Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

The third and last transform in engineering signals and systems.

Z Transform is used for digital signal applications. It would seem to be the most improtant because of its association to digital signals. It maybe if we are into digital signal processing and they are widely used.

When do we know which transform to use and is there a comparison?

It goes like this you surley may have a better answer.

1. Laplace - for systems analysis and design but less for signals
2. Fourier - for signals analysis and design, and less for systems
3. Z - for digital (discrete) signal analysis and design, and systems

At end of this file summaries of all 3 transforms are provided from Linear Systems and Signals 2nd edition by B.P. Lathi. Published by Oxford.

Z transform may be the one transform needed to conquer digital signal processing, and maybe the most profitable.

So some theory and examples first from the Linear Systems and Signals textbook, then on to the tutorial book on using Mathcad Prime for Z transform.

Its NOT likely signals or systems engineer will be focussed on one transform most of the time, similarly for other areas of engineering.

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

# DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

The counterpart of the Laplace transform for discrete-time systems is the $z$-transform. The Laplace transform converts integro-differential equations into algebraic equations. In the same way, the $z$-transforms changes difference equations into algebraic equations, thereby simplifying the analysis of discrete-time systems. The $z$-transform method of analysis of discrete-time systems parallels the Laplace transform method of analysis of continuous-time systems, with some minor differences. In fact, we shall see that *the z-transform is the Laplace transform in disguise*.

The behavior of discrete-time systems is similar to that of continuous-time systems (with some differences). The frequency-domain analysis of discrete-time systems is based on the fact (proved in Section 3.8-3) that the response of a linear, time-invariant, discrete-time (LTID) system to an everlasting exponential $z^n$ is the same exponential (within a multiplicative constant) given by $H[z]z^n$. We then express an input $x[n]$ as a sum of (everlasting) exponentials of the form $z^n$. The system response to $x[n]$ is then found as a sum of the system's responses to all these exponential components. The tool that allows us to represent an arbitrary input $x[n]$ as a sum of (everlasting) exponentials of the form $z^n$ is the $z$-transform.

## 5.1 THE z-TRANSFORM

We define $X[z]$, the direct $z$-transform of $x[n]$, as

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{5.1}$$

where $z$ is a complex variable. The signal $x[n]$, which is the inverse $z$-transform of $X[z]$, can be obtained from $X[z]$ by using the following inverse $z$-transformation:

$$x[n] = \frac{1}{2\pi j} \oint X[z]z^{n-1}\,dz \tag{5.2}$$

The symbol $\oint$ indicates an integration in counterclockwise direction around a closed path in the complex plane (see Fig. 5.1). We derive this $z$-transform pair later, in Chapter 9, as an extension of the discrete-time Fourier transform pair.

As in the case of the Laplace transform, we need not worry about this integral at this point because inverse $z$-transforms of many signals of engineering interest can be found in a $z$-transform table. The direct and inverse $z$-transforms can be expressed symbolically as

$$X[z] = \mathcal{Z}\{x[n]\} \quad \text{and} \quad x[n] = \mathcal{Z}^{-1}\{X[z]\}$$

or simply as

$$x[n] \Longleftrightarrow X[z]$$

Note that

$$\mathcal{Z}^{-1}[\mathcal{Z}\{x[n]\}] = x[n] \quad \text{and} \quad \mathcal{Z}[\mathcal{Z}^{-1}\{X[z]\}] = X[z]$$

## Linearity of the z-Transform

Like the Laplace transform, the $z$-transform is a linear operator. If

$$x_1[n] \Longleftrightarrow X_1[z] \quad \text{and} \quad x_2[n] \Longleftrightarrow X_2[z]$$

then

$$a_1x_1[n] + a_2x_2[n] \Longleftrightarrow a_1X_1[z] + a_2X_2[z] \tag{5.3}$$

The proof is trivial and follows from the definition of the $z$-transform. This result can be extended to finite sums.

## The Unilateral z-Transform

For the same reasons discussed in Chapter 4, we find it convenient to consider the unilateral $z$-transform. As seen for the Laplace case, the bilateral transform has some complications because of nonuniqueness of the inverse transform. In contrast, the unilateral transform has a unique inverse. This fact simplifies the analysis problem considerably, but at a price: the unilateral version can handle only causal signals and systems. Fortunately, most of the practical cases are causal. The more general *bilateral z-transform* is discussed later, in Section 5.9. In practice, the term *z-transform* generally means *the unilateral z-transform*.

In a basic sense, there is no difference between the unilateral and the bilateral $z$-transform. The unilateral transform is the bilateral transform that deals with a subclass of signals starting at $n = 0$ (causal signals). Hence, the definition of the unilateral transform is the same as that of the bilateral [Eq. (5.1)], except that the limits of the sum are from 0 to $\infty$

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{5.4}$$

The expression for the inverse $z$-transform in Eq. (5.2) remains valid for the unilateral case also.

## The Region of Convergence (ROC) of X[z]

The sum in Eq. (5.1) [or (5.4)] defining the direct $z$-transform $X[z]$ may not converge (exist) for all values of $z$. The values of $z$ (the region in the complex plane) for which the sum in Eq. (5.1) converges (or exists) is called the *region of existence*, or more commonly the *region of convergence* (ROC), for $X[z]$. This concept will become clear in the following example.

## EXAMPLE 5.1

Find the $z$-transform and the corresponding ROC for the signal $\gamma^n u[n]$.

By definition

$$X[z] = \sum_{n=0}^{\infty} \gamma^n u[n] z^{-n}$$

Since $u[n] = 1$ for all $n \geq 0$,

$$X[z] = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n \qquad z^{-n} = \frac{1}{z^n}$$

$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots + \cdots \tag{5.5}$$

It is helpful to remember the following well-known geometric progression and its sum:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{if} \quad |x| < 1 \tag{5.6}$$

Use of Eq. (5.6) in Eq. (5.5) yields

$$X[z] = \frac{1}{1 - \frac{\gamma}{z}} \qquad \left|\frac{\gamma}{z}\right| < 1 \qquad \text{multiply by } z \text{ both sides}$$

$$= \frac{z}{z - \gamma} \qquad |z| > |\gamma| \tag{5.7}$$

Observe that $X[z]$ exists only for $|z| > |\gamma|$. For $|z| < |\gamma|$, the sum in Eq. (5.5) may not converge; it goes to infinity. Therefore, the ROC of $X[z]$ is the shaded region outside the circle of radius $|\gamma|$, centered at the origin, in the $z$-plane, as depicted in Fig. 5.1b.

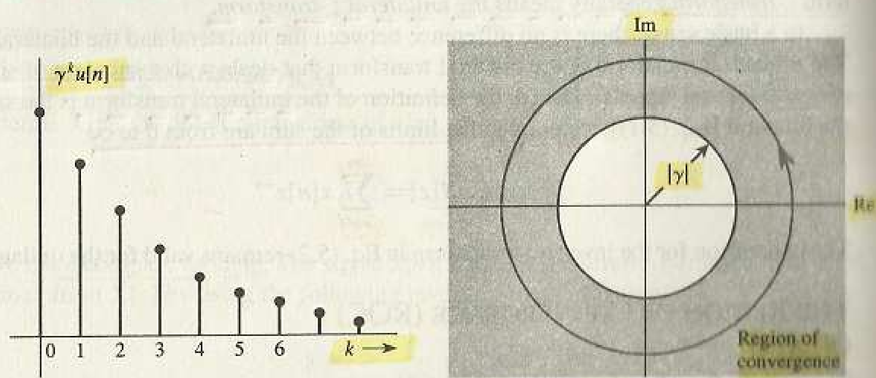



**Figure 5.1** $\gamma^n u[n]$ and the region of convergence of its $z$-transform.

Later in Eq. (5.85), we show that the $z$-transform of another signal $-\gamma^n u[-(n+1)]$ is also $z/(z-\gamma)$. However, the ROC in this case is $|z| < |\gamma|$. Clearly, the inverse $z$-transform of $z/(z-\gamma)$ is not unique. However, if we restrict the inverse transform to be causal, then the inverse transform is unique, namely, $\gamma^n u[n]$.

The ROC is required for evaluating $x[n]$ from $X[z]$, according to Eq. (5.2). The integral in Eq. (5.2) is a contour integral, implying integration in a counterclockwise direction along a closed path centered at the origin and satisfying the condition $|z| > |\gamma|$. Thus, any circular path centered at the origin and with a radius greater than $|\gamma|$ (Fig. 5.1b) will suffice. We can show that the integral in Eq. (5.2) along any such path (with a radius greater than $|\gamma|$) yields the same result, namely $x[n]$.[†] Such integration in the complex plane requires a background in the theory of functions of complex variables. We can avoid this integration by compiling a table of $z$-transforms (Table 5.1), where $z$-transform pairs are tabulated for a variety of signals. To find the inverse $z$-transform of say, $z/(z-\gamma)$, instead of using the complex integration in Eq. (5.2), we consult the table and find the inverse $z$-transform of $z/(z-\gamma)$ as $\gamma^n u[n]$. Because of uniqueness property of the unilateral $z$-transform, there is only one inverse for each $X[z]$. Although the table given here is rather short, it comprises the functions of most practical interest.

The situation of the $z$-transform regarding the uniqueness of the inverse transform is parallel to that of the Laplace transform. For the bilateral case, the inverse $z$-transform is not unique unless the ROC is specified. For the unilateral case, the inverse transform is unique; the region of convergence need not be specified to determine the inverse $z$-transform. For this reason, we shall ignore the ROC in the unilateral $z$-transform Table 5.1.

### Existence of the z-Transform

By definition

$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} \frac{x[n]}{z^n}$$

The existence of the $z$-transform is guaranteed if

$$|X[z]| \le \sum_{n=0}^{\infty} \frac{|x[n]|}{|z|^n} < \infty$$

for some $|z|$. Any signal $x[n]$ that grows no faster than an exponential signal $r_0^n$, for some $r_0$, satisfies this condition. Thus, if

$$|x[n]| \le r_0^n \qquad \text{for some } r_0 \tag{5.8}$$

then

$$|X[z]| \le \sum_{n=0}^{\infty} \left(\frac{r_0}{|z|}\right)^n = \frac{1}{1 - \dfrac{r_0}{|z|}} \qquad |z| > r_0$$

[†]Indeed, the path need not even be circular. It can have any odd shape, as long as it encloses the pole(s) of $X[z]$ and the path of integration is counterclockwise.

Therefore, $X[z]$ exists for $|z| > r_0$. Almost all practical signals satisfy condition (5.8) and are therefore $z$-transformable. Some signal models (e.g., $\gamma^{n^2}$) grow faster than the exponential signal $r_0^n$ (for any $r_0$) and do not satisfy Eq. (5.8) and therefore are not $z$-transformable. Fortunately, such signals are of little practical or theoretical interest. Even such signals over a finite interval are $z$-transformable.

**TABLE 5.1** (Unilateral) z-Transform Pairs

| No. | $x[n]$ | $X[z]$ |
|---|---|---|
| 1 | $\delta[n-k]$ δ[n-k] | $z^{-k}$ z^-k |
| 2 | $u[n]$ | $\dfrac{z}{z-1}$ |
| 3 | $nu[n]$ | $\dfrac{z}{(z-1)^2}$ |
| 4 | $n^2u[n]$ | $\dfrac{z(z+1)}{(z-1)^3}$ |
| 5 | $n^3u[n]$ | $\dfrac{z(z^2+4z+1)}{(z-1)^4}$ |
| 6 | $\gamma^n u[n]$ | $\dfrac{z}{z-\gamma}$ |
| 7 | $\gamma^{n-1}u[n-1]$ | $\dfrac{1}{z-\gamma}$ |
| 8 | $n\gamma^n u[n]$ | $\dfrac{\gamma z}{(z-\gamma)^2}$ |
| 9 | $n^2\gamma^n u[n]$ | $\dfrac{\gamma z(z+\gamma)}{(z-\gamma)^3}$ |
| 10 | $\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$ | $\dfrac{z}{(z-\gamma)^{m+1}}$ |
| 11a | $\lvert\gamma\rvert^n \cos\beta n\, u[n]$ | $\dfrac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$ |
| 11b | $\lvert\gamma\rvert^n \sin\beta n\, u[n]$ | $\dfrac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$ |
| 12a | $r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$ | $\dfrac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$ |
| 12b | $r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n] \quad \gamma=\lvert\gamma\rvert e^{j\beta}$ | $\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$ |
| 12c | $r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$ | $\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$ |

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}}$$

$$\beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert}$$

$$\theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$

### 5.1-1 Finding the Inverse Transform

As in the Laplace transform, we shall avoid the integration in the complex plane required to find the inverse $z$-transform [Eq. (5.2)] by using the (unilateral) transform table (Table 5.1). Many of the transforms $X[z]$ of practical interest are rational functions (ratio of polynomials in $z$), which can be expressed as a sum of partial fractions, whose inverse transforms can be readily found in a table of transform. The partial fraction method works because for every transformable $x[n]$ defined for $n \geq 0$, there is a corresponding unique $X[z]$ defined for $|z| > r_0$ (where $r_0$ is some constant), and vice versa.

INVERSE TRANSFORM BY EXPANSION OF $X[z]$ IN POWER SERIES OF $z^{-1}$

By definition

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n}$$
$$= x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \frac{x[3]}{z^3} + \cdots$$
$$= x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots$$

This result is a power series in $z^{-1}$. Therefore, if we can expand $X[z]$ into the power series in $z^{-1}$, the coefficients of this power series can be identified as $x[0]$, $x[1]$, $x[2]$, $x[3]$, .... A rational $X[z]$ can be expanded into a power series of $z^{-1}$ by dividing its numerator by the denominator. Consider, for example,

$$X[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)}$$
$$= \frac{7z^3 - 2z^2}{z^3 - 1.7z^2 + 0.8z - 0.1}$$

To obtain a series expansion in powers of $z^{-1}$, we divide the numerator by the denominator as follows:

$$\begin{array}{r|l}
 & 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots \\
z^3 - 1.7z^2 + 0.8z - 0.1 & 7z^3 - 2z^2 \\
 & \underline{7z^3 - 11.9z^2 + 5.60z - 0.7} \\
 & 9.9z^2 - 5.60z + 0.7 \\
 & \underline{9.9z^2 - 16.83z + 7.92 - 0.99z^{-1}} \\
 & 11.23z - 7.22 + 0.99z^{-1} \\
 & \underline{11.23z - 19.09 + 8.98z^{-1}} \\
 & 11.87 - 7.99z^{-1}
\end{array}$$

Thus

$$X[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots$$

Therefore

Coefficients → $x[0] = 7, \; x[1] = 9.9, \; x[2] = 11.23, \; x[3] = 11.87, \ldots$

Although this procedure yields $x[n]$ directly, it does not provide a closed-form solution. For this reason, it is not very useful unless we want to know only the first few terms of the sequence $x[n]$.

Next to the tutorial textbook and Mathcad Prime.

## 3.15 Introduction to the $z$-transform

While the Laplace transform is used to transform continuous time systems from the time domain to the frequency domain, the $z$-transform is used to represent discrete-time systems in the $z$-domain. In Chapter 2, we use the convolution integral to calculate the impulse response of a system. Knowing the Laplace transform technique, it is more convenient for us to use that technique to calculate the response of a system. We have seen that the convolution sum requires significant computation. Instead of using the convolution sum to evaluate output response of a system, we can also use the z-transform to that. In this section, we are going to introduce to the $z$-transform. We are going to use the $z$-transform to represent discrete-time systems. We are going to use MathCAD as well to evaluate $z$-transform and plot discrete-time systems responses.

The z-transform of a discrete time system $x(n)$ is defined as $X(z)$, where

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad \text{(Equ.3.10)}$$

Since we are dealing with causal systems, where the value is 0 for $n < 0$, then we should use the unilateral z-transform, where

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \qquad \text{(Equ.3.11)}$$

This equation can be expanded to

$$X(z) = x(n)z^{0} + x(n)z^{-1} + x(n)z^{-2} + x(n)z^{-3} + \cdots$$

Assume that we have a sequence where

$x(n) = 5, 4, 3, 2, 1, 0, 0, 0$

Then, the z-transform of $x(n)$ is

$$X(z) = 5 + 4z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

## 3.16 MathCAD Approach to $z$-Transform

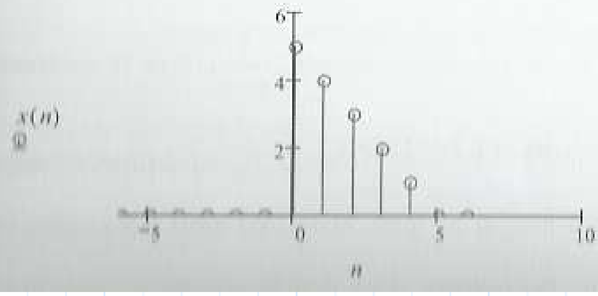
To solve the problem in MathCAD , we have to place the sequence in proper

Assume that we have a sequence where

$x(n) = 5, 4, 3, 2, 1, 0, 0, 0$

format as follows:

$$x(n) = 5\delta(n) + 4\delta(n-1) + 3\delta(n-2) + 2\delta(n-3) + \delta(n-4)$$




By entering the sequence as shown above, we can get an accurate solution by going to the symbolic menu and choosing z-transform. The solution is displayed as follows:

$5\cdot\delta(n) + 4\cdot\delta(n-1) + 3\cdot\delta(n-2) + 2\cdot\delta(n-3) + \delta(n-4)$ has z transform $ztrans(\delta(n), n, z)\cdot\dfrac{(5\cdot z^4 + 4\cdot z^3 + 3\cdot z^2 + 2\cdot z + 1)}{z^4}$

If we use the upper case capital Greek letter "delta" and insert the sequence as

$$x(n) = 5\Delta(n) + 4\Delta(n-1) + 3\Delta(n-2) + 2\Delta(n-3) + \Delta(n-4)$$

we can get the solution without the comment as shown below:

$5\cdot\Delta(n) + 4\cdot\Delta(n-1) + 3\cdot\Delta(n-2) + 2\cdot\Delta(n-3) + \Delta(n-4)$ has z transform $\dfrac{(5\cdot z^4 + 4\cdot z^3 + 3\cdot z^2 + 2\cdot z + 1)}{z^4}$

MathCAD gives a solution simplified to the highest order of $z$. The above solution is simplified by $z^{-4}$. We can also use the insertion from the symbolic palette to insert the expression and the variable whose the transform we want to take. To use the insertion, we can insert the expression inside the place holder to the left and the variable to the right.

$\blacksquare\ ztrans, \blacksquare \rightarrow$ we can do that with the place holder *expression ztrans , variable*

For the above expression, we have

$5\cdot\Delta(n) + 4\cdot\Delta(n-1) + 3\cdot\Delta(n-2) + 2\cdot\Delta(n-3) + \Delta(n-4)\ ztrans, n \rightarrow\ \blacksquare$

The black box in the far right is used to disable the automatic evaluation.

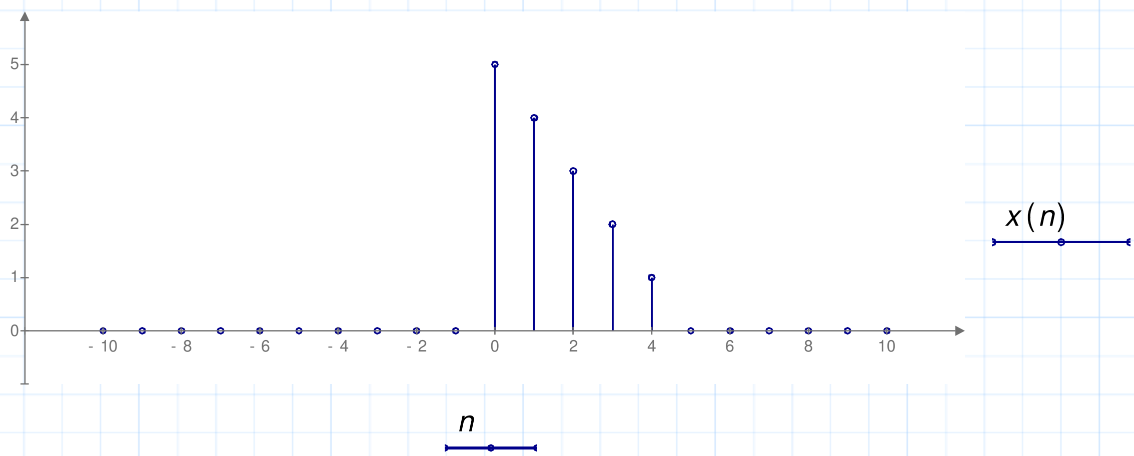
## Example - Demonstration A

$n := -10..10$ This line has to be put in because it does not pick up from the plot x-axis unitl later so if it is NOT put in here n=0 all the way through u(n)

$a := 5$ Amplitude

$\delta(n) := if(n = 0, 1, 0)$

$x(n) := (a)\cdot\delta(n) + (a-1)\cdot\delta(n-1) + (a-2)\cdot\delta(n-2) + (a-3)\cdot\delta(n-3) + (a-4)\cdot\delta(n-4)$




Mathcad Prime 2.0 - Command for Z Transform

To return results that are not fully simplified, use the modifier *raw* after the transform keyword. For example, compare the following results of *ztrans*:

$$x^2 + 3\cdot x \xrightarrow{ztrans} \frac{x\cdot z\cdot(x+3)}{z-1}$$

$$x^2 + 3\cdot x \xrightarrow{ztrans, x, raw} \frac{z^2+z}{(z-1)^3} + \frac{3\cdot z}{(z-1)^2}$$

See next page for the results of of the z transform command

$$x(n) \xrightarrow{ztrans} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{5 \cdot z}{z-1} \\ \frac{4 \cdot z}{z-1} \\ \frac{3 \cdot z}{z-1} \\ \frac{2 \cdot z}{z-1} \\ \frac{z}{z-1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

<---Check with tables. This is correct

$$ztrans(x(n), n, z) \rightarrow \begin{bmatrix} ztrans(0,-10,z) \\ ztrans(0,-9,z) \\ ztrans(0,-8,z) \\ ztrans(0,-7,z) \\ ztrans(0,-6,z) \\ ztrans(0,-5,z) \\ ztrans(0,-4,z) \\ ztrans(0,-3,z) \\ ztrans(0,-2,z) \\ ztrans(0,-1,z) \\ ztrans(5,0,z) \\ ztrans(4,1,z) \\ ztrans(3,2,z) \\ ztrans(2,3,z) \\ ztrans(1,4,z) \\ ztrans(0,5,z) \\ ztrans(0,6,z) \\ ztrans(0,7,z) \\ ztrans(0,8,z) \\ ztrans(0,9,z) \\ ztrans(0,10,z) \end{bmatrix}$$

<--- Values of x(n) match the plot x(n) values

Study the results above, for n each term its transform with the amplitude.

Right vector, the first x(n) results with the amplitude, the second, n, is the nth term shown, the third term z indicates the z transform applied.

Put both the vectors together to get its meaningful result

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

Example - Demonstration B

$clear(x)$

$n := -10 .. 10$ Remember this line has to be entered it defines n in the x(n) equation

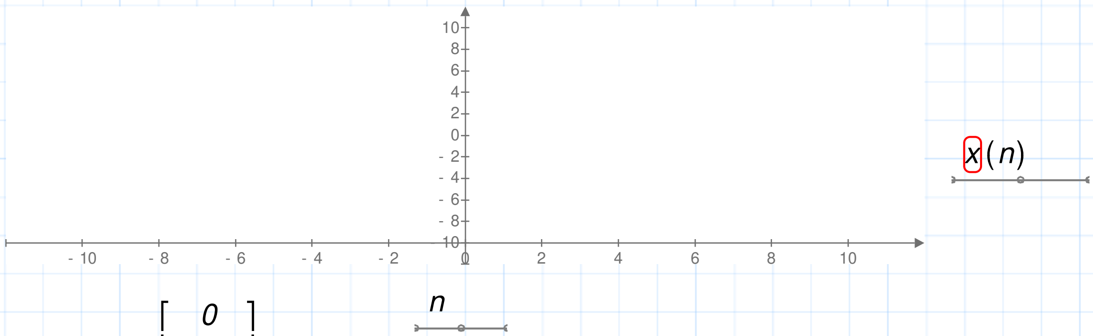
$a := 5$

$\delta(n) := if(n = 0, 1, 0)$

$x1(n) := (a-5)\cdot\delta(n) + (a-4)\cdot\delta(n-1) + (a-3)\cdot\delta(n-2) + (a-2)\cdot\delta(n-3) + (a-1)\cdot\delta(n-4)$

$x2(n) := (a)\cdot\delta(n-5)$

$x(n) := x1(n) + x2(n)$




$$x(n) \xrightarrow{ztrans} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{z}{z-1} \\ \frac{2\cdot z}{z-1} \\ \frac{3\cdot z}{z-1} \\ \frac{4\cdot z}{z-1} \\ \frac{5\cdot z}{z-1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

<---Check with tables. This is correct

$$ztrans(x(n), n, z) \rightarrow \begin{bmatrix} ztrans(0,-10,z) \\ ztrans(0,-9,z) \\ ztrans(0,-8,z) \\ ztrans(0,-7,z) \\ ztrans(0,-6,z) \\ ztrans(0,-5,z) \\ ztrans(0,-4,z) \\ ztrans(0,-3,z) \\ ztrans(0,-2,z) \\ ztrans(0,-1,z) \\ ztrans(0,0,z) \\ ztrans(1,1,z) \\ ztrans(2,2,z) \\ ztrans(3,3,z) \\ ztrans(4,4,z) \\ ztrans(5,5,z) \\ ztrans(0,6,z) \\ ztrans(0,7,z) \\ ztrans(0,8,z) \\ ztrans(0,9,z) \\ ztrans(0,10,z) \end{bmatrix}$$

<--- Values of x(n) match the plot x(n) values

The results are as expected and wrt demonstration A the values in the opposite direction.

Example - PROBLEMATIC!

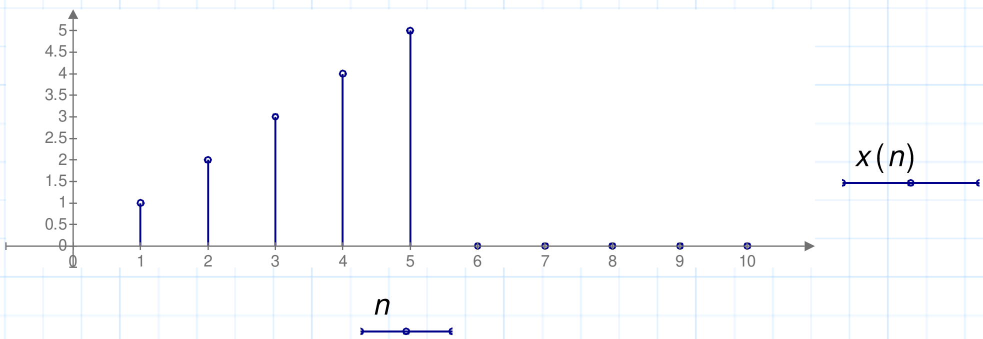
The signal x(n) was formed by the command lines below.
The signal x[n] is expressed as a product of n and gate pulse u[n] - u[n-6]

$clear(u) \qquad clear(x)$

$n := 1..10$

$u(n) := if(n \geq 0, 1, 0)$ Step Function

$x(n) := n \cdot u(n) - n \cdot u(n-6)$




$$x(n) \xrightarrow{ztrans} \begin{bmatrix} \frac{z}{z-1} \\ \frac{2 \cdot z}{z-1} \\ \frac{3 \cdot z}{z-1} \\ \frac{4 \cdot z}{z-1} \\ \frac{5 \cdot z}{z-1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

WRONG! - This is NOT the Z transform for the signal.
Now check the table for the z-transform of n u [n - 6]

Ok so lets read on the right shift (delay), and left shift (advance) in the notes posted on the following page.

Then this example is explained further below.

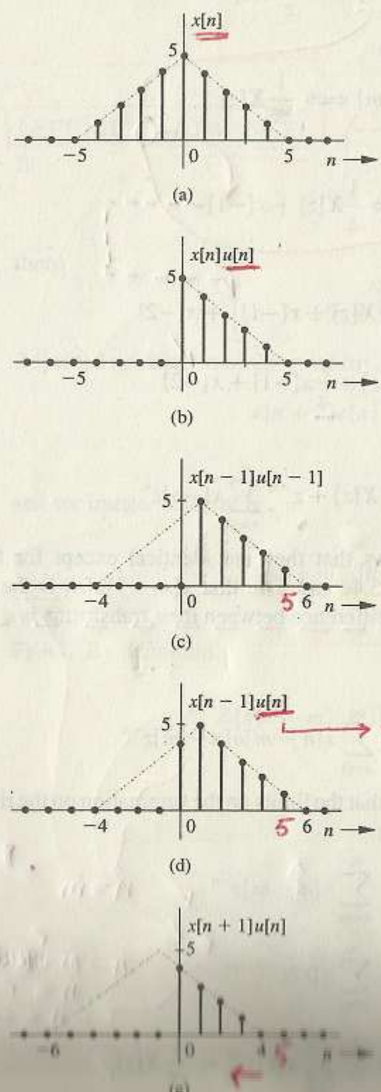
Lets study on some Z transform properties, before going into Z transform system representation.

## 5.2 Some Properties of the $z$-Transform

The $z$-transform properties are useful in the derivation of $z$-transforms of many functions and also in the solution of linear difference equations with constant coefficients. Here we consider a few important properties of the $z$-transform.

In our discussion, the variable $n$ appearing in signals, such as $x[n]$ and $y[n]$, may or may not stand for time. However, in most applications of our interest, $n$ is proportional to time. For this reason, we shall loosely refer to the variable $n$ as time.

In the following discussion of the shift property, we deal with shifted signals $x[n]u[n]$, $x[n-k]u[n-k]$, $x[n-k]u[n]$, and $x[n+k]u[n]$. Unless we physically understand the meaning of such shifts, our understanding of the shift property remains mechanical rather than intuitive or heuristic. For this reason using a hypothetical signal $x[n]$, we have illustrated various shifted signals for $k = 1$ in Fig. 5.4.




→ start at $n=0$ → 1 (step amplitude) OR unit impulse

$x[-1]\,\delta(n)$

Figure 5.4 A signal $x[n]$ and its shifted versions.

**RIGHT SHIFT (DELAY)**

If

$$x[n]u[n] \Longleftrightarrow X[z]$$

then

$$x[n-1]u[n-1] \Longleftrightarrow \frac{1}{z}X[z] \qquad (5.15a)$$

In general,

$$x[n-m]u[n-m] \Longleftrightarrow \frac{1}{z^m}X[z] \qquad (5.15b)$$

Moreover,

$$x[n-1]u[n] \Longleftrightarrow \frac{1}{z}X[z] + x[-1] \qquad (5.16a)$$

Repeated application of this property yields

$$x[n-2]u[n] \Longleftrightarrow \frac{1}{z}\left[\frac{1}{z}X[z] + x[-1]\right] + x[-2]$$

$$= \frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2] \qquad (5.16b)$$

In general, for integer value of $m$

$$x[n-m]u[n] \Longleftrightarrow z^{-m}X[z] + z^{-m}\sum_{n=1}^{m} x[-n]z^n \qquad (5.16c)$$

A look at Eqs. (5.15a) and (5.16a) shows that they are identical except for the extra term $x[-1]$ in Eq. (5.16a). We see from Fig. 5.4c and 5.4d that $x[n-1]u[n]$ is the same as $x[n-1]u[n-1]$ plus $x[-1]\delta[n]$. Hence, the difference between their transforms is $x[-1]$.

**Proof.** For integer value of $m$

$$\mathcal{Z}\{x[n-m]u[n-m]\} = \sum_{n=0}^{\infty} x[n-m]u[n-m]z^{-n}$$

Recall that $x[n-m]u[n-m] = 0$ for $n < m$, so that the limits on the summation on the right-hand side can be taken from $n = m$ to $\infty$. Therefore

$$\mathcal{Z}\{x[n-m]u[n-m]\} = \sum_{n=m}^{\infty} x[n-m]z^{-n}$$

n = m → u(n) → 1

$$= \sum_{r=0}^{\infty} x[r]z^{-(r+m)}$$

n − m = r
n = r + m
−n = −(r + m)

$$= \frac{1}{z^m}\sum_{r=0}^{\infty} x[r]z^{-r} = \frac{1}{z^m}X[z]$$

See Eq 5.1

To prove Eq. (5.16c), we have

$$\mathcal{Z}\{x[n-m]u[n]\} = \sum_{n=0}^{\infty} x[n-m]z^{-n} = \sum_{r=-m}^{\infty} x[r]z^{-(r+m)}$$

$$= z^{-m}\left[\sum_{r=-m}^{-1} x[r]z^{-r} + \sum_{r=0}^{\infty} x[r]z^{-r}\right]$$

$-m \ldots -1 \; 0 \ldots \infty$

discrete

not continuous $\int_{-m}^{\infty}$

$$= z^{-m}\sum_{n=1}^{m} x[-n]z^{n} + z^{-m}X[z]$$

2nd term of eq 5.16c

**LEFT SHIFT (ADVANCE)**

If

$$x[n]u[n] \Longleftrightarrow X[z]$$

then

$x(n-(-1))$

$x(n+1)$

$$x[n+1]u[n] \Longleftrightarrow zX[z] - zx[0] \tag{5.17a}$$

Repeated application of this property yields

$$x[n+2]u[n] \Longleftrightarrow z\{z(X[z]-zx[0]) - x[1]\}$$
$$= z^2X[z] - z^2x[0] - zx[1] \tag{5.17b}$$

and for integer value of $m$

→ see eq. 5.16c Z

$$x[n+m]u[n] \Longleftrightarrow z^m X[z] - z^m \sum_{n=0}^{m-1} x[n]z^{-n} \tag{5.17c}$$

**Proof.** By definition

$$\mathcal{Z}\{x[n+m]u[n]\} = \sum_{n=0}^{\infty} x[n+m]z^{-n}$$

$$= \sum_{r=m}^{\infty} x[r]z^{-(r-m)}$$

$n+m=r$
$n=r-m$
$-n=-(r-m)$

$$= z^m \sum_{r=m}^{\infty} x[r]z^{-r}$$

$$= z^m\left[\sum_{r=0}^{\infty} x[r]z^{-r} - \sum_{r=0}^{m-1} x[r]z^{-r}\right]$$

$$= z^m X[z] - z^m \sum_{r=0}^{m-1} x[r]z^{-r}$$

Here is the explanation for the previous PROBLEMATIC example which was NOT correct!

## EXAMPLE 5.4

Find the $z$-transform of the signal $x[n]$ depicted in Fig. 5.5.

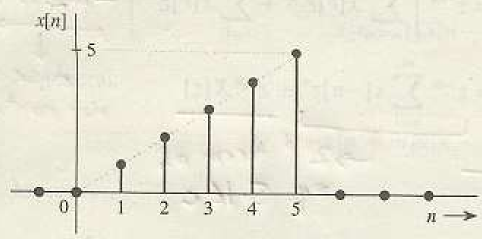



**Figure 5.5**

The signal $x[n]$ can be expressed as a product of $n$ and a gate pulse $u[n]-u[n-6]$. Therefore

$$x[n] = n\{u[n] - u[n-6]\}$$
$$= nu[n] - nu[n-6]$$

We cannot find the $z$-transform of $nu[n-6]$ directly by using the right-shift property [Eq. (5.15b)]. So we rearrange it in terms of $(n-6)u[n-6]$ as follows:

$$x[n] = nu[n] - (n-6+6)u[n-6]$$
$$= nu[n] - (n-6)u[n-6] - 6u[n-6]$$

We can now find the $z$-transform of the bracketed term by using the right-shift property [Eq. (5.15b)]. Because $u[n] \Longleftrightarrow z/(z-1)$

$$u[n-6] \Longleftrightarrow \frac{1}{z^6}\frac{z}{z-1} = \frac{1}{z^5(z-1)}$$

Also, because $nu[n] \Longleftrightarrow z/(z-1)^2$

$$(n-6)u[n-6] \Longleftrightarrow \frac{1}{z^6}\frac{z}{(z-1)^2} = \frac{1}{z^5(z-1)^2}$$

Therefore

$$X[z] = \frac{z}{(z-1)^2} - \frac{1}{z^5(z-1)^2} - \frac{6}{z^5(z-1)}$$
$$= \frac{z^6 - 6z + 5}{z^5(z-1)^2}$$

Convolution.

## CONVOLUTION

Frequency convolution

The time-convolution property states that if[†]

$$x_1[n] \Longleftrightarrow X_1[z] \qquad \text{and} \qquad x_2[n] \Longleftrightarrow X_2[z],$$

then (*time convolution*)

$$x_1[n] * x_2[n] \Longleftrightarrow X_1[z]X_2[z] \tag{5.18}$$

**Proof.** This property applies to causal as well as noncausal sequences. We shall prove it for the more general case of noncausal sequences, where the convolution sum ranges from $-\infty$ to $\infty$.

We have

$$\begin{aligned}\mathcal{Z}\{x_1[n] * x_2[n]\} &= \mathcal{Z}\left[\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]\right]\\ &= \sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]\end{aligned}$$

Interchanging the order of summation, we have

$$\begin{aligned}\mathcal{Z}[x_1[n] * x_2[n]] &= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{n=-\infty}^{\infty} x_2[n-m]z^{-n}\\ &= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{r=-\infty}^{\infty} x_2[r]z^{-(r+m)}\\ &= \sum_{m=-\infty}^{\infty} x_1[m]z^{-m} \sum_{r=-\infty}^{\infty} x_2[r]z^{-r}\\ &= X_1[z]X_2[z]\end{aligned}$$

[†]There is also the frequency-convolution property, which states that if

$$x_1[n]x_2[n] \Longleftrightarrow \frac{1}{2\pi j}\oint X_1[u]X_2\left[\frac{z}{u}\right]u^{-1}\,du$$

## LTID SYSTEM RESPONSE

It is interesting to apply the time-convolution property to the LTID input–output equation $y[n] = x[n] * h[n]$. Since from Eq. (5.14b), we know that $h[n] \Longleftrightarrow H[z]$, it follows from Eq. (5.18) that

$$Y[z] = X[z]H[z] \tag{5.19}$$

Inverse Z transform tutorial textbook example.

Example 3.14 - As shown in notes above

$X(z) := 5 \cdot \left( \frac{z}{z-1} \right)$

$X(z) \xrightarrow{invztrans} 5$ Matches z-transform table CORRECT

By the Linear Systems and Signals textbook by BP Lathi the answer would attach the function u(n) to the result. This makes sense because it associates a step function to it --> 5 u(n). This means for 0 to infinity the amplitude is 5 for all discrete values n = 0 to + infinity. If this is not how its read you tell me?

$X(z) := z^{-2}$

$X(z) \xrightarrow{invztrans} \delta(n-2, 0)$ Matches z-transform table CORRECT
The zero in d(n-2,0)?

$X(z) := 1$

$X(z) \xrightarrow{invztrans} \delta(n, 0)$ Matches z-transform table CORRECT

$X(z) := 5$

$X(z) \xrightarrow{invztrans} 5 \cdot \delta(n, 0)$ Matches z-transform table CORRECT

$X(z) := \frac{1}{(z-1) \cdot z}$

$X(z) \xrightarrow{invztrans} 1 - \delta(n-1, 0) - \delta(n, 0)$ Matches textbook result CORRECT

$X(z) := \frac{1}{(z-1) \cdot z^3}$

$X(z) \xrightarrow{invztrans} 1 - \delta(n-1, 0) - \delta(n-2, 0) - \delta(n-3, 0) - \delta(n, 0)$ Matches textbook result CORRECT

Exercise E5.1 Linear Sys & Sig BP Lathi 2nd ed

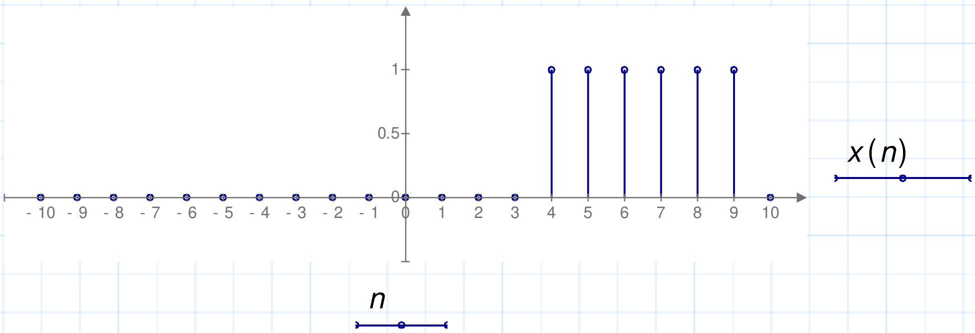
$clear(x)$

$n := -10..10$ Remember this line has to be entered it defines n in the x(n) equation

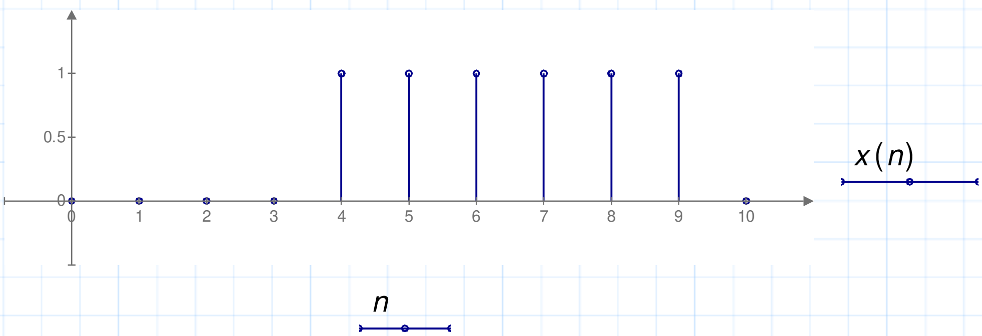
$a := 1$

$\delta(n) := if(n = 0, 1, 0)$

$x(n) := (a)\cdot\delta(n-4) + (a)\cdot\delta(n-5) + (a)\cdot\delta(n-6) + (a)\cdot\delta(n-7) + (a)\cdot\delta(n-8) + (a)\cdot\delta(n-9)$




$n := 0..10$




Continued next page

$$x(n) \xrightarrow{ztrans} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ 0 \end{bmatrix}$$

This is correct REMEMBER the function is discrete x(n) so the results would be a z transform for each discrete value of integer n. If this was a constant value then it would may be possible, its NOT. (1) z/(z-1) = 1 u(n)

As far as the textbook answer this is WRONG? Wait.

Is it because we are to look at it as a signal with a period, with one term defining the z transform? A summation term for a z transform here?

NO!.....but wait.

Go back to the dirac function d(n-4) + d(n-5)... then take the transform of each term and make it into an equation

$$X(z) := z^{-4}$$

$$X(z) \xrightarrow{invztrans} \delta(n-4, 0)$$

This is the inverse of our discrete function d(n-4) is z^ -4 so sum for each term from n = 4 to n = 9

X(z)= z^ -4 + z^ -5 + z^ -6 + z^ -7 + z^ -8 + z^ -9

divide by z^ 9

X(z)= z^ 5 + z^ 4 + z^ 3 + z^ 2 + z^ 1 + z^ 0

X(z)= z^ 5 + z^ 4 + z^ 3 + z^ 2 + z^ 1 + 1 Textbook Answer!

This example proves some expectation of the answer's form is required.

Now we return to our first wrong answer and we see that the textbook answer has a second answer and our answer maybe correct. So lets proceed to show the first answer maybe correct.

(z/(z-1)) informs of a amplitude of 1, so if this is the term for the amplitude at position n, then multiplying it by the transform of d(n-k) = z^ -k, where k is the position on the x-axis. All the terms have amplitude 1, so multiplying z/(z-1) to z^ -k.

Now lets do this for each term:

X(z) = z/(z-1) multiply by each d(n-k) term

X(z) = z/(z-1) (z^ -4 + z^ -5 + z^ -6 + z^ -7 + z^ -8 + z^ -9)    Another Textbook Answer.

Z Transform operations

**TABLE 5.2** *z*-**Transform Operations**

| Operation | $x[n]$ | $X[z]$ |
|---|---|---|
| Addition | $x_1[n] + x_2[n]$ | $X_1[z] + X_2[z]$ |
| Scalar multiplication | $ax[n]$ | $aX[z]$ |
| Right-shifting | $x[n-m]u[n-m]$ | $\frac{1}{z^m}X[z]$ |
| | $x[n-m]u[n]$ | $\frac{1}{z^m}X[z] + \frac{1}{z^m}\sum_{n=1}^{m} x[-n]z^n$ |
| | $x[n-1]u[n]$ | $\frac{1}{z}X[z] + x[-1]$ |
| | $x[n-2]u[n]$ | $\frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2]$ |
| | $x[n-3]u[n]$ | $\frac{1}{z^3}X[z] + \frac{1}{z^2}x[-1] + \frac{1}{z}x[-2] + x[-3]$ |
| Left-shifting | $x[n+m]u[n]$ | $z^m X[z] - z^m \sum_{n=0}^{m-1} x[n]z^{-n}$ |
| | $x[n+1]u[n]$ | $zX[z] - zx[0]$ |
| | $x[n+2]u[n]$ | $z^2X[z] - z^2x[0] - zx[1]$ |
| | $x[n+3]u[n]$ | $z^3X[z] - z^3x[0] - z^2x[1] - zx[2]$ |
| Multiplication by $\gamma^n$ | $\gamma^n x[n]u[n]$ | $X\left[\frac{z}{\gamma}\right]$ |
| Multiplication by $n$ | $nx[n]u[n]$ | $-z\frac{d}{dz}X[z]$ |
| Time convolution | $x_1[n] * x_2[n]$ | $X_1[z]X_2[z]$ |
| Time reversal | $x[-n]$ | $X[1/z]$ |
| Initial value | $x[0]$ | $\lim_{z\to\infty} X[z]$ |
| Final value | $\lim_{N\to\infty} x[N]$ | $\lim_{z\to 1}(z-1)X[z]$ poles of $(z-1)X[z]$ inside the unit circle |

For the next example apply partial fractions.

Example 5.3 Linear Sys & Signals 2n ed

$$X(z) := \frac{2 \cdot z \cdot (3 \cdot z + 17)}{(z - 1) \cdot (z^2 - 6 \cdot z + 25)}$$

$$\frac{2 \cdot z \cdot (3 \cdot z + 17)}{(z - 1) \cdot (z^2 - 6 \cdot z + 25)} \xrightarrow{\substack{parfrac \\ simplified}} 2 \cdot z \cdot (3 \cdot z + 17) \cdot \frac{1}{(z - 1) \cdot (z^2 - 6 \cdot z + 25)}$$

(z^ 2 - 6z + 25) ==> (z - 3 - j4) and (z - 3 + j4)

Poles of X(z) are: 1, 3 - j4, 3 + j4.
So the denomiator term has complex conjugate poles.
To solve this partial factor there are 2 methods:
1. First order factors
2. Quadratic factors
Here quadratic method is used (Easier for me).

Move the z term in the numerator of RHS to the LHS
X(z)/z = [ 2/(z - 1)] + [(Az + B) / (z^ 2 - 6z + 25) ]
Multiply both sides by z
X(z) = [ 2z/(z - 1)] + [z (Az + B) / (z^ 2 - 6z + 25) ]
RHS quadratic term:
Now let z approach infinity z --> infinty
This is a little tricky so visualise it like this
z(Az + B) / ( denominator becomes infinity)
infinity A + B / (infinity) = infinity A/infinity + B/infinity
= A + zero (approaches zero)
RHS term = 2z/(z-1)
= 2 (infinity)/(infinity - 1) = 2 (infinity)/(infinity
= 2
LHS X(z) is X(infinity) = 0
LHS = RHS
0 = 2 + A
A = -2

The red term gives the result 2 in the solution at the end.

Next solve for B?

Substitute A = -2

2(3z + 17)/((z-1)(z^ 2-6z+25)) = [ 2/(z - 1)] + [(-2z + B) / (z^ 2 - 6z + 25) ]

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

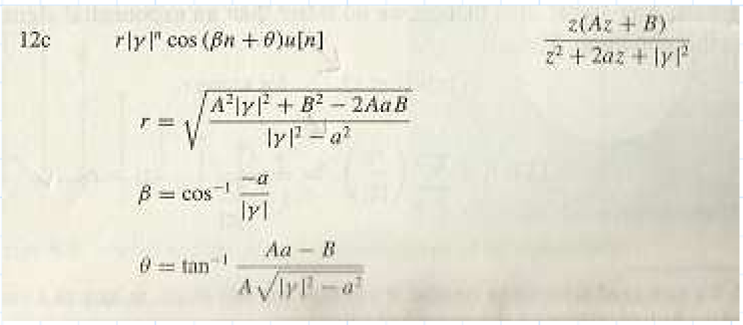
To solve for B, lets make z a convenient value to simplify the terms,
let z = 0, the terms reduce to

-34 /25 = -2 + B/25
B/25 = -34/25 + 50/25 = 16/25
B = 16

Substitute A and B in the equation

X(z)/z = [ 2/(z - 1)] + [(-2z + 16) / (z^ 2 - 6z + 25) ]
Multiply by z both sides

X(z)= [ 2z/(z - 1)] + [z (-2z + 16) / (z^ 2 - 6z + 25) ]


12c $\quad r|\gamma|^n \cos(\beta n + \theta)u[n] \qquad \dfrac{z(Az+B)}{z^2+2az+|\gamma|^2}$

$$r = \sqrt{\frac{A^2|\gamma|^2 + B^2 - 2AaB}{|\gamma|^2 - a^2}}$$

$$\beta = \cos^{-1}\frac{-a}{|\gamma|}$$

$$\theta = \tan^{-1}\frac{Aa - B}{A\sqrt{|\gamma|^2 - a^2}}$$

Use pair 12c from
Z transform table

$A := -2 \qquad B := 16 \qquad \gamma\_s := 25 \qquad \gamma := \sqrt{\gamma\_s} \qquad \gamma = 5 \qquad a := -3$

$$r := \sqrt{\frac{\left(A^2 \cdot \gamma^2 + B^2 - 2 \cdot A \cdot a \cdot B\right)}{\gamma^2 - a^2}} \qquad r = 3.2 \qquad \beta := acos\left(\frac{-(a)}{\gamma}\right) = 0.93\ rad$$

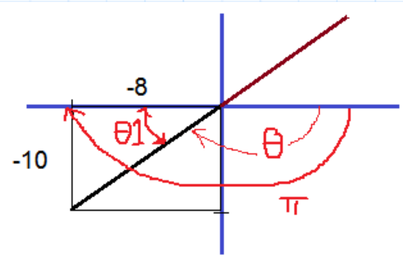
$$hyp := A \cdot \sqrt{\left(\gamma^2 - a^2\right)} = -8 \qquad opp := A \cdot a - B = -10$$

$$\theta 1 := atan\left(\frac{opp}{hyp}\right) = 0.896 \qquad \theta 1 = 0.896 \angle 0\ \pi$$

$\theta := \theta 1 - \pi = -2.246$ Correct (in clockwise ie -ve direction)

$$x(n) := r \cdot |\gamma|^n \cdot cos(\beta \cdot n + \theta)$$

x[n] = [2+ 3.2 (5^ n) cos(0.93n - -2.246)] u[n] Answer




thats a strange 0 deg for the theta1 so now from the figure above, subtracting theta 1 from pi gives the other angle in rad and that is the theta we seek.

Z Transform system representation

### 3.19 z-Transform System Representation

The z-transform can be used to describe a system. A typical system can be represented by the z-transform as shown in Figure 3.37.

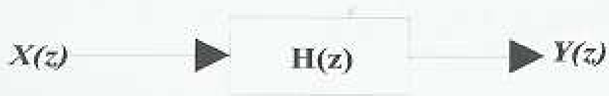



**Figure 3.37:** *A system represented in z-domain*

where

$$H(z) = \frac{Y(z)}{X(z)} \qquad \text{(Equ.3.13)}$$

Equ.3.13 represents the system transfer function in the z-domain. As mentioned previously, knowing the impulse response of the system in the time domain, we can use Equ.3.13 to calculate the output of the system. When we represent a system in the z-domain, we can use the unit delay, which is $z^{-n}$ ; for instance, one delay is represented as $z^{-1}$; for two delays, we use $z^{-2}$, and so. Figure 3.38 shows a system with one delay represented in the z-domain.

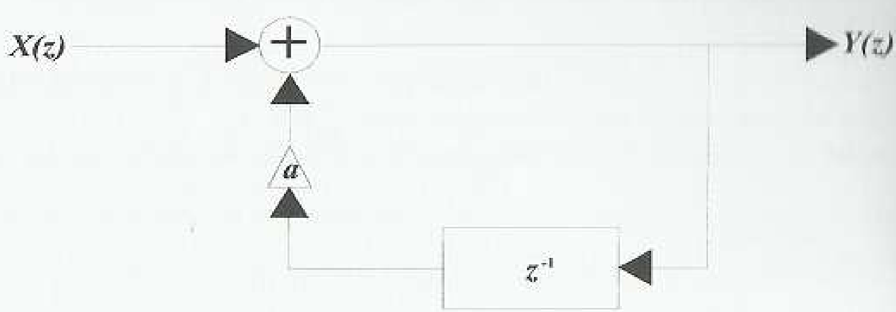



***Figure 3.38:** A system with a unit delay in the z-domain*

The same system representation used in Lapalce Transform, here z instead of s, where z is the discrete time domain.

How to do the inverse Z transform and the transfer function.

**Example 3.15**

Using the z-transform, find the output response of a system where both $x(n)$ and $h(n)$ are represented by the signals below.

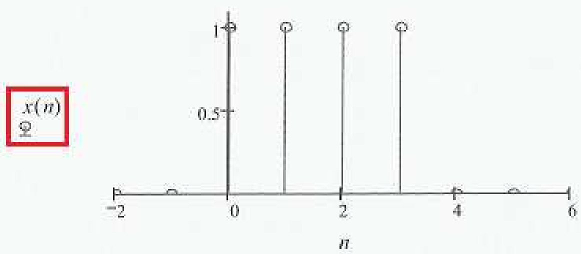



**Figure 3.39**: *Represents the input of a system*

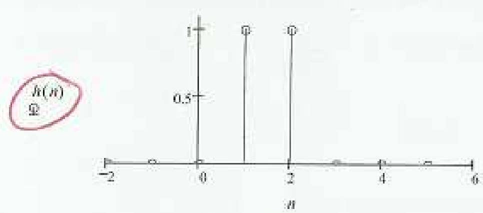



**Figure 3.40**: *Shows the impulse of the system*

Figure 3.39 indicates that $x(n)=\delta(n)+\delta(n-1)+\delta(n-2)+\delta(n-3)$, and Figure 3.40 indicates that $h(n)=\delta(n-1)+\delta(n-2)$. We can use MathCAD to find the z-transform of both $x(n)$ and $h(n)$, as shown below

x(n) $\Delta(n)+\Delta(n-1)+\Delta(n-2)+\Delta(n-3)$ has z transform $\dfrac{(z^3+z^2+z+1)}{z^3}$ ✓

h(n) $\Delta(n-1)+\Delta(n-2)$ has z transform $\dfrac{(z+1)}{z^2}$

Eq 3.13 → $H(z)=\dfrac{Y(z)}{X(z)}$

so $X(z)=\dfrac{z^3+z^2+z^1+1}{z^3}$ and $H(z)=\dfrac{z+1}{z^2}$

Now, to find the output response $y(n)$, we have to take the following steps: first, we multiply both $X(z)$ by $H(z)$ and take the inverse z-transform to get the time response.

$Y(z)=H(z)X(z)$

| Discrete Time | z-Domain |
|---|---|
| $y(n)=x(n)*h(n)$ | $Y(z)=X(z)H(z)$ |

Using MathCAD, we can multiply both terms together and find the inverse z-transform of $Y(z)$. Second, we try to simplify the terms by using **simplify** from the symbolic menu.

$$\frac{(z^3+z^2+z+1)}{z^3}\cdot\frac{(z+1)}{z^2} \quad \text{simplifies to} \quad \frac{(z^3+z^2+z+1)}{z^5}\cdot(z+1)$$

Third, we use **expand** to expand the expression to be suitable for us

$$\frac{(z^3+z^2+z+1)}{z^5}\cdot(z+1) \quad \text{expands to} \quad \frac{(z^4+2\cdot z^3+2\cdot z^2+2\cdot z+1)}{z^5}$$

this gives

$$Y(z)=\frac{z^4+2z^3+2z^3+2z+1}{z^5}$$

Now, we can take the inverse z-transform of $Y(z)$ by using MathCAD; the solution is

$$\frac{(z^4+2\cdot z^3+2\cdot z^2+2\cdot z+1)}{z^5} \quad \text{has inverse z transform} \quad \Delta(n-5)+2\cdot\Delta(n-4)+2\cdot\Delta(n-3)+2\cdot\Delta(n-2)+\Delta(n-1)$$

This solution is the same as $y(n)=\delta(n-5)+2\delta(n-4)+2\delta(n-3)+2\delta(n-2)+\delta(n-1)$.

By looking at the way MathCAD expands the expression above, we realize that it would be better to display the expression in a linear fashion by dividing the terms, which gives

$$Y(z)=z^{-1}+2z^{-2}+2z^{-3}+2z^{-4}+z^{-5}$$

We can use MathCAD to plot the result as shown in Figure 3.41.

$n := -10..10$ Defining the limit fo

$y(n) := \delta(n-5)+2\cdot\delta(n-4)+2\cdot\delta(n-3)+2\cdot\delta(n-2)+\delta(n-1)$ Defining $y(n)$ by copy and paste and change capital delta to small delta

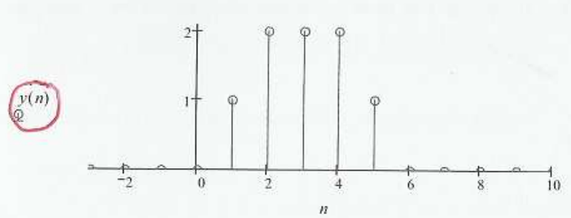



**Figure 3.41**: *Shows the response of the system*

This procedure is the same as doing the Laplace Transform.

Example - From the notes above
- Inverse Transform

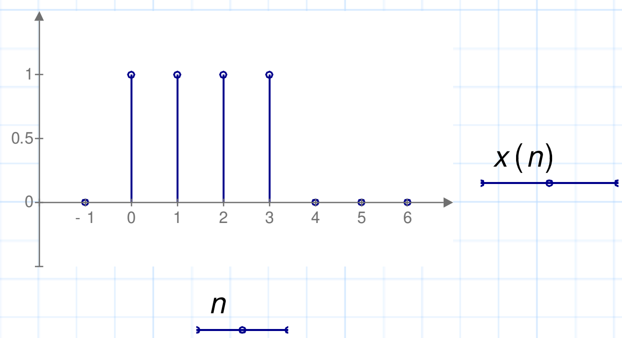
$clear(x)$

$n := -1 .. 6$ Remember this line has to be entered it defines n in the x(n) equation
The range of n is reduced for making the vector later shorter to fit the page

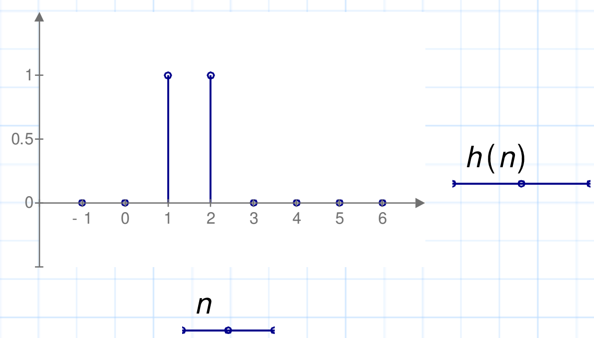
$a := 1$

$\delta(n) := if(n = 0, 1, 0)$

$x(n) := (a)\cdot\delta(n) + (a)\cdot\delta(n-1) + (a)\cdot\delta(n-2) + (a)\cdot\delta(n-3)$




$h(n) := (a)\cdot\delta(n-1) + (a)\cdot\delta(n-2)$




x(n) $\Delta(n) + \Delta(n-1) + \Delta(n-2) + \Delta(n-3)$ has z transform $\dfrac{(z^3 + z^2 + z + 1)}{z^3}$ ✓

h(n) $\Delta(n-1) + \Delta(n-2)$ has z transform $\dfrac{(z+1)}{z^2}$

The Z transforms above can be found as usual from the tables and the above transform then formed in a linear fashion.

$$x(n) \xrightarrow{ztrans} \begin{bmatrix} 0 \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad ztrans(x(n), n, z) \rightarrow \begin{bmatrix} ztrans(0,-1,z) \\ ztrans(1,0,z) \\ ztrans(1,1,z) \\ ztrans(1,2,z) \\ ztrans(1,3,z) \\ ztrans(0,4,z) \\ ztrans(0,5,z) \\ ztrans(0,6,z) \end{bmatrix}$$

Study the results above, for n each term its transform with the amplitude.

Right vector, the first x(n) results with the amplitude, the second, n, is the nth term shown, the third term z indicates the z transform applied.

Put both the vectors together to get its meaningful result

$$h(n) \xrightarrow{ztrans} \begin{bmatrix} 0 \\ 0 \\ \frac{z}{z-1} \\ \frac{z}{z-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad ztrans(h(n), n, z) \rightarrow \begin{bmatrix} ztrans(0,-1,z) \\ ztrans(0,0,z) \\ ztrans(1,1,z) \\ ztrans(1,2,z) \\ ztrans(0,3,z) \\ ztrans(0,4,z) \\ ztrans(0,5,z) \\ ztrans(0,6,z) \end{bmatrix}$$

$clear(X)$ $\qquad$ $clear(H)$

$X(z) := z^{0} + z^{-1} + z^{-2} + z^{-3}$ $\qquad$ Correct d(n-k) --> z^ -k

$H(z) := z^{-2} + z^{-1}$ $\qquad$ Correct d(n-k) --> z^ -k

Nothing wrong with this form of answer, but placing it in a linear form may have its benefit(s), especially when solving equations.

$$X(z) := \frac{\left(z^{3} + z^{2} + z^{1} + 1\right)}{z^{3}}$$

$$H(z) := \frac{\left(z^{1} + z^{0}\right)}{z^{2}} \qquad H(z) := \frac{\left(z^{1} + 1\right)}{z^{2}}$$

Next steps:
1. Multiply X(z) by Y(z)
2. Then the inverse z transform Y(z) = X(z) H(z)

After we multiply lets use a feature of Prime/Mathcad which will further improve the solution in with respect to the formation of the terms.

Apply:

1. Simplify
2. Expand

Y(z) = X(z) Y(z)

$$Y1(z) := \left(\frac{z^3 + z^2 + z^1 + 1}{z^3}\right) \cdot \left(\frac{(z^1 + 1)}{z^2}\right)$$

$$Y1(z) \xrightarrow{simplify} \frac{(z+1)^2 \cdot (z^2+1)}{z^5}$$

$$Y2(z) := \frac{(z+1)^2 \cdot (z^2+1)}{z^5}$$

$$Y2(z) \xrightarrow{expand} \frac{1}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \frac{2}{z^4} + \frac{1}{z^5}$$

Matches textbook

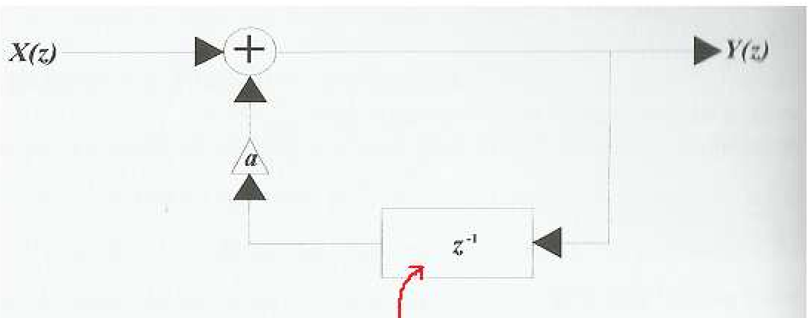


Figure 3.38: A system with a unit delay in the z-domain

See a similarity with the solution and the unit delay in the figure to the right

$$Y2(z) \xrightarrow{invztrans} \delta(n-1,0) + 2\cdot\delta(n-2,0) + 2\cdot\delta(n-3,0) + 2\cdot\delta(n-4,0) + \delta(n-5,0)$$

Is the above correct for the Z transform?

Yes, from the tables.
The 0 in each term would mean its 1 when d(n-k) = d(0) = 1

Next lets plot the result of the inverse Z transform

$$Y2(z) \xrightarrow{invztrans} \delta(n-1,0) + 2\cdot\delta(n-2,0) + 2\cdot\delta(n-3,0) + 2\cdot\delta(n-4,0) + \delta(n-5,0)$$

$clear(y)$

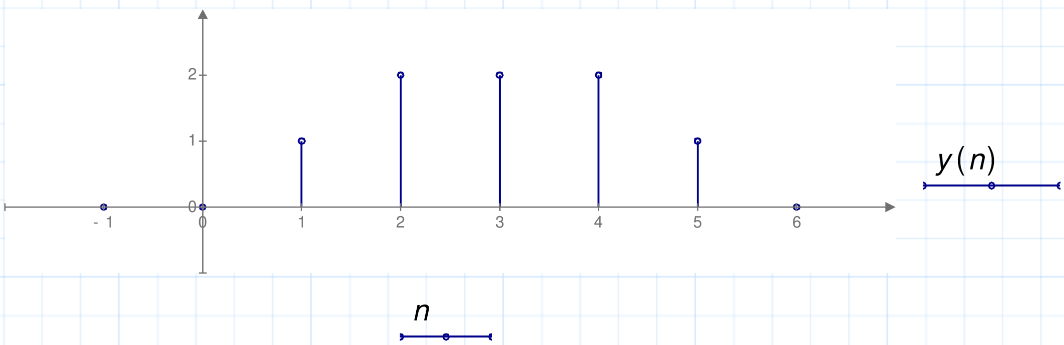
$n := -1 .. 6$

$\delta(n) := if(n = 0, 1, 0)$

Rewrite Y2(z) in 'n' --> y(n)

Make sure the delta d is re-typed if you copied the line from Y2(z) above otherwise it will not plot because the d there maynot be the d for the dirac/delta function.
Here it was retyped because it would not plot for the equation below.

$$y(n) := \delta(n-1) + 2\cdot\delta(n-2) + 2\cdot\delta(n-3) + 2\cdot\delta(n-4) + \delta(n-5)$$




Transfer Function In Z Domain.
(Similar to Laplace Transform Tasks)

## 3.20 Transfer Function in z-Domain

The previous section reveals that the system transfer function in the z-domain represents the output of the system divided by the input. With our knowledge of the Laplace transform, we can show that the system transfer function in the z-domain contains both poles and zeros, where

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots} \qquad \text{(Equ.3.14)}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots} \qquad \text{(Equ.3.15)}$$

From the above equations, we see clearly that the ***b***'s are the poles and ***a***'s are the zeros. With that knowledge, we can plot the pole-zero plot of a z-domain transfer function, as we did for the Laplace transform. We can also show the unit circle within the plot to get more information about the system and to see if the poles and zero's are located inside the circle. This will also give us more information about the region of convergence (ROC).

Similar to the example in Lapalace Transform.

**Example 3.16**

Generate the pole-zero plot of a system transfer function, where

$$H(z)=\frac{z^2-0.4z+0.16}{z^2+0.4z+0.36}$$

We can use MathCAD in several ways to find the poles and zeros of the transfer function. We can either partition the numerator and the denominator, or choose **variable solve** from the symbolics paletter to get the poles and zeros, or use the **poliroots** function. For this example, let's use the **poliroots** function approach.

$$H(z) := \frac{z^2-0.4\cdot z+0.16}{z^2+0.4z+0.36}$$ Defining the transfer function

From the transfer function, we see that

$X(z) := z^2 + 0.4z + 0.36$ Defining the input

$Y(z) := z^2 - 0.4z + 0.16$ Defining the output

Now we can find the root by putting the input and the output in proper order. The higher order goes to the end of the vector.

$$Z := \begin{pmatrix} 0.16 \\ -0.4 \\ 1 \end{pmatrix}$$ Defining the output polynomial as a vector for the zer

$P := \begin{pmatrix} 0.36 \\ 0.4 \\ 1 \end{pmatrix}$ Defining the input polynomial as a vector for the poles

We can use now the **polyroots** function to find the poles and the zeros

$Z := polyroots(Z)$ To find the zeros

$P := polyroots(P)$ To find the poles

$Z = \begin{pmatrix} 0.2 + 0.346i \\ 0.2 - 0.346i \end{pmatrix}$ Result of the zeros

$P = \begin{pmatrix} -0.2 - 0.566i \\ -0.2 + 0.566i \end{pmatrix}$ Result of the poles

Next, we plot the poles and the zeroes for the transfer function

$n := 0..\,length(Z) - 1$ Defining how many zeros

$m := 0..\,length(P) - 1$ Defining how many poles

The pole-zero plot shows the real of each parts **vs** the imaginary of each part. To get a plot identical to the one below, you must go to **plot format** and change your trace to **point.**

Now we will demonstrate the above example in Prime

Example 3.16 Tutorial Textbook.

$H(z) := \dfrac{z^2 - 0.4 \cdot z + 0.16}{z^2 + 0.4 \cdot z + 36}$ Transfer function

$X(z) := \left(z^2 + 0.4 \cdot z + 36\right)$ Input function

$Y(z) := z^2 - 0.4 \cdot z + 0.16$ Output function

Set the coefficients of each function into a vector, for the zeros and poles

$ORIGIN := 1$

$X_{zero\_coef} := \begin{bmatrix} 0.16 \\ -0.4 \\ 1 \end{bmatrix}$ $Y_{pole\_coef} := \begin{bmatrix} 0.36 \\ 0.4 \\ 1 \end{bmatrix}$

Now use the polyroots function to find the poles and zeros.

$X_{zero} := polyroots(X_{zero\_coef})$ $\quad$ $Y_{pole} := polyroots(Y_{pole\_coef})$

$X_{zero} = \begin{bmatrix} 0.2+0.3464j \\ 0.2-0.3464j \end{bmatrix}$ $\quad$ $Y_{pole} = \begin{bmatrix} -0.2-0.5657j \\ -0.2+0.5657j \end{bmatrix}$

Next plot the poles and zeros for the transfer function:

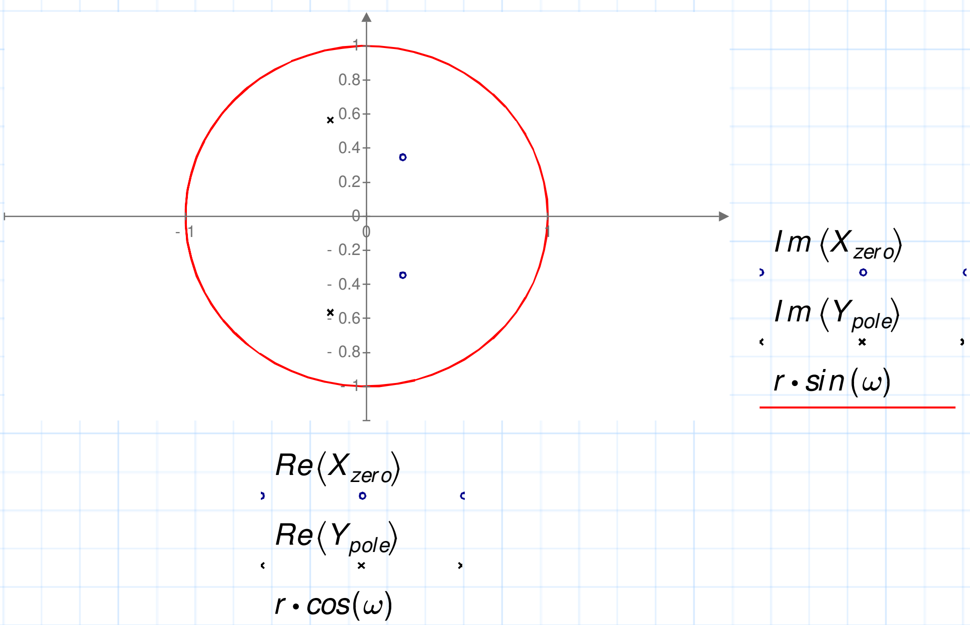
Since the roots were 1 less then number of coefficients (3),
this has to be adjusted for the plot:

$length(X_{zero\_coef}) = 3$ $\quad$ $length(Y_{pole\_coef}) = 3$

$n := 0..length(X_{zero\_coef})$ $\quad$ $m := 0..length(Y_{pole\_coef})$

Now want to show the unit circle in the plot.
To do this define a function for the unit circle as sine and cosine

$r := 1$ radius of circle
$\omega := -4, -3.9..4$




Poles and zeros are in the unit circle - which is the unit delay z^ -1

Frequency response of a system in Z domain

## 5.5 Frequency Response of Discrete-Time Systems

For (asymptotically or BIBO-stable) continuous-time systems, we showed that the system response to an input $e^{j\omega t}$ is $H(j\omega)e^{j\omega t}$ and that the response to an input $\cos \omega t$ is $|H(j\omega)| \cos [\omega t + \angle H(j\omega)]$. Similar results hold for discrete-time systems. We now show that for an (asymptotically or BIBO-stable) LTID system, the system response to an input $e^{j\Omega n}$ is $H[e^{j\Omega}]e^{j\Omega n}$ and the response to an input $\cos \Omega n$ is $|H[e^{j\Omega}]| \cos (\Omega n + \angle H[e^{j\Omega}])$.

The proof is similar to the one used for continuous-time systems. In Section 3.8-3, we showed that an LTID system response to an (everlasting) exponential $z^n$ is also an (everlasting) exponential $H[z]z^n$. This result is valid only for values of $z$ for which $H[z]$, as defined in Eq. (5.14a), exists (converges). As usual, we represent this input–output relationship by a directed arrow notation as

$$z^n \Longrightarrow H[z]z^n \tag{5.42}$$

Setting $z = e^{j\Omega}$ in this relationship yields

$$e^{j\Omega n} \Longrightarrow H[e^{j\Omega}]e^{j\Omega n} \tag{5.43}$$

Noting that $\cos \Omega n$ is the real part of $e^{j\Omega n}$, use of Eq. (3.66b) yields

$$\cos \Omega n \Longrightarrow \text{Re}\,\{H[e^{j\Omega}]e^{j\Omega n}\} \tag{5.44}$$

Expressing $H[e^{j\Omega}]$ in the polar form

$$H[e^{j\Omega}] = |H[e^{j\Omega}]|e^{j\angle H[e^{j\Omega}]} \tag{5.45}$$

Eq. (5.44) can be expressed as

$$\cos \Omega n \Longrightarrow |H[e^{j\Omega}]| \cos (\Omega n + \angle H[e^{j\Omega}])$$

In other words, the system response $y[n]$ to a sinusoidal input $\cos \Omega n$ is given by

$$y[n] = |H[e^{j\Omega}]| \cos (\Omega n + \angle H[e^{j\Omega}]) \tag{5.46a}$$

Following the same argument, the system response to a sinusoid $\cos (\Omega n + \theta)$ is

$$y[n] = |H[e^{j\Omega}]| \cos (\Omega n + \theta + \angle H[e^{j\Omega}]) \tag{5.46b}$$

This result is valid only for BIBO-stable or asymptotically stable systems. The frequency response is meaningless for BIBO-unstable systems (which include marginally stable and asymptotically unstable systems). This follows from the fact that the frequency response in Eq. (5.44) is obtained by setting $z = e^{j\Omega}$ in Eq. (5.42). But, as shown in Section 3.8-3 [Eqs. (3.71)], the relationship (5.42) applies only for values of $z$ for which $H[z]$ exists. For BIBO unstable systems, the ROC for $H[z]$ does not include the unit circle where $z = e^{j\Omega}$. This means, for BIBO-unstable systems, that $H[z]$ is meaningless when $z = e^{j\Omega}$.†

This important result shows that the response of an asymptotically or BIBO-stable LTID system to a discrete-time sinusoidal input of frequency $\Omega$ is also a discrete-time sinusoid of the same frequency. *The amplitude of the output sinusoid is* $|H[e^{j\Omega}]|$ *times the input amplitude, and the phase of the output sinusoid is shifted by* $\angle H[e^{j\Omega}]$ *with respect to the input phase.* Clearly $|H[e^{j\Omega}]|$ is the amplitude gain, and a plot of $|H[e^{j\Omega}]|$ versus $\Omega$ is the amplitude response of the discrete-time system. Similarly, $\angle H[e^{j\Omega}]$ is the phase response of the system, and a plot of $\angle H[e^{j\Omega}]$ versus $\Omega$ shows how the system modifies or shifts the phase of the input sinusoid. Note that $H[e^{j\Omega}]$ incorporates the information of both amplitude and phase responses and therefore is called the *frequency responses* of the system.

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

## 3.21 Frequency Response in the $z$-Domain

In a similar manner to the Laplace transform frequency response, we can also plot the frequency response of a system in the z-domain. The process of plotting the transfer function of a system in z-domain is to take the transfer function and replace $z$ by $e^{j\omega T}$ in the transfer function, where T is the sampling time. As an example, let's plot the frequency response of a digital system, where

$$H(z)=\frac{z}{z-a} \text{ and } a=\frac{1}{2}$$

The sampling time is $T=\frac{2\pi}{\omega_s}$

Example 3.17 - Tutorial Book

$a:=\frac{1}{2}$ defining a constant

$H(z):=\frac{z}{z-a}$ defining a transfer function

$\omega_s:=6\cdot\pi$ sampling frequency

$T:=\frac{2\cdot\pi}{\omega_s}$ sampling rate

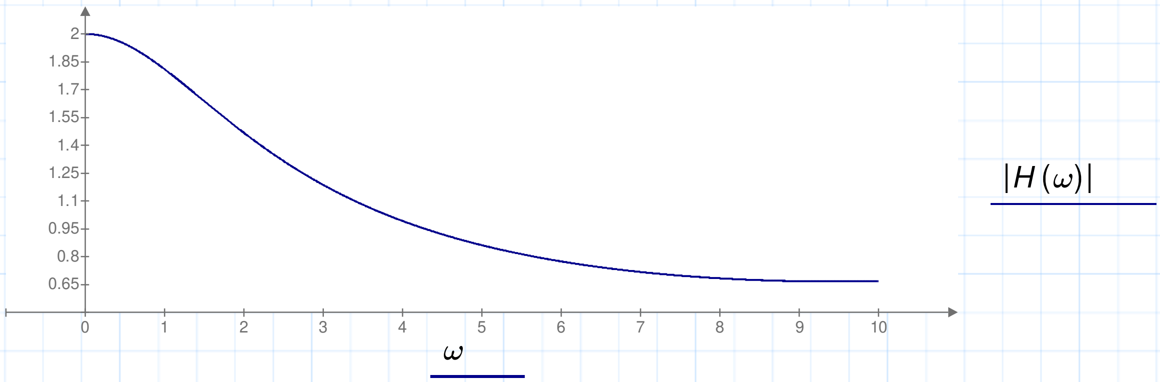
$j:=\sqrt{-1}$

$\omega:=0,0.01..10$

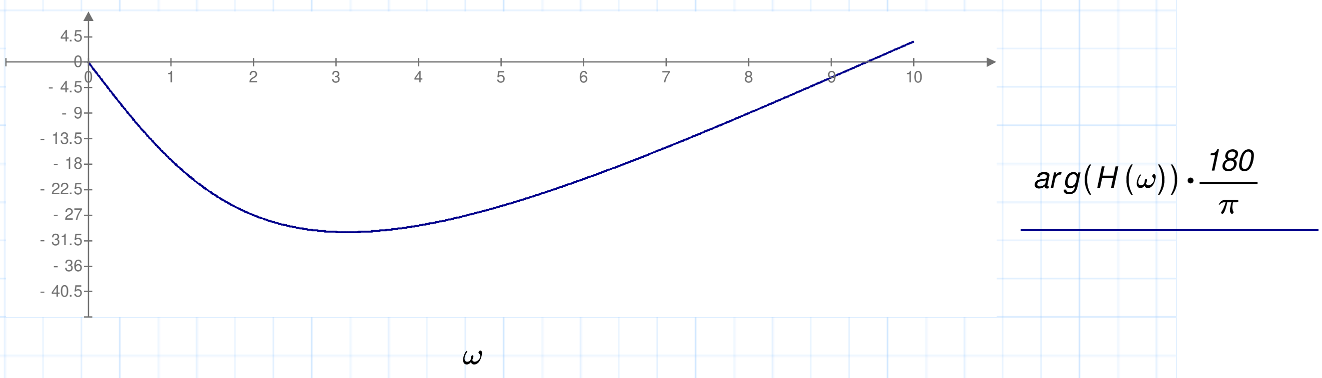
$$H(\omega):=\frac{e^{j\cdot\omega\cdot T}}{e^{j\cdot\omega\cdot T}-\frac{1}{2}}$$

Next the plots:

Plot below the transfer function magnitude of the system

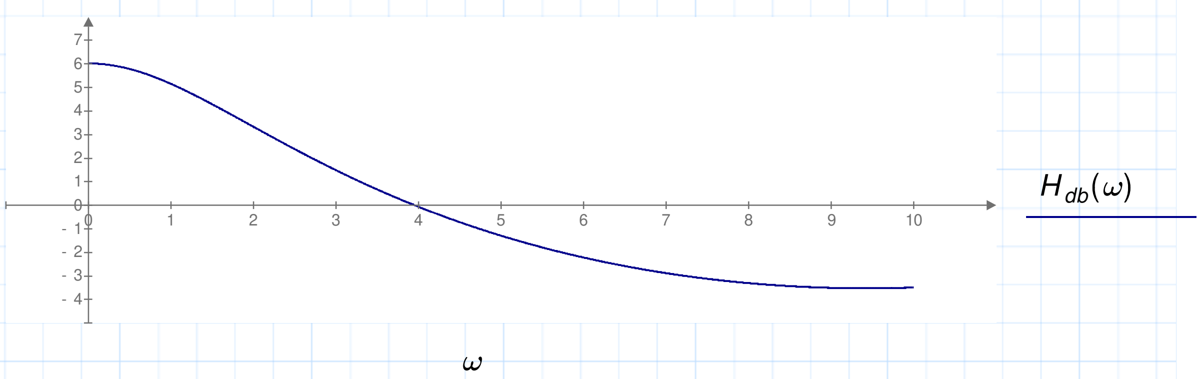



Plot below the transfer function phase angle of the system in degrees (180/pi)




Plot the magnitude in dB

$$H_{db}(\omega) := 20 \cdot \log(|H(\omega)|)$$



Summary of Laplace Transforms:

## 4.12 SUMMARY

This chapter discusses analysis of LTIC (linear, time-invariant, continuous-time) systems by the Laplace transform, which transforms integro-differential equations of such systems into algebraic equations. Therefore solving these integro-differential equations reduces to solving algebraic equations. The Laplace transform method cannot be used for time-varying-parameter systems or for nonlinear systems in general.

The transfer function $H(s)$ of an LTIC system is the Laplace transform of its impulse response. It may also be defined as a ratio of the Laplace transform of the output to the Laplace transform of the input when all initial conditions are zero (system in zero state). If $X(s)$ is the Laplace transform of the input $x(t)$ and $Y(s)$ is the Laplace transform of the corresponding output $y(t)$ (when all initial conditions are zero), then $Y(s) = X(s)H(s)$. For an LTIC system described by an $N$th-order differential equation $Q(D)y(t) = P(D)x(t)$, the transfer function $H(s) = P(s)/Q(s)$. Like the impulse response $h(t)$, the transfer function $H(s)$ is also an external description of the system.

Electrical circuit analysis can also be carried out by using a transformed circuit method, in which all signals (voltages and currents) are represented by their Laplace transforms, all elements by their impedances (or admittances), and initial conditions by their equivalent sources (initial condition generators). In this method, a network can be analyzed as if it were a resistive circuit.

Large systems can be depicted by suitably interconnected subsystems represented by blocks. Each subsystem, being a smaller system, can be readily analyzed and represented by its input–output relationship, such as its transfer function. Analysis of large systems can be carried out with the knowledge of input–output relationships of its subsystems and the nature of interconnection of various subsystems.

LTIC systems can be realized by scalar multipliers, adders, and integrators. A given transfer function can be synthesized in many different ways, such as canonic, cascade, and parallel. Moreover, every realization has a transpose, which also has the same transfer function. In practice, all the building blocks (scalar multipliers, adders, and integrators) can be obtained from operational amplifiers.

The system response to an everlasting exponential $e^{st}$ is also an everlasting exponential $H(s)e^{st}$. Consequently, the system response to an everlasting exponential $e^{j\omega t}$ is $H(j\omega)\,e^{j\omega t}$. Hence, $H(j\omega)$ is the frequency response of the system. For a sinusoidal input of unit amplitude and having frequency $\omega$, the system response is also a sinusoid of the same frequency ($\omega$) with amplitude $|H(j\omega)|$, and its phase is shifted by $\angle H(j\omega)$ with respect to the input sinusoid. For this reason $|H(j\omega)|$ is called the amplitude response (gain) and $\angle H(j\omega)$ is called the phase response of the system. Amplitude and phase response of a system indicate the filtering characteristics of the system. The general nature of the filtering characteristics of a system can be quickly determined from a knowledge of the location of poles and zeros of the system transfer function.

Most of the input signals and practical systems are causal. Consequently we are required most of the time to deal with causal signals. When all signals must be causal, the Laplace transform analysis is greatly simplified; the region of convergence of a signal becomes irrelevant to the analysis process. This special case of the Laplace transform (which is restricted to causal signals) is called the unilateral Laplace transform. Much of the chapter deals with this variety of Laplace transform. Section 4.11 discusses the general Laplace transform (the bilateral Laplace transform), which can handle causal and noncausal signals and systems. In the bilateral transform, the inverse transform of $X(s)$ is not unique but depends on the region of convergence of $X(s)$. Thus the region of convergence plays a very crucial role in the bilateral Laplace transform.

Summary of Fourier Transforms.

## 6.7 SUMMARY

In this chapter we showed how a periodic signal can be represented as a sum of sinusoids or exponentials. If the frequency of a periodic signal is $f_0$, then it can be expressed as a weighted sum of a sinusoid of frequency $f_0$ and its harmonics (the trigonometric Fourier series). We can reconstruct the periodic signal from a knowledge of the amplitudes and phases of these sinusoidal components (amplitude and phase spectra).

If a periodic signal $x(t)$ has an even symmetry, its Fourier series contains only cosine terms (including dc). In contrast, if $x(t)$ has an odd symmetry, its Fourier series contains only sine terms. If $x(t)$ has neither type of symmetry, its Fourier series contains both sine and cosine terms.

At points of discontinuity, the Fourier series for $x(t)$ converges to the mean of the values of $x(t)$ on either sides of the discontinuity. For signals with discontinuities, the Fourier series converges in the mean and exhibits Gibbs phenomenon at the points of discontinuity. The amplitude spectrum of the Fourier series for a periodic signal $x(t)$ with jump discontinuities decays slowly (as $1/n$) with frequency. We need a large number of terms in the Fourier series to approximate $x(t)$ within a given error. In contrast, smoother periodic signal amplitude spectrum decays faster with frequency and we require a fewer number of terms in the series to approximate $x(t)$ within a given error.

A sinusoid can be expressed in terms of exponentials. Therefore the Fourier series of a periodic signal can also be expressed as a sum of exponentials (the exponential Fourier series). The exponential form of the Fourier series and the expressions for the series coefficients are more compact than those of the trigonometric Fourier series. Also, the response of LTIC systems to an exponential input is much simpler than that for a sinusoidal input. Moreover, the exponential form of representation lends itself better to mathematical manipulations than does the trigonometric form. For these reasons, the exponential form of the series is preferred in modern practice in the areas of signals and systems.

The plots of amplitudes and angles of various exponential components of the Fourier series as functions of the frequency are the exponential Fourier spectra (amplitude and angle spectra) of the signal. Because a sinusoid $\cos \omega_0 t$ can be represented as a sum of two exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$, the frequencies in the exponential spectra range from $\omega = -\infty$ to $\infty$. By definition, frequency of a signal is always a positive quantity. Presence of a spectral component of a negative frequency $-n\omega_0$ merely indicates that the Fourier series contains terms of the form $e^{-jn\omega_0 t}$. The spectra of the trigonometric and exponential Fourier series are closely related, and one can be found by the inspection of the other.

In Section 6.5 we discuss a method of representing signals by the generalized Fourier series, of which the trigonometric and exponential Fourier series are special cases. Signals are vectors in every sense. Just as a vector can be represented as a sum of its components in a variety of ways, depending on the choice of the coordinate system, a signal can be represented as a sum of its components in a variety of ways, of which the trigonometric and exponential Fourier series are only two examples. Just as we have vector coordinate systems formed by mutually orthogonal vectors, we also have signal coordinate systems (basis signals) formed by mutually orthogonal signals. Any signal in this signal space can be represented as a sum of the basis signals. Each set of basis signals yields a particular Fourier series representation of the signal. The signal is equal to its Fourier series, not in the ordinary sense, but in the special sense that the energy of the difference between the signal and its Fourier series approaches zero. This allows for the signal to differ from its Fourier series at some isolated points.

Entered by: Karl S Bogha Dhaliwal - Grad Cert Power Systems Protection and Relaying Uni of Idaho. USA.
BSE - Arkansas State U 1990. BSc - USAO Oklahoma 1986.

Summary of Z Transforms

## 5.10 SUMMARY

In this chapter we discussed the analysis of linear, time-invariant, discrete-time (LTID) systems by means of the $z$-transform. The $z$-transform changes the difference equations of LTID systems into algebraic equations. Therefore, solving these difference equations reduces to solving algebraic equations.

The transfer function $H[z]$ of an LTID system is equal to the ratio of the $z$-transform of the output to the $z$-transform of the input when all initial conditions are zero. Therefore, if $X[z]$ is the $z$-transform of the input $x[n]$ and $Y[z]$ is the $z$-transform of the corresponding output $y[n]$ (when all initial conditions are zero), then $Y[z] = H[z]X[z]$. For an LTID system specified by the difference equation $Q[E]y[n] = P[E]x[n]$, the transfer function $H[z] = P[z]/Q[z]$. Moreover, $H[z]$ is the $z$-transform of the system impulse response $h[n]$. We showed in Chapter 3 that the system response to an everlasting exponential $z^n$ is $H[z]z^n$.

We may also view the $z$-transform as a tool that expresses a signal $x[n]$ as a sum of exponentials of the form $z^n$ over a continuum of the values of $z$. Using the fact that an LTID system response to $z^n$ is $H[z]z^n$, we find the system response to $x[n]$ as a sum of the system's responses to all the components of the form $z^n$ over the continuum of values of $z$.

LTID systems can be realized by scalar multipliers, adders, and time delays. A given transfer function can be synthesized in many different ways. We discussed canonical, transposed canonical, cascade, and parallel forms of realization. The realization procedure is identical to that for continuous-time systems with $1/s$ (integrator) replaced by $1/z$ (unit delay).

In Section 5.8, we showed that discrete-time systems can be analyzed by the Laplace transform as if they were continuous-time systems. In fact, we showed that the $z$-transform is the Laplace transform with a change in variable.

The majority of the input signals and practical systems are causal. Consequently, we are required to deal with causal signals most of the time. Restricting all signals to the causal type greatly simplifies $z$-transform analysis; the ROC of a signal becomes irrelevant to the analysis process. This special case of $z$-transform (which is restricted to causal signals) is called the unilateral $z$-transform. Much of the chapter deals with this transform. Section 5.9 discusses the general variety of the $z$-transform (bilateral $z$-transform), which can handle causal and noncausal signals and systems. In the bilateral transform, the inverse transform of $X[z]$ is not unique, but depends on the ROC of $X[z]$. Thus, the ROC plays a crucial role in the bilateral $z$-transform.

End of File.