ME 397- ASBR

Week 4-Lecture 2

a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

FARSHID ALAMBEIGI, Ph.D.

Assistant Professor | Walker Department of Mechanical Engineering

Cockrell School of Engineering | The University of Texas at Austin

The University of Texas at Austin
Mechanical Engineering
Cockrell School of Engineering

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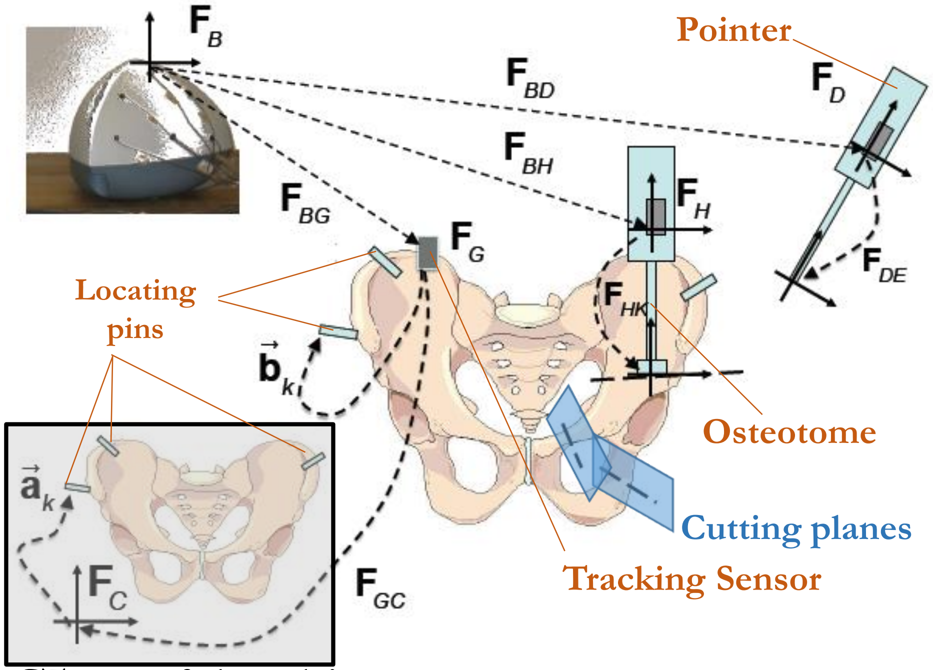
Example: Computer-Assisted Osteotomy

Consider the pelvic osteotomy situation illustrated in the figure. Here we assume that a **three locating pins** have been inserted into the patient's pelvis, and that a CT scan of the pelvis with the pins inserted has been produced. The patient has been placed onto the operating table.

- A magnetic navigation system (here, the Northern Digital Aurora) is present in the room.
- **Two surgical tools** are available:
  - ✓ A probe/pointer device
  - ✓ An osteotome (essentially a fancy chisel) that will be used to cut the pelvis.



CT scan of the pelvis

https://www.youtube.com/watch?v=N8rfMzU4siQ

- 6 DOF Aurora tracking sensors have been attached to the **handle of each tool** and an additional 6 DOF sensor has been **affixed rigidly to the pelvis**. The Aurora is capable of determining the position and orientation of each sensor relative to the Aurora base unit.
- Let $\mathbf{p}_{tip}=\mathbf{p}_{GE}$ be the position of the tip of the pointer tool relative to the reference marker coordinate system $\mathbf{F}_G$.
- Give a formula for computing $\mathbf{p}_{tip}$, based on the available tracking system measurements $\mathbf{F}_{Bx} = (\mathbf{R}_{Bx}, \mathbf{p}_{Bx})$

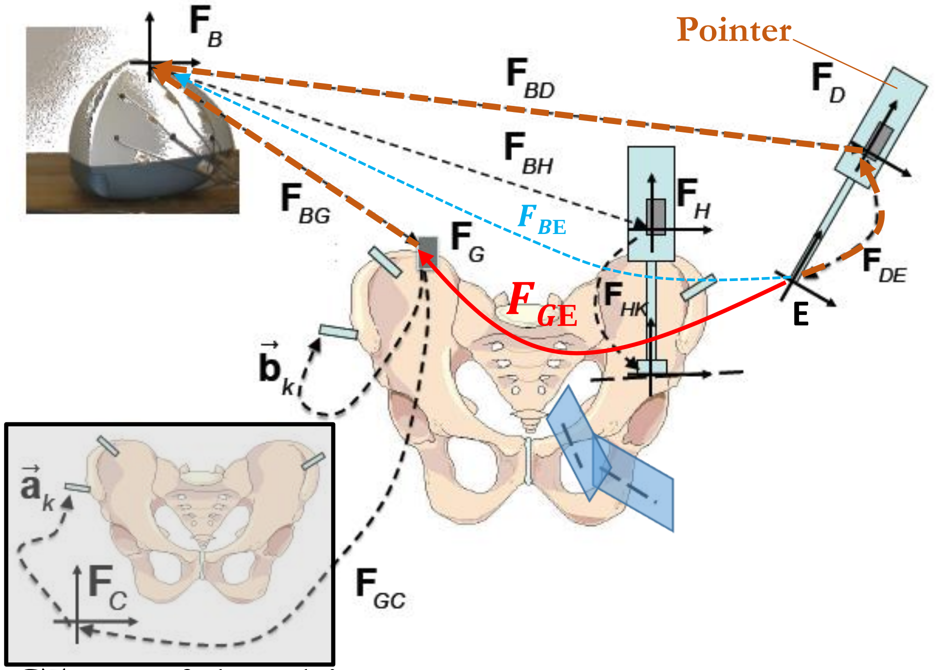
Example is from the Computer Integrated Surgery course, Russell H. Taylor, JHU

Example: Computer-Assisted Osteotomy

$$\overbrace{F_{BG}\, F_{GE}}^{F_{BE}} = \overbrace{F_{BD}\, F_{DE}}^{F_{BE}}$$

$$\Rightarrow \quad F_{GE} = F_{BG}^{-1}\, F_{BD}\, F_{DE}$$

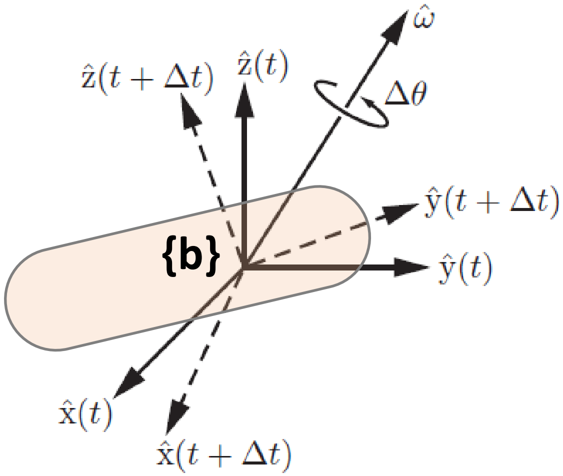
p_{GE} is the last column of F_{GE}



CT scan of the pelvis

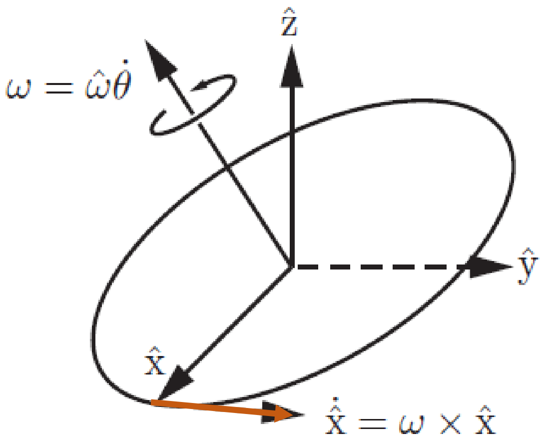
Example is from the Computer Integrated Surgery course, Russell H. Taylor, JHU

Angular Velocities



- Suppose that a **body frame** with **unit** axes $\{\hat{x}, \hat{y}, \hat{z}\}$ is attached to a **rotating body**.
- If we examine the body frame at times **t** and **t+Δt**, **the change in frame orientation** can be described as a rotation of angle $\Delta\theta$ about some unit axis $\hat{\boldsymbol{\omega}}$ passing through the origin.
- The axis $\hat{\boldsymbol{\omega}}$ is **coordinate-free**; it is not **yet** represented in any particular reference frame.
- As **t** approaches zero, the ratio $\Delta\theta/\Delta t$ becomes the **rate of rotation** $\dot{\theta}$, and $\hat{\boldsymbol{\omega}}$ is the **instantaneous axis of rotation**. Hence, angular velocity w is:

$$\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}.$$



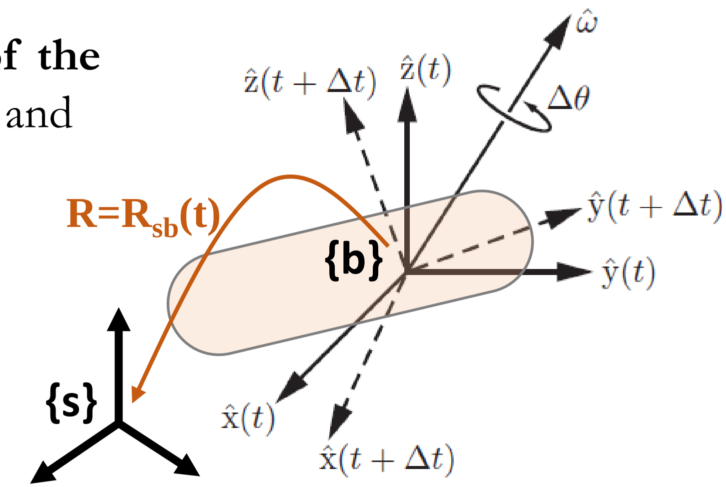
- We can then calculate **linear velocities** as:

$$\begin{aligned} \dot{\hat{x}} &= \mathrm{w} \times \hat{x}, \\ \dot{\hat{y}} &= \mathrm{w} \times \hat{y}, \\ \dot{\hat{z}} &= \mathrm{w} \times \hat{z}. \end{aligned}$$

- To express these equations in **coordinates**, we have to choose a reference frame in which to represent **w** typically the **fixed frame {s} or the body frame {b}.**

Fixed-Frame Angular Velocities

- Let R(t) be the rotation matrix describing the **orientation of the body frame with respect to the fixed frame {s}** at time t and **$\dot{R}(t)$** is its time rate of change.



- Let R(t)= $[\overset{\hat{x}}{r_1}(t);\ \overset{\hat{y}}{r_2}(t);\ \overset{\hat{z}}{r_3}(t)]$ where $\mathbf{r_i}$ is the representation of the corresponding **<u>body frame axis</u>** in the fixed frame {s}.

- At a specific time **t**, let $\boldsymbol{\omega}_s \in \mathbb{R}^3$ be the angular velocity $\boldsymbol{\omega}$ expressed in **fixed-frame** then we have:

$$\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}.$$

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3.$$

OR $$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3]$$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$$a \times b = (a)^{\wedge} b.$$

- We can rewrite $\boldsymbol{\omega}_s \times \boldsymbol{r}_i$ as $[\boldsymbol{\omega}_s]R$, where $[\boldsymbol{\omega}_s]$ is a 3×3 **skewsymmetric** matrix representation of $\boldsymbol{\omega}_s \in \mathbb{R}^3$. Hence:

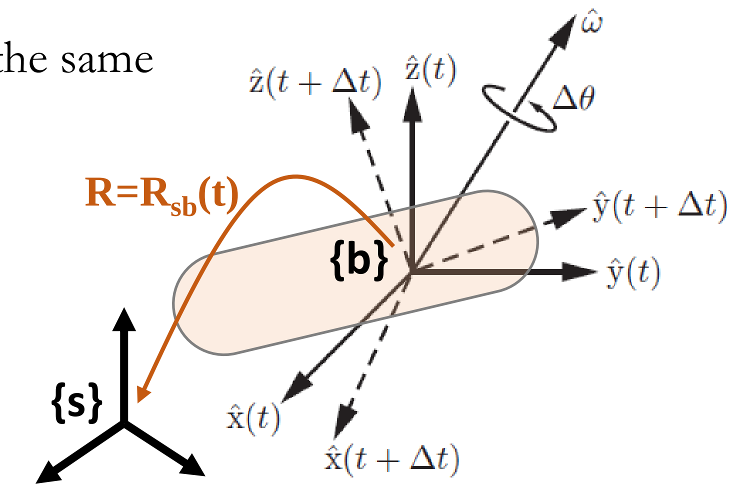
$$[\omega_s]R = \dot{R}$$

$$[\omega_s] = \dot{R}R^{-1}$$

Skew-Symmetric angular **velocity of $\boldsymbol{\omega}$** represented in the **fixed frame**

Time rate of change of the **orientation of the body frame with respect to the fixed frame {s}**

- Let $\boldsymbol{\omega}_s$ **and** $\boldsymbol{\omega}_b$ be two different vector representations of the same angular velocity w expressed in the **fixed and body-frame** coordinates, respectively. Hence: $\omega_s = R_{sb}\omega_b$



$$\Rightarrow \quad \omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^{\mathrm{T}}\omega_s$$

- Let us now use the skew-symmetric operator **[.]** to rewrite this equation in a matrix format:

$$\boxed{R[\omega]R^{\mathrm{T}} = [R\omega]}$$

Given any $\boldsymbol{\omega} \in \mathbb{R}^3$ and $R \in SO(3)$

$$\boxed{[\omega_s] = \dot{R}R^{-1}}$$

$$\boxed{RR^T = R^T R = I.}$$

$$\begin{aligned}
[\omega_b] &= \boxed{[R^{\mathrm{T}}\omega_s]} \\
&= R^{\mathrm{T}}\boxed{[\omega_s]}R \\
&= R^{\mathrm{T}}(\dot{R}\boxed{R^{\mathrm{T}})R} \\
[\omega_b] &= R^{\mathrm{T}}\dot{R} = R^{-1}\dot{R}.
\end{aligned}$$

$$[\omega_s]R = \dot{R}.$$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

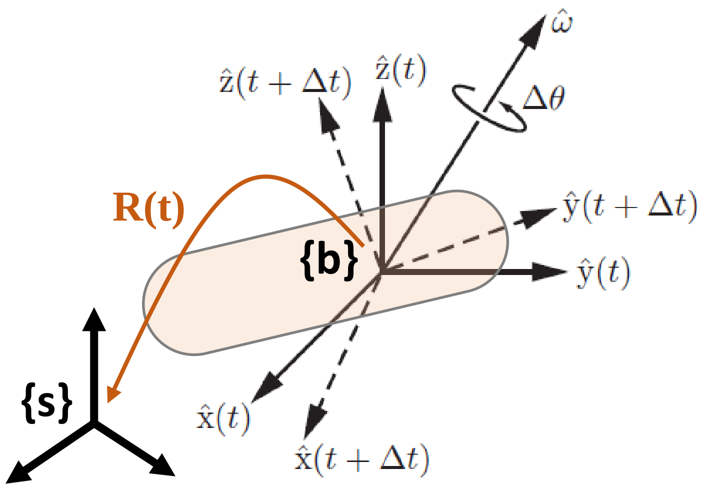
Skew-Symmetric angular velocity of $\boldsymbol{\omega}$ represented in the **body frame**

Time rate of change of the **orientation of the body frame with respect to the fixed frame {s}**

Angular Velocities

- Let R(t) denote the <u>orientation of the rotating frame as seen from the fixed frame</u>. Denote by **w** the angular velocity of the rotating frame. Then

$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b].$$



where $\boldsymbol{\omega}_s \in \mathbb{R}^3$ is the **fixed-frame vector** representation of **w** and $[\boldsymbol{\omega}_s] \in so(3)$ is its 3×3 matrix representation, and where $\boldsymbol{\omega}_b \in \mathbb{R}^3$ is the **body-frame vector** representation of **w** and $[\boldsymbol{\omega}_b] \in so(3)$ is its 3×3 matrix representation.

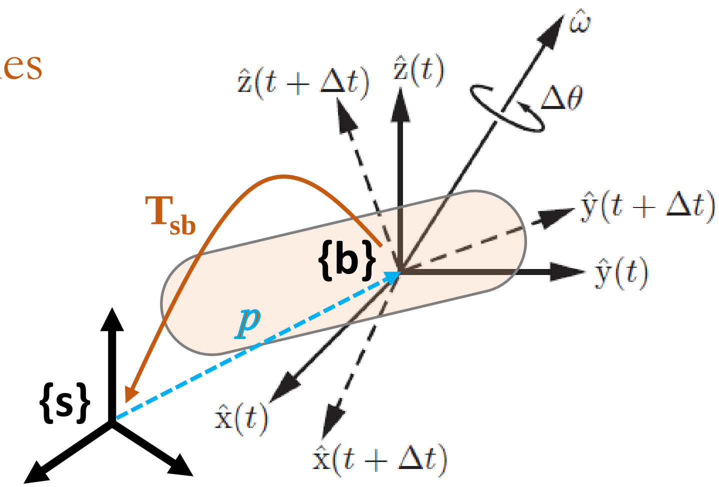
- It is important to note that the **fixed-frame angular velocity** $\boldsymbol{\omega}_s$ does **not** depend on the choice of body frame.
- **<u>Similarly</u>**, the body-frame angular velocity $\boldsymbol{\omega}_b$ does **not** depend on the choice of fixed frame.
- An angular velocity expressed in an **arbitrary frame {d}** can be represented in another frame {c} **<u>if we know the rotation that takes {c} to {d}:</u>**

$$\omega_c = R_{cd}\omega_d.$$

Twists

- We now consider both the linear and angular velocities of a moving frame. Let

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$



- Let us pre-multiply $\dot{T}$ by T^{-1}:

$$\begin{aligned} \boxed{T^{-1}}\boxed{\dot{T}} &= \boxed{\begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}} \boxed{\begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}} \\ &= \begin{bmatrix} \boxed{R^{\mathrm{T}}\dot{R}} & \boxed{R^{\mathrm{T}}\dot{p}} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \boxed{[\omega_b]} & \boxed{v_b} \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

$v_b = R^T\dot{p}$ is the **linear velocity of the origin** of {b} expressed in {b}.

- $T^{-1}\dot{T}$ represents the linear and angular velocities of the moving frame relative to the stationary frame {b} currently aligned with the moving frame (i.e., **current body frame**).

Body Twist

- We define the **spatial velocity in the body frame**, or simply the **body twist $\mathcal{V}_b$** as:

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$$

- As it is convenient to have a **skew-symmetric matrix representation** of an **angular velocity vector**, it is convenient to have a **matrix representation of a twist** as:

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3) \qquad [x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

where **$[\omega] \in$ so(3)** and **$v_b \in \mathbb{R}^3$**.

- The set of all **4×4 matrices** of this form is called **se(3).**

- **se(3)** comprises the **matrix representations** of the twists associated with the rigid-body configurations **SE(3)**.

Spatial Twist

$$\dot{R}R^{-1} = [\omega_s]$$
$$R^{-1}\dot{R} = [\omega_b],$$

- let us now evaluate $\dot{T}T^{-1}$:

$$\begin{aligned} \dot{T}T^{-1} &= \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}R^{\mathrm{T}} & \dot{p} - \dot{R}R^{\mathrm{T}}p \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

where $[\omega_s] \in so(3)$ and $v_s \in \mathbb{R}^3$.

- We can assemble ω_s and v_s into a **six-dimensional twist**:

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6, \qquad \boxed{[\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1}} \in se(3)$$

- We call $\mathcal{V}_s$ the **spatial velocity in the space frame**, or simply the **spatial twist.**

- We can obtain $\mathcal{V}_b$ from $\mathcal{V}_s$ as follows:

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\boxed{\dot{T}} \\ &= T^{-1}\boxed{[\mathcal{V}_s]\,T}. \end{aligned} \qquad [\mathcal{V}_s] = T\,[\mathcal{V}_b]\,T^{-1}$$

Adjoint Representation

$$[\mathcal{V}_s] = T\,[\mathcal{V}_b]\,T^{-1} \Rightarrow [\mathcal{V}_s] = \begin{bmatrix} \boxed{R[\omega_b]R^{\mathrm{T}}} & \boxed{-\boxed{R[\omega_b]R^{\mathrm{T}}}p} + Rv_b \\ 0 & 0 \end{bmatrix}$$

$$\boxed{R[\omega]R^{\mathrm{T}} = [R\omega]}$$

$$\boxed{[\omega]p} = -[p]\omega \text{ for } p, \omega \in \mathbb{R}^3$$

$$\underbrace{\begin{bmatrix} \omega_s \\ v_s \end{bmatrix}}_{\mathcal{V}_s} = \underbrace{\begin{bmatrix} \boxed{R} & 0 \\ \boxed{[p]R} & R \end{bmatrix}}_{[\mathrm{Ad}_T]} \underbrace{\begin{bmatrix} \boxed{\omega_b} \\ v_b \end{bmatrix}}_{\mathcal{V}_b}$$

Math tricks to factor out the \omega_b

- Given T = (R; p)∈ SE(3), its **adjoint** representation $[\mathrm{Ad}_T]$ is

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

- For any $\mathcal{V} \in \mathbb{R}^6$, the adjoint map associated with T is

$$\mathcal{V}' = [\mathrm{Ad}_T]\mathcal{V} \quad \equiv \quad \mathcal{V}' = \mathrm{Ad}_T(\mathcal{V})$$

- In terms of the matrix form $[\mathcal{V}] \in se(3)$ of $\mathcal{V} \in \mathbb{R}^6$: $[\mathcal{V}'] = T[\mathcal{V}]T^{-1}$

Adjoint Properties

- Let $T_1; T_2 \in SE(3)$ and $\mathcal{V} = (\omega; v)$ and $[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$ then,

$$\mathrm{Ad}_{T_1}\left(\mathrm{Ad}_{T_2}(\mathcal{V})\right) = \mathrm{Ad}_{T_1T_2}(\mathcal{V}) \quad or \quad [\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}]\mathcal{V} = [\mathrm{Ad}_{T_1T_2}]\mathcal{V}$$

- Also, for any $T \in SE(3)$ the following holds:

$$[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$$

Proof: Choosing $T_1 = \mathbf{T^{-1}}$ and $T_2 = \mathbf{T}$,

$$\mathrm{Ad}_{T^{-1}}\left(\mathrm{Ad}_T(\mathcal{V})\right) = \mathrm{Ad}_{T^{-1}T}(\mathcal{V}) = \mathrm{Ad}_I(\mathcal{V}) = \mathcal{V}.$$

$$[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}]\mathcal{V} = [\mathrm{Ad}_{T_1T_2}]\mathcal{V}$$

Summary of Twist Results

Given a fixed (space) frame {s}, a body frame {b}, and a differentiable $T_{sb}(t) \in SE(3)$

$$T_{sb}(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

1. Matrix representation of the body twist

$$\boxed{T_{sb}^{-1}\dot{T}_{sb} = [\mathcal{V}_b]} = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

2. Matrix representation of the spatial twist

$$\boxed{\dot{T}_{sb}T_{sb}^{-1} = [\mathcal{V}_s]} = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3)$$

3. The twists $\mathcal{V}_s$ and $\mathcal{V}_b$ are related by

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \boxed{\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \boxed{[\mathrm{Ad}_{T_{sb}}]}\mathcal{V}_b,$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \boxed{\begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix}} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \boxed{[\mathrm{Ad}_{T_{bs}}]}\mathcal{V}_s.$$

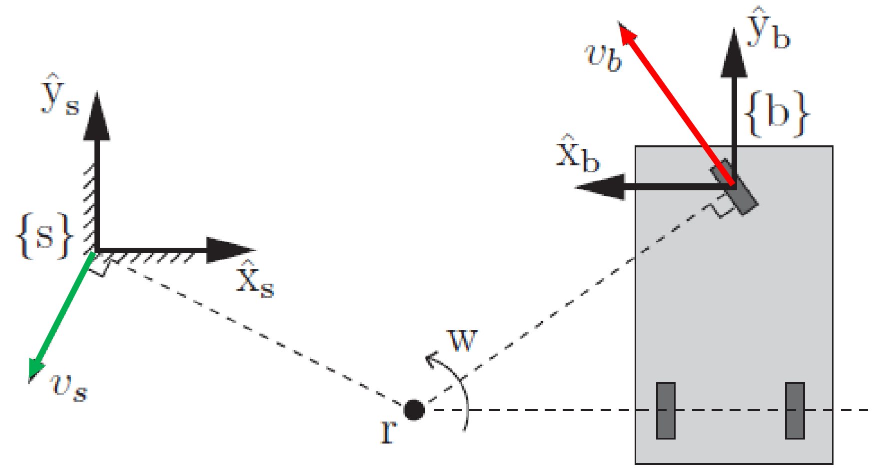
4. For any two frames {c} and {d}, a twist represented as $\mathcal{V}_c$ in {c} is related to its representation $\mathcal{V}_d$ in {d} by:

$$\mathcal{V}_c = [\mathrm{Ad}_{T_{cd}}]\mathcal{V}_d, \qquad \mathcal{V}_d = [\mathrm{Ad}_{T_{dc}}]\mathcal{V}_c$$

Example

Figure shows a top view of a car, with a single steerable front wheel, driving on a plane. The $\hat{\mathbf{z}}_b$**-axis** of **the body frame** {b} is **into the page** and the $\hat{\mathbf{z}}_s$**-axis** of the **fixed frame** {s} is **out of the page**.

- The angle of the front wheel of the car causes the car's motion to be a pure angular velocity w = 2 rad/s about an axis **out of the page** at the **point r** in the plane.

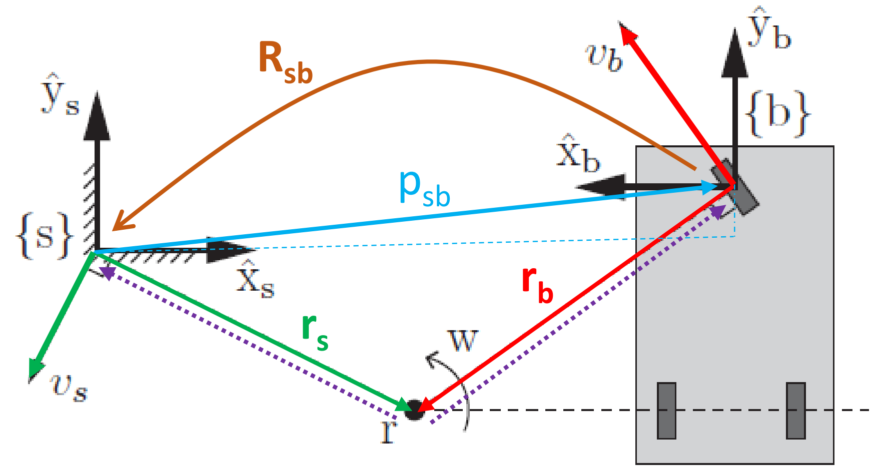


If $\mathbf{r}_s = (2;-1;0)$ or $\mathbf{r}_b = (2;-1.4;0)$, calculate twists ν_s and ν_b and verify them using corresponding adjoints.

Example

$\mathbf{r_s}$ = (2;-1; 0)
$\mathbf{r_b}$ = (2;-1.4; 0)

$\boldsymbol{\omega}_s$ = (0; 0; 2)
$\boldsymbol{\omega}_b$= (0; 0; -2)



$$T_{sb} = \begin{bmatrix} \boxed{R_{sb}} & \boxed{p_{sb}} \\ 0 & 1 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} \widehat{x}_b & \widehat{y}_b & \widehat{z}_b & \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

From the figure we get:

$$\boxed{v_s} = \omega_s \times \boxed{(-r_s)} = r_s \times \omega_s = (-2, -4, 0),$$
$$\boxed{v_b} = \omega_b \times \boxed{(-r_b)} = r_b \times \omega_b = (2.8, 4, 0),$$

$$\Rightarrow \mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \quad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}$$

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} \boxed{R} & 0 \\ \boxed{[p]}\boxed{R} & \boxed{R} \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \boxed{[\mathrm{Ad}_{T_{sb}}]}\mathcal{V}_b$$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

[p] is the **skew-symmetric matrix** representation of vector $\mathrm{p_{sb}}$

References

- Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.

- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**

- Lynch and Park, "*Modern Robotics,*" Cambridge U. Press, 2017, **Chapter 3.**