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# Module 1 Summary

## Response to feedback

Being my first submission, I'm eager to receive feedback on my handwritten documentation, if there's anything I could clean up, proof more succinctly, etc.

## Module Learning Objectives

I certify that I achieved the following learning objectives for the module:

- Apply elementary row operations to systems of linear equations represented in matrix form
- Apply the Gaussian Algorithm to reduce a matrix to row-echelon form
- Express the general solution to a homogeneous equation as a linear combination of sets of basic solutions

## Key Definitions and Theorems

Solutions and Elementary Operations

| Textbook Ref | Theorem/Definition statement | Learning Notes |
|---|---|---|
| **Linear equations** | An equation of the form $$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$ is called a <u>linear equation</u> in the $n$ variables $x_1$, $x_2$, …, $x_n$. With $a_1$, $a_2$, …, $a_n$ referred to as coefficients and $b$ as the constant term (or sometimes 'right-hand side'). | An equation where each term is expressed as either a constant or a variable multiplied by a constant.<br><br>**Variables** \| the unknown values to solve for<br>**Coefficients** \| the multiplying factor for each variable (we assume 1 if only variable and no constant)<br>**Constant Term** \| typically the RHS, this is the constant value which describes what the LHS equates to.<br><br>The relationship described by a linear equation can be represented as a straight line in a 2D cartesian plane, where the change of value for one variable is directly proportional to change in other variables. |

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| **System of linear equations** | A finite collection of linear equations defined with respect to the same variables is called a <u>system of linear equations</u>. | Two or more linear equations with the same variables describe a linear system. This lets us solve problems with multiple unknown values, unknown relationships, related linearly.<br><br>**Solution:** Find the values for each variable which satisfy all equations at the same time. This leads to one of three results:<br><br>**i.** Exactly one solution, one intersecting point on a 2D graph<br><br>**ii.** No solution, inconsistent equations indicating the lines would all be parallel, not intersecting<br><br>**iii.** Infinitely many solutions, where each line is coincident (positioned the same on a 2D plane), meaning the equations are all dependant on each other. |
| **Solutions** | A set of values $s_1, s_2, \dots, s_n$ is a <u>solution</u> to a linear equation if setting $x_1=s_1, x_2=s_2, \dots, x_n=s_n$ allows for the equation to hold (the left-hand side is equal to the right-hand side, $b$).<br><br>A solution to a system of equations requires the solution to hold for all equations in the system. | A value or a set which, when substituted for the variables in the original equation, reads true. The number(s) should each fit in the equation such that the LHS component equals the constant term on the RHS.<br><br>If the equation is not true, meaning that the equation presents an unequal LHS and RHS, then there is no solution.<br><br>A solution to a system of linear equations infers that this must be true for all equations simultaneously, where one result should satisfy every linear relationship in the system. |
| **Augmented matrix** | A system of equations can be represented as an array of numbers consisting of the coefficients and constant terms, e.g., the system<br>$$3x_1+2x_2-x_3=3$$ | A mathematical mechanism to represent a system of linear equations as a 2D collection of numbers. The 2D collection depicts all |

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| | $$-x_1+4x_3=-6$$ Could be represented as the matrix, $$\left[\begin{array}{ccc\|c} 3 & 2 & -1 & 3 \\ -1 & 0 & 4 & -6 \end{array}\right]$$ Which will later be denoted [ A \| **b** ] where A is the array of coefficients and **b** is the column of constants on the right-hand side. | coefficients (all columns except the last) for each equation (rows), followed by their constant term (final column of each row). The resulting matrix depicts each equation in a cols*rows grid, its augmented form. This opens up a lot of possibilities for performing operations on the whole system, providing a compact way for solving systems conveniently via row operation rules. |
| **Definition 1.2** Elementary row operations | The following operations, called <u>elementary row operations</u> on a matrix.<br>I. Interchange two rows.<br>II. Multiply one row by a nonzero number.<br>III. Add a multiple of one row to a different row | These are rules outlining what can be done on a matrix to reduce it, carry it to other forms like REF and RREF<br><br>**i.** Swap two rows<br>**ii.** Multiply a row's elements by x where x != 0<br>**iii.** Multiply a row's elements by a multiple of another row's corresponding elements |

## Gaussian Elimination

| Textbook Ref | Theorem/Definition statement | Notes |
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| **Definition 1.3** Row-Echelon Form (Reduced) | A matrix is said to be in <u>row-echelon form</u> (and will be called a row-echelon matrix) if it satisfies the following three conditions:<br>1. All zero rows (consisting entirely of zeros) are at the bottom.<br>2. The first nonzero entry from the left in each nonzero row is a 1, called the <u>leading 1</u> for that row.<br>3. Each leading 1 is to the right of all leading 1s in the rows above it.<br>A row-echelon matrix is said to be in <u>reduced row-echelon form</u> (and will be called a reduced row-echelon matrix) if, in addition, it satisfies the following condition:<br>4. Each leading 1 is the only nonzero entry in its column. | RREF is a matrix form that aims to simplify matrix that's in REF (not reduced). REF depicts a system in a matrix with an upper-right triangle of values, with the lower-left consisting only of 0s, and leading 1's as the pivot (first value) for each row.<br><br>RREF takes this one step further by satisfying one more condition: Each leading 1 is the only non-zero entry in its row.<br><br>RREF presents an easy-to-read format for interpreting the resulting solutions for each variable. Since each 1 corresponds to a specific variable (x, y, z for e.g., in the above), and the constant term is defined for each in its corresponding final column.<br>A row containing no leading 1 is a free variable, meaning that |

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| | | it can take on any value, and the other variables in the system are expressed in terms of the free variable. |
| **Gaussian Algorithm** | Step 1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form.<br><br>Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it $a$), and move the row containing that entry to the top position.<br><br>Step 3. Now multiply the new top row by $1/a$ to create a leading 1.<br><br>Step 4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.<br><br>This completes the first row, and all further row operations are carried out on the remaining rows.<br><br>Step 5. Repeat steps 1–4 on the matrix consisting of the remaining rows.<br><br>The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros. | GA is used for systematically solving systems of linear equations (as per proven method), taking an augmented matrix directly to RREF. The procedure requires:<br><br>**i.** matrices containing only 0s are already in REF (do nothing)<br>**ii.** the first column from the left containing a non-zero entry has its value denoted as 'a', swap the row containing 'a' with the first row<br>**iii.** divide the entire top row by 'a' to turn the leading element into a 1<br>**iv.** use elementary row operations to make all entries below the leading 1 equal 0 by subtracting multiples of the top row from each row below it<br>**v.** excluding the first row, repeat steps i **- iv** on each resulting matrix<br><br>This results in a system equal to the original, which is easy-to-read and can be interpreted to determine a single, infinitely many, or no solutions at all. |
| **Gaussian Elimination** | To solve a system of linear equations proceed as follows:<br><br>1. Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations.<br><br>2. If a row $[\ 0\ \ 0\ \ 0\ \ \cdots\ 0\ \ 1\ ]$ occurs, the system is inconsistent.<br><br>3. Otherwise, assign the nonleading variables (if any) as parameters, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters. | The algorithm is used to solve a system of linear equations by simplifying the matrix and determining if there's an existing solution. The Gaussian algorithm is applied to bring an augmented matrix to RREF, which allows the relationships to be viewed more compactly.<br><br>The Gaussian Algorithm shows the steps required for RREF reduction, and the elimination process encapsulates the elimination outcome for a system using GA. The algorithm provides the recipe while the elimination process involves using the recipe to achieve the desired outcome. |
| **Definition 1.4** | The rank of matrix A is the number of leading 1s in | The rank of a matrix is |

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| Rank of a matrix | any row-echelon matrix to which A can be carried by row operations. | determined by the number of leading 1s in the REF matrix. This determines the number of linearly independent rows/columns. Since row ops don't change the linear independence of each row, if there are linearly-dependant rows, the REF matrix will contain some rows of zeros. Each row pivot is linearly independent, and the number of pivots determines the matrix rank.<br><br>**Properties:**<br>**i.** Matrix rank is always less or equal to floor(rows, cols)<br>**ii.** Matrix rank is equal to number of linearly independent columns<br>**iii.** Row ops preserve matrix rank<br>**iv.** Matrix rank is equal to the number of matrix column space<br><br>This gives us important insight when it comes to interpreting system solutions. |
| **Theorem 1.2.2**<br>Rank and solutions to a system of linear equations | Suppose a system of *m* equations in *n* variables is consistent, and that the rank of the augmented matrix is *r*.<br><br>1. The set of solutions involves exactly $n - r$ parameters.<br><br>2. If $r < n$, the system has infinitely many solutions.<br><br>3. If $r = n$, the system has a unique solution. | Rank and solutions provide the basis for interpreting the results of solving a system of linear equations. If the system has at least one solution, then it's considered to be consistent.<br><br>Considering a scenario where a system of 'm' equations in 'n' variables is consistent, and the rank of the augmented matrix is r = 1, the set of solutions involves exactly n - r params. In other words, the number of free variables equals the difference between matrix variables total, and matrix rank:<br><br>**i.** When the rank of the cofficient matrix (LHS) is less than the rank of the augmented matrix, there's no solution<br>**ii.** When the rank equals the number of variables, there's a unique solution<br>**iii.** When the rank is less than the number of variables, there |

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| | | are infinitely many solutions<br><br>The system solution nature is therefore determined by the rank of the matrix. |

## Homogeneous Equations

| Textbook Ref | Theorem/Definition statement | Notes |
|---|---|---|
| **Homogeneous system of equations** | A system of equations in the variables $x_1, x_2, \dots, x_n$ is called homogeneous if all the constant terms are zero—that is, if each equation of the system has the form<br><br>$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$ | A system of linear equations is considered homogeneous if each equation's constant term is 0:<br><br>**i.** Homogeneous systems always have at least one solution, the trivial solution<br>**ii.** The trivial solution, where all variables are 0s, is always a solution to a homogeneous system<br>**iii.** Infinitely many solutions are typical for homogeneous systems, expressed with free variables<br>**iv.** Any linear combination of the solution to a homogeneous system is also a solution (it will still solve each equation) |
| **Trivial/non-trivial solutions** | Homogeneous systems always have the solution $x_1 = x_2 = \ldots = x_n = 0$, which is called the trivial solution. Any solution where one variables has a nonzero value is called a nontrivial solution. | The trivial solution contains 0s for each variable, as well as the constant term. This is always a solution to a homogeneous system. Any solution containing at least a single non-zero variable is considered non-trivial. This distinction determines whether variables for one equation in the system are linearly-dependant, and are expressed as a linear combination of the variables. |
| **Definition 1.5** Basic Solutions | The gaussian algorithm systematically produces solutions to any homogeneous linear system, called basic solutions, one for every parameter.<br><br>E.g., where solutions to a system of equations can be | Basic solutions arise from applying the Gaussian Algorithm to a homogeneous linear system. This results in a single particular solution, or a set of solutions with free |

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| | expressed as $s\mathbf{x} + t\mathbf{y}$, then we say $s$ and $t$ are parameters and $\mathbf{x}$ and $\mathbf{y}$ are basic solutions. | variables as params. These free variables can take on any value. When expressing basic solutions with params (free vars), the form is referred to as parametric, or the general solution to the system. This provides clear, readable insight to the structure of the solution. |
| **Theorem 1.3.2** Rank and homogeneous systems | Let A be an $m \times n$ matrix of rank $r$, and consider the homogeneous system in $n$ variables with A as coefficient matrix. Then:<br>1. The system has exactly $n-r$ basic solutions, one for each parameter.<br>2. Every solution is a linear combination of these basic solutions. | For a homogeneous m * n system with r rank and n - r basic solutions, every solution to the homogeneous system can be expressed as a linear combination of the n - r basic solutions. Each basic solution can be scaled by a constant and added together to obtain this.<br><br>The rank of a homogeneous system can be used to determine the existence and nature of a solution, and provides insight to the system characteristics. |

## Summarising your understanding:

The goals of this module revolve around solving systems of linear equations and interpreting their results based on properties like rank and solutions. The goals for this module were:

1) Apply elementary row operations to systems of linear equations represented in matrix form
2) Apply the Gaussian Algorithm to reduce a matrix to row-echelon form
3) Express the general solution to a homogeneous equation as a linear combination of sets of basic solutions

The module was geared towards solving problems with multiple variables and known scalars, revealing the fundamental relationships between each. Given the prerequisite unit topics like algebraic rules, functions, sets, graph curves, etc., this module required a slight paradigm shift to focus on linear relationships in matrix form with solution vectors. For this, it was helpful to think of Gaussian Elimination as a function of sorts, to transform an augmented matrix to RREF. Each operation covered is aimed toward reducing the system to a clear, easily-interpreted form that's equal to the original system.

### Provide examples of performing each of the three elementary row operations and then summarise the different strategies you have learnt for performing Gaussian elimination – when are the different strategies useful?

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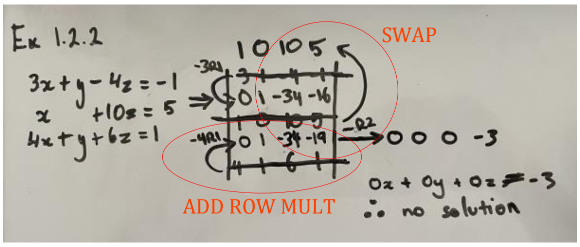
We explored elementary row operations, which provided the stepping-stones for reducing the matrix to REF and RREF. Elementary row operations each have strategic use-cases when reducing a matrix:

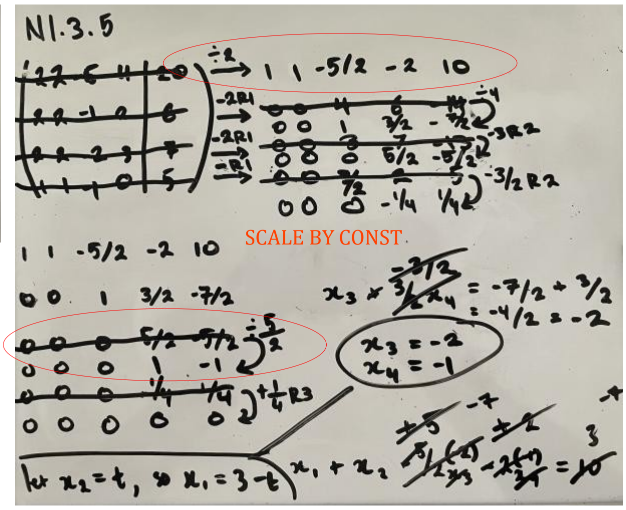
SWAP ROWS | Most useful for getting a non-zero value as the pivot without introducing fractions, good starting point.

MULTIPLY BY SCALAR | Perfect for simplifying and normalising rows, reducing

MULTIPLY BY SCALAR OF ANOTHER ROW | Go-to for eliminating entries below the pivot, very important for Gaussian elimination

Learning to use the fundamental row operations helped build my intuition for the rules and processes of reducing matrices.




These problems highlight the usefulness of each row operation in context. Even though division is not strictly a row operation, it is the same as multiplying a row by constant $1/k$, so it's row scalar multiplication by a constant. Dividing by another fraction is also an example of scalar multiplication, e.g. R / (5/2) = R * 2/5. Addition of a multiple of another row can be expressed as subtraction, since $R_i - xR_j = R_i + (-x)R_j$.

**What is the advantage of Gaussian elimination over a substitution type of approach when you have systems with a large number of variables?**

Gaussian elimination provides a compact method for reducing systems of linear equations to a form that's easy to interpret and solve. It relies on strategically applying row operations to manipulate the augmented matrix, bringing the system to RREF. A substitution approach involves solving for one variable at a time, substituting the values back into each equation, and solving for the next variable. This is rather intensive for larger systems with lots of variables. Since substitution requires solving variables in relation to each other, this can lead to a large stack of sub-problems, each with multi-interdependence. As the number of variables grows, mapping their relationships becomes increasingly complex, making the process longer and less efficient to solve. Gaussian elimination determines a clear starting point for this process and systematically reduces the problem to a point where the relationships are clear and easy to interpret, relying less on reiterating relationships between vars and instead on following a recipe with proven steps.

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**Comment on the relationship between Theorem 1.2.2 and Theorem 1.3.2 (you can consider differences, when each applies etc)**

Additional matrix properties, like rank, give convenient insight for interpreting the final solution for a problem:

NO SOLUTION **|** If the rank of the coefficient matrix (LHS) is less than the rank of the augmented matrix, the system is inconsistent

UNIQUE SOLUTION **|** If the rank of the coefficient matrix equals the rank of the augmented matrix, there exists one solution (set)

INFINITELY MANY SOLUTIONS **|** If the rank of the coefficient matrix is less than the rank of the augmented matrix, there are free variables that can take on any value

Theorem 1.2.2 outlines the role of rank in determining the number of free variables for a system, specifically in cases where a solution exists. Theorem 1.3.2 looks at how this applies to homogeneous systems. For 1.2.2, we unpack a general overview that's applicable to any consistent system, determining the nature of the solution based on rank. When dealing with homogeneous systems, 1.3.2 states that the n – r basic solutions still applies, and that every solution is a linear combination of the n – r basic solutions. These two provide a comprehensive rule-set for system properties (homogeneous and non-homogeneous), based on r rank and n variables.

The module also unpacked Homogeneous systems, where all constant terms are 0 and the trivial solution is always present. The trivial solution is a special case of the general solution, where all variables are 0s. Though trivial, it is always a valid solution. These systems often contain infinitely many solutions, giving rise to free variables of arbitrary values. Any linear combination of the basic solutions is also a solution, so adding multiples of a solution to itself will still satisfy the system (it will still read true).

Summarily, this module provided a comprehensive starting-point for solving linear relationships in systems with multiple unknown variables. By arranging a system into a matrix and reducing the matrix to RREF (through Gaussian elimination), system properties are clearer and more easily interpreted to determine the nature of the solution. Homogeneous systems extend upon this by providing a trivial solution and, generally, infinitely many solutions, which can be expressed as linear combinations of basic solutions.

## Reflecting on the content:

This chapter introduces some interesting mechanisms that draw on a lot of prerequisite information to implement successfully, across the few maths units I've done and my computer-science-dominant background. Being the second time completing this module (previously completed in T2 2024) I was excited to put my preparation to work and start my module assessment from scratch. Matrices are easy to come to terms with as a concept, methodically manipulating them without breaking the system is a rabbit-hole. I have past experience with matrix multiplication and sets, so despite being quite an intensive exercise, the algorithm is satisfying for tying together lots of other concepts like distributive laws and other linear algebraic rules for solution proofing. I get a kick out of filling up my whiteboard with my understanding of how the recipe goes.

Gaussian elimination is a great recipe for performing the necessary reductions in this module, and the technique has some pretty profound implications from a data-science perspective (not just in terms of convenience, it's computationally efficient). So far we've mainly worked with smaller data sets and systems involving <6 variables, so moving forward, reinforcing this technique will be important if we start working with significantly larger systems. I really like how succinctly GA simplifies multi-variable systems. I look forward to drawing more bridges between analysis and homogeneous systems to come to more specific, nuanced conclusions about system behaviour and characteristics.

This module has really helped me solidify a workflow of resources that I can trust for quickly diagnosing specifically what is expected and where I might have diverged if my result isn't aligned with the result-reference. I take a slower, more surgical approach, preparing a mental map of steps I should take for the given problem before building my solution, like how I would go about programming. The more emphasis I put on the intended idea before taking the technical dive, the less overwhelmed I get from unpacking deeply nested problems. Conclusively, I am glad I've dedicated the amount of time I have to chapter 1 of the unit text to make sure my foundation is airtight. I'm ready to push forward and mix these mechanisms with other concepts.

## References

### Module notes, walkthroughs

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### Verification, walkthroughs

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**Calculations, fraction arithmetic**

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