MA540 Project 6

James Savino

April 30, 2021

Problem 1

Statement

Consider the Dittus-Boelter equation

$$Nu = \theta_1 Re^{\theta_2} Pr^{\theta_3}$$

where Nu, Re, and Pr respectively denote the Nusselt, Reynolds, and Prandtl numbers. Reported nominal parameter values are $\theta = [0.023, 0.8, 0.4]$ and data is provided in the file db_data.txt.

Construct the Fisher information matrix and discuss the identifiability of the parameters. Now use DRAM to compute posterior densities for the parameters. Are your pairwise plots consistent with the Fisher information results?

Solution

To construct the Fisher information matrix, the sensitivity matrix, χ must first be constructed. This was done using the complex step method with $h = 1e^{-13}$ to compute the partial derivatives of Nu with respect to eat of the parameters. The Fisher information matrix is then constructed using $F = \chi^T \chi$. The resulting 3x3 matrix is found to be

$$F = 10^8 \times \begin{bmatrix} 25.1 & 5.43 & 2.20 \\ 5.43 & 1.18 & 0.465 \\ 2.20 & 0.465 & 0.214 \end{bmatrix}$$

We then calculate the eigenvalues of the Fisher information matrix in order to gain insight into the identifiability of the parameters. In general, a small eigenvalue will indicate a lower degree of identifiability. For this system, we expect there to be limited identifiability for at least one, likely two parameters because the input values they affect are multiplied by each other for θ_2 and θ_3, and those same input values are multiplied by θ_1. We find the eigenvalues to be $\lambda = 10^5 \times [2.65e4, 2.78, 27.2]$. We note that while none of the eigenvalues are small, there is significant difference between the magnitudes of them, especially for λ_1 and λ_2. We find the associated eigenvector for λ_2 to be

$$\boldsymbol{v} = [-0.210, 0.835, -0.507]$$

Since the value corresponding to θ_2 is the largest value, we can say that it is the least identifiable. We can then repeat the process by fixing θ_2 to its nominal value of 0.8. In doing so, we obtain the following Fisher information matrix:

$$F = 10^8 \times \begin{bmatrix} 25.1 & 2.20 \\ 2.20 & 2.14 \end{bmatrix}$$

We repeat the same process and find that the eigenvalues are $\lambda = 10^6 \times [2.65e3, 2.06]$. This difference in magnitude is not as large as in the three-parameter matrix, so we do not further test the identifiability of the parameters.

Since the analysis from the Fisher information matrix is not definitive – even in the three-parameter matrix, the largest difference in eigenvalues was only on the order of 10^4 – we continue our analysis using DRAM. We calculate the residuals between the data and model to determine σ^2 and multiply that by the inverse of the Fisher information matrix in order to obtain the covariance matrix. We feed σ^2 and V into the pymcmcstat package to obtain more accurate results. We use the nominal parameter values provided as initial guesses and set the number of samples to 15000. The results from the DRAM simulation are summarized in Figures 1, 2, and 3. The parameters are estimated to be $\theta = [0.00509, 0.97, 0.40]$.

In Figure 1, we show the parameter chains. We note that the chains do not do a good job of exploring the space, with strong resistance occurring above or below certain thresholds. This may happen because one or more of the parameters are unidentifiable. We note from the parameter chains that there appears to be an inverse correlation between θ_1 and θ_2. This observation is corroborated by the pairwise plots in Figure 2, which shows a strong negative correlation between the two. We also note that the other parameter combinations also show some degree of correlation, albeit not as strong as shown between θ_1 and θ_2.

To further investigate, we observe the densities for each of the parameters in Figure 3. There is significant skewing or asymmetry in all of the densities, and none appear to be well-sampled. The extremely strong correlation between θ_1 and θ_2 seems to agree with the results from the analysis of the Fisher information matrix. We can fix $\theta_2 = 0.8$ and rerun the DRAM simulation to see if there are still issues with the two-parameter model. The results for this are shown in Figures 4, 5, and 6. We note that the parameter space is more well-explored in the two-parameter model and that the densities are much cleaner with only very mild skewing. There does still appear to be some correlation, but it is nowhere near as strong as in the previous run. It appears that the DRAM simulation agrees with our observations of the Fisher information matrix.

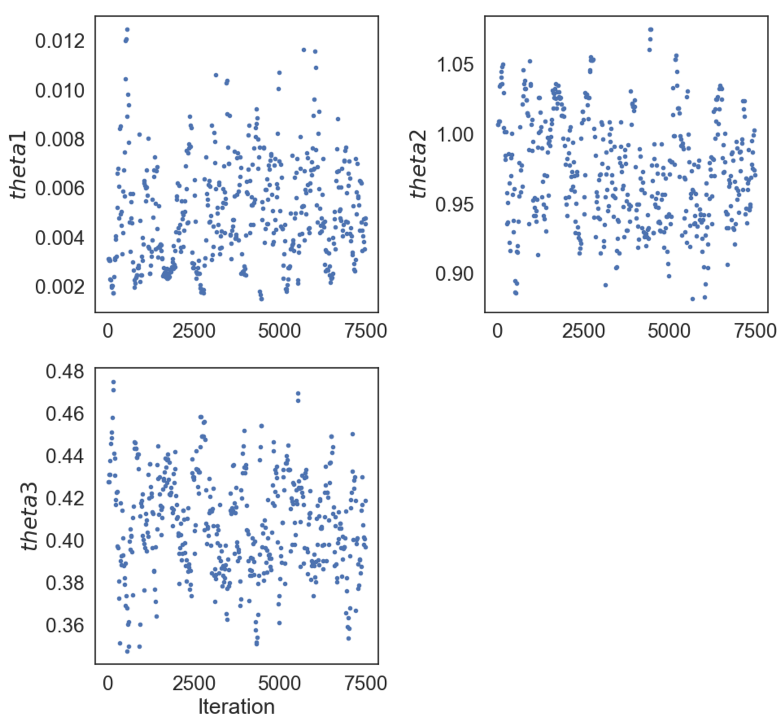

Figure 1: Chains for the three-parameter model.

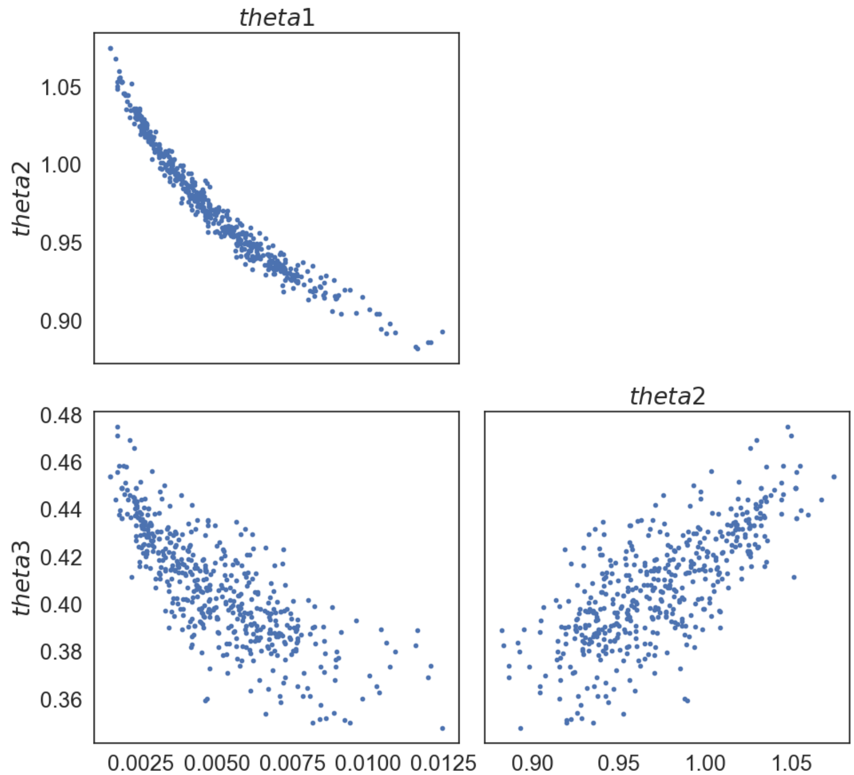

Figure 2: Pairwise plots for the three-parameter model.

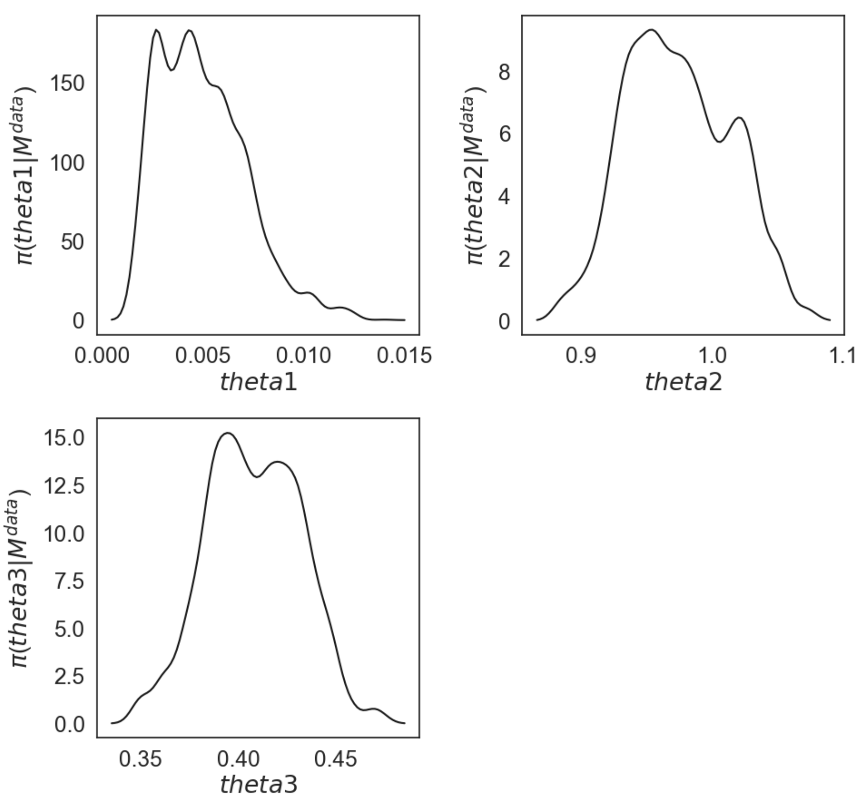

Figure 3: Densities for the three-parameter model.

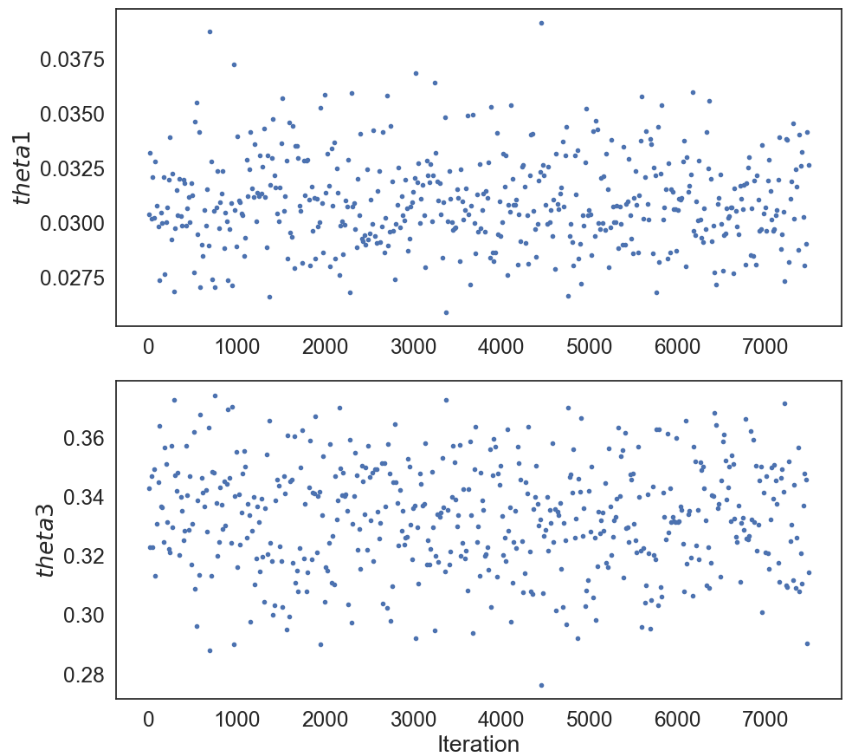

Figure 4: Chains for the two-parameter model.

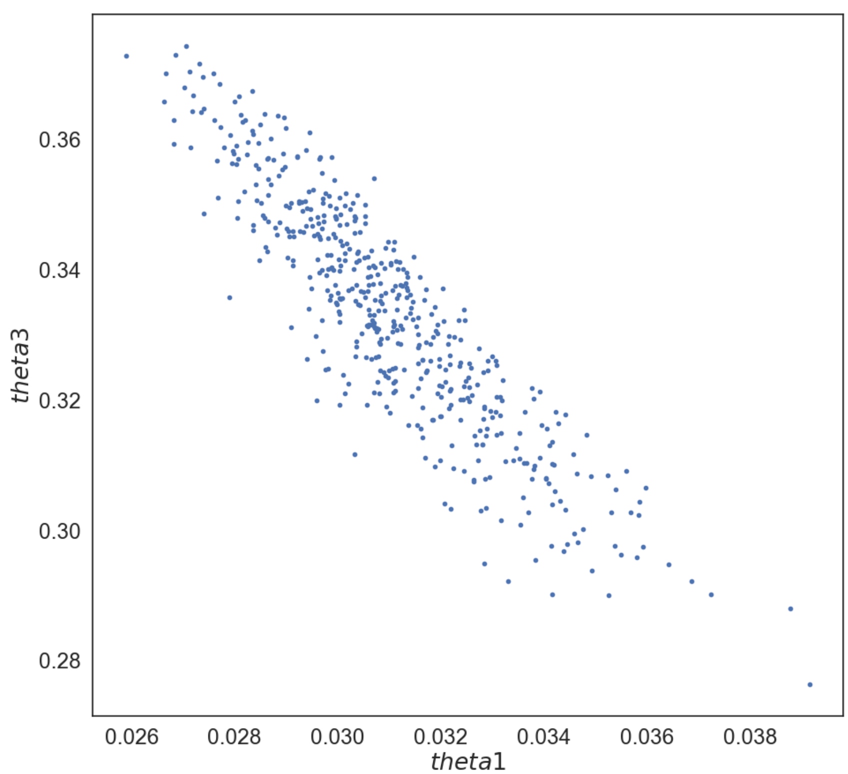

Figure 5: Pairwise plot for the two-parameter model.

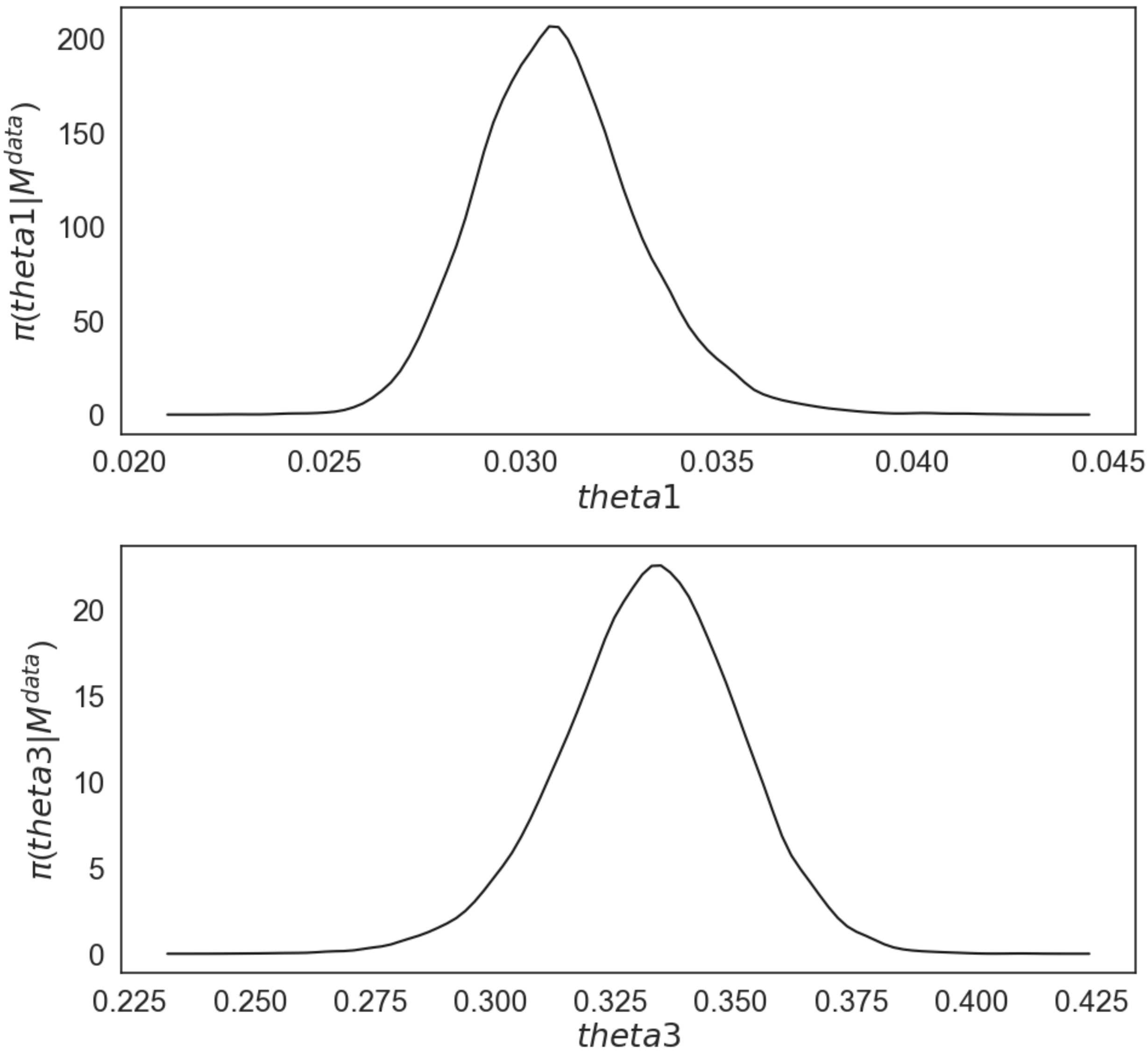

Figure 6: Densities for the two-parameter model.