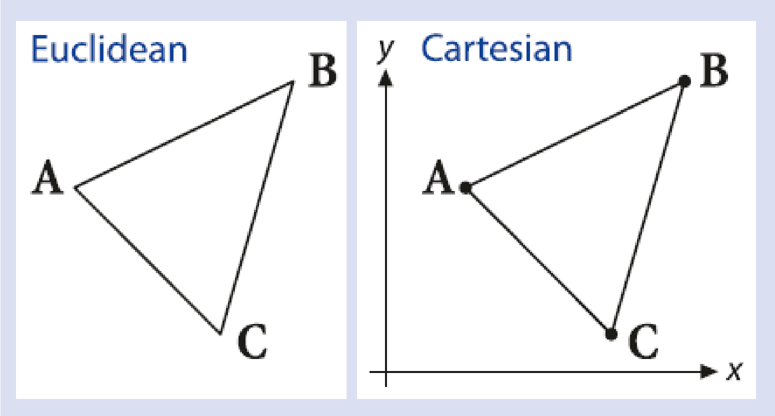
Definitions

- **Euclidean versus Cartesian geometry/Space:**

  - **Euclidean** ***geometry*** is concerned with points and lines and is entirely based on a set of axioms and **makes no use of arithmetic.**
  - Descartes added a coordinate system (2D or 3D) and was then able to describe points, lines and other curves in terms of **algebraic equations**.
  - The Cartesian plane (or space) is the Euclidean plane (or space) with all its axioms and postulates ***plus*** the extra facilities afforded by the added coordinate system.

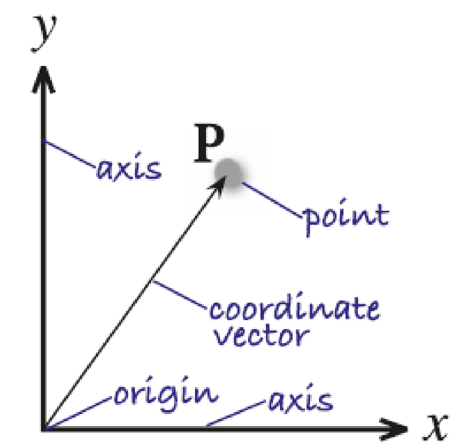


Euclid of Alexandria
(ca. 325 BCE–265 BCE)

René Descartes
(1596–1650)

Definitions

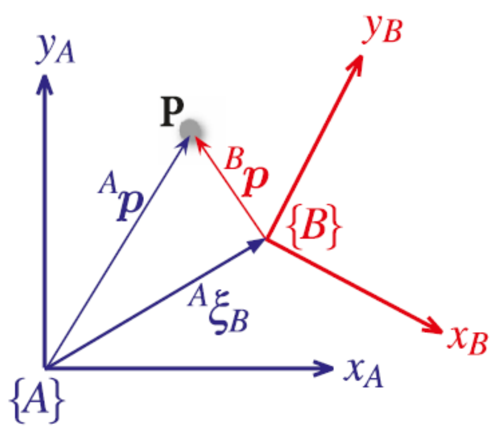
- A **coordinate frame**, or Cartesian coordinate system, is a set of orthogonal axes which intersect at a point known as the origin.



- A **vector** can be described in terms of its components, a **linear combination of unit vectors** which are parallel to the axes of the coordinate frame.

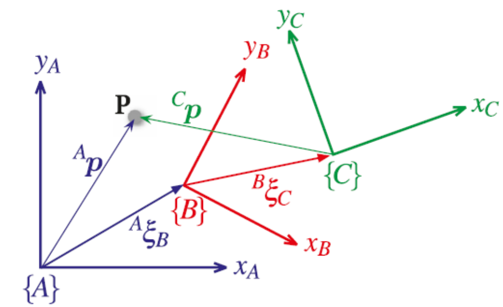
- A **point** is described by a **bound coordinate vector** that represents its *displacement from the origin of a reference coordinate system*.

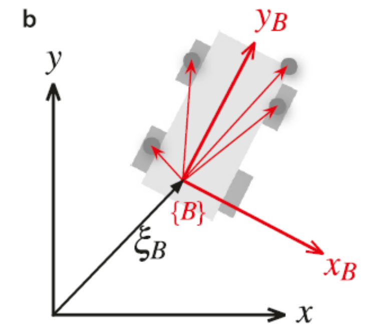
- Points and vectors are **different things** even though they are each described by a tuple of numbers:
  - ✓ We can add vectors but not points (Euclidean points).
  - ✓ A vector has a direction and a magnitude.
  - ✓ The point P can be described by coordinate vectors relative to either frame {A} or {B}.
  - ✓ The difference of two points is a vector,
  - ✓ We can add a vector to a point to obtain another point.

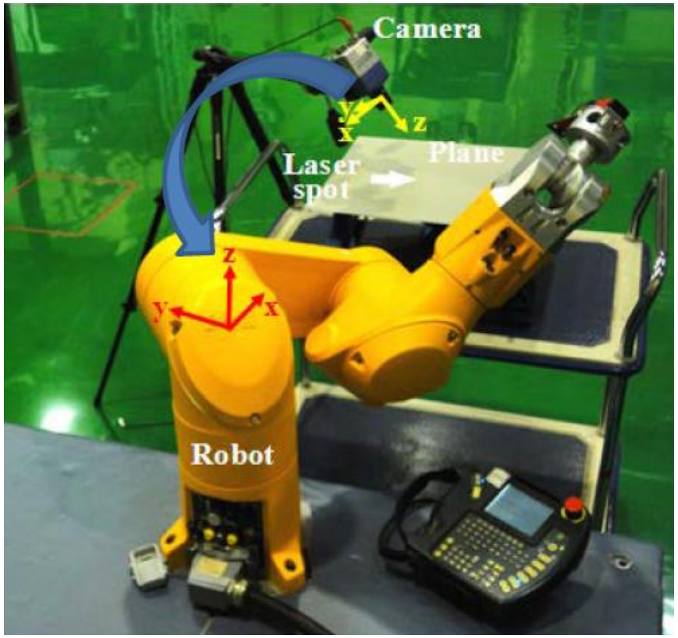


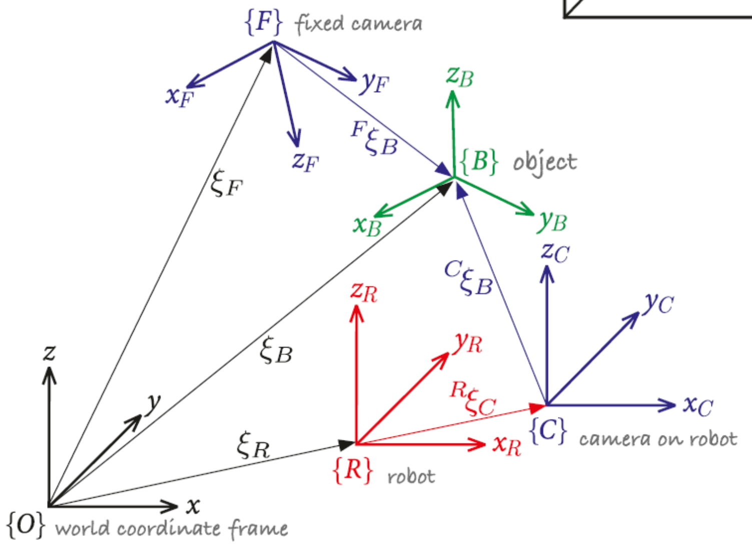
Definitions

- The **point P** can be described by **coordinate vectors** relative to either frame {A}, {B} or {C}.
- The **frames** are described by **relative position and orientation (poses)** ζ.
- An **object**, unlike a point, also has an orientation.
- We can describe the **pose** ζ of the **attached** coordinate frame **to an object** with respect to the **reference coordinate frame.**







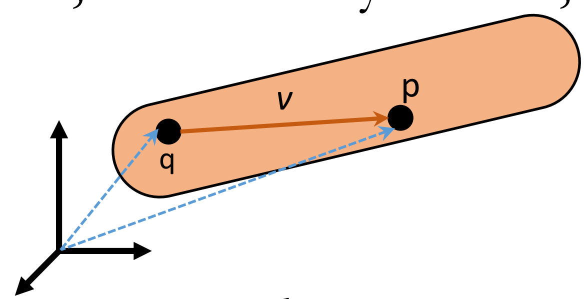


Rigid Body Motion

- ✓ A **Rigid body** is a collection of particles such that the distance between **any two particles** remains fixed, **regardless of any motions of the body or forces exerted on the body.**
- ✓ We loosely define a perfectly rigid body as a completely "undistortable" body.
- ✓ If **p** and **q** are any two points on a rigid body then, as the body moves, **p** and **q** must satisfy:

$$\|p(t) - q(t)\| = \|p(0) - q(0)\| = \text{constant.}$$

where, $\|\vec{v}\|$ is a Euclidean norm, i.e., $\sqrt{v_1^2 + v_2^2 + v_3^2}$



- ✓ A **rigid motion** of an object is a motion which preserves distance between points.
- ✓ The study of **robot kinematics, dynamics, and control** has at its heart the study of the motion of rigid objects.

Rigid Body Transformation

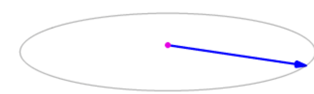
A **mapping** $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a **rigid body transformation** if it satisfies the following properties:

1. (**Necessary condition**) Length is preserved for all points p and q:

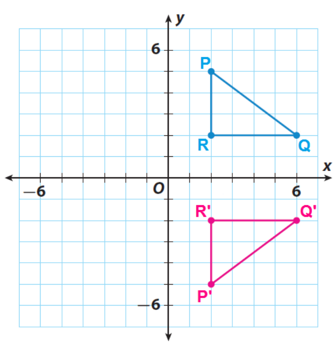
$$\|g(p) - g(q)\| = \|p - q\|$$

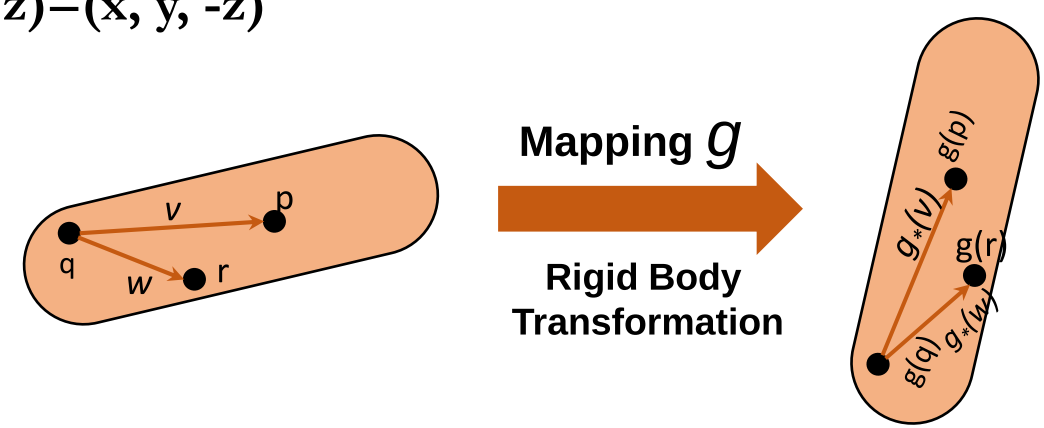
2. (**Sufficient Condition**) The cross product (orientation) is preserved for all vectors ***v*** and ***w***

$$g_*(v \times w) = g_*(v) \times g_*(w)$$



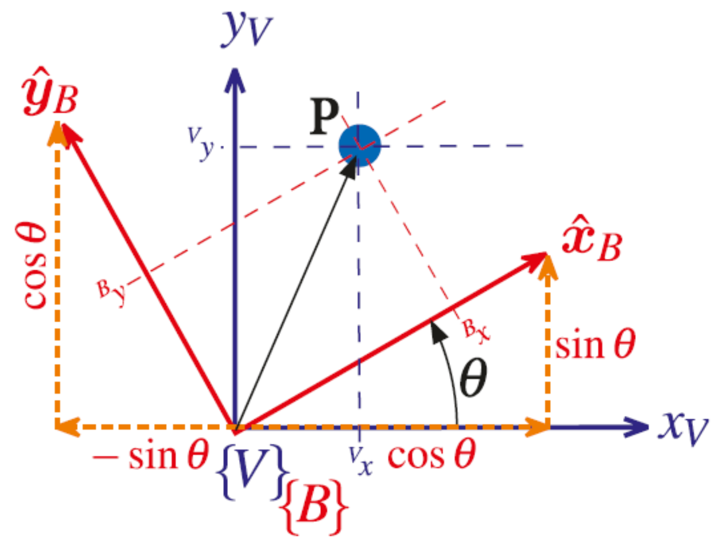
➢ **Is Reflection mapping e.g., g(x, y, z)=(x, y, -z) a rigid body transformation?**





2D Rotation Matrix

- **Goal:** Given known position **P** in the rotated frame {B}, find its position wrt **fixed frame {V}.**
- The point **P** can be considered with respect to the 2D red (Rotated) or blue coordinate frame **with the same origin:** i.e., $^{V}\mathbf{p} = {}^{B}\mathbf{p}$



unit vectors

$$^{B}p = {}^{B}x\,\hat{x}_B + {}^{B}y\,\hat{y}_B = \begin{pmatrix}\hat{x}_B & \hat{y}_B\end{pmatrix}\underbrace{\boxed{\begin{pmatrix}^{B}x \\ ^{B}y\end{pmatrix}}}_{\text{known}}$$

$$^{V}p = {}^{V}x\,\hat{x}_V + {}^{V}y\,\hat{y}_V = \begin{pmatrix}\hat{x}_V & \hat{y}_V\end{pmatrix}\underbrace{\boxed{\begin{pmatrix}^{V}x \\ ^{V}y\end{pmatrix}}}_{\text{unknown}}$$

$$\hat{x}_B = \cos\theta\,\hat{x}_V + \sin\theta\,\hat{y}_V$$
$$\hat{y}_B = -\sin\theta\,\hat{x}_V + \cos\theta\,\hat{y}_V$$

$$\begin{pmatrix}\hat{x}_B & \hat{y}_B\end{pmatrix} = \begin{pmatrix}\hat{x}_V & \hat{y}_V\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

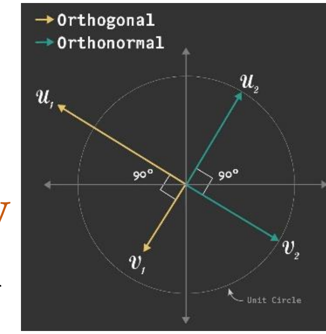
$$^{B}p = \begin{pmatrix}\hat{x}_V & \hat{y}_V\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}^{B}x \\ ^{B}y\end{pmatrix}$$

$^{V}\mathbf{p} = {}^{B}\mathbf{p}$

$$\begin{pmatrix}^{V}x \\ ^{V}y\end{pmatrix} = \boxed{\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}}\begin{pmatrix}^{B}x \\ ^{B}y\end{pmatrix}$$

$$\begin{pmatrix}^{V}x \\ ^{V}y\end{pmatrix} = \boxed{^{V}R_B}\begin{pmatrix}^{B}x \\ ^{B}y\end{pmatrix}$$

Rotation Matrix as an Orthogonal Matrix



- A square matrix whose columns (and rows) are **orthonormal vectors** is an *orthogonal matrix*.
- In other words, a square matrix whose **column vectors r_i** are **mutually perpendicular** and have **magnitude equal to 1** will be an **orthogonal** matrix.

$$r_i^T r_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

- **Rotation Matrices** are examples of **orthogonal matrices** with the following properties:

1. $RR^T = R^T R = I.$
2. $\det R = \pm 1$

$$\det(AB) = \det(A) \times \det(B)$$

$$\det\left(A^\mathsf{T}\right) = \det(A)$$

$$\det\left(I_n\right) = 1$$

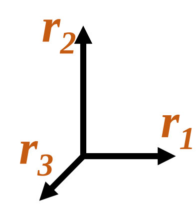
- To determine the sign of the determinant of R, we recall from linear algebra that

$$\det R = r_1^T(r_2 \times r_3)$$

$$\Rightarrow R = r_1^T r_1 = 1$$

For right-handed coordinate frame

$$r_2 \times r_3 = r_1$$



Special Orthogonal Matrices

- We may define the **space of rotation matrices** in $R^{n \times n}$ by

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det R = +1\}$$

- The notation *SO* abbreviates Special Orthogonal.
- Special refers to the fact that det R = +1 rather than ±1.
- We will be primarily interested in n = 3 and n = 2 case (planar rotations).
- Also, the inverse is the same as the transpose, that is, $R^{-1} = R^T$

$$\begin{pmatrix} {}^B x \\ {}^B y \end{pmatrix} = \left({}^V \boldsymbol{R}_B\right)^{-1} \begin{pmatrix} {}^V x \\ {}^V y \end{pmatrix} = \left({}^V \boldsymbol{R}_B\right)^T \begin{pmatrix} {}^V x \\ {}^V y \end{pmatrix} = {}^B \boldsymbol{R}_V \begin{pmatrix} {}^V x \\ {}^V y \end{pmatrix}$$

- SO(3) is a **group** under the operation of matrix multiplication, which means that the product of two rotation matrix is a rotation matrix.

Group Definition

- A set G together with a binary operation $\circ$ defined on <u>elements of G</u> is called a group if it satisfies the following axioms:

1. **Closure:** If $g_1, g_2 \in G$, then $g_1 \circ g_2 \in G$.
2. **Identity:** There exists an identity element, e, such that
   $g \circ e = e \circ g = g$ for every $g \in G$.
3. **Inverse:** For each $g \in G$, there exists a (unique) inverse, $g^{-1} \in G$, such that $g \circ g^{-1} = g^{-1} \circ g = e$.
4. **Associativity:** If $g_1, g_2, g_3 \in G$, then $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

- SO(3) is a **group** under the operation of matrix multiplication

1. If $R_1, R_2 \in SO(3)$, then $R_1R_2 \in SO(3)$. <u>Hint: use definition of SO(3) matrices!</u>
2. The identity matrix is the identity element.
3. Inverse of $R \in SO(3)$ is $R^T \in SO(3)$.
4. The associativity of the group operation follows from the associativity of matrix multiplication; that is, $(R_1R_2)R_3 = R_1(R_2R_3)$.

Rotations are rigid body transformations

A rotation **R ∈ SO(3)** is a rigid body transformation; that is,

1. **R** preserves distance:

$$||Rq - Rp|| = ||q - p|| \;\; for\ all\ q, p \in \mathbb{R}^3$$

2. **R** preserves orientation:

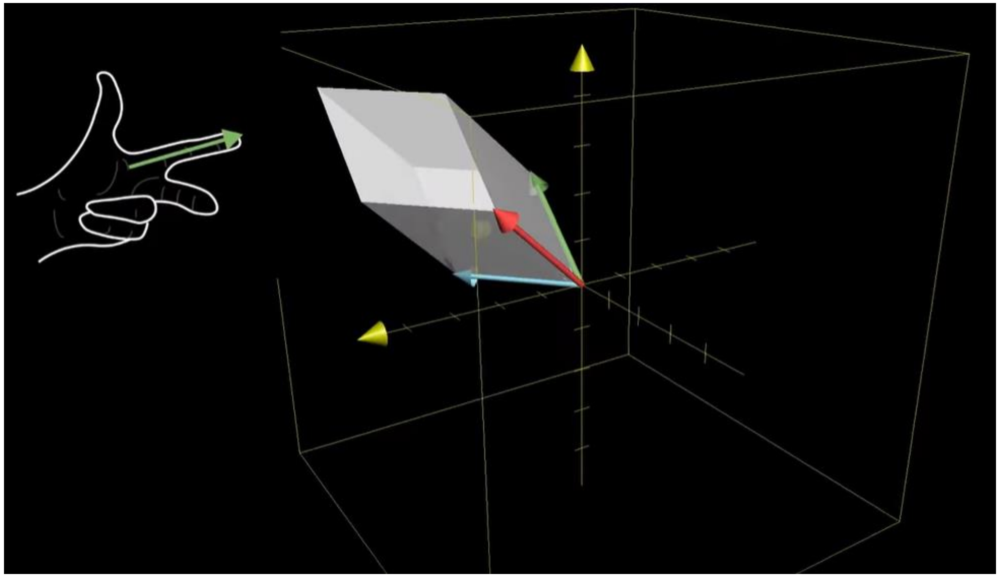
$$R(v \times w) = Rv \times Rw \;\; for\ all\ v, w \in \mathbb{R}^3$$

i.e., the rotation of the cross product of two vectors is the cross product of the rotation of each of the vectors by **R**.

❖ Proof in your **THA1**!

Physical Interpretation of Rotation Matrices

- Geometrically, determinant can be viewed as the volume scaling factor of the linear transformation described by the matrix.
- The determinant is **positive or negative** according to whether the linear mapping **preserves or reverses the orientation of *n*-space**.
- det(R)=+1 means that it's a **rigid body transformation** that does not **change the length** and **orientation**!



https://www.youtube.com/watch?v=Ip3X9LOh2dk

Uses of Rotation Matrices

- There are **three** major uses for a rotation matrix **R**:

  (a) To represent an orientation;
  (b) To change the reference frame in which a vector or a frame is represented (Solved example);
  (c) To rotate a vector or a frame.

- In (a) , **R** is thought of as **representing a frame**;

- In (b) and (c), **R** is thought of as an **operator** that acts on a **vector or frame** (changing its reference frame or rotating it).

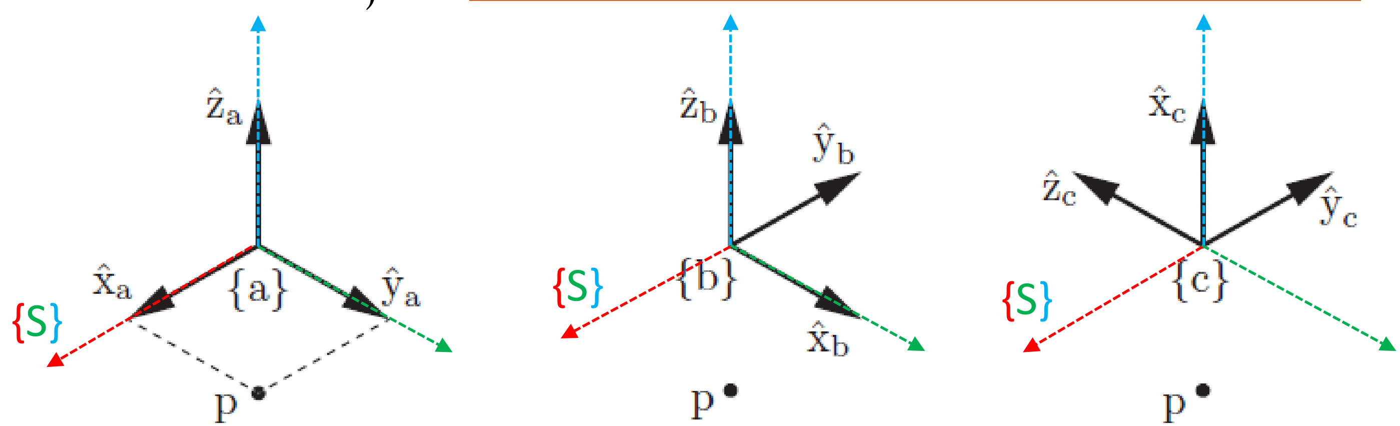
Uses of Rotation Matrices

(a) To represent an orientation:

- ✓ Frames {a}, {b}, and {c} representing <u>the same space with the same origin</u>.
- ✓ <u>RGB color frame</u> is a **fixed space frame {s}**, which is <u>aligned with frame {a}.</u>
- ✓ The orientations of the three frames **relative to {s}** can be written as $\mathbf{R}_f$, which implicitly referring to the <u>orientation of frame</u> { f } <u>relative to the fixed frame</u> {s}, i.e., $\mathbf{R}_{sf}$

$$R_a = \begin{array}{c} \begin{matrix} \hat{x} & \hat{y} & \hat{z} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \quad R_b = \begin{array}{c} \begin{matrix} \hat{x} & \hat{y} & \hat{z} \end{matrix} \\ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \quad R_c = \begin{array}{c} \begin{matrix} \hat{x} & \hat{y} & \hat{z} \end{matrix} \\ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

➢ A rotation matrix is just <u>**a collection of three unit vectors!**</u>



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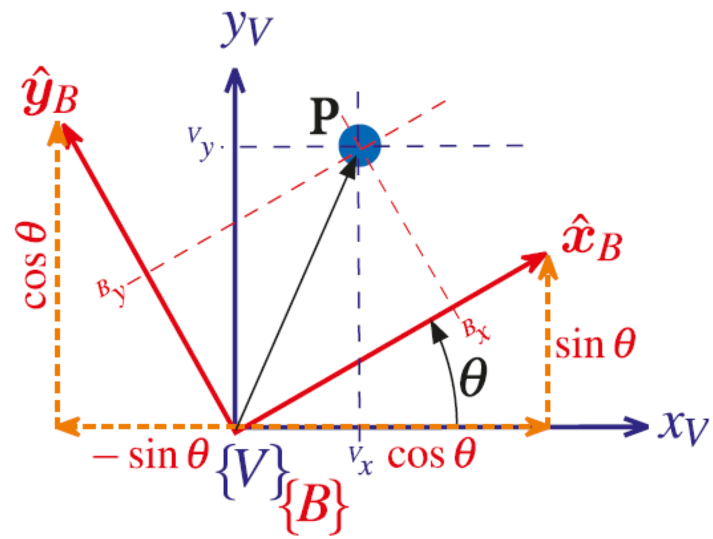


$$\begin{pmatrix} ^{V}x \\ ^{V}y \end{pmatrix} = \boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} ^{B}x \\ ^{B}y \end{pmatrix} \qquad \begin{pmatrix} ^{V}x \\ ^{V}y \end{pmatrix} = \boxed{^{V}R_{B}} \begin{pmatrix} ^{B}x \\ ^{B}y \end{pmatrix}$$

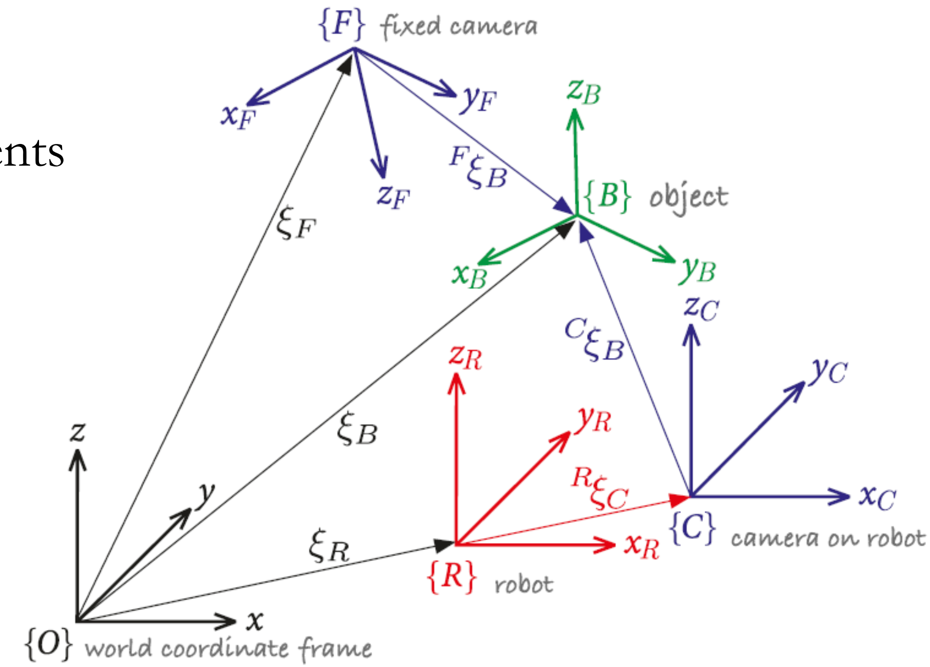
Uses of Rotation Matrices

(b) Changing the reference frame

- If the rotation matrix $\mathbf{R}_{ab}$ represents **the <u>orientation of</u> {b} in {a}** and $\mathbf{R}_{bc}$ represents the **orientation of {c} in {b},** then we have:

$$R_{ac} = R_{ab} R_{bc}.$$

- Notation: $\mathbf{R}_{ab} = {}^{a}\mathbf{R}_{b}$ **both** represents the **orientation of {b} in {a}.**



Uses of Rotation Matrices

(b) Changing the reference frame

- If the rotation matrix $\mathbf{R}_{ab}$ represents the orientation of {b} in {a} and $\mathbf{R}_{bc}$ represents the orientation of {c} in {b}, then we have:

$$R_{ac} = R_{ab} R_{bc}.$$

- $\mathbf{R}_{bc}$ can be viewed as a **representation of the orientation** of {c}.
- While $\mathbf{R}_{ab}$ can be viewed as a **mathematical operator** that changes the reference frame from {b} to {a}.
- A **subscript cancellation rule** helps us to remember this property:

$$R_{ab} R_{bc} = R_{a\cancel{b}} R_{\cancel{b}c} = R_{ac}$$

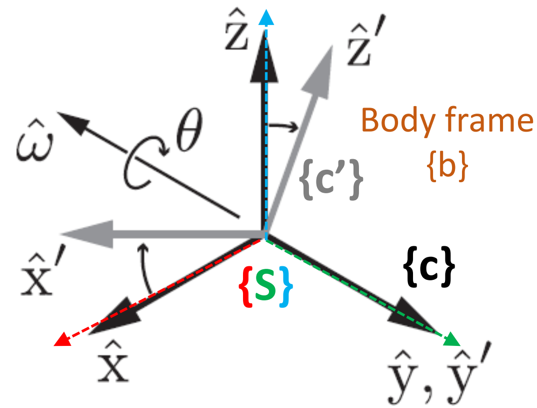
- The **reference frame of a vector** can also be changed by a rotation matrix:

$$R_{ab} p_b = R_{a\cancel{b}} p_{\cancel{b}} = p_a.$$

Uses of Rotation Matrices

(c) Rotating a vector or a frame

- We rotate the frame **{c}** about a unit axis $\hat{\boldsymbol{\omega}}$ by an amount **θ**, the new frame is {c'} and can define it by $\mathbf{R}_{\mathbf{Sc'}}$.
- We can also see **R** as a **rotation operator**, instead of as an orientation, i.e., $R = \mathrm{Rot}(\hat{\omega}, \theta)$



$$\mathrm{Rot}(\hat{\mathrm{x}}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathrm{Rot}(\hat{\mathrm{y}}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

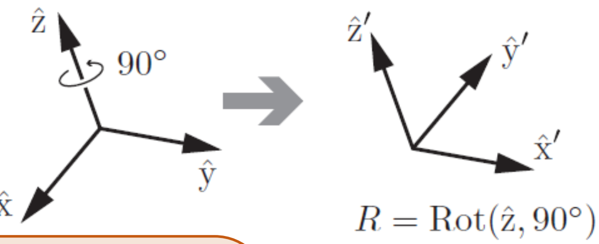
$$\mathrm{Rot}(\hat{\mathrm{z}}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- ***Rot*** is an **independent-to-frame operation** that rotates the orientation **represented by the Identity matrix to the orientation represented by R.**
- We have to specify whether the axis of rotation $\hat{\boldsymbol{\omega}}$ is expressed in **{S}** or **body frame {b},** (e.g., {c} in the figure).

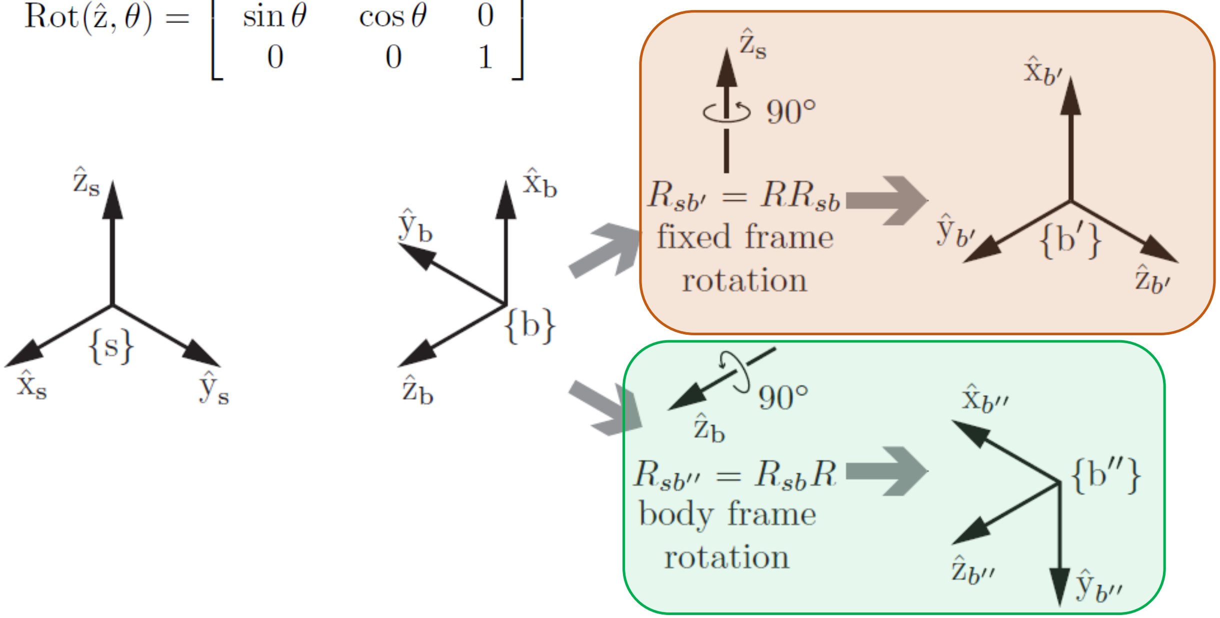
- Depending on our choice, the same numerical $\hat{\boldsymbol{\omega}}$ (and therefore the same numerical R) corresponds to **different rotation axes** in the underlying space!!!

Uses of Rotation Matrices

- The **independent rotation operator** **R = Rot(z^; 90)** gives the orientation of the right-hand frame in the left-hand frame.



$$\mathrm{Rot}(\hat{z},\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

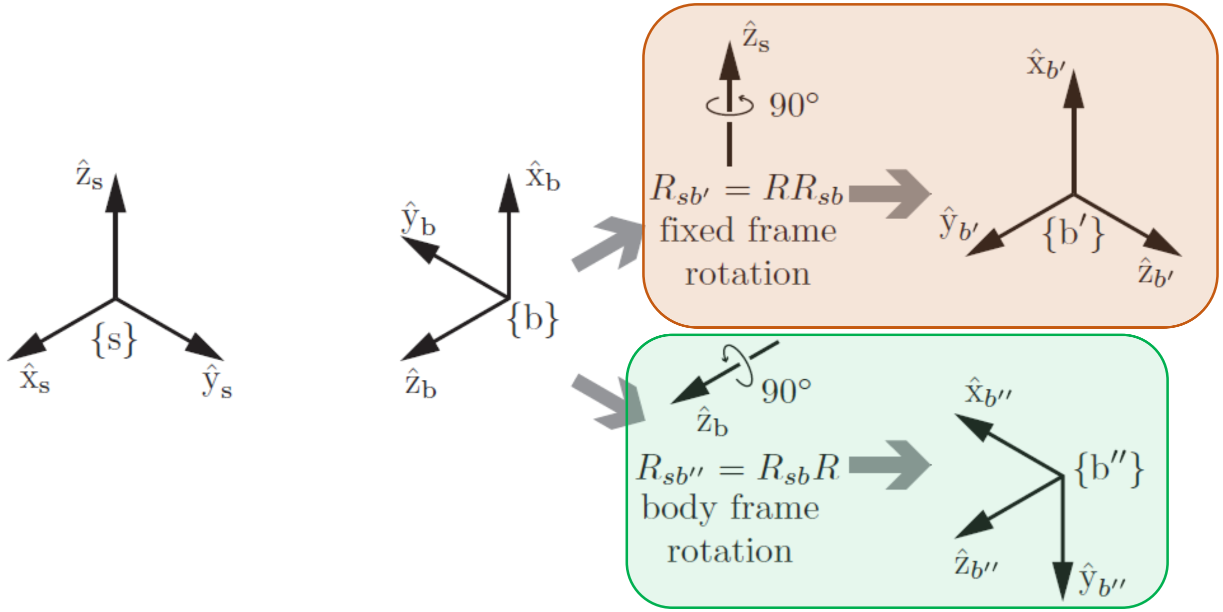


- The quantity RR_{sb} rotates {b} by 90 degree about the **fixed-frame axis z^$_s$** to **{b'}.**
- The quantity $R_{sb}R$ rotates {b} by 90 degree about the **body-frame axis z^$_b$** to **{b"}.**

$$R_{sb'} = \text{rotate\_by\_}R\text{\_in\_\{s\}\_frame}\ (R_{sb}) = \boxed{R}R_{sb}$$

$$R_{sb''} = \text{rotate\_by\_}R\text{\_in\_\{b\}\_frame}\ (R_{sb}) = R_{sb}\boxed{R}.$$

Uses of Rotation Matrices



- In other words, **pre-multiplying** by **R = Rot($\hat{\boldsymbol{\omega}}$; θ)** yields a rotation about an axis $\hat{\boldsymbol{\omega}}$ considered to be in the **fixed frame**, and **post-multiplying by R** yields a rotation about $\hat{\boldsymbol{\omega}}$ considered as being in the **body frame**.

- The quantity RR_{sb} rotates {b} by 90 degree about the **fixed-frame axis z^$_s$** to **{b'}.**
- The quantity $R_{sb}R$ rotates {b} by 90 degree about the **body-frame axis z^$_b$** to **{b''}.**

$$R_{sb'} = \text{rotate\_by\_}R\text{\_in\_\{s\}\_frame}\ (R_{sb}) = RR_{sb}$$
$$R_{sb''} = \text{rotate\_by\_}R\text{\_in\_\{b\}\_frame}\ (R_{sb}) = R_{sb}R.$$

References

- Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.

- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**

- Lynch and Park, "*Modern Robotics*," Cambridge U. Press, 2017, **Chapter 3.**