Module 1 Summary

Response to feedback

If you are resubmitting, include a statement outlining the changes you have made to your submission. This section can be short but should be precise. It is a good idea to quote the feedback you are responding to.

If this is your first submission, include a statement about what part of the lesson review you would most like to receive feedback (and why). Your tutor will take this into consideration when reviewing your work, although they may choose to give you feedback on a different thing if they think it's more appropriate.

Module Learning Objectives

I certify that I achieved the following learning objectives for the module:

- Apply elementary row operations to systems of linear equations represented in matrix form
- Apply the Gaussian Algorithm to reduce a matrix to row-echelon form
- Express the general solution to a homogeneous equation as a linear combination of sets of basic solutions

Key Definitions and Theorems

Solutions and Elementary Operations

Textbook Ref	Theorem/Definition statement	Learning Notes
Linear equations	An equation of the form $$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$ is called a <u>linear equation</u> in the n variables x_1, x_2, … , x_n. With a_1, a_2, … , a_n referred to as coefficients and b as the constant term (or sometimes 'right-hand side').	These are foundational in linear algebra, describing relationships between
System of linear equations	A finite collection of linear equations defined with respect to the same variables is called a <u>system of linear equations</u>.	
Solutions	A set of values s_1, s_2, … , s_n is a <u>solution</u> to a linear equation if setting $x_1=s_1$, $x_2 = s_2$, … , $x_n = s_n$ allows for the equation to hold (the left-hand side is equal to the right-hand side, b).	

	A solution to a system of equations requires the solution to hold for all equations in the system.	
Augmented matrix	A system of equations can be represented as an array of numbers consisting of the coefficients and constant terms, e.g., the system $$3x_1+2x_2-x_3=3$$ $$-x_1+4x_3=-6$$ Could be represented as the matrix, $$\left[\begin{array}{ccc\|c} 3 & 2 & -1 & 3 \\ -1 & 0 & 4 & -6 \end{array}\right]$$ Which will later be denoted [A | **b**] where A is the array of coefficients and **b** is the column of constants on the right-hand side.	
Definition 1.2 Elementary row operations	The following operations, called <u>elementary row operations</u> on a matrix. I. Interchange two rows. II. Multiply one row by a nonzero number. III. Add a multiple of one row to a different row	

Gaussian Elimination

Textbook Ref	Theorem/Definition statement	Notes
Definition 1.3 Row-Echelon Form (Reduced)	A matrix is said to be in <u>row-echelon form</u> (and will be called a row-echelon matrix) if it satisfies the following three conditions: 1. All zero rows (consisting entirely of zeros) are at the bottom. 2. The first nonzero entry from the left in each nonzero row is a 1, called the <u>leading 1</u> for that row. 3. Each leading 1 is to the right of all leading 1s in the rows above it. A row-echelon matrix is said to be in <u>reduced row-echelon form</u> (and will be called a reduced row-echelon matrix) if, in addition, it satisfies the following condition: 4. Each leading 1 is the only nonzero entry in its column.	
Gaussian Algorithm	Step 1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form. Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it *a*), and move the row containing that entry to the top	

	position. Step 3. Now multiply the new top row by $1/a$ to create a leading 1. Step 4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero. This completes the first row, and all further row operations are carried out on the remaining rows. Step 5. Repeat steps 1–4 on the matrix consisting of the remaining rows. The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.	
Gaussian Elimination	To solve a system of linear equations proceed as follows: 1. Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations. 2. If a row $[\,0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1\,]$ occurs, the system is inconsistent. 3. Otherwise, assign the nonleading variables (if any) as parameters, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters.	
Definition 1.4 Rank of a matrix	The rank of matrix A is the number of leading 1s in any row-echelon matrix to which A can be carried by row operations.	
Theorem 1.2.2 Rank and solutions to a system of linear equations	Suppose a system of m equations in n variables is consistent, and that the rank of the augmented matrix is r. 1. The set of solutions involves exactly $n - r$ parameters. 2. If $r < n$, the system has infinitely many solutions. 3. If $r = n$, the system has a unique solution.	

Homogeneous Equations

Textbook Ref	Theorem/Definition statement	Notes
Homogeneous system of equations	A system of equations in the variables $x_1, x_2, \dots, x_n$ is called <u>homogeneous</u> if all the constant terms are zero—that is, if each equation of the system has the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$	

Trivial/non-trivial solutions	Homogeneous systems always have the solution $x_1 = x_2 = \ldots = x_n = 0$, which is called the trivial solution. Any solution where one variables has a nonzero value is called a nontrivial solution.	
Definition 1.5 Basic Solutions	The gaussian algorithm systematically produces solutions to any homogeneous linear system, called basic solutions, one for every parameter. E.g., where solutions to a system of equations can be expressed as $s\mathbf{x} + t\mathbf{y}$, then we say s and t are parameters and $\mathbf{x}$ and $\mathbf{y}$ are basic solutions.	
Theorem 1.3.2 Rank and homogeneous systems	Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then: 1. The system has exactly $n-r$ basic solutions, one for each parameter. 2. Every solution is a linear combination of these basic solutions.	

Summarising your understanding:

- Add notes to the definitions and theorems tables that reflect your understanding of the definition/theorem, how it relates to the learning objectives and things to keep in mind when applying – these notes can be somewhat informal but should be understandable to your OnTrack tutor.

Write a mathematical summary for the module that includes the questions/tasks in the dot points below. You should provide reference to the relevant definitions, theorems and terms in the tables above to help demonstrate your understanding of how the concepts are linked and applied. These should be referred to by name and textbook label (if there is one), e.g., **Theorem 1.3.2** or **Definition 1.4.**

- Provide examples of performing each of the three elementary row operations and then summarise the different strategies you have learnt for performing Gaussian elimination – when are the different strategies useful?

- What is the advantage of Gaussian elimination over a substitution type of approach when you have systems with a large number of variables?

- Comment on the relationship between Theorem 1.2.2 and Theorem 1.3.2 (you can consider differences, when each applies etc)

-

Reflecting on the content:

- Reflect on what you found interesting/difficult in this module, how it relates to previous content covered in this and other units, and which parts helped you to meet each of the learning objectives. This should be written from your personal perspective and not just read as a general summary of the topic.