ME 397- ASBR

Week 6-Lecture 1

a Curiosity NASA/JPLCaltech; **b** Savioke Relay; **c** self driving car, Oxford Univ.;
d Cheetah legged robot, Boston Dynamics

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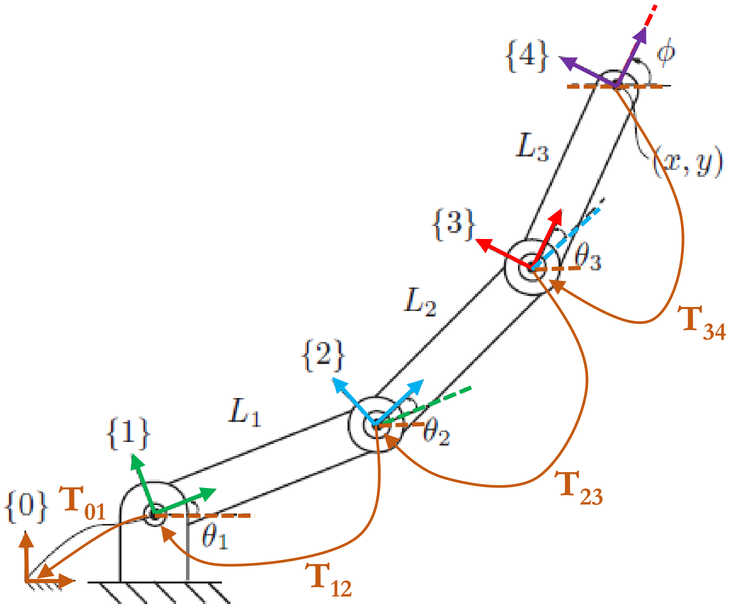
Forward Kinematics

- The **forward kinematics** of a robot refers to the calculation of the **position and orientation** of its end-effector frame **from its joint coordinates**.

- The Geometric forward kinematics problem for a 3R planar open chain:

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3),\\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3),\\
\phi &= \theta_1 + \theta_2 + \theta_3.
\end{aligned}
$$



- A more systematic method of deriving the forward kinematics might involve attaching reference frames {1}, {2} and {3} to each link.

- The forward kinematics can then be written as a **product of four homogeneous transformation matrices**

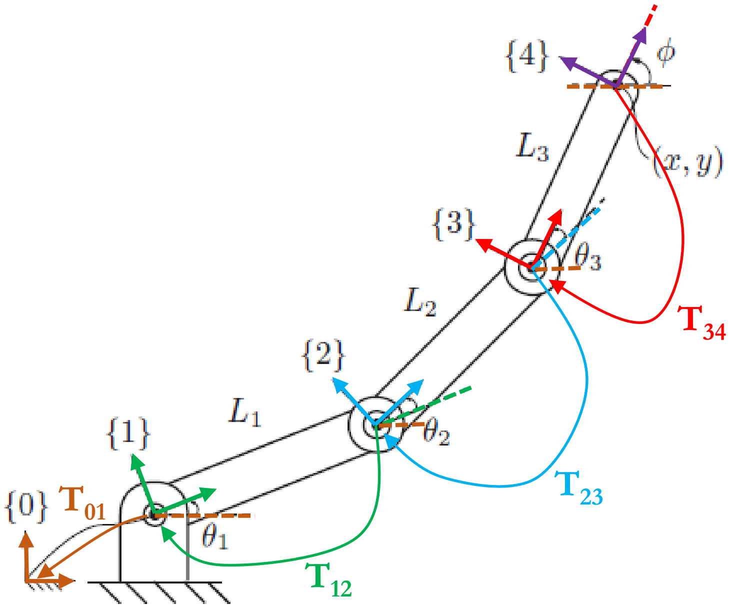
$$T_{04} = T_{01} T_{12} T_{23} T_{34}$$

Forward Kinematics

$$T_{04} = \boxed{T_{01}}\boxed{T_{12}}\boxed{T_{23}}\boxed{T_{34}}$$

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

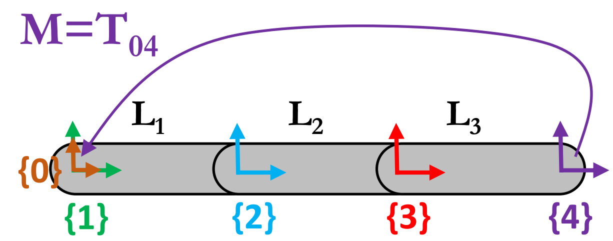


- Observe that $\mathbf{T}_{34}$ is **constant** and that each remaining $\mathbf{T}_{i-1:i}$ depends only on the joint variable $\boldsymbol{\theta}_i$.

- **D**enavit-**H**artenberg parameters (**D-H parameters**) representation of forward kinematics.

- As an alternative approach, let us define **M** to be the **position and orientation** of frame {4} **in** {0} frame when all joint angles are set to zero (the "home or zero" position of the robot), i.e., $\mathbf{T}_{04}$. Then

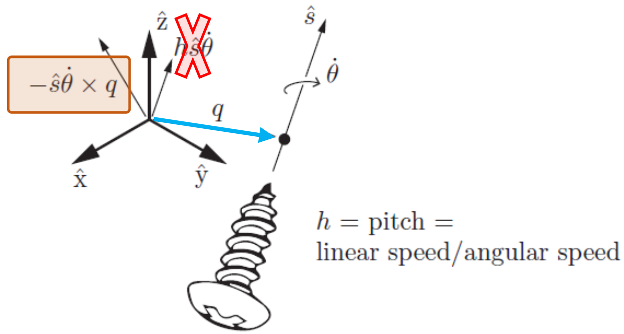
$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

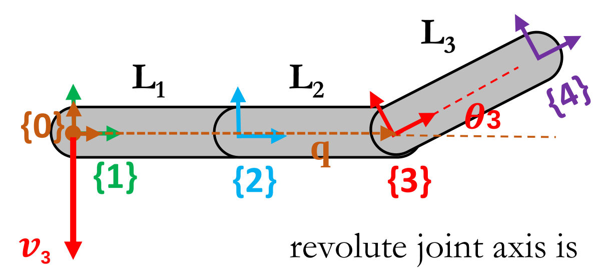


- Each **revolute joint axis** is a **zero-pitch** (2D motion) **screw axis** and its **location** is center of rotation defining **parameter q**.
- If $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are held at their **zero position** then the **screw axis** corresponding to rotating about **joint 3** can be expressed in the {0} frame as

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

$$v_3 = -\omega_3 \times q_3$$
$$q_3 = (L_1 + L_2, 0, 0)$$





revolute joint axis is out of the plane.

Forward Kinematics

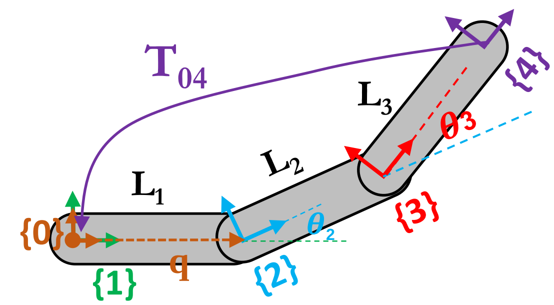
➢ The screw axis S_3 can be expressed in se(3) matrix form as

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

➢ For any $\boldsymbol{\theta}_3$, the matrix exponential representation for screw motions is

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} \underbrace{M}_{\text{original position}} \qquad (\text{for } \theta_1 = \theta_2 = 0)$$

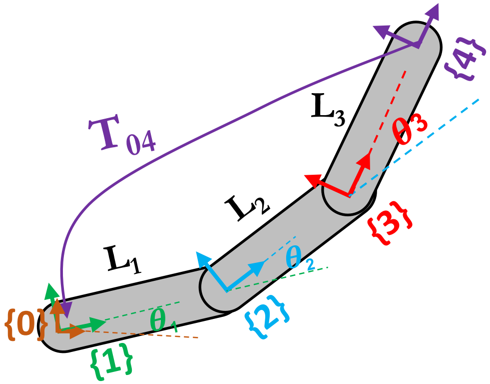
➢ Now, for $\boldsymbol{\theta}_1 = 0$ and **any fixed (but arbitrary)** $\boldsymbol{\theta}_3$, rotation about **joint 2** can be viewed as applying a screw motion to the rigid (link 2)/(link 3) pair, i.e.,



$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = 0) \qquad [\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Forward Kinematics

- Finally, keeping $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$ **fixed**, rotation about **joint 1** can be viewed as applying a screw motion to the entire rigid three-link assembly. We can therefore write, for arbitrary values of ($\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$):



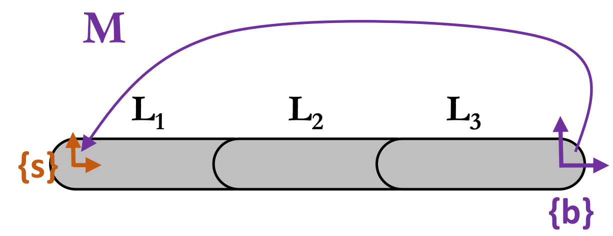
$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

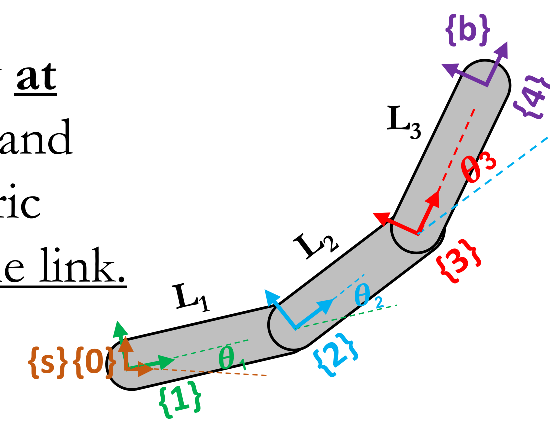
- Thus the **forward kinematics** can be expressed as a product of matrix exponentials, each corresponding to a screw motion.

Product of Exponentials (PoE) Formula

- To use the PoE formula, it is **only necessary** to
  - ✓ Assign a stationary frame {s} (e.g., at the **fixed base of the robot** or anywhere else that is convenient for defining a reference frame)
  - ✓ A frame {b} at the end-effector, described by M when the robot is at its **zero position**.



- It is common to define a frame **at each link**, though, typically **at the joint axis**; these are needed for the **D-H representation** and they are useful for displaying a graphic rendering of a geometric model of the robot and for defining the mass properties of the link.



- Thus, when we are defining the kinematics of an **n-joint** robot, we may either

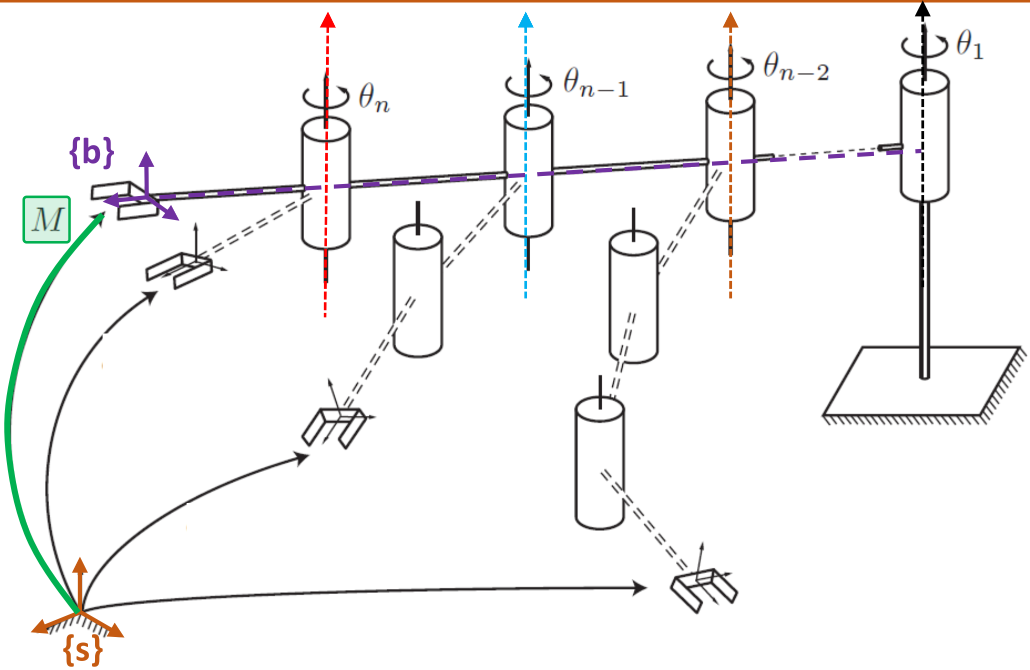
  (1) minimally use the frames {s} and {b} if we are only interested in the kinematics,

  (2) Or refer to {s} as frame {0}, use frames {i} for i = 1:n (the frames for links i at joints i), and use one more frame {n + 1} (corresponding to {b}) at the end-effector.

- The frame {n + 1} (i.e., {b})) is fixed relative to {n}, but it is at a more convenient location to represent the configuration of the end-effector.

Forward Kinematics: Screw Axes in the Base Frame

- Let's consider a general **spatial open chain** consisting of ***n*** **one-dof joints** that are **connected serially**.
- The key concept behind the PoE formula is to regard **each joint** as applying a **screw motion** to all the outward links.
- To apply the **PoE formula**, we must
  1) choose a fixed base frame {s} and an end-effector frame {b} attached to the last link.



  2) Place the robot in its zero position by setting all joint values to zero, with **assigning the direction of positive displacement** (rotation for revolute joints, translation for prismatic joints) for each joint specified.

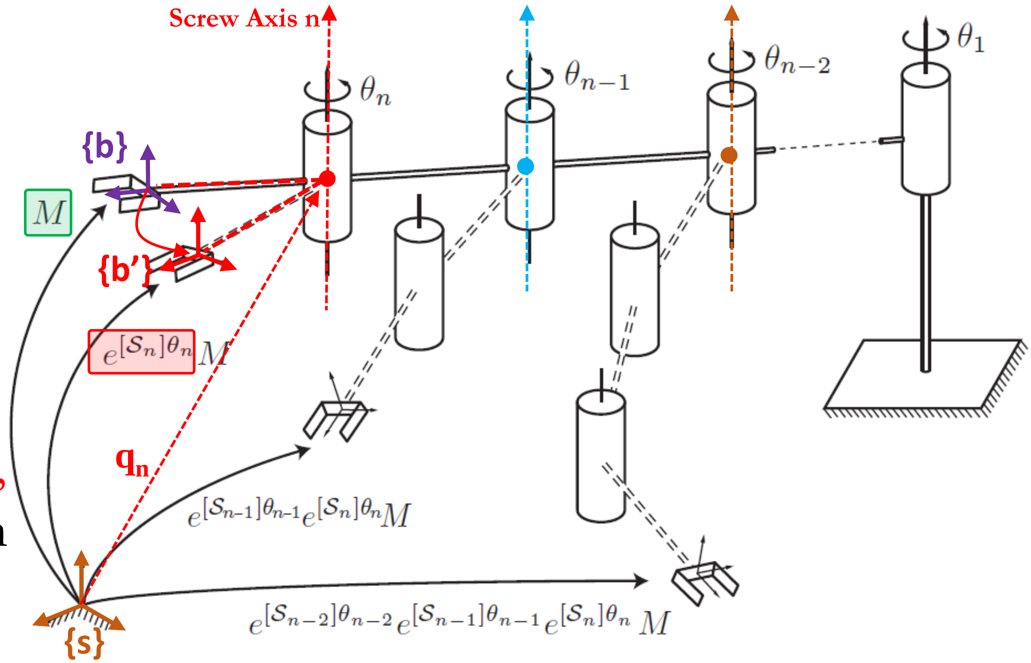
  3) Let **M ∈ SE(3)** denote the configuration of the **end-effector frame relative to the fixed base frame** when the robot is in its **zero position.**

Forward Kinematics: Screw Axes in the Base Frame

4) Now suppose that joint n is displaced to some joint value $\boldsymbol{\theta_n}$. The end-effector frame M then undergoes a displacement of the form

$$T = \boxed{e^{[\mathcal{S}_n]\theta_n}} M,$$

where T ∈ SE(3) is the new configuration of the end-effector frame and $\boldsymbol{S} = (\boldsymbol{\omega}; \boldsymbol{v})$, is the screw axis of joint n as expressed in the fixed base frame.

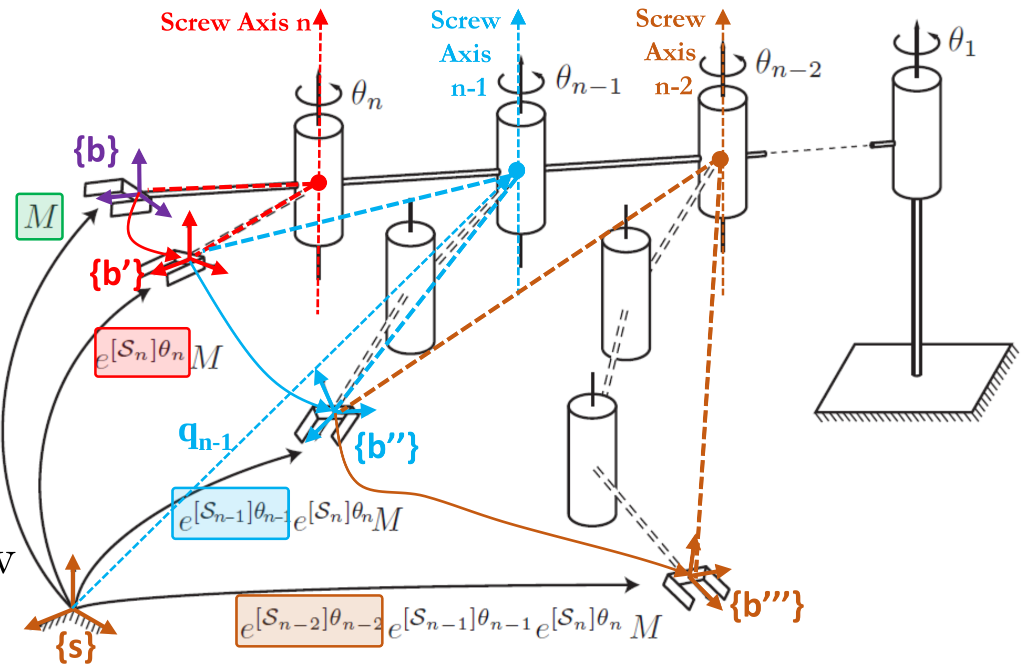


- If joint **n** is **revolute** (corresponding to a screw motion of **zero pitch**) then
  - ✓ $\boldsymbol{\omega_n} \in \mathbb{R}^3$ is a unit vector in the **positive direction of joint axis *n***;
  - ✓ $\mathbf{v_n} = -\boldsymbol{\omega_n} \times \mathbf{q_n}$ with $\mathbf{q_n}$ **any arbitrary point on joint axis n** as written in coordinates in the **fixed base frame**;
  - ✓ $\boldsymbol{\theta_n}$ is the joint angle.
- If joint n is **prismatic** then $\boldsymbol{\omega_n} = 0$,
  - ✓ $\mathbf{v_n} \in \mathbb{R}^3$ **is a unit vector** in the **direction of positive translation**,
  - ✓ $\boldsymbol{\theta_n}$ represents the prismatic extension/retraction.

Forward Kinematics: Screw Axes in the Base Frame

5) If we assume that joint n-1 is also allowed to vary then this has the effect of applying a screw motion to link n-1. The end-effector frame thus undergoes a displacement of the form

$$T = \boxed{e^{[\mathcal{S}_{n-1}]\theta_{n-1}}}\left(e^{[\mathcal{S}_n]\theta_n} M\right)$$



6) Continuing with this reasoning and now allowing **all the joints** ($\boldsymbol{\theta}_1; \ldots ; \boldsymbol{\theta}_n$) to vary, it follows that:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots \boxed{e^{[\mathcal{S}_{n-1}]\theta_{n-1}}}\boxed{e^{[\mathcal{S}_n]\theta_n}}\boxed{M}$$

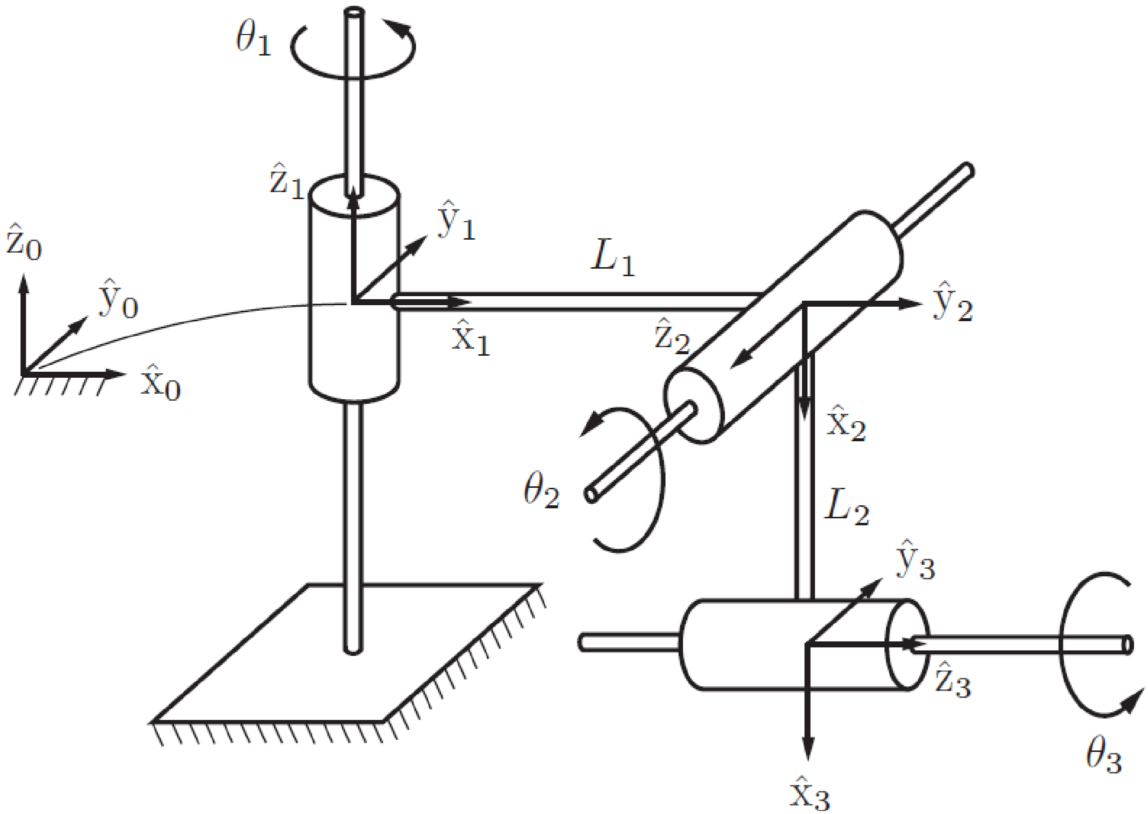
going backwards

- This is the **space form** (i.e., screw axes expressed in the fixed space frame **defining the order of multiplications**) **product of exponentials formula** describing the forward kinematics of an **n-dof open chain.**
- **Unlike the D-H representation,** no link reference frames need to be defined!

Example: 3 Revolute (R) spatial open chain

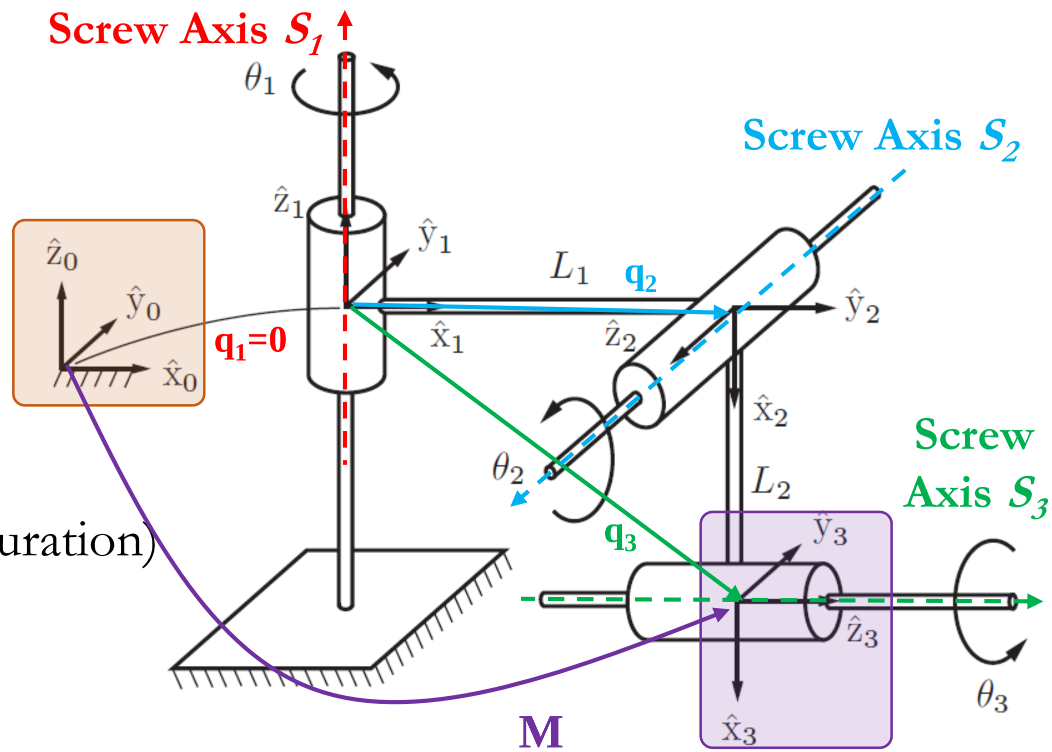
Consider the 3R open chain of the figure shown in **its home position** (all joint variables set equal to zero).

Find the forward kinematics of the robot using the **space form** of the exponential products.



Example: 3R spatial open chain

- We should express all vectors and homogeneous transformations in terms of the **fixed frame**.



Step 1) Choose the fixed frame {0} and end-effector frame {3} as indicated in the figure.

Step 2) By inspection M (Home configuration) can be obtained as:

$$M = \begin{array}{c} \begin{array}{cccc} \hat{x}_3 & \hat{y}_3 & \hat{z}_3 & \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$v = -\omega \times q$

$v_2 = -\omega_2 \times q_2$

$v_3 = -\omega_3 \times q_3$

Step 3) The screw axis $S_1 = (\omega_1; v_1)$, for joint axis 1 is $\omega_1 = (0; 0; 1)$ and $v_1 = (0; 0; 0)$

The screw axis $S_2 = (\omega_2; v_2)$, for joint axis 2 is $\omega_2 = (0; -1; 0)$ and $v_2 = (0; 0; -L_1)$

The screw axis $S_3 = (\omega_3; v_3)$, for joint axis 3 is $\omega_3 = (1; 0; 0)$ and $v_3 = (0; -L_2; 0)$

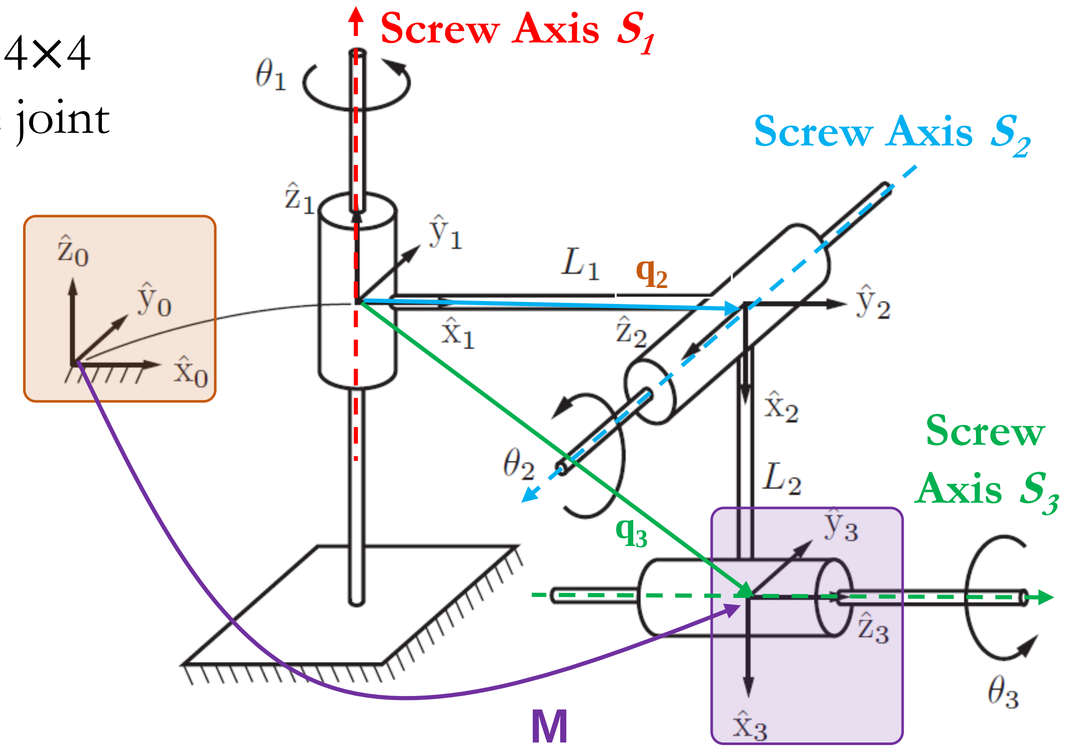
Example: 3R spatial open chain

- In summary, we have the following 4×4 matrix representations for the three joint screw axes $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$:

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



$$v_3 = -\omega_3 \times q_3$$
$$v_2 = -\omega_2 \times q_2$$

- The forward kinematics has the following form:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$$T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$$

where $* =$

$$(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$$

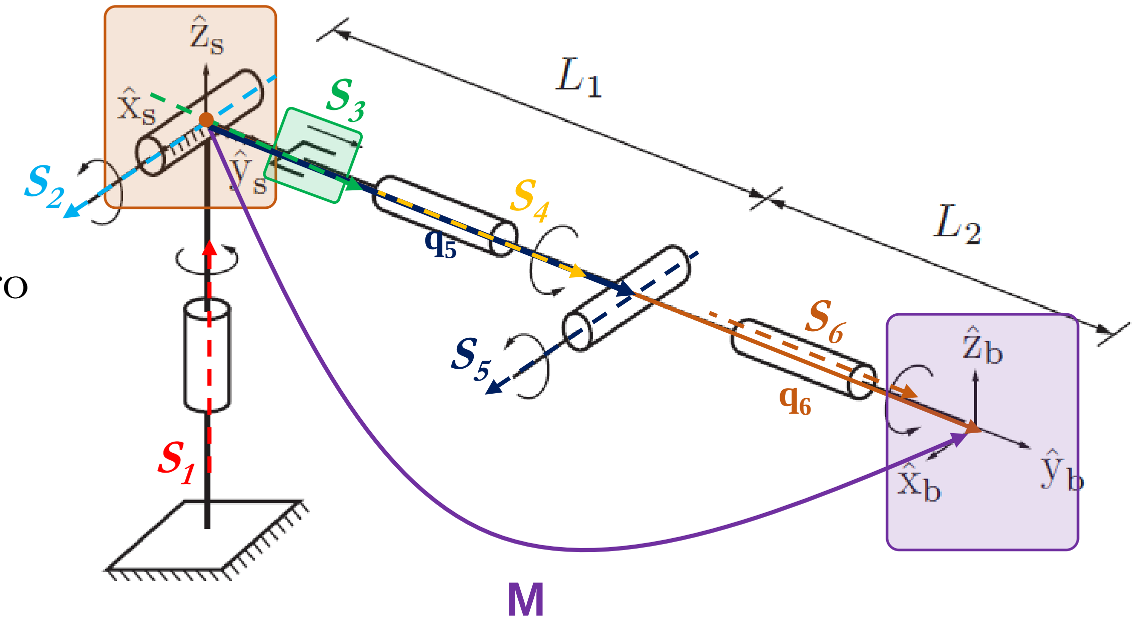
i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,-1,0)$	$(0,0,-L_1)$
3	$(1,0,0)$	$(0,L_2,0)$

Example: RRPRRR spatial open chain

Consider the six-degree-of-freedom RRPRRR spatial open chain of the Figure and find its forward kinematics.

➢ The end-effector frame in the zero position is given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$v = -\omega \times q$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(1,0,0)$	$(0,0,0)$
3	$(0,0,0)$	$(0,1,0)$
4	$(0,1,0)$	$(0,0,0)$
5	$(1,0,0)$	$(0,0,-L_1)$
6	$(0,1,0)$	$(0,0,0)$

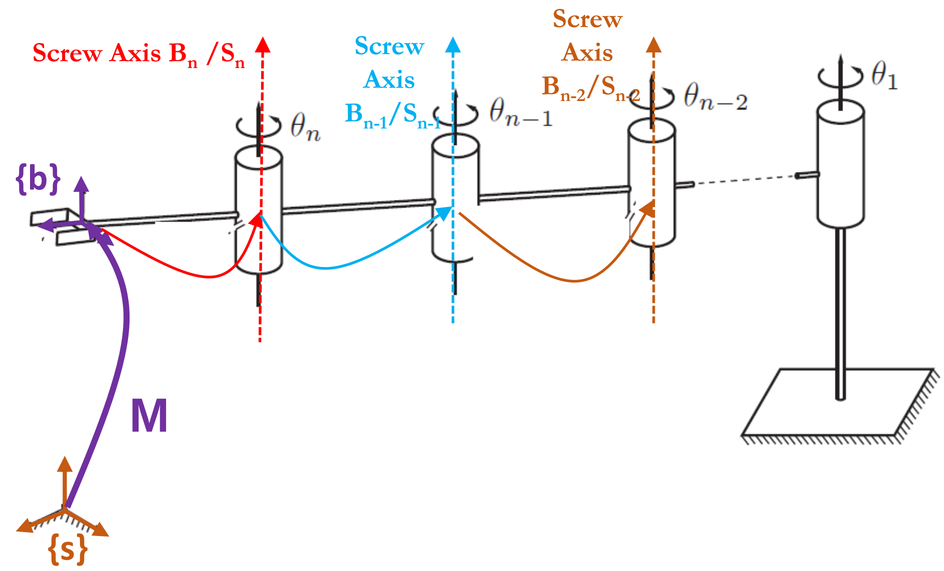
Note that the third joint is prismatic, so that $\boldsymbol{\omega_3} = 0$ and $\boldsymbol{v_3}$ **is a unit vector** in the direction of <u>positive translation</u>.

Forward Kinematics: Screw Axes in the End-Effector Frame

- An alternative form of the product of exponentials formula represents the **joint axes** as screw axes B_i in the end-effector (body) frame

- Step 1 and 2 are similar to the **Base Frame** approach.

- We define screw axes B_i <u>**when the robot is at its zero position**</u>, therefore:



$$T(\theta) = \boxed{M} e^{[\mathcal{B}_1]\theta_1} \cdots \boxed{e^{[\mathcal{B}_{n-1}]\theta_{n-1}}} \boxed{e^{[\mathcal{B}_n]\theta_n}}$$

where each $[B_i] = [Ad_{M^{-1}}] S_i$, **i = 1: … : n**

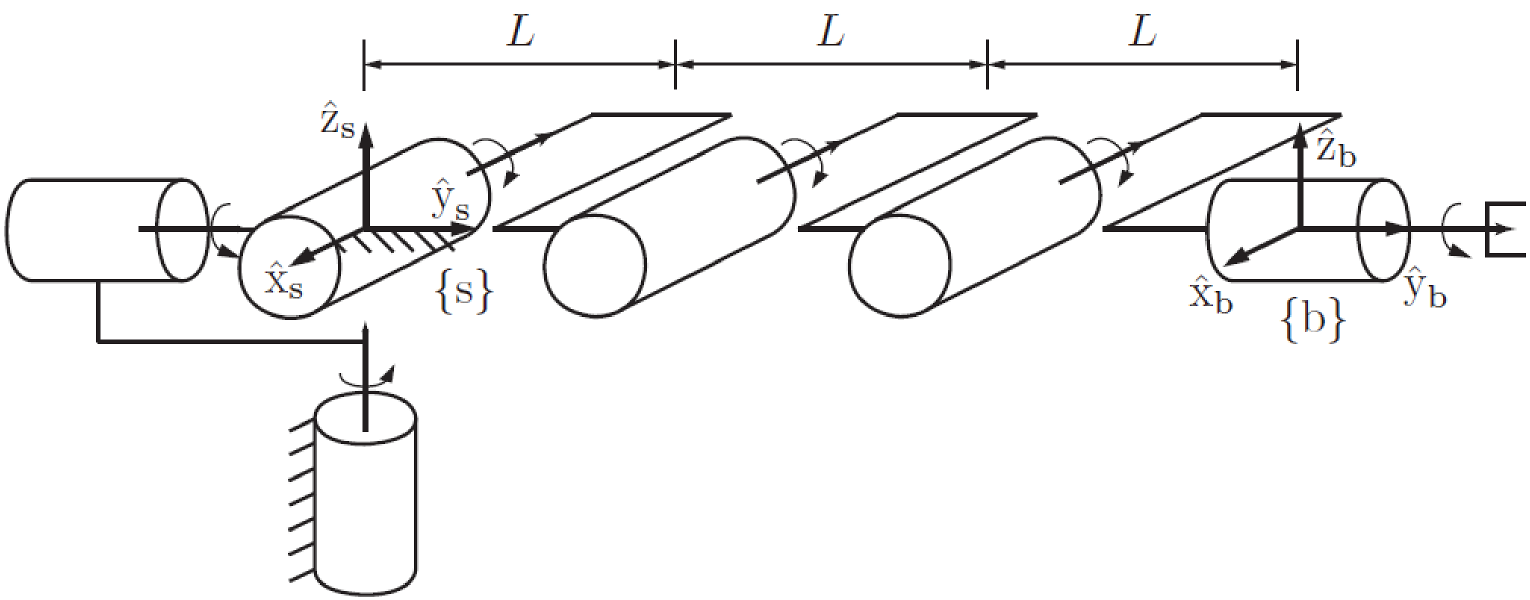
$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \boxed{\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \boxed{[\mathrm{Ad}_{T_{sb}}]} \mathcal{V}_b,$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \boxed{\begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix}} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \boxed{[\mathrm{Ad}_{T_{bs}}]} \mathcal{V}_s.$$

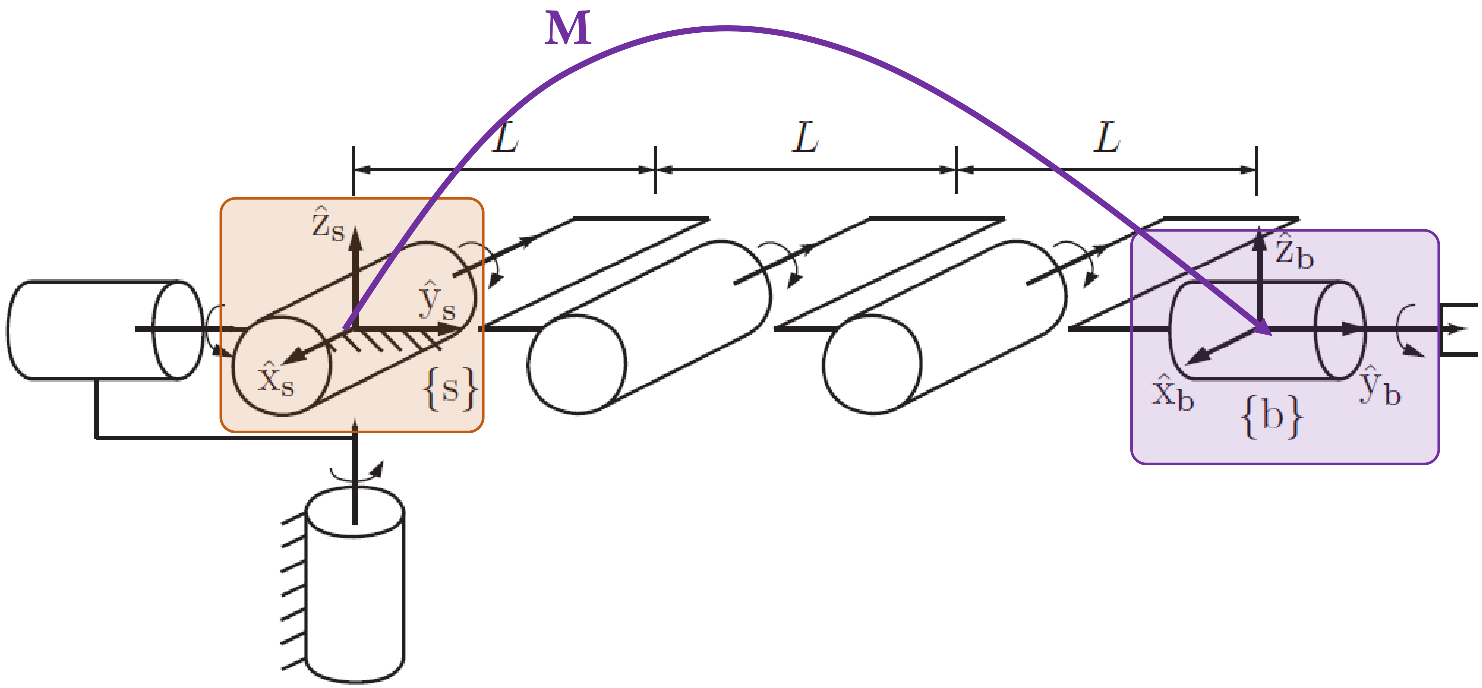
- This is the **body form** of the product of exponentials formula.

Example: 6R spatial open chain

Express **the body form** forward kinematics of the **6R open chain** shown in the Figure.



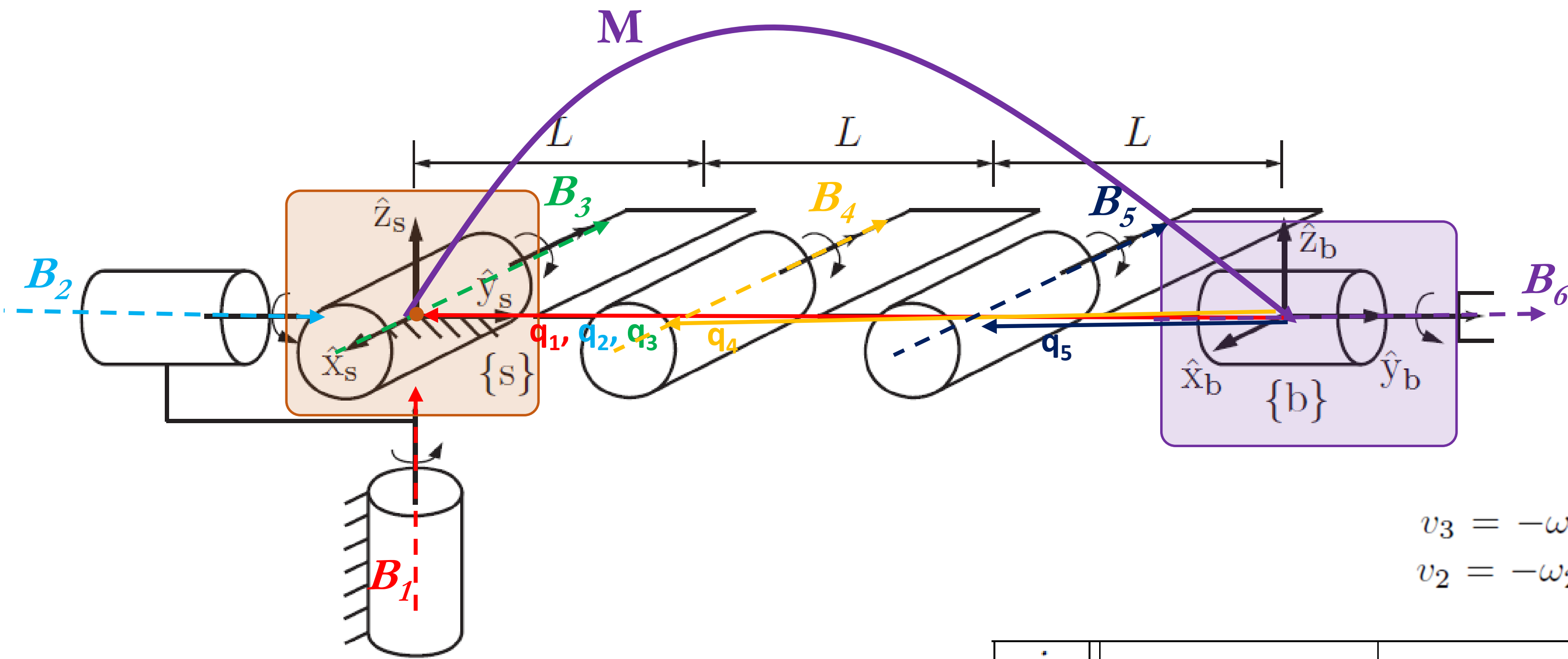
Example: 6R spatial open chain



- M **is still the same as the space form** obtained as the end-effector frame as seen from the fixed frame with the **chain in its zero position**.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: 6R spatial open chain



$$v_3 = -\omega_3 \times q_3$$
$$v_2 = -\omega_2 \times q_2$$

- The **screw axis** for each joint axis, expressed with respect to the end-effector frame in **<u>its zero position</u>**, is given in the table:

i	ω_i	v_i
1	$(0,0,1)$	$(-3L,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,-3L)$
4	$(-1,0,0)$	$(0,0,-2L)$
5	$(-1,0,0)$	$(0,0,-L)$
6	$(0,1,0)$	$(0,0,0)$

$$T(\theta) = \boxed{M}\,\boxed{e^{[\mathcal{B}_1]\theta_1}}\,\boxed{e^{[\mathcal{B}_2]\theta_2}} \cdots \boxed{e^{[\mathcal{B}_6]\theta_6}}.$$

References

- Murray, R.M., Li, Z., Sastry, S.S., "*A Mathematical Introduction to Robotic Manipulation.*", **Chapter 2**.

- Corke, Peter. "Robotics, vision and control: fundamental algorithms in MATLAB®" second, completely revised. Vol. 118. Springer, 2017, **Chapter 2.**

- Lynch and Park, "*Modern Robotics*," Cambridge U. Press, 2017, **Chapter 3.**