HA:

1. First check necessory condition: $\|R(p)-R(q)\|=\|p-q\|$

$R: \mathbb{R}^3 \to \mathbb{R}^3$, $p\in\mathbb{R}^3$, $q\in\mathbb{R}^3$, $R\in\mathbb{R}^{3\times3}$

$R\in SO(3) \Rightarrow RR^T = I$, $\det(R)=1$

The norm for this necessary condition is Euclidean norm.

$$\|R(p)-R(q)\| = \|R(p-q)\|$$

$$=[R(p-q)(p-q)^TR^T]^{1/2}$$

$$(\text{since } (p-q)(p-q)^T\in\mathbb{R})$$

$$=\|p-q\|\cdot[RR^T]^{1/2}$$

$$=\|p-q\|$$

The necessory condition holds.

Next, check the sufficient condition: $R(p\times q)=Rp\times Rq$

By definition: $\begin{cases} R(p\times q)=R(\|p\|\cdot\|q\|\cdot\sin\theta\cdot n) \\ Rp\times Rq=\|Rp\|\cdot\|Rq\|\cdot\sin\theta'\cdot n' \end{cases}$

θ is the angle between p and q

n is a unit vector perpendicular to the surface that contains p and q

We first show $\|Rp\| \cdot \|Rq\| = \|p\| \cdot \|q\|$:

$$\|Rp\| = (Rpp^T R^T)^{1/2} = \|p\|$$

$$\|Rq\| = (Rqq^T R^T)^{1/2} = \|q\|$$

next, we show: $\theta = \theta'$:

By definition: $p \cdot q = \|p\| \cdot \|q\| \cdot \cos\theta$

$$Rp \cdot Rq = \|Rp\| \cdot \|Rq\| \cdot \cos\theta' = \|p\| \cdot \|q\| \cdot \cos\theta'$$

Thus, the problem reduces to show $Rp \cdot Rq = p \cdot q$

$$Rp \cdot Rq = Rp \cdot (Rq)^T = Rp \cdot q^T \cdot R^T \underset{\begin{cases} p \cdot q^T \in \mathbb{R} \\ RR^T = I \end{cases}}{=} p \cdot q^T = p \cdot q$$

Thus, we showed $\theta = \theta'$

We also know that: $\begin{cases} n \perp p, \ n \perp q \\ n' \perp Rp, \ n' \perp Rq \\ \|n\| = \|n'\| = 1 \end{cases}$

$$\left.\begin{array}{l} n \cdot p = n' \cdot Rp = 0 \\ n \cdot q = n' \cdot Rq = 0 \end{array}\right\} \Rightarrow \text{Assuming } p, q \neq 0, \Rightarrow n = Rn'$$

Thus, we proved $R(p \times q) = R(p) \times R(q)$

2. $v = s - r$

$$g(v) = g(s-r) = g(s) - g(r)$$
$$= p + Rs - (p + Rr)$$
$$= Rs - Rr$$
$$= R(s-r)$$
$$= Rv$$

3. a) Let $\omega = [\omega_1 \ \omega_2 \ \omega_3]^T$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\lambda I - \hat{\omega} = \begin{bmatrix} \lambda & -\omega_3 & \omega_2 \\ \omega_3 & \lambda & -\omega_1 \\ -\omega_2 & \omega_1 & \lambda \end{bmatrix}$$

$$\det(\lambda I - \hat{\omega}) = \lambda(\lambda^2 + \omega_1^2) + \omega_3(\lambda\omega_3 - \omega_1\omega_2)$$
$$+ \omega_2(\omega_1\omega_3 + \lambda\omega_2)$$
$$= \lambda^3 + \lambda\omega_1^2 + \lambda\omega_3^2 + \lambda\omega_2^2$$
$$= \lambda(\lambda^2 + \omega_1^2 + \omega_2^2 + \omega_3)^2$$
$$= 0$$

Since $\|\omega\| = 1$,

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$$

$$\lambda(\lambda^2 + 1) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = \hat{i} \quad \lambda_3 = -\hat{i}$$

$$\lambda_1 v_1 = \hat{\omega} v_1$$

$$\hat{\omega} v_1 = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = [\omega_1 \quad \omega_2 \quad \omega_3]^T$$

$$\hat{\omega} v_2 = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix} = \hat{i} \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -\omega_3 + \omega_1 \omega_2 \hat{i} \\ -\hat{i}(1 - \omega_2^2) \\ \omega_1 + \omega_2 \omega_3 \hat{i} \end{bmatrix}$$

v_3 is the complex conjugate of v_2

$$v_3 = \begin{bmatrix} -\omega_3 - \omega_1 \omega_2 \hat{i} \\ i(1 - \omega_2^2) \\ \omega_1 - \omega_2 \omega_3 \hat{i} \end{bmatrix}$$

b) $R = e^{\hat{\omega}\theta}$

From Taylor expansion:

$$R = e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^3}{3!}\hat{\omega}^3 + \cdots$$

The eigenvalue of R satisfies: $R v_i = \lambda_i v_i$

$$R v_i = e^{\hat{\omega}\theta} v_i = I v_i + \hat{\omega} v_i \theta + \frac{\theta^2}{2!}\hat{\omega}^2 v_i + \cdots$$

$$= (I v_i + \lambda_i v_i \theta + \frac{\theta^2}{2!}\hat{\omega}\lambda_i v_i + \cdots)$$

$$= (I + \lambda_i \theta + \frac{\theta^2}{2!}\hat{\omega}\lambda_i v_i + \cdots) v_i$$

$$= (I + \lambda_i \theta + \frac{\theta^2}{2!}\lambda_i^2 v_i + \cdots) v_i$$

$$= e^{\lambda_i \theta} v_i$$

Thus, $e^{\lambda_i \theta}$ is the eigenvalue of R

$\lambda_1 = e^0 = 1$, $\lambda_2 = e^{\hat{i}\theta}$, $\lambda_3 = e^{-\hat{i}\theta}$

v_1 corresponding to λ_0 is $\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

Physical interpretation: the eigenvector corresponding to $\lambda = 1$ is a unit vector pointing in the direction of of ω. This means that the rotation matrix preserves the entity along the axis of rotation, which is expected.

4. a) Since $\hat{\omega}$ is skew symmetric,

$$\omega \times b = \hat{\omega} b$$

$$R^T = \begin{bmatrix} | & | & \\ r_1 & r_2 & \cdots \\ | & | & \end{bmatrix}$$, r_i are the column of R^T

r_i^T are the row of R

$$\omega \times r_i \perp r_i \Rightarrow r_i^T(\omega \times r_i) = 0$$

$$R\hat{\omega}R^T = \begin{bmatrix} r_1^T(\omega \times r_1) & r_1^T(\omega \times r_2) & r_1^T(\omega \times r_3) \\ r_2^T(\omega \times r_1) & r_2^T(\omega \times r_2) & r_2^T(\omega \times r_3) \\ r_3^T(\omega \times r_1) & r_3^T(\omega \times r_2) & r_3^T(\omega \times r_3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -r_3^T\omega & r_2^T\omega \\ r_3^T\omega & 0 & -r_1^T\omega \\ -r_2^T\omega & r_1^T\omega & 0 \end{bmatrix}$$

$$= [R\omega]$$

b) Since $v, w \in \mathbb{R}^3$

$$R(v \times w) = R\hat{v}w$$

$$Rv,\ Rw \in \mathbb{R}^3$$

$$(Rv) \times (Rw) = \widehat{Rv}\, Rw$$

From part (a), $\widehat{Rv} = R\hat{v}R^T$

$$(Rv) \times (Rw) = R\hat{v}R^T R w = R\hat{v}w = R(v \times w)$$

5. a) $Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$

$$P = p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$$

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$
$$p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$$

① check closure under multiplication:

$$Q.P = (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\hat{i} + (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\hat{j} + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\hat{k}$$

$$\|Q.P\| = (q_0^2 + q_1^2 + q_2^2 + q_3^2)\cdot(p_0^2 + p_1^2 + p_2^2 + p_3^2) = 1$$

Thus, $\|Q.P\|$ is also a unit quaternion

② Check Identity under multiplication

For a quaternion e

$e = e_0 + e_1\hat{i} + e_2\hat{j} + e_3\hat{k}$

When $e_1 = e_2 = e_3 = 0$, $e_0 = 1$

$Q.e = e.Q = Q$

③ Check inverse under multiplication

$e = 1$ is the identity element.

$Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{j}$

$P = p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{j}$

Let $P = Q^{-1}$ be the inverse of Q

$$Q.P = (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\hat{i} + (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\hat{j} + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\hat{k} = e$$

$$P = \frac{1}{q_0^2 + q_1^2 + q_2^2 + q_3^2}(q_0 - q_1\hat{i} - q_2\hat{j} - q_3\hat{k})$$

$\|P\| = 1$

Thus, $P = Q^{-1}$ is also a unit quaternion

④ Check associativity under multiplication

Let $Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$

$P = p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$

$n = n_0 + n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

It can be checked using MATLAB: $(Q.P).n = Q.(P.n)$

b) Let $x = x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k}$

$$X = (0, x) = x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k}$$

$$Q = q_0 + q_1\hat{\imath} + q_2\hat{\jmath} + q_3\hat{k}, \quad \|Q\| = 1$$

$$QX = (-\vec{q}\cdot x,\ q_0 x + \vec{q}\times\vec{x})$$

$$Q^* = q_0 - q_1\hat{\imath} - q_2\hat{\jmath} - q_3\hat{k} = (q_0, -\vec{q})$$

$$\underbrace{QX}_{Q}\underbrace{Q^*}_{P} = (-\vec{q}\cdot x\, q_0 + (q_0 x + \vec{q}\times\vec{x})\cdot\vec{q},$$
$$\vec{q}\cdot x\,\vec{q} + q_0(q_0 x + \vec{q}\times\vec{x}) - (q_0 x + \vec{q}\times\vec{x})\times\vec{q})$$

$$-\vec{q}\cdot x\, q_0 + (q_0 x + \vec{q}\times\vec{x})\cdot\vec{q}$$

$$= -\vec{q}\cdot x\, q_0 + q_0 x\cdot\vec{q} + (\vec{q}\times\vec{x})\cdot\vec{q}$$

$$= -\vec{q}\cdot x\, q_0 + \vec{q}\cdot x\, q_0 + (\vec{q}\times\vec{x})\cdot\vec{q}$$

$$= (\vec{q}\times\vec{x})\cdot\vec{q}$$

$$= 0 \quad (\text{since } \vec{q}\times\vec{x} \perp \vec{q})$$

$$\vec{q}\cdot x\,\vec{q} + q_0(q_0 x + \vec{q}\times\vec{x}) + \vec{q}\times(q_0 x + \vec{q}\times\vec{x})$$

$$= (x\cdot\vec{q})\vec{q} + q_0^2 x + q_0(\vec{q}\times\vec{x}) + \vec{q}\times q_0 x + \vec{q}\times(\vec{q}\times x)$$

$$= (x\cdot\vec{q})\vec{q} + q_0^2 x + 2q_0(\vec{q}\times\vec{x}) + \vec{q}(\vec{q}\cdot x) - x(\vec{q}\cdot\vec{q})$$

$$= (q_0^2 - \vec{q}\cdot\vec{q})x + 2(q_0(\vec{q}\times\vec{x}) + (x\cdot\vec{q})\vec{q})$$

6.

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let c denotes cos, and s denotes sin

$$R_{xyz}(\psi, \theta, \phi) = R_x(\phi) R_y(\theta) R_z(\psi)$$

$$= \begin{bmatrix} C_\theta C_\psi & -C_\theta S_\psi & S_\theta \\ C_\phi S_\psi + C_\psi S_\phi S_\theta & C_\phi C_\psi - S_\phi S_\theta S_\psi & -C_\theta S_\phi \\ S_\phi S_\psi - C_\phi C_\psi S_\theta & C_\psi S_\phi + C_\phi S_\theta S_\psi & C_\phi C_\theta \end{bmatrix}$$

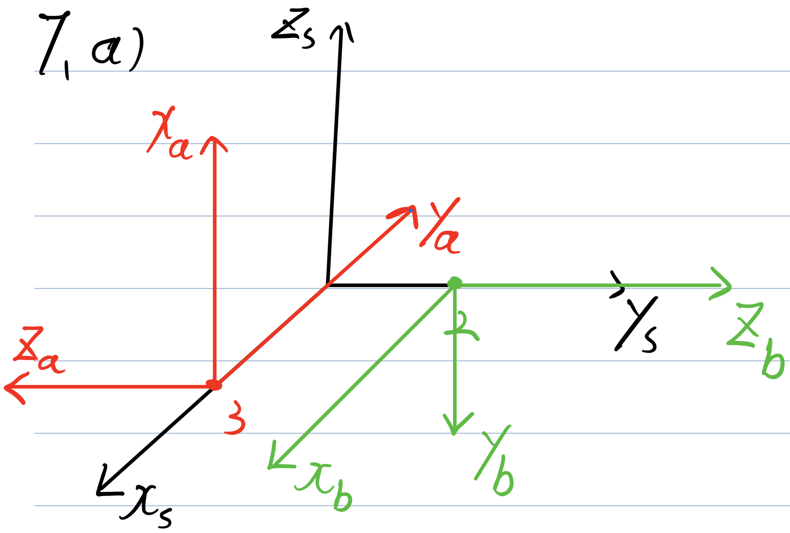
$$\triangleq \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{23} & r_{33} \end{bmatrix}$$

$$\phi = \tan^{-1}(-r_{23}, r_{33})$$

$$\theta = \tan^{-1}\left(r_{13}, \sqrt{r_{23}^2 + r_{33}^2}\right)$$

$$\psi = \tan^{-1}(-r_{12}, r_{11})$$

7, a)



b)

$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$T_{sa} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{sb} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) Given T_{sb}, it represents the frame {b} w.r.t. {s}

T_{sb}^{-1} represents the frame {s} w.r.t. {b}

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T_{sb}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d)

$$T_{as} = T_{sa}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ab} = T_{as} T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e)

$$T_1 = T_{sa} T = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_1 corresponds to a body-fixed transformation of T_{sa}

$$T_2 = T T_{sa} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_2 corresponds to a world-fixed transformation of T_{sa}

f) $p_b = (1, 2, 3)$

$$p_s = p_{sb} + R_{sb} p_b$$

$$\begin{bmatrix} p_s \\ 1 \end{bmatrix} = T_{sb} \begin{bmatrix} p_b \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

$p_s = (1, 5, -2)$

g)

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = T_{sb} \begin{bmatrix} p_s \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

$p' = (1, 5, -2)$

The result should be interpreted as moving the location of the point without changing reference frame.

$$\begin{bmatrix} p'' \\ 1 \end{bmatrix} = T_{sb}^{-1} \begin{bmatrix} p_s \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$p'' = (1, -3, 0)$

The result should be interpreted as changing the coordinate without moving the point.

h) $\mathcal{V}_a = [Ad_{T_{as}}] \mathcal{V}_s$

$$[Ad_{T_{as}}] = \begin{bmatrix} R_{as} & 0 \\ [p_{as}]R_{as} & R_{as} \end{bmatrix}$$

$$R_{as} = R_{sa}^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$p_{as} = (0, 3, 0)^T$

$$[p_{as}] = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} \quad [p_{as}] R_{as} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$[Ad_{T_{as}}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 & -1 & 0 \end{bmatrix}$$

$V_a = (1, -3, -2, -9, 1, -1)$

i)

$$T_{sa} = e^{[S]\theta} \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = e^{[\omega]\theta} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$tr(R) = 0 = 1 + 2\cos\theta$

$\theta = 2.094$

$$\begin{cases} r_{32} - r_{23} = 2\omega_1 \sin\theta = 1 \\ r_{13} - r_{31} = 2\omega_2 \sin\theta = -1 \\ r_{21} - r_{12} = 2\omega_3 \sin\theta = 1 \end{cases} \Rightarrow \begin{cases} \omega_1 = 0.577 \\ \omega_2 = -0.577 \\ \omega_3 = 0.577 \end{cases}$$

$v = G^{-1}(\theta)\, p$

$p = (3, 0, 0)^T$

$$G^{-1}(\theta) = \frac{1}{\theta} I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2$$

$$= \begin{bmatrix} 0.3516 & 0.2257 & 0.3516 \\ -0.3516 & 0.3516 & 0.2257 \\ -0.2257 & -0.3516 & 0.3516 \end{bmatrix}$$

$$v = G^{-1}(\theta) \cdot p = (1.0548, -1.0548, -0.6772)^T$$

$$[S]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1.21 & -1.21 & 2.21 \\ 1.21 & 0 & -1.21 & -2.21 \\ 1.21 & 1.21 & 0 & -1.42 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\|\omega\| = (\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2} = 1$$

$$S = \begin{bmatrix} 0.577 \\ -0.577 \\ 0.577 \\ 1.055 \\ -1.055 \\ -0.677 \end{bmatrix} \qquad \theta = 2.094$$

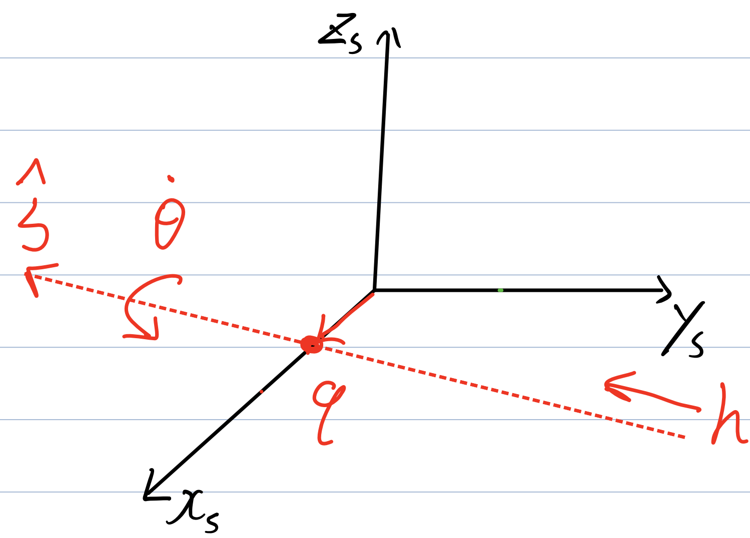
$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$$

$$\hat{s} = \omega / \|\omega\| = (0.577, -0.577, 0.577)$$

$$\dot{\theta} = \|\omega\| = 1$$

$$h = \frac{\hat{\omega}^T v}{\dot{\theta}} = 0.827$$

$$q = \frac{\hat{s} \times v}{\dot{\theta}} = (1, 1, 0)$$



j) $S\theta = (0, 1, 2, 3, 0, 0)$

$\omega\theta = (0, 1, 2)$ $\quad v\theta = (3, 0, 0)$

$\omega = (0, 1/\theta, 2/\theta)$

$$\|\omega\| = \left(\frac{1}{\theta^2} + \frac{4}{\theta^2}\right)^{1/2} = 1 \Rightarrow \theta = \sqrt{5}$$

$$[\omega]\theta = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad v\theta = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$R = e^{[\omega]\theta}$

$$= \begin{bmatrix} C_\theta + \omega_1^2(1-C_\theta) & \omega_1\omega_2(1-C_\theta) - \omega_3 S_\theta & \omega_1\omega_3(1-C_\theta) + \omega_2 S_\theta \\ \omega_1\omega_2(1-C_\theta) + \omega_3 S_\theta & C_\theta + \omega_2^2(1-C_\theta) & \omega_2\omega_3(1-C_\theta) - \omega_1 S_\theta \\ \omega_1\omega_3(1-C_\theta) - \omega_2 S_\theta & \omega_2\omega_3(1-C_\theta) + \omega_1 S_\theta & C_\theta + \omega_3^2(1-C_\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -0.6173 & -0.7037 & 0.3518 \\ 0.7037 & -0.2938 & 0.6469 \\ -0.3518 & 0.6469 & 0.6765 \end{bmatrix}$$

$$p = (I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$$

$$= \begin{bmatrix} 1.055 \\ 1.94 \\ -0.97 \end{bmatrix}$$

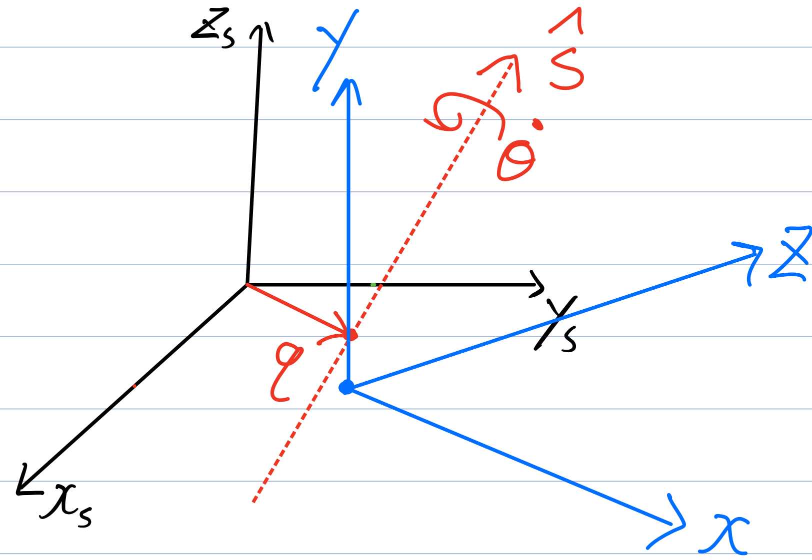
$$T = \begin{bmatrix} -0.62 & -0.70 & 0.35 & 1.06 \\ 0.70 & -0.29 & 0.65 & 1.94 \\ -0.35 & 0.65 & 0.68 & -0.97 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{s} = \frac{\omega}{\|\omega\|} = \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\dot{\theta} = \|\omega\| = 1$$

$$h = \frac{\omega^T \cdot v}{\dot{\theta}} = 0$$

$$q = \frac{\hat{s} \times v}{\dot{\theta}} = (0, 1.2, -0.6)$$



z_s
y
$\hat{s}$
$\dot{\theta}$
z
y_s
q
x_s
x

8. (a) $T_{rs} = T_{ra} T_{ae} T_{es}$

$= T_{ar}^{-1} T_{ea}^{-1} T_{es}$

(b) $p_{es} = (1, 1, 1)$

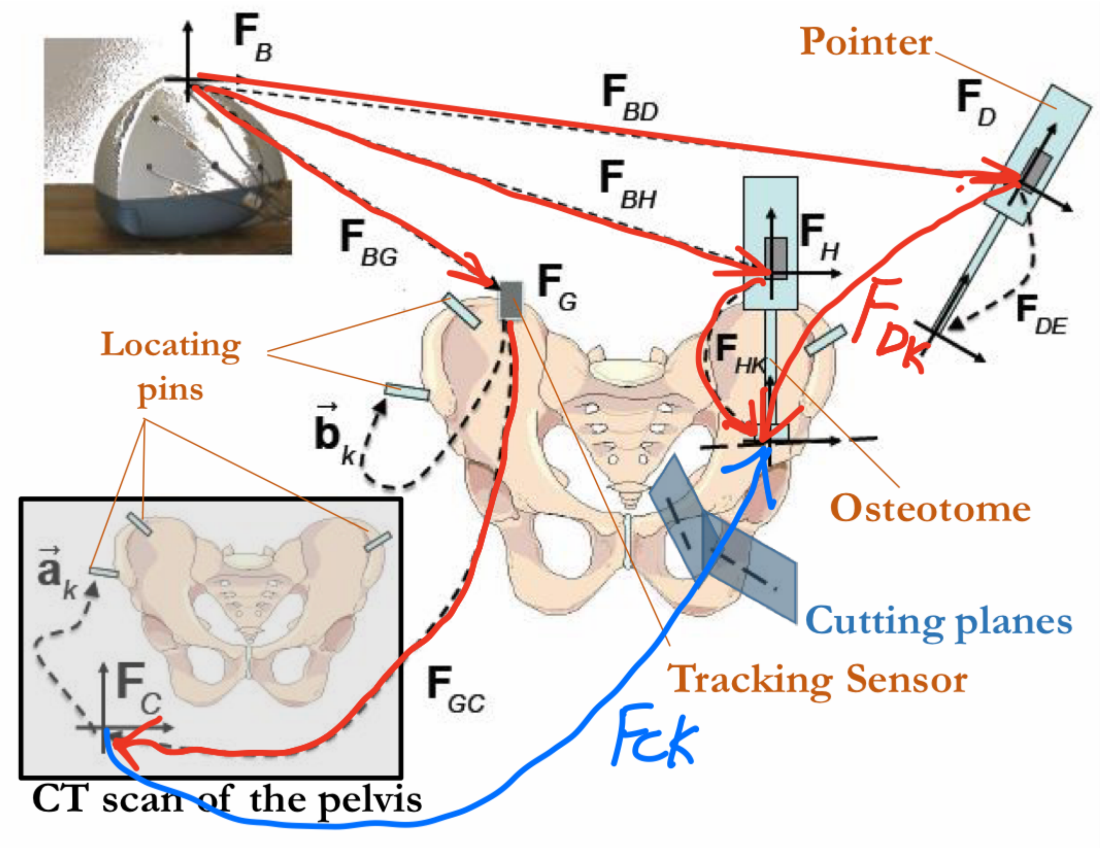
$$\begin{bmatrix} p_{rs} \\ 1 \end{bmatrix} = T_{er}^{-1} \begin{bmatrix} p_{es} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$p_{rs} = (0, 0, 0)$

9.



$$F_{BH} F_{HK} = F_{BG} F_{GC} F_{CK}$$

$$F_{CK} = F_{GC}^{-1} F_{BG}^{-1} F_{BH} F_{HK}$$

References:

1. Lynch, Kevin M., and Frank C. Park. "MODERN ROBOTICS." (2018).
2. http://publish.illinois.edu/ece470-intro-robotics/files/2019/09/08-lecture.pdf
3. http://publish.illinois.edu/ece470-intro-robotics/files/2019/09/09-lecture.pdf