

Homework/Programming Assignment #1

Homework Due: 02/24/2022- 5:00PM

Name/EID: *Bryant Zhou / YZ26659*

Email: *YZ009@utexas.edu*

Signature (required)

I/We have followed the rules in completing this Assignment.

Name/EID: *Jonathan Salfity / JS222238*

Email: *J.Salfity@utexas.edu*

Signature (required)

I/We have followed the rules in completing this Assignment.

Question	Points	Total
HA 1		
HA 2		
HA 3a		
HA 3b		
HA 4a		
HA 4b		
HA 4c		
HA 5a		
HA 5b		
HA 6		
HA 7		
HA 8		
HA 9		
PA 1		
PA 2		
PA 3		

Instruction:

1. Remember that this is a graded homework assignment. It is the equivalent of a **mini take-home exam**.
2. You are to work **alone** or in teams of two and are **not to discuss the problems with anyone other than the TAs or the instructor**.
3. It is open book, notes, and web. But you should cite any references you consult.
4. Unless I say otherwise in class, **it is due before the start of class on the due date** mentioned in the P/H Assignment.
6. **Sign and append** this score sheet as the first sheet of your assignment.
7. Remember to submit your assignment in Canvas.

HA:

1. First check necessary condition: $\|R(p) - R(q)\| = \|p - q\|$

$R: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $p \in \mathbb{R}^3$, $q \in \mathbb{R}^3$, $R \in \mathbb{R}^{3 \times 3}$

$R \in SO(3) \Rightarrow RR^T = I$, $\det(R) = 1$

The norm for this necessary condition is Euclidean norm,

$$\|R(p) - R(q)\| = \|R(p-q)\|$$

$$= [R(p-q)(p-q)^T R^T]^{1/2}$$

(since $(p-q)(p-q)^T \in \mathbb{R}$)

$$= \|p-q\| \cdot [RR^T]^{1/2}$$

$$= \|p-q\|$$

The necessary condition holds.

Next, check the sufficient condition: $R(p \times q) = R(p) \times R(q)$

By definition: $R(p \times q) = R(\|p\| \cdot \|q\| \cdot \sin \theta \cdot n)$

$$R(p) \times R(q) = \|R(p)\| \cdot \|R(q)\| \cdot \sin \theta \cdot n'$$

θ is the angle between p and q

n is a unit vector perpendicular to the surface that contains p and q

We first show $\|R\beta\| \cdot \|Rq\| = \|\beta\| \cdot \|q\|$:

$$\|R\beta\| = (R\beta\beta^T R^T)^{1/2} = \|\beta\|$$

$$\|Rq\| = (Rq q^T R^T)^{1/2} = \|q\|$$

Next, we show: $\theta = \theta'$:

By definition: $\beta \cdot q = \|\beta\| \cdot \|q\| \cdot \cos \theta$

$$R\beta \cdot Rq = \|R\beta\| \cdot \|Rq\| \cdot \cos \theta' = \|\beta\| \cdot \|q\| \cdot \cos \theta'$$

Thus, the problem reduces to show $R\beta \cdot Rq = \beta \cdot q$

$$R\beta \cdot Rq = R\beta \cdot (Rq)^T = R\beta \cdot q^T \cdot R^T = \beta \cdot q^T = \beta \cdot q$$

$$\begin{cases} \beta \cdot q^T \in R \\ RR^T = I \end{cases}$$

Thus, we showed $\theta = \theta'$

We also know that: $\begin{cases} n \perp \beta, n \perp q \\ n' \perp R\beta, n' \perp Rq \\ \|n\| = \|n'\| = 1 \end{cases}$

$$\begin{aligned} n \cdot \beta &= n' \cdot R\beta = 0 \\ n \cdot q &= n' \cdot Rq = 0 \end{aligned} \quad \Rightarrow \text{Assuming } \beta, q \neq 0, \Rightarrow n = Rn'$$

Thus, we proved $R(\beta \times q) = R(\beta) \times R(q)$

$$2. \quad V = S - r$$

$$\begin{aligned} g(V) &= g(S - r) = g(S) - g(r) \\ &= p + R_S - (p + R_r) \end{aligned}$$

$$= R_S - R_r$$

$$= R(S - r)$$

$$= RV$$

3, a) Let $\omega = [\omega_1, \omega_2, \omega_3]^T$

$$\vec{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\lambda I - \vec{\omega} = \begin{bmatrix} \lambda & -\omega_3 & \omega_2 \\ \omega_3 & \lambda & -\omega_1 \\ -\omega_2 & \omega_1 & \lambda \end{bmatrix}$$

$$\det(\lambda I - \vec{\omega}) = \lambda (\lambda^2 + \omega_1^2) + \omega_3 (\lambda \omega_3 - \omega_1 \omega_2)$$

$$+ \omega_2 (\omega_1 \omega_3 + \lambda \omega_2)$$

$$= \lambda^3 + \lambda \omega_1^2 + \lambda \omega_3^2 + \lambda \omega_2^2$$

$$= \lambda (\lambda^2 + \omega_1^2 + \omega_2^2 + \omega_3^2)^2$$

$$= 0$$

$$\text{Since } \|w\| = 1, \\ w_1^2 + w_2^2 + w_3^2 = 1$$

$$\lambda(\lambda^2 + 1) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = i, \quad \lambda_3 = -i$$

$$\lambda_1 V_1 = \tilde{W} V_1$$

$$\tilde{W} V_1 = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} V_1' \\ V_1'' \\ V_1''' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_1 = [w_1 \ w_2 \ w_3]^T$$

$$\tilde{W} V_2 = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} V_2' \\ V_2'' \\ V_2''' \end{bmatrix} = i \begin{bmatrix} V_2' \\ V_2'' \\ V_2''' \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -w_3 + w_1 w_2 i \\ -i(1 - w_2^2) \\ w_1 + w_2 w_3 i \end{bmatrix}$$

V_3 is the complex conjugate of V_2

$$V_3 = \begin{bmatrix} -w_3 - w_1 w_2 i \\ i(1 - w_2^2) \\ w_1 - w_2 w_3 i \end{bmatrix}$$

$$b) R = e^{\hat{w}\theta}$$

From Taylor expansion:

$$R = e^{\hat{w}\theta} = I + \hat{w}\theta + \frac{\theta^2}{2!} \hat{w}^2 + \frac{\theta^3}{3!} \hat{w}^3 + \dots$$

The eigenvalue of R satisfies: $Rv_i = \lambda_i v_i$

$$Rv_i = e^{\hat{w}\theta} v_i = Iv_i + \hat{w}v_i\theta + \frac{\theta^2}{2!} \hat{w}^2 v_i + \dots$$

$$= (Iv_i + \lambda_i v_i\theta + \frac{\theta^2}{2!} \hat{w} \lambda_i v_i + \dots)$$

$$= (I + \lambda_i \theta + \frac{\theta^2}{2!} \hat{w} \lambda_i v_i + \dots) v_i$$

$$= (I + \lambda_i \theta + \frac{\theta^2}{2!} \lambda_i^2 v_i + \dots) v_i$$

$$= e^{\lambda_i \theta} v_i$$

Thus, $e^{\lambda_i \theta}$ is the eigenvalue of R

$$\lambda_1 = e^0 = 1, \quad \lambda_2 = e^{i\theta}, \quad \lambda_3 = e^{-i\theta}$$

v_i corresponding to λ_0 is $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

Physical interpretation: the eigenvector corresponding to $\lambda = 1$ is a unit vector pointing in the direction of $\vec{\omega}$. This means that the rotation matrix preserves the entity along the axis of rotation, which is expected.

4. a) Since $\hat{\omega}$ is skew symmetric,

$$w \times b = \hat{\omega} b$$

$$R^T = \begin{bmatrix} | & | & | \\ r_1 & r_2 & \dots \\ | & | & | \end{bmatrix}, \quad r_i \text{ are the columns of } R^T$$

r_i^T are the rows of R

$$w \times r_i \perp r_i \Rightarrow r_i^T (w \times r_i) = 0$$

$$R \hat{\omega} R^T = \begin{bmatrix} r_1^T (w \times r_1) & r_1^T (w \times r_2) & r_1^T (w \times r_3) \\ r_2^T (w \times r_1) & r_2^T (w \times r_2) & r_2^T (w \times r_3) \\ r_3^T (w \times r_1) & r_3^T (w \times r_2) & r_3^T (w \times r_3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -r_3^T w & r_2^T w \\ r_3^T w & 0 & -r_1^T w \\ -r_2^T w & r_1^T w & 0 \end{bmatrix}$$

$$= [R \hat{\omega}]$$

b) Since $v, w \in \mathbb{R}^3$

$$R(v \times w) = R\hat{v}w$$

$$Rv, Rw \in \mathbb{R}^3$$

$$(Rv) \times (Rw) = \hat{R}\hat{v}Rw$$

From part (a), $\hat{R}\hat{v} = R\hat{v}R^T$

$$(Rv) \times (Rw) = R\hat{v}R^TRw = R\hat{v}w = R(v \times w)$$

5. a) $Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$

$$P = p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$$

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

$$p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$$

① Check closure under multiplication:

$$\begin{aligned} Q \cdot P &= (q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3) + (q_0 p_1 + q_1 p_0 + q_2 p_3 - q_3 p_2)\hat{i} \\ &\quad + (q_0 p_2 - q_1 p_3 + q_2 p_0 + q_3 p_1)\hat{j} + (q_0 p_3 + q_1 p_2 - q_2 p_1 + q_3 p_0)\hat{k} \end{aligned}$$

$$\|Q \cdot P\| = (q_0^2 + q_1^2 + q_2^2 + q_3^2) \cdot (p_0^2 + p_1^2 + p_2^2 + p_3^2) = 1$$

Thus, $\|Q \cdot P\|$ is also a unit quaternion

② Check Identity under multiplication

For a quaternion e

$$e = e_0 + e_1 \hat{i} + e_2 \hat{j} + e_3 \hat{k}$$

When $e_1 = e_2 = e_3 = 0$, $e_0 = 1$

$$Q \cdot e = e \cdot Q = Q$$

③ Check inverse under multiplication

$e = 1$ is the identity element.

$$Q = q_0 + q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k}$$

$$P = p_0 + p_1 \hat{i} + p_2 \hat{j} + p_3 \hat{k}$$

Let $P = Q^{-1}$ be the inverse of Q

$$\begin{aligned} Q \cdot P &= (q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3) + (q_0 p_1 + q_1 p_0 + q_2 p_3 - q_3 p_2) \hat{i} \\ &\quad + (q_0 p_2 - q_1 p_3 + q_2 p_0 + q_3 p_1) \hat{j} + (q_0 p_3 + q_1 p_2 - q_3 p_1 + q_2 p_0) \hat{k} \\ &= e \end{aligned}$$

$$P = \frac{1}{q_0^2 + q_1^2 + q_2^2 + q_3^2} (q_0 - q_1 \hat{i} - q_2 \hat{j} - q_3 \hat{k})$$

$$\|P\| = 1$$

Thus, $P = Q^{-1}$ is also a unit quaternion

④ Check associativity under multiplication

$$\text{Let } Q = q_0 + q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k}$$

$$P = p_0 + p_1 \hat{i} + p_2 \hat{j} + p_3 \hat{k}$$

$$n = n_0 + n_1 \hat{i} + n_2 \hat{j} + n_3 \hat{k}$$

It can be checked using MATLAB: $(Q \cdot P) \cdot n = Q \cdot (P \cdot n)$

b) Let $x = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k}$

$$x = (0, x) = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k}$$

$$Q = q_0 + q_1 \vec{i} + q_2 \vec{j} + q_3 \vec{k}, \quad \|Q\| = 1$$

$$Q \cdot x = (-\vec{q} \cdot x, q_0 x + \vec{q} \times \vec{x})$$

$$Q^* = q_0 - q_1 \vec{i} - q_2 \vec{j} - q_3 \vec{k} = (q_0, -\vec{q})$$

$$\underbrace{Q \cdot Q^*}_{Q \cdot P} = (-\vec{q} \cdot x q_0 + (q_0 x + \vec{q} \times \vec{x}) \cdot \vec{q}, \vec{q} \cdot x \vec{q} + q_0 (q_0 x + \vec{q} \times \vec{x}) - (q_0 x + \vec{q} \times \vec{x}) \times \vec{q})$$

$$-\vec{q} \cdot x q_0 + (q_0 x + \vec{q} \times \vec{x}) \cdot \vec{q}$$

$$= -\vec{q} \cdot x q_0 + q_0 x \cdot \vec{q} + (\vec{q} \times \vec{x}) \cdot \vec{q}$$

$$= -\vec{q} \cdot x q_0 + \vec{q} \cdot x q_0 + (\vec{q} \times \vec{x}) \cdot \vec{q}$$

$$= (\vec{q} \times \vec{x}) \cdot \vec{q}$$

$$= 0 \quad (\text{since } \vec{q} \times \vec{x} \perp \vec{q})$$

$$\vec{q} \cdot x \vec{q} + q_0 (q_0 x + \vec{q} \times \vec{x}) + \vec{q} \times (q_0 x + \vec{q} \times \vec{x})$$

$$= (x \cdot \vec{q}) \vec{q} + q_0^2 x + q_0 (\vec{q} \times \vec{x}) + \vec{q} \times q_0 x + \vec{q} \times (\vec{q} \times \vec{x})$$

$$= (x \cdot \vec{q}) \vec{q} + q_0^2 x + 2q_0 (\vec{q} \times \vec{x}) + \vec{q} (\vec{q} \cdot x) - x (\vec{q} \cdot \vec{q})$$

$$= (q_0^2 - \vec{q} \cdot \vec{q}) x + 2 (q_0 (\vec{q} \times \vec{x}) + (x \cdot \vec{q}) \vec{q})$$

$$6. \quad R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let C denotes \cos , and S denotes \sin

$$R_{xyz}(\psi, \theta, \phi) = R_x(\phi) R_y(\theta) R_z(\psi)$$

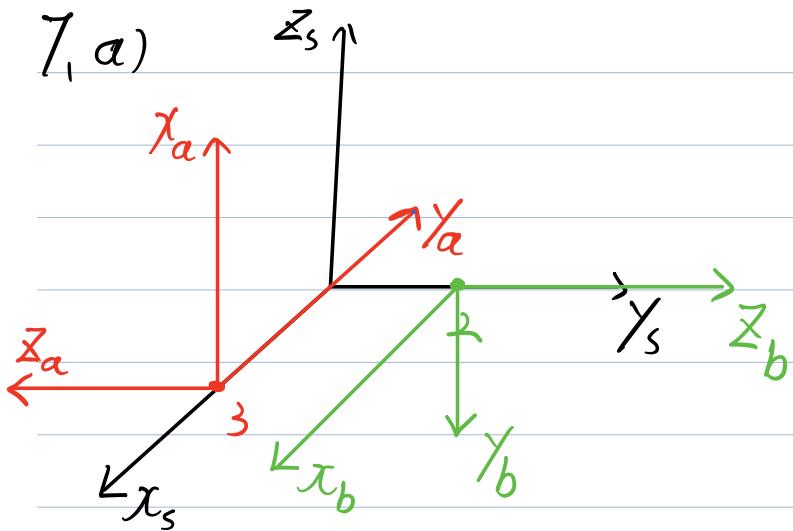
$$= \begin{bmatrix} C_\theta C_\psi & -C_\theta S_\psi & S_\theta \\ C_\phi S_\psi + C_\psi S_\phi S_\theta & C_\phi C_\psi - S_\phi S_\theta S_\psi & -C_\phi S_\theta \\ S_\phi S_\psi - C_\phi C_\psi S_\theta & C_\psi S_\phi + C_\phi S_\theta S_\psi & C_\phi C_\theta \end{bmatrix}$$

$$\equiv \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\phi = \tan^{-1}(-r_{23}, r_{33})$$

$$\theta = \tan^{-1}(r_{13}, \sqrt{r_{23}^2 + r_{33}^2})$$

$$\psi = \tan^{-1}(-r_{12}, r_{11})$$



b)

$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$T_{sa} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{sb} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) Given T_{sb} , it represents the frame $\{b\}$ w.r.t. $\{s\}$
 T_{sb}^{-1} represents the frame $\{s\}$ w.r.t. $\{b\}$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T_{sb}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d)

$$T_{as} = T_{sa}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ab} = T_{as} T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e)

$$\bar{T}_1 = \bar{T}_{sa} T = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

\bar{T}_1 corresponds to a body-fixed transformation
of T_{sa}

$$\bar{T}_2 = \bar{T} \bar{T}_{sa} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

\bar{T}_2 corresponds to a world-fixed transformation
of T_{sa}

$$f) \vec{p}_b = (1, 2, 3)$$

$$\vec{p}_s = \vec{p}_{sb} + R_{sb} \vec{p}_b$$

$$\begin{bmatrix} \vec{p}_s \\ 1 \end{bmatrix} = T_{sb} \begin{bmatrix} \vec{p}_b \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{p}_s = (1, 5, -2)$$

$$g) \begin{bmatrix} \vec{p}' \\ 1 \end{bmatrix} = T_{sb} \begin{bmatrix} \vec{p}_s \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{p}' = (1, 5, -2)$$

The result should be interpreted as moving the location of the point without changing reference frame.

$$\begin{bmatrix} \bar{P}'' \\ 1 \end{bmatrix} = \bar{T}_{sb}^{-1} \begin{bmatrix} P_s \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{P}'' = (1, -3, 0)$$

The result should be interpreted as changing the coordinate without moving the point.

$$h) V_a = [Ad_{T_{as}}] V_s$$

$$[Ad_{T_{as}}] = \begin{bmatrix} R_{as} & 0 \\ [\bar{p}] R_{as} & R_{as} \end{bmatrix}$$

$$R_{as} = R_{sa}^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\bar{p}_{as} = (0, 3, 0)^T$$

$$[\bar{p}_{as}] = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} \quad [\bar{p}_{as}] R_{as} = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$[Ad_{T_{as}}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 & -1 & 0 \end{bmatrix}$$

$$V_a = (1, -3, -2, -9, 1, -1)$$

i)

$$T_{sa} = e^{[s] \theta} \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = e^{[\omega] \theta} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{tr}(R) = 0 = 1 + 2 \cos \theta$$

$$\theta = 2.094$$

$$\begin{cases} r_{32} - r_{23} = 2\omega, \sin \theta = 1 \\ r_{13} - r_{31} = 2\omega_1 \sin \theta = -1 \\ r_{21} - r_{12} = 2\omega_3 \sin \theta = 1 \end{cases} \Rightarrow \begin{cases} \omega_1 = 0.577 \\ \omega_2 = -0.577 \\ \omega_3 = 0.577 \end{cases}$$

$$\begin{aligned} V &= G^{-1}(\theta) p \\ p &= (3, 0, 0)^T \end{aligned}$$

$$G^{-1}(\theta) = \frac{1}{\theta} I - \frac{1}{2} [\omega] + \left(\frac{1}{\theta} - \frac{1}{2} \cot \frac{\theta}{2} \right) [\omega]^2$$

$$= \begin{bmatrix} 0.3516 & 0.2257 & 0.3516 \\ -0.3516 & 0.3516 & 0.2257 \\ -0.2257 & -0.3516 & 0.3516 \end{bmatrix}$$

$$V = G^{-1}(0) \cdot p = (1.0548, -1.0548, -0.6772)^T$$

$$[S]\theta = \begin{bmatrix} [\omega]\theta & V\theta \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1.21 & -1.21 & 2.21 \\ 1.21 & 0 & -1.21 & -2.21 \\ 1.21 & 1.21 & 0 & -1.42 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\|\omega\| = (\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2} = 1$$

$$S = \begin{bmatrix} 0.577 \\ -0.577 \\ 0.577 \\ 1.055 \\ -1.055 \\ -0.677 \end{bmatrix} \quad \theta = 2.094$$

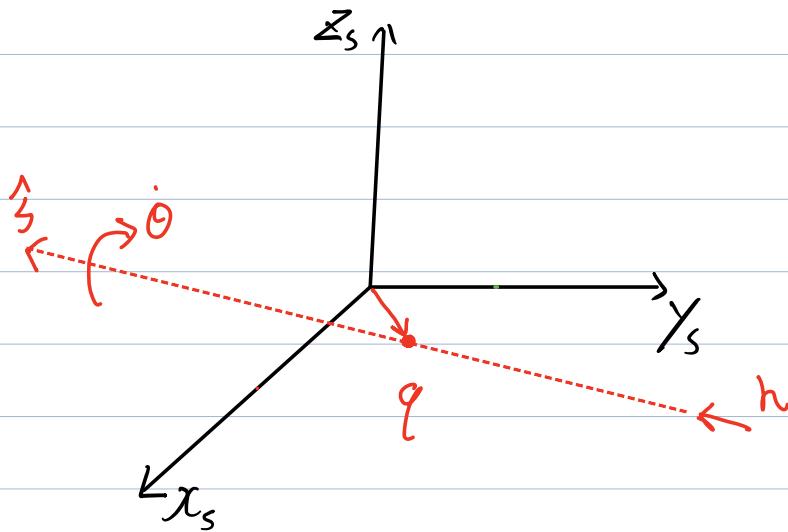
$$V = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\dot{\theta}xg + h\dot{\theta} \end{bmatrix}$$

$$\hat{\zeta} = \frac{\omega}{\|\omega\|} = (0.577, -0.577, 0.577)$$

$$\dot{\theta} = 1/\|\omega\| = 1$$

$$h = \frac{\omega^T V}{\dot{\theta}} = 0.827$$

$$q = \frac{\dot{\zeta} \times V}{\dot{\theta}} = (1, 1, 0)$$



$$j) \zeta\theta = (0, 1, 2, 3, 0, 0)$$

$$\omega\theta = (0, 1, 2) \quad v\theta = (3, 0, 0)$$

$$\omega = (0, 1/\theta, 2/\theta)$$

$$\|\omega\| = \left(\frac{1}{\theta^2} + \frac{4}{\theta^2} \right)^{1/2} = 1 \Rightarrow \theta = \sqrt{5}$$

$$[\omega]_\theta = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad v\theta = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$R = e^{i\omega_0 \theta}$$

$$= \begin{bmatrix} C_0 + \omega_1^2(1-C_0) & \omega_1 \omega_2 (1-C_0) + \omega_3 S_0 & \omega_1 \omega_3 (1-C_0) + \omega_2 S_0 \\ \omega_1 \omega_2 (1-C_0) + \omega_3 S_0 & C_0 + \omega_2^2(1-C_0) & \omega_2 \omega_3 (1-C_0) - \omega_1 S_0 \\ \omega_1 \omega_3 (1-C_0) - \omega_2 S_0 & \omega_2 \omega_3 (1-C_0) + \omega_1 S_0 & C_0 + \omega_3^2(1-C_0) - \end{bmatrix}$$

$$= \begin{bmatrix} -0.6173 & -0.7037 & 0.3518 \\ 0.7037 & -0.2938 & 0.6469 \\ -0.3518 & 0.6469 & 0.6765 \end{bmatrix}$$

$$\vec{p} = (I\theta + (1-\cos\theta)[w] + (\theta - \sin\theta)[w]^2) v$$

$$= \begin{bmatrix} 1.055 \\ 1.94 \\ -0.97 \end{bmatrix}$$

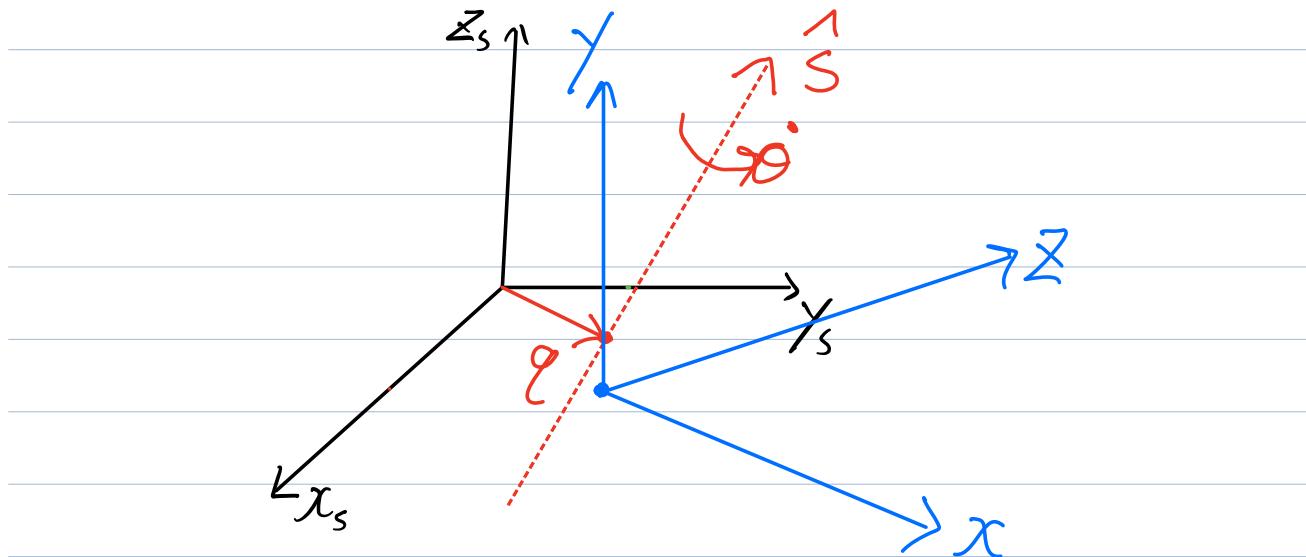
$$T = \begin{bmatrix} -0.62 & -0.70 & 0.35 & 1.06 \\ 0.70 & -0.29 & 0.65 & 1.94 \\ -0.35 & 0.65 & 0.68 & -0.97 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\vec{s}} = \frac{\vec{w}}{\|\vec{w}\|} = (0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$$

$$\dot{\theta} = \|\vec{w}\| = 1$$

$$h = \frac{\vec{w}^\top v}{\dot{\theta}} = 0$$

$$q = \frac{\hat{\vec{s}} \times v}{\dot{\theta}} = (0, 1.2, -0.6)$$



$$8.(a) \bar{T}_{rs} = \bar{T}_{ra} \bar{T}_{ae} \bar{T}_{es}$$

$$= \bar{T}_{ar}^{-1} \bar{T}_{ea}^{-1} \bar{T}_{es}$$

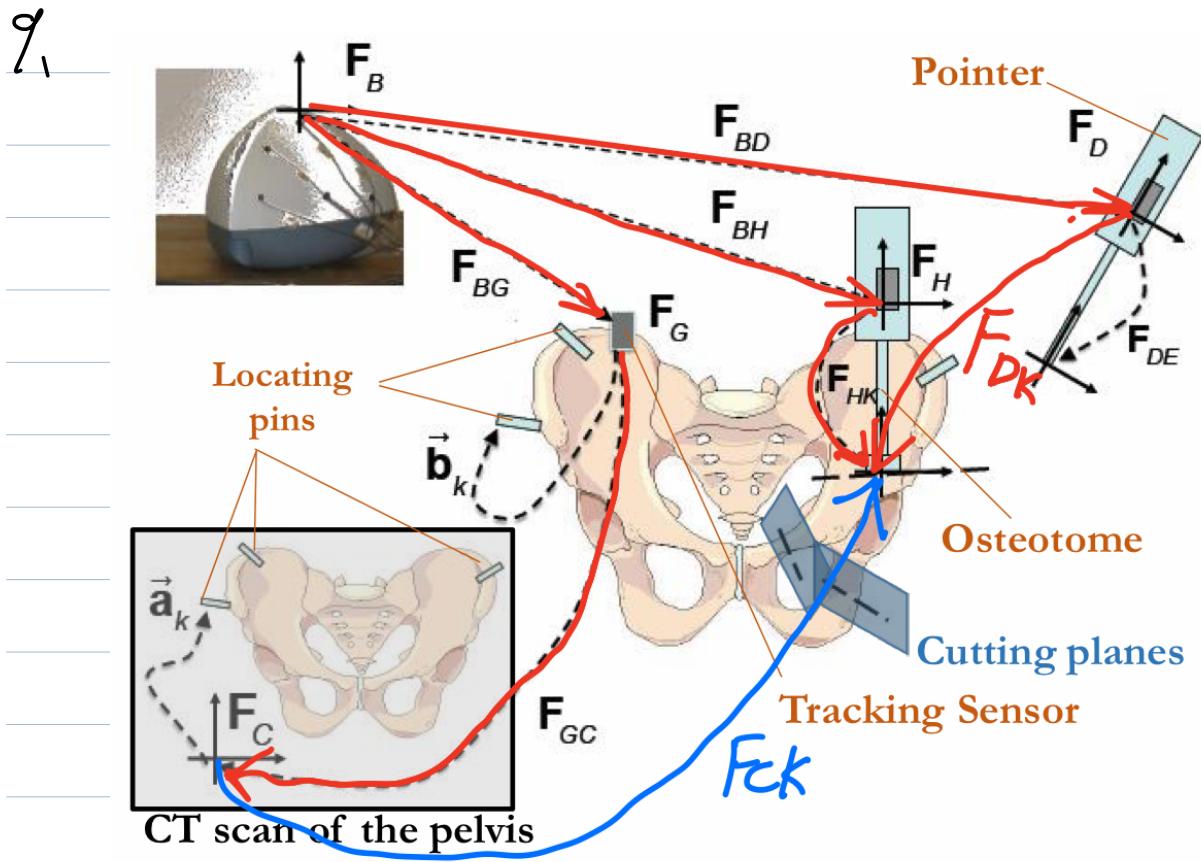
$$(b) \vec{p}_{es} = (1, 1, 1)$$

$$\begin{bmatrix} \vec{p}_{rs} \\ 1 \end{bmatrix} = \bar{T}_{er}^{-1} \begin{bmatrix} \vec{p}_{es} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p}_{rs} = (0, 0, 0)$$



$$F_{BH} F_{HK} = F_{BG} F_{GC} F_{CK}$$

$$F_{CK} = F_{GC}^{-1} F_{BG}^{-1} F_{BH} F_{HK}$$

References:

1. Lynch, Kevin M., and Frank C. Park. "MODERN ROBOTICS." (2018).
 2. <http://publish.illinois.edu/ece470-intro-robotics/files/2019/09/08-lecture.pdf>
 3. <http://publish.illinois.edu/ece470-intro-robotics/files/2019/09/09-lecture.pdf>